BOOLEAN LATTICES, RINGS AND THE EQUIVALENCE OF CATEGORIES.

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AND
THE EQUIVALENCE OF CATEGORIES

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SCOPE AND CONTENTS: This thesis is concerned with showing the relations among Boolean lattices, Boolean rings and Boolean spaces. It establishes that the categories of Boolean lattices and proper Boolean lattice homomorphisms, Boolean spaces and proper continuous maps, Boolean rings and proper ring homomorphisms are equivalent to each other. In the final chapter the notion of a Boolean semi-group is used to obtain an alternate characterization of a Boolean lattice.

PREFACE

Chapter 0 of this thesis is a preliminary chapter in which all the basic concepts and theorems which are needed for the understanding of succeeding chapters are collected. In particular the notions of a Boolean lattice, Boolean space, Boolean ring, Boolean semi-group, categories, functors and various types of proper homomorphisms are introduced and their fundamental properties are studied.

In Chapter I we note the topological properties of the ultrafilter space of a Boolean lattice and establish the main result, namely, the category of Boolean lattices and proper Boolean lattice homomorphisms is equivalent to the category of Boolean spaces and proper continuous maps.

In Chapter II we establish the relation between Boolean lattices and Boolean rings and prove that the category of Boolean lattices and proper Boolean lattice homomorphisms is equivalent to the category of Boolean rings and proper ring homomorphisms.

Chapter III is devoted to using the notion of a Boolean semi-group to arrive at an alternative characterization of a Boolean lattice.

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CHAPTER 0.

PRELIMINARIES.

Introduction: In this chapter we collect together all the basic theorems and definitions which are to be assumed later. In particular we introduce the notions of a Boolean lattice, Boolean ring and Boolean space and discuss some of their fundamental properties.

1. Boolean Lattices.

Let B be a lattice.

Definition 1: B is said to be relatively complemented if given $a \leq x \leq b$, y exists with $x \wedge y = a$, $x \vee y = b$.

Definition 2: B is called distributive if one of the following equivalent laws hold:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for any x, y, z in B.

(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for any x, y, z in B.

We combine the above definitions to introduce the notion of a Boolean lattice.

Definition 3: A relatively complemented distributive lattice with a zero element is called a Boolean lattice.

The following proposition shows that the distributivity condition in a Boolean lattice implies that relative complements are unique.

Proposition 1: In a Boolean lattice B relative complements are unique.

Proof: Let $0 \leq x \leq y$. Suppose there exists u, v with $u \wedge x = v \wedge x = 0$, $u \vee x = v \vee x = y$. Then $u = u \wedge (x \vee u) = u \wedge (x \vee v) = (u \wedge x) \vee (v \wedge u) = (v \wedge x) \vee (v \wedge u) = v \wedge (x \vee u) = v \wedge (x \vee v) = v$. Hence $u = v$ and this completes the proof. Thus in a Boolean lattice we have for $0 \leq x \leq y$, an unique z exists with $x \wedge z = 0$, $x \vee z = y$.

Notation: We will denote this z by $y \sim x$ and read it as the relative complement of x in y.

Remark: In case a Boolean lattice B has an unit e then the unique element $e \sim x$ for any x in B is called the complement of x in B and is denoted by x^1. Hence in every Boolean lattice with unit each element possesses an unique complement. In general a lattice L with o, e is called complemented if for any x in L an element y exists with $x \wedge y = 0$, $x \vee y = e$. Thus every Boolean lattice with unit is in particular complemented. On the other hand any distributive complemented lattice is clearly seen to be a relatively complemented distributive lattice with o, e, that is, a Boolean lattice with unit.

Definition 4: Let B be a Boolean lattice. A non empty subset F of B is called a filter if (1) $a, b \in F$ implies $a \wedge b \in F$ (2) $a \in F$ and $b \geqslant a$ for any b in B implies $b \in F$.

A filter F in B is called proper if F is distinct from B. It is clear that a filter F is proper if and only the zero element of B does not belong to F. We now proceed to introduce the important notion of an ultrafilter.

Definition 5: Let B be any Boolean lattice and F any filter in B. F is called an ultrafilter or a maximal proper filter if F is proper and if G is any filter strictly containing F then $G = B$.

The next proposition shows that every non-trivial Boolean lattice has many ultrafilters. Its proof depends on the axiom of choice.

Proposition 2: (Existence Theorem for Ultrafilters), For any filter F in a Boolean lattice B there exists an ultrafilter U containing F. In fact any filter F in B is equal to the intersection of all ultrafilters containing F.

Proof: (1) We apply Zorn's Lemma. Consider the set $\mathcal{P}$ of all proper filters G in B with F contained in G. Then $\mathcal{P}$ is non-empty for F belongs to $\mathcal{P}$. The set $\mathcal{P}$ is partially ordered by inclusion and is clearly seen to be inductive. Hence by Zorn's Lemma there exists maximal elements in $\mathcal{P}$. Let U in $\mathcal{P}$ be maximal. Then U is a proper filter containing F. Suppose further that there exists another proper filter V containing U; then V belongs to $\mathcal{P}$ which contradicts the maximality of U. Thus U is an ultrafilter containing F. This establishes the first part of the proposition.

(ii) Let $G=\bigcap\{U/U\supseteq F,\ U \text{ an ultrafilter}\}$ where F is any filter in B. It is then clear that G is a filter and that F is any filter in B. Suppose F is strictly contained in G. Then there exists an x in G with x not in F. Take any $y\in F$ and put $H=\{z/z\in B,\ z\geqslant y\sim(y\wedge x)\}$. Note that for any u in F, $u\wedge(y\sim(y\wedge x))\neq o$ for if $u\wedge(y\sim(y\wedge x))=o$ then $u\wedge y\leq x$. Since F is a filter $u\wedge y\in F$ and hence $x\in F$ which contradicts the fact that $x\notin F$. Hence $u\wedge z\neq o$ for any u in F, z in H. Then the filter J generated by the set $F\cup H$ is proper. Let V be an ultrafilter containing J. Then in particular V contains F and $y\sim(y\wedge x)\in V$. This implies that $x\notin V$. This contradicts the fact that $x\in G$. Thus our supposition is false and the proposition is established.

Corresponding to the notion of a filter in a Boolean lattice we now introduce the dual notion of an <u>ideal</u>.

<u>Definition 6</u>: Let B be a Boolean lattice. A non empty set I in B is called an <u>ideal</u> if (1) a, $b \in I$ implies $a \vee b \in I$ and (2) $a \in I$ and $b \leq a$ for any b in B implies $b \in I$.

An ideal I in B is called <u>proper</u> if I is distinct from B. Note that the zero element of B belongs to every ideal in B and hence every ideal I with the operations of B restricted to I forms a Boolean lattice.

<u>Definition 7</u>: An ideal I in a Boolean lattice B is called <u>prime</u> if $x \wedge y \in I$ and $x \notin I$ implies $y \in I$. I is called maximal proper if J is any ideal strictly containing I then $J = B$.

Concerning prime ideals we note the following proposition.

<u>Proposition 3</u>: An ideal P in a Boolean lattice B is prime if and only if for any x, y in B, $x \in P$ or $y \sim (x \wedge y) \in P$.

<u>Proof</u>: (i) Suppose P is a prime ideal in B and take any x, y in B. Clearly $x \wedge (y \sim x \wedge y) = 0 \in P$. Thus since P is prime $x \in P$ or $y \sim x \wedge y \in P$.
(ii) Take an ideal P in B with the property stated. Let $x \wedge y \in P$ and suppose $x \notin P$. This implies by hypothesis that $y \sim x \wedge y \in P$. Also $x \wedge y \in P$. Then since P is an ideal $(y \sim x \wedge y) \vee (x \wedge y) = y \in P$. This completes the proof.

Using Proposition 3 we now establish that the notions of a prime ideal and a maximal ideal in a Boolean lattice are the same.

<u>Proposition 4</u>: An ideal P in a Boolean lattice B is prime if and only if it is maximal.

Proof: (1) Let P be a prime ideal in B. Let M be any ideal strictly containing P. Then there exists an $x \in M$ with $x \notin P$. By Proposition 3 we then have that $y \sim (x \wedge y) \in P$ for any $y \in B$. But $x \in M$ and since M is an ideal $x \wedge y \in M$ and thus $(x \wedge y) \vee (y \sim (x \wedge y)) = y \in M$ for any y in B. That is $M = B$ and P is maximal.

(2) Suppose M is a maximal ideal in B. Let $x \wedge y \in M$ and suppose $x \notin M$. Then since M is maximal the ideal generated by $M \cup \{x\}$ is B. Then $y \leq m \vee x$ for some $m \in M$ since clearly $\{z / z \leq m \vee x,\ m \in M\}$ is the ideal generated by $M \cup \{x\}$. This implies $y = y \wedge (m \vee x) = (y \wedge m) \vee (y \wedge x)$. Now by hypothesis $y \wedge x \in M$ and $y \wedge m \in M$ since M is an ideal. Thus $y \in M$ and this means M is prime. This completes the proof.

In a similar manner as in Proposition 2 one can establish that any proper ideal I is contained in a maximal proper ideal. Concerning ideals in a Boolean lattice one can establish the following proposition:

Proposition 5: Let B be a Boolean lattice and let $\mathcal{J}$ denote the set of all ideals in B. Then $\mathcal{J}$ under set inclusion is a complete distributive lattice and in this lattice $I \vee J = \{a \vee b / a \in I,\ b \in J\}$, and $I \wedge J = \{a \wedge b / a \in I,\ b \in J\}$ where I, J are any ideals in B.

Proof: Let $K = \{a \vee b / a \in I,\ b \in J)$. Take x, y in K. Then $x = a \vee b$, $y = c \vee d$ with a, c in I, b, d in J. Then $x \vee y = (a \vee c) \vee (b \vee d)$ and thus $x \vee y$ is in K. Take any x in K and suppose $z \leq x = a \vee b$ for some z in B. Then $z = z \wedge x = z \wedge (a \vee b) = (z \wedge a) \vee (z \wedge b)$ using the fact that B is distributive. But $z \wedge a \in I$, $z \wedge b \in J$ since $a \in I$, $b \in J$ and they are ideals. Hence $z \in K$. Thus K is an ideal and K clearly contains I, J. But on the other hand if H is any ideal containing I, J then H must clearly contain K.

Hence $K = I \vee J$. Clearly the intersection of any arbitrary collection of ideals in B is again an ideal in B and hence $\mathcal{J}$ is a complete lattice with meet being intersection. It is clear that the set $\{a \wedge b / a \in I, b \in J\}$ is an ideal in B contained in I, J and hence in $I \cap J$. On the other hand if $x \in I \cap J$ then $x \in I$ and $x \in J$ and hence $x \wedge x = x$ belongs to $\{a \wedge b / a \in I, b \in J\}$. Thus $I \wedge J = I \cap J = \{a \wedge b / a \in I, b \in J\}$. It remains to show that $I \cap (J \vee K) = (I \cap J) \vee (I \cap K)$ for any I, J, K. Now $I \cap (J \vee K) \geq (I \cap J) \vee (I \cap K)$ in any lattice. But every element $a \wedge (b \vee c)$ $(a \in I, b \in J, c \in K)$ of $I \cap (J \vee K)$ is an element $(a \wedge b) \vee (a \wedge c)$ of $(I \cap J) \vee (I \cap K)$. Hence the reverse inequality holds and the proof is complete.

Remark: In the above proposition we only required the fact that B be distributive and hence it is true in general for arbitrary distributive lattices.

We now proceed to define the notion of a Boolean lattice homomorphism. In the following B, C will always denote Boolean lattices.

Definition 8: A mapping $f : B \to C$ is called a Boolean lattice homomorphism if (1) $f(x \vee y) = f(x) \vee f(y)$ for any x, y in B.

(2) $f(x \wedge y) = f(x) \wedge f(y)$ for any x, y in B.

(3) $f(y \sim x) = f(y) \sim f(x)$ for any x, y with $0 \leq x \leq y$.

f is called a Boolean lattice isomorphism if in addition f is one to one and onto.

We note that in case B, C have units e_B, e_C respectively then a Boolean lattice homomorphism $f : B \to C$ need not carry e_B into e_C, that is, $f(e_B)$ need not be e_C. The following is an example illustrating this fact.

Example: Let E be any non-empty set and A any non-empty proper subset of E.

Then $\mathcal{P}(E)$, the power class of E, under set inclusion forms a Boolean lattice with unit E. Consider the mapping $f: \mathcal{P}(E) \to \mathcal{P}(E)$ defined by $f(X) = X \cap A$ for any $X \subseteq E$. Then

(i) $f(X \cap Y) = X \cap A \cap Y \cap A = f(X) \cap f(Y)$.

(ii) $f(X \cup Y) = (X \cup Y) \cap A = (X \cap A) \cup (Y \cap A) = f(X) \cup f(Y)$.

(iii) $f(Y - X) = Y \cap \complement_E X \cap A$ with $\phi \subseteq X \subseteq Y$.

On the other hand $f(Y) - f(X) = (Y \cap A) \cap \complement_E (X \cap A) = Y \cap A \cap \complement_E X = f(Y-X)$.
Hence f is a Boolean lattice homomorphism but $f(E) \neq E$.

We distinguish the class of those Boolean lattice homomorphisms which carry the units into the units in the following definition.

<u>Definition 9</u>: Let B, C be Boolean lattices with units e_B, e_C respectively. A Boolean lattice homomorphism $f: B \to C$ is called <u>unitary</u> if $f(e_B) = e_C$.

<u>Remark</u>: A <u>unitary Boolean lattice homomorphism</u> f carries complements into complements, that is, $f(x^1) = (f(x))^1$ for any x in B where X^1 denotes the complement of x in B. This is so for $f(x^1) = f(e_B \sim x) = f(e_B) \sim f(x) = e_C \sim f(x) = (f(x))^1$ since f is unitary. On the other hand if f is a join and meet preserving map which carries complements into complements then f is an unitary Boolean lattice homomorphism.

We now introduce a special class of Boolean lattice homomorphisms which have the pleasant property that when they are restricted to the class of Boolean lattices with unit they correspond precisely to the unitary Boolean lattice homomorphisms.

<u>Definition 10</u>: A Boolean lattice homomorphism $f : B \to C$ is called <u>proper</u> if for any $c \in C$ there exists b in B with $f(b) \geqslant c$

Clearly if B, C have units then any unitary Boolean lattice homomorphism f between B and C is proper for then $f(e_B) = e_C \geqslant c$ for any $c \in C$.

That the converse is true, that is, that the notion of a proper Boolean lattice homomorphism coincides with that of an unitary homomorphism for Boolean lattices with unit is established in the following proposition:

Proposition 6: Let B, C, D be any three Boolean lattices. Let $f: B \to C$, $g: C \to D$ be proper Boolean lattice homomorphisms. Then (1) $g.f: B \to D$ is a proper Boolean lattice homomorphism, that is, the composition of proper maps is proper, (2) If B, C have units e_B, e_C respectively then the proper Boolean lattice homomorphism f is unitary.

Proof: (1) Take any $x \in D$. Since g is proper there exists y in C with $g(y) \geqslant x$. Since f is proper there exists z in B with $f(z) \geqslant y$. Since g is order preserving we have that $(g.f)(z) \geqslant g(y)$ and $g(y) \geqslant x$. Hence $g(f(z)) \geqslant x$. This means that g.f is proper.

(2) Since f is proper there exists a x in B with $f(x) \geqslant e_C$. But $e_B \geqslant x$ and thus $f(e_B) \geqslant f(x)$ since f is order preserving. Thus $f(e_B) \geqslant e_C$. But e_C is the unit of C. Hence $f(e_B) = e_C$ and f is unitary. This completes the proof.

2. Boolean Spaces.

Let E be a topological space.

Definition 11: E is called a zero-dimensional space if the topology on E is generated by the open closed subsets of E. E is called locally compact if each point of E possesses a compact neighbourhood.

Definition 12: E is called totally disconnected if for any two distinct points x, y in E there exists disjoint (open) closed sets A, B of E with $x \in A$, $y \in B$ and $A \cup B = E$.

Proposition 7: A compact T₂ space E is totally disconnected if and only if it is zero dimensional.

Proof: (1) Let E be totally disconnected. Let O be any open set of E containing a point x in E. By the total disconnectedness of E we have that for each point $y \in \complement O$ there exists open closed sets $V(y)$ containing y but not x. But $\complement O$ is compact since E is. Hence there exists a finite number of open closed sets $V(y_1), \ldots, V(y_n)$ with $\bigcup_{i=1}^{n} V(y_i)$ containing $\complement O$. That is, $x \in \bigcap_{i=1}^{n} \complement V(y_i) \subseteq O$. Hence there exists an open closed set contained in O having x as a member. This establishes that E is zero-dimensional.

(2) On the other hand if E is zero-dimensional then E is clearly totally disconnected.

Definition 13: E is called a Boolean space if E is a zero dimensional locally compact Hausdorff space.

Concerning Boolean spaces we prove the following proposition.

Proposition 8: The open and closed subspaces of a Boolean space E are Boolean spaces.

Proof: Since E is zero dimensional the topology on E is generated by open closed sets. Let $\mathcal{R}$ be a collection of open-closed subsets of E which generate the topology on E.

(1) Let T be a closed subset of E with the relative topology. Then T is a Hausdorff space since E is a Hausdorff space. Take any x in T. Then there exists a compact set K in E with $x \in K$. $K \cap T$ is then a compact subset of T containing x. Hence T is locally compact. Moreover, it is clear that $\mathcal{R} \cap T = \{O \cap T / O \in \mathcal{R}\}$ is a collection of relative open closed subsets of T which generate the relative topology of T since $\mathcal{R}$ generates the topology on E. Hence T is a Boolean space.

(2) Let O be any open subset of E. Since any subspace of a Hausdorff space is again Hausdorff, O is a Hausdorff space. Take any x in O. Since E is locally compact there exists a compact neighbourhood K of x. Then $O \cap K$ is a compact subset of O containing x. Hence O is locally compact : Put $\mathcal{R} \cap O = \{R \cap O / R \in \mathcal{R}\}$. Then $\mathcal{R} \cap O$ is a collection of relative open closed subsets of O. Let A be open in O. Then since O is open, A is open in E and hence there exists R in $\mathcal{R}$ with $R \subseteq O$. Then $A \cap R \subseteq O \cap A = A$. Thus $\mathcal{R} \cap O$ generates the relative topology of O. Hence O is a Boolean space. This completes the proof.

We now proceed to define a special class of continuous maps between Boolean spaces.

__Definition 14:__ Let E, F be any two Boolean spaces. A continuous mapping $f : E \rightarrow F$ is called __proper__ if the inverse image of any compact set in F under f is compact in E.

__Remark:__ The notion of a proper continuous map can be defined for arbitrary topological spaces but we have restricted ourselves to Boolean spaces since we are interested in only such spaces.

__Proposition 9:__ Let E be a Boolean space. Then the collection of all the compact open subsets of E is a basis for open sets.

