ESSAYS ON PRICING AND PROMOTION POLICIES OF DIGITAL PLATFORMS

ESSAYS ON PRICING AND PROMOTION POLICIES OF DIGITAL PLATFORMS

By ANOMITRA BHATTACHARYA, M.P.A., M.B.A., B.E.

A Thesis Submitted to the School of Graduate Studies in
Partial Fulfilment of the Requirements for the Degree
Doctor of Philosophy in Business Administration

McMaster University DOCTOR OF PHILOSOPHY in BUSINESS ADMINISTRATION (2024); DeGroote School of Business: McMaster University, Hamilton, Ontario, Canada

TITLE: Essays on Pricing and Promotion Policies of Digital Platforms

AUTHOR: Anomitra Bhattacharya, M.P.A., M.B.A., B.E.

SUPERVISOR: Dr. Ruhai Wu

NUMBER OF PAGES: xix, 246

Lay Abstract

I study vertical segmentation and dynamic promotion policies of two-sided digital platforms.

The first paper studies pricing policy of platforms based on the product/service quality segmentation under two scenarios: the platform setting prices OR the sellers deciding prices independently. In the first case, platform profits increase. Low-quality sellers benefit from segmentation while high-quality sellers suffer. In the second case, if segmentation can verify sellers' unobservable quality, it will lead to a win-win situation for buyers and the platform.

The second paper studies dynamic optimal platform promotion policy under evolution from inception to maturity in the presence of users' interactions/promotions. It finds the optimal promotion depends on the price-cost ratio, the cross-side interactions, and the resultant outcome of ad conversions and users' inherent decay and growth. User decay can be countered with promotion on the same-side initially and on the cross-side in maturity. Incentivizing cross-side user interactions can sustain platform profits.

Abstract

This dissertation investigates the impact of cross-network effects on optimal product/service segmentation pricing and dynamic promotion within digital platforms. Specifically, it examines (1) the implications of vertical segmentation for digital platforms in terms of profitability, sellers' profitability, and buyers' utility, and (2) the optimal dynamic platform promotion alongside user-generated promotion from the embryonic stage to maturity. These topics are addressed through game theoretic models, dynamic programming, and comparative statics to derive managerial insights. The thesis draws upon the literature of economics and marketing, particularly focusing on two-sided platforms, information asymmetry, and dynamic promotion.

The dissertation comprises the following inter-related chapters: (1) Introduction, (2) Literature Review on Digital Platforms and Related Businesses, (3) Vertical Segmentation Implications for Digital Platforms, (4) Optimal Dynamic Platform Promotion Policy under Evolution, and (5) Conclusion.

The introduction section discusses the importance of digital platforms in the modern economy.

The literature review section examines the inception of digital platforms and the differences in marketing strategies, competition, product/service categorization, and business evolution compared to traditional businesses. Gaps in the literature that the thesis aims to address are identified.

Chapter 3 considers two broad situations: when the platform sets the price or when the seller sets the price. Equilibrium outcomes for integrated or segmented markets are derived, along with outcomes under perfect or imperfect information about product/service quality.

Chapter 4 employs dynamic programming to derive Euler equations linking optimal promotion across periods. Three cases are considered: buyers/sellers changing over time, buyers changing but sellers fixed, and a three-period game with buyers/sellers changing over time but platform promotion limited to the first two periods. MATLAB is used to code the dynamic programming model, followed by simulations to derive steady-state outcomes and conduct comparative statics.

The conclusion chapter summarizes the two papers and identifies potential areas for future research.

Acknowledgements

I want to thank the chair of my thesis committee Dr. Ruhai Wu whose guidance and support was invaluable in preparing this work.

I want to thank other members of my thesis committee, Dr. Sourav Ray and Dr. Ashish Pujari, whose suggestions and encouragement helped immensely in this endeavor.

I want to thank many professors with whom I interacted with at McMaster University in the DeGroote School of Business and the departments of economics, statistics and mathematics. I also want to thank the professors at my previous universities for their encouragement and support.

I want to thank my fellow Ph.D. students at the department of marketing at the DeGroote School of Business for their valuable suggestions and help. Special thanks to the generous staffs at the DeGroote School of Business, especially those at the Ph.D. office, for their help and support.

Last but not the least I want to thank my parents and family whose support through this challenging phase of my doctoral studies helped me to persevere.

TABLE OF CONTENTS

List of Tables

List of Figures

List of Abbreviations and Symbols

Abbreviation	Definition
4Ps	Product, Place, Promotion, Price
AMA	American Marketing Association
ET Symposium	Empirical and Theoretical Symposium
UBC	University of British Columbia
MATLAB	MATrix LABoratory, Programming Language by MathWorks
Q.E.D.	Quod Erat Demonstrandum, which was to be demonstrated

Declaration of Academic Achievement

This thesis was prepared under the guidance of my advisor Dr. Ruhai Wu. Other members of my thesis committee, Dr. Sourav Ray and Dr. Debashish Pujari gave their minor suggestions to improve the thesis but were not involved in any technical or explanatory aspect of it.

The thesis also benefitted from suggestions by anonymous reviewers at the American Marketing Association Winter 2019 and Summer 2023 Conferences and from the suggestions by Dr. Mahmut Parlar of the management science department at the DeGroote School of Business, McMaster University, Dr. Neil Bendle of Ivey School of Business, Western University, Canada (presently at University of Georgia, US), and Dr. Charles Weinberg of Sauder School of Business, University of British Columbia.

The introduction, literature review and conclusion chapters of the thesis were conceptualized and written by me. My advisor provided suggestions to improve these chapters.

The chapter on Vertical Segmentation Implications for Digital Platforms was conceptualized by my advisor Dr. Wu. I contributed by bringing my ideas on how different corollaries and propositions can be derived from the main findings. This enriched the paper as the initial conceptualization did not account for all the new findings that can emerge from the models. All technical derivations for the paper as well as the writing were done by me. My advisor provided suggestion in the technical derivation and in the writing of the paper.

The chapter on Optimal Dynamic Platform Promotion Policy under Evolution was conceptualized by me. All technical derivations for the paper, coding, and writing were done by me. My advisor provided suggestions on modifications in the model set up, in testing different findings, and in the writing of the paper.

1. Introduction

Digital platforms are ubiquitous, representing the zeitgeist of our times. They serve as an integral part of our daily lives. Emerging in the early 2000s, digital platforms reflect the prevailing trend of this era, leaving an indelible mark on the global landscape. It is rare for a day to pass without engaging with digital platforms such as Facebook, Uber, Airbnb, Netflix, Spotify, Amazon, eBay, and LinkedIn. As of January 2024, there were 5.35 billion internet users worldwide, with a staggering 5.04 billion actively participating in social media and digital platforms (Statista, 2023). Moreover, it is projected that the global retail e-commerce sales generated by digital platforms will grow at 39 percent from 5.8 trillion USD in 2023 to over 8 trillion USD by the end of 2027 (Statista, 2024). Considering these staggering statistics, it is imperative for businesses and managers to gain a profound understanding of the digital platform economy.

1.1 Digital Platforms

Digital platforms are online systems or applications that act as intermediaries, facilitating interactions, transactions, and the exchange of goods, services, or information among users. These platforms establish governance conditions for these interactions, reducing transaction and production costs while enhancing benefits for users. Users of digital platforms typically include buyers and sellers, but they may also involve third-party service providers. By leveraging digital technologies such as the internet, mobile devices, and cloud computing, digital platforms enable various activities and functionalities, creating value by connecting users with relevant matches. They play a central role in the modern economy, facilitating a wide range of activities from online shopping and entertainment to communication and collaboration. Additionally, digital platforms drive innovation and economic growth (Parker et al., 2016).

1.2 Unique Features of Digital Platforms

Digital platforms are characterized by several key features: global reach, scalability, network effects, aggregation of products, services, and information, data-enabled analytics and personalization, real-time interaction, ecosystem integration, monetization models, and user-generated content, among others. For the purpose of this thesis, I will discuss some aspects of digital platforms.

Digital platforms exhibit network effects, which are non-internalized externalities among end users (Katz and Shapiro, 1985, 1986). These network effects can be categorized into same-side or cross-side network effects.

Same-side network effects arise from user-user interactions on one side of the platform. For instance, there may be a spillover effect among buyers, known as buyer-buyer network effects, or among sellers, referred to as seller-seller network effects.

On the other hand, cross-side network effects benefit an agent from interacting with each agent on the other side (Armstrong, 2006). Cross-side network effects can also be defined as the average benefit per transaction a platform member on one side enjoys with platform members from the other side (Rochet and Tirole, 2006). For example, there are buyer-seller network effects.

Furthermore, digital platforms are characterized by user heterogeneity. While the platform benefits from a variety of buyers and sellers, allowing for diverse transactions, the presence of low-quality sellers can pose challenges. Buyer dissatisfaction with such sellers can tarnish the platform's reputation and impede growth. Platforms can mitigate this risk by implementing quality policies, as seen in app stores and platforms like Airbnb, Amazon, and Uber, which control the quality of services through various policies and can exclude underperforming sellers.

Many digital platforms undergo an evolution from a cold-start phase to becoming behemoths, such as Alibaba and eBay. During this evolution, platforms often employ promotion functions, akin to traditional businesses. Promotions play a critical role in growing the user base, and platforms can target both the buyer and seller sides for promotion.

It's noteworthy that digital platforms often wield monopoly power in local economies. This is evident with Uber and Lyft impacting taxi drivers' businesses[1], and Airbnb disrupting the hotel industry[2]. The debate over breaking up large platforms is complex, as economies of scale and scope benefit customers, suppliers, and workers on these platforms (Hovenkamp, 2020).

1.3 Platform Marketing Strategies

Like traditional businesses, digital platforms require various marketing strategies to manage their operations effectively. One fundamental challenge in marketing is bridging the gap between actual and desired sales. Researchers have extensively explored how the four Ps of marketing - product, place (or distribution), promotion, and price - can be manipulated to address this challenge in traditional businesses. However, the advent of the internet and digital platforms necessitates adaptation of these strategies.

Categorization or segmentation is a crucial strategy that significantly impacts platform profitability. It helps platforms target customers with suitable promotions tailored to specific segments and allows for the adjustment of pricing or products based on various customer

[1] https://laist.com/shows/take-two/the-human-cost-of-uber-and-lyft-life-in-the-dying-taxi-industry
[2] https://globaledge.msu.edu/blog/post/57383/how-airbnb-disrupts-the-hotel-industry#:~:text=Beyond%20revenue%2C%20hotels%20are%20also,increase%20in%20Airbnb%20market%20share

segments. However, the presence of same-side and cross-side network effects makes this process complicated. Furthermore, whether the platform or the sellers, set prices impacts the optimal segmentation pricing strategy of the platform. The complexities involved with platform segmentation strategy require further investigation from a managerial perspective.

Promotion is another key strategy for platforms that influences profitability by attracting new customers. However, the effects of promotion are not instantaneous but are felt over time. There are also same-side and cross-side network effects, as well as user decay, that need to be taken into account while devising promotion strategies for platforms. Platforms must adapt their promotion strategies based on evolving requirements. The dynamic nature of platform promotion in the presence of network effects and user decay necessitates closer scrutiny to devise suitable strategies for managers.

We see that the strategies of segmentation and promotion require suitable adaptation for the digital business platforms of today. This thesis addresses some of these adaptations and optimal marketing strategies regarding segmentation and promotion for digital platforms.

1.4 Thesis Research Topics

Digital platforms offer a myriad of compelling research topics, standing as prominent drivers of modern economies that necessitate comprehensive study to unravel their intricate operations. Researchers address various issues of digital platforms through both theoretical and empirical studies. Theoretical studies help us think rigorously about the workings of the platform, while empirical studies validate or test intuitions gathered from observation or theoretical studies.

Given the enormous user base of digital platforms, a pivotal question arises: How should we manage them? How can managers strategize to extract the maximum benefits from these platforms? In this thesis, I delve into these questions by examining optimal platform pricing under vertical segmentation and optimal platform promotion strategies and their implications for digital platforms under various scenarios.

Pricing and promotion represent two of the most pivotal decisions for digital platforms. Understanding how platforms set prices to generate revenue and subsequently promote themselves to expand their user base is essential. This thesis addresses these core facets of the digital platform economy. Through this exploration, I aim to generate actionable managerial insights for industry practitioners, while also contributing to the marketing and platform literature by delving into unexplored areas.

Chapter three of the thesis examines vertical segmentation pricing policies of digital transaction platforms such as Uber and Airbnb and their implications for platform profit, seller profit, and consumer surplus. Utilizing static models based on game theory, this paper derives optimal pricing for digital platforms considering two conditions: perfect information and information asymmetry. The paper also considers the effect of vertical segmentation on platform pricing strategies when the platform sets the price and when the seller sets the price.

Chapter four of the thesis delves into the optimal promotion policies of platforms like Spotify and Netflix in the presence of cross-side user interactions, tracing the evolution of promotion from the platform's inception to maturity. Using dynamic programming, the paper derives optimal platform promotion considering various scenarios under three cases – a long-term case with both buyers and sellers changing, a short-term case with sellers fixed but buyers changing, and a three-period case consisting of inception, growth, and maturity phase with several periods of the long-term case condensed into one period of the former.

1.5 Structure of the Thesis

The thesis is structured as follows: The next chapter provides a brief survey of the literature on digital platforms, vertical segmentation, and advertising, encompassing key contributions from both economics and marketing. It contrasts research done to address traditional marketing problems with those that arise for platforms. Subsequent chapters delve into the two papers: platform vertical segmentation pricing policies and their implications, and optimal dynamic platform promotion policy under evolution, while the final chapter draws the thesis to a close, offering a discussion of the insights garnered from the two papers and potential avenues for future research. Overall, the thesis contributes to understanding how cross-side network effects affect digital platform outcomes.

2: Literature Review on Digital Platforms and Related Businesses

2.1 Introduction

The literature on platforms began in the early 2000s with the proliferation of the internet. This section reviews the general literature on platforms starting from its inception till the present time. The literature review compares traditional businesses with digital platforms based on the 4Ps of marketing and some other characteristics like competition, product and service categorization, and evolution. I discuss in detail the categorization and promotion literature of digital platforms considering the two papers of the thesis that follow.

2.2 Rationale for the Structure of the Literature Review

Platform business models are characterized by exponential growth driven by the cross-side network effect, where the value for users on one side increases as more users join from the other side. Consequently, early studies in this field have predominantly focused on exploring the ramifications of this network effect. With the widespread adoption and exponential growth of platform business models, understanding how to integrate traditional marketing concerns into platform operations is paramount.

New-age platform businesses fundamentally differ from traditional businesses. Platforms exhibit network effects, both same-side and cross-side, shaping user interactions and driving value creation. Unlike traditional businesses where buyers purchase directly from sellers, platforms serve as intermediaries facilitating transactions. While the core problems in marketing, such as addressing sales gaps, remain consistent across platforms and traditional businesses, the application of the traditional marketing framework, encompassing the 4Ps (product, place, price, and promotion), requires adaptation to suit the unique context of platforms.

This literature review aims to address the application of the 4Ps of marketing within the context of platforms, recognizing the need for modifications to traditional marketing strategies. Additionally, it explores other critical issues in platform marketing, including competition analysis, product and service categorization, and the evolution of platform businesses. Each of these areas will be examined with due consideration of the distinct characteristics and dynamics of platform ecosystems.

2.3 Digital Platforms Literature Inception

The study of network economies and two-sided platforms was initiated by Katz and Shapiro (1985), whose work laid the theoretical foundation for understanding the dynamics of multi-stakeholder interactions. Their introduction of the concept of network effects, or externalities, wherein the value of a good or service increases with the number of other users, was particularly influential. This notion became pivotal for the burgeoning digital platform landscape, as later discussed by subsequent researchers.

In the early 2000s, Evans (2003) expanded upon this groundwork by emphasizing the unique multi-sided nature of platforms in antitrust economics, distinct from traditional business models. Evans et al (2006) further explored how software platforms drove innovation and industry transformation, marking the nascent period of platform literature.

Building upon Katz and Shapiro's seminal idea, Rochet and Tirole (2003, 2006), along with Armstrong (2006), delved into the significant implications of network effects on platform strategies. They distinguished between same-sided and cross-side interactions among users, as well as different types of costs and benefits associated with platform transactions. Meanwhile, Weyl (2010) introduced a novel approach, focusing on the platform's allocation choices rather than pricing structures, thereby simplifying the analysis of network industries.

These foundational works formalized the literature on platforms, primarily focusing on monopoly platforms with buyers and sellers. However, platforms can also be multi-sided, accommodating various stakeholders such as complements. The equilibrium conditions for platforms were approached from two distinct angles: determining optimal pricing given the number of users or determining the optimal number of users based on a suitable pricing structure.

2.4 Characteristics of Digital Platforms

2.4.1 Role of Digital Platforms

Perren and Kozinets (2018) offer insights into the diverse roles of platforms, categorizing them into four types: forums connecting actors, enablers equipping actors, matchmakers pairing actors, and hubs centralizing exchange. They emphasize how platforms transfer responsibility for personal and exchange security to actors, institutions, or governing algorithms. The paper underscores the importance of managing social resources and digital platform algorithms effectively to prevent exploitation, necessitating careful assurances and institutional arrangements.

On the other hand, Parker (2020) argues against regulatory interventions that might undermine the network effects or hinder value creation in platform operations. He advocates for maintaining allocative efficiency, ensuring the fair distribution of value among market participants. This perspective underscores the delicate balance between fostering innovation and ensuring fair competition within the platform economy.

2.4.2 Balancing User Growth on Both Sides

Platforms implement strategies to coordinate users on both the buyer and seller sides considering the network effect.

Caillaud and Jullien (2003) showed how platforms face the "chicken and egg" problem. To attract buyers, platforms must have a critical mass of sellers, but sellers will register with the platform only if they expect a critical mass of buyers. Hagiu (2006) discussed the platform's important role in coordinating expectations on both the buyer and seller sides. The paper found that platforms manage this coordination by implementing pricing that subsidizes one side while the other side makes a profit. Strategies like delaying price announcements and providing substitute products can somewhat mitigate this problem.

Hagiu and Spulber (2013) showed how, sometimes, sellers on platforms can leverage cross-side network effects to garner monopoly power, discouraging the entry of buyers due to the lack of alternative sellers. Hagiu (2007) and Hagiu and Wright (2015) discussed how a platform sometimes functions in the merchant mode and sometimes in the platform mode to manage this demand uncertainty. They found that the problem often depends on control rights and the spillovers across products.

2.4.3 Unique Features of Digital Platforms

There are papers that discuss how digital platforms create value for users. Some papers discuss how platforms learn from reviews to take remedial measures, strengthen network effects, and connect with business clusters. Others discuss how the network effects present in platforms can be asymmetric, with one side of the platform benefiting more than the other. A few papers discuss how user multihoming (i.e. using different platforms) can have profound effect on platforms. Some others discuss how platforms deal with legacy digital resources.

Miracle (1965), McGuinness and Little (1981), and Nowlis and Simonson (1996) discussed the unique characteristics of products in traditional businesses. In contrast to traditional businesses, platforms are characterized by cross-side network effects, which significantly influence potential users' adoption decision.

Chu and Manchanda (2016) demostrated that direct network effects had a limited impact on platform growth in C2C platforms, whereas the cross-side network effect was substantial. There was an asymmetric cross-side network effect, with sellers on the platform having a larger impact on buyer growth compared to the reverse.

Ryu et al. (2023) list the reasons users' switch platforms, arguing that switching is based on the push effect, the pull effect, or the mooring effect. The push effect which drives users away from platforms arises due to several reasons. They include higher prices of products and services that the platform offers, due to a lack of content available on the platform, due to frequent technical issues the platform faces, high delivery costs, or due to lack of trust among users for the platform's functionalities. The pull effect which attracts users to platforms arises due to promotion and recommendation by other users, due to word of mouth promotion or the diversity of content that the platform offers, or due to service line extension and convenience of use offered by the platform. The mooring effect which anchors users to platforms arises due to the convenience of payment options, switching costs, rewards the platform offers, interesting content expectations from users, strength of social networks the platform users enjoys, the convenience of watching content on the platform, and ease of access to individual data.

Eisenmann et al. (2011) discussed that due to network effects and switching costs, sellers entering platforms must offer great products and services to win a large market share. However, a seller in one platform can also enter another platform and combine the functionality what one platform offers with that of another to form a multi-product bundle taking advantage of the shared user relationships between platforms. Such type of sellers, which the authors call envelopers capture market share by preventing an incumbent seller from accessing users, harnessing the network effects that previously protected the incumbent. The two platforms which the seller access can be complements, substitutes, or unrelated in functionality.

Rolland et al. (2018) considered how digital platforms provide organizations with digital investment options over time. However, previous investments in digital infrastructure by these organizations leave a legacy that influences their management of digital platforms. They discussed how organizations face a dilemma to resolve old practices to utilize the full potential of digital platforms, but their eagerness to adopt them could lead to the suboptimal use of legacy digital resources.

2.5 Connecting with Other Networks

Platforms must connect with various networks to leverage synergies. Platforms can counter the exploitation of their resources by providing incentives or entering into exclusive contracts with different user groups.

Markus and Loebbecke (2013) discussed how digital platforms create not only product market segments but also business communities, partner networks, and orchestrators which are large powerful companies at the core of ecosystems. Some digital business processes are standardized with customizable offerings for orchestrators. There are also commoditized platforms that are the same across a business community. There is a benefit to replacing proprietary data exchange rules with open internet-based ones. Different types of platforms include customizable digital platforms, which can be shared by many organizations, and community platforms, tailored for use by a particular business community.

2.6 Digital Platforms Role in Distribution Channels

Now, let's delve into the concept of place or distribution in the marketing mix for digital platforms and how it differs from traditional businesses. Weitz and Jap (1995) discussed a shift in channel management from corporate channel structures and power-based relationships to relationships between firms involving contracts and norms in traditional businesses. In traditional businesses, distribution channel efficiency relies on relationships, whereas for digital platforms, the relationship with other stakeholders takes the form of principal-agent relationships, contracts, or transaction costs.

This strand of literature questions the basis of the platform's existence. Insights from transaction cost theory are used by authors to derive various recommendations. Businesses are keen to adopt platform models to leverage network effect and accelerate business growth. The following papers explain when businesses should operate as platforms and the optimal strategy for new products and services on the platform. They also discuss pitfalls to avoid decline in the platform business.

Hagiu and Wright (2015) analyzed the factors driving organizations to position themselves as multi-sided platforms rather than traditional vertically integrated firms. They found that this characterization depends on whether there is a need to coordinate decisions that generate spillovers on others, for which a vertically integrated firm is best suited, or whether there is a need to motivate unobservable effort by employees and ensure they adapt their decisions to their private information, for which a digital platform is best suited.

Hagiu et al. (2020) explored whether a platform should function as a marketplace or a reseller. They found that the answer to this question is based on whether the control rights over a non-contractible decision, for example, the choice of some marketing activity, are better held by suppliers, in which case the marketplace model applies, or whether the control rights over a non-contractible decision are held by the intermediary, in which case the reseller model applies.

Hagiu and Wright (2020) studied digital platforms' role in getting buyers to try new products or services and their decision on whether to encourage more or less exploration relative to the level induced by sellers on the platform. They found that insufficient exploration is less of a problem than common intuition might suggest. Furthermore, when platforms extract a fixed share of revenues from sellers, there is alignment of interest between the platform’s and the sellers’ in determining the level of exploration for new products of the sellers on the platform.

Zhu and Iansiti (2019) studied the reasons digital platforms face a decline in business performance. They found that the reasons include a decline in network strength for the platform, the lack of a global cluster for the platform, a high incidence of multi-homing and disintermediation by users, and the inability for the platform to connect with other networks.

2.7 Governance of Platforms

There are papers that deal with the question of governance of platforms, taking insights from agency theory and contract theory to come up with various recommendations. These papers explore the incentives and rationale behind how platforms structure their workforce. These papers consider the platform's ability to provide services to users directly or indirectly. The insights form these papers have public policy implications for the role of workers in the gig economy.

Hagiu and Wright (2019) demonstrated that an intermediate classification of workers, between employees and independent contractors, may lead to better outcomes for online platforms. Platforms retain control over some actions, while their workers control others. Hagiu and Wright (2019) considered the problem of choice for the platform. The choice is whether to operate as a firm controlling service by employing professionals or as a platform where agents take control over the provision of their services to customers. They found that the choice depends on the need to balance the two-sided moral hazard problem. This problem arises from investments that only the agents can make and investments that only the firm can make simultaneously. This decision also considers minimizing distortions in decisions that each party could control, such as promoting agents' services, providing training, choosing equipment, and setting prices.

Hagiu and Wright (2019) studied how firms decide to employ professionals and control service delivery or operate as platforms using independent professionals to provide services directly to clients. They found that this decision depends on the principal-agent relationship in which both the principal and the agent must be incentivized to carry out investments that increase the revenue they jointly create.

2.8 Digital Platform Pricing

The pricing strategy of digital platforms is somewhat different from traditional businesses.

Several papers discuss how businesses use price discrimination, such as first-degree, second-degree, and third-degree, to enhance profit. Customer data availability has enabled businesses to have customized prices, and price premium signals are used to resolve quality uncertainty. We also find that traditional business platforms, like magazines, mimic some of the characteristics of digital platforms like 2-sidedness and cross-network effect.

In traditional businesses, the classical pricing literature by Varian (1989) discussed first-degree, second-degree, and third-degree price discrimination. First-degree price discrimination entails charging different prices to different customers. Second-degree price discrimination involves charging a price depending on the product quality. Third-degree price discrimination consists of charging a price depending on customer group characteristics.

Montgomery (1997) discussed the under-utilized information contained in the retailer's store-level scanner data, which could be exploited using new computational techniques for micro-marketing. Mishra et al. (1998) showed how adverse selection, moral hazard problems, and agency problems involving uncertainty about supplier characteristics and product quality can be resolved by means of customer bonds and price premiums serving as signals and supplier incentives.

Kaiser and Wright (2006) empirically tested the price structure of platforms in the magazine industry and found that advertisers value readers more than readers value advertisements. This dynamic resulted in magazines subsidizing cover prices and generating profits primarily from advertisers. The paper also revealed that there is an incentive to subsidize readers to attract additional advertisers, but if demand on the reader side increases, there is an incentive to increase ad rates to exploit the resulting increased demand on the advertising side.

Turning to the literature on digital platform pricing which is somewhat different from traditional businesses, several papers reveal that the optimal strategy for platforms in terms of pricing is to subsidize one side (usually the buyers) and extract higher fees from the sellers, leveraging the understanding that sellers typically value buyers more than buyers value sellers. Transferring benefits to the buyers prevents monopoly and collusive behavior among sellers. Platforms act as intermediaries, eliminating middlemen, offering customers choices to buy from different sellers at one place, thus not only satisfying customer preferences but also lowering costs incurred to them.

The pricing of products and services in our daily lives may be algorithmic or customized, depending on the customer's characteristics. Moreover, platforms can also use dynamic pricing depending on the demand for the products or services. When pricing becomes excessively customized based on the customer's characteristics, it constitutes a case of first-degree price discrimination, where businesses can extract the maximum consumer surplus, leaving consumers worse off.

Hagiu (2006) found that platforms manage expectations on both sides by suitably timing price commitments to charge buyers given sellers' price announcements. Hagiu (2009) discussed how optimal platform pricing consists of extracting more rents from producers/sellers relative to consumers when consumers have a stronger demand for variety, since producers/sellers become less substitutable. The paper also found that the presence of platform competition, consumer preferences for variety, producer/seller market power, and economies of scale in multi-homing make platform price-cutting strategies on the consumer side less effective, and variable fees charged to producers/sellers can serve as a trade-off between producer/seller innovation incentives and the need to reduce the platform holdup problem.

Edelman and Wright (2015) showed that platforms can control the prices sellers charge and profitably raise demand from buyers by eliminating any extra price they face for purchasing through a platform. The paper found that this leads to an increase in retail prices for traditional stores, excessive adoption of platform services, over-investment in benefits to buyers, a reduction in overall consumer surplus, and competition among platforms worsens these problems. Jolivet et al. (2016), through their empirical study demonstrated a significant, positive, and strong impact of seller reputation on prices in e-commerce platforms.

Jullien and Pavan (2019) showed that platforms in which information about users' preferences is dispersed resulted in idiosyncratic uncertainty about user participation decisions which shaped the elasticity of the demands and the equilibrium prices. Wang and Wright (2020) discussed how platform charges raise the possibility of showrooming where consumers search on a platform but then switch and buy directly to take advantage of lower direct prices. The authors discussed that to prevent show-rooming, platforms like Booking.com have adopted price parity clauses where firms need to offer their best prices on the platform.

2.9 Digital Platform Promotion

In the exploration of the literature on promotion in digital platforms, I will first discuss the promotion literature in traditional businesses to contrast it with the literature on promotion in digital platforms.

Rothschild and Gaidis (1981) discussed how behavior that is positively reinforced is more likely to recur than non-reinforced behavior, drawing parallels in marketing, particularly in low-involvement purchase situations and in the development of promotional strategies. Gupta (1988) examined the effectiveness of sales promotion and found that a majority of the sales increase due to promotion comes from brand switching. The paper also found that purchase acceleration in time accounts for a minor portion of the sales increase, whereas stockpiling due to the effect of promotion is negligible.

Zhang and Wedel (2009) found that customized promotion in online and offline stores led to a substantial increase in profit. Loyalty promotions are more profitable in online stores than in offline stores, while the opposite is true for competitive promotions. The research also found that the incremental payoff of individual-level over segment and mass market–level customized promotions is small while low redemption rates impede the success of customized promotions in offline stores.

These papers illustrate the evolution of promotion in traditional businesses, shifting from mass promotion to different types, as businesses recognize the effectiveness of more granular and customized promotion in enhancing profitability.

While the mentioned papers focus on promotion in a static setting, I will now explore the literature on dynamic promotions for traditional businesses. Nerlove and Arrow's (1962) early work in this field discussed optimal advertising policy under dynamic conditions, extending the criteria for maximum net revenue when price and promotion affect demand. Neslin et al. (1995) discussed how retailer and consumer responses influence a manufacturer's optimal advertising and trade promotion plans in a dynamic environment. The study found that the manufacturer's optimal allocation depends on consumer response to advertising, consumer response to retailer promotions, retailer inventory carrying cost, and retailer pass-through behavior. Furthermore, retailer carrying costs and promotion wear out constrain expenditures on trade promotions while advertising and trade deal substitute each other in an optimal plan.

Papatla and Krishnamurthi (1996) studied the dynamic effects of promotions on brand loyalty, customers' price sensitivity, and whether promotional purchases replicate similar promotions. They found that increased purchases with coupons erode brand loyalty, increased price sensitivity, and features and displays' effect on brand choice is reinforced by prior similar purchases. Krishnan and Jain (2006) analyzed the optimal dynamic promotion policy for new products and found that it is determined by advertising effectiveness, discount rate, and the ratio of advertisement to profits.

Bass et al. (2007) recommended that managers dynamically using different themes of advertising like price advertisements versus product advertisements and different versions of an advertisement must consider the interaction effects among the different themes of advertising, the effects of wear, and that of forgetting in the context of an advertising campaign. The researchers found that this model can assist managers in optimizing resource allocation across different ad portfolios, improving scheduling, and significantly boosting demand.

Doganoglu and Klapper (2006) analyzed weekly advertising policies of manufacturing firms in consumer goods markets. They assumed firms engage in persuasive advertising and found that the most important determinant of advertising intensity was goodwill. Elberg et al. (2019) studied the dynamic effects of price promotions with varying discount levels. Their research showed that small firms benefit from heightened promotion sensitivity by using promotions to induce future consideration but when the unit margins are high, heightened promotion sensitivity led to fierce competition, making all firms worse off.

From these papers, it is evident that the strategies for optimal dynamic promotions differ from optimal promotions in the static case, as the firm needs to consider the time value of profits and the evolving nature of various variables over time. Path dependency is highlighted in dynamic promotion. One factor to consider is the wear-out or decay factor, as advertisement effectiveness decays with time. The other factor to consider is the interaction effect between the different types of advertising. Prices and costs can remain constant or change depending on the stochastic nature of the scenario. Introducing additional complexities in modeling though inhibits certain simplifications and insights. The firm in deciding advertising intensity also needs to consider certain factors like goodwill which is a crucial component of a company's overall value but quantifying goodwill can be challenging because it is not a tangible or easily measurable asset like buildings or machinery.

Turning to the literature on promotions for digital platforms, Scheinbaum (2016) considered how digital consumer engagement offers a way for companies to engage consumers with a brand or message on an online or mobile platform where companies need to consider the consumer viewpoint as well as practical considerations. However, intense digital engagement may lead to unwanted consequences, and brands can over-engage, resulting in diminishing returns.

Bruce et al (2017) studied the effects of creative format, messages, and targeting on the performance of digital ads over time. They found that the potency of retargeted ads is dependent on price incentives. Carry-over rates for dynamic formats are greater than static formats, but static formats can still be effective for price ads and retargeting.

Voorveld et al. (2018) found that consumers' engagement with social media platforms influenced their engagement with advertising within these platforms, and there was a context-specific nature of engagement that had its impact on advertising evaluations. Costello and Reczek (2020) distinguished between peer-to-peer brands' use of provider-focused versus platform-focused marketing communications. The authors found that consumers’ willingness to buy and pay is more for the former rather than the latter due to empathy considerations for the individual provider.

Fang et al. (2015) analyzed the direct and indirect effects of buyers and sellers on search advertising revenues in business-to-business electronic platforms. The research revealed that new sellers outbid existing sellers in the mature stage, but the opposite happens at the launch stage because of asymmetry of signal quality at different stages. New buyers tend to click on more search advertisements than existing buyers, particularly in the mature stages. Fong et al. (2019) analyzed the opportunity costs of targeted promotion by platforms and found that targeted ads increased sales of promoted and similar products but could hinder the platform by reducing searches for non-targeted products, leading to variable sales.

Rietveld et al. (2019) studied how platform sponsors can reward successful complements, drive attention to underappreciated complements, and influence consumer's perception through selective promotion. They found that platform sponsors can use promotion to induce and reward strong complements and time promotions to increase sales during slow periods while reducing competitive interactions between complements. The research also revealed that platform sponsors not only promote good complements but strategically invest in them to address complex trade-offs and use selective promotion to manage the ecosystem's value creation as well as its value capture.

These papers reveal that some literature on platform promotion has focused on targeted ads. Much of that sort of promotion uses sophisticated algorithms to target customers based on data collected from various sources, including social media. Here the focus is on the platform's own promotion strategies, but these do not take into account the significant impact of user interactions. This particular strand of literature also considers the strategic impact of advertising on buyers, sellers, and complements. Some of the platform promotion literature deals with static models, while others deals with dynamic ones.

2.10 Digital Platform: Categorization and Segmentation

I now discuss product and service categorization or segmentation by digital platforms and distinguish it from traditional business segmentation. I first discuss segmentation in traditional businesses.

Smith (1956) found that product differentiation and market segmentation were considered as alternative marketing strategies to traditional approaches such as promotion and price changes. Product differentiation strategies can be pursued with or without segmentation but segmentation can be pursued only when differentiation exists. Varian (1989) studied different quality-price combinations to attract various customer segments. Dickson and Ginter (1987) concluded that product differentiation gives a horizontal share of a broad and generalized market while market segmentation gives depth of market position in the defined segments.

Vandenbosch and Weinberg (1995) analyzed product and price competition under two-dimensional vertical differentiations. They found that firms avoid maximum differentiation unlike the one dimensional case. If the range of positioning options is equal for different dimensions, firms choose positions with maximum differentiation on one dimension and minimum differentiation on the other. Bolton and Myers (2003) found price elasticity depend on the quality, type and support for service, and horizontal segments exist for pricing strategies transcending national borders.

Liu and Zhang (2013) showed that when there is dynamic pricing competition between two firms offering vertically differentiated products to strategic customers who maximize utility over time, price skimming arises with the low-quality firm suffering more than the high-quality firm. Commitment to stable pricing improves profits for both firms but if the high-quality firm commits rather than the low-quality one, the profits are higher.

Zeithammer and Thomadsen (2013) found that for price and quality competition in a vertically differentiated duopoly with consumers seeking variety due to diminishing marginal utility, consumer variety seeking can soften or intensify price competition, based on the strength of consumers' preference for variety and the difference in firm qualities. Furthermore, if firms set qualities before competing on price and there are few feasible qualities to restrict variety seeking competition, firms minimally differentiate themselves from each other.

Daughety and Reinganum (2008) proved that when there is imperfect competition and incomplete information in the case of price competition among firms with horizontally and vertically differentiated substitute products, incomplete information about quality signaled by price reduces price competition and imperfect competition reduces the degree firms change prices to signal their types. They also found that low-quality firms and high-quality firms catering to high-value markets prefer incomplete information. The loss to consumers using low-quality products benefits low-quality firms.

These papers show that horizontal differentiation, i.e. offering different product variety and vertical differentiation, i.e. offering different product quality has been used by firms to cater to customers' different tastes and preferences.

Firms have used price discrimination to maximize profits. Third degree price discrimination has been used to charge prices to different segments of customers based on certain characteristics. Second degree price discrimination has been used to charge customers prices depending on the quality of the product. Segmentation based on demographic, psychographic, behavioral, and benefit characteristics, when higher, the better is the match with customers' preferences, the higher the possibility that the firm will be able to extract the maximum consumer surplus and consequently higher is the firm's profit. When there is information asymmetry about the quality of the product, firms gain at the cost of customers with some firms gaining more than others.

I now discuss the product and services categorization literature for digital platforms.

Hagiu (2009) showed that platforms are likely to engage in exclusion of low-quality users when users on one side place more value on the average quality and less value on the total quantity of the users on the other side. Masanell and Hałaburda (2014) argued that it might be rational for platforms to limit the number of applications available on it because even if users prefer application variety, applications also exhibit direct network effects and in its presence, users prefer to consume the same applications to benefit from consumption complementarities.

Dinerstein et al. (2018) focused on analyzing the horizontal differentiation of products (i.e., differences in product characteristics) in platforms. They examined the trade-off associated with consumer's search for the desired products on the platforms while also getting the lowest price offer from sellers. They found that search friction explain why firms selling commodity products have pricing power. Karle et al (2019) found that if product market competition is high, sellers join different platforms resulting in high platform fees. On the other hand, if product market competition is low, sellers agglomerate on a single platform, and a few platforms fight to dominate.

These papers reveal that the interaction between the platform's network effects and users' preferences for variety or quality of products and services produces interesting results. These findings have implications for the profits of the platform as well as the profits of the sellers transacting through it.

2.11 Digital Platform Competition

I discuss now platform competition. But first, let me delve into some papers on competition in traditional businesses.

Alderson (1937) discussed the need for a comprehensive marketing strategy and the leadership role of the American Marketing Association in the face of regulations for competition that have affected the marketing profession. Coughlan (1985) discussed how firms choose prices and the vertical marketing channel to maximize profits in a product-differentiated duopolistic market. The paper found that the integration of the marketing function resulted in greater price competition and lower prices than the use of independent marketing middlemen. Furthermore, the paper showed that integration is negatively associated with the products' substitutability and symmetric channel structures are stable.

Weitz (1985) discussed the framework to analyze competition in terms of the competitors, the intensity of competition, the effect on market evolution and structure, and the effect on the firm's marketing decision and how to maintain a competitive advantage. Ramaswamy et al (1994) outlined a framework for analyzing differences in competitive marketing behavior in established industrial markets distinguishing between retaliatory and cooperative marketing behavior.

Jayachandran et al (1999) studied multimarket competition where the same firms compete against each other in multiple markets. They found that tacit collusion occurs in which firms avoid competition against rivals they meet in multiple markets as multi-market competition increases the familiarity between firms and their ability to deter each other.

These papers reveal the various aspects of competition from price to channels and the role of cooperation and competition that businesses have adopted in a competitive landscape. I now turn to how digital platforms compete in the modern economy.

Economides and Katsamakas (2006) found that when a system based on an open source platform with an independent proprietary application competes with a proprietary system, the proprietary system dominates the open source platform industry both in terms of market share and profitability. Halaburda and Yehezkel (2013) found that platform competition with asymmetric information of users or the imposition of exclusive deals by the platforms results in market failure and lower levels of trade, but multi-homing, where users access several platforms, can solve this problem. Hagiu and Halaburda (2014) showed that platforms with more market power prefer facing more informed users as the price information leads user expectations to be more responsive and hence amplifies the effect of price reductions.

Halaburda et al (2018) showed that matching platforms compete by limiting the number of choices it offers to its customers, while charging higher prices than platforms with unrestricted choice. Hallaburda and Yehezkel (2019) found that beliefs shape platform competition based on the notion of the focal point. Hallaburda et al (2020) found that history matters for dynamic competition among platforms in a market with network externalities.

Bakos and Halaburda (2020) proved that for two-sided platforms, the strategic interdependence between the two sides resulted in the platform maximizing its total profits by subsidizing one of the sides. The authors showed that this result depended on the assumption that at least one side of the market single-homes. They found that when both sides of the platform multi-home (i.e. users access multiple platforms), the strategic interdependence between the two sides of the platform diminished which suggest that the strategy to subsidize one side in order to maximize total profits may have limited application or even may be incorrect.

Johnson (2020) studied dynamic competition between retail platforms in the presence of consumer lock-in, where consumers are tied to a particular platform due to various reasons. The research considered two types of revenue models, one in which the platforms set retail prices and another in which the suppliers set retail prices. The paper found that since platforms have long-term pricing incentives unlike suppliers, the time dependent price path consumers' face depends on the type of revenue model being used. Furthermore, if suppliers set prices instead of platforms, prices are higher in the early periods but lower in the later periods which imply appropriate antitrust enforcement by regulating agencies should consider more than the initial price changes when the industry shifts to the agency model.

These papers indicate that in the presence of network effects, asymmetric information, and users' multi-homing, modeling the competition between platforms becomes complicated. It requires a nuanced approach different from modeling monopoly platforms. Often, major results that hold for monopoly platforms may not hold for platform competition due to strategic behavior of firms.

2.12 Digital Platform Evolution

Several studies have explored how businesses evolve over time.

Achrol (1991) discussed how marketing exchange companies and coalition companies serve as organizing hubs of complex networks of functionally specialized firms in traditional businesses. These networks span organizational systems, with managerial responsibilities also being boundary-spanning, as traditional businesses evolved over time. The paper found that these systems develop over time elaborate relational norms and sophisticated information, political, and quasi-judicial systems.

Let me now discuss the evolution of digital platforms.

Kierzkowski et al (1996) discussed how marketers often approach interactive media through the static, one-way, mass-market broadcast model of traditional media that fails to realize its full potential. The paper emphasized new interactive media approaches like digital marketing as an attractive proposition for various consumer product or service categories. Lamberton and Stephen (2016) discussed the importance of digital, social media, and mobile marketing as a facilitator of individual expression, as a decision support tool, and as a market intelligence source.

Kannan (2017) recommended that research on digital platforms should consider understanding the attributes and characteristics of the technology which has implications for its adoption. The paper also emphasized understanding the associated services and networked products and their versions and how the competitive landscape changes as a result of technological advances. Moreover, the paper stressed the need to understand the complication in implementing marketing strategies due to the fragmentation of media and proliferation of devices and channels with marketing investments and measurement of returns spread across many entities. Eckhardt et al (2019) stressed that research on digital platforms should consider the maturation of these platforms; the paradoxes and dark side of the sharing economy, and be aware of new emerging technologies.

Rangaswamy et al (2020) discussed the role of marketing in helping digital business platforms succeed and how it is derived based on the theory of transaction cost analysis. The primary role for marketing is to increase the number and quality of interactions on a digital business platform while reducing the transaction costs for users and production costs for the platform. The paper also discussed how the interactions and data generated enables value creation and value appropriation on these platforms notwithstanding the challenges that these platforms need to address to cater to the needs of different users. The paper concluded that the role of digital platforms is to coordinate and manage interactions among users on different sides of the platform.

From these papers, it is evident that the realization that digital platforms present an interactive way to connect with customers led to a nuanced approach to marketing. This approach is different from the way traditional businesses handled such activities, emphasizing the ease of data collection and the importance of enhancing customer service quality and managing expectations. The need for balance from the benefits and challenges of technology enabled services was explored.

Liu et al (2021) discussed the role of digital platforms in providing technology-enabled tools that enhance market transparency. These tools include real-time monitoring, ratings, reviews, and grievance redressal for both buyers and sellers. The authors examined the role of digital platforms in affecting moral hazard and service quality. They found that for driver routing choices and efficiency for Uber and taxi drivers, digital platform designs reduce moral hazard but driver selection or differences in driver navigation technologies don't explain the phenomenon. Moreover, they suggested that the technology enabled market designs may not be binding for long routing in times of surge pricing.

Dukes et al (2022), studied skippable ads on digital platforms and found that they may be less effective overall in converting existing viewers to advertisers but bring more viewers to the platform inducing more advertisers. Moreover, the research found that skippable ads are a profitable strategy for a digital platform in the initial and growth stages but not in a saturated market. Farronato et al (2023) found that network effects and product or service differentiation offset each other at the market level so that users are not substantially better off with a combined platform than with two separate ones when platforms merge.

These papers reveal digital platforms act as a middleman verifying certain quality attributes of the products and services. Promotion strategy on these platforms needs to be tailored according to the product life cycle. Platform mergers need several considerations. We see that platforms have evolved from the days of providing a medium of two way communication between buyers and sellers to providing value added service like quality verification.

2.13 Addressing Gap in the Literature on Digital Platforms

The papers discussed in the product and service differentiation and segmentation literature do not reveal managerial implications of pricing strategies for platforms having competing sellers offering different quality products or services to customers having different preferences. My first paper fills this gap in the platform literature by addressing this question. It goes beyond scale or network effect by consider user heterogeneity with the platform or the sellers setting price. Furthermore, it considers users subject to diminishing marginal utility and product substitution effects, along with information asymmetry about the service provided by sellers on the platform which requires platform intervention to verify service quality. The paper thus contributes to the segmentation and price discrimination literature by introducing these for platforms where buyers buy from the platform rather than the seller.

Past research has explored promotion in a static or limited-time dynamic setting for well-established platforms. They dealt with promotions (mostly targeted ads) or interaction of users (network effect). Few researches have delved into the intricacies of platform promotion in realistic dynamic settings where platform and user promotions synergistically evolve from the platform's embryonic stages to maturity. The inherent complexities, coupled with the temporal evolution of relevant variables make this managerially interesting. The second paper seeks to bridge the gap in understanding of platform promotion by examination of promotion for a platform's life cycle. Prior research has scrutinized dynamic pricing for platforms. Prior research also studied the impact of same-sided user interactions on revenue (e.g. Bass model (1969)). This paper studies promotion's interplay with the cross-side network effects in a platform. Dynamic promotion strategy in platforms is a new field. This paper thus contributes to the digital platform and promotion literature in marketing.

3. Vertical Segmentation Implications for Digital Platforms[3]

3.1 Abstract

In this paper, we examine how platforms' vertical segmentation strategies affect the platform economy. Our analytical model reveals a positive cross-side network effect of the variance of seller quality on buyer surplus, providing a theoretical foundation for platforms' vertical segmentation strategies. We examine this vertical segmentation under two scenarios: a) the platform setting the trading prices, and b) sellers deciding prices independently.

In the first scenario, segmentation allows the platform to increase its profit by second-degree price discrimination. Buyer surplus decreases. Low-quality sellers benefit from segmentation, while high-quality sellers suffer unless the quality gap is sufficiently large. In the second scenario, segmentation based on publicly observable sellers' features does not alter the equilibrium outcomes.

However, if a platform's segmentation program can verify and reveal sellers' unobservable quality information, it will reduce the information asymmetry problem between the sellers and the buyers and lead to a win-win solution where both the buyers and the platform gains. Our study thus identifies two important functions of the platform's segmentation strategies: price discrimination and the reduction of information asymmetry.

We further show that any intervention that plays upon the psychology of the buyer and raises her quality preference benefits both the platform and the sellers, as this quality preference

[3] The paper was presented at the ET Symposium, 2019, at UBC, Vancouver, and benefited from useful comments by Charles Weinberg of Sauder, UBC and Neil Bendle of Ivey, Western University, Canada (presently University of Georgia, US). It was also presented at the 2019 AMA Winter Conference at Austin, Texas, and benefitted from useful comments by two anonymous reviewers. We would like to thank all of them for their excellent suggestions.

has a quadratic cross-side network effect that raises both the number of transactions and the optimal trading price.

Furthermore, our analysis reveals that integrated platforms with uniform prices irrespective of the quality of the products do not exist. Segmented platforms with prices based on the quality of the products, and where the platform decides prices, make more profit than platforms where sellers decide prices. These findings provide a rich understanding of platform management.

3.2 Introduction

Two-sided markets or platforms are very popular these days. They enable a large number of buyers and sellers of diverse characteristics to trade in goods and services. Uber and Airbnb are typical examples. These platforms, facing large heterogeneity on both the buyer and the seller sides, often use various categorization rules to segment the market to realize better match outcomes between buyer's preferences and seller's offerings. These platforms often vertically segment the market based on seller quality. For example, Uber has rides for the economy class segmented into UberX, UberXL, and UberSelect based on affordable prices, space, and comfort, as well as rides for the high-end premium class segmented into UberBlack, UberSUV, UberLUX, based on space, style and driver quality. Airbnb's offer for customers ranges from the low end "bed and breakfasts" to "unique homes", "vacation homes", and "boutique hotels". In February of 2018, it launched the high end "premium plus" homes, which are pre-checked by Airbnb staff for quality adherence.

The segmentation by these platforms seems mainly a classification effort to list sellers into categories of different quality standards to better match buyer preferences. Buyers usually are still allowed to, and they do buy from sellers in all categories. However, the segmentation will affect buyers buying behavior given the variety of quality of products and corresponding prices to choose from, sellers' competing strategies, and consequently the profits of the platforms. Thus, segmentation based on product or service quality has become an area of strategic importance for the platform business.

Vertical segmentation is not a novel marketing practice. In shops, departmental stores, and supermarkets, products and services are priced according to their qualities to cater to different buyer segments. Many manufacturers design products targeting specific segments – for example, General Motors has a wide range of compact, midsize, and luxury cars to cater to the needs of its different types of customers. Moreover, some sellers offer different quality-price combinations to differentiate buyers with different quality preferences (Varian 1989).

The existing literature has thoroughly examined vertical segmentation from the perspective of sellers. However, vertical segmentation strategy by a two-sided platform is more complicated as it involves network externality between the buyer and the seller sides. A policy segmenting users on one side does not only restrain the users' trades on this side but also alters the trading strategies of the users on the cross side, which will further influence the users on the original side. It is not clear how vertical segmentation in the presence of sellers' different product quality offerings and buyers' varied preferences along with the presence of network effect affects platform outcomes.

Moreover, the impact of vertical segmentation is also affected by who decides the prices of products and services offered by the platform – the platform itself like in the case of Uber or the sellers like in the case of Airbnb. Because depending on who sets the price, the strategic outcomes of platform profit, seller's profit, and buyer's utility changes. There is a lack of understanding in academic and managerial practices on how to optimize a segmentation policy by platforms to cater to both sellers' different product quality offerings and buyers' different quality preferences given these two arrangements – the platform deciding prices or the sellers deciding prices.

In this paper, we use game theoretical models to explore buyers' shopping decisions and sellers' competition strategies under the platform's segmentation policy. We consider a platform on which sellers with different quality products trade with buyers with different quality preferences. Each buyer can buy multiple units of goods from different sellers. The buyer's utility is characterized by the law of diminishing marginal utility and substitution effect among products from different sellers. The platform segments sellers into two sub-markets according to their product qualities. Buyers can still buy from both sub-markets. However, their purchase decisions are affected by the vertical segmentation policy as the trading prices change.

The equilibrium outcomes illustrate how the vertical segmentation policy affects the sellers' profits, buyers' surplus, and the platform's profit. We examine two common types of platform business models. a) The platform vertically segments sellers and decides the trading price at which sellers sell their products/services to buyers. Transportation network companies such as Uber, Lyft often use this business model. b) The platform vertically segments sellers but the sellers decide their prices in the market independently. Lodging network company Airbnb, travel reservation company Booking.com, and many online marketplaces such as Amazon, Alibaba follow this business model.

Our model elaborates on the cross-side and same-side network effects related to seller quality heterogeneity in detail. Specifically, we show a positive cross-side network effect of the variance of seller quality on buyer surplus. It implies that the platform economy will benefit from increasing seller heterogeneity, providing a theoretical foundation for the platforms' vertical segmentation strategies.

In the business model where the platform sets the trading price(s), segmentation does not alter the market average price, which is a function of the average of buyer preferences and the average of product qualities. However, segmentation allows the platform to increase its profit through second-degree price discrimination. Buyer surplus decreases. Low-quality sellers benefit from segmentation while high-quality sellers suffer unless the quality gap is sufficiently large.

Buyers appreciate quality products and services. Hence, as the mean quality preference (or the average of the quality preference of all buyers in the market) increases, so does the number of transactions. The optimal trading price also increases as it is dependent on the mean quality preference of the buyer. All these results in increasing sellers' profit through a quadratic cross-side network effect.

In the business model where the sellers decide their trading prices, for a buyer, it seems that the market is already vertically differentiated as sellers are offering prices as per their product or service quality. In this market, there is no quality verification of seller's products by the platform and the sellers charge their respective market equilibrium price. But some platforms implement a quality verification mechanism by which they verify the quality of products and services offered by certain sellers on the platform. This is done to help buyers who suffer from an information asymmetry problem as they are not sure whether the quality of products or services is what the sellers claim.

We find that if a platform's segmentation program can verify and reveal sellers' unobservable quality information and in that process further segment the market, it will reduce the information asymmetry problem between the sellers and buyers and lead to a win-win solution. Both buyer surplus and platform profit increases.

The range and variety of buyer preferences matter for the platform business. This is particularly true for sellers who sell low-quality products through the platform. When the platform segments sellers with its quality verification services, suspect high-quality seller's profit decreases while trustworthy high-quality seller's profit increases. Also, the quality verification charge does not affect equilibrium prices or buyer utility. But the platform should initiate the quality verification service only if the cost incurred by it is below a certain threshold level.

Moreover, we find that when sellers do not engage in false claims and report truthfully about the quality of the product or service being offered by them, under segmentation, the platform's profit is higher when it sets the trading prices than when the sellers set the trading prices independently. Both these profits are greater than the profit of the platform in the integrated market with a uniform price.

Our study identifies two important functions of platform segmentation strategies - price discrimination and reducing information asymmetry. An implication for business practitioners is that classifying sellers into detailed categories may not always benefit the platform. It will bring more revenue if and only if it results in larger price variety in the market and enhanced information about product quality for the buyers. Furthermore, we show that platforms with a uniform price are not profitable while the business where the platform decides the price is more profitable than the one where sellers decide the price.

Our research thus contributes to the vertical differentiation literature as well as the literature of platforms by showing how product quality and taste perception affects platform outcomes in the presence of network effects, diminishing marginal utility and product substitution effects. The theoretical findings guide platform managers on a series of pricing and management decisions. The findings also help sellers and buyers to understand the consequent impact of the platform segmentation policy and optimize their trading and entry decisions.

The rest of the paper is organized as follows. Section 3.3 summarizes the related literature and highlights the contribution of our work. Section 3.4 discusses the model setup. We examine how segmentation reshapes the platform economy in the market where the platform sets the trading prices in the following section 3.5 and how it works in the market where sellers set prices independently in section 3.6. In the final section 3.7, we highlight the critical findings of the paper. We first compare platform profits for the integrated market (uniform price irrespective of quality) when platform sets the price, segmented market (price varies according to quality) when platform sets the price and independent market when the seller sets the price depending on their product quality, and derive implications. We then list the other important findings of the paper, discuss the managerial implications and conclude by identifying some limitations of our work and scope for future research.

3.3 Related Literature

We first discuss the literature on segmentation and product differentiation in traditional businesses. After that, we explore the general platform literature and the literature on product and service categorization in platforms and discuss how our paper addresses the gap in the literature.

Horizontal differentiation, i.e. offering different product variety and vertical differentiation, i.e. offering different product quality has been used by firms to cater to customers' different tastes and preferences in traditional businesses. Smith (1956), Dickson and Ginter (1987) considered product differentiation and market segmentation as alternative marketing strategies to promotion and price changes. The need for product differentiation for gaining a horizontal share of a broad and generalized market and the need for segmentation to gain depth of market position in the defined segments was studied. The authors identified that segmentation can be pursued only when differentiation exists. Varian (1989) discussed vertical differentiation, i.e. the need for different price-quality combination to attract differ segments of customers.

Vandenbosch and Weinberg (1995) studied product and price competition in two dimensional vertical differentiation finding that firms avoid maximum differentiation unlike the one dimensional case. They show that if the range of positioning options is equal for different dimensions, firms choose positions with maximum differentiation on one dimension and minimum differentiation on the other. Liu and Zhang (2013) found incidence of price skimming with the low-quality firm suffering more than the high-quality firm when there is dynamic pricing competition between two firms offering vertically differentiated (or quality based differentiation) products to strategic customers maximizing utility over time. The research also showed that commitment to stable pricing improves profits for both firms but if the high-quality firm commits rather than the low-quality one, the profits are higher.

Zeithammer and Thomadsen (2013) found that consumer variety seeking can soften or intensify price competition, based on the strength of the consumers' preference for variety and the difference in firm qualities. The also find that if firms set qualities before competing on price and there are few options on quality to restrict variety seeking competition, firms minimally differentiate themselves from one other. Reinganum and Daughety (2008) showed that when there is information asymmetry about the quality of the product, firms gain at the cost of customers with some firms gaining more than others.

Several seminal papers (e.g., Rochet and Tirole 2003, 2006; Caillaud and Jullien 2001, 2003; Armstrong 2006) initiated the literature of two-sided platforms. They identify a two-sided platform as a unique business model where the network effects among platform users have significant implications on the platform pricing strategies. These papers, as well as other analytical modeling studies (Hagiu 2006, Parker and Van Alstyne 2005, Armstrong and Wright 2007, Ambrus and Argenziano, 2009), focus on the platform economy in terms of the numbers of buyers and sellers. They explore the optimal pricing scheme(s) of a monopoly or two competing platform(s) given exogenous network effects.

To avoid analytical complexity, these models assume linear network effects between homogeneous sellers and buyers, but abstract out the economic mechanisms which generate these network effects. It is not easy to apply the findings of these studies to managerial contexts, where the strengths of network effects are often not readily measurable or observable.

A growing number of recent papers (Economides and Katsamakas 2006, Edelman and Wright 2013, Hałaburda and Yehezkel 2013, Hagiu and Hałaburda 2014) examine the users' transaction decisions on various platforms and investigate the structure of these platform economies. Most of these research assume homogeneous users on both buyer and seller sides, and hence do not consider platform management on the diversity of the transactions.

