ANALYSIS OF CURVES OF MINIMAL ORDER

AN ANALYSIS OF CURVES OF MINIMAL ORDER

AS REGARDS THE

TYPE AND NUMBER OF SINGULAR POINTS

By

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SCOPE AND CONTENTS: The object of this dissertation is to give a classification of curves of minimal order in the real conformal and projective planes with respect to the type and number of singular points. While strongly differentiable curves of minimal order have been studied in detail, little or no research has been done on general differentiable curves of minimal order. The major emphasis lies in the analysis of these curves and the general attack utilizes the notion of the characteristic of a differentiable point. Thus in both the conformal and conical cases, the author obtains valuable information as to the structure of differentiable curves of minimal order in both the conformal and projective planes. It is only left to inquire as to the structure of such curves, if all differentiability restrictions are dropped.

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Introduction

The material of this manuscript is divided into two main parts. The subject matter of the first lies in Sections 1, 2, 3 and deals with conformal geometry; i.e., the geometry of circles. The second part, found in Sections 4, 5 and 6, involves the geometry of conics in the projective plane.

A general introduction to each part will be given here as well as smaller informative additions at the beginning of each section, for the convenience of the reader.

Part I

As this thesis involves the analysis of certain classes of arcs and curves with respect to the geometry of circles, a topology is introduced in Section 1 on the set $\mathfrak{Z}$ of circles in the conformal or inversive plane (which can be regared as the Riemann sphere of complex analysis, cf. 2.5 of [22]). This topology is compact and Hausdorff; cf. 2.2.3. With this topology on $\overline{\mathfrak{Z}}$, limit circles of sequences of circles can be considered with respect to convergence and hence tangent and osculating circles at a point of an arc can be defined; cf. 2.4.

My thesis in this conformal connection is a partial solution of the characterization of all curves $\mathcal{B}_4$ of circular order four in the conformal plane, with regard to type and number of singular points; cf. 2.7.

The conformal proof of a well known theorem about arcs of circular order four is given in 3.1 using methods that correspond to the contraction and expansion theorems of O. Haupt and H. Künneth [12], while 3.2 is an analogous result to that of N. D. Lane and P. Scherk ([4], 3.3) for multiplicities of arcs of circular order four with respect to members of $\overline{\mathcal{L}}$.

The classical four-vertex theorem states that a closed convex curve in the euclidean plane which has continuous curvature everywhere has at least four vertices; i.e., extrema of the curvature. This theorem is supplemented by the result that if this curve has order four then it has exactly four vertices. The four-vertex theorem thus seems to belong to classical euclidean differential geometry. Hence usual proofs of this theorem were worked out in this classical setting; cf. A. Kneser [16] and H. Kneser [17]. The result was extended by W. C. Graustein [18] to any simple closed curve with continuous curvature. Again his proof involved methods of differential calculus.

However, the following considerations of N. D. Lane and P. Scherk show that this approach is not natural. The existence and continuity of the curvature can be interpreted geometrically as the existence and continuity of the osculating circles. At a general point the osculating circle intersects the curve but at an extremum it supports the curve. Also a circular transformation (the basic transformation regarding circles) maps a convex curve onto a curve which may not be convex anymore. However the properties of touching, intersecting and supporting are invariant under circular transformations. As such osculating circles of one curve are mapped to osculating circles of the image curves and the vertices of one curve correspond to vertices of the image curve. Thus the four-vertex theorem and other corresponding results belong rather in conformal differential geometry and the condition of convexity can be replaced by a weaker condition of normality introduced by O. Haupt and H. Künneth [12]; cf. 3.1 S. Mukhopadhyaya [19], [20] seems to be one of the first to consider extrema of curvature, from this "geometric order" viewpoint, as a point of order four, cf. 2.6. Haupt and Künneth also worked with these singular points in a general setting using order characteristics with a fundamental number k (instead of $\overline{\mathfrak{X}}$ where $k = 3$) and a comparison of different kinds of so-called vertices can be found in [12] and [13]. In 3.3 and 3.4 much of Jackson's metric discussion of the four-vertex theorem [8] from an analytic and euclidean framework has been recast into a synthetic and conformal one.

It is well known that a strongly differentiable curve $\mathcal{B}_4$ of order four contains only points with the characteristic

$$(1, 1, 1), (1, 1, 2) \text{ or } (1, 1, 2)_0;$$

cf. 3.3.7, and that such a curve contains exactly four vertices ([12], 4.1.4.3.1). The question can be raised as to the kind of results that can be obtained when the condition of strong differentiability is relaxed to ordinary differentiability for curves $\mathcal{B}_4$ of order four. Then we have more types of differentiable singular points to consider. A number of new results are obtained in 3.3, as regards the number and types of singular points on such a $\mathcal{B}_4$; cf. 3.3.9, 3.3.11, 3.3.13 and 3.3.15. As stated earlier, a strongly differentiable curve $\mathcal{B}_4$ of order four contains at most four vertices. A generalization of this result is derived in Theorem 5 giving the same upper bound for the number of vertices on any differentiable curve $\mathcal{B}_4$ of order four.

It would be nice to be able to drop all differentiability conditions and classify arcs and curves of order four again with respect to type and numbers of singular points.

Part II

In the second half of the thesis arcs and curves of conical order six are analysed with respect to the geometry of conics. With this in mind, a topology is introduced on the set of conics (both degenerate and non-degenerate) in the real projective plane; cf. Section 4. As in the conformal case, this topology is compact and Hausdorff; cf. 4.6. Convergence can be considered with respect to this topology and hence limit conics of sequences of conics are introduced. Using special limit conics; namely, tangent, osculating, superosculating and ultraosculating, conical differentiability of an arc at a point can be defined; cf. 5.3.

An attempt is made to characterize curves of conical order six in the projective plane, with regard to the number and type of conically singular points; cf. 5.6.

In 6.1 conical proofs are given of the general monotony, contraction and expansion theorems of O. Haupt and H. Künneth [12] as applied to arcs of conical order six. Using these results a well-known theorem is obtained as to the number of conically singular points on an arc of conical order six; cf. Theorem 9.

Multiplicities, with respect to the system of conics, for an arc of conical order six are introduced in 6.2 and an analogous result to that of N. D. Lane and K. D. Singh ([10], 4.2) is obtained for such an arc.

Now a curve $\mathcal{B}_6$ of conical order six is either convex or of linear order three; cf. 6.4.1. It is well known that a strongly conically differentiable convex curve $\mathcal{B}_6$ of conical order six contains exactly six conically singular (sextactic) points; cf. Fr. Fabricius-Bjerre [23] and S. Mukhopadhyaya [20]. It is also well known that a strongly conically differentiable curve $\mathcal{B}_6$ of linear order three contains exactly six conically singular points [23]. As in the conformal analysis, one might ask for respective results, if $\mathcal{B}_6$ is only conically differentiable. New results are obtained in 6.4, showing that $\mathcal{B}_6$ contains generally exactly six conically singular points, if $\mathcal{B}_6$ is convex; and contains either exactly four or six, if $\mathcal{B}_6$ is of linear order three.

Again, as in conformal geometry, one would like to classify arcs and curves of conical order six, imposing no differentiability restrictions, with respect to the number and type of conically singular points.

Section 1

A Topology on the Set of Circles in the Inversive Plane

Introduction

Let $\mathfrak{C} = \{C\}$, where C denotes a nondegenerate circle in the real inversive plane π_i. Let $\bar{\mathfrak{C}}$ be the union of $\mathfrak{C}$ and all of the points (considered as point circles) of π_i. Our goal is to introduce a topology on $\bar{\mathfrak{C}}$. We shall do this by introducing a neighbourhood filter at each $C \in \bar{\mathfrak{C}}$.

1.1. For each $C \in \bar{\mathfrak{C}}$, let C_e and C_i be the "exterior" and "interior" respectively of C; the interior of C lying to the left of C. If C is a point circle, then one of these regions is void.

1.1.1. *Let D and D' be two circles with the property that $D \subset D'_e$, $D' \subset D_i$. Then $D'_i \subset D_i$ and $D_e \subset D'_e$.*

Proof. $D \subset D'_e$ implies that $D'_i \cap D = \emptyset$. But $D' \cap D = \emptyset$. Hence $(D'_i \cup D') \cap D = \emptyset$. But then $D'_i \cup D'$ is a closed connected set having no points in common with D. Hence $D'_i \cup D'$ lies totally in either D_i or D_e. However, $D' \subset D_i$. Therefore, $D'_i \subset D_i$.

An analogous argument yields $D_e \subset D'_e$.

<u>1.2</u> Take any $C \in \mathcal{L}$, orient it and let D and E be members of $\mathcal{L}$ such that

$$D \subset C_i, \quad E \subset C_e.$$

Then D and E can be oriented such that

$$C \subset D_e, \quad C \subset E_i.$$

<u>1.2.1</u> Using the above orientations of C, D, and E, let

$$ {}_D^E U_C = \left\{ K \in \mathcal{L} : K \subset D_e \cap E_i \text{ and } K \text{ can be oriented such that } D \subset K_i,\ E \subset K_e \right\}.$$

Let

$$\mathcal{U}_C = \left\{ {}_D^E U_C \right\},$$

where D and E run over all pairs of circles of $\mathcal{L}$ with the above restrictions.

<u>1.2.2</u> $\mathcal{U}_C$ <u>is a filter base.</u>

<u>Proof</u>. Let

$$ {}_{D'}^{E'}U_C \quad \text{and} \quad {}_{D''}^{E''}U_C $$

by any two members of $\mathcal{U}_C$. Let D and E be members of $\mathcal{L}$ such that

$$ D \subset C_i \cap D'_e \cap D''_e; \qquad D', D'' \subset D_i,\ C \subset D_e $$

$$ E \subset C_e \cap E'_i \cap E''_i; \qquad E', E'' \subset E_e,\ C \subset E_i. $$

We now show that

$$ {}_{D}^{E}U_C \subset {}_{D'}^{E'}U_C \cap {}_{D''}^{E''}U_C. $$

Let $K \in {}_{D}^{E}U_C$. Then by definition $D \subset K_i$, $K \subset D_e$. By <u>1.1.1</u>, $D_i \subset K_i$ and $K_e \subset D_e$. Also $D \subset D'_e$ and $D' \subset D_i$. By <u>1.1.1</u> $D'_i \subset D_i$ and $D_e \subset D'_e$. Now

$$ \left.\begin{array}{l} D_i \subset K_i \\ \\ D' \subset D_i \end{array}\right\} \Longrightarrow D' \subset K_i \quad \text{and} \quad \left.\begin{array}{l} D_e \subset D'_e \\ \\ K \subset D_e \end{array}\right\} \Longrightarrow K \subset D'_e . $$

Also, by definition, $K \subset E_i$ and $E \subset K_e$. Then by 1.1.1 $K_i \subset E_i$ and $E_e \subset K_e$. Also $E \subset E'_i$ and $E' \subset E_e$. By 1.1.1, $E_i \subset E'_i$ and $E'_e \subset E_e$. Now

$$\left.\begin{array}{l} E_e \subset K_e \\ \\ E' \subset E_e \end{array}\right\} \Rightarrow E' \subset K_e \quad \text{and} \quad \left.\begin{array}{l} E_i \subset E'_i \\ \\ K \subset E_i \end{array}\right\} \Rightarrow K \subset E'_i .$$

Finally $K \subset D'_e \cap E'_i$, $D' \subset K_i$ and $E' \subset K_e$ imply, by definition, that $K \in {}^{E'}_{D'}U_C$. Similarly $K \in {}^{E''}_{D''}U_C$.

Thus

$$ {}^{E}_{D}U_C \subset {}^{E'}_{D'}U_C \cap {}^{E''}_{D''}U_C .$$

1.2.3 Let C be a point circle and suppose that $C_i = \phi$, say. Let E be a member of $\mathfrak{L}$ such that

$$E \subset C_e .$$

Then E can be oriented such that

$$C \subset E_i .$$

Now let

$$ {}^{E}U_C = \left\{ K \in \mathfrak{L} : \ K \subset E_i \ \text{ and } \ K \ \text{ can be oriented such that } C \subset K_i, \ E \subset K_e \right\} $$

Let

$$ \mathcal{U}_C = \left\{ {}^{E}U_C \right\}, $$

where E runs over all the circles of $\mathfrak{L}$ with the above restrictions.

<u>1.2.4</u> $\mathcal{U}_C$ <u>is a filter base</u>.

<u>Proof</u>. Let

$$ {}^{E'}U_C \quad \text{and} \quad {}^{E''}U_C $$

be any two members of $\mathcal{U}_C$. Let E be a member of $\mathfrak{L}$ such that

$$ E \subset C_e \cap E'_i \cap E''_i; \quad E', E'' \subset E_e, \ C \subset E_i. $$

Then by our choice of E, as in <u>1.2.2</u>,

$$ {}^{E}U_C \subset {}^{E'}U_C \cap {}^{E''}U_C . $$

<u>1.3</u> Let $\mathfrak{F}_C$ be the filter generated by $\mathcal{U}_C$. Consider the family

$$\Theta = (\mathfrak{F}_C)_{C \in \bar{\mathfrak{T}}}$$

of filters on $\bar{\mathfrak{T}}$.

<u>1.3.1</u> <u>For each</u> $C \in \mathfrak{T}$ <u>and all</u> $V \in \mathfrak{F}_C$, <u>there is a</u> $W \in \mathfrak{F}_C$ <u>such that</u> $W \subset V$ <u>and</u> $V \in \mathfrak{F}_K$ <u>for each</u> $K \in W$.

<u>Proof</u>. It is enough to prove that the claim is true for members of the base. Let

$${}_{D'}^{E'}U_C \in \mathcal{U}_C .$$

Take $D, E \in \mathfrak{T}$ with

$$D \subset C_i \cap D'_e,\ C \subset D_e,\ D' \subset D_i$$

$$E \subset C_e \cap E'_i,\ C \subset E'_i,\ E' \subset E_e .$$

As in <u>1.2.2</u>, with this choice of D and E,

$$\bigcup_{D}^{E} U_C \subset \bigcup_{D'}^{E'} U_C$$

and

$$\bigcup_{D'}^{E'} U_C \in \mathcal{U}_K$$

for each

$$K \in \bigcup_{D}^{E} U_C .$$

1.3.2 For each point circle C and all $V \in \mathfrak{F}_C$, there is a $W \in \mathfrak{F}_C$ such that $W \subset V$ and $V \in \mathfrak{F}_K$ for each $K \in W$.

Proof. As in 1.3.1, it is enough to prove that the claim is true for members of the base. Let $C_i = \emptyset$ say, and

$$\bigcup^{E'} U_C \in \mathcal{U}_C .$$

Take $E \in \mathcal{L}$ with

$$E \subset C_e \cap E_i', \; C \subset E_i, \; E' \subset E_e .$$

As in 1.2.4, with this choice of E,

$$ {}^{E}U_C \subset {}^{E'}U_C $$

and

$$ {}^{E'}U_C = {}_C{}^{E'}U_K \varepsilon \mathfrak{U}_K $$

for each

$$ K \in {}^{E}U_C . $$

1.3.3 We combine 1.3.1 and 1.3.2 to obtain:

For each $C \in \mathfrak{C}$ and all $V \in \mathfrak{F}_C$, there is a $W \in \mathfrak{F}_C$ such that $W \subset V$ and $W \in \mathfrak{F}_K$ for each $K \in W$.

1.4 The following theorem is standard ([1], p. 56):

Let X be a set and

$$\Theta = (\mathcal{F}_x)_{x \in X}$$

a family of filters on X indexed by X such that 1.3.3 is satisfied. Then there is a topology $\mathcal{D}$ on X such that $\mathcal{F}_x$ is precisely The neighbourhood system of x with respect to the topology $\mathcal{D}$.

In our case, there is a topology $\mathcal{D}$ on $\bar{\mathfrak{X}}$ such that $\mathcal{F}_C$ is the neighbourhood system at C and U is an open set of $\bar{\mathfrak{X}}$ (i.e. a member of $\mathcal{D}$) if $U \in \mathcal{F}_C$ for all $C \in U$.

We now determine some properties of the topological space $(\bar{\mathfrak{X}}, \mathcal{D})$.

1.5 ($\mathfrak{T}$, $\mathfrak{D}$) satisfies the first and second axioms of countability.

Proof. For all $\overset{E}{\underset{D}{U}}$ Let D and E be determined by three distinct points with rational coordinates.

1.6 $(\overline{\mathcal{L}}, \mathfrak{D})$ is a Hausdorff space.

Proof. Let C_1 and C_2 be two distinct circles of $\mathcal{L}$.

Case (i). $C_1 \cap C_2 = \emptyset$. Then C_1 and C_2 belong to a pencil of the third kind, cf. 2.1, from which one can easily construct disjoint neighbourhoods of C_1 and C_2 (Figure 1).

Case (ii). $C_1 \cap C_2$ is a single point. Then C_1 and C_2 belong to a pencil of the second kind, cf. 2.1, from which one can construct disjoint neighbourhoods of C_1 and C_2 (see Figure 1).

Case (iii). C_1 and C_2 have two points in common. Then C_1 and C_2 determine a pencil of the first kind, cf. 2.1, from which one can construct disjoint neighbourhoods of C_1 and C_2 (see Figure 1).

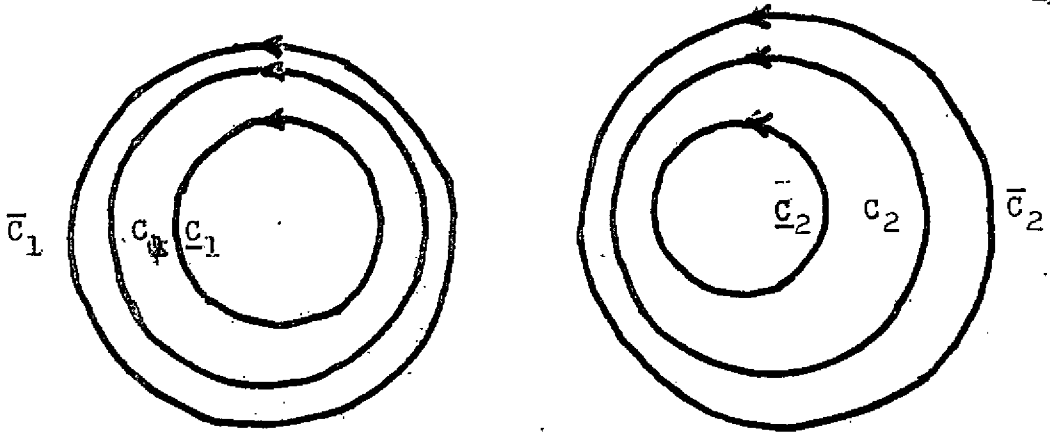
Suppose that one of or both C_1 and C_2 are point circles.

Case (i). Both C_1, C_2 are points circles. Then disjoint neighbourhoods of C_1 and C_2 can easily be constructed (see Figure 2).

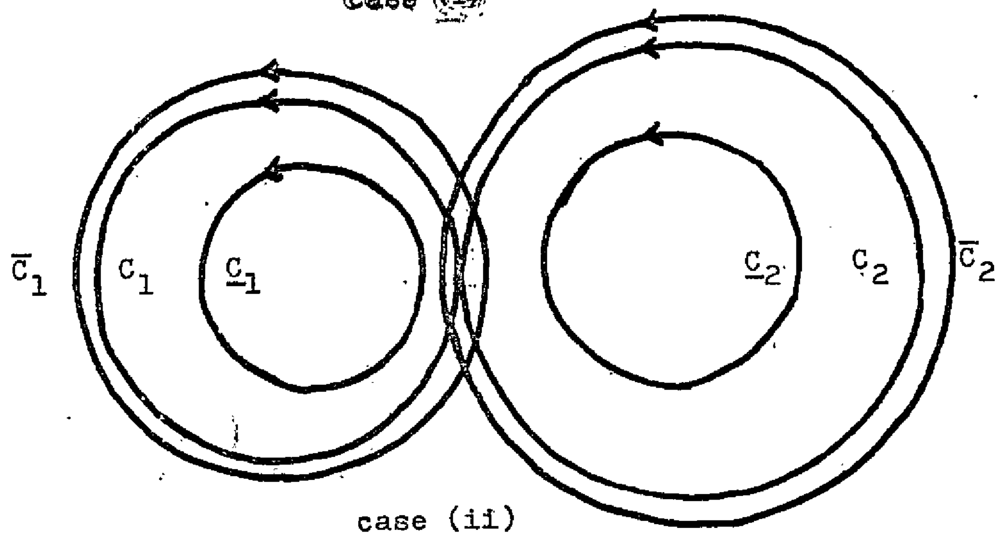
<u>Case (ii)</u>. One of C_1, C_2, say C_1 is a point circle while $C_2 \in \mathcal{L}$. Then either

<u>(a)</u> $C_1 \cap C_2 = \emptyset$. Then disjoint neighbourhoods as in Figure 2 can be constructed.

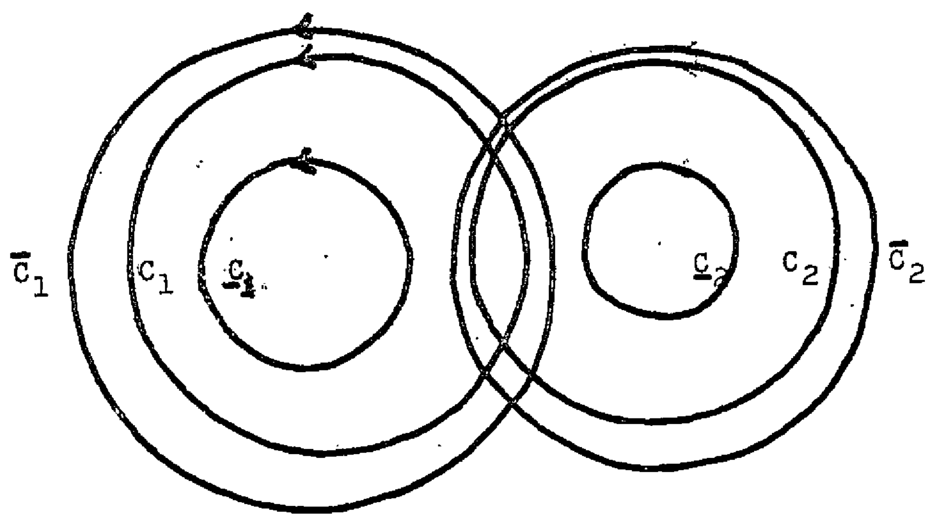
<u>(b)</u> $C_1 \cap C_2 = C_1$. In this case disjoint neighbourhoods as in Figure 2 can be constructed.



case (i)

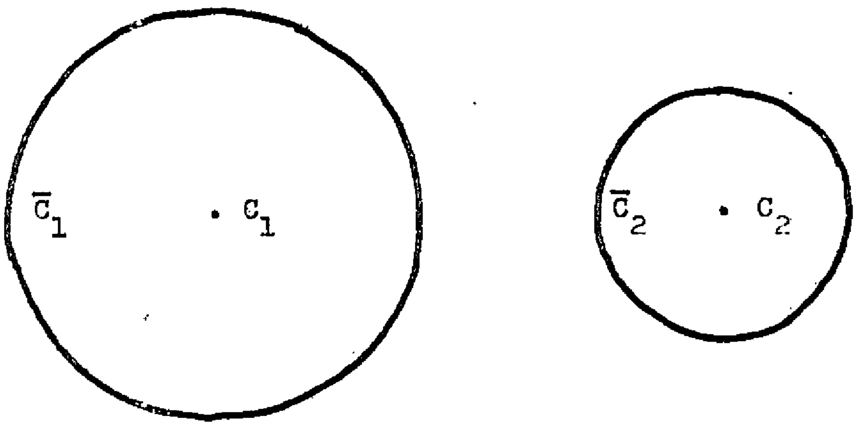


case (ii)

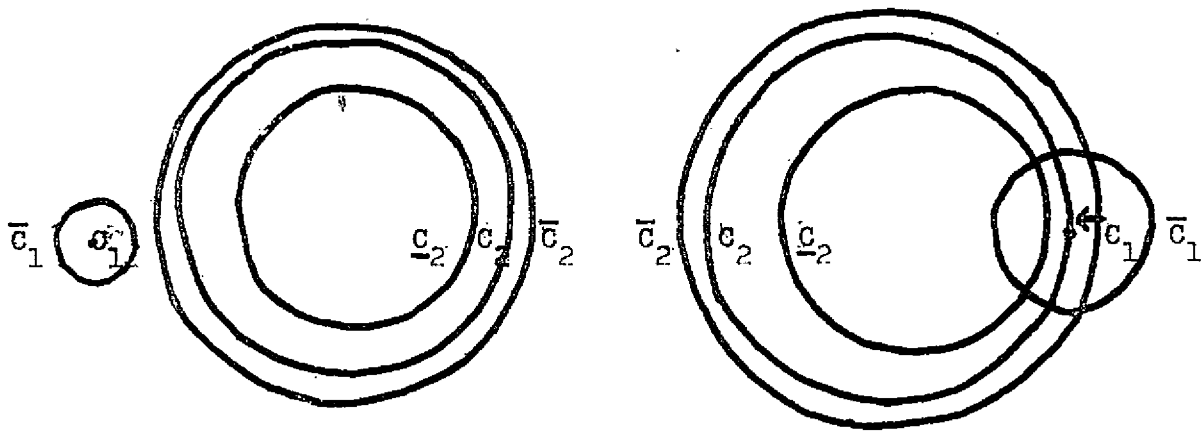


case (iii)

Figure 1



case (i)



(a) (b)

case (ii)

Figure 2

<u>1.7</u> If ${}_{D}^{E}U$ is a base element, then the smallest closed set containing ${}_{D}^{E}U$, denoted by ${}_{D}^{\overline{E}}U$, consists of the following circles of $\mathfrak{L}$; namely, all K such that

$$K \subset (D \cup D_e) \cap (E \cup E_i)$$

$$D \subset K \cup K_i,\ E \subset K \cup K_e.$$

We note that in particular $D, E \in {}_{D}^{\overline{E}}U$.

<u>1.7.1</u> $(\mathfrak{L}, \mathfrak{D})$ <u>is regular.</u>

<u>Proof.</u> Let $C \in \mathfrak{L}$ and let U be any neighbourhood of C. Then there exists a base element

$${}_{D'}^{E'}U_C \subset U.$$

Take circles $D, E \in \mathfrak{L}$ such that

$$D \subset D'_e \cap C_i,\ C \subset D_e,\ D' \subset D_i,$$

$$E \subset E'_i \cap C_e,\ C \subset E_i,\ E' \subset E_e\ .$$

By this choice of D and E, as in 1.2.2,

$$ {}_{D}^{E}U_C $$

is a base neighbourhood of C with

$$ {}_{D}^{\overline{E}}U_C \subset {}_{D'}^{E'}U_C \subset U . $$

Let C be a point circle with $C_i = \phi$ and let U be a neighbourhood of C. Then there exists a base element

$$ {}^{E'}U_C \subset U . $$

Take a circle $E \in \mathcal{J}$ such that

$$ E \subset C_e \cap E_i', \; C \subset E_i, \; E' \subset E_e . $$

By this choice of E, as in 1.2.4,

$$ {}^{E}U_C $$

is a base neighbourhood of C with

$$U_C^{\overline{E}} \subset U_C^{E'} \subset U.$$

Section 2

The Order, Differentiability and Characteristic of Points of an Arc in the Inversive Plane

Introduction

This section is purely a collection of background information with the exception of Theorem 1; cf. 2.2.3. This material is based upon the properties of order, differentiability and characteristic of a point of an arc and can be found in [2] and [4], the work of N.D. Lane and P. Scherk.

2.1 Pencils of Circles.

In the following, P, Q, ..., will denote points in the real inversive plane. The circle through three mutually distinct points P, Q and R is uniquely determined and will be denoted by C(P, Q, R).

The set of all circles that intersect two given circles at right angles form a linear pencil π of circles. A pencil π of the first kind possesses two fundamental points such that π is identical with the set of all circles through these points. A pencil of the second kind has one fundamental point and is identical with the set of those circles that touch a given non-degenerate circle at that point. If π is of the third kind, then any two circles of π are disjoint. For any pencil π and for any point Q which is not a fundamental point of π, there exists a unique circle $C(\pi, Q)$ through Q. We consider the fundamental point of a pencil π of the second kind as a point circle belonging to π.

2.2 Convergence.

In Section 1, we introduced a topology $\mathcal{D}$ on $\bar{\mathcal{L}}$, the set of all circles in the real inversive plane. We have shown that $(\bar{\mathcal{L}}, \mathcal{D})$ is a regular Hausdorff space satisfying the second axiom of countability. With respect to this topology, we can now describe convergence.

2.2.1 A sequence of circles $(C_n)_{n \in N}$ is defined to be convergent to a circle C if for any neighbourhood U of C there exists $n_0 \in N$ such that $C_n \in U$ for all $n > n_0$. We denote this convergence of C_n to C by

$$\lim_{n \in N} C_n = C.$$

2.2.2 $(\bar{\mathcal{L}}, \mathcal{D})$ is a countably compact space.

Proof. Let $p_n \in C_n$ for each $n \in N$, where $(C_n)_{n \in N}$ is an infinite sequence of circles. Then (p_n) is an infinite sequence of points in a compact space. Hence there exists a point p, and a subsequence (p_{nm}) of (p_n) such that $\lim p_{nm} = p$. Let (C_{nm}) be the corresponding subsequence of circles of the sequence (C_n). Let $q_{nm} \in C_{nm}$, $q_{nm} \neq p_{nm}$. By the same argument as above, there exists a point q and a subsequence (q_{nml}) of (q_{nm}) such that $\lim q_{nml} = q$. We can assume that $p \neq q$, for if $\lim p_{nm} = p$ for every sequence of points on (C_{nm}), then the point p, considered as a point circle, will be an accumulation circle of (C_n). By the same argument, we obtain a third point r, distinct from p and q and a subsequence (r_{nmlk}) of points on (C_{nml}) such that $\lim r_{nmlk} = r$. Then the circle determined by p, q and r is an accumulation circle of $(C_n)_{n \in N}$.

<u>2.2.3</u> We combine <u>1.5</u>, <u>1.6</u> and <u>2.2.2</u> to obtain ([1], p. 138):

<u>Theorem 1.</u> $(\mathfrak{T}, \mathfrak{D})$ <u>is a compact Hausdorff space.</u>

2.3 Support and Intersection at a Point of an Arc

Unless otherwise stated, an arc $\mathcal{A}$ [curve $\mathcal{C}$] is the topological image of an interval [circle]. Hence our arcs and curves will be simple. Thus if a sequence of points of that parameter interval converges to a point s, then their image points converge to the image of s. We shall use the same letters $s, t, u, \ldots,$ to denote both the parameter points and their images on $\mathcal{A}$. The end [interior] points of $\mathcal{A}$ are the images of the end [interior] points of the parameter interval. The notation $P \neq s$ will indicate that the points P and s do not coincide.

A neighbourhood of s on $\mathcal{A}$ is the image of a neighbourhood of the parameter s on the parameter interval. If s is an interior point of $\mathcal{A}$, this neighbourhood is decomposed by s into two (open) one-sided neighbourhoods.

Suppose s is an interior point of $\mathcal{A}$. Then we call s a point of support [intersection] with respect to the circle C if a sufficiently small neighbourhood of s is decomposed by s into two one-sided neighbourhoods which lie in the same region [in different regions] bounded by C. C is then called a supporting [intersecting] circle of $\mathcal{A}$ at s. Thus C supports $\mathcal{A}$ at s if $s \notin C$. By definition, the point circle s always supports $\mathcal{A}$ at s.

It can happen that every neighbourhood of s on $\mathcal{A}$ has points $\neq s$ in common with C. Then C neither supports nor intersects $\mathcal{A}$ at s.

2.4 Differentiable and Strongly Differentiable Points.

2.4.1 An arc $\mathcal{Q}$ is said to be (conformally) differentiable at a point p of $\mathcal{Q}$ if it satisfies two conditions:

Condition C I. There exists a point $R \neq p$ such that if the parameter s is sufficiently close to p, then the circle $C(p,s,R)$ through the points p,s and R exists. It converges if s tends to p on $\mathcal{Q}$.

The limiting tangent circle of $\mathcal{Q}$ at p through R is denoted by $C(p^2, R)$.

Condition C I implies [2]:

(i) There is a unique tangent circle $C(p^2, R)$ of $\mathcal{Q}$ at p through each point $R \neq p$ and the union of the set of tangent circles with the point circle p is a pencil of the second kind with the fundamental point p.

(ii) If p is an interior point of $\mathcal{Q}$, then the nontangent circles of $\mathcal{Q}$ through p all intersect $\mathcal{Q}$ at p or all support.

Condition C II. The arc $\mathcal{Q}$ satisfies C I at p and there exists a circle $C(p^3)$ such that

$$\lim_{\substack{s \in \mathcal{A} \\ s \neq p \\ s \to p}} C(p^2, s) = C(p^3).$$

We call $C(p^3)$ the <u>osculating circle of $\mathcal{A}$ at</u> p. $C(p^3)$ may be the point circle p.

Differentiability of $\mathcal{A}$ at an interior point p implies [2]:

<u>(iii)</u> The nonosculating tangent circles of $\mathcal{A}$ at p all intersect $\mathcal{A}$ at p or all support. If $C(p^3) \neq p$, then all of them support.

An arc or curve is said to be <u>differentiable</u> if every point is differentiable.

2.4.2 Strongly Differentiable Points

Let $R \neq p$, $Q \longrightarrow R$ and let s and t converge on $\mathcal{A}$ to p. Then any accumulation circle of the circles $C(s, t, Q)$ is called a <u>general tangent circle</u> of $\mathcal{A}$ at p through R.

<u>Condition C I'</u>. There exists a point $R \neq p$ such that if $Q \longrightarrow R$ and distinct points s and t converge on $\mathcal{a}$ to p, then

$$\lim C(s, t, Q)$$

exists.

Thus this limit circle is the unique general tangent circle of $\mathcal{a}$ at p. Condition C I' implies that the limit circle depends on p and R but not on the choice of the particular sequences s and t. Specializing $Q = R$ and $t = p$ we see that Condition C I' implies Condition C I and that therefore

$$\lim C(s, t, Q) = C(p^2, R).$$

Thus the general tangent circles of a point which satisfies Condition C I' are identical with the ordinary ones.

If three mutually distinct points s, t, u converge on $\mathcal{a}$ to p, then any accumulation circle of the circles $C(s, t, u)$ is called a <u>general osculating circle</u> of $\mathcal{a}$ at p.

<u>Condition C II'</u>. If three mutually distinct points s, t, u converge on $\mathcal{A}$ to p, then

$$\lim C(s, t, u)$$

exists.

Thus this limit circle is the unique general osculating circle of $\mathcal{A}$ at p. Condition C II' does not in general imply C I or C I' ([3], 4).

If Conditions C I' and C II' are both satisfied then $\mathcal{A}$ is said to be <u>strongly differentiable</u> at the point p. An arc or curve is <u>strongly differentiable</u> if every point is strongly differentiable.

Strong differentiability implies ordinary differentiability and the following are also valid ([4], 3):

<u>(i)</u> Let p satisfy Condition C I'. Let $R \neq p$, $Q \longrightarrow R$ and s converge on $\mathcal{A}$ to p. If C_1 is a general tangent circle at s through Q, then

$$\lim C_1 = C(p^2, R).$$

(ii) Suppose $\mathcal{a}$ is strongly differentiable at p. Let the two distinct points s and t converge on $\mathcal{a}$ to p and let C_2 be a general tangent circle at s through t. Then

$$\lim C_2 = C(p^3).$$

(iii) Suppose $\mathcal{a}$ is strongly differentiable at p. Let s converge on $\mathcal{a}$ to p and let C_3 be a general osculating circle at s. Then

$$\lim C_3 = C(p^3).$$

(iv) If at the point p of an arc $\mathcal{a}$ the general tangent circles form a unique pencil τ of the second kind, then $\mathcal{a}$ induces a unique orientation on the circles of τ. In particular $\mathcal{a}$ induces a unique orientation on a general osculating circle at p. If the given condition holds at all points p of $\mathcal{a}$, then the oriented pencil τ varies continuously with p [7].

2.5 Classification of Differentiable Points.

We associate with each differentiable interior point p of an arc $\mathcal{A}$ a characteristic (a_0, a_1, a_2) if $C(p^3) \nmid p$ or $(a_0, a_1, a_2)_0$ if $C(p^3) = p$. The numbers a_0 and a_1 are equal to 1 or 2, while a_2 is equal to 1, 2 or ∞. They have the following properties:

(i) a_0 is even or odd according as the nontangent circles of p support or intersect $\mathcal{A}$ at p.

(ii) $a_0 + a_1$ is even or odd according as the nonosculating tangent circles support or intersect $\mathcal{A}$ at p.

(iii) $a_0 + a_1 + a_2$ is even if $C(p^3)$ supports, odd if $C(p^3)$ intersects, while $a_2 = \infty$ if $C(p^3)$ neither supports nor intersects.

Thus $a_0 + a_1 + a_2$ is even if $C(p^3) = p$. From 2.4.1 (iii), $a_0 = a_1$ if $C(p^3) \nmid p$.

We list the types of differentiable points p of an arc $\mathcal{A}$ (Figure 3). The first eight examples refer to the curves $x = s^n$, $y = s^{n+m}$; the last two refer to $x = s^n$, $y = s^{n+m} \sin \frac{1}{s}$. In all cases we consider the point $s = 0$. Congruences are modulo 2.

CHARACTERISTIC	NON-TANGENT CIRCLES THROUGH p	TANGENT CIRCLES $\neq$ $C(p^3)$	$C(p^3)$		EXAMPLES		
(1,1,1)	intersect	support	$C(p^3) \neq p$	intersects	$n < m$	$n \equiv 1$ $m \equiv 0$	regular point
(1,1,2)	intersect			supports		$n \equiv m$ $\equiv 1$	vertex
(2,2,1)	support			intersects		$n \equiv 0$ $m \equiv 1$	cusp of first kind
(2,2,2)	support			supports		$n \equiv m$ $\equiv 0$	cusp of second kind
$(1,1.2)_0$	intersect	support	point circle		$n > m$	$n \equiv m$ $\equiv 1$	
$(1,2,1)_0$	intersect	intersect				$n \equiv 1$ $m \equiv 0$	
$(2,1,1)_0$	support	intersect				$n \equiv 0$ $m \equiv 1$	
$(2,2,2)_0$	support	support				$n \equiv m$ $\equiv 0$	
$(1,1,\infty)$	intersect	support	neither supports or intersects		$n < m$	$n \equiv 1$	
$(2,2,\infty)$	support					$n \equiv 0$	

FIGURE 3

2.6 Circular Order of a Point

An arc $\mathcal{A}$ is said to be of <u>finite circular order</u> if it has only a finite number of points in common with any circle. If the least upper bound of these numbers is finite, then this number is called the (circular) <u>order</u> of $\mathcal{A}$. The <u>order</u> of a <u>point</u> p of $\mathcal{A}$ is then the minimum of the orders of all neighbourhoods of p on $\mathcal{A}$. Note that the order of a point is ≥ 3.

We list the following results:

<u>(i)</u> Let $\mathcal{A}$ be an arc of finite order. If a circle C intersects $\mathcal{A}$ at s, then every circle sufficiently close to C intersects $\mathcal{A}$ at some point ([4], 2).

<u>(ii)</u> Let p be an end point of an arc $\mathcal{A}$ of finite order. Then $\mathcal{A}$ is differentiable at p([4], 3).

<u>(iii)</u> Let p be a differentiable interior point of an arc $\mathcal{A}$. Suppose that p has the characteristic (a_0, a_1, a_2) or $(a_0, a_1, a_2)_0$. Then the order of p is not less than $a_0 + a_1 + a_2$, ([4], 2).

<u>(iv)</u> An <u>elementary point</u> p of an arc $\mathcal{A}$ is one such that there exists a neighbourhood of p on A which is decomposed by

p into two one-sided neighbourhoods of order three.

Let p be an elementary point of a differentiable arc $\mathcal{a}$. If p has the characteristic (a_0, a_1, a_2) or $(a_0, a_1, a_2)_0$, then the order of p is $a_0 + a_1 + a_2$ ([4], 5).

2.7 Ordinary and Singular Points

A point p of an arc $\mathcal{Q}$ is called <u>ordinary</u> if the order of p is three (the minimal order).

If the order of p is strictly greater than three, p is said to be a <u>singular</u> point.