__Proof:__ Let $\mathcal{L}(E)$ denote the collection of all compact open subsets of E. Take any $x \in E$ and let G be an open subset with $x \in G \subseteq E$. We want to show that there exists an $X \in \mathcal{L}(E)$ with $x \in X \subseteq G$. Since E is locally compact and Hausdorff there exists an open set Y with $x \in Y \subseteq G$ and with the closure ΓY of Y compact. If $Y = \Gamma Y$ we can take $X = Y$. Otherwise take $y \in \Gamma Y \cap \complement Y \neq \emptyset$ an arbitrary point. Then $x \neq y$ and since E is zero-dimensional there exists an open closed neighbourhood $Z(y)$ of y with x not in $Z(y)$. Now $E = Z(y) \cup \complement Z(y)$ and we have $\Gamma Y = Y \cup \bigcup \{\Gamma Y \cap Z(y) \,/\, y \in \Gamma Y \cap \complement Y\}$. Then since ΓY is compact there exists $y_1, \ldots, y_n$ in $\Gamma Y \cap \complement Y$ with $\Gamma Y = Y \cup \bigcup_{i=1}^{n} (\Gamma Y \cap Z(y_i))$. Put $X = Y \cap \bigcap_{i=1}^{n} \complement Z(y_i) = \Gamma Y \cap \bigcap_{i=1}^{n} \complement Z(y_i)$. Note that X is a subset of Y and Y is a subset of G. Hence X is a subset of G. Also $x \in X \subseteq G$. Moreover since $X = X \cap \Gamma Y$, X is closed in ΓY. Thus X is a compact subset of ΓY. But ΓY is compact in E. Thus X is compact open in E and $x \in X \subseteq G$. Hence $\mathcal{L}(E)$ is a basis for open sets and this completes the proof.

We use Proposition 9 in establishing,

<u>Proposition 10:</u> Any compact set K in a Boolean space E is the intersection of all the compact open sets of E containing K.

<u>Proof:</u> Let $\mathcal{R}$ denote the collection of all compact open sets of E. Then because of Proposition 9 the set $\{\complement R \,/\, R \in \mathcal{R}\}$ is a basis for closed sets. Since K is a compact subset in a Hausdorff space K is closed. Hence $K = \bigcap \{\complement R \,/\, K \subseteq \complement R,\ R \in \mathcal{R}\}$. Clearly $K \subseteq \bigcap \{R \,/\, K \subseteq R \in \mathcal{R}\} = T$ say. Take $x \in T$ and suppose x is not in K. Then there exists R_0 in $\mathcal{R}$ with $x \notin \complement R_0$. Take any $y \in K$ then $y \in \complement R_0$. Since $\mathcal{R}$ is a basis for open sets there exists $R(y) \in \mathcal{R}$ with $y \in R(y) \subseteq \complement R_0$. Since K is compact there exists $y_1, \ldots, y_n$ in K with $K \subseteq R(y_1) \cup \cdots \cup R(y_n) \subseteq \complement R_0$. Now each $R(y_i)$ is compact open and hence so is $R(y_1) \cup \cdots \cup R(y_n)$. Now $x \in T$ implies $x \in R(y_1) \cup \cdots \cup R(y_n)$ and hence $x \in \complement R_0$. This contradicts the fact that $x \notin \complement R_0$. This completes the proof.

The following proposition is an immediate consequence of Proposition 10.

<u>Proposition 11</u>: Let E, F be Boolean spaces. A map $f : E \longrightarrow F$ is proper continuous if and only if the inverse image of every compact open set in F under f is compact open in E.

<u>Proof</u>: (1) Let $f : E \longrightarrow F$ be any map with the property stated. Since the collection $\mathcal{R}$ of compact open subsets of F is a basis for open sets of F we have that f is a continuous map. Let K be any compact subset of F. By Proposition 10, $K = \bigcap\{R/K \subseteq R \in \mathcal{R}\}$. Then $f^{-1}(K) = \bigcap\{f^{-1}(R)/K \subseteq R \in \mathcal{R}\}$. But each $f^{-1}(R)$ with $K \subseteq R$ is compact open in E. Hence $f^{-1}(K)$ is a closed subset of a compact set and hence $f^{-1}(K)$ is compact. This means that f is proper continuous.

(2) On the other hand if f is proper continuous then by the very definition of a proper map we have $f^{-1}(K)$ is compact open for any compact open set K of F. This completes the proof.

We conclude this section by showing that the composites of proper continuous maps are again proper continuous.

<u>Proposition 12</u>: Let E, F, G be Boolean spaces. Let $f : E \longrightarrow F$, $g : F \longrightarrow G$ be proper continuous maps. Then the composite map $(g.f) : E \longrightarrow G$ given by $g(f(x)) = (gf)(x)$ is proper continuous.

<u>Proof</u>: Since f, g are continuous maps so is the map $(g.f)$. We check that it is proper. Let K be any compact subset of G. Since g is proper there exists a compact set T of F with $g^{-1}(K) = T$. Since f is proper there exists a compact subset S of E with $f^{-1}(T) = S$. Then $(g.f)^{-1}(K) = f^{-1}(g^{-1}(K)) = f^{-1}(T) = S$. Hence $(g.f)$ is proper. This completes the proof.

3. <u>Boolean Rings.</u>

We introduce the notion of a Boolean ring by distinguishing those rings which have the property mentioned in the following definition.

Definition 15: A Boolean ring is a ring R each of whose elements is idempotent, that is, for each x in R $x^2 = x$. R is said to be a Boolean ring with unit if R is a Boolean ring and there exists an element e in R with xe = ex = x for each x in R. e is called the unit of R.

Remark: It is clear that if a Boolean ring R has a unit e then it is unique for suppose e, e^1 are two units in R. Then $e^1 = ee^1 = e^1e = e$.

In the following proposition we list some of the important algebraic relations satisfied by elements of a Boolean ring.

Proposition 13: Let R be a Boolean ring. Then

(i) x + x = 0 for each x in R, that is, every Boolean ring is of characteristic two.

(ii) If x + y = 0 then x = y for any x, y in R.

(iii) Every Boolean ring is commutative, that is, xy = yx for any x, y in R.

Proof: (i) $(x + x)^2 = x + x$. This implies xx + xx + xx + xx = x + x. Hence x + x = 0 using the fact that $x^2 = x$.

(ii) x + y = 0 implies x + y + y = y. Hence x = y using (i).

(iii) $(x + y)^2 = x + y$. Hence xx + xy + yx + yy = x + y. Hence x + xy + yx + y = x + y. Thus xy + yx = 0. Then xy = yx using (ii).

Let R be an arbitrary ring. Introduce a relation $\leq$ in R as follows: $x \leq y$ if and only if there exists z in R with x = zy. This relation is known as the divisibility relation. In R the divisibility relation is clearly transitive. If R has a unit e then the divisibility relation is also reflexive for x = e.x for any x in R. This also means that $x \leq e$ for any x. Moreover, $0 \leq x$ for any x in R.

In general the divisibility relation is not anti-symmetric. For example, in the ring of integers 1 is <u>divisible by</u> -1 and conversely, yet $1 \neq -1$ In a Boolean ring, however, the divisibility relation becomes a partial order. This becomes apparent when we prove:

<u>Proposition 14</u>: Let R be a Boolean ring. Then $x \leq y$ if and only if $x = xy$.

<u>Proof</u>: Suppose $x = xy$. Then take $z = y$ in the definition of the divisibility relation. Hence $x \leq y$. Conversely suppose $x \leq y$. Then there exists a z in R with $x = zy$. Using the fact that R is a Boolean ring we get $xy = (zy)y = zyy = zy = x$; that is, $x = xy$. This completes the proof.

<u>Proposition 15</u>: In a Boolean ring the divisibility relation is a partial order.

<u>Proof</u>: (i) $x \leq x$ for $x^2 = x$. (ii) $x \leq y$, $y \leq z$ implies $x = xy$, $y = yz$. Hence $x = xy = xyz = xz$, that is, $x \leq z$. (iii) $x \leq y$, $y \leq x$ implies $x = xy = xyz = xy = y$. Hence $\leq$ is a partial order.

We now introduce the notion of an <u>ideal</u> in a Boolean ring. The notion of an ideal in an arbitrary ring is defined similarly but we are interested here only with Boolean rings.

<u>Definition 16</u>: Let R be a Boolean ring. A non-empty subset I of R is called an <u>ideal</u> if

(1) I is a subgroup of R under addition, that is, for any x, y in R, $x - y \in R$.

(2) For any y in R, x in I we have $yx \in I$.

<u>Remark</u>: Since a Boolean ring is commutative as we have seen we do not need to distinguish between left, right or two-sided ideals as one does

for arbitrary rings.

We distinguish certain classes of ideals in a Boolean ring as follows.

Definition 17: Let R be a Boolean ring and I any ideal in R. I is called prime if $xy \in I$ and $x \notin I$ implies $y \in I$. I is called a maximal proper ideal if $I \neq R$ and if J is an ideal strictly containing I then $J = R$.

Analagous to Propositions 3 and 4 is the following proposition which we now establish.

Proposition 16: Let R be a Boolean ring and P any ideal in R.
(1) P is prime if and only if for any $x, y \in R$, $x \in P$ or $y - yx \in P$,
(2) P is prime if and only if P is maximal.

Proof: (1) Suppose P is a prime ideal. Take x, y in R. Then $x(y-yx) = 0$ since R is a Boolean ring. Since $0 \in P$ we have $x(y-yx) \in P$. But P is a prime ideal. Hence $x \in P$ or $y-yx \in P$. On the other hand let P be an ideal with the property stated. Suppose $xy \in P$ but $x \notin P$. Then $y-yx \in P$ and $xy \in P$ and P is an ideal we have $y \in P$.

(2) Suppose P is a prime ideal. Let M be an ideal strictly containing P. Then there exists $x \in M$ with $x \notin P$. $x \notin P$ implies by (1) $y-yx \in P$ for any $y \in R$. Thus $y-yx \in M$ for any $y \in R$. However, M is an ideal and thus $yx \in M$. Hence $y = y-yx+yx$ is in M for any $y \in R$. Hence $M = R$. This means P is a maximal ideal. Conversely suppose M is a maximal ideal of R. Take $xy \in M$ and suppose $x \notin M$. Then since M is maximal the ideal generated by $M \cup \{x\}$ is R. Then, in particular, $y = m + rx$ where $r \in R$. Then $xy = mx + rx$ and since xy, $mx \in M$ and M is an ideal we have $rx \in M$. Thus $y = m + rx \in M$. This means M is prime. This completes the proof.

Precisely as in Proposition 2 one can establish that any proper ideal is contained in a maximal proper ideal. We now proceed to define the general notion of an algebra over a field.

Definition 18: Let K be any field. A set R is called a K - algebra if (1) R is a ring, (2) there is a binary law of composition $K \times R \longrightarrow R$ satisfying:

(i) $a(r + s) = ar + as$ for any $a \in K$, $r, s \in R$.

(ii) $(a + b)r = ar + br$ for any $a, b \in K$, $r \in R$.

(iii) $(ab)r = a(br)$ for any $a, b \in K$, $r \in R$.

(iv) $er = r$ for e the unit of K and any $r \in R$.

(v) $a(rs) = (ar)s = r(as)$ for any $a \in K$, $r, s \in R$.

Consider the set $R_0 = (0, 1)$ and in this set introduce operations of addition and multiplication as follows: Put $0 + 0 = 1 + 1 = 0$, $0 + 1 = 1 + 0 = 1$, $0.1 = 1.0 = 0$, $1.1 = 1$, $0.0 = 0$. It is clear that R_0 under these operations is a Boolean ring. We will refer to it as the two element Boolean ring. In fact we see that R_0 is the field of characteristic two. Let R be any Boolean ring. Define a mapping $f : R_0 \times R \longrightarrow R$ by

$$f(\alpha, x) = \begin{cases} x \text{ if } \alpha = 1 \in R_0 \\ 0 \text{ if } \alpha = 0 \in R_0 \end{cases}$$

It is seen then immediately that R with multiplication as defined is an R_0 - algebra. Hence in the above way any Boolean ring R is an R_0 - algebra.

We now introduce the notion of homomorphism for Boolean rings.

Definition 19: Let R, S be Boolean rings. A mapping $f : R \rightarrow S$ is called a ring homomorphism if (1) $f(x + y) = f(x) + f(y)$ for any x, y in R and (2) $f(xy) = f(x)\, f(y)$ for any x, y in R. In case R, S both have units e_R, e_S respectively a ring homomorphism f from R to S is called unitary if $f(e_R) = e_S$.

Remark: If R, S have units e_R, e_S respectively then clearly any ring homomorphism f from R onto S is necessarily unitary. However, if f is into then f need not be unitary. For example let R_0 be the two element Boolean ring and consider the Boolean ring $R_0 \times R_0$ with addition and multiplication defined in the usual way. Let $f: R_0 \times R_0 \rightarrow R_0 \times R_0$ be defined by $f((a,b)) = (a,0)$. Then f is clearly a ring homomorphism but f is not unitary for $f((1, 1)) = (1,0) \neq (1,1)$.

We now introduce a special class of ring homomorphisms which have the nice property that when defined on the class of Boolean rings with unit they coincide with the unitary ring homomorphisms.

Definition 20: Let R, S be any two Boolean rings. A ring homomorphism f from R into S is called proper if for any $x \in S$ there exists y in R, with $x \leq f(y)$ where $\leq$ stands for the divisibility relation in S.

Let R, S be Boolean rings with units e_R, e_S respectively. Then any unitary ring homomorphism f from into S is clearly proper for then $f(e_R) = e_S \geqslant x$ for any x in S. We now establish that the converse of this statement is also true.

Proposition 17: Let R, S be Boolean rings with units e_R, e_S respectively. Let f from R into S be a proper ring homomorphism. Then f is unitary.

Proof: Since f is proper there exists y in R with $f(y) \geqslant e_S$. Now $e_R \geqslant y$ in R. Then since f is a ring homomorphism we have $f(e_R) \geqslant f(y)$. Since the divisibility relation is transitive we have $f(e_R) \geqslant e_S$. But e_S is the unit of S. Hence $f(e_R) = e_S$, that is, f is unitary.

We conclude this section by establishing that the functional composition of proper ring homomorphisms is again proper.

Proposition 18: Let R, S, T be Boolean rings. Let $f: R \to S$, $g: S \to T$ be proper ring homomorphisms. Then the composite function $g.f : R \to T$ defined by $(g.f)(x) = g(f(x))$ is a proper ring homomorphism.

Proof: Take any $x \in T$. Since g is proper there exists y in S with $g(y) \geqslant x$. Since f is proper there exists z in R with $f(z) \geqslant y$. Thus in all $(g.f)(z) = g(f(z)) \geqslant g(y) \geqslant x$. Hence $g.f$ is proper. That $(g.f)$ carries sums into sums and products into products is clear since f and g do. This completes the proof.

4. Boolean Semi-groups.

In this section we introduce the notion of a Boolean semi-group and establish some of its fundamental properties.

Definition 21: A semi-group is a set G together with a binary associative law of composition which we denote multiplicatively. A semi-group G is called a semi-group with zero element if there exists an element 0 in G with $0x = x0 = 0$ for all $x \in G$.

The zero element of any semi-group (if it exists) is unique for if 0 and 0_1 are two such then $0 = 00_1 = 0_1 0 = 0_1$. A semi-group G is called commutative if its law of composition is also commutative, that is, $xy = yx$ for any x, y in G.

Convention: From now on whenever we speak of a semi-group we will always mean a commutative semi-group with zero element.

Definition 22: A semi-group G is called a Boolean semi-group if there is a unary operation defined on G which attaches to each x in G an element x^1 in G such that (1) $xx^1 = 0$ and (2) $x^1 y = 0$ implies $yx = y$.

The following trivial statement is sometimes useful in proving equalities.

Proposition 19: Let G be a Boolean semi-group. Then $xy^1 = 0$ if and only if $xy = x$.

Proof: Since G is a Boolean semi-group $xy^1 = 0$ implies $xy = x$. Also $x = xy$ implies $xy^1 = xyy^1 = x(yy^1) = x0 = 0$. Hence the proposition.

Let G be any semi-group. In G introduce the following relation: $x \leq y$ if and only if there exists a z in G with $x = yz$. This is the divisibility relation in semi-groups. As in the case of rings the divisibility relation in semi-groups is not a partial order. However, if G is a Boolean semi-group then the divisibility relation is a partial order as the following proposition indicates.

Proposition 20: The divisibility relation in a Boolean semi-group is a partial order.

Proof: (1) $x \leq x$ since $xx^1 = 0$ implies $x = xx$. Hence $\leq$ is reflexive. (2) Suppose $x \leq y$, $y \leq z$. Then since in (1) we see that each element in a Boolean semi-group is idempotent we can say by Proposition 14 that $x \leq y$ if and only if $x = xy$. Thus $x = xy$, $y = yz$. Hence $xz = (xy)z = x(yz) = xy = x$. Thus the transitivity of the relation is established. (3) Suppose $x \leq y$, $y \leq x$. Then $x = xy$ and $y = yx$. Hence $x = y$ and the relation is anti-symmetric. This completes the proof.

Proposition 21: Let G be a Boolean semi-group. Then the transformation $x \longrightarrow x^1$ which attaches to each x in G the element x^1 with $xx^1 = 0$, $xy^1 = 0$ implies $xy = x$ is one to one, onto and order inverting where G is partially ordered by divisibility.

Proof: (1) To show $x \longrightarrow x^1$ is one to one and onto it is enough to show that $x^{11} = (x^1)^1 = x$. We show that $x \leq x^{11}$ and $x^{11} \leq x$. Since G is a Boolean semi-group we have that $x^1 x^{11} = 0$. Hence by Proposition 19, $x^{11} x = x^{11}$ and thus $x^{11} \leq x$ for all x in G. Note that $(x^{11})^1 = (x^1)^{11}$ for $(x^{11})^1 = ((x^1)^1)^1 = (x^1)^{11}$. Now from the first part of the proof we have that $(x^1)^{11} \leq x^1$. Hence $(x^{11})^1 \leq x^1$. Then $x(x^{11})^1 \leq xx^1 = 0$. Therefore $x(x^{11})^1 = 0$, that is, $x \leq x^{11}$. Finally since the divisibility relation is a partial order we have $x = x^{11}$.

(2) Now x y if and only if $xy^1 = 0$. That is by (1) $xy^1 = 0$ if and only if $x^{11}y^1 = 0 = y^1 x^{11}$. Hence $x \leq y$ if and only if $y^1 x^1 = y^1$, that is, if and only if $y^1 \leq x^1$. This completes the proof.

Let G, H be Boolean semi-groups.

Definition 23: A mapping f from G into H is called a Boolean semi-group homomorphism if the following conditions are satisfied:

(1) $f(xy) = f(x)\, f(y)$ for any x, y in G.

(2) $f(x^1) = (f(x))^1$ for any x in G.

We conclude this section by showing that the composite of Boolean semi-group homomorphisms is again a Boolean semi-group homomorphism.

Proposition 22: Let G, H, J be Boolean semi-groups and $f: G \to H$, $g: H \to J$ be Boolean semi-group homomorphisms. Then $(g.f): G \to J$ defined by $(g.f)(x) = g(f(x))$ is a Boolean semi-group homomorphism.

Proof: (1) $(gf)(xy) = g(f(xy)) = g(f(x)f(y)) = (gf)(x)(gf)(y)$, for any x, y in G.

(2) $(gf)(x^1) = g(f(x^1)) = g((f(x))^1) = ((g.f)(x))^1$.

Hence g.f is a Boolean semi-group homomorphism and this completes the proof.

