PSEUDOFREE FINITE GROUP ACTIONS ON 4-MANIFOLDS

by

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ABSTRACT

We prove several theorems about the pseudofree, locally linear and homologically trivial action of finite groups G on closed, connected, oriented 4-manifolds M with non-zero Euler characteristic. In this setting, the $\text{rank}_p(G) \leq 1$, for $p \geq 5$ prime and $\text{rank}(G) \leq 2$, for $p = 2, 3$ (by [19]).

1. If a non-trivial finite group G acts on M in the above way, then $b_1(M) = 0$, and if $b_2(M) \geq 3$, then G must be cyclic and acts semi-freely.

2. If $b_2(M) = 2$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then M must have the same 2-local homology and intersection form as $S^2 \times S^2$. If $G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$, q odd, is non-abelian, then M must have the same q-local homology as $S^2 \times S^2$.

3. If $b_2(M) = 1$ and $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, then M must have the same 3-local homology and intersection form as $\mathbb{CP}^2$. If $G = \mathbb{Z}_q \rtimes \mathbb{Z}_3$, q odd, $3 \nmid q$ is non-abelian, then M must have the same q-local homology as $\mathbb{CP}^2$.

4. If $b_2(M) = 0$ and $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{2^r}$, q odd, $r \geq 2$ is non-abelian, then M must have the same q-local homology as S^4.

We combine these results into two main theorems: Theorem A and Theorem B in Chapter 1. These results strengthen the work done by Edmonds [10], and Hambleton and Pamuk [19]. We remark that for $b_2(M) \leq 2$ there are other examples of finite groups which can act in the above way (see [33] and section 2.4).

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CONTENTS

CHAPTER 1

INTRODUCTION

The goal of this chapter is to provide a brief survey on the pseudofree, locally-linear and homologically trivial actions of finite groups G on closed, connected and oriented topological 4-manifolds, and an outline of our current work. First, we provide the following definitions.

Definition 1.1 Consider a group action $G \times M \to M$. The ***fixed point set*** of a subgroup $H \leq G$ is denoted by $M^H = \{x \in M \mid gx = x, \forall g \in H\}$. The subset $\Sigma = \bigcup_{H \neq \{e\},\, H \leq G} M^H$ is called the ***singular set*** of G.

Definition 1.2 An action is called ***pseudofree*** if the singular set is discrete or equivalently, if it is free on the complement of a discrete set. An action is called ***homologically trivial*** if the induced action on all the homology groups is identity.

Definition 1.3 Let $G_x = \{g \in G | gx = x\}$ be the isotropy subgroup of the G-action. An action is called ***locally linear*** [4][24] if for every point $x \in M$ there exists a G_x-invariant neighbourhood V_x such that V_x is homeomorphic to $\mathbb{R}^n$ and is an orthogonal G_x-space.

Definition 1.4 The ***p*-rank**, of a finite group G, denoted by $\mathrm{rank}_p(G)$ is defined as the maximum rank r of an elementary abelian p-group $(\mathbb{Z}_p)^r \leq G$. The rank of G is defined as $\mathrm{rank}(G) = \max \mathrm{rank}_p(G)$ over all the primes p.

Definition 1.5 Let $p \in \mathbb{Z}$ be a prime. The ***localization of*** $\mathbb{Z}$ ***at the prime ideal*** (p) is defined by

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b \right\}.$$

We say $H_*(M; \mathbb{Z}_{(p)})$ to be the ***p*-local homology** of M.

1.1 BRIEF LITERATURE REVIEW

By the Lefshetz fixed point theorem the only group which acts freely on a 4-dimensional sphere is $\mathbb{Z}_2$. Smith [36], Milnor [34] and Madsen and Wall [27] classified the finite groups which can act on an n-dimensional sphere. We provide a brief discussion on this in Chapter

2. Orlik and Raymond [35] showed that torus actions exist on the four manifolds of the form

$$(\mathbb{CP}^2 \# \ldots \# \mathbb{CP}^2) \# (\overline{\mathbb{CP}}^2 \# \ldots \# \overline{\mathbb{CP}}^2) \# (S^2 \times S^2 \# \ldots \# S^2 \times S^2)$$

and these are the only simply-connected four manifolds which have such actions.

On $\mathbb{CP}^2$, Hambleton and Lee [17] showed that if a finite group acts locally linearly and homologically trivially, then it must be a subgroup of $PGL_3(\mathbb{C})$. Wilczynski [41] obtained a similar result independently. Later in 1995, Hambleton and Lee [18] extended this result to the connected sum of $\mathbb{CP}^2$. They showed that any finite group acting homologically trivially on $\#_i^n \mathbb{CP}^2$ must be of abelian rank ≤ 2. McCooey [32] strengthened the results of [18] to a much larger class of 4-manifolds by studying the symmetries of the singular set. He showed that if G acts effectively, locally linearly and homologically trivially on M with $b_2 \geq 3$, then G must be a subgroup of the torus group, $S^1 \times S^1$.

Edmonds [9][10] showed that if a finite abelian group G acts locally linearly and homologically trivially on a simply connected 4-manifold with either $b_2 \geq 3$, or $b_2 = 2$, but the intersection form is diagonalizable, then the rank of G is at most 2, and G has global fixed points. If $G = \mathbb{Z}_p \times \mathbb{Z}_p$ then there are only $b_2 + 2$ fixed points.

McCooey [31] showed that simply connected four manifolds which admit locally linear and homologically trivial $\mathbb{Z}_p \times \mathbb{Z}_p$ actions are homeomorphic to $S^2 \times S^2$ and connected sums of $\mathbb{CP}^2$, which generalizes the results from the paper by Orlik and Raymond. Fintushel [14] and Yoshida [43] concluded that for the locally linear and homologically trivial S^1-action, the same simply-connected 4-manifolds occur, as in [31]. Subsequently, McCooey [30] also showed that a locally linear, orientation-preserving action of $\mathbb{Z}_p \times \mathbb{Z}_p$ on S^4 is concordant if and only if a certain Kervaire-Arf invariant vanishes.

Cappell and Shaneson [6] studied pseudofree actions of cyclic group actions on spheres. Pseudofree actions are particularly interesting. Groups which act freely, act pseudofreely by suspension. Kwasik and Schultz [25] studied cyclic group actions on S^4 and showed that many of the results in Cappell and Shaneson's work do not extend to the 4-dimensional case.

In 1992, Edmonds and Ewing [12] showed that if a finite cyclic group of prime order acts locally linearly and pseudofreely on a closed, oriented, simply connected 4-manifold, then there are two invariants of the action: the fixed point data and the intersection form. Later in 1998, Edmonds[10] proved that the finite groups which act pseudofreely, locally linearly and homologically trivially on a simply-connected 4-manifold M with $b_2(M) \geq 3$ are the cyclic groups and the action is semifree. We improve this result without assuming the manifolds to be simply-connected.

McCooey [33] gave a complete classification of the groups that act pseudofreely on

$S^2 \times S^2$. The groups are polyhedral groups, cyclic groups $\mathbb{Z}_n$, dihedral groups D_n and $\mathbb{Z}_2 \times \mathbb{Z}_2$, which act locally linearly and homologically trivially as well. The linear pseudofree action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on $\mathbb{CP}^2$ arising from its automorphism group $PGL_3(\mathbb{C})$ is studied in [19] and discussed briefly in Chapter 2 of this thesis. Edmonds [11] and Hambleton [15] independently showed that the finite Dihedral group $D_k = \langle a, b | a^p = b^2 = 1, bab^{-1} = a^{-1} \rangle$ cannot act pseudofreely and locally linearly on S^4, preserving orientation. In particular, Edmonds showed that such actions cannot exist on S^{4k}, while Hambleton proved it for S^{2k}. This result answered the remaining question asked by Kulkarni [23].

Almost all the above work on pseudofree actions is concentrated on simply-connected 4-manifolds. Hambleton and S. Pamuk [19] studied pseudofree actions on 4-manifolds not assuming simply-connectedness.. They showed that if a finite group G acts pseudofreely, locally linearly and homologically trivially on a closed, oriented, connected topological 4-manifold M with non-zero Euler characteristic, then $\text{rank}_p(G) \leq 1$ for $p \geq 5$ and $\text{rank}_p(G) \leq 2$ for $p = 2, 3$.

§ Motivation and Results

Our work in this dissertation was primarily influenced by the paper of Hambleton and Pamuk [19]. Following their results we considered the actions of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ and rank-1 groups on the 4-manifolds without assuming simply-connectedness. In all the cases, we got some restrictions on the Betti-numbers of the 4-manifolds. We summarize the results into two theorems below.

Theorem A: *Let G be a non-trivial finite group acting pseudofreely, locally linearly and homologically trivially on a closed, connected, oriented 4-manifold M with non-zero Euler characteristic. Then $b_1(M) = 0$, and if $b_2(M) \geq 3$, then G must be cyclic and acts semi-freely.*

Theorem B: *Let G be a non-trivial finite group acting pseudofreely, locally linearly and homologically trivially on a closed, connected, oriented 4-manifold M with non-zero Euler characteristic such that $b_2(M) \leq 2$. Then we have the following.*

1. *If $b_2(M) = 2$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then M must have the same 2-local homology and the intersection form as $S^2 \times S^2$. If $G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$, q odd is Dihedral, then M must have the same $q-$local cohomology as $S^2 \times S^2$.*

2. *If $b_2(M) = 1$ and $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, then M must have the same 3-local homology and the intersection form as $\mathbb{CP}^2$. If $G = \mathbb{Z}_q \rtimes \mathbb{Z}_3$, q odd, $3 \nmid q$ is non-abelian, then M must have the same $q-$local cohomology as $\mathbb{CP}^2$.*

3. *If* $b_2(M) = 0$ *and* $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{2^r}$*,* q *odd,* $r \geq 2$*, is non-abelian, then* M *must have the same* $q-$*local homology as* S^4.

§ Ingredients of the dissertation

In general, our work uses the following ingredients:

1. The geometric consequences of pseudofree, locally linear and homologically trivial actions. For example, the Euler characteristic of M has to be positive with some specific values.

2. Calculation of minimal subgroups of certain ranks in order to rule out actions.

3. The Borel construction and comparing the integral-Borel spectral sequence of the manifold to the singular set to get some restrictions.

1.2 OUTLINE

In chapter 2, we discuss basics of group cohomology, Borel Cohomology ($H^*_G(M)$) and Borel spectral sequence. We provide a detailed computation of integral cohomology ring of $\mathbb{Z}_p \times \mathbb{Z}_p$ and its restriction maps. We review the proof of [10, Proposition 2.1] which says that $H^q_G(M) \cong H^q_G(\Sigma)$ for $q \geq 5$, where $\Sigma \subset M$ is the singular set of the G-action. We use this result later to analyze dimensions in the Borel spectral sequence. In the following section, we discuss some known pseudofree actions $S^2 \times S^2$ and $\mathbb{CP}^2$.

In chapter 3, we prove an important proposition (Proposition 3.1) regarding the $\mathbb{Z}_p$-pseudofree action. This result, though not the same, is analogous to a result of Hambleton and Pamuk [19, Proposition 5.1] with mod-p coefficients. Our result is particularly useful since we work with the integral Borel spectral sequence.

In chapter 4, we begin dealing with the pseudofree actions of rank two finite groups. First, we establish some results for the action of $\mathbb{Z}_p \times \mathbb{Z}_p$, for p prime. We introduce an idea of divisibility to deal with the integral Borel spectral sequence. We show that there are restrictions on the Betti-numbers of the manifold. In fact, for most cases the action is ruled out. In case such an action exist, the homology of the manifold either behaves like $S^2 \times S^2$ or $\mathbb{CP}^2$. We also show that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ pseudofree, locally linear and homological trivially action does not exist on $\mathbb{CP}^2 \# \mathbb{CP}^2$.

Chapter 5 deals with the groups of rank one. Using the results from the finite group theory, we restrict to a minimal collection of groups of rank 1, which are given by quaternionic groups, metacyclic groups and cyclic groups. We use coefficients in localized ring $\mathbb{Z}_{(p)}$ in this chapter. We provide slightly modified versions of two propositions by Edmonds [10]. We show that Q_8 cannot act pseudofreely, locally linearly and homologically trivially on manifolds with $b_2 \geq 1$. We compute the group cohomology of the metacyclic groups and show that these groups cannot act on manifolds with $b_2 \geq 3$.

In chapter 6, we prove Theorem A and Theorem B by combining the results of the previous chapters. Theorem A says that if G acts pseudofreely, locally linearly and homologically trivially on M, then in some localized ring coefficients the first homology is zero and if $b_2(M) \geq 3$, then G must be cyclic and acts semi-freely. This result is a generalization of Edmonds [10, Main Theorem]. However, we do not assume the manifold to be simply-connected.

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CHAPTER 2

PRELIMINARIES

In this chapter we summarize some of the background information needed for the thesis.

2.1 COHOMOLOGY OF FINITE GROUPS

The group cohomology can be described in two equivalent ways: algebraically using resolutions [5], or topologically using Eilenberg-MacLane space [5, 2]. In this section, we will provide the topological definition.

Definition 2.1 (Eilenberg-MacLane space) Let G be a group. A CW-complex X is called an Eilenberg-Maclane space of type $K(G,1)$ [5] for a group G, if $\pi_1(X) = G$ and the universal cover of X is contractible.

From the long exact sequence of homotopy groups, this is equivalent to

$$\pi_k(X) \cong \begin{cases} G, & \text{for } k = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.2 $S^1 \cong K(\mathbb{Z},1)$; $S^1 \times S^1 \cong K(\mathbb{Z}\times\mathbb{Z},1)$; $\mathbb{RP}^\infty \cong K(\mathbb{Z}_2,1)$; $L_p^\infty \cong K(\mathbb{Z}_p,1)$, where $L_p^\infty = S^\infty/\mathbb{Z}_p$ is the infinite lens space.

Example 2.3 (Classifying Space of a discrete group) Let G be a discrete group and EG be a free, contractible G space. The orbit space, BG, of the universal principal G-bundle

$$G \hookrightarrow EG \twoheadrightarrow EG/G = BG \tag{2.1}$$

is called the classifying space of G [2].

To see that BG is a $K(G,1)$ space, we use the long exact sequence of the homotopy groups from the covering space (2.1)

$$\to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \cdots \to \pi_1(EG) \to \pi_1(BG) \to \pi_0(G) \to 0.$$

Since G is discrete and EG is contractible, we have $\pi_1(BG) \cong \pi_0(G) = G$ and $\pi_k(BG) = 0$, otherwise.

We will define the Bockstein homomorphism [21] below. This will be used later in the section to compute group cohomology.

Definition 2.4 (Bockstein Homomorphism) Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_p \to 0$. For any space X, we get the long exact sequence

$$\to H^k(X;\mathbb{Z}) \xrightarrow{\times p} H^k(X;\mathbb{Z}) \xrightarrow{j^*} H^k(X;\mathbb{Z}_p) \xrightarrow{\Delta} H^{k+1}(X;\mathbb{Z}) \to .$$

The connecting homomorphism $\Delta : H^k(X;\mathbb{Z}_p) \to H^{k+1}(X;\mathbb{Z})$ is called the Bockstein homomorphism.
Similarly, from the exact sequence $0 \to \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_{p^2} \xrightarrow{j} \mathbb{Z}_p \to 0$. We get the Bockstein homomorphism $\Delta_p : H^k(X;\mathbb{Z}_p) \to H^{k+1}(X;\mathbb{Z}_p)$.

Lemma 2.5 *The following relation holds:* $\Delta_p = j^* \circ \Delta$ *and* $\Delta_p^2 = 0$.

Proof. This can be seen by the long exact sequence above and the induced map on cohomology by the reduction $j : \mathbb{Z} \to \mathbb{Z}_p$. From the commutative diagram below

$$\begin{array}{ccccccc} \xrightarrow{\times p} H^k(X,\mathbb{Z}) & \xrightarrow{j^*} & H^k(X,\mathbb{Z}_p) & \xrightarrow{\Delta} & H^{k+1}(X;\mathbb{Z}) & \xrightarrow{\times p} \\ & & & \searrow^{\Delta_p} & \downarrow j^* & \\ & & & & H^{k+1}(X,\mathbb{Z}_p) & \end{array} \tag{2.2}$$

we get the desired relation.

Now, $\Delta_p^2 = j^* \circ \Delta \circ j^* \circ \Delta$. Since $\Delta \circ j^* = 0$ from the long exact sequence, we get $\Delta_p^2 = 0$. ∎

We are ready to define the topological version of group cohomology.

Definition 2.6 (Group Cohomology) Let G be a discrete group and R be a G-module. The group cohomology of G with coefficients in R, denoted by $H^*(G;R)$ is defined as [1]

$$H^*(G;R) := H^*(K(G,1);\mathcal{R}) = H^*(BG;\mathcal{R}),$$

where $\mathcal{R}$ is the local coefficients on BG associated to the G-module R. Note that, a cohomology group with local coefficients arises when the fundamental group, $\pi_1(BG) = G$ acts on the coefficient module R [13, Chapter V].

Remark 2.7 If G is not a discrete group, then BG may not be a $K(G,1)$-space. In that case, the definition of group cohomology becomes $H^*(G;R) = H^*(K(G,1);\mathcal{R})$.

§ Group Cohomology of $\mathbb{Z}_p \times \mathbb{Z}_p$, p prime

We will be focusing on computing the group cohomology with integral coefficients in details which will be useful in Chapter 4. In the following examples, we assume that G acts trivially on the coefficient module, unless mentioned otherwise.

Example 2.8 1. Let $G = \mathbb{Z}_2$. Now, from the universal covering projection

$$\mathbb{Z}_2 \longrightarrow S^\infty \longrightarrow S^\infty/\mathbb{Z}_2 \cong \mathbb{RP}^\infty,$$

we have $B\mathbb{Z}_2 = \mathbb{RP}^\infty$. So, from the definition of the group cohomology we have

$$H^*(G;\mathbb{Z}) = H^*(\mathbb{RP}^\infty, \mathbb{Z}) = \mathbb{Z}[u]/(2u), \tag{2.3}$$

where $|u| = 2$, the degree of the element u.

For $\mathbb{Z}_2$-coefficients we have

$$H^*(G;\mathbb{Z}_2) = H^*(\mathbb{RP}^\infty, \mathbb{Z}_2) = \mathbb{Z}_2[x], \tag{2.4}$$

where $|x| = 1$ and the cup product is given by the Bockstein homomorphism $\Delta_2(x) = x^2$.

2. Let $G = \mathbb{Z}_p$, p odd prime. Similar to the previous example, from the universal covering projection, we have $BG = S^\infty/\mathbb{Z}_p \cong L_p^\infty$, the infinite dimensional Lens space. Therefore, we have

$$H^*(G;\mathbb{Z}) = H^*(L_p^\infty, \mathbb{Z}) = \mathbb{Z}[u]/(pu), \tag{2.5}$$

where $|u| = 2$.

For the $\mathbb{Z}_p$ coefficients, we have

$$H^*(G;\mathbb{Z}_p) = H^*(L_p^\infty, \mathbb{Z}_p) = \mathbb{Z}_p[y] \otimes \Lambda(x), \tag{2.6}$$

where Λ is the exterior algebra over $\mathbb{Z}_p$, $|y| = 2$, $|x| = 1$, $x^2 = 0$, $\Delta_p(x) = y$ and $\Delta_p(y) = \Delta_p^2(x) = 0$.

Note. The cohomologies of $\mathbb{RP}^\infty$ and L_p^∞ can be found in [21, Thm 3.19, Exm 3.41, 3E.1, 3E.2].

Example 2.9 Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$. Using the Künneth formula, we get that $H^*(G;\mathbb{Z}_p) \simeq H^*(\mathbb{Z}_p;\mathbb{Z}_p) \otimes H^*(\mathbb{Z}_p;\mathbb{Z}_p)$.

Therefore, for $p = 2$, we have

$$H^*(\mathbb{Z}_2 \times \mathbb{Z}_2;\mathbb{Z}_2) = \mathbb{Z}_2[x_1,x_2], \tag{2.7}$$

where $|x_i| = 1$ and $\Delta_2(x_i) = x_i^2$.

For $p > 2$ prime, we have

$$H^*(\mathbb{Z}_p \times \mathbb{Z}_p;\mathbb{Z}_p) = \mathbb{Z}_p[y_1,y_2] \otimes \Lambda(x_1,x_2), \tag{2.8}$$

such that $|y_i| = 2$, $|x_i| = 1$, $x_i^2 = 0$ and $x_1x_2 = -x_2x_1$. The Bockstein relations are given by: $\Delta_p(x_i) = y_i$ and $\Delta_p(y_i) = \Delta_p^2(x_i) = 0$.

Example 2.10 Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, for p prime. Since $H^*(\mathbb{Z}_p;\mathbb{Z}) = \mathbb{Z}_p[u]$ in the positive dimensions, by generalized Künneth formula $H^*(G;\mathbb{Z})$ is of exponent p (smallest positive integer p such that $pg = 0$, for all $g \in H^*(G;\mathbb{Z})$). Therefore, the multiplication map $H^*(G;\mathbb{Z}) \xrightarrow{\times p} H^*(G;\mathbb{Z})$ is zero.

Hence, from the short exact sequence $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_p \to 0$, we get [26]

$$0 \longrightarrow H^k(G;\mathbb{Z}) \xrightarrow{j^*} H^k(G;\mathbb{Z}_p) \xrightarrow{\Delta} H^{k+1}(G;\mathbb{Z}) \longrightarrow 0, \tag{2.9}$$

where the Bockstein $\Delta_p : H^k(G;\mathbb{Z}_p) \to H^{k+1}(G;\mathbb{Z}_p)$ is given by $\Delta_p = \Delta \circ j^*$ as mentioned in lemma 2.5.

Now, from equation (2.9), we have $\operatorname{Im} j^* = \ker \Delta$ and j^* is injective. Therefore, we get $\ker \Delta_p = \ker \Delta = \operatorname{Im} j^* = H^k(G;\mathbb{Z})$.

1. For $p = 2$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, we have $H^*(G;\mathbb{Z}_2) = \mathbb{Z}_2[x_1,x_2]$ such that $|x_i| = 1$ and $\Delta_2(x_i) = x_i^2$, as mentioned in eq (2.7). By the derivation property of the Bockstein homomorphism, we have $\Delta_2(x_i^{2k+1}) = x_i^{2k+2}$ and $\Delta_2(x_i^{2k}) = 0$. We do some explicit computations of $\ker \Delta_2$ below.

   - For $k = 1$, $\ker \Delta_2 = 0$.
   - For $k = 2$, $\Delta_2 : x_i^2 \mapsto 0;\quad x_1x_2 \mapsto x_1x_2^2 + x_1^2x_2$. So, $\ker \Delta_2 = \langle x_1^2, x_2^2 \rangle$.
   - For $k = 3$, $\Delta_2 : x_i^3 \mapsto x_i^4;\ x_1^2x_2 + x_1x_2^2 \mapsto 0$. So, $\ker \Delta_2 = \langle x_1^2x_2 + x_1x_2^2 \rangle$.

Therefore, we see that $\ker \Delta_2$ is generated by x_1^2, x_2^2 and $x_1^2x_2 + x_1x_2^2$. So, we have the following ring isomorphism

$$H^*(\mathbb{Z}_2 \times \mathbb{Z}_2; \mathbb{Z}) \simeq \frac{\mathbb{Z}[u_1, u_2](\mu)}{\left(2u_1, 2u_2, 2\mu, \mu^2 = u_1^2u_2 + u_1u_2^2\right)}, \tag{2.10}$$

given by $j^*(u_i) = x_i^2$, $j^*(\mu) = x_1^2x_2 + x_1x_2^2 = x_1x_2(x_1 + x_2)$. We see that $|u_i| = 2$ and $|\mu| = 3$. Since $(x_1^2x_2 + x_1x_2^2)^2 = x_1^4x_2^2 + x_1^2x_2^4$, we have the relation $\mu^2 = u_1^2u_2 + u_1u_2^2 = u_1u_2(u_1 + u_2)$.

2. For $p \geq 3$ prime and $G = \mathbb{Z}_p \times \mathbb{Z}_p$, we have $H^*(G; \mathbb{Z}_p) = \mathbb{Z}_p[y_1, y_2] \otimes \Delta(x_1, x_2)$ such that $|y_i| = 2$, $|x_i| = 1$ and $x_i^2 = 0$ with the Bockstein $\Delta_p(x_i) = y_i$ and $\Delta_p(y_i) = 0$. Similar to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ calculation, we have

   - For $k = 1$, $\ker \Delta_p = 0$.
   - For $k = 2$, $\ker \Delta_p = \langle y_1, y_2 \rangle$.
   - For $k = 3$, $\ker \Delta_p = \langle x_1y_2 - x_2y_1 \rangle$.

   This gives an ring isomorphism

$$H^*(\mathbb{Z}_p \times \mathbb{Z}_p; \mathbb{Z}) \cong \frac{\mathbb{Z}[u_1, u_2](\mu)}{(2u_1, 2u_2, 2\mu, \mu^2)}, \tag{2.11}$$

   given by $j^*(u_i) = y_i$, $j^*(\mu) = x_1y_2 - x_2y_1$, where $|u_i| = 2$ and $|\mu| = 3$. Using $x_1x_2 = -x_2x_1$, we have $(x_1y_2 - x_2y_1)^2 = 0$. Therefore, we have $\mu^2 = 0$.

2.1.1 Restriction Maps

Let $K < G$ be a subgroup. Then the inclusion map $i : K \hookrightarrow G$ induces a map on the classifying spaces $BK \to BG$ [8, Theorem 8.22]. This induces a map on the cohomologies $H^*(BG; R) \to H^*(BK; R)$. Using the definition of the group cohomology, therefore, we have

$$\mathrm{Res}_K^G : H^*(G; R) \to H^*(K; R).$$

The map Res_K^G is called the *restriction map from G to the subgroup K*.

§ Restriction maps of $H^*(\mathbb{Z}_p \times \mathbb{Z}_p)$

In this section, we are interested in computing the restriction maps $\mathrm{Res}_K^G : H^*(G; \mathbb{Z}) \to H^*(K; \mathbb{Z})$, where $G = \mathbb{Z}_p \times \mathbb{Z}_p$, p odd and $K \cong \mathbb{Z}_p \leq G$. It suffices to compute these maps for $* = 2, 3$. The presentation of G is given by $G = \langle a, b \mid a^p = 1 = b^p, ab = ba \rangle$. There are $(p+1)$-cyclic subgroups given by $\mathcal{K} = \{a\} \cup \{a^i b \mid 1 \le i \le p\}$, where a, b are the generators of G.

Since $H^3(K;\mathbb{Z}) = 0$, it is clear that the restriction map $\mathrm{Res}^G_K : H^3(G;\mathbb{Z}) \to H^3(K;\mathbb{Z})$ is given by $\mathrm{Res}^G_K(\mu) = 0$, for $\mu \in H^3(G;\mathbb{Z})$.

To compute the restriction maps for $* = 2$, we note that, by the universal coefficient theorem $H^1(G;\mathbb{Z}_p) \simeq \mathrm{Hom}\,(G, \mathbb{Z}_p)$ and $H^1(K;\mathbb{Z}_p) \simeq \mathrm{Hom}\,(K, \mathbb{Z}_p)$ for $K < G$. Now, from the computation of $H^*(G;\mathbb{Z}_p)$, we see that $x_1 \in H^1(G;\mathbb{Z}_p)$ is the generator of $\mathrm{Hom}\,(\langle a\rangle, \mathbb{Z}_p)$ and $x_2 \in H^1(G;\mathbb{Z}_p)$ is the generator of $\mathrm{Hom}\,(\langle b\rangle, \mathbb{Z}_p)$.

From the exact sequence $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{j} \mathbb{Z}_p$, we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(G;\mathbb{Z}_p) & \xrightarrow{\Delta} & H^2(G;\mathbb{Z}) & \xrightarrow{\times p} & 0 \\ & & \downarrow{\scriptstyle \mathrm{Res}^G_K} & & \downarrow{\scriptstyle \mathrm{Res}^G_K} & & \\ 0 & \longrightarrow & H^1(K;\mathbb{Z}_p) & \xrightarrow[\Delta]{} & H^2(K;\mathbb{Z}) & \xrightarrow{\times p} & 0, \end{array} \tag{2.12}$$

where $\Delta(x_i) = u_i$ and $\Delta(x) = u$, for $x \in H^1(K, \mathbb{Z}_p)$ and $u \in H^2(K, \mathbb{Z}_p)$ with $K \cong \mathbb{Z}_p < G$.

For $K \cong \langle a\rangle$, we see that, the left vertical map $\mathrm{Res}^G_K(x_1, x_2) = (x_1, 0)$ since $x_1 \in H^1(G;\mathbb{Z}_p)$ is the generator of $\mathrm{Hom}\,(\langle a\rangle, \mathbb{Z}_p)$. Therefore, from the commutative diagram, we have $\mathrm{Res}^G_K(u_1, u_2) = (u, 0)$.

For $K \cong \langle b\rangle$, similarly we have $\mathrm{Res}^G_K(u_1, u_2) = (0, u)$.

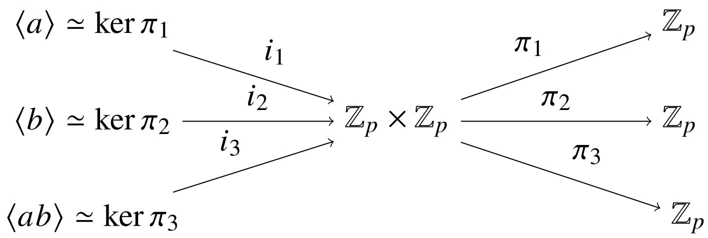
For $K \cong \langle a^i b\rangle$, for $1 \le i \le p-1$, we see that $ix_1 + x_2$ is a generator of $\mathrm{Hom}\,(\langle a^i b\rangle, \mathbb{Z}_p)$. Hence, we have $\mathrm{Res}^G_K(x_1, x_2) = (ix_1, x_2)$. Therefore, from the commutative diagram we have $\mathrm{Res}^G_K(u_1, u_2) = (iu, u)$, for $1 \le i \le p-1$.

The following table lists the restriction maps for $H^*(\mathbb{Z}_p \times \mathbb{Z}_p;\mathbb{Z})$ for the classes u_1, u_2 and μ in the appropriate dimensions:

Table 2.1: Restriction maps of $H^*(\mathbb{Z}_p \times \mathbb{Z}_p;\mathbb{Z})$

$\mathbb{Z}_p \times \mathbb{Z}_p$	$\langle a\rangle$	$\langle b\rangle$	$\langle ab\rangle$	$\cdots$	$\langle a^{p-1}b\rangle$
u_1	u	0	u		$(p-1)u$
u_2	0	u	u		u
μ	0	0	0		0

Remark 2.11 Alternatively, one can compute the restriction maps in the following way. Consider the diagram below. Here we only consider three subgroups.



From the figure above, we get the following relations:

$$\pi_1 i_1 = 1, \quad \pi_2 i_1 = \text{id}, \quad \pi_3 i_1 = \text{id}$$

$$\pi_1 i_2 = \text{id}, \quad \pi_2 i_2 = 1, \quad \pi_3 i_2 = \text{id}$$

$$\pi_1 i_3 = \text{id}, \quad \pi_2 i_3 = \text{id}, \quad \pi_3 i_3 = 1.$$

Therefore, the restriction maps with $u_1 = \pi_1^*(u)$ and $u_2 = \pi_2^*(u)$ are as follows

$$i_1^*(u_1) = 0, \quad i_1^*(u_2) = u, \quad i_1^*(u_1 + u_2) = u$$

$$i_2^*(u_1) = u, \quad i_2^*(u_2) = 0, \quad i_2^*(u_1 + u_2) = u$$

$$i_3^*(u_1) = u, \quad i_3^*(u_2) = u, \quad i_3^*(u_1 + u_2) = u + u = 2u.$$

Therefore, from the above equations, we see that the restriction maps to the subgroups $\langle a \rangle$, $\langle b \rangle$ and $\langle ab \rangle$ are given by $(u_1, u_2) \to (0, u)$, $(u_1, u_2) \to (u, 0)$ and $(u_1, u_2) \to (u, u)$, respectively.

Now, restriction maps to the subgroups $\langle a^i b \rangle$, are therefore given by $(u_1, u_2) \to (iu, u)$, for $1 \le i \le p - 1$.

So, the restriction maps $\text{Res}_K^G : H^2(G) \to H^2(K)$, to the $(p + 1)$ cyclic subgroups $K \simeq \mathbb{Z}_p$ are given by:

$$\text{Res}_K^G : \begin{array}{l} (u_1, u_2) \mapsto (0, u) \\ (u_1, u_2) \mapsto (u, 0) \\ (u_1, u_2) \mapsto (u, u) \\ \vdots \\ (u_1, u_2) \mapsto ((p-1)u, u). \end{array} \tag{2.13}$$

2.1.2 Shapiro's Lemma

Definition 2.12 Let G be a finite group and X be a G-set. Then one can form the free abelian group $\mathbb{Z}X$ generated by X by extending the action of G on X to a $\mathbb{Z}$-linear action of G on $\mathbb{Z}X$ ($\sum_{x\in X}\lambda_x x \mapsto \sum_{x\in X}\lambda_x(gx)$). The resulting G-module $\mathbb{Z}X$ is called a *permutation module* [5, I.3].

Example For every subgroup $H \leq G$, $\mathbb{Z}[G/H]$ is a permutation module, where G acts on the cosets by left translation.

Let $H \leq G$ and M be an H-module. Then one can form an induced module

$$\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

and a coinduced module

$$\mathrm{Coind}_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M).$$

The induced module is completely characterized by $\mathrm{Ind}_H^G M = \bigoplus_{g\in G/H} gM$ [5, Proposition III.5.1]. Taking $M = \mathbb{Z}$, therefore, we have $\mathbb{Z}[G/H] \cong \mathrm{Ind}_H^G\mathbb{Z}$.

If G is finite, or more precisely if $(G : H)$ is finite, then $\mathrm{Ind}_H^G M \cong \mathrm{Coind}_H^G M$ [5, Proposition III.5.9].

Lemma 2.13 (Shapiro's lemma [5, III.6.2]) *Let $H \leq G$ and M is an H-module, then*

$$H_*(H; M) \cong H_*(G; \mathrm{Ind}_H^G M)$$

and

$$H^*(H; M) \cong H^*(G; \mathrm{Coind}_H^G M).$$

If G is finite and $M = \mathbb{Z}$, we have

$$H^*(H;\mathbb{Z}) \cong H^*(G;\mathrm{Coind}_H^G \mathbb{Z}) \cong H^*(G;\mathrm{Ind}_H^G \mathbb{Z}) \cong H^*(G;\mathbb{Z}[G/H]).$$

This will be important for computing dimension bounds in the upcoming chapters.

2.2 FREE ACTIONS ON SPHERES

In this section, we briefly discuss the groups which act freely on spheres.

Theorem 2.14 (Smith [36]) *If a finite group G acts freely on a sphere, then all of its abelian subgroups are cyclic.*

Therefore, if a finite group acts freely on a sphere, then all of its subgroup of order p^2 (for p any prime) must be cyclic. So, Smith's result says that any group containing $\mathbb{Z}_p \times \mathbb{Z}_p$ can not act freely on a sphere. This is called "Smith's p^2 condition".

By [7, Ch XVI, Section 9], if G acts freely on S^n then G has periodic cohomology with period $n + 1$. We shall give a brief proof here following [1].

If G acts freely on S^n, then we have a Gysin sequence

$$\to H^{i+n}(S^n/G) \to H^i(G) \xrightarrow{\smile x} H^{i+n+1}(G) \to H^{i+n+1}(S^n/G) \to$$

where x is the Euler class. Since $H^*(S^n/G) = 0$, for $* > n$, $H^i(G) \xrightarrow{\cong} H^{i+n+1}(G)$ is an isomorphism, for all $i > 0$. This also shows that $\mathbb{Z}_p \times \mathbb{Z}_p$ can not act freely on S^n, since it doesn't have periodic cohomology.

Milnor in [34, Theorem 1] shows that given a 2-periodic map $T : S^n \to S^n$ without fixed points, for every map $f : S^n \to S^n$ of odd degree, there exists a point $x \in S^n$ such that $Tf(x) = f(x)T$. As a corollary to this, we have

Theorem 2.15 (Milnor [34]) *If a finite group G acts freely on an S^n (or manifolds having the mod 2-homology of S^n), then any element of order 2 in G belongs to the centre.*

This says that every subgroup of order $2p$ (for p prime) is cyclic ("the $2p$-condition"). Therefore, the dihedral group D_{2p} can not act freely (topologically) on S^n.

Madsen, Thomas and Wall combined both the statements above and gave sufficient conditions for the free finite group actions on S^n.