A point p of a differentiable arc $\mathcal{Q}$ is called a <u>vertex</u> if p is a point of support of $\mathcal{Q}$ with respect to $C(p^3)$.

Section 3

Arcs and Curves of Circular Order Four in the Inversive Plane.

Introduction.

In this section we shall discuss properties of arcs and curves of circular order four. This larger section is divided into four subsections, 3.1 - 3.4.

In 3.1 and 3.2 we shall consider normal arcs of order four. We restrict our attention to differentiable curves of circular order four in 3.3, for the most part. Finally in 3.4, the discussion centers upon strongly differentiable curves of circular order four.

3.1 Normal Arcs of Order Four.

Introduction

An arc $\mathcal{A}$ is called normal if for each $C \varepsilon \mathcal{L}$, C can be oriented so that the points of $C \cap \mathcal{A}$ lie in the same order on C as they do on $\mathcal{A}$. We note that a curve $\mathcal{C}_4$ of circular order four is always normal ([13], 5).

It is well known that a normal arc $\mathcal{A}_4$ of circular order four is the union of a finite number of arcs of order three; cf. O. Haupt [14] and 4.1.3 of [12]. To derive this result Haupt basically used the so-called "Contraction Theorem"; cf. ([12], 2.4.4), first attributed to Mukhopadhyaya [19], [20] and the "Expansion Theorem"; cf. ([12], 2.4.5). These results generally deal with specific movement of intersection points of arcs with members of classes of so-called "order characteristics"; cf. ([12], 1.1), with a fundamental number k, this number being such that k distinct points uniquely determine one member of the class. The proofs of his results are generally by induction on the fundamental number k.

In conformal geometry the class of order characteristics is the set of all circles and k = 3. It would be of interest to find conformal proofs of corresponding results for normal arcs of order four. With this in mind, 3.1.2, 3.1.4 and Theorem 2 are included for the reader's convenience. Then it is a simple matter to conclude that an end-point of such an arc is ordinary and hence strongly differentiable; cf. 3.1.9 and 3.1.10.

<u>3.1.1</u> <u>Let</u> C_0 <u>be a circle which meets</u> $\mathcal{Q}_4$ <u>at four distinct points</u> a, b, p_1, p_2. <u>Then as</u> t <u>moves monotonically and continuously from</u> a <u>on</u> $\mathcal{Q}_4$, <u>there is a point</u>

$$u \varepsilon C(t, p_1 p_2) \cap \mathcal{Q}_4$$

<u>which moves monotonically and continuously in the opposite direction.</u>

<u>Proof</u>. Without loss of generality we can assume that $p_1 < p_2$. Since $\mathcal{Q}_4$ is of order four, $C_0 = C(a, p_1, p_2)$ intersects $\mathcal{Q}_4$ at a, b, p_1, p_2 and meets $\mathcal{Q}_4$ nowhere else; cf. <u>3.2.2.</u> If t is sufficiently close to a, then $C(t, p_1, p_2)$ will be close to C_0 and will intersect $\mathcal{Q}_4$ at t, p_1, p_2 and at a point u close to b. Also $C(t, p_1, p_2)$ meets $\mathcal{Q}_4$ nowhere else. Thus u depends continuously on t.

It is sufficient to show that t and u move in opposite directions on $\mathcal{Q}_4$ whenever t is close to a. Thus we shall restrict t to a suitably small neighbourhood of a in the following.

If an even [odd] number of points of $\{p_1, p_2\}$ lie between a and b on C_0, then the same number of these points will lie between t and u on $C(t, p_1, p_2)$. Since the distinct circles C_0 and $C(t, p_1, p_2)$ meet exactly at p_1 and p_2, t and u will lie on the same [on opposite sides] of C_0. On the other hand, since $\mathcal{Q}_4 \cap C_0 = \{a, p_1, p_2, b\}$, $\mathcal{Q}_4$ will meet C_0 at an even [odd] number of points between a and b. Also C_0 intersects $\mathcal{A}_4$ at a and b. Hence if t and u move in the same direction on $\mathcal{A}_4$, then t and u will lie on opposite sides [on the same side] of C_0; contradiction.

Remarks (i) The movement of t and u in 3.1.1 can continue as long as none of t, u, p_1, p_2 coincide.

(ii) 3.1.1 remains valid if the arc $\mathcal{A}_4$ is replaced by a curve $\mathcal{L}_4$ of order four.

3.1.2 Let C_0 be a circle which meets $\mathcal{A}_4$ at points $p_0 < q_0 < r_0 < s_0$. If $\mathcal{B}$ is the closed subarc of $\mathcal{A}_4$ between p_0 and s_0, then there exists at least one singular point in the interior of $\mathcal{B}$.

Proof. Consider the parameter interval $I_0 = [p_0, s_0]$. We recall that the same letters are used for the points on the arc as are used for their respective parameter values on the parameter interval. We define a sequence of intervals and a corresponding sequence of circles by induction. Having defined

$$I_n = [p_n, s_n]$$

with $I_n \subset I_0$ and

$$C_n = C_n(p_n, q_n, r_n, s_n)$$

through $p_n < q_n < r_n < s_n$, we define $I_{n+1} \subset I_n$ and C_{n+1} as follows.

Let the length of I_n, denoted $\ell(I_n)$, be ε and let E_i be the point of I_n which is $\frac{i}{8}\varepsilon$ from p_n; $i = 0, 1, \ldots, 8$. One of the following holds:

<u>(i)</u> q_n, r_n lie on the same side of E_4

<u>(ii)</u> q_n, r_n lie on opposite sides of E_4

<u>(iii)</u> $q_n = E_4$ or $r_n = E_4$ (not both since $q_n < r_n$)

If case <u>(i)</u> occurs, suppose that q_n, r_n lie on the same side of E_4 as p_n. Hold p_n, q_n fixed and let t move from r_n toward E_3. If r_n is already between E_3 and E_4, no movement is carried out. Otherwise, by <u>3.1.1</u>, there is a point u which moves from s_n toward E_7. If u reaches E_7 first, then define

$$I_{n+1} = [p_n, u] = [p_n, E_7] = [E_0, E_7]$$

and

$$C_{n+1} = C(p_n, q_n, t, u).$$

If t reaches E_3 first, then hold p_n, $t = E_3$ fixed and let t' move from q_n toward E_2. If q_n is already between E_2 and E_4, no movement is carried out. Otherwise, by <u>3.1.1</u>, there is a point u' which moves from u toward E_7. If u' reaches E_7 first, then define

$$I_{n+1} = [p_n, u'] = [p_n, E_7] = [E_0, E_7]$$

and

$$C_{n+1} = C(p_n, t', t, u').$$

If t' reaches E_2 first, hold $t' = E_2$, $t = E_3$ fixed and let t'' move from $p_n = E_0$ toward E_1. Then there is a point u'' which moves from u' toward E_7. If u'' reaches E_7 first, then define

$$I_{n+1} = [t'', u''] = [t'', E_7] \subset [E_0, E_7]$$

and

$$C_{n+1} = C(t'', t', t, u'').$$

If t'' reaches E_1 first, define

$$I_{n+1} = [t'', u''] = [E_1, u''] \subset [E_1, E_8]$$

and

$$C_{n+1} = C(t'', t', t, u'').$$

If q_n, r_n lie on the same side of E_4 as s_n, then by a symmetric construction we define I_{n+1} and C_{n+1}. We note that

$$\ell(I_{n+1}) \leq \frac{7}{8}\, \ell(I_n).$$

If case <u>(ii)</u> occurs, let F_i be the point on I_n which is $\frac{i}{6}\varepsilon$ from p_n; $i = 0, 1, \ldots, 6$. One defines I_{n+1} and C_{n+1} as follows. Hold p_n, s_n fixed and let t move from q_n toward F_2. If q_n is already between F_2 and F_3, no movement is carried out. Otherwise, by <u>3.1.1</u>, there is a point u which moves from r_n toward F_4. If r_n is already between F_3 and F_4, again no movement is carried out. Without loss of generality, assume that u reaches F_4 first. Now hold p_n, u fixed and let t' move from t toward F_2. Then there is a point u' which moves from s_n toward the point

F_5. If u' reaches F_5 first, then define

$$I_{n+1} = [p_n, u'] = [p_n, F_5] = [F_0, F_5]$$

and

$$C_{n+1} = C(p_n, t', u, u').$$

If t' reaches F_2 first, hold $t' = F_2$, u fixed and let t'' move from p_n toward F_1. Then there is a point u'' which moves from u' toward F_5. If u' reaches F_5 first, then define

$$I_{n+1} = [t'', u''] = [t'', F_5] \subset [F_0, F_5]$$

and

$$C_{n+1} = C(t'', t', u, u'').$$

If t'' reaches F_1 first, then define

$$I_{n+1} = [t'', u''] = [F_1, u''] \subset [F_1, F_6]$$

and

$$C_{n+1} = C(t'', t', u, u'').$$

Then

$$\ell(I_{n+1}) \leq \frac{5}{6} \ell(I_n).$$

If case (iii) occurs, without loss of generality, let $r_n = F_3$. We define I_{n+1} and C_{n+1} as follows. Hold p_n, $r_n = F_3$ fixed and let t move from q_n toward F_2. If q_n is already between F_2 and F_3, no movement is carried out. Otherwise, by 3.1.1, there is a point u which moves from s_n toward F_5. If u reaches F_5 first, then define

$$I_{n+1} = [p_n, u] = [p_n, F_5] = [F_0, F_5]$$

and

$$C_{n+1} = C(p_n, t, r_n, u).$$

If t reaches F_2 first, hold $t = F_2$, $r_n = F_3$ fixed and let t' move from p_n toward F_1. Then there is a point u' which moves from u toward F_5. If u' reaches F_5 first, then define

$$I_{n+1} = [t', u'] = [t', F_5] \subset [F_0, F_5]$$

and

$$C_{n+1} = C(t', t, r_n, u').$$

If t' reaches F_1 first, then define

$$I_{n+1} = [t', u'] = [F_1, u'] \subset [F_1, F_6]$$

and

$$C_{n+1} = C(t', t, r_n, u').$$

Then

$$\ell(I_{n+1}) \leq \frac{5}{6} \ell(I_n).$$

By construction, in each case

$$\ell(I_{n+1}) \leq \frac{7}{8} \ell(I_n)$$

and (I_n) is a nested sequence of closed intervals such that $\ell(I_n) \longrightarrow 0$. Hence

$$\bigcap_n I_n = \{y\}$$

([22], p. 10). This point $y \varepsilon \mathcal{B}$ is the required singular point.

If y is not an interior point of B, then hold q_0 and r_0 fixed and let t move a small distance from p_0 toward q_0 before defining the sequence (I_n). By 3.1.1, there is a point u which moves a small distance from s_0 toward r_0. Now use the interval $[t, u]$ as I_0 and $C(t, q_0, r_0, u)$ as C_0 and construct I_n and C_n as above. This will ensure that we obtain a singular point in $[t, u]$ and hence in the interior of $\mathcal{B}$, as required.

3.1.3 Let $p_0 < q_0 < r_0$ be three points on $\mathcal{A}_4$ and $\mathcal{B}$ the closed subarc of $\mathcal{A}_4$ bounded by p_0 and r_0. Let $a \in \mathcal{A}_4 \setminus \mathcal{B}$. Suppose that there exists a circle through the points a, p_0, q_0, r_0. If $\mathcal{L}_a$ is the system of circles passing through the point a, then there exists at least one $\mathcal{L}_a$ - singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y on $\mathcal{B}$ there exists a circle of $\mathcal{L}_a$ that meets N at least three times.

Proof. Let I_0 be the parameter interval $[p_0, r_0]$ and $C_0 = C(a, p_0, q_0, r_0)$. We define a sequence of intervals and a corresponding sequence of circles by induction. Having defined $I_n = [p_n, r_n]$ with $I_n \subset I_0$ and C_n through a and the points $p_n < q_n < r_n$, we define $I_{n+1} \subset I_n$ and C_{n+1} as follows.

Let $\ell(I_n) = \varepsilon$ and D_i be the point of I_n which is $\frac{i}{4}\varepsilon$ from p_n; $i = 0, \ldots, 4$. Either

(i) $q_n \neq D_2$

or

(ii) $q_n = D_2$.

If case (i) occurs, we can assume, without loss of generality, that q_n lies between D_0 and D_2. Hold a, p_n fixed and let t move from q_n toward D_2. Then by 3.1.1, there is a point u which moves from r_n toward D_3. If u reaches D_3 first, then define

$$I_{n+1} = [p_n, u] = [D_0, D_3]$$

and

$$C_{n+1} = C(a, p_n, t, u).$$

If t reaches D_2 first, then hold a, $t = D_2$ fixed and let t' move from p_n toward D_1. By 3.1.1, there is a point u' which moves from u toward D_3. If u' reaches D_3 first, then define

$$I_{n+1} = [t', u'] = [t', D_3] \subset [D_0, D_3]$$

and

$$C_{n+1} = C(a, t', t, u').$$

If t' reaches D_1 first, then define

$$I_{n+1} = [t', u'] = [D_1, u'] \subset [D_1, D_4]$$

and

and

$$C_{n+1} = C(a, t', t, u').$$

If case <u>(ii)</u> occurs, then we define I_{n+1} and C_{n+1} as in the second paragraph of case <u>(i)</u>.

By construction $\ell(I_{n+1}) \leq \frac{3}{4} \ell(I_n)$ and (I_n) is a nested sequence of closed intervals such that $\ell(I_n) \longrightarrow 0$. Hence

$$\bigcap_n I_n = \{y\}$$

([22], p. 10). This point y on $\mathcal{B}$ is a $\mathcal{L}_a$ - singular point.

<u>3.1.4</u> <u>Let</u> N_1, N_2 <u>be arbitrary neighbourhoods of two singular points</u> z_1, z_2 <u>on</u> $\mathcal{A}_4$. <u>Let</u> $\mathcal{B}$ <u>be the closed subarc of</u> $\mathcal{A}_4$ <u>between</u> z_1 <u>and</u> z_2. <u>If</u> $a \in \mathcal{A}_4 \setminus N_1 \cup \mathcal{B} \cup N_2$, <u>then there exists a circle of</u> $\mathcal{L}_a$ <u>which meets</u> $N_1 \cup \mathcal{B} \cup N_2$ <u>at three distinct points</u>.

<u>Proof</u>. By the definition of a singular point, for each neighbourhood N_i of z_i on $\mathcal{A}_4$ there exists a circle meeting N_i in exactly four intersection points, $i = 1, 2$. Let (x_1, x_2, x_3, x_4) (x_1', x_2', x_3', x_4') be respective quadruplets for z_1 and z_2. Without loss of generality, we may take x_j and x_j', $j = 1, \ldots, 4$, such that

$$x_1 < \dots \leq x_4 < x_1' < \dots < x_4'$$

and $a < x_1$ on $\mathcal{Q}_4$. Since $\mathcal{Q}_4$ is normal, the four-tuples (x_1, x_2, x_3, x_4) and (x_1', x_2', x_3', x_4') lie in the same order on their corresponding circles as they do on $\mathcal{Q}_4$. Now hold x_2, x_3 fixed and let t move on $\mathcal{Q}_4$ from x_4 toward x_3'. Then by 3.1.1, there is a point u which moves from x_1 toward a. Continuing this movement one obtains either

$(i)_1$ t coincides with x_3', while $a < u$

or

$(ii)_1$ u coincides with a, while $t \leq x_3'$.

If case $(ii)_1$ occurs, then we are finished. If case $(i)_1$ occurs, then hold x_2 and $t = x_3'$ fixed and let t' move on $\mathcal{Q}_4$ from x_3 toward x_2'. By 3.1.1, there is a point u' which moves from u toward a with the final result that either

$(i)_2$ t' coincides with x_2', while $a < u$

$(ii)_2$ u' coincides with a, while $t' \leq x_2'$.

If case $(ii)_2$ occurs, then we are finished. If case $(i)_2$ occurs, then hold $t = x_3'$, $t' = x_2'$ fixed and let t'' move from x_2 toward x_1'. Then there is a point u'' which moves from u' toward a. Finally either

$(i)_3$ t'' coincides with x_1', while $a < u''$

if

$(ii)_3$ u" coincides with a, while $t'' \leq x_1'$.

If case $(ii)_3$ occurs, then we are finished. If case $(i)_3$ occurs, then we have a circle which meets $\mathcal{Q}_4$ at u", x_1', x_2', x_3'. However, the points x_1', x_2', x_3' determine a unique circle and this circle also meets $\mathcal{Q}_4$ at x_4'. Thus this circle meets $\mathcal{Q}_4$ at least five times; contradiction. Hence case $(ii)_k$ must occur for some k, $1 \leq k \leq 3$, and the result follows.

3.1.5 Let z_1, z_2 be two singular points of $\mathcal{Q}_4$ and let $a \in \mathcal{Q}_4 \setminus \mathcal{B}$, where $\mathcal{B}$ is the closed subarc of $\mathcal{Q}_4$ between z_1 and z_2. Then there exists at least one $\mathcal{L}_a$ - singular point y on $\mathcal{B}$.

Proof. Let $N_i^{(1)}$ be a neighbourhood of z_i on $\mathcal{Q}_4$ with $a \notin N_i^{(1)}$, i = 1, 2. By 3.1.4, there exists a circle C_0 meeting $\mathcal{Q}_4$ at a and meeting $N_1^{(1)} \cup \mathcal{B} \cup N_2^{(1)}$ at three points $p_0 < q_0 < r_0$. By 3.1.3, there is a $\mathcal{L}_a$-singular point.

$$y^{(1)} \in N_1^{(1)} \cup \mathcal{B} \cup N_2^{(1)}.$$

If $y^{(1)} \in \mathcal{B}$, then we are finished. If $y^{(1)} \notin \mathcal{B}$, then $y^{(1)} \in N_1^{(1)} \setminus \mathcal{B}$, say. Now take suitable smaller neighbourhoods $N_i^{(2)} \subset N_i^{(1)}$ of z_i, i = 1, 2, with $y^{(1)} \notin N_i^{(2)}$ and apply 3.1.4 and 3.1.3 again using the new neighbourhoods $N_i^{(2)}$ of z_i. we obtain a $\mathcal{L}_a$ - singular point

$$y^{(2)} \in N_1^{(2)} \cup \mathcal{B} \cup N_2^{(2)}.$$

If $y^{(2)} \in \mathcal{B}$, then we again take new neighbourhoods $N_i^{(3)}$ of z_i with $y^{(2)} \notin N_i^{(3)}$ and apply 3.1.4 and 3.1.3 to obtain a $\mathcal{b}_a$ - singular point

$$y^{(3)} \in N_1^{(3)} \cup \mathcal{B} \cup N_3^{(3)}.$$

Repeating this process, if necessary, and taking $N_i^{(n)}$ such that $N_i^{(n)}$ converges to z_i, either we obtain a $\mathcal{b}_a$ - singular point $y \in \mathcal{B}$ at some stage or we obtain a sequence $y^{(n)}$ of $\mathcal{b}_a$ - singular points lying outside $\mathcal{B}$ with an accumulation point which coincides with at least one of the z_i, say z_1. But then z_1 is a $\mathcal{b}_a$ - singular point and we have the desired result.

Arguments which are analogous to those used in 3.1.3, 3.1.4 and 3.1.5 yield the following.

3.1.6 Let y_1, y_2 be two points of $\mathcal{Q}_4$ and let $\mathcal{B}$ be the closed subarc of $\mathcal{Q}_4$ between them. Let a_1, a_2 be distinct points of $\mathcal{Q}_4 \setminus \mathcal{B}$. If y_1 and y_2 are $\mathcal{b}_{a_1}$ - singular points, then there exists at least one $\mathcal{b}_{a_1 a_2}$ - singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y there exists a circle passing through a_1, a_2 and meeting N at least twice.

3.1.7 Let y_1, y_2 be two points of $\mathcal{Q}_4$ and let $\mathcal{B}$ be the closed subarc of $\mathcal{A}_4$ between them. Let a_1, a_2, a_3 be mutually distinct points of $\mathcal{A}_4 \setminus \mathcal{B}$. If y_1 and y_2 are $\mathcal{B}_{a_1a_2}$ - singular points, then there exists at least one $\mathcal{B}_{a_1a_2a_3}$ - singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y there exists a circle passing through a_1, a_2, a_3 and meeting N at least once.

3.1.8 We are now ready to give the main result of this section.

Theorem 2: A normal arc $\mathcal{A}_4$ of order four contains at most finitely many singular points.

Proof. Suppose that there are infinitely many singular points on $\mathcal{A}_4$. Take any point a_1 on $\mathcal{A}_4$. Then by 3.1.5, there exist infinitely many $\mathcal{B}_{a_1}$ - singular points on $\mathcal{A}_4$. Take another point a_2 on $\mathcal{A}_4$, $a_1 \neq a_2$. By 3.1.6, there exist infinitely many $\mathcal{B}_{a_1a_2}$ - singular points on $\mathcal{A}_4$. Finally, take another point $\mathcal{A}_3$ on $\mathcal{A}_4$, distinct from a_1 and a_2. By 3.1.7, there exist infinitely many $\mathcal{B}_{a_1a_2a_3}$ - singular points on $\mathcal{A}_4$. But now we have constructed a circle through a_1, a_2, a_3 which meets $\mathcal{A}_4$ at infinitely many points; contradiction.

3.1.9 If p is an end-point of $\mathcal{A}_4$, then p is ordinary.

Proof. Assume p is a singular point. Then for each neighbourhood $N^{(1)}$ of p there exists a circle which meets $N^{(1)}$ four times, say at $p_1 < q_1 < r_1 < s_1$. By 3.1.2, there exists a singular point $y^{(1)}$ in (p_1, s_1). Now take a new smaller neighbourhood $N^{(2)}$ of P with $y^{(1)} \notin N^{(2)}$. By 3.1.2, there exists another singular point $y^{(2)}$ different from $y^{(1)}$. Repeating this process and using 3.1.2, we obtain an infinite number of singular points. This is impossible, by Theorem 2.

3.1.10 In 3.1.9 it was shown that an end-point p of $\mathcal{Q}_4$ is ordinary. Thus there exists a neighbourhood N_3 of p on $\mathcal{Q}_4$ which is of order three. But it is known that $N_3 \cup \{p\}$ is strongly differentiable at p([4], 3.5). Thus, an end-point p of $\mathcal{Q}_4$ is strongly differentiable.

3.2 Multiplicities For Arcs of Order Four

Introduction

In [4] N.D. Lane and P. Scherk introduced multiplicities for open arcs $\mathcal{A}_3$ with one end-point p, counting p [a point q of $\mathcal{A}_3$] three times on $C(p^3)$ [on a general osculating circle of $\mathcal{A}_3$ at q] and twice on any other [general] tangent circle of $\mathcal{A}_3$ at p [at q]. Then they showed that no circle meets $\mathcal{A}_3 \cup p$ more than three times, i.e., the inclusion of p and the introduction of multiplicities do not alter the order of $\mathcal{A}_3$.

In 1.3 of [12] O. Haupt and H. Künneth introduced intersection and support components of continua and derived some interesting results concerning intersection and support properties of arcs and curves with general order characteristics having a certain base number k.

However, since we shall be only interested in the special case where the class of order characteristics is the set of circles in the conformal plane, our attention to arcs of order four will be concerned with the former approach of Lane and Scherk. Correspondingly, in this section we shall prove the following result.

Theorem 3: The order of the open arc $\mathcal{A}_4$, with the possible exception noted in the remark following 3.2.14, is not changed by

(i) the addition of one of the end-points p;

(ii) the introduction of multiplicities of p such that p is counted once, twice and three times, respectively, on a nontangent circle through p, a nonosculating tangent circle through p, the osculating circle $C(p^3)$ of $\mathcal{A}_4 \cup \{p\}$ at p ;

(iii) the introduction of multiplicities at interior points q of $\mathcal{A}_4$ such that q is counted once on any circle through q which is not a general tangent circle at q, twice on any general tangent circle at q which is not a general osculating circle, three times on any general osculating circle at q which intersect $\mathcal{A}_4$ at q and four times on any general osculating circle at q which supports $\mathcal{A}_4$ at q. In the last case q would be a singular point of $\mathcal{A}_4$.

We shall assume in the rest of section 3.2 that $p < s$ for all $s \in \mathcal{A}_4$.

3.2.0 No circle C through four mutually distinct points of $\mathcal{A}_4$ supports $\mathcal{A}_4$ at one of these points and intersects $\mathcal{A}_4$ at another one.

Proof. Suppose C supports $\mathcal{A}_4$ at q_1, intersects $\mathcal{A}_4$ at q_2 and meets $\mathcal{A}_4$ at q_3 and q_4. Then a suitable circle sufficiently close to C through q_2 and q_3 will intersect $\mathcal{A}_4$ at two points near q_1 and at one point near q_2. This contradicts the order of $\mathcal{A}_4$.

3.2.1 No circle C supports $\mathcal{A}_4$ at more than two points.

Proof: Suppose C supports Q_4 at q_1, q_2 and q_3. By 3.2.0, C does not intersect Q_4. Hence

$$Q_4 \subset C \cup C_e,$$

say.

Let L, M, N be three disjoint neighbourhoods on Q_4 of q_1, q_2, q_3, respectively. The end-points of L, M, N lie in C_e. Choose a suitable circle D in C_e which is so close to C that the end-points of L, M, N also lie in D_e. We can orient D such that $C \subset D_i$. On the other hand,

$$C \subset D_i \Longrightarrow q_1, q_2, q_3 \in D_i.$$

Thus D separates q_1, q_2, q_3 from the end-points of L, M, N, respectively, and hence D will intersect L, M, N in not less than two points each. Thus D meets Q_4 more than four times; contradiction.

From 3.2.0 and 3.2.1 we obtain

3.2.2 If a circle C supports Q_4 at t, then C cannot meet Q_4 at more than two further points.

If a circle C meets Q_4 at four distinct points, then all of them are points of intersection.

If a circle C through three mutually distinct points of $\mathcal{Q}_4$ supports $\mathcal{Q}_4$ at one of them, then C intersects $\mathcal{Q}_4$ at the other points.

3.2.3 If a circle C supports $\mathcal{Q}_4$ at s and t, then C does not meet $\mathcal{Q}_4 \cup \{p\}$ again.

Proof. Suppose that C meets $\mathcal{Q}_4 \cup \{p\}$ at a further point u. Then by 3.2.1, u is a point of intersection of $\mathcal{Q}_4$ with C or $u = p$. The first possibility is ruled out by 3.2.2. Thus $u = p$.

Without loss of generality, let $s < t$ on $\mathcal{Q}_4$. Let L, M be disjoint neighbourhoods of $\mathcal{Q}_4$ of s, t, respectively. Also let N be a neighbourhood of p on $\mathcal{Q}_4 \cup \{p\}$, disjoint with L and M. The end-points of L, M will lie in C_e, say. Then for a suitable circle D in C_e which is sufficiently close to C, D will meet $\mathcal{Q}_4$ at a point near p and the end-points of L, M will also lie in D_e. We may orient D such that $C \subset D_i$. Thus $s, t \in D_i$. Therefore D separates s and t from the end-points of L and M, respectively. Hence D will intersect L, M in not less than two points each. D then meets $\mathcal{Q}_4$ at least five times; contradiction.

3.2.4

(i) If a circle through p meets $\mathcal{Q}_4$ at four points, then at most one of them is a point of intersection.

Proof. Suppose that a circle C through p intersects $\mathcal{Q}_4$

at q_1, q_2 and meets $\mathcal{Q}_4$ at two further points r and s. Choose disjoint neighbourhoods L, M, N on $\mathcal{Q}_4 \cup \{p\}$ of p, q_1, q_2, respectively, which do not contain r or s. Then if t converges on L to p, C(r, s, t) converges to C. However, C(r, s, t) will intersect M and N if t is sufficiently close to p. Hence this circle meets $\mathcal{Q}_4$ at least five times; contradiction.

(ii) _If a tangent circle of_ $\mathcal{Q}_4$ _at_ p _meets_ $\mathcal{Q}_4$ _at three points, then at most one of them is a point of intersection._

Proof. Let C be a tangent circle of $\mathcal{Q}_4$ at p intersecting $\mathcal{Q}_4$ at the points q_1, q_2 and meeting $\mathcal{Q}_4$ at a further point r. If t is sufficiently close to p, then C(p, t, r) will be close to C and it will intersect $\mathcal{Q}_4$ at points near q_1 and q_2. This is impossible, by _(i)_.

In the same way _(ii)_ implies

(iii) $C(p^3)$ _intersects_ $\mathcal{Q}_4$ _at most once._

_3.2.5_ _No circle meets_ $\mathcal{Q}_4 \cup \{p\}$ _in more than four points._

Proof. Let C be a circle which meets $\mathcal{Q}_4 \cup \{p\}$ in five mutually distinct points. Because $\mathcal{Q}_4$ is of order four, one of these points must be p while the other four lie on $\mathcal{Q}_4$. If one of these four is a point of support, then a contradiction is obtained in _3.2.2_. Hence all four of these points of $\mathcal{Q}_4$ are points of intersection. But this is impossible, by _(i)_ of _3.2.4_.

3.2.6 No tangent circle of $\mathcal{Q}_4 \cup \{p\}$ at p meets $\mathcal{Q}_4$ in more than two points.

Proof. If a tangent circle of $\mathcal{Q}_4 \cup \{p\}$ at p meets $\mathcal{Q}_4$ at three distinct points, then at least two of these are points of support, by (ii) of 3.2.4. However, this is impossible, by 3.2.3.

3.2.7 No tangent circle of $\mathcal{Q}_4 \cup \{p\}$ at p supports $\mathcal{Q}_4$ at one point and intersects $\mathcal{Q}_4$ at another point.

Proof. Let C be a tangent circle of $\mathcal{Q}_4 \cup \{p\}$ at p which intersects $\mathcal{Q}_4$ at q_1 and supports $\mathcal{Q}_4$ at q_2. Then C does not meet $\mathcal{Q}_4$ elsewhere by 3.2.6. Then if t is sufficiently close to p, $C(p, t, q_1)$ will be close to C and meet $\mathcal{Q}_4$ twice near q_2. This is impossible, by 3.2.5.

3.2.8 $C(p^3)$ cannot support $\mathcal{Q}_4$ at a point q.

Proof. Suppose that $C(p^3)$ supports $\mathcal{Q}_4$ at q. Then, by 3.2.3 and 3.2.7, $C(p^3)$ does not meet $\mathcal{Q}_4$ elsewhere. If t is sufficiently close to p, then $C(p^2, t)$ will be close to $C(p^3)$ and meet $\mathcal{Q}_4$ twice near q. This is impossible, by 3.2.6.

3.2.9 No general osculating circle of $\mathcal{Q}_4$ at q intersects $\mathcal{Q}_4 \setminus \{q\}$ more than once.

Proof. Let C be a general osculating circle of $\mathcal{Q}_4$ at q which intersect $\mathcal{Q}_4 \setminus \{q\}$ at two distinct points r and s. Then, by definition of a general osculating circle, there is a circle D sufficiently close to C which meets $\mathcal{Q}_4$ three times near q. But D will also intersect $\mathcal{Q}_4$ once near r and once near s since D is close to C. This is impossible because $\mathcal{Q}_4$ is of order four.

3.2.10 No general osculating circle of $\mathcal{Q}_4$ at q supports $\mathcal{Q}_4$ at a point $r \neq q$.

Proof. Let C be a general osculating circle of $\mathcal{Q}_4$ at q which supports $\mathcal{Q}_4$ at $r \neq q$. Either

(i) C intersects $\mathcal{Q}_4$ at q

or

(ii) C supports $\mathcal{Q}_4$ at q.

Let

$$N = N' \cup \{q\} \cup N''$$

and L be two small two-sided neighbourhoods of q and r on $\mathcal{Q}_4$ respectively, where N' [N''] precedes [follows] q. Without loss of generality, let $q < r$ and

$$L \setminus \{r\} \subset C_e.$$

In case (ii), 3.2.3 implies that C does not meet $\mathcal{Q}_4$ outside q and r. Hence

$$N'' \subset C_e.$$

Suppose that case (i) occurs. Since C is a general osculating circle of $\mathcal{Q}_4$ at q,

$$C = \lim C(q_n, q_n', q_n'')$$

where the three mutually distinct points q_n, q_n', q_n'' converge on N to q. By taking subsequences $q_{nm}, q_{nm}', q_{nm}''$, if necessary,

$$C = \lim C(q_{nm}, q_{nm}', q_{nm}'')$$

where at least two of the three mutually distinct points $q_{nm}, q_{nm}', q_{nm}''$ converge on N' to q or at least two converge on N'' to q. Now, both $N' \cup \{q\}$ and $N'' \cup \{q\}$ satisfy condition CI' at q; cf. 2.4.2 and 3.1.10. Let D be a circle close to C in

$$C_e \cup \{q\}$$

which supports C at q. Then D will intersect L at least twice. But the end-points of N lie on opposite sides of C. Hence the end-points of N will lie on opposite sides of D. But D is a tangent circle of $N' \cup \{q\}$ or of $N'' \cup \{q\}$ at q, by 2.4.1 (i). Thus either

(a) D supports N at q, intersects $N \setminus \{q\}$ at least once and intersects L at least twice, or

(b) D is a tangent circle of $N' \cup \{q\}$ or of $N'' \cup \{q\}$ at q which intersects $\mathcal{Q}_4$ at q; i.e., a general osculating circle of $\mathcal{Q}_4$ at q.

Both of these are impossible, by 3.2.2 and 3.2.9, respectively.

Suppose that case (ii) occurs. Then

$$N' \subset C_e.$$

We will first show that C is necessarily the osculating circle of $N' \cup \{q\}$ or of $N'' \cup \{q\}$ at q. Suppose that C is neither of the one-sided osculating circles of $\mathcal{Q}_4$ at q. Now

$$C = \lim C(q_n, q_n', q_n'', q_n''')$$

where the four mutually distinct points q_n, q_n', q_n'', q_n''' converge on N to q. Since C is neither of the one-sided osculating circles of $\mathcal{Q}_4$ at q, we can assume, by taking subsequences, if necessary, that at least two of the points q_n, q_n', q_n'', q_n''' converge on N' to q while the other two converge on N'' to q. Thus C is a general tangent circle of both $N' \cup \{q\}$ and $N'' \cup \{q\}$ at q. But since both $N' \cup \{q\}$ and $N'' \cup \{q\}$ satisfy condition CI' at q, then $\mathcal{A}_4$ satisfies condition CI' at q and hence the family of tangent circles of $\mathcal{A}_4$ at q is a pencil of the second kind with fundamental point q. Hence one of the one-sided osculating circles of N at q lies in C_i (see Figure 4). Call this circle K. Without loss of generality, suppose that K is the osculating circle of $N' \cup \{q\}$ at q. Now let $s \in N'$. Then the tangent circle

$$C'(q^2, s)$$

of $N' \cup \{q\}$ at q through s lies in $C_e \cup \{q\}$, since $s \in C_e$. If s converges to q on N', then

$$K = \lim_{s \to q} C'(q^2, s) \subset C \cup C_e,$$

since $N' \cup \{q\}$ is differentiable at q; cf. 3.1.10. This is a contradiction. Hence C is one of the one-sided osculating circle of $\mathcal{A}_4$ at q.

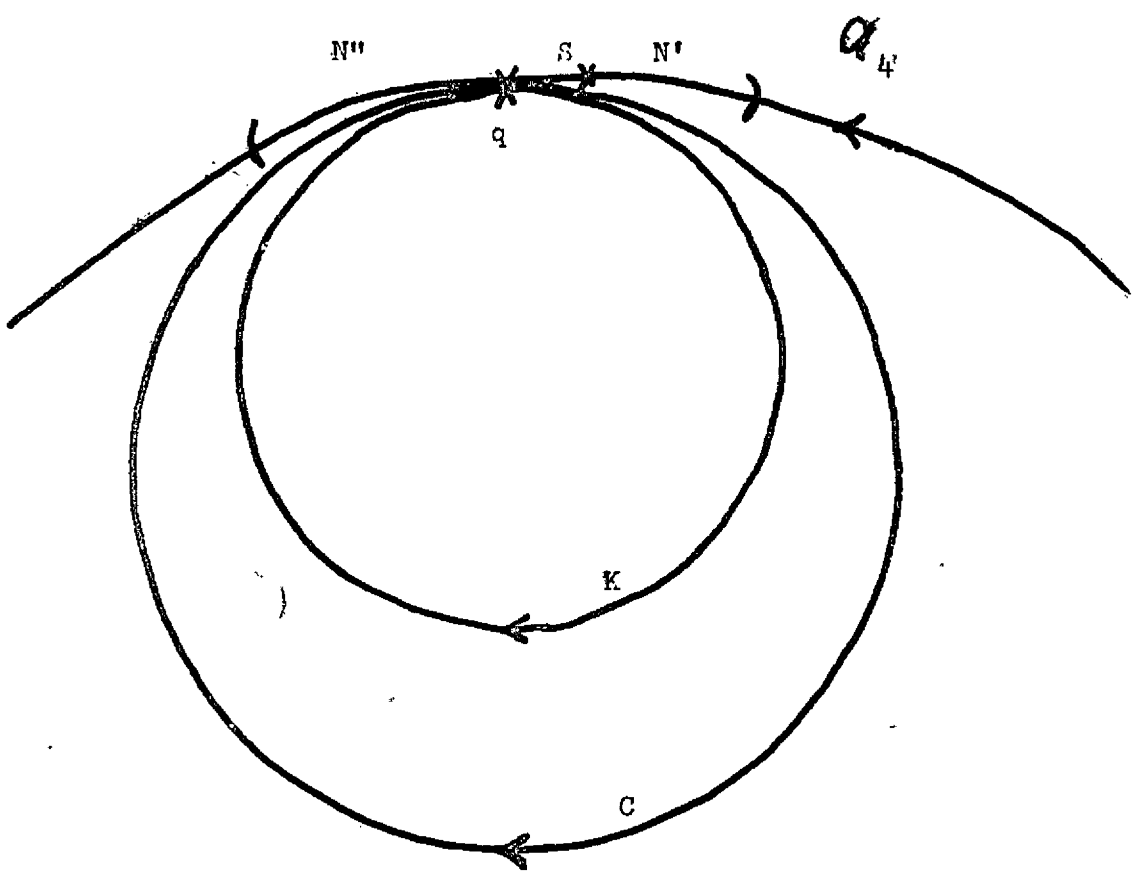


Figure 4

Next C cannot be the osculating circle of $N'' \cup \{q\}$ at q, by 3.2.8. Hence C is the osculating circle of $N' \cup \{q\}$ at q. If t is close to q on N', then the tangent circle $C'(q^2, t)$ of $N' \cup \{q\}$ at q through t will be close to C, by 3.1.10. But $t \in N' \subset C_e$. Hence by 2.4.1 (i),

$$C'(q^2, t) \setminus \{q\} \subset C_e$$

and thus if t is sufficiently close to q on N', $C'(q^2, t)$ will intersect L at least twice. This is impossible, by 3.2.2 and 3.2.9, since $C'(q^2, t)$ either supports $\mathcal{Q}_4$ at q or is a general osculating circle of $\mathcal{Q}_4$ at q.

3.2.11 <u>No general osculating circle of $\mathcal{Q}_4$ at q which supports $\mathcal{Q}_4$ at q can intersect $\mathcal{Q}_4$ again.</u>

Proof. Let C be a general osculating circle of $\mathcal{Q}_4$ at q which supports $\mathcal{Q}_4$ at q and intersects $\mathcal{Q}_4$ at a point $r \neq q$. Since C supports $\mathcal{Q}_4$ at q,

$$C = C_n = \lim C(q_n, q_n', q_n'', q_n''')$$

when the four mutually distinct points q_n, q_n', q_n'', q_n''' converge on $\mathcal{Q}_4$ to q. Since r is a point of intersection of $\mathcal{Q}_4$ with C, for sufficiently large n C_n will be close to C and hence intersect $\mathcal{Q}_4$ at a point close to r. But then C_n meets $\mathcal{Q}_4$ at least five times; contradiction.