5. Categories and Functors.

In this section we introduce the general notion of a <u>category</u>. The definition to be given is motivated from considerations of common properties of collections such as

(i) topological spaces and continuous mappings,

(ii) groups and their group homomorphisms,

(iii) modules and their module homomorphisms, etc.

<u>Definition 24</u>: A <u>category</u> is a class $\mathcal{C}$ of <u>objects</u> in which the following is satisfied: With any pair X, Y in $\mathcal{C}$ there is associated a set $H(X,Y)$ called <u>the set of maps</u> $f: X \to Y$ such that for any three objects X, Y, Z in $\mathcal{C}$ there is given a mapping $H(X,Y) \times H(Y,Z) \to H(X,Z)$ denoted by $(f,g) \to (g.f)$ which satisfies:

(1) If $f: X \to Y$, $g: Y \to Z$, $h: Z \to T$ then $h.(g.f) = (h.g.).f$.

(2) For each X in $\mathcal{C}$ there exists a map e_X in $H(X,X)$ such that $e_X . f = f$ for all $f \in H(Y,X)$ and $f . e_X = f$ for all $f \in H(X,Y)$.

An element in $H(A,A)$ for any A in $\mathcal{C}$ is called an <u>identity map</u>. The objects of $\mathcal{C}$ are in one to one, onto correspondence $A \to H(A,A)$ with the set of <u>identities</u>. We now introduce the notion of a <u>functor</u> as maps between categories.

<u>Definition 25</u>: Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let T be a function which maps the objects of $\mathcal{C}$ into the objects of $\mathcal{D}$ and, in addition, assigns to each map f in $\mathcal{C}$ a map $T(f)$ in $\mathcal{D}$. The map T is called a <u>covariant functor</u> from $\mathcal{C}$ to $\mathcal{D}$ if it satisfies the following conditions:

(1) If $f \in H(A,B)$ then $T(f) \in H(T(A),T(B))$ for any A, B in $\mathcal{C}$.

(2) If $e_A \in H(A,A)$ then $T(e_A) = e_{T(A)}$ for any object A in $\mathcal{C}$.

(3) If $f \in H(A,B)$, $g \in H(B,C)$ for any A,B,C in $\mathcal{C}$ then $T(g.f) = T(g)T(f)$.

The map T is called a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ if the above conditions are replaced by

(1^1) If $f \in H(A,B)$ then $T(f) \in H(T(B),T(A))$

(2^1) If $e_A \in H(A,A)$ then $T(e_A) = e_{T(A)}$

(3^1) If $f \in H(A,B)$, $g \in H(B,C)$ then $T(g.f) = T(f)\,T(g)$ where A,B,C are any objects in $\mathcal{C}$.

If T is a functor from $\mathcal{C}$ to $\mathcal{D}$ and S is a functor from $\mathcal{D}$ to $\mathcal{E}$ then they may be composed in the obvious manner to form a functor ST from $\mathcal{C}$ to $\mathcal{E}$. If T, S have same (opposite) variance then ST is covariant (contravariant). In view of preoperty (2) above we see that a functor T is completely determined by the function T defined for maps only. Thus a covariant functor is essentially a homomorphism of the maps in $\mathcal{C}$ to the maps in $\mathcal{D}$ subject to the condition that identities be mapped to identities. One functor that always exists is the identity functor $I_{\mathcal{C}}$ defined from $\mathcal{C} \longrightarrow \mathcal{C}$ which keeps each object and map of $\mathcal{C}$ fixed. In the ensuing chapters we will see examples of contravariant and covariant functors.

We now proceed to define transformations between functors.

Definition 24: Let T and S be two covariant functors from the category $\mathcal{C}$ to the category $\mathcal{D}$. A function Γ which assigns to each object $C \in \mathcal{C}$ a map $\Gamma(C) \in \mathcal{D}$ such that

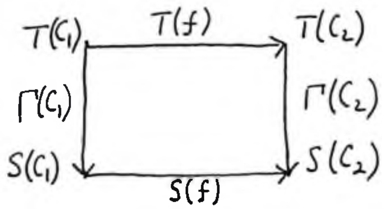
(1) $\Gamma(C) : T(C) \longrightarrow S(C)$

(2) If $f : C_1 \longrightarrow C_2$ then $\Gamma(C_2)\,T(f) = S(f)\,\Gamma(C_1)$

is called a natural transformation of the functor T into the functor S. In case T and S are contravariant functors the condition (2) is replaced by

(2^1) if $f \in H(C_1, C_2)$ then $\Gamma(C_1)\, T(f) = S(f)\, \Gamma(C_2)$.
If the map $\Gamma(C)$ has an inverse $\Gamma'(C)$ such that $\Gamma(C)\, \Gamma'(C)$ and $\Gamma'(C)\, \Gamma(C)$ are the identity maps for each $C \in \mathcal{C}$ then Γ is called a <u>natural equivalence</u> of the functors T and S.

Condition (2) implies that commutativity hold in the following diagram:



It is clear that the notion of <u>natural equivalence</u> introduced above is an equivalence relation. We use this notion to define equivalence between categories as follows:

<u>Definition 25</u>: Let $\mathcal{C}$ and $\mathcal{D}$ be categories. $\mathcal{C}$ and $\mathcal{D}$ are said to be <u>equivalent</u> if there exists a covariant (or contravariant) functor $S : \mathcal{C} \to \mathcal{D}$ and a covariant (or contravariant) functor $T : \mathcal{D} \to \mathcal{C}$ such that the composite functor $ST : \mathcal{D} \to \mathcal{D}$ is <u>naturally equivalent</u> to the identity functor $I_{\mathcal{D}}$ on $\mathcal{D}$ and the composite functor $TS : \mathcal{C} \to \mathcal{C}$ is <u>naturally equivalent</u> to the identity functor $I_{\mathcal{C}}$ on $\mathcal{C}$.

In the above we have introduced the notion of proper maps between Boolean lattices, Boolean spaces, Boolean rings and Boolean semi-groups. In each case we have established that the composite of proper maps is again proper. This fact enables us to give several examples of categories.

<u>EXAMPLES OF CATEGORIES.</u>

(1) The first example of a category is composed of Boolean lattices and Boolean lattice homomorphisms. The objects are Boolean lattices and the

maps are Boolean lattice homomorphisms.

(2) The class $\mathcal{C}$ of Boolean lattices and proper Boolean lattice homomorphisms forms a category.

(3) The class $\mathcal{D}$ consisting of Boolean spaces and proper continuous maps constitutes a category.

(4) The class $\mathcal{E}$ of Boolean rings and ring homomorphisms constitutes a category.

(5) The class $\mathcal{E}'$ consisting of Boolean rings and proper ring homomorphisms forms a category.

(6) As a last example of a category we note that the class $\mathcal{G}$ consisting of Boolean semi-groups and Boolean semi-group homomorphisms constitute a category.

Remark: In the ensuing chapters we will establish that the categories $\mathcal{C}$ and $\mathcal{D}$ of examples (2) and (3) are _equivalent_, and that $\mathcal{C}$ is _equivalent_ to the category $\mathcal{E}'$. It should be noted that the notion of _proper homomorphisms_ is such that when one, for example, restricts the objects in $\mathcal{C}$ to Boolean lattices with unit and the objects in $\mathcal{E}'$ to Boolean rings with unit, the proper Boolean lattice homomorphisms become unitary Boolean lattice homomorphisms and the proper ring homomorphisms become unitary. Hence when we establish that the categories $\mathcal{C}$ and $\mathcal{E}'$ are _equivalent_ we automatically establish that the categories of Boolean lattices with unit and unitary homomorphisms is equivalent to the category of Boolean rings with unit and unitary homomorphisms.

CHAPTER I.

BOOLEAN LATTICES AND BOOLEAN SPACES.

1. The Ultrafilter Space of a Boolean Lattice.

In this paragraph we study the consequences of introducing a topology on the set of all ultrafilters of a Boolean lattice B. The corresponding topological space that results will be referred to as the ultrafilter space of B. The main result we will obtain here is that the ultrafilter space of a Boolean lattice is a Boolean space. We will also characterize the open, closed, and compact open sets of the ultrafilter space of B in terms of the ideals and filters in B.

Let B be a Boolean lattice and let $\Omega = \Omega(B)$ denote the set of all the ultrafilters of B. For each a in B define (1) $\Omega(a) = \{U/a \in U \in \Omega\}$. Let $\mathcal{R} = \{\Omega(a)/a \in B\}$. Then $\mathcal{R}$ has the properties which enables it to be used as a basis for a topology on Ω : namely (i) $\Omega(a) \cap \Omega(b) = \Omega(a \wedge b)$, that is, $\mathcal{R}$ is closed under intersection (ii) $\bigcup \mathcal{R} = \Omega$. Property (ii) is obvious. We check (i). Note that $a \wedge b \in U$ if and only if $a \in U$ and $b \in U$ since U is a filter. This establishes (i). Let $O(\mathcal{R})$ denote the topology generated by $\mathcal{R}$. We will speak of the topological space $(\Omega, O(\mathcal{R}))$ as the ultrafilter space of B. For any ideal I in B define:

(2) $\mathcal{M}(I) = \{U/U \in \Omega,\ U \cap I \neq \phi\}$

(3) $\mathcal{D}(I) = \{U/U \in \Omega,\ U \cap I = \phi\}$

For each filter F in B define:

(4) $\mathcal{E}(F) = \{U/U \in \Omega \ , \ U \supseteq F\}$

The main properties of the ultrafilter space of B are established in the following theorem.

Theorem 1 : Let B be a Boolean lattice and Ω its ultrafilter space. Then

(i) Ω is a Boolean space.

(ii) Ω is compact if and only if B has a unit.

(iii) The character of Ω (i.e. the least cardinal number belonging to a basis in Ω) is equal to the cardinality of B if B is infinite.

(iv) Let X be any subset of Ω . Then the closure of X in Ω , denoted by ΓX, is $\{U/U \in \Omega \ , \ U \subseteq \bigcup_{V \in X} V\}$

(v) The compact open sets of Ω are precisely the sets Ω (a).

(vi) The compact sets of Ω are precisely the sets $\mathcal{E}$ (F).

(vii) The open sets of Ω are precisely the sets $\mathcal{M}$(I).

(viii) The closed sets of Ω are precisely the sets $\mathcal{D}$(I).

Proof: (i) We establish first that the sets Ω(a) are compact. Let $\Omega(a) \subseteq \bigcup_{x \in L} \Omega(x)$ for some subset L of B but suppose Ω(a) strictly contains $\Omega(x_1) \cup \ldots \cup \Omega(x_n)$ for any finite number n with $\{x_1, \ldots, x_n\} \subseteq L$ Let I be the ideal generated by L in B. Then $I = \{y/y \in B, \ y \leq \vee Y$ with $Y \subseteq L$ and Y finite$\}$. Hence by the definition of I and our supposition $a > y$ for each y in I. Let $H = \{z/z = a \sim y, \ y \in I\}$. Since I is an ideal, H is a proper filter basis with a in H. Let U be an ultrafilter containing H. By construction $a \in U$ and $y \notin U$ for any y in I. Hence $U \notin \bigcup_{x \in L} \Omega(x)$. This, however, is a contradiction for $U \in \Omega(a)$ and $\Omega(a) \subseteq \bigcup_{x \in L} \Omega(x)$. Hence our supposition is false and the sets Ω (a) are compact. Take any $U \in \Omega$

and pick an arbitrary point $a \in U$. Then $\Omega(a)$ is a compact basic open neighbourhood of U. Hence each point of Ω has a compact neighbourhood, that is, it is locally compact. Let U, V be any two distinct points of Ω. Then there exists $a \in U$ with $a \notin V$. $a \notin V$ implies since V is maximal proper that there exists a $b \in V$ with $b \wedge a = 0$. Then $\Omega(a)$, $\Omega(b)$ are basic open neighbourhoods of U, V respectively and these neighbourhoods are clearly disjoint. Hence Ω is a Hausdorff space. In a Hausdorff space any compact set is closed. The set $\mathcal{R} = \{\Omega(a) / a \in B\}$ which generates the topology of Ω is thus composed of open closed sets. Hence Ω is a zero dimensional space. Collecting all the results together we have established that Ω is a Boolean space.

(ii) Suppose B has a unit e. Then $\Omega = \Omega(e)$ and by (i) $\Omega(e)$ is compact. Hence Ω is compact. Conversely suppose Ω is compact. Then there exists some finite subset A of B with $\Omega = \bigcup\{\Omega(a) / a \in A\}$. Let $\vee A = b$. Then $\Omega = \Omega(b)$. Hence $b \in U$ for each U in Ω. This means b must be the unit of B since the unit is the only element which belongs to every ultrafilter of B. Hence B has a unit.

(iii) Let B be infinite. Let $\mathcal{R} = \{\Omega(a) / a \in B\}$ and consider the map f: $B \longrightarrow \mathcal{R}$ given by $f(a) = \Omega(a)$. This map is clearly onto. We show that it is also one to one. Take $a \neq b$ in B and without loss of generality suppose $0 \neq a \nleq b$. Let F be the filter generated by a and let U be an ultrafilter containing F. Then $U \in \Omega(a)$ but $U \notin \Omega(b)$. That is $\Omega(a) \neq \Omega(b)$. Hence f is one to one. Since $\mathcal{R}$ is a basis for the topology on Ω we now have that the character of Ω is less equal the cardinality of $B = |\mathcal{R}|$ On the other hand, suppose $(G_\alpha)_{\alpha \in I}$ is a basis for Ω. Then each $\Omega(a)$

being open is the union of appropriate G_α. The compactness of $\Omega(a)$ yields indices $\alpha_1,\dots,\alpha_n$ with $\Omega(a) = \bigcup_{i=1}^{n} G_{\alpha_i}$. Since every $\Omega(a)$ is the union of a finite number of G's the cardinal number of $\mathcal{R}$, does not exceed that of the basis in question using the fact that B is infinite. That is $/\mathcal{R}/ = /B/ \leq /\{G_\alpha/\alpha\in I\}/$. Hence the character of Ω is equal to the cardinality of B.

Remark: In case B is a finite Boolean lattice then $/B/ = 2^n$ for some natural number $n \geq 0$. Hence B has n minimal non-zero elements say $a_1,\dots,a_n$. Each ultrafilter in B is principal and is generated by one of the elements $a_1,\dots,a_n$. Then Ω has n elements; every subset of Ω is open and a basis for the topology on Ω say the set $\{\Omega(a_1),\dots,\Omega(an)\}$ is composed of n elements. Thus in the case B is finite the character of Ω is strictly less than the cardinality of B.

(iv) Let X be any subset of Ω. By the definition of closure we have $\Gamma X = \{U/U\in\Omega,\ \Omega(a)\cap X \neq \phi \text{ for each } a\in U\}$.
Let $Y = \{U/U\in\Omega,\ U\subseteq \bigcup_{V\in X} V\}$. We must show that $\Gamma X = Y$. Suppose $U\in\Gamma X$. Then $\Omega(a)\cap X\neq\phi$ for each $a\in U$. Hence for each a in U there exists a V in X with $a\in V$. This implies that for each $a\in U$, $a\in\bigcup_{V\in X} V$, which means that $U\in Y$. On the other hand suppose $U\in Y$. Then for each $a\in U$ there exists a $V\in X$ with $a\in V$. This means that $U\in\Gamma X$. Therefore $\Gamma X = Y$.

(v) In (i) we showed that the sets $\Omega(a)$ were compact. These sets are open by definition. We now establish the converse. Let O be any compact open set of Ω. Then O open implies $O = \bigcup\{\Omega(a)/a\in A\}$ for some suitable subset A of B. Since O is compact there exists $a_1,\dots,a_n$ in A with $O = \bigcup_{i=1}^{n}\Omega(a_i) = \Omega(a_1 \vee\dots\vee a_n)$. Let $b = a_1\vee\dots\vee a_n$. Then $O = \Omega(b)$ as required.

(vi) We now show that sets of the form $\mathcal{E}(F)$ are compact. We note trivially that $\mathcal{E}(F) = \bigcap_{a \in F} \Omega(a)$. Since each $\Omega(a)$ is a closed set we have that $\mathcal{E}(F)$ is closed. Also $\mathcal{E}(F) \subseteq \Omega(a)$ for each a in F. Hence $\mathcal{E}(F)$ is a closed subset of a compact set and hence $\mathcal{E}(F)$ is compact. Conversely, let T be any compact subset of Ω. T compact implies, since Ω is a Boolean space by (i), that $T = \bigcap \{\Omega(a)/T \subseteq \Omega(a)\}$ applying (v) and Proposition 10 of the previous chapter. Let $F = \{a/a \in B, T \subseteq \Omega(a)\}$. It is clear that F is a filter in B and $\mathcal{E}(F) = \{U/U \in \Omega, U \supseteq F\} = \bigcap_{a \in F} \Omega(a)$. Hence $T = \mathcal{E}(F)$.

(vii) We first show that any set of the form $\mathcal{M}(I)$ is open. By definition $\mathcal{M}(I) = \{U/U \in \Omega, U \cap I \neq \phi\}$. We show that $\mathcal{M}(I) = \bigcup_{a \in I} \Omega(a)$. Take any $U \in \mathcal{M}(I)$. Then there exists $a \in I$ with $a \in U$, that is, $U \in \Omega(a)$. Hence U is in the right hand side. On the other hand take any ultrafilter U in the right hand side. This implies there exists $a \in I$ with $a \in U$, that is, $U \cap I \neq \phi$. Hence our claim is established and $\mathcal{M}(I)$ is open. Conversely let O be any open set for Ω. Let $I = \{a/a \in B, \Omega(a) \subseteq O\}$. Then I is non-empty set, for example $o \in I$, and I is clearly an ideal. Also since O is open and the sets $\Omega(a)$ is a basis we have $O = \bigcup_{a \in I} \Omega(a)$. We now show that $O = \mathcal{M}(I)$. Take any $U \in O$. Then $U \in \Omega(a)$ for some $a \in I$. Hence $U \cap I \neq \phi$ and thus $U \in \mathcal{M}(I)$. On the other hand if $U \in \mathcal{M}(I)$ then there exists $a \in I$ with $U \in \Omega(a) \subseteq O$. This establishes that the sets $\mathcal{M}(I)$ are precisely the open sets of Ω.

(viii) The sets $\mathcal{D}(I)$ are now seen to be precisely the closed subsets of Ω since for each ideal I in B, $\mathcal{D}(I) = C_\Omega \mathcal{M}(I)$. This completes the proof of the theorem.

Theorem 1 has given us an insight into the topological features of the ultrafilter space Ω of B in terms of the lattice theoretic notions of filters and ideals. A fact of additional interest is given in the next proposition.

Proposition 1 : Let B be any Boolean lattice and Ω its ultrafilter space. Let F, G, $\mathcal{R}$, partially ordered by inclusion, be the lattices of open, closed and compact open sets respectively. Let $\mathcal{I}$ denote the lattice of all ideals in B. Then

(i) Each of the lattices, F, G are lattice isomorphic to $\mathcal{I}$, the isomorphism f from $\mathcal{I}$ to F being given by $f(I) = \mathcal{M}(I)$.

(ii) The lattice $\mathcal{R}$ is isomorphic to B by the mapping $f: B \rightarrow \mathcal{R}$ given by $f(a) = \Omega(a)$.