Theorem 2.16 (Madsen-Thomas-Wall [27]) *A finite group G acts freely on S^n if and only if G satisfies both the p^2 and $2p$ conditions for all primes p.*

Therefore, the classification of groups G with either the p^2 condition or the $2p$ condition, provides candidates for groups acting freely on S^n. Now, by Artin and Tate [7, Chap XII, Sec 11, Theorem 11.6], the following statements are equivalent (i) a finite group G has periodic cohomology, (ii) every abelian subgroup of G is cyclic and (iii) every Sylow p-subgroup of G is either quaternion ($p = 2$) or cyclic (p odd).

§ Free actions on S^3

In the context of our thesis, it is useful to analyze free action of groups on S^3.

1. From the above discussions, we have seen that $\mathbb{Z}_p \times \mathbb{Z}_p$ can not act freely on S^3. Therefore, groups with rank ≥ 2 can not act freely on S^3.

2. From theorem 2.16 and the results by Artin and Tate above, we see that the quaternions ($Q_{2^k} \subset SU(2) = S^3$, for $k \geq 3$) act freely on S^3. The presentation of the quaternion group is given by $Q_{4k} = \langle a, b | a^k = b^2, a^{2k} = 1, ab = ba^{-1} \rangle$.

3. The metacyclic groups (see Section 5.3) $\mathbb{Z}_q \rtimes \mathbb{Z}_{2^r}$, q odd, $r \geq 2$, have 4-periodic cohomology. Therefore, these groups act freely on S^3. The action factors through $\mathbb{Z}_2$.

Interested readers are encouraged to read the survey paper by Hambleton [16] and [4, Chapter III, Section 8] for more details regarding the free actions on spheres.

2.3 BOREL CONSTRUCTION

Let G be a finite group. As discussed in Chapter 1, the orbit space BG of the principal G-bundle $G \to EG \xrightarrow{\pi} EG/G = BG$ is the classifying space of G. The group G acts freely on EG and EG is contractible.

Let M be a closed, connected and oriented topological G-manifold. Denote $M_G = M \times_G EG = (M \times EG)/G$.

2.3.1 Borel Cohomology

Consider the associated bundle to $EG \to BG$ with fibre M as following

$$M \xrightarrow{p} M \times_G EG \longrightarrow BG \tag{2.14}$$

where $p(m) = (m, b)$ is map inclusion by fibre and $(m, b) \mapsto \pi(b) \in BG$. The total space M_G is called the homotopy quotient. The cohomology $H^*_G(M) := H^*(M_G)$ is called the Borel cohomology of M. Given a closed invariant subspace $A \subset M$, the relative cohomology is defined analogously $H^*_G(M, A) = H^*(M_G, A_G)$. We have the following exact sequence

$$\to H^k_G(M, A) \to H^k_G(M) \to H^k_G(A) \to H^{k+1}_G(M) \to .$$

Given $g \in H^*(BG)$ and $m \in H^*_G(M)$, we define $gm = \pi^*(g) \smile m$. This makes $H^*_G(M)$ a module over $H^*(BG)$. With these definitions, H^*_G is a cohomology theory with usual properties. For more details, the readers are encouraged to read [39, Chapter III, Section 1].

2.3.2 Borel Spectral Sequence

The Leray-Serre spectral sequence of the Borel fibration $M \to M_G \to BG$ is called the Borel Spectral sequence. We adapt the Serre spectral sequence theorem [22, Theorem

5.15] for the Borel Spectral sequence as follows

Theorem 2.17 *Given the Borel fibration $M \to M_G \to BG$ and the trivial action of $\pi_1(BG) = G$ on $H^*(M)$, there is a cohomological spectral sequence $\{E_r^{*,*}, d_r\}$ with the E_2-page*

$$E_2^{k,l} = H^k(BG; H^l(M)) \equiv H^k(G; H^l(M)) \implies H^{k+l}(M_G) \equiv H_G^{k+l}(M)$$

converging to $H_G^{k+l}(M)$ such that

1. $d_r^{k,l} : E_r^{k,l} \to E_r^{k+r,l-r+1}$ *and* $E_{r+1}^{k,l} = \ker d_r^{k,l} / \operatorname{Im} d_r^{k-r,l+r-1}$ *at* $E_r^{k,l}$,

2. *for $n \geq 0$, $H_G^n(M)$ admits a decreasing filtration*

$$0 = F^{n+1,-1} \subset F^{n,0} \subset F^{n-1,1} \subset \cdots \subset F^{1,n-1} \subset F^{0,n} = H_G^n(M)$$

   where the stable terms are given by

$$E_\infty^{p,q} \cong F^{p,q} / F^{p+1,q-1}.$$

Remark 2.18 If $\pi_1(BG) = G$, does not act trivially on $H^*(M)$, we have to consider the local coefficients when computing the E_2-terms.

Remark 2.19 The terms E_∞^* are called the associated graded modules of $H_G^*(M)$. From the theorem above, we see that if we are working with vector spaces (or if the cohomologies are taken with coefficients in fields), by counting dimensions, we have an isomorphism

$$H_G^n(M) \cong \bigoplus_{k+l=n} E_\infty^{p,q}.$$

For arbitrary module, we must deal with the extension problems.

We say a spectral sequence *collapses* on E_r-page if the differentials $d_r = 0$. In such a case, we have $E_\infty = E_r$. The goal of the spectral sequence is to find the E_∞ terms and if possible, recover $H_G^*(M)$.

Following figure shows the E_∞-terms along with the filtration of $H_G^*(M)$ for $* = 4$.

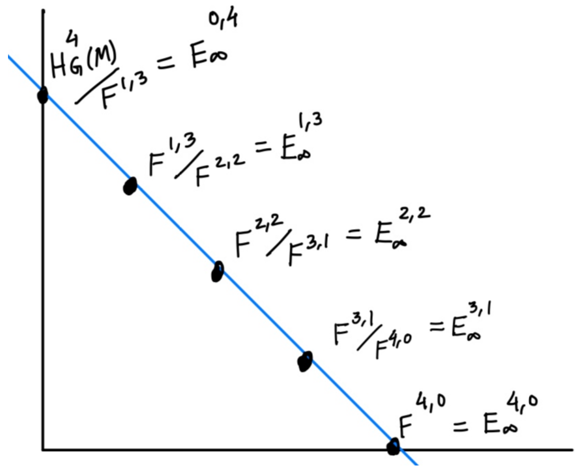


Figure 2.1: Diagonal entries of E_∞

§ Edge Homomorphisms

Considering $H^k(M) = E_2^{0,k}$, we have the following inclusions from the spectral sequence

$$E_\infty^{0,k} = E_r^{0,k} = \ker d_{r-1}^{0,k} \subseteq E_{r-1}^{0,k} \subseteq \cdots \subseteq E_3^{0,k} \subseteq E_2^{0,k} = H^k(M). \tag{2.15}$$

From the filtration in theorem 2.17, we see that $H_G^k(M) \twoheadrightarrow H_G^k(M)/F^{1,k-1} = E_\infty^{0,k}$. Therefore, we have

$$H_G^k(M) \twoheadrightarrow E_\infty^{0,k} \hookrightarrow E_2^{0,k}.$$

The above composition induced by $M \xrightarrow{p} M_G$ is called the edge homomorphism. We will use this later in Chapter 4.

There is another edge homomorphism on the x-axis. Note that, we have the following sequence of projections

$$H^k(G) = E_2^{k,0} \twoheadrightarrow \operatorname{coker} d_2^{k,0} = E_3^{k,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{k,0} \subseteq H_G^k(M).$$

From theorem 2.17, we see the bottom of the filtration is $E_\infty^{k,0}$. Therefore, we have

$$H^k(G) \twoheadrightarrow E_\infty^{k,0} \hookrightarrow H_G^k(M).$$

The composition, induced by $M_G \xrightarrow{\pi} BG$ is another edge homomorphism.

§ Multiplicativity and Derivation

Recall that, we have the standard cup product structure on

$$H^k(G;H^l(M)\times H^r(G;H^s(M)) \to H^{k+r}(G;H^{l+s}(M))$$

given by

$$(f \smile f')(\sigma) = f({}_k\sigma) \smile f'(\sigma_r),$$

where $\sigma : \Delta^{k+r} \to BG$ is a $(k+r)$-simplex; ${}_k\sigma$ is the front k-face of σ, defined by $\Delta^k \hookrightarrow \Delta^{k+r} \to BG$; σ_r is the back r-face of σ defined by $\Delta^r \hookrightarrow \Delta^{k+r} \to BG$; $f \in \mathrm{Hom}\,(C_k(BG), H^l(M))$ and $f' \in \mathrm{Hom}\,(C_r(BG), H^s(M))$ are cochains.

Therefore, this induces a cup product structure on the E_2-page of the spectral sequence. However, this cup product structure comes with a sign. The following theorem [37] describes the cup product structure.

Theorem 2.20 *The cup product structure on E_2-page $E_2^{k,l} \times E_2^{r,s} \to E_2^{k+r,l+s}$ is $(-1)^{lr}$ times the standard cup product structure described above.*

Furthermore, each differential d_r is a derivation, satisfying

$$d(xy) = d(x)y + (-1)^{k+l}xd(y),$$

for $x \in E_r^{k,l}$, therefore, inducing a cup product structure on E_{r+1}.

The product structure on E_∞ is the one induced from the products of E_r and coincides with the cup product structure $H_G^*(M) \times H_G^*(M) \to H_G^{*+*}(M)$.

Proof of the theorem above can be found in [22], [37] and [29].

§ Naturality

This is an important property of the spectral sequence. Suppose that two Borel fibrations $M \to M_G \to BG$ and $M' \to M'_{G'} \to BG'$, and a map f between them

$$\begin{array}{ccccc} M & \longrightarrow & M_G & \longrightarrow & BG \\ \downarrow & & \downarrow \tilde{f} & & \downarrow f \\ M' & \longrightarrow & M'_{G'} & \longrightarrow & BG' \end{array} \tag{2.16}$$

are given. Then we have the following

1. The induced maps f_r^* on E_r-pages commutes with the differentials as follows

$$\begin{array}{ccc} E_r^{k,l}(M) & \xrightarrow{f_r^*} & E_r^{k,l}(M') \\ \downarrow d_r^{k,l} & & \downarrow d_r^{k,l} \\ E_r^{k+r,l-r+1}(M) & \xrightarrow{f_r^*} & E_r^{k+r,l-r+1}(M'). \end{array} \tag{2.17}$$

2. The map $\tilde{f}^* : H_G^*(M) \to H_{G'}^*(M')$ preserves filtration, therefore, inducing a map on E_∞.

The proof of the naturality can be found in [22].

Now, we prove the following lemma as a consequence of Borel's Localization theorem [39, Theorem III.3.13].

Lemma 2.21 *Let $G = (\mathbb{Z}_p)^n$ for $p \geq 2$ and $n > 0$ and M be a 4-dimensional G-manifold. If the fixed set M^G is non empty, then image of any differential hitting the bottom horizontal line is zero in the Borel spectral sequence.*

Proof. As a corollary [19, Corollary 2.3], [39, Proposition III.3.14] to Borel's Localization theorem, we have $M^G \neq \emptyset$ if and only if $j^* : H_G^*(pt) \to H_G^*(M)$ is injective, where $j : M \to \{pt\}$.

Since the Borel spectral sequence of $\{pt\}$ collapses on E_2-page, we have $H_G^*(pt) = H^2(G)$. By naturality, we have the following diagram on the E_2-page

$$\begin{array}{ccc} 0 = E_2^{k,1}(pt) & \longrightarrow & E_2^{k,1}(M) \\ \downarrow 0 & & \downarrow d_2^{k,1} \\ H^2(G) = E_2^{k+2,0}(pt) & \overset{j^*}{\hookrightarrow} & E_2^{k+2,0}(M) = H^2(G). \end{array} \tag{2.18}$$

Suppose that $d_2^{k,1} \neq 0$. Then $E_3^{k+2,0}(M) \cong H^2(G)/\operatorname{Im} d_2^{k,1}$, for any $k > 0$. Therefore, j^* cannot be injective, a contradiction. Hence, $d_2^{k,1} = 0$. Similarly, we see that image of any higher order differential to the bottom line is zero. ■

Remark 2.22 As an application of spectral sequence, one can compute the cohomology of the total space. In chapter 5, we show the calculation of group cohomology of meta cyclic groups using spectral sequence. Cartan and Eilenberg [7] used spectral sequence to show that if G acts on sphere, then it has periodic group cohomology. Interested readers are encouraged to read [37], [29], [39], [7] and [22] for a detailed account in spectral sequence and its applications.

2.3.3 Edmonds' result

Definition 2.23 (Alexander-Spanier Cohomology) We briefly provide the definition of Alexander-Spanier cohomology. For a detailed account of this theory see [37].

- Let X be a topological space and P be an R-module. Let $C^q(X)$ be the module of all the functions from X^{q+1} to P. Defining a suitable coboundary δ makes $(C^q(X), \delta)$ a cochain complex.
- An element $\phi \in C^q(X)$ is called locally zero if there is a covering $\mathcal{U}$ of X such that ϕ vanished on any $(q+1)$-tuple of X^{q+1} which lies in some element of $\mathcal{U}$. The subset of $C^q(X)$ of all the locally zero functions is a submodule and is denoted by $C_0^q(X)$.
- Consider the cochain subcomplex $(C_q^0(X), \delta)$. Define $\overline{C}^*(X)$ to be the quotient cochain complex of $C^*(X)$ by $C_0^*(X)$. The Cohomology module of $\overline{C}^*(X)$ of degree q is denoted by $\overline{H}^q(X; P)$ and is called Alexander-Spanier cohomology.

The following is a variation of Vietoris-Begle theorem stated in [37, Theorem 6.9.15].

Theorem 2.24 (Vietoris-Begle Mapping theorem) *Let $f : X' \to X$ be a closed continuous surjective map between paracompact Hausdorff spaces. Assume that there is $n \geq 0$ such that the reduced cohomology $\tilde{\overline{H}}^q(f^{-1}(x); G) = 0$ for all $x \in X$ and $q < n$. Then $\overline{H}^q(X; G) \to \overline{H}^q(X'; G)$ is an isomorphism for $q < n$ and an injection for $q = n$.*

Now we state and briefly discuss the proof of the following result by Edmonds [10] [9]. A variant of the proof can be found in [32] as well.

Theorem 2.25 *Let M be a closed, connected and oriented G-manifold of dimension n. Let Σ be the singular set of the G-action. Then, we have the following isomorphism*

$$H_G^q(M) \cong H_G^q(\Sigma), \quad \textit{for } q > n.$$

Proof. From the projection $\tilde{j} : M \times EG \to M$, we have $j : M_G \to M/G \equiv M^*$, where $j([x, e]) = x^*$, for $x \in M$. Similarly, for the singular set $\Sigma \subset M$, we have $j : \Sigma_G \to \Sigma^*$. Since the equivalence relation is given by $(xg \sim ge)$, the fibre $j^{-1}(x^*) \simeq BG_x$, where G_x is the isotropy group.

Taking $x^* \in M^* - \Sigma^*$, we see that $j^{-1}(x^*)$ is contractible. Therefore, by the Vietoris-Begle mapping theorem, $\overline{H}_G^*(M - \Sigma) \cong \overline{H}^*(M^* - \Sigma^*)$. Now, for paracompact Hausdorff spaces the Alexander-Spanier cohomology and Singular cohomology are isomorphic by [37, Cor 6.9.5]. Therefore, we have $H_G^*(M - \Sigma) \cong H^*(M^* - \Sigma^*)$. By excision, we have $H_G^*(M, \Sigma) \cong H^*(M^*, \Sigma^*)$.

Now, from the long exact sequence of the pair (M, Σ), we have

$$\longrightarrow H_G^q(M,\Sigma) \longrightarrow H_G^q(M) \longrightarrow H_G^q(\Sigma) \longrightarrow H_G^{q+1}(M,\Sigma) \longrightarrow \qquad (2.19)$$

Since, $H_G^q(M, \Sigma) \cong H^q(M^*, \Sigma^*)$ for $q > n = \dim M$, we have the desired isomorphism $H_G^q(M) \cong H_G^q(\Sigma)$, for $q > \dim M$. ∎

2.4 EXAMPLES OF PSEUDOFREE ACTIONS

In this section, we discuss some examples of known pseudofree actions.

2.4.1 Pseudofree Actions on $S^2 \times S^2$

McCooey in [33] classified the groups that act pseudofreely, locally linearly and homologically trivially on $S^2 \times S^2$. McCooey classified the group extensions induced by a finite group action on $S^2 \times S^2$. Let G acts on $S^2 \times S^2$, which induces an action on $H_2(S^2 \times S^2)$. Therefore, we get a representation of G, $\phi : G \to GL(2, \mathbb{Z})$. Since ϕ must respect the intersection form of $S^2 \times S^2$, we have $\phi(g)$ of the form $\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Therefore,

$$\phi(g) \in \left\langle \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

This gives the following extension of G by K

$$1 \to K \to G \xrightarrow{\phi} Q \to 1,$$

where K acts trivially on homology and $Q \subset \mathbb{Z}_2 \times \mathbb{Z}_2$. McCooey used the above extension to classify the groups. For the linear case, we have the following exact sequence [33, Lemma 3.1]

$$1 \to \mathrm{SO}(3) \times \mathrm{SO}(3) \to W \xrightarrow{\phi} \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1,$$

where $W = \{A \in \mathrm{SO}(6) \mid A(S^2 \times S^2) = S^2 \times S^2\}$ is the group of linear actions on $S^2 \times S^2$. By McCooey [33, Lemma 3.1], $W \cong (\mathrm{SO}(3) \times \mathrm{SO}(3) \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$. The following is the main theorem of the paper, which classifies the groups acting on $S^2 \times S^2$. We only note the groups that acts pseudofreely, locally linearly and homologically trivially.

Theorem 2.26 ([33, Theorem 3.8]) *Let G acts pseudofreely, locally linearly and homologically trivially on $S^2 \times S^2$, then G is given by Tet $\cong A_4$, Oct $\cong S_4$, Ics $\cong A_5$, $\mathbb{Z}_n$, and D_n.*

The following is an example of a $\mathbb{Z}_3$ pseudofree action on $S^2 \times S^2$. We provide $\mathbb{Z}_2 \times \mathbb{Z}_2$ pseudofree action on $S^2 \times S^2$ in Chapter 4 (example 4.45).

Example 2.27 Let $G = \mathbb{Z}_3$ acts on $S^2 \times S^2$ by $g \cdot (x, y) = (gx, g^2 y)$, where g acts on S^2 by a rotation of angle $2\pi/3$ around $z-$axis. Therefore, the action has four isolated (north and south poles) as fixed points.

2.4.2 Pseudofree Action on $\mathbb{CP}^2$

The following well known example of $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on $\mathbb{CP}^2$ can be found in [19] and [17]. Let $S, T \in \mathbb{Z}_3 \times \mathbb{Z}_3$ be the generators such that

$$S(z_1, z_2, z_3) = (z_1, \omega z_2, \omega z_3)$$

and

$$T(z_1, z_2, z_3) = (z_2, z_3, z_1),$$

where ω is the cube root of unity.
The fixed point data is computed as follows.

- Let $T(z_1, z_2, z_3) = (z_2, z_3, z_1)$. Therefore, $z_2 = \lambda z_1$, $z_3 = \lambda z_2$ and $z_1 = \lambda z_3$, for $\lambda \in S^1$. Substituting, we get $z_2 = \lambda z_1$, $z_3 = \lambda^2 z_1$ and $z_1 = \lambda^3 z_1$. Therefore, $\lambda^3 = 1$. Hence, the fixed points for T are

$$(1, 1, 1); \quad (1, \omega, \omega^2); \quad (1, \omega^2, \omega).$$

- Similarly, for S, the fixed points are given by

$$(1, 0, 0); \quad (0, \omega, 0); \quad (0, 0, \omega^2).$$

- For ST, we see that

$$ST = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega \\ \omega^2 & 0 & 0 \end{pmatrix}.$$

The fixed points are given by

$$(1,1,\omega^2);\quad (1,\omega^2,1);\quad (\omega^2,1,1).$$

- Now, S^2T is given by

$$S^2T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \omega^2 \\ \omega & 0 & 0 \end{pmatrix}.$$

Therefore, the fixed points are

$$(1,1,\omega);\quad (\omega,1,1);\quad (1,\omega,1).$$

So, the singular set of the action consists of 12 points, arranged in 4 triangles each fixed by one of the 4 subgroups of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The fixed vertices of a triangle for a subgroup is rotated by the other subgroups in one orbit of size 3. This can be seen in the following example.

Consider the fixed set of T and the action of S. So, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}, \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}. \tag{2.20}$$

CHAPTER 3

AN IMPORTANT RESULT ABOUT $\mathbb{Z}_p$ ACTION

In this chapter we prove an analogous version of [19, Proposition 5.1] in the Borel spectral sequence with integral coefficients. This result will be extremely useful in the coming chapters. First, we define the Betti-numbers of a manifold below.

§ The Betti-numbers

Let $b_i(M)$ denote the integral or rational Betti-number of M which is defined by

$$b_i(M) = \dim_{\mathbb{Q}} H_i(M;\mathbb{Q}). \tag{3.1}$$

Note that, $b_3(M) = \dim H_3(M;\mathbb{Q}) = H^1(M;\mathbb{Q}) = b_1(M)$, and $b_0(M) = b_4(M) = 1$. Whenever it is clear, we will use b_i to denote the Betti-number of M.

We will prove the following proposition in this chapter.

Proposition 3.1 *Let $G = \mathbb{Z}_p$, for p prime, act locally-linearly, pseudofreely and homologically trivially on a closed, connected, orientable topological 4-manifold, M with non-zero Euler characteristic. Let F be the fixed set of G. Then in the Borel spectral sequence with integral coefficients,*

(i) the differentials, $d_2^{k,3}$ are injective for $k \geq 1$;

(ii) the reduced mod-p differential, $\bar{d}_2^{0,3} : E_2^{0,3} \otimes \mathbb{F}_p \to E_2^{2,2}$ is injective

(iii) the differentials, $d_2^{k,2}$ are surjective for $k \geq 0$, even and are 0 for $k \geq 1$ odd;

(iv) all the higher order differentials $d_r = 0$ for $r \geq 3$, and

(v) the Betti-numbers satisfy $b_2(M) \geq 2b_1(M)$, implying $\chi(M) > 0$.

3.1 BOREL COHOMOLOGY OF *M* WITH INTEGRAL COEFFICIENTS

Recall that, the integral cohomology of $G = \mathbb{Z}_p$ is given by

$$H^i(G;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i \geq 1,\ odd \\ \mathbb{Z}_p & i \geq 2,\ even. \end{cases} \quad \text{and} \quad H^*(G;\mathbb{Z}) = \mathbb{Z}[u],\ \text{with } pu = 0, |u| = 2. \tag{3.2}$$

To compute the Borel cohomology, we first specify some notations regarding the torsion subgroups as follows.

§ Torsion in $H_1(M)$ and $H^*(M)$

Since M is closed, the homology is finitely generated. Therefore, $H_i(M;\mathbb{Z}) = \mathbb{Z}^{b_i} \oplus T_i$, where $T_i = \text{Tors}\,(H_i(M))$ denote the torsion part of $H_i(M;\mathbb{Z})$. We define

$$t_1 := \dim_{\mathbb{F}_p} \left(T_1 \otimes \mathbb{F}_p\right), \tag{3.3}$$

where $T_1 = \text{Tors}\,(H_1(M;\mathbb{Z}))$.

By the Universal Coefficient theorem, we have $H^i(M;\mathbb{Z}) = \mathbb{Z}^{b_i} \oplus T_{i-1}$. Therefore, $H^1(M;\mathbb{Z})$ is torsion-free and $\text{Tors}\,(H^2(M;\mathbb{Z})) = T_1$. Now, by Poincaré duality, $\text{Tors}\,(H^3(M;\mathbb{Z})) = \text{Tors}\,(H_1(M;\mathbb{Z})) = T_1$. Therefore, we have the following

$$\dim_{\mathbb{F}_p} \left(H^l(M) \otimes \mathbb{F}_p\right) = \begin{cases} 1, & l = 0, 4 \\ b_1, & l = 1 \\ b_2 + t_1, & l = 2 \\ b_1 + t_1, & l = 3. \end{cases} \tag{3.4}$$

3.1.1 Computing dimensions of the E_2-page in the Borel Spectral Sequence

By the Künneth theorem, we have

$$H^k(G, H^l(M)) = H^k(G) \otimes H^l(M) \oplus \text{Tor}\,(H^{k+1}(G), H^l(M)). \tag{3.5}$$

Therefore, using the group cohomology of $G = \mathbb{Z}_p$ and the Künneth formula, we have

$$E_2^{k,l} := H^k(G, H^l(M)) = \begin{cases} H^l(M) & k = 0, \\ H^k(G) \otimes H^l(M) & k > 0, \text{ even} \\ \text{Tor}\left(H^{k+1}(G), H^l(M)\right) & k > 0, \text{ odd.} \end{cases} \tag{3.6}$$

Note that, $E_2^{k,l}$ is a $\mathbb{F}_p$-vector space except for $k = 0$.

Lemma 3.2 *For $k > 0$, $E_2^{k,l} \cong E_2^{k+2,l}$.*

Proof. For $k > 0$ even, the cup product $\smile u : H^k(G) \cong H^{k+2}(G)$ induces the isomorphism $\smile u \otimes \text{id} : H^k(G) \otimes H^l(M) \cong H^{k+2}(G) \otimes H^l(M)$.

Similarly, for k odd, we see that the isomorphism $\smile u : H^{k+1}(G) \cong H^{k+3}(G)$, induces an isomorphism $\text{Tor}\left(H^{k+1}(G), H^l(M)\right) \cong \text{Tor}\left(H^{k+3}(G), H^l(M)\right)$. ∎

§ Dimensions as $\mathbb{F}_p$-vector spaces

For $\boldsymbol{k > 0}$, **odd**, the dimensions of the Tor terms are given by

$$\begin{aligned} \dim_{\mathbb{F}_p}(\text{Tor}\,(H^{k+1}(G), H^l(M))) &= \dim_{\mathbb{F}_p} \text{Tor}\,(\mathbb{Z}_p, \mathbb{Z}^{b_l} \oplus T_{l-1}) \\ &= \dim_{\mathbb{F}_p} \text{Tor}\,(\mathbb{Z}_p, T_{l-1}) \\ &= \begin{cases} 0, & l = 0, 1, 4 \\ t_1, & l = 2, 3. \end{cases} \end{aligned} \tag{3.7}$$

For $\boldsymbol{k > 0}$, **even**, the $\mathbb{F}_p$-dimensions of $H^k(G) \otimes H^l(M)$ is given by

$$\begin{aligned} \dim_{\mathbb{F}_p}(H^k(G) \otimes H^l(M)) &= \dim_{\mathbb{F}_p}(\mathbb{Z}_p \otimes H^l(M)) \\ &= \begin{cases} 1, & l = 0, 4 \\ b_1, & l = 1 \\ b_2 + t_1, & l = 2 \\ b_1 + t_1, & l = 3. \end{cases} \end{aligned} \tag{3.8}$$

Remark 3.3 Since for $k = 0$, $E_2^{0,l}$ is not a vector space, we compute the dimension of $E_2^{0,l}$ after reduction mod-p. Therefore, we define

$$\dim_{\mathbb{F}_p} E_2^{0,l} := \dim_{\mathbb{F}_p}(H^l(M) \otimes \mathbb{F}_p). \tag{3.9}$$

Using (3.6), (3.7) and (3.8), we tabulate the dimensions of $E_2^{k,l}$ for $k \geq 0$ below

Table 3.1: *Dimension of $E_2^{k,l}$ as $\mathbb{F}_p$-vector space*

$E_2^{k,l}$	$k = 0$	$k > 0$, odd	$k > 0$, even
$E_2^{k,0}, E_2^{k,4}$	1	0	1
$E_2^{k,1}$	b_1	0	b_1
$E_2^{k,2}$	$b_2 + t_1$	t_1	$b_2 + t_1$
$E_2^{k,3}$	$b_1 + t_1$	t_1	$b_1 + t_1$

where the dimensions for $k = 0$ are treated as a $\mathbb{F}_p$-vector space as defined in the remark 3.3.

3.2 PROOF OF PROPOSITION 3.1

Proof. In this section, we prove proposition 3.1. We will consider the differentials in each page of the Borel spectral sequence.

3.2.1 Differentials in the E_2-page

§ 3A The differentials $d_2^{k,1} : E_2^{k,1} \to E_2^{k+2,0}$ are zero

The differentials $d_2^{k,1}$ are zero for $k \geq 1$ odd, since $E_2^{k,1} = 0$, for k odd.

For even $k \geq 0$, since the fixed set F is non-empty, by the result of Borel (lemma 2.21), the differentials hitting the zero horizontal line are zero.

For completeness, we will prove this here as well. Take $x \in F$ and consider the long-exact sequence of the pair $(M, \{x\})$, with $i \geq 0$

$$\to H^i(M, \{x\}\,;\mathbb{Z}) \xrightarrow{j^*} H^i(M;\mathbb{Z}) \xrightarrow{i^*} H^i(\{x\};\mathbb{Z}) \to H^{i+1}(M, \{x\}\,;\mathbb{Z}) \to,$$

induced from the maps $\{x\} \overset{i}{\hookrightarrow} M$ and $(M, \emptyset) \xrightarrow{j} (M, \{x\})$.

From the long exact sequence, we get $H^0(M, \{x\}) = 0$ and $H^i(M, \{x\}) \underset{\cong}{\xrightarrow{j^*}} H^i(M)$, isomorphism for $i \geq 1$.

Now, by naturality, we get the following commutative diagram:

$$\begin{array}{ccc} H^k(G, H^1(M,\{x\})) = E_2^{k,1}(M,\{x\}) & \xrightarrow{0} & E_2^{k+2,0}(M,\{x\}) = H^{k+2}(G, H^0(M,\{x\})) \\ \downarrow \cong & & \downarrow 0 \\ E_2^{k,1}(M) & \xrightarrow[d_2^{k,1}]{} & E_2^{k+2,0}(M). \end{array} \tag{3.10}$$

Since $H^0(M,\{x\}) = 0$, the top horizontal map and right vertical maps are zero. Now, since $H^1(M,\{x\}) \cong H^1(M)$, we have an isomorphism on the left vertical map. Therefore, from the diagram, we get $d_2^{k,1} = 0$, for $k \geq 0$.

§ 3B The differentials $d_2^{k,4} : E_2^{k,4} \to E_2^{k+2,3}$ are zero

For $k \geq 1$ odd, the differentials are zero, since $H^k(G) = 0$, for k odd.

For $k \geq 0$ even, consider the map of pairs $(M, \emptyset) \to (M, M - x)$, where $x \in F$ a fixed point of the action. This induces the long exact sequence of pairs

$$\to H^i(M, M - \{x\}\,;\mathbb{Z}) \xrightarrow{j^*} H^i(M\,;\mathbb{Z}) \xrightarrow{i^*} H^i(M - \{x\}\,;\mathbb{Z}) \to .$$

Now, using excision we get

$$H^i(M, M - x\,;\mathbb{Z}) \cong H^i(D^4, D^4 - 0) = \begin{cases} \mathbb{Z}, & i = 4 \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Now, since M is oriented, we have $H^4(M, M - \{x\}) \cong H^4(M)$. Therefore, from the long exact sequence, we get $H^4(M - \{x\}) = 0$ and $H^i(M - \{x\}) \cong H^i(M)$, for $i \leq 3$. By the naturality of the spectral sequence, we get the following commutative diagram between $H^*_G(M)$ and $H^*_G(M - x)$

$$\begin{array}{ccc} E_2^{k,4}(M) & \xrightarrow{d_2^{k,4}} & E_2^{k+2,3}(M) \\ \downarrow 0 & & \downarrow \cong \\ 0 = H^k(G, H^4(M - \{x\})) = E_2^{k,4}(M - x) & \xrightarrow[d_2^{k,4}=0]{} & E_2^{k+2,3}(M - x). \end{array} \tag{3.12}$$

Since $H^4(M - \{x\}) = 0$, the left vertical map is zero. The right vertical map is an isomorphism, because $H^3(M - \{x\}) \cong H^3(M)$. Therefore, from the commutativity of the diagram, the top horizontal map $d_2^{k,4} = 0$, for $k \geq 0$.

§ 3C The differentials $d_2^{k,2} : E_2^{k,2} \to E_2^{k+2,1}$ are zero, for $k \geq 1$ odd

For $k \geq 1$ odd, the maps $d_2^{k,2} = 0$, since $E_2^{\text{odd},1} = H^{\text{odd}}(G; H^1(M)) = 0$.

§ 3D The remaining differentials in the E_2-page

We denote the remaining differentials in E_2-page as follows

$$\operatorname{coker} d_2^{2k,2} = \mathcal{R}_{2k}; \quad \ker d_2^{k,3} = \mathcal{L}_k, \quad \text{for } k \geq 1. \tag{3.13}$$

The following lemma shows that the above kernels and cokernels are two-periodic. This will simplify the notation.

Lemma 3.4 *For $k \geq 1$, $\mathcal{R}_{2k} \cong \mathcal{R}_{2k+2}$, and $\mathcal{L}_k \cong \mathcal{L}_{k+2}$.*

Proof. We will show $\mathcal{R}_2 \cong \mathcal{R}_4$. The proof for the rest $\mathcal{R}_{2k} \cong \mathcal{R}_{2k+2}$, for $k \geq 2$ is same. Consider the following commutative diagram of the E_2-page of $H_G^*(M)$ for $G = \mathbb{Z}_p$, where the vertical sequences are short exact

$$\begin{array}{ccc} E_2^{2,2} & \xrightarrow[\cong]{\smile u} & E_2^{4,2} \\ \Big\downarrow {\scriptstyle d_2^{2,2}} & & \Big\downarrow {\scriptstyle d_2^{4,2}} \\ y \in E_2^{4,1} & \xrightarrow[\smile u]{\cong} & E_2^{6,1} \ni uy \\ \Big\downarrow & & \Big\downarrow \\ [y] \in \mathcal{R}_2 = \operatorname{coker} d_2^{2,2} & \xrightarrow[\smile u]{} & \operatorname{coker} d_2^{4,2} = \mathcal{R}_4 \ni [uy] \end{array} \tag{3.14}$$

In the diagram above, the cup product $\smile u$ is an isomorphism between the E_2-pages. This induces a map $\smile u : [y] \mapsto [uy]$ between the cokernels $\mathcal{R}_2$ and $\mathcal{R}_4$, making the bottom square commutative. The map can be easily checked to be well-defined. From the bottom square, it follows that it is a surjection. To show that it is one-to-one, take $[y] \mapsto 0$. Then by surjectivity $\exists uy$ such that $uy \mapsto [0]$. Therefore, there exists $\tilde{y} \in E_2^{4,2}$ mapping onto uy.

Now, by the top commutative square, we get $y \in \operatorname{Im} d_2^{2,2}$. Hence, $[y] = 0 \in \mathcal{R}_2$, showing that the bottom horizontal map is injective.

To show that $\mathcal{L}_k \cong \mathcal{L}_{k+2}$, for $k > 0$, we use the following commutative diagram

$$\begin{array}{ccc} \mathcal{L}_k = \ker d_2^{k,3} & \longrightarrow & \ker d_2^{k+2,3} = \mathcal{L}_{k+2} \\ \big\downarrow & & \big\downarrow \\ E_k^{k,3} & \xrightarrow{\cong} & E_2^{k+2,3} \\ \big\downarrow{\scriptstyle d_2^{k,3}} & & \big\downarrow{\scriptstyle d_2^{k+2,3}} \\ E_k^{k+2,2} & \xrightarrow{\cong} & E_2^{k+4,2}, \end{array} \tag{3.15}$$

where the two middle isomorphisms are induced by the cup product $\smile u$, as described in Lemma 3.2. Therefore, the top horizontal map is isomorphic. ■

Notation: Using the previous lemma, for $k \geq 1$, we denote

$$\begin{aligned} R_2 &:= \dim \operatorname{coker} d_2^{2,2} \cong \dim \mathcal{R}_{2k}, \\ L_1 &:= \dim \ker d_2^{1,3} \cong \dim \mathcal{L}_{2k-1}, \\ L_2 &:= \dim \ker d_2^{2,2} \cong \dim \mathcal{L}_{2k}. \end{aligned} \tag{3.16}$$

3.2.2 Differentials in E_3-page:

First, we prove the following lemma.

Lemma 3.5 *Let $x \in F$ be a fixed point of the action of $G = \mathbb{Z}_p$. Then*

(i) $E_3^{k,l}(M) \cong E_3^{k,l}(M, \{x\})$, if $d_2^{k,l}(M) = 0 = d_2^{k,l}(M, x)$, for some k and $l \geq 1$,

(ii) $E_3^{k,l}(M) \cong E_3^{k,l}(M - x)$, if $d_2^{k,l}(M) = 0 = d_2^{k,l}(M - x)$, for some k and $l \leq 3$.