A series of papers study the governance of platforms. The authors of these papers discuss condition for a platform to function as a marketplace or as a reseller (Hagiu and Wright, 2015), the economic trade-offs that compel organizations to position themselves as a multi-sided platform (Hagiu and Wright, 2015), how much revenue and control should an agent be given by a platform (Hagiu and Wright, 2018), how a firm decide whether to employ people or to operate as a platform outsourcing the service operations for clients (Hagiu and Wright, 2019), whether people employed by platforms can be categorized as independent contractors or employees (Hagiu and Wright, 2019), reasons platforms face decline in business (Zhu and Iansiti, 2019), the extent to which platform should enable the entry of untested new products and sellers (Hagiu and Wright, 2020), and conditions under which a firm can profitably turn itself into a platform by allowing rival products and services to be traded on it alongside its products and services (Hagiu et al, 2020).

Other papers deal with strategic pricing – Jullien and Pavan, 2019, study information management and pricing in platforms, Jolivet et al, 2016, discusses reputation management and prices in the e-market, Wang and Wright, 2020, considers the aspect of showrooming and price parity clauses in search platforms, Johnson (2023) examine platform design when sellers use pricing algorithms. Some papers address other strategic aspects of online platforms. Correia Da Silva et al, 2019, discuss horizontal mergers between multisided platforms. Hallaburda and Yehezkel, 2019, consider how beliefs shape platform competition based on the notion of a partial focality in the presence of network effects. Hallaburda et al, 2020, studies the dynamic competition among platforms in a market with network externalities. While these papers address important issues, they do not particularly deal with the strategic case of vertical segmentation in platforms.

Dukes and Gal-Or (2003) analyze competition between platforms when there are exclusive contracts. Considering a monopoly platform, Nocke, Peitz, and Stahl (2007), Galeotti and Moraga-Gonz´alez (2009), and Gomes (2014) analyzed platform ownership, search, and optimal auction design respectively. Belleflamme and Toulemonde (2009) analyzed competition between for-profit and not-for-profit platforms. These papers do not consider the seller competition and its effect on the market structure. Karle et al (2019) consider the competition between sellers by showing that product market concentration and platform fees are negatively correlated and provide a rationale for several homogeneous platforms using segmentation when product market competition is high.

Some papers consider user heterogeneity on platforms. Weyl (2010) discusses applications of monopoly pricing by platforms with user heterogeneity to regulation, market power, and merger analysis. Dinerstein et al (2018), focusing on the products' horizontal differentiation (difference in produce characteristics), examined the trade-off associated with buyer's search for desired products on platforms while also getting the lowest price offer from sellers. They found that search friction explain why firms selling commodity products have pricing power. But these papers while addressing important questions do not provide managerially relevant recommendations for dealing with the platform economy.

There are a few other papers studying product diversity in the platform economy. Casadesus-Masanell and Hałaburda (2014) investigate product selections in a platform economy. They find that it may be rational for platforms to limit the number of applications available in spite of users preferring application variety since applications exhibit direct network effects due to which users prefer to consume the same applications to benefit from consumption complementarities. They focus on the products' horizontal differentiation (difference in product or service characteristics) but do not address the quality difference.

Lin et al. (2011) examine the innovation and price competition between two quality-differentiated sellers on a platform. Parker and Van Alstyne (2010) study how platforms intervene in the innovation process by app developers. The number of sellers on a platform is exogenously given in the above two papers. Wu and Lin (2014) introduce quality heterogeneity into a Salop circular city model to capture the comprehensive competition in product characteristics, quality, and price in a platform business.

Accommodating vertical differentiation and quality have received some attention in the platform literature. Hagiu (2009) finds that platforms are likely to engage in exclusion of low-quality users when users on one side place more value on the average quality and less value on the total quantity of the users on the other side. Exclusion also depends on the proportion of high-quality users on the other side and their cost advantage in joining the platform relative to low-quality users.

Hermalin (2016) showed that the optimal quality threshold level of products is higher when the platform charges the sellers than when it charges the buyers. Zennyo (2016) finds that when the advantage of product variety dominates the disadvantage of lower quality, the platform dealing in lower quality products and services can enjoy greater profit than its rival.

But these papers do not reveal managerially relevant implications on pricing strategies of platforms in the presence of competition among the sellers offering different quality products to customers having different preferences. Our paper fills this gap in platform literature by addressing this question in the presence of network complexities, diminishing marginal utility, and product substitution effects.

Our paper also contributes to the segmentation and price discrimination literature. We know that segmentation is based on demographic, psychographic, behavioral, and benefit (Haley, 1968) characteristics. Stigler (1992), Varian (1989), and Tirole (1988) contributed to the rich literature on price discrimination by discussing extensively the nature and conditions necessary for their occurrences, and the various types of price discrimination.

Price discrimination has been studied in several contexts - in medicine (Kessel, 1958), inter-temporally (Stokey, 1979), through coupons (Narasimhan, 1984, Bester and Petrakis, 1996), in trade (Knetter, 1989), in-network competition (Laffont et al, 1998), in advance selling (Xie and Shugan, 2001), concerning purchasing history (Acquisti and Varian, 2005), with bundling (Geng et al, 2005 and Sankaranarayanan, 2007), in coupons and rebates (Chen et al, 2005 and Lu and Moorthy, 2007), and the context of e-commerce (Hinz et al, 2011).

Our paper differs from these by incorporating price discrimination in the case of platforms. Here buyers buy from the platform rather than the seller. In some cases, the platform sets the price and offer price discrimination to match product quality with varied buyer preferences. In other cases, sellers set the price depending on their product quality levels and sell through the platform. In such cases, to the buyer, the market seems vertically differentiated even without the platform's intervention.

And finally, we have cases where not only do the sellers vertically differentiate themselves but the platform further segments the market by intervening. Since buyers are multi-homing and can buy from different market segments with different product quality offerings, price discrimination through the platform is a special case of second-degree price discrimination.

3.4 Model Setup

Suppose there are m number of sellers and n number of buyers on the platform (m and n are large numbers with $m < n$). Sellers sell similar products with different qualities. Specifically, seller j sells a product of quality v_j, where $v_j \in \{v_h, v_l\}$, and $v_h > v_l$. Among all sellers, a proportion t_s of them sell products of high quality v_h, and $1 - t_s$ of them sell products of low quality v_l.

Buyer i has a quality preference θ_i, where $\theta_i \in \{\theta_h, \theta_l\}$ and $\theta_h > \theta_l$. A proportion t_b of the buyers have a high-quality preference θ_h, and $1 - t_b$ of them have a low-quality preference θ_l. The quality preference of the buyer is a measure of the appreciation of quality by her.

We assume that the sellers and the platform know the sellers' product quality information but do not have information about the buyers' quality preferences. The proportion of high product quality sellers and the proportion of buyers with high product quality preference are public knowledge.

Similar to the utility function used in Daughety and Reinganum (2008), we define the utility function of a buyer who buys a *non-negative* quantity from each of the sellers as –

$$u_i = \sum_{j=1}^{m}[\theta_i v_j q_{ij} - \frac{\beta}{2} q_{ij}^2 - \frac{\gamma}{2}\sum_{j \neq j'} q_{ij} q_{ij'} - q_{ij} p_j] \ldots\ldots \quad (1)$$

where p_j is the unit trading price of the products of seller j, and q_{ij} is the quantity of the products bought by buyer i from seller j.

β is the coefficient defining diminishing marginal utility and γ represents the product substitution effect among products from different sellers. For simplicity, we assume that these values are the same for all products and buyers, i.e., the coefficient of substitution between any two goods is the same.

As in Daughety and Reinganum (2008), we have $0 < \gamma < \beta < 1$. This utility function well defines buyers' choices when they can buy multiple products from multiple sellers. Both diminishing marginal utility and substitution effect between different sellers are considered.

To simplify the model, we assume the sellers' production cost is zero. The platform charges a δ fraction of the price paid by the buyer as commission charge and hence $(1 - \delta)$ of the fraction of the sales revenue is kept by the sellers.

It is worth noting that our model is applicable not only to markets with buyers engaging in multiple transactions with various sellers, but also markets where consumers choose only one service provider for each transaction, such as Uber and Airbnb. In these latter markets, consumers may potentially engage with all service providers over time. Moreover, in our model, the consumption variable q_i is treated as a continuous variable, which can be interpreted as a propensity of the transaction in platforms like Uber and Airbnb.

3.5 Segmentation when the Platform Sets the Trading Price

In this section, we study how vertical segmentation affects the market outcomes in a market where the platform sets the trading prices. Most transportation platforms (i.e. Uber, Lyft, Didi), decide the trading prices rather than let the service providers decide them. Other examples include logistics platforms like Postmates and online platforms for residential cleaning, installation, and home services like Handy. These companies also set the prices of the services they provide.

Shutterstock, a NY based company, is a platform for people to exchange photography, footage, music and editing tools. While content providers retain copyright over their photos, music, etc. which they contribute to the platform, Shutterstock sets the rates for the contents downloaded by subscribers. Moreover, along with horizontal segmentation, the platform also vertically segments product offerings with curated collections of images, footages, and premium music carefully chosen by experts. In this section, we develop a game-theoretical model to examine the platform's vertical segmentation strategy when it decides the trading price between sellers and buyers.

3.5.1 Integrated Market where the Platform Sets a Uniform Price

We first consider a benchmark case where the platform does not segment the market. All sellers, although with different qualities, trade in the same market. The platform sets a uniform price p in the market irrespective of the quality of the product. When Uber launched its business, it implemented a uniform rate for all drivers even though they had different types of cars. It later segmented the market according to the types of cars. The setup is pictorially depicted in Figure 3.1 (see Appendix A). The game is composed of two stages. In the first stage, the platform sets a trading price p^{int} (*int* represents integrated market) at which all sellers sell their products. In the second stage, buyers make their purchase decisions. We solve the equilibrium market outcome by backward induction. Given p^{int}, the buyers optimize their purchases $\{q_{ij}\}$ from different sellers to maximize their utilities. Anticipating the demands $\{q_{ij}\}$, the platform then optimizes the trading price p^{int} to maximize its profit.

Lemma 1: Given the trading price p^{int}, the demand for the product from seller j from buyer i is as follows.

$q_{ij}^{int} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{\gamma m}{\beta+\gamma(m-1)}\tilde{v}\right) - \frac{\beta-\gamma}{\beta+\gamma(m-1)}p^{int}\right]$ *where* $\tilde{v} = t_s v_h + (1-t_s)v_l$, *is the average product quality among the sellers*[4].

(Please see Appendix A for the proofs to all Lemmas, Corollaries, and Propositions)

Lemma 1 shows that a buyer's demand for the product from a seller depends on the seller's relative quality to the market average quality, the buyer's quality preference, and the market price. Here the trading price is set by the platform rather than by the sellers. A buyer would like to buy more products/services from a high-quality seller than a low-quality seller

[4] Since $\theta_h > \theta_l$ and $v_h > v_l$, the condition for a positive demand function irrespective of types of buyers or products must satisfy $\theta_l\left(v_l - \frac{\gamma m}{\beta+\gamma(m-1)}\tilde{v}\right) > \frac{\beta-\gamma}{\beta+\gamma(m-1)}p^{int}$ when we substitute $i = j = l$. This condition is satisfied when $\theta_h = \theta_l$, which occurs when buyers have no difference in preference.

given the product/service have the same price as it gives her more utility (since buyer's utility is increasing in product/service quality). Consequently, the demand for high-quality products/services is much higher than those of low quality.

Given the individual demands, the platform owner can estimate the aggregate demand and adjust the trading price to maximize its profit.

Proposition 1: The optimal trading price for the platform is $p^{int} = \frac{\tilde{\theta}\tilde{v}}{2}$, *and its profit is* $\pi_p^{int} = \frac{\delta mn}{\beta+\gamma(m-1)}(\frac{\tilde{\theta}^2\tilde{v}^2}{4})$. *where* $\tilde{v} = t_s v_h + (1-t_s)v_l$, *is the average product quality among the sellers, and* $\tilde{\theta} = t_b\theta_h + (1-t_b)\theta_l$ *is the average preference for quality among the buyers in the market.*

Proposition 1 illustrates the optimal trading price and the platform's profit[5] in the equilibrium market outcome. When the platform does not segment the market, the optimal price and its profit depend only on the average seller quality and average buyer preference. Even though users behave differently in the market (i.e., high-quality sellers sell more than low-quality sellers as the price is the same for both types of goods) and buyers with high-quality preference buys more than those with low-quality preference (as the demand and utility function for a buyer is increasing in her preference level θ_i), the heterogeneity among the users seems irrelevant to the optimal price and the platform profit. This is because the platform faces an aggregate demand which depends only on the average seller quality and the average buyer preference. It is also

[5] Note that the proportion of the high quality sellers t_s that are allowed to transact through the platform can also be modeled as an endogenous platform decision. We can then differentiate the profit function with respect to t_s to find the optimal proportion of high and low quality sellers that should be allowed to transact through the platform to maximize the platform's profit.

evident from the platform's profit expression that the profit increases with *n*, the number of buyers using the platform. It also increases with *m,* the number of sellers using the platform[6].

Corollary 1: The profit of a seller is $\pi_j^{int} = \frac{(1-\delta)n}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) + \frac{(1-\delta)n(v_j-\tilde{v})}{\beta-\gamma}\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right)$ *where* $v_j \in \{v_l, v_h\}$. *Here* $v_h > \tilde{v} > v_l$.

Corollary 1 shows that a seller's profit[7] is composed of an average profit level $\frac{(1-\delta)n}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right)$, and an adjustment $\frac{(1-\delta)n(v_j-\tilde{v})}{\beta-\gamma}\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right)$ which is linearly correlated to the gap between its product quality and the market average product quality. High-quality sellers earn more profit than the market average as they sell more and low-quality sellers earn less profit as their products are less attractive. Corollary 1 also presents buyers' network effects on sellers' profits side. Both the number of buyers and their quality preference levels have a positive network effect on the seller's profit. Sellers' profit is a linear function of the number of buyers. As more buyers join the platform, sellers' profits increase.

The average quality preference of the buyers ($\tilde{\theta}$) benefits the sellers' profits in two ways. As buyers' quality preferences increases, they will consume more products (since demand is increasing in buyer preference). Each seller sells more products. Meanwhile, the platform to maximize its profit will raise the trading price which increases the seller's markup. In general, the average buyer quality preference has a quadratic cross-side network effect on the sellers' profits.

---

6 $\frac{\partial \pi_p^{int}}{\partial m} = \left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right)\delta n \frac{\beta-\gamma}{[\beta+\gamma(m-1)]^2} > 0$ as all terms are positive

7 Since $\theta_h > \theta_l$ and $v_h > v_l$, the condition for a positive profit function irrespective of types of buyers or products must satisfy $\frac{(1-\delta)n}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) + \frac{(1-\delta)n(v_l-\tilde{v})}{\beta-\gamma}\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right) > 0$ when we substitute $i = j = l$. This condition is satisfied when $\theta_h = \theta_l$, which occurs when buyers have no difference in preference.

The same-side network effects on the seller's profit are from the number of sellers as well as the average quality offered by all the sellers in the market. Consistent with existing literature, our result shows that the same side network effect from the number of sellers is negative – i.e. the greater is the competition from other sellers, the lesser is the profit for a particular seller[8].

If the seller is a low-quality seller, the same side network effect is negative from the average quality of the product offered in the market. This is because the market average quality is higher than the low quality. On the other hand, if the seller is a high-quality seller, the same side network effect from the average quality of the product offered in the market is negative only if the quality difference between the high and low-quality product is below a certain level (or the ratio of the low and high quality is above a certain level)[9]. This arises because when the quality difference between the low and high-quality product is sufficiently low, buyers are faced with a more tempting trade-off between quality and price while buying a product.

Corollary 2: The utility of a buyer is $u_i^{int} = \frac{m}{2}\left[\frac{\tilde{v}^2}{\beta+\gamma(m-1)}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 + \frac{1}{\beta-\gamma}\theta_i^2 Var(v)\right]$ *where* $Var(v) = t_s v_h^2 + (1-t_s)v_l^2 - \{t_s v_h + (1-t_s)v_l\}^2 = t_s(1-t_s)(v_h - v_l)^2$, *which represents the quality heterogeneity among the sellers in the market.*

The buyer utility function has two components. Similar to the analysis of the seller side, Corollary 2 shows a linear positive cross-side network effect on the buyers' surplus in the form of the number of the sellers. The first component of the utility function reveals a quadratic positive cross-side network effect in the form of the average product quality among the sellers. It also reveals that a buyer's utility is determined by the relative quality preference of the buyer to

---

[8] $\frac{\partial \pi_j^{int}}{\partial m} = -\frac{(1-\delta)n\gamma}{[\beta+\gamma(m-1)]^2}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) < 0$

[9] $\frac{\partial \pi_j^{int}}{\partial \tilde{v}} = \frac{(1-\delta)\tilde{\theta}^2 n[(\beta+\gamma(m-1))(v_j-\tilde{v})-m\gamma\tilde{v}]}{2(\beta-\gamma)(\beta+\gamma(m-1))}$. Now since $v_l < \tilde{v}$, $\frac{\partial \pi_j^{int}}{\partial \tilde{v}} < 0\ if\ v_j = v_l$. But we know that $v_h > \tilde{v}$. If $v_j = v_h$, $\frac{\partial \pi_j^{int}}{\partial \tilde{v}} = \frac{(1-\delta)\tilde{\theta}^2 n[(v_h-v_l)(1-t_s)(\beta-\gamma+2m\gamma)-m\gamma v_h]}{(\beta-\gamma)(\beta+\gamma(m-1))} < 0$ if $\frac{v_l}{v_h} > 1 - \frac{m\gamma}{(1-t_s)(\beta-\gamma+2m\gamma)}$ and vice versa

the average quality preference of all buyers which is a measure of how different their choices are from all others in the market.

Corollary 2 also shows that the variance of product quality among sellers has a positive cross-side network effect on the buyers' surplus. Buyers are buying both high and low-quality products. Their utility is increasing in both their individual preference as well as the quality of the product. If the difference in product quality that is offered in the market increases, buyers have more choices to align their preferences to their quality of choice. Consequently, they shift their buying from low type quality products to more high type quality products given that the price remains the same for both types in an integrated market. Hence, their utility increases.

As there is no direct competition among the buyers, the number of buyers does not generate any same-side network effect on the buyers' surplus. However, the average quality preference of the buyers has an indirect negative same-side network effect (provided the buyer's preference is above a certain level)[10]. When the average quality preference increases, the platform will raise the trading price and all buyers suffer a reduction in surplus.

3.5.2 Segmented Market where the Platform Sets Different Prices

We now consider the case where the platform segments high-quality sellers and low-quality sellers into two sub-markets and sets different trading prices. Buyers are multi-homing, i.e. buying from both sub-markets. Most transportation network companies, e.g. Uber, Lyft, Didi, now adopt this business model. Drivers are categorized into several classes based on the quality of the vehicles from economy to luxury cars. The network companies set different

[10] $\frac{\partial u_i^{int}}{\partial \tilde{\theta}} = -\frac{m}{2}\left[\frac{\tilde{v}^2}{[\beta+\gamma(m-1)]}\right]\left(\theta_i - \frac{\tilde{\theta}}{2}\right) < 0 \; if \; \theta_i > \frac{\tilde{\theta}}{2}$, i.e. if the concerned buyer's preference is greater than half the average preference of all buyers

mileage rates for rides in different classes. Buyers can use drivers from all classes, depending on their services, prices, and availability.

The setup is pictorially depicted in Figure 3.2 (see Appendix A). The model for this case is composed of two stages. In the first stage, the platform segments the sellers into two sub-markets - a sub-market of high-quality sellers and that of low-quality sellers. It sets the trading prices p_h^{seg} and p_l^{seg} for the two sub-markets respectively (*seg* represents the segmented market). In the second stage, the buyers buy in both submarkets. We again solve the model by backward induction. We first investigate the buyers' consumption decisions.

Lemma 2: When the market is segmented into high-quality sellers and low-quality sellers, and different prices p_h^{seg} and p_l^{seg} are assigned for transactions, the demand for the products from seller j from a buyer i is

$$q_{ij}^{seg} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_h^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]\ if\ seller\ j\ is\ of\ high\ quality$$

$$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_l^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]\ if\ seller\ j\ is\ of\ low\ quality$$

where $\tilde{p}^{seg} \equiv t_s p_h^{seg} + (1-t_s)p_l^{seg}$, is the mean of the prices for all sellers in both sub-markets and $\tilde{v} = t_s v_h + (1-t_s)v_l$.

Lemma 2 shows that a buyer's demand[11] for the product from a seller depends on the seller's relative quality and the buyer's quality preference. It also depends on the gap between the seller's price and the mean of all sellers' prices in the market.

---

[11] Since $\theta_h > \theta_l$ and $v_h > v_l$, the condition for a positive demand function irrespective of types of buyers or products must satisfy $\theta_l\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_l^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right) > 0$ when we substitute $i = j = l$. This condition is satisfied when $\frac{\theta_h}{\theta_l} < 1 + \frac{1}{t_b}$. Hence the ratio of buyer's preferences matters when doing business through a platform.

Anticipating the individual demands, the platform owner can predict the aggregate demand and optimize the trading prices accordingly to maximize its profit.

Proposition 2: When the platform segments the market into high-quality submarket and low-quality submarket, it sets the optimal trading prices as $p_h^{seg} = \frac{\tilde{\theta} v_h}{2}$ *and* $p_l^{seg} = \frac{\tilde{\theta} v_l}{2}$ *respectively. The platform's total profit from the two submarkets is* $\pi_p^{seg} = \frac{\delta mn}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2 \tilde{v}^2}{4}\right) + \frac{\delta mn\tilde{\theta}^2}{4(\beta-\gamma)} Var(v)$*, which is higher than that from an integrated market.*

Proposition 2 illustrates the platform's optimal pricing policy under segmentation. It is interesting to note that under such a policy, the average of the trading prices of all sellers across the market is $\tilde{p}^{seg} \equiv t_s p_h + (1 - t_s) p_l = \frac{\tilde{\theta}\tilde{v}}{2}$, which is equal to the platform's optimal price in the integrated market. In other words, when the platform segments the market, it sets a high price for high-quality sellers and a low price for low-quality sellers, but the mean of the sellers' prices in the market remains the same as that before segmentation.

Proposition 2 proves that the platform will benefit from market segmentation. The platform's profit increases compared to the integrated market by an amount proportional to the variance of the sellers' product qualities. In the integrated market, because of the uniform price, even high-quality products/services which should be priced higher are offered at the same price as low-quality products/services. Buyers benefit in such circumstances by buying more high-quality products/services while the platform loses revenue.

But when the platform sets price according to the quality of the product/services, buyers lose the benefit of procuring high-quality products/services at prices the same as low-quality products/services while the platform can avoid the loss incurred by selling high-quality products/services at lower prices. Consequently, the platform's profit increases.

Moreover, the more is the heterogeneity among the sellers in terms of product/service quality offerings, the higher is the opportunity for the platform to segment the market and price product/service offerings according to their quality. Consequently, the lesser is the possibility of products/services of higher quality being offered at lesser prices. So the segmentation policy of the platform decreases buyer's surplus (or in this case buyer's utility) and increases producer's surplus (or in this case the platform profit).

In the case where the platform can set the trading prices for the sellers, segmentation policy is essentially a second-degree price discrimination strategy. Since the buyer's type (θ_h, θ_l) is unknown to the price setter (i.e. the platform), segmented prices are set to incentivize buyers to "self-select" the product of the matching quality tier. The platform's profit is proportional to the aggregate sellers' side profit. Hence, although the products are traded between third-party heterogeneous sellers and buyers, the platform can optimize the pricing scheme for the products with different quality by the segmentation policy to maximize the aggregate seller side profit.

Proposition 3: When the platform segment the market, seller j's profit is $\pi_j^{seg} = \frac{(1-\delta)n\tilde{\theta}^2 v_j}{4(\beta-\gamma)}\left[v_j - \frac{\gamma m\tilde{v}}{\beta+\gamma(m-1)}\right]$. *Sellers with low product quality earn more profit from the segmented market than that from the integrated market. Sellers with high product quality earn more profit if and only if* $\frac{v_l}{v_h} \leq 1 - \frac{m\gamma}{(1-t_s)(\beta-\gamma+2m\gamma)}$.

It is worth noticing that when the platform optimizes the segmentation pricing, it also maximizes the total seller side profit[12]. However, an individual seller's profit does not

[12] Since $\theta_h > \theta_l$ and $v_h > v_l$, the condition for a positive demand function irrespective of the types of buyers or products must satisfy $v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)} > 0$. This in turn yields $\frac{v_h}{v_l} < \frac{\beta-\gamma}{m\gamma t_s} + 1$. Hence the ratio of the product quality matters for sellers when doing business through a platform.

necessarily increase. Proposition 3 presents an asymmetric change in individual sellers' profits. While low-quality sellers always benefit from segmented pricing, high-quality sellers' profits increase only when the quality difference among the sellers is sufficiently large (i.e. the ratio between the low quality and the high quality is sufficiently small). This is due to competition between and within the different groups of sellers and the resulting market elasticity.

Specifically, in the integrated market, the platform acting like a monopoly sets a uniform trading price to maximize the total seller-side profit. However, sellers compete with each other and the demand is highly elastic at this "monopoly" price. Hence, when the platform segments the market and lowers the trading price for the low-quality sellers, the low-quality sellers' sales increase, and their profit margin decreases. As the market is elastic at the uniform trading price, lowering the trading price for low-quality sellers leads to more profit gain from increased sales than profit loss from the reduced profit margin. Therefore, low-quality sellers benefit from the price segmentation policy.

On the other hand, because of the elasticity of the market, when the high-quality seller's trading price increases, their sales drop dramatically. Hence their profits decrease when their trading price rises in the elastic market.

However, if the gap between the high quality and the low quality is large, the substitution effect between the high-quality products and the low-quality products becomes small. So does the demand elasticity for high-quality products. The high-quality sellers may then benefit from the rising profit margin even at the higher trading price[13].

---

[13] Note that the seller's profit is positive only when $v_j > \frac{\gamma m \tilde{v}}{\beta+\gamma(m-1)}$. Since, $v_h > \tilde{v}$ and $v_l < \tilde{v}$ and $\frac{\gamma m}{\beta+\gamma(m-1)} < 1$, the high-quality seller's profit is always positive while the low-quality seller's profit is positive when $\frac{v_h}{v_l} < \frac{\beta-\gamma}{m\gamma t_s} + 1$. Moreover, the condition for high-quality sellers to make more profit in a segmented market than an integrated market is $\frac{v_l}{v_h} \leq 1 - \frac{m\gamma}{(1-t_s)(\beta-\gamma+2m\gamma)}$. These two conditions are simultaneously satisfied when $t_s <$

We see that in the case of Uber, economy car services like UberX, UberXL, and UberSelect benefit from the segmentation with high demand from customers who prefer an economy ride, but high-end luxury car services like UberBlack, UberSUV, UberLUX also benefit by serving certain niche customers who would prefer such luxury rides and are willing to pay the extra price for that service. Hence there is an incentive for high-end cars and drivers to categorize themselves as offering high-quality service at high prices to distinguish themselves from the rest.

Proposition 4: When the platform segments the market, a buyer's utility is $u_i^{seg} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 var(v)\right]$, *which is smaller than the utility in an integrated market.*

Proposition 4 shows that when the platform segments the market, every buyer's utility decreases. Specifically, the decreased amount is relevant to the variance of product quality among sellers. As we have explained in Corollary 2, in the integrated market, buyers gain a surplus from seller quality heterogeneity as they can optimize the consumption between high- and low-quality sellers.

In the segmented market, buyers can still gain a surplus from adjusting consumptions between the two types of sellers. However, compared with the case of the integrated market, buyers gain a lower per unit surplus from high-quality products as their price increases. Moreover, buyers now consume more low-quality products and less high-quality products due to

---

$\frac{\beta-\gamma}{\beta-\gamma+m\gamma}$ (see proof of proposition 3). Hence, if the proportion of high-quality sellers is below a certain level $\frac{\beta-\gamma}{\beta-\gamma+m\gamma}$, low-quality sellers can make a positive profit in a segmented market along with high-quality sellers making more profit in a segmented market compared to the integrated market. Hence, the proportion of high-quality sellers matter in the profit outcomes for all types of sellers in a segmented market.

the price difference between the two types of products. Therefore, buyers' surplus becomes smaller.

Table 3.1 compares the outcomes of the integrated and segmented market. As discussed earlier, the platform profit increases in the segmented case due to a better match between the buyer's product quality preferences and seller's product quality offerings. The platform can extract more surplus form the buyer in the segmented market as compared to the integrated one.

Table 3.1 Comparing Outcomes of Integrated and Segmented Market (Platform sets Price)

	Integrated	Segmented
equilibrium price	$\frac{\tilde{\theta}\tilde{v}}{2}$	$\tilde{p} = \frac{\tilde{\theta}\tilde{v}}{2}$, $p_j = \frac{\tilde{\theta} v_j}{2}$ (j= h, l)
platform profit	$\frac{\delta mn}{\beta+\gamma(m-1)}(\frac{\tilde{\theta}^2\tilde{v}^2}{4})$ <	$\frac{\delta mn}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right)+\frac{\delta mn\tilde{\theta}^2}{4(\beta-\gamma)}Var(v)$
seller's profit (low quality)	$\frac{(1-\delta)n}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right)+\frac{(1-\delta)n(v_l-\tilde{v})}{\beta-\gamma}\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right)$ <	$\frac{(1-\delta)n\tilde{\theta}^2 v_l}{4(\beta-\gamma)}\left[v_l-\frac{\gamma m\tilde{v}}{\beta+\gamma(m-1)}\right]$
seller's profit (high quality)	$\frac{(1-\delta)n}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right)+\frac{(1-\delta)n(v_h-\tilde{v})}{\beta-\gamma}\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right)$ <	$\frac{(1-\delta)n\tilde{\theta}^2 v_h}{4(\beta-\gamma)}\left[v_h-\frac{\gamma m\tilde{v}}{\beta+\gamma(m-1)}\right]$ if $\frac{v_l}{v_h}\leq 1-\frac{m\gamma}{(1-t_s)(\beta-\gamma+2m\gamma)}$
buyer's utility	$\frac{m}{2}\left[\frac{\tilde{v}^2}{\beta+\gamma(m-1)}\left(\theta_i-\frac{\tilde{\theta}}{2}\right)^2+\frac{1}{\beta-\gamma}\theta_i^2 Var(v)\right]$ >	$\frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i-\frac{\tilde{\theta}}{2}\right)^2+\frac{1}{(\beta-\gamma)}\left(\theta_i-\frac{\tilde{\theta}}{2}\right)^2 var(v)\right]$

Note that $Var(v) = t_s(1-t_s)(v_h-v_l)^2$

3.6 Segmentation when Sellers set Prices Independently

In some industries such as the lodging industry, travel reservations, and many online marketplaces such as Amazon, Alibaba, etc., platforms often allow the sellers to set prices by themselves. Booking.com, one of the largest travel e-commerce companies in the world, connects travelers with the world's largest selection of lodgings - from apartments, vacation homes, family-run bed and breakfasts, 5-star luxury resorts, tree houses, to even igloos. It allows sellers on its website to choose their own rules from prices, policies, to rules for guests and charges a 15% commission for its service. Airbnb is an online marketplace and hospitality service brokerage company headquartered in San Francisco. The quality of the accommodations on Airbnb varies in terms of location, size, neighborhood environment, indoor decoration, furniture, and other devices availability, host hospitality, etc. The platform, although providing details of the accommodation quality, does not set the prices for individual hosts. Hosts set the prices by themselves. Buyers in the market perceive hosts' accommodation "qualities" from room descriptions and customer reviews and make their consumption decisions.

Generally, in these types of markets, sellers with high-quality offerings ask for higher prices. Thus we see that products or services have been vertically differentiated by independently set prices. Along with that, many of these platforms use "elite" programs to categorize high-end sellers.

Airbnb launched its Premium Plus program as "a new selection of only the highest quality homes with hosts known for great reviews and attention to detail". The program is a selection of homes verified for quality and design by Airbnb. To be a part of it, the hosts must meet "Superhost" level hospitality standards, and the homes must meet a checklist of requirements. In return, not only can the hosts (or the seller in our case) charge a higher price but they also get additional support from Airbnb in terms of higher visibility in buyer search results, highlighted features, and customer support.

Similarly, Amazon distinguishes "Premium Sellers" who "are committed to providing customers with the highest standard of customer service." Amazon also has the Prime program for sellers who commit to fulfilling orders within two days delivery period at no additional charge to customers. Amazon gives such sellers access to the required transportation solutions to help meet the high bar for customer experience. Prospective sellers need to qualify for this program by fulfilling certain requirements. In return, they get access to the most loyal customers of Amazon along with being recommended as the default option among sellers when buyers buy their products.

Essentially, these platforms intervene to further segment the market even though the sellers have vertically differentiated themselves by setting prices according to their product quality. In this section, we use game theoretical models to explore the economic intuition behind this type of segmentation policy where sellers set prices and the platform intervenes or does not intervene to further segment the market and examine its influence on the market outcome.

Similar to the model setting in section 3, there are the $(1 - t_s)$ proportion of low-quality sellers and the t_s proportion of high-quality sellers. We assume all low-quality sellers have quality v_l. But, α proportion of the high-quality sellers are *suspect high-quality sellers* with true quality $(v_h - d)$, and $(1 - \alpha)$ proportion of them are *trustworthy high-quality sellers* with quality v_h.

We consider the same buyer utility function as that in section 3. As before, we assume the sellers' production cost is zero and the platform charges δ fraction of the price paid by the buyer as commission charge and hence the $(1 - \delta)$ fraction of the sales revenue is kept by the sellers. Different from section 3.4, sellers set the trading prices by themselves. Therefore, the game is composed of two stages - in stage 1, sellers set the trading price by anticipating the future demand under competition, and in stage 2 buyers make purchase decisions.

In section 3.6.1, we examine a benchmark model where all sellers trade in the market without segmentation. In section 3.6.2, we study the case where the platform segments the trustworthy high-quality sellers through a premium seller program. The comparison of the market outcomes between the two cases will illustrate the motivation of the platform's segmentation policy.

3.6.1 Integrated Market when Sellers set Prices Independently

The setup is pictorially depicted in Figure 3.3 (see Appendix A). Without market segmentation by the platform, buyers cannot differentiate trustworthy high-quality sellers and suspect high-quality sellers. Thus, all high-quality sellers have the same expected product quality $(v_h - \alpha d)$ (see proof of $Lemma$ 3), and they set the same trading price p_h^{int}. The low-quality sellers have the same product quality v_l and their decision on the trading price is p_l^{int} (*int* for integrated market). Therefore, buyers face two different types of products with different expected qualities and different prices. Their consumption decisions are presented as follows.

Lemma 3: When the market is not segmented and sellers determine their trading prices independently, the demand for seller j from buyer i is

$$q_{ij}^{int} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - \alpha d - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_h^{int} - \frac{m\gamma\tilde{p}^{int}}{\beta+\gamma(m-1)}\right)\right] \ if \ \ j \ is \ a \ high \ quality \ seller$$

$$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_l^{int} - \frac{m\gamma\tilde{p}^{int}}{\beta+\gamma(m-1)}\right)\right] \ if \ j \ is \ a \ low \ quality \ seller$$

Where $\tilde{v} = t_s(v_h - \alpha d) + (1 - t_s)v_l$ *is the average of the sellers' quality, and* $\tilde{p}^{int} \equiv t_s p_h^{int} + (1 - t_s)p_l^{int}$*, is the average of the prices set by all sellers in the market.*

Lemma 3, like lemma 1 and 2 shows the buyer's demand[14] function and has similar implications. Here, buyers perceive the quality differences between high-quality and low-quality

[14] Since $\theta_h > \theta_l$ and $v_h > v_l$, the condition for a positive demand function irrespective of types of buyers or products must satisfy $\theta_l\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_l^{int} - \frac{m\gamma\tilde{p}^{int}}{\beta+\gamma(m-1)}\right) > 0$ when we substitute $i = j = l$. This condition is satisfied when $\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right)\left[\frac{\beta+m\gamma-\gamma}{2\beta+2m\gamma-3\gamma}\tilde{\theta} - \theta_l\right] < \frac{m\gamma\tilde{v}\tilde{\theta}(\beta-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)(\beta+m\gamma-\gamma)}$. Since the right-hand side is

sellers and face different prices when making purchase decisions. Essentially, high-quality and low-quality sellers are vertically differentiated, even without the platform's segmentation policy. However, as buyers cannot distinguish trustworthy high-quality sellers from suspect high-quality sellers due to information asymmetry, the two types of high-quality sellers adopt the same pricing strategy.

Proposition 5: When the market is not segmented by the platform and sellers determine their prices independently, the optimal price of a high-quality seller, irrespective of whether she is a suspect or a trustworthy high-quality seller, is

$$p_h^{int} = \frac{\widetilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d)-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_h^{int} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\widetilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d)-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$

The optimal price of a low-quality seller is

$$p_l^{int} = \frac{\widetilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_l-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_l^{int} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\widetilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_l-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$

Proposition 5 present the sellers' optimal prices and profits[15] in the equilibrium outcome. The seller's market equilibrium price is a product of average buyer preference and a linear function of the market average quality and expected product quality as perceived by the buyers.

---

positive, the inequality is satisfied if the left-hand side is negative, i.e. if $\left(v_l-\frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right)>0$ and $\left[\frac{\beta+m\gamma-\gamma}{2\beta+2m\gamma-3\gamma}\widetilde{\theta}-\theta_l\right]<0$. Now $\left(v_l-\frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right)>0$ is necessary for a positive profit function for sellers. From $\left[\frac{\beta+m\gamma-\gamma}{2\beta+2m\gamma-3\gamma}\widetilde{\theta}-\theta_l\right]<0$ we get $\frac{\theta_h}{\theta_l}<\left[1+\left(\frac{\beta+m\gamma-2\gamma}{\beta+m\gamma-\gamma}\right)\left(\frac{1}{t_b}\right)\right]$ which shows the ratio of buyer's preferences matters in platform business.

[15] Since $\theta_h>\theta_l$ and $v_h>v_l$, the condition for a positive profit function irrespective of types of buyers or products must satisfy $(\beta+m\gamma-\gamma)v_l-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}>0$. This yields $\frac{(v_h-\alpha d)}{v_l}<\frac{(\beta-\gamma)(2\beta+m\gamma-3\gamma)}{m\gamma t_s}+1$. Hence, the ratio of product quality matters for sellers in platform business.

Essentially, the difference between a product's expected quality and the market average quality determines the product's price. Seller's profits are a quadratic function of their price. The pricing strategies depicted here is a type of pooling equilibrium.

Higher expected quality for a seller not only allows her to charge a higher price and gain a higher profit margin but also makes her products more attractive and generates more sales even at a higher price, resulting in the quadratic effect. There is a positive cross-side network effect on profits in terms of the number of buyers. For both low and high-quality sellers, there is a negative same-side network effect in terms of the market average quality offered by all the sellers[16]. As the market average quality rises, the seller's profit suffers.

Given the equilibrium prices, the platform's profit and buyer surplus are discussed in Corollaries 3 and 4.

Corollary 3: In the market where sellers determine their prices independently, the platform's profit is $\pi_p^{int} = \delta m n \tilde{\theta}^2 \left[\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)\tilde{v}^2}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2} [t_s(v_h - \alpha d - \tilde{v})^2 + (1-t_s)(v_l - \tilde{v})^2] \right]$ *if it does not use the market segmentation policy.*

Corollary 3 shows that the platform's profit is associated with an average seller quality level, as well as the variance of the sellers' qualities. Recall that the platform charges δ fraction of the price paid by the buyer. Since the optimal price charged by the sellers as well as the aggregate demand the platform faces depends on the average seller quality and the average buyer preference, the platform profit expression contains the quadratic terms of these entities.

[16] $\frac{\partial \pi_l^{int}}{\partial \tilde{v}} = -\frac{2(1-\delta)n}{(\beta-\gamma)} \frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))} \frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2} \left[(\beta+m\gamma-\gamma)(v_l) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma} \right] \frac{(\beta+m\gamma-2\gamma)m\gamma}{2\beta+m\gamma-3\gamma} < 0$ (similarly for high quality sellers)

Like in Proposition 2, the more is the heterogeneity among the sellers in terms of product/service quality offerings, the higher is the segmentation of the market and product/service offerings are priced according to their quality (unlike in Proposition 2, here the sellers rather than the platform are pricing their products/services). Consequently, the lesser is the possibility of products/services of higher quality being offered at lesser prices.

When there is a gap between the high and low-quality product/service (or the market is more segmented), high-quality sellers' profits increases and low-quality seller's profits shrink. This is because buyers appreciate quality and are willing to pay the price for their desired quality. The better matching of buyer's preference with seller's product/service quality offerings results in an overall increase in total seller-side profit because the profit gains by high-quality sellers surpass the profit loss by the low-quality sellers. As the platform's profit is proportional to the total seller-side profit, it will benefit from the increasing quality heterogeneity.

Moreover, note that $t_s(v_h - \alpha d - \tilde{v})^2 + (1 - t_s)(v_l - \tilde{v})^2$ represents the quality variance based on the distribution of the expected sellers' qualities. As the quality difference between suspect high-quality sellers and trustworthy high-quality sellers is unobservable and unverifiable, it doesn't affect the equilibrium outcome.

Corollary 4: When the platform does not segment the market and the sellers set prices independently, buyer i's utility is

$$u_i^{int} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 [t_s(v_h - \alpha d - \tilde{v})^2 + (1 - t_s)(v_l - \tilde{v})^2]\right]$$

Corollary 4 shows a linear positive cross-side network effect on the buyers' surplus in the form of the number of the sellers, and a quadratic positive cross-side network effect in the form of the average product quality among the sellers. It also reveals that a buyer's utility is determined by the relative quality preference of the buyer to the average quality preference of all buyers. The variance of the expected seller quality product offerings also affects buyer surplus. Similar to the discussion in Section 3, it can be shown that the average quality preference of the buyers has an indirect negative same-side network effect.

3.6.2 Segmented Market when Sellers Set Prices Independently

We now consider the platform's potential segmentation policy in the market where the sellers set prices independently. There are three different types of sellers in our model - trustworthy high-quality sellers with quality v_h, suspect high-quality sellers with quality $(v_h - d)$, and low-quality sellers with quality v_l. Theoretically, the platform can execute a segmentation policy to separate out the high-quality sellers and low-quality sellers, between whom the quality difference is evident to the buyers from the difference in prices.

For instance, an online marketplace can separate branded products from those with no brand reputation like in the case of Alibaba which segments its platform into Taobao (for C2C business) and Tmail (for branded B2C business). This segmentation does not provide additional information to buyers. Prices are a signal of quality and branded products are naturally priced higher than non-branded ones as they signal higher quality.

On the other hand, platforms often use segmentation policy to distinguish trustworthy high-quality sellers from suspect high-quality sellers, when the quality difference between them is not easily perceptible to buyers. As we have mentioned earlier, many platforms have introduced a premium seller program by which the trustworthy high-quality seller is identified. We will address the impacts of these two types of segmentation separately[17].

[17] Some sellers not marked as premium sellers could be as good as premium sellers but buyers just do not know whether that is true in advance. Such sellers are probably risk-averse to verify quality. Not signaling quality through

3.6.2.1 Segmenting Sellers with an Observable Quality Difference

When sellers set prices independently, all the information on observed quality differences is reflected in the equilibrium prices. Even though the sellers sell their products in an integrated market, they appear to be vertically differentiated to the buyers. When the platform segments the market into high quality and low-quality sellers, buyers' purchase decisions are not affected as they can perceive the quality difference from the difference in prices of the products and they are free to shop for both types of products depending on their budget. No new information about the products is presented to the buyers by the segmentation that the platform performs.

Hence, sellers also do not have any incentive to change their pricing strategy from the equilibrium outcome. This is a trivial case in terms of theoretical modeling. However, the managerial implication is that if a platform's segmentation policy does not bring additional information to buyers, that segmentation is not worthwhile. Segmentation will result in enhanced platform profit only if that segmentation provides buyers with more information to better match their preferences with product quality.

3.6.2.2 Segmenting sellers with an Unobservable Quality Difference

The platform segments the sellers into "premium sellers" and "normal sellers." It verifies trustworthy high-quality sellers and marks them as premium sellers. The suspected high-quality sellers and low-quality sellers are categorized as normal sellers. Price signals quality. As buyers can observe the quality difference between the suspect high-quality sellers and low-quality sellers in the normal seller category through price differences, the sellers are essentially vertically differentiated into three classes in the market.

---

the verification process means these sellers are not sure of the relative quality of their product compared to other high-quality sellers.

The three classes of sellers set their prices $as\ p_{th}^{seg}, p_{sh}^{seg}$, and p_l^{seg} respectively (*seg* means segmented market, *th* represents trustworthy high-quality, *sh* represents suspect high-quality). The platform incurs a cost *C* per seller to verify the seller's type and charges a price C_v from each verified high-quality product seller for the verification service. Like before, α proportion of the high-quality sellers are *suspect high-quality sellers* with true quality $(v_h - d)$, and $(1-\alpha)$ proportion of them are *trustworthy high-quality sellers* with quality v_h.

Also, $(v_h - d) < v_l$. The suspect high-quality sellers use small differences in quality to cheat the system. These sellers incur a quality flaw d due to various reasons. The product/service quality is often overstated. Imperfections in them might be intentionally skipped by sellers. For example, a host on Airbnb may not be as hospitable as is advertised on the website; an Amazon seller could use a cheap delivery service which is not always reliable. Buyers suffer in such instances. Product description and customer reviews do not provide sufficiently detailed and comprehensive data to reveal these quality flaws.

The model setup for this case is depicted in Figure 3.4 (see Appendix A). The buyers' demand functions are described in Lemma 4.

Lemma 4: When the platform segment sellers into premium and normal sellers, the demand for seller j from buyer i is

$$q_{ij}^{seg} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_{th}^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]\ if\ j\ is\ a\ trustworthy\ high\ quality\ seller$$

$$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - d - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_{sh}^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]\ if\ j\ is\ a\ suspect\ high\ quality\ seller$$

$$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_l^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]\ if\ j\ is\ a\ low\ quality\ seller$$

where $\tilde{v} = t_s(v_h - \alpha d) + (1 - t_s)v_l$ *is the average of the sellers' quality, which is the same as that in Lemma 3, and* $\tilde{p}^{seg} \equiv (1-\alpha)t_s p_{th}^{seg} + \alpha t_s p_{sh}^{seg} + (1-t_s)p_l^{seg}$, *is the mean of the prices set by all sellers in the market.*

The demand[18] function takes a similar form as previous cases. Anticipating the demand functions, the three types of sellers optimize their prices to maximize their profits. Their equilibrium prices and profits are given in Proposition 6

Proposition 6: When the platform segment sellers into premium and normal sellers, the optimal price of a trustworthy high-quality seller is

$$p_{th}^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta + m\gamma - \gamma)v_h - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_{th}^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta + m\gamma - \gamma)v_h - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2 - C_v$*;*

The optimal price of a suspect high-quality seller is

$$p_{sh}^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta + m\gamma - \gamma)(v_h - d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_{sh}^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta + m\gamma - \gamma)(v_h - d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$*;*

The optimal price of a low-quality seller is

$$p_{l}^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta + m\gamma - \gamma)v_l - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_{l}^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta + m\gamma - \gamma)v_l - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$

The segmentation policy solves the information asymmetry problem between high-quality sellers and buyers, differentiating the trustworthy and suspect high-quality sellers. When the trustworthy high-quality sellers are verified by the premium seller program, they sell at a higher price and gain more profit and suspect high-quality sellers sell at a lower price and earn less. The

[18] The condition for a positive demand function irrespective of types of buyers or products is similar to the integrated market in Lemma 3

pricing strategies depicted here is a type of separating equilibrium. We see that the quality verification charge does not affect the equilibrium prices or the profits of suspect high-quality and low-quality sellers.

Comparing *Proposition 5* and *Proposition 6*, we find that low-quality sellers' prices and profits[19] are not affected by the segmentation policy as we assume buyers are risk-neutral. The average of all the sellers' prices in the market is also not affected. Suspect high-quality seller's profit decreases as they are no more able to take advantage of the information asymmetry. Trustworthy high-quality seller's profit increases if the verification charge is below a certain level. The increase in profit is again due to a better match between buyer preferences and product quality offerings.

Thus, high-quality sellers will adopt the verification service if the price charged by the platform for verification is lower than a certain threshold value. The lower is this price charged by the platform the higher is the profit of the seller (see *Proposition 6* proof for derivations).

Proposition 7: In the market where sellers determine their prices independently, when the platform segment sellers into premium and normal sellers, the platform's profit is $\pi_p^{seg} = \delta m n \tilde{\theta}^2 \left[\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2}\tilde{v}^2 + \frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}\left[t_s(v_h - \alpha d - \tilde{v})^2 + (1-t_s)(v_l - \tilde{v})^2\right] + t_s\alpha(1-\alpha)\frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}d^2\right] + [m t_s(1-\alpha)(C_v - C)]$. *This profit is greater than its profit without the segmentation policy even without the revenue from the verification service.*

The platform's profit expression has two components – the first from the transactions after implementing the segmentation policy and the second from the verification service. The platform's profit increases in this case compared to the integrated case even without considering

[19] The condition for a positive profit function irrespective of types of buyers or products is similar to the integrated market in Proposition 5. Note that low-quality seller's price, as well as the average price of all the sellers in the market, is not affected by the segmentation policy.

the revenue earned from the verification service. The platform's profit increases with the segmentation policy due to the increase in the total seller side profit.

As the sellers' profits are a quadratic function of their identified quality, separating trustworthy high-quality sellers from suspect high-quality sellers brings more profit gains from the former than the profit loss from the latter. Hence, further vertical differentiation of the sellers results in more profits to the seller side, and consequently more benefit to the platform. Moreover, the platform uses verification service as an additional tool to increase profit. Since the platform's profit increases with segmentation, the platform can launch the premium sellers program even if the verification service incurs a net loss.

Corollary 5: The platform makes a profit from the verification service if the cost C incurred by it is less than the price charged for verification C_v*, which should be below the threshold*

$$\frac{\alpha d\tilde{\theta}^2(1-\delta)(1-\alpha)mnt_s}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(2v_h-\alpha d)-\frac{2(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{(2\beta+m\gamma-3\gamma)}\right]$$

In essence, the platform will try to maximize its profit by increasing the verification charge to the point that trustworthy high-quality sellers do not find it worthwhile anymore to use the service as it is not profitable for them. And then we are left with a market situation where suspect high-quality sellers will again start to enjoy more profit by cheating the system. The trustworthy high-quality sellers seeing a decline in profit will again start to think about adopting the verification service. The platform realizing that it has lost its extra revenue from verification will lower the verification charge.

Thus a cat and mouse game will continuously play out over time with first the platform increasing verification charges, trustworthy high-quality sellers then rejecting the service, the platform then decreasing the charges, and the same sellers now adopting the service. The platform will figure out if such a policy is profit-enhancing or a stable verification charge is better[20]. If the platform adopts the stable policy, the verification charge should satisfy

$$C < C_v < \frac{\alpha d\tilde{\theta}^2(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(2v_h-\alpha d)-\frac{2(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{(2\beta+m\gamma-3\gamma)}\right]$$

While the trustworthy high-quality seller would like to have the price charged for verification by the platform as low as possible to maximize her profit, the platform wants the verification price charged to the high-quality trustworthy seller to be above the cost of verification to make a profit from the service. This leads to tension. Nevertheless, the platform may still consider adopting the verification service even if it incurs loss through this service as long as its net profit which is the sum of its revenue from the transactions in the segmented market plus the profit/loss from the verification is above the profit earned in the integrated market.

The complex expression on the right-hand side of this inequality above indicates that this verification charge is dependent on the number of buyers and sellers, the proportion of high-quality sellers, the proportion of trustworthy high-quality sellers, the quality difference between a suspect and trustworthy high-quality sellers, as well as complex interactions between the coefficient of diminishing marginal utility, coefficient of substitution effect and the respective product quality.

Proposition 8: In the market where sellers determine their prices independently, when the platform segment sellers into premium and normal sellers, buyer i's utility is

$$u_i^{seg} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i-\frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2+\frac{1}{(\beta-\gamma)}\left(\theta_i-\frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2[t_s(v_h-\alpha d-\tilde{v})^2+(1-t_s)(v_l-\tilde{v})^2]+\left(\theta_i-\frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 t_s\alpha(1-\alpha)d^2\right]$$

[20] Frequent changes in verification price generally send a wrong signal to potential high-quality trustworthy sellers who want to join the platform.

This is greater than her utility without the segmentation policy.

The platform's segmentation policy benefits not only the seller side but also each buyer. The premium program identifies the true high-quality seller and brings more price-quality options for the buyer to choose from. The reduction in information asymmetry provides more variety to buyers when they make consumption decisions. Essentially, it ensures buyers get better value for their money spent. Hence, buyers now consume more from trustworthy high-quality sellers than from the suspect high-quality sellers. Since the buyer's utility is increasing in product quality, buyer surplus increases due to a better match. It is also to be noted that the quality verification charge does not affect buyer utility expression.

We present below the comparison of the integrated and segmented market in tabular form in Table 3.2. As discussed earlier, the platform's profit and the buyer's utility increases in the segmented case due to a better match between the buyer's product quality preferences and the seller's product quality offerings. Low-quality seller's profit remains unchanged as their equilibrium price remains unchanged since the platform does not segment low-quality sellers. Trustworthy high-quality sellers profit increases in the segmented case while suspect high-quality seller's profit decreases.

Table 3.2 Comparing Outcomes of Integrated and Segmented Market (Sellers sets Price)

	Integrated	Segmented
equilibrium price	$p_h^{int} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$	$p_{th}^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_h - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$ $p_{sh}^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)(v_h-d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$
	$p_l^{int} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_l - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$ **=**	$p_l^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_l - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$

platform profit	$\delta mn\tilde{\theta}^2 \left[\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)\tilde{v}^2}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}[t_s(v_h-\alpha d-\tilde{v})^2+(1-t_s)(v_l-\tilde{v})^2]\right]$ <	$\delta mn\tilde{\theta}^2 \left[\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2}\tilde{v}^2 + \frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}[t_s(v_h-\alpha d-\tilde{v})^2+(1-t_s)(v_l-\tilde{v})^2] + t_s\alpha(1-\alpha)\frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}d^2\right] + mt_s(1-\alpha)(C_v-C)$
seller's profit (low quality)	$\frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_l-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$ =	$\frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_l-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$
seller's profit (high quality)	$\pi_h^{int}=\frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d)-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$ < $\pi_h^{int}=\frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d)-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$ >	$\pi_{th}^{seg}=\frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_h-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2-C_v$ (if C_v is below a certain level) $\pi_{sh}^{seg}=\frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-d)-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$
buyer's utility	$\frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i-\frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2+\frac{1}{(\beta-\gamma)}\left(\theta_i-\frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2[t_s(v_h-\alpha d-\tilde{v})^2+(1-t_s)(v_l-\tilde{v})^2]\right]$ <	$\frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i-\frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2+\frac{1}{(\beta-\gamma)}\left(\theta_i-\frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2[t_s(v_h-\alpha d-\tilde{v})^2+(1-t_s)(v_l-\tilde{v})^2]+\left(\theta_i-\frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 t_s\alpha(1-\alpha)d^2\right]$

3.7 Critical Findings, Managerial Implications, and Conclusion

3.7.1 Comparing Platform Profits under Different Scenarios

We compare the platform profit for the case when the platform decides the price (i.e. the case of the integrated market with uniform price and the case of the segmented market with different prices depending on the quality of the product) and when sellers decide the price (we call them independent here) with the condition that sellers do not make false claims about their product or service quality. This comparison is an offshoot of our work on vertical segmentation pricing policy by online platforms but reveals some very interesting results.

Proposition 9: Under segmentation, when sellers do not make false claims about product or service quality, the platform's profit is higher when it sets the trading prices than when the sellers set the trading prices independently. Both these profits are greater than the profit of the platform in the integrated market with a uniform price.

In most markets, the platforms do not set trading prices directly. Instead, the sellers on the platforms determine their prices. This could be due to various reasons such as the nature of the goods or service being traded, industry tradition, information asymmetry regarding how to price the product or service, etc. However, if the platform can decide the price, *Proposition 9* shows that the platform's profit increases.

That is because when the sellers set prices independently, competition among them significantly lowers down their profit margins and hence the total seller-side profit. But when the platform sets prices, it can acquire a monopoly profit. We do know that platforms that decide the price and segments the market exists and we have discussed quite a few examples earlier. We also know now that their profits are higher than platforms where sellers decide the price depending on the quality of products offered. But we do not know the reason why such arrangements exist in the economy – i.e. why some platforms choose to operate in a segmented way and some independently. That is left as an interesting topic for future research.

Proposition 9 also shows that an integrated platform, where the platform sets a uniform price irrespective of product quality, cannot exist in the economy. It is not profitable to have an integrated platform in the economy. It is therefore not a surprise that Uber started as an integrated platform but soon moved towards a segmented one which enabled it to garner higher profits.

3.7.2 Critical Findings

In this paper, we used game theoretical models to examine several vertical segmentation policies that are commonly adopted by platform businesses. In these scenarios, buyers are multi-homing — i.e. shopping freely in all submarkets, after the sellers are segmented into different submarkets or channels. The segmentation policies benefit the platform but their impact on the sellers and buyers is different.

Specifically, we examined two types of markets - the market where the platform set trading prices for sellers and that where sellers set prices independently. Our results show that in the first case, vertical segmentation is fundamentally a price discrimination tool to gain more surpluses from the buyer side. Different from the classical price discrimination strategy, the platform's segmentation pricing policy benefits low-quality sellers but may reduce the profits of high-quality sellers.

In the market where sellers determine their prices, we show that the function of segmentation policy is to reduce information asymmetry and further differentiate the sellers. Different from the first case, this segmentation policy will result in a better match and benefit both the platform and the buyers.

We find that the optimal price and profit of the platform in an integrated market (i.e. a market without segmentation and a uniform price regardless of the quality of products/services) when the platform decides price depends on the average seller quality and the average buyer preference and the platform user heterogeneity does not seem to matter. In a segmented market when the platform decides the price, the optimal price for each segment depends on the respective quality of that segment, and the average buyer preference.

Different from previous literature, we show that the variance of product quality among sellers has a positive cross-side network effect on buyer surplus. If the difference in product quality increases, buyers optimize their consumption from low type quality products to high type quality products. This in turn increases platform profit.

