Gröbner Geometry for Hessenberg Varieties

GRÖBNER GEOMETRY FOR HESSENBERG VARIETIES

By MICHAEL GEOFFREY ASHCROFT CUMMINGS, BSc

A thesis submitted to the School of Graduate Studies
in partial fulfillment of the requirements for the degree
Master of Science

McMaster University
© Mike Cummings, April 2024

Master of Science, 2024
McMaster University
Department of Mathematics & Statistics
Hamilton, Ontario, Canada

TITLE: Gröbner Geometry for Hessenberg Varieties
AUTHOR: Mike Cummings, BSc
SUPERVISOR: Prof. Jenna Rajchgot
PAGES: ix, 66

Lay Abstract

Algebraic varieties provide a generalization of curves in the plane, such as parabolas and ellipses. One such family of these varieties are called Hessenberg varieties, and they are known to have connections to other areas of pure and applied mathematics, including to numerical linear algebra, combinatorics, and geometric representation theory.

In this thesis, we view Hessenberg varieties as a collection of subvarieties, called coordinate charts, and study the computational geometry of each coordinate chart. Although this is a local approach, we recover global geometric data on Hessenberg varieties. We also provide a partial positive answer to an open question in the area.

Abstract

We study Hessenberg varieties in type A via their local defining equations, called patch ideals. We focus on two main classes of Hessenberg varieties: those associated to a regular nilpotent operator and to those associated to a semisimple operator.

In the setting of regular semisimple Hessenberg varieties, which are known to be smooth and irreducible, we determine that their patch ideals are triangular complete intersections, as defined by Da Silva and Harada. For semisimple Hessenberg varieties, we give a partial positive answer to a conjecture of Insko and Precup that a given family of set-theoretic local defining ideals are radical.

A regular nilpotent Hessenberg Schubert cell is the intersection of a Schubert cell with a regular nilpotent Hessenberg variety. Following the work of the author with Da Silva, Harada, and Rajchgot, we construct an embedding of the regular nilpotent Hessenberg Schubert cells into the coordinate chart of the regular nilpotent Hessenberg variety corresponding to the longest-word permutation in Bruhat order. This allows us to use work of Da Silva and Harada to conclude that regular nilpotent Hessenberg Schubert cells are also local triangular complete intersections.

Acknowledgements

I am grateful to have worked under the supervision of Jenna Rajchgot. Her unwavering support, guidance, and positivity has made this thesis possible and enjoyable. Moreover, I thank Cam Franc, Megumi Harada, and Adam Van Tuyl for their collective support throughout my degree. Sergio Da Silva and Thái Thành Nguyễn were always willing to take time to answer questions and for that, I thank them as well. I also thank Hiraku Abe, Erik Insko, and Martha Precup for answering several questions throughout the past couple years.

My studies were partially funded by an NSERC CGS-M, an Ontario Graduate Scholarship, and a Milos Novotny Fellowship.

Many of the results in this thesis arose from computations in *Macaulay2*, so I thank Dan Grayson, Mike Stillman, and the larger *Macaulay2* community for making this possible.

Last, but certainly not least, I thank my friends and family for their love and consistent support. Although to exhaustively enumerate each of you without omission is an impossible exercise which I do not attempt here, I appreciate each and every one of your irreplaceable contributions.

Contents

List of Figures

Index of Symbols

$\mathrm{Mat}_n(\mathbb{K})$, $\mathbb{K}$-vector space of $n \times n$ matrices, page 23

N, regular nilpotent operator, page 26

$\mathcal{N}_w$, coordinate chart in $\mathbf{GL}_n(\mathbb{K})/B$ containing $w \in \mathfrak{S}_n$, page 17

$p^w_{k,\ell}$, regular semisimple generator, page 45

$P_{w,h}$, regular semisimple patch ideal, page 45

ψ_w, homomorphism of rings $\mathbb{K}[\mathbf{x}_{w_0}] \to \mathbb{K}[\mathbf{z}_w]$, page 35

$q^w_{k,\ell}$, semisimple generator, the (k,ℓ)-th entry of $(wM)^{-1}\mathsf{S}(wM)$, page 50

$\widetilde{q}^w_{k,\ell}(i)$, modified generator for $K_{w,h}$ derived from $q^w_{k,\ell}$, page 53

R, regular semisimple operator (in Jordan canonical form), page 45

S, semisimple operator (in Jordan canonical form), page 50

$\mathbf{S}$, subgroup of $\mathbf{GL}_n(\mathbb{K})$ of elements of the form diag$(s, s^2, \ldots, s^n)$, page 22

$\mathfrak{S}_n$, group of permutations on n letters, page 17

Z_w, variables of $\mathbb{K}[\mathbf{x}_{w_0}]$ that map to zero under ψ_w, page 34

Chapter 1

Introduction

The flag variety consists of sequences of nested vector subspaces. Hessenberg varieties are subvarieties of the full flag variety that satisfy an additional inclusion relation depending on a linear operator and an integer valued function called a Hessenberg function. Since their introduction in 1992 by De Mari, Procesi, and Shayman [DPS92] they have been widely studied for the following reasons. For one, Hessenberg varieties can be viewed as generalizations of other varieties, such as Peterson varieties [IY12], toric varieties associated to Weyl chambers [DPS92], and Springer fibers [Yun17]. At the same time, the study of Hessenberg varieties makes connections to other areas of research, including algebraic combinatorics [SW12; Gua16; BC18], geometric representation theory [Spr76; Fun03; Tym08], and Schubert calculus [AT10; HT11; Dre15; IT16].