Proof.

(i) Since $d_2^{k,l}(M) = 0 = d_2^{k,l}(M, \{x\})$, we get $E_3^{k,l} = E_2^{k,l}/\mathrm{Im}\, d_2^{k-2,l-1}$, for both M and $(M, \{x\})$. Now, consider the following commutative diagram of the exact sequences

$$\begin{array}{ccccccc}
\ker d_2^{k-2,l-1}(M,\{x\}) & \hookrightarrow & E_2^{k-2,l-1}(M,\{x\}) & \xrightarrow{d_2^{k-2,l-1}} & E_2^{k,l}(M,\{x\}) & \twoheadrightarrow & E_3^{k,l}(M,\{x\}) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \pi^* \\
\ker d_2^{k-2,l-1}(M) & \hookrightarrow & E_2^{k-2,l-1}(M) & \xrightarrow[d_2^{k-2,3}]{} & E_2^{k,l}(M) & \twoheadrightarrow & E_3^{k,l}(M)
\end{array} \tag{3.17}$$

Since $H^l(M) \cong H^l(M, \{x\})$, for $1 \leq l \leq 4$, the middle two vertical maps are isomorphisms. Therefore, the left vertical map on the kernels is an isomorphism. Hence, by the commutative diagram above the right vertical map $E_3^{k,l}(M, \{x\}) \cong E_3^{k,l}(M)$, is an isomorphism.

(ii) Since $H^l(M) \cong H^l(M - x)$, for $0 \leq l \leq 3$, we have

$$E_2^{k,l}(M) = H^k(G, H^l(M)) \cong H^k(G, H^l(M - x)) = E_2^{k,l}(M - x).$$

Therefore, in the following commutative diagram, the middle two vertical maps are isomorphisms, for $1 \leq l \leq 3$

$$\begin{array}{ccccccc}
\ker d_2^{k-2,l-1}(M) & \longrightarrow & E_2^{k-2,l-1} & \xrightarrow{d_2^{k-2,l-1}} & E_2^{k,l} & \longrightarrow & E_3^{k,l}(M) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow i^* \\
\ker d_2^{k-2,l-1}(M - \{x\}) & \longrightarrow & E_2^{k-2,l-1} & \xrightarrow[d_2^{k-2,l-1}]{} & E_2^{k,l} & \longrightarrow & E_3^{3,2}(M - \{x\}).
\end{array} \tag{3.18}$$

Therefore, arguing similarly, the right most vertical map can be seen to be an isomorphism.

This proves the assertion of the lemma. ■

§ 3E The differentials $d_3^{k,2} : E_3^{k,2} \to E_3^{k+3,0}$ are zero

For $k \geq 0$ **even**, we have $E_3^{k,0}(M) = E_2^{k,0}(M)$, since $d_2^{k,1} = 0$ from the previous section 3.2.1. Therefore, $E_3^{\mathrm{odd},0}(M) = 0$, since $H^{\mathrm{odd}}(G, \mathbb{Z}) = 0$. Hence, $d_3^{k,3} = 0$, for $k \geq 0$ even.

For $k \geq 1$ **odd**, we proceed as follows. Our aim is to show that $d_3^{k,2}$ is a zero differential.

We will compare the spectral sequences of M and $(M, \{x\})$ induced by the isomorphism $j^* : H^i(M, \{x\}) \to H^i(M)$ for $i \geq 1$, and $H^0(M, \{x\}) = 0$.

First, for the pair $(M, \{x\})$, we show the following

$$E_3^{k,2}(M, \{x\}) = E_2^{k,2}(M, \{x\})/\operatorname{Im} d_2^{k-2,3}(M, \{x\}), \text{for } k \geq 1 \text{ odd.} \tag{3.19}$$

Since $d_2^{k,2}(M) = 0$ for $k \geq 1$ odd (as $E_2^{\text{odd},\,1}(M) = 0$, see 3C), we have $E_3^{k,2} = E_2^{k,2}/\operatorname{Im} d_2^{k-2,3}$. Now, using the isomorphism $H^l(M, \{x\}) \to H^l(M)$, for $l \geq 1$, we get $E_2^{k,l}(M) \cong E_2^{k,l}(M, \{x\})$, for $l \geq 1$. Hence, $d_2^{k,2}(M, \{x\}) = 0$ for $k \geq 1$ odd, since $d_2^{k,2}(M) = 0$, proving equation (3.19). Therefore, by Lemma 3.5, $E_3^{k,2}(M) \cong E_3^{k,2}(M, \{x\})$, for $k \geq 1$ odd.

Now, by comparing the spectral sequences, we have the following commutative diagram

$$\begin{array}{ccc} E_3^{k,2}(M, \{x\}) = E_2^{k,2}(M, \{x\})/\operatorname{Im} d_2^{k-2,3}(M, \{x\}) & \xrightarrow{\ 0\ } & E_3^{k+3,0}(M, \{x\}) = 0 \\ \Big\downarrow{\scriptstyle \pi^*, \cong} & & \Big\downarrow{\scriptstyle 0} \\ E_3^{k,2}(M) = E_2^{k,2}(M)/\operatorname{Im} d_2^{k-2,3}(M) & \xrightarrow[d_3^{k,2}]{} & E_3^{k+3,0}(M). \end{array} \tag{3.20}$$

Now, $E_3^{q,0}(M,x) = E_2^{q,0}(M,x) = H^q(G, H^0(M,x)) = 0$, for any $q \geq 0$, since $H^0(M, \{x\}) = 0$. Therefore, the right vertical map and the top horizontal map of (3.20) are zero. As mentioned in the paragraph above, the left vertical map $E_3^{k,2}(M, \{x\}) \to E_3^{k,2}(M)$ is an isomorphism, by Lemma 3.5.

Hence, from the commutative diagram above the lower horizontal map $d_3^{k,2}(M) = 0, \forall k \geq 0$.

§ 3F The differentials $d_3^{k,4} : E_3^{k,4} \to E_3^{k+3,2}$ are zero

For $\boldsymbol{k \geq 1}$ **odd**, the maps $d_3^{k,4} = 0$, since $H^{\text{odd}}(G, H^4(M; \mathbb{Z})) = 0$.

For $\boldsymbol{k \geq 0}$ **even**, consider a fixed point $x \in F$. Since the spectral sequence is of period 2, it suffices to concentrate on $\boldsymbol{k = 0}$.

Note that, $E_3^{3,2}(M) = E_2^{3,2}(M)/\operatorname{Im} d_2^{1,3}(M)$, since $d_2^{3,2} = 0$ (as the codomain $E_2^{5,1} = 0$). Similarly, we also have $d_2^{3,2}(M-x) = 0$. Therefore, $E_3^{3,2}(M-x) = E_2^{3,2}(M-x)/\operatorname{Im} d_2^{1,3}(M-x)$.Hence, by Lemma 3.5, we have $E_3^{3,2}(M) \cong E_3^{3,2}(M-x)$.

Now, by naturality, we have the following commutative diagram of spectral sequences of M and $M - \{x\}$,

$$\begin{array}{ccc} E_3^{0,4}(M) & \xrightarrow{d_3^{0,4}} & E_2^{3,2}(M)/\mathrm{Im}\, d_2^{1,3}(M) = E_3^{3,2}(M) \\ \Big\downarrow 0 & & \Big\downarrow \cong \\ 0 = E_3^{0,4}(M - x) & \xrightarrow[0]{} & E_2^{3,2}(M)/\mathrm{Im}\, d_2^{1,3}(M - x) = E_3^{3,2}(M - x). \end{array} \tag{3.21}$$

Since $H^4(M - \{x\}) = 0$, $E_2^{0,4}(M - x) = 0$, hence $E_3^{0,4}(M - x) = 0$. Therefore, the left vertical map and the bottom horizontal map are zero. The isomorphism of the right vertical map, $E_3^{3,2}(M) \to E_3^{3,2}(M - x)$, comes from lemma 3.5. Hence, by the commutative diagram the top horizontal map, $d_3^{0,4}(M) = 0$.

As mentioned earlier, the spectral sequence is 2-periodic. Therefore, the argument above works for any $k > 0$ even, resulting $d_3^{k,4}(M) = 0$.

§ 3G The remaining differentials $d_3^{k,3} : E_3^{k,3} \to E_3^{k+3,1}$

For $k \geq 0$ **even**, the differentials $d_3^{k,3} = 0$, since $E_3^{\mathrm{odd},1} = 0$.

For k **odd**, let us assume $\dim \ker d_3^{k,3} = \mathcal{L}'_k$. We will show that $\mathcal{L}'_1 \cong \mathcal{L}'_3$. The argument for $\mathcal{L}'_k \cong \mathcal{L}'_{k+2}$, for $k \geq 1$ odd, is same.

Since $d_2^{k,4} = 0$, we have $E_3^{k,3} \cong \ker d_2^{k,3}$. Therefore, we get the following commutative diagram

$$\begin{array}{ccc} \mathcal{L}'_1 = \ker d_3^{1,3} & \longrightarrow & \ker d_2^{3,3} = \mathcal{L}'_3 \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{L}_1 = \ker d_2^{1,3} = E_3^{1,3} & \xrightarrow[\cong]{} & E_3^{3,3} = \ker d_2^{3,3} = \mathcal{L}_3 \\ \Big\downarrow d_2^{1,3} & & \Big\downarrow d_3^{3,3} \\ \mathcal{R}_2 = \mathrm{coker}\, d_2^{2,2} = E_3^{4,1} & \xrightarrow[\cong]{} & E_3^{6,1} = \mathrm{coker}\, d_2^{4,2} = \mathcal{R}_4, \end{array} \tag{3.22}$$

The horizontal maps in the bottom square are isomorphisms by remark 3.2. Therefore, the top horizontal map is an isomorphism. Hence, $\mathcal{L}'_1 \cong \mathcal{L}'_3$.

Notation: We denote

$$L'_1 := \dim\ker d_3^{1,3} = \dim\ker d_3^{k,3}, \quad \text{for } k \text{ odd.} \tag{3.23}$$

3.2.3 Differentials in the E_4 and E_5-pages

§ 3H The differentials $d_4^{k,3} : E_4^{k,3} \to E_4^{k+4,0}$ are zero

For **$k \geq 1$ odd**, $d_4^{k,3} = 0$, since $H^{\text{odd}}(G, H^0(M, \mathbb{Z})) = 0$.

For **$k \geq 0$ even**, let $x \in F$ be a fixed point. Then we see that $H^i(M, \{x\} \cong H^i(M)$, for $i \geq 1$ and $H^0(M, \{x\}) = 0$. Again to simplify the notation, will use $k = 2$. For $k > 2$ even and $k = 0$, the argument is the same.

By comparing the spectral sequences, we get,

$$\begin{array}{ccc} E_4^{2,3}(M,\{x\}) & \xrightarrow{\;0\;} & E_4^{6,0}(M,\{x\}) = 0 \\ \downarrow \cong & & \downarrow 0 \\ E_4^{2,3}(M) & \xrightarrow[d_4^{2,3}]{} & E_4^{6,0}(M). \end{array} \tag{3.24}$$

The right vertical map and the top horizontal map are zero, since $H^0(M\,\{x\}) = 0$. Therefore, it is enough to show that the left vertical map is an isomorphism, so that $d_4^{2,4} = 0$.

The isomorphism of the left vertical map can be seen from the following. First, note that, $E_4^{2,3}(M,\{x\}) = E_3^{2,3}(M,\{x\}) = \ker d_2^{2,3}(M,\{x\})/\operatorname{Im} d_2^{0,4}(M,\{x\})$ and $E_4^{2,3}(M) = E_3^{2,3}(M) = \ker d_2^{2,3}(M)$, since $d_2^{2,4}(M) = 0 = d_3^{2,4}(M)$. Now, by naturality, consider the following commutative diagram of the E_2-pages of M and $(M,\{x\})$ to show that

$d_2^{0,4}(M, \{x\}) = 0$,

$$\begin{array}{ccc} H^0(G, H^4(M, \{x\})) & \xrightarrow{d_2^{0,4}(M,\{x\})} & H^2(G, H^3(M, \{x\})) \\ \downarrow \cong & & \downarrow \cong \\ H^0(G, H^4(M)) & \xrightarrow[d_2^{0,4}(M)=0]{} & H^2(G, H^3(M)) \end{array} \tag{3.25}$$

Since $d_2^{0,4}(M) = 0$ and the vertical maps are isomorphisms, we get the top horizontal map $d_2^{0,4}(M, \{x\}) = 0$. Therefore, we have $E_4^{2,3}(M, \{x\}) = E_3^{2,3}(M, \{x\}) = \ker d_2^{2,3}(M, \{x\})$.

Now, from the following commutative diagram,

$$\begin{array}{ccccccc} E_4^{2,3}(M, \{x\}) = \ker d_2^{2,3}(M, \{x\}) & \hookrightarrow & E_2^{2,3}(M, \{x\}) & \xrightarrow{d_2^{2,3}} & E_2^{3,2}(M, \{x\}) & \longrightarrow & \\ \downarrow & & \downarrow \cong & & \downarrow \cong & & \\ E_4^{2,3}(M) = \ker d_2^{2,3}(M) & \hookrightarrow & E_2^{2,3}(M) & \xrightarrow[d_2^{2,3}]{} & E_2^{3,2}(M) & \longrightarrow & \end{array} \tag{3.26}$$

we see that $E_4^{2,3}(M, x) \cong E_4^{2,3}(M)$.

For $k = 0$, since $E_4^{0,3}(M, \{x\}) = E_3^{0,3}(M, \{x\}) = \ker d_2^{0,3}(M, \{x\})$, argument follows similar to (3.26).

§ 3I The differentials $d_4^{k,4} : E_4^{k,4} \to E_4^{k+4,1}$ are zero

For $k \geq 1$ **odd**, we have $d_4^{k,4} = 0$, since $E_4^{k,4} = 0$, for k odd.

For $k \geq 0$ **even**, we proceed as follows. Let $x \in F$ be a fixed point. As seen earlier, the inclusion map $i : M - \{x\} \hookrightarrow M$ induces an isomorphism $H^i(M) \cong H^i(M - \{x\})$, for $i \leq 3$, and $H^4(M - \{x\}) = 0$.

We consider the following commutative diagram of spectral sequences. We will use

$k = 0$ to simplify notation. The proof is same for $k > 0$.

$$\begin{array}{ccc} E_4^{0,4}(M) & \xrightarrow{d_4^{0,4}} & E_3^{4,1}(M)/\operatorname{Im} d_3^{1,3}(M) = E_4^{4,1}(M) \\ \Big\downarrow 0 & & \Big\downarrow \cong,\, i^* \\ 0 = E_4^{0,4}(M-\{x\}) & \xrightarrow[0]{} & E_3^{4,1}(M-\{x\})/\operatorname{Im} d_3^{1,3}(M-\{x\}) = E_4^{4,1}(M-x) \end{array} \tag{3.27}$$

We have $E_4^{0,4}(M-\{x\}) = E_3^{0,4}(M-\{x\}) = E_2^{0,4}(M-\{x\}) = 0$. Therefore, the left vertical map and bottom horizontal map are zero. If we can show that the right vertical map is an isomorphism, then we are done, that is, $d_4^{0,4} = 0$. Therefore, by periodicity of the spectral sequence, we get $d_4^{k,4} = 0, \forall k \geq 0$.

To show that $E_4^{4,1}(M) \cong E_4^{4,1}(M-x)$, we consider the following commutative diagram of the short exact sequences, where $E_4^{4,1} = E_3^{4,1}/\operatorname{Im} d_3^{1,3}$, since $d_3^{4,1} = 0$ for both M and $M-x$,

$$\begin{array}{ccccccc} \ker d_3^{1,3}(M) & \hookrightarrow & E_3^{1,3}(M) & \xrightarrow{d_3^{1,3}} & E_3^{4,1}(M) & \longrightarrow & E_4^{4,1}(M) \\ \Big\downarrow \cong & & \Big\downarrow \cong & & \Big\downarrow \cong & & \Big\downarrow \\ \ker d_3^{1,3}(M-\{x\}) & \hookrightarrow & E_3^{1,3}(M-\{x\}) & \xrightarrow[d_3^{1,3}]{} & E_3^{4,1}(M-\{x\}) & \longrightarrow & E_4^{4,1}(M-\{x\}) \end{array} \tag{3.28}$$

Note that, $d_2^{4,1}(M-x) = 0$, since $d_2^{4,1}(M) = 0$ (by 3B) and $E_2^{4,1}(M) \cong E_2^{4,1}(M-x)$ (as $H^1(M) \cong H^1(M-x)$). Therefore, by Lemma 3.5, $E_3^{4,1}(M) \cong E_3^{4,1}(M-x)$.

To see that, $E_3^{1,3}(M) \cong E_3^{1,3}(M-x)$, we use the following commutative diagram

$$\begin{array}{ccccccc} E_3^{1,3}(M) = \ker d_2^{1,3}(M) & \longrightarrow & E_2^{1,3} & \longrightarrow & E_2^{3,2} & \longrightarrow & \\ \Big\downarrow & & \Big\downarrow \cong & & \Big\downarrow \cong & & \\ E_3^{1,3}(M-x) = \ker d_2^{1,3}(M-x) & \longrightarrow & E_2^{1,3} & \longrightarrow & E_2^{3,2} & \longrightarrow & \end{array} \tag{3.29}$$

where the middle two vertical maps are isomorphisms, since $H^l(M) \cong H^l(M-x)$, for $l < 4$. Therefore, the map on the kernels is an isomorphism. Note that, since nothing hits the (1,3)-position from above, $E_3^{1,3} = \ker d_2^{1,3}$, for both M and $M-x$.

Therefore, the middle two vertical maps of (3.28) are isomorphisms. So, the left most vertical map on kernels is an isomorphism. Hence, the right vertical map $E_4^{4,1}(M) \to E_4^{4,1}(M-x)$ is an isomorphism.

Hence, $d_4^{0,4}(M)$ is zero, from the diagram (3.27). So, by the periodicity of the spectral sequence, $d_4^{k,4} = 0$, for $k \geq 2$ even.

§ 3J The differentials $d_5^{k,4} : E_5^{k,4} \to E_5^{k+5,0}$ are zero

Note that, since $d_4^{k,4} = d_3^{k,4} = d_2^{k,4} = 0$, we have $E_5^{k,4} = E_2^{k,4}$. The d_5 differentials are zero, since $H^{\text{odd}}(G,\mathbb{Z}) = 0$.

§ Summary: the remaining non-vanishing differentials

We showed that the differentials in the integral Borel spectral sequence are zero except for $d_2^{k,2}$, for $k \geq 0$ even; $d_2^{k,3}$, for $k \geq 0$ and $d_3^{k,3}$, for $k \geq 1$ odd.

In the next section, we will use the dimension bound of the Borel cohomology of the fixed set to get conclusions on the remaining differentials.

3.2.4 Dimension bound

Following Edmonds [10, Proposition 2.1], we have an isomorphism $H_G^q(M;\mathbb{Z}) \xrightarrow{\cong} H_G^q(F;\mathbb{Z})$, for $q > 4$. Now, since F is a discrete set, all the differentials $d_r(F) = 0$, for $r \geq 2$. Hence, in the Borel spectral sequence we have $H_G^q(F,\mathbb{Z}) = H^q(G; H^0(F;\mathbb{Z}))$. Since there are $\chi(M)$ number of fixed points, we have

$$\dim_{\mathbb{F}_p} H^q(G; H^0(F;\mathbb{Z})) = \chi(M) \dim_{\mathbb{F}_p} H^q(G;\mathbb{Z}), \quad \text{for } q \geq 5.$$

Since, for $q \geq 5$, $E_\infty^{q-i,i}$ are $\mathbb{F}_p$-vector spaces, the dimension count becomes,

$$\begin{aligned}
\sum_{i=0}^{4} \dim_{\mathbb{F}_p} E_\infty^{q-i,i} &= \dim_{\mathbb{F}_p} H_G^q(M;\mathbb{Z}) \;\; (\text{since } E_\infty^{q-i,i} \text{ are vector spaces}) \\
&= \dim_{\mathbb{F}_p} H^q(G; H^0(F;\mathbb{Z})) \\
&= \begin{cases} 0 & q \geq 5 \; odd, \\ \chi(M) & q \geq 6 \; even. \end{cases}
\end{aligned} \tag{3.30}$$

Therefore, $\dim_{\mathbb{F}_p} E_\infty^{q-i,i} = 0$, for each $i = 0, 1, 2, 3, 4$, for $q \geq 5$, odd.

§ Dimensions on the odd lines in E_∞-page

From (3.16) and (3.23), we recall the dimensions of the kernels and co-kernels of the remaining non-vanishing differentials.

$$\dim \operatorname{coker} d_2^{2,2} = R_2; \qquad \dim \ker d_2^{1,3} = L_1$$
$$\dim \ker d_2^{2,3} = L_2; \qquad \dim \ker d_3^{1,3} = L_1'.$$

In the following diagram, we show the remaining non-zero differentials involving the 5-line of the Borel spectral sequence along with their kernels and cokernels of the E_2 and E_3-pages.

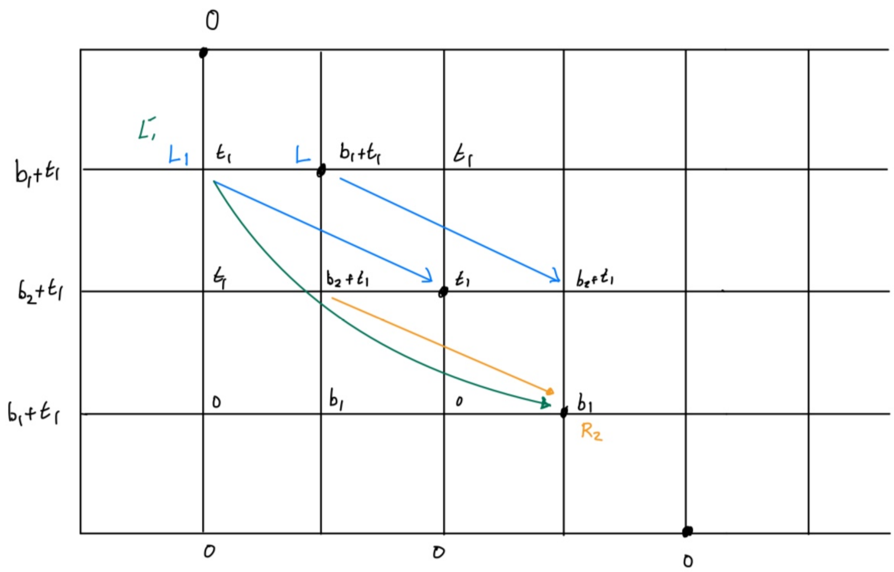


Figure 3.1: E_2 & E_3-page of $H_G^*(M)$

We compute the dimensions of the E_∞-page below.

- Since $d_2^{4,1} = 0$ and $\operatorname{coker} d_2^{2,2} = R_2$, we get $\dim E_3^{4,1} = R_2$. Now, $\dim E_3^{1,3} = \ker d_2^{1,3} = L_1$, since nothing hits the (1,3)-position. So, $\dim\left(\operatorname{Im} d_3^{1,3}\right) = L_1 - L_1'$, where $L_1' = \dim\ker d_3^{1,3}$. Hence,

$$\dim E_4^{4,1} = \dim E_3^{4,1} - \dim\left(\operatorname{Im} d_3^{1,3}\right) = R_2 - L_1 + L_1'.$$

  This is the E_∞-term.

- Since $E_2^{1,3} = t_1$ and $\dim\ker d_2^{1,3} = L_1$, we have $\dim\left(\operatorname{Im} d_2^{1,3}\right) = t_1 - L_1$. So,

$$\dim E_3^{3,2} = \dim E_2^{3,2} - \dim\left(\operatorname{Im} d_2^{1,3}\right) = t_1 - t_1 + L_1 = L_1.$$

  This is the E_∞-term.

- Since $d_3^{2,3} = 0 = d_4^{2,3}$, we have

$$\dim E_\infty^{2,3} = \dim E_3^{2,3} = \dim\ker d_2^{2,3} = L_2.$$

- From the spectral sequence, $\dim E_\infty^{5,0} = 0 = \dim E_\infty^{0,5}$.

In the following table 3.2, we tabulate the dimension data of the 5-line.

Table 3.2: *Dimension counts for 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	0	b_1	t_1	$b_1 + t_1$	0
E_3	0	R_2	L_1	L_2	0
$E_\infty = E_4$	0	$R_2 - L_1 + L_1'$	L_1	L_2	0

Remark 3.6 We note the $\mathbb{F}_p - dimension$ of $E_3^{2,2}$, which will be useful to prove $b_2 \geq 2b_1$. Using the mod-p reduction differential, $\bar{d}_2^{0,3}$, we see that

$$\dim E_3^{2,2} = \ker d_2^{2,2} - \operatorname{Im} \bar{d}_2^{0,3} = (b_2 + t_1) - R_2 - \operatorname{Im} \bar{d}_2^{0,3}. \tag{3.31}$$

3.2.5 Conclusions

Now using the dimension bound, $\dim E_\infty^{5-i,i} = 0$ for $i = 0, 1, 2, 3, 4$ and the table 3.2, we get the following conclusions: Therefore, from Table 3.2 we have,

(i) From the (3,2)-position, $L_1 = 0 \implies \dim\ker d_2^{1,3} = 0$. Therefore, $d_2^{1,3}$ is injective. To show that, $d_2^{k,3}$ is injective, for $k \geq 1$ odd, we use the following commutative diagram

$$\begin{CD} E_2^{1,3} @>{\cong}>> E_2^{3,3} \\ @V{d_2^{1,3}}VV @VV{d_2^{3,3}}V \\ E_2^{3,2} @>>{\cong}> E_2^{5,2}. \end{CD} \tag{3.32}$$

The top and bottom horizontal maps are isomorphisms due the isomorphism on Tor terms as mentioned in lemma 3.2. Therefore, from the commutative diagram above, the right vertical map is injective. Repeating this, we get the injectivity of $d_2^{k,3}$, for $k \geq 1$ odd.

Note that, since $d_2^{1,3}$ is injective, $E_3^{1,3} = 0$. Therefore, $L_1' = 0$.

(ii) From the (2,3)-position, $L_2 = 0 \implies \dim\ker d_2^{2,3} = 0$. Therefore, $d_2^{2,3}$ is injective. Hence, by the following commutative diagram

$$\begin{CD} E_2^{2,3} @>{\cong}>> E_2^{4,3} \\ @V{d_2^{2,3}}VV @VV{d_2^{4,3}}V \\ E_2^{4,2} @>>{\cong}> E_2^{6,2}. \end{CD} \tag{3.33}$$

and arguing similarly as above, we get $d_2^{k,3}$ to be injective, for each $k \geq 2$, even. The top and bottom isomorphisms are induced by the cup product as mentioned in Lemma 3.2.

(iii) From the (4,1)-position, $\operatorname{coker} d_2^{2,2} = R_2 = 0$, since $L_1 = 0$ and $L_1' = 0$. Therefore, we see that $d_2^{2,2}$ is surjective. Now, using the following commutative diagram,

$$\begin{CD} E_2^{2,2} @>{\cong}>> E_2^{4,2} \\ @V{d_2^{2,2}}VV @VV{d_2^{4,2}}V \\ E_2^{4,1} @>>{\cong}> E_2^{6,1}. \end{CD} \tag{3.34}$$

we show that $d_2^{4,2}$ is surjective, and repeating the argument we see that $d_2^{k,2}$ to be surjective for $k \geq 2$ even.

(iv) To prove that $d_2^{0,2}$ is surjective, we use the following commutative diagram

$$\begin{array}{ccc} H^0(G;H^2(M,\mathbb{Z})) & \xrightarrow{\smile u} & H^2(G;H^2(M,\mathbb{Z})) \\ \Big\downarrow d_2^{0,2} & & \Big\downarrow d_2^{2,2} \\ H^2(G;H^1(M,\mathbb{Z})) & \xrightarrow[\smile u]{\cong} & H^4(G;H^1(M,\mathbb{Z})). \end{array} \tag{3.35}$$

The bottom horizontal map is isomorphic, since the Tate cohomology $\hat{H}^q(G,A) \cong \hat{H}^{q+2}(G,A)$ is isomorphic, and for $q \geq 1$, $\hat{H}^q(G,A) = H^q(G,A)$ (by definition), where A is an abelian group (see [20]).

The top horizontal map is a surjection, since the following composition

$$H^0(G,A) \twoheadrightarrow \hat{H}^0(G,A) \cong \hat{H}^2(G,A) = H^2(G,A)$$

is surjective, where $\hat{H}^0(G,A) = H^0(G,A)/NA$ with norm map N. Therefore, we see that $d_2^{0,2}$ is surjective.

(v) Note that, the differential $d_2^{0,3}$ can never be injective. However, the differential reduction mod-p, $\bar{d}_2^{0,3} : H^3(M) \otimes_{\mathbb{Z}} \mathbb{Z}_p \to H^2(G,H^3(M))$ is injective. To see this, we look at the following commutative diagram

$$\begin{array}{ccc} H^0(G;H^3(M)) \otimes G & \xrightarrow[\cong]{\smile u} & H^2(G;H^3(M)) \\ \Big\downarrow \bar{d}_2^{0,3} & & \Big\downarrow d_2^{2,3} \\ H^2(G;H^2(M)) & \xrightarrow[\smile u]{\cong} & H^4(G;H^2(M)). \end{array} \tag{3.36}$$

Note that, $H^0(G,H^3(M)) \otimes G = H^3(M) \otimes G$ and $H^2(G;H^3(M)) \cong H^2(G) \otimes H^3(M)$, by Künneth theorem. The map $x \otimes a \mapsto au \otimes x$ is surjective, and hence an isomorphism, where $a \in G$ and $x \in H^3(M)$. Therefore, the top horizontal cup product is an isomorphism. Similarly, the bottom horizontal cup product is an isomorphism as well.

Therefore, from the commutative diagram above, we see that $\bar{d}_2^{0,3}$ is injective.

Remark 3.7 Alternatively, one can use the following commutative diagram to show that $\bar{d}_2^{0,3}$ is injective

$$\begin{CD} H^3(M)\otimes \mathbb{F}_p @>{\bar{d}_2^{0,3}}>> H^2(G, H^2(M)) \\ @V{\cong}VV @VVV \\ H^3(M;\mathbb{F}_p) @>>{\text{inj}}> H^2(G, H^2(M;\mathbb{F}_p)). \end{CD} \tag{3.37}$$

The left vertical map is isomorphism by Künneth formula and $\mathrm{Tor}\,(H^4(M)=\mathbb{Z}, G) = 0$. The bottom map is injective due to [19, Proposition 5.1].

§ $b_2 \geq 2b_1$

Since $\bar{d}_2^{0,3}$ is injective and $d_2^{2,2}$ is surjective, $\dim(\mathrm{Im}\,\bar{d}_2^{0,3}) = b_1 + t_1$ and $\ker d_2^{2,2} = b_1$. Therefore, by remark 3.6 and equation 3.31, we have

$$\dim E_3^{2,2} = b_2 + t_1 - b_1 - b_1 - t_1 = b_2 - 2b_1.$$

As the dimension is non-negative, we must have $b_2 \geq 2b_1$.

This proves the assertions of the Proposition 3.1. ■

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CHAPTER 4

RANK TWO FINITE GROUP ACTIONS

In this chapter, we rule out pseudofree, locally linear and homologically trivial action by the groups of rank 2 except for a few cases with low betti-numbers. Since it is enough to rule out the actions of minimal subgroups, we consider $\mathbb{Z}_p \times \mathbb{Z}_p$, for p prime, as the rank 2 minimal subgroup.

We outline the chapter below. In the first section, we establish some useful results for the action of $\mathbb{Z}_p \times \mathbb{Z}_p$, p prime. In the remaining sections, we rule out the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 3$ prime using the Borel spectral sequence.

Throughout the chapter, we will use the Borel spectral sequence with integral cohomology, unless mentioned otherwise.

4.1 SOME USEFUL RESULTS FOR $\mathbb{Z}_p \times \mathbb{Z}_p$ ACTION FOR p PRIME

In this section, we are interested in the locally-linear, pseudo-free and homologically-trivial action of $G = \mathbb{Z}_p \times \mathbb{Z}_p$, for $p = 2$ or an odd prime on a closed, connected, oriented 4-manifold M with $\chi(M) \neq 0$. First, we recall the integral group cohomology of G below.

For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, we have

$$H^*(\mathbb{Z}_2 \times \mathbb{Z}_2; \mathbb{Z}) = \mathbb{Z}(u_1, u_2)[\mu], \tag{4.1}$$

where $|u_i| = 2, 2u_i = 0 = 2\mu$, $|\mu| = 3$ and $\mu^2 = u_1 u_2(u_1 + u_2)$.

For $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 3$ prime, we have

$$H^*(\mathbb{Z}_p \times \mathbb{Z}_p; \mathbb{Z}) = \mathbb{Z}(u_1, u_2)[\mu], \tag{4.2}$$

where $|u_i| = 2, pu_i = 0 = p\mu$, $|\mu| = 3$ and $\mu^2 = 0$.

Lemma 4.1 *Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, for $p \geq 2$ prime. If G acts locally linearly, pseudofreely and homologically trivially on a closed, connected, oriented 4-manifold M with $\chi(M) \neq 0$, then*

(i) G cannot have a global fixed point, and

(ii) each of the $(p+1)$-cyclic subgroups, $H \cong \mathbb{Z}_p$ has $\chi(M)$ many fixed points.

Proof. (i) Suppose G has a global fixed point x. Since the action is pseudofree, x is isolated. Now, since the action is locally linear, we get G_x invariant 4-ball around x, on which G acts freely on the ball, particularly on the sphere S^3. However, this is a contradiction, since G cannot act freely on S^3, due to Smith's p^2 condition [36].

(ii) Take $h \in H$ such that $h \neq 1$. Since H is cyclic and the action is pseudofree, we have $|M^H| = |M^h| = \chi(M^h)$. By Lefschetz fixed-point theorem, we have

$$\begin{aligned} |M^h| = \chi(M^h) = \Lambda_h &= \sum_{i=0}^{\dim M} (-1)^i \operatorname{Tr}\left[H_*(h) : H_*(M;\mathbb{Q}) \to H_*(M;\mathbb{Q})\right] \\ &= \sum_{i=0}^{\dim M} (-1)^i \operatorname{Tr}\, \mathrm{id}_{H_*(M;\mathbb{Q})} \ \text{(since homologically trivial)} \qquad (4.3) \\ &= \sum_{i=0}^{\dim M} \dim_{\mathbb{Q}} H_*(M;\mathbb{Q}) = \chi(M). \end{aligned}$$

This proves the assertion. ■

§ Dimension of $H^q_G(M)$ for $q > 4$

In this section, we are interested in computing the dimension (as $\mathbb{F}_p$-vector space) of the Borel cohomology $H^q_G(M)$, for $q \geq 5$ with $G = \mathbb{Z}_p \times \mathbb{Z}_p$, without computing the Borel spectral sequence.

The action of G does not have a global fixed point by the previous lemma. However, each of the $(p+1)$ cyclic subgroups $K \cong \mathbb{Z}_p$ has $\chi(M)$ many fixed points and they are permuted freely in $\chi(M)/p$ orbits by the other subgroups. Let

$$a = \frac{\chi(M)}{p}.$$

We see that $H^q_G(M) \overset{\cong}{\longmapsto} H^q_G(\Sigma)$, for $q > 4$ [10, Proposition 2.1], where Σ is the singular set of $G-$action. Note that, the singular set is given by $\Sigma = \bigcup_i (G/K_i)^a$. Therefore, for

$q > 4$, we have

$$\begin{aligned}
\dim H^q_G(M) = \dim H^q_G(\Sigma) &= \dim H^q(G; H^0(\Sigma)), \quad \text{(since } d_2 = 0\text{)} \\
&= \sum_{i=1}^{p+1} \dim H^q(G; \mathbb{Z}[G/K_i])^a, \\
&= \sum_{i=1}^{p+1} \dim H^q(G; \mathrm{Ind}^G_{K_i}(\mathbb{Z}))^a, \\
&= \sum_{i=1}^{p+1} \dim H^q(K_i, \mathbb{Z})^a \quad \text{(Shapiro's lemma)} \\
&= \begin{cases} 0, & \text{for } q \geq 5 \text{ odd} \\ (p+1)a = (p+1)\dfrac{\chi(M)}{p}, & \text{for } q \geq 5 \text{ even,} \end{cases}
\end{aligned} \tag{4.4}$$

where $\mathrm{Ind}^G_{K_i}(\mathbb{Z}) \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[K_i]} \mathbb{Z} \cong \mathbb{Z}[G/K_i]$.