<u>3.2.12</u> <u>No general osculating circle of $\mathcal{Q}_4$ at q which supports $\mathcal{Q}_4$ at q meets $\mathcal{Q}_4 \cup \{p\}$ again.</u>

<u>Proof.</u> Let C be a general osculating circle of $\mathcal{Q}_4$ at q which supports $\mathcal{Q}_4$ at q and meets $\mathcal{Q}_4 \cup \{p\}$ at a further point u. By <u>3.2.10</u> and <u>3.2.11</u>, u = p and C does not meet $\mathcal{Q}_4 \setminus \{q\}$. As in the proof of <u>3.1.10</u>, one can show that C is necessarily one of the one-sided osculating circles of $\mathcal{Q}_4$ at q. Let

$$N = N' \cup \{q\} \cup N'' \quad [L]$$

be a small two-sided [one-sided] neighbourhood of q [p] on $\mathcal{Q}_4 \cup \{p\}$. Without loss of generality, let

$$\mathcal{Q}_4 \setminus \{q\} \subset C_e.$$

Then $N', N'' \subset C_e$.

Suppose that C is the osculating circle of $N' \cup \{q\}$ at q. If t is sufficiently close to q on N', the tangent circle

$$C'(q^2, t)$$

of $N' \cup \{q\}$ at q through t will be close to C. By <u>2.4.1</u> (i),

$$C'(q^2, t) \setminus \{q\} \subset C_e$$

and thus if t is sufficiently close to q on N', $C'(q^2, t)$ will meet L at least once. But $C'(q^2, t)$ must meet N with an even multiplicity. But then we have one of

(a) $C'(q^2, t)$ supports $\mathcal{Q}_4$ at q, t and meets L at least once,

(b) $C'(q^2, t)$ supports $\mathcal{Q}_4$ at q, intersects $N \setminus \{q\}$ at two points t, r and meets L at least once,

(c) $C'(q^2, t)$ intersects $\mathcal{Q}_4$ at q, t and meets L at least once.

But these situations are impossible, by 3.2.3, 3.2.2, 3.2.9 and 3.2.10.

Similarly C cannot be the osculating circle of $N'' \cup \{q\}$ at q.

3.2.13 No general osculating circle of $\mathcal{Q}_4$ at q meets $\mathcal{Q}_4 \cup \{p\} \setminus \{q\}$ more than once.

Proof. Suppose that C is a general osculating circle of $\mathcal{Q}_4$ at q which meets $\mathcal{Q}_4 \cup \{p\}$ at two further points $r < s$. Then by 3.2.9 and 3.2.10, $r = p$ and s is a point of intersection of $\mathcal{Q}_4$ with C. By 3.2.11, C intersects $\mathcal{Q}_4$ at q. But then we construct a circle D, as in 3.2.10 case (i), which either supports $\mathcal{Q}_4$ at q and meets $\mathcal{Q}_4$ at three further points or is a general osculating circle of $\mathcal{Q}_4$ at q which meets $\mathcal{Q}_4$ at two further points. Both of these situations are impossible, by 3.2.2, 3.2.9 and 3.2.10.

3.2.14 Let C be a general osculating circle of $\mathcal{A}_4$ at q which meets $\mathcal{A}_4 \cup \{p\}$ at p. By 3.2.12 and 3.2.13, C intersects $\mathcal{A}_4$ at q and does not meet $\mathcal{A}_4$ elsewhere. Now

$$C = \lim C(q_n, q_n', q_n'')$$

where each of the circles $C(q_n, q_n', q_n'')$ has a natural orientation with the points q_n, q_n', q_n'' in the order in which they occur on $\mathcal{A}_4$. Hence the orientations of $C(q_n, q_n', q_n'')$, (we take subsequence if necessary) induce an orientation on the limit circle C. Then we have the following result (see Figure 5(a)).

The oriented circle C cannot be a tangent circle of $\mathcal{A}_4 \cup \{p\}$ at p in the same direction.

Proof. Let

$$N = N' \cup \{q\} \cup N'' \quad [L]$$

be a small two-sided [one-sided] neighbourhood of q [p] on $\mathcal{A}_4 \cup \{p\}$. Without loss of generality let $N' \subset C_e$. Then the entire subarc $\mathcal{B}$ of $\mathcal{A}_4$ bounded by p and q lies in C_e.

Since C is a general osculating circle of $\mathcal{A}_4$ at q, C is either the tangent circle $C'(q^2, p)$ of $N' \cup \{q\}$ at q through p or the tangent circle $C''(q^2, p)$ of $N'' \cup \{q\}$ at q through p. Let $C = C'(q^2, p)$, say. Then

$$\lim_{\substack{s' \varepsilon N' \\ s' \to q}} C(q, s', p) = C.$$

Next suppose that $C(q, s', p)$ meets $\mathcal{B}$ again at a point t. Then

$$C(p, t, q) = C(q, s', p)$$

and as $s' \longrightarrow q$, $t \longrightarrow p$. However, the end-points of N lie on opposite sides of C. Thus the end-points of N will lie on opposite sides of $C(p, t, q)$ for t close to p on L. Therefore $C(p, t, q)$ meets N with an odd multiplicity. But $C(p, t, q)$ meets N at q and s'. Thus $C(p, t, q)$ meets N at least three times and we have a circle $C(p, t, q)$ meeting $\mathcal{A}_4 \cup \{p\}$ at least five times. This is impossible, by 3.2.5. Hence $C(q, s', p)$ does not meet again.

But $\mathcal{B} \subset C_e$. Thus

$$\mathcal{B} \setminus N' \subset C(q, s', p)_e \quad \text{(see Figure 5(b)).}$$

Let $t \varepsilon L$. The end-point f of L lies in C_e. Hence f lies in $C(p, t, q)_e$. Also $C(p, t, q)$ meets $C(q, s', p)$ only at p and q. But $t \varepsilon \mathcal{B} \setminus N'$. Hence

$$t \varepsilon C(q, s', p)_e$$

and thus the arc of $C(p, t, q)$ containing t between p and q lies in $C(q, s', p)_e$. Thus

$$s' \varepsilon C(p, t, q)_i.$$

But

$$f \varepsilon C(p, t, q)_e$$

and $$s' \varepsilon C(p, t, q)_i$$

imply that the arc of $\mathcal{Q}_4$ between f and s' meets $C(p, t, q)$ with an odd multiplicity and hence at least once, say at s. Thus

$$C(p, t, q) = C(q, s, p)$$

as $t \longrightarrow p$, $s \longrightarrow q$. We now proceed as in the preceding paragraph to obtain a circle that meets $\mathcal{Q}_4 \cup \{p\}$ at least five times; contradiction.

<u>Remark:</u> The possible exception to <u>Theorem 3</u> occurs as follows.

Let q and p be end-points of an arc $\mathcal{Q}_4$ of order four. Then the osculating circle of $\mathcal{Q}_4 \cup \{q\}$ at q, which has a unique orientation induced by $\mathcal{Q}_4$, can also be a tangent circle of $\mathcal{Q}_4 \cup \{p\}$ at p.

The possibility seems still to exist if q is an interior point and p an end-point of $\mathcal{Q}_4$. The problem arises when an oriented general osculating circle C is a tangent circle of $\mathcal{Q}_4 \cup \{p\}$ at p in the opposite direction (see Figure 5, (d)).

This problem seems to be analogous to the involving the addition of end-points to open arcs of order three.

Let $\mathcal{Q}_3$ be an open arc of order 3 with end-points p, q. Also let C be a tangent circle of $\mathcal{Q}_4 \cup \{p\}$ at p and a tangent circle of $\mathcal{Q}_4 \cup \{q\}$ at q. Then

<u>(a)</u> the order of $\mathcal{Q}_3 \cup \{p\} \cup \{q\}$ is increased to order four if C is tangent to $\mathcal{Q}_3 \cup \{p\} \cup \{q\}$ at p and q in the same direction (see Figure 5(d)).

However,

<u>(b)</u> the order of $\mathcal{Q}_3$ is unchanged by the addition of both end-points if C is tangent to $\mathcal{Q}_3 \cup \{p\} \cup \{q\}$ at p and q in the opposite direction (see Figure 5(e)).

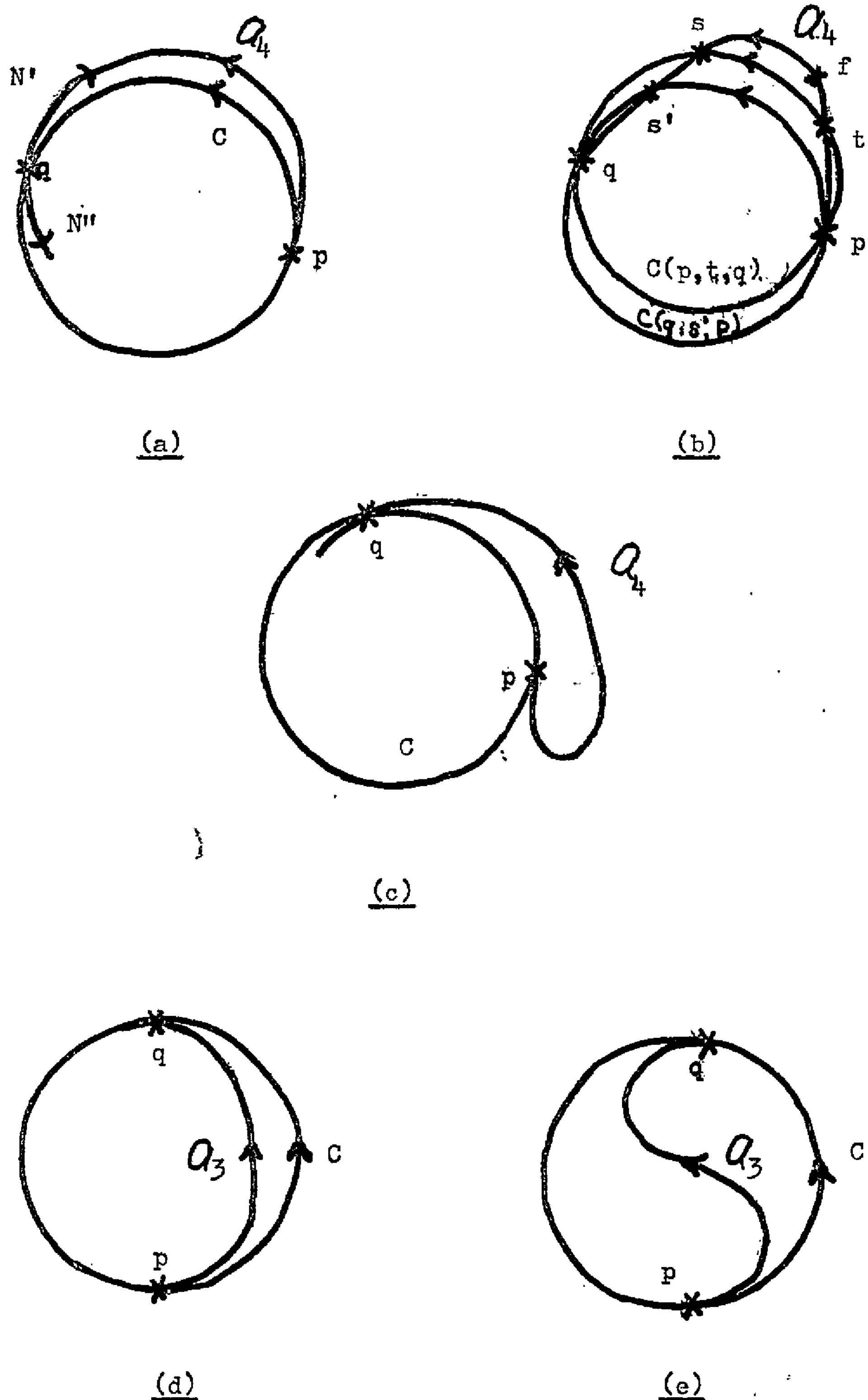


Figure 5

3.3 Differentiable Curves of Order Four

Introduction.

In this section let $\mathcal{B}_4$ be a curve of order four. We shall assume here for the most part that $\mathcal{B}_4$ is conformally differentiable; cf. 2.4.1. By 2.6 (iii), the characteristic of a point of $\mathcal{B}_4$ is one of

$$(1, 1, 1), (1, 1, 2), (1, 1, 2)_0, (1, 2, 1)_0, (2, 1, 1)_0.$$

By Theorem 2, $\mathcal{B}_4$ contains at most finitely many singular points. Hence each singular point is elementary; cf. 2.6 (iv). But then each singular point is a vertex; cf. 2.7. Moreover by 2.6 (iv), the only possible singular points are those whose characteristic is one of the last four of the five listed above.

In 3.3.1 we derive a result, namely Theorem 4, which is very helpful in studying the classification of differentiable curves $\mathcal{B}_4$ of order four in regard to types and numbers of singular points. An euclidean proof was originally given by P. Erdös and can be found in [8].

In 3.3.5 and 3.3.6, singular points p of $\mathcal{B}_4$ with the characteristic $(2, 1, 1)_0$ and $(1, 2, 1)_0$, respectively, are discussed as regards the induced orientation on the osculating circle $C(p^3) = p$ of $\mathcal{A}_4$ at p.

In 3.3.9 it is shown that $\mathcal{B}_4$ contains at most one point with the characteristic $(2, 1, 1)_0$ and we use Theorem 4 to show that $\mathcal{B}_4$ contains at most two points with the characteristic $(1, 2, 1)_0$. If $\mathcal{B}_4$ contains a point with the characteristic $(2, 1, 1)_0$, then $\mathcal{B}_4$ contains at most one point with the characteristic $(1, 2, 1)_0$; cf. 3.3.11. $\mathcal{B}_4$ contains an even or odd number of singular points according to the existence of no points or exactly one point on $\mathcal{B}_4$, respectively, with the characteristic $(1, 2, 1)_0$; cf. 3.3.13 and 3.3.15.

It is well known that a strongly differentiable curve $\mathcal{B}_4$ of order four contains at most four vertices ([12], 4.1.4.3). Here we shall show that the weaker condition of ordinary differentiability on $\mathcal{B}_4$ yield the same result; cf. Theorem 5.

3.3.1 First we shall give a conformal proof of a theorem which is very useful as well as being of some interest in its own right ([5]).

Theorem 4: Let $\mathcal{R}$ be a closed simply connected region of the real inversive plane bounded by a Jordan curve $\mathcal{J}$, and let $\mathcal{J}$ be divided into three closed arc $\mathcal{Q}_1, \mathcal{Q}_2 \mathcal{Q}_3$. Then there exists a circle contained in $\mathcal{R}$ and having points in common with all three arcs.

Proof. Let S_i be the set of circles lying in $\mathcal{R}$ which have a point in common with $\mathcal{Q}_i$, i = 1, 2, 3. We include in S_i the point circles of $\mathcal{Q}_i$. The set S_i are closed and connected. Since $S_i \cap S_j \neq \emptyset$, $S_1 \cup S_2$ is a closed connected set and so is $S = S_1 \cup S_2 \cup S_3$.

Let P be any fixed point, $P \notin \mathcal{R}$. Let φ be the mapping: $S \rightarrow \mathcal{R}$ which takes a non-degenerate circle C of S into that point of $\mathcal{R}$ which is the image of P under inversion in the circle C. If C is a point circle of S, take $\varphi(C) = C$. The mapping is a homeomorphism and both φ and φ^{-1} take closed connected sets into closed connected sets. Also $\varphi[S] = \mathcal{R}$.

It is well known that the set of points of $\mathcal{R}$ is unicoherent (i.e., if $\mathcal{R}$ is written as a union of two closed connected sets $\mathcal{R}_1$ and $\mathcal{R}_2$, then $\mathcal{R}_1 \cap \mathcal{R}_2$ is also closed and connected). Hence S is also unicoherent.

Suppose that $S_1 \cap S_2 \cap S_3 = \emptyset$. Then $S_3 \cap S_1$ and $S_3 \cap S_2$ are disjoint. They are also non-empty. Hence

$$S_3 \cap (S_1 \cup S_2) = (S_3 \cap S_1) \cup (S_3 \cap S_2)$$

consists of two non-empty disjoint closed sets and is therefore not connected. This contradicts the unicoherence of S. Hence there is some circle C in $\mathcal{R}$ that has points in common with each of the arcs $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$.

Remark: $\mathcal{J}$ divides the inversive plane into two closed simply connected regions bounded by $\mathcal{J}$. By this theorem, there exist two circles, one in $\mathcal{J}_i \cup \mathcal{J}$ and one in $\mathcal{J}_e \cup \mathcal{J}$ which have points in common with the three arcs $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$.

The following results are special cases of 3.2.1 and 3.2.12, respectively.

3.3.2 Let $\mathcal{R}$ be a closed region bounded by a curve $\mathcal{C}_4$ of order four. Then any circle lying in $\mathcal{R}$ has at most two points in common with $\mathcal{C}_4$.

3.3.3 Let $\mathcal{C}_4$ be a differentiable curve of order four. Then the osculating circle of $\mathcal{C}_4$ at any vertex has no further points in common with $\mathcal{C}_4$.

Next we give a result for an interior point of an arc that does not involve order but is needed to obtain 3.3.11.

3.3.4 Let p be a once differentiable cusp point of an arc $\mathcal{A}$. Then all circles $\neq$ p which support $\mathcal{A}$ at p lie locally on the same side of $\mathcal{A}$ outside p.

Proof. Since p is a cusp point, there is at least one circle K which supports $\mathcal{A}$ at p. Let K' be any circle, $K' \neq K$, which supports $\mathcal{A}$ at p.

Suppose that K and K' lie locally on opposite sides of $\mathcal{A}$ in a small neighbourhood N of p. Then

$$(K \setminus \{p\}) \cap N \quad \text{and} \quad (K' \setminus \{p\}) \cap N$$

are separated by $\mathcal{A}$. Hence K' supports K at p. Therefore K' belongs to a pencil of circles τ of the second kind, the members of which touch the circle K at p. Then there are nontangent cirles of $\mathcal{A}$ at p which intersect K and K' at p and hence intersect $\mathcal{A}$ at p also. This is impossible, since all the nontangent circles support $\mathcal{A}$ at a cusp point.

3.3.5 Let p be a point of a differentiable curve $\mathcal{C}_4$ of order four with the characteristic $(2, 1, 1)_0$. Then by 3.3.4, we may assume that each circle that supports $\mathcal{C}_4$ at p lies locally on the same side of $\mathcal{C}_4$ outside p, say in $\mathcal{C}_{4i}$. Let

$$N = N' \cup \{p\} \cup N''$$

be a small two-sided neighbourhood of p on $\mathcal{C}_4$ where N' [N"] is a preceding [proceeding] neighbourhood of p.

We know that the osculating circles of $N' \cup \{p\}$, $N'' \cup \{p\}$ and $\mathcal{C}_4$ at p are all equal to the point circle p since N is differentiable at p and p has characteristic $(2, 1, 1)_0$.

We would like to know in what manner $N' \cup \{p\}$ and $N'' \cup \{p\}$ induce orientations on their common osculating circle

$$C'(p^3) = C''(p^3) = p.$$

Let s', t' be two points on N' with s' > t' (see Figure 6). Then a natural orientation is induced on C(t', s', p) with the points t', s', p in the order in which they occur on $\mathcal{C}_4$. Now C(t', s', p) supports $\mathcal{C}_4$ at p. Otherwise, C(t', s', p) would be a tangent circle of $\mathcal{C}_4$ at p which intersects $\mathcal{C}_4$ at p and hence would be a general osculating circle of $\mathcal{C}_4$ at p. This is impossible, by 3.2.9. We have assumed that C(t', s', p) lies locally in $\mathcal{C}_{4i}$ outside p. Also C(t', s', p) cannot meet $\mathcal{C}_4$ again, by 3.2.2. Hence the arc of C(t', s', p) between s' and p which does not contain t' lies in $\mathcal{C}_{4i}$. Because of the natural orientation induced on C(t', s', p), the arc of N' between s' and p lies in $C(t', s', p)_e$. Now let $s' \longrightarrow p$ on N'. Then

$$\lim_{\substack{s' \to p \\ s' \in N'}} C(t', s', p) = C'(p^2, t'),$$

the tangent circle of $N' \cup p$ at p through t' and a natural orientation is induced on $C'(p^2, t')$ such that the arc of N' between t' and p lies in $C'(p^2, t')_i$.

Now let $t' \longrightarrow p$ on N'. Then

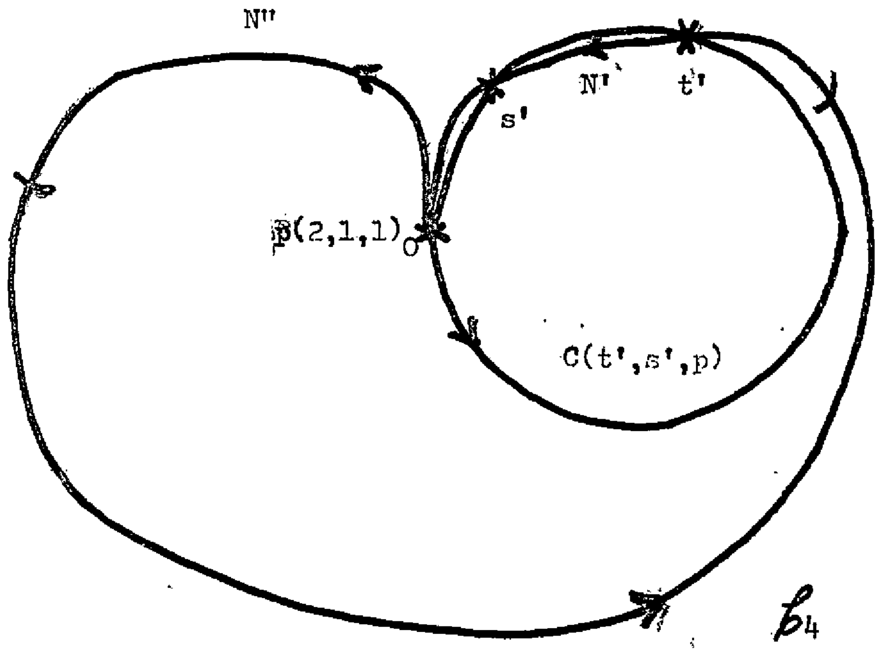
$$\lim_{\substack{t' \to p \\ t' \in N'}} C'(p^2, t') = C'(p^3) = p$$

and

$$\lim_{\substack{t' \to p \\ t' \in N'}} C'(p^2, t')_i = \emptyset.$$

Thus a natural orientation is induced on the osculating circle $C'(p^3) = p$ of $N' \cup \{p\}$ at p such that $C'(p^3)_i = \emptyset$.

Similarly it can be seen that a natural orientation is induced on the osculating circle $C''(p^3) = p$ of $N'' \cup \{p\}$ at p such that $C''(p^3)_i = \emptyset$ (see Figure 7).



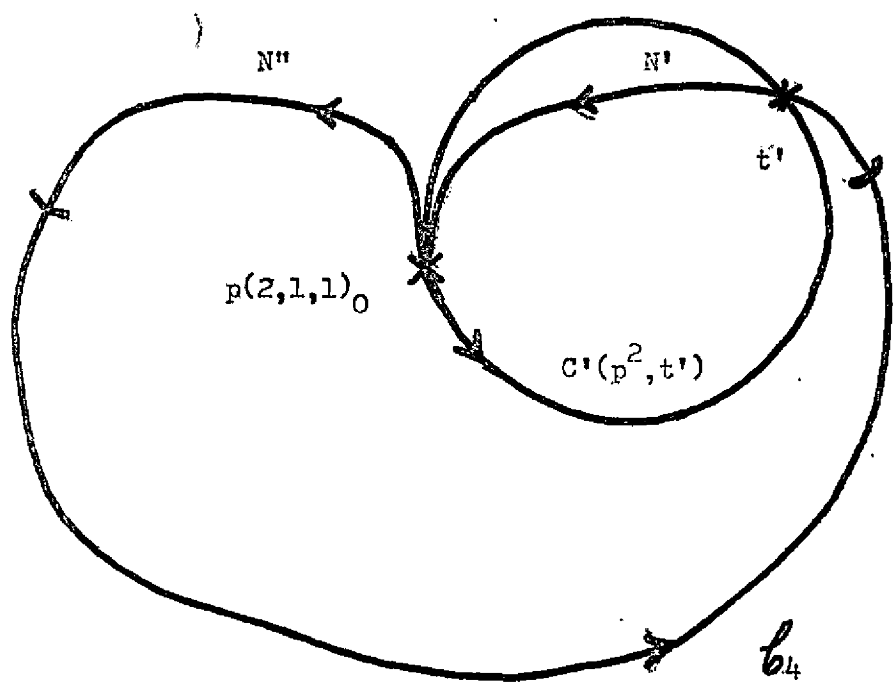


Figure 6

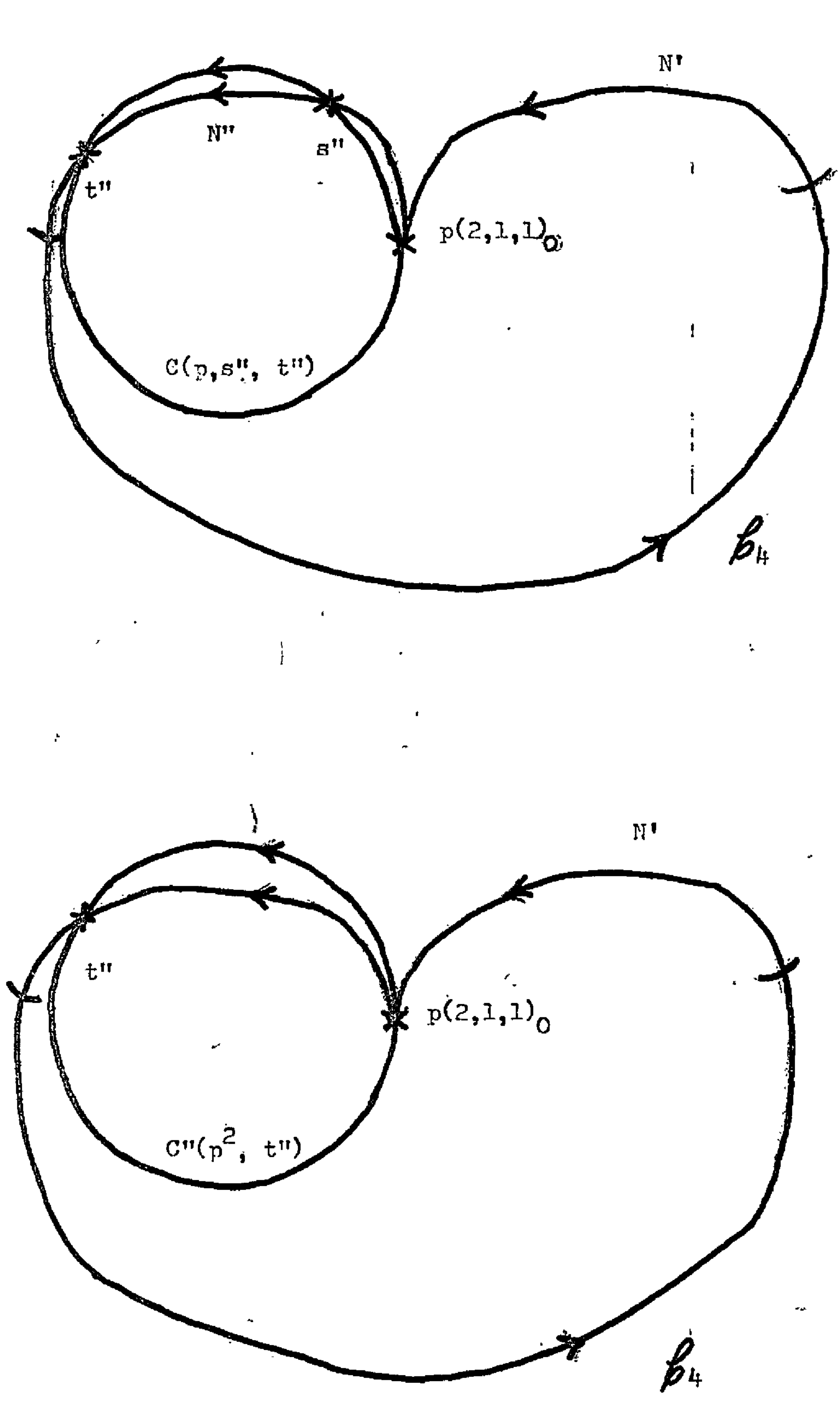


Figure 7

3.3.6 Let p be a point of $\mathcal{C}_4$ with the characteristic $(1, 2, 1)_0$. Let

$$N = N' \cup \{p\} \cup N''$$

be a small neighbourhood of p on $\mathcal{C}_4$ where N' [N''] is a preceding [proceeding] neighbourhood of p of order three; cf. 3.1.9.

We know, as in 3.3.5, that the osculating circles of $N' \cup \{p\}$, $N'' \cup \{p\}$ and $\mathcal{C}_4$ at p are all equal to the point circle p. But in what manner does $N' \cup \{p\}$ [$N'' \cup \{p\}$] induce an orientation on its osculating circle $C'(p^3)$ [$C''(p^3)$]?

Take points $s' > t'$ on N' (see Figure 8). Consider $C(t', s', p)$. $C(t', s', p)$ does not meet N' again since $N' \cup \{p\}$ is of order three. Also $C(t', s', p)$ is not a tangent circle of $N' \cup \{p\}$ at p. Suppose, for example, that the arc of $C(t', s', p)$ between s' and p which does not contain t' lies in $\mathcal{C}_{4e}$. We can choose t' and s' so close to p, and hence $C(t', s', p)$ so close to the point circle p, that $C(t', s', p)$ meets N'' at exactly one point r''. Then a natural orientation is induced on $C(t', s', p)$ by taking the points t', s', p in the order in which they occur on $\mathcal{C}_4$. Because of this induced orientation, the arc of N' between s' and p lies in $C(t', s', p)_i$. Now let $s' \longrightarrow p$ on N'. Then

$$\lim_{\substack{s' \to p \\ s' \in N'}} C(t', s', p) = C'(p^2, t'),$$

the tangent circle of $N' \cup \{p\}$ at p through t' and a natural orientation is induced on $C'(p^2, t')$ such that the arc of N' between t' and p lies in $C'(p^2, t')_e$.

Now let $t' \longrightarrow p$ on N'. Then

$$\lim_{\substack{t' \to p \\ t' \in N'}} C'(p^2, t') = C'(p^3) = p$$

and

$$\lim_{\substack{t' \to p \\ t' \in N'}} C'(p^2, t')_e = \emptyset.$$

Thus a natural orientation is induced on the osculating circle $C'(p^3) = p$ of $N' \cup \{p\}$ at p such that $C'(p^3)_e = \emptyset$.

Now take $t'' > s''$ on N'' (see Figure 9). Consider $C(p, s'', t'')$. A natural orientation is induced on $C(p, s'', t'')$ such that the arc of N'' between p and s'' lies in $C(p, s'', t'')_e$.

Let $s'' \longrightarrow p$ on N''. Then

$$\lim_{\substack{s'' \to p \\ s'' \in N''}} C(p, s'', t'') = C''(p^2, t''),$$

the tangent circle of $N'' \cup \{p\}$ at p through t'' and a natural orientation is induced on $C''(p^2, t'')$ such that the arc of N'' between p and t'' lies in $C''(p^2, t'')_i$.

Now let $t'' \longrightarrow p$ on N''. Then

$$\lim_{\substack{t'' \to p \\ t'' \varepsilon N''}} C''(p^2, t'') \doteq C''(p^3) = p$$

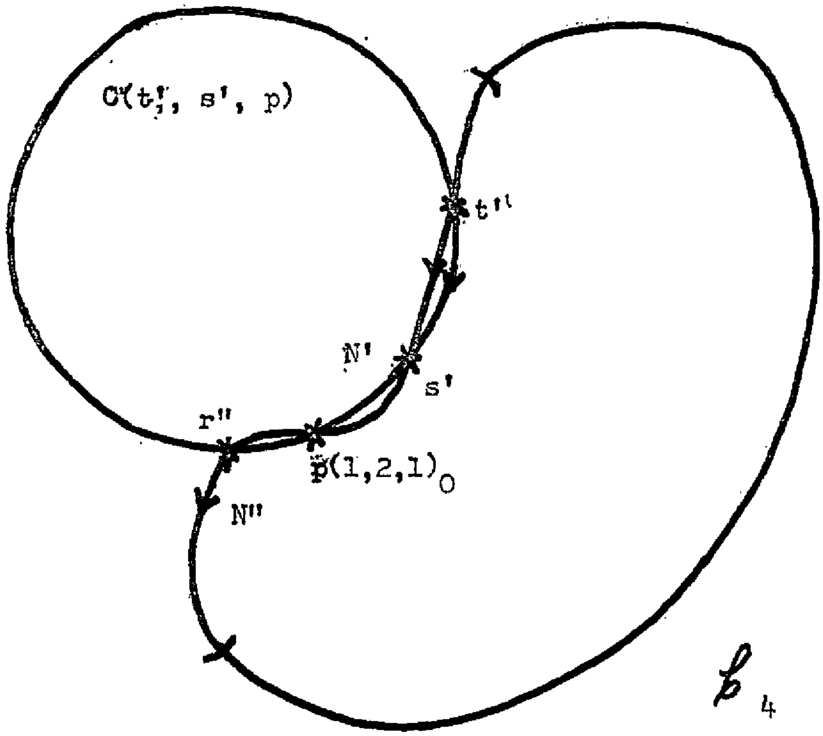
and

$$\lim_{\substack{t'' \to p \\ t'' \varepsilon N''}} C''(p^2, t'')_i = \emptyset.$$

Thus a natural orientation is induced on the osculating circle $C''(p^3) = p$ of $N'' \cup \{p\}$ at p such that $C''(p^3)_i = \emptyset$.

We note that in 3.3.5 when p was of type $(2, 1, 1)_0$, the natural orientation of $C^*(p^3) = p$ was the same as the induced orientation of $C''(p^3) = p$. However, if p has characteristic $(1, 2, 1)_0$ as above, we see that the induced orientation of $C'(p^3) = p$ is opposite to the natural one induced on $C''(p^3) = p$.

Thus the natural orientation induced by $\mathcal{C}_4$ on circles through the points of N' and N'' is discontinuous at p in the case where the characteristic of p is $(1, 2, 1)_0$.



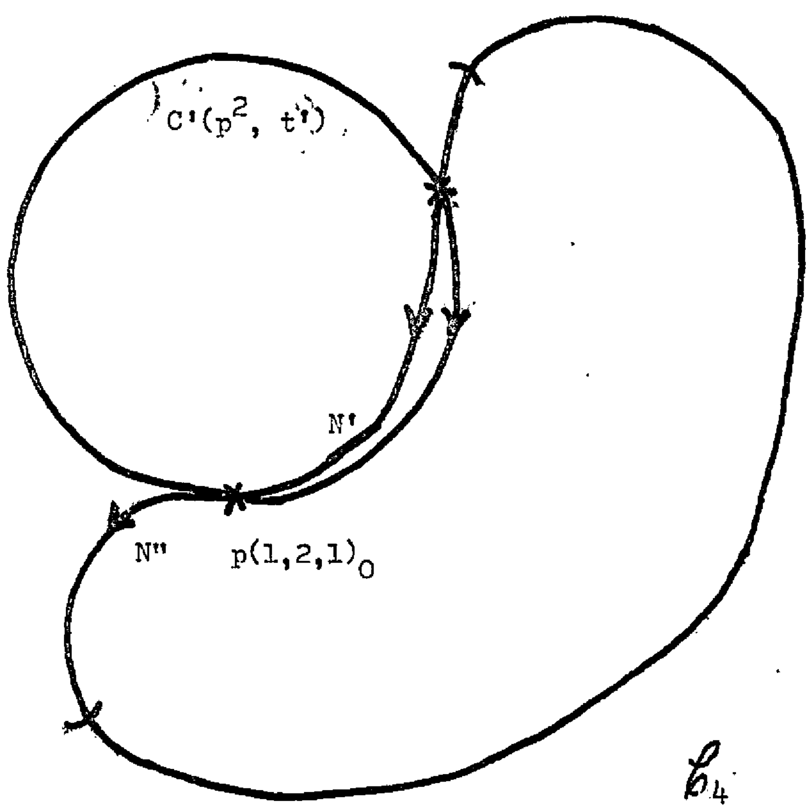
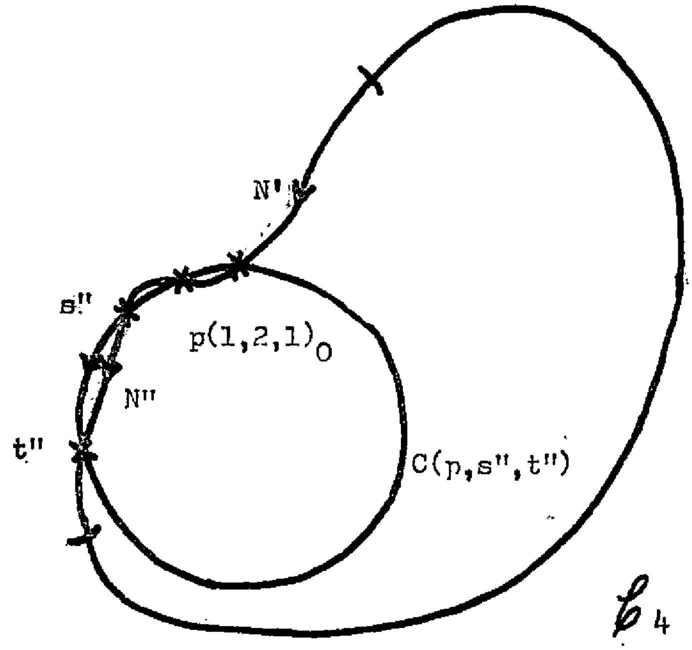


Figure. 8



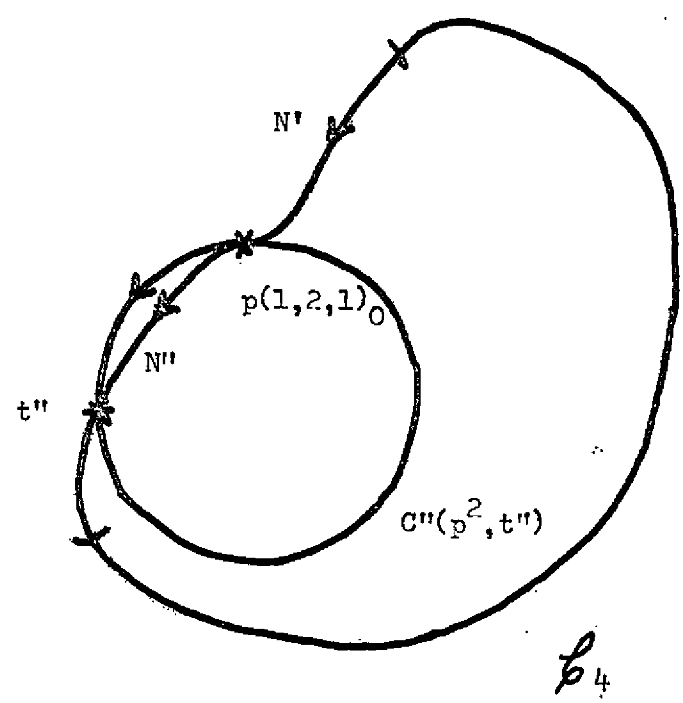


Figure 9

3.3.7 We next state a well known result ([4], Theorem 5).

Let p be a conformally elementary point of an arc $\mathcal{A}$ with the characteristic (a_0, a_1, a_2) or $(a_0, a_1, a_2)_0$. Then

(i) p satisfies condition CI'; cf 2.4.2, iff it satisfies condition CI; cf. 2.4.1, and $a_0 = 1$.

(ii) $\mathcal{A}$ is strongly differentiable at p iff it is differentiable at p and $a_0 = a_1 = 1$.

By 3.1.9, each point of a normal arc $\mathcal{A}_4$ [curve $\mathcal{C}_4$] of order four is elementary.

If p is a differentiable point of an arc $\mathcal{A}_4$ [$\mathcal{C}_4$] with the characteristic (1, 1, 2) or $(1, 1, 2)_0$, then by the above result, $\mathcal{A}_4$ [$\mathcal{C}_4$] is strongly differentiable at p.

3.3.8 We combine 2.4.2 (iv) and 3.3.7 to obtain:

Let p be a differentiable point of $\mathcal{C}_4$ with the characteristic (1, 1, 2) or $(1, 1, 2)_0$. Then the natural orientations induced on the one-sided osculating circles of $\mathcal{C}_4$ at p are identical.

3.3.9 Let $\mathcal{C}_4$ be a differentiable curve of order four. Then $\mathcal{C}_4$ contains at most one point with the characteristic $(2, 1, 1)_0$.