Proof: (i) Consider $f: \mathcal{I} \rightarrow F$ given by $f(I) = \mathcal{M}(I)$. For any I, J in $\mathcal{I}$ we have by Proposition 5 in the previous chapter that $I \vee J = \{a \vee b / a \in I,\ b \in J\}$, $I \wedge J = \{a \wedge b / a \in I,\ b \in J\}$. We now establish

(1) $\mathcal{M}(I) \cup \mathcal{M}(J) = \mathcal{M}(I \vee J)$.

(2) $\mathcal{M}(I) \cap \mathcal{M}(J) = \mathcal{M}(I \wedge J)$.

Regarding (1) : Take any $U \in \mathcal{M}(I) \cup \mathcal{M}(J)$. Then $U \cap I \neq \phi$ or $U \cap J \neq \phi$ Hence $U \cap (I \cup J) \neq \phi$ and thus $U \cap (I \vee J) \neq \phi$. Therefore $U \in \mathcal{M}(I \vee J)$. On the other hand take $U \in \mathcal{M}(I \vee J)$. Then $U \cap (I \vee J) \neq \phi$. Then there exists $a \in U$ with $a = s \vee t$ for some s in I, t in J. Since U is an ultrafilter either s or t is in U. Thus either $U \cap I \neq \phi$ or $U \cap J \neq \phi$. Therefore $U \in \mathcal{M}(I) \cup \mathcal{M}(J)$. This shows (1).

Regarding (2) : Take $U \in \mathcal{M}(I) \cap \mathcal{M}(J)$. Then $U \cap I$ and $U \cap J$ are both non-empty. Pick $a \in U \cap I$, $b \in U \cap J$. Then $a \wedge b \in I \wedge J$ and $a \wedge b \in U$. Hence

$U \cap (I \wedge J) \neq \phi$. Thus $U \in \mathcal{M}(I \wedge J)$. Conversely take any $U \in \mathcal{M}(I \wedge J)$. Then there exists $a \in U$ with $a = s \wedge t$ where $s \in I$, $t \in J$. Then $s \in U \cap I \neq \phi$ and $t \in U \cap J \neq \phi$. Thus $U \in \mathcal{M}(I) \cap \mathcal{M}(J)$. This proves (2). (1) and (2) together imply that the mapping f is a lattice homomorphism. Moreover f is clearly onto. Next take I, J in $\mathcal{I}$ with $I \neq J$. Then without loss of generality there exists $b \in I$ with $b \notin J$. Now $\mathcal{M}(I) = \bigcup\{\Omega(a)/a \in I\}$ and $\mathcal{M}(J) = \bigcup\{\Omega(a)/a \in J\}$. Since $b \notin J$, $\Omega(b)$ is not contained in $\mathcal{M}(J)$ but is contained in $\mathcal{M}(I)$. Hence $\mathcal{M}(I) \neq \mathcal{M}(J)$. Thus f is one to one. This establishes that f is a lattice isomorphism.

(ii) Consider the map $g: B \to \mathcal{R}$ given by $g(a) = \Omega(a)$. By part (i) we have that $\Omega(a \wedge b) = \Omega(a) \cap \Omega(b)$, $\Omega(a \vee b) = \Omega(a) \cup \Omega(b)$ since $\Omega(a) = \mathcal{M}((a))$ where (a) is the principal ideal generated by a. Hence g is a lattice homomorphism. In Theorem 1 we established that the mapping g was one to one and onto. Hence g is a lattice isomorphism.

<u>It follows directly from Proposition 1 that the lattice $\mathcal{R}$ of all the compact open subsets of Ω is a Boolean lattice.</u>

2. <u>The Boolean lattice of a Boolean Space.</u>

In the last section we saw that the ultrafilter space of any Boolean lattice is a Boolean space. We now proceed to establish the counterpart of Theorem 1.

<u>Theorem 2</u> : Let E be a Boolean space. Then

(i) The set $\mathcal{L}(E)$ of compact open subsets of E partially ordered by set inclusion is a Boolean lattice in which meet is set intersection, join is set union and Boolean complement is relative complement.

(ii) The character of E is equal to the cardinality of $\mathcal{L}(E)$ if E is infinite.

(iii) The ultrafilter space Ω of $\mathcal{L}(E)$ is homeomorphic to E, the homeomorphism $f: E \longrightarrow \Omega$ being given by $f(x) = F(x)$ where $F(x) = \{X / x \in X \in \mathcal{L}(E)\}$.

(iv) The filters in the Boolean lattice $\mathcal{L}(E)$ are precisely the sets $F(K) = \{V / K \subseteq V \in \mathcal{L}(E)\}$ where K ranges over all the compact subsets of E.

(v) The ideals in $\mathcal{L}(E)$ are given precisely by the sets $I(O) = \{V / O \supseteq V \in \mathcal{L}(E)\}$ where O ranges over all the open subsets of E.

Proof: (i) Take any X, Y in $\mathcal{L}(E)$. Then since E is a Hausdorff space X, Y are closed. Thus $X \cap Y$ is a closed subset of a compact set X and hence $X \cap Y$ is compact. Also $X \cap Y$ is open since X, Y are open. Thus $X \cap Y \in \mathcal{L}(E)$. Also since X, Y are compact open so is $X \cup Y$. Next take any X, Y in $\mathcal{L}(E)$ with $\phi \subseteq X \subseteq Y$. Then $Y - X = Y \cap \complement X$ is an open-closed subset of the compact set Y. Hence $Y - X$ is compact open and hence belongs to $\mathcal{L}(E)$. Since set intersection and set union distribute over each other we have that $\mathcal{L}(E)$ is a relatively complemented, distributive lattice. Also $\phi \subseteq X$ for any $X \in \mathcal{L}(E)$ and ϕ is compact open. Hence $\mathcal{L}(E)$ is a Boolean lattice.

(ii) Let E be infinite. Let $\{(G_\alpha) / \alpha \in I\}$ be any basis for open sets of E. Since by Proposition 9 of the previous chapter $\mathcal{L}(E)$ is a basis for open sets of E the character of E is less equal the cardinality of $\mathcal{L}(E)$. Let X be any member of $\mathcal{L}(E)$. X open implies X is the union of a suitable number of G's. The compactness of X then yields indices $\alpha_1, \ldots \alpha_n$ with $X = \bigcup_{i=1}^{n} G_{\alpha_i}$. Hence every member of $\mathcal{L}(E)$ is the union of a finite number of elements from the basis under consideration. Since E is infinite this means $|\mathcal{L}(E)| \leq |\{G_\alpha / \alpha \in I\}|$. Hence the character of E is equal to the cardinality of $\mathcal{L}(E)$.

Remark: In case E is a finite Boolean space then the Boolean lattice $\mathcal{L}(E)$ is finite, say $|\mathcal{L}(E)| = 2^n$ for some $n \geq 0$. Then $\mathcal{L}(E)$ has n minimal non-zero members say $X_1, \ldots, X_n$ and every member of $\mathcal{L}(E)$ is a union of suitable $X_1, \ldots, X_n$. By Proposition 9 in the previous chapter $\mathcal{L}(E)$ is a basis for open sets. Hence $X_1, \ldots, X_n$ is also a basis for E and the character of E in this case is then strictly less the cardinality of $\mathcal{L}(E)$.

(iii) Consider the mapping $f: E \rightarrow \Omega$ given by $f(x) = F(x)$ where $F(x)$ is as stated. $F(x)$ is clearly a filter in $\mathcal{L}(E)$. Let F be any filter in $\mathcal{L}(E)$ with $F(x) \subseteq F$. Take any X in F. Let $\mathcal{G} = \{Y \cap X / Y \in F(x)\}$. Then $\mathcal{G}$ is a system of closed sets and is a proper filter basis. Also each member of $\mathcal{G}$ is contained in the compact set X. Hence $\bigcap \mathcal{G} = \bigcap_{x \in Y} Y \cap X \neq \phi$. Now $\bigcap_{x \in Y} Y = \{x\}$ for take any a in E with $a \neq x$. Then since E is a Hausdorff space which has $\mathcal{L}(E)$ for a basis there exists compact open neighbourhoods W of a and Y of x with $W \cap Y = \phi$. This implies $a \notin Y$ and hence $a \notin \bigcap_{x \in Y} Y$. Thus $\bigcap_{x \in Y} Y = \{x\}$. This means, $\bigcap \mathcal{G} = \{x\} \cap X \neq \phi$, that is, $x \in X$. Hence $X \in F(x)$ and $F \subseteq F(x)$, that is, $F = F(x)$. Therefore $F(x)$ is an ultrafilter and hence belongs to Ω, the ultrafilter space of $\mathcal{L}(E)$. Take x, y two distinct points of E. Then since E is a Hausdorff space and $\mathcal{L}(E)$ is a basis for open sets, there exists X, Y in $\mathcal{L}(E)$ with $x \in X$ and $y \notin X$. Hence $F(x) \neq F(y)$ and f is one to one. Next let F be any ultrafilter in $\mathcal{L}(E)$. Then by a similar argument as above $\bigcap_{X \in F} X \neq \phi$. Take any point $x \in \bigcap_{X \in F} X$. Then clearly $F \subseteq F(x)$. But F is maximal proper and hence $F = F(x)$. This shows that the map f is onto.

Take any $X \in \mathcal{L}(E)$. Then $f(X) = \left\{ F(x)/x \in X \right\} = \left\{ F(x)/X \in F(x) \right\} = \Omega(X)$, a basic open set in Ω. Hence f is one to one, onto and carries basic open sets of E into basic open sets of Ω. Hence f is a homeomorphism.

(iv) Let K be any compact subset of E. Then $F(K)$ as stated is certainly a filter in $\mathcal{L}(E)$. On the other hand let F be any filter in $\mathcal{L}(E)$. Put $M = \left\{ x/x \in E, F(x) \supseteq F \right\}$. The set M is the inverse image under the mapping f of part (iii) of the set $K = \left\{ F(x)/F(x) \in \Omega, F(x) \supseteq F \right\}$. By Theorem 1, K is compact and since f is a homeomorphism we have that M is compact. Moreover $F(M) = F$. To establish this, take any $V \in F$ and then since $x \in M$ implies $F(x) \supseteq F$ we have $V \in F(x)$. This means $x \in V$ for any $x \in M$. Hence $M \subseteq V$. Therefore $V \in F(x)$. On the other hand, take any $V \in F(M)$. Then $x \in V$ for each $x \in M$. That is $V \in F(x)$ for each $x \in M$ and thus $V \in \bigcap_{x \in M} F(x)$. But by Proposition 2 of the previous chapter we have $\bigcap_{x \in M} F(x) = F$. Hence $V \in F$. This establishes (iv).

(v) For any open set O, $I(O)$ is certainly an ideal in $\mathcal{L}(E)$. On the other hand, let I be any ideal in $\mathcal{L}(E)$. Put $O = \bigcup_{X \in I} X$. Then O is an open subset of E and we show $I(O) = I$. Take any $V \in I$. Then $V \subseteq O$ trivially and hence $V \in I(O)$. Conversely take any $V \in I(O)$. Then $V \subseteq O \subseteq \bigcup_{X \in I} X$. The compactness of V implies there exists $X_1, \ldots, X_n$ in I with $V \subseteq X_1 \cup \ldots \cup X_n$. Since I is an ideal $X_1 \cup \ldots \cup X_n \in I$ and thus $V \in I$. Hence the sets $I(O)$ describe precisely the ideals in $\mathcal{L}(E)$. This completes the proof.

Theorem 2 has given us a complete description of the ideals and filters in the Boolean lattice of compact open subsets of a Boolean space E in terms of the topological properties of E. Theorems 1 and 2 together have shown that the ultrafilter space of any Boolean lattice is a Boolean space and conversely that any Boolean space E gives rise to a

Boolean lattice whose ultrafilter space is homeomorphic to E. Finally we have the following proposition analogous to Proposition 1. We state it here without proof since its proof is given in the same manner as Proposition 1.

Proposition 2 : Let E be any Boolean space and let $\mathcal{L}(E)$ be its lattice of compact open sets. Then (i) the ideal lattice $\mathcal{I}$ of $\mathcal{L}(E)$ is lattice isomorphic to the lattice of all open subsets of E, the isomorphism being given by attaching to any open set O of E the ideal I(O) in $\mathcal{I}$. (ii) the sublattice in $\mathcal{I}$ consisting of all the principal ideals of $\mathcal{L}(E)$ is a Boolean lattice isomorphic to $\mathcal{L}(E)$.

3. Equivalence of Categories.

Let $\mathcal{C}$ denote the category whose objects are Boolean lattices and whose maps are proper Boolean lattice homomorphisms. Let $\mathcal{D}$ denote the category whose objects are Boolean spaces and whose maps are proper continuous mappings. In this section we establish that the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

Let B, C be any two Boolean lattices and let $\Omega(B)$, $\Omega(C)$ denote their respective ultrafilter spaces. Let $f: B \to C$ be a proper Boolean lattice homomorphism. Then f gives rise to a map $f_\Omega : \Omega(C) \to \Omega(B)$ defined as follows:

$$f_\Omega(U_C) = f^{-1}(U_C) = \{x / x \in B,\ f(x) \in U_C\}$$

for any U_C in $\Omega(C)$. Since f is a proper homomorphism and U_C is a filter, $f^{-1}(U_C) \neq \phi$. Moreover $f^{-1}(U_C)$ is an ultrafilter in B since f is a Boolean lattice homomorphism and U is an ultrafilter in C. Thus $f_\Omega(U_C)$ is a member of $\Omega(B)$. Take any basic open set $\Omega(a)$ in $\Omega(B)$.

Then $f_\Omega^{-1}(\Omega(a)) = f_\Omega^{-1}\{U_B/a\in U_B\} = \{U_C/f(a)\in U_C\} = \Omega(f(a))$, a basic open set in $\Omega(C)$ where $f^{-1}(U_C) = U_B$ an ultrafilter in B with $a\in U_B$. Hence the inverse image under f_Ω of any basic open set in $\Omega(B)$ is basic open in $\Omega(C)$. Hence f_Ω is a continuous map. Since the set of all $\Omega(a)$ as a ranges through B characterize the compact open sets of $\Omega(B)$ we have by Proposition 11 of the previous chapter that f is proper continuous.

On the other hand let E, F be any two Boolean spaces and let $\mathcal{L}(E)$, $\mathcal{L}(F)$ denote their respective Boolean lattices of compact open sets. Let $f : E\rightarrow F$ be a proper continuous map. Then f gives rise to a map $f_{\mathcal{L}}: \mathcal{L}(F)\rightarrow\mathcal{L}(E)$ defined by $f_{\mathcal{L}}(X) = f^{-1}(X)$ for any X in $\mathcal{L}(F)$. Since X is compact open and f is proper continuous we have that $f^{-1}(X)$ is compact open and hence belongs to $\mathcal{L}(E)$. We proceed to show that f is a proper Boolean lattice homomorphism. For any X, Y in $\mathcal{L}(F)$ we have:

(1) $f_{\mathcal{L}}(X\cap Y) = f^{-1}(X\cap Y) = f^{-1}(X)\cap f^{-1}(Y) = f_{\mathcal{L}}(X)\cap f_{\mathcal{L}}(Y)$.

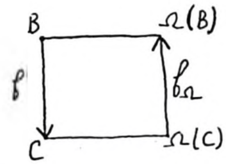
(2) $f_{\mathcal{L}}(X\cup Y) = f^{-1}(X\cup Y) = f^{-1}(X)\cup f^{-1}(Y) = f_{\mathcal{L}}(X)\cup f_{\mathcal{L}}(Y)$.

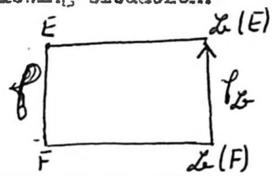
If X, $Y\in\mathcal{L}(F)$ with $\phi\subseteq X\subseteq Y$ then

(3) $f_{\mathcal{L}}(Y-X) = f^{-1}(Y-X) = f^{-1}(Y) - f^{-1}(X) = f_{\mathcal{L}}(Y) - f_{\mathcal{L}}(X)$.

Hence $f_{\mathcal{L}}$ is a Boolean lattice homomorphism. Let X in $\mathcal{L}(E)$ be arbitrary. Then $f(X) = Y$ is a compact set in F since f is continuous. Take a compact open set Z in F with Z containing Y. Such a Z exists since Y is compact and $\mathcal{L}(F)$ is a basis for open sets of F. Then Z is in $\mathcal{L}(F)$ and $f_{\mathcal{L}}(Z) = f^{-1}(Z)$ contains $f^{-1}(Y) \supseteq X$. Hence $f_{\mathcal{L}}$ is a proper Boolean lattice homomorphism.

Diagrammatically we have the following situation:





The main result of this section is the following theorem.

<u>Theorem 3</u> : The correspondences $\left.\begin{array}{l} B \longrightarrow \Omega(B) \\ f \longrightarrow f_\Omega \end{array}\right\}$ and $\left.\begin{array}{l} E \longrightarrow \mathcal{Z}(E) \\ f \longrightarrow f_{\mathcal{Z}} \end{array}\right\}$

are contravariant functors $T : \mathcal{C} \longrightarrow \mathcal{D}$ and $S : \mathcal{D} \longrightarrow \mathcal{C}$ which establish the equivalence of the categories $\mathcal{C}$ and $\mathcal{D}$.

<u>Proof</u>: (i) Let A, B, C be Boolean lattices. The map T is certainly well defined. Also we have,

(1) if $f \in H(A,B)$ then $T(f) = f_\Omega \in H(\Omega(B), \Omega(A)) = H(T(B), T(A))$.

(2) if $o_A \in H(A,A)$ then $T(o_A) = (o_A)_\Omega \in H(\Omega(A), \Omega(A)) = o_{T(A)}$.

(3) Finally suppose $f \in H(A,B)$ $g \in H(B,C)$. We must show that $T(gf) = T(f)\,T(g)$. Now $T(gf) = (gf)_\Omega$ and $T(f)\,T(g) = f_\Omega\, g_\Omega$. Hence we need to show that $(gf)_\Omega(U_C) = (f_\Omega \cdot g_\Omega)(U_C) = f_\Omega(g_\Omega(U_C))$ for any U_C in $\Omega(C)$. Now $(gf)_\Omega(U_C) = (gf)^{-1}(U_C) = \{x / x \in A,\ g(f(x)) \in U_C\}$ and $(f_\Omega \cdot g_\Omega)(U_C) = f^{-1}(g^{-1}(U_C))$. Take any y in $(g.f)_\Omega(U_C)$. Then $g(f(y)) = u$ for some $u \in U_C$. This means $f(y) \in g^{-1}(u)$, that is, $y \in f^{-1}(g^{-1}(u))$. Hence $y \in f^{-1}(g^{-1}(U_C)) = (f_\Omega \cdot g_\Omega)(U_C)$. On the other hand take any $x \in (f_\Omega \cdot g_\Omega)(U_C)$. Then $x \in (f^{-1}g^{-1})(u)$ for some $u \in U_C$. This implies $x \in f^{-1}(v)$ where $v \in g^{-1}(u)$. Hence $f(x) = v \in g^{-1}(U_C)$. Thus $g(f(x)) = u \in U_C$. Therefore $x \in (gf)_\Omega(U_C)$. Hence we have established that $(gf)_\Omega(U_C) = (f_\Omega \cdot g_\Omega)(U_C)$ for any $U_C \in \Omega(C)$. Hence $T(gf) = T(f)\,T(g)$, that is, T is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.