In this thesis, we study type A Hessenberg varieties locally, via their local defining ideals, called patch ideals. The approach of using patches to study distinguished subvarieties of the flag variety dates back at least to the study of Schubert varieties and was such a widespread approach that a primary source is not known to the author. For more on the background of patches and patch ideals, see [IY12, Example 3]. Insko and Yong were the first to use patch ideals to study Hessenberg varieties. Using this approach, in 2012, they gave a combinatorial description of the singular locus of a family of regular nilpotent Hessenberg varieties called Peterson varieties [IY12]. Other instances of studying Hessenberg varieties via their patches include [ADGH18; IP19; AFZ20; ITW20; AI22; EPS22; Ata23; DH23; CDHR24].

We now outline the structure of this thesis. After providing the necessary background in Chapter 2, we begin Chapter 3 by discussing results from Da Silva and Harada [DH23] concerning the patches of regular nilpotent Hessenberg varieties at the longest-word permutation in Bruhat order. The main result from their work is that the local defining ideals in this setting are triangular complete intersections, as in Definition 2.3.5. The remainder of Chapter 3 discusses work of the author with Da Silva, Harada, and Rajchgot [CDHR24] involving regular nilpotent Hessenberg Schubert cells, intersections of regular nilpotent Hessenberg varieties with Schubert cells. Our main result is the following.

Theorem (Theorem 3.2.13)**.** *Regular nilpotent Hessenberg Schubert cells are local triangular complete intersections.*

Our proof, in Section 3.2, constructs an embedding of regular nilpotent Hessenberg Schubert cells into the coordinate chart of regular nilpotent Hessenberg varieties that was studied by Da Silva and Harada. Then in Section 3.3 we discuss applications of the above result, including recovering—in type A—the following result of Tymoczko [Tym07].

Corollary (Theorem 3.3.2)**.** *Regular nilpotent Hessenberg varieties has an affine paving by the set of regular nilpotent Hessenberg Schubert cells.*

We then turn our attention to regular semisimple and other (non-regular) semisimple Hessenberg varieties in Chapter 4. Regular semisimple Hessenberg varieties have been known to be smooth since their introduction [DPS92, Theorem 6] and the corresponding Gröbner geometry is correspondingly well-behaved. Although, to the knowledge of the author, this result is not in the literature, it will not be surprising to experts.

Theorem (Theorem 4.1.9)**.** *Regular semisimple Hessenberg varieties are local triangular complete intersections.*

Underlying the work thus far is a translation from the classical definition of the Hessenberg variety to one from which we may derive the local defining ideals. This translation guarantees that the local ideals set-theoretically cut out the Hessenberg variety patches but it is not guaranteed that these ideals are radical, and hence also scheme-theoretically define the patches of Hessenberg varieties. In the regular nilpotent cases above, this was shown by Insko and Yong [IY12] for Peterson varieties and Abe, DeDieu, Galetto, and Harada for all indecomposable regular nilpotent Hessenberg varieties [ADGH18]. Although some other cases are known (for instance, [AFZ20; ITW20; EPS22]), many other cases remain open, including the semisimple case. This was conjectured to hold by Insko and Precup.

Conjecture ([IP19, Conjecture 5.4])**.** *The natural set-theoretic defining ideals of semisimple Hessenberg varieties are radical.*

They also provided a sketch of a proof for the case of the semisimple Hessenberg variety associated to the Hessenberg function $(2, 3, \dots, n-1, n, n)$. Using our work from the regular semisimple case we give a positive answer of Insko and Precup's conjecture in a couple special cases depending on the patch and Hessenberg function. The main result of Section 4.2 is a positive answer for Insko and Precup's conjecture in the case that the semisimple operator $\mathsf{S} : \mathbb{K}^n \to \mathbb{K}^n$ has exactly $n-1$ eigenvalues. The machinery underpinning this result is an algorithm that rewrites the natural generators for the set-theoretic ideal, resulting in a generating set with relatively prime initial terms with respect to a lexicographic monomial order.

Theorem (Theorem 4.2.10)**.** *Insko and Precup's conjecture holds for Hessenberg varieties in* $\mathbb{K}^n$ *associated to semisimple operators with* $n-1$ *eigenvalues.*

We conclude this thesis in Chapter 5 by discussing the literature on Hessenberg varieties with a focus on patch ideals. We also review some problems that remain open and potential future research directions about Hessenberg varieties.

Chapter 2

Background

2.1 Commutative algebra basics

In this section we will establish some basic ideas in commutative algebra. These objects will appear later in Chapter 3 and Chapter 4 as tools and applications of our main results. Our main reference will be [Eis04] and, for this section, we assume only that the reader is familiar with algebra to the level of a first undergraduate course. For our purposes, we do not need the broader language of modules, so we will prefer the language of ideals and quotient rings. As polynomial rings form the setting for the later chapters, we will make the convention for this chapter that all rings are commutative with multiplicative identity.

A nonzero element r in a (commutative) ring R is a **zero divisor** if there exists a nonzero $s \in R$ such that $rs = 0$. If there is no such s, we say that r is a **nonzero divisor** and rings with no zero divisors are called **integral domains**.

Definition 2.1.1 ([Eis04, Chapter 10.3]). Let R be a commutative ring. A sequence $r_1, \dots, r_n$ of elements in R is a **regular sequence** if $\langle r_1, \dots, r_n \rangle$ is a proper ideal and the image of r_{i+1} in the quotient $R/\langle r_1, \dots, r_i \rangle$ is a nonzero divisor for all $i = 1, \dots, n-1$.

The **depth** of a ring is the maximal length of a regular sequence [Mat86, Theorem 16.7]. A nice setting is when this agrees with its dimension, which we will discuss later in this section.

Example 2.1.2. If R is an integral domain, then any sequence with exactly one term is a regular sequence.