Moreover, the average quality preference of the buyers not only increases the number of transactions but also raises the optimal trading price which is set by the platform thus increasing sellers' profit through a quadratic cross-side network effect. This means as the average quality preference increases, market equilibrium price increases. In addition, buyers are buying more as they appreciate the quality of the products.

We further show that whenever the platform uses quality verification service to categorize premium sellers, there is an upper limit to how much they can charge for that service so that sellers still have an incentive to use that service. Platforms may sometimes use the verification service even if it is incurring loss at rendering that service as long as the net profit from transactions plus the verification is greater than the case where no such service is used.

Finally, we show that an integrated platform business with uniform prices irrespective of the quality of the product is not viable. Moreover, a segmented platform business model where the platform decides the price is more profitable than one where sellers decide the price of products/services. Yet, both these types of platforms exist in the economy. It remains an open question of how or under what circumstances a platform can structure itself in the former way rather than the latter.

3.7.3 Managerial Implications

We discuss here the managerial implications of our work.

Sellers who are planning to enter platforms need to be mindful of their product/service quality relative to the competition.

Our analysis suggests that a segmentation policy can bring about more quality and price heterogeneity for the buyers to choose from as well as reduce information asymmetry regarding the quality of products being sold. This helps in a better match between buyer preferences and product/service quality offerings resulting in enhanced revenues for the platform. Furthermore, when the platform decides the price, compared to the one where sellers decide prices, the monopoly power brings the platform more profit.

Targeting the average buyer preference and average product quality amidst all the variation in terms of tastes or quality seems to be the optimal strategy for platform management in an integrated market. In a segmented market, targeting the average buyer preference and respective quality of that segment is the optimal strategy.

Giving buyers more choice in terms of quality benefits the platform.

Any intervention by the sellers or the platform like advertisements that play upon the psychology of the buyer and enhances her average quality preference benefits both the sellers and the platform.

Segmentation with the quality verification service that reduces information asymmetry for the buyers benefits the platform in terms of enhanced revenue. It also benefits buyers as there is a better match between what they want and what they get.

3.7.4 Conclusion and Future Research

As this paper is among the first few papers on vertical segmentation strategy by online platforms, the research focuses on its predominant functions. Quite a few potential extensions of our research can be expected to gain a more comprehensive understanding of the platform segmentation strategy. For instance, we only consider the cases of matured platform business where the size of the users remains constant. A segmentation policy can interact with platform network externalities and function differently in a rapidly-growing platform business where the numbers of users change quickly.

Our analysis results are also subject to some other assumptions, i.e. buyers are perfectly multi-homing after the segmentation, the substitution effects within and between different submarkets are assumed to be the same, the coefficient of diminishing marginal utility is same for all buyers and product variety, buyers and sellers are risk-neutral, etc. Interesting results might appear if we relax those assumptions.

4. Optimal Dynamic Platform Promotion Policy under Evolution[21]

4.1 Abstract

Two-sided digital platforms often use promotion to attract users despite user-generated interactions or promotions which also attract users to the platform. While the number of platform users on both the buyer and seller sides varies over time due to cross-side interactions and user dissatisfaction, the platform's promotion efforts interact with this organic user variation and lead to complex consequences for platform profitability. In this study, we develop dynamic programming models to explore the evolution of a platform's optimal promotion policy from its inception till maturity.

Specifically, we examine three models representing different business scenarios: (i) a most general scenario model to investigate the platform's long-run promotion strategies when the number of buyers and sellers changes over time, (ii) a simplified model to simulate the platform's short-run promotion efforts on the buyer side while the number of sellers keeps constant, and (iii) a three-period game representing the platform's inception, growth, and maturity phases to understand the platform's promotion evolution in the inception and growth periods.

The analytic models demonstrate a few insightful findings. First of all, the platform's optimal promotion efforts in any period are dependent on five key components: the ratio between the platform's prices charged to users and its promotion cost, the substitution effect of promotion between sequential periods, the cross-side effect of promotions and user interactions, user

[21] This paper was presented at the 2023 American Marketing Association Summer Conference, San Francisco, CA, and benefitted from useful comments by two anonymous reviewers and Prof. Mahmut Parlar of the DeGroote School of Business, McMaster University. We would like to thank them for their suggestions.

organic retention rate after decay due to dissatisfaction and growth of users from cross-side interactions.

We find that the optimal promotion strategy for the platform to counter user decay on one side is to have more promotion to this side in the initial stages of the platform evolution. At maturity, less promotion is needed on that side and more on the other side. Contrary to intuition, an increase in the initial number of users on one side may compel the platform to input more promotion on this side and less on the other side in the initial stages of the platform evolution.

For a traditional business, when the potential market of users increases, more promotion is required to attract some of those users. However, if the potential market of users on one side of the platform increases (assuming all these potential users transact through the platform), the platform may require lesser promotion on that side. Instead, it may require more promotion on the other side in the initial phase of the platform evolution. This is due to the increase in the cross-side interaction enabled by the increase in the number of potential users which ultimately reduces the platform's effort.

While the platform's long-run promotion strategy is gradual, path-dependent, and shaped by the cross-side network effect, its promotions in the short run are often characterized by investment spikes. The platform often invests substantial amounts of promotion effort in the initial periods to quickly build up a critical mass of users.

Bass model of same-side user interactions require front-loading of promotion to counter user decay. We find that seeding in a little bit of platform promotion attract some users who in turn bring in more users through cross-sided interactions. Platforms can free-ride on this cross-sided interaction to counter user decay. Platforms should therefore incentivize cross-side interactions, as they reduce the need for platform promotion efforts.

4.2 Introduction

In the contemporary digital landscape, many two-sided platforms, including Spotify, Netflix, Skillshare, LinkedIn, Amazon Prime, Upwork, Scribd, Adobe Creative Cloud, and Microsoft 365, strategically leverage promotional techniques to engage both buyers and sellers, thereby stimulating transactions. For example, Netflix committed approximately 2.53 billion USD to marketing endeavors in 2022, whereas Spotify allocated nearly 1.6 billion Euros for marketing and sales within the same period (Statista, March 2023). These substantial investments underscore the importance of understanding, from a managerial perspective, the workings of platform promotion in a dynamic setting characterized by intricate variables and interdependencies.

The multifaceted nature of platform promotion is marked by a range of factors and interactions. One notable aspect is the presence of cross-side network effects, which are generated from interaction between present buyers with prospective sellers or present sellers with prospective buyers, potentially attracting prospective users (buyers or sellers) to join the platform in the future. Alongside cross-side network effects, there are also same-side network effects, which are generated when present buyers interact with prospective buyers or present sellers interact with prospective sellers, also influencing prospective users (buyers or sellers) to join the platform in the future.

In addition to the network effects which help platforms to increase their user base, platforms also engage in promotional campaigns directed at prospective users on both the buying and the selling sides, which might entice prospective users to join the platform. These buyers and sellers, in turn, affect the decision of prospective buyers and sellers to join the platform in the future. The interaction of platform promotion and cross-side network effects, along with the sequential interaction of users, shapes the platform's user base on both sides. Thus, we see that the effects of promotion on one side ripple through to the other side in subsequent periods. This cyclic interplay between promotion efforts and users' decisions to join shapes the intricate dynamics of the platform ecosystem.

While promotion attracts users to the platform, not all users experience satisfaction. Consequently, some users may disengage and depart from the platform. Thus, the platform must consider this natural or organic decay in users to determine the optimal promotion required on both sides. To counter an increase in decay on one side, the platform may employ different strategies depending on the stage of evolution it is in.

Given the costs associated with promotion, the primary objective for platforms becomes optimizing promotional strategies to ensure sustained profitability over time. This leads to questions: What constitutes an optimal promotion policy for platforms? How should these platforms evolve and adapt their promotional strategies over time to effectively meet their objectives?

Taking a case in point, consider Spotify, a prominent subscription-based music streaming platform. Spotify's spending on sales and marketing increased from \$111 million to \$1.57 billion from 2013 to 2022 (Statista, 2023), with a significant portion allocated to advertisement costs. Spotify's user base grew from 18 million to 226 million from 2015 to 2023 (Statista, 2023), while subscription fees remained stable with minor changes. During this period, Spotify's dynamic promotion attracted new users to the platform on both the user and content provider side. Meanwhile, Spotify users interacted with prospective users by sharing music preferences and playlists on social media, thereby attracting new users to the platform through same-side network effects. However, some users on both sides experienced dissatisfaction and left the platform, resulting in a net decay on each side, stemming from organic user growth and decay. Moreover, Spotify users also attract prospective users like artists, musicians, and content creators through the cross-side network effect. All these processes are dynamic, with Spotify's promotion attracting new users who, in turn, attract more new users in the future.

Thus, Spotify's promotional landscape is not one-dimensional; it leverages both its own promotional campaigns and the activities initiated by its users. Moreover, Spotify's promotion continues to evolve, in terms of both the number of users and the amount spent on promotion to attract new users.

Consider Netflix, a global streaming powerhouse renowned for its diverse repertoire of movies, TV shows, documentaries, and original content. The case of Netflix demonstrates a dynamic evolution of promotional strategies. Netflix's spending on advertising increased from $533 million to $1.59 billion from 2014 to 2022 (Statista, 2023). Netflix's user base grew from 24 million to 209 million from 2011 to 2023, while subscription fees remained stable with minor changes (particularly for basic services[22]). During this period, Netflix's dynamic promotion attracted new users to the platform on both the user and content provider sides. Meanwhile, Netflix's users engaged with prospective users through word-of-mouth recommendations, ratings, and endorsements, attracting new users to the platform through same-side network effects. However, there is also user churn on both sides, with some users leaving the platform, resulting in a net decay effect on each side stemming from organic user growth and decay. Furthermore, Netflix users attract prospective users like content providers and production

[22] https://flixed.io/netflix-price-hikes

companies through the cross-side network effect. All these processes are dynamic, with Netflix's promotion attracting new users who, in turn, attract more new users in the future.

Similar to Spotify, Netflix's users influence prospective users through both the same-side and cross-side interactions. Thus, Netflix's promotional landscape thrives on the synergies of its in-house promotional endeavors and the organic promotional activities generated by its users. Moreover, this process is evolving in terms of the number of users and the amount spent on promotion to attract new users.

The case studies of Netflix and Spotify collectively underscore a salient phenomenon: the symbiotic relationship between user-initiated promotion and platform-led promotion. Users engaging in self-promotion not only accrue individual benefits but also contribute to the platform's growth by attracting new users. However, the platforms also actively engage in promotional efforts to bolster their user bases. In essence, the platform faces tradeoff decisions on several dimensions – how much promotion to do considering its costliness, whether to rely on user-generated promotion or its own, and how to manage this promotion as the platform evolves from its inception phase to the growth phase to maturity.

Prior research has predominantly explored platform strategies, including promotion, within static or limited-time dynamic settings, often focusing on well-established platforms. However, scant theoretical research delves into the intricacies of platform promotion strategies in realistic dynamic settings, wherein platform and user promotions synergistically evolve from the platform's embryonic stages to maturity. The inherent complexities, coupled with the temporal evolution of relevant variables, render the analysis of platform promotion within dynamic contexts imperative from a managerial standpoint. This paper seeks to bridge this gap in understanding by offering a comprehensive examination of platform promotion within a dynamic framework.

In this paper, we present a dynamic model of platform promotion that captures the evolving dynamics of user numbers, taking into account a few key factors: (i) The user growth catalyzed by the platform's promotional efforts targeting prospective users (ii) The net decay effect resulting from organic user growth from interactions between current and prospective same-side users and the organic user decay due to dissatisfaction, and (iii) The organic user growth resulting from interactions between current and prospective cross-side users. This tension between user growth and decay persists as the platform transitions from its inception stage to maturity.

We assume the platform charges users subscription fees, which remain stable under normal economic conditions. The marginal cost of promotion is consistent for both buyers and sellers, with existing agreements ensuring stability in these costs. These costs also include administrative expenses. For both buyers and sellers, there exists a fixed market potential, representing the maximum number of users the platform can potentially attain. We assume that given adequate amount of proportion, this potential can be reached. The profit of the platform is the revenue earned from users minus the promotion costs. The ultimate objective of the platform is then to maximize its net present value of profit.

Using dynamic optimization techniques and comparative statics for comprehensive analysis, we explore the impact of promotion across different periods on users (buyers/sellers) within the platform ecosystem which offers valuable insights for managers to tailor promotion strategies according to the platform's lifecycle stage.

By examining the Euler equations that balances the benefit and cost of one extra unit of additional promotion, we observe that promotion at any given time is influenced by the previous period's promotion efforts, as well as the number and promotion activities targeting cross-side users. Significantly, we identify five key components influencing promotion within a specific period: (i) the ratio of the price charged to users and the marginal cost of the quadratic promotion effort (ii) the substitution effect of promotion between sequential periods (iii) the cross-side effect of promotions and interactions (iv) the user retention rate after decay due to dissatisfaction and (v) the organic growth of users from cross-side interactions.

In our analysis, we explore three distinct models to examine optimal promotion strategies for the platform under different scenarios. The first model devises a platform’s optimal promotion strategy for long-term consideration where buyers and sellers exhibit variability in their behaviors. In the second model, we delve into the platform’s optimal short-term promotion strategy for buyers, when the number of sellers remains stable over a short period. The third model examines a three-period game, which simulates the inception, growth, and maturity trajectory in a condensed form. Promotion occurs during the first two periods, followed by cessation in the final period due to a change in ownership or managerial responsibility.

From the three cases, we find that platform promotion, coupled with user engagement initiatives, fosters a significant increase in user adoption, resulting in enhanced profitability. However, as platforms mature, they tend to reach a state of equilibrium where growth becomes constrained by competitive pressures and user fatigue. Additionally, we uncover that the effectiveness of promotion is influenced by the platform's historical trajectory, as highlighted by the pivotal role played by initial conditions. Such factors significantly shape the platform's long-term profitability.

In the first case, our findings reveal that as a platform matures, there is a decreased need for promotion on the side facing user decay and an increased necessity for promotion on the other side to counteract this user decay. As the potential user market expands, the platform recognizes the economic infeasibility of extensive promotion. Instead, it capitalizes on the inherent attractiveness of cross-side interactions, effectively 'free-riding' on them. Additionally, actively engaged users who attract others result in a saving of platform promotion efforts. This suggests that platforms may opt to attract more engaged users via their marketing strategies. Bass model of same-side user interactions recommend front loading of promotion to counter user decay. We find that seeding a little bit of promotion attract users who in turn attract more users through cross-side network effect thus countering user decay.

In the second case where the number of sellers remain fixed, our analysis reveals that the optimal promotion strategy for buyers remains unaffected by increases in the buyer's total market potential. Moreover, as the net decay for buyers intensifies, the platform reduces its promotion on the buyer side. Similarly, an increase in the cross-side network effect also results in reduced promotion targeting buyers. This phenomenon occurs because, with a fixed number of sellers, the platform is unable to fully leverage the benefits of the cross-side network effect within a limited timeframe. We also find that platform promotion often leads to sporadic spikes, aimed at intensifying promotional activities to attract a critical mass of buyers.

In the context of the third case, promotion effort during the second period is dependent only on the prices charged on platform users and promotion cost, regardless of the initial number of users or the market size. In contrast, optimal promotional efforts during the first period depend on various factors, including user prices, promotional costs, market potential, and the initial number of users. We find that if the initial cross-side network effect is above a certain threshold, a larger market potential of users on one side leads to decreased initial promotion effort on that side. Instead, there is increased initial stage promotion effort on the other side. Moreover, an increase in the initial number of users on one side necessitates more promotion on that side and less on the other during the platform's initial stages of evolution, while an increase in cross-side users results in reduced promotion on that side. Furthermore, an increase in net decay on one side results in increased initial promotion on that side and decreased initial promotion on the other side. Conversely, a rise in the cross-side network effect leads to reduced initial promotion. The phenomena in the third case arise due to the strategic decision-making ability influenced by backward induction possessed by platform managers, as there is no promotion in the last stage. Consequently, the first stage promotion decision becomes crucial.

Our study's managerial implications highlight the importance of fostering cross-side network effects, such as present and prospective user interactions, through incentivization strategies. Engaged users who attract others are advantageous for platforms, as they reduce the platform's promotion efforts and enable access to a larger potential market with minimal promotion. Conversely, unengaged users, yielding insufficient revenue, represent a suboptimal outcome for platforms. They should be screened out following a cost-benefit analysis. Platforms must remain vigilant to the influence of initial conditions during the inception phase, as these determinants significantly shape future profitability trajectories.

Managers can address user decay on one side with heightened promotion on that side during the initial stages and reduced promotion during maturity. Alongside, there should be additional promotion on the other side to attract users on that side. This would lead to broadening the choices for prospective users on the side facing user decay and facilitate their adoption of the platform. In the short run, platform managers can recommend intensive promotion to build a critical mass of users, but a rise in net user decay or cross-side network effect on one side may necessitate decreased promotion on that side.

Contrary to intuition, managers may find it beneficial to increase promotion on the side with the initially higher number of users during the early stages of the platform's evolution. This strategy is rooted in understanding both current and future requirements, leveraging the cross-side network effect effectively. But when the cross-side network effect is significant and net organic decay is manageable, reduced promotion is needed for the side with the larger potential market. Instead, it becomes crucial to allocate more promotion efforts to the opposite side during the initial stages. This approach aims to capitalize on the dynamics of the cross-side network effect while ensuring effective growth and engagement on both sides.

While the model discussed in this paper focuses on subscription platforms, the insights derived extend beyond this specific context to transactional platforms characterized by consistent average user revenues over time. Our findings can be generalized due to the stability of average transactional revenue per user and constant subscription fees per user, resulting in a consistent total revenue stream per user for the platform over time.

Prior research has predominantly examined dynamic pricing and the impact of same-sided user interactions on revenue. Our investigation bridges this gap in the literature by scrutinizing dynamic promotion strategies over a platform's lifecycle and their interaction with cross-side network effects. In doing so, we contribute to both the platform and advertisement literature.

The paper is structured as follows: Section 2 provides an in-depth exploration of the existing literature on digital platforms and promotion. Section 3 presents the construction of our theoretical model, focusing on the dynamics of platform promotion. In Section 4, we analyze the optimal platform promotion strategy when both buyers and sellers exhibit variability. Section 5 examines the optimal platform promotion strategy in scenarios where sellers are fixed and buyers vary. Section 6 delves into the optimal platform promotion strategy within a three-period lifespan, where promotion occurs in the initial two periods, followed by its absence in the concluding period. This case mirrors the inception-growth-maturity trajectory, condensed into three periods, illustrating scenarios such as short-lived platforms, limited-time promotions, product testing, events, and pilot projects. Finally, Section 7 summarizes the essence of the paper, discusses its managerial implications, acknowledges limitations, and suggests avenues for future research.

4.3 Related Literature

In this section, we first discuss the general literature on two-sided platforms and network effect. Following that, we discuss the literature on platform promotions and relevant studies concerning firm's dynamic promotion strategies.

4.3.1 General Literature on Platforms

The literature on platforms traces its roots back to the seminal work of Katz and Shapiro (1985) who introduced the concept of network effect or network externality. This idea suggests that the benefit a user derives from a good or service is positively influenced by the number of other users. Evans (2003) and Evans et al (2006) expanded upon this notion by discussing the multi-sided nature of platforms, which differs from traditional businesses, and how they drive innovation and transform industries by facilitating interactions among multiple stakeholders, resulting in synergies.

Building on Katz and Shapiro's (1985) work, Rochet and Tirole (2003, 2006), as well as Armstrong (2006) formally established the literature on platforms. They demonstrated how network effects among users have significant implications for platform strategies. Concerning themselves with two sided monopoly platforms consisting of a buyer and a seller side, they distinguished two types of network effects – same-side interactions and cross-side interactions among users (buyers and sellers) as well as two types of costs and benefits – per-transaction and fixed. Given the number of users, these papers derived the equilibrium price for optimal profitability of monopoly platforms.

Weyl (2010) took a different approach by considering the monopoly platform's problem in terms of allocation choce, i.e. given a suitable pricing structure to attract users, what is the optimal number of users on the platform. He simplified and generalized the analysis of network industries, showing that critical properties of platforms depend on the source of user heterogeneity. The key finding was that monopoly platforms can use tariffs to avoid coordination failure and implement any desired allocation.

Caillaud and Jullien (2003) highlighted the "chicken and egg" problem faced by platforms: to attract buyers, a platform must have a critical mass of sellers, but sellers will only register with the platform if they expect a critical mass of buyers. Subsequent work by Hagiu (2006, 2007), Hagiu and Spulber (2013), and Hagiu and Wright (2015) explored how platforms implement strategies to coordinate users on both the buyer and seller sides, considering the network effect.

Chu and Manchanda (2016) found that direct network effects had limited impact on platform growth for C2C platforms, whereas the cross-side network effects were substantial. There was an asymmetric cross-side network effect, with sellers on the platform having a larger impact on buyer growth compared to the reverse.

4.3.2 Platform Promotion Literature

The literature on two-sided platforms and network effects laid the foundation for understanding the dynamics of platform promotion. Scheinbaum (2016) explores how consumer engagement on online or mobile platforms offers companies a means to interact with consumers, cautioning against over-engagement which may yield diminishing returns. Bruce et al. (2017) investigate the impact of creative format, messaging, and targeting on digital ad performance over time, highlighting the effectiveness of retargeted promotions contingent upon price incentives. They note that while dynamic formats often yield higher carry-over rates compared to static formats, the latter can still be effective for price promotions and retargeting.

Voorveld et al. (2018) reveal that customers' engagement with social media platforms influences their responsiveness to promotions within these platforms, emphasizing the context-specific nature of engagement and its impact on advertising evaluations. Costello and Reczek (2020) distinguish between provider-focused and platform-focused marketing communications in peer-to-peer brands, finding that consumers exhibit greater willingness to buy and pay when empathy considerations for individual providers are emphasized.

Fang et al. (2015) analyze the direct and indirect effects of buyers and sellers on search advertising revenues in B2B electronic platforms, uncovering nuances such as new sellers outbidding existing ones in mature stages but the reverse occurring at launch stages due to signal quality asymmetry. Fong et al. (2019) delve into the opportunity costs of targeted promotions, noting their potential to boost sales of promoted and similar products while potentially hindering non-targeted product searches, leading to variable sales.

Rietveld et al. (2019) discuss how platform sponsors can leverage promotions to reward successful complements, drive attention to underappreciated ones, and influence consumer perceptions through selective advertisement. They underscore the strategic investment in complements and the use of selective advertisement to manage ecosystem value creation and capture.

Overall, this body of literature predominantly focuses on targeted and search ads, often employing sophisticated algorithms to target customers based on data from various sources, including social media. However, it tends to overlook the significant impact of user interactions. Some research deals with the network effect in platforms as discussed earlier. These papers consider the users interaction effect on platforms which can be considered a sort of promotion with no direct involvement of the platform. Further research could explore the strategic implications of platform advertising on buyers, sellers, and complements within the platform ecosystem in the presence of user interactions.

4.3.3 Dynamic Promotion Strategies

We now delve into the literature on dynamic promotion strategies, which are characterized by repeated promotions over time, contrasting with static models commonly used in traditional business settings.

In the early works within this domain, Nerlove and Arrow (1962) extended the Dorfman-Steiner (1954) criteria to explore optimal advertising policies under dynamic conditions, considering how present promotions influence future demand. Papatla and Krishnamurthi (1996) examined the dynamic effects of promotions on brand loyalty and customers' price sensitivity, revealing that increased coupon purchases may erode brand loyalty while reinforcing the impact of features and displays on brand choice.

Krishnan and Jain (2006) demonstrated that the optimal dynamic promotion policy for new products hinges on factors such as advertising effectiveness, discount rates, and the advertisement-to-profits ratio. Doganoglu and Klapper (2006) analyzed weekly advertising policies in consumer goods markets, highlighting the significance of firm goodwill in determining advertising intensity, particularly in persuasive advertising contexts.

Bass et al. (2007) emphasized the importance of dynamically using different advertising types and versions, considering interaction effects, wear-out, and forgetting within advertising campaigns. They argued that such a nuanced approach can optimize resource allocation and scheduling, leading to improved demand.

These papers reveal that the strategies for optimal dynamic promotions are a bit different from the optimal promotion in the static case as the firm needs to consider the time value of the profits and the evolving nature of various variables with time. Path dependency is highlighted in dynamic promotion. One factor that deserved consideration is the wear out or decay factor as advertisement effectiveness decays with time. The other important factor is the interaction effect between different types of advertising. The firm in deciding advertising intensity also needs to consider certain factors like goodwill which is a crucial component of a company's overall value. But quantifying goodwill can be challenging because it is not a tangible asset which can be easily measured.

4.3.4 Gap in the Literature

It's evident that while promotion strategies in traditional businesses have embraced dynamic models, the literature on platform-based promotion has predominantly leaned towards static or limited-time dynamic models. This discrepancy is notably reflected in the emphasis on targeted ads, often driven by algorithms, within platform promotion literature. Unlike traditional business promotion, which encompasses a range of dynamic strategies, including varying ad formats and timing, platform promotion studies tend to concentrate on targeted ad campaigns. Furthermore, this literature often segregates considerations for platform-initiated promotions, user-generated promotions, and the impact of network effects, treating them as distinct entities rather than interconnected components of a comprehensive promotion strategy.

Despite the breadth of research on platform advertising, there is a notable gap in the marketing literature regarding the theoretical study of how platform promotion efforts on users acting in tandem with users' promotion evolve over time especially from the platform's initial phase of operation till maturity, and how as a result platform profitability also changes with time. This paper aims to address this gap, providing insights applicable to a wide range of platforms undergoing evolution in its operation. Platform promotion along with user interaction in a dynamic context is new area that this paper attempts to address.

4.4 Model Setup

We develop three models to examine platform's optimal promotion strategies in different scenarios. In each model, we consider a multi-period game. The games in the three models share the same framework with different settings in time frame and seller variation. Specifically, in the first model, we consider a T-period game (T being large) where the number of buyers and sellers vary in each period, and each period denotes a month. In the second model, we consider a T-period game (T being large), in which sellers remains constant while buyers vary in each period, and each period denotes an hour. In the third model, we consider a three-period game, where buyers and sellers vary in each period with the platform launching promotions in the first two periods but not in the third period, and each period denotes several years.

The first model depicts platforms' general promotion concerns for long-term success. The second model delves into the platforms' optimal short-term promotion strategy on the buyer side, when the number of sellers remains fixed as it takes time to verify seller's quality. The third model is a three-period game, which depicts the platform's inception-growth-maturity trajectory in a condensed form with promotion only in the first two periods due to a change in ownership or managerial responsibility.

The three models share the following common settings –

$N_t^B,\ N_t^S$ are buyers and sellers on the platform at time t. $N_0^B,\ N_0^S$ are buyers and sellers on the platform at time $t = 0$. They can be considered naive buyers and sellers who join the platform out of curiosity or goodwill (maybe they are acquaintances of the platform owner). $\overline{N^B},\ \overline{N^S}$ are the total numbers of potential buyers and sellers in the market. We assume that this potential pool of buyers and sellers do not change during the time period under consideration. $\left(\overline{N^B} - N_t^B\right)$, $(\overline{N^S} - N_t^S)$ denote the prospective buyers and sellers at any time t. The prospective buyers and sellers denote those from the potential pool of buyers and sellers who are not transacting through the platform at any moment whom the platform wants to attract by targeting them with promotion.

$P^B,\ P^S$ are price charged to buyers and sellers respectively which remain constant for the time period under consideration. $P^B,\ P^S$ can be thought of as registration, subscription, or membership fees which remain unchanged for relatively long periods of time. To simplify the analytical framework, we assume stable prices, a condition met when the economic environment remains unaffected by major shocks.

Historical data indicates that substantial changes in pricing occur infrequently (e.g. Spotify has made few changes to its subscription fees[23] since inception). The pricing considered in this paper is thus based on the number of users and not on the transactions. We know that while platform pricing decision takes time, promotion is more dynamic. The platform is conservative in changing prices frequently as it affects a large number of users. Moreover, this

[23] https://www.theverge.com/2023/7/24/23805364/spotify-us-price-increase-10-99-a-month-9-99-month-twelve-years

pricing perspective facilitates an in-depth exploration of the dual dynamics of user and platform promotions as the platform matures. While dynamic pricing and growth influenced by the same-sided effect (for e.g. the diffusion model of Bass (1969)) have been extensively investigated, the focal point of this paper is the intricate interplay between dynamic promotion and platform evolution, as well as how the cross-side network effects influence promotional strategies.

$A_t^B,\ A_t^S$ are the platform promotion efforts per user to attract buyers and sellers which can be considered conversion factors $(0 \leq A_t^B,\ A_t^S \leq 1)$. $A_0^B,\ A_0^S$ are the platform promotion effort at time $t = 0$ which may or may not be zero. The cost for promotion per user at time t is $c\left(A_t^i\right)^2, i \in (B, S),$ where c is the marginal cost of the quadratic promotion effort for both buyers and sellers which remain constant. Pre-decided contracts by the platform with vendors for managing promotion for the coming time period means marginal cost of promotion is constant for the time period under consideration. All administrative costs are assumed to be included in this promotion cost as the primary purpose of the digital platform is to attract and retain customers with promotion.

β is the discount factor for the time value of money $(0 < \beta < 1)$. Present users (buyers or sellers on the platform) connect with prospective users (sellers or buyers on the platform) on the cross-side through surveys, feedbacks, targeted ads, reviews, and other forms of interactions. These represent cross-side network effects or benefits (as depicted in Armstrong (2006) but with a modification as these denote interactions between present and prospective cross-side users unlike present cross-side users) independent of the platform's promotion effort. $\eta^B,\ \eta^S$ are coefficients that describe how present sellers/buyers interact with prospective buyers/sellers for the platform with $0 < \eta^B,\ \eta^S < 1$.

User dissatisfaction results in decay. For buyers, this could be caused by various factors including prices being charged, unavailability of products, influences of other buyers in relating their negative experience, etc. For sellers, this could be caused by a lack of adequate sales, influences of other sellers in relating their negative experience, presence of competitors on the platform, etc. On the other hand, there is user growth due to the same-sided interaction between present and prospective buyers or present and prospective sellers.

As depicted in Chu and Manchanda (2016), the same-sided direct network effect on the growth of users of the platform is negligible. We therefore avoid any independent term to denote the growth of users due to the same-sided direct network effects and include it within the terms for decay (hence it is the net decay after taking into account growth due to the direct network effect). This simplifies our analysis without compromising on the rigor. Hence, δ^B, δ^S $(0 \leq \delta^B, \delta^S \leq 1)$ denote buyers' and sellers' net decay coefficients after taking into account organic growth due to the same-sided interaction between present and prospective users. δ^B, δ^S thus represent the same-sided effects.

$\delta^B,\ \delta^S,\ \eta^B,\ \eta^S,\ \overline{N^B},\ \overline{N^S},\ P^B,\ P^S,\ \beta,\ c$ are given. Note that given the parameter values, the following conditions hold as they seem natural for most operating platforms: $\overline{N^B} \gg N_0^B \gg 1$, $\overline{N^S} \gg N_0^S \gg 1$, as the market potential of users is much larger than the initial number of users which again is larger than single users on the platform. Similar to Chu and Manchanda (2016) we assume there are significant cross-side network effects. Hence, $\eta^B,\ \eta^S$ have values which signify a moderate to high cross-side network effect. Moreover, $\eta^B > \eta^S$, as the cross-side network effect of present sellers interacting with prospective buyers is larger than the cross-side network effect of present buyers interacting with prospective sellers (as found in Chu and Manchanda (2016)). $\overline{N^B} \gg \overline{N^S}$, $N_0^B > N_0^S$ as the market potential for buyers is much larger than the market potential of sellers and the initial number of naive buyers joining the platform is larger than the initial number of naïve sellers.

For the analytical derivation of results, we also assume the following relations –

$P^S > P^B > c$. This seems reasonable as the price charged to sellers is generally much greater than buyers which should be greater than c, the marginal cost of the quadratic promotion effort. $\delta^B \geq \delta^S$ as the decay for buyers tends to be larger than sellers (buyers have a high churn rate while sellers tend to wait to see how the business prospects unfolds before quitting).

$N^B > N^S$ (although $N^B,\ N^S$ are equilibrium or steady state values which emerge from the analysis, since the market potential for buyers $\overline{N^B}$ is much larger than the market potential of sellers $\overline{N^S}$, it is reasonable to state that the steady state number of buyers is likely to be greater than the steady state number of sellers). $A^B < A^S$ (given the earlier defined relationships and assumptions, this is likely when we consider the expression for the steady state values for A^B and A^S). This also seems reasonable given the number of buyers in steady state is greater than the number of sellers in steady state with the platform having a symmetric promotion structure on buyers and sellers.

A table summarizing the meaning, magnitude assumptions, and justifications of key variables and parameters discussed in the model setup is given below in Table 3.3.

Table 3.3 Summary of the meaning, magnitude assumptions, and justifications of key variables and parameters discussed in the model setup

Items	Meanings	Magnitude Assumptions	Justifications
T	Denotes the total time period for a T period game	T is large	We want to solve for steady state values of the dynamic programming model
B, S	Buyers, sellers	NA	NA
N_t^B, N_t^S	Buyers and sellers on the platform at time t	Magnitude are determined from the optimization process	NA
N^B, N^S	Steady state values for the number of buyers and sellers respectively	$N^B > N^S$	Since the market potential for buyers $\overline{N^B}$ is much larger than the market potential of sellers $\overline{N^S}$, it is reasonable to state that the steady state number of buyers is likely to be greater than the steady state number of sellers

$\overline{N^B}, \overline{N^S}$	Total number of potential buyers and sellers in the market	$\overline{N^B} \gg \overline{N^S}$	Market potential for buyers is generally much larger than the market potential of sellers
N_0^B, N_0^S	Initial number of naïve buyers and sellers on the platform	$\overline{N^B} \gg N_0^B \gg 1$, $\overline{N^S} \gg N_0^S \gg 1$, $N_0^B > N_0^S$	The market potential of users is much larger than the initial number of users which again is larger than single users on the platform, initial number of naive buyers joining the platform is larger than the initial number of naïve sellers
$(\overline{N^B} - N_t^B)$, $(\overline{N^S} - N_t^S)$	Denotes the prospective buyers and sellers at any time *t*	Magnitude are determined from the optimization process	NA
P^B, P^S	Price charged to buyers and sellers respectively which remain constant for the time period under consideration	$P^S > P^B$	The prices can be thought of as registration, subscription, or membership fees which remain unchanged for relatively long periods of time. Price charged to sellers is generally much greater than buyers.
c	Marginal cost of the quadratic promotion effort for both buyers and sellers which remain constant	$P^S > P^B > c$	Pre-decided contracts by the platform with vendors for managing promotion implies marginal cost of promotion is constant. Cost of promotion should be less than the price charged to users for a business to be profitable
A_t^B, A_t^S	Platform promotion efforts per user to attract buyers and sellers	$0 \leq A_t^B,\ A_t^S \leq 1$	Platform promotion efforts can be considered conversion factors
A^B, A^S	Steady state values for the promotion effort on buyers and sellers	$A^B < A^S$	Given $N^B > N^S$, this is likely when we consider the expression for the steady state values for A^B and A^S
$c(A_t^i)^2$, $i \in (B, S)$	Cost for promotion per user at time *t* where *c* is the marginal cost of the quadratic promotion effort for both buyers and sellers	NA	The quadratic cost captures the concept of increasing marginal costs and diminishing returns to scale
β	Discount factor for the time value of money	$0 < \beta < 1$	The platform therefore needs to maximize the net present value of its profit considering the time value of money
η^B, η^S	Coefficients that describe how present sellers/buyers interact with prospective buyers/sellers for the platform	η^B, η^S have values which signify a moderate to high cross-side network effect, $1 > \eta^B > \eta^S > 0$	There are significant cross-side network effects, and the cross-side network effect of present sellers interacting with prospective buyers is larger than the cross-side network effect of present buyers interacting with prospective sellers (Chu and Manchanda (2016))
δ^B, δ^S	Denote buyers' and sellers' net decay coefficients after taking into account organic growth due to the same-sided interaction between present and prospective users	$1 > \delta^B \geq \delta^S > 0$	Decay coefficients range from zero to one, decay for buyers tends to be larger than sellers (buyers have a high churn rate while sellers tend to wait to see how the business prospects unfolds before quitting)

We then model the transition equation for the numbers of buyers and sellers between periods t and $t + 1$ as –

$$N_{t+1}^B = (1 - \delta^B)N_t^B + (\overline{N^B} - N_t^B)(A_t^B + \eta^B N_t^S) \quad \text{.......................................} \quad (1)$$

$$N_{t+1}^S = (1 - \delta^S)N_t^S + (\overline{N^S} - N_t^S)(A_t^S + \eta^S N_t^B) \quad \text{.......................................} \quad (2)$$

Consider equation 1 which depicts the number of buyers on the platform at time $t + 1$ (equation 2 for sellers have similar interpretation). The transition equation shows that the number of buyers in the next period N_{t+1}^B is determined by the following factors –

(i) The net decay $\delta^B N_t^B$ (after considering growth due to the same-sided interaction of present and prospective buyers) in the number of buyers in period t. Hence, the number of buyers remaining for period $t+1$ is $(1-\delta^B)N_t^B$

(ii) The addition of buyers $\left(\overline{N^B}-N_t^B\right)A_t^B$ due to the platform's promotion effort A_t^B on prospective buyers $(\overline{N^B}-N_t^B)$ in period t

(iii) The addition of buyers $(\overline{N^B}-N_t^B)\eta^B N_t^S$ due to the platform's present sellers' N_t^S independent effort to attract prospective buyers $(\overline{N^B}-N_t^B)$ in period t which is dependent on the cross-side network effects η^B

The platform's operations are depicted in Figure 4.1 below[24].

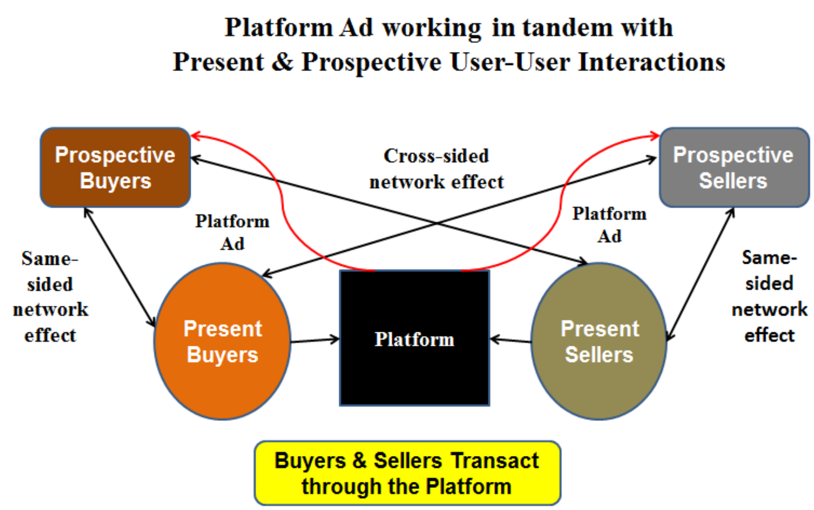


Figure 4.1: Platform Promotion Working in Tandem with

Present and Prospective User-User Interactions

[24] Note that although there is the possibility of interaction of prospective buyers and prospective sellers, they hardly contribute to the user growth on the platform. Prospective sellers would like to know the taste and preferences of current buyers on the platform before entering the platform. Similarly, prospective buyers would like to know the product and service offerings of current sellers on the platform before deciding to transact. Moreover, slightly different from the platform literature (e.g. Armstrong (2006)), the same-sided interaction in this paper refers to the interaction between present and prospective users on the same side (i.e. present buyer – prospective buyer or present seller – prospective seller) rather than the interaction between present same–sided users (i.e. present buyer – present buyer or present seller – present seller). Similarly, the cross-side interaction in this paper refers to the interaction between present and prospective users on the cross-side (i.e. present buyer – prospective seller or present seller – prospective buyer) rather than the interaction between the present cross–side users.

The following formula gives the platform's profit –

$$\pi_t^{platform} = P^B N_t^B + P^S N_t^S - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2 - c\left(\overline{N^S} - N_t^S\right)(A_t^S)^2 \quad \text{………….………. (3)}$$

This platform profit is the revenue collected from buyers and sellers minus the cost of promotion to prospective buyers and sellers in period t. Given the time discount factor β, the platform wants to maximize the net present value of its profit over a time frame.

In assuming the transition equations and the platform's profit function, we decided to keep the model simple and parsimonious. The platform's objective can then be written as –

$$Max_{\,A_t^B,\ A_t^S} \sum_0^T \beta^t \{P^B N_t^B + P^S N_t^S - c\,(\overline{N^S} - N_t^S)(A_t^S)^2 - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2\} \ s.t.$$

$$N_{t+1}^B = (1-\delta^B)N_t^B + (\overline{N^B} - N_t^B)(A_t^B + \eta^B N_t^S) \ \text{ and}$$

$$N_{t+1}^S = (1-\delta^S)N_t^S + (\overline{N^S} - N_t^S)(A_t^S + \eta^S N_t^B) \quad \text{…………………………………………. (4)}$$

Given that all variables change over time, the tools of dynamic programming are best suited to solve this problem. Note that choosing promotion effort on buyers and sellers at time t is equivalent to choosing the number of buyers and sellers at time $t+1$. State variables are N_t^B, N_t^S. Choice variables are $A_t^B, A_t^S \geq 0$. The Bellman Equation is –

$$V(N_t^B, N_t^S) = Max_{\,A_t^B,\ A_t^S}\{P^B N_t^B + P^S N_t^S - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2 - c\left(\overline{N^S} - N_t^S\right)(A_t^S)^2 +$$
$$\beta V(N_{t+1}^B, N_{t+1}^S) + \lambda_t\left[(1-\delta^B)N_t^B + \left(\overline{N^B} - N_t^B\right)(A_t^B + \eta^B N_t^S) - N_{t+1}^B\right] + \mu_t\left[(1-\delta^S)N_t^S +\right.$$
$$\left.\left(\overline{N^S} - N_t^S\right)(A_t^S + \eta^S N_t^B) - N_{t+1}^S\right] + \rho_t A_t^B + \sigma_t A_t^S\} \quad \text{………………………….……………. (5)}$$

where $\lambda, \mu, \rho, \sigma$ are Lagrange multipliers and V is the value function.

4.5 Platform Promotion when Buyers and Sellers Varies

We first consider a dynamic scenario where both buyers and sellers vary over time which adds complexity to the platform promotion strategy. This situation is indeed more reflective of real-world scenarios where the user base on both sides of the platform is subject to change. It represents the most general scenario for a platform to design long-run promotion. In this case, both the number of buyers and sellers are subject to change in each time period. This introduces an additional layer of uncertainty and variability to the platform's user base. The platform needs to adapt its promotion efforts to cater to changing user dynamics. With varying numbers of buyers and sellers, the platform needs to strike a balance between the supply and demand sides of the marketplace. This requires continuous monitoring of user trends to ensure that both sides of the platform remain engaged and active.

The interaction between buyers and sellers becomes even more critical in this dynamic environment. Cross-side network effects, where the growth of one side positively influences the other, can play a significant role in driving user engagement and organic growth. The challenge lies in optimizing platform promotion efforts to ensure that both sides of the platform continue to grow and interact. The platform must decide how much promotion to allocate to each side, taking into account factors like the rate of user decay, cross-side network effects, and the cost-effectiveness of promotion.

The dynamic nature of this scenario creates a feedback loop where the platform's promotion efforts in one period can impact the user base in subsequent periods. This implies that the platform's actions have ripple effects over time, making the decision-making process more intricate. Overall, the dynamic scenario of platform promotion with varying buyers and sellers underscores the need for a sophisticated and adaptive approach. The platform must continuously evaluate and refine its promotion efforts to ensure sustained growth, user engagement, and the maximization of platform profit as well as benefits for all participants.

We consider a *T*-period game (*T* being large) where the number of buyers and sellers vary in each period, and each period denotes a month. We solve the model via dynamic programming algorithm. Given that *T* is large, we can solve for steady state values.

Proposition 1: The platform's optimal promotion efforts and the numbers of its buyers and sellers at period $t+1$ are defined by the following equations.

$$A_{t+1}^{B} = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A_t^B + 2\eta^S \left(A_{t+1}^S\right)\left(\overline{N^S} - N_{t+1}^S\right) + \left[(1-\delta^B) - \eta^B N_{t+1}^S\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^B) - \eta^B N_{t+1}^S\right]$$

$$A_{t+1}^{S} = \left[\left(\frac{P^S}{c}\right) - \left(\frac{2}{\beta}\right) A_t^S + 2\eta^B \left(A_{t+1}^B\right)\left(\overline{N^B} - N_{t+1}^B\right) + \left[(1-\delta^S) - \eta^S N_{t+1}^B\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^S) - \eta^S N_{t+1}^B\right]$$

(Please refer to Appendix B for Figures, Tables, and Proofs related to Lemmas, Propositions, and Corollaries)

Proposition 1 reveals the "Euler equation" which relates the costs and benefits of one extra unit of promotion. The "Euler equation" can be rearranged to express promotion effort at time $t+1$ as a function of the promotion effort in time t as depicted in proposition 1. If the platform faces a chance of termination of its operation in every period, the Euler equation can be modified[25].

[25] A practical challenge in business practice is that a platform is unsure when it reaches maturity in its operations (and soon will decline) or there is still space for business growth. What if the platform is unsure whether a certain period is the end of its operation? Let α be the termination probability of the platform's operation for each period. The platform's optimal promotion efforts and the numbers of its buyers and sellers in this stochastic horizon model at period $t+1$ are defined by the following equations:

$$A_{t+1}^{B} = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{(1-\alpha)\beta}\right) A_t^B + 2\eta^S \left(A_{t+1}^S\right)\left(\overline{N^S} - N_{t+1}^S\right) + \left[(1-\delta^B) - \eta^B N_{t+1}^S\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^B) - \eta^B N_{t+1}^S\right]$$

$$A_{t+1}^{S} = \left[\left(\frac{P^S}{c}\right) - \left(\frac{2}{(1-\alpha)\beta}\right) A_t^S + 2\eta^B \left(A_{t+1}^B\right)\left(\overline{N^B} - N_{t+1}^B\right) + \left[(1-\delta^S) - \eta^S N_{t+1}^B\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^S) - \eta^S N_{t+1}^B\right]$$

Given the symmetric nature of promotion on buyers and sellers, let us consider the promotion effort on buyers (the promotion effort on sellers has similar interpretation). The promotion effort on buyers is given by the equation which can be divided into five parts – price-cost ratio, substitution effect of promotion between sequential periods, cross-side effect of promotions and interactions, user retention rate, and organic growth.

$$A_{t+1}^{B} = \left[\underbrace{\left(\frac{P^{B}}{c}\right)}_{\text{price-cost ratio}} - \underbrace{\left(\frac{2}{\beta}\right)A_t^{B}}_{\text{substitution effect between sequential periods}} + \underbrace{2\eta^{S}\left(A_{t+1}^{S}\right)\left(\overline{N^{S}} - N_{t+1}^{S}\right)}_{\text{cross-side effects of promotions and interactions}} + \left[\underbrace{(1-\delta^{B})}_{\text{retention rate}} - \underbrace{\eta^{B}N_{t+1}^{S}}_{\text{organic growth}}\right]^{2}\right]^{\frac{1}{2}} + \left[\underbrace{(1-\delta^{B})}_{\text{retention rate}} - \underbrace{\eta^{B}N_{t+1}^{S}}_{\text{organic growth}}\right]$$

(a) *Price-Cost Ratio:* The promotion effort is affected by the ratio between the price charged to users and the marginal cost of the quadratic promotion effort $\left(\frac{P^B}{c}\right)$ (the price-cost ratio)

(b) *Substitution Effect between Sequential Periods:* The expression $\left(\frac{2}{\beta}\right)A_t^B$ considers the promotion effort from the same side in the previous period adjusted by the discount factor. This reflects the platform's calculation of the cost and benefit of investing in extra promotion.

(c) *Cross-Side Effects of Promotions and Interactions:* The promotion effort also takes into account the cross-side effect of promotion on the remaining cross-side users, along with the cross-side interaction between prospective users and the existing user base $\eta^S(A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right)$. This highlights the platform's strategic consideration of optimal

---

Here, β is replaced by $\beta(1-\alpha)$ in the Euler equation connecting the optimal promotion effort between two subsequent periods. Another way to incorporate a sudden end of the platform's operation is to use Bayesian updating in each period which takes into account previous information to update the probability that the platform's operation will end at a certain period.

promotion on one side while considering the cross-side users and promotions, while leveraging the network externalities to attract more users once the initial promotion efforts are in place. An example is how platforms like Spotify strategically collaborate with influencers and media outlets, taking advantage of user-generated content sharing on social media to foster engagement and community.

(d) *Retention Rate:* The optimal platform promotion on prospective users in any period is dependent on the number of users remaining on the platform in the previous period. This is the retention rate. Given that some users leave the platform due to dissatisfaction which is denoted by the decay coefficient δ^B, $(1-\delta^B)$ is the retention rate.

(e) *Organic Growth:* The optimal platform promotion also takes into account the organic growth $\eta^B N_{t+1}^S$ due to cross-side interactions independent of the platform's promotion. Organic growth reduces the platform promotion effort. $(1-\delta^B)-\eta^B N_{t+1}^S$ can therefore be considered as the organic fluctuation of users.

Overall, this multifaceted approach to promotion reflects the platform's strategic decision-making process. By considering price-cost dynamics, cross-side promotion and network effects, user interaction and organic growth, and retention rate, the platform aims to optimize its promotion efforts for sustainable growth and user engagement while minimizing costs.

We explore the steady-state outcome where the platform's promotion strategy achieves a fixed level and the number of buyers and sellers also achieve a mature level. Specifically, in the steady state, $A_{t+1}^B = A_t^B = A^B$, $N_{t+1}^B = N_t^B = N^B$, $A_{t+1}^S = A_t^S = A^S$, $N_{t+1}^S = N_t^S = N^S$.

Proposition 2: The steady-state values of promotion effort, number of buyers and sellers, and platform profit are given by:

$$A^B = \left[\left(\frac{P^B}{c}\right) + 2\eta^S (A^S)\left(\overline{N^S} - N^S\right) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right],$$

$$A^S = \left[\left(\frac{P^S}{c}\right) + 2\eta^B (A^B)\left(\overline{N^B} - N^B\right) + \left[(1-\delta^S) - \frac{1}{\beta} - \eta^S N^B\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^S) - \frac{1}{\beta} - \eta^S N^B\right],$$

$$N^B = \frac{\overline{N^B}(A^B + \eta^B N^S)}{[A^B + \eta^B N^S + \delta^B]}, \quad N^S = \frac{\overline{N^S}(A^S + \eta^S N^B)}{[A^S + \eta^S N^B + \delta^S]},$$

$$\pi_{ss} = \frac{P^B \overline{N^B}(A^B + \eta^B N^S) - c\delta^B \overline{N^B}(A^B)^2}{[A^B + \eta^B N^S + \delta^B]} + \frac{P^S \overline{N^S}(A^S + \eta^S N^B) - c\delta^S \overline{N^S}(A^S)^2}{[A^S + \eta^S N^B + \delta^S]}$$

The steady state represents a saturated or mature stage of the platform where it maintains a constant level of promotion to uphold the status quo. This involves balancing the natural net decay of users with their organic increase due to cross-side network effects and the influence of platform promotion.

Analyzing comparative statics on steady-state values can be intricate because changes in one variable can impact others. Since the steady state values of promotion effort (A^B, A^S) and the number of users (N^B, N^S) are interdependent, changing any parameter $\left(\delta^B, \delta^S, \eta^B, \eta^S, \overline{N^B}, \overline{N^S}, P^B, P^S, \beta, c\right)$ or variable (A^B, A^S, N^B, N^S) would impact other variables. Hence, comparative statics analysis by simply taking the derivative would not suffice in this case.

We first tried to understand the behavior of the system by giving values to the parameters and checking how the values of the variables change as we change one parameter value at a time. Then, by using total derivative and partial derivative, we build a system of equations, solve it by Cramer's rule, and then argue about the sign of the numerator and the denominator of each member of the solution set given the assumptions and plausible conditions. By this process, we perform the comparative statics for this complicated system.

Corollary 1: As the decay coefficient for buyers (δ^B) *increases, the promotion effort on the buyers' side* (A^B) *decreases, while the promotion efforts on the seller-side* (A^S) *increase. Similarly, as the decay coefficient for sellers* (δ^S) *increases, the promotion effort on the seller-side* (A^S) *decreases, and the promotion efforts on the buyers' side* (A^B) *increase.*

This analysis exposes an inherent asymmetry in the platform's promotion strategies. As the net decay coefficient increases on one side, the promotional effort on that side decreases while it increases on the opposite side. When one side experiences substantial net decay, the platform recognizes that investing significant resources in promotional endeavors on that side to attract reluctant potential users or those previously dissatisfied is less efficient. Instead, it becomes more cost-effective for the platform to direct its advertising and promotional efforts toward the opposite side. This strategy capitalizes on the cross-side network effect, leveraging it to bolster user numbers on the side grappling with high user decay.

In essence, the platform takes into account the effectiveness of its promotion efforts and the advantageous effects of cross-side interactions to formulate its strategic approach. An exemplar case to illustrate this phenomenon is the dating platform Tinder. While Tinder advertises to both genders, in many cases, when one gender's user count dwindles due to significant user decay, the platform shifts its advertising focus to the other gender. As more users from the other gender join the platform, it becomes more appealing to the gender confronting high decay. This cycle continues, with the platform leveraging cross-side network externalities to rectify the imbalance stemming from user decay by strategically advertising on the side opposite to the one experiencing decay.

Thus, platforms can effectively mitigate the impact of user decay by capitalizing on cross-side network interactions to redirect promotional efforts where they can yield optimal results. This underscores the strategic nature of promoting user growth in the face of asymmetric user decay.

Corollary 2: As the potential market size $\overline{N^B}$ *or* $\overline{N^S}$ *increases, the promotion efforts* A^B *and* A^S *decrease. Conversely, when the market size decreases, the promotion efforts increase.*

Amidst the presence of cross-side user-generated interactions, a larger market size serves as a compensatory factor for reduced promotion coverage, achieving this by facilitating more interactions between buyers and sellers, ultimately driving increased transactions. This assumes all the users from the potential market will ultimately transact through the platform. With a larger pool of prospective buyers or sellers, the platform is inclined to allocate less effort towards promoting to these potential users, as the price-cost ratio remains unchanged.

In essence, as the pool of potential users grows, the platform recognizes that it's economically impractical to extensively invest in self-promotion to attract users within this larger market segment. Instead, it harnesses the power of cross-side interactions, which act as an intrinsic means of attraction. In this context, the platform capitalizes on the intrinsic pull of cross-side interactions, leveraging them to decrease its own promotion endeavors. This strategic approach enables the platform to accommodate a larger potential market by effectively capitalizing on the cross-side network effect. The platform thus benefits from a form of "free riding" on this cross-side interaction, which significantly aids in attracting a broader user base.

The robust cross-side network effect empowers a platform to enact substantial business growth with minimal promotion requirements.

This observation underscores the strategic importance of selecting a potential target market prudently. By doing so, the platform can foster a substantial user base with minimal promotional efforts. Notably, engaged users play a pivotal role in reducing the platform's need for aggressive promotion, indicating potential room for employing user screening strategies based on cost-benefit analysis. Such screening policies could encompass aspects like pricing adjustments or background checks, contingent upon the specific requirements of the business.

An intriguing question arises: why do platforms continue to engage in promotion when they could potentially rely on the cross-side network effect and user-generated promotion to propel business growth? One plausible explanation is the need to swiftly accumulate a significant level of profit. Platform businesses often strive to deliver noteworthy results to investors and shareholders who might not have the patience to await gradual organic growth.

Corollary 3: As the cross-side interaction coefficients η^B and η^S increase, the promotion efforts A^B and A^S decrease. Conversely, as these coefficients decrease, the promotion efforts increase.

With an elevation in the organic cross-side interaction coefficient $(\eta^B,\ \eta^S)$, the number of users and, consequently, the profitability of the platform experiences a boost. Concurrently, the platform's promotion efforts directed at users, and thus the associated costs, diminish. This implies that the most valuable users for the platform are those who not only fulfill the demands of the opposite side but also actively participate in interaction activities independently, without the need for platform intervention. User-generated interactions contribute to reducing the platform's promotion endeavors. This underscores the significance of the platform's user population.

The success of the platform is greatly influenced by users who not only engage in interactions that cater to the needs of the other side but also foster independent interaction activities. In scenarios where users on one side are actively and intrinsically involved in interactions, this allure attracts potential users on the other side, generating a robust cross-side network effect. Thus, the platform can rely less on its own promotional efforts, as the inherent appeal of the engaged users becomes more pivotal than the platform's promotional endeavors.

Bass model of same-side interaction recommends front loading of promotion to counter user decay in the later stages. By seeding in a little bit of promotion which attracts some users who in turn attracts other users through cross-side interaction, the platform leverages this cross-side interaction to counter user decay. An increase in cross-side interaction thus benefits the platform to address user decay.

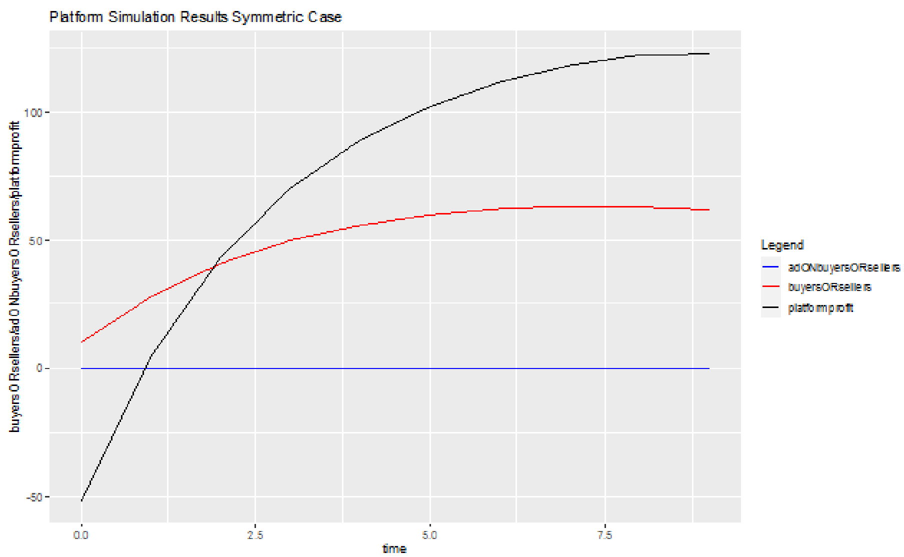
Corollary 4: When there are increases in the discount factor for the time value of money (β), *or in the prices charged to buyers and sellers* (P^B, P^S) *or a decrease in promotion cost* c, *the promotion efforts* A^B *and* A^S *experience an augmentation. Conversely, decreases in these factors lead to reduced promotion efforts.*

This analysis underscores the concept of path dependency. An escalation in the discount factor (β) or prices (P^B, P^S), or a reduction in the cost of promotion (c), leads to heightened steady-state promotion efforts. Consequently, the economic landscape, target market, timing, and initial conditions surrounding the initiation of the platform business hold substantial significance.