(ii) Let E, F, G be Boolean spaces.

(1) if $f \in H(E,F)$ then $S(f) = f_{\mathcal{L}} \in H(\mathcal{L}(F), \mathcal{L}(E))$.

(2) if $e_E \in H(E,E)$ then $S(e_E) \in H(\mathcal{L}(E), \mathcal{L}(E)) = e_{S(E)}$.

(3) Finally suppose $f \in H(E,F)$ and $g \in H(F,G)$. We must show that $S(gf) = S(f)\ S(g)$. Thus we need to establish that $(gf)_{\mathcal{L}}(X) = f_{\mathcal{L}}(g_{\mathcal{L}}(X))$ for any X in $\mathcal{L}(G)$. Now $(gf)_{\mathcal{L}}(X) = (gf)^{-1}(X) = \{x/x \in E,\ g(f(x)) \in X\}$ and $f_{\mathcal{L}}(g_{\mathcal{L}}(X)) = f^{-1}(g^{-1}(X)) = \{f^{-1}(g^{-1}(x))/x \in X\}$. Take any $y \in (gf)_{\mathcal{L}}(X)$. Then $g(f(y)) = x$ for some $x \in X$ and thus $y \in g^{-1}(f^{-1}(x))$ for some $x \in X$. Therefore $y \in (gf)_{\mathcal{L}}(X)$. On the other hand if $y \in (f_{\mathcal{L}} \cdot g_{\mathcal{L}})(X)$, then $y = f^{-1}(g^{-1}(x))$ for some $x \in X$. Hence $g(f(y)) \in X$ which implies $y \in (gf)_{\mathcal{L}}(X)$. This shows that $S(gf) = S(f)S(g)$, that is, S is a contravariant functor from $\mathcal{D}$ to $\mathcal{C}$.

(iii) Since S, T are contravariant functors, the composites ST and TS are covariant functors on $\mathcal{C}$, $\mathcal{D}$ respectively. Let $I_{\mathcal{C}}$, $I_{\mathcal{D}}$ denote the identity functors on $\mathcal{C}$, $\mathcal{D}$ respectively. In order to establish that the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent we must show that there is a natural equivalence Γ_1 of the functors ST and $I_{\mathcal{C}}$ and a natural equivalence Γ_2 of the functors TS and $I_{\mathcal{D}}$. In Proposition 1 we saw that any Boolean lattice B is isomorphic to $\mathcal{L}(\Omega(B))$ the isomorphism $i(B)$ being given by $i(B)(a) = \Omega(a)$. Define a map Γ_1 which assigns to each $B \in \mathcal{C}$ the map $i_B: I_{\mathcal{C}}(B) \longrightarrow ST(B)$. Let B, C be in $\mathcal{C}$ and let $f : B \longrightarrow C$ be a proper homomorphism. We show that $i_C\, f = (f_\Omega)_{\mathcal{L}}\, i_B$. Take any $a \in B$. Then $(i_C f)(a) = i_C(f(a)) = \Omega(f(a))$, the compact open set in $\Omega(C)$ determined by $f(a)$. On the other hand we have $(f_\Omega)_{\mathcal{L}}\, i_B(a) = (f_\Omega)_{\mathcal{L}}(\Omega(a)) = f_\Omega^{-1}(\Omega(a)) = \Omega(f(a))$. Hence $i_C f = (f_\Omega)_{\mathcal{L}}\, i_B$. Thus Γ_1 is a natural transformation of the functor $I_{\mathcal{C}}$ into the functor ST. Also for each

$B \in \mathcal{C}$, the map $i(B)$ being one to one and onto has an inverse. Hence is a natural equivalence of the functors ST and $I_{\mathcal{C}}$. We now show that there is a natural equivalence of the functors TS and $I_{\mathcal{D}}$. For each $E \in \mathcal{D}$ let $i(E)$ denote the homeomorphism between E and $\Omega(\mathcal{L}(E))$ given as in Theorem 2. Consider the map Γ_2 which assigns to each E in $\mathcal{D}$ the map $i(E) : I_{\mathcal{D}}(E) \longrightarrow TS(E)$. Then take any E, F in $\mathcal{D}$ and let $f : E \longrightarrow F$ be a proper continuous map. We now show that $i_F f = (f_{\mathcal{L}})_{\Omega}\, i_E$. Take any x in E. Then $i_F(f(x)) = F(f(x)) = \{Y/f(x) \in Y \in \mathcal{L}(F)\}$. On the other hand $(f_{\mathcal{L}})_{\Omega}\, i_E(x) = (f_{\mathcal{L}})_{\Omega}\, (F(x))$.

$$= f_{\mathcal{L}}^{-1}\{Y/x \in Y \in \mathcal{L}(E)\}$$
$$= \{Z/f(x) \in Z \in \mathcal{L}(F)\}$$

Hence $i_F f = (f_{\mathcal{L}})_{\Omega}\, i_E$. Since for each E in $\mathcal{D}$, $i(E)$ has an inverse $i^{-1}(E)$ with $i(E)i^{-1}(E)$ and $i^{-1}(E)i(E)$ the identity maps, Γ_2 is a natural equivalence between the functors TS and $I_{\mathcal{D}}$. Hence the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent. This completes the proof.

As a direct consequence of Theorem 3 we observe that a Boolean lattice is determined up to isomorphism by its ultrafilter space, that is, two Boolean lattices are isomorphic if and only if their ultrafilter spaces are homeomorphic. Moreover the group of automorphisms of any Boolean lattice is isomorphic to the group of homeomorphisms of its ultrafilter space.

4. The Spaces $\Omega(I)$ and $\Omega(B/I)$.

Given a Boolean lattice B and its ultrafilter space $\Omega(B)$, one might wonder what relation the ultrafilter space of an ideal I in B considered as a Boolean lattice bears to the space $\Omega(B)$? We shall see in the following that the open subspaces of Ω (B) correspond precisely to the ultrafilter spaces of ideal I in B.

Let B be a Boolean lattice and let I be any ideal in B. Consider the open set $\mathcal{M}(I) = \{ U / U \cap I \neq \phi \}$ corresponding to this ideal I.

Notation : For any ultrafilter U in I, let $[U]$ denote the filter generated by U in B.

One has the following lemma:

Lemma 1 : Let B be a Boolean lattice and I any ideal in B. Let $\Omega(I)$ denote the ultrafilter space of I considered as a Boolean lattice. The map $f : \Omega(I) \rightarrow \mathcal{M}(I)$ given by $f(U) = [U]$ is one-to-one and onto where U is in $\Omega(I)$.

Proof: Take any U in $\Omega(I)$. Then note that $[U]$ is in $\mathcal{M}(I)$. Since $U \subseteq I$ we have $U \cap I \neq \phi$. We only need to show that $[U]$ is an ultrafilter in B. Take a filter V in B with $[U] \subset V$. Then there exists an $x \in V$ with $x \notin [U]$. Take any y in B. If $y \in U$ then $y \in V$. If $y \notin U$ then there exists z in U with $y \wedge z = 0$. Also x not in $[U]$ implies x not in U. Hence there exists z_1 in U with $z_1 \wedge x = 0$. Now $y = y \vee (z_1 \wedge x) = (y \vee z_1) \wedge (y \vee x)$. Now both $(y \vee x)$ and $y \vee z_1$ belong to V. Hence $y \in V$. Thus in either case $V = B$. Hence $[U]$ is an ultrafilter in B and thus $[U]$ is a member of $\mathcal{M}(I)$. Moreover since U is an ultrafilter in I we have $[U] \cap I = U$. Take U, V in $\Omega(I)$ with $U \neq V$. Then $[U] \neq [V]$ for suppose $[U] = [V]$. Then $[U] \cap I = U$, $[V] \cap I = V$ and $U = V$ which contradicts $U \neq V$. Hence f is one-to-one. Take any $U \in \mathcal{M}(I)$. Then $U \cap I \neq \phi$ is a filter in I and $[U \cap I] = U$. It is sufficient to show $U \subseteq [U \cap I]$. Take $x \in U$ and suppose $x \notin [U \cap I]$. This implies $x \not\geq z$ for any $z \in U \cap I$. If $x \wedge z = o$ then this would contradict the fact that U is proper. Hence $x \wedge z \neq o$ and also $x \wedge z \in U \cap I$. Also $x \geqslant x \wedge z$ and this contradicts the fact that $x \ngeq z$ for any $z \in U \cap I$. Hence $U = [U \cap I]$, and this means $U \cap I$ is an ultrafilter in I. Hence f is onto. This completes the proof.

Using Lemma 1 we arrive at the answer to the question posed at the beginning of this section. We have the following theorem:

Theorem 4 : Let B be a Boolean lattice and I any ideal in B. Then the ultrafilter space $\Omega(I)$ of I is homeomorphic to $\mathcal{M}(I)$, the map $f : \Omega(I) \to \mathcal{M}(I)$ given by $f(U) = [U]$. Conversely let E be any Boolean space and O any open subspace of E. Then the Boolean lattice $\mathcal{Z}(O)$ is an ideal in the Boolean lattice $\mathcal{Z}(E)$.

Proof: (i) By Lemma 2 the map $f : \Omega(I) \to \mathcal{M}(I)$ given by $f(U) = [U]$ is one-to-one and onto. First note that every set $\Omega(a) \subseteq \mathcal{M}(I)$ is compact open in $\mathcal{M}(I)$ since $\mathcal{M}(I)$ is open. Moreover since the $\Omega(a)$ are precisely the compact open subsets of $\Omega(B)$, the $\Omega(a) \subseteq \mathcal{M}(I)$ are precisely the compact open subsets of $\mathcal{M}(I)$ and these form a basis for $\mathcal{M}(I)$. Also note that $\mathcal{M}((a)) = \Omega(a)$ where (a) denotes the principal ideal generated by a. Hence we get $\Omega(a) \subseteq \mathcal{M}(I)$ if and only if $a \in I$. Take a basic open set $\Omega(a) \subseteq \mathcal{M}(I)$. Then,

$f^{-1}(\Omega(a)) = f^{-1}\{U/a \in U \cap I\} = \{U \cap I/a \in U \cap I\}$ which is a basic open set in $\Omega(I)$. On the other hand take a basic open set $\Omega(a) \subseteq \Omega(I)$. Then $a \in I$ and

$$f(\Omega(a)) = f\{U/U \in I,\ a \in U\}$$
$$= \{[U]/a \in U\} \cap \mathcal{M}(I) \text{ which is open in } \mathcal{M}(I).$$

Hence f is a homeomorphism.

(ii) We showed in the previous chapter that any open subset O of E is a Boolean space. Now $\mathcal{L}(O) = \{X/X \subseteq O,\ X \text{ compact open}\}$. Now by Theorem 2 we have that $\mathcal{L}(O)$ is an ideal in $\mathcal{L}(E)$ for X compact in O implies X compact in E.

Theorem 4 allows us to deduce that <u>the ultrafilter spaces of ideals in a given Boolean lattice B are, up to homeomorphisms, precisely the open subspaces of the ultrafilter space $\Omega(B)$.</u>

Let B be a Boolean lattice and I any ideal in B. We conclude this section by showing that the ultrafilter space $\Omega(B/_I)$ of the quotient lattice $B/_I$ correspond precisely to the closed sets $\mathcal{D}(I)$ of $\Omega(B)$. Let $\nu: B \longrightarrow B/_I$ be the natural homomorphism and let the ultrafilters in B/I be denoted by $\overline{U}, \overline{V}$, etc.

We then have the following theorem:

<u>Theorem 5</u> : Let B be any Boolean lattice and I any ideal in B. Then the ultrafilter space $\Omega(B/_I)$ of the quotient lattice $B/_I$ is homeomorphic to the closed set $\mathcal{D}(I)$, the homeomorphism $f: \Omega(B/_I) \longrightarrow \mathcal{D}(I)$ being given by $f(U) = \nu^{-1}((U))$.

<u>Proof</u>: Take any ultrafilter $\overline{U}$ in $B/_I$. Then $\nu^{-1}((\overline{U})) = U$ say is an ultrafilter in B. We must show U is in $\mathcal{D}(I)$. Since ν is a Boolean lattice homomorphism $\nu^{-1}((\overline{U}))$ is an ultrafilter in B. We must show $U \cap I = \phi$ Take any $a \in U$. Then $\nu(a) \in \overline{U}$, that is, $\nu(a) \neq \overline{O}$. Hence $a \notin I$ and thus $U \cap I = \phi$. f is clearly one-to-one. Take U in $\mathcal{D}(I)$, then $U \cap I = \phi$ Then $\nu(U) = \overline{U}$ is an ultrafilter in B/I and since ν is onto we have $f(\overline{U}) = \nu^{-1}((\overline{U})) = U$. Hence f is onto. Take any open set in $\mathcal{D}(I)$. This is of the form $\mathcal{M}(J) \cap \mathcal{D}(I)$ for some ideal J in B.

Then $f^{-1}((\mathcal{M}(J) \cap \mathcal{D}(I)))$

$= f^{-1}(\mathcal{M}(J)) \cap \Omega(B/_I)$

$= \{ \bar{U}/\bar{U} \cap \nu(J) \neq \phi \} \cap \Omega(B/_I)$ which is an open set of $\Omega(B/_I)$.

Hence f is continuous. On the other hand take an open subset $\mathcal{M}(\bar{I})$ of $\Omega(B/_I)$. Then we have, $f(\mathcal{M}(\bar{I})) = f\{\bar{U}/\bar{U} \cap \bar{I} \neq \phi\}$

$= \{ U/U \cap I \neq \phi \}$ where $U = \nu^{-1}(\bar{U})$

Hence $f(\mathcal{M}(\bar{I})) = \mathcal{M}(I)$. Hence f is a homeomorphism. This completes the proof.

5. Illustrative Example.

An example of a compact Boolean space is the Cantor ternary set or Cantor discontinuum D endowed with the relative topology of the reals. It is well known that each element of D which lies in the closed interval $[0, 1]$ can be written in the form $\sum_{n=0}^{\infty} \frac{a_n}{3^n}$ where $a_n = 0$ or 2. We will now establish that D with the topology mentioned is a compact totally disconnected space. The following proposition mentions a property of D which will be useful in achieving our goal.

Proposition 3: Let D be the Cantor ternary set. Let $x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ $y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$ be two distinct points of D with $|x - y| < 3^{-r}$ for some integer r. Then $a_n = b_n$ for at least the first r terms.

Proof: Since the points x, y are distinct there exists a smallest integer m for which $a_n \neq b_n$, that is, $a_n = b_n$ for $n = 1, 2, \ldots m-1$. In particular if $x<y$ we have $|x - y| = \left| \sum_{n=r}^{\infty} \frac{a_n - b_n}{3^n} \right| \geqslant 3^{-m}$

Now suppose s r and $a_n = b_n$ for exactly the first s terms. Then by the property just stated $|x - y| \geqslant 3^{-s}$ and since $s < r$ we have $3^{-s} > 3^{-r}$. Thus $|x - y| > 3^{-r}$ which is a contradiction to the hypothesis that $|x - y| < 3^{-r}$.

Hence our supposition is false and the proposition is established.

We now establish that D is a Boolean space.

Proposition 4 : The Cantor ternary set D endowed with the relative topology of the reals is a compact Boolean space.

Proof: By Proposition 7 of the previous chapter it is enough to show D is compact and totally disconnected. Since D is a subset of the reals, to show D compact it is enough to show D is closed and bounded. D is a bounded set since each element of D lies between the real numbers 0, 1. In order to show that D is closed it is sufficient to show that the following condition is satisfied: If $\inf_{x \in D} |x - z| = 0$ for some real number z then z is an element of D. Take such a real number z. Then since the greatest lower bound of $\{x - z \mid x \in D\}$ is zero there exists a subsequence $(x_n - z)_{n \in N}$ tending to zero. Thus to each integer r there corresponds a x_r in D with $|x_r - z| < 3^{-r-1}$. If $r < s$ then we have, $|x_r - x_s| \leq |x_r - z| + |z - x_s| < 3^{-r-1} + 3^{-s-1} < 3^{-r}$.
Say $x_r = \sum_{n=1}^{\infty} \frac{a_{rn}}{3^n}$ Then by Proposition 3 $a_{rn} = a_{sn}$ for at least $n = 1, \ldots, r$. Define $w = \sum_{r=1}^{\infty} \frac{a_{rr}}{3^r}$ Then we have that the first r terms in the expansion of w and x_r coincide. That is, $|w - x_r| < 3^{-r}$. Thus for each r we have, $|w - z| \leq |w - x_r| + |x_r - z| < 3^{-r} + 3^{-r-1} < 3^{-r+1}$. Hence $|w - z| \to 0$ as $r \to \infty$. Thus $w = z$. But by its very definition w is an element in D. Hence $z \in D$. Thus D is closed and we have already noted that D is bounded. Therefore D is compact. We now show that D is totally disconnected. Take x, y in D with, say $x < y$. Then by the proof of Proposition 3 there exists a smallest integer r for which $a_n \neq b_n$ where $x = \sum_1^{\infty} \frac{a_n}{3^n}$, $y = \sum_1^{\infty} \frac{b_n}{3^n}$. We now define $z = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$

where $C_n = \begin{cases} a_n = b_n & \text{for } n = 1,\ldots,r-1. \\ 0 & \text{for } n = r \\ 2 & \text{for } n > r. \end{cases}$

Then clearly $x \leq z < y$. Put $A = \{ p/p \in D, \ p \leq z \}$ and $B = \{ p/p \in D, \ p > z \}$. Then clearly $x \in A$, $y \in B$, $A \cap B = \phi$, $A \cup B = D$. Also A is clearly closed subset of D. Hence D is totally disconnected. In all D is a compact Boolean space. This completes the proof.

CHAPTER II.

BOOLEAN LATTICES AND BOOLEAN RINGS.

Introduction : In this chapter we exhibit the connection between Boolean lattices and Boolean rings. The main result we will establish here is that the category of Boolean lattices and proper Boolean lattice homorphisms is equivalent to the category of Boolean rings and proper ring homomorphisms.

1. The Boolean Ring of a Boolean Lattice.

Let B be any Boolean lattice and I any ideal in B. Let $B/_I$ denote the quotient lattice. Introduce two binary operations in the set B as follows (i) $x + y = (x \sim x \wedge y) \vee (y \sim x \wedge y)$ for any x, y in B (ii) $x \cdot y = x \wedge y$ for any x, y in B. We will denote the triple $(B, +, \cdot)$ for short by $\mathcal{R}(B)$. We establish the following general theorem about $\mathcal{R}(B)$.

Theorem 1 : (i) $\mathcal{R}(B)$ is a Boolean ring in which the zero of B is the zero of $\mathcal{R}(B)$.

(ii) B has an unit if and only if $\mathcal{R}(B)$ has a unit, and the unit of B is the unit of $\mathcal{R}(B)$ and conversely.

(iii) The ideals in $\mathcal{R}(B)$ are given precisely by the sets $\mathcal{R}(I)$ where I is an ideal in B.

(iv) The prime ideals in $\mathcal{R}(B)$ are given precisely by the sets $\mathcal{R}(I)$ where I is a prime ideal in B.

(v) The maximal ideals in $\mathcal{R}(B)$ are given precisely by the sets $\mathcal{R}(I)$ where I is a maximal ideal in B.