Let $\mathbb{K}$ denote a field of arbitrary characteristic and consider the polynomial ring $R = \mathbb{K}[x, y, z]$. The sequence x, y, z is regular. Indeed, $y \cdot f = 0$ in $R/\langle x \rangle$ if and only if x divides f, and similarly $z \cdot g = 0$ in $R/\langle x, y \rangle$ if and only if each term of g is divisible by either x or y. In contrast, the sequence xy, yz in R is not regular since x is nonzero in $R/\langle xy \rangle$ yet $x \cdot yz = (xy)z = 0$ in $R/\langle xy \rangle$.

As we will see shortly, the notion of a regular sequence gives rise to the notion of a complete intersection ideal. Though to define a complete intersection, we first discuss local rings and localization.

A **local ring** is a commutative ring with exactly one maximal ideal. We will denote by $(R, \mathfrak{m})$ a local ring and its maximal ideal.

Example 2.1.3. Any field is a local ring with unique maximal ideal $\langle 0 \rangle$. Furthermore, if p is any prime integer and n is any positive integer, then the ring $\mathbb{Z}/p^n\mathbb{Z}$ is local whose maximal ideal is the set of multiples of p.

Many commutative algebra problems can be simplified to the local setting via *localization*. Some such problems involve local properties; properties of rings that can be verified at its localizations. We outline this process now, following [Eis04, Chapter 2.1].

Let R be a ring. A multiplicatively closed subset S is a subset of R that is closed under multiplication. (All ideals are multiplicatively closed subsets, but the converse is false.) The **localization of** R **at** S is the set of tuples $S^{-1}R = \{(r, s) \mid r \in R, s \in S\}$ under the equivalence relation $(r, s) \sim (r', s')$ if there exists some $t \in S$ such that $t(rs' - r's) = 0$ in R. If R is an integral domain, this reduces to $(r, s) \sim (r', s')$ if and only if $rs' = r's$. We will often suggestively write elements (r, s) equivalently as r/s. There is a canonical inclusion $R \hookrightarrow S^{-1}R$ given by $r \mapsto r/1$. Lastly, so that $S^{-1}R$ has the structure of a ring, we equip it with addition and multiplication defined by

$$r/s + r'/s' = (rs' + r's)/ss' \quad \text{and} \quad (r/s)(r'/s') = rr'/ss'.$$

For a simple example, notice that $\mathbb{Q} = (\mathbb{Z}^*)^{-1}\mathbb{Z}$. More generally, if P is a prime ideal of R, then $S = R \setminus P$ is a multiplicatively closed subset. In this case, we denote by R_P the localization $S^{-1}R$. We conclude this paragraph with the fact that localization is not a misnomer; it does indeed yield a local ring.

Proposition 2.1.4. *The localization $S^{-1}R$ is a local ring.*

Proof. We will argue that the subset $\mathfrak{m}$ of $S^{-1}R$ consisting of elements r/s with $r \in R \setminus S$ is the unique maximal ideal of $S^{-1}R$. From the above definition, it is clear that $\mathfrak{m}$ forms an ideal of $S^{-1}R$. Moreover, to see that it is the unique maximal ideal, we show that any element of $S^{-1}R$ not in $\mathfrak{m}$ is a unit of $S^{-1}R$. Indeed, let $r/s \in S^{-1}R$ with $r, s \in S$. Then s/r is also in $S^{-1}R$ and $(r/s)(s/r) = rs/sr$. Because R is commutative, we have that $(r/s)(s/r) = rs/rs = 1$. ∎

To define the notion of a complete intersection, we need to define a final pair of additional concepts. The **Krull dimension** of a ring R, denoted $\dim R$, is the supremum of the lengths of nested chains of prime ideals. Note that we count the chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_s$ as having length s, the number of inclusions, rather than the number of ideals. When there is no ambiguity, we will omit "Krull" and just say **dimension**.

In general, the depth of a Noetherian local ring is bounded above by its dimension [Eis04, Proposition 18.2]. In the case that depth and dimension agree, we say that the ring is **Cohen-Macaulay**. We note that a polynomial ring $R[x]$ is Cohen-Macaulay if and only if R is Cohen-Macaulay [Eis04, Proposition 18.9]. The next example computes the dimension of a polynomial ring; the interested reader can verify that it agrees with the depth. We will say that an ideal I of a Noetherian ring R is **Cohen-Macaulay** if the quotient R/I, localized at any prime, is a Cohen-Macaulay ring.

Cohen-Macaulayness is a central property in commutative algebra, with applications to geometry and combinatorics. For instance, local Cohen-Macaulay rings are equidimensional [Eis04, Corollary 18.11] and have no embedded primes (this follows from the Unmixedness Theorem [Eis04, Corollary 18.14]). They were also used to prove the upper-bound conjecture for spheres [Sta75].

Example 2.1.5. A field $\mathbb{K}$ of any characteristic has dimension 0 since the only proper ideal in $\mathbb{K}$ is $\langle 0\rangle$. The polynomial ring R over $\mathbb{K}$ in n indeterminates has dimension n. Indeed, the chain $\langle 0\rangle \subset \langle x_1\rangle \subset \langle x_1, x_2\rangle \subset \cdots \subset \langle x_1, \ldots, x_n\rangle$ has length n, so the dimension is at least n. To bound the dimension above by n, see, for instance, [Wat12, Theorem 3.4].

A **regular local ring** is a Noetherian local ring $(R, \mathfrak{m})$ such that $\mathfrak{m}$ can be generated by exactly $\dim R$ elements. The principal ideal theorem (see [Eis04, Theorem 10.2]) guarantees that $\dim R$ is a lower bound on the number of generators of $\mathfrak{m}$, so the property of regular local says that this lower bound is attained.