The interplay of these factors is pivotal as prices and the discount factor are intrinsically tied to these elements.

Platform's Evolution: There is no analytical solution for the value function and policy function of the platform's dynamic optimization problem. We employed the “Euler equations” to establish connections between promotional efforts directed at buyers or sellers across consecutive periods. These equations serve as the foundation for graphically portraying solutions, a representation found in the figure below. It's worth noting that this illustration is intended to convey the overall trends of the curves. For computational ease, we depict the symmetric case, the curves in the asymmetric case is somewhat similar. We find that in scenarios where both buyers and sellers are subject to changes, the graphs illustrating platform profit, user counts, and promotional efforts exhibit gradual and smooth transitions.

The graphical representation, as depicted in Figure 4.2 below highlights that minimal promotional efforts are required when the platform attains its peak profitability. The inherent decline in user numbers is counterbalanced by the inherent increase facilitated by the cross-side network effect—namely, the collaborative interaction of sellers or buyers with their counterparts on the other side. We see the gradual ascendancy in platform profit and number of users, accompanied by a gradual decline in promotion efforts. The evolving cross-side network effect over time contributes to the smoother convergence of these curves in the long run. It's worth noting that the period of optimal platform profit might not align with the period of maximum number of users due to the presence of promotion costs.



Note that the blue line is the promotion effort and not the horizontal axis. Promotion effort (or the conversion factor) ranges between zero and one, while platform profit and number of users can have high values.

Figure 4.2: Platform's Profit, Number of Users, and Promotion on Users with Time when both Buyers and Sellers Varies

4.6 Platform Promotion on Buyers with Fixed Sellers

In the context of the repeated game spanning from time 0 to T periods, we now focus on the simpler scenario where the number of sellers (N^S) remains fixed, which holds true in the short run. While the number of transacting buyers frequently fluctuates, the process of verifying and approving new sellers before they can engage in transactions takes time. Consequently, despite potential sellers interacting with existing buyers on the platform, the number of new sellers joining the platform during the short run is zero. Additionally, there exists certain inertia among sellers to remain on the platform during this timeframe. New sellers typically prefer to observe the platform's performance for a period before making a decision to leave. Thus, in the short run, the number of sellers can be reasonably considered to be fixed.

We consider a T-period game (T being large), in which sellers remains fixed while buyers vary in each period, and each period denotes an hour. Given that each time period is an hour, we can still solve for steady state when the number of time periods T is large. This model is a variation of the most general model of platform promotion discussed in section 4.5. The primary objective of the platform is then to maximize the expression:

$$Max_{A_t^B} \sum_0^T \beta^t \{P^B N_t^B + P^S N_t^S - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2\} \ s.t.$$

$$N_{t+1}^B = (1-\delta^B)N_t^B + (\overline{N^B} - N_t^B)(A_t^B + \eta^B N^S) \qquad \text{………………………………………….. (6)}$$

where the state variable is N_t^B and the choice variable is A_t^B. The approach for solving this problem closely resembles that used when both buyers and sellers exhibit varying patterns over time. We use dynamic programming algorithm to solve the model. We first explore the platform's dynamic promotion decision in each period.

Proposition 3: The optimal platform promotion on buyers with a fixed number of sellers in period $t+1$ is given by the equation:

$$A_{t+1}^B = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right)A_t^B + [(1-\delta^B) - \eta^B N^S]^2\right]^{\frac{1}{2}} + [(1-\delta^B) - \eta^B N^S]$$

The term $[(1-\delta^B) - \eta^B N^S]$ is a constant as the number of sellers remains fixed. From the expression for the optimal promotion on buyers, we see that the promotion equation remains unaffected by the number of buyers at any given time. Note that the promotion here is on a per user basis. The optimal promotion effort takes into account the number of buyers which keeps on changing with time during the optimization process to determine the optimal promotion value for each period for each buyer. On the other hand, since the number of sellers is fixed, it acts as a parameter in the optimization process and is found in the expression for the optimum promotion on each buyer for each period. The platform by this process thus maximizes its net revenue for each period.

The equation presented in Proposition 3 illustrates that the platform's promotion effort towards buyers in the next period is influenced by the difference between the price charged to buyers per the unit cost of the quadratic advertising investment (the price-cost ratio or markup) and the promotion effort in the current period, adjusted by the discount factor. This difference represents the benefit for the platform, driving the promotion effort in the following period in anticipation of attracting more customers.

A higher $\frac{P^B}{c}$ ratio results in a greater promotion effort in the subsequent period. This implies that the price charged to buyers per the marginal cost of the quadratic promotion effort (referred to as the price-cost ratio) must be high enough to warrant investment in the subsequent period. A higher discount factor (β) corresponds to an increased promotion effort in the next period.

As the decay coefficient increases, the optimal promotion effort on buyers decreases. The seemingly counterintuitive relationship between the same-sided customer decay parameter (δ^B) and the promotion effort can be explained by considering that as the customer decay increases the remaining population targeted for promotion reduces, resulting in a decrease in promotion. In such cases, the platform may need to modify its strategy due to insufficient customer retention.

With an increase in the number of sellers (N^S), the required promotion for the next period decreases. More sellers offer customers more choices, leading to a higher likelihood of transactions and consequently reducing the necessity for promotion efforts. Furthermore, a higher cross-side parameter (η^B), which determines how the potential buyers interact with the current sellers to become actual buyers on the platform in the following period, results in a reduced need for platform promotion effort. The efforts of sellers to attract buyers contribute to this effect.

The managerial implication is that in the short run, the platform in each period will consider the benefit and the cost of the next period promotion. This net benefit, i.e. the benefit minus the cost, depends on the price-cost ratio and the promotion in the previous period adjusted by the discount factor.

Since each time period denotes an hour and the number of time periods *T* is large, we can solve for steady state values. If the platform achieves a steady state in the short run, we can explore the steady-state outcome where the platform's promotion effort remains fixed and the number of buyers achieves a mature fixed level. Specifically, in the steady state $A_{t+1}^B = A_t^B = A^B$, $N_{t+1}^B = N_t^B = N^B$.

Proposition 4: The number of buyers, platform's promotion effort, and platform profit with a fixed number of sellers in the steady state are as follows: $N^B = \frac{\overline{N^B}(A^B+\eta^B N^S)}{A^B+\eta^B N^S+\delta^B}$,

$$A^B = \left[\left(\frac{P^B}{c}\right) + \left\{(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right\}^2\right]^{\frac{1}{2}} + \left\{(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right\},$$

$$\pi_{ss} = \frac{\overline{N^B}\left[P^B(A^B+\eta^B N^S) - c\delta^B (A^B)^2\right]}{A^B+\eta^B N^S+\delta^B} + P^S N^S$$

Even though the model in this section describes the platform's short-run promotion strategy on the buyer side, the promotion dynamics can still quickly converge into a steady state. The steady state represents a relatively stable situation of the platform's evolution path where it satisfies the current users' demand and has no ambition for further or long-term growth. The time-invariant steady-state values provide an opportunity for comparative statics.

Corollary 5: When the number of sellers is held constant, the steady-state number of buyers follows these relationships: $\frac{dN^B}{dA^B} > 0$, $\frac{dN^B}{d\overline{N^B}} > 0$, $\frac{dN^B}{d\eta^B} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{\partial \eta^B}\right) + \left(\frac{\partial N^B}{\partial \eta^B}\right) > 0$, $\frac{dN^B}{d\delta^B} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{\partial \delta^B}\right) + \left(\frac{\partial N^B}{\partial \delta^B}\right) < 0$

The steady-state number of buyers increases with an increase in the steady-state promotion effort, the market potential size of the number of buyers, and the cross-side coefficient representing how present sellers attract prospective buyers. Conversely, it decreases as the net decay coefficient increases. In summary, cross-side attractions in each period contribute to a larger number of buyers in the steady state, while the same-side user churn every period restrains the numbers.

Corollary 6: When the number of sellers remains fixed, the steady-state promotion effort adheres to the following relationships: $\frac{dA^B}{d\delta^B} < 0$, $\frac{dA^B}{d\eta^B} < 0$, $\frac{dA^B}{d\overline{N^B}} = 0$

The increase in the cross-side network effect results in reduced promotion targeting buyers. Notably, the optimal promotion for buyers is unaffected by the buyer's total market potential. It reveals that the platform's promotion, even in a monopoly market, is restrained by its own capability (cost and seller-side attraction). An increase in market size of potential buyers cannot tempt the platform to spend more to promote buyers. As the net decay coefficient for buyers' increases, the platform engages in less promotion toward buyers. In the short run, the platform does not have much opportunity to leverage the cross-side network effect due to the fixed number of seller although the number of buyers is varying. With an escalation in buyer decay, the platform allocates fewer promotional efforts to buyers during the steady state, as such efforts become less profitable.

Platform's Evolution: Unfortunately, there is no closed-form analytical solution of the value function and policy function of the platform's dynamic optimization problem. Instead, we use a numerical simulation to demonstrate the evolution process of the platform's dynamic promotion strategy. Graphical representations of our findings are provided in Figure 4.3 below. We intentionally selected parameter values to differentiate organic decay (stemming from dissatisfaction) from organic growth (stemming from seller/buyer interaction). It's important to note that this illustration serves to portray the general behavior, with parameter values chosen for computational convenience. The grid of 100 for promotion efforts corresponds to a factor of 1, representing the maximum conversion factor for promotion.

We see from Figure 4.3 given below that when the number of sellers is held constant, short-term promotion activities lead to spikes in promotion efforts, platform profits, and the number of buyers. The platform's profit in relation to promotion rapidly increases before gradually stabilizing. The number of buyers exhibits a similar pattern due to the initial promotion spike. Over time, the promotion effort experiences a rapid ascent but then quickly declines, eventually settling at a low, steady value. While high promotion efforts are necessary in the short run to establish a critical mass, afterward, a combination of limited platform promotion and organic growth suffices to sustain profitability. As the market approaches its full potential, the promotion effort on buyers escalates gradually and peaks, reflecting the necessity for heightened promotion to attract the remaining hesitant buyers.

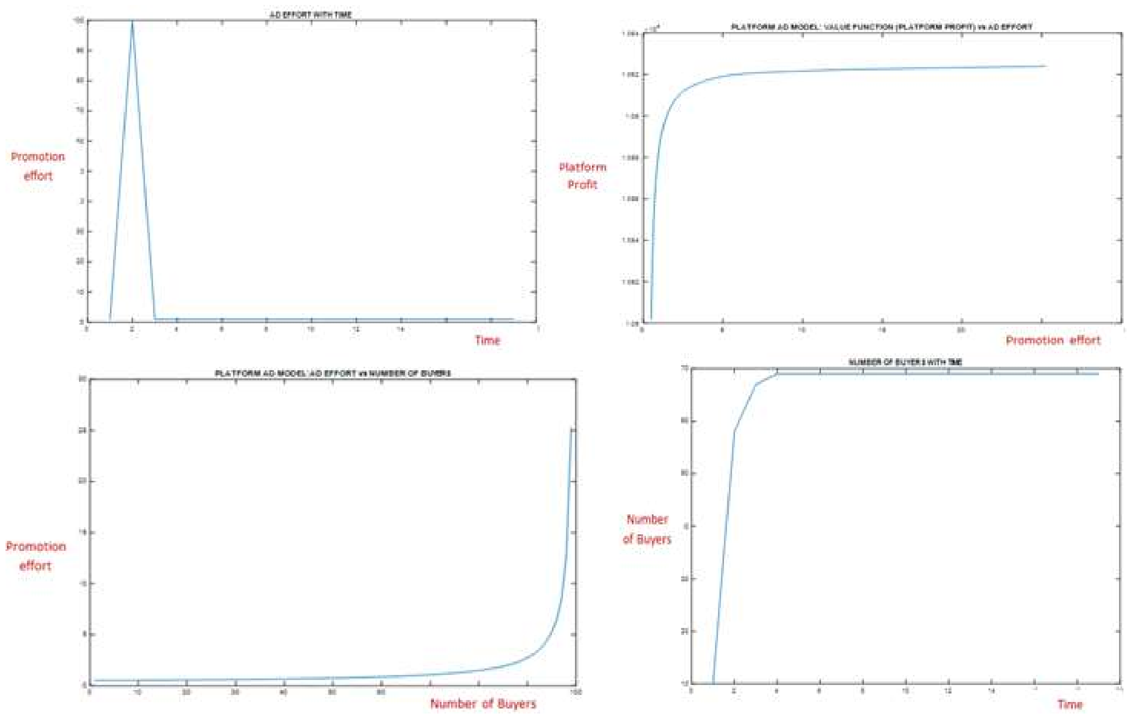


Figure 4.3: Evolution of the Platform's Promotion Effort, Platform's Profit, and the Number of Buyers, when the Number of Sellers are Fixed

4.7 Platform Promotion for a Three-Period Lifespan

We now delve into the scenario where the longer-term repeated game discussed in section 4.5 is condensed into a three-period framework. This model is also a variation of the most general model of platform promotion discussed in section 4.5. In essence, several time periods from the former case are encompassed within a single period of the latter (each period denotes several years) with no promotion in the final period due to a change of ownership or managerial responsibility. The trajectory of the platform now encompasses a start phase, a growth phase, and a mature phase. This situation also closely resembles the context of short-lived platforms, which are designed for limited-time promotions, product testing, events, or pilot projects.

The start phase denotes the inception period of the platform, the growth phase denotes the period when platform users and profitability grow rapidly, and the mature phase denotes the period when user growth and platform profitability stabilizes. The platform is not able to grow continuously due to competitive forces, inertia, and user saturation. It is to be noted that in this three-period framework, promotion occurs during the first two stages, with no promotion in the final phase.

The duration of each phase within this condensed framework can significantly vary, contingent upon the platform's ability to adapt to market dynamics and provide value to users. Platform managers possess strategic decision-making authority, informed by the realization that the platform's final phase may not necessitate promotional efforts. This three-period lifecycle can be effectively represented as a game unfolding across three time periods (designated as 0, 1, and 2).

In this scenario, both buyers and sellers undergo variations over the stipulated timeframe. To maintain consistency in notation across the three cases that we consider in section 4.5, 4.6 and 4.7, we denote the first period of the three-period cycle as period 0 rathern than period 1, the second period as period 1 rather than period 2, and the third period as period 2 rather than period 3.

From the perspective of the platform's promotional campaign, it's notable that the promotional effort during the final period ($A_2^i,$ where $i \in (B, S)$) is set to zero. Furthermore, the initial number of users (N_0^B, N_0^S) are established and known at the outset.

Proposition 5: The optimal platform promotional strategy involves initial and intermediate stage promotion efforts denoted as $A_0^i \; and \; A_0^i \; respectively, where \;\; i \in (B, S)$. *Specifically, these values are* $A_1^B = \frac{\beta P^B}{2c}$, $A_1^S = \frac{\beta P^S}{2c}$,

$$A_0^B =$$

$$\frac{2c\beta\left[P^B+\beta\eta^S\overline{N^S}P^S+\beta(1-\delta^B)P^B-\frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2-\beta(\eta^BP^B+\eta^SP^S)[(1-\delta^S)N_0^S+\eta^S(\overline{N^S}-N_0^S)N_0^B]\right]-\beta^3(\overline{N^S}-N_0^S)(P^B\eta^B+P^S\eta^S)\left[P^S+\beta\eta^B\overline{N^B}P^B+\beta(1-\delta^S)P^S-\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-\beta(\eta^BP^B+\eta^SP^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]\right]}{[4c^2-\beta^4(\eta^BP^B+\eta^SP^S)^2(\overline{N^B}-N_0^B)(\overline{N^S}-N_0^S)]}$$

$$A_0^S =$$

$$\frac{\beta^3(\overline{N^B}-N_0^B)(\eta^BP^B+\eta^SP^S)\left[P^B+\beta\eta^S\overline{N^S}P^S+\beta(1-\delta^B)P^B-\frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2-\beta(\eta^BP^B+\eta^SP^S)[(1-\delta^S)N_0^S+\eta^S(\overline{N^S}-N_0^S)N_0^B]\right]-2c\beta\left[P^S+\beta\eta^B\overline{N^B}P^B+\beta(1-\delta^S)P^S-\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-\beta(\eta^BP^B+\eta^SP^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]\right]}{[\beta^4(\eta^BP^B+\eta^SP^S)^2(\overline{N^B}-N_0^B)(\overline{N^S}-N_0^S)-4c^2]}$$

This solution demonstrates that once the platform acknowledges the absence of the last period promotion, it optimizes its promotional strategy for earlier periods using the backward induction method. In this scenario, the second period promotion on users is half the discounted price-cost ratio and is independent of the initial number of users or the potential market size. This relation emerges from maximizing the revenue over time. It reveals that the platform keeps the optimal second period promotion relatively simple deciding on the benefits and costs of one extra unit of promotion. Since the platform maximizes the net present value of its revenue, any future earnings and costs are discounted.

The backward induction process implies the platform's most strategic decision is the first period promotion. Given no promotion in the third period and the optimal promotion in the second period, the platform decides its optimal promotion in the first period. The function of the first period promotion is thus a complex function of user prices, promotion costs, market potential, and the initial number of users. From it, we derive several intuitions as follows.

Corollary 7: The change in the first period promotion with respect to the change in the market potential for buyers is such that $\frac{\partial A_0^B}{\partial N^B} < 0$ *and* $\frac{\partial A_0^S}{\partial N^B} > 0$ *provided the initial cross-side network effect* $\eta^B N_0^S > 1$. *Similar trends apply to sellers.*

If the initial cross-side network effect is greater than a certain one, a larger market potential for buyers leads to a decreased initial promotion effort targeted at buyers. This counter intuitive phenomenon arises due to the anticipation of the role of the cross-side network effect leading to a significant organic user growth over time. However, a larger market potential for buyers leads to an increased initial promotion effort for sellers, assuming the initial cross-side network effect is greater than one. Recognizing the potential of an expanded buyer market motivates the platform to intensify advertising efforts on the seller side. This strategy aims to enhance the platform's attractiveness to prospective buyers in the upcoming period by capitalizing on the higher number of sellers. Consequently, the platform aims to leverage the organic growth fueled by the cross-side network effect to foster user base expansion.

Thus we see that there is an asymmetric effect of a large potential buyer market on initial period promotion for buyers compared to sellers. For buyers, the initial promotion with an increase in the potential market for buyers decreases while for sellers, the initial promotion with an increase in the potential market for buyers' increases. The asymmetry arises because a large potential market for buyers conveys that buyers would be available as future customers on the platform in the future and hence requires less cajoling while to satisfy the needs of the increased availability of buyers in the future, the platform needs more sellers and hence responds with a larger promotion effort at sellers.

A similar phenomenon occurs when there is a larger potential market for sellers. The initial promotion effort on sellers decreases with an increase in the potential market for sellers while the initial promotion effort on buyers increases with an increase in the potential market for sellers.

Corollary 8: The net decay coefficients induce initial promotion efforts for buyers, resulting in $\frac{\partial A_0^B}{\partial \delta^B} > 0$, $\frac{\partial A_0^B}{\partial \delta^S} < 0$. *Similar outcomes apply to sellers.*

In the context of the longer term platform promotion when both buyers and sellers vary, during steady-state conditions, we observed that an increase in the net decay coefficient for buyers leads to a decrease in the promotion efforts targeting buyers and an increase in the promotion efforts on sellers. However, in a three-stage game where promotion efforts cease in the final period, we find that a rise in the decay coefficient for buyers is associated with an increase in the initial promotion effort on buyers, while an increase in the decay coefficient for sellers corresponds to a decrease in the initial promotion effort. Similar trends are evident for the seller side.

The result that the promotion effort on the buyer-side increases with an increase in the net decay coefficient for buyers is not surprising as the platform wants to counter the increased decay with enhanced promotion to ensure future profitability. When the net decay coefficient for sellers increases, the platform realizes that this would result in lesser number of sellers on the platform in the future and consequently lesser number of buyers on the platform as it becomes less attractive for buyers. Then the platform optimizes on costly promotion by reducing it on the buyer side. A similar pattern follows on the seller side.

Consequently, the optimal strategy for the platform to counteract decay on one side during the initial stages involves increased promotion for that side. Conversely, if decay intensifies on the other side, the optimal approach is to reduce promotion for this side. This outcome suggests that it could be a reasonable course of action for the platform to counteract user decay on one side by bolstering promotion efforts for that side, while simultaneously decreasing efforts on the other side during the initial stages.

Corollary 9: The effect of the cross-side network effect on promotion in the first period is such that $\frac{\partial A_0^B}{\partial \eta^B} < 0$ *if* $\left[\frac{\{(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S\}}{\{(1-\delta^S)N_0^S+\eta^S(\overline{N^S}-N_0^S)N_0^B\}}\right] > \sqrt{\left[\frac{(\overline{N^B}-N_0^B)}{(\overline{N^S}-N_0^S)}\right]}$.

The same condition applies for $\frac{\partial A_0^B}{\partial \eta^S} < 0$ *along with*

$$c < \frac{\beta\left(1-\frac{N_0^S}{\overline{N^S}}\right)\left\{\beta(\eta^B P^B+\eta^S P^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]+\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-P^S-\beta\eta^B\overline{N^B}P^B-\beta(1-\delta^S)P^S\right\}}{2\left[N_0^B\left\{\eta^B\left(\frac{P^B}{P^S}\right)+\eta^S\right\}\left(1-\frac{N_0^S}{\overline{N^S}}\right)-1\right]}.$$

Similar outcomes apply to sellers.

Increased cross-side network effect results in a decrease in promotion efforts directed at buyers during the initial period, as long as the cost of promotion remains below a threshold and the ratio of the net organic growth for buyers and sellers exceeds the square root of the ratio of prospective buyers and sellers in the initial period. This conclusion also applies for sellers.

$\left[\frac{\{(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S\}}{\{(1-\delta^S)N_0^S+\eta^S(\overline{N^S}-N_0^S)N_0^B\}}\right] > \sqrt{\left[\frac{(\overline{N^B}-N_0^B)}{(\overline{N^S}-N_0^S)}\right]}$ condition indicates that the platform judges the organic growth that is possible on the two sides compared to the prospective users that needs to be attracted. If that organic growth prospect is above a certain threshold $\sqrt{\left[\frac{(\overline{N^B}-N_0^B)}{(\overline{N^S}-N_0^S)}\right]}$, and the cross-side network effects (η^B, η^S) increases, the platform, to optimize promotion spending, reduces its initial period promotion.

$$c < \frac{\beta\left(1-\frac{N_0^S}{\overline{N^S}}\right)\left\{\beta(\eta^B P^B+\eta^S P^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]+\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-P^S-\beta\eta^B\overline{N^B}P^B-\beta(1-\delta^S)P^S\right\}}{2\left[N_0^B\left\{\eta^B\left(\frac{P^B}{P^S}\right)+\eta^S\right\}\left(1-\frac{N_0^S}{\overline{N^S}}\right)-1\right]}$$ condition

indicate the platform does a cost-benefit analysis of how much promotion to do in the initial period given the cost involved to do promotion in the second period and decides to lower its promotion in the first period if the marginal cost of promotion is low enough to have adequate promotion in the second period, considering also that there is no promotion in the third period.

Corollary 9 thus indicates that when there is ample potential for organic growth through cross-side interactions and the cost of promotion is sufficiently low to allow adequate promotion in the second period $\left(A_1^B = \frac{\beta P^B}{2c},\ A_1^S = \frac{\beta P^S}{2c}\right)$, it becomes optimal for the platform to reduce promotion in the first period to take benefit of the cross-side network effect.

Corollary 10: The promotion in the first period concerning changes in the initial number of users follows $\frac{\partial A_1^B}{\partial N_0^B} > 0$ *and* $\frac{\partial A_1^B}{\partial N_0^S} < 0$. *Similar outcomes apply to sellers.*

With an influx of buyers at the initial stage, a business may feel good and conclude that not much promotion is required on buyers. But we find that an increase in the number of buyers in the initial stage requires that the platform enhances its promotion on prospective buyers in that period. This counterintuitive insight arises from the expectation of having fewer buyers in the subsequent period. Since the market potential of buyers is fixed, more buyers initially imply lesser buyers in the following periods. The platform desiring to maximize the sum total of net revenue after promotion expenditure collected over time needs to consider the future in deciding its promotion strategies. The promotion on buyers in spite of having a larger number of buyers in the initial period is to ensure sustained profitability of the platform over time.

Conversely, a higher number of sellers in the initial period lead the platform to reduce its promotion directed at prospective buyers in the same period. This adjustment is based on the platform's rationale that the presence of more sellers during this period will attract buyers in the subsequent periods. Consequently, the platform desiring to optimize promotion spending to enhance profitability reduces promotion on buyers in the initial period. Thus, the platform takes into account the cross-side network advantages associated with an increased number of sellers in the initial stage to attract buyers in the subsequent periods to fine-tuning its promotion strategy toward buyers.

This promotion strategy also draws our attention to the chicken and egg problem which we discussed in the literature review section. Without a sufficient number of sellers, buyers will not join the platform and without a sufficient number of buyers, sellers will not join the platforms. The platform through its promotion strategies tries to ensure this balance between the number of buyers and the number of sellers is maintained by attracting a critical mass of buyers and sellers to join the platform in each period.

4.8 Managerial Implications and Conclusion

4.8.1 Main Findings

This paper explored dynamic optimal platform promotion strategies from inception to maturity, considering promotions from user generated interactions. We developed a model of platform users, which vary from one period to the next, influenced by factors such as the (a) platform promotion among prospective users (b) organic decay from dissatisfaction after considering growth due to same-sided user-user interactions, and (c) organic growth from cross-side interactions between current and prospective users. To analyze these dynamics, we employed dynamic optimization techniques to derive 'Euler equations' that connect the optimal promotion effort of one period to the next. These equations were solved using the value function iteration algorithm implemented in MATLAB. We derived steady state results, conducted simulations by changing variable values and observing its effect on other variables, and did comparative static analysis using total differential, partial differential and Cramer’s Rule.

We considered three distinct cases: (i) A along run case with both buyers and sellers varying where each time period denotes a month (ii) a short-run scenario where sellers remain fixed but buyers vary (due to the time required for seller quality verification) where each time period denotes an hour (iii) a long run case condensed into three periods (start, growth, maturity) with promotion in the first two periods (allowing managers strategic decision-making ability over promotion extent and timing) where each time period denotes several years.

We found that promotion in a given period within the dynamic process is contingent on five key components: the ratio of price charged to platform users and the marginal cost of quadratic promotion effort, substitution effect of promotion between sequential periods, cross-side effect of promotions and interactions, user retention rate, and organic growth.

Our findings suggest that during the initial stages, increased promotion is necessary on the side experiencing user decay. However, as platforms mature, countering user decay on one side entails reducing promotion on that side and increasing it on the other. Larger potential markets require less promotion, benefiting from cross-side user interactions. Engaged users enhance cross-side network effects, thereby reducing the need for extensive platform promotion.

Nevertheless, platforms may still engage in advertising to achieve quick profitability, particularly under stakeholder pressure. Although platforms may aspire to serve a large market with minimal promotion relying on cross-side network effect, our research demonstrates that platforms ultimately reach a steady state and do not exhibit continuous growth in the long run. Seeding in a little bit of promotion attract users who in turn attracts more users through cross-side network effect thus helping platforms counter user decay. Platform promotion is path dependent with long-term promotion gradually influenced by cross-side network effects.

In the short-run case where sellers remain fixed but buyers vary, our findings reveal that promotion is marked by intermittent spikes aimed at establishing a critical mass of users. Additionally, we observed that an increase in the cross-side network effect on one side led to more users on that side and reduced platform promotion for that side. Conversely, an increase in net decay resulted in a decrease in users on that side and a decrease in promotion efforts on that side. This asymmetry stems from the fixed number of sellers, which inhibits the platform from fully capitalizing on the cross-side network effect within a short timeframe, despite the variability in the number of buyers.

Furthermore, our investigation uncovered insights into a three-period game, which serves as a representation of long-term platform operations condensed into three periods. We found that promotion on users during the second period equates to half the discounted price-cost ratio, irrespective of initial user numbers or potential market size but promotion during the first period is determined by a complex interplay of factors including user prices, promotion costs, market potential, and the initial number of users. Our analysis revealed that the presence of more initial same-sided users leads to higher initial period promotion, whereas a higher number of initial cross-side users results in reduced promotion. Additionally, a larger market potential of users on one side correlates with decreased initial period promotion on that side and increased promotion on the other side. Moreover, an increase in net decay on one side necessitates more initial period promotion on that side and less on the other side, whereas an increase in the cross-side network effect leads to reduced initial period promotion.

While past literature has extensively examined platform dynamic pricing and user growth resulting from same-sided effects for mature platforms utilizing static models or limited-time dynamic models, our contribution lies in understanding the optimal dynamic and steady-state promotion decisions made by platforms as they progress from inception to maturity, and how cross-side network effects influence these decisions throughout the platform's lifecycle.

4.8.2 Managerial Implications

The managerial implications of our findings are multifaceted. Firstly, when addressing initial user decay on one side, platforms can effectively counteract it by increasing promotion on that side while decreasing it on the other side. Conversely, in mature stages, user decay necessitates less promotion on the same side and more on the other side. Engaged users play a crucial role in reducing the need for platform promotion, as they attract other users and enable the platform to cater to a larger potential market with minimal promotion efforts. However, less engaged users can be costly if they fail to generate sufficient revenue. In such cases, platforms could implement screening strategies following thorough cost-benefit analyses.

In the short run, a strategy of intense promotion may be necessary to build up a critical mass of users. However, an increase in net decay or cross-side network effects on one side may require decreased promotion on that side. Moreover, an increase in initial users on one side may call for more promotion on that side and less on the other during the initial stages. Conversely, a greater market potential of users on one side leads to lesser initial period promotion on that side and more initial period promotion on the other side.

To enhance long-term profitability, platforms should focus on incentivizing content sharing, fostering positive interactions, and promoting community engagement. Additionally, selecting the right target market, initiation timing, and location are crucial factors that can optimize profitability in the long run.

4.8.3 Limitations and Scope for Future Research

In our analysis, we relied on plausible assumptions regarding the levels of the cross-side network effect and net organic decay to prove certain inequalities for the comparative statics results. However, future research could explore the validity of our analysis under varying levels of these factors and identify potential threshold values where our results may become invalid. Additionally, we made assumptions about fixed values for net decay and cross-side interaction, which could complicate the analysis if these parameters were to change. Further research could investigate how the variables in our models behave with changing parameters using bifurcation analysis.

A major assumption in our paper was the fixed prices (subscription fees). While this simplification facilitated a detailed exploration of platform promotion, it may not fully capture the complexities of real-world scenarios where prices can fluctuate with environmental uncertainties. Future research could investigate platforms considering changing prices and varying promotional efforts, as platforms seek to maximize long-term revenues. This investigation could present interesting challenges and provide valuable insights into the dynamics of platform promotion strategies.

5. Conclusion

Digital platforms are a burgeoning business model in the modern economy. Due to cross-side and same-side network effects, platforms' marketing strategies are more complicated traditional businesses. In this thesis, I completed a thorough literature review on platform marketing strategy and two studies on platform segmentation and promotion strategies.

The literature review discussed the inception of the digital platform literature, the unique features of digital platforms, their role in the modern economy, how digital platforms balanced user growth on the buyer and the seller side, how they connect with other networks, the governance of platforms, their role in distribution channels, the pricing and promotion strategies of digital platforms, segmentation or categorization policies of digital platforms, competition among platforms, and the evolution of the digital platform business. I then justified how the two papers of the thesis fill the gap in the marketing literature on digital platforms.

I then explored two important facets of the digital platform economy in the two papers: (i) vertical segmentation pricing policy and its implications for digital platforms and (ii) optimal promotion policy for platforms from inception till maturity under the presence of user-generated promotion or cross-side network effects. Below, I summarize what we have learned from these two papers, the contribution of the thesis to the literature, and the future scope for research.

5.1 Summary of the First Paper

The first paper discussed the significance of vertical segmentation in contemporary two-sided markets or platforms like Uber and Airbnb. These platforms facilitate transactions between numerous buyers and sellers with diverse characteristics, offering goods and services. Vertical segmentation, where products or services are categorized based on their quality or attributes, is a common strategy employed by such platforms to better match buyer preferences with seller offerings.

The paper examined the impact of platforms' vertical segmentation strategies on the platform economy. The analytical models uncovered a positive cross-side network effect of seller quality variance on buyer surplus, laying the theoretical groundwork for platforms' vertical segmentation strategies. The research investigated this segmentation under two scenarios: (a) where the platform sets trading prices, and (b) where sellers independently determine prices.

In the first scenario, segmentation enables the platform to boost its profit through second-degree price discrimination, albeit at the expense of reduced buyer surplus. Low-quality sellers benefit from segmentation, while high-quality sellers may suffer unless the quality gap is significant. Under the second scenario, segmentation based on publicly observable seller features does not change the equilibrium outcomes.

The research found, if a platform's segmentation program can verify and reveal sellers' unobservable quality information, it can mitigate the information asymmetry between sellers and buyers, leading to a mutually beneficial outcome for both parties. The study highlighted two critical functions of platform segmentation: price discrimination and information asymmetry reduction.

Furthermore, the paper demonstrated that interventions that influence buyer psychology to prioritize quality benefit both the platform and sellers. This quality preference exerts a quadratic cross-side network effect, increasing transaction volume and optimal trading price.

Additionally, the analysis revealed that integrated platforms offering uniform prices regardless of product quality do not exist. Segmented platforms, where prices correspond to product quality and are determined by the platform, generate higher profits compared to platforms where sellers set prices. These findings offer valuable insights into platform management.

5.2 Summary of the Second Paper

In the second paper, I discussed the optimal dynamic promotion policy for digital platforms like Spotify and Netflix from their inception till maturity in the presence of user-generated promotions or interactions, which also draw users to the platform. Such platforms face difficult decisions on how much promotion to do considering it is costly, whether to rely on own or user-generated promotion, and how to manage promotion as the platform evolves.

As the number of users on both the buyer and seller sides fluctuates due to cross-side interactions and user dissatisfaction, the platform's promotion efforts interact with this organic user variation, leading to complex consequences for platform profitability. This study developed dynamic programming models to explore the evolution of a platform's optimal promotion policy from its inception to maturity.

Specifically, the paper examined three models representing different business scenarios: (i) a most general scenario model to investigate the platform's long-run promotion strategies when the number of buyers and sellers changes over time, (ii) a simplified model to simulate the platform's short-run promotion efforts on the buyer side while the number of sellers remains constant, and (iii) a three-period game representing the platform's inception, growth, and maturity phases to understand the platform's promotion evolution in the inception and growth periods.

The analytic models revealed several insightful findings. Firstly, the platform's optimal promotion efforts in any period depend on five key components: the ratio between the platform's prices charged to users and its promotion cost, the substitution effect of promotion between sequential periods, the cross-side effect of promotions and user interactions, user organic retention rate after decay due to dissatisfaction, and the growth of users from cross-side interactions.

The research found that the optimal promotion strategy for the platform to counter user decay on one side is to increase promotion to this side in the initial stages of the platform's evolution. At maturity, promotion on this side should decrease while increasing on the other side. Contrary to intuition, an increase in the initial number of users on one side may lead the platform to invest more in promotion on this side and less on the other side in the initial stages of the platform's evolution.

For a traditional business, when the potential market of users increases, more promotion is typically required to attract some of those users. However, the study found that if the potential market of users on one side of the platform increases, the platform may require less promotion on that side. Instead, it may need more promotion on the other side in the initial phase of the platform's evolution due to the increase in cross-side interaction enabled by the larger number of potential users, ultimately reducing the platform's promotional effort.

While the platform's long-run promotion strategies are gradual, path-dependent, and shaped by the cross-side network effect, the research found that its promotions in the short run are often characterized by investment spikes. The platform often invests substantial promotion efforts in the initial periods, to quickly build up a critical mass of users.

The study recommended platforms incentivize cross-side interactions, as they reduce the need for platform promotion efforts and counter user decay.

5.3 Contribution

The thesis contributes to pricing and promotion, two important aspects of digital platform research in marketing. It examined the role of vertical segmentation pricing and its implications for digital platforms, and the role of optimal dynamic platform promotion in the presence of user-generated promotion and its implications. It thus contributes to both the digital platform literature and the literature on pricing and promotions in marketing and points to some new areas where future research could be explored.

5.4 Scope for Future Research

Future research could explore the relaxation of assumptions made in the first paper and analyze the outcomes. This could be in the form of addressing the implications for vertical segmentation pricing in digital platforms, when buyers are multi-homing or using multiple platforms, when buyers have different coefficients of diminishing marginal utility, when substitution effects within and between different submarkets are different, when buyers and sellers are risk-neutral, when the scenario is dynamic, or when there is entry cost for users.

On the other hand, modifications in the second paper could lead to new avenues of research. For example, future research can examine potential threshold values for the cross-side network effect or potential threshold values for user decay for the results to become invalid. Future research can also examine varying levels of cross-side network effect and user decay based on user characteristics and how that affects the results. Furthermore, the introduction of stochasticity with varying prices in the model framework poses an interesting challenge that researchers can pursue. Given that various variables are interconnected in a platform setting, empirical analysis of any question would involve structural equation modeling and estimation by the maximum likelihood principle.

Bibliography

Achrol, R.S., 1991. Evolution of the marketing organization: new forms for turbulent environments. *Journal of marketing*, *55*(4), pp.77-93.

Acquisti, A. and Varian, H.R., 2005. Conditioning prices on purchase history. *Marketing Science*, *24*(3), pp.367-381.

Ambrus, A. and Argenziano, R., 2009. Asymmetric networks in two-sided markets. *American Economic Journal: Microeconomics*, *1*(1), pp.17-52.

Alderson, W., 1937. A marketing view of competition. *Journal of Marketing*, *1*(3), p.189.

Armstrong, M., 2006. Competition in two-sided markets. *The RAND journal of economics*, *37*(3), pp.668-691.

Armstrong, M. and Wright, J., 2007. Two-sided markets, competitive bottlenecks and exclusive contracts. *Economic Theory*, *32*, pp.353-380.

Bakos, Y. and Halaburda, H., 2020. Platform competition with multihoming on both sides: Subsidize or not?. *Management Science*, *66*(12), pp.5599-5607.

Bass, F.M., 1969. A new product growth for model consumer durables. *Management science*, *15*(5), pp.215-227.

Bass, F.M., Bruce, N., Majumdar, S. and Murthi, B.P.S., 2007. Wearout effects of different advertising themes: A dynamic Bayesian model of the advertising-sales relationship. *Marketing Science*, *26*(2), pp.179-195.

Belleflamme, P. and Toulemonde, E., 2009. Negative intra-group externalities in two-sided markets. *International Economic Review*, *50*(1), pp.245-272.

Bester, H. and Petrakis, E., 1996. Coupons and oligopolistic price discrimination. *International journal of industrial organization*, *14*(2), pp.227-242.

Bolton, R.N. and Myers, M.B., 2003. Price-based global market segmentation for services. *Journal of Marketing*, *67*(3), pp.108-128.

Bruce, N.I., Murthi, B.P.S. and Rao, R.C., 2017. A dynamic model for digital advertising: The effects of creative format, message content, and targeting on engagement. *Journal of marketing research*, *54*(2), pp.202-218.

Caillaud, B. and Jullien, B., 2001. Chicken and egg: Competing matchmakers. *Available at SSRN 278562*.

Caillaud, B. and Jullien, B., 2003. Chicken & egg: Competition among intermediation service providers. *RAND journal of Economics*, pp.309-328.

Casadesus-Masanell, R. and Hałaburda, H., 2014. When does a platform create value by limiting choice?. *Journal of Economics & Management Strategy*, *23*(2), pp.259-293.

Chen, Y., Moorthy, S. and Zhang, Z.J., 2005. Research note—price discrimination after the purchase: Rebates as state-dependent discounts. *Management Science*, *51*(7), pp.1131-1140.

Chu, J. and Manchanda, P., 2016. Quantifying cross and direct network effects in online consumer-to-consumer platforms. *Marketing Science*, *35*(6), pp.870-893.

Correia-da-Silva, J., Jullien, B., Lefouili, Y. and Pinho, J., 2019. Horizontal mergers between multisided platforms: Insights from Cournot competition. *Journal of Economics & Management Strategy*, *28*(1), pp.109-124.

Costello, J.P. and Reczek, R.W., 2020. Providers versus platforms: Marketing communications in the sharing economy. *Journal of Marketing*, *84*(6), pp.22-38.

Coughlan, A.T., 1985. Competition and cooperation in marketing channel choice: Theory and application. *Marketing science*, *4*(2), pp.110-129.

Daughety, A.F. and Reinganum, J.F., 2008. Imperfect competition and quality signaling. *The RAND Journal of Economics*, *39*(1), pp.163-183.

Dickson, P.R. and Ginter, J.L., 1987. Market segmentation, product differentiation, and marketing strategy. *Journal of marketing*, *51*(2), pp.1-10.

Dinerstein, M., Einav, L., Levin, J. and Sundaresan, N., 2018. Consumer price search and platform design in internet commerce. *American Economic Review*, *108*(7), pp.1820-1859.

Doganoglu, T. and Klapper, D., 2006. Goodwill and dynamic advertising strategies. *Quantitative Marketing and Economics*, *4*, pp.5-29.

Dorfman, R. and Steiner, P.O., 1954. Optimal advertising and optimal quality. *The American Economic Review*, *44*(5), pp.826-836.

Dukes, A. and Gal–Or, E., 2003. Negotiations and exclusivity contracts for advertising. *Marketing Science*, *22*(2), pp.222-245.

Dukes, A., Liu, Q. and Shuai, J., 2022. Skippable ads: Interactive advertising on digital media platforms. *Marketing Science*, *41*(3), pp.528-547.

Eckhardt, G.M., Houston, M.B., Jiang, B., Lamberton, C., Rindfleisch, A. and Zervas, G., 2019. Marketing in the sharing economy. *Journal of Marketing*, *83*(5), pp.5-27.

Economides, N. and Katsamakas, E., 2006. Two-sided competition of proprietary vs. open source technology platforms and the implications for the software industry. *Management science*, *52*(7), pp.1057-1071.

Edelman, B. and Wright, J., 2015. Price coherence and excessive intermediation. *The Quarterly Journal of Economics*, *130*(3), pp.1283-1328.

Eisenmann, T., Parker, G. and Van Alstyne, M., 2011. Platform envelopment. *Strategic management journal*, *32*(12), pp.1270-1285.

Elberg, A., Gardete, P.M., Macera, R. and Noton, C., 2019. Dynamic effects of price promotions: Field evidence, consumer search, and supply-side implications. *Quantitative Marketing and Economics*, *17*, pp.1-58.

Evans, D.S., 2003. The antitrust economics of multi-sided platform markets. *Yale J. on Reg.*, *20*, p.325.

Evans, D.S., Hagiu, A. and Schmalensee, R., 2006. Software platforms. *Industrial Organization and the Digital Economy*, *31*.

Fang, E., Li, X., Huang, M. and Palmatier, R.W., 2015. Direct and indirect effects of buyers and sellers on search advertising revenues in business-to-business electronic platforms. *Journal of Marketing Research*, *52*(3), pp.407-422.

Farronato, C., Fong, J. and Fradkin, A., 2024. Dog eat dog: Balancing network effects and differentiation in a digital platform merger. *Management Science*, *70*(1), pp.464-483.

Fong, N., Zhang, Y., Luo, X. and Wang, X., 2019. Targeted promotions on an e-book platform: Crowding out, heterogeneity, and opportunity costs. *Journal of Marketing Research*, *56*(2), pp.310-323.

Galeotti, A. and Moraga-González, J.L., 2009. Platform intermediation in a market for differentiated products. *European Economic Review*, *53*(4), pp.417-428.

Geng, X., Stinchcombe, M.B. and Whinston, A.B., 2005. Bundling information goods of decreasing value. *Management science*, *51*(4), pp.662-667.

Gomes, R., 2014. Optimal auction design in two-sided markets. *The RAND Journal of Economics*, *45*(2), pp.248-272.

Gupta, S., 1988. Impact of sales promotions on when, what, and how much to buy. *Journal of Marketing research*, *25*(4), pp.342-355.

Hagiu, A., 2006. Pricing and commitment by two-sided platforms. *The RAND Journal of Economics*, *37*(3), pp.720-737.

Hagiu, A., 2007. Merchant or two-sided platform?. *Review of Network Economics*, *6*(2).

Hagiu, A., 2009. Quantity vs. quality and exclusion by two-sided platforms. *Harvard Business School Strategy Unit Working Paper*, (09-094).

Hagiu, A., 2009. Multi-sided platforms: From microfoundations to design and expansion strategies. *Harvard Business School Strategy Unit Working Paper*, (09-115).

Hagiu, A., 2009. Two-sided platforms: Product variety and pricing structures. *Journal of Economics & Management Strategy*, *18*(4), pp.1011-1043.

Hagiu, A. and Spulber, D., 2013. First-party content and coordination in two-sided markets. *Management Science*, *59*(4), pp.933-949.

Hagiu, A. and Hałaburda, H., 2014. Information and two-sided platform profits. *International Journal of Industrial Organization*, *34*, pp.25-35.

Hagiu, A. and Wright, J., 2015. Marketplace or reseller?. *Management Science*, *61*(1), pp.184-203.

Hagiu, A. and Wright, J., 2015. Multi-sided platforms. *International Journal of Industrial Organization*, *43*, pp.162-174.

Hagiu, A. and Wright, J., 2018. *Platform minimum requirements*. Technical report, Working Paper.

Hagiu, A. and Wright, J., 2019. Controlling vs. enabling. *Management Science*, *65*(2), pp.577-595.

Hagiu, A. and Wright, J., 2019. The status of workers and platforms in the sharing economy. *Journal of Economics & Management Strategy*, *28*(1), pp.97-108.

Hagiu, A. and Wright, J., 2020. Platforms and the exploration of new products. *Management Science*, *66*(4), pp.1527-1543.

Hagiu, A., Jullien, B. and Wright, J., 2020. Creating platforms by hosting rivals. *Management Science*, *66*(7), pp.3234-3248.

Hałaburda, H. and Yehezkel, Y., 2013. Platform competition under asymmetric information. *American Economic Journal: Microeconomics*, *5*(3), pp.22-68.

Halaburda, H., Jan Piskorski, M. and Yıldırım, P., 2018. Competing by restricting choice: The case of matching platforms. *Management Science*, *64*(8), pp.3574-3594.

Halaburda, H. and Yehezkel, Y., 2019. Focality advantage in platform competition. *Journal of Economics & Management Strategy*, *28*(1), pp.49-59.

Halaburda, H., Jullien, B. and Yehezkel, Y., 2020. Dynamic competition with network externalities: how history matters. *The RAND Journal of Economics*, *51*(1), pp.3-31.

Haley, R.I., 1968. Benefit segmentation: A decision-oriented research tool. *Journal of marketing*, *32*(3), pp.30-35.

Hermalin, B.E., 2016. Platform-intermediated trade with uncertain quality. *Journal of Institutional and Theoretical Economics (JITE)/Zeitschrift für die gesamte Staatswissenschaft*, pp.5-29.

Hinz, O., Hann, I.H. and Spann, M., 2011. Price discrimination in e-commerce? An examination of dynamic pricing in name-your-own price markets. *Mis quarterly*, pp.81-98.

Hovenkamp, H., 2020. Antitrust and platform monopoly. *Yale LJ*, *130*, p.1952.

Jayachandran, S., Gimeno, J. and Varadarajan, P.R., 1999. The theory of multimarket competition: A synthesis and implications for marketing strategy. *Journal of Marketing*, *63*(3), pp.49-66.

Johnson, J.P., 2020. The agency and wholesale models in electronic content markets. *International Journal of Industrial Organization*, *69*, p.102581.

Johnson, J.P., Rhodes, A. and Wildenbeest, M., 2023. Platform design when sellers use pricing algorithms. *Econometrica*, *91*(5), pp.1841-1879.

Jolivet, G., Jullien, B. and Postel-Vinay, F., 2016. Reputation and prices on the e-market: Evidence from a major french platform. *International Journal of Industrial Organization*, *45*, pp.59-75.

Jullien, B. and Pavan, A., 2019. Information management and pricing in platform markets. *The Review of Economic Studies*, *86*(4), pp.1666-1703.

Kaiser, U. and Wright, J., 2006. Price structure in two-sided markets: Evidence from the magazine industry. *International Journal of Industrial Organization*, *24*(1), pp.1-28.

Kannan, P.K., 2017. Digital marketing: A framework, review and research agenda. *International journal of research in marketing*, *34*(1), pp.22-45.

Karle, H., Peitz, M. and Reisinger, M., 2020. Segmentation versus agglomeration: Competition between platforms with competitive sellers. *Journal of Political Economy*, *128*(6), pp.2329-2374.

Katz, M.L. and Shapiro, C., 1985. Network externalities, competition, and compatibility. *The American economic review*, *75*(3), pp.424-440.

Katz, M.L. and Shapiro, C., 1986. Technology adoption in the presence of network externalities. *Journal of political economy*, *94*(4), pp.822-841.

Kessel, R.A., 1958. Price discrimination in medicine. *The Journal of Law and Economics*, *1*, pp.20-53.

Kierzkowski, A., McQuade, S., Waitman, R. and Zeisser, M., 1996. Marketing to the digital consumer. *The mckinsey quarterly*, (3), p.4.

Knetter, M.M., 1989. Price discrimination by US and German exporters. *The American Economic Review*, *79*(1), pp.198-210.

Krishnan, T.V. and Jain, D.C., 2006. Optimal dynamic advertising policy for new products. *Management Science*, *52*(12), pp.1957-1969.

Laffont, J.J., Rey, P. and Tirole, J., 1998. Network competition: II. Price discrimination. *The RAND Journal of Economics*, pp.38-56.

Lamberton, C. and Stephen, A.T., 2016. A thematic exploration of digital, social media, and mobile marketing: Research evolution from 2000 to 2015 and an agenda for future inquiry. *Journal of marketing*, *80*(6), pp.146-172.

Lin, M., Li, S. and Whinston, A.B., 2011. Innovation and price competition in a two-sided market. *Journal of management information systems*, *28*(2), pp.171-202.

Liu, Q. and Zhang, D., 2013. Dynamic pricing competition with strategic customers under vertical product differentiation. *Management Science*, *59*(1), pp.84-101.

Liu, M., Brynjolfsson, E. and Dowlatabadi, J., 2021. Do digital platforms reduce moral hazard? The case of Uber and taxis. *Management Science*, *67*(8), pp.4665-4685.

Lu, Q. and Moorthy, S., 2007. Coupons versus rebates. *Marketing Science*, *26*(1), pp.67-82.

Markus, M.L. and Loebbecke, C., 2013. Commoditized digital processes and business community platforms: New opportunities and challenges for digital business strategies. *Mis Quarterly*, *37*(2), pp.649-653.

McGuinness, N.W. and Little, B., 1981. The influence of product characteristics on the export performance of new industrial products. *Journal of Marketing*, *45*(2), pp.110-122.

Miracle, G.E., 1965. Product characteristics and marketing strategy. *Journal of Marketing*, *29*(1), pp.18-24.

Mishra, D.P., Heide, J.B. and Cort, S.G., 1998. Information asymmetry and levels of agency relationships. *Journal of marketing Research*, *35*(3), pp.277-295.

Montgomery, A.L., 1997. Creating micro-marketing pricing strategies using supermarket scanner data. *Marketing science*, *16*(4), pp.315-337.

Narasimhan, C., 1984. A price discrimination theory of coupons. *Marketing Science*, *3*(2), pp.128-147.

Nerlove, M. and Arrow, K.J., 1962. Optimal advertising policy under dynamic conditions. *Economica*, pp.129-142.

Neslin, S.A., Powell, S.G. and Schneider Stone, L., 1995. The effects of retailer and consumer response on optimal manufacturer advertising and trade promotion strategies. *Management Science*, *41*(5), pp.749-766.

Nocke, V., Peitz, M. and Stahl, K., 2007. Platform ownership. *Journal of the European Economic Association*, *5*(6), pp.1130-1160.

Nowlis, S.M. and Simonson, I., 1996. The effect of new product features on brand choice. *Journal of marketing research*, *33*(1), pp.36-46.

Papatla, P. and Krishnamurthi, L., 1996. Measuring the dynamic effects of promotions on brand choice. *Journal of Marketing research*, *33*(1), pp.20-35.

Parker, G.G. and Van Alstyne, M.W., 2005. Two-sided network effects: A theory of information product design. *Management science*, *51*(10), pp.1494-1504.

Parker, G.G., Van Alstyne, M.W. and Choudary, S.P., 2016. *Platform revolution: How networked markets are transforming the economy and how to make them work for you*. WW Norton & Company.

Parker, G., Petropoulos, G. and Van Alstyne, M.W., 2020. Digital platforms and antitrust.

Perren, R. and Kozinets, R.V., 2018. Lateral exchange markets: How social platforms operate in a networked economy. *Journal of Marketing*, *82*(1), pp.20-36.

Ramaswamy, V., Gatignon, H. and Reibstein, D.J., 1994. Competitive marketing behavior in industrial markets. *Journal of Marketing*, *58*(2), pp.45-55.

Rangaswamy, A., Moch, N., Felten, C., Van Bruggen, G., Wieringa, J.E. and Wirtz, J., 2020. The role of marketing in digital business platforms. *Journal of Interactive Marketing*, *51*(1), pp.72-90.

Rietveld, J., Schilling, M.A. and Bellavitis, C., 2019. Platform strategy: Managing ecosystem value through selective promotion of complements. *Organization Science*, *30*(6), pp.1232-1251.

Rochet, J.C. and Tirole, J., 2003. Platform competition in two-sided markets. *Journal of the european economic association*, *1*(4), pp.990-1029.

Rochet, J.C. and Tirole, J., 2006. Two-sided markets: a progress report. *The RAND journal of economics*, *37*(3), pp.645-667.

Rolland, K.H., Mathiassen, L. and Rai, A., 2018. Managing digital platforms in user organizations: The interactions between digital options and digital debt. *Information Systems Research*, *29*(2), pp.419-443.

Rothschild, M.L. and Gaidis, W.C., 1981. Behavioral learning theory: Its relevance to marketing and promotions. *Journal of marketing*, *45*(2), pp.70-78.

Ryu, J., Han, S.L. and Park, T., 2023. An exploratory study of consumer switching behavior in platform businesses: A mixed methods approach. *Journal of Consumer Behaviour*, *22*(6), pp.1399-1414.

Sankaranarayanan, R., 2007. Innovation and the durable goods monopolist: The optimality of frequent new-version releases. *Marketing Science*, *26*(6), pp.774-791.

Scheinbaum, A.C., 2016. Digital engagement: opportunities and risks for sponsors: Consumer-Viewpoint and practical considerations for marketing via mobile and digital platforms. *Journal of Advertising Research*, *56*(4), pp.341-345.

Smith, W.R., 1956. Product differentiation and market segmentation as alternative marketing strategies. *Journal of marketing*, *21*(1), pp.3-8.

Stigler, G.J., 1992. Law or economics?. *The Journal of Law and Economics*, *35*(2), pp.455-468.

Stokey, N.L., 1979. Intertemporal price discrimination. *The Quarterly Journal of Economics*, *93*(3), pp.355-371.

Tirole, J., 1988. *The theory of industrial organization*. MIT press.

Vandenbosch, M.B. and Weinberg, C.B., 1995. Product and price competition in a two-dimensional vertical differentiation model. *Marketing Science*, *14*(2), pp.224-249.

Varian, H.R., 1989. Price discrimination. *Handbook of industrial organization*, *1*, pp.597-654.

Voorveld, H.A., Van Noort, G., Muntinga, D.G. and Bronner, F., 2018. Engagement with social media and social media advertising: The differentiating role of platform type. *Journal of advertising*, *47*(1), pp.38-54.

Wang, C. and Wright, J., 2020. Search platforms: Showrooming and price parity clauses. *the RAND Journal of Economics*, *51*(1), pp.32-58.

Weitz, B.A., 1985. Introduction to special issue on competition in marketing. *Journal of Marketing Research*, *22*(3), pp.229-236.

Weitz, B.A. and Jap, S.D., 1995. Relationship marketing and distribution channels. *Journal of the academy of Marketing Science*, *23*, pp.305-320.

Weyl, E.G., 2010. A price theory of multi-sided platforms. *American Economic Review*, *100*(4), pp.1642-1672.

Wu, R. and Lin, M., 2017. Seller Heterogeneity in a Competitive Marketplace. *Available at SSRN 2167604*.

Xie, J. and Shugan, S.M., 2001. Electronic tickets, smart cards, and online prepayments: When and how to advance sell. *Marketing Science*, *20*(3), pp.219-243.

Zeithammer, R. and Thomadsen, R., 2013. Vertical differentiation with variety-seeking consumers. *Management Science, 59*(2), pp.390-401.

Zennyo, Y., 2016. Competition between vertically differentiated platforms. *Journal of Industry, Competition and Trade*, *16*(3), pp.309-321.

Zhang, J. and Wedel, M., 2009. The effectiveness of customized promotions in online and offline stores. *Journal of marketing research*, *46*(2), pp.190-206.

Zhu, F. and Iansiti, M., 2019. Why Some Platforms Thrives... and Others Don't What Alibaba, Tencent, and Uber teach us about networks that flourish. The five characteristcs that make the difference. *Harvard Business Review*, *97*(1), pp.118-125.