(vi) The quotient rings of $\mathcal{R}(B)$ are precisely of the form $\mathcal{R}(B/_I)$ where I is an ideal in B.

Proof: (i) Let $\Omega = \Omega(B)$ denote the set of all ultrafilters in B and for each $a \in B$ let $\Omega(a) = \{U/a \in U \in \Omega\}$. For each a in B define h_a : $\Omega \longrightarrow R_0 = \{0, 1\}$ by $h_a(U) = \begin{cases} 0 & U \notin \Omega(a) \\ 1 & U \in \Omega(a) \end{cases}$ where R_0 is the two element Boolean ring. Then h_a is the characteristic function of $\Omega(a)$. Let $\mathcal{L}(B)$ denote the set of all these characteristic functions, that is, let $\mathcal{L}(B) = \{h_a/a \in B\}$. We first show that $\mathcal{L}(B)$ forms a Boolean ring under functional addition and multiplication. To this end we check that the set $\mathcal{L}(B)$ is closed under the operations just stated. That the remaining properties hold which makes $\mathcal{L}(B)$ into a ring is then quite clear. Take any U in Ω. Then $(h_a + h_b)(U) = h_a(U) + h_b(U)$. Thus $(h_a + h_b)(U) = 1$ if and only if $U \in \Omega(a) + \Omega(b)$ where '+' denotes symmetric difference. We now show that $\Omega(a) + \Omega(b) = \Omega(a + b)$ for any a, b in B. $U \in \Omega(a+b)$ if and only if $a \sim a \wedge b$ or $b \sim a \wedge b \in U$. This is the case if and only if $a \in U$ and $b \notin U$ or $a \notin U$ and $b \in U$. Thus $U \in \Omega(a+b)$ if and only if $U \in \Omega(a) + \Omega(b)$. Hence $(h_a + h_b)$ is 1 on $\Omega(a+b)$ and 0 otherwise, that is, $h_a + h_b = h_{a+b}$ for any a, b in B. Thus $\mathcal{L}(B)$ is closed under functional addition. Now $(h_a\, h_b)(U) = h_a(U)\, h_b(U) = \begin{cases} 0 & \text{if } U \in \Omega(a+b) \\ 1 & \text{if } U \in \Omega(a \wedge b) \end{cases}$ while

$$h_{ab}(U) = \begin{cases} 0 & \text{if } U \notin \Omega(a \wedge b) \\ 1 & \text{if } U \in \Omega(a \wedge b) \end{cases}$$

. It is now clear that $h_{ab} = h_a \cdot h_b$, that is, $\mathcal{L}(B)$ is closed under functional addition. Clearly, $h_o + h_a = h_{o+a} = h_a$; hence h_o is the zero of $\mathcal{L}(B)$ where o is the zero of B. Also $(h_a)^2 = h_{a2} = h_a$, that is, $\mathcal{L}(B)$ is a Boolean ring. Consider the map $f : \mathcal{L}(B) \longrightarrow \mathcal{R}(B)$ given by $f(h_a) = a$. Since the elements of B, $\mathcal{R}(B)$ are the same f is clearly onto. Also a = b implies $\Omega(a) = \Omega(b)$ from which it follows that $h_a = h_b$. Hence f is one to one. Moreover, we have $f(h_a + h_b) = f(h_{a+b}) = a + b = f(h_a) + f(h_b)$ and $f(h_a h_b) = f(h_{ab}) = ab = f(h_a)f(h_b)$, that is, $\mathcal{R}(B)$ is ring isomorphic to the Boolean ring $\mathcal{L}(B)$. Hence $\mathcal{R}(B)$ is a Boolean ring.

(ii) Suppose B has unit e. Then $e \geqslant x$ for each x in B, that is, $e \wedge x = x$ for each x in B. Hence $ex = x$ for each x in $\mathcal{R}(B)$, that is e is the unit of $\mathcal{R}(B)$. On the other hand, suppose $\mathcal{R}(B)$ has an unit e. Then $ex = x$ for each x in $\mathcal{R}(B)$. This means, $e \wedge x = x$ for each x in B. Hence $e \geqslant x$ for each x in B, and e is then also the unit of B.

(iii) Let I be any ideal in B. Take any x, y in $\mathcal{R}(I)$. Then x, y are in I and hence $x \vee y \in I$ and $x \wedge y \in I$. Hence since I is an ideal in B, $(x \vee y) \sim (x \wedge y) \in I$. But $x - y = (x \vee y) \sim (x \wedge y)$. Hence $x - y$ is in $\mathcal{R}(I)$. Take any x in $\mathcal{R}(B)$ and y in $\mathcal{R}(I)$. Then $x \wedge y \leq y \in I$, and thus $x \wedge y \in I$ which means $xy \in \mathcal{R}(I)$. Therefore $\mathcal{R}(I)$ is an ideal in $\mathcal{R}(B)$ where I is an ideal in B. On the other hand let J be any ideal in the Boolean ring $\mathcal{R}(B)$. Then J is a subset of B and we show J is an ideal in B. Take any x, y in J. Then $x + y = (x \vee y) \sim (x \wedge y)$ is in (J) and $xy \in$ (J). Thus $((x \vee y) \sim (x \wedge y)) \vee (x \wedge y) \in J$, that is, $x \vee y \in J$. Take any $x \in \mathcal{R}(B)$, $y \in J$. Then $xy \in J$ and hence $x \wedge y \in J$ considered as a subset of B. Hence J is an ideal in the lattice theoretic sense and taking J in this sense it is clear that $\mathcal{R}(J) = J$. Hence the ideals in $\mathcal{R}(B)$ are precisely of the form $\mathcal{R}(I)$ where I is an ideal in B.

(iv) Let I be a prime ideal in B. We know that $\mathcal{R}(I)$ is an ideal in $\mathcal{R}(B)$. We must show that $\mathcal{R}(I)$ is prime. Since I is a prime ideal in B we have that $x \wedge y$ in I implies x in I or y in I. This means that xy in $\mathcal{R}(I)$ implies x in $\mathcal{R}(I)$ or y in $\mathcal{R}(I)$. Hence $\mathcal{R}(I)$ is a prime ideal in $\mathcal{R}(B)$. On the other hand let J be any prime ideal of $\mathcal{R}(B)$. In view of (iii) it is enough to show that J is prime when considered as a subset of B. Since J is prime in $\mathcal{R}(B)$ we have that xy in J implies x in J or y in J. Hence $x \wedge y$ in J implies x in J or y in J considered as an ideal in B. This establishes (iv).

(v) Let I be a maximal ideal in B. Then ~~by (v)~~ by Proposition 4 the set of all maximal ideals in B is equal to the set of all prime ideals in B and by Proposition 16 of the previous chapter the set of all maximal ideals in $\mathcal{R}(B)$ is equal to the set of all prime ideals in $\mathcal{R}(B)$. Hence by part (iv) the maximal ideals of $\mathcal{R}(B)$ are precisely the sets $\mathcal{R}(I)$ where I ranges over the set of maximal ideals of B.

(vi) To establish (vi) we show $\mathcal{R}(B/_I) = \mathcal{R}(B)/\mathcal{R}(I)$. Denote the elements of $B/_I$ by $\bar{x}$ where x is in B. For any two elements x, y in B put $x \equiv y$ if and only if $(x \sim x \wedge y)$ and $(y \sim x \wedge y)$ belongs to I. $\equiv$ is an equivalence relation and x denotes the equivalence class determined by x under this relation. The quotient ring $\mathcal{R}(B)/\mathcal{R}(I)$ consists of elements $[x]$ where $[x]$ is the equivalence class determined by x under the equivalence relation : $x \sim y$ if and only if $x - y$ is in $\mathcal{R}(I)$. Now $x \sim y$ if and only if $x + y$ is in $\mathcal{R}(I)$ since $\mathcal{R}(B)$ is a Boolean ring. Now $x + y = (x \sim x \wedge y) \vee (y \sim x \wedge y)$. Hence $x \equiv y$ if and only if $x \sim y$. Then since B and $\mathcal{R}(B)$ have the same elements, $\mathcal{R}(B/_I)$ and $\mathcal{R}(B)/\mathcal{R}(I)$

have the same elements, that is, $\bar{x} = [x]$. Moreover $\bar{x} + \bar{y} = \overline{x + y} = [x + y] = [x] + [y]$ and $\bar{x}\,\bar{y} = \overline{xy} = [xy] = [x]\cdot[y]$. Hence $\mathcal{R}(B/_I) = \mathcal{R}(B)/\mathcal{R}(I)$ and the theorem is established.

Theorem 1 has shown that a Boolean lattice B can be regarded as a Boolean ring by defining operations of addition and multiplication on the underlying set B in terms of the lattice operations of B. Moreover these operations are defined in such a manner that the ring theoretic ideals of $\mathcal{R}(B)$ correspond precisely to the lattice theoretic ideals of B. In the next section we show that in a similar way a Boolean ring can be regarded as a Boolean lattice.

2. The Boolean Lattice of a Boolean Ring.

We now proceed to establish the counterpart of Theorem 1.

Theorem 2 : Let R be any Boolean ring. Then (i) R under the divisibility relation is a Boolean lattice which we denote by $\mathcal{B}(R)$. In $\mathcal{B}(R)$, the zero of the ring is the zero of $\mathcal{B}(R)$, $x \wedge y = xy$, $x \vee y = x + y + xy$ and if $0 \leq x \leq y$ then $y \sim x = y - x$.

(ii) R has an unit if and only if $\mathcal{B}(R)$ has an unit.

(iii) The ideals in $\mathcal{B}(R)$ are given precisely by the sets $\mathcal{B}(I)$ where I is an ideal in R.

(iv) The prime ideals in $\mathcal{B}(R)$ are given precisely by the sets $\mathcal{B}(I)$ where I is a prime ideal in R.

(v) The maximal ideals in $\mathcal{B}(R)$ are given precisely by the sets $\mathcal{B}(I)$ where I is a maximal ideal in R.

(vi) The quotient lattices of $\mathcal{B}(R)$ are precisely of the form $\mathcal{B}(R/_I)$ where I is an ideal in R.

Proof: (i) The divisibility relation does partially order R. We show that $xy = \inf.\{x,y\}$, $x+y+xy = \sup.\{x,y\}$ under the divisibility relation. Now $xy \leq x,y$ for $x(xy) = xy$ and $y(xy) = xy$. Thus xy is a lower bound for the set $\{x,y\}$. Let s be less equal x,y. Then $sx = xy = x$ and thus $sxy = sy^2 = sy = s$. Hence $s \leq xy$. Thus $xy = \inf.\{x,y\}$. Clearly $x+y+xy \geqslant x,y$ for $x(x+y+xy) = x^2+xy+xy = x$ and $y(x+y+xy) = y$ using the fact that R is a Boolean ring. If $z \geqslant x,y$ then $x+y = zx+zy$ and $x+y+xy = z(x+y+xy)$, that is, $z \geqslant x+y+xy$. Hence $x+y+xy$ is the least upper bound of $\{x,y\}$ under the divisibility relation. Hence R under divisibility is a lattice with meet and join as stated.

We now show that the operations meet and join distribute over each other. Now, $x \wedge (y \vee z) = x(y+z+yz) = xy+xz+xyz = (x \wedge y) \vee (x \wedge z)$. Take the zero of R. Then $ox = o$ for each x in R, that is, $o \leq x$ for each x in R. Hence R under divisibility is a distributive lattice with zero. Take $o \leq x \leq y$ and put $z = y-x$. Then $z \wedge x = (y-x)x = 0$ and $x \vee z = y$ for $x \leq y$. Hence $y = x+y+xy = x+z+zx$ for $z = y-x$. Hence $y = x \vee z$. Hence R under divisibility is a relatively complemented distributive lattice with zero, that is, a Boolean lattice.

(ii) Suppose R has an unit e. Then $xe = x$ for each x in R. Hence $x \wedge e = x$ for each x in $\mathcal{B}(R)$. Thus e is also the unit of $\mathcal{B}(R)$. On the other hand let $\mathcal{B}(R)$ have unit e. Then $e \geqslant x$ for each x in $\mathcal{B}(R)$, that is $ex = x$ for each x in R. Hence e is the unit of R.

(iii) Let I be any ideal in R. Then $\mathcal{B}(I)$ is a subset of $\mathcal{B}(R)$. Take any x,y in $\mathcal{B}(I)$. Then $x,y \in I$ since I and $\mathcal{B}(I)$ have the same elements. Since I is an ideal in R we have $x+y = (x \vee y) \sim (x \wedge y)$ and xy are in I. Thus $x+y+xy = x \vee y$ is in I and hence in $\mathcal{B}(I)$. Next let $x \in \mathcal{B}(I)$ and suppose $y \leq x$. Then $yx = y$ and since $x \in I$ and I is an ideal in R we have $yx \in I$. Hence $y \in \mathcal{B}(I)$. Thus $\mathcal{B}(I)$ is an ideal in $\mathcal{B}(R)$. On the other hand let J be any ideal of $\mathcal{B}(R)$. We show that J is also an ideal of R. Take x,y in $\mathcal{B}(R)$; then $x \vee y = x+y+xy$ and $x \wedge y$ belongs to J. Hence $(x \vee y) \sim (x \wedge y) = x+y$ belongs to J. Take any x in R and y in J. Then $x \wedge y \in J$ and hence $xy \in J$ considered as a subset of R. Hence J is an ideal in R and clearly $\mathcal{B}(J) = J$.

(iv) Let I be a prime ideal in R. Then by (iii) $\mathcal{B}(I)$ is an ideal in $\mathcal{B}(R)$. Since I is prime in R we have that if $xy \in I$ then $x \in I$ or $y \in I$. Hence $xy \in \mathcal{B}(I)$ implies $x \in \mathcal{B}(I)$ or $y \in \mathcal{B}(I)$. Thus $\mathcal{B}(I)$ is a prime ideal in $\mathcal{B}(R)$. On the other hand let J be any prime ideal of $\mathcal{B}(R)$. Then $x \wedge y$ in J implies x in J or y in J. Thus xy in J implies x in J or y in J. Hence J is prime when considered as an ideal in R and thus (iv) is established.

(v) By Proposition 16 the set of prime ideals in R is equal to the set of maximal ideals in R and by Proposition 4 the set of prime ideals in $\mathcal{B}(R)$ is equal to the set of maximal ideals in $\mathcal{B}(R)$. Hence by part (iv) we get that $\mathcal{B}(I)$ ranges through the maximal ideals of $\mathcal{B}(R)$ as I ranges through the maximal ideals of R.

(vi) To establish (vi) we show that $\mathcal{B}(R/_I) = \mathcal{B}(R)/\mathcal{B}_{(I)}$ where I is an ideal in R. For any two elements x,y in R, put $x \equiv y$ if and only if $x-y \in I$. This relation is an equivalence relation on R and the equivalence classes determined by this relation constitute the elements of $R/_I$. That is $R/_I = \left\{ \overline{x} \,/\, \overline{x} \text{ equivalence class determined by } x \text{ under } \equiv \right\}$

In $R/_I$ addition and multiplication is defined as follows: $\overline{x} + \overline{y} = \overline{x+y}$,

$\bar{x}.\bar{y} = \overline{xy}$. The quotient lattice $\mathcal{B}(R)/\mathcal{B}(I)$ consists of elements $[x]$ where $[x]$ is the equivalence class determined by x in $\mathcal{B}(R)$ under the following equivalence relation : $x \setminus y$ if and only if $(x \sim x \wedge y)$ and $(y \sim x \wedge y)$ are in $\mathcal{B}(I)$. Operations, of meet, join and relative complements in $\mathcal{B}(R)/\mathcal{B}(I)$ is as follows: $[x] \wedge [y] = [x \wedge y]$, $[x] \vee [y] = [x \vee y]$, and if $o \leq x \leq y$ then $[y] \sim [x] = [y \sim x]$. It is now clear that $x \setminus y$ in B if and only if $x-y \in I$ and since the elements of R and $\mathcal{B}(R)$ are the same we have $\bar{x} = [x]$. Moreover, $\bar{x} \wedge \bar{y} = \overline{x \wedge y} = [x \wedge y] = [x] \wedge [y]$, $\bar{x} \vee \bar{y} = \overline{x \vee y} = [x \vee y] = [x] \vee [y]$, $\bar{x} \sim \bar{y} = \overline{x \sim y} = [x \sim y] = [x] \sim [y]$, if $o \leq y \leq x$. Hence $\mathcal{B}(R/_I) = \mathcal{B}(R)/_{\mathcal{B}(I)}$ and the theorem is established.

3. <u>Equivalence of Categories</u>.

Let $\mathcal{C}$ denote the category whose objects are <u>Boolean lattices</u> and whose maps are <u>proper Boolean lattice homomorphisms</u>. Let $\mathcal{D}$ denote the category whose objects are <u>Boolean rings</u> and whose maps are <u>proper ring homomorphisms</u>. In this section we establish that the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

Let B, C be any two Boolean lattices and let $f : B \longrightarrow C$ be any proper Boolean lattice homomorphism. Then f gives rise to a map $f_R : \mathcal{R}(B) \longrightarrow \mathcal{R}(C)$ defined by $f_R(x) = f(x)$ for any x in $\mathcal{R}(B)$. The map f_R has the following properties:

(i) $f_R(x+y) = f((x \sim x \wedge y) \vee (y \sim x \wedge y)) = (f(x) \sim f(x) \wedge f(y)) \vee (f(y) \sim f(x) \wedge f(y))$

$= (f_R(x) \sim f_R(x) \wedge f_R(y)) \vee (f_R(y) \sim f_R(x) \wedge f_R(y)) = f_R(x) + f_R(y)$.

(ii) $f_R(xy) = f(x \wedge y) = f(x) \wedge f(y) = f_R(x)\, f_R(y)$.

(iii) Take any z in $\mathcal{R}(C)$. Then z is in C and since f is proper there exists an x in B with $f(x) \geqslant z$, that is, $f_R(x)$ is divisible by z in $\mathcal{R}(C)$. Thus f_R is a proper ring homomorphism.

On the other hand let R, S be any two Boolean rings and let $f : R \longrightarrow S$ be a proper ring homomorphism. Then f gives rise to a map $f_B : \mathcal{B}(R) \longrightarrow \mathcal{B}(S)$ defined by $f_B(x) = f(x)$ for any x in $\mathcal{B}(R)$. The map f_B has the following properties:

(i) $f_B(x \wedge y) = f(xy) = f(x)f(y) = f_B(x) \wedge f_B(y)$

(ii) $f_B(x \vee y) = f(x+y+xy) = f(x)+f(y)+f(x)f(y) = f_B(x) + f_B(y) +$

$$+ f_B(x)f_B(y) = f_B(x) \vee f_B(y).$$

(iii) $f_B(y \sim x) = f(y-x) = f(y) - f(x) = f_B(y) \sim f_B(x)$ for any x,y with $0 \leq x \leq y$.

(iv) Take any z in $\mathcal{B}(S)$. Then z is in S and since f is a proper ring homomorphism there exists an x in R with $f(x) \geqslant z$. Hence $f_B(x) \geqslant z$.

In all this means that f_B is a proper Boolean lattice homomorphism between the Boolean lattices associated with the Boolean rings R and S. We now establish the main result of this section.