Proof. Suppose that $\mathcal{C}_4$ contains at least two points q_1, q_2 with characteristic $(2, 1, 1)_0$. Let $r \in \mathcal{C}_4$, $r \neq q_1, q_2$ and $K = K(q_1, q_2, r)$ be the circle determined by q_1, q_2 and r. By 3.2.2, at most one of q_1, q_2 and r is a point of support of K with $\mathcal{C}_4$.

If K supports $\mathcal{C}_4$ at r, then K intersects $\mathcal{C}_4$ at q_1, q_2. Since q_1 has characteristic $(2, 1, 1)_0$, then K is a tangent circle of $\mathcal{C}_4$ at q_1 which intersects $\mathcal{C}_4$ there. Hence K is a general osculating circle of $\mathcal{C}_4$ at q_1 which meets $\mathcal{C}_4$ at r and q_2. This is impossible, by 3.2.9 and 3.2.10.

If K supports $\mathcal{C}_4$ at q_1, say, then K will intersect $\mathcal{C}_4$ at q_2 and hence K will be a general osculating circle of $\mathcal{C}_4$ at q_2. Again this is impossible, by 3.2.10.

Thus q_1, q_2 and r are all points of intersection of K with $\mathcal{C}_4$. But again K will then be a general osculating circle of $\mathcal{C}_4$ at q_1 which meets $\mathcal{C}_4$ at q_2 and r. This is a contradiction, by 3.2.9. Hence we have the required result.

3.3.10 $\mathcal{C}_4$ contains at most two points with the characteristic $(1, 2, 1)_0$.

Proof. Suppose $\mathcal{C}_4$ contains three points p_1, p_2, p_3 with characteristic $(1, 2, 1)_0$. Then these points divide $\mathcal{C}_4$ into three closed arcs. Hence by <u>Theorem 4</u>, there exists a circle K lying in $\mathcal{C}_4 \cup \mathcal{C}_{4_i}$, having points in common with all three arcs. By <u>3.3.2</u>, one of the p_i, say p, is a point of contact of K with $\mathcal{C}_4$. Hence K supports $\mathcal{C}_4$ at p. But this possibility is excluded by the characteristic of p, since both the nonosculating tangent circles and the nontangent circles of $\mathcal{C}_4$ at p intersect $\mathcal{C}_4$ at p.

<u>3.3.11</u> <u>If $\mathcal{C}_4$ contains a point with the characteristic $(2,1,1)_0$, then at most one point of $\mathcal{C}_4$ has the characteristic $(1, 2, 1)_0$.</u>

<u>Proof</u>. Let p be a point of $\mathcal{C}_4$ with characteristic $(2, 1, 1)_0$ and q_1, q_2 points of $\mathcal{C}_4$ with characteristic $(1, 2, 1)_0$. Then p, q_1, q_2 divide $\mathcal{C}_4$ into three closed arcs. By <u>Theorem 4</u>, there exists a circle K lying in $\mathcal{C}_4 \cup \mathcal{C}_{4_i}$, having points in common with all three arcs. By <u>3.3.2</u>, one of the points p, q_1, q_2 is a point of contact of K with $\mathcal{C}_4$. This point of K cannot be either q_1 or q_2. Otherwise, K would support $\mathcal{C}_4$ at this point. But this situation is excluded by the characteristic, since both the nonosculating tangent circles and the nontangent circles of $\mathcal{C}_4$ intersect $\mathcal{C}_4$ at a point of type $(1, 2, 1)_0$. Hence p must be the point of contact of K with $\mathcal{C}_4$.

However, by the remark following <u>Theorem 4</u>, there exists a circle K' lying in $\mathcal{C}_4 \cup \mathcal{C}_{4_e}$, having points in common with all three arcs.

As before, neither q_1 or q_2 is a point of contact of K' with $\mathcal{C}_4$. Hence p is again the point of contact of K' with $\mathcal{C}_4$. But this is impossible, by 3.3.4.

3.3.12 Here we introduce a concept of monotony of an arc $\mathcal{A}$. We shall denote a general osculating circle of $\mathcal{A}$ at a point p by C(p).

$\mathcal{A}$ is said to be monotone if $\mathcal{A}$ induces a unique orientation on the general osculating circles at each point of $\mathcal{A}$ such that if $p < q$ on $\mathcal{A}$,

$$C(p) \subset C(q)_i \quad \text{and} \quad C(q) \subset C(p)_e$$

or

$$C(p) \subset C(q)_e \quad \text{and} \quad C(q) \subset C(p)_i .$$

We note the following results:

(i) Arcs of order three are monotone ([6], 4).

(ii) Let each interior point of an arc $\mathcal{A}_4$ of order four be ordinary. Then the closed arc $\overline{\mathcal{A}}_4$ is monotone.

Proof. Each interior point of $\mathcal{A}_4$ is ordinary. Also the end-points of $\mathcal{A}_4$ are ordinary, by 3.1.9. Hence each [interior] point of the arc possesses a [two-sided] neighbourhood of order three. But each of these neighbourhoods is monotone, by (i). By taking the union of these neighbourhoods one obtains the monotony of $\overline{\mathcal{A}}_4$.

<u>3.3.13</u> <u>Let a differentiable curve</u> $\mathcal{C}_4$ <u>of order four contain exactly one point with the characteristic</u> $(1, 2, 1)_0$. <u>Then</u> $\mathcal{C}_4$ <u>contains altogether an odd number of singular points.</u>

<u>Proof.</u> Let p be the point of $\mathcal{C}_4$ with the characteristic $(1, 2, 1)_0$. Suppose that $\mathcal{C}_4$ contains an even number of singular points altogether. We know that the number of singular points is finite, by <u>Theorem 2</u>.

Let the other singular points of $\mathcal{C}_4$ be $q_1 < q_2 < \ldots < q_{2n+1}$ where $n \geq 0$. Without loss of generality, let p lie between q_{2n+1} and q_1 on $\mathcal{C}_4$, if $n \geq 1$. Let

$$N_p = N'_p \cup \{p\} \cup N''_p \; [N_{q_j} = N'_{q_j} \cup \{q_j\} \cup N''_{q_j}]$$

be a small two-sided neighbourhood of p $[q_j]$ on $\mathcal{C}_4$, where N'_p $[N'_{q_j}]$ is a preceding neighbourhood of p $[q_j)$ and N''_p $[N''_{q_j}]$ is a proceeding neighbourhood of p $[q_j]$ on $\mathcal{C}_4$, $j = 1, 2, \ldots, 2n+1$.

Without loss of generality, let the natural orientation induced on the osculating circle $C''(p^3) = p$ of $N'' \cup \{p\}$ at p be such that $C''(p^3)_i = \emptyset$. By <u>3.3.12</u> (ii), the closed arc of $\mathcal{C}_4$ between p and q_1 is monotone. Hence the osculating circle $C'(p_1^3)$ of $N'_{q_1} \cup \{q_1\}$ at q_1 is such that

$$C'(q_1^3) \subset C''(p^3)_e \quad \text{and} \quad C''(p^3) \subset C'(q_1^3)_i.$$

Thus by 3.3.3, $\mathcal{C}_4 \setminus \{q_1\} \subset C'(q_1^3)_i$. Now q_1 has characteristic (1, 1, 2), $(1, 1, 2)_0$ or $(2, 1, 1)_0$. But then by 3.3.5 and 3.3.8, the osculating circle $C''(q_1^3)$ $(= C'(q_1^3))$ of $N''_{q_1} \cup \{q_1\}$ at q_1 has the same induced orientation as $C'(q_1^3)$. Hence

$$\mathcal{C}_4 \setminus \{q_1\} \subset C''(q_1^3)_i.$$

Next, the arc of $\mathcal{C}_4$ between q_1 and q_2 is monotone, by 3.3.12 (ii). Thus

$$C'(q_2^3) \subset C''(q_1^3)_i \quad \text{and} \quad C''(q_1^3) \subset C'(q_2^3)_e,$$

where $C'(q_2^3)$ is the osculating circle of $N'_{q_2} \cup \{q_2\}$ at q_2. Thus by 3.3.3,

$$\mathcal{C}_4 \setminus \{q_2\} \subset C'(q_2^3)_e,$$

and again by 3.3.5 and 3.3.8, the osculating circle $C''(q_2^3)$ $(= C'(q_2^3))$ of $N''_{q_2} \cup \{q_2\}$ at q_2 has the same induced orientation as $C'(q_2^3)$. Hence

$$\mathcal{C}_4 \setminus \{q_2\} \subset C''(q_2^3)_e.$$

Continuing in this manner we obtain for each j with j odd

$$\mathcal{C}_4 \setminus \{q_j\} \subset C''(q_j^3)_i$$

and for each j with j even

$$\mathcal{C}_4 \setminus \{q_j\} \subset C''(q_j^3)_e.$$

In particular,

$$p \in \mathcal{C}_4 \setminus \{q_{2n+1}\} \subset C''(q_{2n+1}^3)_i.$$

But again by 3.3.12 (ii), the arc of $\mathcal{C}_4$ between q_{2n+1} and p is monotone. Therefore

$$C'(p^3) \subset C''(q_{2n+1}^3)_i \quad \text{and} \quad C''(q_{2n+1}^3) \subset C'(p^3)_e.$$

However we assumed that $C''(p^3)_i = \emptyset$. By 3.3.6, $C'(p^3)_e = \emptyset$. Thus $q_{2n+1} \in C'(p^3)_i$ and finally

$$C''(q_{2n+1}^3) \subset C'(p^3)_i.$$

This is a contradiction and we have the required result.

Corollary. Let a differentiable curve $\mathcal{C}_4$ of order four contain exactly one point with characteristic $(1, 2, 1)_0$. Then $\mathcal{C}_4$ contains an odd number of singular points ≥ 3.

Proof. Let $\mathcal{A}_4 = \mathcal{C}_4 \setminus \{p\}$, where p is the point with characteristic $(1, 2, 1)_0$. Suppose that $\mathcal{A}_4$ contains no singular points. Then by 3.3.12 (ii), $\overline{\mathcal{A}_4} = \mathcal{B}_4$ is monotone. Now let

$$N_p = N' \cup \{p\} \cup N''$$

be a small two-sided neighbourhood of p on $\mathcal{B}_4$. Since $\mathcal{B}_4$ is monotone

$$C'(p^3) \subset C''(p^3)_e \quad \text{and} \quad C''(p^3) \subset C'(p^3)_i$$

or

$$C'(p^3) \subset C''(p^3)_i \quad \text{and} \quad C''(p^3) \subset C'(p^3)_e,$$

where $C'(p^3)$ $[C''(p^3)]$ is the osculating circle of $N' \cup \{p\}$ $[N'' \cup \{p\}]$ at p. But p is a differentiable point of $\mathcal{B}_4$. Hence

$$C'(p^3) = C''(p^3)$$

and thus neither of the two conditions stated above is satisfied. Thus our assumption that $\mathcal{A}_4$ contained no singular points is incorrect. Hence $\mathcal{A}_4$ contains at least one singular point and therefore $\mathcal{B}_4$ contains at least two singular points. By 3.3.13, $\mathcal{B}_4$ contains an odd number of singular points ≥ 3.

Remark: We notice in the proof of the above corollary that no use of the characteristic of p was made in obtaining at least two singular points on $\mathcal{B}_4$. We state this result here.

3.3.14 Let $\mathcal{C}_4$ be a differentiable curve of order four. Then $\mathcal{C}_4$ contains at least two singular points.

Proof. If $\mathcal{C}_4$ contains no singular points, then $\mathcal{C}_4$ is monotone by 3.3.12 (ii), and we obtain a contradiction as in the proof of 3.3.13 Corollary, by taking any point q on $\mathcal{C}_4$. Thus $\mathcal{C}_4$ contains at least one singular point, say p.

But then as in the proof of 3.3.13 Corollary, $\mathcal{Q}_4 = \mathcal{C}_4 \setminus \{p\}$ contains at least one singular point and we obtain the result.

3.3.15 By using methods which are similar to those employed in 3.3.13, we obtain:

Let $\mathcal{C}_4$ be a differentiable curve of order four containing no points with the characteristic $(1, 2, 1)_0$. Then $\mathcal{C}_4$ contains altogether an even number of singular points ≥ 2.

3.3.16 We are now ready to prove

Theorem 5: A differentiable curve $\mathcal{C}_4$ of order four contains at most four vertices.

Proof. Let us assume that $\mathcal{C}_4$ contains five vertices p_γ, $\gamma = 1, 2, \ldots, 5$.

Case (i). We allow here that the points p_γ have only the characteristics (1, 1, 2) or $(1, 1, 2)_0$. By 3.3.5, $\mathcal{C}_4$ induces a unique orientation on $C(p_\gamma^3)$ for each γ. We also note that by 3.3.15, $\mathcal{C}_4$ must contain at least six vertices in this case. However, we shall not need this latter result here.

Without loss of generality, there are at least three vertices, say q_j, j = 1, 2, 3, such that locally outside q_j

$$C(q_j^3) \subset \mathcal{C}_{4_i}$$

and

$$\mathcal{C}_4 \subset C(q_j)_e.$$

By 3.3.3,

$$C(q_j^3) \subset \mathcal{C}_{4_i} \cup \{q_j\}$$

and

$$\mathcal{C}_4 \subset C(q_j^3)_e \cup \{q_j\}.$$

Now the points q_j divide $\mathcal{C}_4$ into three closed arcs. By Theorem 4, there exists a circle K lying in $\mathcal{C}_4 \cup \mathcal{C}_{4_i}$, having points in common with all three arcs. By 3.3.2, one of the q_j, say p, is a point of contact of K with $\mathcal{C}_4$. Since nontangent circles intersect $\mathcal{C}_4$ at p, K is a tangent circle of $\mathcal{C}_4$ at p. (see Figure 10).

Next, K and $C(p^3)$ belong to the pencil of tangent circles τ of $\mathscr{C}_4$ at p where τ is a pencil of the second kind with fundamental point p. Now

$$\mathscr{C}_4 \subset C(p^3)_e \cup \{p\}$$

and K has another point of contact on $\mathscr{C}_4$ outside p. Thus

$$K \subset C(p^3)_e \cup \{p\} .$$

Also $\mathscr{C}_4$ induces a continuous orientation on τ . Hence

$$C(p^3) \subset K_i \cup \{p\} .$$

As in 1.1.1,

$$K \subset C(p^3)_e \cup \{p\}$$

and

$$C(p^3) \subset K_i \cup \{p\}$$

imply

$$C(p^3)_i \subset K_i .$$

On the other hand, let s be close to p on $\mathcal{C}_4$. Then

$$C(p^2, s) \subset K_e \cup \{p\}.$$

Thus by the continuous orientation on τ,

$$K \subset C(p^2, s)_i \cup \{p\}$$

and

$$C(p^2, s) \subset K_e \cup \{p\}$$

imply

$$K_i \subset C(p^2, s)_i.$$

Now let $s \longrightarrow p$ on $\mathcal{C}_4$. Thus

$$K_i \subset \lim_{s \to p} C(p^2, s)_i = C(p^3)_i.$$

This is a contradiction.

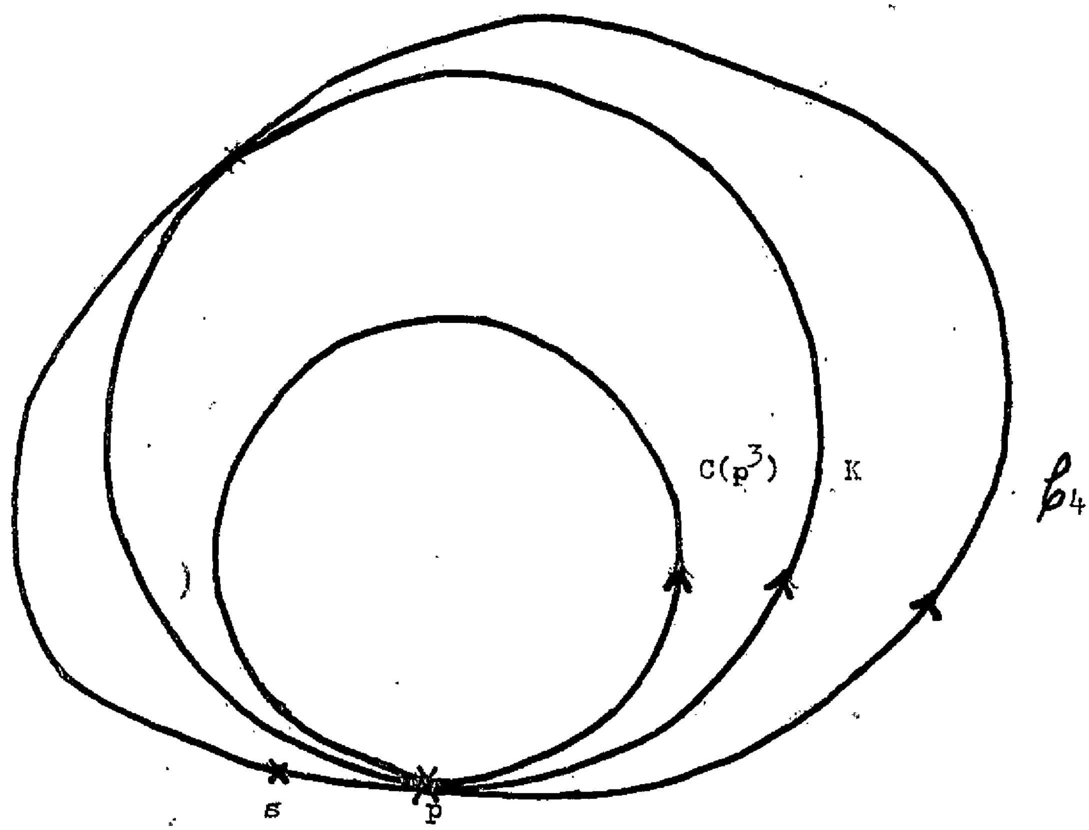


Figure 10

Case (ii) We allow as vertices only points with the characteristics $(1, 1, 2)$, $(1, 1, 2)_0$ or $(1, 2, 1)_0$. We assume that at least one of the vertices has the characteristic $(1, 2, 1)_0$; otherwise we are back to Case (i). We label these points type α, β and δ, respectively.

We first notice that by 3.3.10, at most two of the p_γ can be of type δ. Hence at least three of the five vertices p_γ are of type α or β. But then without loss of generality, there are two of these three, say q_j, $j = 1, 2$, such that

$$C(q_j^3) \subset \mathcal{C}_{4_i} \cup \{q_j\}$$

and

$$\mathcal{C}_4 \subset C(q_j^3)_e \cup \{q_j\}.$$

Take a point q of type δ. By Theorem 4, there exists a circle $\overline{K}$ lying entirely in $\mathcal{C}_4 \cup \mathcal{C}_{4_i}$, having points in common with the three arcs determined by q, q_1, q_2. But one of these points is a point of contact of $\overline{K}$ with $\mathcal{C}_4$, by 3.3.2. Also the point q cannot be a point of contact as was shown in the proof of 3.3.10. Hence one of the q_j is a point of contact of $\overline{K}$ with $\mathcal{C}_4$ and we proceed as in Case (i) to obtain a contradiction.

Case (iii) We allow finally p_γ to have any of the following characteristics

$$(1, 1, 2),\ (1, 1, 2)_0,\ (1, 2, 1)_0 \text{ or } (2, 1, 1)_0.$$

By 3.3.9, at most one of the vertices is of type $(2, 1, 1)_0$ and if $\mathscr{C}_4$ contains a vertex of this type, then at most one of the vertices has the characteristic $(1, 2, 1)_0$, by 3.3.11.

Let us assume, for the moment, that $\mathscr{C}_4$ contains no points with the characteristic $(1, 2, 1)_0$. Then all of the p_γ are of type $(1, 1, 2)$, $(1, 1, 2)_0$ or $(2, 1, 1)_0$. Let us also assume that $\mathscr{C}_4$ contains at least one and then exactly one point of type $(2, 1, 1)_0$; otherwise we have Case (i). By 3.3.15, $\mathscr{C}_4$ contains an even number of vertices. Hence $\mathscr{C}_4$ contains at least six vertices. But exactly one has the characteristic $(2, 1, 1)_0$. Hence the other five have the characteristic $(1, 1, 2)$ or $(1, 1, 2)_0$. But then we proceed as in Case (i) to reach a contradiction.

Finally we assume that $\mathscr{C}_4$ contains exactly one point p with the characteristic $(2, 1, 1)_0$ and exactly one point q with the characteristic $(1, 2, 1)_0$. Hence there are at least three vertices of type $(1, 1, 2)$ or $(1, 1, 2)_0$. But then, without loss of generality, at least two of the vertices of these types, say q_j, $j = 1, 2$, are such that

$$C(q_j^3) \subset \mathscr{C}_{4_i} \cup \{q_j\}$$

and

$$\mathscr{C}_4 \subset C(q_j^3)_e \cup \{q_j\} .$$

But again the points q_1, q_2 and q determine three closed arcs of $\mathscr{C}_4$ and we proceed as in Case (ii) to reach a contradiction.

Since the only possible vertices of $\mathscr{C}_4$ have the characteristics

$$(1, 1, 2),\ (1, 1, 2)_0,\ (1, 2, 1)_0 \text{ or } (2, 1, 1)_0,$$

the theorem is proved.

3.4 Strongly Differentiable Curves of Order Four

Introduction.

In 3.3 we showed that a differentiable curve $\mathcal{C}_4$ of order four contains at most four vertices. Under the restriction that $\mathcal{C}_4$ is strongly differentiable, here we use Theorem 4 of 3.3 to show that $\mathcal{C}_4$ contains at least four vertices. This is the well known four-vertex theorem for curves of order four; cf. Theorem 6.

Then these two theorems combine to give a conformal proof of the main result that a strongly differentiable curve $\mathcal{C}_4$ of order four contains exactly four vertices; cf. Theorem 7.

First we need the following result.

3.4.1 Let $\mathcal{Q}_4 = \bar{\mathcal{Q}}_4$ be a strongly differentiable closed arc of order four with end-points p, q and K an oriented circle satisfying

(a) K belongs to the oriented pencils of tangent circles of $\mathcal{Q}_4$ at p and q,

(b) $\mathcal{Q}_4 \cap K = \{p, q\}$.

Then there exists at least one singular point in the interior of $\mathcal{Q}_4$.

Proof. Without loss of generality, assume that $\mathcal{Q}_4 \subset K \cup K_e$. Suppose that each interior point of $\mathcal{Q}_4$ is ordinary. Then by 3.3.12 (ii), $\mathcal{Q}_4$ is monotone.

But K is the tangent circle $C(p^2, q)$ of $\mathcal{Q}_4$ at p through q. Let $s \in \mathcal{Q}_4$, $s \neq p, q$. Thus $s \in K_e$. By 2.4.2 (iv),

$$C(p^2, s) \subset K_e \cup \{p\} \quad \text{and} \quad K \subset C(p^2, s)_i \cup \{p\}.$$

This statement is true for all $s \in \mathcal{Q}_4$, $s \neq p, q$. Let $s \longrightarrow p$ on $\mathcal{Q}_4$. By 3.2.8, $C(p^3) \neq K$. Hence

$$C(p^3) \subset K_e \cup \{p\} \quad \text{and} \quad K \subset C(p^3)_i \cup \{p\}.$$

However $q \in K$. Thus $q \in C(p^3)_i$ and the monotony of $\mathcal{Q}_4$ yields

$$C(q^3) \subset C(p^3)_i \quad \text{and} \quad C(p^3) \subset C(q^3)_e.$$

Similarly K is the tangent circle $C(q^2, p)$ of $\mathcal{Q}_4$ at q through p. Let $s' \varepsilon \mathcal{Q}_4$, $s' \neq p, q$. Thus $s' \varepsilon K_e$. By 2.4.2 (iv),

$$C(q^2, s') \subset K_e \cup \{q\} \quad \text{and} \quad K \subset C(q^2, s')_i \cup \{q\}.$$

But this statement is true for all $s' \varepsilon \mathcal{Q}_4$, $s' \neq p, q$. Let $s' \longrightarrow q$ on $\mathcal{Q}_4$. By 3.2.8, $C(q^3) \neq K$. Hence

$$C(q^3) \subset K_e \cup \{q\} \quad \text{and} \quad K \subset C(q^3)_i \cup \{q\}.$$

But $p \varepsilon K$. Thus $p \varepsilon C(q^3)_i$ and the monotony of $\mathcal{Q}_4$ yields

$$C(p^3) \subset C(q^3)_i \quad \text{and} \quad C(q^3) \subset C(p^3)_e.$$

This is a contradiction.

Hence the assumption that each interior point of $\mathcal{Q}_4$ is ordinary was incorrect. Thus there exists at least one singular point in the interior of $\mathcal{Q}_4$.

3.4.2

Theorem 6: A strongly differentiable curve $\mathcal{C}_4$ of order four contains at least four vertices.

Proof. Let $\mathcal{C}_4$ be separated into any three closed arcs. By Theorem 4, there exists a circle $\underline{K}$ lying in $\mathcal{C}_4 \cup \mathcal{C}_{4i}$, having points in common with all three arcs. By 3.3.2, $\underline{K}$ meets $\mathcal{C}_4$ at exactly two points q_1, q_2, say. Now $\mathcal{C}_4$ is strongly differentiable. Thus $\underline{K}$ is a tangent circle of $\mathcal{C}_4$ at q_1 and q_2.

By 3.4.1, there exists at least one singular point in the open arc $q_1 q_2$ of $\mathcal{C}_4$ and at least one singular point in the open arc $q_2 q_1$ of $\mathcal{C}_4$. In 3.3 we remarked that each singular point is actually a vertex when we consider differentiable curves $\mathcal{C}_4$ of order four. Thus q_1 and q_2 are vertices of $\mathcal{C}_4$.

Now suppose that the open arcs $q_1 q_2$ and $q_2 q_1$ of $\mathcal{C}_4$ contain no singular points. By 3.3.12 (ii), the closed arcs $\overline{q_1 q_2}$ and $\overline{q_2 q_1}$ are monotone. Since q_1 and q_2 are vertices, the osculating circles $C(q_1^3)$, $C(q_2^3)$ of $\mathcal{C}_4$ at q_1, q_2, respectively, support $\mathcal{C}_4$ and do not meet $\mathcal{C}_4$ again, by 3.3.3. If

$$\mathcal{C}_4 \subset C(q_1^3)_i \cup \{q_1\} \quad \text{and} \quad \mathcal{C}_4 \subset C(q_2^3)_i \cup \{q_2\}$$

or

$$\mathcal{C}_4 \subset C(q_1^3)_e \cup \{q_1\} \quad \text{and} \quad \mathcal{C}_4 \subset C(q_2^3)_e \cup \{q_2\} ,$$

then neither of the conditions of 3.3.12 for the monotony of the arcs $\overline{q_1 q_2}$ and $\overline{q_2 q_1}$ can be satisfied. Hence either

$$\mathscr{C}_4 \subset C(q_1{}^3)_i \cup \{q_1\} \quad \text{and} \quad \mathscr{C}_4 \subset C(q_2{}^3)_e \cup \{q_2\}$$

or

$$\mathscr{C}_4 \subset C(q_1{}^3)_e \cup \{q_1\} \quad \text{and} \quad \mathscr{C}_4 \subset C(q_2{}^3)_i \cup \{q_2\} .$$

Without loss of generality, we shall assume that the first situation occurs.

Now ℓ_2, q_2, ℓ_1 divide $\mathscr{C}_4$ into three closed arcs. By the remark following Theorem 4, there exists a circle $\overline{K}$ lying in $\mathscr{C}_4 \cup \mathscr{C}_{4e}$, having points in common with all three arcs. Again by 3.3.2, one of the points ℓ_2, q_2, ℓ_1 is a point of contact of $\overline{K}$ with $\mathscr{C}_4$.

If q_2 is a point of contact of $\overline{K}$ with $\mathscr{C}_4$, then the other point of contact m lies in $\overline{\ell_1 \ell_2}$ on $\mathscr{C}_4$. But then by 3.4.1, there exist singular points r_1, r_2 in the open arcs q_2m and mq_2 of $\mathscr{C}_4$, respectively. At least one of r_1, r_2 is distinct from q_1. This contradicts our assumption that the open arcs q_1q_2 and q_2q_1 contain no singular points. Hence ℓ_1 or ℓ_2 is a point of contact of K with $\mathscr{C}_4$.

Next we note that not both ℓ_1 and ℓ_2 are points of contact of $\overline{K}$ with $\mathscr{C}_4$. Otherwise, K and $\overline{K}$ support each other at ℓ_1 and ℓ_2 and thus are identical. This is impossible by our choice of K and $\overline{K}$.

If ℓ_1, say, is a point of contact of $\overline{K}$ with $\mathscr{C}_4$, then the other point of contact m lies strictly between ℓ_2 and q_2 on $\mathscr{C}_4$ (see Figure 11). But we assumed that

$$\mathscr{C}_4 \subset C(q_2{}^3)_e \cup \{q_2\}.$$

Now the open arc $q_2 \ell_1$ is a subarc of the monotone arc q_2q_1 and hence is itself monotone. Thus $q_2 \ell_1 \subset C(\ell_1{}^3)_i$. $C(\ell_1{}^3) \not\ni \underline{K}, \overline{K}$, by 3.2.8. Also $C(\ell_1{}^3) \not\ni \ell_1$. Otherwise, ℓ_1 would be a singular point lying between q_2 and q_1; contradiction. But as in the proof of 3.4.1,

$$C(\ell_1{}^3) \subset (\underline{K}_e \quad \overline{K}_i) \cup \{\ell_1\}.$$

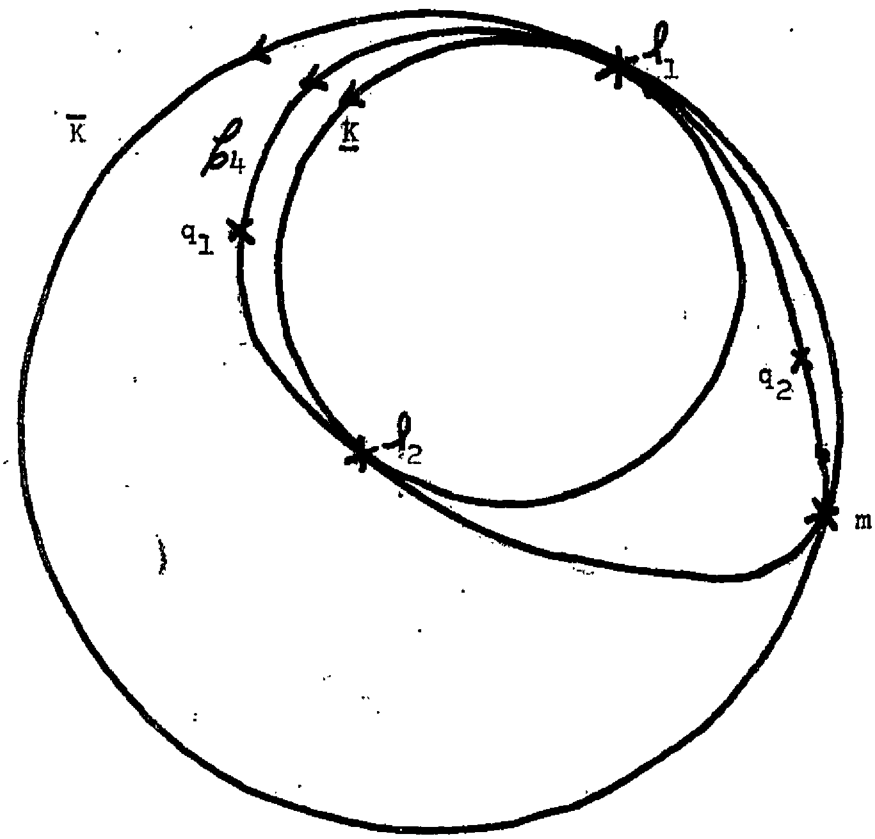


Figure 11

Hence $C(\ell_1^3)$ intersects the open arc $\ell_2 m$ at least once and intersects the open arc mq_2 at least once. This is impossible by 3.2.4 (iii).

If ℓ_2 is a point of contact of $\bar{K}$ with $\mathcal{C}_4$, then m lies strictly between q_2 and ℓ_1 on $\mathcal{C}_4$. A similar argument shows that $C(\ell_2^3)$ will intersect the open subarcs $q_2 m$ and $m\ell_1$ of $\mathcal{C}_4$ at least once. Again this is impossible by 3.2.4 (iii).

Thus our assumption that the open arcs $q_1 q_2$ and $q_2 q_1$ of $\mathcal{C}_4$ contain no singular points is incorrect. Thus there exists at least one singular point in the interior of $q_1 q_2$ or $q_2 q_1$ on $\mathcal{C}_4$. Hence $\mathcal{C}_4$ contains at least three singular points. By 3.3.7 and 3.3.15, $\mathcal{C}_4$ contains at least four singular points. But again by 3.3, each of these singular points is actually a vertex and we have the desired result.

3.4.3 We combine Theorem 5 and Theorem 6 to obtain

Theorem 7: *A strongly differentiable curve $\mathcal{C}_4$ of order four contains exactly four vertices.*

Section 4

A Topology on the Set of Conics in the Projective Plane

4.1. Let $\Gamma = \{\gamma\}$, where γ denotes a non-degenerate conic in the real projective plane π_p. Let $\overline{\Gamma}$ be the union of Γ and the following so called degenerate conics: a pair of lines, a double line (a line counted twice), a double line segment (a line segment counted twice) or a point.

<u>4.2.</u> A non-degenerate conic has a well defined <u>interior</u> γ_i and an <u>exterior</u> γ_e. Two distinct lines define a conic γ, which decomposes the projective plane into two homeomorphic disjoint regions which we may denote by γ_i and γ_e. Two points are said to be separated by a non-degenerate conic or a pair of distinct lines γ if and only if one of the points lies in γ_i and the other in γ_e. For a line or a line segment γ only one of γ_i, γ_e is non-void.

4.3. Five distinct points, no three of which are collinear, determine a unique non-degenerate conic. If three of the five points lie on a line $\mathcal{L}$, which does not pass through the remaining two, then is a unique pair of lines through them, viz., the line $\mathcal{L}$ and the line joining the other two points. If exactly four of the five points lie on a line $\mathcal{L}$, which does not contain the fifth point, there are infinitely many conics through these five points, viz., $\mathcal{L}$ and any other line through the fifth point. If all the five points lie on a line $\mathcal{L}$, there are infinitely many conics through them, viz. $\mathcal{L}$ together with any other line, the double line coincident with $\mathcal{L}$, and any double line segment on $\mathcal{L}$ containing the five points.

4.4. It is possible to introduce a topology $\mathcal{D}'$ on Γ as was done in Section 1 on the set of circles in the inversive plane.

A neighbourhood of a non-degenerate conic γ is the set of conics which lie in the region outside a non-degenerate conic

$$\varepsilon \subset \gamma_i$$

and inside a non-degenerate conic.

$$n \subset \gamma_e.$$

A neighbourhood of a pair of distinct lines γ is the set of conics which separate (and thus lie in the common exterior of) two non-degenerate conics which are separated by γ.

A neighbourhood of a double line γ is the set of conics which separate γ from a non-degenerate conic which does not meet γ.

A neighbourhood of a double segment [a point conic] γ is the set of conics which lie in the interior of a non-degenerate conic which contains γ in its interior.

4.5. A sequence of conics $(\gamma_n)_{n\in N}$ converges to a conic γ if for any neighbourhood $\mathcal{U}$ of γ there exists $n_0 \in N$ such that $\gamma_n \in \mathcal{U}$ for all $n > n_0$. We denote this convergence of γ_n to γ by

$$\lim_{n\in N} \gamma_n = \gamma .$$

It is well-known that $(\bar{\Gamma}, \mathcal{B}')$ is countably compact ([9], 2.1). In addition the following results are analogous to those in 1.5, 1.6 and 1.7.

4.5.1 $(\bar{\Gamma}, \mathcal{B}')$ _satisfies the first and second axioms of countability_.

4.5.2 $(\bar{\Gamma}, \mathcal{B}')$ _is a Hausdorff space_.

4.5.3 $(\bar{\Gamma}, \mathcal{B}')$ _is a regular space_.

<u>4.6</u> Combining the results of <u>4.5</u>, we obtain ([1], p. 138):

<u>Theorem 8</u>: $(\overline{\Gamma}, \mathfrak{B}')$ <u>is a compact Hausdorff space</u>.

Section 5

The Order, Differentiability and Characteristic of a Point of an Arc in the Projective Plane

Introduction

This section is comprised of basic background material which is fundamental in the analysis of arcs and curves of conical order six. The results are generally due to the research of N.D. Lane, K.D. Singh and P. Scherk and can be found in [3], [9], [10] and [11].

In 5.3.1 the concept of a conically differentiable point is introduced; while in 5.3.3, strongly conically differentiable points of an arc are studied. A characteristic is associated with each conically differentiable interior point of an arc in 5.4. One of the first men to study the concept of characteristic was P. Scherk [15] and his basic ideas were used by N.D. Lane and K.D. Singh in [4] to introduce the conical characteristic of a conically differentiable point. Using the characteristic one can list the different kinds of conically differentiable interior points of an arc; cf. 5.4.1.

In 5.5 the conical order of a point on an arc is defined. The geometric notion of order seems to have been first studied extensively by S. Mukhopadhyaya in [19] and [20]. A more general notion of order was introduced by O. Haupt and H. Künneth in [12], who used many of the ideas of Mukhopadhyaya. With the concept of the order of a point on an arc, one can define ordinary and singular points; cf. 5.6. Some of the earlier work involving singular points was done by Mukhopadhyaya and W. Blaschke [21] in the consideration of sextactic points.

<u>5.1</u> The three-parameter family of non-degenerate conics which touch a line $\mathfrak{J}$ at a point p is denoted by τ. If no three of the points P, Q, R, S are collinear and Q, R, S do not lie on $\mathfrak{J}$, the conic of t through $Q, R,$ and S is denoted by $\gamma(\tau; Q, R, S)$.

<u>5.1.1</u> If $\gamma \varepsilon \tau$, $\varphi = \varphi(\gamma)$ denotes the two-parameter subfamily of τ which consists of those conics of τ which have at least three-point contact with γ at p; φ_R is the subfamily of φ through $R \notin \mathfrak{J}$. Let $\varphi_p(\gamma)$ denote the subfamily of $\varphi(\gamma)$, each of whose members have at least four-point contact with γ at p.

5.2 The concepts of arc, curve and neighbourhood of a point can be defined in the projective plane as they were defined in 2.3 for the inversive plane.

5.2.1 Suppose s is an interior point of an arc $\mathcal{Q}$. Then we call s a point of support [intersection] with respect to a conic $\gamma \epsilon \overline{\Gamma}$ if a sufficiently small neighbourhood of s on $\mathcal{Q}$ is decomposed by s into two one-sided neighbourhoods which lie in the same region [in different regions] bounded by γ. γ is then called a supporting [intersecting] conic of $\mathcal{Q}$ at s.

5.3 Differentiable and Strongly Differentiable Points.

5.3.1 A point p on $\mathcal{A}$ is said to be conically differentiable if it satisfies four conditions:

Condition PI. If the parameter s is sufficiently close to p, $s \neq p$, the line ps is uniquely determined. It converges as s tends to p ([9], 4.2).

The limit straight line $\mathcal{T}$ is the ordinary tangent of $\mathcal{A}$ at p. Condition PI implies:

(i) If $\mathcal{A}$ satisfies Condition PI at an interior point p, then the non-degenerate, non-tangent conics through p all intersect $\mathcal{A}$ at p or all of them support ([9], 4.11).

Condition PII. Let $\mathcal{A}$ satisfy PI at p and let Q and R be any fixed points, $Q \notin \mathcal{T}$, $R \notin \mathcal{T}$; p, Q, R not collinear. If s is close to p, $s \neq p$, the unique tangent conic $\gamma(p^2, s, Q, R)$ of $\mathcal{A}$ at p through Q, R and s converges as s tends to p ([9], 5.1).

The limiting osculating conic of $\mathcal{A}$ at p through Q and R is denoted by $\gamma(p^3, Q, R)$. The family of all the osculating conics of $\mathcal{A}$ at p is denoted by σ.

(ii) If Condition PII holds for two points Q, R such that p, Q, R are not collinear and $Q \notin \mathcal{T}$, $R \notin \mathcal{T}$, then it holds for every such pair of points ([9], 5.4).

(iii) Let PII hold at p. If the arc $\mathcal{Q}$ intersects $\mathcal{T}$ at p, then the conics of σ are degenerate ([9], 5.5).