Example 2.1.6. Let $\mathbb{K}$ be a field of any characteristic and set $R = \mathbb{K}[x_1, \ldots, x_n]$. The ideal $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$ is maximal so in particular, $\mathfrak{m}$ is prime. Recall that we denote by $R_\mathfrak{m}$ the localization $(R \setminus \mathfrak{m})^{-1}R$. Proposition 2.1.4 and [Eis04, Corollary 2.3] guarantee that $R_\mathfrak{m}$ is local and Noetherian, respectively. The maximal ideal $\overline{\mathfrak{m}}$ of $S^{-1}R$, as described in the proof of Proposition 2.1.4, is generated by $x_1, \ldots, x_n$. (More precisely, $\overline{\mathfrak{m}}$ is generated by the images of $x_1, \ldots, x_n$ in $R_\mathfrak{m}$ under the canonical inclusion $R \hookrightarrow R_\mathfrak{m}$.) By a similar argument as that given in Example 2.1.5, we conclude that $\dim R_\mathfrak{m} = n$, and that $R_\mathfrak{m}$ is regular local.

The same justification applies to show that the localization of a polynomial ring at any prime ideal is a regular local ring.

Rings that are regular local are necessarily Cohen–Macaulay [Mat86, Theorem 17.8]. A Noetherian ring is **regular** if its localization at any prime ideal is a regular local ring. For instance, a polynomial ring over a field is a regular ring.

Definition 2.1.7 ([Eis04, Chapter 18.5])**.** A **complete intersection ring** is the quotient $R/\langle r_1, \ldots, r_n\rangle$ of a regular ring R by a regular sequence $r_1, \ldots, r_n$ in R. In this setting, we say that the ideal $\langle r_1, \ldots, r_n\rangle$ is a **complete intersection**.

In Chapters 3 and 4, we will study quotients R/I where R is a polynomial ring. Example 2.1.6 and the discussion that followed assures us that polynomial rings are regular, so we will need only satisfy the regular sequence part of Definition 2.1.7.

We conclude this section with a discussion on Hilbert series, following [MS05, Chapter 8] and [Pee11, Chapter 2]. A **graded ring** is a ring R with a direct sum decomposition $R = \bigoplus_{i\in\mathbb{Z}} R_i$ into countably-many (additive) abelian groups R_i satisfying $R_iR_j \subset R_{i+j}$ for all integers i and j. Elements of each R_i are called **homogeneous**. That the decomposition is a direct sum allows us to uniquely write any $f \in R$ as a sum of its **homogeneous components**, say $f = \sum_{i\in I} f_i$, where $I \subseteq \mathbb{Z}$ is a finite subset and $f_i \in R_i$ for all i.

The following is a prototypical example of a graded ring.

Example 2.1.8 (Standard grading on polynomial rings)**.** Consider the polynomial ring $R = \mathbb{K}[x_1, \dots, x_n]$. To each indeterminate x_i assign a **degree** of 1. The monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ thus has degree $|\alpha| = \sum_{i=1}^n \alpha_i$. We define each R_i to be the $\mathbb{K}$-vector space spanned by monomials of degree i. This yields the decomposition $R = \bigoplus_{i\geq 0} R_i$.

We say that an ideal $I \subseteq R$ is **homogeneous** if it has a generating set consisting of homogeneous elements. Some of our work in Chapter 3 will be to find non-standard gradings with respect to which the ideals in this chapter are homogeneous.

We now restrict to the case that R denotes the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ where $\mathbb{K}$ is a field of arbitrary characteristic, equipped with a $\mathbb{Z}$ grading satisfying $R_0 = \mathbb{K}$. Since $R_0R_i \subseteq R_i$ for all i, it follows that each R_i is a $\mathbb{K}$-vector space. In the following definition, we denote by $\dim_{\mathbb{K}}(R_i)$ the dimension of R_i as a $\mathbb{K}$-vector space, to distinguish it from the Krull dimension.

Definition 2.1.9 ([MS05, Definition 8.14])**.** Let $R = \bigoplus_{i\in\mathbb{Z}} R_i$ be a graded ring. The **Hilbert function** is the map $\mathbb{Z} \to \mathbb{N}$ given by $i \mapsto \dim_{\mathbb{K}} R_i$. This is encoded in the **Hilbert series** of R, defined as the formal series

$$HS_R(t) := \sum_{i\in\mathbb{Z}} (\dim_{\mathbb{K}} R_i)t^i.$$

We conclude this section with an example of a Hilbert series computation and a lemma that gives a formula for the Hilbert series of a complete intersection.

Example 2.1.10. Equip $R = \mathbb{K}[x]$ with the standard grading. For each $i \geq 0$, a basis for R_i is $\{x^i\}$, so the Hilbert series of R is

$$HS_R(t) = \sum_{i\geq 0} 1 \cdot t^i = \frac{1}{(1-t)}.$$

More generally, if $\deg x = d$, then $H_R(t) = 1/(1-t^d)$. Now if R denotes the polynomial ring in the n-many variables $x_1, \dots, x_n$, its Hilbert series is given by

$$HS_R(t) = \prod_{i=1}^n \frac{1}{(1-t^{\deg x_i})}.$$

A proof is given in [MS05, Lemma 8.16].

The following lemma will be used later in both Chapter 3 and Chapter 4. In the setting of those chapters, this result, paired with the computation in the previous example, yield an explicit formula for the Hilbert series.