Appendices

Appendix A

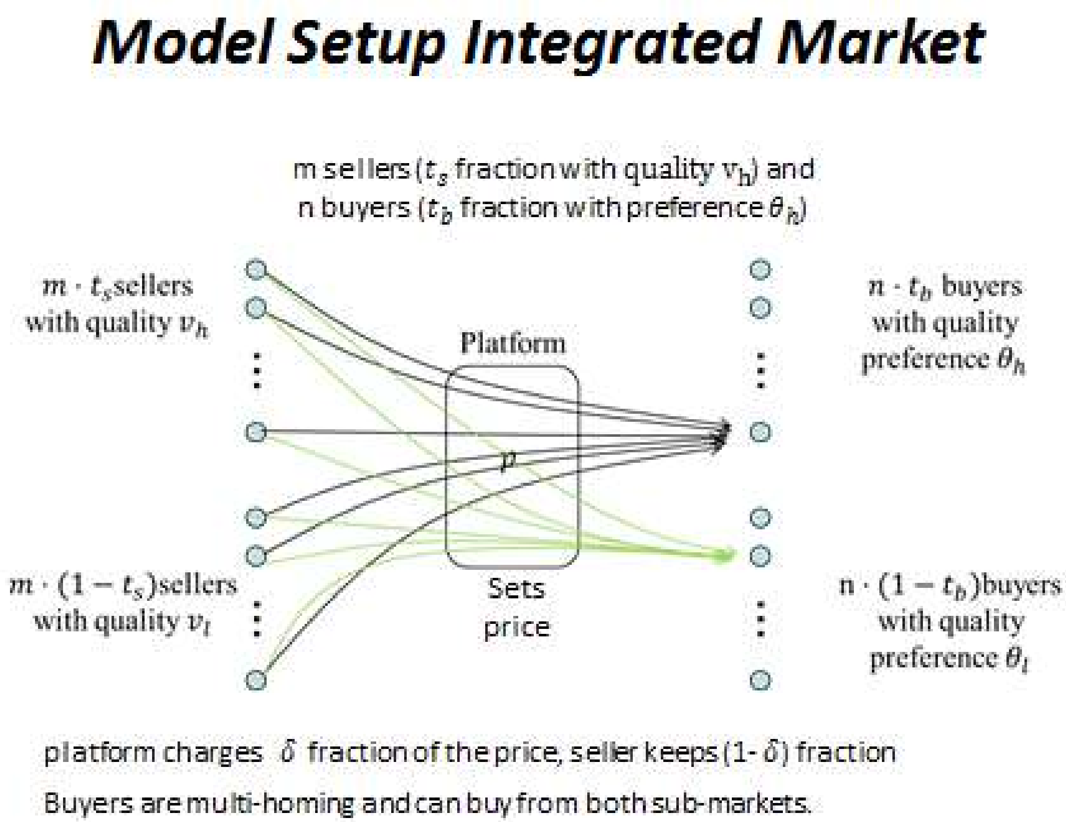


Figure 3.1 Model Setup for the Integrated Market (Platform sets price)

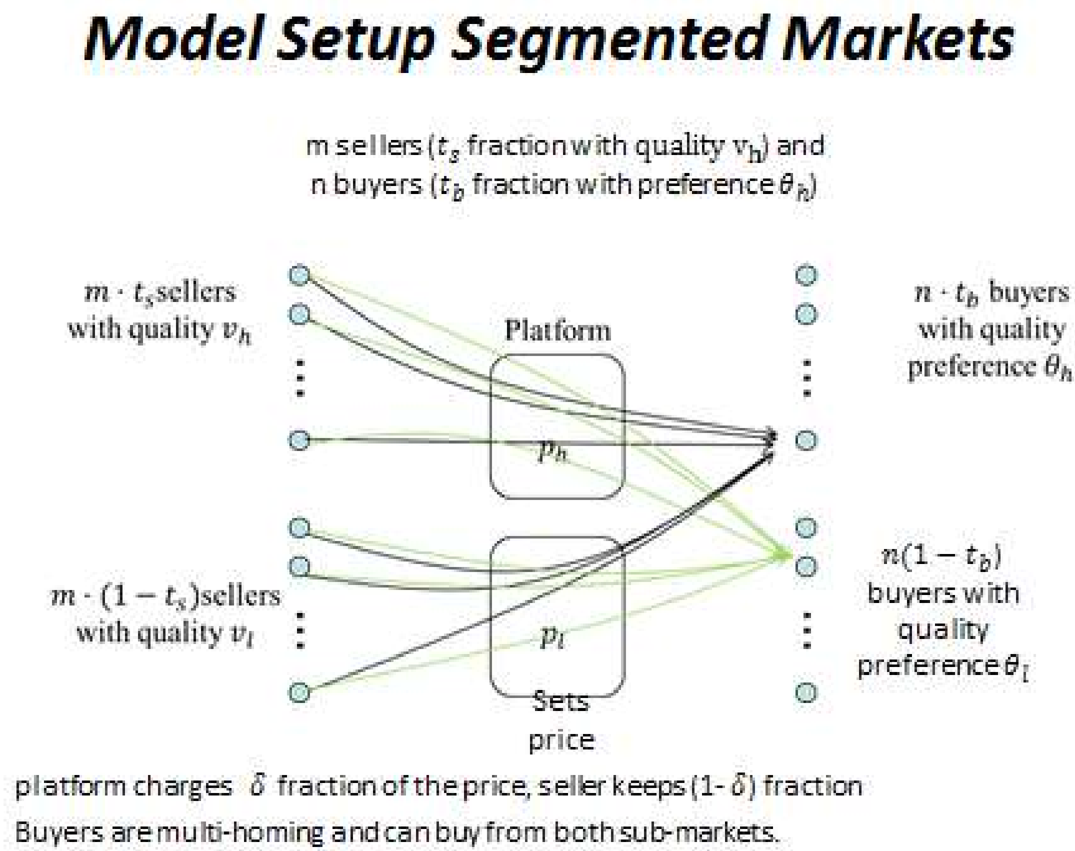


Figure 3.2 Model Setup for the Segmented Market (Platform sets price)

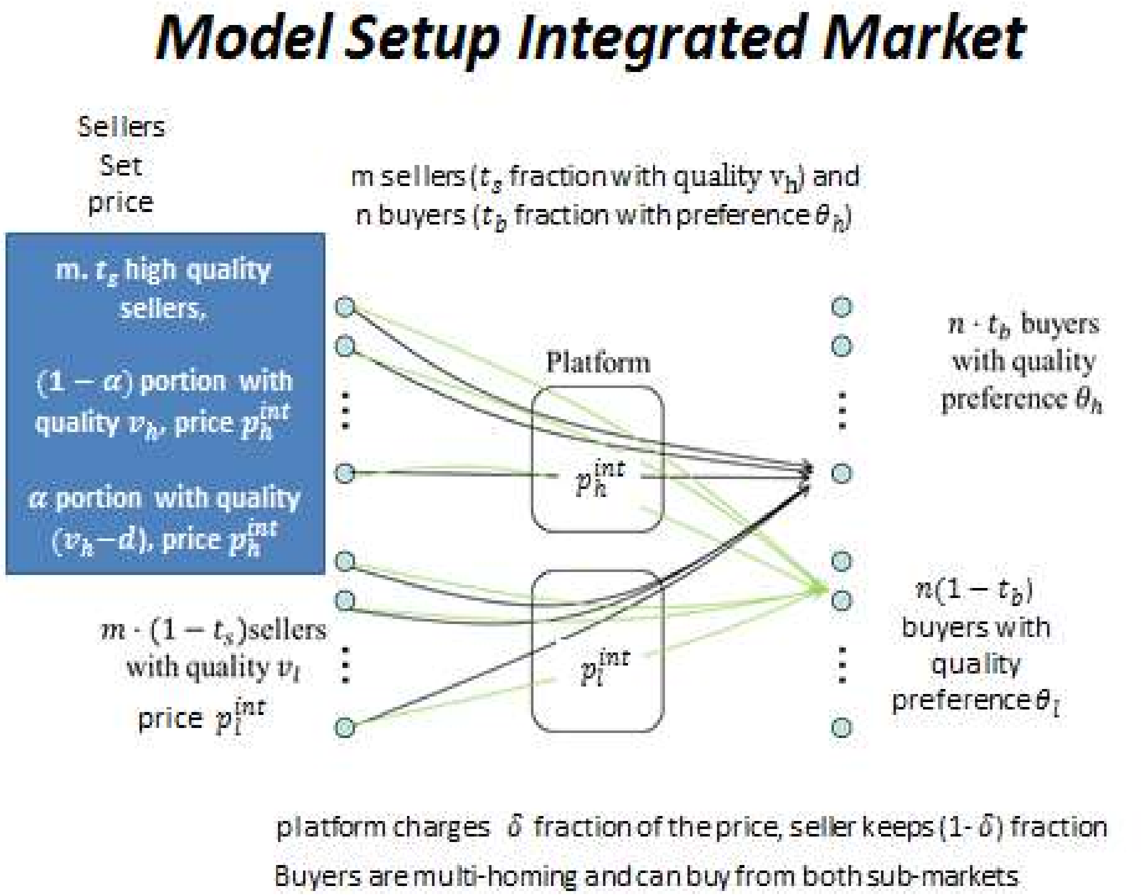


Figure 3.3 Model Setup for the Integrated Market (Sellers sets price)

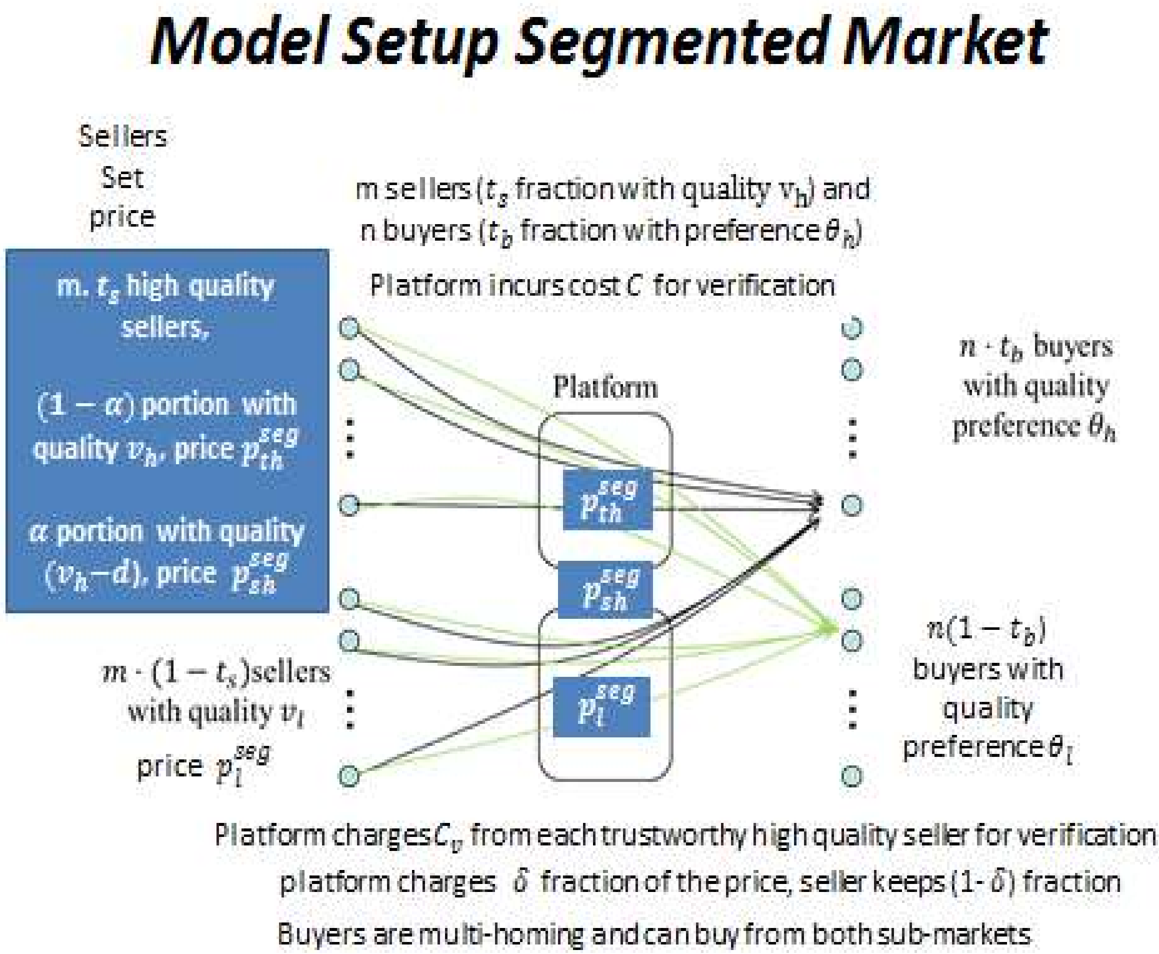


Figure 3.4 Model Setup for the Segmented Market (Sellers sets price)

Proof of Lemma 1

$u_i = \sum_{j=1}^{m} \ [q_{ij}\theta_i v_j - \frac{\beta}{2} q_{ij}^2 - \frac{\gamma}{2} \sum_{j \neq j'} \ q_{ij} q_{ij'} - q_{ij} p]$ ………………………………………. (1)

F.O.C. w.r.t q_{ij} we have -

$\theta_i v_j - \beta q_{ij} - \gamma \sum_{j \neq j'} \ q_{ij'} - p = 0$ ………………………………………………….……..…… (2)

Note that differentiating the term $\sum_{j=1}^{m} \ \left(\frac{\gamma}{2} \sum_{j \neq j'} \ q_{ij} q_{ij'}\right)$ w.r.t. q_{ij} gives $\gamma \sum_{j \neq j'} \ q_{ij'}$ and not $\frac{\gamma}{2} \sum_{j \neq j'} \ q_{ij'}$

From (2) we have -

$\theta_i v_j - (\beta - \gamma) q_{ij} - \gamma \sum q_{ij} - p = 0$ which gives -

$q_{ij} = \frac{\theta_i v_j - \gamma \sum q_{ij} - p}{\beta - \gamma}$ ……………………………………………………………...………. (3)

Summing (2) for all m we have –

$\theta_i \sum v_j - \beta \sum q_{ij} - \gamma (m-1) \sum q_{ij} - mp = 0$ which gives -

$\sum q_{ij} = \frac{[\theta_i m (t_s v_h + (1 - t_s) v_l) - mp]}{[\beta + \gamma (m-1)]} = \frac{m[\theta_i \tilde{v} - p]}{[\beta + \gamma (m-1)]}$

[Since $\sum v_j = \ m(t_s v_h + (1 - t_s) v_l)$]

Substituting $\sum q_{ij}$ value in (3) we have –

$q_{ij} = \frac{1}{\beta - \gamma} \left[\theta_i v_j - \frac{\gamma m \theta_i \tilde{v}}{[\beta + \gamma (m-1)]} + \frac{m \gamma p}{[\beta + \gamma (m-1)]} - p\right]$ Denoting the $p \ here \ as \ p^{int}$, we have –

$q_{ij}^{int} = \frac{1}{\beta-\gamma}\left[\theta_i(v_j - \frac{\gamma m\tilde{v}}{[\beta+\gamma(m-1)]}) - \frac{(\beta-\gamma)}{[\beta+\gamma(m-1)]}p^{int}\right]$ .... (4)

Q.E.D.

Proof of Proposition 1

The platform profit can be written as -

$\pi_p^{int} = \delta\sum_{i=1}^{n} \sum_{j=1}^{m} q_{ij}p = \delta mnp[t_s t_b q_{hh} + t_s(1-t_b)q_{lh} + (1-t_s)t_b q_{hl} + (1-t_s)(1-t_b)q_{ll}]$

$\pi_p^{int} = \frac{\delta mnp}{\beta-\gamma}\left[-\frac{(\beta-\gamma)p}{[\beta+\gamma(m-1)]} + t_s t_b\theta_h v_h + t_s(1-t_b)\theta_l v_h + (1-t_s)t_b\theta_h v_l + (1-t_s)(1-t_b)\theta_l v_l - \frac{t_b m\gamma\theta_h\tilde{v}}{[\beta+\gamma(m-1)]} - \frac{(1-t_b)m\gamma\theta_l\tilde{v}}{[\beta+\gamma(m-1)]}\right]$, where p is p^{int}

$\pi_p^{int} = \frac{\delta mnp}{\beta-\gamma}\left[-\frac{(\beta-\gamma)p}{[\beta+\gamma(m-1)]} + t_s v_h\tilde{\theta} + (1-t_s)v_l\tilde{\theta} - \frac{m\gamma\tilde{v}\tilde{\theta}}{[\beta+\gamma(m-1)]}\right]$

$\pi_p^{int} = \frac{\delta mnp}{\beta-\gamma}\left[-\frac{(\beta-\gamma)p}{[\beta+\gamma(m-1)]} + \tilde{\theta}\tilde{v}\left(1 - \frac{m\gamma}{[\beta+\gamma(m-1)]}\right)\right]$

$\pi_p^{int} = \frac{\delta mnp}{[\beta+\gamma(m-1)]}(\tilde{\theta}\tilde{v} - p)$ .... (5)

$\pi_p^{int} = \delta p[\frac{mn(\tilde{\theta}\tilde{v}-p)}{\beta+\gamma(m-1)}]$ where the term in square brackets is the aggregate demand.

Differentiating (5) w.r.t. p we have –

$p^{int} = \frac{\tilde{\theta}\tilde{v}}{2}$ Q.E.D. .... (6)

Substituting this value of p in equation 5, we get the profit of the platform as -

$\pi_p^{int} = \frac{\delta mnp^2}{[\beta+\gamma(m-1)]} = \frac{\delta mn}{[\beta+\gamma(m-1)]}(\frac{\tilde{\theta}^2\tilde{v}^2}{4})$ ……………………………………………...……..….. (7)

Q.E.D.

Proof of Corollary 1

Seller′sprofit $= \pi_j^{int} = p(1-\delta)\sum_{i=1}^{n} q_{ij} = p(1-\delta)n[t_b q_{hj} + (1-t_b)q_{lj}]$

Recall there are nt_b buyers with a high-quality preference θ_h, and $n(1-t_b)$ buyers with a low-quality preference θ_l. So we have by substituting the values of θ_i for $i = h$ and $i = l$ in q_{ij}^{int}

$\pi_j^{int} = \frac{(1-\delta)pn}{\beta-\gamma}\left[t_b\theta_h v_j + (1-t_b)\theta_l v_j - \frac{\gamma m\tilde{v}(t_b\theta_h+(1-t_b)\theta_l)}{[\beta+\gamma(m-1)]} - \frac{(\beta-\gamma)p}{[\beta+\gamma(m-1)]}\right]$

Substituting the value of p $which\ is$ p^{int} we have

$\pi_j^{int} = \frac{(1-\delta)pn}{\beta-\gamma}\left[\tilde{\theta}v_j - \frac{m\gamma\tilde{v}\tilde{\theta}}{[\beta+\gamma(m-1)]} - \frac{(\beta-\gamma)\tilde{\theta}\tilde{v}}{2[\beta+\gamma(m-1)]}\right]$ So we have -

$\pi_j^{int} = \frac{(1-\delta)p^2 n}{\beta-\gamma}\left[\frac{\beta-\gamma}{[\beta+\gamma(m-1)]} + 2\left(\frac{v_j}{\tilde{v}} - 1\right)\right]$ where $p = \frac{\tilde{\theta}\tilde{v}}{2}$ ……………………………….……. (8)

This can also be written as -

$\pi_j^{int} = \frac{(1-\delta)n}{[\beta+\gamma(m-1)]}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) + \frac{(1-\delta)n}{\beta-\gamma}\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right)(v_j - \tilde{v})$ ……………………………..……...…...…. (9)

Q.E.D

Proof of Corollary 2

$u_i^{int} = \sum_{j=1}^{m} q_{ij}\left[\theta_i v_j - \frac{\beta}{2}q_{ij} - \frac{\gamma}{2}\sum_{j\neq j'} q_{ij'} - p\right]$

$= \sum_{j=1}^{m}\left(\frac{q_{ij}}{2}\right)[(\theta_i v_j - \beta q_{ij} - \gamma\sum_{j\neq j'} q_{ij'} - p) + \theta_i v_j - p]$ Using the F.O.C. from (2) we have

$u_i^{int} = \sum_{j=1}^{m} \frac{q_{ij}}{2}\left(\theta_i v_j - p\right)$ ………………………………………………………..…...…..…. (10)

Substituting the value of q_{ij} we have

$$u_i^{int} = \sum_{j=1}^{m} \frac{(\theta_i v_j - p)}{2}\left(\frac{1}{\beta-\gamma}\right)\left[\theta_i v_j - \frac{\gamma m \theta_i \tilde{v}}{[\beta+\gamma(m-1)]} - \frac{(\beta-\gamma)p}{[\beta+\gamma(m-1)]}\right]$$

$$u_i^{int} = \sum_{j=1}^{m} \frac{(\theta_i v_j - p)}{2}\left(\frac{1}{\beta-\gamma}\right)\left[\theta_i v_j - \frac{\gamma m \theta_i \tilde{v}}{[\beta+\gamma(m-1)]} - \frac{(\beta-\gamma)p}{[\beta+\gamma(m-1)]} + \theta_i \tilde{v} - \theta_i \tilde{v}\right]$$

$$u_i^{int} = \sum_{j=1}^{m} \frac{(\theta_i v_j - p)}{2}\left[\frac{\theta_i \tilde{v} - p}{[\beta+\gamma(m-1)]} + \frac{1}{\beta-\gamma}(\theta_i v_j - \theta_i \tilde{v})\right]$$

$$u_i^{int} = \frac{1}{2}\sum_{j=1}^{m} \left[\frac{1}{[\beta+\gamma(m-1)]}\left\{\theta_i^2 v_j \tilde{v} - \theta_i\left(v_j + \tilde{v}\right)p + p^2\right\} + \frac{1}{\beta-\gamma}\{\theta_i^2 v_j^2 - \theta_i^2 v_j \tilde{v} + \theta_i \tilde{v} p - \theta_i v_j p\}\right]$$

Now $\sum_{j=1}^{m} v_j = m(t_s v_h + (1-t_s)v_l) = m\tilde{v}$, $\sum_{j=1}^{m} v_j^2 = m(t_s v_h^2 + (1-t_s)v_l^2)$ and $p = \frac{\tilde{\theta}\tilde{v}}{2}$

So we have -

$$u_i^{int} = \frac{1}{2}\left[\frac{1}{[\beta+\gamma(m-1)]}\left\{m\theta_i^2\tilde{v}^2 - m\theta_i\tilde{\theta}\tilde{v}^2 + \frac{m}{4}\tilde{\theta}^2\tilde{v}^2\right\} + \frac{1}{\beta-\gamma}\{m\theta_i^2(t_s v_h^2 + (1-t_s)v_l^2) - m\theta_i^2\tilde{v}^2\}\right]$$

As $Var(v) = \frac{1}{m}\sum_{j=1}^{m} v_j^2 - \left(\frac{1}{m}\sum_{j=1}^{m} v_j\right)^2 = t_s v_h^2 + (1-t_s)v_l^2 - \tilde{v}^2$, we have

$u_i^{int} = \frac{m}{2}\left[\frac{\tilde{v}^2}{[\beta+\gamma(m-1)]}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 + \frac{\theta_i^2}{\beta-\gamma}Var(v)\right]$ ……..……………………………...…..…..……. (11)

Q.E.D.

Proof of Lemma 2

Due to multi-homing, buyers now buy from both segments of the market – i.e. they buy from both high type and low type sellers and pay different prices according to the quality of the product.

$$u_i = \sum_{j=1}^{m} [q_{ij}\theta_i v_j - \frac{\beta}{2} q_{ij}^2 - \frac{\gamma}{2} \sum_{j \neq j'} q_{ij} q_{ij'} - q_{ij} p_j] \quad \ldots\ldots (12)$$

F.O.C. w.r.t. q_{ij} –

$$\theta_i v_j - \beta q_{ij} - \gamma \sum_{j' \neq j} q_{ij'} - p_j = 0 \quad \ldots\ldots (13)$$

$$q_{ij} = \frac{1}{\beta - \gamma} [\theta_i v_j - \gamma \sum_{j=1}^{m} q_{ij} - p_j] \quad \ldots\ldots (14)$$

Summing all the m equations like (13) we have –

$\theta_i \sum_{j=1}^{m} v_j - \beta \sum_{j=1}^{m} q_{ij} - \gamma(m-1) \sum_{j=1}^{m} q_{ij} - \sum_{j=1}^{m} p_j = 0$ which gives

$$\sum_{j=1}^{m} q_{ij} = \frac{1}{\beta + \gamma(m-1)} (\theta_i \sum_{j=1}^{m} v_j - \sum_{j=1}^{m} p_j) \quad \ldots\ldots (15)$$

But $\sum_{j=1}^{m} v_j = m\tilde{v}$ and

$\sum_{j=1}^{m} p_j = m[t_s p_h + (1 - t_s) p_l] = m\tilde{p}$

So we have –

$$\sum_{j=1}^{m} q_{ij} = \frac{m}{\beta + \gamma(m-1)} (\tilde{v} - \tilde{p}) \quad \ldots\ldots (16)$$

Simplifying (14) and (16) we have –

$$q_{ij}^{seg} = \frac{1}{\beta - \gamma} \left[\theta_i \left(v_j - \frac{m\gamma\tilde{v}}{\beta + \gamma(m-1)} \right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta + \gamma(m-1)} \right) \right] \quad \ldots\ldots (17)$$

$q_{ij}^{seg} = \{\frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_h^{seg} - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$ if seller j is of high quality

$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_l^{seg} - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$ if seller j is of low quality

Q.E.D.

Proof of Proposition 2

$\pi_p^{seg} = \delta\sum_{i=1}^{n}\ \sum_{j=1}^{m}\ q_{ij}p_j = \delta mn[t_s t_b q_{hh}p_h + t_s(1-t_b)q_{lh}p_h + (1-t_s)t_b q_{hl}p_l +$

$(1-t_s)(1-t_b)q_{ll}p_l]$

$\pi_p^{seg} = \frac{\delta mn}{\beta-\gamma}[t_s p_h\{t_b\left(\theta_h v_h - \frac{m\gamma\theta_h\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) + (1-t_b)(\theta_l v_h - \frac{m\gamma\theta_l\tilde{v}}{\beta+\gamma(m-1)} - p_h +$

$\frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)})\} + (1-t_s)p_l\left\{t_b\left(\theta_h v_l - \frac{m\gamma\theta_h\tilde{v}}{\beta+\gamma(m-1)} - p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) + (1-t_b)\left(\theta_l v_l - \frac{m\gamma\tilde{v}\theta_l}{\beta+\gamma(m-1)} -\right.\right.$

$\left.\left.p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right\}]$

$\pi_p^{seg} = \frac{\delta mn}{\beta-\gamma}\left[t_s p_h\left\{\tilde{\theta}v_h - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right\} + (1-t_s)p_l\left\{\tilde{\theta}v_l - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)} - p_l +\right.\right.$

$\left.\left.\frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right\}\right]$ ………………………………………………………………..…..…...……. (18)

We need to take derivative w.r.t. p_h and p_l. So we replace $\tilde{p}$ with p_h and p_l.

$\pi_p^{seg} = \frac{\delta mn}{\beta-\gamma}\left[\tilde{\theta}\{t_s p_h v_h + (1-t_s)p_l v_l\} - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)}\{t_s p_h + (1-t_s)p_l\} - \{t_s p_h^2 + (1-\right.$

$\left.t_s)p_l^2\} + \frac{m\gamma}{\beta+\gamma(m-1)}\{t_s p_h + (1-t_s)p_l\}^2\right]$

F.O.C. w.r.t. p_h after canceling out the common term t_s –

$\frac{\delta mn}{\beta-\gamma}\left[\tilde{\theta}v_h - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)} - 2p_h + \frac{2m\gamma}{\beta+\gamma(m-1)}\{t_s p_h + (1-t_s)p_l\}\right] = 0$ ……………………......…. (19)

F.O.C. w.r.t. p_l after canceling out the common term $(1-t_s)$ –

$$\frac{\delta mn}{\beta-\gamma}\left[\tilde{\theta}v_l-\frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)}-2p_l+\frac{2m\gamma\{t_sp_h+(1-t_s)p_l\}}{\beta+\gamma(m-1)}\right]=0 \quad (20)$$

From (19) and (20) we have –

$\frac{(\beta-\gamma)\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)}-2\tilde{p}+\frac{2m\gamma\tilde{p}}{\beta+\gamma(m-1)}=0$ which gives

$$\tilde{p}^{seg}=\frac{\tilde{\theta}\tilde{v}}{2} \quad (21)$$

Q.E.D.

Substituting $\tilde{p}=\frac{\tilde{\theta}\tilde{v}}{2}$ in (19) we have –

$$p_h^{seg}=\frac{\left[\tilde{\theta}v_h-\frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)}+\frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)}\right]}{2}=\frac{\tilde{\theta}v_h}{2} \quad (22)$$

Similarly, we have –

$$p_l^{seg}=\frac{\tilde{\theta}v_l}{2} \quad (23)$$

So we have –

$$p_j^{seg}=\frac{\tilde{\theta}v_j}{2}. \quad (24)$$

Q.E.D

$$\pi_p^{seg}=\frac{\delta mn}{\beta-\gamma}\tilde{\theta}^2\left[\frac{t_sv_h^2}{2}+\frac{(1-t_s)v_l^2}{2}-\frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{\tilde{v}^2}{2}\right)-\frac{1}{4}\{t_sv_h^2+(1-t_s)v_l^2\}+\frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{\tilde{v}^2}{4}\right)\right]$$

$\pi_p^{seg}=\frac{\delta mn}{4(\beta-\gamma)}\tilde{\theta}^2\left[\widetilde{v^2}-\frac{m\gamma}{\beta+\gamma(m-1)}\tilde{v}^2\right]$ Now using the fact that $\widetilde{v^2}=t_sv_h^2+(1-t_s)v_l^2$

$and\ \tilde{v}^2 = \{t_s v_h + (1 - t_s)v_l\}^2$ which leads to $\widetilde{v^2} = \tilde{v}^2 + t_s(1 - t_s)(v_h - v_l)^2$

$\pi_p^{seg} = \frac{\delta mn}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) + \frac{\delta mn\tilde{\theta}^2}{4(\beta-\gamma)} t_s(1 - t_s)(v_h - v_l)^2$ …………………………………..…. (25)

Now $\tilde{v} = t_s v_h + (1 - t_s)v_l$ and $t_s(1 - t_s)(v_h - v_l)^2 = Var(v)$. So we have $\widetilde{v^2} = \tilde{v}^2 + var(v)$

$\pi_p^{seg} = \frac{\delta mn}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) + \frac{\delta mn\tilde{\theta}^2}{4(\beta-\gamma)} Var(v)$. …………………………...………………….…… (26)

Q.E.D

Now $\pi_p^{int} = \frac{\delta mn}{[\beta+\gamma(m-1)]}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right)$

So we can see that $\pi_p^{seg} > \pi_p^{int}$ ………………………………………………….…….………. (27)

Q.E.D.

Proof of Proposition 3

Sellers profit in a segmented market is given by

Seller's profit $=\pi_j^{seg} = (1 - \delta)\sum_{i=1}^{n} p_j q_{ij} = p_j(1 - \delta)n[t_b q_{hj} + (1 - t_b)q_{lj}]$

Substituting the values of $q_{ij}^{seg}\ for\ i = h$ and $i = l$ we have

$$\pi_j^{seg} = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}\tilde{v}}{2}\right)\right)\right]$$

$$\pi_j^{seg} = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}v_j - p_j - \frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}\tilde{v}}{2}\right)\right] = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\frac{\tilde{\theta}v_j}{2} - \frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}\tilde{v}}{2}\right)\right]$$

as $p_j^{seg} = \frac{\tilde{\theta} v_j}{2}$ So we have

$$\pi_j^{seg} = \frac{(1-\delta)n\tilde{\theta}^2 v_j}{4(\beta-\gamma)}\left[v_j - \frac{m\gamma}{\beta+\gamma(m-1)}\tilde{v}\right]. \quad \text{……} (28)$$

Q.E.D

Note that the seller's profit is positive only when $v_j > \frac{\gamma m \tilde{v}}{\beta+\gamma(m-1)}$. Since $v_h > \tilde{v}$ *and* $v_l < \tilde{v}$ *and* $\frac{\gamma m}{\beta+\gamma(m-1)} < 1$, the high-quality seller's profit is always positive while the low-quality seller's profit is positive when $\frac{v_h}{v_l} < \frac{\beta-\gamma}{m\gamma t_s} + 1$.

Now we have

$$\pi_j^{int} = \frac{(1-\delta)n}{[\beta+\gamma(m-1)]}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) + \frac{(1-\delta)n}{\beta-\gamma}\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right)(v_j - \tilde{v})$$

Comparing segmented market seller's profit with the integrated market we have

$\pi_j^{seg} \gtrless \pi_j^{int}$ if

$$\frac{(1-\delta)n\tilde{\theta}^2 v_j}{4(\beta-\gamma)}\left[v_j - \frac{\gamma m\tilde{v}}{\beta+\gamma(m-1)}\right] \gtrless (1-\delta)n\left(\frac{\tilde{\theta}^2\tilde{v}}{2}\right)\left[\frac{\tilde{v}}{2(\beta+\gamma(m-1))} - \frac{\tilde{v}-v_j}{\beta-\gamma}\right]$$

Simplifying this inequality we get

$$(v_j - \tilde{v})\left[(v_j - \tilde{v})(\beta + \gamma(m-1)) - m\gamma\tilde{v}\right] \gtrless 0$$

If $v_j = v_l$, $(v_l - \tilde{v}) < 0$ and

$\left[(v_l - \tilde{v})(\beta + \gamma(m-1)) - m\gamma\tilde{v}\right] = v_l[t_s(\beta - \gamma + 2m\gamma) - m\gamma] - v_h[t_s(\beta - \gamma + 2m\gamma)] <$

$v_h[t_s(\beta - \gamma + 2m\gamma) - m\gamma] - v_h[t_s(\beta - \gamma + 2m\gamma)]$ (as $v_l < v_h$)

$= -m\gamma v_h$.

So $\left[(v_l - \tilde{v})\big(\beta + \gamma(m-1)\big) - m\gamma\tilde{v}\right]$ is less than zero. Hence

$(v_l - \tilde{v})\left[(v_l - \tilde{v})\big(\beta + \gamma(m-1)\big) - m\gamma\tilde{v}\right] > 0$.

Hence $\pi_l^{seg} > \pi_l^{int}$.

If $v_j = v_h$, $(v_h - \tilde{v}) > 0$ and

$\left[(v_h - \tilde{v})\big(\beta + \gamma(m-1)\big) - m\gamma\tilde{v}\right] = (v_h - v_l)(1-t_s)(\beta - \gamma + 2m\gamma) - m\gamma v_h \geq 0$ if

$\frac{v_l}{v_h} \leq 1 - \frac{m\gamma}{(1-t_s)(\beta-\gamma+2m\gamma)}$.

So $\pi_h^{seg} \geq \pi_h^{int}$ if $\frac{v_l}{v_h} \leq 1 - \frac{m\gamma}{(1-t_s)(\beta-\gamma+2m\gamma)}$ and vice versa.

Q.E.D.

If the proportion of high-quality sellers, t_s, is below a certain level ($t_s < \frac{\beta-\gamma}{\beta-\gamma+m\gamma}$), low-quality sellers can make a positive profit in a segmented market, while high-quality sellers can make more profit in a segmented market compared to the integrated market.

Earlier, we got a condition for positive seller's profit for low-quality sellers in a segmented market - $\frac{v_h}{v_l} < \frac{\beta-\gamma}{m\gamma t_s} + 1$.

And now we get a condition for high-quality sellers to make more profit in a segmented market than an integrated market - $\frac{v_l}{v_h} \leq 1 - \frac{m\gamma}{(1-t_s)(\beta-\gamma+2m\gamma)}$ which can be written as

$\frac{v_h}{v_l} \geq \frac{(1-t_s)(\beta-\gamma+2m\gamma)}{[(1-t_s)(\beta-\gamma+2m\gamma)-m\gamma]}$

Both these conditions are simultaneously satisfied when

$\frac{(1-t_s)(\beta-\gamma+2m\gamma)}{[(1-t_s)(\beta-\gamma+2m\gamma)-m\gamma]} < \frac{\beta-\gamma}{m\gamma t_s} + 1$ Simplifying this leads to

$t_s < \frac{\beta-\gamma}{\beta-\gamma+m\gamma}$.

Q.E.D.

Proof of Proposition 4

$$u_i^{seg} = \sum_{j=1}^{m} q_{ij}[\theta_i v_j - \frac{\beta}{2}q_{ij} - \frac{\gamma}{2}\sum_{j\neq j'} q_{ij'} - p_j] = \sum_{j=1}^{m} \left(\frac{q_{ij}}{2}\right)[(\theta_i v_j - \beta q_{ij} - \gamma\sum_{j\neq j'} q_{ij'} - p_j) + \theta_i v_j - p_j]$$

Using the F.O.C. we can write this as –

$$u_i^{seg} = \sum_{j=1}^{m} \frac{q_{ij}}{2}\left(\theta_i v_j - p_j\right) \quad \text{(29)}$$

$$u_i^{seg} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} [\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - (\frac{\tilde{\theta}v_j}{2} - \frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}\tilde{v}}{2}\right)]\left(\theta_i v_j - p_j\right)$$

Now $p_j^{seg} = \frac{\tilde{\theta}v_j}{2}$. So we have

$$u_i^{seg} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} \left[\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right)v_j\right]$$

$$u_i^{seg} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} \left[\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2\left(v_j^2 - \frac{m\gamma\tilde{v}v_j}{\beta+\gamma(m-1)}\right)\right]$$

$u_i^{seg} = \frac{m}{2(\beta-\gamma)}\sum_{j=1}^{m} \left[\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2\left(t_s v_h^2 + (1-t_s)v_l^2 - \frac{m\gamma\tilde{v}(t_s v_h + (1-t_s)v_l)}{\beta+\gamma(m-1)}\right)\right]$ where we have used the fact that $\sum_{j=1}^{m} v_j = m\tilde{v}$ and $\sum_{j=1}^{m} v_j^2 = m(t_s v_h^2 + (1-t_s)v_l^2)$. So we have

$u_i^{seg} = \frac{m}{2(\beta-\gamma)}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 \sum_{j=1}^{m} \left[\widetilde{v^2} - \frac{m\gamma\tilde{v}^2}{\beta+\gamma(m-1)}\right]$. Now using the fact that $\widetilde{v^2} = t_s v_h^2 + (1-t_s)v_l^2$ and $\tilde{v}^2 = \{t_s v_h + (1-t_s)v_l\}^2$ which leads to $\widetilde{v^2} = \tilde{v}^2 + t_s(1-t_s)(v_h - v_l)^2$

$$u_i^{seg} = \frac{m\tilde{v}^2}{2(\beta+\gamma(m-1))}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 + \frac{m}{2(\beta-\gamma)}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 t_s(1-t_s)(v_h - v_l)^2$$

Since $t_s(1-t_s)(v_h - v_l)^2 = Var(v)$ we have

$$u_i^{seg} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 Var(v)\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

Q.E.D.

But we have

$$u_i^{int} = \frac{m}{2}\left[\frac{\tilde{v}^2}{\beta+\gamma(m-1)}\left(\theta_i - \frac{\tilde{\theta}}{2}\right)^2 + \frac{1}{\beta-\gamma}\theta_i^2 Var(v)\right]$$

Comparing term by term u_i^{seg} and u_i^{int} we have

$u_i^{int} > u_i^{seg}$

Q.E.D.

Proof of Lemma 3

There are mt_s high type sellers of which a $(1-\alpha)$ fraction is the actual high type with quality v_h (they also claim quality as v_h) and a α fraction who claim they are high type but have actual quality $v_h - d$. There are $m(1-t_s)$ low type sellers with quality v_l.

The expected quality for the high type of product is $v_{he} = (1-\alpha)v_h + \alpha(v_h - d) = (v_h - \alpha d)$

The market average quality is $\tilde{v} = t_s v_{he} + (1 - t_s)v_l = t_s(v_h - \alpha d) + (1 - t_s)v_l$.

The market price for the product with quality $(v_h - \alpha d)$ is p_h^{int} and v_l is p_l^{int} .

$\tilde{p}^{int} \equiv t_s p_h^{int} + (1 - t_s)p_l^{int}$

The demand function is just like the segmented case in section 3 (please see proof of Lemma 2)

$$q_{ij}^{int} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$$

where $v_j = (v_h - \alpha d)$ for high type sellers or $v_j = v_l$ for low type sellers.

Substituting the various values we have

$$q_{ij}^{int} = \{\frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - \alpha d - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_h^{int} - \frac{m\gamma\tilde{p}^{int}}{\beta+\gamma(m-1)}\right)\right] \text{ if } j \text{ is a high} - \text{quality seller}$$

$$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_l^{int} - \frac{m\gamma\tilde{p}^{int}}{\beta+\gamma(m-1)}\right)\right] \text{ if } j \text{ is a low} - \text{quality seller}$$

……………………………………………………………….……….. (31)

Q.E.D.

Proof of Proposition 5

The demand function is of the form

$$q_{ij}^{int} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$$

where $v_j = (v_h - \alpha d)$ for high type sellers or $v_j = v_l$ for low type sellers and

$p_j = p_h^{int}$ or p_h^{int}.

The seller's profit function is

Seller's profit = $\pi_j^{int} = (1-\delta)\sum_{i=1}^{n} p_j q_{ij} = p_j(1-\delta)n[t_b q_{hj} + (1-t_b)q_{lj}]$

Substituting the values of q_{ij}^{int} for $i = h$ and $i = l$ we have

$\pi_j^{int} = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$ where $\tilde{p}^{int} \equiv t_s p_h^{int} + (1-t_s)p_l^{int}$

F.O.C. w.r.t. p_j

$$\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) - p_j\left[1 - \frac{\gamma}{\beta+\gamma(m-1)}\right] = 0$$

where we have used the fact that $\tilde{p} = \frac{\sum_{j=1}^{m} p_j}{m}$ $and \frac{\partial\tilde{p}}{\partial p_j} = \frac{1}{m}$

Summing up the F.O.C. of *m* sellers and canceling *m*, the common factor we have

$\tilde{\theta}\left(\tilde{v} - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(\tilde{p} - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) - \tilde{p}\left[1 - \frac{\gamma}{\beta+\gamma(m-1)}\right] = 0$ This gives

$\frac{\tilde{\theta}\tilde{v}(\beta-\gamma)}{\beta+\gamma(m-1)} = \tilde{p}\left[\frac{\beta-\gamma}{\beta+\gamma(m-1)} + \frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]$ This gives

$$\tilde{p} = \frac{\tilde{\theta}\tilde{v}(\beta-\gamma)}{2\beta+m\gamma-3\gamma} \quad (32)$$

Substituting this value in the F.O.C. condition we have

$p_j = \frac{1}{2-\frac{\gamma}{\beta+\gamma(m-1)}}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) + \frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{(\beta-\gamma)\tilde{\theta}\tilde{v}}{2\beta+m\gamma-3\gamma}\right)\right]$ Simplifying this gives

$$p_j^{int} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_j - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right] \quad (33)$$

So we have

$$p_h^{int} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d)-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

$$p_l^{int} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_l-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Seller's profit is

$$\pi_j^{int} = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j-\frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right)-\left(p_j-\frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$$

Using F.O.C. and simplifying, we have

$$\pi_j^{int} = \frac{(1-\delta)np_j^2}{\beta-\gamma}\left[1-\frac{\gamma}{\beta+\gamma(m-1)}\right]$$

So we have

$$\pi_j^{int} = \frac{(1-\delta)n}{(\beta-\gamma)}\left[\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_j-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$$

where $j = h \; or \; l$ and $v_j = (v_h-\alpha d)$ or v_l ………………………………………………. (34)

Hence we have

$$\pi_h^{int} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d)-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$$

$$\pi_l^{int} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_l-\frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$$

Q.E.D.

Proof of Corollary 3

The demand function is

$q_{ij}^{int} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$

The platform profit is

$\pi_p^{int} = \delta\sum_{i=1}^{n}\ \sum_{j=1}^{m}\ q_{ij}p_j = \delta mn[t_s t_b q_{hh}p_h + t_s(1-t_b)q_{lh}p_h + (1-t_s)t_b q_{hl}p_l +$

$(1-t_s)(1-t_b)q_{ll}p_l]$ where $p_h = p_h^{int},\ p_l = p_l^{int},\ \tilde{p}^{int} \equiv t_s p_h^{int} + (1-t_s)p_l^{int}$,

$v_j = (v_h - \alpha d)$ for high type sellers or $v_j = v_l$ for low type sellers and

$\tilde{v} = t_s(v_h - \alpha d) + (1-t_s)v_l$.

Substituting the values of the demand function we have

$\pi_p^{int} = \frac{\delta mn}{\beta-\gamma}[t_s p_h\{t_b\left(\theta_h(v_h - \alpha d) - \frac{m\gamma\theta_h\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) + (1-t_b)(\theta_l(v_h - \alpha d) -$

$\frac{m\gamma\theta_l\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)})\} + (1-t_s)p_l\left\{t_b\left(\theta_h v_l - \frac{m\gamma\theta_h\tilde{v}}{\beta+\gamma(m-1)} - p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) +\right.$

$\left.(1-t_b)\left(\theta_l v_l - \frac{m\gamma\theta_l\tilde{v}}{\beta+\gamma(m-1)} - p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right\}]$

$\pi_p^{int} = \frac{\delta mn}{\beta-\gamma}\left[t_s p_h\left\{\tilde{\theta}(v_h - \alpha d) - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right\} + (1-t_s)p_l\left\{\tilde{\theta}v_l - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)} -\right.\right.$

$\left.\left.p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right\}\right]$

$\pi_p^{int} = \frac{\delta mn}{\beta-\gamma}\left[t_s p_h\tilde{\theta}(v_h - \alpha d) + (1-t_s)p_l\tilde{\theta}v_l - \frac{m\gamma\tilde{\theta}\tilde{v}\tilde{p}}{\beta+\gamma(m-1)} - \{t_s p_h^2 + (1-t_s)p_l^2\} + \frac{m\gamma\tilde{p}^2}{\beta+\gamma(m-1)}\right]$

$\pi_p^{int} = \frac{\delta mn}{\beta-\gamma}\left[t_s p_h\tilde{\theta}(v_h - \alpha d) + (1-t_s)p_l\tilde{\theta}v_l - \{t_s p_h^2 + (1-t_s)p_l^2\} + \frac{m\gamma\tilde{p}(\tilde{p}-\tilde{\theta}\tilde{v})}{\beta+\gamma(m-1)}\right]$

Substituting the values of p_h, p_l *and* $\tilde{p}$ from *Proposition* 5 and simplifying we have –

$$\pi_p^{int} = \frac{\delta mn\tilde{\theta}^2}{\beta-\gamma}\Bigg[\frac{\beta+m\gamma-\gamma}{2\beta+2m\gamma-3\gamma}(t_s(v_h-\alpha d)^2+(1-t_s)v_l^2) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}^2}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)} -$$

$$\left(\frac{\beta+m\gamma-\gamma}{2\beta+2m\gamma-3\gamma}\right)^2(t_s(v_h-\alpha d)^2+(1-t_s)v_l^2) - \frac{(\beta+m\gamma-2\gamma)^2m^2\gamma^2\tilde{v}^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)^2} +$$

$$\frac{2(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)m\gamma\tilde{v}^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)} - \frac{m\gamma(\beta-\gamma)(\beta+m\gamma-2\gamma)\tilde{v}^2}{(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)}\Bigg]$$

Now we have

$$(t_s(v_h-\alpha d)^2+(1-t_s)v_l^2) = \tilde{v}^2 + Var(v\ ),$$

$$Var(v\ ) = t_s(1-t_s)(v_h-\alpha d-v_l)^2\ .$$

Also, we have

$$t_s(v_h-\alpha d-\tilde{v})^2+(1-t_s)(v_l-\tilde{v})^2 = t_s(1-t_s)(v_h-\alpha d-v_l)^2 = Var(v)$$

So we have –

$$\pi_p^{int} = \frac{\delta mn\tilde{\theta}^2}{\beta-\gamma}\Bigg[\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}Var(v\ ) + \tilde{v}^2\Bigg\{\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2} +$$

$$\frac{m\gamma^2(\beta+m\gamma-2\gamma)}{(2\beta+m\gamma-3\gamma)(2\beta+2m\gamma-3\gamma)^2} - \frac{m\gamma(\beta-\gamma)(\beta+m\gamma-2\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} - \frac{m^2\gamma^2(\beta+m\gamma-2\gamma)^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)^2}\Bigg\}\Bigg]$$

$$\pi_p^{int} = \frac{\delta mn\tilde{\theta}^2}{\beta-\gamma}\left[\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}Var(v) + \tilde{v}^2\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)^2(2\beta+2m\gamma-3\gamma)^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)}\right]$$

$$\pi_p^{int} = \delta mn\tilde{\theta}^2\tilde{v}^2\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \delta mn\tilde{\theta}^2\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}Var(v\ )$$

Substituting the value of $Var(v)$, we have

$$\pi_p^{int} = \delta mn\tilde{\theta}^2\tilde{v}^2\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \delta mn\tilde{\theta}^2\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}[t_s(v_h-\alpha d-\tilde{v})^2 +$$

$$(1-t_s)(v_l-\tilde{v})^2]$$

$\pi_p^{int} = \delta mn\tilde{\theta}^2 \left[\tilde{v}^2 \frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}[t_s(v_h - \alpha d - \tilde{v})^2 + \right.$

$\left.(1-t_s)(v_l - \tilde{v})^2]\right]$ ………………………………………………………………………….. (35)

Q.E.D.

Proof of Corollary 4

The demand function is

$q_{ij}^{int} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$

where $p_j = p_h or\ p_l,\ p_h = p_h^{int},\ p_l = p_l^{int},\ \tilde{p}^{int} \equiv t_s p_h^{int} + (1-t_s)p_l^{int}$,

$v_j = (v_h - \alpha d)$ for high type sellers or $v_j = v_l$ for low type sellers and

$\tilde{v} = t_s(v_h - \alpha d) + (1-t_s)v_l$.

Using the F.O.C. condition we have

$u_i^{int} = \sum_{j=1}^{m} \frac{q_{ij}}{2}\left(\theta_i v_j - p_j\right)$ Substituting the values for q_{ij} we have

$u_i^{int} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} \left(\theta_i v_j - p_j\right)\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$

$u_i^{int} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} \left(\theta_i v_j - p_j\right)\left[\left(\theta_i v_j - p_j\right) - \frac{m\gamma}{\beta+\gamma(m-1)}(\theta_i\tilde{v} - \tilde{p})\right]$

Substituting the values for $\tilde{p}$ from $Proposition$ 5 we have

$u_i^{int} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} \left(\theta_i v_j - p_j\right)\left[\left(\theta_i v_j - p_j\right) - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\left(\theta_i - \frac{\tilde{\theta}(\beta-\gamma)}{2\beta+m\gamma-3\gamma}\right)\right]$

Substituting the value of p_j from $Proposition$ 5 we have -

$$u_i^{int} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} [\left\{\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right) v_j + \frac{m\gamma(\beta+m\gamma-2\gamma)\tilde{\theta}\tilde{v}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}^2 - \left(\frac{m\gamma\tilde{v}}{\beta+m\gamma-\gamma}\right)\left\{\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right\}\left\{\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right) v_j + \frac{m\gamma(\beta+m\gamma-2\gamma)\tilde{\theta}\tilde{v}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}]$$

Now we have

$$(t_s(v_h - \alpha d)^2 + (1-t_s)v_l^2) = \tilde{v}^2 + Var(v\ ),$$

$$Var(v\ ) = t_s(1-t_s)(v_h - \alpha d - v_l)^2\ .$$

Also, we have

$$t_s(v_h - \alpha d - \tilde{v})^2 + (1-t_s)(v_l - \tilde{v})^2 = t_s(1-t_s)(v_h - \alpha d - v_l)^2 = Var(v)$$

$and\ \sum_{j=1}^{m} [v_j] = m\tilde{v}$. So we have

$$u_i^{int} = \frac{m}{2(\beta-\gamma)}\left[\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 Var(v) + \left\{\theta_i - \frac{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}^2 \tilde{v}^2 - \left(\frac{m\gamma\tilde{v}^2}{\beta+m\gamma-\gamma}\right)\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)\left\{\theta_i - \frac{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}\right]$$

$$u_i^{int} = \frac{m}{2(\beta-\gamma)}\left[\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 Var(v) + \left(\frac{\beta-\gamma}{\beta+m\gamma-\gamma}\right)\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 \tilde{v}^2\right]$$

$$u_i^{int} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 Var(v)\right]$$

Substituting the value of $Var(v)$, we have

$$u_i^{int} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 [t_s(v_h - \alpha d - \tilde{v})^2 + (1-t_s)(v_l - \tilde{v})^2]\right] \quad \text{..........} (36)$$

Q.E.D.

Proof of Lemma 4

In this case, the platform verifies the quality of the high type seller and hence segments the market.

So we have mt_s high type sellers of which $(1-\alpha)$ fraction is the actual high type with quality v_h and price p_{th}^{seg} (they also claim quality as v_h)

α fraction of high type sellers claimed their product quality as a high type but was found to have quality $(v_h - d)$ after verification, they have price p_{sh}^{seg} .

There are $m(1-t_s)$ low type sellers with quality v_l and price p_l^{seg}.

So we have $v_j = v_h$ or $(v_h - d)$ or v_l.

The market average quality is $\tilde{v} = t_s[(1-\alpha)v_h + \alpha(v_h - d)] + (1-t_s)v_l = t_s(v_h - \alpha d) + (1-t_s)v_l$.

Hence the market average quality does not change from the integrated case.

The market average price is $\tilde{p}^{seg} \equiv (1-\alpha)t_s p_{th}^{seg} + \alpha t_s p_{sh}^{seg} + (1-t_s)p_l^{seg}$. We will refer to this price as $\tilde{p}$.

The demand function like Lemma 3 is

$$q_{ij}^{seg} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$$

Substituting the various values we have

$q_{ij}^{seg} = \{\frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_{th}^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]$ if j is a trustworthy high − quality seller

$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_h - d - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_{sh}^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]$ if j is a suspect high − quality seller

$= \frac{1}{\beta-\gamma}\left[\theta_i\left(v_l - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_{l}^{seg} - \frac{m\gamma\tilde{p}^{seg}}{\beta+\gamma(m-1)}\right)\right]$ if j is a low − quality seller

……………………………………………………...……………….. (37)

Q.E.D.

Proof of Proposition 6

The demand function is

$$q_{ij}^{seg} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$$

where $v_j = v_h \; or \; (v_h - d)$ for high type sellers and $v_j = v_l$ for low type sellers and

$p_j = p_{th}^{seg}$ or p_{sh}^{seg} or p_l^{seg} and $\tilde{p} = \tilde{p}^{seg} \equiv (1-\alpha)t_s p_{th}^{seg} + \alpha t_s p_{sh}^{seg} + (1-t_s)p_l^{seg}$

$\tilde{v} = t_s[(1-\alpha)v_h + \alpha(v_h - d)] + (1-t_s)v_l = t_s(v_h - \alpha d) + (1-t_s)$

The platform incurs a cost C to verify the seller's type and charges a price C_v from each verified high-quality product seller for the verification service.

Seller's profit = $\pi_j^{seg} = (1-\delta)\sum_{i=1}^{n} p_j q_{ij}$

$= p_j(1-\delta)n[t_b q_{hj} + (1-t_b)q_{lj}]$ if j is a suspect high or low − quality seller

$= p_j(1-\delta)n\left[t_b q_{hj} + (1-t_b)q_{lj}\right] - C_v$ if j is a trustworthy high − quality seller

Substituting the values of q_{ij}^{seg} for $i = h$ and $i = l$ we have

$\pi_j^{seg} = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$ if j is a suspect high or low − quality seller

$\frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right] - C_v$ if j is a trustworthy high − quality seller

F.O.C. w.r.t. p_j

$$\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) - p_j\left[1 - \frac{\gamma}{\beta+\gamma(m-1)}\right] = 0$$

where we have used the fact that $\tilde{p} = \frac{\sum_{j=1}^{m} p_j}{m}$ and $\frac{\partial\tilde{p}}{\partial p_j} = \frac{1}{m}$

Summing up the F.O.C. of *m* sellers and canceling *m*, the common factor we have

$\tilde{\theta}\left(\tilde{v} - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(\tilde{p} - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) - \tilde{p}\left[1 - \frac{\gamma}{\beta+\gamma(m-1)}\right] = 0$ This gives

$\frac{\tilde{\theta}\tilde{v}(\beta-\gamma)}{\beta+\gamma(m-1)} = \tilde{p}\left[\frac{\beta-\gamma}{\beta+\gamma(m-1)} + \frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]$ This gives

$\tilde{p} = \frac{\tilde{\theta}\tilde{v}(\beta-\gamma)}{2\beta+m\gamma-3\gamma}$ which is similar to the case with no quality verification (*Proposition 5*)

Substituting this value in the F.O.C. condition we have

$p_j = \frac{1}{2-\frac{\gamma}{\beta+\gamma(m-1)}}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) + \frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{(\beta-\gamma)\tilde{\theta}\tilde{v}}{2\beta+m\gamma-3\gamma}\right)\right]$ Simplifying this gives

$p_j^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_j - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$ ………………………………...……… (38)

Seller's profit is

$\pi_j^{seg} = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$ or

$= \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right] - C_v$

Using F.O.C. and simplifying, we have

$\pi_j^{seg} = \frac{(1-\delta)np_j^2}{\beta-\gamma}[1 - \frac{\gamma}{\beta+\gamma(m-1)}]$ which has a similar form as the case with no quality verification

Or

$\pi_j^{seg} = \frac{(1-\delta)np_j^2}{\beta-\gamma}[1 - \frac{\gamma}{\beta+\gamma(m-1)}] - C_v$

So we have

$\pi_j^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)}\left[\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_j - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$ where $j = sh$ or l

$\pi_j^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)}\left[\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_j - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2 - C_v$ where $j = th$

…………………..……………………...………… (39)

Hence we have

The optimal price of a trustworthy high-quality seller is

$$p_{th}^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_h - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_{th}^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_h - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2 - C_v$

Now the profit of a high-quality seller (suspect or trustworthy) in the integrated case as depicted in *Proposition 5* is

$$\pi_h^{int} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$$

So we have $\pi_{th}^{seg} > \pi_h^{int}$ if

$$\frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_h - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2 - C_v >$$

$$\frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2 \text{ i.e. if}$$

$$C_v < \frac{\alpha d\tilde{\theta}^2(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(2v_h-\alpha d) - \frac{2(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{(2\beta+m\gamma-3\gamma)}\right] \quad \text{………..……. (40)}$$

Hence the trustworthy high-quality seller would like to have as low a verification charge as possible with a threshold beyond which verification is not profitable.

The optimal price of a suspect high-quality seller is

$$p_{sh}^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)(v_h-d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_{sh}^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))}\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$

Now the profit of a high-quality seller (suspect or trustworthy) in the integrated case as is

$$\pi_h^{int} = \frac{(1-\delta)n}{(\beta-\gamma)} \frac{(\beta+m\gamma-2\gamma)}{(\beta+\gamma(m-1))} \frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(v_h-\alpha d) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$$

Comparing the two results, we see that the suspect seller's profit decreases as $\alpha < 1$.

The optimal price of a low-quality seller is

$$p_l^{seg} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_l - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

Her profit is $\pi_l^{seg} = \frac{(1-\delta)n}{(\beta-\gamma)} \frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)} \frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_l - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2$

Comparing with *Proposition 5*, we see that the low-quality seller's profit remains the same.