Remark 4.2 From the calculations above, we see that

$$H^q_G(\Sigma) = \begin{cases} (p+1)\dfrac{\chi(M)}{p} & \text{for } q \text{ even} \\ 0 & \text{for } q \geq 1 \text{ odd.} \end{cases} \tag{4.5}$$

Lemma 4.3 *In the integral Borel spectral sequence* $\dim E^{q-k,k}_\infty = 0$*, for* $q \geq 5$ *odd and* $0 \leq k \leq 4$.

Proof. From the odd-line dimension bound above

$$\dim H^q_G(\Sigma) = \sum_{k=0}^{4} \dim E^{q-k,k}_\infty = 0, \quad \text{for } q \geq 5, \text{ odd}$$

$$\Longrightarrow \dim E^{q-k,k}_\infty = 0, \quad \text{for each } k = 0, 1, 2, 3, 4, \; q \text{ odd} \tag{4.6}$$

This shows that dimensions on the odd lines are zero, for $q \geq 5$. ■

Recall that, from 2.13, the restriction maps $\mathrm{Res}^G_K : H^2(G) \to H^2(K)$, to the $(p+1)$

cyclic subgroups $K \simeq \mathbb{Z}_p$ are given by:

$$\begin{aligned}
(u_1, u_2) &\mapsto (0, u) \\
(u_1, u_2) &\mapsto (u, 0) \\
(u_1, u_2) &\mapsto (u, u) \\
&\vdots \\
(u_1, u_2) &\mapsto ((p-1)u, u).
\end{aligned} \tag{4.7}$$

Lemma 4.4 *The restriction map* $\bigoplus_K \mathrm{Res}^G_K : H^2(G) \to \bigoplus_K H^2(K)$ *is injective.*

Proof. From the equation (2.13), we get the following matrix for the map $\bigoplus_K \mathrm{Res}^G_K$,

$$\bigoplus_K \mathrm{Res}^G_K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ & \vdots \\ (p-1) & 1 \end{bmatrix}.$$

Since, both the columns are linearly independent for $p \geq 2$ prime, the restriction $\bigoplus_K \mathrm{Res}^G_K$ is injective. ∎

4.1.1 Essential Cohomology

In this section, we recall the definition of Essential cohomology, and prove an important lemma (Lemma 4.11) regarding the essential cohomology below, which says that the image of a differential with the codomain in 0-horizontal line ($E_r^{k,0}$) in the Borel spectral sequence must be contained in the essential ideal modulo indeterminacy.

Definition 4.5 Let G be any finite group and $R = \mathbb{Z}$ or a field k such that the characteristic of k divides $|G|$. Let $x \in H^n(G, R)$ be a cohomology class. Then x is called an *Essential element* if the restriction maps to H, $\mathrm{Res}^G_H(x) = 0$, for every proper subgroup H of G. In other words, we have $x \in \bigcup_{e \neq H < G} \ker \mathrm{Res}^G_H$.

The essential elements form a graded ideal and is called *Essential ideal* [3], denoted by $\mathrm{Ess}^n(G) \subseteq H^n(G,R)$.

Example 4.6 For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, with the mod-2 coefficients, the Essential ideal is generated by the monic polynomials [3, Lemma 2.2] in $H^*(G;\mathbb{F}_2) = \mathbb{Z}_2[x_1,x_2]$, for $|x_i| = 1$ and $x_i \mapsto x_i^2$ under the Bockstein homomorphism. The Essential cohomology is generated by $\mathrm{Ess}^3(G;\mathbb{F}_2) = x_1x_2(x_1+x_2) = \gamma$.

In this thesis, we are mostly interested in the integral coefficients. We prove the following lemma.

Lemma 4.7 *For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ the Essential cohomology,* $\mathrm{Ess}^*(G;\mathbb{Z})$ *is generated by $\langle\mu\rangle$, where $\mu \in H^3(G;\mathbb{Z})$ is the generator.*

***Proof*.** First, note that $H^k(G;\mathbb{Z}) \xrightarrow{\times p} H^k(G;\mathbb{Z})$ is a zero map, for all $k \geq 1$, since $H^k(G)$ is a $\mathbb{Z}_p$-vector space. Therefore, from the Bockstein sequence $\mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{j_p} \mathbb{Z}_p$, we see that $H^k(G;\mathbb{Z}) \to H^k(G;\mathbb{Z}_p)$ is injective. Similarly, $H^k(K;\mathbb{Z}) \to H^k(K;\mathbb{Z}_p)$ is injective, for each $K \cong \mathbb{Z}_p < G$ and $k > 0$. Now consider, the following commutative diagram

$$\begin{array}{ccc} \ker(\oplus_K \mathrm{Res}_k) = \mathrm{Ess}^k(G;\mathbb{Z}) & \xhookrightarrow[j_p^*]{\text{injective}} & \mathrm{Ess}^k(G;\mathbb{Z}_p) = \ker(\oplus_K \mathrm{Res}_k) \\ \big\downarrow{\scriptstyle \text{inclusion}} & & \big\downarrow{\scriptstyle \text{inclusion}} \\ H^k(G;\mathbb{Z}) & \xhookrightarrow[j_p^*]{\text{injective}} & H^k(G;\mathbb{Z}_p) \\ \big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} & & \big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} \\ \bigoplus_K H^k(K;\mathbb{Z}) & \xhookrightarrow[\text{injective}]{} & \bigoplus_K H^k(K;\mathbb{Z}_p). \end{array} \tag{4.8}$$

From the diagram above, we see that $\mathrm{Ess}^*(G;\mathbb{Z}) \hookrightarrow \mathrm{Ess}^*(G;\mathbb{Z}_p)$ is injective. Note that, $\mu \in \mathrm{Ess}^3(G;\mathbb{Z})$, since $H^3(K;\mathbb{Z}) = 0$, for $K \cong \mathbb{Z}_p < G$.

The reduction mod-2 homomorphism on μ is given by $j_2^*(\mu) = \gamma = x_1x_2(x_1+x_2)$, where $x_i \in H^1(G;\mathbb{F}_2)$ are the generators. The other reduction mod-2 homomorphisms are given by $u_i \mapsto x_i^2$ (see eq. (2.10)). Let $x \in \mathrm{Ess}^k(G)$, for some $k \geq 3$. From the commutative diagram above, $j_p^*(x) \in \mathrm{Ess}^k(G;\mathbb{F}_p)$. By Aksu and Green (example 4.6), $\mathrm{Ess}^k(G;\mathbb{F}_p)$ is generated by $\langle\gamma\rangle$. Hence $j_2^*(x) \in \langle\gamma\rangle$. Since j_2^* is an injection and $j_2^*(\mu) = \gamma$, we must have $x \in \langle\mu\rangle$. ∎

For $G = \mathbb{Z}_p \times \mathbb{Z}_p$, p odd prime, the description of the Essential cohomology with mod-p coefficients, is given by Aksu and Green [3, Theorem 1.1, 1.2]. For p odd prime, first, we recall that the mod-p cohomology of G, which is given by

$$H^*(G;\mathbb{F}_p) = \mathbb{F}_p[u_1,u_2] \otimes \Lambda(x_1,x_2),$$

where $u_i \in H^2(G;\mathbb{F}_p)$ and $x_i \in H^1(G;\mathbb{F}_p)$ with $x_i^2 = 0$. We quote the theorem provided by Hambleton and Pamuk [19] below, which a special case of the general result.

Theorem 4.8 (Aksu and Green) *For $G = \mathbb{Z}_p \times \mathbb{Z}_p$, p odd prime, the essential cohomology* $\mathrm{Ess}\,(G)$ *is the smallest ideal in $H^*(G,\mathbb{F}_p)$ containing x_1x_2 and closed under the action of the Steenrod algebra. Moreover, as a module over $\mathbb{F}_p[u_1,u_2]$, the essential ideal is free on the set of Mùi generators.*

Example 4.9 For $G = \mathbb{Z}_p \times \mathbb{Z}_p$ with p odd prime, the Múi generators are given by

$$\gamma_1 = x_1x_2;\ \gamma_2 = x_1u_1 - x_2u_2;\ \gamma_3 = x_1u_2^p - x_2u_1^p;\ \gamma_4 = u_1u_2^p - u_2u_1^p.$$

Note that, $\gamma_2 = \Delta_p(\gamma_1)$ and $\gamma_4 = \Delta_p(\gamma_3)$, where Δ_p is the Bockstein homomorphism.

Example 4.10 For $G = \mathbb{Z}_p \times \mathbb{Z}_p$, with the integral coefficients, we see that $\mu \in \mathrm{Ess}^3(G) \subseteq H^3(G)$ is an essential element, since $\mathrm{Res}_K(\mu) = 0$, for each $K \cong \mathbb{Z}_p$. However, the integral Essential cohomology is not entirely generated by $\langle\mu\rangle$. For $p = 3$, consider $\gamma_4 = u_1u_2^3 - u_2u_1^3 \in \mathrm{Ess}^8(G;\mathbb{Z})$ but $\gamma_4 \notin \langle u\rangle$.

Lemma 4.11 *Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 2$ prime, acts pseudofreely, locally linearly and homologically trivially on M. Then in the Borel spectral sequence,* $\mathrm{Im}\, d_2^{k,1} \subseteq \mathrm{Ess}^{k+2}(G) \subseteq H^{k+2}(G)$*, for all $k \geq 0$. Moreover,* $\mathrm{Im}\, d_r^{k,r-1} \subseteq \overline{\mathrm{Ess}}^*(G) \subseteq E_r^{k+r,0}$*, where* $\overline{\mathrm{Ess}}^*(G)$ *is* $\mathrm{Ess}^*(G)$ *modulo the indeterminacy from the previous differentials, for all $r \geq 2$, $k \geq 0$.*

Proof. We compare the spectral sequences $H_G^*(M)$ and $H_G^*(\Sigma)$, where Σ is the singular set of the action of G on M. Since Σ consists isolated points (pseudofree action), the spectral sequence collapses in the E_2-page, having only the 0-horizontal line. Therefore, we have $E_\infty^{k,0}(\Sigma) = E_2^{k,0}(\Sigma) = H^k(G, H^0(\Sigma))$. From the commutative diagram below with $r \geq 2$,

$$\begin{array}{ccc} E_r^{k,r-1}(M) & \xrightarrow{\ \phi\ } & E_r^{k,r-1}(\Sigma)\ (=0) \\ \Big\downarrow{\scriptstyle d_r^{k,r-1}} & & \Big\downarrow{\scriptstyle 0} \\ E_r^{k+r,0}(M) & \xrightarrow[\ \phi\ ]{} & E_r^{k+r,0}(\Sigma), \end{array} \tag{4.9}$$

we get $\phi(d_r^{k,r-1}(x)) = 0$, for $x \in H^k(G; H^1(M))$, implying, $\operatorname{Im} d_r^{k,r-1} \subseteq \ker\phi$, where ϕ is induced from the inclusion $\Sigma \hookrightarrow M$.

We know that $M^K \neq \emptyset$ for $K \cong \mathbb{Z}_p \leq G$. Therefore, by the result of Borel (lemma 2.21) or Proposition 3.1, no differential can hit the bottom line $E_r^{k,0}(K)$, for $r \geq 2$. So, we have

$$E_\infty^{k,0}(K, M) \cong E_2^{k,0}(K, M) = H^k(K, \mathbb{Z}).$$

Now, we consider the following commutative diagram. For $k \geq 0, r \geq 2$, with $H^0(M) = \mathbb{Z}$, we get

$$\begin{array}{ccc} E_r^{k+r,0}(G,M) \supseteq \operatorname{Im} d_r^{k,r-1} & \xrightarrow{\quad \phi \quad} & E_r^{k+r,0}(G,\Sigma) = H^{k+r}(G, H^0(\Sigma)) \\ \Big\downarrow {\scriptstyle \bigoplus_K \operatorname{Res}_K} & & \Big\downarrow {\scriptstyle \cong} \\ \bigoplus_K H^{k+r}(K,\mathbb{Z}) \cong E_r^{k+r,0}(K,M) & \xhookrightarrow[\quad \Delta \quad]{} & \bigoplus_K H^{k+r}(K,\mathbb{Z})^{\chi(M)/p}, \end{array} \tag{4.10}$$

where Δ is the diagonal embedding. The right vertical map is an isomorphism due to the following sequence of isomorphisms:

$$H^{k+r}(G, H^0(\Sigma)) = \oplus_K H^{k+r}(G, \mathbb{Z}[G/K])^{\chi(M)/p} \cong \oplus_K H^{k+r}(K, \mathbb{Z})^{\chi(M)/p},$$

where the first equality comes from the fact that, each of the $(p+1)$-subgroups K has $\chi(M)$ many isolated fixed points which is permuted by G/K into $\chi(M)/p$ orbits. The second isomorphism is due to Shapiro's lemma. See lemma 4.1 and eq 4.4 for details.

Since $\operatorname{Im} d_r^{k,r-1} \subseteq \ker\phi$ and Δ is injective (diagonal embedding), we see from diagram (4.10) that $\operatorname{Im} d_r^{k,r-1} \subseteq \bigcap_K \ker \operatorname{Res}_K$. Therefore. $\operatorname{Im} d_r^{k,r-1} \subseteq \overline{\operatorname{Ess}}^k(G)$, for all $k \geq 0$ and $r \geq 2$. ■

Remark 4.12 Note that, Hambleton - Pamuk [19] also showed that $\operatorname{Im} d_r^{k,r-1} \subseteq \operatorname{Ess}^*(G)$, with mod-$p$ coefficients.

4.1.2 Dimension of E_2-page as $\mathbb{F}_p$-vector spaces

The following notations are similar to that of the Borel cohomology of M discussed in Section 3.1. However, for convenience we reproduce it here.

§ Torsion in $H_1(M)$ and $H^*(M)$

We define

$$t_1 := \dim_{\mathbb{F}_p}\left(T_1 \otimes \mathbb{F}_p\right), \tag{4.11}$$

where $T_1 = \text{Tors}\,(H_1(M))$, the torsion part of $H_1(M)$. Since $H^i(M) = \mathbb{Z}^{b_i} \oplus \text{Tors}\,(H_{i-1}(M))$, we have $\text{Tors}\,(H^2(M)) = T_1$. By Poincaré duality, $\text{Tors}\,(H^3(M)) = \text{Tors}\,(H_1(M)) = T_1$. Therefore,

$$\dim_{\mathbb{F}_p}\left(H^l(M) \otimes \mathbb{Z}_p\right) = \begin{cases} 1, & l = 0,4 \\ b_1, & l = 1 \\ b_2 + t_1, & l = 2 \\ b_1 + t_1, & l = 3. \end{cases} \tag{4.12}$$

§ Dimensions

By the Künneth formula, we have

$$E_2^{k,l} = H^k(G, H^l(M)) = H^k(G) \otimes H^l(M) \oplus \text{Tor}\,(H^{k+1}(G), H^l(M)).$$

Now, for $k > 0$, $H^k(G)$ is an $\mathbb{F}_p$-vector space. Therefore,

$$H^k(G) \otimes H^l(M) = \bigoplus_{\dim H^k(G)} \mathbb{Z}_p \otimes H^l(M).$$

Hence, for $k > 0$,

$$\dim_{\mathbb{F}_p} H^k(G) \otimes H^l(M) = \dim H^k(G) \times \dim(H^l(M) \otimes \mathbb{Z}_p). \tag{4.13}$$

For the dimension of the Tor term, we see that,

$$\text{Tor}\,(H^{k+1}(G), H^l(M)) = \bigoplus_{\dim H^{k+1}(G)} \text{Tor}\,\left(\mathbb{Z}_p, H^l(M)\right) \tag{4.14}$$

Hence, we have

$$\begin{aligned} \dim_{\mathbb{F}_p}\left[\text{Tor}\,\left(H^{k+1}(G), H^l(M)\right)\right] &= \dim H^{k+1}(G) \times \dim \text{Tor}\,\left(\mathbb{Z}_p, H^l(M)\right) \\ &= \begin{cases} 0 & \text{for } l = 0,4 \\ \dim H^{k+1}(G) \times t_1 & \text{for } l = 1,2. \end{cases} \end{aligned} \tag{4.15}$$

So, using the Künneth formula, equations (4.13) and (4.14),the $\mathbb{F}_p$-dimension of $E_2^{k,l}$, for $k > 0$, is given by

$$\dim E_2^{k,l} = \dim H^k(G) \times \dim \left(H^l(M) \otimes \mathbb{Z}_p\right) + \dim H^{k+1}(G) \times \dim \left(\mathrm{Tor}\,(\mathbb{Z}_p, H^l(M))\right)$$
$$= \begin{cases} \dim H^k(G) & \text{for } l = 0, 4 \\ \dim H^k(G) \cdot b_1 & \text{for } l = 1 \\ \dim H^k(G) \cdot (b_2 + t_1) + \dim H^{k+1}(G) \cdot t_1 & \text{for } l = 2 \\ \dim H^k(G) \cdot (b_1 + t_1) + \dim H^{k+1}(G) \cdot t_1 & \text{for } l = 3 \end{cases} \tag{4.16}$$

Remark 4.13 Note that, for $k = 0$, $E_2^{0,l} = H^l(M)$ is not a $\mathbb{F}_p$-vector space. So, we define,

$$\begin{aligned} \dim_{\mathbb{F}_p} E_2^{0,l} &:= \dim_{\mathbb{F}_p} \left(E_2^{0,l} \otimes \mathbb{Z}_p\right) \\ &= \dim_{\mathbb{F}_p} \left(H^l(M) \otimes \mathbb{Z}_p\right) \\ &= \begin{cases} 1, & l = 0, 4 \\ b_1, & l = 1 \\ b_2 + t_1, & l = 2 \\ b_1 + t_1, & l = 3 \end{cases} . \end{aligned} \tag{4.17}$$

4.1.3 Some results regarding the Borel spectral sequence

In this section, we prove some useful results regarding the differentials in the integral Borel spectral sequence. Throughout the section, $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts pseudofreely, locally linearly and homologically trivially on a closed, connected and oriented 4-manifold M, with $\chi(M) \neq 0$.

Lemma 4.14 *Let G act as above on a 4-manifold M, then in the Borel spectral sequence with integral coefficients $d_2^{k,1} = 0$.*

Proof. Since $H^1(M)$ is torsion free, we have $H^k(G; H^1(M)) \cong H^k(G) \otimes H^1(M)$. Let $\{\alpha_i\}_{1 \le i \le b_1'}$ be a basis for $H^1(M)$. Now, we have $H^k(G; H^1(M)) = \bigoplus_i H^k(G)\alpha_i$. Therefore, by the derivation property of the differentials, we have,

$$d_2^{k,1}(H^k(G; H^1(M))) = \bigoplus_i H^k(G) d_2^{0,1}(\alpha_i).$$

Hence, it is sufficient to show that $d_2^{0,1} = 0$.

Recall that, the essential cohomology of G is generated by $\langle\mu\rangle$ (example 4.10). So, we have $\mathrm{Ess}^2(G) = 0$. Therefore, by lemma 4.11, $d_2^{0,1} = 0$.Hence, by the above mentioned multiplicative property, we get $d_2^{k,1} = 0$, for all $k \geq 0$. ∎

Lemma 4.15 *In the integral Borel spectral sequence, the reduced mod-p differential* $\bar{d}_2^{0,3} : E_2^{0,3} \otimes \mathbb{Z}_p \to E_2^{2,1}$ *is injective.*

Proof. We denote $\bar{E}_2^{0,3} = E_2^{0,3} \otimes \mathbb{Z}_p$. Also, note that $E_2^{0,3} = H^3(M)$. Now, consider the following commutative diagram,

$$\begin{array}{ccc}
\bar{E}_2^{0,3}(G,M) = H^3(M) \otimes \mathbb{Z}_p & \xrightarrow{\bar{d}_2^{0,3}} & H^2(G,H^2(M)) = E_2^{2,2}(G,M) \\
\Big\downarrow{\scriptstyle \mathrm{Res}_K \otimes \mathrm{id}, \cong} & & \Big\downarrow{\scriptstyle \mathrm{Res}_K} \\
\bar{E}_2^{0,3}(K,M) = H^3(M) \otimes \mathbb{Z}_p & \xhookrightarrow[\bar{d}_2^{0,3}]{} & H^2(K,H^2(M)) = E_2^{2,2}(K,M).
\end{array} \tag{4.18}$$

Now, the restriction map $\mathrm{Res}_K : E_2^{0,3}(G) \to E_2^{0,3}(K)$ is an isomorphism, since $E_2^{0,3}(G) = H^3(M) = E_2^{0,3}(K)$. Therefore, after tensoring with $\mathbb{Z}_p$, the left vertical map, Res_K, in the diagram is an isomorphism. The bottom horizontal map $\bar{d}_2^{0,3}$ is injective due to proposition 3.1. Therefore, by following the diagram, we see that the top horizontal map $\bar{d}_2^{0,3}$ is injective. ∎

§ Injectivity of $d_2^{1,3}$

In the section, we show the injectivity of the differential $d_2^{1,3}$. To see this, we use Proposition 3.1. By naturality, we have the commutative diagram below:

$$\begin{array}{ccc}
H^1(G,H^3(M)) & \xrightarrow{d_2^{1,3}} & H^3(G,H^2(M)) \\
\Big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} & & \Big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} \\
\bigoplus_K H^1(K,H^3(M)) & \xhookrightarrow[d_2^{1,3}]{} & \bigoplus_K H^3(K,H^2(M)).
\end{array} \tag{4.19}$$

The bottom horizontal map is injective due to Proposition 3.1. *Therefore, to show the injectivity of the top horizontal map, we only need to analyse the injectivity of the left vertical map of diagram 4.19.*

By naturality of the Künneth formula, consider the following commutative diagram for any $k \geq 0$:

$$\begin{array}{ccccc} H^k(G) \otimes H^3(M) & \hookrightarrow & H^k(G, H^3(M)) & \twoheadrightarrow & \mathrm{Tor}\,(H^{k+1}(G), H^3(M)) \\ \downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} & & \downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} & & \downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} \\ \bigoplus_K H^k(K) \otimes H^3(M) & \hookrightarrow & \bigoplus_K H^k(K, H^3(M)) & \twoheadrightarrow & \bigoplus_K \mathrm{Tor}\,(H^{k+1}(K), H^3(M)). \end{array} \tag{4.20}$$

In the diagram 4.20 above, the middle vertical map $\oplus_K \mathrm{Res}_K$ is the map of our interest. *We would like to investigate the injectivity of that map for $k = 1$.* First, prove the following lemma for the right most vertical map in diagram (4.20).

Lemma 4.16 *The map $\oplus_K \mathrm{Res}_K : \mathrm{Tor}\,(H^2(G), H^3(M)) \to \oplus \mathrm{Tor}\,(H^2(K), H^3(M))$ is injective.*

Proof. We know that $H^2(G) = \{u_1, u_2\}$. Recall that, the restriction map, $\oplus_K \mathrm{Res}_K : H^2(G) \to \oplus_K H^2(K)$ is injective, by lemma 4.4.

Now, consider the short-exact sequence with $f = \oplus_K \mathrm{Res}_K$

$$0 \longrightarrow H^2(G) \overset{f}{\hookrightarrow} \bigoplus_K H^2(K) \twoheadrightarrow \bigoplus_K H^2(K)/f(H^2(G)) \longrightarrow 0. \tag{4.21}$$

Tensoring with $H^3(M)$ over $\mathbb{Z}$, we get following long-exact sequence,

$$0 \to \mathrm{Tor}\,(H^2(G), H^3(M)) \hookrightarrow \mathrm{Tor}\,(\oplus_K H^2(K), H^3(M)) \to \cdots, \tag{4.22}$$

where $\mathrm{Tor} = \mathrm{Tor}_1^{\mathbb{Z}}$, and $\mathrm{Tor}_{n\geq 2}^{\mathbb{Z}} = 0$ as we are dealing with finite abelian groups.

Since Tor commutes with direct product, from the injectivity of the map in long-exact sequence (4.22), we get the desired result

$$\oplus_K \mathrm{Res}_K : \mathrm{Tor}\,(H^2(G), H^3(M)) \to \bigoplus_K \mathrm{Tor}\,(H^2(K), H^3(M))$$

is injective. ∎

Lemma 4.17 *The differential $d_2^{1,3}$ is injective.*

Proof. For $k = 1$, $H^1(G) = 0 = H^1(K)$. Therefore, from the Künneth diagram (4.20), we get

$$\begin{array}{ccc} H^1(G, H^3(M)) & \xrightarrow{\cong} & \mathrm{Tor}\,(H^2(G), H^3(M)) \\ \downarrow {\scriptstyle \oplus_K \mathrm{Res}_K} & & \downarrow {\scriptstyle \oplus_K \mathrm{Res}_K} \\ \bigoplus_K H^1(K, H^3(M)) & \xrightarrow[\cong]{} & \bigoplus_K \mathrm{Tor}\,(H^2(K), H^3(M)), \end{array} \tag{4.23}$$

where the isomorphisms of two horizontal maps comes from the Künneth formula, and the injectivity of the right vertical map comes from lemma 4.16. Therefore using the commutativity of the diagram, we get the left vertical map $H^1(G, H^3(M)) \to \bigoplus_K H^1(K, H^3(M))$ to be injective.

Therefore, using the following diagram and Proposition 3.1

$$\begin{array}{ccc} H^1(G, H^3(M)) & \xrightarrow{d_2^{1,3}} & H^3(G, H^2(M)) \\ \downarrow {\scriptstyle \oplus_K \mathrm{Res}_K} & & \downarrow {\scriptstyle \oplus_K \mathrm{Res}_K} \\ \bigoplus_K H^1(K, H^3(M)) & \xhookrightarrow[d_2^{1,3}]{} & \bigoplus_K H^3(K, H^2(M)). \end{array}$$

we have the desired result. ■

4.1.4 Divisibility

Recall that, from the spectral sequence we have

$$E_\infty^{0,k} = E_{r'}^{0,k} \subseteq \cdots E_{r+1}^{0,k} \subseteq E_r^{0,k} \subseteq \cdots \subseteq E_3^{0,k} \subseteq E_2^{0,k} = H^k(M),$$

for some $r' > r \geq 2$, where $E_r^{0,k} = \ker d_{r-1}^{0,k}$. Let $T_r^{0,k}$ be the torsion subgroup of $E_r^{0,k}$. Since $T_{r+1}^{0,k} \subseteq T_r^{0,k} \subseteq \cdots T_2^{0,k}$, we have

$$\mathbb{Z}^s \cong E_\infty^{0,k}/T_\infty^{0,k} \subseteq E_r^{0,k}/T_r^{0,k} \subseteq \cdots \subseteq E_3^{0,k}/T_3^{0,k} \subseteq H^k(M)/T_2^{0,k} \cong \mathbb{Z}^s,$$

where each $E_r^{0,k}/T_r^{0,k} \cong \mathbb{Z}^s$, for $r \geq 2, r = \infty$, is a finitely generated free abelian group of rank $s > 0$. For simplicity, we will write $T \equiv T_r^{0,k}$, for any r.

Definition 4.18 Let $y \in H^k(M)/T$ be a primitive element. We say *the divisibility of* y at the E_{r-1}-page is the smallest integer $n > 0$ such that ny survives onto the E_r-page. If ny survives onto the E_∞-page, we say n is the *maximum divisibility* of y.

Example 4.19 Let $(z_1, z_2) \in E_2^{0,3}/T \cong \mathbb{Z}^2$. Consider the map $d_2^{0,3} : (z_1, z_2) \mapsto (\alpha_1, \alpha_2)$. Then $(2z_1, 2z_2) \in \ker d_2^{0,3} = E_3^{0,3}$. Therefore, both z_1 and z_2 have divisibilities 2 in E_2-page.

Let $d_3^{0,3}(2z_1, 2z_2) = (0, \mu)$. Therefore, $(2z_1, 4z_2) \in \ker d_3^{0,3} = E_\infty^{0,3}$. So, the maximum divisibilities of z_1 and z_2 are 2 and 4, respectively.

4.1.5 The p^*, q^* and j^* maps

Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 2$ prime. By Lemma 4.1 using Lefschetz fixed point theorem, each $K \cong \mathbb{Z}_p < G$ has $\chi(M)$ many fixed points. Since the action is locally linear (Def 1.2), for every $x \in \Sigma$, we have a G_x-invariant 4-ball $B^4(x)$. Let $V = \cup_{x\in\Sigma} B^4(x)$ be the G-invariant collection of such 4-balls.

Denote $M^* = M/G$, for any G-space M and $M_0 = M - V$. We consider the following sequence of maps,

$$M \xrightarrow{p} M \times_G EG \xrightarrow{j} M^*, \quad q = j \circ p \tag{4.24}$$

where $q = j \circ p : M \to M^*$ is a $|G|$-fold covering branched at $(p+1)\chi(M)$ singular points.

Since G acts pseudofreely on M, for each ball $B^4(x)$ the subgroups $K \cong \mathbb{Z}_p$ act freely on $S^3 = \partial B^4(x)$. So, we have

$$\partial V^* = \begin{cases} \cup_{x\in\Sigma} \mathbb{RP}^3, & \text{for } G = \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \cup_{x\in\Sigma} L_{p,q}, & \text{for } G = \mathbb{Z}_p \times \mathbb{Z}_p,\ p \geq 3 \text{ prime.} \end{cases} \tag{4.25}$$

where $L_{p,q} = S^3/\mathbb{Z}_p$ is the lens space with p, q coprime.

Similarly, we have

$$V^* = \begin{cases} \cup_{x\in\Sigma} C(\mathbb{RP}^3), & \text{for } G = \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \cup_{x\in\Sigma} C(L_{p,q}), & \text{for } G = \mathbb{Z}_p \times \mathbb{Z}_p,\ p \geq 3 \text{ prime.} \end{cases} \tag{4.26}$$

Lemma 4.20 *The map $q : M_0 \to M_0^*$ is a regular $|G|$-fold covering.*

Proof. We see that $q^{-1}([x]) = \{gx \mid g \in G, x \in M - \Sigma\}$. If $g_1 x = g_2 x$, then we have $g_1^{-1} g_2$ fixes x. Since the action is pseudo-free (free in the complement of Σ), we have $g_1^{-1} g_2 = \text{id}$ i.e., $g_1 = g_2$. Therefore, $|\, q^{-1}([x]) \,| = |G|$. ■

We have the following maps induced on cohomology.

(i) The map $M \xrightarrow{p} M \times_G EG$ is an inclusion by fibre. The induced map on cohomology is therefore $p^* : H^k_G(M) \to H^k(M)$. We have defined p^* in chapter 2 as edge homomorphism. For convenience, we reiterate it here. First consider the filtration of $H^k_G(M)$ below

$$0 = F^{k+1} \subseteq F^k_k \subseteq \cdots \subseteq F^k_1 \subseteq F^k_0 = H^k_G(M), \tag{4.27}$$

where $E^{l,k-l}_\infty \simeq F^k_l / F^k_{l+1}$. Therefore, $E^{0,k}_\infty \simeq H^k_G(M)/F^k_1$, is a subquotient of $H^k_G(M)$. Therefore, p^* factors through $E^{0,k}_\infty$ as in the following diagram

$$\begin{array}{ccc} & H^k_G(M) & \\ {\scriptstyle \pi}\swarrow & & \searrow{\scriptstyle p^*} \\ E^{0,k}_\infty = E^{0,k}_r & \xhookrightarrow[i]{} & E^{0,k}_2 = H^k(M). \end{array} \tag{4.28}$$

Since there are no incoming differentials, the inclusion map $E^{0,k}_\infty = E^{0,k}_r \hookrightarrow \cdots \hookrightarrow E^{0,k}_3 \hookrightarrow E^{0,k}_2$ is given by the inclusion of $\ker d^{0,k}_r \hookrightarrow E^{0,k}_2$ for some $r \geq 2$. Note that $E^{0,k}_\infty = E^{0,k}_r$ if $r \geq 5$.

(ii) The map $j^* : H^k(M^*) \to H^k_G(M)$ induced by the projection $j : M \times_G EG \to M^*$ is defined by the composition $q^* = p^* \circ j^* : H^k(M^*) \to H^k(M)$ as seen in the following commutative diagram

$$\begin{array}{ccc} H^k(M^*) & \xrightarrow{j^*} & H^k_G(M), \\ {\scriptstyle q^*}\searrow & & \swarrow{\scriptstyle p^*} \\ & H^k(M) & \end{array} \tag{4.29}$$

Lemma 4.21 *Suppose that $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts pseudofreely on M with $q : M \to M^*$ as above. Then, we have the following commutative diagram* (4.30)

$$\begin{array}{ccc} H^k(M) & \xleftarrow{q^*} & H^k(M^*) \\ \simeq \Big\| & & \Big\| \simeq \\ H^k(M_0, \partial M_0) & \xleftarrow{q^*} & H^k(M^*_0, \partial M^*_0) \\ {\scriptstyle \cap [M_0, \partial M_0] \simeq}\Big\downarrow & & \Big\downarrow{\scriptstyle \cap p^2[M^*_0, \partial M^*_0]} \\ H_{4-k}(M_0) & \xrightarrow[q_*]{} & H_{4-k}(M^*_0). \end{array} \tag{4.30}$$

Proof. The vertical isomorphisms in the top square comes from the following argument. Since $V = \cup_{x\in\Sigma} B^4(x)$ is contractible, from the long exact sequence of the pair (M,V) we have $H^k(M) \cong H^k(M,V)$, for $k = 2,3,4$. Now, by excision, we have $H^k(M,V) \cong H^k(M_0, \partial M_0)$, where $M_0 = M - V$, as mentioned above. Therefore, we have $H^k(M) \cong H^k(M_0, \partial M_0)$.

For $M^* = M/G$, we have a similar argument. Given V as above, by eq (4.26) and (4.25), V^* is a cone and ∂V^* is either a real projective space or a lens space. Since cone of a space is contractible, by long exact sequence of the pair (M^*, V^*), we have $H^k(M^*) \cong H^k(M^*, V^*)$, for $k = 2,3,4$. Again, by excision we have $H^k(M^*, V^*) \cong H^k(M_0^*, \partial M_0^*)$. Therefore, for $k = 2,3,4$, we have $H^k(M^*) \cong H^k(M_0^*, \partial M_0^*)$.

For the bottom square of the diagram (4.30), we see that the left isomorphism is the Poincaré-Lefschetz duality. The right vertical map follows from the following. Let $u^* \in H^k(M_0^*, \partial M_0^*)$. Using the bottom square we get

$$u^* \mapsto q^*(u^*) \to q^*(u^*) \cap [M_0, \partial M_0] \mapsto q_*\big(q^*(u^*) \cap [M_0, \partial M_0]\big).$$

Now, using the naturality of cap product, we get

$$q_*\big(q^*(u^*) \cap [M_0, \partial M_0]\big) = u^* \cap q_*[M_0, \partial M_0]. \tag{4.31}$$

Since q is $|G|$-fold covering projection on M_0 (lemma 4.20), we have $q_*[M_0, \partial M_0] = p^2[M_0^*, \partial M_0^*]$. Therefore,

$$u^* \cap q_*[M_0, \partial M_0] = u^* \cap p^2[M_0^*, \partial M_0^*], \tag{4.32}$$

which gives the desired right vertical map, $p^2[M_0^*, \partial M_0^*]$ in the bottom square of (4.30). ∎

Computing p^*, q^* and j^* would be one of our main focus in the later sections.

4.1.6 Computations of some homologies

Let $M_0 = M - V$, where $V = \cup_{x\in\Sigma} B^4(x)$, G-invariant collection of 4-balls centred at the isolated singular points, as described in Section 4.1.5.

From the long exact sequence of the pair (M, M_0), we get

$$\longrightarrow H_{k+1}(M, M_0) \longrightarrow H_k(M_0) \longrightarrow H_k(M) \longrightarrow H_k(M, M_0) \longrightarrow \cdots$$

Now, $H_k(M, M_0) \simeq H_k(V, \partial V) \simeq H^{4-k}(V)$ by excision and Poincaré-Lefschetz duality. Therefore, for $k = 0,1,2,3$, we have $H_k(M, M_0) = 0$. Hence, from the long exact sequence above

$$H_k(M) \simeq H_k(M_0), \text{ for } k = 0, 1, 2. \tag{4.33}$$

§ Cartan-Leray Spectral sequence

Since G acts freely on M_0, we have a homology spectral sequence of the regular covering map $M_0 \to M_0^*$ such that

$$E^2_{pq} \cong H_p(G, H_q(M_0)) \implies H_{p+q}(M_0^*). \tag{4.34}$$

This is called the Cartan-Leray spectral sequence [5, Chapter VII, Theorem 7.9].