Lemma 2.1.11 ([Pee11, Exercise 16.4])**.** *Let $f_1, \ldots, f_r$ be a regular sequence in the ring $R = \mathbb{K}[x_1, \ldots, x_n]$, so $r \leq n$. Then the Hilbert series of $S := R/\langle f_1, \ldots, f_r \rangle$ is*

$$HS_S(t) = HS_R(t) \prod_{i=1}^{r} \left(1 - t^{\deg f_i}\right).$$

2.2 Gröbner bases and degeneration

The theory of Gröbner bases was introduced Bruno Buchberger's in 1965 PhD thesis (see [Buc06]), and were named after his supervisor, Wolfgang Gröbner. Introduced as a tool to study residue classes, they permeate modern computational algebraic geometry as a tool to answer questions such as ideal membership [CLO15, Chapter 2.8] and has applications to optimization in integer programming [CLO05, Chapter 8.1], robotics, and automatic geometric theorem proving [CLO15, Chapter 8], among others. Our main reference for this section will be [CLO15].

Throughout this section and throughout the rest of this thesis, let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic and fix a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. Recall that R is Noetherian so every ideal in R can be represented as $\langle f_1, \ldots, f_r \rangle$ for some generators $f_1, \ldots, f_r \in R$. For the sake of notation, we make the convention of representing the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ by x^α. If $f \in R$ is a polynomial, we may write it as the sum $f = \sum_\alpha c_\alpha x^\alpha$ where each c_α is in $\mathbb{K}$. The **terms** of f are the products $c_\alpha x^\alpha$, while each x^α, omitting the coefficient, is a **monomial**.

We first introduce the notion of a *monomial order* on R. A total order on R is a transitive relation such that for each pair of monomials x^α and x^β in R, exactly one of the following holds: $x^\alpha > x^\beta$, $x^\alpha = x^\beta$, or $x^\alpha = x^\beta$.

Definition 2.2.1 ([CLO15, Chapter 2.2, Definition 1])**.** A **monomial order** on R is a total ordering $<$ on the monomials of R such that:

(i) for any monomials x^α, x^β, and x^γ in R, if $x^\alpha > x^\beta$, then $x^\alpha \cdot x^\gamma > x^\beta \cdot x^\gamma$;

(ii) $<$ is a well-ordering on the monomials of R, that is, every nonempty subset of monomials of R has a least element.

We next give examples of monomial orders. For proofs that they satisfy Definition 2.2.1, we refer the reader to [CLO15, Chapter 2.2].

Definition 2.2.2. We state the definitions in terms of the exponent vectors in $\mathbb{Z}^n_{\geq 0}$.

(i) **Lexicographic (lex) order**. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be exponent vectors. We say that $\alpha >_{\text{lex}} \beta$ in lex order (and equivalently, that $x^\alpha >_{\text{lex}} x^\beta$) if the first nonzero entry in the vector $\alpha - \beta$ is positive.

(ii) **Graded lexicographic (grlex) order**. Again let $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$. Denote by $|\alpha|$ the sum of the entries in α, and similarly for $|\beta|$. We say that $\alpha >_{\text{grlex}} \beta$ if $|\alpha| > |\beta|$, or if both $|\alpha| = |\beta|$ and $\alpha >_{\text{lex}} \beta$.

Example 2.2.3. Let $R = \mathbb{K}[x, y]$ and consider the monomials x and y^2. We have that $x >_{\text{lex}} y^2$ while $y^2 >_{\text{grlex}} x$.

If the monomial order is clear from context, we may omit the subscript. In later chapters, we will focus on lexicographic monomial orders.

Now fix a monomial order $<$ and consider a polynomial $f \in R$. The **initial term** of f with respect to $<$, written $\text{in}_<(f)$, is the largest term of f with respect to $<$.

Definition 2.2.4. Fix a monomial order $<$ on R and consider an ideal $I \subseteq R$. The **initial ideal** is the set of initial terms of elements of I. That is, $\text{in}_<(I) \coloneqq \{\text{in}_<(f) \mid f \in I\}$.

The initial ideal is indeed an ideal, and moreover, it is a monomial ideal, meaning that it has a generating set consisting of monomials. For a proof, we refer the reader to [CLO15, Chapter 2.5, Proposition 3]. The following example motivates the necessity for *Gröbner basis* and the idea that not all generating sets are created equal. In particular, the initial ideal is not necessarily the ideal generated by the initial terms of the generators.

Example 2.2.5. Let $R = \mathbb{K}[x, y, z]$ and consider the ideal $I = \langle x - y, x - z\rangle$. Suppose $<$ denotes the lexicographic order $x > y > z$. Then $\text{in}_<(x - y) = \text{in}_<(x - z) = x$. Yet $y-z$ is in I, so $\text{in}_<(y-z) = y$ is in the initial ideal. So $\langle \text{in}_<(x-y), \text{in}_<(x-z)\rangle \subsetneq \text{in}_<(I)$.

A Gröbner basis is a special generating set for an ideal such that the initial terms of its elements generate the initial ideal.

Definition 2.2.6 ([CLO15, Chapter 2.5, Definition 5])**.** Let I be an ideal of R and fix a monomial order $<$. A subset $\{g_1, \ldots, g_r\}$ of I, different from $\{0\}$, is said to be a **Gröbner basis for I with respect to** $<$, if $\text{in}_<(I) = \langle \text{in}_<(g_1), \ldots, \text{in}_<(g_r)\rangle$.

In the setting of Example 2.2.5, a Gröbner basis for I is $\{x - y, y - z\}$. A Gröbner basis is necessarily a generating set for an ideal [CLO15, Chapter 2.5, Corollary 6], however Gröbner bases, with respect to a fixed monomial order, are not unique. However, *reduced* Gröbner bases are unique, up to a choice of monomial order [CLO15, Chapter 2.7, Theorem 5].

Definition 2.2.7 ([CLO15, Chapter 2.7, Definition 4])**.** Let I be an ideal of R and $<$ a monomial order. A Gröbner basis $\{g_1, \ldots, g_r\}$ for I with respect to $<$ is **reduced** if, for each $i = 1, \ldots, r$,

(i) $\text{in}_<(g_i)$ is monic, and,
(ii) no monomial of g_i is in the ideal $\text{in}_<(\langle g_1, \ldots, \hat{g}_i, \ldots, g_r\rangle)$.