(iv) If PII holds at p, then σ is one of the following families ([9], 5.6).

Type 1. σ is a subfamily Q of τ which consists of all the conics of τ which have at least three-point contact at p with any particular member of σ; cf. 5.1.1.

Type 2. σ consists of the pairs of distinct lines through p, both of them different from $\mathcal{T}$.

Type 3. σ consists of the pairs of lines one of which is $\mathcal{T}$ while the other does not pass through p.

(v) If $\mathcal{Q}$ satisfies PII at an interior point p, then the conics of $\tau - \sigma$ all support $\mathcal{Q}$ at p, except when p is of Type 2 and $\mathcal{Q}$ intersect $\mathcal{T}$ at p, in which case they all intersect $\mathcal{Q}$ at p ([9], 5.10).

Condition PIII. $\mathcal{Q}$ satisfies PII at p and if $Q \notin \mathcal{T}$, then $\gamma(p^3, s, Q)$ converges as s tends to p on $\mathcal{Q}$.

The limiting superosculating conic of $\mathcal{Q}$ at p through Q is denoted by $\gamma(p^4, Q)$. The family of all the superosculating conics of $\mathcal{Q}$ at p is denoted by ρ.

(vi) If Condition PIII holds for a single point $Q \notin \mathcal{T}$, then it holds for all such points ([9], 6.3).

(vii) If PIII holds at p, then ρ is one of the following ([9], 6.4).

Type 1(a). ρ is a subfamily φ_p of σ which consists of those conics of σ which have four-point contact at p with a particular conic of σ .

Type 1(b). ρ consists of all pairs of lines through p, one of which is $\mathcal{J}$.

In Types 2 and 3, PIII is satisfied automatically.

(viii) If $\mathcal{Q}$ satisfies PIII at an interior point p, then the conics of $\sigma - \rho$ all support $\mathcal{Q}$ at p or all of them intersect.

Condition PIV. $\mathcal{Q}$ satisfies PIII at p and the superosculating conic $\gamma(p^4, s)$ converges as s tends to p on $\mathcal{Q}$.

The limiting ultraosculating conic of $\mathcal{Q}$ at p is denoted by $\gamma(p^5)$.

(ix) If $\mathcal{Q}$ satisfies PIV at p, then $\gamma(p^5)$ is non-degenerate (Type 1a(i)), or the point conic p(Type 1a(ii)), or the double line on $\mathcal{J}$ (Type 1a(iii)).

In the remaining cases, Types 1b, 2 and 3, PIV is satisfied automatically and $\gamma(p^5)$ is the double line on $\mathcal{J}$ ([9], 7) .

(x) If $\mathcal{Q}$ satisfies PIV at an interior point p, then the superosculating conics $\neq \gamma(p^5)$ all support $\mathcal{Q}$ or all intersect ([9], 7.2).

Remarks (1) We shall adopt the convention that the double line on $\mathcal{J}$ supports $\mathcal{Q}$ at p, even if $\mathcal{Q}$ crosses $\mathcal{J}$ at p.

(2) If the conic γ consists of a pair of distinct lines through p, the arc $\mathcal{a}$ will be said to support [intersect] γ at p, if there exist one-sided neighbourhoods of p on $\mathcal{a}$ which lie in the same region [in different regions] with respect to γ.

5.3.2 Suppose that no three of P, Q, R, S are collinear. We call γ a general tangent conic of $\mathcal{a}$ at p if there exists a sequence of quintuples of mutually distinct points s_n, t_n, Q_n R_n, S_n such that s_n and T_n converge on $\mathcal{a}$ to p, $Q_n \longrightarrow Q$, $R_n \longrightarrow R$, $S_n \longrightarrow S$ and the conic $\gamma(s_n, t_n, Q_n, R_n, S_n)$ through these points converges to γ.

Remark: If $\mathcal{a}$ satisfies PI at p, every tangent conic of $\mathcal{a}$ is a general tangent conic. The converse need not be true. For example, a cusp point satisfying PI has general tangent conics other than the ordinary tangent conics ([10], 2.1).

Suppose that p, Q, R are not collinear. We call γ a general osculating conic of $\mathcal{a}$ at p if there exists a sequence of quintuples of mutually distinct points s_n, t_n, u_n, Q_n, R_n such that s_n, t_n, u_n converge on $\mathcal{a}$ to p, $Q_n \longrightarrow Q$, $R_n \longrightarrow R$ and the conic $\gamma(s_n, t_n, u_n, Q_n, R_n)$ converges to γ.

As in the remark above if $\mathcal{a}$ satisfies PII every osculating conic of $\mathcal{a}$ is a general osculating conic, but the converse is not necessarily true.

We call γ a <u>general superosculating conic</u> of $\mathcal{A}$ at p if there exists a sequence of sets of mutually distinct points s_n, t_n, u_n, v_n, Q_n such that s_n, t_n, u_n, v_n converge to p on $\mathcal{A}$, $Q_n \longrightarrow Q$, $Q \neq p$, and a conic $\gamma(s_n, t_n, u_n, v_n, Q_n)$ through these points converges to γ.

Finally we call γ a <u>general ultraosculating conic</u> of $\mathcal{A}$ at p if there exists a sequence of sets of mutually distince points s_n, t_n, u_n, v_n, w_n such that s_n, t_n, u_n, v_n, w_n converge to p on $\mathcal{A}$ and a conic $\gamma(s_n, t_n, u_n, v_n, w_n)$ through these points converges to γ.

We shall need the following results later ([10], 2).

(i) If γ is a non-degenerate general osculating conic of $\mathcal{A}$ at p, then every member of the family $\varphi_p(\gamma)$ is also a general osculating conic of $\mathcal{A}$ at p; cf. 5.1.1.

(ii) If γ is a non-degenerate general superosculating conic of $\mathcal{A}$ at p, then every member of the family $\varphi_p(\gamma)$ is also a general superosculating conic of $\mathcal{A}$ at p; cf. 5.1.1.

5.3.3 Strongly Differentiable Points.

A point p on $\mathcal{A}$ is said to be <u>strongly conically differentiable</u> if it satisfies four conditions:

<u>Condition PI'</u>. If the parameters s and t are sufficiently close to the parameter p, $s \neq t$, the straight line determined by s and t converges as s and t tend to p.

In particular, if we take $t = p$, we see that PI' implies PI.

<u>Condition PII'</u>. A satisfies Condition PI' at p and there exist two distinct points Q and R, not collinear with p, which do not lie on a general tangent of $\mathcal{A}$ at p with the following properties. If s, t, u are mutually distinct and lie sufficiently close to p on $\mathcal{A}$, the conic $\gamma(s, t, u, Q, R)$ is uniquely defined. It converges as s, t, u converge to p.

We note that if Condition PII' holds for two points Q and R (thus, p, Q, R are not collinear; $Q \notin \mathcal{T}$, $R \notin \mathcal{T}$), then it holds for every such pair of points ([10], <u>3.3</u>).

<u>Condition PIII'</u>. $\mathcal{A}$ satisfies PII' at p, and there exists a point $R \notin \mathcal{T}$, with the following properties. If s, t, u, v are mutually distinct and lie sufficiently close to p, the conic $\gamma(s, t, u, v, R)$ is uniquely defined. It converges as s, t, u, v converge to p.

If Condition PIII' is satisfied at a point p of Type 1 or 3 for one point R (thus $R \notin \mathcal{T}$), then it is satisfied for every point not on $\mathcal{T}$. We note that if $\mathcal{A}$ satisfies PII at an interior point p of type 2, then $\mathcal{A}$ does not satisfy Condition PIII' at p ([10], <u>3.4</u>).

<u>Condition PIV'</u>. $\mathcal{A}$ satisfies PIII' at p $\gamma(s, t, u, v, w)$ is uniquely defined and converges if the mutually distinct points s, t, u, v, w converge on $\mathcal{A}$ to p.

5.4. A Classification of Conically Differentiable Points.

We associate with each conically differentiable interior point p of an arc $\mathcal{Q}$ a characteristic (similar to that introduced for conformally differentiable points) $(a_0, a_1, a_2, a_3, a_4; k)$, k = 1a(i), 1a(ii), 1a(iii), 1b, 2 or 3. The numbers a_i are equal to 1 or 2; i = 0, 1, 2, 3, 4. They are determined as follows:

(i) a_0 is even or odd according as the nontangent conics through p all support $\mathcal{Q}$ at p all or intersect; cf. 5.3.1(i).

(ii) $a_0 + a_1$ is even or odd according as the nonosculating tangent conics of $\mathcal{Q}$ at p all support $\mathcal{Q}$ at p or all intersect cf. 5.3.1 (v).

(iii) $a_0 + a_1 + a_2$ is even or odd according as the non-superosculating osculating conics of $\mathcal{Q}$ at p support or intersect; cf. 5.3.1 (viii).

(iv) $a_0 + a_1 + a_2 + a_3$ is even or odd according as the nonultra-osculating superosculating conics of $\mathcal{Q}$ at p all support $\mathcal{Q}$ at p or all intersect; cf. 5.3.1(x).

(v) $a_0 + a_1 + a_2 + a_3 + a_4$ is even or odd according as the ultraosculating conic $\gamma(p^5)$ of $\mathcal{Q}$ at p supports or intersects $\mathcal{Q}$ at p.

Remark. It may turn out, for example, that the nonsuper-osculating osculating conics of $\mathcal{Q}$ at p say, do not support or intersect. We will exclude these types of conically differentiable points.

<u>5.4.1.</u> We list the types of conically differentiable points p of an arc $\mathcal{A}$ here. Examples of these types of points can be found in [11].

<u>Points having no cusp</u>

Points having no cusp		Cusps	
(1,1,1,1,1; 1a(i))	$\gamma(p^5)$ intersects $\mathcal{A}$ at p	(2,2,2,2,1; 1a(i))	cusps of the second kind
(1,1,1,1,2; 1a(i))	a small neighbourhood of p on $\mathcal{A}\setminus\{p\}$ lies in $\gamma(p^5)_i$ or $\gamma(p^5)_e$	(2,2,2,2,2; 1a(i))	cusps of the second kind
(1,1,1,1,2; 1a(ii))	$\gamma(p^5) = p$	(2,2,2,2,2; 1a(ii))	cusps of the second kind
(1,1,1,1,2; 1a(iii))	$\gamma(p)$ = double line on $\mathcal{T}$	(2,2,2,2,2; 1a(iii))	cusps of the second kind
(1,1,1,2,1; 1b)	conics of σ intersect $\mathcal{A}$ at p	(2,2,1,1,2; 1b)	cusps of the second kind
(1,1,2,1,1; 1b)	a small neighbourhood of p on $\mathcal{A}\setminus\{p\}$ lies in the interior or exterior of a conic of σ	(2,2,2,2,2; 1b)	cusps of the second kind
(1,1,2,1,1; 2) (1,1,2,1,1; 3)	$\mathcal{A}$ does not cross $\mathcal{T}$ at p	(2,2,2,2,2; 2) (2,2,2,2,2; 3)	cusps of the second kind
(1,2,1,2,2; 2) (1,1,1,1,2; 3)	$\mathcal{A}$ crosses $\mathcal{T}$ at p	(2,1,1,1,1; 2) (2,2,1,2,1; 3)	cusps of the first kind

5.5. Linear and Conical Order of a Point.

Analogously to 2.6, we introduce the concepts of linear and conical order of an arc $\mathcal{Q}$. An arc $\mathcal{Q}$ is said to be of finite conical order [finite linear order] if it has only a finite number of points in common with any conic [line]. If the least upper bound of these numbers is finite, then it is called the conical order [linear order] of $\mathcal{Q}$. The conical order [linear order] of a point p of $\mathcal{Q}$ is then the minimum of the conical [linear] orders of all neighbourhoods of p on $\mathcal{Q}$. In the case of conical order, the order of a point is ≥ 5. In the case of linear order, the order of a point is ≥ 2.

We note the following results:

(i) An end-point p of an arc $\mathcal{Q}$ of finite conical order satisfies PII. If p is of Type 1, then $\mathcal{Q}$ satisfies PIII, and if p is of Type 1(a), then $\mathcal{Q}$ satisfies PIV ([10], 4.1).

(ii) Let p be a conically differentiable interior point of an arc $\mathcal{Q}$. Suppose that p has the characteristic $(a_0, a_1, a_2, a_3, a_4; R)$. Then the conical order of p is not less than $a_0 + a_1 + a_2 + a_3 + a_4$ ([11], Theorem 1).

(iii) A conically elementary point of an arc $\mathcal{Q}$ is a point which decomposes a neighbourhood of p on $\mathcal{Q}$ into two one-sided neighbourhoods of conical order five.

Let p be a conically elementary point of a differentiable arc $\mathcal{A}$. If p has the characteristic $(a_0, a_1, a_2, a_3, a_4; k)$, then the conical order of p is $a_0 + a_1 + a_2 + a_3 + a_4$ ([11], <u>Theorem</u> 2).

5.6. Ordinary and Singular Points.

A point p of an arc $\mathcal{Q}$ is called conically ordinary [linearly ordinary] if the conical order [linear order] of p is five [two].

If the conical order [linear order] of a point p on $\mathcal{Q}$ is strictly greater than five [two], p is said to be a conically singular [linearly singular] point.

A point p of a conically differentiable arc is said to be a vertex if p is a point of support with respect to $\gamma(p^5)$, the ultraosculating conic of $\mathcal{Q}$ at p.

Section 6

Arcs and Curves of Conical Order Six in the Projective Plane

Introduction

This section parallels the analysis of arcs and curves of circular order four done in Section 3. Here we shall investigate some of the properties of arcs and curves of conical order six in the projective plane.

We first consider general arcs of conical order six in 6.1 and 6.2. In 6.3 we obtain important monotony results for conically differentiable convex arcs of conical order six. Conically differentiable curves of order six are analysed in 6.4; while our attention is restricted to strongly conically differentiable curves of order six in 6.5.

6.1 Convex Arcs of Conical Order Six

Introduction

It is well known that an arc $\mathcal{A}_6$ of conical order six is the union of a finite number of arcs of conical order five; cf. O Haupt and H. Künneth ([12], 4.1.3) and Fr. Fabricius-Bjerre [23]. The latter's methods involved the consideration of arcs in higher dimensional spaces and the use of properties involving projections of such arcs to the plane. We have already mentioned, in the introduction to 3.1, how the contraction and expansion theorems of Haupt and Künneth simplified the analysis of normal arcs of order k + 1 with respect to the system of "order characteristics" with the fundamental number k, proofs using induction on k. Again we should acknowledge the work of S. Mukhopadhyaya [19], [20] who first studied the process of contraction.

In our considerations the system of order characteristics is the set of all conics (both non-degenerate and degenerate) with k = 5. It might be of some interest to develop proofs for such results as those given above strictly from a conical point of view. Thus sections 6.1.2, 6.1.4 and Theorem 9 have been included for completeness. Hence it can be concluded that an end-point of an arc of conical order six is ordinary and with one possible exception strongly conically differentiable; cf. 6.1.11 and 6.1.13.

6.1.0 Let $\mathcal{A}_6$ be an open convex arc of conical order six. We first note that if $\mathcal{L}$ is a line intersecting $\mathcal{A}_6$ at points s,u and $\mathcal{M}$ another line intersecting $\mathcal{A}_6$ at points t, v with $s < t < u < v$ on $\mathcal{A}_6$, then these two lines comprise a conic γ_0 which cannot be oriented with $s < t < u < v$ on γ_0. However, we do have a corresponding type of normality condition (cf. 3.1) for convex arcs $\mathcal{A}_6$ of conical order six.

Let γ_0 intersect $\mathcal{A}_6$ in six points. Then γ_0 can be oriented so that these points lie on the same order on γ_0 as they do on $\mathcal{A}_6$.

Proof. By taking another line L_∞, if necessary, we can assume that the convex arc $\mathcal{A}_6$ does not meet L_∞.

Let the points of intersection of γ_0 with $\mathcal{A}_6$ be r, s, t, u, v, w with $r < s < t < u < v < w$ on $\mathcal{A}_6$. Let α be the family of conics which pass through s, t, u, v. Then α is decomposed into three subfamilies $\alpha_1, \alpha_2, \alpha_3$ by

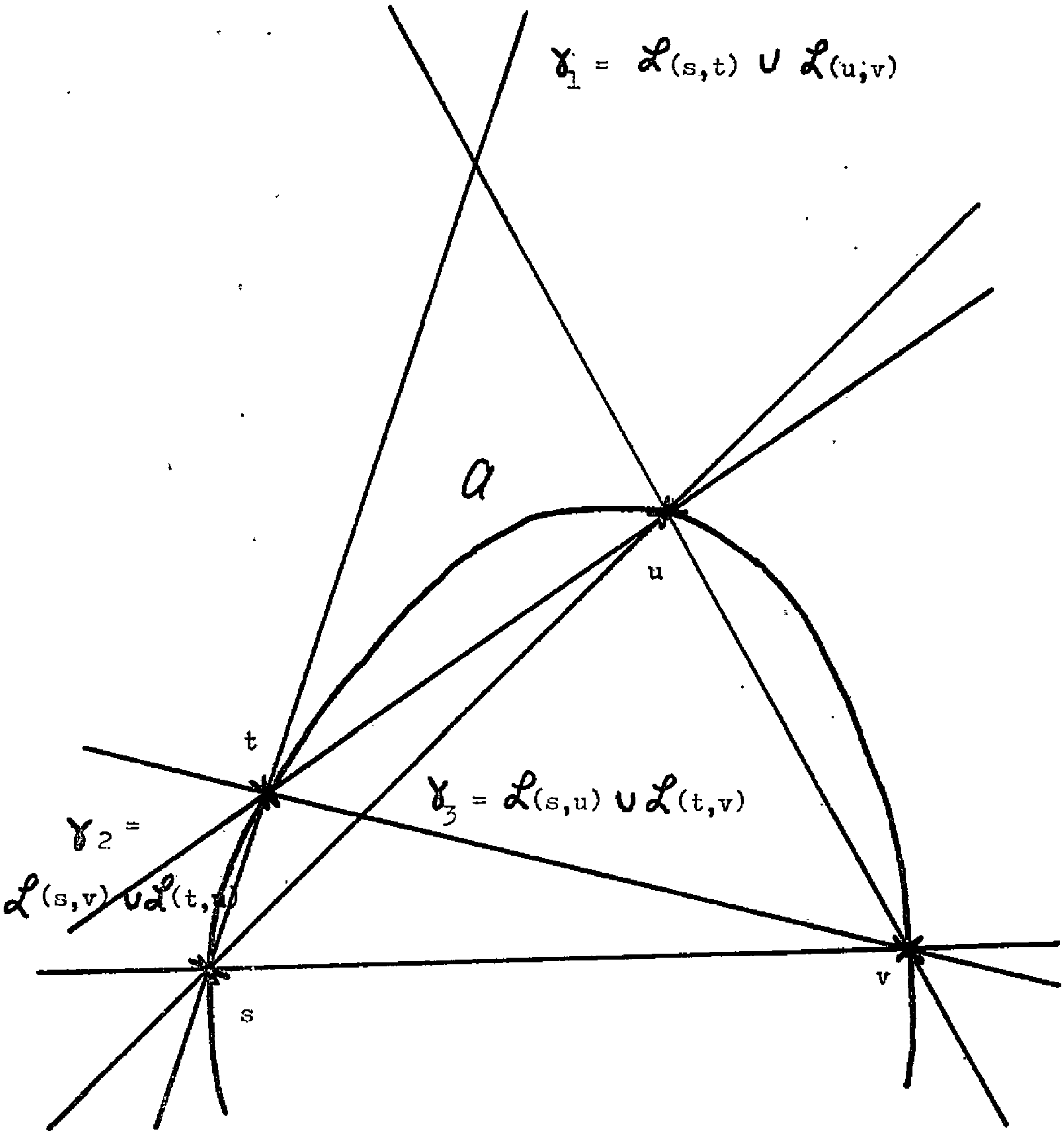
$$\gamma_1 = \mathcal{L}(s, t) \cup \mathcal{L}(u, v),$$
$$\gamma_2 = \mathcal{L}(s, v) \cup \mathcal{L}(t, u),$$
$$\gamma_3 = \mathcal{L}(s, u) \cup \mathcal{L}(t, v),$$

where α_1, say, is bounded by γ_1 and γ_2, α_2 by γ_1 and γ_3, and α_3 and γ_2 (see Figure 12).

In ([11]; 4.6) it is shown that s, t, u, v lie on each conic of α_1 in the indicated order. Thus s, t, u, v lie on γ_0 in the same order as on $\mathcal{Q}_6$. One should notice that each conic that meets $\mathcal{Q}_6$ again is a member of α_1.

Now repeat the above argument using the family β of conics which pass through r, s, t, u. Then r, s, t, u will lie in the same order on γ_0 as they do on $\mathcal{Q}_6$. Hence r, s, t, u, v lie in the same order on γ_0 as they do on $\mathcal{Q}_6$.

Finally, repeat the argument using the family δ of conics which pass through t, u, v, w. Then t, u, v, w lie in the same order on γ_0 as they do on $\mathcal{Q}_6$. Thus r, s, t, u, v, w lie in the indicated order on γ_0 and we have the desired result.



$\gamma_1 = \mathcal{L}(s,t) \cup \mathcal{L}(u,v)$
a
u
t
$\gamma_3 = \mathcal{L}(s,u) \cup \mathcal{L}(t,v)$
$\gamma_2 =$
$\mathcal{L}(s,v) \cup \mathcal{L}(t,u)$
s
v

Figure 12

6.1.1. Let γ be a conic which meets $\mathcal{A}_6$ at six distinct points a, b, p_1, p_2, p_3, p_4. Then as t moves monotonically and continuously from a on $\mathcal{A}_6$, there is a point

$$u \in \gamma(t, p_1, p_2, p_3, p_4) \cap \mathcal{A}_6$$

which moves monotonically and continuously from b in the opposite direction.

Proof. Without loss of generality, we assume that $p_1 < p_2 < p_3 < p_4$. Since $\mathcal{A}_6$ is of order six,

$$\gamma_0 = \gamma(a, p_1, p_2, p_3, p_4)$$

intersects $\mathcal{A}_6$ at a, b, p_1, p_2, p_3, p_4 and meets $\mathcal{A}_6$ nowhere else, by 6.2.2. If t is sufficiently close to a, then

$$\gamma(t, p_1, p_2, p_3, p_4)$$

will be close to γ_0 and will intersect $\mathcal{A}_6$ at t, p_1, p_2, p_3, p_4 and at a point u close to b. Also

$$\gamma(t, p_1, p_2, p_3, p_4)$$

meets $\mathcal{A}_6$ nowhere else. Thus u depends continuously on t.

It is sufficient to show that t and u move in opposite directions on $\mathcal{Q}_6$ whenever t is close to a. Thus we shall restrict t to a suitably small neighbourhood of a in the following.

If an even [odd] number of points of $\{p_1, p_2, p_3, p_4\}$ lie between a and b on γ_0, then the same number of these points will lie between t and u on

$$\gamma(t, p_1, p_2, p_3, p_4).$$

Since the distinct conics γ_0 and $\gamma(t, p_1, p_2, p_3, p_4)$ meet exactly at p_1, p_2, p_3 and p_4, t and u will lie on the same side [on opposite sides] of γ_0.

On the other hand, since

$$\mathcal{Q}_6 \cap \gamma_0 = \{a, p_1, p_2, p_3, p_4, b\},$$

$\mathcal{Q}_6$ will meet γ_0 at an even [odd] number of points between a and b. Hence if t and u move in the same direction on $\mathcal{Q}_6$, then t and u will lie on opposite sides [on the same side] of γ_0; contradiction.

Remarks. (i) The movement of t and u in 6.1.1 can continue as long as

(1) neither t nor u coincide with one of p_1, p_2, p_3, p_4.

(2) t and u do not coincide with each other,

(3) neither t nor u coincides with an end-point of $\mathcal{A}_6$.

(ii) 6.1.1 remains valid if the arc $\mathcal{A}_6$ is replaced by a convex circle $\mathcal{B}_6$ of conical order six.

We note that the proof of 6.1.1 is completely analogous to that of 3.1.1. By using the method of the proofs of 3.1.2, 3.1.3, 3.1.4 and 3.1.5 we obtain the following results.

6.1.2 Let γ_0 be a conic which meets $\mathcal{A}_6$ at points $p_0 < q_0 < r_0 < s_0 < t_0 < u_0$. If $\mathcal{B}$ is the closed subarc of $\mathcal{A}_6$ between p_0 and u_0, then there exists at least one singular point in the interior of $\mathcal{B}$.

Proof. A short systematic proof of this result using 12 equal subintervals of

$$I = [p_0, u_0]$$

can be given.

Divide the parameter interval

$$I = [p_0, u_0]$$

into 12 equal subintervals with the end-points A_i; $i = -6, -5, \ldots, 0, 1, \ldots, 6$. The goal is to construct a conic which passes through six points of either the interval $[A_{-6}, A_5]$ or the interval $[A_{-5}, A_6]$ of $\mathcal{A}_6$.

In the following, put $i = 1, 2, 3, 4, 5$ in turn. Suppose that at the i^{th} step, only $i-1$ of the points of $\gamma \cap I$ lie in the closed subinterval $[A_{-i}, A_i]$ of $\mathcal{A}_6$ and that there are points of $\gamma \cap I$ on both sides of $[A_{-i}, A_i]$. Then we move the two points which lie outside, adjacent to, and on opposite sides of $[A_{-i}, A_i]$ toward this interval, using 6.1.1, while keeping the other four points of $\gamma \cap I$ fixed. Eventually, at least one of these moving points reaches $[A_{-i}, A_i]$. If necessary, we proceed with the next step.

6.1.3 Let $p_0 < q_0 < r_0 < s_0 < t_0$ be five points on $\mathcal{A}_6$ and $\mathcal{B}$ the closed subarc of $\mathcal{A}_6$ bounded by p_0 and t_0. Let the point $a \in \mathcal{A}_6 \setminus \mathcal{B}$. Suppose that there exists a conic through the points $a, p_0, q_0, r_0, s_0, t_0$. If γ_a is the system of conics passing through the point a, then there exists at least one γ_a-singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y on $\mathcal{B}$ there exists a conic of γ_a that meets N at least five times.

Proof. A systematic proof, similar to that of 6.1.2, may be given.

6.1.4 Let N_1, N_2 be arbitrary neighbourhoods of two singular points z_1, z_2 on $\mathcal{A}_6$. Let $\mathcal{B}$ be the closed subarc of $\mathcal{A}_6$ between z_1 and z_2. If

$$a \in \mathcal{A}_6 \setminus N_1 \cup \mathcal{B} \cup N_2,$$

then there exists a conic which meets $\mathcal{A}_6$ at a and at five distinct points of $N_1 \cup \mathcal{B} \cup N_2$.

6.1.5 Let z_1, z_2 be two singular points of $\mathcal{A}_6$ and let $a \in \mathcal{A}_6 \setminus \mathcal{B}$, where $\mathcal{B}$ is the closed subarc of $\mathcal{A}_6$ between z_1 and z_2. Then there exists at least one γ_a-singular point y on $\mathcal{B}$.

6.1.6 Let y_1, y_2 be two points of $\mathcal{A}_6$ and let $\mathcal{B}$ be the closed subarc of $\mathcal{A}_6$ between them. Let a_1, a_2 be distinct points of $\mathcal{A}_6 \setminus \mathcal{B}$. If y_1 and y_2 and γ_{a_1}-singular points, then there exists at least one $\gamma_{a_1a_2}$-singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y on $\mathcal{B}$ there exists a conic passing through a_1, a_2 and meeting N at least four times.

6.1.7. Let y_1, y_2 be two points of $\mathcal{A}_6$ and let $\mathcal{B}$ be the closed subarc of $\mathcal{A}_6$ between them. Let a_1, a_2, a_3 be mutually distinct points of $\mathcal{A}_6 \setminus \mathcal{B}$. If y_1 and y_2 are $\gamma_{a_1a_2}$-singular points, then there exists at least one $\gamma_{a_1a_2a_3}$-singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y

on $\mathcal{B}$ there exists a conic passing through a_1, a_2, a_3 and meeting N at least three times.

6.1.8 Let y_1, y_2 be two points of $\mathcal{a}_6$ and let $\mathcal{B}$ be the closed subarc of $\mathcal{a}_6$ between them. Let a_1, a_2, a_3, a_4 be mutually distinct points of $\mathcal{a}_6 \setminus \mathcal{B}$. If y_1 and y_2 are $\gamma_{a_1a_2a_3}$-singular points, then there exists at least one $\gamma_{a_1a_2a_3a_4}$-singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y on $\mathcal{B}$ there exists a conic passing through a_1, a_2, a_3, a_4 and meeting N at least twice.

6.1.9. Let y_1, y_2 be two points of $\mathcal{a}_6$ and let $\mathcal{B}$ be the closed subarc of $\mathcal{a}_6$ between them. Let a_1, a_2, a_3, a_4, a_5 be mutually distinct points of $\mathcal{a}_6 \setminus \mathcal{B}$. If y_1 and y_2 are $\gamma_{a_1a_2a_3a_4}$-singular points, then there exists at least one $\gamma_{a_1a_2a_3a_4a_5}$-singular point y on $\mathcal{B}$; i.e., for any neighbourhood N of y on $\mathcal{B}$ there exists a conic passing through a_1, a_2, a_3, a_4, a_5 and meeting N at least once.

6.1.10 Now we prove the main result of this section.

Theorem 9: A convex arc $\mathcal{a}_6$ of conical order six contains at most finitely many singular points.

<u>Proof.</u> Assume that there are infinitely many singular points on $\mathcal{A}_6$. Let a_1 be any point on $\mathcal{A}_6$. Then by <u>6.1.5</u>, there are infinitely many γ_{a_1}-singular points on $\mathcal{A}_6$. Take another point a_2 on $\mathcal{A}_6$, $a_1 \neq a_2$. By <u>6.1.6</u>, there exist infinitely many $\gamma_{a_1 a_2}$-singular points on $\mathcal{A}_6$. Take another point a_3 on $\mathcal{A}_6$ with a_1, a_2, a_3 mutually distinct. By <u>6.1.7</u>, $\mathcal{A}_6$ contains infinitely many $\gamma_{a_1 a_2 a_3}$-singular points. By taking another point a_4 on $\mathcal{A}_6$ distinct form a_1, a_2, a_3 and applying <u>6.1.8</u>, we obtain infinitely many $\gamma_{a_1 a_2 a_3 a_4}$-singular points on $\mathcal{A}_6$. Finally let a_5 be a point of $\mathcal{A}_6$ distinct from a_1, a_2, a_3, a_4. By <u>6.1.9</u>, $\mathcal{A}_6$ contains infinitely many $\gamma_{a_1 a_2 a_3 a_4 a_5}$-singular points. But then we have constructed a conic passing through a_1, a_2, a_3, a_4, a_5 which meets $\mathcal{A}_6$ at infinitely many points; contradiction.

<u>Corollary.</u> <u>An arc</u> $\mathcal{A}_6$ <u>of conical order six contains at most finitely many singular points.</u>

<u>Proof.</u> In <u>6.4.1</u> it is shown that $\mathcal{A}_6$ is either convex or of linear order three. By <u>3.2.3</u> of [12], such an arc is the union of finitely many convex arcs. Using <u>Theorem 9</u> we obtain the desired result.

<u>6.1.11</u> <u>If</u> p <u>is an end-point of</u> $\mathcal{A}_6$, <u>then</u> p <u>is ordinary.</u>

Proof. If p is a singular point, then for each neighbourhood $N^{(1)}$ of p there exists a conic which meets $N^{(1)}$ six times, say at $p_1 < q_1 < r_1 < s_1 < t_1 < u_1$. By 6.1.2, there exists a singular point $y^{(1)}$ in (p_1, u_1). Now take a new smaller neighbourhood $N^{(2)}$ of p with $y^{(1)} \notin N^{(2)}$. By 6.1.2, there exists another singular point $y^{(2)} \varepsilon N^{(2)}$ with $y^{(2)} \neq y^{(1)}$. Repeating this process and using 6.1.2, we obtain an infinite numberof singular points on $\mathcal{Q}_6$. This is impossible, by Theorem 9.

6.1.12 In 6.1.11 it was shown that an end-point p of $\mathcal{Q}_6$ is ordinary. Hence there eixsts a neighbourhood N_5 of p on $\mathcal{Q}_6$ which is of order five. But it is known that $N_5 \cup \{p\}$ is strongly conically differentiable at p unless p is of Type 2 ([10], 5.5).

Thus an end-point p of $\mathcal{Q}_6$ is strongly conically differentiable with the exception noted above.

6.1.13 In 6.1.11 and 6.1.12 we assumed that $\mathcal{Q}_6$ was convex. However, as in 6.4.1, arcs of conical order six are either

(i) convex arcs

or

(ii) arcs of linear order three.

But is is well known that such arcs contain at most finitely many linearly singular points ([12], 3.2.3). Thus if p is an end point of $\mathcal{Q}_6$, then p has a one-sided convex neighbourhood on $\mathcal{Q}_6$.

Hence we obtain the following result.

Let $\mathcal{A}_6$ be an arc of conical order six with an end point p. Then

(a) p is ordinary

(b) $\mathcal{A}_6 \cup \{p\}$ is strongly conically differentiable at p if p is not of Type 2.

6.2 Multiplicities for Arcs of Conical Order Six

Introduction

Multiplicities for open arcs $\mathcal{Q}_5$ of conical order five with one end-point p were introduced by N.D. Lane and K. D. Singh in [10], counting p once, twice, three times, four times and five times, respectively, on a non-tangent conic through p a non-osculating tangent conic at p, a non-superosculating osculating conic at p, a non-ultraosculating superosculating conic at p and the ultraosculating conic $\gamma(p^5)$ at p and counting an interior point q of $\mathcal{Q}_5$ once on any conic through q which is not a general tangent conic at q, twice [three times; four times] on any general tangent [osculating; superosculating] conic at q which is not a general osculating [superosculating; ultraosculating] conic at q and five times on any general ultraosculating conic at q. Then it was shown that no conic meets $\mathcal{Q}_5 \cup \{p\}$ more than five times; i.e., the inclusion of p and the introduction of multiplicities do not alter the conical order of $\mathcal{Q}_5$.

In this section the above result will be generalized with the exceptions noted below, to an arc $\mathcal{Q}_6$ of conical order six.

<u>Theorem 10</u>: The conical order of the open arc $\mathcal{Q}_6$ is not changed, with the exceptions observed in the remark following 6.2.27, by

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Introduction

Multiplicities for open arcs $\mathcal{Q}_5$ of conical order five with one end-point p were introduced by N.D. Lane and K. D. Singh in [10], counting p once, twice, three times, four times and five times, respectively, on a non-tangent conic through p a non-osculating tangent conic at p, a non-superosculating osculating conic at p, a non-ultraosculating superosculating conic at p and the ultraosculating conic $\gamma(p^5)$ at p and counting an interior point q of $\mathcal{Q}_5$ once on any conic through q which is not a general tangent conic at q, twice [three times; four times] on any general tangent [osculating; superosculating] conic at q which is not a general osculating [superosculating; ultraosculating] conic at q and five times on any general ultraosculating conic at q. Then it was shown that no conic meets $\mathcal{Q}_5 \cup \{p\}$ more than five times; i.e., the inclusion of p and the introduction of multiplicities do not alter the conical order of $\mathcal{Q}_5$.

In this section the above result will be generalized with the exceptions noted below, to an arc $\mathcal{Q}_6$ of conical order six.

Theorem 10: The conical order of the open arc $\mathcal{Q}_6$ is not changed, with the exceptions observed in the remark following 6.2.27, by

(i) the addition of one of the end-points p;

(ii) the introduction of multiplicities at p, as above; or

(iii) the introduction of multiplicities at interior points q of $\mathcal{Q}_6$, as above. The point q is counted five times on any general ultraosculating conic at q that intersects $\mathcal{Q}_6$ at q and q is counted six times on any general ultraosculating conic at q that supports $\mathcal{Q}_6$ at q. In this last case q is a conically singular point.

Remark. It is assumed that $p < s$ for all $s \in \mathcal{Q}_6$.

<u>6.2.1 No conic γ supports $\mathcal{Q}_6$ at more than three points.</u>

<u>Proof.</u> Suppose γ supports $\mathcal{Q}_6$ at q_1, q_2, q_3 and q_4. If there is another point s which is a point of intersection of γ with $\mathcal{Q}_6$, we may assume that s does not lie between q_1 and q_2, say, on $\mathcal{Q}_6$. Then a suitable conic γ_0 sufficiently close to γ through q_3 and q_4 will intersect $\mathcal{Q}_6$ at two points near q_1, at two points near q_2 and at one point near s. This is impossible.

Hence $\mathcal{Q}_6 \subset \gamma \cup \gamma_e$, say. Let L_1, L_2, L_3 and L_4 be four disjoint neighbourhoods on $\mathcal{Q}_6$ of q_1, q_2, q_3 and q_4, respectively. Choose a conic γ' in γ_e which is close to γ. Since the end-points of L_1, L_2, L_3 and L_4 lie in γ_e, they will also lie in γ'_e. We can orient γ' such that $\gamma \subset \gamma'_i$. Then $q_1, q_2, q_3, q_4 \in \gamma'_i$. Thus γ' separates q_1, q_2, q_3, q_4 from the end-points of L_1, L_2, L_3, L_4 respectively. γ' will intersect each of L_1, L_2, L_3, L_4 in not less than two points. Thus $\gamma' \cap \mathcal{Q}_6$ contains more than six points; contradiction.

<u>6.2.2 If a conic γ supports $\mathcal{Q}_6$ at a point</u> t, <u>then γ cannot meet $\mathcal{Q}_6$ at more than four further points.</u>

<u>Proof</u>. Suppose that γ meets $\mathcal{a}_6$ at q_1, q_2, q_3, q_4, q_5 and supports $\mathcal{a}_6$ at t. Then at least one of the q_i; $i = 1, 2, \ldots, 5$, say q_1, is a point of intersection, by <u>6.2.1</u>. But then a conic γ' sufficiently close to γ through q_2, q_3, q_4, q_5 will intersect $\mathcal{a}_6$ at two points near t and at one point near q_1. Hence γ' meets $\mathcal{a}_6$ at least sevem times; contradiction.

<u>6.2.3.</u> <u>If a conic γ supports $\mathcal{a}_6$ at s and t, then γ does not meet $\mathcal{a}_6 \cup \{p\}$ at more than two further points</u>.

<u>Proof</u>. Suppose that γ meets $\mathcal{a}_6 \cup \{p\}$ at three further points q_1, q_2, q_3. Then either one of the q_i, say q_1, is p or none of the q_i is p. In the second case by <u>6.2.1</u>, one of the q_i, say q_1, is a point of intersection. Then in either case, as in previous arguments, a suitable conic γ' sufficiently close to γ through q_2 and q_3 will meet $\mathcal{a}_6$ twice near s, twice near t and once near q_1. Thus γ' meets $\mathcal{a}_6$ at least seven times; contradiction.

<u>6.2.4</u>

<u>(i)</u> <u>If a conic through p meets $\mathcal{a}_6$ at six points, then at most one of them is a point of intersection</u>.

<u>Proof</u>. Suppose that a conic γ through p intersects $\mathcal{a}_6$ at q_1, q_2 and meets $\mathcal{a}_6$ at four further points r, s, u, v. Choose disjoint neighbourhoods L, L_1, L_2 of p, q_1, q_2, respectively,

which do not contain r, s, u or v. Then if t converges in L to p, $\gamma(r, s, u, v, t)$ converges to γ. However $\gamma(r, s, u, v, t)$ intersects L_1 and L_2 if t is sufficiently close to p. Hence this conic meets $\mathcal{Q}_6$ in not less than seven points; contradiction.

(ii) If a tangent conic of $\mathcal{Q}_6$ at p meets $\mathcal{Q}_6$ at five points, then at most one of them is a point of intersection.

Proof. Let γ be a tangent conic of $\mathcal{Q}_6$ at p intersecting $\mathcal{Q}_6$ at the points q_1, q_2 and meeting $\mathcal{Q}_6$ at further points r, s, u. If t is sufficiently close to p, then $\gamma(p, r, s, u, t)$ will be close to γ and it will intersect $\mathcal{Q}_6$ at points near q_1 and q_2. This is impossible, by (i).

In the same way we obtain the following.

(iii) If an osculating conic of $\mathcal{Q}_6$ at p meets $\mathcal{Q}_6$ at four points, then at most one of them is a point of intersection.