Now, from $H_1(M) \simeq H_1(M_0)$, we have $E^2_{0,1} = H_1(M_0) \simeq \mathbb{Z}^{b_1} \oplus T_1$, where T_1 is the torsion subgroup of $H_1(M)$. Now, $E^\infty_{0,1} = E^2_{0,1}$ and $E^\infty_{1,0} = E^2_{1,0}$, since it is a first quadrant sequence. Therefore, as an associated graded module, we have $\mathcal{G}H_1(M_0^*) = E^\infty_{0,1} \oplus E^\infty_{1,0}$. From the filtration, we have

$$E^\infty_{0,1} = F_{0,1} \subset F_{1,0} = H_1(M_0^*).$$

Therefore, $E^\infty_{1,0} \cong H_1(M_0^*)/E^\infty_{0,1}$. Since $E^\infty_{0,1} = H_1(M_0) = \mathbb{Z}^{b_1} \oplus T_1$ and $E^\infty_{1,0} = H_1(G) = \mathbb{Z}_p \times \mathbb{Z}_p$, we have $\mathbb{Z}_p^2 \cong H_1(M_0^*)/(\mathbb{Z}^{b_1} \oplus T_1)$.

Therefore, we get the following exact sequence, which is used later

$$0 \longrightarrow H_1(M_0) \longrightarrow H_1(M_0^*) \longrightarrow H_1(G) \longrightarrow 0. \tag{4.35}$$

Now, we will show the above sequence is split. Consider the pair (M^*, M_0^*). By excision, we have $H_k(M^*, M_0^*) \cong H_k(V^*, \partial V^*)$, where V^* is a cone and ∂V^* is either $\cup_{x\in\Sigma}\mathbb{RP}^3$ or $\cup_{x\in\Sigma} L_{p,q}$ as mentioned in (4.26) and (4.25). Considering the long exact sequence of $(V^*, \partial V^*)$, we have

$$\to H_k(V^*) \to H_k(V^*, \partial V^*) \to H_{k-1}(\partial V^*) \to .$$

Since $H_2(\partial V^*) = 0$ and $H_3(V^*) = 0$, we have $H_3(V^*, \partial V^*) = 0 = H_3(M^*, \partial M_0^*)$.

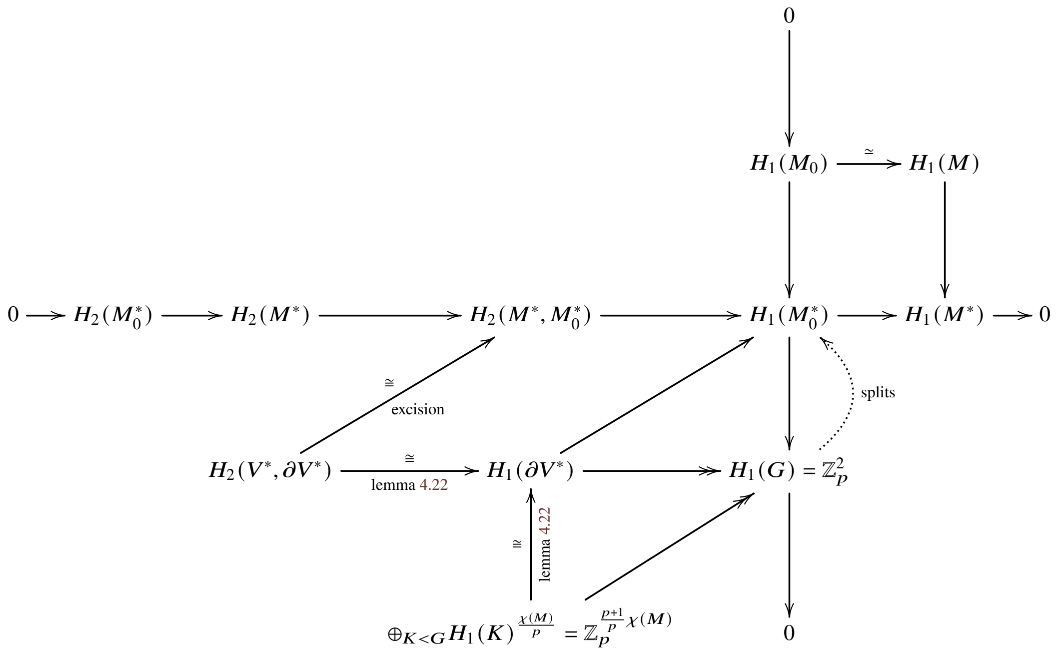
Since $H_0(V^*) \cong H_0(\partial V^*)$, we have $H_1(V^*, \partial V^*) = 0 = H_1(M^*, M_0^*)$. Now, using $k = 2$ in the above sequence and the fact that $\mathrm{Fix}(K)$ of each $(p+1)$ cyclic subgroup K is permuted freely in $\chi(M)/p$ orbits by other subgroups, we get (see eq. (4.4))

$$H_2(V^*, \partial V^*) = H_1(\partial V^*) \cong \oplus_{K<G} H_1(K)^{\chi(M)/p} = \mathbb{Z}_p^{\frac{p+1}{p}\chi(M)}.$$

This proves the following lemma.

Lemma 4.22 $H_2(M_0, M_0^*) \cong H_2(V^*, \partial V^*) \cong H_1(\partial V^*) \cong \mathbb{Z}_p^{\frac{p+1}{p}\chi(M)}$.

Now, we consider the following diagram. The middle sequence is the long exact sequence of (M^*, M_0^*). The vertical short exact sequence is from the spectral sequence of covering as described in eq (4.35).



(4.36)

Using Lemma 4.22, we have $H_1(\partial V^*) \cong \mathbb{Z}_p^{\frac{p+1}{p}\chi(M)}$. Since $H_1(G) = \mathbb{Z}_p^2$, we see that $H_1(\partial V^*)$ surjects onto $H_1(G)$. This gives us the lower commutative triangle.

We also have $H_2(M^*, M_0^*) \cong H_1(\partial V^*)$ by Lemma 4.22. Therefore, we get the middle commutative triangle. Since $H_1(\partial V^*)$ surjects onto $H_1(G)$, using this middle commutative triangle, we see that the vertical short exact sequence splits.

Therefore, we have $H_1(M_0^*) \cong H_1(M_0) \oplus H_1(G) \cong \mathbb{Z}^{b_1} \oplus \mathbb{Z}_p^2 \oplus T_1$.

This proves the following lemma.

Lemma 4.23 *If $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts pseudofreely, locally linearly and homologically trivially on a closed, connected, oriented 4-manifold with $\chi(M) \neq 0$, then $H_1(M_0^*) = H_1(M_0) \oplus H_1(G) = \mathbb{Z}^{b_1} \oplus T_1 \oplus \mathbb{Z}_p^2$, where T_1 is the torsion subgroup of $H_1(M)$.*

Remark 4.24 If M has torsion free homology, that is $T_1 = 0$. Then we have

$$H_1(M_0^*) = \mathbb{Z}^{b_1} \oplus \mathbb{Z}_p^2. \tag{4.37}$$

4.1.7 Some results for the manifolds with torsion-free homology

In this section, we prove lemmas involving injectivity of the differential $d_2^{2,3}$, where the homology of the manifold is torsion-free.

Lemma 4.25 *Let the homology of M be torsion free. Then the restriction map $\bigoplus_K \mathrm{Res}_K : H^2(G, H^3(M)) \to \bigoplus_K H^2(K, H^3(M))$ is injective.*

Proof. Since the homology of M is torsion free, all the Tor terms are zero. So, from the Künneth formula in 4.20, we get $H^2(G, H^3(M)) \cong H^2(G) \times H^3(M)$. Therefore, by naturality, we have

$$
\begin{array}{ccc}
H^2(G) \otimes H^3(M) & \xhookrightarrow{\cong} & H^2(G, H^3(M)) \\
\Big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} & & \Big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} \\
\bigoplus_K H^2(K) \otimes H^3(M) & \xhookrightarrow[\cong]{} & \bigoplus_K H^2(K, H^3(M)).
\end{array}
\tag{4.38}
$$

The top and bottom horizontal maps in the square are isomorphisms from short exact sequence of Künneth theorem. The restriction map $\oplus_K \mathrm{Res}_K : H^2(G) \to \oplus_K H^2(K)$ is injective, as discussed in lemma 4.16. Now, similar to equation (4.21), we have short exact sequence $0 \to H^2(G) \overset{f}{\hookrightarrow} \oplus_K H^2(K) \to \operatorname{coker} f \to 0$. Since $H^3(M)$ is torsion-free, the Tor terms $\mathrm{Tor}(-, H^3(M))$ are zero. Therefore, after tensoring the above short exact sequence with $H^3(M)$, we get the left vertical map to be injective. ∎

Lemma 4.26 *Let M be a manifold with torsion-free homology. Then the differential $d_2^{2,3}$ is injective.*

Proof. Consider the following commutative diagram using the naturality of spectral sequence

$$
\begin{array}{ccc}
H^2(G, H^3(M)) & \xrightarrow{d_2^{2,3}} & H^{2+2}(G, H^2(M)) \\
\Big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} & & \Big\downarrow{\scriptstyle \oplus_K \mathrm{Res}_K} \\
\bigoplus_K H^2(K, H^3(M)) & \xhookrightarrow[d_2^{2,3}]{} & \bigoplus_K H^{2+2}(K, H^2(M)).
\end{array}
\tag{4.39}
$$

The lower horizontal map is injective due to Proposition 3.1 and the left vertical map is injective due to the previous lemma 4.25. Hence, by the commutativity the top horizontal map is injective. ∎

Lemma 4.27 *Let M be a 4-manifold such that the homology is torsion-free. Then, in the Borel spectral sequence, we must have $d_2^{0,4}(w) = 0$, for $w \in H^4(M)$.*

Proof. Consider the following diagram

$$\begin{array}{ccc} H^4(M) \simeq H^0(G, H^4(M)) & \xrightarrow{\quad d_2^{0,4} \quad} & H^2(G, H^3(M)) \\ \big\downarrow {\scriptstyle \oplus_K \mathrm{Res}_K} & & \big\downarrow {\scriptstyle \oplus_K \mathrm{Res}_K} \\ \bigoplus_K H^4(M) \simeq \bigoplus_K H^0(K, H^4(M)) & \xrightarrow[\quad d_2^{0,4}(=0) \quad]{} & \bigoplus_K H^2(K, H^3(M)). \end{array} \tag{4.40}$$

The bottom horizontal map is zero due to Proposition 3.1. The right vertical map is injective by lemma 4.25. The left vertical map $\oplus_K \mathrm{Res}_K : H^4(M) \to \oplus_K H^4(M)$ is a diagonal embedding, hence injective. Therefore, from the commutative diagram above, the top horizontal map, $d_2^{0,4}$ must be zero, proving our assertion. ■

Lemma 4.28 *If $H_1(M)$ is torsion-free and $d_2^{3,2}$ is surjective, then so is $d_2^{0,2}$.*

Proof. Since M is torsion-free, $E_2^{k,l} \simeq H^k(G) \otimes H^l(M)$. Consider the following commutative diagram

$$\begin{array}{ccc} E_2^{0,2} & \xrightarrow{\smile \mu \otimes \mathrm{id}} & E_2^{3,2} \\ \big\downarrow {\scriptstyle d_2^{0,2}} & & \big\downarrow {\scriptstyle d_2^{3,2}} \\ E_2^{2,1} & \xrightarrow[\smile \mu \otimes \mathrm{id}]{\simeq} & E_2^{5,1} \end{array} \tag{4.41}$$

where $d_2(\mu z_i) = \mu\, d_2(z_i)$, $\mu \in H^3(G)$ and $z_i \in H^2(M) \simeq E_2^{0,2}$. The top horizontal map is surjective, since $\smile \mu : H^0(G) \to H^3(G)$ is surjective. For the similar reason, the bottom horizontal map is surjective. Moreover, since $\dim E_2^{2,1} = \dim E_2^{5,1} = 2b_1'$, the bottom horizontal map is an isomorphism. Now, since the right vertical map is given to be surjective, from the commutative diagram $d_2^{0,2}$ is onto. ■

4.1.8 Summary

We summarize the most important results of the section 4.1 below. The group $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts pseudofreely, locally linearly and homologically trivially on 4-manifold M with $\chi(M) \neq 0$. So, we have

(i) $\dim H^q_G(M) = \begin{cases} 0, & \text{for } q \geq 5, \text{ odd} \\ \frac{p+1}{p}\chi(M), & \text{for } q \geq 5, \text{ even.} \end{cases}$

(ii) $\operatorname{Im} d_r^{k,r-1}$ is contained in $\operatorname{Ess}^*(G)$ modulo indeterminacy [lemma 4.11].

(iii) The differentials $d_2^{k,1} = 0$, for $k \geq 0$, the mod-p differential $\bar{d}_2^{0,3}$ and $d_2^{1,3}$ are injective [lemma 4.14, 4.15, 4.17].

(iv) If M has torsion-free homology, then $d_2^{2,3}$ is injective [lemma 4.26].

(v) If M has torsion-free homology, then $d_2^{0,4}(w) = 0$, for $w \in H^4(M)$ [lemma 4.27].

4.2 ACTION OF $\mathbb{Z}_2 \times \mathbb{Z}_2$

Assumption: Throughout, the section, we assume that $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts pseudofreely, locally linearly and homologically trivially on a closed, connected, oriented topological 4-manifold, M with $\chi(M) \neq 0$. We work with Borel spectral sequence with the integral coefficients.

The goal of this section is to rule out such actions of G on M, except for certain cases with low betti-numbers. The main idea is to analyze the Borel spectral sequence with the integral coefficients and get some conclusions on the action of G in terms of the betti-numbers.

§ Integral Group cohomology and the essential elements of G

Recall that the integral group cohomology of G is given by

$$H^*(G, \mathbb{Z}) = \mathbb{Z}(u_1, u_2)[\mu], \tag{4.42}$$

with $2u_i = 0$, $\mu^2 = u_1u_2(u_1 + u_2)$, $|u_i| = 2$ and $|\mu| = 3$, where $|\cdot|$ represents the degree of an element.

Recall that (equation (2.13)) the restriction maps $\text{Res}^G_K : H^2(G) \to H^2(K)$, for each of the three subgroups $K \cong \mathbb{Z}_2$ are given by

$$\begin{aligned} (u_1, u_2) &\to (u, 0), \\ (u_1, u_2) &\to (0, u), \\ (u_1, u_2) &\to (u, u). \end{aligned} \tag{4.43}$$

Also recall that, $\mu \in \text{Ess}^3(G)$, since $\text{Res}^G_K(\mu) = 0$, for every subgroup K of G. We now prove the following lemma.

Lemma 4.29 *In the Borel spectral sequence with the integral coefficients, the codimension of* $\text{Ess}^k(G) \subseteq H^k(G, \mathbb{Z}))$ *is 3, for* $k \geq 6$*, even, and is 0 for* $k \geq 3$*, odd.*

Proof. We claim that the even dimensional essential cohomology $\text{Ess}^{2q}(G)$, is generated by μ^2, for $q \geq 3$. By the Lemma 4.7, the entire essential ideal is generated by μ. Therefore, an even degree essential element can be represented by μx, where x is an odd degree element of $H^*(G)$. Since all the odd degree elements are generated by μ, we see that x is a multiple of μ. Hence, $\text{Ess}^{2q}(G) = \langle \mu^2 \rangle$.

The cup product $\smile \mu^2 : H^{2q-6}(G) \to H^{2q}(G)$, for $q \geq 3$, is a ring monomorphism, since $\mu^2 x \neq 0$ for $x \neq 0$ for any $x \in \mathbb{Z}_2[u_1, u_2]$. From $\dim H^{2q-6}(G) = q - 2$ and

$\dim H^{2q}(G) = q+1$, we see that coker $(\smile \mu^2) = 3$. Since $\operatorname{Ess}^{2q}(G) = \langle \mu^2 \rangle$, we have codim $\operatorname{Ess}^{2q}(G) = 3$.

Now, the odd degree cohomology algebra is generated by μ. Therefore, $H^{2q+1}(G) = \operatorname{Ess}^{2q+1}(G)$, for $q \geq 1$. Hence, codim $\operatorname{Ess}^{2q+1}(G) = 0$. ■

Corollary 4.30 $\dim E_\infty^{2q,0} \geq 3$ *for* $q \geq 3$.

Proof. By the previous lemma, $\operatorname{codim} \operatorname{Ess}^{2q}(G) = 3$. By lemma 4.11, image of a differential must be contained in the essential cohomology. This proves our assertion. ■

4.2.1 Dimensions in Borel Cohomology of $\mathbb{Z}_2 \times \mathbb{Z}_2$

The dimensions of $E_2^{k,l}$ are given by the equations (4.16) and (4.17), where $E_2^{0,l}$ is considered to be $E_2^{0,l} \otimes \mathbb{Z}_p$, as mentioned in the remark 4.13 in section 4.1.2. We reproduce the dimensions of $E_2^{k,l}$ below

$$\dim E_2^{k,l} = \begin{cases} \dim H^k(G) & \text{for } l = 0,4 \\ \dim H^k(G) \cdot b_1 & \text{for } l = 1 \\ \dim H^k(G) \cdot (b_2 + t_1) + \dim H^{k+1}(G) \cdot t_1 & \text{for } l = 2 \\ \dim H^k(G) \cdot (b_1 + t_1) + \dim H^{k+1}(G) \cdot t_1 & \text{for } l = 3 \end{cases} \tag{4.44}$$

For completeness, we show the calculation of the dimension of $E_2^{3,2}$ below

$$\begin{aligned} \dim E_2^{3,2} &= (b_1 + t_1) \dim H^3(G) + \dim H^4(G) t_1 \\ &= (b_1 + t_1) \cdot 1 + 3t_1 = b_1 + 4t_1. \end{aligned}$$

Since we will mostly be interested in the 5-line of the spectral sequence, the dimensions of $E_2^{k,5-k}$ have been tabulated below:

Table 4.1: *Dimensions of* $E_2^{k,5-k}$

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0

§ BSS diagram

The spectral sequence diagram of $H_G^*(M)$ is given by

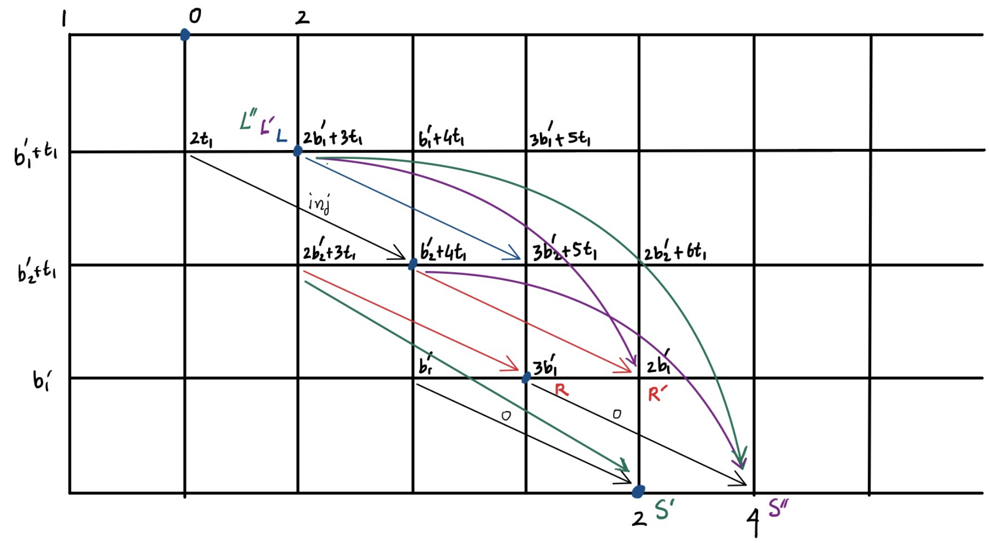


Figure 4.1: E_2 page of $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathbb{Z}$-coefficients

Remark 4.31 We will use the following notations.

- $R = \dim \operatorname{coker} d_2^{2,2}$; $R' = \dim \operatorname{coker} d_2^{3,2}$.
- $L = \dim \ker d_2^{2,3}$; $L' = \dim \ker d_3^{2,3}$; $L'' = \dim \ker d_4^{2,3}$.
- $S' = \dim \operatorname{coker} d_3^{2,2}$; $S'' = \dim \operatorname{coker} d_3^{3,2}$.

Remark 4.32 Since the reduced differential $\overline{d_2^{0,3}}$ is injective, the image is given by $\operatorname{Im} d_2^{0,3} = b_1 + t_1$. Now, $R' = \dim \operatorname{coker} d_2^{2,2}$ and $\dim E_2^{4,1} = 3b_1$. Therefore,

$$\dim \ker d_2^{2,2} = 2b_1 + 3t_1 - 3b_1 + R.$$

Hence, the dimension of $E_3^{2,2}$ is given by

$$\begin{aligned} \dim E_3^{2,2} &= \dim \ker d_2^{2,2} - \dim \operatorname{Im} \overline{d_2^{0,3}} \\ &= 2b_2 - 4b_1 + 2t_1 + R. \end{aligned} \tag{4.45}$$

Remark 4.33 ($3 \le S'' \le 4$) There is only one essential element, μ^2 in (6,0) position and $\dim E_2^{6,0} = 4$. Since the image of differentials in the 0-horizontal line must be contained in the essential cohomology (by lemma 4.11), we have $3 \le S'' \le 4$. In fact, since S'' is an integer, it must equal to either 3 or 4.

By the dimension calculations in (4.4), we have

$$\dim H_G^q(\Sigma) = \begin{cases} 0, & \textit{for } q \textit{ odd} \\ \frac{3}{2}\chi(M), & \textit{for } q \textit{ even.} \end{cases} \tag{4.46}$$

Remark 4.34 ($b_2 - 2b_1$ is even) From the dimension bound above, we see that $2 \mid \chi(M)$. Therefore, $\chi(M)$ must be even. Since $\chi(M) = b_2 - 2b_1 + 2$, we must have $b_2 - 2b_1$ even.

Lemma 4.35 *Let $b_2 = 2b_1$. Then $\dim E_\infty^{q-k,k} = 0$, for $q \geq 6$ even, for $k = 1, 2, 3, 4$.*

Proof. Since $b_2 = 2b_1$, we have $\chi(M) = 2$. From the even line dimension bound in (4.46), we have

$$\dim E_\infty^{q,0} + \sum_{k=1}^{4} \dim E_\infty^{q-k,k} = 3.$$

Using $\dim E_\infty^{q,0} \geq 3$ from corollary 4.30, we get $\sum_{k=1}^4 \dim E_\infty^{q-k,k} = 0$. This implies that $\dim E_\infty^{q-k,k} = 0$, for each $k = 1, 2, 3, 4$. Moreover, from the equation above, we see that $E_\infty^{q,0} = 3$, for $q \geq 6$, even. ■

4.2.2 Calculating $q^* : H^k(M^*) \to H^k(M)$ maps for manifolds with torsion-free homology

Notation: Let $f : V \to W$ be a homomorphism of finitely generated free abelian groups with $\text{rank}(V) = \text{rank}(W) = n$. Using Smith normal form, there exist bases for V and W such that f is a diagonal matrix. Suppose that, for a choice of diagonal basis $f = \text{diag}(a_1, a_2, \ldots, a_n)$, $a_i \in \mathbb{Z}$. Then we use the following notation for f

$$f = (a_1, a_2, \ldots, a_n). \tag{4.47}$$

Suppose that $f = \text{diag}(a, a, \ldots, a)$, for $a \in \mathbb{Z}$. Then we will use the notation

$$f = (\times a). \tag{4.48}$$

Now, we will compute the $q^* : H^k(M^*) \to H^k(M)$ maps defined in section 4.1.5 below, for $k = 2$ and 3.

§ For $k = 3$:

We have the following commutative diagram by lemma 4.21

$$\begin{array}{ccccccc}
H^3(M) & \xleftarrow{q^*(\times 4)} & H^3(M^*) & & & & \\
\| \simeq & & \| \simeq & & & & \\
H^3(M_0, \partial M_0) & \xleftarrow{q^*(\times 4)} & H^3(M_0^*, \partial M_0^*) & & & & \\
\downarrow \cap [M_0, \partial M_0] \simeq & & \downarrow \cap 4[M_0^*, \partial M_0^*] & & & & \\
0 \to H_1(M_0) & \xrightarrow[q_*(\times 1)]{} & H_1(M_0^*) & \longrightarrow & H_1(G) & \longrightarrow & 0,
\end{array} \tag{4.49}$$

where $H^3(M) \simeq \mathbb{Z}^{b_1'} \simeq H_1(M_0)$ (since $t_1 = 0$) and $H_1(G) \simeq \mathbb{Z}_2^2$. The bottom sequence is split exact, as discussed in eq (4.36) of section 4.1.6. Therefore, $H_1(M_0^*) = \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2$. Hence, we have the following exact sequence

$$0 \longrightarrow \mathbb{Z}^{b_1} \xrightarrow{q_*} \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^2 \longrightarrow 0.$$

Since coker $q_* = \mathbb{Z}_2^2$, we must have q_* as a multiplication by 1. So, from the commutative diagram, we must have q^* to be multiplication by 4.

§ For $k = 2$:

First, we prove the following lemma.

Lemma 4.36 *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts pseudofreely, locally linearly and homologically trivially on M with torsion free homology. Suppose that, $b_2 - 2b_1 = 2$ that is, $\chi(M) = 4$, or $b_2 = 2b_1$, that is $\chi(M) = 2$. Then $H^2(M^*)$ is torsion free.*

***Proof*.** Let $c = \frac{3}{2}\chi(M)$. Therefore, $c = 6$ or 3, for $b_2 - 2b_1 = 2$ or $b_2 = 2b_1$, respectively. Now, consider the long exact sequence of the pair (M^*, M_0^*)

$$0 \to H_2(M_0^*) \to H_2(M^*) \to H_2(M^*, M_0^*) \to H_1(M_0^*) \to H_1(M^*) \to 0. \tag{4.50}$$

Now, $H_2(M_0, M_0^*) = \mathbb{Z}_2^c$ by Lemma 4.22. By Lemma 4.23,

$$H_1(M_0^*) = \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2.$$

We also have, Tors $(H^2(M^*)) = \text{Tors}\,(H_1(M^*)) = T$ and $H_2(M_0^*) \cong H^2(M^*)$. Therefore,

$$H_2(M_0^*) = \mathbb{Z}^{b_2} \oplus T.$$

Since $H_2(M^*) \cong H_2(M_0^*, \partial M_0^*) \cong H^2(M_0^*)$, and Tors $(H^2(M_0^*)) = \text{Tors}\,(H_1(M_0^*)) = \mathbb{Z}_2^2$, we have

$$H_2(M^*) = \mathbb{Z}^{b_2} \oplus \mathbb{Z}_2^2.$$

Now, substituting all the values of the cohomologies in the sequence above (4.50), we get

$$0 \to \mathbb{Z}^{b_2} \oplus T \xrightarrow{g} \mathbb{Z}^{b_2} \oplus \mathbb{Z}_2^2 \xrightarrow{f} \mathbb{Z}_2^c \xrightarrow{h} \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2 \xrightarrow{l} \mathbb{Z}^{b_1} \oplus T \to 0, \tag{4.51}$$

where $c = 6$ or 3.

Since h surjects $\mathbb{Z}_2^c$ onto $\mathbb{Z}_2^2$ by (4.35) and (4.36), there is not enough room for T in the last term. Therefore, $T = 0$. Hence, $H^2(M^*)$, $H_2(M_0^*)$ and $H_1(M^*)$ are torsion-free. ∎

Now, to compute $q^* : H^2(M^*) \to H^2(M)$, consider the following commutative diagram

$$\begin{array}{ccccccc}
H^2(M) & \xleftarrow{q^*(2,4,4,\dots)} & H^2(M^*) & & & & \\
\| \simeq & & \| \simeq & & & & \\
H^2(M_0, \partial M_0) & \xleftarrow{q^*(2,4,4,\dots)} & H^2(M_0^*, \partial M_0^*) & & & & \\
\cap\,[M_0,\partial M_0] \simeq \downarrow & & \downarrow \cap\, 4[M_0^*,\partial M_0^*] & & & & \\
H_2(M_0) & \xrightarrow[q_*(2,1,1,\dots)]{} & H_2(M_0^*) & \longrightarrow & H_2(G) & \longrightarrow & 0.
\end{array} \tag{4.52}$$

The bottom sequence comes from the Cartan-Leray spectral sequence of the covering projection $M_0 \to M_0^*$. Using the filtration, we have

$$E_{0,2}^{\infty} = F_{0,2} \subset F_{1,1} \subset F_{2,0} = H_2(M_0^*),$$

where $E_{2,0}^{\infty} \cong H_2(M_0^*)/F_{1,1}$. Now, since $H_2(M_0) = H_2(M)$ is torsion-free and $E_{2,1}^2, E_{3,0}^3$ are $\mathbb{Z}_2$-vector spaces, no differential hits $E_{0,2}^2 = H_2(M_0)$. Therefore, $E_{0,2}^{\infty} = H_2(M_0) = \mathbb{Z}^{b_2}$. So, we get

$$E_{2,0}^{\infty} \cong H_2(M_0^*)/F_{1,1}.$$

Therefore, $H_2(M_0^*)$ surjects onto $E_{2,0}^{\infty}$. Now, $d_{2,0}^2 : E_{2,0}^2 \to E_{0,1}^2$ is zero, since $E_{2,0}^2 = H_2(G) = \mathbb{Z}_2$ and $E_{0,1}^2$ is torsion-free. Therefore, $E_{2,0}^{\infty} = H_2(G)$. So, $H_2(M_0^*)$ surjects onto $H_2(G)$, and we get the bottom sequence in diagram (4.52), which is non-split.

Now, we substitute the values of the cohomologies $H^2(M) \simeq \mathbb{Z}^{b_2} \simeq H_2(M_0)$, $H^2(M^*) \simeq \mathbb{Z}^{b_2} \simeq H_2(M_0^*)$ and $H_2(G) \simeq \mathbb{Z}_2$. Therefore, from the bottom sequence q_* is multiplication by 2 on one basis element. Hence, from the lower square in (4.52), we have $q^* : (z_1^*, z_2^*, z_3^*, \dots) \mapsto (2z_1, 4z_2, 4z_3, \dots)$.

4.2.3 4th order Exact Sequence with arbitrary torsion in homology

Considering the pair (M, Σ), we get the following short-exact sequence

$$0 \longrightarrow H^4(M^*) \xrightarrow{j^*} H^4_G(M) \longrightarrow H^4_G(\Sigma) \longrightarrow 0, \tag{4.53}$$

where $H^3_G(\Sigma) = 0$ and $H^5(M^*) = 0$. Now, we have $H^4(M^*) \simeq \mathbb{Z}$ and $H^4_G(\Sigma) \simeq \mathbb{Z}_2^{\frac{3}{2}\chi(M)}$. Therefore, (4.53) becomes,

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j^*} H^4_G(M) \longrightarrow \mathbb{Z}_2^{\frac{3}{2}\chi(M)} \longrightarrow 0. \tag{4.54}$$

Notation: For the rest of the article, we denote w to be the generator of $H^4(M) \cong \mathbb{Z}$.

Lemma 4.37 *For all $r > 1$, $i \geq 2$, we must have $d_r^{0,4}(2^i w) = 0$, that is, the maximum divisibility of w is 4. In other words, we can only have $d_r^{0,4}(w) \neq 0$ and $d_{r+1}^{0,k}(2w) = 0$, for some $r > 1$.*

Proof. The q^* map is a multiplication by 4, which is determined by the following commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \simeq H^4(M) & \xleftarrow{\quad q^* \quad} & H^4(M^*) \simeq \mathbb{Z} \\ \simeq \Big\| & & \Big\| \simeq \\ H^4(M_0, \partial M_0) & \xleftarrow{q^*\ (\times 4)} & H^4(M_0^*, \partial M_0^*) \\ \cap\, [M_0, \partial M_0] \simeq \Big\downarrow & & \Big\downarrow \cap\, 4[M_0^*, \partial M_0^*] \\ \mathbb{Z} \simeq H_0(M_0) & \xrightarrow[\text{id}]{\quad\quad} & H_0(M_0^*) \simeq \mathbb{Z}. \end{array} \tag{4.55}$$

Suppose that, $d_r^{0,4}(2^2 w) \neq 0$, for some $r > 1$ and $d_{r'>r}^{0,4}(2^3 w) = 0$. Therefore, the maximum divisibility of w is 2^3, since $2^3 w \in \ker d_r^{0,4}$. So, $p^*(\bar{w}) = 8w$, where p^* is the edge map defined in (4.28). This contradicts the commutative diagram of p^*, q^* and j^* below

$$\begin{array}{ccc} H^k(M^*) & \xrightarrow{\quad j^* \quad} & H^k_G(M), \\ & {}_{q^*(\times 4)}\searrow \quad \swarrow_{p^*(\times 8)} & \\ & H^k(M) & \end{array}$$

with q^* multiplication by 4. This proves the claim. ■

Lemma 4.38 *Let $b_2 = 2b_1$ and $d_2^{0,4}(w) = 0$. Then, for any $r \geq 3$, we can not have $d_r^{0,4}(w) \neq 0$ with $d_{r'}^{0,4}(2w) = 0$, for $r' > r$.*

Proof. For $b_2 = 2b_1$, we get $\chi(M) = 2$. Therefore, $H^4_G(\Sigma) = \mathbb{Z}_2^3$. Now, from the Borel spectral sequence, we have

$$\begin{aligned} \mathcal{G}H^4_G(M) &= E_\infty^{0,4} \oplus E_\infty^{2,2} \oplus E_\infty^{3,1} \oplus E_\infty^{4,0} \\ &= \mathbb{Z} \oplus E_\infty^{2,2} \oplus E_\infty^{3,1} \oplus (\mathbb{Z}_2)^3\,, \end{aligned} \tag{4.56}$$

where $E_\infty^{1,3} = 0$, since $d_2^{1,3}$ is injective by lemma 4.17. Now, using the exact sequence (4.54), we have

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j^*} \mathbb{Z} \oplus E_\infty^{2,2} \oplus E_\infty^{3,1} \oplus \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^3 \longrightarrow 0, \tag{4.57}$$

where $E_\infty^{1,3} = 0$, since $d_2^{1,3}$ is injective.

Now, from the diagram 4.55 in the previous lemma 4.37, we see that q^* is multiplication by 4. Since $d_r^{0,4}(w) \neq 0$ and the higher order differentials are zero for some $r \geq 3$, the map p^*, is multiplication by 2.

Therefore, by the commutative diagram of p^*, q^* and j^* in (4.29),

$$\begin{array}{ccc} H^4(M^*) & \xrightarrow{\quad j^*(\times 2) \quad} & H^4_G(M), \\ & {}_{q^*(\times 4)}\searrow \quad \swarrow_{p^*(\times 2)} & \\ & H^4(M) & \end{array}$$

j^* must be multiplication by 2. However, this contradicts the exact sequence (4.57), since $(\mathbb{Z}_2)^4 \subseteq \operatorname{coker} j^*$. ■

4.2.4 Analysing dimension counts

First, we prove the following lemma.

Lemma 4.39 *In the Borel spectral sequence of M with the integral coefficients, the following cases can not happen:*

(i) $d_5^{0,4} = 0$ *and* $\dim E_3^{2,2} \leq 1$*;*

(ii) $E_3^{2,2} = 0$.

Proof. We consider the possible differentials that can hit $E_\infty^{5,0}$. The differential $d_2^{3,1} = 0$, by lemma 4.14 . By lemma 4.17 , $d_2^{1,3}$ is injective. Therefore, $E_3^{1,3} = 0$, hence $d_4^{1,3} = 0$. Therefore, the differentials $d_3^{2,2}$ and $d_5^{0,4}$ are the only differentials that can hit the (5,0)-position.

Now, $\dim E_2^{5,0} = 2$, and both the elements, $\{\mu u_1, \mu u_2\}$ are essential elements. By the dimension bound, we must have $\dim E_\infty^{5,0} = 0$.

So, if $\dim E_3^{2,2} \leq 1$ and $d_5^{0,4} = 0$, the $E_\infty^{5,0}$ can not be killed off completely, which is a contradiction to the dimension bound.

For the second case, $d_3^{2,2} = 0$, since $E_3^{2,2} = 0$. Now, $\dim \operatorname{Im} E_5^{0,4} \leq 1$. Therefore, $d_5^{0,4}$ can kill only one element in the (5,0)-position, contradicting the dimension bound. ∎

To analyse the dimension count on 5-line, we divide the argument into several cases depending on the image of the generator $w \in H^0(G, H^4(M))$ under the correct differentials, considering the divisibilities. We will prove Lemma 4.44 by showing that the only possibility for the image of w is $d_2(w) = 0$, $d_3(w) \neq 0$ and $d_{r\geq 4}(2w) = 0$. This will give a constraint on the betti numbers and the torsion: $b_2 = 2$, $b_1 = 0$ and $t_1 = 0$, where t_1 is the $\mathbb{F}_2$-dimension of torsion of $H_1(M)$, as defined in (4.11).