Q.E.D.

Proof of Proposition 7

Since the profit of the platform is proportional to the aggregate profit of the seller side and since the platform incurs a cost C to verify the seller's type and charges a price C_v from each verified high-quality seller we have

$$\pi_p^{seg} = \sum_1^m \frac{\delta n}{(\beta-\gamma)}\left[\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)v_j - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]^2 + mt_s(1-\alpha)(C_v - C)$$

Let $X = (\beta+m\gamma-\gamma), Y = \frac{(\beta+m\gamma-2\gamma)m\gamma}{2\beta+m\gamma-3\gamma}$

We have mt_s high type sellers of which $(1-\alpha)$ fraction is the actual high type with quality v_h and price p_{th}^{seg} (they also claim quality as v_h). α fraction of high type sellers claimed quality high type but were found to have quality $(v_h - d)$. They have price p_{sh}^{seg}. There are $m(1-t_s)$ low type sellers with quality v_l and price p_l.

Now $v_j = v_h$ or $(v_h - \alpha d)$ for high type sellers and $v_j = v_l$ for low type sellers and

$$\tilde{v} = t_s[(1-\alpha)v_h + \alpha(v_h - d)] + (1-t_s)v_l = t_s(v_h - \alpha d) + (1-t_s)v_l$$

So $\tilde{v}$ is the same as in the case with no quality verification. So we have

$$\pi_p^{seg} = \frac{\delta mn}{(\beta-\gamma)}\left[\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}[t_s(1-\alpha)\{Xv_h - Y\tilde{v}\}^2 + t_s\alpha\{X(v_h-\alpha d) - Y\tilde{v}\}^2 + (1-t_s)\{Xv_l - Y\tilde{v}\}^2] + mt_s(1-\alpha)(C_v - C)$$

$$\pi_p^{seg} = K[t_s(1-\alpha)\{X(v_h-\alpha d) - Y\tilde{v} + X\alpha d\}^2 + t_s\alpha\{X(v_h-\alpha d) - Y\tilde{v} - X(1-\alpha)d\}^2 + (1-t_s)\{Xv_l - Y\tilde{v}\}^2] + mt_s(1-\alpha)(C_v - C) \quad \text{where } K = \frac{\delta mn}{(\beta-\gamma)}\left[\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}$$

$$\pi_p^{seg} = K[t_s\{X(v_h-\alpha d) - Y\tilde{v}\}^2 + (1-t_s)\{Xv_l - Y\tilde{v}\}^2 + t_s\alpha(1-\alpha)X^2d^2] + mt_s(1-\alpha)(C_v - C)$$

$$\pi_p^{seg} = K[X^2\{t_s(v_h-\alpha d)^2 + (1-t_s)v_l^2\} + Y^2\tilde{v}^2 - 2XY\tilde{v}^2 + t_s\alpha(1-\alpha)X^2d^2] + mt_s(1-\alpha)(C_v - C)$$

$$\pi_p^{seg} = K[X^2\{\tilde{v}^2 + Var(v)\} + Y^2\tilde{v}^2 - 2XY\tilde{v}^2 + t_s\alpha(1-\alpha)X^2d^2] + mt_s(1-\alpha)(C_v - C)$$

where we have used the fact that $(t_s(v_h-\alpha d)^2 + (1-t_s)v_l^2) = \tilde{v}^2 + Var(v)$,

$Var(v) = t_s(1-t_s)(v_h - \alpha d - v_l)^2$ and $\tilde{v} = t_s(v_h - \alpha d) + (1-t_s)v_l$.

Also, we have

$$t_s(v_h - \alpha d - \tilde{v})^2 + (1-t_s)(v_l - \tilde{v})^2 = t_s(1-t_s)(v_h - \alpha d - v_l)^2 = Var(v)$$

So we have

$$\pi_p^{seg} = K[X^2Var(v) + \tilde{v}^2\{X - Y\}^2 + t_s\alpha(1-\alpha)X^2d^2] + mt_s(1-\alpha)(C_v - C) \quad (41)$$

Putting the values and simplifying we have

$$\pi_p^{seg} = \frac{\delta mn}{(\beta-\gamma)}\left[\frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right]\frac{\tilde{\theta}^2}{(2\beta+2m\gamma-3\gamma)^2}\Big[(\beta+m\gamma-\gamma)^2Var(v) + \frac{(\beta-\gamma)^2(2\beta+2m\gamma-3\gamma)^2}{(2\beta+m\gamma-3\gamma)^2}\tilde{v}^2 +$$

$$t_s\alpha(1-\alpha)(\beta+m\gamma-\gamma)^2d^2\Big] + mt_s(1-\alpha)(C_v - C)$$

$$\pi_p^{seg} = \frac{\delta mn\tilde{\theta}^2(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}Var(v) + \frac{\delta mn\tilde{\theta}^2(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2}\tilde{v}^2 + \delta mn\tilde{\theta}^2t_s\alpha(1-$$

$$\alpha)\frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}d^2 + mt_s(1-\alpha)(C_v - C)$$

Substituting the value of $Var(v)$, we have

$$\pi_p^{seg} = \delta mn\tilde{\theta}^2\Big[\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2}\tilde{v}^2 + \frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}[t_s(v_h - \alpha d - \tilde{v})^2 +$$

$$(1-t_s)(v_l - \tilde{v})^2] + t_s\alpha(1-\alpha)\frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}d^2\Big] + mt_s(1-\alpha)(C_v - C) \quad (42)$$

Now, the platform will use the verification service as an additional tool to increase profit

Comparing with the result of Corollary 4 we have

$$\pi_p^{int} = \delta mn\tilde{\theta}^2\Big[\tilde{v}^2\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}[t_s(v_h - \alpha d - \tilde{v})^2 +$$

$$(1-t_s)(v_l - \tilde{v})^2]\Big]$$

$$\pi_p^{seg} - \pi_p^{int} = \left[\delta mn\tilde{\theta}^2t_s\alpha(1-\alpha)\frac{(\beta+m\gamma-2\gamma)(\beta+m\gamma-\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}d^2\right] + [mt_s(1-\alpha)(C_v - C)]$$

The expression in the first square bracket is positive. So segmentation increases platform profit even without the revenue from the verification service. Hence we have $\pi_p^{seg} - \pi_p^{int} > 0$, hence

$\pi_p^{seg} > \pi_p^{int}$ ……………………………………………………………………………… (43)

Q.E.D.

Proof of Corollary 5

Note that the platform revenue from the verification service is $mt_s(1-\alpha)(C_v - C)$. Using equation (40) and noting that if the platform wants to make a profit from the verification service, we have the below inequality condition -

$$C < C_v < \frac{\alpha d\tilde{\theta}^2(1-\delta)n}{(\beta-\gamma)}\frac{(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}\left[(\beta+m\gamma-\gamma)(2v_h-\alpha d)-\frac{2(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{(2\beta+m\gamma-3\gamma)}\right] \quad \text{Q.E.D.}$$

Proof of Proposition 8

$v_j = v_h \ or \ (v_h - \alpha d)$ for high type sellers and $v_j = v_l$ for low type sellers and

$\tilde{v} = t_s[(1-\alpha)v_h + \alpha(v_h - d)] + (1-t_s)v_l = t_s(v_h - \alpha d) + (1-t_s)v_l$

Note that the average quality $\tilde{v}$ is the same as the case with no quality verification.

Here $p_j = p_{th}$, p_{sh} or p_l.

The buyer's utility is

$u_i^{seg} = \sum_{j=1}^{m} \frac{q_{ij}}{2}(\theta_i v_j - p_j)$ Substituting the values for q_{ij} from $Lemma$ 4 we have

$$u_i^{seg} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} (\theta_i v_j - p_j)\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$$

$$u_i^{seg} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} \left(\theta_i v_j - p_j\right)\left[\left(\theta_i v_j - p_j\right) - \frac{m\gamma}{\beta+\gamma(m-1)}(\theta_i\tilde{v} - \tilde{p})\right]$$

Substituting the values for $\tilde{p}$ we have

$$u_i^{seg} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} \left(\theta_i v_j - p_j\right)\left[\left(\theta_i v_j - p_j\right) - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\left(\theta_i - \frac{\tilde{\theta}(\beta-\gamma)}{2\beta+m\gamma-3\gamma}\right)\right]$$

Substituting the value of p_j from Proposition 6 we have -

$$u_i^{seg} = \frac{1}{2(\beta-\gamma)}\sum_{j=1}^{m} [\left\{\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)v_j + \frac{m\gamma(\beta+m\gamma-2\gamma)\tilde{\theta}\tilde{v}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}^2 - \left(\frac{m\gamma\tilde{v}}{\beta+m\gamma-\gamma}\right)\left\{\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right\}\left\{\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)v_j + \frac{m\gamma(\beta+m\gamma-2\gamma)\tilde{\theta}\tilde{v}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}]$$

Taking the sum over m sellers we have

$$\sum_{j=1}^{m} [v_j^2] = m[t_s(1-\alpha)v_h^2 + t_s\alpha(v_h-d)^2 + (1-t_s)v_l^2] = m(\tilde{v}^2 + Var(v))$$

$$Var(v) = t_s(1-t_s)(v_h - \alpha d - v_l)^2 + t_s\alpha(1-\alpha)d^2$$

$$\tilde{v} = t_s(v_h - \alpha d) + (1-t_s)v_l$$

Since $t_s(v_h - \alpha d - \tilde{v})^2 + (1-t_s)(v_l - \tilde{v})^2 = t_s(1-t_s)(v_h - \alpha d - v_l)^2$ we have

$$Var(v) = t_s(v_h - \alpha d - \tilde{v})^2 + (1-t_s)(v_l - \tilde{v})^2 + t_s\alpha(1-\alpha)d^2$$

$$Also\ \sum_{j=1}^{m} [v_j] = m\tilde{v}$$

Hence we have

$$u_i^{seg} = \frac{m}{2(\beta-\gamma)}\left[\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 Var(v) + \left\{\theta_i - \frac{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}^2 \tilde{v}^2 - \left(\frac{m\gamma\tilde{v}^2}{\beta+m\gamma-\gamma}\right)\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)\left\{\theta_i - \frac{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)}\right\}\right]$$

$$u_i^{seg} = \frac{m}{2(\beta-\gamma)}\left[\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 Var(v) + \left(\frac{\beta-\gamma}{\beta+m\gamma-\gamma}\right)\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 \tilde{v}^2\right]$$

$$u_i^{seg} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 Var(v)\right]$$

Substituting the value of $Var(v)$, we have

$$u_i^{seg} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 [t_s(v_h - \alpha d - \tilde{v})^2 + \right.$$

$$\left.(1-t_s)(v_l - \tilde{v})^2] + \left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 t_s\alpha(1-\alpha)d^2\right] \quad \text{……………………..…………. (44)}$$

Now utility in the case of no quality verification is given in Corollary 4

$$u_i^{int} = \frac{m}{2}\left[\frac{\tilde{v}^2}{(\beta+\gamma(m-1))}\left(\theta_i - \frac{(\beta-\gamma)\tilde{\theta}}{(2\beta+m\gamma-3\gamma)}\right)^2 + \frac{1}{(\beta-\gamma)}\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 [t_s(v_h - \alpha d - \tilde{v})^2 + \right.$$

$$\left.(1-t_s)(v_l - \tilde{v})^2]\right]$$

Since the average quality $\tilde{v}$ is the same for both cases, we can easily compare the utility values.

Since the term $\left(\theta_i - \frac{(\beta+m\gamma-\gamma)\tilde{\theta}}{(2\beta+2m\gamma-3\gamma)}\right)^2 t_s\alpha(1-\alpha)d^2 > 0$ we can conclude that

$$u_i^{seg} > u_i^{int} \quad \text{……………………………………………………………………..….. (45)}$$

Q.E.D.

Proof of Proposition 9

In the independent market, the sellers decide their prices rather than the platform. All high type sellers have optimal price p_h^* and all low type sellers have optimal price p_l^*.

In this case, sellers report their quality truthfully through their price.

The demand function like the segmented case is

$$q_{ij}^{ind} = \frac{1}{\beta-\gamma}\left[\theta_i\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right]$$

where *ind* denotes independent

The seller's profit like the segmented case is

Seller's profit $=\pi_j^{ind} = (1-\delta)\sum_{i=1}^{n} p_j q_{ij} = p_j(1-\delta)n[t_b q_{hj} + (1-t_b)q_{lj}]$

Substituting the values of q_{ij}^{ind} for $i = h$ and $i = l$ we have

$$\pi_j^{ind} = \frac{(1-\delta)np_j}{\beta-\gamma}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right] \text{ where } [t_s p_h + (1-t_s)p_l] = \tilde{p}$$

F.O.C. w.r.t. p_j

$$\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(p_j - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) - p_j\left[1 - \frac{\gamma}{\beta+\gamma(m-1)}\right] = 0$$

where we have used the fact that $\tilde{p} = \frac{\sum_{j=1}^{m} p_j}{m}$ and $\frac{\partial\tilde{p}}{\partial p_j} = \frac{1}{m}$

Summing up the F.O.C. of *m* sellers and canceling *m*, the common factor we have

$$\tilde{\theta}\left(\tilde{v} - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) - \left(\tilde{p} - \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) - \tilde{p}\left[1 - \frac{\gamma}{\beta+\gamma(m-1)}\right] = 0 \text{ This gives}$$

$$\frac{\tilde{\theta}\tilde{v}(\beta-\gamma)}{\beta+\gamma(m-1)} = \tilde{p}\left[\frac{\beta-\gamma}{\beta+\gamma(m-1)} + \frac{\beta+m\gamma-2\gamma}{\beta+\gamma(m-1)}\right] \text{ This gives}$$

$$\tilde{p}^{ind} = \frac{\tilde{\theta}\tilde{v}(\beta-\gamma)}{2\beta+m\gamma-3\gamma}$$

Substituting this value in the F.O.C. condition we have

$p_j = \frac{1}{2-\frac{\gamma}{\beta+\gamma(m-1)}}\left[\tilde{\theta}\left(v_j - \frac{m\gamma\tilde{v}}{\beta+\gamma(m-1)}\right) + \frac{m\gamma}{\beta+\gamma(m-1)}\left(\frac{(\beta-\gamma)\tilde{\theta}\tilde{v}}{2\beta+m\gamma-3\gamma}\right)\right]$ Simplifying this gives

$$p_j^{*\,ind} = \frac{\tilde{\theta}}{2\beta+2m\gamma-3\gamma}\left[(\beta+m\gamma-\gamma)v_j - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}}{2\beta+m\gamma-3\gamma}\right]$$

where $j = h\ or\ l.$ So we have

$p_j^* = \frac{\tilde{\theta}(\beta+m\gamma-\gamma)v_j}{2\beta+2m\gamma-3\gamma} - \frac{m\gamma(\beta+m\gamma-2\gamma)\tilde{p}}{(2\beta+2m\gamma-3\gamma)(\beta-\gamma)} = \frac{1}{2\beta+2m\gamma-3\gamma}\left[\tilde{\theta}(\beta+m\gamma-\gamma)v_j - \frac{m\gamma(\beta+m\gamma-2\gamma)\tilde{p}}{(\beta-\gamma)}\right]$ So

$$p_j^{*\,ind} = \frac{1}{2\beta+2m\gamma-3\gamma}\left[\tilde{\theta}(\beta+m\gamma-\gamma)v_j - \frac{m\gamma(\beta+m\gamma-2\gamma)\tilde{p}}{(\beta-\gamma)}\right]$$

The platform profit is

$$\pi_p^{ind} = \delta\sum_{i=1}^{n}\sum_{j=1}^{m} q_{ij}p_j = \delta mn[t_s t_b q_{hh}p_h + t_s(1-t_b)q_{lh}p_h + (1-t_s)t_b q_{hl}p_l + (1-t_s)(1-t_b)q_{ll}p_l]$$

$$\pi_p^{ind} = \frac{\delta mn}{\beta-\gamma}[t_s p_h\left\{t_b\left(\theta_h v_h - \frac{m\gamma\theta_h\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) + (1-t_b)\left(\theta_l v_h - \frac{m\gamma\theta_l\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right\} + (1-t_s)p_l\left\{t_b\left(\theta_h v_l - \frac{m\gamma\theta_h\tilde{v}}{\beta+\gamma(m-1)} - p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right) + (1-t_b)\left(\theta_l v_l - \frac{m\gamma\tilde{v}\theta_l}{\beta+\gamma(m-1)} - p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right)\right\}]$$

$$\pi_p^{ind} = \frac{\delta mn}{\beta-\gamma}\left[t_s p_h\left\{\tilde{\theta}v_h - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)} - p_h + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right\} + (1-t_s)p_l\left\{\tilde{\theta}v_l - \frac{m\gamma\tilde{\theta}\tilde{v}}{\beta+\gamma(m-1)} - p_l + \frac{m\gamma\tilde{p}}{\beta+\gamma(m-1)}\right\}\right]$$

$$\pi_p^{ind} = \frac{\delta mn}{\beta-\gamma}\left[t_s p_h\tilde{\theta}v_h + (1-t_s)p_l\tilde{\theta}v_l - \frac{m\gamma\tilde{\theta}\tilde{v}\tilde{p}}{\beta+\gamma(m-1)} - \{t_s p_h^2 + (1-t_s)p_l^2\} + \frac{m\gamma\tilde{p}^2}{\beta+\gamma(m-1)}\right]$$

$$\pi_p^{ind} = \frac{\delta mn}{\beta-\gamma}\left[t_s p_h \tilde{\theta} v_h + (1-t_s)p_l \tilde{\theta} v_l - \{t_s p_h^2 + (1-t_s)p_l^2\} + \frac{m\gamma\tilde{p}(\tilde{p}-\tilde{\theta}\tilde{v})}{\beta+\gamma(m-1)}\right]$$

Substituting the values of p_h, p_l and $\tilde{p}$ and simplifying we have –

$$\pi_p^{ind} = \frac{\delta mn\tilde{\theta}^2}{\beta-\gamma}\left[\frac{\beta+m\gamma-\gamma}{2\beta+2m\gamma-3\gamma}(t_s v_h^2 + (1-t_s)v_l^2) - \frac{(\beta+m\gamma-2\gamma)m\gamma\tilde{v}^2}{(2\beta+2m\gamma-3\gamma)(2\beta+m\gamma-3\gamma)} - \right.$$

$$\left(\frac{\beta+m\gamma-\gamma}{2\beta+2m\gamma-3\gamma}\right)^2 (t_s v_h^2 + (1-t_s)v_l^2) - \frac{(\beta+m\gamma-2\gamma)^2 m^2\gamma^2\tilde{v}^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)^2} + \frac{2(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)m\gamma\tilde{v}^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)} -$$

$$\left.\frac{m\gamma(\beta-\gamma)(\beta+m\gamma-2\gamma)\tilde{v}^2}{(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)}\right]$$

Now $(t_s v_h^2 + (1-t_s)v_l^2) = \widetilde{v^2} = \tilde{v}^2 + var(v)$ So we have –

$$\pi_p^{ind} = \frac{\delta mn\tilde{\theta}^2}{\beta-\gamma}\left[\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}var(v) + \tilde{v}^2\left\{\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2} + \right.\right.$$

$$\left.\left.\frac{m\gamma^2(\beta+m\gamma-2\gamma)}{(2\beta+m\gamma-3\gamma)(2\beta+2m\gamma-3\gamma)^2} - \frac{m\gamma(\beta-\gamma)(\beta+m\gamma-2\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} - \frac{m^2\gamma^2(\beta+m\gamma-2\gamma)^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)^2}\right\}\right]$$

$$\pi_p^{ind} = \frac{\delta mn\tilde{\theta}^2}{\beta-\gamma}\left[\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}var(v) + \tilde{v}^2\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)^2(2\beta+2m\gamma-3\gamma)^2}{(2\beta+2m\gamma-3\gamma)^2(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)}\right]$$

$$\pi_p^{ind} = \delta mn\tilde{\theta}^2\tilde{v}^2\frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \delta mn\tilde{\theta}^2\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2}var(v) \quad \ldots\ldots\ldots\ldots\ldots (46)$$

But $\pi_p^{seg} = \frac{\delta mn}{\beta+\gamma(m-1)}\left(\frac{\tilde{\theta}^2\tilde{v}^2}{4}\right) + \frac{\delta mn\tilde{\theta}^2}{4(\beta-\gamma)}Var(v)$ from equation (26)

$$\pi_p^{seg} - \pi_p^{ind} =$$

$$\frac{\delta mn\tilde{\theta}^2\tilde{v}^2}{(\beta+m\gamma-\gamma)}\left[\frac{1}{4} - \frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(2\beta+m\gamma-3\gamma)^2}\right] + \frac{\delta mn\tilde{\theta}^2}{(\beta-\gamma)}var(v)\left[\frac{1}{4} - \frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(2\beta+2m\gamma-3\gamma)^2}\right]$$

$\pi_p^{seg} - \pi_p^{ind} = \frac{\delta mn\tilde{\theta}^2\tilde{v}^2}{(\beta+m\gamma-\gamma)}\left[\frac{\gamma^2(m-1)^2}{4(2\beta+m\gamma-3\gamma)^2}\right] + \frac{\delta mn\tilde{\theta}^2}{(\beta-\gamma)}var(v)\left[\frac{\gamma^2}{4(2\beta+2m\gamma-3\gamma)^2}\right] > 0$ as

both the first and second terms are positive. Hence we have –

$\pi_p^{seg} > \pi_p^{ind}$ i.e.

$\pi_{platform}^{segmented} > \pi_{platform}^{independent}$ ………………………………………………………………….. (47)

From equation (7) and (46) we have $\pi_{platform}^{independent} > \pi_{platform}^{integrated}$ if

$\delta mn\tilde{\theta}^2\tilde{v}^2 \frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(\beta+m\gamma-\gamma)(2\beta+m\gamma-3\gamma)^2} + \delta mn\tilde{\theta}^2 \frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2} var(v) > \frac{\delta mn}{[\beta+\gamma(m-1)]}(\frac{\tilde{\theta}^2\tilde{v}^2}{4})$ i.e. if

$\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2} var(v) > \frac{\tilde{v}^2}{(\beta+m\gamma-\gamma)}[\frac{1}{4} - \frac{(\beta+m\gamma-2\gamma)(\beta-\gamma)}{(2\beta+m\gamma-3\gamma)^2}]$ i.e. if

$\frac{(\beta+m\gamma-\gamma)(\beta+m\gamma-2\gamma)}{(\beta-\gamma)(2\beta+2m\gamma-3\gamma)^2} var(v) > \frac{\tilde{v}^2}{(\beta+m\gamma-\gamma)}\left[\frac{\gamma^2(m-1)^2}{4(2\beta+m\gamma-3\gamma)^2}\right]$

But $t_s(1-t_s)(v_h-v_l)^2 = var(v)$ and $\tilde{v} = t_s v_h + (1-t_s)v_l,$ so we have -

$t_s(1-t_s)\left(\frac{v_h}{v_l}-1\right)^2 > \left(t_s\left(\frac{v_h}{v_l}\right) + (1-t_s)\right)^2 \left[\frac{\gamma^2(m-1)^2(2\beta+2m\gamma-3\gamma)^2}{4(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)^2}\right]\left[\frac{(\beta-\gamma)}{(\beta+m\gamma-2\gamma)}\right]$

Let $K = \left[\frac{\gamma^2(m-1)^2(2\beta+2m\gamma-3\gamma)^2}{4(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)^2}\right],\ L = \left[\frac{(\beta-\gamma)}{(\beta+m\gamma-2\gamma)}\right],\ r = \frac{v_h}{v_l}$

It is easy to see that $K > 0$ and $L > 0$ ($as\ K$ consists of all squared terms and $\beta > \gamma$)

Now $1-\sqrt{K} = 1-\left[\frac{\gamma\,(m-1)\,(2\beta+2m\gamma-3\gamma)}{2(2\beta+m\gamma-3\gamma)\,(\beta+m\gamma-\gamma)}\right] = \frac{4(\beta-\gamma)^2+4m\gamma(\beta-\gamma)+\gamma^2(m-1)}{2(2\beta+m\gamma-3\gamma)\,(\beta+m\gamma-\gamma)} > 0$. So $K < 1$ and

$1-L = 1-\left[\frac{(\beta-\gamma)}{(\beta+m\gamma-2\gamma)}\right] = \frac{\gamma(m-1)}{(\beta+m\gamma-2\gamma)} > 0$ as m is large. Hence $L < 1$. So we have

$0 < K < 1$ and $0 < L < 1$ which means $0 < KL < 1$. So we have -

$t_s(1-t_s)(r-1)^2 > \left(t_s r + (1-t_s)\right)^2 KL$ This gives after simplification –

$t_s[(1-t_s)-t_s KL]r^2 + r[-2t_s(1-t_s)(1+KL)] + (1-t_s)[t_s-(1-t_s)KL] > 0$

Let $X = t_s[(1-t_s) - t_sKL]$, $Y = [-2t_s(1-t_s)(1+KL)]$, $Z = (1-t_s)[t_s - (1-t_s)KL]$

Note that $Y < 0$ as both K and L are greater than zero.

The inequality can be written as $Xr^2 + Yr + Z > 0$ .... (48)

So we have $\pi_{platform}^{independent} > \pi_{platform}^{integrated}$ if $Xr^2 + Yr + Z > 0$

The equation $Xr^2 + Yr + Z = 0$ has 2 roots $(\frac{-Y-\sqrt{Y^2-4XZ}}{2X}, \frac{-Y+\sqrt{Y^2-4XZ}}{2X})$ for the value of r.

So $Xr^2 + Yr + Z > 0$ implies $X\left[r - \left(\frac{-Y-\sqrt{Y^2-4XZ}}{2X}\right)\right]\left[r - \left(\frac{-Y+\sqrt{Y^2-4XZ}}{2X}\right)\right] > 0$

If $X > 0$, then $r > \frac{-Y+\sqrt{Y^2-4XZ}}{2X}$ or $r < \frac{-Y-\sqrt{Y^2-4XZ}}{2X}$ for the above inequality to hold.

If $X < 0$, then $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} < r < \frac{-Y-\sqrt{Y^2-4XZ}}{2X}$ for the above inequality to hold.

where

$X = t_s[(1-t_s) - t_sKL]$, $\quad Y = [-2t_s(1-t_s)(1+KL)]$, $\quad Z = (1-t_s)[t_s - (1-t_s)KL]$

Now the discriminant of the equation $Xr^2 + Yr + Z = 0$ is $Y^2 - 4XZ = 4t_s(1-t_s)KL > 0$ (as $K > 0$ and $L > 0$) and X, Y and Z are real numbers. So the roots of the equation $Xr^2 + Yr + Z = 0$ are real.

Case 1 -

If $X > 0$, i.e. $t_s[(1-t_s) - t_sKL] > 0$, i.e. $t_s < \frac{1}{KL+1}$

Now $\left(-Y - \sqrt{Y^2 - 4XZ}\right) = 2t_s(1-t_s)(1+KL) - \sqrt{4t_s(1-t_s)KL} > 0$ if

$t_s(1-t_s) > \frac{KL}{(1+KL)^2}$.

Let $\frac{1}{KL+1} - t_s = (1-\mathrm{t_s}) - \frac{KL}{KL+1} = \eta > 0$ (since $t_s < \frac{1}{KL+1}$)

Then $t_s(1-t_s) = \left(\frac{1}{KL+1} - \eta\right)\left(\frac{KL}{KL+1} + \eta\right) = \frac{KL}{(1+KL)^2} + \eta\left(\frac{1-KL}{KL+1}\right) - \eta^2$. Now

$t_s(1-t_s) > \frac{KL}{(1+KL)^2}$ simplifies to $\eta\left(\frac{1-KL}{KL+1}\right) - \eta^2 > 0,\ \text{i.e. } \eta < \left(\frac{1-KL}{KL+1}\right),\ \text{i.e. } t_s >$ $\frac{KL}{KL+1}$ which is not true since $t_s < \frac{1}{KL+1}$. So $\left(-Y - \sqrt{Y^2 - 4XZ}\right) < 0$.

But since $X > 0,\ \frac{-Y-\sqrt{Y^2-4XZ}}{2X} < 0$.

But $r = \frac{v_h}{v_l} > 1$ because $v_h > v_l$. So $r < \frac{-Y-\sqrt{Y^2-4XZ}}{2X}$ is not possible.

Now $\left(-Y + \sqrt{Y^2 - 4XZ}\right) = 2t_s(1-t_s)(1+KL) + \sqrt{4t_s(1-t_s)KL} > 0$

(as both terms are positive)

Hence $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} > 0$ as $X > 0$. So the condition we are left with is if $X > 0, r > \frac{-Y+\sqrt{Y^2-4XZ}}{2X}$, then $Xr^2 + Yr + Z > 0$ and hence $\pi_{platform}^{independent} > \pi_{platform}^{integrated}$.

Note that if $X > 0$, the graph of $Xr^2 + Yr + Z = 0$ is a parabola that opens up.

Now $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} = \frac{\left[(1-t_s)(1+KL) + \sqrt{KL\left(\frac{1-t_s}{t_s}\right)}\right]}{[(1-t_s) - t_s KL]}$ and $K = \left[\frac{\gamma^2(m-1)^2(2\beta+2m\gamma-3\gamma)^2}{4(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)^2}\right],\ L = \left[\frac{(\beta-\gamma)}{(\beta+m\gamma-2\gamma)}\right]$

Also $0 < K < 1$ and $0 < L < 1, 0 < KL < 1$ (as shown earlier)

Then $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} > 1$ if $X(X+Y+Z) < 0,\ \text{i.e. if } X+Y+Z < 0$ (as $X > 0$).

Substituting the values of X, Y, Z and simplifying we have -

$X + Y + Z < 0$ implies $- KL < 0$ which is true (as $0 < KL < 1$). Hence $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} > 1$.

Now since $r = \frac{v_h}{v_l} > 1$ because $v_h > v_l$, so $r > \frac{-Y+\sqrt{Y^2-4XZ}}{2X}$ is a possible condition and hence the condition described above is also possible. So we have –

If $X > 0, r > \frac{-Y+\sqrt{Y^2-4XZ}}{2X}$, then $Xr^2 + Yr + Z > 0$ and hence $\pi_{platform}^{independent} > \pi_{platform}^{integrated}$.

Case 2 -

If $X < 0$, then $\mathrm{t_s}[(1 - t_s) - t_s KL] < 0$, i.e. $t_s > \frac{1}{KL+1}$

If $X < 0$, then $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} < r < \frac{-Y-\sqrt{Y^2-4XZ}}{2X}$ for the inequality $Xr^2 + Yr + Z > 0$ to hold.

$-Y + \sqrt{Y^2 - 4XZ} > 0$ as shown earlier.

Now $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} < 0$ as $X < 0$. But we know $r > 0$ as it is the ratio $\frac{v_h}{v_l}$.

So $\frac{-Y+\sqrt{Y^2-4XZ}}{2X} < r$ does not give any new condition for r.

Now $\left(-Y - \sqrt{Y^2 - 4XZ}\right) = 2t_s(1 - t_s)(1 + KL) - \sqrt{4t_s(1 - t_s)KL} < 0$ if

$t_s(1 - t_s) < \frac{KL}{(1+KL)^2}$.

Let $\frac{1}{KL+1} - t_s = (1 - \mathrm{t_s}) - \frac{KL}{KL+1} = \mu < 0$ (since $t_s > \frac{1}{KL+1}$)

Then $t_s(1 - t_s) = \left(\frac{1}{KL+1} - \mu\right)\left(\frac{KL}{KL+1} + \mu\right) = \frac{KL}{(1+KL)^2} + \mu\left(\frac{1-KL}{KL+1}\right) - \mu^2$. Now

$t_s(1-t_s) < \frac{KL}{(1+KL)^2}$ simplifies to $\mu\left(\frac{1-KL}{KL+1}\right) - \mu^2 < 0$, i.e. if $\mu < \left(\frac{1-KL}{KL+1}\right)$, i.e. if $t_s > \frac{KL}{KL+1}$.

Now $X < 0$ if $t_s > \frac{1}{KL+1}$, so if $t_s > \frac{KL}{KL+1}$, X is still less than zero. Now if

$-Y - \sqrt{Y^2 - 4XZ} < 0$, then $\frac{-Y-\sqrt{Y^2-4XZ}}{2X} > 0$ as $X < 0$. So we have -

If $X < 0, 0 < r < \frac{-Y-\sqrt{Y^2-4XZ}}{2X}$, then $Xr^2 + Yr + Z > 0$ and hence $\pi_{platform}^{independent} > \pi_{platform}^{integrated}$.

Note that if $X < 0$, the graph of $Xr^2 + Yr + Z = 0$ is a parabola that opens downward.

Now $\frac{-Y-\sqrt{Y^2-4XZ}}{2X} = \frac{2t_s(1-t_s)(1+KL)-\sqrt{4t_s(1-t_s)KL}}{2t_s[(1-t_s)-t_sKL]}$ and

$K = \left[\frac{\gamma^2(m-1)^2(2\beta+2m\gamma-3\gamma)^2}{4(2\beta+m\gamma-3\gamma)^2(\beta+m\gamma-\gamma)^2}\right]$, $L = \left[\frac{(\beta-\gamma)}{(\beta+m\gamma-2\gamma)}\right]$

Also $0 < K < 1$ and $0 < L < 1, 0 < KL < 1$ (as shown earlier).

Then $\frac{-Y-\sqrt{Y^2-4XZ}}{2X} < 1$ if $(2X+Y) + \sqrt{Y^2-4XZ} < 0$ (given $X < 0$), i.e. if $t_s > \frac{1}{1+KL}$ (after substituting the values of X, Y, and Z and simplifying). But this condition is true as $X < 0$. Hence we have $\frac{-Y-\sqrt{Y^2-4XZ}}{2X} < 1$. But we know that $r = \frac{v_h}{v_l} > 1$ because $v_h > v_l$, hence $r < \frac{-Y-\sqrt{Y^2-4XZ}}{2X}$ is not possible. Hence the statement that If $X < 0, 0 < r < \frac{-Y-\sqrt{Y^2-4XZ}}{2X}$, then $Xr^2 + Yr + Z > 0$ and hence $\pi_{platform}^{independent} > \pi_{platform}^{integrated}$ does not hold.

Hence we are only left with the conclusion that -

$X > 0, r > \frac{-Y+\sqrt{Y^2-4XZ}}{2X}$, then $Xr^2 + Yr + Z > 0$ and hence $\pi_{platform}^{independent} > \pi_{platform}^{integrated}$.

Hence we have –

$$\pi_{platform}^{independent} > \pi_{platform}^{integrated} \quad \text{……………………………………………………………………. (49)}$$

Combining equations (48) and (49) we have

$$\pi_{platform}^{segmented} > \pi_{platform}^{independent} > \pi_{platform}^{integrated} \quad \text{……………………………………………………. (50)}$$

Q.E.D.

Appendix B

Proof of Proposition 1

The Bellman Equation is –

$$V(N_t^B, N_t^S) = Max_{A_t^B, A_t^S}\{P^B N_t^B + P^S N_t^S - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2 - c\left(\overline{N^S} - N_t^S\right)(A_t^S)^2 + \beta V(N_{t+1}^B, N_{t+1}^S) + \lambda_t\left[(1-\delta^B)N_t^B + \left(\overline{N^B} - N_t^B\right)(A_t^B + \eta^B N_t^S) - N_{t+1}^B\right] + \mu_t\left[(1-\delta^S)N_t^S + \left(\overline{N^S} - N_t^S\right)(A_t^S + \eta^S N_t^B) - N_{t+1}^S\right] + \rho_t A_t^B + \sigma_t A_t^S\}$$

where $\lambda, \mu, \rho, \sigma$ are Lagrange multipliers and V is the value function.

F.O.C. w.r.t. A_t^B –

$-2c\left(\overline{N^B} - N_t^B\right)(A_t^B) + \lambda_t\left(\overline{N^B} - N_t^B\right) + \rho_t = 0$ which gives $\lambda_t = 2c\,A_t^B - \frac{\rho_t}{(\overline{N^B} - N_t^B)}$

F.O.C. w.r.t. A_t^S –

$-2c\left(\overline{N^S} - N_t^S\right)(A_t^S) + \mu_t\left(\overline{N^S} - N_t^S\right) + \sigma_t = 0$ which gives $\mu_t = 2c\,A_t^S - \frac{\sigma_t}{(\overline{N^S} - N_t^S)}$

F.O.C. w.r.t. N_{t+1}^B – $\beta V'(N_{t+1}^B, N_{t+1}^S) = \lambda_t$

F.O.C. w.r.t. N_{t+1}^S – $\beta V'(N_{t+1}^B, N_{t+1}^S) = \mu_t$

F.O.C. w.r.t. λ_t – $(1-\delta^B)N_t^B + \left(\overline{N^B} - N_t^B\right)(A_t^B + \eta^B N_t^S) = N_{t+1}^B$

F.O.C. w.r.t. μ_t – $(1-\delta^S)N_t^S + \left(\overline{N^S} - N_t^S\right)(A_t^S + \eta^S N_t^B) = N_{t+1}^S$

Envelope condition w.r.t. N_t^B and N_t^S are –

$$\frac{\partial V}{\partial N_t^B} = V'_{N_t^B}(N_t^B, N_t^S) = P^B + c\,(A_t^B)^2 + \lambda_t[(1-\delta^B) - (A_t^B + \eta^B N_t^S)] + \mu_t \eta^S\left(\overline{N^S} - N_t^S\right)$$

$$\frac{\partial V}{\partial N_t^S} = V'_{N_t^S}(N_t^B, N_t^S) = P^S + c\,(A_t^S)^2 + \lambda_t \eta^B\left(\overline{N^B} - N_t^B\right) + \mu_t[(1-\delta^S) - (A_t^S + \eta^S N_t^B)]$$

Update one period in the future envelope condition w.r.t. N_t^B–

$$V'(N_{t+1}^B, N_{t+1}^S) = P^B + c\,(A_{t+1}^B)^2 \; + \lambda_{t+1}[(1-\delta^B) - (A_{t+1}^B + \eta^B N_{t+1}^S)] + \mu_{t+1}\eta^S\left(\overline{N^S} - N_{t+1}^S\right)$$

Update one period in the future envelope condition w.r.t. N_t^S–

$$V'(N_{t+1}^B, N_{t+1}^S) = P^S + c\,(A_{t+1}^S)^2 + \lambda_{t+1}\eta^B\left(\overline{N^B} - N_{t+1}^B\right) + \mu_{t+1}[(1-\delta^S) - (A_{t+1}^S + \eta^S N_{t+1}^B)]$$

Equating the values of $V'(N_{t+1}^B, N_{t+1}^S)$, the Euler equation for buyers is then –

$$P^B + c\,(A_{t+1}^B)^2 \; + \lambda_{t+1}[(1-\delta^B) - (A_{t+1}^B + \eta^B N_{t+1}^S)] + \mu_{t+1}\eta^S\left(\overline{N^S} - N_{t+1}^S\right) = \frac{\lambda_t}{\beta}$$

Since $\lambda_t = 2c\,A_t^B - \frac{\rho_t}{(\overline{N^B} - N_t^B)}$ and $\mu_t = 2c\,A_t^S - \frac{\sigma_t}{(\overline{N^S} - N_t^S)}$, the Euler equation for buyers is –

$$P^B + c\,(A_{t+1}^B)^2 + \left[2c\,(A_{t+1}^B) - \frac{\rho_{t+1}}{(\overline{N^B} - N_{t+1}^B)}\right][(1-\delta^B) - (A_{t+1}^B + \eta^B N_{t+1}^S)] + \left[2c\,(A_{t+1}^S) - \frac{\sigma_{t+1}}{(\overline{N^S} - N_{t+1}^S)}\right]\eta^S\left(\overline{N^S} - N_{t+1}^S\right) = \frac{2c\,(A_t^B)}{\beta} - \frac{\rho_t}{\beta(\overline{N^B} - N_t^B)}$$

This gives the Euler equation for buyers as –

$$P^B - c\,(A_{t+1}^B)^2 + 2c\,(A_{t+1}^B)[(1-\delta^B) - \eta^B N_{t+1}^S] + 2c\,(A_{t+1}^S)\eta^S\left(\overline{N^S} - N_{t+1}^S\right) + \left[\frac{\rho_t}{\beta(\overline{N^B} - N_t^B)} - \frac{\rho_{t+1}}{(\overline{N^B} - N_{t+1}^B)}\{(1-\delta^B) - (A_{t+1}^B + \eta^B N_{t+1}^S)\} - \frac{\sigma_{t+1}}{(\overline{N^S} - N_{t+1}^S)}\left\{\eta^S\left(\overline{N^S} - N_{t+1}^S\right)\right\}\right] = \frac{2c\,(A_t^B)}{\beta} \quad \text{i.e.}$$

$$\left(\frac{\beta}{2}\right)\left(\frac{P^B}{c}\right) - \left(\frac{\beta}{2}\right)(A_{t+1}^B)^2 + \beta(A_{t+1}^B)[(1-\delta^B) - \eta^B N_{t+1}^S] + \beta\eta^S(A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right) + \left(\frac{\beta}{2c}\right)\left[\frac{\rho_t}{\beta(\overline{N^B} - N_t^B)} - \frac{\rho_{t+1}}{(\overline{N^B} - N_{t+1}^B)}\{(1-\delta^B) - (A_{t+1}^B + \eta^B N_{t+1}^S)\} - \eta^S\sigma_{t+1}\right] = A_t^B$$

$(A_{t+1}^B)^2 - 2(A_{t+1}^B)[(1-\delta^B) - \eta^B N_{t+1}^S] = \left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A_t^B + 2\eta^S (A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right) +$

$\left(\frac{1}{c}\right)\left[\frac{\rho_t}{\beta\left(\overline{N^B} - N_t^B\right)} - \frac{\rho_{t+1}}{\left(\overline{N^B} - N_{t+1}^B\right)}\{(1-\delta^B) - (A_{t+1}^B + \eta^B N_{t+1}^S)\} - \eta^S \sigma_{t+1}\right]$

When $A_t^B > 0,\ \rho_t = 0$. When $A_t^S > 0,\ \sigma_t = 0$. When $A_t^B,\ A_t^S > 0$, we have

$$\left(\frac{\beta}{2}\right)\left(\frac{P^B}{c}\right) \quad - \left(\frac{\beta}{2}\right)(A_{t+1}^B)^2 + \beta(A_{t+1}^B)[(1-\delta^B) - \eta^B N_{t+1}^S] + \beta\eta^S (A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right) = A_t^B$$

This equation can also be written as –

$(A_{t+1}^B)^2 - 2(A_{t+1}^B)[(1-\delta^B) - \eta^B N_{t+1}^S] = \left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A_t^B + 2\eta^S (A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right)$ i.e.

$(A_{t+1}^B)^2 - 2(A_{t+1}^B)[(1-\delta^B) - \eta^B N_{t+1}^S] + [(1-\delta^B) - \eta^B N_{t+1}^S]^2 = \left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A_t^B +$

$2\eta^S (A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right) + [(1-\delta^B) - \eta^B N_{t+1}^S]^2$ i.e.

$A_{t+1}^B = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A_t^B + 2\eta^S (A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right) + [(1-\delta^B) - \eta^B N_{t+1}^S]^2\right]^{\frac{1}{2}} + [(1-\delta^B) -$

$\eta^B N_{t+1}^S]$ which is the Euler equation for buyers

Equating the values of $V'(N_{t+1}^B, N_{t+1}^S)$, the Euler equation for sellers is then –

$$P^S + c\,(A_{t+1}^S)^2 + \lambda_{t+1}\eta^B\left(\overline{N^B} - N_{t+1}^B\right) + \mu_{t+1}[(1-\delta^S) - (A_{t+1}^S + \eta^S N_{t+1}^B)] = \frac{\mu_t}{\beta}$$

Since $\lambda_t = 2c\,(A_t^B) - \frac{\rho_t}{\left(\overline{N^B} - N_t^B\right)}$ and $\mu_t = 2c\,(A_t^S) - \frac{\sigma_t}{\left(\overline{N^S} - N_t^S\right)}$, the Euler equation for sellers is –

$P^S + c\,(A_{t+1}^S)^2 + \left[2c\,(A_{t+1}^B) - \frac{\rho_{t+1}}{\left(\overline{N^B} - N_{t+1}^B\right)}\right]\eta^B\left(\overline{N^B} - N_{t+1}^B\right) + \left[2c\,(A_{t+1}^S) - \frac{\sigma_{t+1}}{\left(\overline{N^S} - N_{t+1}^S\right)}\right][(1-$

$\delta^S) - (A_{t+1}^S + \eta^S N_{t+1}^B)] = \frac{2c\,(A_t^S)}{\beta} - \frac{\sigma_t}{\beta\left(\overline{N^S} - N_t^S\right)}$

This gives the Euler equation for seller as –

$$P^S - c\,(A^S_{t+1})^2 + 2c\,(A^B_{t+1})\eta^B\big(\overline{N^B} - N^B_{t+1}\big) + 2c\,(A^S_{t+1})[(1-\delta^S) - \eta^S N^B_{t+1}] + \left[\frac{\sigma_t}{\beta(\overline{N^S} - N^S_t)} - \frac{\sigma_{t+1}}{(\overline{N^S} - N^S_{t+1})}\{(1-\delta^S) - (A^S_{t+1} + \eta^S N^B_{t+1})\} - \eta^B \rho_{t+1}\right] = \frac{2c\,(A^S_t)}{\beta} \quad \text{i.e.}$$

$$\left(\frac{\beta}{2}\right)\left(\frac{P^S}{c}\right) - \left(\frac{\beta}{2}\right)(A^S_{t+1})^2 + \beta(A^S_{t+1})[(1-\delta^S) - \eta^S N^B_{t+1}] + \beta\eta^B(A^B_{t+1})\big(\overline{N^B} - N^B_{t+1}\big) + \left(\frac{\beta}{2c}\right)\left[\frac{\sigma_t}{\beta(\overline{N^S} - N^S_t)} - \frac{\sigma_{t+1}}{(\overline{N^S} - N^S_{t+1})}\{(1-\delta^S) - (A^S_{t+1} + \eta^S N^B_{t+1})\} - \eta^B \rho_{t+1}\right] = A^S_t \qquad \text{i.e.}$$

$$(A^S_{t+1})^2 - 2(A^S_{t+1})[(1-\delta^S) - \eta^S N^B_{t+1}] = \left(\frac{P^S}{c}\right) - \left(\frac{2}{\beta}\right)A^S_t + 2\eta^B(A^B_{t+1})\big(\overline{N^B} - N^B_{t+1}\big) + \left(\frac{1}{c}\right)\left[\frac{\sigma_t}{\beta(\overline{N^S} - N^S_t)} - \frac{\sigma_{t+1}}{(\overline{N^S} - N^S_{t+1})}\{(1-\delta^S) - (A^S_{t+1} + \eta^S N^B_{t+1})\} - \eta^B \rho_{t+1}\right]$$

When $A^B_t > 0,\ \rho_t = 0$. When $A^S_t > 0,\ \sigma_t = 0$.

When $A^B_t,\ A^S_t > 0$, we have

$$\left(\frac{\beta}{2}\right)\left(\frac{P^S}{c}\right) - \left(\frac{\beta}{2}\right)(A^S_{t+1})^2 + \beta(A^S_{t+1})[(1-\delta^S) - \eta^S N^B_{t+1}] + \beta\eta^B(A^B_{t+1})(1 - N^B_{t+1}) = A^S_t$$

This Equation can also be written as –

$$(A^S_{t+1})^2 - 2(A^S_{t+1})[(1-\delta^S) - \eta^S N^B_{t+1}] + [(1-\delta^S) - \eta^S N^B_{t+1}]^2 = \left(\frac{P^S}{c}\right) - \left(\frac{2}{\beta}\right)A^S_t + 2\eta^B(A^B_{t+1})(1 - N^B_{t+1}) + [(1-\delta^S) - \eta^S N^B_{t+1}]^2 \qquad \text{i.e.}$$

$$A^S_{t+1} = \left[\left(\frac{P^S}{c}\right) - \left(\frac{2}{\beta}\right)A^S_t + 2\eta^B(A^B_{t+1})(1 - N^B_{t+1}) + [(1-\delta^S) - \eta^S N^B_{t+1}]^2\right]^{\frac{1}{2}} + [(1-\delta^S) - \eta^S N^B_{t+1}] \qquad \text{which is the Euler equation for sellers}$$

Q.E.D.

Proof of Proposition 2

$A_{t+1}^B = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A_t^B + 2\eta^S (A_{t+1}^S)\left(\overline{N^S} - N_{t+1}^S\right) + [(1-\delta^B) - \eta^B N_{t+1}^S]^2\right]^{\frac{1}{2}} + [(1-\delta^B) - \eta^B N_{t+1}^S]$

At steady state $A_{t+1}^B = A_t^B = A^B$, $N_{t+1}^B = N_t^B = N^B$, $A_{t+1}^S = A_t^S = A^S$, $N_{t+1}^S = N_t^S = N^S$

$A^B = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A^B + 2\eta^S (A^S)\left(\overline{N^S} - N^S\right) + [(1-\delta^B) - \eta^B N^S]^2\right]^{\frac{1}{2}} + [(1-\delta^B) - \eta^B N^S]$ i.e.

$(A^B)^2 - 2A^B[(1-\delta^B) - \eta^B N^S] = \left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A^B + 2\eta^S (A^S)\left(\overline{N^S} - N^S\right)$ i.e.

$(A^B)^2 - 2A^B\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right] = \left(\frac{P^B}{c}\right) + 2\eta^S (A^S)\left(\overline{N^S} - N^S\right)$ i.e.

$A^B = \left[\left(\frac{P^B}{c}\right) + 2\eta^S (A^S)\left(\overline{N^S} - N^S\right) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]$

Similarly for sellers we have

$A^S = \left[\left(\frac{P^S}{c}\right) + 2\eta^B (A^B)\left(\overline{N^B} - N^B\right) + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]$

The steady state values of buyers and sellers are derived just like in proposition 2. Hence we have $N^B = \frac{\overline{N^B}(A^B + \eta^B N^S)}{[A^B + \eta^B N^S + \delta^B]}$, $N^S = \frac{\overline{N^S}(A^S + \eta^S N^B)}{[A^S + \eta^S N^B + \delta^S]}$.

$\pi_{ss} = P^B N^B + P^S N^S - c\left(\overline{N^B} - N^B\right)(A^B)^2 - c\left(\overline{N^S} - N^S\right)(A^S)^2$ i.e.

$$\pi_{ss} = \frac{P^B\overline{N^B}(A^B+\eta^B N^S)}{[A^B+\eta^B N^S+\delta^B]} + \frac{P^S\overline{N^S}(A^S+\eta^S N^B)}{[A^S+\eta^S N^B+\delta^S]} - c\left[\overline{N^B} - \frac{\overline{N^B}(A^B+\eta^B N^S)}{[A^B+\eta^B N^S+\delta^B]}\right](A^B)^2 - c\left[\overline{N^S} - \frac{\overline{N^S}(A^S+\eta^S N^B)}{[A^S+\eta^S N^B+\delta^S]}\right](A^S)^2$$

$$\pi_{ss} = \frac{P^B\overline{N^B}(A^B+\eta^B N^S)}{[A^B+\eta^B N^S+\delta^B]} + \frac{P^S\overline{N^S}(A^S+\eta^S N^B)}{[A^S+\eta^S N^B+\delta^S]} - c\left[\frac{\overline{N^B}\delta^B}{[A^B+\eta^B N^S+\delta^B]}\right](A^B)^2 - c\left[\frac{\overline{N^S}\delta^S}{[A^S+\eta^S N^B+\delta^S]}\right](A^S)^2 \text{ i.e.}$$

$$\pi_{ss} = \frac{\overline{N^B}\left[P^B(A^B+\eta^B N^S) - c\delta^B (A^B)^2\right]}{[A^B+\eta^B N^S+\delta^B]} + \frac{\overline{N^S}\left[P^S(A^S+\eta^S N^B) - c\delta^S (A^S)^2\right]}{[A^S+\eta^S N^B+\delta^S]}$$

Q.E.D.

Proof of Corollary 1

$A^B = f_1(P^B, c, \eta^S, \overline{N^S}, \eta^B, \delta^B, \beta, A^S, N^S)$, $A^S = f_2(P^S, c, \eta^B, \overline{N^B}, \eta^S, \delta^S, \beta, A^B, N^B)$, $N^B = f_3(\overline{N^B}, \eta^B, \delta^B, A^B, N^S)$, $N^S = f_4(\overline{N^S}, \eta^S, \delta^S, A^S, N^B)$ where $f_1. f_2, f_3, f_4$ are functions.

Whether $\frac{dA^B}{d\delta^s}, \frac{dA^S}{d\delta^S}, \frac{dN^B}{d\delta^S}, \frac{dN^S}{d\delta^S}$ is $+$ ve or $-$ ve can be determined only after obtaining their value which is possible by solving the system of four equations given by

$$\frac{dA^B}{d\delta^s} = \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{d\delta^S}\right) + \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{d\delta^S}\right), \quad \frac{dA^S}{d\delta^S} = \left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{d\delta^S}\right) + \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{d\delta^S}\right) + \frac{\partial A^S}{\partial \delta^S}$$

$$\frac{dN^S}{d\delta^S} = \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{d\delta^S}\right) + \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{d\delta^S}\right) + \left(\frac{\partial N^S}{\partial \delta^S}\right), \quad \frac{dN^B}{d\delta^S} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{d\delta^S}\right) + \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{d\delta^S}\right) \text{ i.e.}$$

$$[1]\left(\frac{dA^B}{d\delta^s}\right) - \left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^S}{d\delta^S}\right) + [0]\left(\frac{dN^B}{d\delta^S}\right) + \left[\frac{\eta^S A^S + \eta^B\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}}} + \eta^B\right]\left(\frac{dN^S}{d\delta^S}\right) = 0 \quad \text{...(a)}$$

$$[0]\left(\frac{dA^B}{d\delta^s}\right)-\left[\frac{\overline{N^S}\delta^s}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left(\frac{dA^S}{d\delta^S}\right)-\left[\frac{\overline{N^S}\eta^S\delta^s}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left(\frac{dN^B}{d\delta^S}\right)+[1]\left(\frac{dN^S}{d\delta^S}\right)=-\left[\frac{\overline{N^S}\left(A^S+\eta^S N^B\right)}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]$$

……..………………………... (b)

$$-\left[\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^B}{d\delta^S}\right)+[1]\left(\frac{dA^S}{d\delta^S}\right)+$$

$$\left[\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right]\left(\frac{dN^B}{d\delta^S}\right)+[0]\left(\frac{dN^S}{d\delta^S}\right)=$$

$$-\left[\frac{\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+1\right]$$

……..……………………………… (c)

$$-\left[\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left(\frac{dA^B}{d\delta^S}\right)+[0]\left(\frac{dA^S}{d\delta^S}\right)+[1]\left(\frac{dN^B}{d\delta^S}\right)-\left[\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left(\frac{dN^S}{d\delta^S}\right)=0$$

…………. (d)

Now by Cramer's Rule, we have $\left(\frac{dA^B}{d\delta^s}\right)=\frac{D_1}{D}$, $\left(\frac{dA^S}{d\delta^S}\right)=\frac{D_2}{D}$, $\left(\frac{dN^B}{d\delta^S}\right)=\frac{D_3}{D}$, $\left(\frac{dN^S}{d\delta^S}\right)=\frac{D_4}{D}$ where

$$D=$$

$$\begin{bmatrix}
1 & -\left[\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\eta^B\right] \\
0 & -\left[\frac{\overline{N^S}\delta^s}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & -\left[\frac{\overline{N^S}\eta^S\delta^s}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & 1 \\
-\left[\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}\right] & 1 & \left[\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\
-\left[\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right] & 0 & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]
\end{bmatrix}$$

$$D=\left[1-\left(\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right)\left(\frac{\overline{N^S}\eta^S\delta^s}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right)\right]\left[1-\left\{\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\right]+$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^S}\delta^s}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left[\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}+\frac{\overline{N^B}\delta^B\eta^S}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]+$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}-\right.$$

$$\left.\left\{\frac{\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\overline{N^S}\delta^s}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\right]$$

Now $1-a\,fraction>0$. Hence the first term is positive provided $\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}<1$ and $\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}<1$ which is plausible given $P^S>P^B>c,\ N^B\gg 1,\ \eta^B<1,\ A^B\leq 1$.

Also $N^S\gg 1, \beta<1, 1>\eta^B>\eta^S,\ \delta^B<1,\ A^S\leq 1,\ \left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]<0$,

$\eta^S A^S$ is small and $\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]<0$. Hence D is $+$ ve.