(iv) If a superosculating conic of $\mathcal{Q}_6$ at p meets $\mathcal{Q}_6$ at three points, then at most one of them is a point of intersection.

(v) $\gamma(p^5)$ intersects $\mathcal{Q}_6$ at most once.

6.2.5 No conic meets $\mathcal{Q}_6 \cup \{p\}$ in more than six points.

Proof. Let γ be a conic which meets $\mathcal{A}_6 \cup \{p\}$ in seven mutually distinct points. Since $\mathcal{A}_6$ is of conical order six, one of these points must be p while the other six lie on $\mathcal{A}_6$. These six points are all points of intersection of γ with $\mathcal{A}_6$, by 6.2.2. But this is impossible, by (i) of 6.2.4.

Corollary. No conic through p which supports $\mathcal{A}_6$ at a point s can meet $\mathcal{A}_6$ at four further points.

6.2.6 If a conic γ supports $\mathcal{A}_6$ at s, t and u, then γ does not meet $\mathcal{A}_6 \cup \{p\}$ again.

Proof. Suppose that γ meets $\mathcal{A}_6 \cup \{p\}$ at a further point v. Then by 6.2.1, v is a point of intersection of $\mathcal{A}_6$ with γ or v = p.

In either case a suitable conic γ' through v and sufficiently close to γ will intersect $\mathcal{A}_6$ twice near s, twice near t and twice near u. This is impossible, by 6.2.5.

Corollary. No tangent conic of $\mathcal{A}_6 \cup \{p\}$ at p supports $\mathcal{A}_6$ at more than two points.

6.2.7 No tangent conic of $\mathcal{A}_6 \cup \{p\}$ at p meets $\mathcal{A}_6$ in more than four points.

Proof. If a tangent conic of $\mathcal{Q}_4 \cup \{p\}$ at p meets $\mathcal{Q}_6$ at five distinct points, then at least four of these are points of support, by (ii) of 6.2.4. However, this is impossible, by 6.2.1.

Corollary 1. No tangent conic of $\mathcal{Q}_6 \cup \{p\}$ at p supports $\mathcal{Q}_6$ at two points and intersects $\mathcal{Q}_6$ at a further point.

Corollary 2. No tangent conic of $\mathcal{Q}_6 \cup \{p\}$ at p supports $\mathcal{Q}_6$ at one point and intersects $\mathcal{Q}_6$ at three further points.

6.2.8 No osculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p meets $\mathcal{Q}_6$ in more than three points.

Proof. If an osculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p meets $\mathcal{Q}_6$ at four distinct points, then at least three of these are points of support, by (iii) of 6.2.4. However, this is impossible, by 6.2.6.

Corollary 1. No osculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p supports $\mathcal{Q}_6$ at more than one point.

Corollary 2. No osculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p supports $\mathcal{Q}_6$ at one point and intersects $\mathcal{Q}_6$ at more than one point.

<u>6.2.9</u> <u>No superosculating conic of</u> $\mathcal{Q}_6 \cup \{p\}$ <u>at</u> p <u>meets</u> $\mathcal{Q}_6$ <u>more than twice</u>.

<u>Proof</u>. If a superosculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p meets $\mathcal{Q}_6$ at three points, then at least two of these are points of support, by <u>(iv)</u> of <u>6.2.4</u>. However, this is impossible, by <u>Corollary 1</u> of <u>6.2.8.</u>

<u>Corollary 1</u>. No superosculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p supports $\mathcal{Q}_6$ more than once.

<u>Corollary 2</u>. No superosculating conic of $\mathcal{Q}_6 \cup \{p\}$ which supports $\mathcal{Q}_6$ at one point can meet $\mathcal{Q}_6$ again.

<u>6.2.10</u> $\gamma(p^5)$ cannot meet $\mathcal{Q}_6$ more than once.

<u>Proof</u>. If $\gamma(p^5)$ meets $\mathcal{Q}_6$ at two points, then at least one of these is a point of support, by <u>(v)</u> of <u>6.2.4</u>. However, this is impossible, by <u>Corollary 2</u> of <u>6.2.9</u>.

<u>Corollary</u>. $\gamma(p^5)$ cannot support $\mathcal{Q}_6$ at any point.

<u>6.2.11</u> No general osculating conic of $\mathcal{Q}_6$ at q intersects $\mathcal{Q}_6 \setminus \{q\}$ more than three times.

Proof. Let γ be a general osculating conic of Q_6 at q which intersects $Q_6 \setminus \{q\}$ at four points r_1, r_2, r_3, r_4. By the definition of a general osculating conic, there is a conic γ' sufficiently close to γ and that γ' meets Q_6 three times near q and once each near r_1, r_2, r_3, r_4. Altogether γ' meets Q_6 at least seven times; contradiction.

6.2.12 No general osculating conic of Q_6 at q supports Q_6 more than once.

Proof. Let γ be a general osculating conic of Q_6 at q which supports Q_6 at r and s. Then by 5.3.2 (i), a conic of $\varphi(\gamma)$ sufficiently close to γ will be a general osculating conic of Q_6 at q which intersects Q_6 twice near r and twice near s. This is impossible, by 6.2.11.

Similarly one obtains the following.

6.2.13

(a) No general osculating conic of Q_6 at q which supports Q_6 at a point $r \neq q$ can meet $Q_6 \cup \{p\}$ at more than one further point.

(b) No general osculating conic of Q_6 at q can meet $Q_6 \cup \{p\}$ at more than three further points.

6.2.14 *No general superosculating conic of $\mathcal{Q}_6$ at q intersects $\mathcal{Q}_6$ more than twice.*

Proof. The proof is analogous to 6.2.11.

6.2.15 *No general superosculating conic of $\mathcal{Q}_6$ at q supports $\mathcal{Q}_6 \setminus \{q\}$ more than once.*

Proof. This is a sqecial case of 6.2.12, since every superosculating conic of $\mathcal{Q}_6$ at q is a general osculating conic of $\mathcal{Q}_6$ at q.

Corollary. *No general superosculating conic of $\mathcal{Q}_6$ at q which supports $\mathcal{Q}_6$ at a point r $\neq$ q can meet $\mathcal{Q}_6 \cup \{p\}$ again.*

Proof. The proof is analogous to 6.2.13 (a) using 5.3.2 (ii) and 6.2.15.

6.2.16 *No general superosculating conic of $\mathcal{Q}_6$ at q can meet $\mathcal{Q}_6 \cup \{p\}$ at more than two other points.*

Proof. The proof is analogous to 6.2.13 (b) using 5.3.2 (ii).

6.2.17 *No general ultraosculating conic of $\mathcal{Q}_6$ at q intersect $\mathcal{Q}_6 \setminus \{q\}$ more than twice.*

Proof. The proof is analogous to 6.2.11.

6.2.18

(a) No general ultraosculating conic of $\mathcal{Q}_6$ at q which intersects $\mathcal{Q}_6$ at q can support $\mathcal{Q}_6$ at a point $r \neq q$.

Proof. Let γ be a general ultraosculating conic of $\mathcal{Q}_6$ at q intersecting $\mathcal{Q}_6$ at q which supports $\mathcal{Q}_6$ at r. Since γ intersects $\mathcal{Q}_6$ at q, the end-points of a small neighbourhood N of q on $\mathcal{Q}_6$ will lie on opposite sides of γ. Let γ' have four-point contact with γ at q (cf. 5.1.1) and be sufficiently close to γ so that the end-points of N will still lie on opposite sides of γ' and γ' will intersect $\mathcal{Q}_6$ twice near r. Thus γ' meets N with an odd multiplicity. Hence γ' is either a general superosculating conic of $\mathcal{Q}_6$ at q (cf. 5.3.2 ii) which meets N at another point or a general superosculating conic of $\mathcal{Q}_6$ at q which intersects $\mathcal{Q}_6$ at q; i.e., a general ultra-osculating conic of $\mathcal{Q}_6$ at q. But these situations are impossible, by 6.2.16 and 6.2.17.

(b) No general ultraosculating conic of $\mathcal{Q}_6$ at q which intersects $\mathcal{Q}_6$ at q can meet $\mathcal{Q}_6 \cup \{p\}$ at more than one other point.

Proof. Let γ be a general ultraosculating conic of $\mathcal{Q}_6$ at q intersecting $\mathcal{Q}_6$ at q which meets $\mathcal{Q}_6 \cup \{p\}$ at two further points r, s. By 6.2.17 and 6.2.18 (a), one of these points, say r, is a point of intersection of $\mathcal{Q}_6$ with γ and the other point s = p.

As in (a), a suitable conic γ' having four-point contact with γ at q which lies sufficiently close to γ will intersect $\mathcal{Q}_6$ at a point near r and a point near p. Again as in (a), γ' meets a small neighbourhood N of q with an odd multiplicity. Hence γ' is either a general superosculating conic of $\mathcal{Q}_6$ at q which meets N at another point or a general ultraosculating conic of $\mathcal{Q}_6$ at q. This is impossible, by 6.2.16 and 6.2.17.

6.2.19 No general ultraosculating conic of $\mathcal{Q}_6$ at q which supports $\mathcal{Q}_6$ at q can meet $\mathcal{Q}_6 \cup \{p\}$ again.

Proof. Let γ be a general ultraosculating conic of $\mathcal{Q}_6$ at q which supports $\mathcal{Q}_6$ at q and meets $\mathcal{Q}_6 \cup \{p\}$ at a further point u. Then there are three possibilities:

(a) γ intersects $\mathcal{Q}_6$ at u;
(b) γ supports $\mathcal{Q}_6$ at u; or
(c) γ meets $\mathcal{Q}_6 \cup \{p\}$ at u = p.

If case (a) occurs, then as in 6.2.17, a conic γ' can be constructed close to γ meeting $\mathcal{Q}_6$ six times near q and once near u; contradiction.

Suppose that case (b) occurs. Then γ cannot meet $\mathcal{Q}_6$ except at q and u, by the Corollary to 6.2.15. Without loss of generality, let

$$\mathcal{Q}_6 \setminus \{q, u\} \subset \gamma_e .$$

Also let

$$N = N' \cup \{q\} \cup N'' \; [L]$$

be a small two sided neighbourhood of q [u] on $\mathcal{Q}_6$. We claim that γ is one of the one-sided ultraosculating conics of $\mathcal{Q}_6$ at q. Otherwise, γ is a general tangent conic of both $N' \cup \{q\}$ and $N'' \cup \{q\}$ at q and hence $\mathcal{Q}_6$ satisfies Condition PI at q (cf. 5.3.3), by 6.1.12. Thus γ is a tangent conic of $\mathcal{Q}_6$ at q and the family of tangent conics of $\mathcal{Q}_6$ at q all touch the tangent line at q. Let η be the one-sided ultraosculating conic of $\mathcal{Q}_6$ at q which lies in γ_i with the exception of q. Now if $s \in N$, then $s \in \gamma_e$. Thus the superosculating conic of N at q through s is blocked by γ as s converges to q and hence cannot converge to η ; contradiction. Thus γ is one of the one-sided ultraosculating conics of $\mathcal{Q}_6$ at q, say of $N' \cup \{q\}$.

Let s' be close to q on N'. Then the superosculating conic γ' of $N' \cup \{q\}$ at q through s' will be close to γ and lie in γ_e with the exception of q. Hence γ' intersects L at two points near u. This is impossible, by 6.2.14.

Finally suppose that case <u>(c)</u> occurs. Let

$$N = N' \cup \{q\} \cup N'' \; [L]$$

be a small two-sided [one-sided] neighbourhood of q [p] on $\mathcal{Q}_6 \cup \{p\}$. By the method of <u>(b)</u>, we can construct a conic γ' which is a general superosculating conic of $\mathcal{Q}_6$ at q intersecting N at s' and L at one point s. But then γ' must meet N with an even multiplicity. Thus γ' either intersects $\mathcal{Q}_6$ at q or meets N at another point. Both are impossible, by <u>6.2.18</u> (b) and <u>6.2.16</u>.

<u>6.2.20</u> <u>No general osculating conic of</u> $\mathcal{Q}_6$ <u>at</u> q <u>can be a general superosculating conic of</u> $\mathcal{Q}_6$ <u>at</u> r <u>which supports</u> $\mathcal{Q}_6$ <u>at</u> r.

<u>Proof.</u> Let γ be a general osculating conic of $\mathcal{Q}_6$ at q which is a general superosculating conic of $\mathcal{Q}_6$ at r and supports $\mathcal{Q}_6$ at r. Let

$$N = N' \cup \{q\} \cup N'' \; [L = L' \cup \{r\} \cup L'']$$

be a small two-sided nieghbourhood of q [r] on $\mathcal{Q}_6$. Either γ intersects $\mathcal{Q}_6$ at q or γ supports $\mathcal{Q}_6$ at q.

Finally suppose that case <u>(c)</u> occurs. Let

$$N = N' \cup \{q\} \cup N'' \; [L]$$

be a small two-sided [one-sided] neighbourhood of q [p] on $\mathcal{Q}_6 \cup \{p\}$. By the method of <u>(b)</u>, we can construct a conic γ' which is a general superosculating conic of $\mathcal{Q}_6$ at q intersecting N at s' and L at one point s. But then γ' must meet N with an even multiplicity. Thus γ' either intersects $\mathcal{Q}_6$ at q or meets N at another point. Both are impossible, by <u>6.2.18</u> (b) and <u>6.2.16</u>.

<u>6.2.20</u> <u>No general osculating conic of</u> $\mathcal{Q}_6$ <u>at</u> q <u>can be a general superosculating conic of</u> $\mathcal{Q}_6$ <u>at</u> r <u>which supports</u> $\mathcal{Q}_6$ <u>at</u> r.

<u>Proof</u>. Let γ be a general osculating conic of $\mathcal{Q}_6$ at q which is a general superosculating conic of $\mathcal{Q}_6$ at r and supports $\mathcal{Q}_6$ at r. Let

$$N = N' \cup \{q\} \cup N'' \; [L = L' \cup \{r\} \cup L'']$$

be a small two-sided nieghbourhood of q [r] on $\mathcal{Q}_6$. Either γ intersects $\mathcal{Q}_6$ at q or γ supports $\mathcal{Q}_6$ at q.

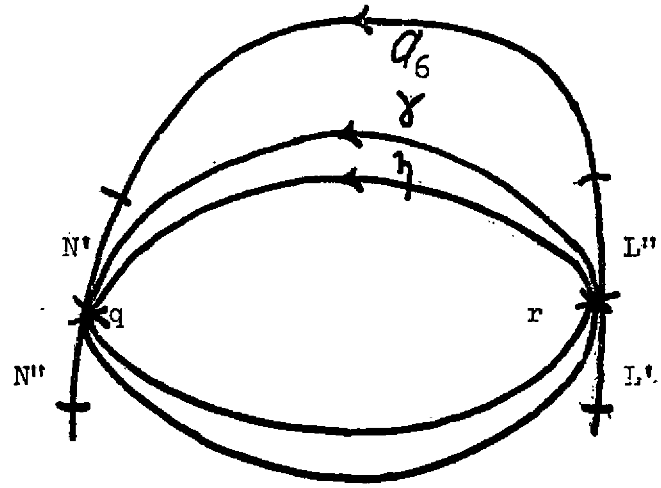
Suppose that γ intersects $\mathcal{Q}_6$ at q. Since γ is a general osculating conic of $\mathcal{Q}_6$ at q, then γ is a general tangent conic of $N' \cup \{q\}$ or of $N'' \cup \{q\}$ at q, say the former. But $N' \cup \{q\}$ satisfies Condition PI' at q (cf. 5.3.3), by 6.1.12. Thus γ is a tangent conic of $N' \cup \{q\}$ at q. Let $t' \in N'$, close to q. Then the conic γ' through q, t' and having three-point contact with γ will be close to γ. But the end-points of N[L] lie on opposite sides [on the same side] of γ. Thus the end-points of N[L] lie on opposite sides [on the same side] of γ'. But γ' meets N [L] at q and t' [at r]. Thus γ' will meet N[L] at a further point. Also γ' is a general osculating conic of $\mathcal{Q}_6$ at r, by 5.3.2 (i). This is impossible, by 6.2.11 and 6.2.13.

Suppose that γ supports $\mathcal{Q}_6$ at q (see Figure 13(i)). Then γ is a general superosculating conic of $\mathcal{Q}_6$ at both r and q supporting $\mathcal{Q}_6$ at these points. By the corollary to 6.2.15, γ does not meet $\mathcal{Q}_6$ again. Without loss of generality, let

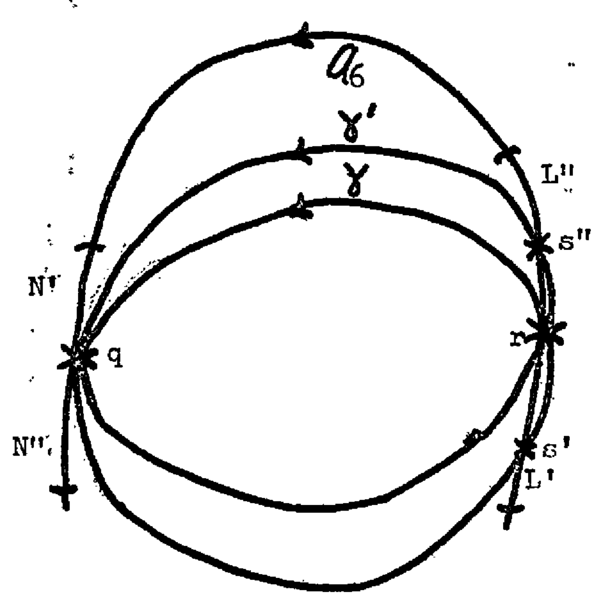
$$\mathcal{Q}_6 \setminus \{r, q\} \subset \gamma_e.$$

We now claim that γ is one of the one-sided osculating conics of $\mathcal{Q}_6$ at r in the family of conics which support each other at q. If not, then γ, being a general superosculating conic of $\mathcal{Q}_6$ at r, must be a general tangent conic of $L' \cup \{r\}$ and $L'' \cup \{r\}$ at r. Hence $\mathcal{Q}_6$ satisfies Condition PI' at r and the family of tangent conics of $\mathcal{Q}_6$ at r all touch the tangent line at r. Let η be the one-sided osculating conic of L at r, in the family of conics which support each other at q, that lies in γ_i with the exception of the points q and r. Let $s \in L$. Then $s \in \gamma_e$. Thus the tangent conic of L at r through s and supporting γ at q is blocked by γ as s converges to r and hence cannot converge to η; contradiction. Thus γ is one of the one-sided osculating conics of $\mathcal{Q}_6$ at r, say of $L' \cup \{r\}$, in the family of conics which support each other at q.

Let s' be close to r on L'. Then the conic γ' which supports γ at r and q through s' is close to γ. Now the end-points of L lie on the same side of γ. Thus the end-points of L lie on the same side of γ'. Thus γ' meets L at a further point s". By 6.2.3, γ' cannot meet $\mathcal{Q}_6$ outside the points q, r, s', s" (see Figure 13 (ii)). Now the end-points of N lie in γ_e. Thus $N \setminus \{q\}$ lies in γ'_e and γ' supports $\mathcal{Q}_6$ at q. But by the methods of the preceding paragraph, γ will be one of the one-sided osculating conics of $\mathcal{Q}_6$ at q in the family of conics which support each other at r. As u tends to q on N, the tangent conic of N at q through u and supporting γ is blocked by γ' and hence cannot converge to γ; contradiction.



(i)



(ii)

Figure 13

<u>6.2.21</u> <u>No general osculating conic of $\mathcal{Q}_6$ at q can be a general osculating conic of $\mathcal{Q}_6$ at r meeting $\mathcal{Q}_6 \cup \{p\}$ again.</u>

<u>Proof.</u> Let γ be a general osculating conic of $\mathcal{Q}_6$ at q which is a general osculating conic of $\mathcal{Q}_6$ at r and meets $\mathcal{Q}_6 \cup \{p\}$ at a further point u. By <u>6.2.20,</u> γ intersects $\mathcal{Q}_6$ at q and r. Let

$$N = N' \cup \{q\} \cup N'' \quad [L]$$

be a small two-sided neighbourhood of q [r] on $\mathcal{Q}_6$. Now γ is either a general tangent conic of $N' \cup \{q\}$ or of $N'' \cup \{q\}$ at q, since it is a general osculating conic of $\mathcal{Q}_6$ at q. Without loss of generality, let γ be a general tangent conic of $N' \cup \{q\}$ at q. But, $N' \cup \{q\}$ satisfies Condition PI', by <u>6.1.12</u>. Hence γ is a tangent conic of $N' \cup \{q\}$ at q. Let s' be close to q on N'. Then the conic γ' which supports γ at r and passes through q, s', u is close to γ. But γ' must meet both N and L with an odd multiplicity. Hence γ' meets N at another point while γ' either supports $\mathcal{Q}_6$ at r meeting L at another point or is a general osculating conic of $\mathcal{Q}_6$ at r. But these situations are impossible, by <u>6.2.2</u>, <u>6.2.11</u> and the corollary following <u>6.2.5</u>.

<u>6.2.22</u> <u>No general osculating conic of $\mathcal{Q}_6$ at q which is a tangent conic of $\mathcal{Q}_6 \cup \{p\}$ at p can intersect $\mathcal{Q}_6 \setminus \{q\}$</u>

more than once.

Proof. Let γ be a general osculating conic of $\mathcal{Q}_6$ at q which is a tangent conic of $\mathcal{Q}_6 \cup \{p\}$ at p and intersects $\mathcal{Q}_6$ at s and t. Now let r be close to p on $\mathcal{Q}_6$. Then the conic γ' having three-point contact with γ at q and passing through p, r will be close to γ. By 5.3.2 (i), γ' is a general osculating conic of $\mathcal{Q}_6$ at q and will intersect $\mathcal{Q}_6$ at points close to s and t, since it is close to γ. This is impossible, by 6.2.13 (b).

6.2.23 No general superosculating conic of $\mathcal{Q}_6$ at q which is a tangent conic of $\mathcal{Q}_6 \cup \{p\}$ at p can meet $\mathcal{Q}_6$ elsewhere.

Proof. Let γ be a general superosculating conic of $\mathcal{Q}_6$ at q which is a tangent conic of $\mathcal{Q}_6 \cup \{p\}$ and meets $\mathcal{Q}_6$ at a further point u. Then u is a point of intersection of γ with $\mathcal{Q}_6$, by 6.2.16. Also γ supports $\mathcal{Q}_6$ at q, by the Corollary following 6.2.18. Let s be close to p on $\mathcal{Q}_6$. Then the conic γ' having three-point contact with γ at q and passing through the points q, s will be close to γ. Hence γ' intersects $\mathcal{Q}_6$ at a point on $\mathcal{Q}_6$ close to u. But γ' is a general osculating conic of $\mathcal{Q}_6$ at q (cf. 5.3.2 (i)) which must meet $\mathcal{Q}_6$ with an even multiplicity near q. Hence γ' also meets $\mathcal{Q}_6$ at a point near q. This is impossible, by 6.2.13 (b).

<u>6.2.24</u> <u>No general osculating conic of</u> $\mathcal{O}_6$ <u>at</u> q <u>which is a tangent conic of</u> $\mathcal{O}_6 \cup \{p\}$ <u>at</u> p <u>can support</u> $\mathcal{O}_6$ <u>at</u> r.

<u>Proof.</u> Let γ be a general osculating conic of $\mathcal{O}_6$ at q which is a tangent conic of $\mathcal{O}_6 \cup \{p\}$ at p and supports $\mathcal{O}_6$ at r. Then a suitable conic γ' close to γ supporting at q and p will intersect $\mathcal{O}_6$ twice near r. But γ intersects $\mathcal{O}_6$ at q, by <u>6.2.23</u>. Thus γ' must meet $\mathcal{O}_6$ with an odd multiplicity near q. Hence γ' is a general osculating conic of $\mathcal{O}_6$ at q or γ' supports $\mathcal{O}_6$ at q and meets $\mathcal{O}_6$ at another point close to q. This is impossible, by <u>6.2.22</u> and by Corollary 2 following <u>6.2.7</u>.

<u>6.2.22</u> and <u>6.2.24</u> imply the following.

<u>6.2.25</u> <u>No general osculating conic of</u> $\mathcal{O}_6$ <u>at</u> q <u>which is a tangent conic of</u> $\mathcal{O}_6 \cup \{p\}$ <u>at</u> p <u>can meet</u> $\mathcal{O}_6 \setminus \{q\}$ <u>more than once</u>.

<u>6.2.26</u> <u>No general superosculating conic of</u> $\mathcal{O}_6$ <u>at</u> q <u>which supports</u> $\mathcal{O}_6$ <u>at</u> q <u>can be an osculating conic of</u> $\mathcal{O}_6 \cup \{p\}$ <u>at</u> p.

<u>Proof.</u> Let γ be a general superosculating conic of $\mathcal{O}_6$ at q, supporting $\mathcal{O}_6$ at q, which is an osculating conic of $\mathcal{O}_6 \cup \{p\}$ at p. Let

$$N = N' \cup \{q\} \cup N'' \ [L]$$

be a small two-sided [one-sided] neighbourhood of q [p] on $\mathcal{Q}_6 \cup \{p\}$. Now γ cannot meet $\mathcal{Q}_6$ elsewhere, by 6.2.23. Without loss of generality, let

$$\mathcal{Q}_6 \setminus \{q\} \subset \gamma_e.$$

Now as in the proof of the second part of 6.2.20, γ is one of the one-sided osculating conics of $\mathcal{Q}_6$ at q in the family of conics that support each other at q.

Let s be close to p on L. Then the conic γ' which supports γ at p, q and passes through s is a tangent conic of $\mathcal{Q}_6 \cup \{p\}$ at p and is close to γ (see Figure 14). Since $s \in \mathcal{Q}_6 \setminus \{q\}$, $s \in \gamma_e$. Thus

$$\gamma' \setminus \{p, q\} \subset \gamma_e.$$

Now the end-poonts of N lie in γ_e. Hence the end-points of N lie in γ'_e.

Next suppose that γ' does not support $\mathcal{Q}_6$ at q. Then γ' is a general osculating conic of $\mathcal{Q}_6$ at q in the family of conics which support each other at q. But γ' must meet N with an even multiplicity and hence meets N at another point.

This is impossible, by 6.2.25. Thus γ' supports $\mathcal{Q}_6$ at q.

Also γ' does not intersect $N \setminus \{q\}$. Otherwise γ' must meet N with an even multiplicity and would intersect N again. This is impossible, by Corollary 2 of 6.2.7. Finally γ' does not support $N \setminus \{q\}$ at any point, by Corollary 1 following 6.2.7.

Now we proceed as in the last few lines of the last paragraph of the proof of 6.2.20 to obtain a contradiction.

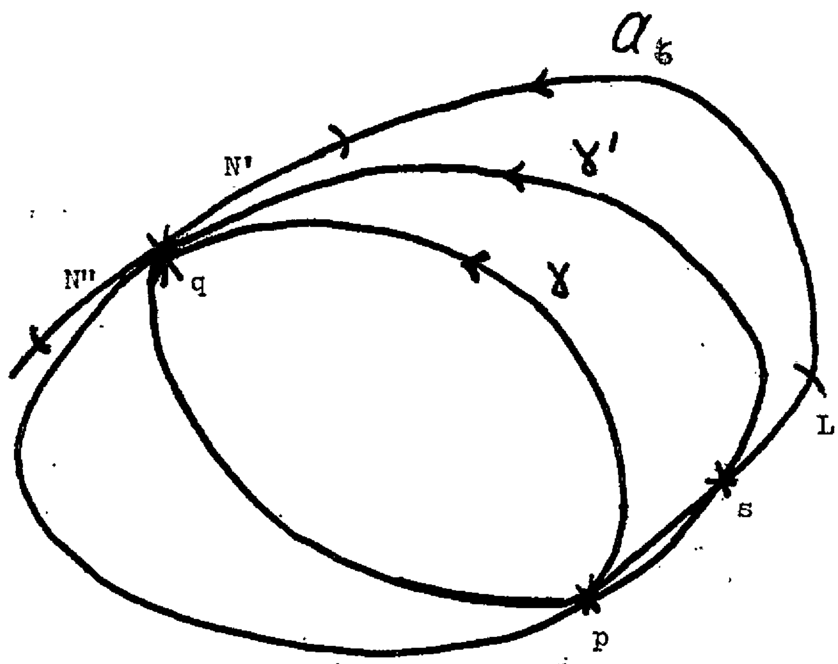


Figure 14

<u>6.2.27</u>. <u>No general osculating conic of</u> $\mathcal{Q}_6$ <u>at</u> q <u>can be a superosculating conic of</u> $\mathcal{Q}_6 \cup \{p\}$ <u>at</u> p <u>if this conic is tangent to</u> $\mathcal{Q}_6 \cup \{p\}$ <u>in the same direction at both</u> q <u>and</u> p.

<u>Proof</u>. Let γ be a general osculating conic of $\mathcal{Q}_6$ at q which is a superosculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p. Also assume that γ is tangent to $\mathcal{Q}_6 \cup \{p\}$ in the same direction at both q and p (see Figure 15(i)).

Let s be close to p on $\mathcal{Q}_6$. Let γ' be the osculating conic of $\mathcal{Q}_6 \cup \{p\}$ at p through s and q. Then γ' intersects γ at p, q and is close to γ. Now the end-points of a small two-sided neighbourhood N of q on $\mathcal{Q}_6$ lie on opposite sides of γ, since γ intersects $\mathcal{Q}_6$ at q; cf. <u>6.2.26</u>. Thus the end-points of N lie on opposite sides of γ', since γ' is close to γ.

Now γ' cannot support $\mathcal{Q}_6$ at q. Otherwise it would meet N again, since it must meet N with an odd multiplicity. This is impossible, by Corollary 2 following <u>6.2.8</u>. Thus γ' intersects $\mathcal{Q}_6$ at q. Also γ' cannot support $N \setminus \{q\}$ at any point, by Corollary 2 following 2 following <u>6.2.8</u>. Finally γ' does not intersect $N \setminus \{q\}$ at any point. Otherwise, γ' must intersect $N \setminus \{q\}$ at still another point. This is also impossible, by <u>6.2.8</u>. Thus γ' intersects N at q and meets N nowhere else.

Next, since γ is tangent to $\mathcal{Q}_6 \cup \{p\}$ in the same direction at p and q, γ' lies (in some sense) between the general osculating conic γ of $\mathcal{Q}_6$ at q and the arc N of $\mathcal{Q}_6$ itself.

Finally γ, being a general osculating conic of $\mathcal{Q}_6$ at q is a general tangent conic of $N' \cup \{q\}$ or or $N'' \cup \{q\}$ at q, if

$$N = N' \cup \{q\} \cup N''.$$

Without loss of generality, let γ be a general tangent conic of $N' \cup \{q\}$ at q. But $N' \cup \{q\}$ satisfies Condition PI', by <u>6.1.12</u>. Thus γ is a tangent conic of $N' \cup \{q\}$ at q. Let $s \in N'$; hence $s \in \gamma'_e$. Thus the conic passing through s, q and having three-point contact with γ at p is blocked by γ' as s converges on N' to q and hence cannot converge to γ; contradiction.

<u>Remark</u>. A similar problem seems to arise here in the conical analysis for multiplicities of arcs $\mathcal{Q}_6$ of order six as the one which occured for the circular case concerning multiplicities of arcs $\mathcal{Q}_4$ of order four; cf. <u>3.2.14</u>.

It seems to be possible to have

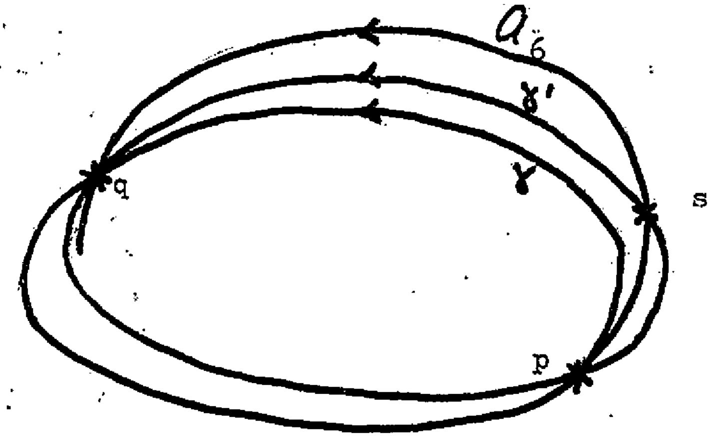
(a) a general osculating conic γ of $\mathcal{A}_6$ at q which is a superosculating conic of $\mathcal{A}_6 \cup \{p\}$ at p;

(b) a general ultraosculating conic γ of $\mathcal{A}_6$ at q which is a tangent conic of $\mathcal{A}_6 \cup \{p\}$ at p (of course γ would have to intersect $\mathcal{A}_6$ at q, by 6.2.19); or

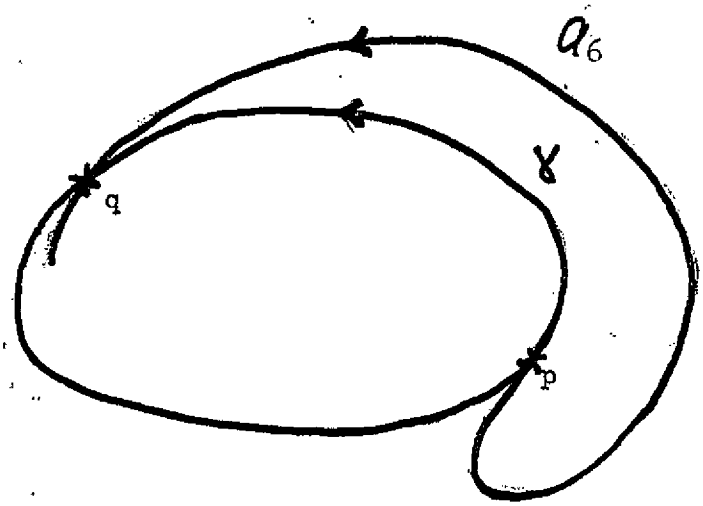
(c) a general ultraosculating conic γ of $\mathcal{A}_6$ at q which is a general osculating conic of $\mathcal{A}_6$ at r (γ must intersect $\mathcal{A}_6$ at both q and r, by 6.2.19 and 6.2.20);

if γ is not tangent to $\mathcal{A}_6 \cup \{p\}$ in the same direction at p, q for (a) and (b) or if γ is not tangent to $\mathcal{A}_6 \cup \{p\}$ in the same direction at q and r for (c) (see Figure 15(ii)). These exceptions are not possible, if $\mathcal{B}_6$ is convex.

The author would appreciate any research which would either rule out these possibilities or give examples of the existence of such arcs of conical order six.



(i)



(ii)

Figure 15

6.3 Monotony Theorems for Conically Differentiable Covex Arcs of Conical Order Six

Introduction

A "Monotony Theorem" is derived by O. Haupt and H. Künneth ([12], 2.3) for arcs of finite order with respect to a system of order characteristics with **fundamental** number k. A statement and proof of a corresponding monotony result for the circular case was given in 3.1.1 for arcs of order four. A similar result for the conical case and arcs of conical order six was obtained in 6.1.1.

In this section we shall derive a generalization of 6.1.1 under the assumption that $\mathcal{Q}_6$ is a strongly conically differentiable convex arc of conical order six. In 6.3.7 the monotony results 6.3.2-6.3.6 are extended to conically differentiable convex arcs $\mathcal{Q}_6$, as long as $\mathcal{Q}_6$ contains no points of Type 2; cf. 5.3.1.

These results will be very useful in the analysis of conically differentiable convex curves of conical order six; cf. 6.4.6 and 6.4.7.

6.3.1 In the following it is assumed, unless otherwise stated, that $\mathcal{A}_6$ is an open strongly differentiable convex arc of conical order six. It will become evident that $\mathcal{A}_6$ could be replaced by a strongly conically differentiable convex curve $\mathcal{C}_6$ without affecting the validity of the results.

6.3.2 Let $p_1 \leq p_2 \leq p_3 \leq p_4$ be four points on $\mathcal{A}_6$. Let γ_0 be a conic which passes through these points and meets $\mathcal{A}_6$ six times altogether counting multiplicities. Call the other two points a and b. Then as t moves monotonically and continuously from a in one direction on $\mathcal{A}_6$, there is a point

$$u \in \gamma(t, p_1, p_2, p_3, p_4) \cap \mathcal{A}_6$$

which moves monotonically and continuously from b in the opposite direction.

Proof. Since $\mathcal{A}_6$ is of conical order six, γ_0 meets $\mathcal{A}_6$ at a, b, p_1, p_2, p_3, p_4, and nowhere else. There are a number of cases, depending upon the coincidence of one or more of these points. In all cases, if t is distinct from and close to a, then $\gamma(t, p_1, p_2, p_3, p_4)$ is close to $\gamma(a, p_1, p_2, p_3, p_4) = \gamma_0$ since $\mathcal{A}_6$ is strongly conically differentiable. Hence $\gamma(t, p_1, p_2, p_3, p_4)$ meets $\mathcal{A}_6$ at a point u close to b. Also $u \neq b$; otherwise, $\gamma(t, p_1, p_2, p_3, p_4)$ meets $\mathcal{A}_6$ more than six times, counting multiplicities and this is a contradiction.

Similarly, $\gamma(t, p_1, p_2, p_3, p_4)$ can meet $\mathcal{A}_6$ nowhere else. Thus u depends continuously on t.

Because of the continuity of the movement of u, it is sufficient to show that t and u move in opposite directions on $\mathcal{A}_6$ whenever t is close to a. We will give proofs for the cases in which the points p_1, p_2, p_3, p_4 are mutually distinct. Similar arguments can be used for the cases in which one or more of p_1, p_2, p_3, p_4 coincide. We shall assume, without loss of generality, that $a \leq b$ on $\mathcal{A}_6$.

(i) All of a, b, p_1, p_2, p_3, p_4 are distinct. This is 6.1.1.

(ii) Let $b = p_i$ for some i, $1 \leq i \leq 4$, $a \neq b$, $a \neq p_j$, $1 \leq j \leq 4$.

Then γ_0 intersects $\mathcal{A}_6$ at a and p_j, $j \neq i$, $1 \leq j \leq 4$ and supports $\mathcal{A}_6$ at $b = p_i$. The subarc $\mathcal{A}_6'$ of $\mathcal{A}_6$ between a and b contains either an even or odd number of points of

$$\{p_1, p_2, p_3, p_4\} \setminus \{p_i\}.$$

Suppose that $\mathcal{A}_6'$ contains an odd number of points of the above set (the following argument can be slightly modified to take care of the even number case). Let this number be σ. Then $\mathcal{A}_6'' = \mathcal{A}_6 \setminus \mathcal{A}_6'$ will contain an even number e of the points

$$\{p_1, p_2, p_3, p_4\} \setminus \{p_i\},$$

where $\theta + e = 3$. Since t is close to a and u is close to b, then the same number of these points will lie on the respective arcs γ', γ'' of $\gamma(t, p_1, p_2, p_3, p_4)$ between t and u.

If u moves away from b on $\mathcal{Q}_6''$ as t moves away from a on $\mathcal{Q}_6'$, then t and u will lie on opposite sides of γ_0. Hence $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 with an odd multiplicity on both arcs γ, γ'' of $\gamma(t, p_1, p_2, p_3, p_4)$ between t and u. On the arc γ'', $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 with an odd multiplicity $\geq e$; i.e., $\geq e + 1$. On the other arc γ'', $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 at least θ times and at the additional point p_i; i.e., $\geq \theta + 1$.

If u moves away from b on $\mathcal{Q}_6'$ as t moves away from a on $\mathcal{Q}_6''$, then t and u will lie on the same side of γ_0, since γ_0 intersects $\mathcal{Q}_6$ at a and supports $\mathcal{Q}_6$ at $b = p_i$. Hence $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 with an even multiplicity on both arcs γ', γ'' of $\gamma(t, p_1, p_2, p_3, p_4)$ between t and u. On the arc γ', $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 with and even multiplicity $\geq \theta$; i.e., $\geq \theta + 1$. On the other arc γ'', $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 at least e times and at the additional point p_i; i.e., $\geq e + 1$.

In both cases $\gamma(t, p_1p, p_2, p_3, p_4)$ meets γ_0 altogether at least

$$(e + 1) + (\theta + 1) = 5$$

times. Hence

$$\gamma_0 = \gamma(t, p_1, p_2, p_3, p_4)$$

and this conic meets $\mathcal{Q}_6$ more than six times; contradiction.