Buchberger's algorithm [CLO15, Chapter 2.7] takes a generating set for an ideal and computes a (reduced) Gröbner basis with respect to a fixed monomial order. However, the details of the algorithm are not necessary for this thesis. Instead, we will show that the naturally arising generators for the ideals in the later chapters form a Gröbner basis with respect to a convenient (lexicographic) monomial order. The following lemma will be our main tool to that end.

Lemma 2.2.8 ([CLO15, Chapter 2.9, Proposition 4]). *Let $\mathcal{G} = \{g_1, \ldots, g_r\}$ be a subset of an ideal I of $R = \mathbb{K}[x_1, \ldots, x_n]$ and $<$ a monomial order on R. Suppose that the monomials in the list $\{\mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_r)\}$ are relatively prime. Then $\mathcal{G}$ is a Gröbner basis for I with respect to $<$.*

The following lemma says that the Hilbert series of a quotient ring R/I is equal to the Hilbert series of the quotient $R/\mathrm{in}_<(I)$. For a statement of this lemma in the language of our thesis, see [CDHR24, Lemma 5.9].

Lemma 2.2.9 ([Eis04, Theorem 15.26]). *Let I be an ideal of R generated by polynomials that are homogeneous with respect to a positive $\mathbb{Z}$-grading on R. Fix a monomial order $<$. Then, $HS_{R/I}(t) = HS_{R/in_<(I)}(t)$.*

The process of taking initial ideals is an example of a degeneration technique, insofar as some properties, such as a Hilbert series, are invariant under replacing I by $\mathrm{in}_<(I)$. Initial ideals are standard theory in computational algebraic geometry. We now consider another degeneration technique, called *geometric vertex decomposition* introduced in 2009 by Knutson, Miller, and Yong [KMY09]. As the name suggests, geometric vertex decomposition gives a generalization of the notion of a vertex decomposable simplicial complex [PB80], but this is beyond the scope of this thesis.

Let $\mathbb{K}$ be a field of arbitrary characteristic and write $\mathbf{x}$ for a nonempty collection of indeterminates $\{x_1, \ldots, x_n\}$. Define $R = \mathbb{K}[\mathbf{x}]$ and pick an indeterminate $y = x_j$. Any polynomial $f \in R$ can be written in the form $f = \sum_{d=0}^{D} y^d g_d$, where g_D is nonzero and each g_d is a polynomial in $\mathbb{K}[\mathbf{x} \setminus y]$. The **initial y-form** of f is the polynomial $\mathrm{in}_y(f) = y^D g_D$; that is, the sum of the terms of f involving the highest power of y. If I is an ideal of R, then the **ideal of initial y forms**, denoted $\mathrm{in}_y(I)$, is the ideal of initial y-forms of elements of I. That is, $\mathrm{in}_y(I) = \langle \mathrm{in}_y(f) \mid f \in I \rangle$. A monomial order $<$ is said to be **y-compatible** if $\mathrm{in}_<(f) = \mathrm{in}_<(\mathrm{in}_y(f))$ for all polynomials f in R. That is, if and only if the diagram in Figure 2.1 commutes. One such order is the lexicographic order where $y > x_i$ for all $i \neq j$.

Definition 2.2.10 ([KMY09, Section 2.1]). Fix an ideal $I \subseteq R$ and a y-compatible monomial order $<$ on R. Construct a Gröbner basis $\{y^{d_i} q_i + r_i \mid i = 1, \ldots, r\}$ of I with respect to $<$ such that, y does not divide any term of q_i and y^{d_i} does not divide any term of r_i, for each i. Define ideals $C_{y,I} \coloneqq \langle q_i \mid i = 1, \ldots, r \rangle$ and $N_{y,I} \coloneqq \langle q_i \mid d_i = 0 \rangle$. If $\mathrm{in}_y(I) = C_{y,I} \cap (N_{y,I} + \langle y \rangle)$, then we say that I has a **geometric vertex decomposition** with respect to y.

$$\begin{tikzcd} R \arrow[r, "\mathrm{in}_y"] \arrow[dr, "\mathrm{in}_<"'] & R \arrow[d, "\mathrm{in}_<"] \\ & R \end{tikzcd}$$

Figure 2.1: Commutative diagram for y-compatible monomial orders $<$.

We make the following two remarks, following the notation of the definition and the preceding paragraph. First, requiring that y^{d_i} does not divide any term of r_i is equivalent to requiring that $\mathrm{in}_y(y^{d_i}q_i + r_i) = y^{d_i}q_i$. Secondly, the ideal of initial y-forms can be constructed from the above Gröbner basis (computed with respect to a y-compatible order) as $\mathrm{in}_y(I) = \langle y^{d_i}q_i \mid i = 1, \dots, r\rangle$.

As in the case of degeneration via initial ideals, geometric vertex decomposition can be used to compute Hilbert series [KMY09, Theorem 2.1(e)], although we do not require this result for this thesis. What is of interest to us, is the following recursive definition from Klein and Rajchgot, which builds on the work of Knutson, Miller, and Yong. To introduce this recursive definition we first require the notion of an *unmixed* ideal.

The **ideal quotient** of ideals I and J of R is the ideal $I : J := \{r \in R \mid rJ \subseteq I\}$. Here, rJ denotes the ideal of products of the form rj for $j \in J$. If I is principal, say $I = \langle f \rangle$, then we will write $f : J$, and similarly when J is principal.

Definition 2.2.11 ([AM69, Chapter 1] and [Lan65, Chapter VI.4])**.** Let R be a ring and fix $x \in R$. The **annihilator** of x is the ideal $\mathrm{Ann}(x) := (0 : x)$.