$D_1 =$

$$\begin{bmatrix}
0 & -\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\eta^B\right] \\
-\left[\frac{\overline{N^S}(A^S+\eta^S N^B)}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -\left[\frac{\overline{N^S}\delta^s}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -\left[\frac{\overline{N^S}\eta^S\delta^s}{(A^S+\eta^S N^B+\delta^S)^2}\right] & 1 \\
-\left[\frac{\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+1\right] & 1 & \left[\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\
0 & 0 & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]
\end{bmatrix}$$

$$D_1=\left[\frac{A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left[\left\{\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\overline{N^S}\delta^s}{(A^S+\eta^S N^B+\delta^S)^2}\right\}+\right.$$

$$\left.\left(\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right)\left(\frac{\overline{N^S}\eta^S\delta^s}{(A^S+\eta^S N^B+\delta^S)^2}\right)-\left\{\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\right]+$$

$$\left[\frac{\overline{N^S}(A^S+\eta^S N^B)}{(A^S+\eta^S N^B+\delta^S)^2}\right]\left\{\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\left[\left\{\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\frac{1}{\beta}-\eta^S N^B\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}+1\right]$$

All other terms being positive, consider two remaining terms of the above expression.

$$-\left[\frac{A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left\{\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}+$$

$$\left[\frac{\overline{N^S}(A^S+\eta^S N^B)}{(A^S+\eta^S N^B+\delta^S)^2}\right]\left\{\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}=$$

$$\frac{-A^S\eta^S(\overline{N^S}-N^S)(A^S+\eta^S N^B+\delta^S)^2+\overline{N^S}(A^S+\eta^S N^B)(\eta^B A^B+\eta^S A^S)\left\{A^S-\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]\right\}}{(A^S+\eta^S N^B+\delta^S)^2\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}$$

Now $N^S=\frac{\overline{N^S}(A^S+\eta^S N^B)}{[A^S+\eta^S N^B+\delta^S]}$, hence $\overline{N^S}=\frac{N^S(A^S+\eta^S N^B+\delta^S)}{(A^S+\eta^S N^B)}$, hence $\overline{N^S}-N^S=\frac{N^S\delta^S}{(A^S+\eta^S N^B)}$

The denominator is positive. The numerator is

$$-A^S\eta^S(\overline{N^S}-N^S)(A^S+\eta^S N^B+\delta^S)^2+\overline{N^S}(A^S+\eta^S N^B)(\eta^B A^B+\eta^S A^S)\left\{A^S-\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]\right\}$$ i.e.

$$-N^S(A^S+\eta^S N^B+\delta^S)\left[(1-\delta^S)-\frac{1}{\beta}\right]+N^S(A^S+\eta^S N^B+\delta^S)\left[-A^S\left\{\eta^S\delta^S+\frac{\eta^S(\delta^S)^2}{(A^S+\eta^S N^B)}\right\}+(\eta^B A^B+\eta^S A^S)(A^S+\eta^S N^B)\right]$$

Now the first term is +ve as $(1-\delta^S)-\frac{1}{\beta}<0$. Consider the expression in square brackets of the second term. Now $N^B\gg 1,\ A^S\leq 1,\ A^B\leq 1,\ \delta^S<\delta^B<1,\ \eta^S<\eta^B<1$ with $\eta^B,\ \eta^S$ being sufficiently large. Moreover, at steady state, $A^S>A^B$ as the number of sellers is much less than the number of buyers and therefore ad on each seller is more than ad on each buyer. Hence, it is plausible this expression is positive. Hence the numerator is positive. Hence $D_1 > 0$. Hence

$\left(\frac{dA^B}{d\delta^s}\right) = \frac{D_1}{D} > 0.$

$D_2 =$

$$\begin{bmatrix} 1 & 0 & 0 & \left[\frac{\eta^S A^S + \eta^B\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right) + 2\eta^S(A^S)(\overline{N^S} - N^S) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}}} + \eta^B\right] \\ 0 & -\left[\frac{\overline{N^S}(A^S+\eta^S N^B)}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -\left[\frac{\overline{N^S}\eta^S\delta^s}{(A^S+\eta^S N^B+\delta^S)^2}\right] & 1 \\ -\left[\frac{\eta^B(\overline{N^B} - N^B)}{\left[\left(\frac{P^S}{c}\right) + 2\eta^B(A^B)(\overline{N^B} - N^B) + \left[(1-\delta^S) - \frac{1}{\beta} - \eta^S N^B\right]^2\right]^{\frac{1}{2}}}\right] & -\left[\frac{\left[(1-\delta^S) - \frac{1}{\beta} - \eta^S N^B\right]}{\left[\left(\frac{P^S}{c}\right) + 2\eta^B(A^B)(\overline{N^B} - N^B) + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}} + 1\right] & \left[\frac{\eta^B A^B + \eta^S\left[(1-\delta^S) - \frac{1}{\beta} - \eta^S N^B\right]}{\left[\left(\frac{P^S}{c}\right) + 2\eta^B(A^B)(\overline{N^B} - N^B) + \left[(1-\delta^S) - \frac{1}{\beta} - \eta^S N^B\right]^2\right]^{\frac{1}{2}}} + \eta^S\right] & 0 \\ -\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & 0 & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \end{bmatrix}$$

$D_2 = \frac{numerator}{denominator}$ where

$denominator = \left[\left(\frac{P^B}{c}\right) + 2\eta^S(A^S)(\overline{N^S} - N^S) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}} \left[\left(\frac{P^S}{c}\right) + 2\eta^B(A^B)(\overline{N^B} - N^B) + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]^2\right]^{\frac{1}{2}} (A^B + \eta^B N^S + \delta^B)(A^B + \eta^B N^S)(A^S + \eta^S N^B + \delta^S)(A^S + \eta^S N^B)$. The denominator is positive.

$numerator = \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right][-N^B\delta^B N^S\,\eta^B\{(\eta^B A^B + \eta^S A^S)(A^S + \eta^S N^B) + A^S\eta^S\delta^s\} + A^S(A^S + \eta^S N^B + \delta^S)(A^S + \eta^S N^B)(A^B + \eta^B N^S + \delta^B)(A^B + \eta^B N^S)] + (A^B + \eta^B N^S + \delta^B)(A^S + \eta^S N^B + \delta^S)(A^S + \eta^S N^B)[N^B\delta^B\eta^B(\eta^B A^B + \eta^S A^S) - A^B A^S(A^B + \eta^B N^S)] - N^B\delta^B N^S\eta^S A^S[(\eta^B A^B + \eta^S A^S)(A^S + \eta^S N^B) + \delta^s\eta^S A^S]$ i.e.

$N^B N^S \left[\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]\left[-\delta^B\,\eta^B\{(\eta^B A^B + \eta^S A^S)(A^S + \eta^S N^B) + A^S\eta^S\delta^s\} + A^S(A^S + \eta^S N^B + \delta^S)\left(\frac{A^S}{N^B} + \eta^S\right)(A^B + \eta^B N^S + \delta^B)\left(\frac{A^B}{N^S} + \eta^B\right)\right] + \left(\frac{A^B}{N^S} + \eta^B + \frac{\delta^B}{N^S}\right)(A^S +\right.$

$\eta^S N^B + \delta^S)\left(\frac{A^S}{N^B} + \eta^S\right)[N^B \delta^B \eta^B(\eta^B A^B + \eta^S A^S) - A^B A^S(A^B + \eta^B N^S)] - \delta^B \eta^S A^S[(\eta^B A^B +$

$\eta^S A^S)(A^S + \eta^S N^B) + \delta^s \eta^S A^S]\Big]$. Now $\frac{A^B}{N^S}, \frac{A^S}{N^B} \approx 0$ as $A^B, A^S \leq 1,\ N^B \gg 1,\ N^S \gg 1$.

Hence the expression is approximately equal to

$N^B N^S N^B \Big[\Big[(1 - \delta^B) - \frac{1}{\beta} - \eta^B N^S\Big]\Big[-\delta^B \eta^B \left\{(\eta^B A^B + \eta^S A^S)\left(\frac{A^S}{N^B} + \eta^S\right) + A^S \eta^S \delta^s\right\} +$

$A^S\left(\frac{A^S}{N^B} + \eta^S + \frac{\delta^S}{N^B}\right)\eta^S(A^B + \eta^B N^S + \delta^B)\eta^B\Big] + \eta^B\left(\frac{A^S}{N^B} + \eta^S + \frac{\delta^S}{N^B}\right)\eta^S[N^B \delta^B \eta^B(\eta^B A^B +$

$\eta^S A^S) - A^B A^S(A^B + \eta^B N^S)] - \delta^B \eta^S A^S\left[(\eta^B A^B + \eta^S A^S)\left(\frac{A^S}{N^B} + \eta^S\right) + \frac{\delta^s \eta^S A^S}{N^B}\right]\Big]$

But $\frac{\delta^s \eta^S A^S}{N^B} \approx 0$. As $\delta^s, \eta^S, A^S < 1, N^B \gg 1$. Using similar reasoning as earlier, the expression reduces to approximately

$(N^B)^2 N^S \eta^S \Big[\Big[(1 - \delta^B) - \frac{1}{\beta} - \eta^B N^S\Big][-\delta^B \eta^B\{(\eta^B A^B + \eta^S A^S) + A^S \delta^s\} + A^S \eta^S(A^B +$

$\eta^B N^S + \delta^B)\eta^B] + \eta^B \eta^S[N^B \delta^B \eta^B(\eta^B A^B + \eta^S A^S) - A^B A^S(A^B + \eta^B N^S)] - \delta^B A^S[(\eta^B A^B +$

$\eta^S A^S)\eta^S]\Big]$ i.e.

$(N^B)^2 (N^S)^2 \eta^S \Big[\Big[\frac{(1-\delta^B)}{N^S} - \frac{1}{\beta N^S} - \eta^B\Big][-\delta^B \eta^B\{(\eta^B A^B + \eta^S A^S) + A^S \delta^s\} + A^S \eta^S(A^B +$

$\eta^B N^S + \delta^B)\eta^B] + \eta^B \eta^S\left[\frac{N^B}{N^S}\delta^B \eta^B(\eta^B A^B + \eta^S A^S) - A^B A^S\left(\frac{A^B}{N^S} + \eta^B\right)\right] - \delta^B A^S\left[\frac{(\eta^B A^B + \eta^S A^S)\eta^S}{N^S}\right]\Big]$

Now $N^S \gg 1,\ \delta^B < 1,\ \beta < 1,\ 1 > \eta^B > \eta^S,\ A^B \leq 1,\ A^S \leq 1,\ \frac{(1-\delta^B)}{N^S} \approx 0,\ \frac{A^B}{N^S} \approx 0$.

$\delta^B A^S \left[\frac{(\eta^B A^B+\eta^S A^S)\eta^S}{N^S}\right] \approx 0$. Hence the expression reduces to approximately

$$(N^B)^2(N^S)^2\eta^S(\eta^B)^2\Bigg[-[-\delta^B\{(\eta^B A^B+\eta^S A^S)+A^S\delta^s\}+A^S\eta^S(A^B+\eta^B N^S+\delta^B)]+$$

$$\eta^S\left[\frac{N^B}{N^S}\delta^B(\eta^B A^B+\eta^S A^S)-A^B A^S\right]\Bigg] \text{ i.e.}$$

$$(N^B)^2(N^S)^3\eta^S(\eta^B)^2\Bigg[-\left[-\delta^B\left\{\frac{(\eta^B A^B+\eta^S A^S)}{N^S}+\frac{A^S\delta^s}{N^S}\right\}+A^S\eta^S\left(\frac{A^B}{N^S}+\eta^B+\frac{\delta^B}{N^S}\right)\right]+$$

$$\eta^S\left[\frac{N^B}{N^S}\frac{\delta^B(\eta^B A^B+\eta^S A^S)}{N^S}-\frac{A^B A^S}{N^S}\right]\Bigg]$$

Now $N^S \gg 1,\ \delta^S < 1,\ \delta^B < 1,\ \beta < 1,\ 1 > \eta^B > \eta^S,\ A^B \leq 1,\ A^S \leq 1,\ \frac{(\eta^B A^B+\eta^S A^S)}{N^S} \approx 0,$ $\frac{A^S\delta^s}{N^S} \approx 0,\ \frac{A^B}{N^S} \approx 0,\ \frac{\delta^B}{N^S} \approx 0,\ \frac{A^B A^S}{N^S} \approx 0$ Hence the expression reduces to approximately

$$(N^B)^2(N^S)^3(\eta^S)^2(\eta^B)^2\left[-A^S\eta^B+\left[\frac{N^B}{N^S}\frac{\delta^B(\eta^B A^B+\eta^S A^S)}{N^S}\right]\right] \text{ i.e.}$$

$$(N^B)^2(N^S)^3(\eta^S)^2(\eta^B)^3 A^S\left[-1+\left[\delta^B\frac{N^B}{(N^S)^2}\left(\frac{A^B}{A^S}+\frac{\eta^S}{\eta^B}\right)\right]\right]$$

Now $\delta^B < 1,\ \beta < 1,\ 1 > \eta^B > \eta^S,\ A^B < A^S \leq 1$ as at steady state, the number of sellers is much less than the number of buyers and therefore ad on each seller is more than ad on each buyer. $N^B \gg N^S \gg 1$ but if $(N^S)^2 > N^B$, then $-1+\left[\delta^B\frac{N^B}{(N^S)^2}\left(\frac{A^B}{A^S}+\frac{\eta^S}{\eta^B}\right)\right] < 0.$

Hence the numerator is negative. Hence $D_2 < 0$. Hence $\left(\frac{dA^S}{d\delta^S}\right) = \frac{D_2}{D} < 0.$

Q.E.D.

Proof of Corollary 2

$A^B = f_1\left(A^S, N^S, \overline{N^S}, \delta^B, \eta^B, \eta^S, P^B, c, \beta\right)$; $\quad A^S = f_2\left(A^B, N^B, \overline{N^B}, \delta^S, \eta^S, \eta^B, P^S, c, \beta\right)$;

$N^B = f_3\left(A^B, \overline{N^B}, N^S, \eta^B, \delta^B\right)$; $N^S = f_4\left(A^S, \overline{N^S}, N^B, \eta^S, \delta^S\right)$ where f_1, f_2, f_3, f_4 are functions.

Whether $\frac{dA^B}{d\overline{N^B}}, \frac{dA^S}{d\overline{N^B}}, \frac{dN^B}{d\overline{N^B}}, \frac{dN^S}{d\overline{N^B}}$ is $+$ ve or $-$ ve can be determined only after obtaining their value which is possible by solving the system of four equations given by

$$\frac{dA^B}{d\overline{N^B}} = \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{d\overline{N^B}}\right) + \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{d\overline{N^B}}\right); \quad \frac{dA^S}{d\overline{N^B}} = \left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{d\overline{N^B}}\right) + \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{d\overline{N^B}}\right) + \left(\frac{\partial A^S}{\partial \overline{N^B}}\right);$$

$$\frac{dN^B}{d\overline{N^B}} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{d\overline{N^B}}\right) + \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{d\overline{N^B}}\right) + \left(\frac{\partial N^B}{\partial \overline{N^B}}\right); \quad \frac{dN^S}{d\overline{N^B}} = \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{d\overline{N^B}}\right) + \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{d\overline{N^B}}\right)$$

Now, we can write the 4 equations as –

$$[1]\left(\frac{dA^B}{d\overline{N^B}}\right) - \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{d\overline{N^B}}\right) + [0]\left(\frac{dN^B}{d\overline{N^B}}\right) - \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{d\overline{N^B}}\right) = 0$$

$$-\left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{d\overline{N^B}}\right) + [1]\left(\frac{dA^S}{d\overline{N^B}}\right) - \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{d\overline{N^B}}\right) + [0]\left(\frac{dN^S}{d\overline{N^B}}\right) = \left(\frac{\partial A^S}{\partial \overline{N^B}}\right)$$

$$-\left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{d\overline{N^B}}\right) + [0]\left(\frac{dA^S}{d\overline{N^B}}\right) + [1]\left(\frac{dN^B}{d\overline{N^B}}\right) - \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{d\overline{N^B}}\right) = \left(\frac{\partial N^B}{\partial \overline{N^B}}\right)$$

$$[0]\left(\frac{dA^B}{d\overline{N^B}}\right) - \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{d\overline{N^B}}\right) - \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{d\overline{N^B}}\right) + [1]\left(\frac{dN^S}{d\overline{N^B}}\right) = 0 \quad \text{i.e.}$$

$$[1]\left(\frac{dA^B}{d\overline{N^B}}\right) - \left[\frac{\eta^S(\overline{N^S} - N^S)}{\left[\left(\frac{P^B}{c}\right) + 2\eta^S(A^S)(\overline{N^S} - N^S) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^S}{d\overline{N^B}}\right) + [0]\left(\frac{dN^B}{d\overline{N^B}}\right) +$$

$$\left[\frac{\eta^S A^S + \eta^B\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right) + 2\eta^S(A^S)(\overline{N^S} - N^S) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}}} + \eta^B\right]\left(\frac{dN^S}{d\overline{N^B}}\right) = 0 \quad \text{………………………...…… (e)}$$

$$-\left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^B}{d\overline{N^B}}\right)+[1]\left(\frac{dA^S}{d\overline{N^B}}\right)+$$

$$\left[\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right]\left(\frac{dN^B}{d\overline{N^B}}\right)+[0]\left(\frac{dN^S}{d\overline{N^B}}\right)=$$

$$\left[\frac{\eta^B A^B}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(f)}$$

$$-\left[\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left(\frac{dA^B}{d\overline{N^B}}\right)+[0]\left(\frac{dA^S}{d\overline{N^B}}\right)+[1]\left(\frac{dN^B}{d\overline{N^B}}\right)-\left[\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left(\frac{dN^S}{d\overline{N^B}}\right)=\frac{\left(A^B+\eta^B N^S\right)}{\left[A^B+\eta^B N^S+\delta^B\right]}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(g)}$$

$$[0]\left(\frac{dA^B}{d\overline{N^B}}\right)-\left[\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left(\frac{dA^S}{d\overline{N^B}}\right)-\left[\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left(\frac{dN^B}{d\overline{N^B}}\right)+[1]\left(\frac{dN^S}{d\overline{N^B}}\right)=0 \ldots\ldots\ldots\ldots\ldots \text{(h)}$$

Now by Cramer's Rule, $\frac{dA^B}{d\overline{N^B}}=\frac{D_1}{D},\frac{dA^S}{d\overline{N^B}}=\frac{D_2}{D},\frac{dN^B}{d\overline{N^B}}=\frac{D_3}{D},\frac{dN^S}{d\overline{N^B}}=\frac{D_4}{D}$ where

$D=$

$$\begin{bmatrix} 1 & -\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\eta^B\right] \\ -\left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 1 & \left[\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\ -\left[\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right] & 0 & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right] \\ 0 & -\left[\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & -\left[\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & 1 \end{bmatrix}$$

$$D=\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left\{\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right\}\left[-\left\{\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right\}\left\{\frac{\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}+\right.$$

$$\left.\left\{\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\right]+\left[1-\left\{\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right\}\left\{\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right\}\right]\left[1-\right.$$

$$\left\{\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\Bigg] +$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left[\left\{\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}+\left\{\frac{\overline{N^B}\delta^B\eta^S}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right\}\right]$$

Now $1-a\ fraction>0$. Hence all terms from the second term onwards are positive provided $\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}<1,\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}<1,\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}<1,\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}<1$ which is plausible given $\eta^B,\eta^S<1,\delta^B,\delta^S<1,P^S>P^B>c,N^B>N^S\gg 1,\beta<1,A^B,A^S\leq 1$.

Let us examine the first term. $\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]<0$ and $A^S\leq 1,\ \eta^S<1$ and small, hence $\eta^S A^S+\ \eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]<0,$ hence the first term >0. Hence $D>0$.

$D_1=$

$$\begin{bmatrix} 0 & -\left[\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\ \eta^B\right] \\ \left[\frac{\eta^B A^B}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 1 & \left[\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\ \frac{\left(A^B+\eta^B N^S\right)}{\left[A^B+\eta^B N^S+\delta^B\right]} & 0 & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right] \\ 0 & -\left[\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & -\left[\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & 1 \end{bmatrix}$$

$D_1=$

$$\left\{\frac{N^S\delta^S\left[\eta^S\left(\eta^S N^B+\delta^S\right)-\eta^B A^B\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}\left(A^S+\eta^S N^B+\delta^S\right)\left(A^S+\eta^S N^B\right)}\right\}\left\{\frac{\eta^B\left(\delta^B A^B-\eta^S A^S N^S\right)-\eta^S A^B A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}\left[A^B+\eta^B N^S+\delta^B\right]}\right\}-$$

$$\left[\left\{\frac{\eta^S A^S+\eta^B A^B}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\eta^S\left(A^B+\eta^B N^S\right)}{\left[A^B+\eta^B N^S+\delta^B\right]}\right\}+\right.$$

$$\left.\left\{\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right\}\left\{\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right\}\right]$$

Now $N^S = \frac{\overline{N^S}(A^S+\eta^S N^B)}{[A^S+\eta^S N^B+\delta^S]}$. Hence $\overline{N^S} = \frac{N^S[A^S+\eta^S N^B+\delta^S]}{(A^S+\eta^S N^B)}$.

Hence $\overline{N^S} - N^S = \frac{N^S[A^S+\eta^S N^B+\delta^S]}{(A^S+\eta^S N^B)} - N^S = \frac{N^S\delta^S}{(A^S+\eta^S N^B)}$.

Since $A^B \leq 1, A^S \leq 1,\ \eta^B < 1, \eta^S < 1, \delta^B < 1, \delta^S < 1, N^B \gg 1, N^S \gg 1.$

$\eta^B(\delta^B A^B - \eta^S A^S N^S) - \eta^S A^B A^S < 0$ and $\eta^S(\eta^S N^B + \delta^S) - \eta^B A^B > 0.$ Hence $D_1 < 0$. Then

$\frac{dA^B}{d\overline{N^B}} = \frac{D_1}{D} < 0.$

$D_2 =$

$$\begin{bmatrix} 1 & 0 & 0 & \left[\frac{\eta^S A^S+\eta^B\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\eta^B\right] \\ -\left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & \left[\frac{\eta^B A^B}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & \left[\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\ -\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & \frac{(A^B+\eta^B N^S)}{[A^B+\eta^B N^S+\delta^B]} & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & 0 & -\left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & 1 \end{bmatrix}$$

$$D_2 = \left[\left\{\frac{\eta^B A^B}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\} - \left\{\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}} + \eta^S\right\}\left\{\frac{(A^B+\eta^B N^S)}{[A^B+\eta^B N^S+\delta^B]}\right\}\right] -$$

$$\left[\frac{\overline{N^S}\,\overline{N^B}N^S\eta^S\delta^S\eta^B\left[\delta^B A^B\left\{\frac{\eta^S A^S}{N^S}+\eta^B\left[\frac{(1-\delta^B)}{N^S}-\frac{1}{\beta N^S}-\eta^B\right]\right\}+\left(1-\frac{N^B}{\overline{N^B}}\right)\left(\frac{A^B}{N^S}+\eta^B\right)(\eta^S A^S+\eta^B A^B)(A^B+\eta^B N^S+\delta^B)\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}(A^S+\eta^S N^B+\delta^S)^2(A^B+\eta^B N^S+\delta^B)^2}\right]$$

The denominator of the second term is positive. Let us examine its numerator.

$\overline{N^S N^B} N^S \eta^S \delta^S \eta^B \left[\delta^B A^B \left\{\frac{\eta^S A^S}{N^S} + \eta^B \left[\frac{(1-\delta^B)}{N^S} - \frac{1}{\beta N^S} - \eta^B\right]\right\} + \left(1 - \frac{N^B}{\overline{\overline{N^B}}}\right)\left(\frac{A^B}{N^S} + \eta^B\right)(\eta^S A^S + \eta^B A^B)(A^B + \eta^B N^S + \delta^B)\right]$ Since $\eta^S < 1,\ A^B \leq 1,\ A^S \leq 1, \delta^B < 1,\ \beta < 1$ but large, $N^S \gg 1$

$\frac{\eta^S A^S}{N^S} \approx 0,\ \frac{(1-\delta^B)}{N^S} \approx 0,\ \frac{1}{\beta N^S} \approx 0,\ \frac{A^B}{N^S} \approx 0$. Hence, the numerator is approximately equal to

$\overline{N^S N^B} N^S \eta^S \delta^S \eta^B \left[\delta^B A^B \{\eta^B[-\eta^B]\} + \left(1 - \frac{N^B}{\overline{\overline{N^B}}}\right)(\eta^B)(\eta^S A^S + \eta^B A^B)(A^B + \eta^B N^S + \delta^B)\right]$ i.e.

$\overline{N^S N^B} N^S \eta^S \delta^B A^B \delta^S (\eta^B)^3 \left[-1 + \left(1 - \frac{N^B}{\overline{\overline{N^B}}}\right)\left(\frac{\eta^S A^S}{\eta^B A^B} + 1\right)\left(\frac{A^B}{\delta^B} + \frac{\eta^B N^S}{\delta^B} + 1\right)\right]$ Now

$\delta^B < 1, A^B < A^S \leq 1, \eta^S < \eta^B < 1, N^S \gg 1, \eta^B$ not too low, δ^B not too high, $\eta^B,\ \eta^S,\ \delta^B$ close in magnitude, $\overline{N^B} > N^B$, hence $\left[-1 + \left(1 - \frac{N^B}{\overline{\overline{N^B}}}\right)\left(\frac{\eta^S A^S}{\eta^B A^B} + 1\right)\left(\frac{A^B}{\delta^B} + \frac{\eta^B N^S}{\delta^B} + 1\right)\right] > 0$. Hence the second term is negative as it has a minus sign before it.

Let us now examine the numerator of the first term after simplification.

$\eta^B A^B[A^B + \eta^B N^S + \delta^B] - \left\{\eta^B A^B + \eta^S \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]\right\}(A^B + \eta^B N^S) -$

$\eta^S (A^B + \eta^B N^S)\left[\left(\frac{P^S}{c}\right) + 2\eta^B (A^B)\left(\overline{N^B} - N^B\right) + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]^2\right]^{\frac{1}{2}} = K$ (say)

$A^S = \left[\left(\frac{P^S}{c}\right) + 2\eta^B (A^B)\left(\overline{N^B} - N^B\right) + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]$, so

$\left[\left(\frac{P^S}{c}\right) + 2\eta^B (A^B)\left(\overline{N^B} - N^B\right) + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]^2\right]^{\frac{1}{2}} = A^S - \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]$

Hence we have

$K = \eta^B A^B[A^B + \eta^B N^S + \delta^B] - \left\{\eta^B A^B \ + \ \eta^S \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]\right\}(A^B + \eta^B N^S) -$

$\eta^S(A^B + \eta^B N^S)\left\{A^S - \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]\right\}$ i.e.

$K = \eta^B[\delta^B A^B - \eta^S N^S A^S] - \eta^S A^B A^S$

Now $N^S \gg 1,\ A^S \leq 1,\ A^B \leq 1,\ \delta^B < 1,\ \eta^S < 1,\ \eta^B < 1$. Hence the term inside the square bracket is negative. Hence $K < 0$. Hence $D_2 < 0$. Hence $\frac{dA^S}{d\overline{N^B}} = \frac{D_2}{D} < 0$.

Q.E.D.

Proof of Corollary 3

Since $A^B = f_1\left(A^S, N^S, \overline{N^S}, \delta^B,\ \eta^B, \eta^S, P^B, c, \beta\right)$; $\qquad A^S = f_2\left(A^B, N^B, \overline{N^B}, \delta^S, \eta^S, \eta^B, P^S, c, \beta\right)$; $N^B = f_3\left(A^B, \overline{N^B}, N^S, \eta^B, \delta^B\right)$; $N^S = f_4\left(A^S, \overline{N^S}, N^B, \eta^S, \delta^S\right)$ where f_1, f_2, f_3, f_4 are functions. whether $\frac{dA^B}{d\eta^B}, \frac{dA^S}{d\eta^B}, \frac{dN^B}{d\eta^B}, \frac{dN^S}{d\eta^B}$ is + ve or − ve can be determined only after obtaining their value which is possible by solving the system of four equations given by

$\frac{dA^B}{d\eta^B} = \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{d\eta^B}\right) + \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{d\eta^B}\right) + \left(\frac{\partial A^B}{\partial \eta^B}\right)$; $\frac{dA^S}{d\eta^B} = \left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{d\eta^B}\right) + \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{d\eta^B}\right) + \left(\frac{\partial A^S}{\partial \eta^B}\right)$;

$\frac{dN^B}{d\eta^B} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{d\eta^B}\right) + \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{d\eta^B}\right) + \left(\frac{\partial N^B}{\partial \eta^B}\right)$; $\frac{dN^S}{d\eta^B} = \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{d\eta^B}\right) + \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{d\eta^B}\right)$

Now, we can write the 4 equations as –

$$[1]\left(\frac{dA^B}{d\eta^B}\right) - \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{d\eta^B}\right) + [0]\left(\frac{dN^B}{d\eta^B}\right) - \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{d\eta^B}\right) = \left(\frac{\partial A^B}{\partial \eta^B}\right)$$

$$-\left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{d\eta^B}\right) + [1]\left(\frac{dA^S}{d\eta^B}\right) - \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{d\eta^B}\right) + [0]\left(\frac{dN^S}{d\eta^B}\right) = \left(\frac{\partial A^S}{\partial \eta^B}\right)$$

$$-\left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{d\eta^B}\right) + [0]\left(\frac{dA^S}{d\eta^B}\right) + [1]\left(\frac{dN^B}{d\eta^B}\right) - \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{d\eta^B}\right) = \left(\frac{\partial N^B}{\partial \eta^B}\right)$$

$[0]\left(\frac{dA^B}{d\eta^B}\right) - \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{d\eta^B}\right) - \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{d\eta^B}\right) + [1]\left(\frac{dN^S}{d\eta^B}\right) = 0$ i.e.

$$[1]\left(\frac{dA^B}{d\eta^B}\right) - \left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^S}{d\eta^B}\right) + [0]\left(\frac{dN^B}{d\eta^B}\right) +$$

$$\left[\frac{\eta^S A^S+\eta^B\left\{(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}} + \eta^B\right]\left(\frac{dN^S}{d\eta^B}\right) =$$

$$-N^S\left[\frac{\left\{(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}} + 1\right] \quad \text{.. (i)}$$

$$-\left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^B}{d\eta^B}\right) + [1]\left(\frac{dA^S}{d\eta^B}\right) +$$

$$\left[\frac{\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\}}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}} + \eta^S\right]\left(\frac{dN^B}{d\eta^B}\right) + [0]\left(\frac{dN^S}{d\eta^B}\right) =$$

$$\left[\frac{A^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] \quad \text{.. (j)}$$

$$-\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left(\frac{dA^B}{d\eta^B}\right) + [0]\left(\frac{dA^S}{d\eta^B}\right) + [1]\left(\frac{dN^B}{d\eta^B}\right) - \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left(\frac{dN^S}{d\eta^B}\right) = \left[\frac{\overline{N^B}N^S\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]$$

.... (k)

$$[0]\left(\frac{dA^B}{d\eta^B}\right) - \left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right]\left(\frac{dA^S}{d\eta^B}\right) - \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right]\left(\frac{dN^B}{d\eta^B}\right) + [1]\left(\frac{dN^S}{d\eta^B}\right) = 0 \quad \text{..................... (l)}$$

Now by Cramer's Rule, we have $\left(\frac{dA^B}{d\eta^B}\right) = \frac{D_1}{D}$, $\left(\frac{dA^S}{d\eta^B}\right) = \frac{D_2}{D}$, $\left(\frac{dN^B}{d\eta^B}\right) = \frac{D_3}{D}$, $\left(\frac{dN^S}{d\eta^B}\right) = \frac{D_4}{D}$ where

$D =$

$$\begin{bmatrix} 1 & -\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^S A^S+\eta^B\left\{(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\eta^B\right] \\ -\left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 1 & \left[\frac{\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\}}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\ -\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & 0 & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & -\left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -\left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & 1 \end{bmatrix}$$

$$D = \left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right]\left[\left\{\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}-\right.$$

$$\left.\left\{\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\}}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\right]+\left[1-\left\{\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\left\{\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\right]\left[1-\right.$$

$$\left.\left\{\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\right]+$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left[\left\{\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}+\left\{\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\right]$$

Now $1 - a\ fraction > 0$. The second term is positive provided

$\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2} < 1, \frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2} < 1, \frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}} < 1, \frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}} < 1$ which is

plausible given $\eta^B, \eta^S < 1, \delta^B, \delta^S < 1, P^S > P^B > c, N^B > N^S \gg 1, \beta < 1, A^B, A^S \leq 1.$

The third term is also positive. Consider the first term.

Now $N^B \gg 1,\ \beta < 1,\ A^B \leq 1,\ \delta^S < 1,\ \eta^S < 1,\ \eta^B < 1.$ $(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}$ is large.

Given that $\eta^B,\ \eta^S$ are relatively large, we have $\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\} < 0.$

Hence the first term is positive. Hence $D > 0$.

$D_1 =$

$$\begin{bmatrix} -N^S\left[\frac{\left\{(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+1\right] & -\left[\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^S A^S+\eta^B\left\{(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\eta^B\right] \\ \left[\frac{A^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 1 & \left[\frac{\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\}}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\ \left[\frac{\overline{N^B}N^S\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right] & 0 & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right] \\ 0 & -\left[\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & -\left[\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right] & 1 \end{bmatrix}$$

$$D_1 = -\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left[\left\{\frac{A^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}-\right.$$

$$\left.\left\{\frac{\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\}}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\overline{N^B}N^S\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right\}\right]-$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^B}N^S\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left[\left\{\frac{A^B}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\overline{N^S}\delta^S\eta^B}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right\}+\right.$$

$$\left.\left\{\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\right]-\left\{\frac{N^S A^B}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\left[1-\left\{\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right\}\left\{\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right\}\right]\left[1-\right.$$

$$\left.\left\{\frac{N^B\delta^B\eta^S\delta^S}{\left(A^B+\eta^B N^S\right)\left(A^S+\eta^S N^B\right)\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\right]$$

Now $1-a\ fraction > 0$. The third term is negative provided $\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2} < 1, \frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2} < 1, \frac{N^B\delta^B\eta^S\delta^S}{\left(A^B+\eta^B N^S\right)\left(A^S+\eta^S N^B\right)\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}} < 1$ which is plausible given

$\eta^B, \eta^S < 1, \delta^B, \delta^S < 1, P^S > P^B > c, N^B > N^S \gg 1, \beta < 1, A^B, A^S \leq 1.$

The second term is also negative. Consider the first term.

Now $N^B \gg 1,\ \beta < 1,\ A^B \leq 1,\ \delta^S < 1,\ \eta^S < 1,\ \eta^B < 1. (1-\delta^S)-\eta^S N^B-\frac{1}{\beta}$ is large

$\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\} < 0.$ Hence the first term is negative also. Hence

$D_1 < 0$. Hence $\frac{D_1}{D} < 0$. Hence $\frac{dA^B}{d\eta^B} < 0$.

$$D_2 = \begin{bmatrix} 1 & -N^S\left[\frac{\left\{(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+1\right] & 0 & \left[\frac{\eta^S A^S+\eta^B\left\{(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+\eta^B\right] \\ -\left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & \left[\frac{A^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & \left[\frac{\eta^B A^B+\eta^S\left\{(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right\}}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right] & 0 \\ -\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & \left[\frac{\overline{N^B}N^S\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & 1 & -\left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & 0 & -\left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & 1 \end{bmatrix}$$

$D_2 = \frac{numerator}{denominator}$ where

$denominator = \left[\left(\frac{P^S}{c}\right) + 2\eta^B(A^B)\left(\overline{N^B} - N^B\right) + \left[(1-\delta^S) - \eta^S N^B - \frac{1}{\beta}\right]^2\right]^{\frac{1}{2}} \left[\left(\frac{P^B}{c}\right) + 2\eta^S(A^S)\left(\overline{N^S} - N^S\right) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}} (A^B + \eta^B N^S)(A^B + \eta^B N^S + \delta^B)(A^S + \eta^S N^B)(A^S + \eta^S N^B + \delta^S) > 0$. Now the numerator is

$NR = \left(\overline{N^B} - N^B\right)\left[N^B N^S \delta^B \delta^S \eta^S \left\{A^B \eta^B \left(1 - \delta^B - \frac{1}{\beta}\right) - \eta^B N^S(\eta^B A^B + \eta^S A^S)\right\} + A^B(A^B + \eta^B N^S)(A^B + \eta^B N^S + \delta^B)(A^S + \eta^S N^B)(A^S + \eta^S N^B + \delta^S)\left\{A^B - (1-\delta^B) + \frac{1}{\beta}\right\}\right] + N^B N^S \delta^B(\eta^B A^B + \eta^S A^S)\left\{(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right\}(A^S + \eta^S N^B)(A^S + \eta^S N^B + \delta^S)$

$= (A^S + \eta^S N^B)(A^S + \eta^S N^B + \delta^S)\left[\left(\overline{N^B} - N^B\right)A^B(A^B + \eta^B N^S)(A^B + \eta^B N^S + \delta^B)\left\{A^B - (1-\delta^B) + \frac{1}{\beta}\right\} + N^B N^S \delta^B(\eta^B A^B + \eta^S A^S)\left\{(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right\}\right] + \left(\overline{N^B} - N^B\right)N^B N^S \delta^B \delta^S \eta^S \left\{A^B \eta^B \left(1 - \delta^B - \frac{1}{\beta}\right) - \eta^B N^S(\eta^B A^B + \eta^S A^S)\right\}$

Now $N^B = \frac{\overline{N^B}(A^B + \eta^B N^S)}{[A^B + \eta^B N^S + \delta^B]}$. Hence, $\overline{N^B} - N^B = \frac{N^B(A^B + \eta^B N^S + \delta^B)}{(A^B + \eta^B N^S)} - N^B = \frac{N^B \delta^B}{(A^B + \eta^B N^S)}$.

Consider the expression in square brackets of the first term.

$\left(\overline{N^B} - N^B\right)A^B(A^B + \eta^B N^S)(A^B + \eta^B N^S + \delta^B)\left\{A^B - (1-\delta^B) + \frac{1}{\beta}\right\} + N^B N^S \delta^B(\eta^B A^B +$

$\eta^S A^S)\left\{(1-\delta^B)\ -\frac{1}{\beta} -\ \eta^B N^S\right\} = \left\{(1-\delta^B)\ -\ \frac{1}{\beta}\right\}\{N^S(\eta^B A^B + \eta^S A^S) - N^B\delta^B A^B(A^B +$

$\eta^B N^S + \delta^B)\} + N^B\delta^B A^B(A^B + \eta^B N^S + \delta^B)A^B - N^S(\eta^B A^B + \eta^S A^S)\,\eta^B N^S$

Hence the numerator is

$NR = (A^S + \eta^S N^B)(A^S + \eta^S N^B + \delta^S)\left[\left\{(1-\delta^B)\ -\ \frac{1}{\beta}\right\}\{N^S(\eta^B A^B + \eta^S A^S) - N^B\delta^B A^B(A^B +\right.$

$\left.\eta^B N^S + \delta^B)\} + N^B\delta^B A^B(A^B + \eta^B N^S + \delta^B)A^B - N^S(\eta^B A^B + \eta^S A^S)\,\eta^B N^S\right] +$

$\frac{N^B\delta^B}{(A^B+\eta^B N^S)} N^B N^S \delta^B \delta^S \eta^S \left\{A^B\eta^B\left(1 - \delta^B - \frac{1}{\beta}\right) - \eta^B N^S(\eta^B A^B + \eta^S A^S)\right\}$ i.e.

$NR = \left(1 - \delta^B - \frac{1}{\beta}\right)\left[(A^S + \eta^S N^B)(A^S + \eta^S N^B + \delta^S)\{N^S(\eta^B A^B + \eta^S A^S) - N^B\delta^B A^B(A^B +\right.$

$\left.\eta^B N^S + \delta^B)\} + \frac{N^B\delta^B}{(A^B+\eta^B N^S)} N^B N^S \delta^B\delta^S\eta^S A^B\eta^B\right] + (A^S + \eta^S N^B)(A^S + \eta^S N^B +$

$\delta^S)[N^B\delta^B A^B(A^B + \eta^B N^S + \delta^B)A^B - N^S(\eta^B A^B + \eta^S A^S)\,\eta^B N^S] -$

$\frac{N^B\delta^B}{(A^B+\eta^B N^S)} N^B N^S \delta^B\delta^S\eta^S\eta^B N^S(\eta^B A^B + \eta^S A^S)$ i.e.

$NR = (N^B)^2 N^S\left[\left(1 - \delta^B - \frac{1}{\beta}\right)\left[\left(\frac{A^S}{N^B} + \eta^S\right)\left(\frac{A^S}{N^B} + \eta^S + \frac{\delta^S}{N^B}\right)\left\{(\eta^B A^B + \eta^S A^S) - N^B\delta^B A^B\left(\frac{A^B}{N^S} +\right.\right.\right.$

$\left.\left.\eta^B + \frac{\delta^B}{N^S}\right)\right\} + \frac{\delta^B}{(A^B+\eta^B N^S)}\delta^B\delta^S\eta^S A^B\eta^B\Big] + \left(\frac{A^S}{N^B} + \eta^S\right)\left(\frac{A^S}{N^B} + \eta^S + \frac{\delta^S}{N^B}\right)\left[N^B\delta^B A^B\left(\frac{A^B}{N^S} + \eta^B +\right.\right.$

$\left.\left.\frac{\delta^B}{N^S}\right)A^B - (\eta^B A^B + \eta^S A^S)\,\eta^B N^S\right] - \frac{\delta^B}{(A^B+\eta^B N^S)}\delta^B\delta^S\eta^S\eta^B N^S(\eta^B A^B + \eta^S A^S)\Big]$

Now $\frac{A^S}{N^B}, \frac{\delta^S}{N^B}, \frac{A^B}{N^S}, \frac{\delta^B}{N^S} \approx 0$ as $A^B, A^S, \delta^B, \delta^S < 1;\quad N^B, N^S \gg 1$. Hence we have

$NR \approx (N^B)^2 N^S \left[\left(1-\delta^B-\frac{1}{\beta}\right) N^B \left\{\eta^S\eta^S\left(\frac{(\eta^B A^B+\eta^S A^S)}{N^B}-\delta^B A^B \eta^B\right)+\right.\right.$

$\left.\frac{\delta^B}{N^B(A^B+\eta^B N^S)}\delta^B\delta^S\eta^S A^B\eta^B\right\}+\eta^S\eta^S\{N^B\delta^B A^B\eta^B A^B-(\eta^B A^B+\eta^S A^S)\,\eta^B N^S\}-$

$\left.\frac{\delta^B}{(A^B+\eta^B N^S)}\delta^B\delta^S\eta^S\eta^B N^S(\eta^B A^B+\eta^S A^S)\right]$ Now

$\frac{(\eta^B A^B+\eta^S A^S)}{N^B}\approx 0$ and $\frac{\delta^B}{N^B(A^B+\eta^B N^S)}\delta^B\delta^S\eta^S A^B\eta^B\approx 0$ as $A^B, A^S, \delta^B, \delta^S, \eta^B, \eta^S < 1;\ \ N^B \gg 1$. So

$NR \approx (N^B)^2 N^S\eta^B\eta^S\left[\left(1-\delta^B-\frac{1}{\beta}\right)N^B\{-\eta^S\delta^B A^B\}+\eta^S\{N^B\delta^B A^B A^B-(\eta^B A^B+\eta^S A^S)N^S\}-\right.$

$\left.\frac{\delta^B}{(A^B+\eta^B N^S)}\delta^B\delta^S N^S(\eta^B A^B+\eta^S A^S)\right]$ i.e.

$$NR \approx (N^B)^3 N^S\eta^B\eta^S\delta^B\eta^S A^B\left[-\left(1-\delta^B-\frac{1}{\beta}\right)+\left\{A^B-\frac{\left(\eta^B+\frac{\eta^S A^S}{A^B}\right)N^S}{\delta^B N^B}\right\}-\frac{\delta^B}{\left(\frac{A^B}{N^S}+\eta^B\right)}\frac{\delta^S\left(\frac{\eta^B}{\eta^S}+\frac{A^S}{A^B}\right)}{N^B}\right]$$

Now $\frac{A^B}{N^S}\approx 0$ as $A^B < 1, N^S \gg 1$. Hence the expression reduces to approximately

$NR \approx (N^B)^3 N^S\eta^B\eta^S\delta^B\eta^S A^B\left[-\left(1-\delta^B-\frac{1}{\beta}\right)+A^B-\frac{(\eta^B A^B+\eta^S A^S)}{N^B A^B}\left\{\frac{N^S}{\delta^B}+\frac{\delta^B\delta^S}{\eta^B\eta^S}\right\}\right]$ i.e.

$$NR \approx (N^B)^3 (N^S)^2\eta^B\eta^S\delta^B\eta^S A^B\left[-\frac{\left(1-\delta^B-\frac{1}{\beta}\right)}{N^S}+\frac{A^B}{N^S}-\frac{(\eta^B A^B+\eta^S A^S)}{N^B A^B}\left\{\frac{1}{\delta^B}+\frac{\delta^B\delta^S}{N^S\eta^B\eta^S}\right\}\right]$$

Now $A^B, A^S, \delta^B, \delta^S, \eta^B, \eta^S < 1;\ \ N^S \gg 1$. $\beta < 1$ but close to 1.

Hence $\frac{\left(1-\delta^B-\frac{1}{\beta}\right)}{N^S}\approx 0,\ \frac{A^B}{N^S}\approx 0,\ \frac{\delta^B\delta^S}{N^S\eta^B\eta^S}\approx 0$. Hence we have

$NR \approx (N^B)^3 N^S\eta^B\eta^S\delta^B\eta^S A^B\left[-\frac{(\eta^B A^B+\eta^S A^S)}{N^B A^B}\left\{\frac{1}{\delta^B}\right\}\right]$ i.e.

$NR \approx -(N^B)^2 N^S\eta^B\eta^S\eta^S(\eta^B A^B+\eta^S A^S) < 0$

Hence $NR < 0$. Hence $D_2 < 0$. Hence $\left(\frac{dA^S}{d\eta^B}\right) = \frac{D_2}{D} < 0$.

Q.E.D.

Proof of Corollary 4

$\frac{dA^B}{d\beta} = \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{d\beta}\right) + \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{d\beta}\right) + \left(\frac{\partial A^B}{\partial \beta}\right)$; $\frac{dA^S}{d\beta} = \left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{d\beta}\right) + \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{d\beta}\right) + \left(\frac{\partial A^S}{\partial \beta}\right)$;

$\frac{dN^B}{d\beta} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{d\beta}\right) + \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{d\beta}\right)$; $\frac{dN^S}{d\beta} = \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{d\beta}\right) + \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{d\beta}\right)$ i.e.

$$[1]\left(\frac{dA^B}{d\beta}\right) - \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{d\beta}\right) + [0]\left(\frac{dN^B}{d\beta}\right) - \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{d\beta}\right) = \left(\frac{\partial A^B}{\partial \beta}\right)$$

$$\left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{d\beta}\right) - [1]\left(\frac{dA^S}{d\beta}\right) + \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{d\beta}\right) + [0]\left(\frac{dN^S}{d\beta}\right) = -\left(\frac{\partial A^S}{\partial \beta}\right)$$

$$\left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{d\beta}\right) + [0]\left(\frac{dA^S}{d\beta}\right) - [1]\left(\frac{dN^B}{d\beta}\right) + \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{d\beta}\right) = 0$$

$[0]\left(\frac{dA^B}{d\beta}\right) + \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{d\beta}\right) + \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{d\beta}\right) - [1]\left(\frac{dN^S}{d\beta}\right) = 0$ Hence we have

$$[1]\left(\frac{dA^B}{d\beta}\right) - \left[\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S\left(A^S\right)\left(\overline{N^S}-N^S\right)+\left[\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^S}{d\beta}\right) + [0]\left(\frac{dN^B}{d\beta}\right) +$$

$$\left[\frac{\eta^S A^S + \eta^B\left[\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S\left(A^S\right)\left(\overline{N^S}-N^S\right)+\left[\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}} + \eta^B\right]\left(\frac{dN^S}{d\beta}\right) = \left(\frac{1}{\beta^2}\right)\left[\frac{\left[\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S\left(A^S\right)\left(\overline{N^S}-N^S\right)+\left[\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}} + 1\right];$$

$$\left[\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B\left(A^B\right)\left(\overline{N^B}-N^B\right)+\left[\left(1-\delta^S\right)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^B}{d\beta}\right)-[1]\left(\frac{dA^S}{d\beta}\right)-\left[\frac{\eta^B A^B+\eta^S\left[\left(1-\delta^S\right)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B\left(A^B\right)\left(\overline{N^B}-N^B\right)+\left[\left(1-\delta^S\right)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+\eta^S\right]\left(\frac{dN^B}{d\beta}\right)+[0]\left(\frac{dN^S}{d\beta}\right)=-\left(\frac{1}{\beta^2}\right)\left[\frac{\left[\left(1-\delta^S\right)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B\left(A^B\right)\left(\overline{N^B}-N^B\right)+\left[\left(1-\delta^S\right)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+1\right];$$

$$\left[\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left(\frac{dA^B}{d\beta}\right)+[0]\left(\frac{dA^S}{d\beta}\right)-[1]\left(\frac{dN^B}{d\beta}\right)+\left[\frac{\overline{N^B}\eta^B\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left(\frac{dN^S}{d\beta}\right)=0;$$

$$[0]\left(\frac{dA^B}{d\beta}\right)+\left[\frac{\overline{N^S}\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left(\frac{dA^S}{d\beta}\right)+\left[\frac{\overline{N^S}\eta^S\delta^S}{\left(A^S+\eta^S N^B+\delta^S\right)^2}\right]\left(\frac{dN^B}{d\beta}\right)-[1]\left(\frac{dN^S}{d\beta}\right)=0;$$

By Cramer's Rule, we have $\frac{dA^B}{d\beta}=\frac{D_1}{D}$ where

$D=$

$$\begin{bmatrix} 1 & -\left[\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] \\ \left[\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & -1 & -\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 0 \\ \left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & 0 & -1 & \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & \left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -1 \end{bmatrix}$$

$$D=-\left[1-\left\{\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\right]\left[1-\left\{\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\right]-\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right]-\left\{\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left[\frac{\eta^S\left(\overline{N^S}-N^S\right)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]-$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left\{\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\left[-\left\{\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}+\right.$$

$$\left.\left\{\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\right]$$

The second and third terms are negative. Now $1-a\,fraction>0$. The first term is negative provided $\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}<1, \frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}<1, \frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}<1, \frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}<1$ which is plausible given $\eta^B,\eta^S<1,\delta^B,\delta^S<1,P^S>P^B>c,N^B>N^S\gg 1,\beta<1,A^B,A^S\leq 1.$

$N^B\gg 1,\ \beta<1,\ A^B\leq 1,\ \delta^S\leq 1,\ \eta^S<1,\ \eta^B<1,\ \eta^B,\ \eta^S$ are relatively large. Hence, $\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]<0$. Hence, $D<0$.

$D_1=$

$$\begin{bmatrix}
\left(\frac{1}{\beta^2}\right)\left[\frac{\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}+1\right] & -\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] \\
-\left(\frac{1}{\beta^2}\right)\left[\frac{\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}+1\right] & -1 & -\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 0 \\
0 & 0 & -1 & \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\
0 & \left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -1
\end{bmatrix}$$

$$D_1=-\left(\frac{1}{\beta^2}\right)\left[\frac{A^B}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left[\left\{1-\left(\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right)\left(\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right)\right\}+\left\{\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\right]+$$

$\left(\frac{1}{\beta^2}\right)\left[\frac{A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right][K]$ where the first term is negative as $1-a\,fraction>0$ as

$\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}<1, \frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}<1$ which is plausible given

$\eta^B,\eta^S<1,\delta^B,\delta^S<1,P^S>P^B>c,N^B>N^S\gg 1,\beta<1,A^B,A^S\leq 1.$ Now

$$K = \frac{-N^S\delta^S(A^B+\eta^B N^S+\delta^B)(A^S+\eta^S N^B+\delta^S)[(A^B+\eta^B N^S)(A^S+\eta^S N^B)(A^B+\eta^B N^S+\delta^B)\{\eta^S(\eta^S N^B+\delta^S)-\eta^B A^B\}-\eta^S N^B\eta^B\delta^B N^S\eta^S\delta^S]}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}(A^B+\eta^B N^S+\delta^B)^2(A^S+\eta^S N^B+\delta^S)^2(A^S+\eta^S N^B)^2(A^B+\eta^B N^S)}$$

The denominator is positive. Consider the expression inside the square brackets of the numerator.

$$(A^B+\eta^B N^S)(A^S+\eta^S N^B)(A^B+\eta^B N^S+\delta^B)\{\eta^S(\eta^S N^B+\delta^S)-\eta^B A^B\}-\eta^S N^B\eta^B\delta^B N^S\eta^S\delta^S$$

$$N^B N^S\left[\left(\frac{A^B}{N^S}+\eta^B\right)\left(\frac{A^S}{N^B}+\eta^S\right)(A^B+\eta^B N^S+\delta^B)\{\eta^S(\eta^S N^B+\delta^S)-\eta^B A^B\}-\eta^S\eta^B\delta^B\eta^S\delta^S\right]$$

Now $\frac{A^B}{N^S}\approx 0,\ \frac{A^S}{N^B}\approx 0$ as $A^B\le 1, A^S\le 1,\ N^B\gg 1,\ N^S\gg 1$.

The expression reduces to approximately

$N^B N^S\eta^B\eta^S[(A^B+\eta^B N^S+\delta^B)\{\eta^S(\eta^S N^B+\delta^S)-\eta^B A^B\}-\eta^S\delta^B\delta^S]$ i.e.

$$(N^B)^2 N^S\eta^B\eta^S\left[(A^B+\eta^B N^S+\delta^B)\left\{\eta^S\left(\eta^S+\frac{\delta^S}{N^B}\right)-\frac{\eta^B A^B}{N^B}\right\}-\frac{\eta^S\delta^B\delta^S}{N^B}\right]$$

Now $\frac{\delta^S}{N^B}\approx 0,\ \frac{\eta^B A^B}{N^B}\approx 0,\ \frac{\eta^S\delta^B\delta^S}{N^B}\approx 0$ as $A^B\le 1,\ \delta^B<1,\ \delta^S<1,\ 1>\eta^B>\eta^S,\ N^B\gg 1.$

Hence the expression reduces to approximately $(N^B)^2 N^S\eta^B\eta^S[(A^B+\eta^B N^S+\delta^B)(\eta^S)^2]>0.$

Hence $K<0.$ Hence $D_1<0.$ Hence $\frac{dA^B}{d\beta}=\frac{D_1}{D}>0.$

Let $\frac{P^B}{c}=x,\ \frac{P^S}{c}=y$

$\frac{dA^B}{dx}=\left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{dx}\right)+\left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{dx}\right)+\left(\frac{\partial A^B}{\partial x}\right);\ \frac{dA^S}{dx}=\left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{dx}\right)+\left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{dx}\right);$

$$\frac{dN^B}{dx} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{dx}\right) + \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{dx}\right);\ \frac{dN^S}{dx} = \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{dx}\right) + \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{dx}\right)\ \text{i.e.}$$

$$[1]\left(\frac{dA^B}{dx}\right) - \left(\frac{\partial A^B}{\partial A^S}\right)\left(\frac{dA^S}{dx}\right) + [0]\left(\frac{dN^B}{dx}\right) - \left(\frac{\partial A^B}{\partial N^S}\right)\left(\frac{dN^S}{dx}\right) = \left(\frac{\partial A^B}{\partial x}\right)$$

$$\left(\frac{\partial A^S}{\partial A^B}\right)\left(\frac{dA^B}{dx}\right) - [1]\left(\frac{dA^S}{dx}\right) + \left(\frac{\partial A^S}{\partial N^B}\right)\left(\frac{dN^B}{dx}\right) + [0]\left(\frac{dN^S}{dx}\right) = 0$$

$$\left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{dx}\right) + [0]\left(\frac{dA^S}{dx}\right) - [1]\left(\frac{dN^B}{dx}\right) + \left(\frac{\partial N^B}{\partial N^S}\right)\left(\frac{dN^S}{dx}\right) = 0$$

$$[0]\left(\frac{dA^B}{dx}\right) + \left(\frac{\partial N^S}{\partial A^S}\right)\left(\frac{dA^S}{dx}\right) + \left(\frac{\partial N^S}{\partial N^B}\right)\left(\frac{dN^B}{dx}\right) - [1]\left(\frac{dN^S}{dx}\right) = 0. \quad \text{Hence we have}$$

$$[1]\left(\frac{dA^B}{dx}\right) - \left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^S}{dx}\right) + [0]\left(\frac{dN^B}{dx}\right) +$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dN^S}{dx}\right) = \left[\frac{1}{2\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]$$

$$\left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dA^B}{dx}\right) - [1]\left(\frac{dA^S}{dx}\right) - \left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left(\frac{dN^B}{dx}\right) + [0]\left(\frac{dN^S}{dx}\right) = 0$$

$$\left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left(\frac{dA^B}{dx}\right) + [0]\left(\frac{dA^S}{dx}\right) - [1]\left(\frac{dN^B}{dx}\right) + \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left(\frac{dN^S}{dx}\right) = 0$$

$$[0]\left(\frac{dA^B}{dx}\right) + \left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right]\left(\frac{dA^S}{dx}\right) + \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right]\left(\frac{dN^B}{dx}\right) - [1]\left(\frac{dN^S}{dx}\right) = 0$$

By Cramer's Rule, we have $\frac{dA^B}{dx} = \frac{D_1}{D}$ and $\frac{dA^S}{dx} = \frac{D_2}{D}$ where

$D =$

$$\begin{bmatrix} 1 & -\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] \\ \left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & -1 & -\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 0 \\ \left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & 0 & -1 & \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & \left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -1 \end{bmatrix}$$

$$D = -\left[1-\left\{\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right\}\right]\left[1-\right.$$

$$\left.\left\{\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\right]-\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right]\left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right]\left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right]-$$

$$\left\{\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]-$$

$$\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left\{\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\left[-\left\{\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\eta^B A^B+\eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}+\right.$$

$$\left.\left\{\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\right]$$

Now $N^B \gg 1,\ \beta < 1,\ A^B \leq 1,\ \delta^S < 1,\ \eta^S < 1,\ \eta^B < 1$. A^B is small at steady state as ad on each buyer is less given the large number of buyers. $1 - a\ fraction > 0$, hence the first term is negative if $\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2} < 1, \frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2} < 1, \frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}} < 1, \frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}} < 1$ which is plausible given $\eta^B, \eta^S < 1, \delta^B, \delta^S < 1, P^S > P^B > c, N^B > N^S \gg 1, \beta < 1, A^B, A^S \leq 1$. The second and third terms are negative. The fourth term is negative as as $\eta^B A^B + \eta^S\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right] < 0$ (as discussed earlier), hence $D < 0$.

$D_1 =$

$$\begin{bmatrix} \left[\frac{1}{2\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & -\left[\frac{\eta^S(\overline{N^S}-N^S)}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] \\ 0 & -1 & -\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 0 \\ 0 & 0 & -1 & \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & \left[\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -1 \end{bmatrix}$$

$$D_1 = -\left[\frac{1}{2\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\left[\left\{1-\left(\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right)\left(\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right)\right\}+\right.$$

$$\left.\left\{\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right\}\left\{\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right\}\left\{\frac{\overline{N^S}\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right\}\right]$$

$1 - a\ fraction > 0.$ $\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2} < 1, \frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2} < 1,$ which is plausible given $\eta^B, \eta^S < 1, \delta^B, \delta^S < 1, P^S > P^B > c, N^B > N^S \gg 1, \beta < 1, A^B, A^S \leq 1$. Hence, $\left\{1-\left(\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right)\left(\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right)\right\} > 0.$

Hence, $D_1 < 0$. Hence $\frac{dA^B}{dx} = \frac{D_1}{D} > 0$. But $\frac{P^B}{c} = x$. Hence $\frac{dA^B}{dP^B} > 0$ and $\frac{dA^B}{dc} < 0$.