Thus $\gamma(t, p_1, p_2, p_3, p_4)$ meets $\mathcal{Q}_6$ at a point u which moves monotonically and continuously on $\mathcal{Q}_6$ in the opposite direction to that of t.

<u>(iii)</u> Let $a = b \neq p_j$, $1 \leq j \leq 4$.

Then γ_0 intersects $\mathcal{Q}_6$ at p_j, $j = 1, \ldots, 4$ and supports $\mathcal{Q}_6$ at $a = b$. Let N be a suitably small two-sided neighbourhood of a on $\mathcal{Q}_6$. The end-points of N lie on the same side of γ_0, say in γ_{0_e}, since γ_0 supports $\mathcal{Q}_6$ at a. We can assign a continuous orientation to the conics through p_1, p_2, p_3, p_4 near γ_0. Hence if t is sufficiently close to a, the end-points of N will also lie in $\gamma(t, p_1, p_2, p_3, p_4)_e$.

Without loss of generality, let $p_1 < a = b < p_2$. Now $t \in \gamma_{0_e}$ and hence the arc of $\gamma(t, p_1, p_2, p_3, p_4)$ between p_1 and p_2 which contains t lies in γ_{0_e}. By the continuous orientation of the conics through p_1, p_2, p_3, p_4 near γ_0, the arc of γ_0 between p_1 and p_2 which contains a lies in $\gamma(t, p_1, p_2, p_3, p_4)_i$. In particular, $a \in \gamma(t, p_1, p_2, p_3, p_4)_i$. Hence each of the one-sided neighbourhoods of a that make up N will intersect

$\gamma(t, p_1, p_2, p_3, p_4)$, once at t and once at u. Also $\gamma(t, p_1, p_2, p_3, p_4)$ does not meet $\mathcal{Q}_6$ again.

Thus $\gamma(t, p_1, p_2, p_3, p_4)$ meets $\mathcal{Q}_6$ at a point u which moves monotonically and continuously on $\mathcal{Q}_6$ in the opposite direction to that of t.

<u>(iv)</u> Let $a = p_i$ for some i, $1 \leq i \leq 4$, $a \neq b$ and $b \neq p_j$, $1 \leq j \leq 4$.

Then γ_0 intersects $\mathcal{Q}_6$ at b and p_j, $j \neq i$, $i \leq j \leq 4$ and supports $\mathcal{Q}_6$ at $a = p_i$. This case is identical to case <u>(ii)</u> with b replaced by a and can be dealt with in a similar manner.

<u>(v)</u> Let $a = p_i$ for some i, $1 \leq i \leq 4$, $b = p_j$ for some j, $j = i$, $i \leq j \leq 4$.

Then γ_0 intersects $\mathcal{Q}_6$ at p_k, $k \neq i, j$, $1 \leq k \leq 4$ and supports $\mathcal{Q}_6$ at $a = p_i$ and at $b = p_j$. The subarc $\mathcal{Q}_6'$ of $\mathcal{Q}_6$ between a and b contains either an even or odd number of points of

$$\{p_1, p_2, p_3, p_4\} \setminus \{p_i, p_j\}.$$

Suppose that $\mathcal{Q}_6'$ contains an odd number and hence one of the points of the above set. Then $\mathcal{Q}_6'' = \mathcal{Q}_6 \setminus \mathcal{Q}_6'$ will contain the other point of this set. Since t is close to a and u is close to b, the same number of these points (namely one) will lie on the respective arcs γ', γ'' of $\gamma(t, p_1, p_2, p_3, p_4)$ between t and u.

If u moves away from b on $\mathcal{a}_6''$ [$\mathcal{a}_6'$] as t moves away from a on $\mathcal{a}_6'$ [$\mathcal{a}_6''$], then t and u will lie on opposite sides of γ_0. Hence $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 with an odd multiplicity on both arcs γ', γ'' of $\gamma(t, p_1, p_2, p_3, p_4)$. On the arc γ'[γ''], $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 at least once and at the additional point p_i and hence ≥ 3 times. On the arc γ'' [γ'], $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 at least once and at the additional point p_j; i.e., ≥ 2 times. Hence $\gamma(t, p_1, p_2, p_3, p_4)$ meets γ_0 altogether at least five timees. Thus

$$\gamma_0 = \gamma(t, p_1, p_2, p_3, p_4)$$

and this conic meets $\mathcal{a}_6$ at least seven times; contradiction. The assumption that the subarc $\mathcal{a}_6'$ of $\mathcal{a}_6$ contains an even number of points of

$$\{p_1, p_2, p_3, p_4\} \setminus \{p_i, p_j\}$$

similarly leads to a contradiction.

Thus $\gamma(t, p_1, p_2, p_3, p_4)$ meets $\mathcal{a}_6$ at a point u which moves monotonically and continuously on $\mathcal{a}_6$ in the opposite direction to that of t.

<u>(vi)</u> Let $a = b = p_i$ for some i, $1 \leq i \leq 4$.

Then γ_0 intersects $\mathcal{Q}_6$ at $a = b = p_i$ and at the points p_j, $1 \leq j \leq 4$. Let N' be a suitably small two-sided neighbourhood of $b = a$ on $\mathcal{Q}_6$. Then by the continuity of u, there exists a small two-sided neighbourhood N of $a = b$ on $\mathcal{Q}_6$ such that if $t \in N$, then $u \in N'$. Let $t \in N$, $t \neq a$. Then $u \neq b = a$, as was shown in the proof of the continuity of u.

Suppose u and t lie on the same side of a on N' [$u = t$] Without loss of generality if $u \neq t$, let t lie between a and u on N'. By case <u>(i)</u> [<u>(iii)</u>], as t' moves monotonically and continuously from t towards a on N, there is a point

$$u' \in \gamma(t, p_1, p_2, p_3, p_4) \cap \mathcal{Q}_6$$

which moves from u in the opposite direction on N'. Thus u' cannot converge to a as t' tends to a; contradiction.

Hence $\gamma(t, p_1, p_2, p_3, p_4)$ meets $\mathcal{Q}_6$ at a point u which moves monotonically and continuously on $\mathcal{Q}_6$ in the opposite direction to that of t.

<u>6.3.3</u> <u>Let $p_1 \leq p_2 \leq p_3$ be three points on</u> $\mathcal{Q}_6$. Let γ_0 <u>be a tangent conic of</u> $\mathcal{Q}_6$ <u>at a point</u> a <u>which passes through these points and meets</u> $\mathcal{Q}_6$ <u>six times altogether counting multiplicites. Call the other point</u> b. <u>Then as</u> t <u>moves monotonically and</u>

<u>continuously from</u> a <u>in one direction on</u> $\mathcal{a}_6$, <u>there is a point</u>

$$u \in \gamma(t^2, p_1, p_2, p_3) \cap \mathcal{a}_6$$

<u>which moves monotonically and continuously from</u> b <u>in the opposite direction.</u>

<u>Proof</u>. Since $\mathcal{a}_6$ is of order six, γ_0 meets $\mathcal{a}_6$ at the points a, b, p_1, p_2, p_3 and nowhere else. Again we have a number of cases depending upon the coincidence of one or more of these points. In all cases, it t is distinct from and close to a, then $\gamma(t^2, p_1, p_2, p_3)$ is close to γ_0 since $\mathcal{a}_6$ is strongly conically differentiable. Thus $\gamma(t^2, p_1, p_2, p_3)$ meets $\mathcal{a}_6$ at a point u close to b. Also $\gamma(t^2, p_1, p_2, p_3)$ can meet $\mathcal{a}_6$ nowhere else. Thus u depends continuously on t.

Because of the continuity of u, it is sufficient to show that t and u move in opposite directions on $\mathcal{a}_6$ whenever t is close to a. Again we will give proofs for the cases in which the points p_1, p_2, p_3 are mutually distinct. Similar arguments can be used to prove the monotony property for the cases in which one or more of the points p_1, p_2, p_3 coincide.

<u>(i)</u> All of a, b, p_1, p_2, p_3 are distinct.

Then γ_0 intersects $\mathcal{a}_6$ at b, p_1, p_2, p_3 and supports $\mathcal{a}_6$ at a. By <u>6.3.2</u> (iii), if t is distinct from and close to a, then $\gamma(t, p_1, p_2, p_3, b)$ intersects $\mathcal{a}_6$ at a point q on the opposite side of and close to a. Let r converge from q through a to t on $\mathcal{Q}_6$. Then

$$\lim_{r \to t} \gamma(r, t, p_1, p_2, p_3) = \gamma(t^2, p_1, p_2, p_3)$$

By 6.3.2 (i), as r moves monotonically and continuously from q through a to t on $\mathcal{Q}_6$, there is a point

$$u_r \in \gamma(r, t, p_1, p_2, p_3) \cap \mathcal{Q}_6$$

which moves monotonically and continuously from b in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction from b on $\mathcal{Q}_6$ as t does from a.

(ii) Let $b = p_i$ for some i, $1 \leq i \leq 3$, a = b, $a = p_j$, $1 \leq j \leq 3$.

Then γ_0 is a nonosculating tangent conic of $\mathcal{Q}_6$ at a and $b = p_i$. Hence γ_0 intersects $\mathcal{Q}_6$ at the points p_j, j = i, and supports $\mathcal{Q}_6$ at a and b. By 6.3.2 (v), as t moves monotonically and continuously from a on $\mathcal{Q}_6$, $\gamma(t, a, p_1, p_2, p_3)$ intersects $\mathcal{Q}_6$ at a point q which moves monotonically and continuously from b in the opposite direction. It t is close to a, then q is close to b. Now let r converge from a to t on $\mathcal{Q}_6$. Then

$$\lim_{r \to t} \gamma(r, t, p_1, p_2, p_3) = \gamma(t^2, p_1, p_2, p_3).$$

By 6.3.2 (i), as r moves monotonically and continuously from a to t, then there is a point

$$u_r \varepsilon \gamma(r, t, p_1, p_2, p_3) \cap \mathcal{A}_6$$

which moves monotonically and continuously from q in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction on $\mathcal{A}_6$ to that of t.

(iii) Let $a = b \neq p_j$, $1 \leq j \leq 3$.

Then γ_0 is a nonsuperosculating osculating conic of $\mathcal{A}_6$ at $a = b$. Hence γ_0 intersects $\mathcal{A}_6$ at $a = b$ and at the points p_j, $1 \leq j \leq 3$. By 6.3.2 (vi), if t is close to and distinct from a, $\gamma(t, p_1, p_2, p_3, a)$ intersects $\mathcal{A}_6$ at a point q on the opposite side of and close to a. Let r converge from a to t on $\mathcal{A}_6$. Then

$$\lim_{r \to t} \gamma(r, t, p_1, p_2, p_3) = \gamma(t^2, p_1, p_2, p_3).$$

By 6.3.2 (i), as r moves monotonically and continuously from a to t, there is a point,

$$u_r \in \gamma(r, t, p_1, p_2, p_3) \cap \mathcal{O}_6$$

which moves monotonically and continuously from q in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction on $\mathcal{O}_6$ to that of t.

(iv) Let $a = p_i$ for some $i, 1 \leq i \leq 3$, $a \neq b$ and $b \neq p_j, 1 \leq j \leq 3$.

Then γ_0 is a nonsuperosculating osculating conic of $\mathcal{O}_6$ at $a = p_i$. Hence γ_0 intersects $\mathcal{O}_6$ at $a = p_i$, b and p_j, $1 \leq j \leq 3$. By 6.3.2, if t moves monotonically and continuously from a on $\mathcal{O}_6$, then $\gamma(t, a, p_1, p_2, p_3)$ intersects $\mathcal{O}_6$ at a point q which moves monotonically and continuously from b in the opposite direction. If t is close to a, then q is close to b. Let r converge from a to t on $\mathcal{O}_6$. Then

$$\lim_{r \to t} \gamma(r, t, p_1, p_2, p_3) = \gamma(t^2, p_1, p_2, p_3).$$

By 6.3.2 (iv), as r moves monotonically and continuously from a to t, there is a point

$$u_r \in \gamma(r, t, p_1, p_2, p_3) \cap \mathcal{O}_6$$

which moves monotonically and continuously from q in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction on $\mathcal{A}_6$ to that of t.

<u>(v)</u> Let $a = p_i$ for some i, $1 \leq i \leq 3$, $b = p_j$ for some j, $j \neq i$, $1 \leq j \leq 3$.

Then γ_0 is simultaneously a nonsuperosculating osculating conic of $\mathcal{A}_6$ at a and a nonosculating tangent conic of $\mathcal{A}_6$ at b. Hence γ_0 intersects $\mathcal{A}_6$ at a, p_k, $k \neq j$, $1 \leq k \leq 3$ and supports $\mathcal{A}_6$ at b. By <u>6.3.2</u>, if t moves monotonically and continuously from a on $\mathcal{A}_6$, $\gamma(t, a, p_1, p_2, p_3)$ intersects $\mathcal{A}_6$ at a point q which moves monotonically and continuously from b in the opposite direction. If t is close to a, then q is close to b. Let r converge from a to t on $\mathcal{A}_6$. Then

$$\lim_{r \to t} \gamma(r, t, p_1, p_2, p_3) = \gamma(t^2, p_1, p_2, p_3).$$

By <u>6.3.2</u> (iv), as r moves monotonically and continuously from a to t on $\mathcal{A}_6$, there is a point,

$$u_r \in \gamma(r, t, p_1, p_2, p_3) \cap \mathcal{A}_6$$

which moves monotonically and continuously from q in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction on $\mathcal{Q}_6$ to that of t.

(vi) Let $a = b = p_i$ for some i, $1 \leq i \leq 3$.

Then γ_0 is a nonultraosculating superosculating conic of $\mathcal{Q}_6$ at a which intersects $\mathcal{Q}_6$ at p_j, $j \neq i$, $1 \leq j \leq 3$ and supports $\mathcal{Q}_6$ at a. Let N' be a small two-sided neighbourhood of a on $\mathcal{Q}_6$. Then by the continuity of u, there exists a small neighbourhood N of a such that if $t \in N$, $\gamma(t^2, p_1, p_2, p_3)$ meets $\mathcal{Q}_6$ at $u \in N'$. Let $t \in N$, $t \neq a$.

Firstly $u \neq a$. Otherwise let t' move from t toward a on N. Then by 6.3.3 (ii), there is a point

$$u' \in \gamma(t'^2, p_1, p_2, p_3) \cap \mathcal{Q}_6$$

which moves monotonically and continuously from $b = a$ toward t on N. Hence the points t' and u' must coincide at some position $\bar{t} \in N$. By 6.3.3 (iii), as we continue the monotone and continuous movement of t' from $\bar{t}$ toward a on N, u' moves in the opposite direction from $\bar{t}$ on N'. Thus u' cannot converge to a as t' tends to a.

Suppose u and t lie on the same side of a on N' (without loss of generality if $u \neq t$, let t lie between a and u) [$u = t$]. Then by 6.3.3 (i) [(iii)], as t' moves monotonically and continuously from t toward a on N, there is a point

$$u' \in \gamma(t'^2, p_1, p_2, p_3) \cap \mathcal{Q}_6$$

which moves from u in the opposite direction on N'. Thus u' cannot converge to a as t' tends to a; contradiction.

Hence $\gamma(t^2, p_1, p_2, p_3)$ meets $\mathcal{Q}_6$ at a point u which moves monotonically and continuously on $\mathcal{Q}_6$ in the opposite direction to that of t.

6.3.4 Let $p_1 \leq p_2$ be two points on $\mathcal{Q}_6$. Let γ_0 be an osculating conic of $\mathcal{Q}_6$ at a point a which passes through these points and meets $\mathcal{Q}_6$ six times altogether counting multiplicities. Call the other point b. Then as t moves monotonically and continuously from a in one direction on $\mathcal{Q}_6$, there is a point

$$u \in \gamma(t^3, p_1, p_3) \cap \mathcal{Q}_6$$

which moves monotonically and continuously from b in the opposite direction.

Proof. Since $\mathcal{Q}_6$ is of order six, γ_0 meets $\mathcal{Q}_6$ at the points a, b, p_1, p_2 and mowhere else. As in 6.3.2 and

6.3.3 we have a number of cases depending upon the coincidence of one or more of these points.

In each case if t is close to and distinct from a, $\gamma(t^3, p_1, p_2)$ is close to γ_0 since $\mathcal{a}_6$ is strongly conically differentiable. Thus $\gamma(t^3, p_1, p_2)$ meets $\mathcal{a}_6$ at a point u close to b. Also $\gamma(t^3, p_1, p_2)$ meets $\mathcal{a}_6$ nowhere else. Hence u depends continuously on t.

Again it is sufficient to show that t and u move in opposite directions on $\mathcal{a}_6$ whenever t is close to a. Again we give proofs for the cases in which p_1 is distinct from p_2. Similar arguments can be used to obtain the monotony of u for the cases in which p_1 coincides with p_2.

(i) All of a, b, p_1, p_2 are distinct.

Then γ_0 intersects $\mathcal{a}_6$ at all of these points. By 6.3.3 (iii), if t is distinct form and close to a, $\gamma(t^2, p_1, p_2, b)$ intersects $\mathcal{a}_6$ at a point q on the opposite side of and close to a. Let r converge from q through a to t on $\mathcal{a}_6$. Then

$$\lim_{r \to t} \gamma(r, t^2, p_1, p_2) = \gamma(t^3, p_1, p_2).$$

By 6.3.3 (i), as r moves monotonically and continuously from q through a to t on $\mathcal{a}_6$, there is a point

$$u_r \in \gamma(r, t^2, p_1, p_2) \cap \mathcal{Q}_6$$

which moves monotonically and continuously from b in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction from b on $\mathcal{Q}_6$ as t does from a.

Remark. We notice that the proof of 6.3.4 (i) is completely analogous to that of 6.3.3 (i). Similar arguments as in 6.3.3 (ii), (iii), (iv), (v), (vi), respectively, allow us to obtain 6.3.4 for the cases

(ii) Let $b = p_i$ for some i, $i = 1$ or 2, $a \neq b$, a distinct from p_j, $j = 1,2$.

(iii) Let $a = b \neq p_j$, $j = 1,2$.

(iv) Let $a = p_i$ for some i, $i = 1$ or 2, $a \neq b$, $b \neq p_j$, $j = 1, 2$.

(v) Let $a = p_i$ for some i, $i = 1$ or 2, $b = p_j$, $j \neq i$, $j = 1$ or 2.

(vi) Let $a = b = p_i$ for some i, $i = 1$ or 2.

6.3.5 Let p be a point on $\mathcal{Q}_6$. Let γ_0 be a superosculating conic of $\mathcal{Q}_6$ at a point a which passes through p and meets $\mathcal{Q}_6$ six times altogether counting multiplicities. Call the other point b.

<u>Then as</u> t <u>moves monotonically and continuously from</u> a <u>in one direction on</u> $\mathcal{Q}_6$, <u>there is a point</u>

$$u \varepsilon \gamma(t^4, p) \cap \mathcal{Q}_6$$

<u>which moves monotonically and continuously from</u> b <u>in the opposite direction.</u>

<u>Proof.</u> Since $\mathcal{Q}_6$ is of order six, γ_0 meets $\mathcal{Q}_6$ at the points a, b, p and nowhere else. We have again a number of cased depending upon the coincidence of one or more of these points. If t is close to and distinct from a, then $\gamma(t^4, p)$ is close to γ_0, since $\mathcal{Q}_6$ is strongly conically differentiable. Thus $\gamma(t^4, p)$ meets $\mathcal{Q}_6$ at a point u close to b. Also $\gamma(t^4, p)$ can meet $\mathcal{Q}_6$ nowhere else. Thus u depends continuously on t.

It suffices to show that t and u move in opposite directions on $\mathcal{Q}_6$ whenever t is close to a.

<u>(i)</u> All of a, b, p are distinct.

Then γ_0 intersects $\mathcal{Q}_6$ at each of the points b, p and supports $\mathcal{Q}_6$ at a. By <u>6.3.4</u> (iii), if t is close to and distinct from a, $\gamma(t^3, p, b)$ intersects $\mathcal{Q}_6$ at a point q on the opposite side of and close to a. Let r converge from q through a to t. Then

$$\lim_{r \to t} \gamma(r, t^3, p) = \gamma(t^4, p).$$

By 6.3.4 (i), as r moves monotonically and continuously from q through a to t on $\mathcal{a}_6$, there is a point

$$u_r \in \gamma(r, t^3, p) \cap \mathcal{a}_6$$

which moves monotonically and continuously from b in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction from b on $\mathcal{a}_6$ as does t from a.

Remark. Again we notice that the proof of 6.3.5 (i) is analogous to 6.3.4 (i). By using methods similar to those used in 6.3.4 (ii), (iii), (iv), (v), (vi), we obtain 6.3.5 for the cases

(ii) Let $b = p \nmid a$.

(iii) Let $a = b \nmid p$.

(iv) Let $a = p \nmid b$.

(v) Let $a = p = b$.

6.3.6 Let γ_0 be an ultraosculating conic of $\mathcal{a}_6$ at a point a which meets $\mathcal{a}_6$ six times altogether counting multiplicities. Call the other point b. Then as t moves monotonically and continuously from a in one direction on $\mathcal{a}_6$, there is a point

$$u \in \gamma(t^5) \cap \mathcal{a}_6$$

which moves monotonically and continuously from b in the opposite direction.

Proof. Since $\mathcal{a}_6$ is of order six, γ_0 meets $\mathcal{a}_6$ at the points a, b and nowhere else. Since $\mathcal{a}_6$ is strongly conically differentiable, if t is close to a, then $\gamma(t^5)$ is close to γ_0. Thus $\gamma(t^5)$ meets $\mathcal{a}_6$ at a point u close to b. Also $\gamma(t^5)$ meets $\mathcal{a}_6$ nowhere else. Hence u depends continuously on t.

Again it now suffices to show that t and u move in opposite directions on $\mathcal{a}_6$, whenever t is close to a.

(i) Let $a \neq b$.

Then γ_0 intersects $\mathcal{a}_6$ at a and b. By 6.3.5 (iii), if t is distinct from and close to a, $\gamma(t^4, b)$ intersects $\mathcal{a}_6$ at a point q on the opposite side of and close to a. Let r converge from q through a to t on $\mathcal{a}_6$. Then

$$\lim_{r \to t} \gamma(r, t^4) = \gamma(t^5).$$

By 6.3.5 (i), as r moves monotonically and continuously from q through a to t on $\mathcal{O}_6$, there is a point

$$u_r \varepsilon \ \gamma(r, t^4) \cap \mathcal{O}_6$$

which moves monotonically and continuously from b in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction from b on $\mathcal{O}_6$ as t does from a.

(ii) Let $a = b$.

Then γ_0 is the ultraosculating conic of $\mathcal{O}_6$ at a and supports $\mathcal{O}_6$ at this point. By 6.3.5 (v), as t moves monotonically and continuously from a on $\mathcal{O}_6$, then $\gamma(t^4, a)$ intersects $\mathcal{O}_6$ at a point q which moves monotonically and continuously from b in the opposite direction. If t is close to a, then q is close to b. Let r converge from a to t on $\mathcal{O}_6$. Then

$$\lim_{r \to t} \gamma(r, t^4) = \gamma(t^5).$$

By 6.3.5 (i), as r moves monotonically and continuously from a to t on $\mathcal{O}_6$, there is a point

$$u_r \varepsilon \ \gamma(r, t^4) \cap \mathcal{O}_6$$

which moves monotonically and continuously from q in the opposite direction. Thus

$$u = \lim_{r \to t} u_r$$

moves in the opposite direction on $\mathcal{Q}_6$ to that of t.

6.3.7 . We note here that the results 6.3.2 - 6.3.6 can be obtained even if $\mathcal{Q}_6$ is only a conically differentiable convex arc of order six, as long as we add the restriction that $\mathcal{Q}_6$ contains no points of Type 2; cf. 5.3.1. For by Theorem 9 , $\mathcal{Q}_6$ contains only a finite number s_i of singular points, $i = 1, 2, \ldots, n$. Each point $p \neq s_i$ is a strongly conically differentiable point since it is differentiable and ordinary ([10], 6). Also $\mathcal{Q}_6$ is strongly differentiable at s_i from either side, since s_i is not of Type 2; cf. 6.1.12

Thus if t is close to a on $\mathcal{Q}_6$, then

$$\gamma(t, p_1, p_2, p_3, p_4),\ \gamma(t^2, p_1, p_2, p_3),\ \gamma(t^3, p_1, p_2),$$
$$\gamma(t^4, p),\ \gamma(t^5)$$

are close to

$$\gamma(a, p_1, p_2, p_3, p_4),\ \gamma(a^2, p_1, p_2, p_3),\ \gamma(a^3, p_1, p_2),$$
$$\gamma(a^4, p),\ \gamma(a^5),$$

respectively. Hence we obtain the continuity of u. The monotony of u follows exactly as was shown in the proofs of <u>6.3.2</u> - <u>6.3.6</u>.

6.4 Conically Differentiable Curves of Order Six.

Introduction

A curve $\mathcal{C}_6$ of conical order six is either convex or of linear order three; cf. 6.4.1.

If $\mathcal{C}_6$ is convex and strongly conically differentiable, then $\mathcal{C}_6$ contains exactly six conically singular points; cf. S. Mukhopadhyaya [20] and Fr. Fabricius-Bjerre [23]. In the literature the term "sextactic point" is used in this context. Adapting some of the methods of Mukhopadhyaya and using the results of 6.3, this theorem can be extended to a conically differentiable convex curve $\mathcal{C}_6$, if points of Type 2 are not allowed; cf. 6.4.6 and 6.4.7.

Let $\mathcal{C}_6$ be of linear order three. There are four possible types of such a curve as regards number and kind of linearly singular points; cf. O. Haupt and H. Künneth ([12], 3). These cases are listed in 6.4.2. Now if $\mathcal{C}_6$ is also conically differentiable, two of these cases cannot occur; cf. 6.4.2. In 6.4.5 it is shown that the linearly singular points with the characteristic (1,2) and (2,1) are conically singular points, having the conical characteristic (1, 1, 1, 1, 2; 3) and (2, 1, 1, 1, 1; 2), respectively.

It is well known that a strongly conically differentiable curve $\mathcal{C}_6$ of linear order three contains exactly six singular points; cf. Fr. Fabricius-Bjerre [23]. In 6.4.9 and 6.4.10 this result is extended to a conically differentiable curve $\mathcal{B}_6$ of linear order three that contains three inflection points.

If the curve $\mathcal{B}_6$ of linear order three is only conically differentiable then it is possible for $\mathcal{B}_6$ to have only two linearly singular points; cf. 6.4.2 (b). In this case $\mathcal{B}_6$ contains exactly four conically singular points; cf. 6.4.11 and 6.4.12.

6.4.1 One kind of degenerate conic is the double line; cf. 4.1. Let $\mathcal{B}_6$ be a curve of conical order six. Then

$\mathcal{B}_6$ has linear order at most three.

Otherwise, a line meeting $\mathcal{B}_6$ n times (n > 3), considered as a conic will meet $\mathcal{B}_6$ 2n times counting multiplicities. Hence either

(i) $\mathcal{B}_6$ is convex

or

(ii) $\mathcal{B}_6$ is of linear order three.

6.4.2 However, we already know the structure of curves $\mathcal{B}$ of linear order three in the projective plane ([12], 3). There are four possibilities.

(a) $\mathcal{B}$ is decomposed into three convex arcs by three linearly singular points; namely, three points of inflection. Thus all of the other points of $\mathcal{B}$ are linearly ordinary (see Figure 16). If $\mathcal{B}$ is linearly differentiable, these points have the linear characteristic (1, 2) and (1, 1), respectively, [15].

(b) $\mathcal{B}$ is decomposed into two convex arcs by two linearly singular points; namely, a corner shaped like a thorn or a cusp of the first kind and an inflection point. Then all of the other points of $\mathcal{B}$ are linearly ordinary (see Figure 17). If $\mathcal{B}$ is linearly differentiable, these points have the linear characteristic (2,1), (1, 2) and (1, 1), respectively.

(c) $\mathcal{B}$ is decomposed into two convex arcs by two linearly singular points; namely, a corner shaped like a thorn or a cusp of the first kind and a corner shaped like a beak. Thus all of the other points of $\mathcal{B}$ are linearly ordinary (see Figure 18). If $\mathcal{B}$ is linearly differentiable, then a beak shaped corner will be a cusp of the second kind with the linear characteristic (2,2). But then such a point is of linear order at least four ([15], 4.1). This is impossible, since $\mathcal{B}$ is of linear order three. Thus case (c) cannot occur if $\mathcal{B}$ is linearly differentiable.

(d) $\mathcal{B}$ is decomposed into three convex arcs by three beak-like corners. Then all other points of $\mathcal{B}$ are linearly ordinary (see Figure 19). If $\mathcal{B}$ is linearly differentiable, these singular points would be cusps of the second kind and they would have the characteristic (2, 2). Hence as before, case (d) cannot occur.

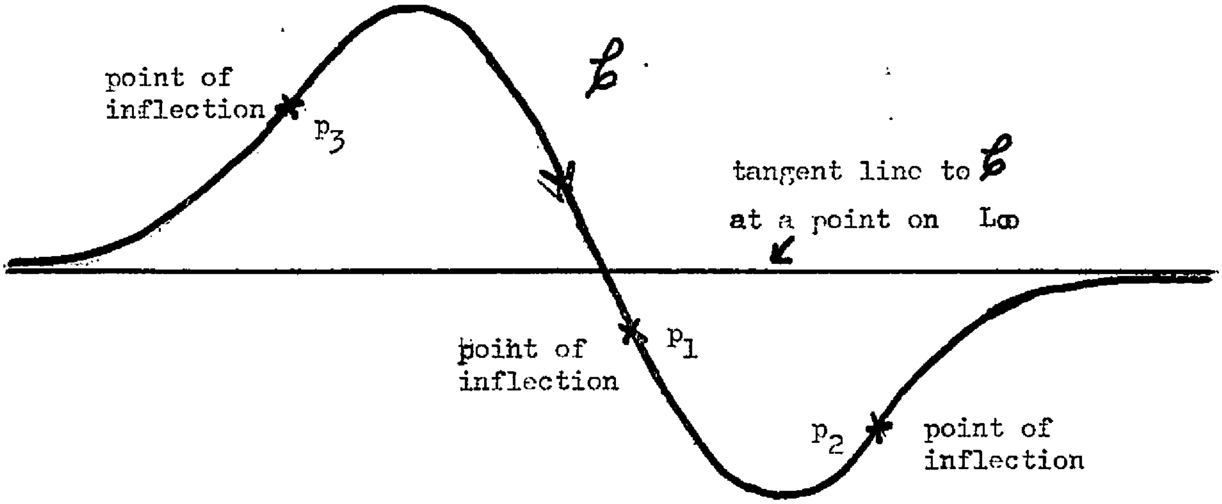

point of
inflection
p_3
$\mathcal{b}$
tangent line to $\mathcal{b}$
at a point on L_∞
point of
inflection
p_1
p_2
point of
inflection

Figure 16

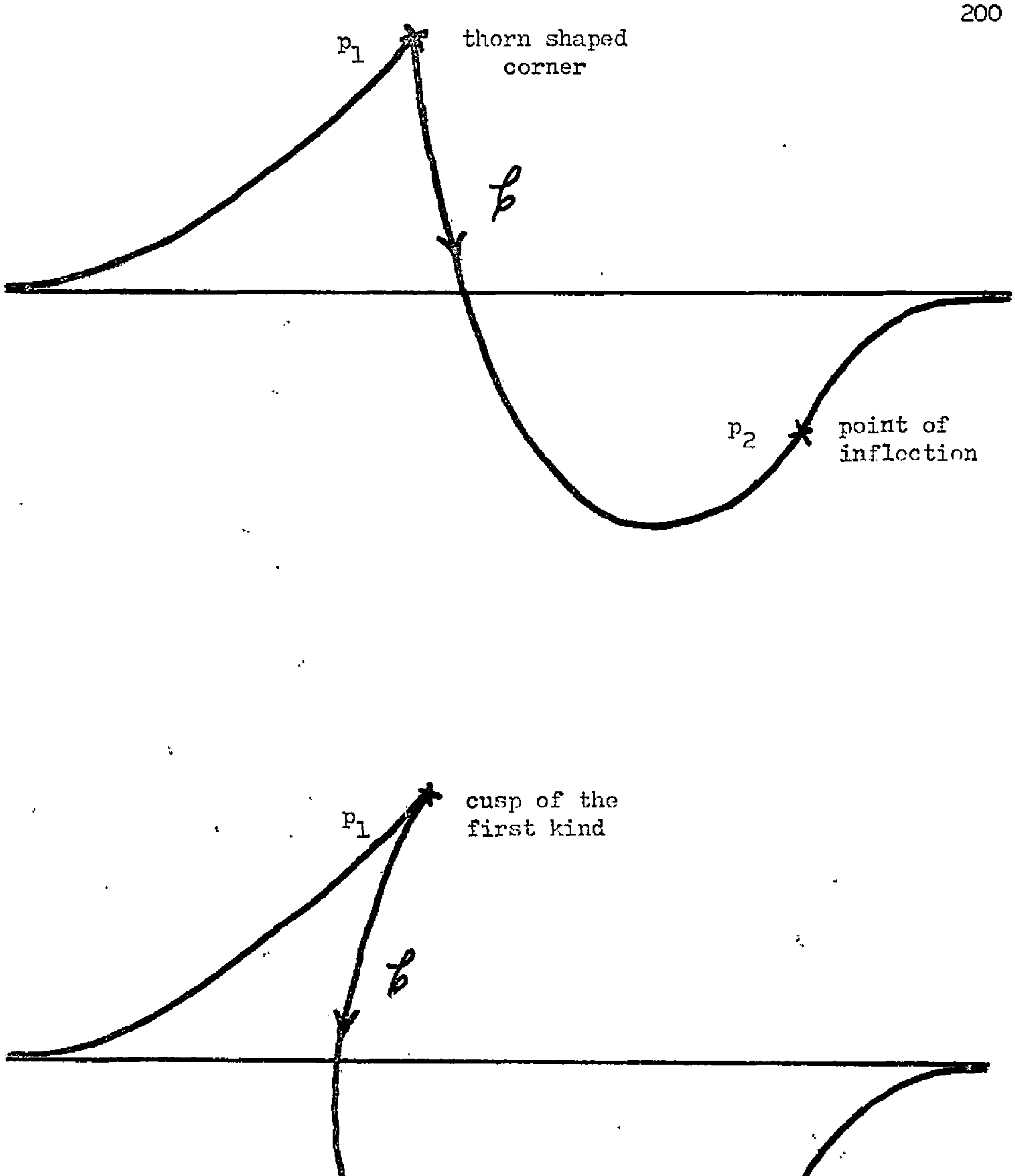

p1
thorn shaped corner
p2
point of inflection
p1
cusp of the first kind
p2
point of inflection

Figure 17

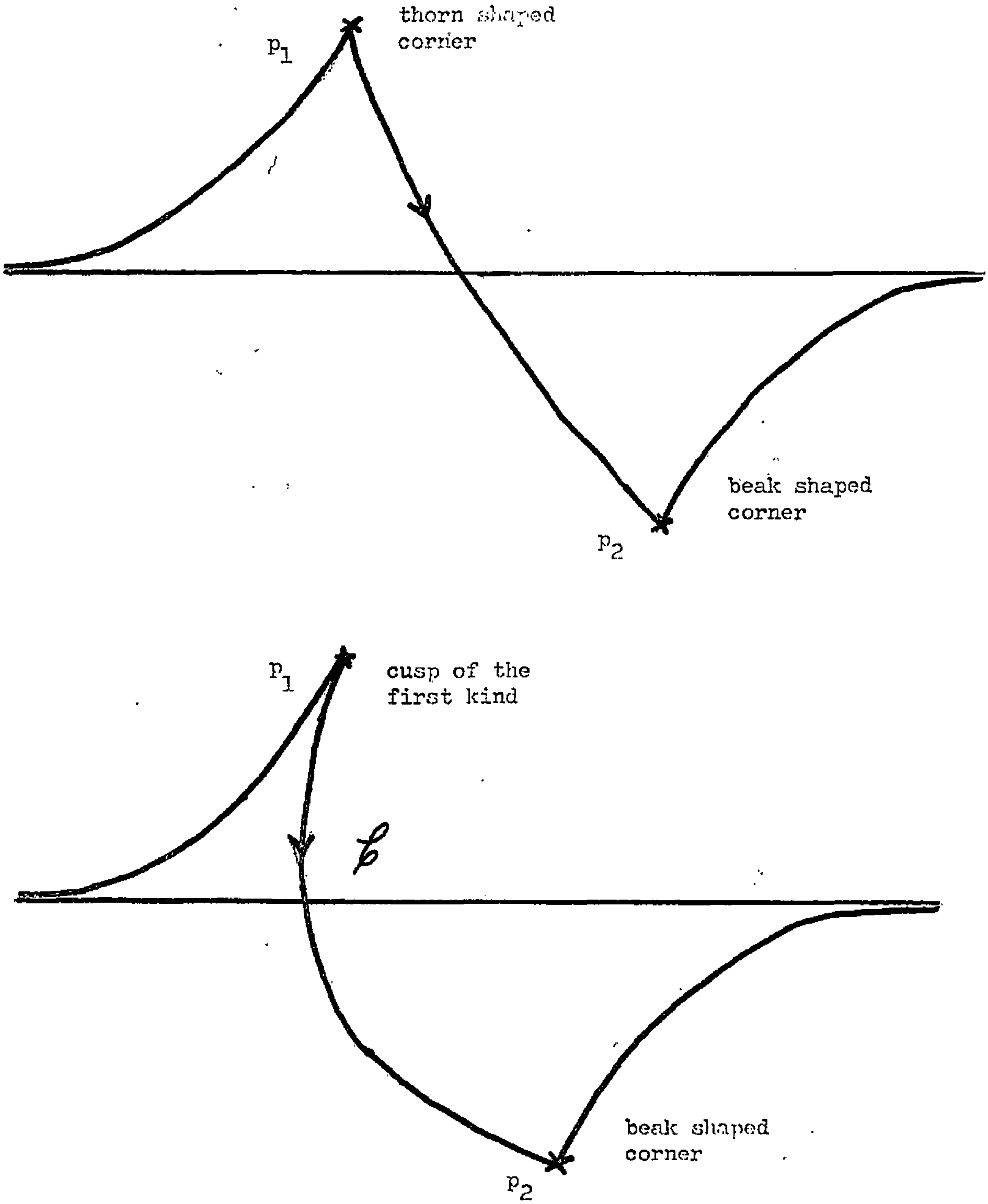
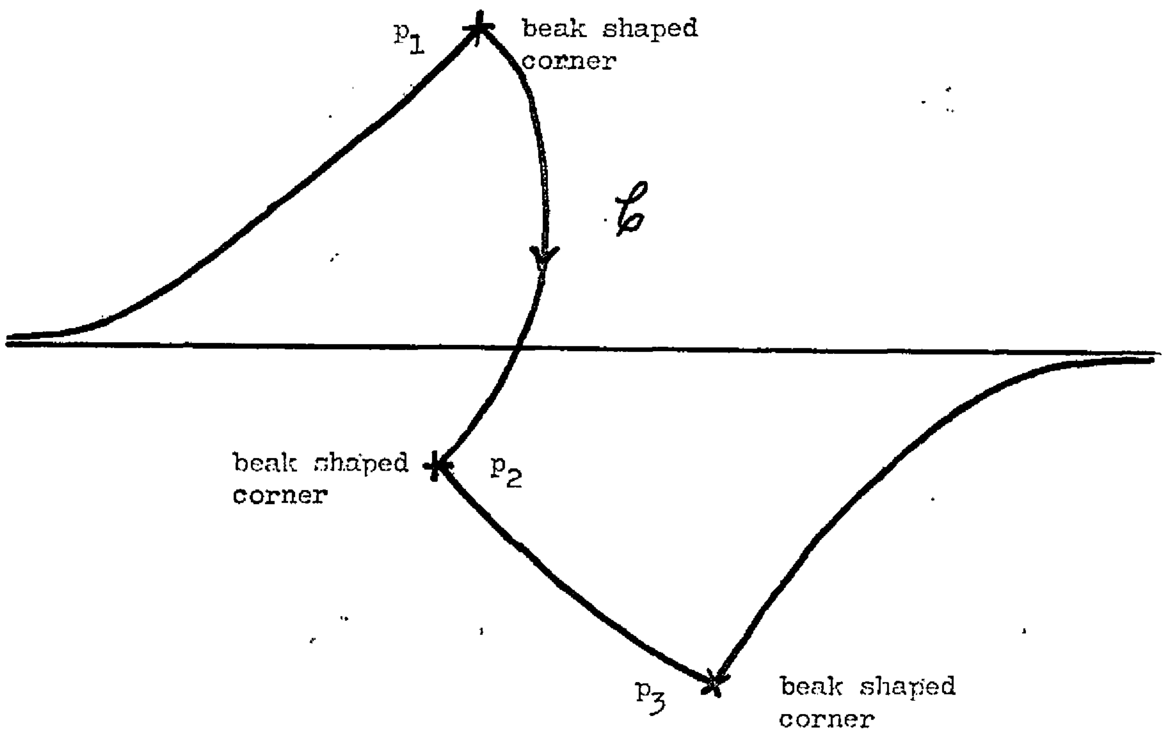

thorn shaped corner
p_1
beak shaped corner
p_2
p_1
cusp of the first kind
beak shaped corner
p_2

Figure 18



p_1
beak shaped corner
$\mathcal{C}$
beak shaped corner
p_2
p_3
beak shaped corner

Figure 19

6.4.3 *Let $\mathcal{B}_6$ be a curve of conical order six. Then $\mathcal{B}_6$ contains at most finitely many singular points.*

Proof. If $\mathcal{B}_6$ is convex, then by *Theorem 9*, $\mathcal{B}_6$ contains only finitely many singular points.