Let I be an ideal of R. A prime ideal $P \subseteq R$ is **associated** to I if $P = \mathrm{Ann}(f)$ for some $f \in R/I$. We denote by $\mathrm{Ass}(I)$ the set of all such primes.

We say that an ideal is unmixed if the quotient ring of each associated prime has the same Krull dimension.

Definition 2.2.12 ([Mat86, Chapter 17])**.** An ideal I of R is **unmixed** if each associated prime P of I satisfies $\dim(R/I) = \dim(R/P)$.

With this in hand, we now introduce the notion of a geometrically vertex decomposable ideal.

Definition 2.2.13 ([KR21, Definition 2.7])**.** An unmixed ideal I of $R = \mathbb{K}[\mathbf{x}]$ is **geometrically vertex decomposable** if any of the following hold:

(i) $I = R$;

(ii) I is generated by a (possibly empty) subset of indeterminates;

(iii) I has a geometric vertex decomposition with respect to some $y \in \mathbf{x}$ and both ideals $C_{y,I}$ and $N_{y,I}$ are geometrically vertex decomposable in $\mathbb{K}[\mathbf{x} \setminus y]$.

We take the (standard) convention that the ideals $\langle 0 \rangle$ and $\langle 1 \rangle$ of the base field $\mathbb{K}$ are geometrically vertex decomposable.

For our purposes, we will need not worry about the unmixedness property, nor in fact, the definition of an ideal being geometrically vertex decomposable. Rather, we will introduce a stronger version of geometric vertex decomposability, followed by a lemma that has a similar flavour to Lemma 2.2.9. Subsequently, we will list some applications of the geometric vertex decomposability property.

Definition 2.2.14 ([KR21, Definition 2.16]). Let I be an ideal of R and denote by y the largest variable of R with respect to a lexicographic monomial order $<$. We say that I is **$<$-compatibly geometrically vertex decomposable** if it satisfies Definition 2.2.13 upon replacing item (iii) with:

(iii') I has a geometric vertex decomposition with respect to y and the contractions of $C_{y,I}$ and $N_{y,I}$ to $\mathbb{K}[\mathbf{x}\backslash y]$ are $\prec$-compatibly geometrically vertex decomposable with respect to $\prec$, the naturally induced order on $\mathbb{K}[\mathbf{x} \setminus y]$ from $<$.

An ideal that is $<$-compatibly geometrically vertex decomposable with respect to a lexicographic order $<$ is necessarily also geometrically vertex decomposable. The converse does not hold [KR21, Example 2.16]. But as a result, we have the following sufficient condition.

Lemma 2.2.15 ([DH23, Lemma 3.6]). *Let I be an ideal of R. If there exists a lexicographic order $<$ such that the initial ideal $\mathrm{in}_<(I)$ is generated by indeterminates, then I is geometrically vertex decomposable.*

Klein and Rajchgot showed that this indeed provides a generalization of the notion of a vertex decomposable simplicial complex [KR21, Proposition 2.9]. The next theorem summarizes some of their other results.

Theorem 2.2.16 ([KR21]). *A geometrically vertex decomposable ideal is radical. An ideal that is both geometrically vertex decomposable and homogeneous is Cohen-Macaulay and G-linked to a complete intersection.*

That geometric vertex decomposition relates to liaison is an observation from Klein and Rajchgot [KR21]. Indeed, this provides a new tool for answering an open question that asks whether every arithmetically Cohen-Macaulay projective subscheme is G-linked to a complete intersection [KMMNP01, Question 1.6]. The reader can compare, for instance, [KR21, Corollary 4.3] with [CDH05, Proposition 5.1].

Knutson, Miller, and Yong showed that the Hilbert series can be constructed recursively via geometric vertex decompositions [KMY09, Theorem 2.1(e)]. More recently, Nguyễn, Rajchgot, and Van Tuyl connected geometrically vertex decomposable ideals to other commutative algebraic invariants, namely, the Castelnuovo-Mumford regularity, the multiplicity, and a-invariant [NRV23].

2.3 Frobenius splitting

Frobenius splitting was introduced by Mehta and Ramanathan in the 1980s as a tool in their study of Schubert varieties [MR85]. From there, local and global Frobenius splitting tools developed independently (see, for instance, [BK05] and [Hun96]) yet discovered many of the same results. In this section, we develop the notions of Frobenius splitting and compatibly split ideals, derive a sufficient condition for an ideal to be compatibly split, and discuss applications of being compatibly split. For a survey on Frobenius splitting and its history, see [SZ15].

In this section, we will denote by $\mathbb{K}$ an algebraically closed field of characteristic p and R a polynomial ring over $\mathbb{K}$. In later sections, we again allow $\mathbb{K}$ to denote an arbitrary algebraically closed field as many of our other arguments work over any characteristic. Our main reference for this section will be [BK05], though we restrict our language to the setting of polynomial rings over finite fields.

Definition 2.3.1. A map φ is a Frobenius splitting of R if and only if for all f and g in R, the following properties hold:

(i) $\varphi(f+g) = \varphi(f) + \varphi(g)$,
(ii) $\varphi(f^p g) = f\varphi(g)$,
(iii) $\varphi(1) = 1$.

Denote by φ a Frobenius splitting on R. An ideal I of R is **compatibly split with respect to** φ if $\varphi(I) \subseteq I$. The following proposition says that an ideal is radical if it is compatibly split and, moreover, that being Frobenius split is preserved under taking sums and intersections.