$d_2(w) = 0 = d_3(w)$

Case 1 ($d_2(w) = 0 = d_3(w)$, $d_4(w) = 0$, $d_5(w) \neq 0$) The 5-line dimension counts are given in Table 4.2 below

Table 4.2: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	L	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L'	0
E_5	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L''	0
$E_\infty = E_6$	$S' - 1$	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L''	0

Therefore, from the dimension bound on the 5-line, we have

$$S' = 1, \quad R = 0, \quad b_2 - 2b_1 + 2t_1 + R' + S'' = 4, L'' = 0. \tag{4.58}$$

Since $S'' \geq 3$, we have $b_2 - 2b_1 + 2t_1 + R' \leq 1$. Therefore, we must have $b_2 = 2b_1$, $t_1 = 0$ and $R' \leq 1$. This is not possible due to either lemma 4.38 or lemma 4.39.

Remark 4.40 Alternatively, we can see the following: the equalities give $\dim E_3^{2,2} = 2b_2' - 4b_1 + 2t_1 + R = 0$. By lemma 4.39, this is not possible.

Case 2 ($d_2(w) = 0 = d_3(w)$, $d_4(w) \neq 0$, $d_5(2w) = 0$) The dimension count of the 5-line goes as follows.

Table 4.3: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	L	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L'	0
$E_\infty = E_5$	S'	$R - 1$	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L''	0

From 5-line we get,

$$S' = 0, \quad R = 1, \quad b_2 - 2b_1 + 2t_1 + R' + S'' = 4, L'' = 0. \tag{4.59}$$

Since $S'' \geq 3$ by remark 4.33, we have $b_2 - 2b_1 + 2t_1 + R' \leq 1$. Since, $b_2 - 2b_1$ is even, we must have $b_2 = 2b_1$, $t_1 = 0$ and $R' \leq 1$. This case is not possible due to lemma 4.38.

Remark 4.41 Alternatively, we can see the following: the equalities give $\dim E_3^{2,2} = 2b_2' - 4b_1 + 2t_1 + R = 1$. By lemma 4.39, this is not possible.

Case 3 ($d_2(w) = 0 = d_3(w),\ d_4(w) \neq 0,\ d_5(2w) \neq 0$) The dimension count of the 5-line goes as follows.

Table 4.4: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	L	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L'	0
E_5	S'	$R - 1$	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L''	0
$E_\infty = E_6$	$S' - 1$	$R - 1$	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L''	0

From 5-line we get,

$$S' = 1, \quad R = 1, \quad b_2 - 2b_1 + 2t_1 + R' + S'' = 4. \tag{4.60}$$

Since $S'' \geq 3$, we have $b_2 - 2b_1 + 2t_1 + R' \leq 1$. Therefore, we must have

$$b_2 = 2b_1,\ t_1 = 0 \text{ and } R' \leq 1$$

. Note that, since there is no 2-torsion in $H^*(M)$, we have $H^k(G, H^l(M)) \simeq H^k(G) \otimes H^l(M)$.

$R' = 1$: From our notation, $R' = \dim \operatorname{coker} d_2^{3,2}$, where $\operatorname{coker} d_2^{3,2} \subset E_2^{5,1}$. The positions which can hit the (5, 1)-position are (2, 3) and (1, 4). Since $E_2^{1,4} = 0$, and by lemma 4.26, $d_2^{2,3}$ is injective, nothing hits the (5,1)-position after E_3-page. Now, since $d_2^{k,1} = 0$ by lemma 4.14, we have $\dim E_\infty^{5,1} = \dim E_3^{5,1} = R'$. But $R' = 1$ contradicts lemma 4.35, which says, for $b_2 = 2b_1$, $\dim E_\infty^{5,1} = 0$.

$R' = 0$: This means $d_2^{3,2}$ is surjective. Since $t_1 = 0$, by lemma 4.28, $d_2^{0,2}$ is surjective. Note that, $\dim E_2^{0,2} = \dim E_2^{2,1} = 2b_1$. Therefore, we can write the following map modulo some change of basis

$$\left(z_i, z_{i+b_1'}\right) \xmapsto{d_2^{0,2}} (u_1\alpha_i, u_2\alpha_i), \quad 1 \leq i \leq b_1'. \tag{4.61}$$

We now explore the differential $d_2^{2,2} : E_2^{2,2} \to E_2^{4,1}$. Note that, since $t_1 = 0$,

$E_2^{2,2} \simeq H^2(G) \otimes H^2(M)$ and $E_2^{4,1} \simeq H^4(G) \otimes H^1(M)$. Using the derivation property of the differential and (4.61), we get the following $d_2^{2,2}$ maps, for $1 \leq i \leq b_1$

$$\begin{aligned} u_1 z_i &\longrightarrow u_1^2 \alpha_i, \\ u_2 z_i &\longrightarrow u_1 u_2 \alpha_i, \\ u_1 z_{i+b_1} &\longrightarrow u_1 u_2 \alpha_i, \\ u_2 z_{i+b_1} &\longrightarrow u_2^2 \alpha_i. \end{aligned} \tag{4.62}$$

We see that, each basis element $u_1^2\alpha_i$, $u_1u_2\alpha_i$, $u_2^2\alpha_i$ of $E_2^{4,1}$ has pre-image under the map $d_2^{2,2}$, for each $1 \leq i \leq b_1'$. Therefore, $d_2^{2,2}$ is surjective, given that $d_2^{0,2}$ is surjective. However, this contradicts with $\dim \operatorname{coker} d_2^{2,2} = R = 1$.

$d_2(w) = 0,\ d_3(w) \neq 0$

Case 4 ($d_2(w) = 0,\ d_3(w) \neq 0,\ d_{r\geq 4}(2w) = 0$) In this case, we get the following table.

Table 4.5: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	L	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4 - 1$	L'	0
$E_\infty = E_5$	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 5$	L''	0

Therefore, using the dimension bound of 5-line, we get the following bounds:

$$S' = 0; \quad R = 0 \quad b_2 - 2b_1 + 2t_1 + R' + S'' = 5, \quad L'' = 0. \tag{4.63}$$

From $S'' \geq 3$, we get $b_2 - 2b_1 + 2t_1 + R' \leq 2$. Since $b_2 - 2b_1$ is even, $b_2 - 2b_1 = 0$ or 2. Now, $b_2 = 2b_1$ is not possible due to lemma 4.38. Therefore, the only possibility is

$$b_2 - 2b_1 = 2,\ t_1 = 0,\ R' = 0.$$

Now, q^* is multiplication by 4 from the commutative diagram 4.55, and p^* is multiplication by 2, since $d_3(w) \neq 0$ and $d_{r\geq 4}(w) = 0$. Therefore, j^* must be multiplication by 2 using the following commutative diagram of p^*, q^*, j^*:

$$\begin{array}{ccc} H^4(M^*) & \xrightarrow{j^*(\times 2)} & H^4_G(M), \\ & {\scriptstyle q^*(\times 4)} \searrow \quad \swarrow {\scriptstyle p^*(\times 2)} & \\ & H^4(M) & \end{array} \tag{4.64}$$

Now, we compute $H^4_G(M)$. Since $\dim E_\infty^{2,2} = 2b_2 - 4b_1 + 2t_1 + R + S' - 2$, using the values $b_2 - 2b_1 = 2, t_1 = 0, R' = 0$ and $S' = 0$ from (4.63), we get $\dim E_\infty^{2,2} = 2$. Therefore, $E_\infty^{2,2} = \mathbb{Z}_2^2$.

There is no essential element in (4,0)-position, Therefore, $E_\infty^{4,0} = \mathbb{Z}_2^3$. Since $d_2^{1,3}$ is injective, $E_\infty^{1,3} = 0$.

Therefore, from the 4th order exact sequence of the pair (M, Σ) in (4.53), we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j^*(\times 2)} \mathbb{Z} \oplus \mathbb{Z}_2^2 \oplus E_\infty^{3,1} \oplus \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^6 \longrightarrow 0, \tag{4.65}$$

where the $j^* : H^4(M^*) \to H^4_G(M)$ is multiplication by 2 as discussed in the previous paragraph.

Since $t_1 = 0$, $E_2^{1,2} = 0$. Therefore, $E_\infty^{3,1} = E_3^{3,1}$. Now, from the above short-exact sequence, we must have $E_\infty^{3,1} = 0$. Hence, we need $d_3^{0,3}$ to be surjective.

Consider a basis $\{\beta_i\} \subset H^3(M)$, for $1 \leq i \leq b_1$. Since $\overline{d_2^{0,3}}$ is injective, we have $E_3^{0,3} = \{2\beta_i\}$, for $1 \leq i \leq b_1$. Now, $\dim E_3^{3,1} = \dim E_2^{3,1} = b_1 = \dim E_3^{0,3}$. Therefore, $E_\infty^{0,3} = \{4\beta_i\}$, since $d_3^{0,3}$ is surjective. So, p^* is multiplication by 4.

The q^* map is multiplication by 4. It follows from the diagram 4.49. We recall the diagram here as well.

$$\begin{array}{ccccccc} & & H^3(M) & \xleftarrow{q^*(\times 4)} & H^3(M^*) & & \\ & & \simeq \Vert & & \Vert \simeq & & \\ & & H^3(M_0, \partial M_0) & \xleftarrow{q^*(\times 4)} & H^3(M_0^*, \partial M_0^*) & & \\ & & {\scriptstyle \cap [M_0, \partial M_0] \simeq} \downarrow & & \downarrow {\scriptstyle \cap 4[M_0^*, \partial M_0^*]} & & \\ 0 & \longrightarrow & H_1(M_0) & \xrightarrow[q_*(\times 1)]{} & H_1(M_0^*) & \longrightarrow H_1(G) \longrightarrow & 0, \end{array} \tag{4.66}$$

where the bottom sequence is split exact (see eqn 4.35).

Hence, from the commutative diagram of p^*, q^* and j^* (4.29), for $k = 3$ we get

$$\begin{array}{ccc} H^3(M^*) & \xrightarrow{j^*(\times 1)} & H^3_G(M), \\ & {\scriptstyle q^*(\times 4)}\searrow \quad \swarrow{\scriptstyle p^*(\times 4)} & \\ & H^3(M) & \end{array} \tag{4.67}$$

Therefore, j^* is a multiplication by 1.

Now, using $H^1_G(\Sigma) = 0 = H^3_G(\Sigma)$ we get the following long-exact sequence from the pair (M, Σ)

$$0 \longrightarrow H^2(M^*) \xrightarrow{j^*} H^2_G(M) \longrightarrow H^2_G(\Sigma) \longrightarrow H^3(M^*) \xrightarrow{j^*(\times 1)} H^3_G(M) \longrightarrow 0. \tag{4.68}$$

We compute $j^* : H^2(M^*) \to H^2_G(M)$ below. First, we compute the Borel cohomologies $H^2_G(\Sigma), H^2_G(M)$ and $H^3_G(M)$.

$\boldsymbol{H^2_G(\Sigma)}$ **:** Since $\chi(M) = 4$, by (4.46), we have $\dim H^2_G(\Sigma) = 6$. So,

$$H^2_G(\Sigma) = \mathbb{Z}_2^6.$$

$\boldsymbol{H^2_G(M)}$ **:** Note that, $E_\infty^{1,1} = E_2^{1,1} = H^1(G) \otimes H^1(M) = 0$. Since $d_2^{0,1} = 0$, the (2,0)-position is never hit. Therefore, $E_\infty^{2,0} = \mathbb{Z}_2^2$. Hence,

$$H^2_G(M) = \mathbb{Z}^{b_2} \oplus \mathbb{Z}_2^2.$$

$\boldsymbol{H^3_G(M)}$ **:** Since $R' = 0$, $d_2^{3,2}$ is surjective. So, $d_2^{0,2}$ is surjective by lemma 4.28. Therefore, we get $E_\infty^{2,1} = 0$. The differential $d_2^{0,2}$ can be written by a suitable change of basis, as follows

$$d_2^{0,2}: \quad (z_1, z_2) \mapsto (0,0); \quad (z_{2i-1}, z_{2i}) \mapsto (u_1\alpha_{i-1}, u_2\alpha_{i-1}), \quad \text{for } 2 \le i \le b_1 + 1. \tag{4.69}$$

Note that, $d_3^{0,2}$ must be non-zero. Otherwise, if $(z_1, z_2) \mapsto (0,0)$ under $d_3^{0,2}$, then $d_3^{2,2} : E_3^{2,2} \to E_3^{5,0}$ is a zero differential using the derivation property on $E_3^{2,2} = \{\mu z_1, \mu z_2\}$. This contradicts the surjectivity of $d_3^{2,2}$ (as $S' = 0$). Therefore, for the $d_3^{0,2}$ differential, we have then following map. In the remark 4.42 below, we discuss why this is the only possible choice. So,

$$d_3^{0,2}: \quad z_1 \mapsto 0; \quad z_2 \mapsto \mu; \quad 2z_i \mapsto 0, \quad 3 \le i \le b_2. \tag{4.70}$$

Note that, if z_1 and z_2 both map to μ, we can make a basis change $(z_1 + z_2, z_2) \mapsto (0,\ \mu)$. Therefore, $E_\infty^{3,0} = 0$. We see that $E_\infty^{0,3} = \mathbb{Z}^{b_1}$ and $E_\infty^{1,2} = H^1(G) \otimes H^2(M) = 0$. Hence,

$$H_G^3(M) = E_\infty^{0,3} = \mathbb{Z}^{b_1}.$$

Now, from the $d_3^{0,2}$ differential above, we have $E_\infty^{0,2} = \ker d_3^{0,2} = \{z_1, 2z_2, 2z_3, 2z_4, \dots\}$. Therefore,

$$p^*(\bar{z}_1, \bar{z}_2, \bar{z}_3, \dots) = (z_1, 2z_2, 2z_3, \dots),$$

where $\{\bar{z}_1, \bar{z}_2, \bar{z}_3, \dots, \bar{z}_{b_2}\}$ is a free basis for $H_G^2(M) = \mathbb{Z}^{b_2} \oplus \mathbb{Z}_2^2$. Now, from the diagram 4.52 (shown below in (4.71) as well)

$$\begin{array}{ccccccc}
\mathbb{Z}^{b_2} \simeq H^2(M) & \xleftarrow{q^*\times(2,4,4,\dots)} & H^2(M^*) \simeq \mathbb{Z}^{b_2} & & & & \\
\| \simeq & & \| \simeq & & & & \\
H^2(M_0, \partial M_0) & \xleftarrow{q^*\times(2,4,4,\dots)} & H^2(M_0^*, \partial M_0^*) & & & & \\
\downarrow \cap [M_0, \partial M_0] \simeq & & \downarrow \cap 4[M_0^*, \partial M_0^*] & & & & \\
H_2(M_0) & \xrightarrow[q_*\times(2,1,1,\dots)]{} & H_2(M_0^*) & \longrightarrow & H_2(G) & \longrightarrow & 0,
\end{array} \tag{4.71}$$

q^* is multiplied by $(2, 4, 4, 4, \dots)$. The bottom sequence is non-split, which comes from the Cartan-Leray spectral sequence of the covering. Since $H_2(M_0) \simeq H_2(M)$ and $H_2(M)$ is torsion free, we have $H^2(M^*) \simeq H_2(M_0^*) \simeq \mathbb{Z}^{b_1}$. Since $H_1(M^*)$ is torsion free (by remark 4.24), $H^2(M^*) \simeq H_2(M_0^*) \simeq \mathbb{Z}^{b_2}$.

Therefore, $j^* : H^2(M^*) \to H_G^2(M)$ is multiplication by 2 on all elements by the commutative diagram below

$$\begin{array}{ccc}
H^2(M^*) & \xrightarrow{j^*\ (\times 2)} & H_G^2(M), \\
 & \searrow^{\times(2,4,4\dots)\ q^*} \quad \swarrow_{p^*\ \times(1,2,2,\dots)} & \\
 & H^2(M) &
\end{array} \tag{4.72}$$

Now, $H^3(M^*) \simeq H_1(M_0^*) \simeq \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2$. Substituting all the cohomologies in the short-exact sequence (4.68), we get

$$0 \longrightarrow \mathbb{Z}^{b_2} \xrightarrow{j^*(\times 2)} \mathbb{Z}^{b_2} \oplus \mathbb{Z}_2^2 \longrightarrow \mathbb{Z}_2^6 \xrightarrow{\partial^*} \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2 \xrightarrow{j^*(\times 1)} \mathbb{Z}^{b_1} \longrightarrow 0. \tag{4.73}$$

From the sequence above, we get $\mathbb{Z}_2^{b_2+2} = \operatorname{coker} j^* = \ker \partial^* = \mathbb{Z}_2^4$. Therefore, $b_2 = 2$, and from $b_2 - 2b_1 = 2$ we have $b_1 = 0$.

So, we see that for this case it might be possible to have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on M with $b_2 = 2$, $b_1 = 0$ and $t_1 = 0$. We will come back this case later.

Remark 4.42 In this remark, we show that the $d_2^{0,2}$ differential in (4.69) is the only possibility. Recall that, from equation (4.69), the $d_2^{0,2}$ is given by

$$d_2^{0,2}: \quad (z_1, z_2) \mapsto (0,0); \quad (z_{2i-1}, z_{2i}) \mapsto (u_1\alpha_{i-1}, u_2\alpha_{i-1}), \quad \text{for } 2 \le i \le b_1 + 1. \tag{4.74}$$

For $2 \le i \le b_1 + 1$, by derivation property, $d_2^{2,2}$ is given by

$$\begin{aligned} u_1 z_{2i-1} &\longrightarrow u_1^2 \alpha_{i-1}, \\ u_2 z_{2i-1} &\longrightarrow u_1 u_2 \alpha_{i-1}, \\ u_1 z_{2i} &\longrightarrow u_1 u_2 \alpha_{i-1}, \\ u_2 z_{2i} &\longrightarrow u_2^2 \alpha_{i-1}. \end{aligned} \tag{4.75}$$

Hence dim $E_3^{2,2} = 4$, we have $E_3^{2,2} = \{u_1 z_1, u_2 z_1, u_1 z_2, u_2 z_2\}$.

Now, to show that, $d_3^{0,2}$ can not be the following maps.

(i) Consider the following map for $d_3^{0,2}$

$$d_3^{0,2}: \quad (z_1, z_2) \mapsto (0,0); \quad 2z_3 \mapsto \mu; \quad 2z_{i \ge 4} \mapsto 0. \tag{4.76}$$

We will show that with this map $E_\infty^{5,0}$ cannot be killed off completely, which is a contradiction to the odd line dimension bound. Since $E_3^{2,2} = \{u_1 z_1, u_2 z_1, u_1 z_2, u_2 z_2\}$, using multiplicativity we get $d_3^{2,2}(E_3^{2,2}) = 0$.

Now, since $t_1 = 0$, $E_3^{1,2} = H^1(G) \otimes H^2(M) = 0$. Since by lemma 4.14, no differential can hit the bottom line, we have $d_2^{3,1} = 0$. By assumption, $d_5^{0,4}(w) = 0$. Therefore, $E_\infty^{5,0}$ can not be killed off completely, contradicting the 5-line dimension bound. Hence, we can not have $d_3^{0,2}$ maps in (4.76).

(ii) The maps

$$z_1 \mapsto 0; \quad z_2 \mapsto \mu; \quad 2z_3 \mapsto \mu; \quad 2z_{i \ge 4} \mapsto 0. \tag{4.77}$$

By the change of basis $z_2' \mapsto z_2 + 2z_3$, $d_3^{0,2}$ in (4.77) becomes the map in (4.76). Therefore, we cannot have this differential either.

Case 5 ($d_2(w) = 0,\ d_3(w) \neq 0,\ d_4(2w) \neq 0,\ d_5(4w) = 0$) We get the following table.

Table 4.6: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	L	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4 - 1$	L'	0
$E_\infty = E_5$	S'	$R - 1$	$b_2 - 2b_1 + 2t_1 + R' + S'' - 5$	L''	0

Therefore, we get the following equations from the 5-line bound:

$$S' = 0; \quad R = 1 \quad b_2 - 2b_1 + 2t_1 + R' + S'' = 5, \quad L'' = 0. \tag{4.78}$$

So, we have $b_2 - 2b_1 + 2t_1 + R' \leq 2$, since $S'' \geq 3$. Since $b_2 - 2b_1$ is even, we get the following cases.

(i) $\boldsymbol{b_2 = 2b_1}$: From the $b_2 - 2b_1 + 2t_1 + R' \leq 2$, we get the following cases.

(a) $\underline{t_1 = 0,\ R' \leq 2}$: Since $R = 1$, we have $\dim E_3^{2,2} = 2b_2 - 4b_1 + 2t_1 + R = 1$. Now, $d_5^{0,4}(4w) = 0$ by the maximum divisibility condition of w being 2 (lemma 4.37). Therefore, by lemma 4.39, $E_\infty^{5,0}$ position can not be killed, contradicting the dimension bound.

(b) $\underline{t_1 = 1,\ R' = 0}$: We have $\dim E_\infty^{2,2} = 2b_2 - 4b_1 + 2t_1 + R - 2 + S'$. Using the values of t_1, S' and R, we get $\dim E_\infty^{2,2} = 1$. Since $d_2^{1,3}$ is injective, $E_\infty^{1,3} = 0$. $E_\infty^{4,0} = \mathbb{Z}_2^3$, since $\mathrm{Ess}^4(G) = 0$. Therefore, we have $H_G^4(M) = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus E_\infty^{3,1} \oplus \mathbb{Z}_2^3$. Therefore, we get the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j^*} \mathbb{Z} \oplus \mathbb{Z}_2 \oplus E_\infty^{3,1} \oplus \mathbb{Z}_2^3 \longrightarrow \mathbb{Z}_2^3 \longrightarrow 0, \tag{4.79}$$

However, this is not possible, since there is not enough torsion for the cokernel.

(ii) $\boldsymbol{b_2 - 2b_1 = 2}$: Again, using $b_2 - 2b_1 + 2t_1 + R' \leq 2$, we get the following values: $t_1 = 0$ and $R' = 0$.

Since $R' = 0$, $d_2^{3,2}$ is surjective. Therefore, by lemma 4.28, $d_2^{0,2}$ is surjective. By a change of basis, we have

$$(z_1, z_2) \mapsto (0,0); \quad (z_{2i-1}, z_{2i}) \mapsto (u_1\alpha_{i-1}, u_2\alpha_{i-1}), \quad \text{for } 2 \le i \le b_1 + 1. \tag{4.80}$$

Now, for $2 \le i \le b_1 + 1$, by the multiplicativity of the differential $d_2^{2,2}$ we have

$$\begin{aligned} u_1 z_{2i-1} &\longrightarrow u_1^2\alpha_{i-1}, \\ u_2 z_{2i-1} &\longrightarrow u_1u_2\alpha_{i-1}, \\ u_1 z_{2i} &\longrightarrow u_1u_2\alpha_{i-1}, \\ u_2 z_{2i} &\longrightarrow u_2^2\alpha_{i-1}. \end{aligned} \tag{4.81}$$

Hence, $d_2^{2,2}$ is surjective, contradicting $R = 1$, where $R = \operatorname{coker} d_2^{2,2}$. Note that, this argument is similar to the case $R' = 0$ of Case 3.

Case 6 ($d_2(w) = 0$, $d_3(w) \neq 0$, $d_4(2w) = 0$, $d_5(2w) \neq 0$) The following is the table for the dimension counts of the 5-line for this case.

Table 4.7: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	L	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' - 1 + S'' - 4$	L'	0
E_5	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 5$	L''	0
$E_\infty = E_6$	$S' - 1$	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 5$	L''	0

By the dimension bound of the 5-line, we get

$$S' = 1; \quad R = 0; \quad b_2 - 2b_1 + 2t_1 + R' + S'' = 5; \quad L'' = 0. \tag{4.82}$$

Since $S'' \ge 3$, we have $b_2 - 2b_1 + 2t_1 + R' \le 2$. Therefore, we get the following three cases:

(i) $\boldsymbol{b_2 = 2b_1}$: From the $b_2 - 2b_1 + 2t_1 + R' \le 2$, we get the following cases.

(a) $\underline{t_1 = 0,\ R' \le 2}$: Since $R = 0$, we have $\dim E_3^{2,2} = 2b_2 - 4b_1 + 2t_1 = 0$. Therefore, by lemma 4.39, we can not kill all the elements in (5, 0)-position, contradicting the dimension bound of 5-line.

(b) $\underline{t_1 = 1,\ R' = 0}$: We have $\dim E_\infty^{2,2} = 2b_2 - 4b_1 + 2t_1 + R - 2 + S'$. Using the values of t_1, S' and R, we get $\dim E_\infty^{2,2} = 1$. The argument is exactly same as the case $b_2 = 2b_1, t_1 = 0, R' = 0$ in Case 5 (see equation 4.79).

(ii) $\boldsymbol{b_2 - 2b_1 = 2}$: From $b_2 - 2b_1 + 2t_1 + R' \leq 2$, we get $t_1 = 0$ and $R' = 0$. Since $R' = 0$, $d_2^{3,2}$ is surjective. Since $t_1 = 0$, by lemma 4.28 the differential $d_2^{0,2}$ is surjective. Therefore, by a suitable change of basis $d_2^{0,2}$ can be written as maps given in equation (4.80), i.e.,

$$d_2^{0,2} :(z_1, z_2) \mapsto (0,0); \quad (z_{2i-1}, z_{2i}) \mapsto (u_1\alpha_{i-1}, u_2\alpha_{i-1}), \quad \text{for } 2 \leq i \leq b_1 + 1.$$

Now, by cupping with u_1, u_2 and using the derivation property, we see that $d_2^{2,2}$ is surjective and is given by the same maps in (4.81). Therefore, for $2 \leq i \leq b + 1$

$$d_2^{2,2} : \quad \begin{aligned} u_1z_{2i-1} &\longrightarrow u_1^2\alpha_{i-1}, \\ u_2z_{2i-1} &\longrightarrow u_1u_2\alpha_{i-1}, \\ u_1z_{2i} &\longrightarrow u_1u_2\alpha_{i-1}, \\ u_2z_{2i} &\longrightarrow u_2^2\alpha_{i-1}. \end{aligned}$$

We see that $E_3^{2,2}$ consists of the elements $\{u_1z_1,\ u_2z_1,\ u_1z_2,\ u_2z_2\}$. Since $\operatorname{coker} d_3^{2,2} \equiv S' = 1$, only one element in $E_3^{5,0} \equiv E_2^{5,0} = \{u_1\mu, u_2\mu\}$ is hit by at least one element from $E_3^{2,2}$. Without loss of generality, let us assume that $u_1z_1 \mapsto u_1\mu$, under $d_3^{2,2}$. Now, by the derivation property $d_3^{2,2}(u_1z_1) = u_1d_3^{0,2}(z_1)$. Therefore, we must have $d_3^{0,2}(z_1) = \mu$.

Now, this gives $d_3^{2,2}(u_2z_1) = u_2d_3^{0,2}(z_1) = u_2\mu$. Therefore, $d_3^{2,2}$ is surjective onto $E_3^{5,0}$, contradicting $S' = 1$.

$d_2(w) \neq 0$

Case 7 ($d_2(w) \neq 0, d_{r\geq 3}(2w) = 0$) The following is the table for the dimension counts of the 5-line for this case.

Table 4.8: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	L	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L'	0
$E_\infty = E_5$	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 4$	L''	0

By the dimension bound of 5-line and using $S' \geq 3$, we get

$$S' = 0; \quad R = 0; \quad b_2 - 2b_1 + 2t_1 + R' \leq 1; \tag{4.83}$$

Therefore, we have $b_2 = 2b_1,\ t_1 = 0,\ R' \leq 1$. However, this is not possible due to lemma 4.27.

Remark 4.43 Note that the argument for the cases,

$$d_2(w) \neq 0,\ d_3(2w) = 0,\ d_4(2w) \neq 0$$

and

$$d_2(w) \neq 0,\ d_3(2w) = 0,\ d_4(2w) \neq 0,\ d_5(2w) \neq 0$$

is same as the previous case, since the (3, 2)-position is not being hit. So, we get $t_1 = 0$. Therefore, $d_2(w) \neq 0$ can not happen by lemma 4.27, contradicting the assumption $d_4(w) \neq 0$.

$d_2(w) \neq 0,\ d_3(w) \neq 0$

The cases with $d_{r\geq 4}(2w) \neq 0$ can not happen because of maximum divisibility of w being 2 by lemma 4.37. Therefore, we are left with the following case.

Case 8 ($d_2(w) \neq 0, d_3(2w) \neq 0, d_{r\geq 4}(2w) = 0$) The following is the table for the dimension counts of the 5-line for this case.

Table 4.9: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	$L - 1$	0
E_4	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 1$	L'	0
E_5	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 1$	L''	0
$E_\infty = E_6$	S'	R	$b_2 - 2b_1 + 2t_1 + R' + S'' - 5$	L''	0

By the dimension bound of 5-line and using $S' \geq 3$, we get

$$S' = 0; \quad R = 0; \quad b_2 - 2b_1 + 2t_1 + R' \leq 2. \tag{4.84}$$

From the above equation we see that $t_1 \leq 1$. Therefore, we get the following cases based on the values of t_1.

(i) $\boldsymbol{t_1 = 0}$ **:** This case can not happen due to lemma 4.27.

(ii) $\boldsymbol{t_1 = 1}$ **:** In this case, we have

$$b_2 = 2b_1,\ t_1 = 1,\ R' = 0.$$

We will consider the following long exact sequence of the pair (M^*, M_0^*).

$$0 \longrightarrow H_2(M_0^*) \longrightarrow H_2(M^*) \longrightarrow H_2(M^*, M_0^*) \longrightarrow H_1(M_0^*) \longrightarrow H_1(M^*) \longrightarrow 0. \tag{4.85}$$

Note that $H_k(M^*, M_0^*) \simeq H_k(V^*, \partial V^*) = 0$, for $k = 1, 3$. Since $H_1(V^*) = 0$ and $H_0(\partial V^*) \simeq H_0(V^*)$, we have $H_1(V^*, \partial V^*) = 0$. Similarly, $H_3(V^*, \partial V^*) = 0$, since

$H_3(V^*) = 0 = H_2(\partial V^*)$. Now, from $H_2(M^*, M_0^*) \simeq H_2(V^*, \partial V^*) \simeq H_1(\partial V^*) \simeq \mathbb{Z}_2^3$, since $\chi(M) = 2$, for $b_2 = 2b_1$.

Notation: We denote $(k; a, b, \dots)$ to represent the abelian group $\mathbb{Z}^k \oplus \mathbb{Z}_{2^a} \oplus \mathbb{Z}_{2^b} \oplus \cdots$. Since in our case, $t_1 = 1$, we denote

$$H_1(M) = (b_1; r).$$

As discussed in section 4.1.6, from the spectral sequence of the regular covering $M_0 \to M_0^*$, we get the following exact sequence in homology, which is split:

$$0 \to H_1(M_0) \to H_1(M_0^*) \to H_1(G) \to 0.$$

Therefore, we have

$$H_1(M_0^*) = (b_1; r, 1, 1),$$

since $H_1(M) \simeq H_1(M_0) = (b_1; r)$ and $H_1(G) \simeq G = (1, 1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

By excision and Poincaré-Lefschetz duality $H_2(M_0^*) \simeq H_2(M_0^*, \partial M_0^*) \simeq H^2(M^*, V^*) \simeq H^2(M^*)$. Since, Tors $(H^2(M^*)) =$ Tors $(H_1(M^*)$, we have

$$H_2(M_0^*) = (b_2; r_1).$$

Now, Tors $(H^2(M_0^*) =$ Tors $(H_1(M_0^*))$. Therefore, using $H_2(M^*) \simeq H^2(M_0^*)$, we have

$$H_2(M^*) \simeq (b_2; r, 1, 1).$$

Substituting, all the values in the exact sequence (4.85), we have,

$$0 \longrightarrow (b_2; r_1) \xrightarrow{g} (b_2; r, 1, 1) \xrightarrow{f} (1, 1, 1) \xrightarrow{h} (b_1; r, 1, 1) \xrightarrow{l} (b_1; r_1) \longrightarrow 0. \tag{4.86}$$

From the sequence, we see that $r_1 \le r$.

1. $\boldsymbol{r = r_1}$ **:** In this case, we have

$$0 \longrightarrow (b_2; r) \xrightarrow{g} (b_2; r, 1, 1) \xrightarrow{f} (1, 1, 1) \xrightarrow{h} (b_1; r, 1, 1) \xrightarrow{l} (b_1; r) \longrightarrow 0. \tag{4.87}$$

   (a) Suppose that the free parts go isomorphically. Therefore, $(b_2; r) \to (b_2; r)$ maps isomorphically and coker $g = \mathbb{Z}_2^2$. So, Im $h = \mathbb{Z}_2$. However, ker $l = \mathbb{Z}_2^2$, making the sequence non-exact.

(b) Suppose, there is some divisibility in the free part, that means, $b_2 \to b_2$ is some multiplication map. Note that, there can be no divisibility in the map l, since it has to be a surjection. If the divisibility in $b_2 \to b_2$ is n, then we have $\operatorname{coker} g = \mathbb{Z}_n \oplus \mathbb{Z}_2^2$. Since $\operatorname{coker} g = \ker h$ and we are only dealing with 2-torsion, the divisibility n must be 2. Therefore, $\operatorname{coker} g = (1,1,1) = \ker h$. So, l is an isomorphism, which is impossible.

2. $\boldsymbol{r_1 < r}$ **:** Assuming that the free parts map isomorphically in g, we see that $\operatorname{coker} g = (r - r_1, 1, 1)$. Again, $r - r_1$ can be at most 1, since we are only dealing with 2-torsion, in domain of h. Now, the argument is same as 1(b) in the previous case.

   The divisibility in free parts make the situation worse, since there is too much torsion in the cokernel of g.

Now, recall that, from Case 4 above, we had a possibility of $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on manifolds with $b_2 = 2$, $b_1 = 0$ and $t_1 = 0$. Since $t_1 = \dim_{\mathbb{F}_2}(T \otimes \mathbb{F}_2)$, as defined in (4.11), we have the following lemma.

Lemma 4.44 *Let M be a closed, connected, 4-manifold with $\chi(M) \neq 0$. If $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts pseudofreely, homologically trivially and locally linearly on M, then with coefficients in the localized ring $\mathbb{Z}_{(2)}$, $b_2(M) = 2$ and $H_1(M; \mathbb{Z}_{(2)}) = 0$.*

Note that, the manifolds $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ has the same cohomology groups but different cohomology rings. In the following section, we will improve the above lemma by considering $\mathbb{Z}_2 \times \mathbb{Z}_2$ actions on these two manifolds.

4.2.5 Examples and Non examples

It is known by McCooey [33] that $\mathbb{Z}_2 \times \mathbb{Z}_2$ admits a pseudofree, homologically trivial and locally linear action on $S^2 \times S^2$. For convenience, we provide the action below.

Example 4.45 (Action on $\boldsymbol{S^2 \times S^2}$) Let γ_1, γ_2 and γ_3 denote rotations by π along the x, y and z axes, respectively. Therefore, $\langle \gamma_i \rangle \cong \mathbb{Z}_2$, for each i, and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{\mathrm{id}, \gamma_1, \gamma_2, \gamma_3\}$.

Now, consider the diagonal action of $\gamma_i \in G$ on M: $\gamma(u, v) = (\gamma u, \gamma v)$, where γ_i is one of the rotations. We see that each such group $\langle \gamma_i \rangle$ has four fixed points, mainly the poles along the axis. However, the whole group G does not have a global fixed point. Hence, the action is pseudofree. Since $\chi(M) = 4$, by Lefschetz fixed point theorem this action is homologically trivial (see lemma 4.1).

To see the differentials in Borel spectral sequence with the integral cohomology, first recall that $H^*(G) = \mathbb{Z}[u_1, u_2](\mu)$ where $|u_i| = 2$, $|\mu| = 3$ and $\mu^2 = u_1 u_2 (u_1 + u_2)$. Also,

$H^*(S^2 \times S^2) = \mathbb{Z}[z_1, z_2]/(z_1^2, z_2^2)$ and the intersection form is given by $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. This defines a cup product structure on $S^2 \times S^2$. Let $w \in H^4(S^2 \times S^2) \cong \mathbb{Z}$, then $w = z_1 z_2$, for $z_1, z_2 \in H^2(S^2 \times S^2)$.

Since H^1 and H^3 are zero for the manifold, we only have three horizontal lines in the spectral sequence.