Theorem 3 : The two correspondences $\left.\begin{array}{l} B \longrightarrow \mathcal{R}(B) \\ f \longrightarrow f_R \end{array}\right\}$ and $\left.\begin{array}{l} R \longrightarrow \mathcal{B}(R) \\ f \longrightarrow f_B \end{array}\right\}$

are covariant functors $T : \mathcal{C} \longrightarrow \mathcal{D}$ and $S : \mathcal{D} \longrightarrow \mathcal{C}$ which establish the equivalence of the categories $\mathcal{C}$ and $\mathcal{D}$.

Proof: (i) Let A, B, C be Boolean lattices.

(a) if $f \in H(A,B)$ then $T(f) = f_R \in H(\mathcal{R}(A), \mathcal{R}(B)) = H(T(A),T(B))$.

(b) if $e_A \in H(A,A)$ then $T(e_A) = (e_A)_R \in H(\mathcal{R}(A), \mathcal{R}(A))$.

(c) Finally suppose $f \in H(A,B)$, $g \in H(B,C)$.

We must show that $T(gf) = T(g)T(f)$. Now $T(gf) = (gf)_R$ and $T(g)T(f) = g_R f_R$. But $(gf)_R(x) = (gf)(x) = g(f(x))$ and $(g_R f_R)(x) = g(f(x))$ for any x in $\mathcal{R}(A)$. Hence $T(gf) = T(g)T(f)$, that is, T is a covariant functor from $\mathcal{C}$ to $\mathcal{D}$.

(ii) Let P, Q, R be Boolean rings.

(a) if $f \in H(P,Q)$ then $S(f) = f_B \in H(\mathcal{B}(P), \mathcal{B}(Q)) = H(S(P), S(Q))$.

(b) if $e \in H(P,P)$ then $S(e) = e_B \in H(\mathcal{B}(P), \mathcal{B}(P))$.

(c) Finally let $f \in H(P,Q)$, $g \in H(Q,R)$.

We must show that $S(gf) = S(g)S(f)$. Now $(gf)_B(x) = (gf)(x) = g(f(x))$ whereas $S(g)S(f) = g_B f_B$ and $(g_B f_B)(x) = g(f(x))$ for any x in $\mathcal{B}(P)$. Hence $S(gf) = S(g)S(f)$. Thus S is a covariant functor from $\mathcal{D}$ to $\mathcal{C}$.

(iii) Since T, S are covariant functors, the composites ST and TS are covariant functors on $\mathcal{C}$, $\mathcal{D}$ respectively. Let $I_{\mathcal{C}}$, $I_{\mathcal{D}}$ denote the identity functors on $\mathcal{C}$, $\mathcal{D}$ respectively. We show that $ST = I_{\mathcal{C}}$, $TS = I_{\mathcal{D}}$. This will establish that the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent. To this end we prove,

(1) $B = \mathcal{B}(\mathcal{R}(B))$ for any Boolean lattice and

(2) $R = \mathcal{R}(\mathcal{B}(R))$ for any Boolean ring R.

To show (1) : Put $R = \mathcal{R}(B)$, $\overline{B} = \mathcal{B}(R)$. Let the operations in $\overline{B}$ be denoted by $\overline{\wedge}$, $\overline{\vee}$, $\overline{\sim}$ and zero of $\overline{B}$ by $\overline{o}$. We want to show $B = \overline{B}$. B and $\overline{B}$ have the same elements namely the elements of R. Thus $o = \overline{o}$. Also, $x \overline{\wedge} y = xy = x \wedge y$ for any x, y; $x \overline{\vee} y = x+y+xy$. We now have that $x+y = (x \vee y) \sim (x \wedge y) = (x \vee y) - xy$. Thus $x \overline{\vee} y = x+y+xy = (x \vee y) - xy + xy = x \vee y$. Let $o \leq x \leq y$. Then $y \overline{\sim} x = y - x = y \sim x$. Hence $B = \overline{B}$. This proves (1). We now show (2). Let $B = \mathcal{B}(R)$ and $\overline{R} = \mathcal{R}(B)$. Let the operations in R be denoted by $\overline{+}$, $\overline{\cdot}$ and zero of $\overline{R}$ by $\overline{0}$. We want to show $R = \overline{R}$. R and $\overline{R}$ have the same elements namely the elements of B. Now $x \overline{+} y = (x \sim x \wedge y) \vee (y \sim x \wedge y) = (x-xy) \vee (y-xy) = (x-xy) + (y-xy) + (x-xy)(y-xy) = x + y$. Hence $x \overline{+} y = x + y$ for all x in B; $x \overline{\cdot} y = x \wedge y = xy$. Hence $R = \overline{R}$. Thus $(ST)(B) = \mathcal{B}(\mathcal{R}(B)) = B$ for any Boolean lattice B and take any $f \in H(A,B)$ say, then $(ST)(f) = (f_R)_B$ has for its domain the domain of f namely A and $(f_R)(x) = f(x)$ for any x in B. Hence (ST) is the identity functor on $\mathcal{C}$, that is, $ST = I_{\mathcal{C}}$. On the other hand $(TS)(R) = \mathcal{R}(\mathcal{B}(R)) = R$ for any Boolean ring R. Also if $f \in H(P,R)$ say then $(TS)(f) = (f_B)_R$ has for its domain P and $(f_B)_R(x) = f(x)$ for any x in P. Hence (TS) is the identity functor on $\mathcal{D}$, that is, $TS = I_{\mathcal{D}}$. Thus we have established that the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent.

4. Adjunction of Unit.

In this paragraph we describe a method of imbedding a given Boolean lattice B into a Boolean lattice B^* with unit and a given Boolean ring R into a Boolean ring R^* with unit. These imbeddings will be essentially unique in a sense to be made precise below. We will show, moreover, that if we imbed B in B^* and R in R^* then $\mathcal{R}(B^*)$ is ring isomorphic to R^* and $\mathcal{B}(R^*)$ is Boolean lattice isomorphic to B^*.

Let $R_o = \{0, 1\}$ be the two element Boolean ring and let R be any Boolean ring. Consider R as an algebra over R_o and let $R^* = \{(\alpha, x) / \alpha \in R_o, x \in R\}$. In the set R^* introduce the following two binary operations:

(1) $(\alpha, x) + (\beta, y) = (\alpha + \beta, x+y)$ for arbitrary elements in R .

(2) $(\alpha, x) \cdot (\beta, y) = (\alpha\beta, \alpha y + \beta x + xy)$ for any two elements in R .

One now has the following proposition.

Proposition 1 : R^* under the operations (1), (2) is a Boolean ring with unit and contains R as a subring. R^* is essentially unique in the following sense: If S is a Boolean ring with unit containing R as a subring then S contains a Boolean ring T isomorphic to R^* and containing R as a subring.

Proof : It is clear that R^* under the operations mentioned does form a ring. It is a Boolean ring for $(\alpha, x)(\alpha, x) = (\alpha^2, \alpha x + \alpha x + x^2) = (\alpha, x)$ for any element of R^*. R^* has an unit, namely, (1,0), for $(1,0)(\alpha, x) = (\alpha, 0 \cdot \alpha + 0 \cdot x + 1 \cdot x) = (\alpha, x)$ and similarly $(\alpha, x)(1,0) = (\alpha, x)$. The set $\{(0,x) / x \in R\}$ is contained in R^* and is clearly seen to be ring isomorphic to R, the isomorphism being given by x to $(0, x)$. We now show that R^* is unique in the sense indicated. Let S be any Boolean ring with unit e containing R as a subring. Consider the map f : $R^* \longrightarrow S$ given by $f((\alpha, x)) = f(\alpha) + x$ where $f(\alpha) = 0$ if $\alpha = 0$ and $f(\alpha) = e$ if $\alpha = 1$. We show that f from R^* onto $T = \{x + f(\alpha) / x \in R\}$ is a ring isomorphism. We first show that f is one-to-one. Take (α, x), (β, y) in R^* with $(\alpha, x) \neq (\beta, y)$.

Case 1 : $\alpha \neq \beta$, $x = y$. Without loss of generality say $\alpha = 0$, $\beta = 1$. Then $f((\alpha, x)) = x$ and $f(\beta, y) = e + y$ and $x \neq e + y$ for e is not in R.

Case 2 : $\alpha = \beta$, $x \neq y$. Suppose $\alpha = \beta = 0$. Then $f((\alpha, x)) = x$, $f(\beta, y) = y$ and $x \neq y$. Suppose $\alpha = \beta = 1$. Then $f((\alpha, x)) = e + x$, $f((\beta, y)) = e + y$ and $e + x \neq e + y$ for $x \neq y$.

Case 3 : $\alpha \neq \beta$, $x \neq y$. Without lost of generality suppose $\alpha = 0$, $\beta = 1$. Then $f(\alpha, x) = x$, $f((\beta, y)) = e + y$ and $x \neq e + y$ for e is not in R.

Hence f is one to one. It is clear that f is onto T for take $x + f(\alpha)$ in T. Then (α, x) is in R^* and $f(\alpha, x) = f(\alpha) + x$. Moreover, we have

$f((\alpha, x)+(\beta, y)) = f((\alpha+\beta, x+y)) = f(\alpha+\beta)+x+y = f(\alpha)+x+f(\beta)+y =$
$f((\alpha, x)) + f((\beta, y))$ and,
$f((\alpha, x)(\beta, y)) = f((\alpha\beta, \alpha y+\beta x+xy)) = f(\alpha)f(\beta)+\alpha y+\beta x+xy$
$= f((\alpha, x))\, f((\beta, y))$. Hence f is a ring homomorphism. Thus R^* is isomorphic to T and T contains $R = \{f(o) + x \mid x \in R\}$ as a subring and this completes the proof.

We now describe a method of imbedding a Boolean lattice B into a Boolean lattice with unit. By Stone's representation theorem B is isomorphic to a Boolean lattice of subsets of some set X. Take $\mathcal{F} = \{S_x \mid S_x \subseteq X, x \in B\}$ a Boolean lattice of subsets isomorphic to B, the isomorphic correspondence being given by $x \longrightarrow S_x$. Let $Y = \bigcup_{x\in B} S_x$ and put $G = \{\complement_Y S_x / x \in B\}$. Let $\mathcal{F}^* = G \cup \mathcal{F}$. We now establish,

<u>Proposition 2</u> : $\mathcal{F}^*$, partially ordered by set inclusion, is a Boolean lattice with unit Y containing B. $\mathcal{F}^*$ is essentially unique in the following sense : If C is a Boolean lattice with unit containing B as a sub-Boolean lattice then C contains a Boolean lattice A with unit isomorphic to $\mathcal{F}^*$ and containing B.

<u>Proof</u>: Take any S_x in $\mathcal{F}$ and $\complement S_y$ in G. Then $S_x \cap \complement_Y S_y = S_{x\sim x\wedge y}$ $= S_x - (S_x \cap S_y)$. Thus $\mathcal{F}^*$ is closed under intersection. Moreover, it is clear that the set $\mathcal{F}^*$ is closed with respect to taking complements in Y. Also, by the definition of Y each element of $\mathcal{F}^*$ is contained in Y. Hence $\mathcal{F}^*$ is a Boolean lattice with unit Y. Now $\mathcal{F}^*$ contains $\mathcal{F}$ which is Boolean lattice isomorphic to B. Thus by identification $\mathcal{F}^*$ contains B as a Boolean lattice. We now show that $\mathcal{F}^*$ is unique in the sense indicated. Let C be a Boolean lattice with unit e containing B. Consider the map $f : \mathcal{F}^* \longrightarrow C$ given by $f(M) = \begin{cases} x \text{ if } M = S_x \\ e\sim x \text{ if } M = \complement_Y S_x \end{cases}$.

Then f is one to one, for take $M \neq N$ in $\mathcal{F}^*$.

__Case 1__ : $M = S_x$, $N = S_y$ say with $x \neq y$. Then $f(M) = x$, $f(S_y) = y$ and $x \neq y$.

__Case 2__ : $M = \complement_y S_x$, $N = S_y$. Then $f(M) = e \sim x$ and $f(N) = y$ and $e \sim x \neq y$ for e is not in B.

__Case 3__ : $M = \complement_y S_x$, $N = \complement_y S_y$ and then $x \neq y$ for $M \neq N$. Thus $e \sim x \neq e \sim y$, that is, $f(M) \neq f(N)$. Hence in all cases $f(M) \neq f(N)$ and f is one to one.

f is clearly onto the set $A = \{x / x \in B\} \cup \{e \sim x / x \in B\}$.

Also (i) $f(S_x \cap S_y) = f(S_{x \wedge y}) = x \wedge y = f(S_x) \cap f(S_y)$

(ii) $f(\complement_y S_y \cap S_x) = f(S_{x \sim x \wedge y}) = x \sim x \wedge y = (e \sim y) \wedge x = f(\complement_y S_y) \cap f(S_x)$.

(iii) $f(\complement_y S_x \cap \complement_y S_y) = f(\complement_y (S_x \cup S_y)) = e \sim (x \vee y) = (e \sim x) \wedge (e \sim y)$
$= f(\complement_y S_x) \cap f(\complement_y S_y)$.

Thus $f(M \cap N) = f(M) \cap f(N)$ for any M, N in $\mathcal{F}^*$. Similarly $f(M \cup N) = f(M) \cup f(N)$ for any M, N in $\mathcal{F}^*$. Finally we have, $f(Y) = f(\complement_y S_o) = e \sim o = e$. Hence f is a unitary Boolean lattice homomorphism which is also one to one and onto, that is, f is an isomorphism. Moreover, A contains B as a Boolean sublattice, completing the proof.

__Notation__ : For any Boolean lattice B let B^* denote the Boolean lattice with unit into which B can be imbedded as described in Proposition 2. Similarly for any Boolean ring R let R^* denote the Boolean ring with unit into which R can be imbedded as described in Proposition 1.

We now state and prove in the following theorem the result announced earlier.

__Theorem 4__ : Let B be a Boolean lattice and $\mathcal{R}(B)$ its Boolean ring. Then $\mathcal{R}(B^*)$ is ring isomorphic to $\mathcal{R}(B)^*$. Conversely let R be any Boolean ring and $\mathcal{B}(R)$ its Boolean lattice. Then $\mathcal{B}(R^*)$ is lattice isomorphic to $\mathcal{B}(R)^*$.

Proof: (i) Let B, B^* be given. Since B^* contains an isomorphic copy of B we have by Theorem 1 that $\mathcal{R}(B^*)$ contains an isomorphic copy of the Boolean ring $\mathcal{R}(B)$. However, by Proposition 1 $\mathcal{R}(B)^*$ is the smallest Boolean ring containing $\mathcal{R}(B)$. Hence again by Proposition 1, $\mathcal{R}(B)$ is contained isomorphically as a subring in $\mathcal{R}(B^*)$. By Theorem 3 $\mathcal{B}(\mathcal{R}(B)) = B$. Hence $\mathcal{B}(\mathcal{R}(B)^*)$ contains an isomorphic copy of the Boolean lattice B. Then applying Proposition 2 we have that $\mathcal{B}(\mathcal{R}(B)^*)$ contains an isomorphic copy of B^*. Hence $\mathcal{R}(B^*)$ is contained isomorphically as a subring within $\mathcal{R}(B)^*$. Hence we get $\mathcal{R}(B^*)$ is ring isomorphic to $\mathcal{R}(B)^*$.

(ii) Let R, R^* be given. Since R^* contains an isomorphic copy of R we have that $\mathcal{B}(R^*)$ contains an isomorphic copy of the lattice $\mathcal{B}(R)$. Then by Proposition 2 we have $\mathcal{B}(R^*)$ contains an isomorphic copy of $\mathcal{B}(R)^*$ which is the smallest Boolean lattice containing $\mathcal{B}(R)$. Now, on the other hand, $\mathcal{B}(R)^*$ contains an isomorphic copy of $\mathcal{B}(R)$. Hence $\mathcal{R}(\mathcal{B}(R))$ is contained isomorphically as a Boolean ring in $\mathcal{R}(\mathcal{B}(R)^*)$. By Theorem 3 $\mathcal{R}(\mathcal{B}(R)) = R$. Hence by Proposition 1 $\mathcal{R}(\mathcal{B}(R)^*)$ contains an isomorphic copy of R^*. Thus $\mathcal{B}\mathcal{R}(\mathcal{B}(R)^*)$ contains an isomorphic copy of $\mathcal{B}(R^*)$. But $\mathcal{B}(\mathcal{R}(\mathcal{B}(R)^*)) = \mathcal{B}(R)^*$ by Theorem 3. Hence in all $\mathcal{B}(R^*)$ is lattice isomorphic to $\mathcal{B}(R)^*$ and the theorem is established.

5. Free Boolean Lattices and Boolean Rings.

In this paragraph we introduce the notions of absolute and relative freeness for Boolean lattices and Boolean rings. We show that relatively free Boolean lattices correspond precisely to relatively free

Boolean rings and that free Boolean lattices correspond precisely to free Boolean rings.

Let R be any ring. By an extension ring of R we mean any ring S which contains R as a subring. Let B be any Boolean lattice. By an extension lattice of B we mean any Boolean lattice C which contains B as a sub-Boolean lattice. We now make the following definitions:

Definition 1 : An extension Boolean ring R^* of a Boolean ring R is said to be relatively free over R with X a free set of generators over R if and only if any mapping $f_0 : X \longrightarrow S$ where S is any extension Boolean ring of R extends to an unique ring homomorphism $f : R^* \longrightarrow S$ with f restricted to R being the identity mapping.

Correspondingly we make the following definition:

Definition 2 : Let B be a Boolean lattice.

An extension Boolean lattice B^* of B is said to be free over B with X a free set of generators over B if and only if any mapping $f_0 : X \rightarrow C$ where C is any extension Boolean lattice of B extends uniquely to a Boolean lattice homomorphism $f : B^* \longrightarrow C$ with f restricted to B being the identity mapping.

The following is an example of a relatively free Boolean ring.

Example : Let R be a Boolean ring and X any set. Let $R[X]$ denote the polynomial ring over R in the set of indeterminates X.

Let J be the ideal in $R[X]$ generated by the set $G = \{ x_1^2 \cdots x_n^2 - x_1 \cdots x_n, / x_i \in X, \ n \geqslant 1 \}$. Let $f_0 : X \longrightarrow S$ be any mapping where S is any extension Boolean ring of R. Then f_0 extends to an unique ring homomorphism $f : R[X] \longrightarrow S$ with f restricted to R the identity. Let K be the kernel of f. We note that $K \cap R = 0$ for $f(R) = R$ and thus $f(r) = 0$ where $r \in K \cap R$

implies r = 0. We now show that the ideal J is contained in K. For this purpose it is sufficient to show G is contained in K. Now $f(x_1^2 \cdots x_n^2 - x_1 \cdots x_n)$ $= f(x_1)^2 \cdots f(x_n)^2 - f(x_1) \cdots f(x_n) = 0$ since $f(x)$ lies in the Boolean ring S. Hence as claimed G is a subset K. Thus f gives rise to an unique ring homomorphism g from the quotient ring $R[X]/_J$ into S with $g \cdot \nu = f$ where ν is the natural homomorphism from $R[X]$ to $R[X]/_J$. Let $r \in R$ be arbitrary. Then $\nu(r) = 0$ implies $r \in R \cap J$. But $R \cap J = 0$. Hence $r = 0$ and ν is one-to-one on R. Thus $R[X]/_J$ is an extension ring of R with the property that any mapping $f_0 : X \to S$ gives rise to an unique ring homomorphism $g : R[X]/_J \to S$ with g restricted to R the identity mapping. We now show that $R[X]/_J$ is a Boolean ring. Denote the elements of $R[X]/_J$ by $\overline{p}$ where p is a polynomial in $R[X]$. Then $\overline{p} = p + J$ and to show $(\overline{p})^2 = \overline{p}$ we need to show $p^2 - p \in J$. Since $p \in R[X]$ we have $p = \sum_{i,k}' a_{i_1 \cdots i_k} x_1^{i_1} \cdots x_k^{i_k}$ with $a_{i_1 \cdots i_k}$ in R and x_i in X. Hence,
$$p^2 = \sum_{i,k}' \left(a_{i_1 \cdots i_k} x_1^{i_1} \cdots x_k^{i_k}\right)^2 = \sum_{i,k}' a_{i_1 \cdots i_k} x_1^{2i_1} \cdots x_k^{2i_k},$$
using the fact that R is a Boolean ring. Thus to show $p^2 - p$ is in J it is enough to show that $x_1^{2i_1} \cdots x_k^{2i_k} - x_1^{i_1} \cdots x_k^{i_k}$ is in J. The latter, however, belong to G and hence to J. Thus $R[X]/_J$ is a relatively free Boolean ring over R with X a free set of generators.