$D_2 =$

$$\begin{bmatrix} 1 & \left[\frac{1}{2\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] & 0 & \left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)(\overline{N^S}-N^S)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] \\ \left[\frac{\eta^B(\overline{N^B}-N^B)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 0 & -\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 0 \\ \left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & 0 & -1 & \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & 0 & \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -1 \end{bmatrix}$$

$D_2 =$

$$-\left[\frac{1}{2\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\begin{bmatrix} \left[\frac{\eta^B\left(\overline{N^B}-N^B\right)}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & -\left[\frac{\eta^B A^B+\eta^S A^S}{\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)\left(\overline{N^B}-N^B\right)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}\right] & 0 \\ \left[\frac{\overline{N^B}\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] & -1 & \left[\frac{\overline{N^B}\eta^B\delta^B}{(A^B+\eta^B N^S+\delta^B)^2}\right] \\ 0 & \left[\frac{\overline{N^S}\eta^S\delta^S}{(A^S+\eta^S N^B+\delta^S)^2}\right] & -1 \end{bmatrix}$$

$$D_2 = -\left[\frac{1}{2\left[\left(\frac{P^B}{c}\right)+2\eta^S(A^S)\left(\overline{N^S}-N^S\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right]\frac{\eta^B(\overline{N^B}-N^B)(A^B+\eta^B N^S+\delta^B)^2(A^S+\eta^S N^B+\delta^S)^2-\eta^B(\overline{N^B}-N^B)\overline{N^B}\eta^B\delta^B\overline{N^S}\eta^S\delta^S-\overline{N^B}\delta^B(\eta^B A^B+\eta^S A^S)(A^S+\eta^S N^B+\delta^S)^2}{(A^B+\eta^B N^S+\delta^B)^2(A^S+\eta^S N^B+\delta^S)^2\left[\left(\frac{P^S}{c}\right)+2\eta^B(A^B)(\overline{N^B}-N^B)+\left[(1-\delta^S)-\eta^S N^B-\frac{1}{\beta}\right]^2\right]^{\frac{1}{2}}}$$

Consider the expression

$\eta^B\left(\overline{N^B}-N^B\right)(A^B+\eta^B N^S+\delta^B)^2(A^S+\eta^S N^B+\delta^S)^2-\eta^B\left(\overline{N^B}-N^B\right)\overline{N^B}\eta^B\delta^B\overline{N^S}\eta^S\delta^S-$

$\overline{N^B}\delta^B(\eta^B A^B+\eta^S A^S)(A^S+\eta^S N^B+\delta^S)^2$ Now

$N^B=\frac{\overline{N^B}(A^B+\eta^B N^S)}{[A^B+\eta^B N^S+\delta^B]}$, $N^S=\frac{\overline{N^S}(A^S+\eta^S N^B)}{[A^S+\eta^S N^B+\delta^S]}$, hence $\overline{N^B}=\frac{N^B(A^B+\eta^B N^S+\delta^B)}{(A^B+\eta^B N^S)}$ and $\overline{N^S}=$

$\frac{N^S(A^S+\eta^S N^B+\delta^S)}{(A^S+\eta^S N^B)}$. $\overline{N^B}-N^B=\frac{N^B(A^B+\eta^B N^S+\delta^B)}{(A^B+\eta^B N^S)}-N^B=\frac{\delta^B N^B}{(A^B+\eta^B N^S)}$. Hence the expression is

$\eta^B\frac{\delta^B N^B}{(A^B+\eta^B N^S)}(A^B+\eta^B N^S+\delta^B)^2(A^S+\eta^S N^B+\delta^S)^2-$

$\eta^B\frac{\delta^B N^B}{(A^B+\eta^B N^S)}\frac{N^B(A^B+\eta^B N^S+\delta^B)}{(A^B+\eta^B N^S)}\eta^B\delta^B\frac{N^S(A^S+\eta^S N^B+\delta^S)}{(A^S+\eta^S N^B)}\eta^S\delta^S-\frac{N^B(A^B+\eta^B N^S+\delta^B)}{(A^B+\eta^B N^S)}\delta^B(\eta^B A^B+$

$\eta^S A^S)(A^S+\eta^S N^B+\delta^S)^2$ i.e.

$\frac{\delta^B N^B N^B N^S(A^B+\eta^B N^S+\delta^B)(A^S+\eta^S N^B+\delta^S)}{(A^B+\eta^B N^S)}\Big[\eta^B\left(\frac{A^B}{N^S}+\eta^B+\frac{\delta^B}{N^S}\right)\left(\frac{A^S}{N^B}+\eta^S+\frac{\delta^S}{N^B}\right)-$

$\eta^B\frac{\eta^B\delta^B}{(A^B+\eta^B N^S)}\frac{\eta^S\delta^S}{(A^S+\eta^S N^B)}-\left(\frac{\eta^B A^B}{N^S}+\frac{\eta^S A^S}{N^S}\right)\left(\frac{A^S}{N^B}+\eta^S+\frac{\delta^S}{N^B}\right)\Big]$

Now $\frac{A^B}{N^S}\approx 0$, $\frac{\delta^B}{N^S}\approx 0$, $\frac{A^S}{N^B}\approx 0$, $\frac{\delta^S}{N^B}\approx 0$, $\frac{\eta^B A^B}{N^S}\approx 0$, $\frac{\eta^S A^S}{N^S}\approx 0$

as $A^B\le 1$, $A^S\le 1$, $\delta^B<1$, $\delta^S<1$, $\eta^S<\eta^B<1$, $N^B\gg 1$, $N^S\gg 1$.

Hence the expression is approximately

$$\frac{\delta^B N^B N^B N^S \eta^B \eta^S (A^B+\eta^B N^S+\delta^B)(A^S+\eta^S N^B+\delta^S)}{(A^B+\eta^B N^S)}\left[\eta^B - \frac{\eta^B \delta^B}{(A^B+\eta^B N^S)}\frac{\delta^S}{(A^S+\eta^S N^B)}\right] \text{ i.e.}$$

$$\frac{\delta^B N^B N^B N^S \eta^B \eta^S (A^B+\eta^B N^S+\delta^B)(A^S+\eta^S N^B+\delta^S)}{(A^B+\eta^B N^S)}\left[\eta^B - \frac{\frac{\eta^B \delta^B}{N^S}}{\left(\frac{A^B}{N^S}+\eta^B\right)}\frac{\frac{\delta^S}{N^B}}{\left(\frac{A^S}{N^B}+\eta^S\right)}\right]$$

But $\frac{\eta^B \delta^B}{N^S} \approx 0, \frac{\delta^S}{N^B} \approx 0, \frac{A^B}{N^S} \approx 0, \frac{A^S}{N^B} \approx 0$ as $A^B \le 1,\ A^S \le 1,\ \delta^B < 1,\ \delta^S < 1,\ \eta^S < \eta^B < 1,$ $N^B \gg 1,\ N^S \gg 1.$ Hence the expression reduces to approximately

$$\frac{\delta^B N^B N^B N^S \eta^B \eta^S (A^B+\eta^B N^S+\delta^B)(A^S+\eta^S N^B+\delta^S)}{(A^B+\eta^B N^S)}[\eta^B] > 0.$$

Hence $D_2 < 0$. Hence $\frac{dA^S}{dx} = \frac{D_2}{D} > 0$. But $\frac{P^B}{c} = x$. Hence $\frac{dA^S}{dP^B} > 0$ and $\frac{dA^S}{dc} < 0$.

Similarly, $\frac{dA^S}{dP^S} > 0,\ \frac{dA^B}{dP^S} > 0$ and $\frac{dA^B}{dc} < 0, \frac{dA^S}{dc} < 0$.

Q.E.D.

Proof of Proposition 3

$$Max_{A_t^B} \sum_0^T \beta^t \{P^B N_t^B + P_t^S N^S - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2\}\ s.t.$$

$$N_{t+1}^B = (1-\delta^B)N_t^B + (\overline{N^B} - N_t^B)(A_t^B + \eta^B N^S) \text{ and } A_t^B \ge 0$$

The Bellman Equation is –

$$V(N_t^B) = Max_{A_t^B}\{P^B N_t^B + P^S N^S - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2 + \beta V(N_{t+1}^B) + \lambda_t\big[(1-\delta^B)N_t^B +$$
$$\left(\overline{N^B} - N_t^B\right)(A_t^B + \eta^B N^S) - N_{t+1}^B\big] + \rho_t A_t^B\}$$

where λ, ρ are Lagrange multipliers and V is the value function.

F.O.C. w.r.t. A_t^B –

$-2c\left(\overline{N^B} - N_t^B\right)(A_t^B) + \lambda_t\left(\overline{N^B} - N_t^B\right) + \rho_t = 0$ which gives

$\lambda_t = 2c\, A_t^B - \frac{\rho_t}{(\overline{N^B} - N_t^B)}$

F.O.C. w.r.t. N_{t+1}^B –

$\beta V'(N_{t+1}^B) = \lambda_t$

F.O.C. w.r.t. λ_t –

$(1-\delta^B)N_t^B + \left(\overline{N^B} - N_t^B\right)(A_t^B + \eta^B N^S) = N_{t+1}^B$

Envelope condition w.r.t. N_t^B–

$\frac{\partial V}{\partial N_t^B} = V'_{N_t^B}(N_t^B) = P^B + c\,(A_t^B)^2 + \lambda_t[(1-\delta^B) - (A_t^B + \eta^B N^S)]$

Update one period in the future envelope condition w.r.t. N_t^B–

$V'(N_{t+1}^B) = P^B + c\,(A_{t+1}^B)^2 + \lambda_{t+1}[(1-\delta^B) - (A_{t+1}^B + \eta^B N^S)]$

The Euler equation for buyers is then –

$P^B + c\,(A_{t+1}^B)^2 + \lambda_{t+1}[(1-\delta^B) - (A_{t+1}^B + \eta^B N^S)] = \frac{\lambda_t}{\beta}$

Since $\lambda_t = 2c\, A_t^B - \frac{\rho_t}{(\overline{N^B} - N_t^B)}$, the Euler equation for buyers is –

$P^B + c\,(A_{t+1}^B)^2 + \left[2c\,(A_{t+1}^B) - \frac{\rho_{t+1}}{(\overline{N^B} - N_{t+1}^B)}\right][(1-\delta^B) - (A_{t+1}^B + \eta^B N^S)] = \frac{2c\left(A_t^B\right)}{\beta} - \frac{\rho_t}{\beta\left(\overline{N^B} - N_t^B\right)}$

This gives the Euler equation for buyers as –

$$P^B - c\,(A_{t+1}^B)^2 + 2c\,(A_{t+1}^B)[(1-\delta^B) - \eta^B N^S] + \left[\frac{\rho_t}{\beta(\overline{N^B} - N_t^B)} - \frac{\rho_{t+1}}{(\overline{N^B} - N_{t+1}^B)}\{(1-\delta^B) - (A_{t+1}^B + \eta^B N^S)\}\right] = \frac{2c\,(A_t^B)}{\beta} \quad \text{i.e.}$$

$$\left(\frac{\beta}{2}\right)\left(\frac{P^B}{c}\right) - \left(\frac{\beta}{2}\right)(A_{t+1}^B)^2 + \beta(A_{t+1}^B)[(1-\delta^B) - \eta^B N^S] + \left(\frac{\beta}{2c}\right)\left[\frac{\rho_t}{\beta(\overline{N^B} - N_t^B)} - \frac{\rho_{t+1}}{(\overline{N^B} - N_{t+1}^B)}\{(1-\delta^B) - (A_{t+1}^B + \eta^B N^S)\}\right] = A_t^B \quad \text{i.e.}$$

$$(A_{t+1}^B)^2 - 2(A_{t+1}^B)[(1-\delta^B) - \eta^B N^S] = \left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right)A_t^B + \left(\frac{1}{c}\right)\left[\frac{\rho_t}{\beta(\overline{N^B} - N_t^B)} - \frac{\rho_{t+1}}{(\overline{N^B} - N_{t+1}^B)}\{(1-\delta^B) - (A_{t+1}^B + \eta^B N^S)\}\right]$$

When $A_t^B > 0, \rho_t = \rho_{t+1} = 0$.

Hence we have –

$$\left(\frac{\beta}{2}\right)\left(\frac{P^B}{c}\right) - \left(\frac{\beta}{2}\right)(A_{t+1}^B)^2 + \beta(A_{t+1}^B)[(1-\delta^B) - \eta^B N^S] = A_t^B \quad \text{i.e.}$$

$$(A_{t+1}^B)^2 - 2(A_{t+1}^B)[(1-\delta^B) - \eta^B N^S] = \left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right)A_t^B \text{ i.e.}$$

$$(A_{t+1}^B)^2 - 2(A_{t+1}^B)[(1-\delta^B) - \eta^B N^S] + [(1-\delta^B) - \eta^B N^S]^2 = \left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right)A_t^B + [(1-\delta^B) - \eta^B N^S]^2 \quad \text{i.e.}$$

$$A_{t+1}^B = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right)A_t^B + [(1-\delta^B) - \eta^B N^S]^2\right]^{\frac{1}{2}} + [(1-\delta^B) - \eta^B N^S]$$

Q.E.D.

Proof of Proposition 4

$$A_{t+1}^B = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A_t^B + [(1-\delta^B) - \eta^B N^S]^2\right]^{\frac{1}{2}} + [(1-\delta^B) - \eta^B N^S]$$

$A_{t+1}^B = A_t^B = A^B,\ \ N_{t+1}^B = N_t^B = N^B$ at steady state. Hence

$$A^B = \left[\left(\frac{P^B}{c}\right) - \left(\frac{2}{\beta}\right) A^B + [(1-\delta^B) - \eta^B N^S]^2\right]^{\frac{1}{2}} + [(1-\delta^B) - \eta^B N^S] \text{ i.e.}$$

$$(A^B)^2 - 2\left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right] A^B = \left(\frac{P^B}{c}\right) \text{ i.e.}$$

$$A^B = \left[\left(\frac{P^B}{c}\right) + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]^2\right]^{\frac{1}{2}} + \left[(1-\delta^B) - \frac{1}{\beta} - \eta^B N^S\right]$$

Also $N_{t+1}^B = (1-\delta^B)N_t^B + (\overline{N^B} - N_t^B)(A_t^B + \eta^B N_t^S)$ In steady state

$$N^B = (1-\delta^B)N^B + (\overline{N^B} - N^B)(A^B + \eta^B N^S) \text{ i.e. } N^B = \frac{\overline{N^B}(A^B + \eta^B N^S)}{A^B + \eta^B N^S + \delta^B}$$

$$\pi_{platform} = P^B N_t^B + P^S N_t^S - c\left(\overline{N^B} - N_t^B\right)(A_t^B)^2 \qquad \text{In steady state}$$

$$\pi_{ss} = P^B N^B + P^S N^S - c\left(\overline{N^B} - N^B\right)(A^B)^2 \qquad \text{Substituting value of } N^B = \frac{\overline{N^B}(A^B + \eta^B N^S)}{A^B + \eta^B N^S + \delta^B}$$

$$\pi_{ss} = \frac{\overline{N^B}\left[P^B(A^B + \eta^B N^S) - c\delta^B (A^B)^2\right]}{A^B + \eta^B N^S + \delta^B} + P^S N^S$$

Q.E.D.

Proof of Corollary 5

$$\frac{dN^B}{dA^B} = \left(\frac{\partial N^B}{\partial A^B}\right) = \frac{\overline{N^B}\delta^B}{\left(A^B + \eta^B N^S + \delta^B\right)^2} > 0,\ \ \frac{dN^B}{d\overline{N^B}} = \left(\frac{\partial N^B}{\partial \overline{N^B}}\right) = \frac{(A^B + \eta^B N^S)}{A^B + \eta^B N^S + \delta^B} > 0,$$

$$\frac{dN^B}{d\eta^B} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{\partial \eta^B}\right) + \left(\frac{\partial N^B}{\partial \eta^B}\right) = -N^S\left[\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left[\frac{\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]}{\left[\left(\frac{P^B}{c}\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] > 0 \text{ as}$$

$$\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right] < 0 \quad (\text{note } \beta < 1)$$

$$\frac{dN^B}{d\delta^B} = \left(\frac{\partial N^B}{\partial A^B}\right)\left(\frac{dA^B}{\partial \delta^B}\right) + \left(\frac{\partial N^B}{\partial \delta^B}\right) = -\left[\frac{\overline{N^B}\delta^B}{\left(A^B+\eta^B N^S+\delta^B\right)^2}\right]\left[\frac{A^B}{\left[\left(\frac{P^B}{c}\right)+\left[(1-\delta^B)-\frac{1}{\beta}-\eta^B N^S\right]^2\right]^{\frac{1}{2}}}\right] - \frac{\overline{N^B}\left(A^B+\eta^B N^S\right)}{\left(A^B+\eta^B N^S+\delta^B\right)^2} < 0$$

Q.E.D.

Proof of Corollary 6

$$\frac{dA^B}{d\delta^B} = \frac{\partial A^B}{\partial \delta^B} = -\left[\frac{\left\{\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+\left\{\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right\}^2\right]^{\frac{1}{2}}}+1\right] = -\frac{A^B}{\left[\left(\frac{P^B}{c}\right)+\left\{\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right\}^2\right]^{\frac{1}{2}}} < 0\,.$$

$$\frac{dA^B}{d\,\eta^B} = \frac{\partial A^B}{\partial\,\eta^B} = -N^S\left[\frac{\left\{\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right\}}{\left[\left(\frac{P^B}{c}\right)+\left\{\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right\}^2\right]^{\frac{1}{2}}}+1\right] = -\frac{A^B N^S}{\left[\left(\frac{P^B}{c}\right)+\left\{\left(1-\delta^B\right)-\frac{1}{\beta}-\eta^B N^S\right\}^2\right]^{\frac{1}{2}}} < 0\,.$$

Q.E.D.

Proof of Proposition 5

Given: $N_0^B, N_0^S, A_2^B = 0, A_2^S = 0, N_2^B \leq \overline{N^B}, N_2^S \leq \overline{N^S}$

Note that choosing promotion on users in a period is also equivalent to choosing the number of users in the next period. The platform's problem can then be defined as

$Max_{A_0^B,\ A_0^S,\ A_1^B,\ A_1^S,\ N_1^B,\ N_1^S,\ N_2^B,\ N_2^S} \{P^B N_0^B + P^S N_0^S - c\left(\overline{N^B} - N_0^B\right)(A_0^B)^2 - c\left(\overline{N^S} - N_0^S\right)(A_0^S)^2\} + \beta\{P^B N_1^B + P^S N_1^S - c\left(\overline{N^B} - N_1^B\right)(A_1^B)^2 - c\left(\overline{N^S} - N_1^S\right)(A_1^S)^2\} + \beta^2\{P^B N_2^B + P^S N_2^S\}\ s.t.$

$N_2^B = (1-\delta^B)N_1^B + (\overline{N^B} - N_1^B)(A_1^B + \eta^B N_1^S)$ and $N_2^S = (1-\delta^S)N_1^S + (\overline{N^S} - N_1^S)(A_1^S + \eta^S N_1^B)$

$N_1^B = (1-\delta^B)N_0^B + (\overline{N^B} - N_0^B)(A_0^B + \eta^B N_0^S)$ and $N_1^S = (1-\delta^S)N_0^S + (\overline{N^S} - N_0^S)(A_0^S + \eta^S N_0^B)$

Assuming interior solutions, the Lagrangian can then be written as

$L = \{P^B N_0^B + P^S N_0^S - c\left(\overline{N^B} - N_0^B\right)(A_0^B)^2 - c\left(\overline{N^S} - N_0^S\right)(A_0^S)^2\} + \beta\{P^B N_1^B + P^S N_1^S - c\left(\overline{N^B} - N_1^B\right)(A_1^B)^2 - c\left(\overline{N^S} - N_1^S\right)(A_1^S)^2\} + \beta^2\{P^B N_2^B + P^S N_2^S\} + \lambda_1\{(1-\delta^B)N_0^B + \left(\overline{N^B} - N_0^B\right)(A_0^B + \eta^B N_0^S) - N_1^B\} + \lambda_2\{(1-\delta^S)N_0^S + \left(\overline{N^S} - N_0^S\right)(A_0^S + \eta^S N_0^B) - N_1^S\} + \lambda_3\{(1-\delta^B)N_1^B + \left(\overline{N^B} - N_1^B\right)(A_1^B + \eta^B N_1^S) - N_2^B\} + \lambda_4\{(1-\delta^S)N_1^S + \left(\overline{N^S} - N_1^S\right)(A_1^S + \eta^S N_1^B) - N_2^S\}$ where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are Lagrange multipliers.

$\frac{\partial L}{\partial A_0^B} = -2c\left(\overline{N^B} - N_0^B\right)A_0^B + \lambda_1\left(\overline{N^B} - N_0^B\right) = 0$ i.e. $\lambda_1 = 2cA_0^B$

$\frac{\partial L}{\partial A_0^S} = -2c\left(\overline{N^S} - N_0^S\right)A_0^S + \lambda_2\left(\overline{N^S} - N_0^S\right) = 0$ i.e. $\lambda_2 = 2cA_0^S$

$\frac{\partial L}{\partial A_1^B} = -2\beta c\left(\overline{N^B} - N_1^B\right)A_1^B + \lambda_3\left(\overline{N^B} - N_1^B\right) = 0$ i.e. $\lambda_3 = 2\beta cA_1^B$

$\frac{\partial L}{\partial A_1^S} = -2\beta c\left(\overline{N^S} - N_1^S\right)A_1^S + \lambda_4\left(\overline{N^S} - N_1^S\right) = 0$ i.e. $\lambda_4 = 2\beta cA_1^S$

$\frac{\partial L}{\partial N_1^B} = \beta P^B + \beta c(A_1^B)^2 - \lambda_1 + \lambda_3\{(1-\delta^B) - (A_1^B + \eta^B N_1^S)\} + \lambda_4 \eta^S\left(\overline{N^S} - N_1^S\right) = 0$ i.e.

$\beta P^B + \beta c(A_1^B)^2 - 2cA_0^B + 2\beta c A_1^B\{(1-\delta^B) - (A_1^B + \eta^B N_1^S)\} + 2\beta c A_1^S \eta^S\left(\overline{N^S} - N_1^S\right) = 0$ i.e.

$$A_1^B = \left[\frac{P^B}{c} - \frac{2}{\beta}A_0^B + 2A_1^S\eta^S\left(\overline{N^S} - N_1^S\right) + \{(1-\delta^B) - \eta^B N_1^S\}^2\right]^{\frac{1}{2}} + \{(1-\delta^B) - \eta^B N_1^S\}$$

Similarly, simplifying $\frac{\partial L}{\partial N_1^S}$ we get

$$A_1^S = \left[\frac{P^S}{c} - \frac{2}{\beta}A_0^S + 2A_1^B\eta^B\left(\overline{N^B} - N_1^B\right) + \{(1-\delta^S) - \eta^S N_1^B\}^2\right]^{\frac{1}{2}} + \{(1-\delta^S) - \eta^S N_1^B\}$$

$\frac{\partial L}{\partial N_2^B} = \beta^2 P^B - \lambda_3 = 0$ i.e. $\lambda_3 = \beta^2 P^B$. But earlier, we found $\lambda_3 = 2\beta c A_1^B$. Hence $\beta^2 P^B = 2\beta c A_1^B$ i.e. $A_1^B = \frac{\beta P^B}{2c}$. Similarly $A_1^S = \frac{\beta P^S}{2c}$.

Using the value of $A_1^B = \frac{\beta P^B}{2c}$ and $A_1^S = \frac{\beta P^S}{2c}$ in the equation linking A_0^B and A_1^B we have

$(A_1^B)^2 - 2A_1^B\{(1-\delta^B) - \eta^B N_1^S\} = \frac{P^B}{c} - \frac{2}{\beta}A_0^B + 2A_1^S\eta^S\left(\overline{N^S} - N_1^S\right)$ i.e.

$$(\eta^B P^B + \eta^S P^S)N_1^S + \frac{2c}{\beta^2}A_0^B = \frac{P^B}{\beta} + \eta^S\overline{N^S}P^S + (1-\delta^B)P^B - \frac{\beta}{c}\left(\frac{P^B}{2}\right)^2$$

Using the value of $A_1^B = \frac{\beta P^B}{2c}$ and $A_1^S = \frac{\beta P^S}{2c}$ in the equation linking A_0^S and A_1^S we have

$(A_1^S)^2 - 2A_1^S\{(1-\delta^S) - \eta^S N_1^B\} = \frac{P^S}{c} - \frac{2}{\beta}A_0^S + 2A_1^B\eta^B\left(\overline{N^B} - N_1^B\right)$ i.e.

$$(\eta^B P^B + \eta^S P^S)N_1^B + \frac{2c}{\beta^2}A_0^S = \frac{P^S}{\beta} + \eta^B\overline{N^B}P^B + (1-\delta^S)P^S - \frac{\beta}{c}\left(\frac{P^S}{2}\right)^2$$

Now we have $N_2^B = (1-\delta^B)N_1^B + (\overline{N^B} - N_1^B)(A_1^B + \eta^B N_1^S)$ and

$N_2^S = (1-\delta^S)N_1^S + (\overline{N^S} - N_1^S)(A_1^S + \eta^S N_1^B)$ and

$N_1^B = (1-\delta^B)N_0^B + (\overline{N^B} - N_0^B)(A_0^B + \eta^B N_0^S)$ and $N_1^S = (1-\delta^S)N_0^S + (\overline{N^S} - N_0^S)(A_0^S + \eta^S N_0^B)$

Using the value of N_1^S in the equation for A_0^B we have

$(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + (\overline{N^S} - N_0^S)(A_0^S + \eta^S N_0^B)\right] + \frac{2c}{\beta^2}A_0^B = \frac{P^B}{\beta} + \eta^S \overline{N^S} P^S + (1-\delta^B)P^B - \frac{\beta}{c}\left(\frac{P^B}{2}\right)^2$ i.e.

$2cA_0^B + \beta^2(\eta^B P^B + \eta^S P^S)\left(\overline{N^S} - N_0^S\right)A_0^S = \beta P^B + \beta^2 \eta^S \overline{N^S} P^S + \beta^2(1-\delta^B)P^B - \frac{\beta^3}{c}\left(\frac{P^B}{2}\right)^2 - \beta^2(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]$ ……………………………………………. (m)

Using the value of N_1^B in the equation for A_0^S we have

$(\eta^B P^B + \eta^S P^S)\left[(1-\delta^B)N_0^B + (\overline{N^B} - N_0^B)(A_0^B + \eta^B N_0^S)\right] + \frac{2c}{\beta^2}A_0^S = \frac{P^S}{\beta} + \eta^B \overline{N^B} P^B + (1-\delta^S)P^S - \frac{\beta}{c}\left(\frac{P^S}{2}\right)^2$ i.e.

$\beta^2(\eta^B P^B + \eta^S P^S)\left(\overline{N^B} - N_0^B\right)A_0^B + 2cA_0^S = \beta P^S + \beta^2 \eta^B \overline{N^B} P^B + \beta^2(1-\delta^S)P^S - \frac{\beta^3}{c}\left(\frac{P^S}{2}\right)^2 - \beta^2(\eta^B P^B + \eta^S P^S)\left[(1-\delta^B)N_0^B + \eta^B\left(\overline{N^B} - N_0^B\right)N_0^S\right]$ ……………………………………………. (n)

We can then solve for A_0^B, A_0^S.

Multiplying equation (m) by $\beta^2\left(\overline{N^B} - N_0^B\right)(P^B\eta^B + P^S\eta^S)$ and equation (n) by $2c$, and subtracting (n) from (m), we have –

$$\left[\beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right) - 4c^2\right]A_0^S = \beta^2\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S)\left[\beta P^B + \beta^2\eta^S\overline{N^S}P^S + \beta^2(1-\delta^B)P^B - \frac{\beta^3}{c}\left(\frac{P^B}{2}\right)^2 - \beta^2(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]\right] - 2c\left[\beta P^S + \beta^2\eta^B\overline{N^B}P^B + \beta^2(1-\delta^S)P^S - \frac{\beta^3}{c}\left(\frac{P^S}{2}\right)^2 - \beta^2(\eta^B P^B + \eta^S P^S)\left[(1-\delta^B)N_0^B + \eta^B\left(\overline{N^B} - N_0^B\right)N_0^S\right]\right]$$ i.e.

$$A_0^S = \frac{\beta^3\left(\overline{N^B}-N_0^B\right)(\eta^B P^B+\eta^S P^S)\left[P^B+\beta\eta^S\overline{N^S}P^S+\beta(1-\delta^B)P^B-\frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2-\beta(\eta^B P^B+\eta^S P^S)\left[(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right]\right]-2c\beta\left[P^S+\beta\eta^B\overline{N^B}P^B+\beta(1-\delta^S)P^S-\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-\beta(\eta^B P^B+\eta^S P^S)\left[(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right]\right]}{\left[\beta^4(\eta^B P^B+\eta^S P^S)^2\left(\overline{N^B}-N_0^B\right)\left(\overline{N^S}-N_0^S\right) - 4c^2\right]}$$

Multiplying equation (m) by $2c$ and equation (n) by $\beta^2\left(\overline{N^S} - N_0^S\right)(P^B\eta^B + P^S\eta^S)$, and subtracting (n) from (m), we have

$$\left[4c^2 - \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\right]A_0^B = 2c\beta\left[P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]\right] - \beta^3\left(\overline{N^S} - N_0^S\right)(P^B\eta^B + P^S\eta^S)\left[P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^B)N_0^B + \eta^B\left(\overline{N^B} - N_0^B\right)N_0^S\right]\right]$$ i.e.

$$A_0^B = \frac{2c\beta\left[P^B+\beta\eta^S\overline{N^S}P^S+\beta(1-\delta^B)P^B-\frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2-\beta(\eta^B P^B+\eta^S P^S)\left[(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right]\right]-\beta^3\left(\overline{N^S}-N_0^S\right)(P^B\eta^B+P^S\eta^S)\left[P^S+\beta\eta^B\overline{N^B}P^B+\beta(1-\delta^S)P^S-\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-\beta(\eta^B P^B+\eta^S P^S)\left[(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right]\right]}{\left[4c^2 - \beta^4(\eta^B P^B+\eta^S P^S)^2\left(\overline{N^B}-N_0^B\right)\left(\overline{N^S}-N_0^S\right)\right]}$$

Q.E.D.

Proof of Corollary 7

$A_0^B = \frac{D_1}{D}$, $A_0^S = -\frac{D_2}{D}$ where

$$D_1 = 2c\beta\left[P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]\right] - \beta^3\left(\overline{N^S} - N_0^S\right)(P^B\eta^B + P^S\eta^S)\left[P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^B)N_0^B + \eta^B\left(\overline{N^B} - N_0^B\right)N_0^S\right]\right]$$

$$D_2 = \beta^3\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S)\left[P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]\right] - 2c\beta\left[P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^B)N_0^B + \eta^B\left(\overline{N^B} - N_0^B\right)N_0^S\right]\right]$$

$$D = \left[4c^2 - \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\right]$$

Suppose $\mathrm{D} > 0$. Then since $\mathrm{A}_0^\mathrm{B} > 0, \mathrm{A}_0^\mathrm{S} > 0$, $A_0^B = \frac{D_1}{D}$, $A_0^S = -\frac{D_2}{D}$, $\mathrm{D}_1 > 0$ and $D_2 < 0$.

But if $\mathrm{D} > 0$, $c > \frac{\left[\beta^2(\eta^B P^B + \eta^S P^S)\sqrt{\left[\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\right]}\right]}{2}$.

This is not possible as $\overline{N^B} \gg N_0^B$, $\overline{N^S} \gg N_0^S$, $\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)$ is a very large number. Consequently, $\frac{\left[\beta^2(\eta^B P^B + \eta^S P^S)\sqrt{\left[\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\right]}\right]}{2}$ becomes large. The unit cost of promotion c cannot be that large. Hence $D < 0$. Since $\mathrm{A}_0^\mathrm{B} > 0$ and $\mathrm{A}_0^\mathrm{S} > 0, \mathrm{D}_1 < 0$ and $D_2 > 0$.

Since $\mathrm{D} < 0$, $\mathrm{c} < \frac{\left[\beta^2(\eta^B P^B+\eta^S P^S)\sqrt{[(\overline{N^B}-N_0^B)(\overline{N^S}-N_0^S)]}\right]}{2}$. Let

$X = P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\big[(1-\delta^S)N_0^S +$

$\eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\big]$ and

$Y = P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\big[(1-\delta^B)N_0^B +$

$\eta^B\left(\overline{N^B} - N_0^B\right)N_0^S\big]$. Then we have

$D_2 = \beta^3\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S)X - 2c\beta Y > 0$ and

$D_1 = 2c\beta X - \beta^3\left(\overline{N^S} - N_0^S\right)(P^B\eta^B + P^S\eta^S)Y < 0$

Given that $\frac{P^B}{P^S} < 1$, $N_0^B \gg 1$, $N_0^S \gg 1$, $\eta^B < 1$, $\eta^S < 1$, $N_0^B \ll \overline{N^B}$, $N_0^S \ll \overline{N^S}$, $\delta^B < 1$, $\delta^S <$ 1, $\beta < 1$, $P^B < P^S$, $\overline{N^B} \gg \overline{N^S}$, $N_0^B \gg N_0^S$, and η^B, η^S being relatively large,

it is reasonbale to assume that $X < 0$ and $Y < 0$.

$\frac{\partial A_0^B}{\partial \overline{N^B}} = \frac{[-\beta^3(\overline{N^S}-N_0^S)(P^B\eta^B+P^S\eta^S)\{\beta\eta^B P^B-\beta\eta^B(\eta^B P^B+\eta^S P^S)N_0^S\}]D+\beta^4(\eta^B P^B+\eta^S P^S)^2(\overline{N^S}-N_0^S)D_1}{\left[4c^2-\beta^4(\eta^B P^B+\eta^S P^S)^2(\overline{N^B}-N_0^B)(\overline{N^S}-N_0^S)\right]^2} =$

$\frac{D_3}{\left[4c^2-\beta^4(\eta^B P^B+\eta^S P^S)^2(\overline{N^B}-N_0^B)(\overline{N^S}-N_0^S)\right]^2}$ where the denominator is positive and

$D_3 = -\beta^3\left(\overline{N^S} - N_0^S\right)(P^B\eta^B + P^S\eta^S)\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\}D + \beta^4(\eta^B P^B +$ $\eta^S P^S)^2\left(\overline{N^S} - N_0^S\right)D_1$

Now $\mathrm{D} < 0$, $\mathrm{D}_1 < 0$, $\mathrm{A}_0^{\mathrm{B}} > 0$. Now we have

$\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\} = \beta\eta^B\{P^B - (\eta^B P^B + \eta^S P^S)N_0^S\} = \beta\eta^B\{P^B(1-\eta^B N_0^S) - \eta^S P^S N_0^S\}$. Now $P^B(1-\eta^B N_0^S) - \eta^S P^S N_0^S < 0$ if $\eta^B N_0^S > 1$ which is plausible given η^B is relatively large and $N_0^S >> 1$. Since $D < 0$, we have

$-\beta^3\left(\overline{N^S} - N_0^S\right)(P^B\eta^B + P^S\eta^S)\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\}D < 0$ if $\eta^B N_0^S > 1$. Now

$D_3 = -\beta^3\left(\overline{N^S} - N_0^S\right)(P^B\eta^B + P^S\eta^S)\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\}D + \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^S} - N_0^S\right)D_1$

The first term is less than zero and the second term is also less than zero since $\mathrm{D}_1 < 0$. Hence $D_3 < 0$. Hence $\frac{\partial A_0^B}{\partial \overline{N^B}} < 0$ if $\eta^B N_0^S > 1$.

$$\frac{\partial A_0^S}{\partial \overline{N^B}} = -\frac{\left[\beta^3(\eta^B P^B + \eta^S P^S)\left\{P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]\right\} - 2c\beta\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\}\right](D) + \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^S} - N_0^S\right)D_2}{\left[\beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right) - 4c^2\right]^2}$$

The denominator is positive. The numerator is

$\overline{D}_3 = \left[\beta^3(\eta^B P^B + \eta^S P^S)\left\{P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]\right\} - 2c\beta\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\}\right](D) + \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^S} - N_0^S\right)D_2$ (say) Now

$X = P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right] < 0$ and

$Y = P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^B)N_0^B + \eta^B\left(\overline{N^B} - N_0^B\right)N_0^S\right] < 0$ and

$D_2 = \beta^3\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S)X - 2c\beta Y > 0$ and

$D_1 = 2c\beta X - \beta^3\left(\overline{N^S} - N_0^S\right)(\eta^B P^B + \eta^S P^S)Y < 0$ and

$D = \left[4c^2 - \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\right] < 0$

$\overline{D}_3 = \left[\beta^3(\eta^B P^B + \eta^S P^S)\left\{P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\left[(1-\delta^S)N_0^S + \eta^S\left(\overline{N^S} - N_0^S\right)N_0^B\right]\right\} - 2c\beta\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\}\right](D) + \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^S} - N_0^S\right)D_2$ i.e.

$\overline{D}_3 = [\beta^3(\eta^B P^B + \eta^S P^S)X - 2c\beta\{\beta\eta^B P^B - \beta\eta^B(\eta^B P^B + \eta^S P^S)N_0^S\}]\left[4c^2 - \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\right] + \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^S} - N_0^S\right)\left[\beta^3\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S)X - 2c\beta Y\right]$ i.e.

$\overline{D}_3 = 4c^2\beta^3(\eta^B P^B + \eta^S P^S)X - 8c^3\beta^2\eta^B\{P^B - (\eta^B P^B + \eta^S P^S)N_0^S\} + 2c\beta^6\eta^B(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\{P^B - (\eta^B P^B + \eta^S P^S)N_0^S\} - 2c\beta^5(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^S} - N_0^S\right)Y$

$\overline{D}_3 = 2c\beta^2(\eta^B P^B + \eta^S P^S)\left[2c\beta X - \beta^3\left(\overline{N^S} - N_0^S\right)(\eta^B P^B + \eta^S P^S)Y\right] + 2c\beta^2\eta^B\{P^B - (\eta^B P^B + \eta^S P^S)N_0^S\}\left[\beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right) - 4c^2\right]$ i.e.

$\overline{D}_3 = 2c\beta^2(\eta^B P^B + \eta^S P^S)D_1 - 2c\beta^2\eta^B\{P^B - (\eta^B P^B + \eta^S P^S)N_0^S\}D$

But $D < 0$ and $D_1 < 0$ and $P^B - (\eta^B P^B + \eta^S P^S)N_0^S < 0$ (shown earlier).

Hence we have $\overline{D}_3 < 0$. Hence $\frac{\partial A_0^S}{\partial \overline{N^B}} > 0$ if $\eta^B N_0^S > 1$.

Q.E.D.

Proof of Corollary 8

$$\frac{\partial A_0^B}{\partial \delta^B} = \frac{2c\beta[-\beta P^B] - \beta^3(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)[\beta(\eta^B P^B + \eta^S P^S)N_0^B]}{\left[4c^2 - \beta^4(\eta^B P^B + \eta^S P^S)^2(\overline{N^B} - N_0^B)(\overline{N^S} - N_0^S)\right]} = -\frac{2c\beta^2 P^B + \beta^4(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)^2 N_0^B}{D}$$

Since $D < 0$, $\frac{\partial A_0^B}{\partial \delta^B} > 0$.

$$\frac{\partial A_0^B}{\partial \delta^S} = \frac{2c\beta[\beta(\eta^B P^B + \eta^S P^S)N_0^S] + \beta^4(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)P^S}{D}$$ Since $D < 0$, $\frac{\partial A_0^B}{\partial \delta^S} < 0$.

Q.E.D.

Proof of Corollary 9

$\frac{\partial A_0^B}{\partial \eta^B} = \frac{D_4}{D^2}$ where

$$D_4 = \Big[2c\beta\big[-\beta P^B\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}\big] - \beta^3(\overline{N^S} - N_0^S)P^B\Big[P^S + \beta\eta^B\overline{N^B}P^B +$$

$$\beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\}\Big] -$$

$$\beta^3(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)\big[\beta\overline{N^B}P^B - \beta P^B\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\} -$$

$$\beta(\eta^B P^B + \eta^S P^S)(\overline{N^B} - N_0^B)N_0^S\big]\Big]D + 2\beta^4 P^B(\eta^B P^B + \eta^S P^S)(\overline{N^B} - N_0^B)(\overline{N^S} - N_0^S)D_1$$

Let $M = \Bigg[2c\beta\big[-\beta P^B\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}\big] - \beta^3(\overline{N^S} - N_0^S)P^B\Big[P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\}\Big] - \beta^3(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)\big[\beta\overline{N^B}P^B - \beta P^B\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\} - \beta(\eta^B P^B + \eta^S P^S)(\overline{N^B} - N_0^B)N_0^S\big]\Bigg]$ i.e.

$M = -2c\beta^2 P^B\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\} - \beta^3(\overline{N^S} - N_0^S)P^B Y - \beta^4(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)\overline{N^B}P^B + \beta^4(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)P^B\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\} + \beta^4(\overline{N^S} - N_0^S)(\overline{N^B} - N_0^B)N_0^S(P^B\eta^B + P^S\eta^S)^2$ where

$Y = P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\big[(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\big]$

But $Y < 0$, hence $-\beta^3(\overline{N^S} - N_0^S)P^B Y > 0$. So M consists of 3 positive terms and 2 negative terms where we conjecture that

$\beta^4(\overline{N^S} - N_0^S)(\overline{N^B} - N_0^B)N_0^S(P^B\eta^B + P^S\eta^S)^2 > \beta^4(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)\overline{N^B}P^B$ and $\beta^4(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)P^B\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\} > 2c\beta^2 P^B\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}$. Consider the first inequality.

$\beta^4(\overline{N^S} - N_0^S)(\overline{N^B} - N_0^B)N_0^S(P^B\eta^B + P^S\eta^S)^2 > \beta^4(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)\overline{N^B}P^B$ i.e. $\left(1 - \frac{N_0^B}{\overline{N^B}}\right)N_0^S\left(\eta^B + \eta^S\left(\frac{P^S}{P^B}\right)\right) > 1$. This is true as $P^S \gg P^B, N_0^B \ll \overline{N^B}$, and $N_0^S \gg 1$.

Now consider the second inequality.

$\beta^4\left(\overline{N^S}-N_0^S\right)\left(P^B\eta^B+P^S\eta^S\right)P^B\left\{(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right\}>2c\beta^2P^B\left\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right\}$ which reduces to

$$c<\frac{\beta^2\left(P^B\eta^B+P^S\eta^S\right)\left(\overline{N^S}-N_0^S\right)}{2}\left[\frac{\left\{(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right\}}{\left\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right\}}\right].$$

Recall, since $D<0$, $c<\frac{\left[\beta^2\left(\eta^BP^B+\eta^SP^S\right)\sqrt{\left[\left(\overline{N^B}-N_0^B\right)\left(\overline{N^S}-N_0^S\right)\right]}\right]}{2}$. Now if

$$\left[\frac{\left\{(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right\}}{\left\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right\}}\right]\left(\overline{N^S}-N_0^S\right)>\sqrt{\left[\left(\overline{N^B}-N_0^B\right)\left(\overline{N^S}-N_0^S\right)\right]}$$

i.e. if $\left[\frac{\left\{(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right\}}{\left\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right\}}\right]>\sqrt{\left[\frac{\left(\overline{N^B}-N_0^B\right)}{\left(\overline{N^S}-N_0^S\right)}\right]}$ then

$$c<\frac{\beta^2\left(P^B\eta^B+P^S\eta^S\right)\left(\overline{N^S}-N_0^S\right)}{2}\left[\frac{\left\{(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right\}}{\left\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right\}}\right]$$

Hence $M>0$ if the above condition holds. Since $D<0$ and $D_1<0$, and $M>0$, $D_4<0$. Hence

$\frac{\partial A_0^B}{\partial\eta^B}<0$ if $\left[\frac{\left\{(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right\}}{\left\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right\}}\right]>\sqrt{\left[\frac{\left(\overline{N^B}-N_0^B\right)}{\left(\overline{N^S}-N_0^S\right)}\right]}$. Now

$\frac{\partial A_0^B}{\partial\eta^S}=\frac{D_5}{D^2}$ where

$D_5=\Big[2c\beta\left[\beta\overline{N^S}P^S-\beta P^S\left\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\right\}-\beta\left(\eta^BP^B+\eta^SP^S\right)\left(\overline{N^S}-\right.\right.$

$\left.N_0^S\right)N_0^B\Big]-\beta^3P^S\left(\overline{N^S}-N_0^S\right)\Big\{P^S+\beta\eta^B\overline{N^B}P^B+\beta(1-\delta^S)P^S-\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-\beta\left(\eta^BP^B+\right.$

$\eta^S P^S)\left[(1-\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\right]\Big\}-\beta^3\left(\overline{N^S}-N_0^S\right)(P^B\eta^B+P^S\eta^S)\big[-\beta P^S\{(1-$

$\delta^B)N_0^B+\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\}\big]\Bigg]D+\left[2\beta^4 P^S(\eta^B P^B+\eta^S P^S)\left(\overline{N^B}-N_0^B\right)\left(\overline{N^S}-N_0^S\right)\right]D_1$ i.e.

$D_5=\big[2c\beta\big[\beta\overline{N^S}P^S-\beta P^S\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\}-\beta(\eta^B P^B+\eta^S P^S)\big(\overline{N^S}-$

$N_0^S\big)N_0^B\big]-\beta^3 P^S\left(\overline{N^S}-N_0^S\right)Y+\beta^4 P^S\left(\overline{N^S}-N_0^S\right)(P^B\eta^B+P^S\eta^S)\{(1-\delta^B)N_0^B+\eta^B\big(\overline{N^B}-$

$N_0^B\big)N_0^S\}\big]D+\left[2\beta^4 P^S(\eta^B P^B+\eta^S P^S)\left(\overline{N^B}-N_0^B\right)\left(\overline{N^S}-N_0^S\right)\right]D_1$ where

$Y=P^S+\beta\eta^B\overline{N^B}P^B+\beta(1-\delta^S)P^S-\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-\beta(\eta^B P^B+\eta^S P^S)\big[(1-\delta^B)N_0^B+$

$\eta^B\left(\overline{N^B}-N_0^B\right)N_0^S\big]$. Let

$N=2c\beta\big[\beta\overline{N^S}P^S-\beta P^S\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\}-\beta(\eta^B P^B+\eta^S P^S)\big(\overline{N^S}-$

$N_0^S\big)N_0^B\big]-\beta^3 P^S\left(\overline{N^S}-N_0^S\right)Y+\beta^4 P^S\left(\overline{N^S}-N_0^S\right)(P^B\eta^B+P^S\eta^S)\{(1-\delta^B)N_0^B+\eta^B\big(\overline{N^B}-$

$N_0^B\big)N_0^S\}$ i.e.

$N=2c\beta^2\overline{N^S}P^S-2c\beta^2 P^S\{(1-\delta^S)N_0^S+\eta^S\left(\overline{N^S}-N_0^S\right)N_0^B\}-2c\beta^2(\eta^B P^B+\eta^S P^S)\big(\overline{N^S}-$

$N_0^S\big)N_0^B-\beta^3 P^S\left(\overline{N^S}-N_0^S\right)Y+\beta^4 P^S\left(\overline{N^S}-N_0^S\right)(P^B\eta^B+P^S\eta^S)\{(1-\delta^B)N_0^B+\eta^B\big(\overline{N^B}-$

$N_0^B\big)N_0^S\}$

Since $Y<0$, hence $-\beta^3 P^S\left(\overline{N^S}-N_0^S\right)Y>0$.

Now N has 3 positive terms and 2 negative terms where we conjecture that

$2c\beta^2\overline{N^S}P^S-\beta^3 P^S\left(\overline{N^S}-N_0^S\right)Y>2c\beta^2(\eta^B P^B+\eta^S P^S)\left(\overline{N^S}-N_0^S\right)N_0^B$ and

$\beta^4 P^S(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\} > 2c\beta^2 P^S\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}$. Consider the first inequality.

$2c\beta^2\overline{N^S}P^S - \beta^3 P^S(\overline{N^S} - N_0^S)Y > 2c\beta^2(\eta^B P^B + \eta^S P^S)(\overline{N^S} - N_0^S)N_0^B$ i.e. it reduces to

$c < \frac{\beta\left(1-\frac{N_0^S}{\overline{N^S}}\right)\left\{\beta(\eta^B P^B+\eta^S P^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]+\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-P^S-\beta\eta^B\overline{N^B}P^B-\beta(1-\delta^S)P^S\right\}}{2\left[N_0^B\left\{\eta^B\left(\frac{P^B}{P^S}\right)+\eta^S\right\}\left(1-\frac{N_0^S}{\overline{N^S}}\right)-1\right]}$. Now since

$\mathrm{D} < 0,\ \mathrm{c} < \frac{\left[\beta^2(\eta^B P^B+\eta^S P^S)\sqrt{[(\overline{N^B}-N_0^B)(\overline{N^S}-N_0^S)]}\right]}{2}$. Consider

$\frac{\beta\left(1-\frac{N_0^S}{\overline{N^S}}\right)\left\{\beta(\eta^B P^B+\eta^S P^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]+\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-P^S-\beta\eta^B\overline{N^B}P^B-\beta(1-\delta^S)P^S\right\}}{\left[N_0^B\left\{\eta^B\left(\frac{P^B}{P^S}\right)+\eta^S\right\}\left(1-\frac{N_0^S}{\overline{N^S}}\right)-1\right]} < \left[\beta^2(\eta^B P^B + \eta^S P^S)\sqrt{[(\overline{N^B} - N_0^B)(\overline{N^S} - N_0^S)]}\right]$ i.e.

$\frac{\left(1-\frac{N_0^S}{\overline{N^S}}\right)\left\{\beta(\eta^B P^B+\eta^S P^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]+\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-P^S-\beta\eta^B\overline{N^B}P^B-\beta(1-\delta^S)P^S\right\}}{\left[N_0^B\left\{\eta^B\left(\frac{P^B}{P^S}\right)+\eta^S\right\}\left(1-\frac{N_0^S}{\overline{N^S}}\right)-1\right]} < \left[\beta\left(\eta^B\left(\frac{P^B}{P^S}\right) + \eta^S\right)\sqrt{[(\overline{N^B} - N_0^B)(\overline{N^S} - N_0^S)]}\right]$

Now $\frac{P^B}{P^S} < 1,\ N_0^B \gg 1, N_0^S \gg 1, \eta^B < 1, \eta^S < 1, N_0^B \ll \overline{N^B},\ N_0^S \ll \overline{N^S}, \delta^B < 1, \delta^S < 1$. Hence the above inequality condition is plausible. Hence

$c < \frac{\beta\left(1-\frac{N_0^S}{\overline{N^S}}\right)\left\{\beta(\eta^B P^B+\eta^S P^S)[(1-\delta^B)N_0^B+\eta^B(\overline{N^B}-N_0^B)N_0^S]+\frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2-P^S-\beta\eta^B\overline{N^B}P^B-\beta(1-\delta^S)P^S\right\}}{2\left[N_0^B\left\{\eta^B\left(\frac{P^B}{P^S}\right)+\eta^S\right\}\left(1-\frac{N_0^S}{\overline{N^S}}\right)-1\right]}$ is plausible.

Hence the first inequality is plausible. Consider now the second inequality.

$\beta^4 P^S(\overline{N^S} - N_0^S)(P^B\eta^B + P^S\eta^S)\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\} > 2c\beta^2 P^S\{(1-\delta^S)N_0^S +$

$\eta^S(\overline{N^S} - N_0^S)N_0^B\}$ i.e.

$$c < \frac{\beta^2(P^B\eta^B + P^S\eta^S)(\overline{N^S} - N_0^S)}{2}\left[\frac{\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\}}{\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}}\right].$$

Recall, since $\mathrm{D} < 0$, $\mathrm{c} < \frac{\left[\beta^2(\eta^B P^B + \eta^S P^S)\sqrt{[(\overline{N^B} - N_0^B)(\overline{N^S} - N_0^S)]}\right]}{2}$. Now if

$\left[\frac{\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\}}{\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}}\right](\overline{N^S} - N_0^S) > \sqrt{[(\overline{N^B} - N_0^B)(\overline{N^S} - N_0^S)]}$ i.e. if

$\left[\frac{\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\}}{\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}}\right] > \sqrt{\left[\frac{(\overline{N^B} - N_0^B)}{(\overline{N^S} - N_0^S)}\right]}$ i.e. then

$c < \frac{\beta^2(P^B\eta^B + P^S\eta^S)(\overline{N^S} - N_0^S)}{2}\left[\frac{\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\}}{\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}}\right]$ and the second inequality is also satisfied.

Hence $N > 0$.

Since $\mathrm{D} < 0$ and $\mathrm{D}_1 < 0$, and $\mathrm{N} > 0$, $\mathrm{D}_5 < 0$. Hence $\frac{\partial A_0^B}{\partial \eta^S} < 0$.

Hence we have –

$\frac{\partial A_0^B}{\partial \eta^S} < 0$ if $\left[\frac{\{(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S\}}{\{(1-\delta^S)N_0^S + \eta^S(\overline{N^S} - N_0^S)N_0^B\}}\right] > \sqrt{\left[\frac{(\overline{N^B} - N_0^B)}{(\overline{N^S} - N_0^S)}\right]}$ and

$$c < \frac{\beta\left(1 - \frac{N_0^S}{\overline{N^S}}\right)\left\{\beta(\eta^B P^B + \eta^S P^S)[(1-\delta^B)N_0^B + \eta^B(\overline{N^B} - N_0^B)N_0^S] + \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - P^S - \beta\eta^B\overline{N^B}P^B - \beta(1-\delta^S)P^S\right\}}{2\left[N_0^B\left\{\eta^B\left(\frac{P^B}{P^S}\right) + \eta^S\right\}\left(1 - \frac{N_0^S}{\overline{N^S}}\right) - 1\right]}.$$

Q.E.D.

Proof of Corollary 10

$X = P^B + \beta\eta^S\overline{N^S}P^S + \beta(1-\delta^B)P^B - \frac{\beta^2}{c}\left(\frac{P^B}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\big[(1-\delta^S)N_0^S +$

$\eta^S\big(\overline{N^S} - N_0^S\big)N_0^B\big] < 0$ and

$Y = P^S + \beta\eta^B\overline{N^B}P^B + \beta(1-\delta^S)P^S - \frac{\beta^2}{c}\left(\frac{P^S}{2}\right)^2 - \beta(\eta^B P^B + \eta^S P^S)\big[(1-\delta^B)N_0^B +$

$\eta^B\big(\overline{N^B} - N_0^B\big)N_0^S\big] < 0$ and

$D_1 = 2c\beta X - \beta^3\big(\overline{N^S} - N_0^S\big)(P^B\eta^B + P^S\eta^S)Y < 0$ and

$D_2 = \beta^3\big(\overline{N^B} - N_0^B\big)(\eta^B P^B + \eta^S P^S)X - 2c\beta Y > 0$ and

$D = \big[4c^2 - \beta^4(\eta^B P^B + \eta^S P^S)^2\big(\overline{N^B} - N_0^B\big)\big(\overline{N^S} - N_0^S\big)\big] < 0$

$A_0^B = \frac{D_1}{D},\ A_0^S = -\frac{D_2}{D}$

Hence $A_0^B = \frac{D_1}{D} = \frac{-D_1}{-D} = \frac{\beta^3\left(\overline{N^S}-N_0^S\right)\left(P^B\eta^B+P^S\eta^S\right)Y-2c\beta X}{\beta^4\left(\eta^B P^B+\eta^S P^S\right)^2\left(\overline{N^B}-N_0^B\right)\left(\overline{N^S}-N_0^S\right)-4c^2}$

$$\frac{\partial A_0^B}{\partial N_0^B} = \frac{\left[\beta^3\left(\overline{N^S}-N_0^S\right)\left(P^B\eta^B+P^S\eta^S\right)\left\{-\beta\left(\eta^B P^B+\eta^S P^S\right)\left(1-\delta^B-\eta^B N_0^S\right)\right\}+2c\beta^2\left(\eta^B P^B+\eta^S P^S\right)\eta^S\left(\overline{N^S}-N_0^S\right)\right](-D)+\{-D_1\}\beta^4\left(\overline{N^S}-N_0^S\right)\left(\eta^B P^B+\eta^S P^S\right)^2}{D^2}$$

The denominator is positive. The numerator N is

$N = \big[\beta^3\big(\overline{N^S} - N_0^S\big)(P^B\eta^B + P^S\eta^S)\{-\beta(\eta^B P^B + \eta^S P^S)(1-\delta^B - \eta^B N_0^S)\} + 2c\beta^2(\eta^B P^B +$

$\eta^S P^S)\eta^S\big(\overline{N^S} - N_0^S\big)\big](-D) + \{-D_1\}\beta^4\big(\overline{N^S} - N_0^S\big)(\eta^B P^B + \eta^S P^S)^2$ i.e.

$N = \beta^2\big(\overline{N^S} - N_0^S\big)(\eta^B P^B + \eta^S P^S)[\beta^2(\eta^B P^B + \eta^S P^S)\{(1-\delta^B - \eta^B N_0^S)D - D_1\} - 2c\eta^S D]$

Since $D < 0,\ D_1 < 0,\ N > 0$ if $1 - \delta^B - \eta^B N_0^S < 0$ which is likely given η^B although less than one is sufficiently large and $N_0^S \gg 1$. Hence $\frac{\partial A_0^B}{\partial N_0^B} > 0$.

$$\frac{\partial A_0^B}{\partial N_0^S} = \frac{\left[-\beta^3 Y - \beta^4 \eta^B \left(\overline{N^S} - N_0^S\right)\left(\overline{N^B} - N_0^B\right)\left(\eta^B P^B + \eta^S P^S\right) + 2c\beta^2\left(1 - \delta^S - \eta^S N_0^B\right)\right](-D) + \{-D_1\}\beta^4\left(\overline{N^B} - N_0^B\right)\left(\eta^B P^B + \eta^S P^S\right)}{\frac{D^2}{\left(\eta^B P^B + \eta^S P^S\right)}}$$

The denominator is positive. The numerator M is

$$M = \left[-\beta^3 Y - \beta^4 \eta^B \left(\overline{N^S} - N_0^S\right)\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S) + 2c\beta^2(1 - \delta^S - \eta^S N_0^B)\right](-D) +$$
$$\{-D_1\}\beta^4\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S) \quad \text{i.e.}$$

$$M = D\left[\beta^4 \eta^B \left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)(\eta^B P^B + \eta^S P^S) - 2c\beta^2(1 - \delta^S - \eta^S N_0^B)\right] - 2c\beta^2 D_2$$

where $D_2 = \beta^3\left(\overline{N^B} - N_0^B\right)(\eta^B P^B + \eta^S P^S)X - 2c\beta Y > 0$ and

$D = \left[4c^2 - \beta^4(\eta^B P^B + \eta^S P^S)^2\left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)\right] < 0$. Now $(1 - \delta^S - \eta^S N_0^B) < 0$ as $N_0^B \gg 1$ and $\eta^S < 1$ but relatively large. Hence $\beta^4 \eta^B \left(\overline{N^B} - N_0^B\right)\left(\overline{N^S} - N_0^S\right)(\eta^B P^B + \eta^S P^S) - 2c\beta^2(1 - \delta^S - \eta^S N_0^B) > 0$. Since $D < 0$ and $D_2 > 0$, $M < 0$. Hence if $M < 0$, $\frac{\partial A_0^B}{\partial N_0^S} < 0$.

Q.E.D.