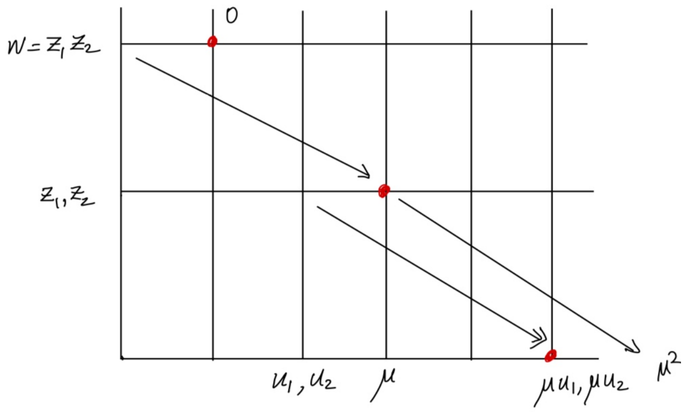


Figure 4.2: E_3-page of $H_G^*(M)$

We consider the d_3 differential on z_1, z_2. Since the image of this differential hitting the bottom horizontal line, by lemma 4.11, $\operatorname{Im} d_3$ must be contained in Ess$^*(G)$. Note that, Ess$^*(G)$ is generated by μ as discussed in 4.29.

Let $d_3(z_1) = \mu$ and $d_3(z_2) = 0$. Therefore, $d_3(w) = d_3(z_1 z_2) = \mu z_2$. The differential $d_3^{2,2}$ mapping onto the (5, 0)-position is surjective such that $u_i z_1 \to \mu u_i$. The differential $d_3^{3,2}$ maps μz_1 to $\mu^2 \in E_3^{6,0}$. The remaining element $\mu z_2 \in E_3^{3,2}$ is killed by w.

Since there are no essential elements in (5, 0) position, w does not have any secondary differential. This provides the full picture for the spectral sequence. Therefore, the spectral sequence collapses and $E_\infty = E_4$ satisfying the dimension bound in eq (4.4) and lemma 4.3.

Now, we will consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action of $M = \mathbb{CP}^2 \# \mathbb{CP}^2$. Note that, a similar argument will work for $M = \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

Lemma 4.46 *The group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ cannot act pseudofreely, locally linearly and homologically trivially on $M = \mathbb{CP}^2 \# \mathbb{CP}^2$.*

Proof. Since $\mathbb{CP}^2 \# \mathbb{CP}^2$ has same cohomology group structure as $S^2 \times S^2$, the Borel spectral sequence has only three horizontal lines. The intersection form of M is given by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Therefore, the cup product structure varies from that of $S^2 \times S^2$, and this is crux of this lemma. Taking $z_1, z_2 \in H^2(M)$ and $w \in H^4(M)$, we have $w = z_1^2$. Assuming

$$d_3^{0,2}(z_1) = \mu \quad \text{and} \quad d_3^{0,2}(z_2) = 0, \tag{4.88}$$

we get $d_3(w) = 2z_1 d_3(z_1) = 0$.

This contradicts Case 4, where we got the conditions on betti-numbers ($b_2 = 2, b_1 = 0, t_1 = 0$), only from $d_2(w) = 0, d_3(w) \neq 0$ ans $d_{r\geq 4}(2w) = 0$. Whereas, by analyzing the spectral sequence we get $d_3(w) = 0$ for the same configuration of betti-numbers.

Alternatively, one can also show that the Borel spectral sequence does not satisfy the dimension bound in lemma 4.3. The E_3-page of the spectral sequence is shown below In fact, the dimension bound violates at (3, 2)-position. The element μz_2 survives in

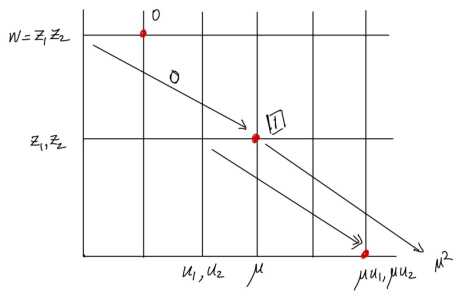


Figure 4.3: E_3-page of $H_G^*(\mathbb{CP}^2 \# \mathbb{CP}^2)$

E_∞-page. ∎

4.2.6 Proof of Theorem 4.47

We combine the results of this section and prove the following theorem about the $\mathbb{Z}_2 \times \mathbb{Z}_2$ pseudofree action.

Theorem 4.47 *Let M be a closed, connected, 4-manifold with $\chi(M) \neq 0$. If $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts pseudofreely, homologically trivially and locally linearly on M, then $b_2 = 2$, $H_1(M; \mathbb{Z}_{(2)}) = 0$ and M must have the intersection form of $S^2 \times S^2$.*

Proof. Combining lemma 4.44, example 4.45 and lemma 4.46 we have the desired result. ∎

4.3 ACTION OF $\mathbb{Z}_p \times \mathbb{Z}_p$, FOR PRIMES $p \geq 3$

The goal of this section is to prove the following theorem.

Theorem 4.48 *Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 3$ prime, acts locally linearly, pseudofreely and homologically trivially on a closed, connected and orientable 4-manifold M with non-zero Euler characteristic. Then, for $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 5$ prime, there does not exist an action on M.*

For $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, M must have $b_2(M) = 1$ and $H_1(M; \mathbb{Z}_{(3)}) = 0$, .

Remark 4.49 We provide a new proof of the result by Hambleton-Pamuk [19] which says that such $\mathbb{Z}_p \times \mathbb{Z}_p$ action does not exist, for $p \geq 5$.

Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$, $p \geq 3$ prime and M be the manifold as above. We recall a few facts and notations about $\mathbb{Z}_p \times \mathbb{Z}_p$.

- The integral cohomology of G is given by

$$H^*(G, \mathbb{Z}) = \mathbb{Z}[u_1, u_2](\mu)/(pu_1, pu_2, p\mu, \mu^2),$$

  where $|u_i| = 2$ and $|\mu| = 3$.

- By (2.13) the restriction maps $\mathrm{Res}^G_K : H^2(G) \to H^2(K)$, to the $(p+1)$ cyclic subgroups $K \simeq \mathbb{Z}_p$ are given by:

$$\begin{aligned}
(u_1, u_2) &\mapsto (0, u) \\
(u_1, u_2) &\mapsto (u, 0) \\
(u_1, u_2) &\mapsto (u, u) \\
&\vdots \\
(u_1, u_2) &\mapsto (u, (p-1)u).
\end{aligned} \tag{4.89}$$

- By lemma 4.4, $\oplus_K \mathrm{Res}^G_K : H^2(G) \to \oplus_K H^2(K)$ is injective.

- The element $\mu \in H^3(G)$ is an essential element, since $\oplus_K \mathrm{Res}^G_K : H^3(G) \to \oplus_K H^3(K)$ is zero, for $K \cong \mathbb{Z}_p < G$. Since $\mu^2 = 0$, there is no essential elements in the even dimension. The essential ideal in the odd dimension is generated by $\langle \mu \rangle$.

- By lemma 4.1, we see that each subgroup has $\chi(M)$ number of fixed points. So, $\chi(M) > 0$. Now, using the fact that $H^q_G(M) \simeq H^q_G(\Sigma)$, for $q > 4$, the dimension bound becomes (see eq (4.4))

$$\dim H^q_G(\Sigma) = \begin{cases} 0, & \textit{for } q \textit{ odd} \\ \frac{p+1}{p}\chi(M). & \textit{for } q \textit{ even.} \end{cases} \tag{4.90}$$

- b_i denotes the i-th rational (integral) betti numbers of M; $t_1 = \dim_{\mathbb{F}_p}(T \otimes \mathbb{F}_p)$, where T is the torsion part of $H_1(M;\mathbb{Z})$. For a detailed calculation of dimensions of E_2-page, see section 4.1.2.
- $\operatorname{Im} d_r^{k,r-1}$ is contained in $\operatorname{Ess}^*(G)$ modulo indeterminacy [lemma 4.11].
- The differentials $d_2^{k,1} = 0$, for $k \geq 0$, the mod-p differential $\bar{d}_2^{0,3}$ and $d_2^{1,3}$ are injective [lemma 4.14, 4.15, 4.17].
- If M has torsion-free homology, then $d_2^{2,3}$ is injective [lemma 4.26].
- If M has torsion-free homology, then $d_2^{0,4}(w) = 0$, for $w \in H^4(M)$ [lemma 4.40].

Lemma 4.50 *In the integral Borel spectral sequence $H^*_G(M)$, the differential $d_3^{3,2} = 0$.*

Proof. As discussed above, there are no essential elements in $H^6(G) = E_2^{6,0}$, since $\mu^2 = 0$. Since $\operatorname{Im} d_3^{3,2}$ must be contained in essential cohomology modulo indeterminacy (lemma 4.11), the assertion follows. ■

Following simple observations can be made from the dimension bound.

Lemma 4.51 *For $p > 3$, we must have $b_2 - 2b_1 > 1$, and for $p \geq 3$ $b_2 - 2b_1 \geq 1$, importantly $b_2 \neq 2b_1$.*

Proof. From the dimension bound above (4.90), we see that $p|\chi(M)$. Since $\chi(M) = b_2 - 2b_1 + 2$, the result follows immediately when $p > 3$.

If $b_2 - 2b_1 = 0$, then for $p \geq 3$, p does not divide $\chi(M) = 2$. The assertion follows. ■

Similar to lemma 4.37, we have the following lemma for $\mathbb{Z}_p \times \mathbb{Z}_p$, p odd prime.

Lemma 4.52 *The maximum divisibility of w with a non-zero differential is p^2. Therefore, in the Borel spectral sequence, we can only have $d_r(w) \neq 0$ and $d_r(pw) \neq 0$.*

Proof. The proof is essentially the same. For convenience, we reproduce it here. Due to lemma 4.21 and the commutative diagram (4.30), we have

$$
\begin{array}{ccc}
\mathbb{Z} \simeq H^4(M) & \xleftarrow{\;q^*\;} & H^4(M^*) \simeq \mathbb{Z} \\
\simeq \Big\| & & \Big\| \simeq \\
H^4(M_0, \partial M_0) & \xleftarrow{\;q^*\ (\times\, p^2)\;} & H^4(M_0^*, \partial M_0^*) \\
\cap\, [M_0, \partial M_0] \simeq \Big\downarrow & & \Big\downarrow \cap\, p^2 [M_0^*, \partial M_0^*] \\
\mathbb{Z} \simeq H_0(M_0) & \xrightarrow[\text{id}]{} & H_0(M_0^*) \simeq \mathbb{Z},
\end{array}
\tag{4.91}
$$

where the bottom map is an isomorphism due to the spectral sequence of the covering $M_0 \to M_0^*$.

From the bottom square, we see that q^* is a map multiplication by p^2. Suppose that w has a maximum divisibility p^3. Then the edge homomorphism $p^* : H_G^*(M) \to H^*(M)$ is multiplication by p^3. Since q^* is multiplication by p^2, this contradicts the commutative diagram of $q^* = p^* \circ j^*$ in (4.29). ■

Lemma 4.53 *Let $G = \mathbb{Z}_p \times \mathbb{Z}_p$ acts on M with $b_2 - 2b_1 = 1$. Suppose that homology of M is torsion free. Then $H^2(M^*)$ is torsion free.*

Proof. Consider the long exact sequence of the pair (M^*, M_0^*)

$$0 \longrightarrow H_2(M_0^*) \longrightarrow H_2(M^*) \longrightarrow H_2(M^*, M_0^*) \longrightarrow H_1(M_0^*) \longrightarrow H_1(M^*) \longrightarrow 0. \tag{4.92}$$

Now, $H_2(M_0, M_0^*) = \mathbb{Z}_2^4$ by lemma 4.22. By lemma 4.23, $H_1(M_0^*) = \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2$.

We also have, Tors $(H^2(M^*)) =$ Tors $(H_1(M^*)) = T_1$ and $H_2(M_0^*) \cong H^2(M^*)$. Therefore, $H_2(M_0^*) = \mathbb{Z}^{b_2} \oplus T_1$.

Similarly, since $H_2(M^*) \cong H^2(M_0^*)$ and Tors $(H^2(M_0^*) =$ Tors $(H_1(M_0^*) = \mathbb{Z}_2^2$, we have $H_2(M^*) = \mathbb{Z}^{b_2} \oplus \mathbb{Z}_2^2$.

Substituting all the values in the above sequence we get

$$0 \to \mathbb{Z}^{b_2} \oplus T_1 \xrightarrow{\;g\;} \mathbb{Z}^{b_2} \oplus \mathbb{Z}_2^2 \xrightarrow{\;f\;} \mathbb{Z}_2^4 \xrightarrow{\;h\;} \mathbb{Z}^{b_1} \oplus \mathbb{Z}_2^2 \xrightarrow{\;l\;} \mathbb{Z}^{b_1} \oplus T_1 \to 0. \tag{4.93}$$

Since f maps $\mathbb{Z}_2^2$ into $\mathbb{Z}_2^4$ and h surjects onto $\mathbb{Z}_2^2$, there is no room for T_1 in the last term. ■

4.3.1 Proof of Theorem 4.48

***Proof*.** Similar to $\mathbb{Z}_2 \times \mathbb{Z}_2$, we will focus on the various differentials on the generator $w \in H^4(G)$ to analyse the sequence.

§ Analysing the dimension counts

Remark 4.54 We recall the notations from remark 4.31 below.

- $R = \dim \operatorname{coker} d_2^{2,2}$; $R' = \dim \operatorname{coker} d_2^{3,2}$.
- $L = \dim \ker d_2^{2,3}$; $L' = \dim \ker d_3^{2,3}$; $L'' = \dim \ker d_4^{2,3}$.
- $S' = \dim \operatorname{coker} d_3^{2,2}$

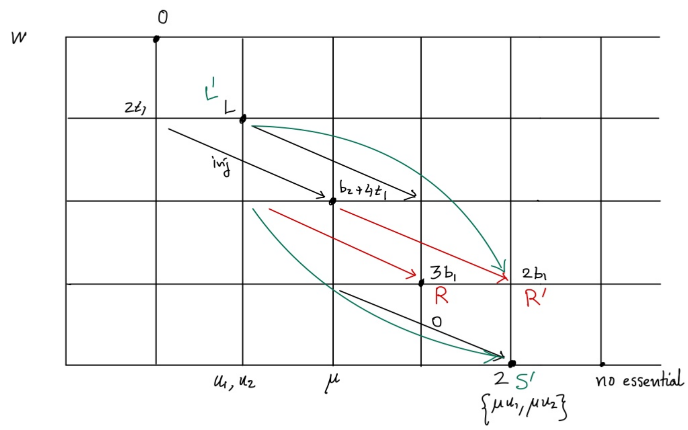


Figure 4.4: E_3-page of $H^*_G(M)$

Case 1 ($d_2(w) \neq 0$; $d_{r\geq 3}(p(w)) = 0$) The dimension count of the 5-line goes as follows.

Table 4.10: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	$L - 1$	0
$E_\infty = E_4$	S'	R	$b_2 - 2b_1 + 2t_1 + R'$	L'	0

From the dimension bound we get,

$$S' = 0, \quad R = 0, \quad b_2 - 2b_1 + 2t_1 + R' = 0. \tag{4.94}$$

Since $b_2 - 2b_1 + 2t_1 + R' = 0$, we must have $b_2 - 2b_1 = 0$, $t_1 = 0$ and $R' = 0$. This is a contradiction due to lemma 4.51, as $p \geq 3$ can not divide $\chi(M) = 2$.

Remark 4.55 *Note that, in the discussion above, the (3, 2)-position is the crucial one. Since there is no essential element in (6, 0)-position, $d_3^{3,2} = 0$, as mentioned in lemma 4.50. Therefore, the only way for the dimension of $E_\infty^{3,2}$ to be different from $b_2 - 2b_1 + 2t_1 + R'$ is, if there is a differential from (0, 4)-position in E_3-page.*

Due to the previous remark, any case with $d_3(nw) = 0$, for $n = 1, p$, has $\dim E_\infty^{3,2} = b_2 - 2b_1 + 2t_1 + R'$. Therefore, by 5-line dimension bound and lemma 4.51, these cases can not occur.

Therefore, we are left with the following set of cases where $d_3(nw) \neq 0$ for $n = 1, p$ for $p \geq 3$ odd.

Case 2 ($d_2(w) \neq 0, d_3(pw) \neq 0, d_4(p^2w) = d_5(p^2w) = 0$) The dimension count of the 5-line goes as follows.

Table 4.11: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	$L - 1$	0
$E_\infty = E_4$	S'	R	$b_2 - 2b_1 + 2t_1 + R' - 1$	L'	0

From 5-line we get,

$$S' = 0, \quad R = 0, \quad b_2 - 2b_1 + 2t_1 + R' = 1. \tag{4.95}$$

Since $b_2 - 2b_1 \geq 1$ for $p \geq 3$, from $b_2 - 2b_1 + 2t_1 + R' = 1$ above, we must have $t_1 = 0$. However, this contradicts lemma 4.27, which says that for a manifold with torsion-free homology $d_2(w) = 0$.

Case 3 ($d_2(w) = 0, d_3(w) \neq 0, d_4(pw) \neq 0, d_5(p^2w) = 0$) We have the following for the dimension count

Table 4.12: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	$L - 1$	0
$E_\infty = E_4$	S'	$R - 1$	$b_2 - 2b_1 + 2t_1 + R' - 1$	L'	0

From the dimension bound, we see that $R = 1$, $b_2 - 2b_1 = 1$, $t_1 = 0$ and $R' = 0$. Since $R' = 0$, $d_2^{3,2}$ is surjective. Therefore, by lemma 4.28, we have $d_2^{0,2}$ is surjective. Since $b_2 - 2b_1 = 1$, we have $p = 3$, since $p|\chi(M)$. Hence, by a suitable change of basis, we can write

$$d_2^{0,2} : z_1 \mapsto 0; \quad (z_{2i}, z_{2i+1}) \mapsto (u_1\alpha_i, u_2\alpha_i), \quad \text{for } 1 \leq i \leq b_1, \tag{4.96}$$

where z_i's and α_i's are the free basis of $H^2(M)$ and $H^1(M)$, respectively. Hence, $d_2^{2,2}$ is given by, for $1 \leq i \leq b_1$,

$$\begin{aligned} u_1 z_{2i} &\longrightarrow u_1^2\alpha_i, \\ u_2 z_{2i} &\longrightarrow u_1u_2\alpha_i, \\ u_1 z_{2i+1} &\longrightarrow u_1u_2\alpha_i, \\ u_2 z_{2i+1} &\longrightarrow u_2^2\alpha_i, \end{aligned}$$

making it a surjection which contradicts $R = 1$, since $R = \dim\operatorname{coker} d_2^{2,2}$.

Case 4 ($d_2(w) = 0, d_3(w) \neq 0, d_4(pw) = 0, d_5(pw) \neq 0$) The dimension table is given by

Table 4.13: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	$L - 1$	0
$E_\infty = E_4$	$S' - 1$	R	$b_2 - 2b_1 + 2t_1 + R' - 1$	L'	0

From the dimension bounds, we have $R = 0 = R' = t_1$ and $b_2 - 2b_1 = 1$. Similar to the previous case, we therefore have $p = 3$. Again, using lemma 4.28, the differential $d_2^{0,2}$ is surjective with $z_1 \to 0$, as in equation (4.96). Therefore, $u_1z_1, u_2z_1 \in \ker d_2^{2,2}$, i.e., they survive in E_3-page. Now, $E_3^{2,2}$ consists u_iz_1, for $i = 1, 2$. Since $E_\infty^{5,0}$ must vanish, the differential $d_3^{2,2}$ must hit one of $\{\mu u_1, \mu u_2\} = E_3^{5,0} = E_2^{5,0}$. If $\mu_1 u_1$ is being hit by $d_3^{2,2}$, we $d_3^{0,3}(z_1) = \mu$, making $d_3^{2,2}$ surjective.

Hence, there is nothing left at $(5,0)$ for d_5 differential to hit, a contradiction to $d_5(pw) \neq 0$.

Case 5 ($d_2(w) = 0, d_3(w) \neq 0, d_4(pw) = 0, d_5(pw) = 0$) The dimension table is given by

Table 4.14: *Dimension count of 5-line*

Page	(5,0)	(4,1)	(3,2)	(2,3)	(1,4)
E_2	2	$3b_1$	$b_2 + 4t_1$	$2b_1 + 3t_1$	0
E_3	2	R	$b_2 - 2b_1 + 2t_1 + R'$	$L - 1$	0
$E_\infty = E_4$	S'	R	$b_2 - 2b_1 + 2t_1 + R' - 1$	L'	0

From the dimension bound, we see that $b_2 - 2b_1 = 1$, $t_1 = 0$ and $R' = 0$. Since $b_2 - 2b_1 = 1$, we must have $p = 3$. Now we consider the case $p = 3$ that is a $\mathbb{Z}_3 \times \mathbb{Z}_3$ action. By lemma 4.21 we have the following commutative diagram

$$\begin{array}{ccc}
\mathbb{Z} \simeq H^4(M) & \xleftarrow{\quad q^* \quad} & H^4(M^*) \simeq \mathbb{Z} \\
\| \simeq & & \| \simeq \\
H^4(M_0, \partial M_0) & \xleftarrow{\quad q^*\ (\times 9) \quad} & H^4(M_0^*, \partial M_0^*) \\
\downarrow \cap [M_0, \partial M_0] \simeq & & \downarrow \cap 9[M_0^*, \partial M_0^*] \\
\mathbb{Z} \simeq H_0(M_0) & \xrightarrow[\text{id}]{\qquad} & H_0(M_0^*) \simeq \mathbb{Z}.
\end{array} \tag{4.97}$$

where q^* is multiplication by 9 (for the commutativity of the bottom square). The isomorphism in the bottom map comes from the spectral sequence of the covering $M_0 \to M_0^*$.

Since $d_3(w) \neq 0$ and $d_{r \geq 4}(w) = 0$, the divisibility of w is 3. Hence, $p^* : H^4_G(M) \to H^4(M)$ is multiplication by 3. Therefore, by the commutativity of $q^* = p^* \circ j^*$ (see diagram 4.72), j^* is multiplication by 3.

Note that, $H^4(M^*) \simeq \mathbb{Z}$. Since $\chi(M) = 3$, by remark 4.2, $H^4_G(\Sigma) = \mathbb{Z}_3^4$. Now, we have the following 4th order short-exact sequence of the pair (M, Σ) as in (4.53),

$$0 \longrightarrow \mathbb{Z} \xrightarrow{j^*(\times 3)} H^4_G(M) \longrightarrow \mathbb{Z}_3^4 \longrightarrow 0. \tag{4.98}$$

The mod-3 differential $\bar{d}_2^{0,3}$ is injective (lemma 4.15). The differential $d_2^{2,2}$ is surjective, since $R = 0$ by the dimension bound calculations above. Therefore, $E_3^{2,2} = 2b_2 - 3b_1 - b_1 = 2$, since $b_2 - 2b_1 = 1$. Since $S' = 0$ by the dimension bound, $d_3^{2,2}$ must surject onto $E_3^{5,0}$, which is essential and has dimension 2. Hence, $E_\infty^{2,2} = 0$. Since the (4,0)-position does not contain any essential elements, nothing hits this position. Therefore, $E_\infty^{4,0} = (\mathbb{Z}_3)^3$.

Now, from the filtration of $H^4_G(M)$

$$0 \subset E_\infty^{4,0} \subset F^{3,1} = F^{2,2} = F^{1,3} \subset F^{0,4} = H^4_G(M)$$

we see that $\mathbb{Z}_3^3 = E_\infty^{4,0} \subset H^4_G(M)$. Therefore, we must have $E_\infty^{3,1} = 0$, otherwise we will have $\mathbb{Z}_3^r \subset F^{3,1}$ for some $r \geq 4$. This contradicts the exact sequence above.

So, the (3,1)-position must be hit by (0,3)-position in E_3 page. Now, the mod-3 differential $\bar{d}_2^{0,3}$ is injective (lemma 4.15). Since $t_1 = 0$, we have $E_2^{1,2} = 0$, so is $d_2^{1,2} = 0$. Now, since $\dim_{\mathbb{F}_3} E_3^{0,3} = b_1 = E_3^{3,1} = b_1$ and $d_3^{0,3}$ must surject onto $E_3^{3,1}$, we have $p^* : H^3_G(M) \to H^3(M)$ is multiplication by 9.

Now, from the following commutative diagram, we see that

$$\begin{array}{ccccccc}
H^3(M) & \xleftarrow{q^*(\times 9)} & H^3(M^*) & & & & \\
\| \simeq & & \| \simeq & & & & \\
H^3(M_0, \partial M_0) & \xleftarrow{q^*(\times 9)} & H^3(M_0^*, \partial M_0^*) & & & & \\
\downarrow {\cap [M_0, \partial M_0]} \simeq & & \downarrow {\cap 9[M_0^*, \partial M_0^*]} & & & & \\
H_1(M_0) & \xrightarrow[q_*(\times 1)]{} & H_1(M_0^*) & \longrightarrow & H_1(G) & \longrightarrow & 0.
\end{array} \tag{4.99}$$

The bottom spectral sequence come from the regular covering of $M_0 \to M_0^*$ as discussed in 4.35, and it is split. Therefore, q^* is multiplication by 9. So, by commutativity $q^* = p^* \circ j^*$ in 4.29, we must have j^* multiplication by 1.

Now, since $R' = 0$, $d_2^{3,2}$ is surjective. So, by lemma 4.28 $d_2^{0,2}$ is surjective. Since $b_2 - 2b_1 = 1$, by a suitable basis change we have the following map

$$d_2^{0,2} : \; z_1 \mapsto 0; \tag{4.100}$$

and the rest of the $2b_1$ elements $\{z_i\}$ for $2 \leq i \leq 2b_1$ maps surjectively onto the $2b_1$ elements in $E_2^{2,1}$.

Hence, $d_3^{0,3}(z_i) = \mu$ and $d_3^{0,3}(3z_i) = 0$, for $2 \leq i \leq 2b_1$. By a similar argument as that to remark 4.42, we can not have $d_3^{0,3}(z_1) = 0$, $d_3^{0,3}(3z_2) = \mu$ and $d_3^{0,3}(z_{i\geq 3}) = 0$. Essentially, this contradicts the dimension bound at the (5,0) position, since $d_3^{3,2} = 0$ in this case.

Therefore, $d_3^{0,3}$ is given by eq 4.100. So, p^* is multiplication by 3.

Now, consider the following diagram

$$
\begin{array}{ccccccccc}
 & & H^2(M) & \xleftarrow{q^*(3,9,9,\dots)} & H^2(M^*) & & & & \\
 & & \| \simeq & & \| \simeq & & & & \\
 & & H^2(M_0, \partial M_0) & \xleftarrow{q^*(3,9,9,\dots)} & H^2(M_0^*, \partial M_0^*) & & & & \\
 & & \downarrow \cap [M_0, \partial M_0] \simeq & & \downarrow \cap 9[M_0^*, \partial M_0^*] & & & & \\
H_3(G) & \longrightarrow & H_2(M_0) & \xrightarrow[q_*(3,1,1,\dots)]{} & H_2(M_0^*) & \longrightarrow & H_2(G) & \longrightarrow & 0,
\end{array}
\tag{4.101}
$$

where the bottom sequence comes from the Cartan-Leray spectral sequence of the covering and is non split. Now, $H^2(M_0) \cong H^2(M) = \mathbb{Z}^{b_2}$ and $H_2(G) = \mathbb{Z}_3$. Since $H^2(M^*)$ is torsion-free, $H_2(M_0^*) = \mathbb{Z}^{b_2}$. Therefore, from the bottom square we get q^* as multiplication by $(3, 9, 9, 9, \dots)$. So, by the commutativity $q^* = p^* \circ j^*$, we get $j^* = \times(1, 3, 3, 3, \dots)$.

Now, consider the long exact sequence of the pair (M, Σ) with $H_G^2(\Sigma) = \mathbb{Z}_3^4$ below

$$
0 \longrightarrow \mathbb{Z}^{b_2} \xrightarrow{j^*(1,3,3,\dots)} H_G^2(M) \longrightarrow \mathbb{Z}_3^4 \longrightarrow \mathbb{Z}^{b_1} \oplus \mathbb{Z}_3^2 \longrightarrow H_G^3(M) \longrightarrow 0,
\tag{4.102}
$$

where $H^3(M^*) = H_1(M_0) = \mathbb{Z}^{b_1} \oplus \mathbb{Z}_3^2$.

We have $E_\infty^{3,0} = \mathbb{Z}_3^3$. Since $E_\infty^{1,1} = 0$, we have the following filtration

$$\mathbb{Z}_3^2 = F^{0,2} = F^{1,1} \subset F^{0,2} = H_G^2(M).$$

Therefore, $E_\infty^{0,2} \cong H_G^2(M)/\mathbb{Z}_3^2$. Since $E_\infty^{0,2} = \mathbb{Z}^{b_2}$, we must have $H_G^2(M) \cong \mathbb{Z}^{b_2} \oplus \mathbb{Z}_3^2$.

Now, $E_\infty^{3,0} = 0$, since $d_3^{3,0}(z_1)$ maps onto μ, as discussed above. Since $d_2^{0,2}$ is surjective (see (4.100)), $E_\infty^{2,1} = 0$. Now, $E_\infty^{1,2} = 0$, since there is no torsion. Therefore, from the filtration of $H^3_G(M)$

$$0 = F^{3,0} = F^{2,1} = F^{1,2} \subset F^{0,3} = H^3_G(M)$$

we have $H^3_G(M) = E^{0,3}_\infty = \mathbb{Z}^{b_1}$.

Substituting all the values in the long exact sequence of the pair (M, Σ) above, we get the following sequence

$$0 \to \mathbb{Z}^{b_2} \xrightarrow{j^*(1,3,3,\dots)} \mathbb{Z}^{b_2} \oplus \mathbb{Z}_3^2 \longrightarrow \mathbb{Z}_3^4 \xrightarrow{\partial^*} \mathbb{Z}^{b_1} \oplus \mathbb{Z}_3^2 \xrightarrow{j^*(\times 1)} \mathbb{Z}^{b_1} \to 0. \quad (4.103)$$

Now, dim coker $[j^*(1,3,3,\dots)] = b_2 - 1 + 2 = b_2 + 1$ and dim ker$[j^*(\times 1)] = 2$. Therefore, from exactness, $b_2 = 1$, and hence $b_1 = 0$, since $b_2 - 2b_1 = 1$.

This proves the assertion in theorem 4.48. ∎

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CHAPTER 5

RANK ONE FINITE GROUP ACTIONS

In the previous chapter, we have dealt with pseudofree actions of rank 2 finite groups. One of the goals of this chapter is to prove the following theorem about the locally linear, pseudofree and homologically trivial actions by groups of rank 1.

Theorem 5.1 *Let G be a finite group of rank one, acting pseudofreely, locally linearly and homologically trivially on a closed, connected and oriented 4-manifold M with $\chi(M) \neq 0$ then $b_1(M) = 0$, and if $b_2(M) \geq 3$, then G must be cyclic and acts semifreely.*

For $b_2(M) \leq 2$, pseudofree actions exist. For example, the action of the quaternion group Q_8 on S^4 by suspension (remark 5.6). Proposition 5.9 provides possible actions by the metacyclic groups (see definition 5.8).

First, we use some group theoretical arguments to reduce to certain minimal rank 1 groups.

5.1 REDUCTION TO MINIMAL RANK 1 GROUPS

Let G be a finite group with rank one. Recall that $\operatorname{rank}(G) = \max_p \operatorname{rank}_p(G)$, where $\operatorname{rank}_p(G) := \{\max r \mid (\mathbb{Z}_p)^r \leq G\}$. These groups are classified in terms of the quotient by maximal normal subgroups of odd order into six types given by Wolf [42, pp. 179, 195-8] and Wall [40, p. 389].

To eliminate many of these groups, it is enough to consider some minimal cases. Every rank one finite group contains one or more of the minimal groups discussed below.

1. Let G be a p-group. Then, since $\operatorname{rank}(G) = 1$, every abelian subgroup is cyclic. Therefore, by Cartan and Eilenberg [7, XII.11.6], G is either a cyclic or a generalized quaternion group,

   (a) for $p = 2$, G is either cyclic, or a generalized quarternion 2-group,

   (b) for p odd, G is cyclic.

2. Suppose that, G is not a p-group.

   Consider a q-Sylow subgroup, $Q \subset \operatorname{Syl}_q(G)$, where $|Q| = q^s$, for some $s \geq 1$.

Now, consider the normalizer $N_G(Q) \leq G$. Since $Q \trianglelefteq N_G(Q)$, we have an extension

$$1 \to Q \hookrightarrow N_G(Q) \twoheadrightarrow K \to 1,$$

where $K = N_G(Q)/Q$ and $q \nmid |K|$. So, we have $N_G(Q) \simeq Q \rtimes K$. Since Q is a Sylow q-subgroup, it is either cyclic or generalized quaternion group, we have

(a) for $q = 2$, $N_G(Q) \simeq Q_{2^n} \rtimes K$.

(b) for $q = 2$ or odd, $N_G(Q) \simeq \mathbb{Z}_{q^s} \rtimes K$.

Consider a p-Sylow subgroup of G, $P \leq K$, where p and q are coprime. Therefore, we have the subgroups $\mathbb{Z}_{q^s} \rtimes \mathbb{Z}_{p^r} \leq N_G(Q) \leq G$.

Now, considering the semidirect product above, we have the conjugation action

$$\alpha : \mathbb{Z}_{p^r} \longrightarrow \mathrm{Aut}(\mathbb{Z}_{q^s}) \simeq \mathbb{Z}_{q^{s-1}} \oplus \mathbb{Z}_{q-1}.$$

Since p and q are coprime, we must have $\mathrm{Im}\,\alpha \leq \mathbb{Z}_{q-1}$. Therefore, we have the action

$$\alpha : \mathbb{Z}_{p^r} \longrightarrow \mathrm{Aut}(\mathbb{Z}_q) \simeq \mathbb{Z}_{q-1}. \tag{5.1}$$

So, we can assume that the groups $H = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, p, q coprime, to be the minimal groups contained in $N_G(Q)$. We also want $\mathrm{Im}\,\alpha \neq \mathrm{id}$, otherwise, H is cyclic. Now,

(i) for $p = 2$, we have the groups $\mathbb{Z}_q \rtimes \mathbb{Z}_{2^r}$, with q odd (since p and q are coprime) and $\mathrm{Im}\,\alpha \neq \mathrm{id}$,

(ii) for p odd, we have the groups $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, with $\mathrm{Im}\,\alpha \neq \mathrm{id}$. Now,

(A) for $q = 2$, we have a cyclic group,

(B) for q odd, we have the metacylic groups $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, such that p and q are relatively prime, and $\mathrm{Im}\,\alpha \neq \mathrm{id}$.

To summarize, given a finite group G of rank 1, we only need to consider the action of the minimal subgroups of G which are cyclic, generalized quaternion or metacyclic groups, to rule out the action of G, wherever possible. In the following sections, we will rule out pseudofree actions of generalized quaternions and metacylic groups for certain 4-manifolds.

5.2 ACTION OF QUATERNION 2-GROUPS

In this section, we consider the generalized quaternion groups (Q_{2^n}, for $n \geq 3$) in (1a) and the groups $Q_{2^n} \rtimes K$ in (2a) from the previous section. The minimal subgroup in both the above groups is given by the quaternion 2-group, $Q_8 = \langle a, b \mid a^4 = e, a^2 = b^2, ba = a^{-1}b \rangle$. We will show that Q_8 cannot act pseudofreely on except on certain 4-manifolds using Borel cohomology with localized coefficients, $\mathbb{Z}_{(2)}$.

Recall that the cohomology ring of Q_8 with $\mathbb{F}_2$-coefficients is given by [1, Lemma 2.10]

$$H^*(Q_8, \mathbb{F}_2) = \mathbb{Z}_2[x_1, y_1, z_4],$$

where $x_1^2 + x_1y_1 + y_1^2 = 0 = x_1^2y_1 + x_1y_1^2$. The cohomology ring of Q_8 with integral coefficients is given by [26, §4]

$$H^*(Q_8, \mathbb{Z}) = \mathbb{Z}[x_2, y_2, z_4],$$

with $8z_4 = 2x_2 = 2y_2 = x_2^2 = y_2^2 = 0$ and $x_2y_2 = 4z_4$.

In the following section, we will discuss the cohomology with coefficients in the localized ring and show that the restriction map $H^2(Q_8; \mathbb{Z}_{(2)}) \to H^2(\mathbb{Z}_2; \mathbb{Z}_{(2)})$ is zero.

5.2.1 Cohomology with coefficients in localized ring

Definition 5.2 The localized ring is defined by $\mathbb{Z}_{(p)} := \left\{ \dfrac{a}{b} \mid p \text{ and } b \text{ are coprime} \right\} \subseteq \mathbb{Q}$.