Concerning relatively free Boolean rings and Boolean lattices one has the following general theorem.

<u>Theorem 5</u> : Let B be a Boolean lattice and let B^* be a relatively free extension of B with X a free set of generators. Then $\mathcal{R}(B^*)$ is a relatively free extension of $\mathcal{R}(B)$ with X a free set of generators. Conversely let R^* be a Boolean ring which is a relatively free extension of a Boolean ring R with X a free set of generators. Then $\mathcal{B}(R^*)$ is a

relatively free extension of $\mathcal{B}(R)$ with X a free set of generators.

Proof : (1) Let R be any extension Boolean ring of $\mathcal{R}(B)$. Let $f_o : X \to R$ be any mapping. Then f_o is also a mapping of X into $\mathcal{B}(R)$. Since B^* is a relatively free extension over B with X a free set of generators f_o extends uniquely to a Boolean lattice homomorphism $f : B^* \longrightarrow \mathcal{B}(R)$ with f restricted to B the identity map. f gives rise to a map $f_R : \mathcal{R}(B^*) \to R$ defined by $f_{R}(x) = f(x)$ for any x, noting that $\mathcal{R}(\mathcal{B}(R)) = R$. We saw in section 3 that f_R is a ring homomorphism and clearly f_R restricted to $\mathcal{R}(B)$ is the identity on $\mathcal{R}(B)$ since f is the identity on B. Moreover, f_R is unique since f is unique. Hence $\mathcal{R}(B^*)$ is a relatively free extension over $\mathcal{R}(B)$ with X a free set of generators.

(2) Let C be any extension lattice of $\mathcal{B}(R)$. Let $f_o : X \to C$ be any mapping. Then f_o is also a mapping of $X \longrightarrow \mathcal{R}(C)$. f_o gives rise to an unique ring homomorphism $f : R^* \longrightarrow \mathcal{R}(C)$ with f restricted to R the identity since R^* is a relatively free extension over R. f gives rise to a map $f_B : \mathcal{B}(R^*) \longrightarrow \mathcal{B}(\mathcal{R}(C)) = C$ defined by $f_B(x) = x$ for any x. Now, we have seen in section 3 that f_B is a Boolean lattice homomorphism. Moreover f_B is unique since f is and f_B restricted to $\mathcal{B}(R)$ is the identity map since f restricted to R is the identity map. This completes the proof.

We have thus shown that relatively free extensions over a Boolean lattice B correspond precisely to relatively free extensions over $\mathcal{R}(B)$. We now proceed to introduce the notion of absolute freeness for Boolean lattices and Boolean rings.

Definition 3 : A Boolean ring R is said to be free with X a free set of generators if any mapping $f_o : X \to S$ where S is any other Boolean ring extends uniquely to a ring homomorphism $f : R \to S$. A Boolean lattice B is said to be free with X a free set of generators if any mapping $f_0 :$ $X \longrightarrow C$ where C is any other Boolean lattice extends uniquely to a Boolean lattice homomorphism $f : B \to C$.

Theorem 6 : Let R be a free Boolean ring with X a free set of generators. Then $\mathcal{B}(R)$ is a free Boolean lattice with X a free set of generators. Conversely let B be a free Boolean lattice with X a free set of generators. Then $\mathcal{R}(B)$ is a free Boolean ring with X a free set of generators.

Proof : (1) Let C be any Boolean lattice. Let $f_0 : X \to C$ be any mapping. Then f_0 is also a mapping of X into $\mathcal{R}(C)$ since C and $\mathcal{R}(C)$ are the same set. Since R is a free Boolean ring with X a free set of generators f_0 extends to an unique ring homomorphism $f : R \to \mathcal{R}(C)$. f gives rise to a map $f_B : \mathcal{B}(R) \to C$ defined by $f_B(x) = f(x)$ for any x. We saw in Theorem 3 that f_B is a Boolean lattice homomorphism and f_B is unique since f is unique. Hence $\mathcal{B}(R)$ is a free Boolean lattice with X a free set of generators.

(2) Let R be any Boolean ring. Let $f_0 : X \longrightarrow R$ be any mapping. Then f_0 is also a mapping from X to $\mathcal{B}(R)$. Since B is a free Boolean lattice with X a free set of generators f_0 extends to an unique Boolean lattice homomorphism $f : B \to \mathcal{B}(R)$. Now f gives rise to a map $f_R : \mathcal{R}(B) \to R$ defined by $f_{\mathcal{R}}(x) = f(x)$ for any x. Then we have seen earlier that f_R is a ring homomorphism and f_R is unique since f is unique. This completes the proof.

CHAPTER III.

BOOLEAN SEMI-GROUPS.

Introduction : In this chapter we describe how the notion of a Boolean semi-group can be used to give an alternate characterization of a Boolean lattice with unit. We will conclude this chapter by proving that the category of Boolean lattices with unit and unitary Boolean lattice homomorphisms and the category of Boolean semi-groups and Boolean semi-group homomorphisms are equivalent.

1. Boolean Semi-groups and Boolean Lattices.

The main result we establish in this section is the following:

Theorem 1 : Any Boolean semi-group G is a Boolean lattice with unit under the divisibility relation. In this Boolean lattice the zero of the semi-group G is the zero of the Boolean lattice, the priming operation of G is the complementation in the Boolean lattice, $x \wedge y = xy$, $x \vee y = (x^1 \ y^1)^1$. Conversely any Boolean lattice B with unit is a Boolean semi-group under meet and complementation.

Proof : We establish the theorem in several steps.

(i) We first show that a Boolean semi-group G partially ordered by divisibility is a lattice in which $x \wedge y = xy$, $x \vee y = (x^1 y^1)^1$ and that in this lattice 0 of the G is the zero of the lattice and 0^1 is the unit of the lattice.

To show $xy = x \wedge y$ we must show (i) $xy \leq x,y$ and

(ii) if $z \leq x,y$ then $z \leq xy$. Now $(xy)x^1 = (xx^1)y = oy = o$.

Hence by Proposition 14 $(xy)x = xy$. Hence $xy \leq x$. Now $xy = yx \leq y$. Thus $xy \leq x,y$ as required. Next take z such that $z \leq x,y$. Then $zx = z$, $zy = z$. Thus $z(xy) = (zx)y = zy = z$. Hence $z \leq xy$. Therefore $xy = \inf.\{x,y\}$.

Next we show that $x \vee y = (x^1y^1)^1$ for any x, y in G. We first show that $x, y \leq (x^1y^1)^1$. By Proposition 16 we have $x^{11} = x$, $y^{11} = y$. Now $x^1y^1 \leq x^1, y^1$. Thus applying Proposition 16 once more we have $x,y \leq (x^1y^1)^1$. Thus $(x^1y^1)^1$ is an upper bound for x,y. Take any z in G such that $x,y \leq z$. Then we have $z^1 \leq x^1, y^1$. Hence $z^1 \leq x^1y^1$. Applying Proposition 16 once more we have $(x^1y^1)^1 \leq z$. Hence $x \vee y = (x^1y^1)^1$. Hence G is a lattice under the divisibility relation with meet and join as stated. Since $ox = o$ for each x in G o is a lower bound for each x in G. Hence o is the zero element of the lattice. To show that o^1 (where o is the zero element of the lattice) is the unit of the lattice G we only need to show that $x \leq o^1$ for all $x \in G$. That is we must show $x \vee o^1 = o^1$. But we know that $o^{11} = o$ and $x \vee o^1 = (x^1o^{11})^1 = (x^1o)^1 = o^1$. Hence o^1 is the unit of the lattice G.

(2) Now we show that the lattice G with o, o^1 as zero and unit is complemented. Since G is a Boolean semi-group $x \wedge x^1 = xx^1 = o$ for any $x \in G$. Also $o^1 = (x \wedge x^1)^1 = (x^{11} \wedge x^1)^1 = x \vee x^1$. Hence each x in G has for its complement the element x^1 where ^{1} is the priming operation of G. Hence the lattice G is complemented,

(3) We now show that the complemented lattice G is distributive. In any lattice we always have, $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$ and hence in particular for G. It remains to show that $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$.

To this end it is sufficient to show that

$$x \wedge (y \vee z) \wedge (x \wedge y)^1 \wedge (x \wedge z)^1 = o$$

We first establish that $x \wedge z = x \wedge (x^1 \vee z)$.

(a) $x \wedge z \leq x \wedge (x^1 \vee z)$ for $x \wedge z \leq x,\ x^1 \vee z$. Thus it remains to show

(b) $x \wedge (x^1 \vee z) \leq x \wedge z$. We see that $x \wedge (x^1 \vee z) \leq x$. Next $x \wedge (x^1 \vee z) \leq z$ for $x \wedge (x^1 \vee z) \wedge z^1 = (x \wedge z^1) \wedge (x^1 \vee z) = (x \wedge z^1) \wedge (x \wedge z^1)^1 = o$. Thus $x \wedge z = x \wedge (x^1 \vee z)$. Finally using what we have just established we get,

$$\begin{aligned} & x \wedge (y \vee z) \wedge (x \wedge y)^1 \wedge (x \wedge z)^1 \\ = \ & x \wedge (y \vee z) \wedge (x^1 \vee y^1) \wedge (x^1 \vee z^1) \\ = \ & (y \vee z) \wedge x \wedge y^1 \wedge (x^1 \vee z^1) \\ = \ & (y \vee z) \wedge (x \wedge z^1) \wedge y^1 \\ = \ & (y \vee z) \wedge (y^1 \wedge z^1) \wedge x \\ = \ & (y^1 \wedge z^1)^1 \wedge (y^1 \wedge z^1) \wedge x = o \wedge x = 0. \end{aligned}$$

Hence $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. That is, the Boolean semi-group G under the divisibility relation is a complemented distributive lattice. Therefore G is a Boolean lattice with unit with meet and join as stated.

Conversely, let B be any Boolean lattice with unit e. Then trivially the triple $(B, \wedge, 1)$ forms a Boolean semi-group and clear the zero of the Boolean lattice is the zero of the semi-group, $(B, \wedge, 1)$. This completes the proof.

2. Equivalence of Categories :

Let $\mathcal{C}$ denote the category whose objects are Boolean lattices with unit and whose maps are unitary Boolean lattice homomorphisms. Let $\mathcal{G}$ denote the category whose objects are Boolean semi-groups and whose maps are Boolean semi-group homomorphisms. In this section we establish that the categories $\mathcal{C}$ and $\mathcal{G}$ are equivalent. Employing the correspondence developed in Theorem 1 denote by $\mathcal{B}(G)$ the Boolean lattice associated with the Boolean semi-group G. Similarly for any Boolean lattice B with unit denote the associated Boolean semi-group by $\mathcal{G}(B)$. We now observe the following : Let $f : G \longrightarrow H$ be a Boolean semi-group homomorphism. Then f gives to a map $f_B : \mathcal{B}(G) \longrightarrow \mathcal{B}(H)$ defined by $f_B(x) = f(x)$ for any $x \in \mathcal{B}(G)$. The mapping f_B has the following properties :

(1) $f_B(x \wedge y) = f(xy) = f(x)f(y) = f_B(x) \wedge f_B(y)$ for any x,y in $\mathcal{B}(G)$.

(2) $f_B(x \vee y) = f((x^1y^1)^1) = (f(x^1y^1))^1 = ((f(x))^1(f(y))^1)^1 = f_B(x) \vee f_B(y)$.

(3) $f_B(x^1) = f(x^1) = (f(x))^1$ for any x in $\mathcal{B}(G)$.

Hence f_B is an unitary Boolean lattice homomorphism between the Boolean lattices $\mathcal{B}(G)$, $\mathcal{B}(H)$. On the other hand let B, C be Boolean lattices with unit and let $f : B \longrightarrow C$ be an unitary Boolean lattice homomorphism. Then f gives rise to a map $f_G : \mathcal{G}(B) \longrightarrow \mathcal{G}(C)$ defined by $f_G(x) = f(x)$ for any x in $\mathcal{G}(B)$. f_G has the following properties :

(1) $f_G(xy) = f(x \wedge y) = f(x) \wedge f(y) = f_G(x)\ f_G(y)$

(2) $f_G(x^1) = f(x^1) = f(x))^1 = (f_G(x))^1$.

Hence f_G is a Boolean semi-group homomorphism.

The main result of this section is the following theorem:

<u>Theorem 2</u> : The two correspondences $\left.\begin{matrix} B \longrightarrow \mathcal{G}(B) \\ f \longrightarrow f_G \end{matrix}\right\}$ and $\left.\begin{matrix} G \longrightarrow \mathcal{B}(G) \\ f \longrightarrow f_B \end{matrix}\right\}$ are covariant functors $S : \mathcal{C} \longrightarrow \mathcal{G}$ and $T : \mathcal{G} \longrightarrow \mathcal{C}$ which establish the equivalence of the categories $\mathcal{C}$ and $\mathcal{G}$.

<u>Proof</u> : (i) Let A, B, C be any three Boolean lattices with unit. The map S is clearly well-defined. Also,

(a) if $f \in H(A,B)$ then $S(f) = f_G \in H(\mathcal{G}(A), \mathcal{G}(B)) = H(S(A), S(B))$

(b) if $e_A \in H(A,A)$ then $S(e_A) = (e_A)_G \in H(\mathcal{G}(A), \mathcal{G}(A)) = e_{S(A)}$

(c) Finally suppose $f \in H(A,B)$, $g \in H(B,C)$.

We must show $S(g\ f) = S(g)T(f)$. Now $S(gf) = (gf)_G$ and $S(g)S(f) = g_G f_G$. Hence we must show that $(gf)_G(x) = (g_G f_G)(x)$ for any $x \in \mathcal{G}(A)$. But $(gf)_G(x) = (gf)(x) = g(f(x))$ for any x and $(g_G\ f_G)(x) = g_G(f_G(x)) = g_G(f(x)) = g(f(x))$ for any x. Hence $S(gf) = S(g)S(f)$, that is, S is a covariant functor from $\mathcal{C}$ to $\mathcal{G}$.

(ii) Let G, H, F be any three Boolean semi-groups.

(a) if $f \in H(G,H)$ then $T(f) = f_B \in H(\mathcal{B}(G), \mathcal{B}(H)) = H(T(G), T(H))$

(b) if e_G $H(G,G)$ then $T(e_G) \in H(\mathcal{B}(G), \mathcal{B}(G)) = e_{T(G)}$

(c) Finally suppose $f \in H(G,H)$, $g \in H(H,F)$.

We must show that $T(gf) = T(g)\ T(f)$. Now $T(gf) = (gf)_B$ and $T(g)T(f) = (g_B\ f_B)$. But $(gf)_B(x) = (gf)(x) = g(f(x))$ for any x and $(g_B\ f_B)(x) = g_B(f(x) = g(f(x))$. Hence $T(gf) = T(g)T(f)$; that is, T is a covariant functor from $\mathcal{G}$ to $\mathcal{C}$.

(iii) Since S and T are covariant functors the composites ST and TS are covariant functors on $\mathcal{G}$, $\mathcal{C}$ respectively. Let $I_{\mathcal{C}}$, $I_{\mathcal{G}}$ denote the identity functors on $\mathcal{C}$, $\mathcal{G}$ respectively. We show that $TS = I_{\mathcal{C}}$ and $ST = I_{\mathcal{G}}$. This will then establish that the categories $\mathcal{C}$ and $\mathcal{G}$ are equivalent. To this end we prove,

(1) $B = \mathcal{B}(\mathcal{G}(B))$ and (2) $G = \mathcal{G}(\mathcal{B}(G))$ where B is any Boolean lattice with unit and G is any Boolean semi-group. To show (1) : Put $G = \mathcal{G}(B)$, $\overline{B} = \mathcal{B}(G)$. Let the operations in B be denoted by $\overline{\wedge}$, $\overline{\vee}$, $\overline{/}$ and let the zero of $\overline{B}$ be denoted by $\overline{o}$. We want to show $B = \overline{B}$. B and $\overline{B}$ have the

same elements namely the elements of G. Thus $o = \bar{o}$. Also $x \bar{\wedge} y = xy$ $= x \wedge y$ for any x,y. $x \bar{\vee} y = (x^1 y^1)^1 = (x^1 \wedge y^1)^1 = x \vee y$ and clearly $x^{\bar{1}} = x^1$. Hence $B = \bar{B}$.

To show (2) : Let $B = \mathcal{B}(G)$ and $\bar{G} = \mathcal{G}(B)$. Let the operations in G be denoted by $\bar{\cdot}$, $\bar{/}$ and the zero of G by $\bar{o}$. We want to show $G = \bar{G}$. G and $\bar{G}$ have the same elements namely the elements of B. Now $x \bar{\cdot} y =$ $x \wedge y = xy$ for any x,y. Also $x^{\bar{1}} = x^1$ for any x. Hence $G = \bar{G}$.

Thus $(TS)(B) = \mathcal{B}(\mathcal{G}(B)) = B = I_{\mathcal{C}}(B)$ for any Boolean lattice B with unit. Also take any unitary Boolean lattice homomorphism $f : B \to C$. Then $(TS)(f) = T(f_G) = (f_G)_B$ and $I_{\mathcal{C}}(f) = f$. But $(f_G)_B$ has as its domain $\mathcal{B}(\mathcal{G}(B)) = B$ which is the domain of f and $(f_G)_B(x) = f(x)$ for any x in B. Hence $TS = I_{\mathcal{C}}$. On the other hand, $(ST)(G) = \mathcal{G}(\mathcal{B}(G))$ $= G = I_{\mathcal{G}}(G)$ for any Boolean semi-group G. Let $f \in H(G,F)$ be arbitrary. Then $I_G(f) = f$ and $(ST)(f) = (f_B)_G$. Again $(f_B)_G$ has as its domain $\mathcal{G}(\mathcal{B}(G)) = G$ which is the domain of f and $(f_B)_G(x) = f_B(x) = f(x)$ for any $x \in G$. Hence $ST = I_{\mathcal{G}}$. Therefore the categories $\mathcal{C}$ and $\mathcal{G}$ are equivalent. This completes the proof.

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