If $\mathcal{B}_6$ is of linear order three, then $\mathcal{B}_6$ is the union of either two or three convex arcs, by 6.4.2. But then each of these convex arcs contains only finitely many singular points. Thus $\mathcal{B}_6$ contains at most finitely many singular points.

6.4.4 Next we introduce a concept of monotony of an arc $\mathcal{A}$ in the conical case analogous to that which was used in 3.3.12 for the circular case. We shall denote a general ultraosculating conic of $\mathcal{A}$ at a point p by $\gamma(p)$.

$\mathcal{A}$ is said to be *monotone* if $\mathcal{A}$ induces a unique orientation on the general ultraosculating circles at each point of $\mathcal{A}$ such that if $p < q$ on $\mathcal{A}$,

$$\gamma(p) \subset \gamma(q)_i \quad \text{and} \quad \gamma(q) \subset \gamma(p)_e$$

or

$$\gamma(p) \subset \gamma(q)_e \quad \text{and} \quad \gamma(q) \subset \gamma(p)_i .$$

Again we have results which are analogous to those in 3.3.12, (i) and (ii).

(i) Arcs of conical order five are monotone ([10], 7).

(ii) Suppose that each interior point of an arc $\mathcal{A}_6$ of conical order six is ordinary. Then the closed arc $\bar{\mathcal{A}}_6$ is monotone.

Proof. Each interior point of $\mathcal{A}_6$ is ordinary. Also the end-points of $\mathcal{A}_6$ are ordinary, by 6.1.13 (a). Hence each [interior] point of $\bar{\mathcal{A}}_6$ possesses a [two-sided] neighbourhood of conical order five. But each of these neighbourhoods is monotone; by (i). By taking the union of these neighbourhoods one obtains the monotony of $\bar{\mathcal{A}}_6$.

6.4.5 Let us restrict our attention in the rest of Section 6.4 to a conically differentiable curve $\mathcal{B}_6$ of conical order six.

By 6.4.3, $\mathcal{B}_6$ contains only finitely many singular points. Thus each singular point is elementary; cf. 5.5.

Thus by 5.5 (iii), each singular point of $\mathcal{B}_6$ has exactly one "2" in its characteristic; then each of the other digits is "1".

Also, if a curve intersects its tangent at a conically differentiable point p, then the osculating conics of the curve at p are degenerate ([9], Theorem 4). Thus the ultraosculating conic of the curve at p is the double line on the tangent of the curve at p.

Hence, for a conically differentiable curve $\mathcal{B}_6$ of conical order six, we have the following:

(i) An inflection point with linear characteristic (1, 2) has the conical characteristic (1, 1, 1, 1, 2; 3).

(ii) A cusp of the first kind with linear characteristic (2, 1) has the conical characteristic (2, 1, 1, 1, 1; 2).

6.4.6 Let $\mathcal{B}_6$ be a convex curve which has no points of Type 2. Then $\mathcal{B}_6$ contains at least six conically singular points.

Proof. Since $\mathcal{B}_6$ is of order six, there exists a conic γ_0 which intersects $\mathcal{B}_6$ six times. γ_0 is non-degenerate owing to the convexity of $\mathcal{B}_6$. Let this six consectutive points be p_i, $i = 0, 1, \ldots, 5$.

(i). Let $\gamma p_2p_3p_4p_5$ be the family of conics through the points p_2, p_3, p_4, p_5. Then

$$\gamma_0 \in \gamma p_2p_3p_4p_5.$$

Keeping p_2, p_3, p_4, p_5 fixed, by 6.3.2 and 6.3.7, as t moves monotonically and continuously from p_0 toward p_1 on $\mathcal{B}_6$, there is a point

$$u \in \gamma(t, p_2, p_3, p_4, p_5) \cap \mathcal{B}_6$$

which moves monotonically and continuously from p_1 toward p_0 in the opposite direction on $\mathcal{B}_6$. Hence they must coincide at some point $p_{0,1}$ between p_0 and p_1 on $\mathcal{B}_6$. This point $p_{0,1}$ is a $\gamma_{p_2p_3p_4p_5}$-singular point; cf. 6.1.8, and

$$\gamma(p_{0,1}^{\;2}, p_3, p_4, p_5)$$

is the tangent conic of $\mathcal{B}_6$ at $p_{0,1}$ passing through the points p_2, p_3, p_4, p_5.

In this manner, by considering $\gamma_{p_ip_{i+1}p_{i+2}p_{i+3}}$ we obtain a point $p_{i+4,\, i+5}$ between p_{i+4} and p_{i+5} which is $\gamma_{p_ip_{i+1}p_{i+2}p_{i+3}}$-singular and

$$\gamma(p_{i+4,\, i+5}^{\;2}, p_{i+1}, p_{i+2}, p_{i+3})$$

is the tangent conic of $\mathcal{B}_6$ at $p_{i+4,\, i+5}$ passing through the points $p_i, p_{i+1}, p_{i+2}, p_{i+3}$; $i = 0, \ldots, 5$. Here the subscripts are to be interpreted modulo 6.

(2) Let $\gamma_{p_3p_4p_5}$ be the subfamily of conics of $\gamma_{p_2p_3p_4p_5}$ passing through the points p_3, p_4, p_5. Now

$$\gamma(p_{0,1}^2, p_3, p_4, p_5) \in \gamma_{p_3p_4p_5}$$

and intersects $\mathcal{B}_6$ at p_2. Also

$$\gamma(p_{1,2}^2, p_4, p_5, p_0) \in \gamma_{p_3p_4p_5}.$$

Let t move monotonically and continuously from $p_{0,1}$ on $\mathcal{B}_6$ toward $p_{1,2}$. Then by 6.3.3 and 6.3.7, there is a point

$$u \in \gamma(t^2, p_3, p_4, p_5) \cap \mathcal{B}_6$$

which moves monotonically and continuously from p_2 toward $p_{0,1}$ in the opposite direction on $\mathcal{B}_6$.

Suppose t reaches $p_{1,2}$ before u does. Then we obtain a tangent conic at $p_{1,2}$ passing through the points u, p_3, p_4, p_5. However, $\gamma(p_{1,2}^2, p_4, p_5, p_0)$ also meets $\mathcal{B}_6$ at p_3. Thus

$$\gamma(p_{1,2}^2, p_3, p_4, p_5) = \gamma(p_{1,2}^2, p_4, p_5, p_0)$$

and this conic would then meet $\mathcal{B}_6$ at least seven times, counting multiplicities; contradiction. Thus u and t coincide at a point $p_{0,1,2}$ between $p_{0,1}$ and $p_{1,2}$ on $\mathcal{B}_6$. This point $p_{0,1,2}$ is a $\gamma_{p_3p_4p_5}$-singular point; cf. 6.1.7 and

$$\gamma(p_{0,1,2}^{3},\ p_4,\ p_5)$$

is the osculating conic of $\mathcal{B}_6$ at $p_{0,1,2}$ passing through the points p_3, p_4 and p_5.

In this manner, by considering $\gamma_{p_i p_{i+1} p_{i+2}}$ we obtain a point $p_{i+3,i+4,i+5}$ between $p_{i+5,i+4}$ and $p_{i+4,i+5}$ which is $\gamma_{p_i p_{i+1} p_{i+2}}$-singular and

$$\gamma(p_{i+3,i+4,i+5}^{3},\ p_{i+1},\ p_{i+2})$$

is the osculating conic of $\mathcal{B}_6$ at $p_{i+3,i+4,i+5}$ passing through the points p_i, p_{i+1} and p_{i+2}; $i = 0, \ldots, 5$.

<u>(3)</u> Let $\gamma_{p_4 p_5}$ be the subfamily of conics of $\gamma_{p_3 p_4 p_5}$ passing through the points p_4 and p_5. Now

$$\gamma(p_{0,1,2}^{3},\ p_4,\ p_5) \in \gamma_{p_4 p_5}$$

and intersects $\mathcal{B}_6$ at p_3. Also

$$\gamma(p_{1,2,3}^{3},\ p_5,\ p_0) \in \gamma_{p_4 p_5}.$$

Let t move monotonically and continuously from $p_{0,1,2}$ on $\mathcal{B}_6$ toward $p_{1,2,3}$. Then by <u>6.3.4</u> and <u>6.3.7</u>, there is a point

$$u \varepsilon \ \gamma(t^3, p_4, p_5) \cap \mathscr{B}_6$$

which moves monotonically and continuously from P_3 toward $p_{0,1,2}$ in the opposite direction on $\mathscr{B}_6$.

As in (2), t cannot reach $p_{1,2,3}$ before u does. Thus u and t coincide at a point $p_{0,1,2,3}$ between $p_{0,1,2}$ and $p_{1,2,3}$ on $\mathscr{B}_6$. This point $p_{0,1,2,3}$ is a $\gamma_{p_4 p_5}$-singular point, cf. 6.1.6 and

$$\gamma(p_{0,1,2,3}^{\ 4}, p_5)$$

is the superosculating conic of $\mathscr{B}_6$ at $p_{0,1,2,3}$ passing through the points p_4 and p_5.

In this manner, by considering $\gamma_{p_i p_{i+1}}$ we obtain a point $p_{i+2,i+3,i+4,i+5}$ between $p_{i+2,i+3,i+4}$ and $p_{i+3,i+4,i+5}$ which is $\gamma_{p_i p_{i+1}}$-singular and

$$\gamma(p_{i+2,i+3,i+4,i+5}^{\ 4}, p_{i+1})$$

is the superosculating conic of $\mathscr{B}_6$ at $p_{i+2,i+3,i+4,i+5}$ passing through the points p_i and p_{i+1}; $i = 0, \ldots, 5$.

(4) Let γ_{p_5} be the subfamily of conics of $\gamma_{p_4 p_5}$ passing through the point p_5. Now

$$\gamma(p_{0,1,2,3}{}^{4}, p_5) \in \gamma_{p_5}$$

and intersects $\mathscr{B}_6$ at p_4. Also

$$\gamma(p_{1,2,3,4}{}^{4}, p_0) \in \gamma_{p_5}$$

Let t move monotonically and continuously from $p_{0,1,2,3}$ on $\mathscr{B}_6$ toward $p_{1,2,3,4}$. Then by 6.3.5 and 6.3.7, there is a point

$$u \in \gamma(t^4, p_5) \cap \mathscr{B}_6$$

which moves monotonically and continuously from p_4 toward $p_{0,1,2,3}$ in the opposite direction on $\mathscr{B}_6$.

Again as in (2), t cannot reach $p_{1,2,3,4}$ before u does. Thus u and t coincide at a point $p_{0,1,2,3,4}$ between $p_{0,1,2,3}$ and $p_{1,2,3,4}$ on $\mathscr{B}_6$. This point $p_{0,1,2,3,4}$ is a γ_{p_5}-singular point; cf. 6.1.5 and

$$\gamma(p_{0,1,2,3,4}{}^{5})$$

is the ultraosculating conic of $\mathscr{B}_6$ at $p_{0,1,2,3,4}$ passing through the point p_5.

In this manner, by considering γ_{p_i} we obtain a point $p_{i+1,i+2,i+3,i+4,i+5}$ between $p_{i+1,i+2,i+3,i+4}$ and $p_{i+2,i+3,i+4,i+5}$

on $\mathcal{B}_6$ which is γ_{p_i}-singular and

$$\gamma(p^{5}_{i+1,i+2,i+3,i+4,i+5})$$

is the ultraosculating conic of $\mathcal{B}_6$ at $p_{i+1,i+2,i+3,i+4,i+5}$ passing through p_i; $i = 0, \ldots, 5$.

(5) Now $\gamma(p_{0,1,2,\overset{5}{3},4})$ intersects $\mathcal{B}_6$ at p_5 and $\gamma(p_{1,2,3,\overset{5}{4},5})$ intersects $\mathcal{B}_6$ at p_0. Let t move monotonically and continuously from $p_{0,1,2,3,4}$ on $\mathcal{B}_6$ toward $p_{1,2,3,4,5}$. Then by 6.3.6 and 6.3.7, there is a point,

$$u \in \gamma(t^5) \cap \mathcal{B}_6$$

which moves monotonically and continuously from p_5 toward $p_{0,1,2,3,4}$ in the opposite direction on $\mathcal{B}_6$.

Again t cannot reach $p_{1,2,3,4,5}$ before u does. Thus u and t coincide at a point $p_{0,1,2,3,4,5}$ between $p_{0,1,2,3,4}$ and $p_{1,2,3,4,5}$ on $\mathcal{B}_6$. This point is a singular point and

$$\gamma(p^{5}_{0,1,2,3,4,5})$$

is the ultraosculating conic of $\mathcal{B}_6$ at $p_{0,1,2,3,4,5}$.

In this way we obtain a singular point $p_{i,i+1,i+2,i+3,i+4,i+5}$

between $p_{i,i+1,i+2,i+3,i+4}$ and $p_{i+1,i+2,i+3,i+4,i+5}$ on $\mathcal{B}_6$; $i = 0, \ldots, 5$. Thus $\mathcal{B}_6$ contains at least six singular points.

<u>6.4.7</u> <u>Let</u> $\mathcal{B}_6$ <u>be a conically differentiable convex curve with no points of Type 2. Then</u> $\mathcal{B}_6$ <u>contains at most six conically singular points.</u>

<u>Proof.</u> Suppose that $\mathcal{B}_6$ contains at least seven conically singular points, say $s_1 < s_2 < \ldots < s_7$ on $\mathcal{B}_6$.

<u>(1)</u> Let p_1 be any point on the open arc of $\mathcal{B}_6$ between s_7 and s_1. Now s_1 is a conically singular point. Thus $\gamma(s_1^5)$ meets $\mathcal{B}_6$ nowhere else. By <u>6.3.6</u> and <u>6.3.7</u>, as t moves monotonically and continuously from s_1 toward s_2 on $\mathcal{B}_6$, $\gamma(t^5)$ meets $\mathcal{B}_6$ at a point u which moves monotonically and continuously from s_1 in the opposite direction. But s_2 is also a singular point. Thus as t converges to s_2, u converges to s_2. Hence there exists a point s_{12} between s_1 and s_2 on $\mathcal{B}_6$ such that $\gamma(s_{12}^5)$ meets $\mathcal{B}_6$ at p_1. This point s_{12} is a γ_{p_1}-singular point.

Similarly we obtain points s_{23} between s_2 and s_3, s_{34} between s_3 and s_4, s_{45} between s_4 and s_5, s_{56} between s_5 and s_6 and s_{67} between s_6 and s_7 which are γ_{p_1}-singular points.

(2) Let p_2 be any point on the open arc of $\mathcal{B}_6$ between s_{67} and s_{12}, $p_2 \neq p_1$. Now

$$\gamma(s_{12}^5) = \gamma(s_{12}^4, p_1)$$

meets $\mathcal{B}_6$ at s_{12}, p_1 and nowhere else. By 6.3.5 and 6.3.7, as t moves monotonically and continuously from s_{12} toward s_{23} on $\mathcal{B}_6$, $\gamma(t^4, p_1)$ meets $\mathcal{B}_6$ at a point u which moves monotonically and continuously from s_{12} in the opposite direction. But s_{23} is also a γ_{p_1}-singular point. Thus as t converges to s_{23}, u converges to s_2. Hence there exists a point s_{123} between s_{12} and s_{23} on $\mathcal{B}_6$ such that $\gamma(s_{123}^4, p_1)$ meets $\mathcal{B}_6$ at p_2. This point s_{123} is a $\gamma_{p_1p_2}$-singular point.

Similarly, we obtain points s_{234} between s_{23} and s_{34}, s_{345} between s_{34} and s_{45}, s_{456} between s_{45} and s_{56} and s_{567} between s_{56} and s_{67} which are $\gamma_{p_1p_2}$-singular points.

(3) Let p_3 be any point on the open arc of $\mathcal{B}_6$ between s_{567} and s_{123}, distinct from p_1 and p_2. Now

$$\gamma(s_{123}^4, p_1) = \gamma(s_{123}^3, p_1, p_2)$$

meets $\mathcal{B}_6$ at s_{123}, p_1, p_2 and nowhere else. By 6.3.4 and 6.3.7, as t moves monotonically and continuously from s_{123} toward s_{234}

on $\mathcal{B}_6$, $\gamma(t^3, p_1, p_2)$ meets $\mathcal{B}_6$ at a point u which moves monotonically and continuously from s_{123} in the opposite direction. But s_{234} is also a $\gamma_{p_1p_2}$-singular point. Thus as t converges to s_{234}, u converges to s_{234}. Hence there exists a point s_{1234} between s_{123} and s_{234} on $\mathcal{B}_6$ such that $\gamma(s_{1234}^3, p_1, p_2)$ meets $\mathcal{B}_6$ at p_3. This point s_{1234} is a $\gamma_{p_1p_2p_3}$-singular point.

Similarly we obtain points s_{2345} between s_{234} and s_{345}, s_{3456} between s_{345} and s_{456} and s_{4567} between s_{456} and s_{567} which are $\gamma_{p_1p_2p_3}$-singular points.

<u>(4)</u> Let p_4 be any point on the open arc of $\mathcal{B}_6$ between s_{4567} and s_{1234}, distinct from p_1, p_2 and p_3. Now

$$\gamma(s_{1234}^3, p_1, p_2) = \gamma(s_{1234}^2, p_1, p_2, p_3)$$

meets $\mathcal{B}_6$ at s_{1234}, p_1, p_2, p_3 and nowhere else. By <u>6.3.3</u> and <u>6.3.7</u>, as t moves monotonically and continuously from s_{1234} toward s_{2345} on $\mathcal{B}_6$, $\gamma(t^2, p_1, p_2, p_3)$ meets $\mathcal{B}_6$ at a point u which moves monotonically and continuously from s_{1234} in the opposite direction. But s_{2345} is also a $\gamma_{p_1p_2p_3}$-singular point. Thus as t converges to s_{2345}, u converges to s_{2345}. Hence there exists a point s_{12345} between s_{1234} and s_{2345} on $\mathcal{B}_6$ such that $\gamma(s_{12345}, p_1, p_2, p_3)$ meets $\mathcal{B}_6$ at p_4. This point s_{12345} is a $\gamma_{p_1p_2p_3p_4}$-singular point.

Similarly, we obtain points s_{23456} between s_{2345} and s_{3456} and s_{34567} between s_{3456} and s_{4567} which are $\gamma_{p_1p_2p_3p_4}$-singular points.

<u>(5)</u> Let p_5 be any point on the open arc between s_{34567} and s_{12345}, distinct from p_1, p_2, p_3 and p_4. Now

$$\gamma(s^2_{12345}, p_1, p_2, p_3) = \gamma(s_{12345}, p_1, p_2, p_3, p_4)$$

meets $\mathcal{B}_6$ at $s_{12345}, p_1, p_2, p_3, p_4$ and nowhere else. By <u>6.3.2</u> and <u>6.3.7</u>, as t moves monotonically and continuously from s_{12345} toward s_{23456} on $\mathcal{B}_6$, $\gamma(t, p_1, p_2, p_3, p_4)$ meets $\mathcal{B}_6$ at a point u which moves monotonically and continuously from s_{12345} in the opposite direction. By s_{23456} is also a $\gamma_{p_1p_2p_3p_4}$-singular point. Thus as t converges to s_{23456}, u converges to s_{23456}. Hence there exists a point s_{123456} between s_{12345} and s_{23456} on $\mathcal{B}_6$ such that $\gamma(s_{123456}, p_1, p_2, p_3, p_4)$ meets $\mathcal{B}_6$ at p_5.

Similarly we obtain a point s_{234567} between s_{23456} and s_{34567} on $\mathcal{B}_6$ such that $\gamma(s_{234567}, p_1, p_2, p_3, p_4)$ meets $\mathcal{B}_6$ at p_5. But then

$$\gamma(s_{123456}, p_1, p_2, p_3, p_4) = \gamma(s_{234567}, p_1, p_2, p_3, p_4)$$

and this conic meets $\mathcal{B}_6$ at least seven times; contradiction.

Thus $\mathscr{B}_6$ contains at most six conically singular points.

<u>6.4.8</u> Let $\mathscr{B}_6$ be of linear order three. Now $\mathscr{B}_6$ satisfies Condition PI at each point since it is conically differentiable; cf. <u>5.3.1</u>. But as was pointed out in <u>6.4.2</u>, a cusp of the second kind has the linear characteristic (2, 2) and then such a point is of linear order at least four, which is impossible. Thus cases (c) and (d) of <u>6.4.2</u> cannot occur.

<u>6.4.9</u> <u>Let case</u> (a) <u>of</u> 6.4.2 <u>occur. Then</u> $\mathscr{B}_6$ <u>contains at least six conically singular points.</u>

<u>Proof.</u> Let p_1, p_2, p_3 be the three inflection points of $\mathscr{B}_6$. By <u>6.4.5</u> (i), each of these points is a conically singular point with the conical characteristic (1, 1, 1, 1, 2; 3) and the ultraosculating conic $\gamma(p_i^5)$ of $\mathscr{B}_6$ at p_i is the double line on the tangent $\mathscr{T}_{p_i}$ of $\mathscr{B}_6$ at p_i; i = 1, 2, 3. We note that

$$\gamma(p_1^5) \cap \gamma(p_2^5) \neq \emptyset$$

since two distinct lines in the projective plane intersect.

Now suppose that there are no conically singular points on the convex open arc p_1p_2 between p_1 and p_2 on $\mathscr{B}_6$. Then the closed arc p_1p_2 is monotone, by <u>6.4.4</u>(ii). In particular

$$\gamma(p_1^5) \cap \gamma(p_2^5) = \emptyset.$$

This is a contradiction. Hence we obtain the existence of a conically singular point q_1 on p_1p_2.

Similarly, there exist conically singular points q_2, q_3 on the open arcs p_2p_3, p_3p_1 of $\mathcal{B}_6$, respectively. We conclude that $\mathcal{B}_6$ contains at least six conically singular points, if case (a) of 6.4.2 occurs.

6.4.10 Let case (a) of 6.4.2 occur. Then $\mathcal{B}_6$ contains at most six conically singular points.

Proof. Suppose that $\mathcal{B}_6$ contains at least seven conically singular points. Then as in 6.4.9, the three inflection points p_1, p_2, p_3 are singular. Withoug loss of generality, there are at least two conically singular points $q_1 < q_2$ on the convex open arc p_1p_2 between p_1 and p_2 on $\mathcal{B}_6$. We may assume, by taking another line as L_∞, if necessary, that the arc $\overline{p_1p_2}$ does not meet L_∞. The tangent line supports $\mathcal{B}_6$ at each point p of p_1p_2 and hence lies locally to the right of $\mathcal{B}_6$ (with the exception of p) at p, say.

Next, we note that at no interior point of the arc p_1p_2 is the ultraosculating conic the double line on the tangent at that particular point. Otherwise, $\mathcal{B}_6$ being of odd linear order, the tangent line at such a point must meet $\mathcal{B}_6$ with an odd multiplicity. But the tangent supports $\mathcal{B}_6$ at this point and hence must intersect $\mathcal{B}_6$ at exactly one other point. But then the ultraosculating conic, being the double line on the tangent, meets $\mathcal{B}_6$ more than six times counting multiplicities; contradiction. Thus the characteristic of the points q_1, q_2 is either (1, 1, 1, 1, 2; 1a(i)) or (1, 1, 1, 1, 2; 1a(ii)); cf. 5.4.

We now show that the assumption of at least two singular points q_1, q_2 on the open arc p_1p_2 would imply the existence of a singular point q on p_1p_2 such that $\gamma(q^5)$ lies locally to the right of $\mathcal{B}_6$ (with the exception of q) at q. If either $\gamma(q_1^5)$ or $\gamma(q_2^5)$ lie locally to the right of $\mathcal{B}_4$ at q_1 or q_2, respectively, then we have such a q. Hence we can assume that neither q_1 nor q_2 have the desired property. Then we have the following three possibilities:

(a) $\gamma(q_1^5)$ and $\gamma(q_2^5)$ are both non-degenerate and lie locally to the left of $\mathcal{B}_6$ at q_1 and q_2, respectively (see Figure 20 (a));

(b) one of $\gamma(q_1^5)$, $\gamma(q_2^5)$ say $\gamma(q_2^5)$ is non-degenerate and lies locally to the left of $\mathcal{B}_6$ at q_2, while $\gamma(q_1^5) = q_1$ (see Figure 20 (b)); or

(c) $\gamma(q_1^5) = q_1$ and $\gamma(q_2^5) = q_2$ (see Figure 20 (c)).

But now we claim the existence of a new singular point $\bar{q}$ on the open arc q_1q_2. Otherwise, q_1q_2 is monotone by 6.4.4 (ii); i.e.,

$$\gamma(q_1^5) \subset \gamma(q_2^5)_i \quad \text{and} \quad \gamma(q_2^5) \subset \gamma(q_1^5)_e$$

or

$$\gamma(q_1^5) \subset \gamma(q_2^5)_e \quad \text{and} \quad \gamma(q_2^5) \subset \gamma(q_1^5)_i.$$

In particular,

$$\left.\begin{array}{l} q_1 \in \gamma(q_2^5)_i \quad \text{and} \quad q_2 \in \gamma(q_1^5)_e \\ \text{or} \\ q_1 \in \gamma(q_2^5)_e \quad \text{and} \quad q_2 \in \gamma(q_1^5)_i. \end{array}\right\} \quad (*)$$

Now if t_1, u_1, v_1 $[t_2, u_2, v_2]$ are close to q_1 $[q_2]$ on p_1, p_2, then

$$\gamma(q_1^2, t_1, u_1, v_1) \; [\, \gamma(q_2^2, t_2, u_2, v_2)]$$

and the arc p_1p_2 touches the tangent $\mathfrak{T}_{q_1}$ $[\mathfrak{T}_{q_2}]$ at q_1 $[q_2]$ from the same side ([11], 4.3) and hence lies to the left of $\mathfrak{T}_{q_1}$ $[\mathfrak{T}_{q_2}]$. By letting t_1, u_1, v_1 $[t_2, u_2, v_2]$ converge to q_1 $[q_2]$ the limit conic $\gamma(q_1^5)$ $[\gamma(q_2^5)]$ and the arc p_1p_2 touches the tangent $\mathfrak{T}_{q_1}$ $[\mathfrak{T}_{q_2}]$ from the same side; i.e., to the left of $\mathfrak{T}_{q_1}$ $[\mathfrak{T}_{q_2}]$. But the convex arc p_1p_2 induces a natural orientation of $\gamma(q_1^5)$ [$\gamma(q_2^5)$] with the result that $\mathscr{B}_6$ lies locally to the right of $\gamma(q_1^5)$ [$\gamma(q_2^5)$] as $\gamma(q_1^5)$ [$\gamma(q_2^5)$] was assumed to lie locally to the left of $\mathscr{B}_6$. But $\gamma(q_1^5)$ [$\gamma(q_2^5)$] does not meet $\mathscr{B}_4$ again. Thus

$$\mathscr{B}_6 \setminus \{q_1\} \subset \gamma(q_1^5)_e \quad [\mathscr{B}_6 \setminus \{q_2\} \subset \gamma(q_2^5)_e$$

If $\gamma(q_1^5) = q_1$ [$\gamma(q_2^5) = q_2$], then

$$\gamma(q_1^5)_i = \emptyset \; [\; \gamma(q_2^5)_i = \emptyset].$$

In particular, regardless of cases <u>(a)</u>, <u>(b)</u> or <u>(c)</u> we have

$$q_2 \in \gamma(q_1^5)_e \text{ and } q_1 \in \gamma(q_2^5)_e.$$

This contradicts (*). Hence we obtain the existence of a singular point $\overline{q}$ on the open arc q_1q_2. If $\overline{q}$ is such that $\gamma(\overline{q}^5)$ lies locally to the right of $\mathscr{B}_6$ (with the exception of $\overline{q}$) at $\overline{q}$, then we have the required singular point of the third paragraph. If $\overline{q}$ is not such a point, then the above method will yield a singular point between q_1 and $\overline{q}$ on p_1p_2. By repeating this argument, if necessary, we obtain a singular point q of the desired type or we obtain an infinite sequence of singular points. This last possibility cannot occur, by <u>Theorem 9</u>. Thus we obtain a conically singular point q on the open arc p_1p_2 with the property that $\gamma(q^5)$ is non-degenerate and lies locally to the right of $\mathscr{B}_6$ (with the exception of q) at q (see Figure 20 (d)).

Now in the same manner as was shown for each of the points q_1, q_2, $\gamma(q^5)$ lies locally to the left of the tangent $\mathcal{T}_q$ of $\mathcal{B}_6$ at q. But $\gamma(q^5)$ lies locally to the right of $\mathcal{B}_6$ and $\mathcal{T}_q$ supports $\mathcal{B}_6$ at q, hence intersects $\mathcal{B}_6$ at exactly one point m. This point m is not on p_1p_2 since $q \varepsilon p_1p_2$ and p_1p_2 is convex. Thus one arc of $\gamma(q^5)$ from q is trapped in the region bounded by the arc $q\,p_2\,m$ of $\mathcal{B}_6$ and one of the arcs qm of $\mathcal{T}_q$. $\gamma(q^5)$, being a closed curve must meet the arc $q\,p_2\,m$ of $\mathcal{B}_6$ or meet the arc q m of $\mathcal{T}_q$. The first possibility cannot exist since $\mathcal{B}_6$ is of conical order six. The latter possibility implies that $\gamma(q^5)$ is the double line on $\mathcal{T}_q$ which was ruled out in paragraph two of the proof.

Thus our assumption that $\mathcal{B}_6$ contains at least seven conically singular points is invalid and we have the required result.

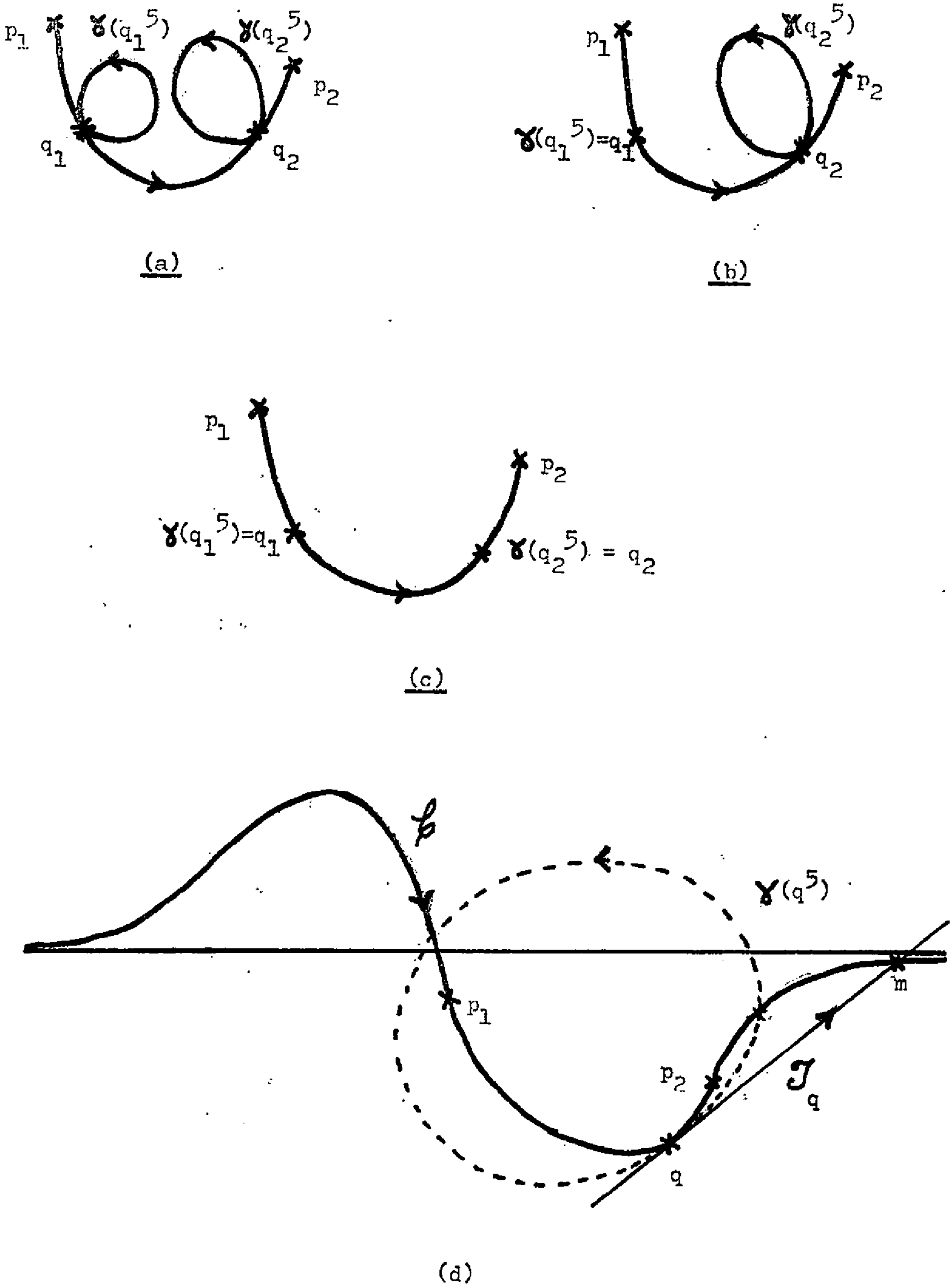

p_1
$\gamma(q_1^5)$
$\gamma(q_2^5)$
p_2
q_1
q_2
(a)
p_1
$\gamma(q_2^5)$
p_2
$\gamma(q_1^5)=q_1$
q_2
(b)
p_1
p_2
$\gamma(q_1^5)=q_1$
$\gamma(q_2^5) = q_2$
(c)
$\gamma(q^5)$
p_1
p_2
$\mathcal{T}_q$
q
m
(d)

Figure 20

<u>6.4.11</u> <u>Let case (b) of 6.4.2 occur. Then $\mathscr{B}_6$ contains at least four conically singular points.</u>

<u>Proof.</u> Let p_1 be the cusp of the first kind and p_2 the point of inflection of $\mathscr{B}_6$. By <u>6.4.5</u> (ii), p_1 is a conically singular point with the characteristic (2, 1, 1, 1, 1; 2). By <u>6.4.5</u> (i) p_2 is a singular point with the characteristic (1, 1, 1, 1, 2; 3). The ultraosculating conic $\gamma(p_1^5)$ [$\gamma(p_2^5)$] of $\mathscr{B}_6$ at p_1 [p_2] is the double line on the tangent $\mathcal{T}_{p_1}$ [$\mathcal{T}_{p_2}$] of $\mathscr{B}_6$ at p_1 [p_2]. Hence

$$\gamma(p_1^5) \cap \gamma(p_2^5) \neq \emptyset,$$

as in <u>6.4.9</u>.

We obtain singular points q_1, q_2 on the open arcs p_1p_2, p_2p_1 of $\mathscr{B}_6$, respectively, in exactly the same manner as in <u>6.4.9</u>. Hence $\mathscr{B}_6$ contains at least four conically singular points, as required.

<u>6.4.12</u> <u>If case (b) of 6.4.2 occurs, then $\mathscr{B}_6$ contains at most four singular points.</u>

<u>Proof.</u> Suppose that $\mathscr{B}_6$ contains at least five conically singular points. Then, as in <u>6.4.11</u>, the cusp of the first kind

p_1 is a conically singular point along with the inflection point p_2. Without loss of generality, there are at least two conically singular points q_1, q_2 on the convex arc p_1p_2 between p_1 and p_2 on $\mathcal{B}_6$. We may assume, by taking another line as L_∞, if necessary, that the arc p_1p_2 does not meet L_∞.

We now proceed exactly as in 6.4.10 to obtain a contradiction. Thus $\mathcal{B}_6$ contains at most four singular points, if case (b) of 6.4.2 occurs.

6.4.13 We summarize the results of this section in the following theorem.

Theorem 11: Let $\mathcal{B}_6$ be a curve of conical order six. Then we have the following results.

(1) $\mathcal{B}_6$ contains at most finitely many conically singular points.

(2) If $\mathcal{B}_6$ is a convex conically differentiable curve with no points of Type 2, then $\mathcal{B}_6$ contains exactly six conically singular points.

(3) If case (a) of 6.4.2 occurs, a conically differentiable curve $\mathcal{B}_6$ contains exactly six conically singular points.

(4) If case (b) of 6.4.2 occurs, then a conically differentiable curve $\mathcal{B}_6$ contains exactly four conically singular points.

Corollary. We have the following results for a curve $\mathcal{B}_6$ of conical order six.

(1) $\mathcal{B}_6$ is decomposed by the finitely many singular points into finitely many arcs of conical order five.

(2) If $\mathcal{B}_6$ is a convex conically differentiable curve with no points of Type 2, then $\mathcal{B}_6$ is decomposed by the singular points into six arcs of conical order five.

(3) If case (a) of 6.4.2 occurs, then a conically differentiable curve $\mathcal{B}_6$ is decomposed by the singular points into six arcs of conical order five.

(4) If case (b) of 6.4.2 occurs, then a conically differentiable curve $\mathcal{B}_6$ is decomposed by the singular points into four arcs of conical order five.

6.5 Strongly Conically Differentiable Curves of Order Six

Introduction

In this short section our attention is restricted to a strongly conically differentiable curve $\mathcal{C}_6$ of conical order six. In 6.5.2 it is shown that $\mathcal{C}_6$ contains exactly six singular points, if $\mathcal{C}_6$ is convex; while in 6.5.3 it is shown that $\mathcal{C}_6$ contains exactly six singular points, if $\mathcal{C}_6$ is of linear order three. These two results are both well known; cf. [20] and [23]. However, proofs are included for completeness and the convenience of the reader.

6.5.1 In 6.5 we shall assume that $\mathscr{C}_6$ is a strongly conically differentiable curve of conical order six. As in 6.4.1, we have two cases:

(i) $\mathscr{C}_6$ is convex

(ii) $\mathscr{C}_6$ is of linear order three.

Using the proofs of 6.4.6 and 6.4.7 we have the following result.

6.5.2 Let $\mathscr{C}_6$ be a convex curve. Then $\mathscr{C}_6$ contains exactly six conically singular points.

Remark. In 6.5.2, $\mathscr{C}_6$ is strongly conically differentiable. Hence the points of Type 2 are automatically excluded since these points are not strongly differentiable.

6.5.3 Let $\mathscr{C}_6$ be of linear order three. Now $\mathscr{C}_6$ satisfies Condition PI', since $\mathscr{C}_6$ is strongly conically differentiable. But a cusp of the first kind has linear characteristic (2,1) and does not satisfy Condition PI' ([11], 1.3). Thus case (b) of 6.4.2 cannot occur and (a) is the only possibility. We combine 6.4.9 and 6.4.10 to obtain

$\mathscr{C}_6$ contains exactly six conically singular points.

6.5.4 We summarize the results of this section in the following theorem.

Theorem 12: Let $\mathcal{C}_6$ be a strongly conically differentiable curve of conical order six. Then $\mathcal{C}_6$ contains exactly six conically singular points and $\mathcal{C}_6$ is decomposed by these singular points into six arcs of conical order five.

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