We note that $H^*(-, \mathbb{Z}_{(p)}) \simeq H^*(-) \otimes \mathbb{Z}_{(p)}$, for $p = 2$ or prime, since $\mathbb{Z}_{(p)}$ is a flat $\mathbb{Z}$-module.

Definition 5.3 We recall the torsion in $H^*(M, \mathbb{Z}_{(p)})$. First note that, $\text{Tors}\,(H_1(M; \mathbb{Z}_{(p)})) = \text{Tors}\,(H_1(M) \otimes \mathbb{Z}_{(p)}$. Now, suppose that $\text{Tors}\,(H_1(M)) = T_p \oplus T_{p'}$, where p and p' are coprime. Therefore, as a $\mathbb{F}_p-$vector space

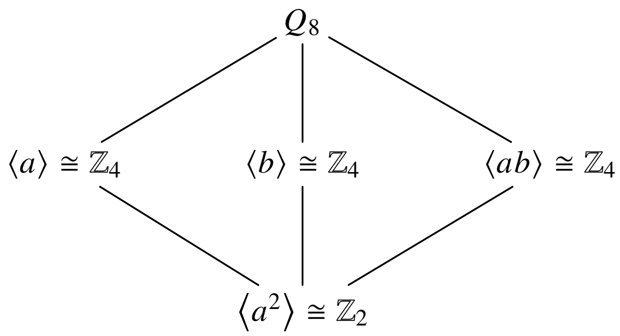
$$\dim_{\mathbb{F}_p} \left(\text{Tors}\,(H_1(M; \mathbb{Z}_{(p)})\right) = \dim_{\mathbb{F}_p} \left(\text{Tors}\,(H_1(M) \otimes \mathbb{Z}_{(p)}) \otimes \mathbb{Z}_p\right) = t_1,$$

since $\mathbb{Z}_{p^r} \otimes \mathbb{Z}_{(p)} \simeq \mathbb{Z}_{p^r}$, and $\mathbb{Z}_{p'} \otimes \mathbb{Z}_{(p)} = 0$, for p and p' coprime.

Lemma 5.4 *The restriction map* $\text{Res}^{Q_8}_{\mathbb{Z}_2} : H^2(Q_8, \mathbb{Z}_{(2)}) \to H^2(\mathbb{Z}_2, \mathbb{Z}_{(2)})$ *is zero.*

Proof. It suffices to show that the restriction map is zero in integral coefficients i.e.,

$H^2(Q_8) \to H^2(\mathbb{Z}_2)$ is zero. The subgroup structure of Q_8 shown in the following diagram



(5.2)

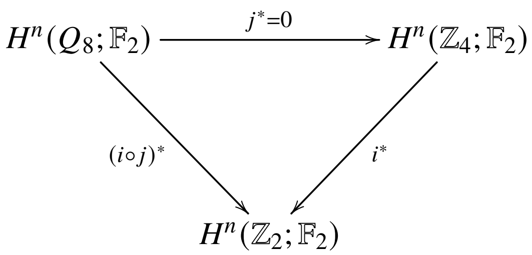
Note that, the cohomology ring of $\mathbb{Z}_4$ is given by [26], $H^*(\mathbb{Z}_4, \mathbb{F}_2) = \Lambda[u_1] \otimes \mathbb{F}_2[v_2]$, where $u_1^2 = 0$. From mod-2 group cohomology ring of Q_8, we have generators $x_1 \in \operatorname{Hom}(\langle a \rangle, \mathbb{Z}_2)$ and $y_1 \in \operatorname{Hom}(\langle b \rangle, \mathbb{Z}_2)$. Therefore, we get the following restriction maps for $H^1(Q_8; \mathbb{F}_2)$.

Table 5.1: *Restriction of $H^1(Q_8; \mathbb{F}_2)$ onto each $\mathbb{Z}_4$ subgroups*

Q_8	$\langle a \rangle$	$\langle b \rangle$	$\langle ab \rangle$
x_1	u_1	0	u_1
y_1	0	u_1	u_1

Since u_1 is an exterior element, we see that $x_1^2 \mapsto 0$ and $y_1^2 \mapsto 0$ under the restriction map onto each of the $\mathbb{Z}_4$ subgroups. Therefore, the restriction $\operatorname{Res}^{Q_8}_{\mathbb{Z}_4} : H^2(Q_8; \mathbb{F}_2) \to H^2(\mathbb{Z}_4; \mathbb{F}_2)$ is zero.

Now, from the inclusion of subgroups $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4 \hookrightarrow Q_8$, we have the following commutative diagram in cohomology



(5.3)

Since the restriction map $j^* : H^2(Q_8; \mathbb{F}_2) \to H^2(\mathbb{Z}_4; \mathbb{F}_2)$ is zero, the restriction onto $\mathbb{Z}_2$-subgroup $(i \circ j)^* : H^2(Q_8; \mathbb{F}_2) \to H^2(\mathbb{Z}_2; \mathbb{F}_2)$ is zero.

Therefore, using the commutativity of reduction mod 2

$$\begin{array}{ccc} H^2(Q_8;\mathbb{Z}) & \xrightarrow{\text{reduction}} & H^2(Q_8;\mathbb{F}_2) \\ \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}=0} \\ H^2(\mathbb{Z}_2;\mathbb{Z}) & \xrightarrow[\simeq]{} & H^2(\mathbb{Z}_2;\mathbb{F}_2), \end{array} \tag{5.4}$$

we see that the left vertical restriction map $H^2(Q_8,\mathbb{Z}) \to H^2(\mathbb{Z}_2,\mathbb{Z})$ is zero. ∎

5.2.2 Modified version of Edmonds' result

First, we note a modified version of Theorem 5.2 in [10]. In the original paper, the proof has been done for simply-connected 4-manifolds, however, the proof applies just as well for $H_1(M;\mathbb{Z}_{(2)}) = 0$ in a similar way. For completeness, we prove it through the proposition below.

Proposition 5.5 *If a finite group G acts semifreely and homologically-trivially on a closed, connected and oriented 4-manifold M such that $H_1(M;\mathbb{Z}_{(|G|)}) = 0$ and $b_2(M) \geq 1$, then G is cyclic,*

Proof of Proposition 5.5. Suppose the fixed set of G, M^G contains a 2-dimensional component. Since the action is semi-freely, G acts freely on the normal 2-plane, $\nu(M^G)$. Therefore, for any $x \in M^G$, we have a free action of G on the boundary of the normal 2-disk neighbourhood, S^1. Therefore, G must have period 2. Hence, G must be cyclic.

Therefore, M^G must be finite, containing isolated points. Since the action is semifree, $M^G = M^K$, for cyclic subgroups $K \leq G$. Now, the action is homologically trivial and locally linear. Therefore, M^K contains $\chi(M)$ number isolated points by Lefschetz fixed point theorem (see Lemma 4.1).

Since G has a global fixed point, it acts freely on S^3. Therefore, G must have cohomology with period 4 or 2. But the groups with period 2 are cyclic [28, 38].

For the groups with period 4, we first consider the integral group cohomology of G [10, Proposiion 5.1],

$$H^i(G;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, \\ G/[G,G] & \text{for } i > 0, i = 4k+2, \\ \mathbb{Z}_{|G|} & \text{for } i > 0, i = 4k, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

Note that, the group cohomology with the coefficients in $\mathbb{Z}_{(|G|)}$ is given by $H^i(G;\mathbb{Z}_{(|G|)}) = H^i(G) \otimes \mathbb{Z}_{(|G|)}$. From [10], we know that $H^i_G(M;\mathbb{Z}_{(|G|)}) \simeq H^i_G(M^G;\mathbb{Z}_{(|G|)})$, for $i > 4$. Since $H_1(M;\mathbb{Z}_{(|G|)}) = 0$, the Borel cohomology, $H^*_G(M;\mathbb{Z}_{(|G|)})$ is concentrated on the even dimensions. Therefore, the sequence collapses.
Now, considering the 12-line, we get

$$\begin{aligned} \left|H^{12}_G(M;\mathbb{Z}_{(|G|)})\right| &= \left|H^{12}(G;\mathbb{Z}_{(|G|)})\right| \times \left|H^{10}(G;\mathbb{Z}_{(|G|)}) \otimes H^2(M;\mathbb{Z}_{(|G|)})\right| \times \left|H^8(G;\mathbb{Z}_{(|G|)})\right| \\ &= |G| \times \left|G/[G,G]\right|^{b_2} \times |G| \end{aligned} \tag{5.6}$$

For the fixed set, we have

$$\begin{aligned} \left|H^{12}_G(M^G;\mathbb{Z}_{(|G|)})\right| &= \left|H^{12}(G;H^0(M^G;\mathbb{Z}_{(|G|)}))\right| \\ &= |G|^{b_2+2}. \end{aligned} \tag{5.7}$$

Therefore, comparing equations (5.6) and (5.7), and using $H^{12}_G(M;\mathbb{Z}_{(|G|)}) \simeq H^{12}_G(M^G;\mathbb{Z}_{(|G|)})$, for $4k+2 > 4$, we get

$$|G| = \left|G/[G,G]\right|.$$

Hence, $[G,G]$ is trivial, implying that G is abelian. Thus, G is cyclic, since every abelian periodic group is cyclic. ■

Remark 5.6 Note that, pseudofree and semifree actions of Q_8 on manifolds with $b_2 = 0$ may exist in the following way. Since Q_8-acts freely on S^3, one can get a pseudofree action of Q_8 on S^4 by suspension.

5.2.3 Proof of Theorem 5.7

In this section, we prove the following theorem regarding the Q_8-action.

Theorem 5.7 *Let Q_8 acts pseudofreely, locally linearly and homologically trivially on a closed, connected and oriented 4-manifold M. Then,*

1. *$H_1(M;\mathbb{Z}_{(2)}) = 0$, and*

2. *if $b_2(M) \geq 1$, then the action of Q_8 does not exist. Therefore, groups containing Q_8 cannot act on such manifolds in this way.*

We divide the proof into the following three steps:

(Step 1) We show that $H_1(M;\mathbb{Z}_{(2)}) = 0$.

(Step 2) We show that the pseudofree action of Q_8 is a semifree action.

(Step 3) Using Proposition 5.5, which is a modified version of [10, Theorem 5.2], we conclude the result.

Proof. (Step 1) Comparing the E_2-page of the Borel cohomology of Q_8 with $\mathbb{Z}_2$ (as $\mathbb{Z}_2$-vector spaces), we have the following commutative diagram:

$$\begin{array}{ccc}
E_2^{0,3}(Q_8) = H^3(M;\mathbb{Z}_{(2)}) & \xrightarrow{\simeq} & H^3(M;\mathbb{Z}_{(2)}) = E_2^{0,3}(\mathbb{Z}_2) \\
\downarrow{\scriptstyle \otimes \mathbb{Z}_2} & & \downarrow{\scriptstyle \otimes \mathbb{Z}_2} \\
\bar{E}_2^{0,3}(Q_8) = H^3(M;\mathbb{Z}_{(2)}) \otimes \mathbb{Z}_2 & \xrightarrow{\simeq} & H^3(M;\mathbb{Z}_{(2)}) \otimes \mathbb{Z}_2 = \bar{E}_2^{0,3}(\mathbb{Z}_2) \\
\downarrow{\scriptstyle \bar{d}_2^{0,3}} & & \downarrow{\scriptstyle \bar{d}_2^{0,3};\ \mathrm{inj}} \\
E_2^{2,2}(Q_8) = H^2(Q_8;H^2(M;\mathbb{Z}_{(2)})) & \xrightarrow[0]{} & H^2(\mathbb{Z}_2;H^2(M;\mathbb{Z}_{(2)})) = E_2^{2,2}(\mathbb{Z}_2).
\end{array} \tag{5.8}$$

Note that, $E_2^{0,3} = H^0(-;H^3(M;\mathbb{Z}_{(2)})) \simeq H^3(M;\mathbb{Z}_{(2)})$. The right vertical map in the bottom square is injective due to Proposition 3.1. Since $H^2(Q_8;H^2(M;\mathbb{Z}_{(2)})) = H^2(Q_8;\mathbb{Z}_{(2)}) \otimes H^2(M;\mathbb{Z}_{(2)})$ and $H^2(\mathbb{Z}_2;H^2(M;\mathbb{Z}_{(2)})) = H^2(\mathbb{Z}_2;\mathbb{Z}_{(2)}) \otimes H^2(M;\mathbb{Z}_{(2)})$ by lemma 5.4, the bottom horizontal map is zero. Therefore, from the bottom square of the commutative diagram above, we see that $\bar{E}_2^{0,3}(Q_8) = 0$. Since $\dim_{\mathbb{F}_2} \bar{E}_2^{0,3}(Q_8) = b_1 + t_1 = 0$, we have $E_2^{0,3}(Q_8) = H^3(M;\mathbb{Z}_{(2)}) = 0 = H_1(M;\mathbb{Z}_{(2)})$, by Poincare duality.

(Step 2) For the singular set Σ, we observe that $\mathrm{Fix}\langle a\rangle \subseteq \mathrm{Fix}\langle a^2\rangle$ and $\mathrm{Fix}\langle b\rangle \subseteq \mathrm{Fix}\langle a^2\rangle$, where a, b are the generators of Q_8 and $\langle a\rangle \simeq \mathbb{Z}_4 \simeq \langle b\rangle$, $\langle a^2\rangle \simeq \mathbb{Z}_2$ are the cyclic subgroups of Q_8. Now, since the action is homologically-trivial, fixed set of each cyclic subgroup contain $\chi(M)$ number of isolated points. Therefore, combining with inclusion of fixed sets, we have $\mathrm{Fix}\langle a\rangle = \mathrm{Fix}\langle a^2\rangle = \mathrm{Fix}\langle b\rangle$. Now, since $\mathrm{Fix}\langle a\rangle = \mathrm{Fix}\langle b\rangle$, we have $\mathrm{Fix}\langle ab\rangle = \mathrm{Fix}\langle a\rangle$. Hence, the action is semifree.

(Step 3) Therefore, by Proposition 5.5 (the modified version of [10, Theorem 5.2]), we see that groups acting semifreely on 4-manifolds, M, with $b_2 \geq 1$ must be cyclic. Hence, Q_8 cannot have such an action on M.

Hence, the generalized quaternion groups and $Q_{2^n} \rtimes K$ cannot act on 4-manifolds with $b_2 \geq 1$. ∎

5.3 ACTION OF THE METACYCLIC GROUPS

As discussed in section 5.1, we consider the action of the Metacyclic groups, $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, where q is odd and p, q are coprime. We recall the definition of such groups below.

Definition 5.8 The metacyclic groups are given by $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, where q and p are coprime and we have the following extension

$$1 \longrightarrow \mathbb{Z}_q \longrightarrow G \longrightarrow \mathbb{Z}_{p^r} \longrightarrow 1 \tag{5.9}$$

and the presentation $\left\langle a, b \mid a^q = 1 = b^{p^r}, bab^{-1} = a^k \right\rangle$, where $k^{p^r} \equiv 1 \pmod{q}$. To see the presentation, note that $a = b^{p^r} a\, b^{-p^r} = a^{k^{p^r}}$. This implies, $k^{p^r} \equiv 1 \pmod{q}$. We also want to have $k \not\equiv 1 \pmod{q}$, so that the group is not abelian. The above presentation can also be seen by the induced action

$$\mathbb{Z}_{p^r} \xrightarrow{\alpha} \mathrm{Aut}(\mathbb{Z}_q) \simeq \mathbb{Z}_{q-1}, \text{ where } \alpha(b)(a) = bab^{-1} = a^k.$$

In the following section, we discuss the Group Cohomology of the Metacyclic groups.

5.3.1 Group Cohomology of the Metacyclic groups

To find the integral group cohomology of G, we use Serre spectral sequence. Since the cohomologies are concentrated in the even dimensions, the spectral sequence collapses in E_2-page. Therefore, $E_\infty = E_2$. Hence, we get $H^{i+j}(G) = E_2^{i,j} = H^i(\mathbb{Z}_{p^r}, H^j(\mathbb{Z}_q))$. So, we have

$$E_2^{i,j} = H^i(\mathbb{Z}_{p^r}; H^j(\mathbb{Z}_q)) = \begin{cases} \mathbb{Z}_{p^r} & i = \text{even},\ j = 0, \\ H^0(\mathbb{Z}_{p^r}; H^j(\mathbb{Z}_q)) = \left(H^j(\mathbb{Z}_q)\right)^{\mathbb{Z}_{p^r}} & i = 0,\ j = \text{even}, \\ 0 & \text{otherwise}. \end{cases} \tag{5.10}$$

To compute, $H^0(\mathbb{Z}_p; H^j(\mathbb{Z}_q))$, we proceed as follows. The automorphism $a \mapsto a^k$ on $\mathbb{Z}_q$, induces the multiplication map $a \longrightarrow ka$ on $H_1(\mathbb{Z}_q)$. Since Hom $(H_{2m}(\mathbb{Z}_q), \mathbb{Z}) = 0$, by Universal Coefficient theorem, we have $H^{2m}(\mathbb{Z}_q) = \text{Ext}\,(H_{2m-1}(\mathbb{Z}_q), \mathbb{Z})$, for $m \geq 1$. Since the multiplication map of groups induces the multiplication map on Ext groups, we have the multiplication map $a \mapsto ka$ on $H^2(\mathbb{Z}_q)$. Therefore, we get the map multiplication map $a^{2m} \mapsto k^m a^{2m}$ on $H^{2m}(\mathbb{Z}_q)$ induced by cup product.

From the discussion above, the fixed set is given by

$$\left(H^{2l}(\mathbb{Z}_q)\right)^{\mathbb{Z}_{p^r}} = \left\{a \mid k^l a \equiv a \pmod{q}\right\}, \text{ for } l > 0.$$

Now, consider $b^{p^{r-1}} \in \mathbb{Z}_p \leq \mathbb{Z}_{p^r}$. Therefore, the induced action becomes $\alpha(b^{p^{r-1}})a = a^k$, for $a \in \mathbb{Z}_q$. This says, $k^p \equiv 1 \pmod{q}$. Therefore, from $k^l \equiv 1 \pmod{q}$ above, we get $l = mp$, for $m > 0$. Hence, the cohomology of G is given by

$$H^i(G; \mathbb{Z}) = \begin{cases} 0 & \text{for } i \text{ odd}, \\ \mathbb{Z}_{p^r} \oplus \mathbb{Z}_q & \text{for } i = 2mp, \\ \mathbb{Z}_{p^r} & \text{for } i \text{ even and } i \neq 2mp, \\ \mathbb{Z} & \text{for } i = 0, \end{cases} \tag{5.11}$$

which is $2p$-periodic.

In the following sections, we will consider the action of following Metacyclic groups separately:

(Type I) $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, for p, q odd, $r \geq 1$, p, q coprime and the action $\text{Im}\,\alpha \neq \text{id} \subset \mathbb{Z}_{q-1}$,

(Type II) $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{2^r}$, for q odd, $r \geq 2$ and the action $\text{Im}\,\alpha \neq \text{id} \subset \mathbb{Z}_{q-1}$,

(Type III) $G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$, for q odd and the action $\text{Im}\,\alpha \neq \text{id} \subset \mathbb{Z}_{q-1}$,

Note that, if the action is trivial, we would have abelian groups.

Now, we prove the following slightly modified version of [10, Theorem 7.2]. The proof of this proposition is essentially same as that of Edmond's article.

Proposition 5.9 *Let $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}, q$ odd, be a non-abelian metacyclic group acting pseudofreely, locally-linearly and homologically-trivially on a closed, connected and oriented 4-manifold M with $H_1(M, \mathbb{Z}_{(q)}) = 0$. Then*

(1) for (Type I) groups, $b_2(M) = 1$, $p = 3$ and $r = 1$,

(2) for (Type II) groups, $b_2(M) = 0$,

(3) for (Type III) groups, $b_2(M) = 2$.

Proof. We will use the fact that the Borel Cohomologies $H^n_G(M; \mathbb{Z}_{(q)})$ and $H^n_G(\Sigma; \mathbb{Z}_{(q)})$ are isomorphic for $n > 4$. Since the Borel spectral sequence of $H^*_G(M; \mathbb{Z}_{(q)})$ contains only three (0, 2, 4) horizontal lines and the cohomology is concentrated in the even lines, the spectral sequence collapses in the E_2-page and $E_\infty = E_2$. The following table lists the graded module $H^n_G(M; \mathbb{Z}_{(q)})$ in Borel cohomology.

Now, with the coefficients in $\mathbb{Z}_{(q)}$, the group cohomology G becomes $H^*(G; \mathbb{Z}_{(q)}) = H^*(G) \otimes \mathbb{Z}_{(q)}$. Since $\mathbb{Z}_q \otimes \mathbb{Z}_{(q)} \simeq \mathbb{Z}_q$ and $\mathbb{Z}_{p^r} \otimes \mathbb{Z}_{(q)} = 0$, for p, q coprime, from eqn (5.11) we get

$$H^i(G; \mathbb{Z}_{(q)}) = \begin{cases} \mathbb{Z}_{(q)} & \text{for } i = 0, \\ \mathbb{Z}_q & \text{for } i = 2mp, \\ 0 & \text{otherwise.} \end{cases} \tag{5.12}$$

We also recall that, if G acts freely on S^n, then the minimal period of $H^*(G)$ divides $n + 1$.

(1) Consider the (Type I) groups, G, where p is odd, $r \geq 1$ and p, q are coprimes. Since the cohomology of G has period $2p$, G cannot have period 4. Therefore, G cannot act freely on S^3. So, $M^G = \emptyset$.

Now, since the action is pseudofree and homologically-trivial, each cyclic subgroup K has $\chi(M)$ many fixed points which is permuted by G/K. For the $G/\mathbb{Z}_q$, these $\chi(M)$ points are permuted by $\mathbb{Z}_{p^r}$ in $x_p = \dfrac{\chi(M)}{p^r}$ orbits.

Therefore, for the singular set, we get

$$\begin{aligned} H^n(G, H^0(\Sigma; \mathbb{Z}_{(q)})) &= H^n(G, \mathbb{Z}_{(q)}[G/\mathbb{Z}_q])^{x_p}, \\ &\simeq H^n(\mathbb{Z}_q; \mathbb{Z}_{(q)})^{x_p}, \\ &= (\mathbb{Z}_q)^{x_p}, \quad \text{for } n \text{ even.} \end{aligned} \tag{5.13}$$

So, the order is given by $|H^n_G(\Sigma; \mathbb{Z}_{(q)})| = q^{x_p}$, for n even.

Now, we tabulate the graded groups $\mathcal{G}H^n_G(M; \mathbb{Z}_{(q)})$, in the following table.

Table 5.2: *The graded groups* $\mathcal{G}H^n_G(M;\mathbb{Z}_{(q)})$

Value	$H^n(G;H^0(\mathbb{Z}_{(q)}))$	$H^{n-2}(G,H^2(M;\mathbb{Z}_{(q)}))$	$H^{n-4}(G;H^4(M;\mathbb{Z}_{(q)})$
$n=4p^r$	$\mathbb{Z}_q$	0	0
$n=4p^r+2$	0	$\mathbb{Z}_q\otimes H^2(M;\mathbb{Z}_{(q)})$	0

Therefore, computing the order, we have the following:

$$|\mathcal{G}H^n_G(M)| = \begin{cases} q & n=4p^r, \\ q^{b_2} & n=4p^r+2. \end{cases} \tag{5.14}$$

Since $H^n_G(M;\mathbb{Z}_{(|G|)}) \simeq H^n_G(\Sigma;\mathbb{Z}_{(|G|)})$ for $n>4$, by comparing the order of the groups from equation (5.13) and (5.14) we get

$$\begin{aligned} x_p &= 1, \quad \text{from } n=4p^r, \\ x_p &= b_2, \quad \text{from } n=4p^r-2. \end{aligned} \tag{5.15}$$

Therefore, from above equation, we get $b_2 = x_p = 1$.

So, the Euler characteristic is given by $\chi(M)=3$. From $x_p = \dfrac{\chi(M)}{p^r} = 1$, we see that $p^r=3$. Therefore, p and r must satisfy $p=3$ and $r=1$.

(2) Consider the (Type II) groups $G=\mathbb{Z}_q \rtimes \mathbb{Z}_{2^r}$, where q is odd and $r\geq 2$. Since, G can act freely on S^3, it might have a global fixed point. The computation of the Borel cohomology of $H^n_G(\Sigma;\mathbb{Z}_{(q)})$ is follows:

$$\begin{aligned} H^n(G,H^0(\Sigma;\mathbb{Z}_{(q)})) &= H^n(G,\mathbb{Z}_{(q)}[G/\mathbb{Z}_q])^{x_2} \oplus H^n(G,\mathbb{Z}_{(q)}[G/G])^{x_1} \\ &\simeq H^n(\mathbb{Z}_q;\mathbb{Z}_{(q)})^{x_2} \oplus H^n(G;\mathbb{Z}_{(q)})^{x_1}, \\ &= \begin{cases} (\mathbb{Z}_q)^{x_2}\oplus(\mathbb{Z}_q)^{x_1} & \text{for } n=4m>0, \\ (\mathbb{Z}_q)^{x_2} & \text{for } n \text{ even}, n\neq 4m, \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{5.16}$$

where $x_2 = \dfrac{\chi(M)}{2^r}$ is the number of orbits corresponding to $G/\mathbb{Z}_q$ (the isotropy subgroup $\mathbb{Z}_{2^r}$)and x_1 is the number of orbits corresponding to the isotropy group of

G.

Therefore, the order $|H^n_G(\Sigma;\mathbb{Z}_{(q)})|$ is given by

$$|H^n_G(\Sigma;\mathbb{Z}_{(q)})| = \begin{cases} q^{x_2+x_1} & \text{for } n = 4m > 0, \\ q^{x_2} & \text{for } n \text{ even, } n \neq 4m > 0. \end{cases} \tag{5.17}$$

Using the localized group cohomology of G from eqn (5.12), we tabulate the graded groups $\mathcal{G}H^n_G(M;\mathbb{Z}_{(q)})$ below.

Table 5.3: *The graded groups* $\mathcal{G}H^n_G(M;\mathbb{Z}_{(q)})$

Value	$H^n(G;H^0(\mathbb{Z}_{(q)}))$	$H^{n-2}(G,H^2(M;\mathbb{Z}_{(q)}))$	$H^{n-4}(G;H^4(M;\mathbb{Z}_{(q)})$
$n = 4\cdot 2^r$	$\mathbb{Z}_q$	0	$\mathbb{Z}_q$
$n = 4\cdot 2^r + 2$	0	$\mathbb{Z}_q \otimes H^2(M;\mathbb{Z}_{(q)})$	0

Therefore, the order is given by

$$|\mathcal{G}H^n_G(M)| = \begin{cases} q^2 & n = 4\cdot 2^r, \\ q^{b_2} & n = 4\cdot 2^r + 2. \end{cases} \tag{5.18}$$

Using $H^n_G(M;\mathbb{Z}_{(|G|)}) \simeq H^n_G(\Sigma;\mathbb{Z}_{(|G|)})$ for $n > 4$ and comparing the order of the groups from equation (5.16) and (5.18) we get

$$\begin{aligned} q^{x_1+x_2} = q^2 &\implies x_1 + x_2 = 2, \quad \text{from } n = 4p^r, \\ q^{x_2} = q^{b_2} &\implies x_2 = b_2, \quad \text{from } n = 4p^r + 2. \end{aligned} \tag{5.19}$$

From the equation above, we see that $b_2 = x_2 \leq x_1 + x_2 = 2$. So, we have the following cases

(a) $b_2 = x_2 = 1; x_1 = 1$: From, $\chi(M) = 3$. This contradicts the facts that $2^r|\chi(M)$.

(b) $b_2 = x_2 = 2; x_1 = 0$. In this case, $x_2 = 2 = \chi(M)/2^r$. Since $\chi(M) = 2 + 2 = 4$, we have $2^r = 2$, implying $r = 1$. This contradicts our assumption of $r \geq 2$.

Therefore, the only possibility is $b_2 = 0$.

(3) Consider the (Type III) groups $G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$, for q odd. These are also the Dihedral groups. Note that, these groups cannot act freely on S^3 by Milnor's $2p$-condition. Therefore, there is no global fixed point for the action. So, we can just substitute $x_1 = 0$ in eqn (5.19) and get that $b_2 = x_2 = 2$, where $x_2 = \frac{\chi(M)}{2}$.

This proves the assertion in Propostion 5.9. ■

In the section below, we will prove the following theorem.

Theorem 5.10 *The groups containing the non-abelian metacyclic groups $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, q odd, $p = 2$ or odd, p, q coprime, cannot act pseudofreely, locally-linearly and homologically-trivially on a closed, connected and oriented 4-manifold M unless $H_1(M;\mathbb{Z}_{(q)}) = 0$.*
In that case, we must have $b_2(M) = 1$ and $G = \mathbb{Z}_q \rtimes \mathbb{Z}_3$, or $b_2(M) = 2$ and $G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$.

5.3.2 Proof of Theorem 5.10

First, we prove the following proposition below.

Proposition 5.11 *Suppose, $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, q odd, $p = 2$ or odd, p, q coprime is non-abelian and acts pseudofreely, locally linearly and homologically-trivially on a closed, connected, oriented 4-manifold. Then $H_1(M;\mathbb{Z}_{(q)}) = 0$.*

Proof. We will use Borel spectral sequence with the localized coefficients, $\mathbb{Z}_{(q)}$ to compare $H^n_G(M;\mathbb{Z}_{(q)})$ and $H^n_K(M;\mathbb{Z}_{(q)})$, where $K = \mathbb{Z}_q \leq G$. By naturality of spectral sequence, we have the following commutative diagram between E_2- pages of $H^*_G(M,\mathbb{Z}_{(q)})$ and $H^*_{\mathbb{Z}_q}(M,\mathbb{Z}_{(q)})$ after reduction,

$$\begin{array}{ccc} H^3(M,\mathbb{Z}_{(q)})\otimes\mathbb{Z}_q = \bar{E}_2^{0,3}(G) & \xrightarrow{\ \simeq\ } & \bar{E}_2^{0,3}(K) = H^3(M,\mathbb{Z}_{(q)})\otimes\mathbb{Z}_q \\ \Big\downarrow{\scriptstyle d_2=0} & & \Big\downarrow{\scriptstyle \bar{d}_2^{0,3}\ \text{inj}} \\ 0 = E_2^{2,2}(G) & \xrightarrow[0]{\qquad} & E_2^{2,2}(K) = \mathbb{Z}_q\otimes H^2(M;\mathbb{Z}_{(q)}) \end{array} \tag{5.20}$$

Note that, with the localized coefficients $\mathbb{Z}_{(q)}$, we get $E_2^{2,2}(G) = H^2(G,\mathbb{Z}_{(q)})\otimes H^2(M,\mathbb{Z}_{(q)}) = \mathbb{Z}_{p^r}\otimes H^2(M,\mathbb{Z}_{(q)}) = 0$, since $H^2(G,\mathbb{Z}_{(q)}) = H^2(G,\mathbb{Z})\otimes\mathbb{Z}_{(q)} = \mathbb{Z}_{p^r}\otimes\mathbb{Z}_{(q)} = 0$. Therefore, the bottom horizonal map $E_2^{2,2}(G) \longrightarrow E_2^{2,2}(\mathbb{Z}_q)$ and the left vertical maps are zero. Hence, from the above diagram, we must have $H^3(M,\mathbb{Z}_{(q)}) = 0 = H_1(M,\mathbb{Z}_{(q)})$. ■

Proof of Theorem 5.10. If $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$, q odd, $p = 2$ or odd, p, q coprime, acts in the usual way, then by Proposition 5.11, $H_1(M; \mathbb{Z}_{(q)} = 0$. Therefore, using Proposition 5.9, we have the assertion. ■

5.4 PROOF OF THEOREM 5.1

To finish the chapter, we prove the main theorem (Theorem 5.1) of this chapter. The theorem is restated below:

Let G be a finite group of rank one, acting pseudofreely, locally linearly and homologically trivially on a closed, connected and oriented 4-manifold M with $\chi(M) \neq 0$ *then* $b_1(M) = 0$*, and if* $b_2(M) \geq 3$*, then G must be cyclic and acts semifreely.*

Proof. Given a finite group G of rank one, we have the reduction to the minimal subgroups: cyclic, generalized quaternions (Q_{2^n}, $n \geq 3$) and metacyclic groups (see section 5.1).

By theorems 5.7 and 5.10, $H_1(M; R) = 0$, where R is either $\mathbb{Z}_{(2)}$ or $\mathbb{Z}_{(q)}$, for q odd. Therefore, $b_1(M) = 0$.

Theorem 5.7 rules out actions of the generalized quaternion 2-groups for $b_2(M) \geq 1$. The action of the non-abelian metacyclic groups is ruled out by Theorem 5.10 for $b_2(M) \geq 3$. Therefore, we are left with the cyclic groups. Now, a pseudofree action of a cyclic group is semifree by Lefschetz fixed point theorem, with the singular set $\Sigma = M^G$ consisting $\chi(M)$ many points. This proves our assertion. ■

CHAPTER 6

PROOF OF THE MAIN THEOREMS

In this chapter, we collect the previous results and prove the two main theorems (Theorem A and Theorem B) stated in Chapter 1. We restate the theorems below for convenience.

6.1 PROOF OF THEOREM A

Theorem A: *Let G be a non-trivial finite group acting pseudofreely, locally linearly and homologically trivially on a closed, connected, oriented 4-manifold M with non-zero Euler characteristic. Then $b_1(M) = 0$, and if $b_2(M) \geq 3$, then G must be cyclic and acts semi-freely.*

Proof. For the rank two groups, by Theorems 4.47 and 4.48, we have $H_1(M;\mathbb{Z}_{(2)}) = 0 = H_1(M;\mathbb{Z}_{(3)})$. For the rank one groups, we have $H_1(M;\mathbb{Z}_{(2)}) = 0$, for $G = \mathbb{Q}_8$ and $H_1(M;\mathbb{Z}_{(q)}) = 0$, for $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{p^r}$ by Theorems 5.7 and 5.10. Therefore, $b_1(M) = 0$.

If $b_2(M) \geq 3$, the action of rank two groups are ruled out by the Theorems 4.47 and 4.48. Now, by Theorem 5.1, we see that G must be cyclic and acts semifreely. ■

6.2 PROOF OF THEOREM B

Theorem B: *Let G be a non-trivial finite group acting pseudofreely, locally linearly and homologically trivially on a closed, connected, oriented 4-manifold M with non-zero Euler characteristic such that $b_2(M) \leq 2$. Then we have the following.*

1. *If $b_2(M) = 2$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, then M must have the same 2-local homology and the intersection form as $S^2 \times S^2$. If $G = \mathbb{Z}_q \rtimes \mathbb{Z}_2$, q odd is Dihedral, then M must have the same $q-$local cohomology as $S^2 \times S^2$.*

2. *If $b_2(M) = 1$ and $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, then M must have the same 3-local homology and the intersection form as $\mathbb{CP}^2$. If $G = \mathbb{Z}_q \rtimes \mathbb{Z}_3$, q odd, $3 \nmid q$ is non-abelian, then M must have the same $q-$local cohomology as $\mathbb{CP}^2$.*

3. *If* $b_2(M) = 0$ *and* $G = \mathbb{Z}_q \rtimes \mathbb{Z}_{2^r}$, *q odd,* $r \geq 2$, *is non-abelian, then M must have the same q−local homology as* S^4.

Proof. By Theorem 4.47, Theorem 4.48 and Theorem 5.10, we get (1) and (2). Using Proposition 5.9, we get (3). ■

Remark 6.1 Hambleton and Pamuk [19, Theorem B] showed that

If a finite group G acts pseudofreely, locally linearly and homologically trivially on a closed, connected and oriented 4-manifold M with $\chi(M) \neq 0$, *then* $\mathrm{rank}_p(G) \leq 1$ *for* $p \geq 5$ *and* $\mathrm{rank}_p(G) \leq 2$ *for* $p = 2, 3$.

Our result in Theorem A, generalizes Edmonds' result [10, Main Theorem] to the non simply-connected 4-manifolds for the manifolds with $b_2 \geq 3$.

The results 1 and 2 in Theorem B, refine the above theorem by Hambleton and Pamuk for the rank 2 cases by ruling out the 4-manifolds on which the $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$ cannot act pseudofreely, locally linearly and homologically trivially. We also provide a new proof of the fact that there is no such action of $\mathbb{Z}_p \times \mathbb{Z}_p$ for $p \geq 5$ on M.

For the rank 1 cases, it identifies the minimal groups and rules out 4-manifolds which cannot be acted upon in the above way, improving the result of Hambleton and Pamuk.

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