Proposition 2.3.2 ([BK05, Section 1.2])**.** *Suppose that φ is a Frobenius splitting of R and fix ideals I and J of R. Then,*

(i) R contains no nilpotent elements;
(ii) if φ compatibly splits I, then I is radical and φ compatibly splits its prime components;
(iii) if φ compatibly splits both I and J, then it also compatibly splits $I+J$ and $I \cap J$.

The prime components of an ideal I are the prime ideals $P_1, \dots, P_r$ of R satisfying $\sqrt{I} = P_1 \cap \cdots \cap P_r$ [CLO15, Chapter 4.6, Theorem 6]. These correspond to the irreducible components of an algebraic variety.

Definition 2.3.3 ([BK05, Definition 1.3.5])**.** The **trace map** Tr on $R = \mathbb{K}[x_1, \dots, x_n]$ is defined on monomials m of R by:

$$\mathrm{Tr}(m) = \begin{cases} \sqrt[p]{m \prod x_i} \Big/ \prod x_i & \text{if } m \prod x_i \text{ is a } p\text{-th power} \\ 0 & \text{otherwise,} \end{cases}$$

and extended additively to all of R.

It turns out that every Frobenius splitting φ can be realized as $\varphi(g) = \text{Tr}(fg)$ for some $f \in R$ [BK05, Chapter 1.3]. The following is a sufficient condition for an ideal to be compatibly split with respect to a convenient Frobenius splitting. It is immediate from the definitions and we include a proof for completeness. This result appears in the literature, for instance, as [DH23, Theorem 5.8].

Lemma 2.3.4. *Let $I = \langle f_1, \dots, f_r \rangle$ be an ideal of R and write $F = f_1 \cdots f_r$. Suppose that there exists a lexicographic order $<$ such that $in_<(F)$ is a squarefree product of variables. If u denotes the product of the variables that do not divide $in_<(F)$, then $\varphi : g \mapsto Tr((uF)^{p-1}g)$ is a Frobenius splitting of R that compatibly splits I.*

Proof. The trace map is additive and $\mathbb{K}$-linear by construction, so it remains to check that $\varphi(1) = 1$ and $\varphi(I) \subseteq I$. For the former, it is a straightforward computation to see that $\varphi(1) = \text{Tr}((uF)^{p-1}) = 1$. Indeed, the only monomial in uF^{p-1} that Tr does not map to 0 is $u^{p-1} \cdot \text{in}_<(F^{p-1})$, which is exactly the $(p-1)$-th power of the product of all the variables in R. We conclude that φ is a Frobenius splitting of R.

To see that φ compatibly splits I, let $h \in I$ and write $h = q_1 f_1 + \cdots + q_r f_r$ for $q_1, \dots, q_s \in R$. Then,

$$\varphi(h) = \text{Tr}\left((uF)^{p-1}h\right) = \sum_{i=1}^{r} \text{Tr}\left((uF)^{p-1}q_i f_i\right) = \sum_{i=1}^{r} f_i \, \text{Tr}\left((u\hat{F_i})^{p-1}q_i\right) \in I,$$

where $\hat{F_i} = f_1 \cdots \hat{f_i} \cdots f_r$ for each i. The final equality uses the fact that Tr is $\mathbb{K}$-linear. (This is straightforward for the case of monomials, and this case is sufficient because $(f+g)^p = f^p + g^p$ for any $f, g \in R$, since R has characteristic p [DF04, Chapter 7, Section 4, Exercise 26].) Hence I is compatibly split with respect to φ. ■

We conclude this section with a lemma that we will make use of in later chapters. It strengthens the notion of a complete intersection introduced in Section 2.1 and has connections to the Frobenius splitting discussed in this section and geometric vertex decomposition in Section 2.2. The lemma that follows from this definition is clear to experts, but to the knowledge of the author, first appeared in the literature in [DH23]. Recall that the initial monomial of a polynomial (with respect to a given monomial order) is its initial term divided by its corresponding coefficient.

Definition 2.3.5 ([DH23, Definition 3.3])**.** Let I be an ideal of $R = \mathbb{K}[x_1, \dots, x_n]$ where $\mathbb{K}$ is a field of arbitrary characteristic. Fix a monomial order $<$ on R. We say that I is a **triangular complete intersection of height** r **with respect to** $<$ if there exists an ordered list of generators $\{f_1, \dots, f_r\}$ for I satisfying:

(i) the list of initial monomials consists of distinct indeterminates, and,
(ii) for each i, the initial monomial of f_i does not appear in any f_j with $j > i$.

When the monomial order and/or height are either clear or unimportant, we may drop them and simply say **triangular complete intersection**.

Height of an ideal I, more generally, is defined to be the length of the longest chain of prime ideals contained in I. In their paper, Da Silva and Harada required for their definition that I be a complete intersection of height r, however, they noted in [DH23, Remark 3.8] that these assumptions can be removed; any ideal with a generating set satisfying (i) and (ii) is necessarily a complete intersection. In the following lemma, we provide a proof of this fact. The argument we provide removes from their argument the requirement for the generators to be homogeneous.

Lemma 2.3.6 ([CDHR24; DH23]). *Let $\mathbb{K}$ be an algebraically closed field of arbitrary characteristic and $R = \mathbb{K}[x_1, \dots, x_n]$. Suppose that the ideal I of R is a triangular complete intersection with respect to the lexicographic term order $<$ and $\{f_1, \dots, f_r\}$ are the generators of I satisfying Definition 2.3.5. Then,*

(i) I is a complete intersection,

(ii) I is prime,

(iii) the height of I is r,

(iv) we have an isomorphism of varieties $\mathbb{V}(I) \cong \mathbb{A}^{n-r}_{\mathbb{K}}$,

(v) $\{f_1, \dots, f_r\}$ is a Gröbner basis for I with respect to $<$,

(vi) I is geometrically vertex decomposable,

(vii) if $\mathbb{K}$ is an algebraically closed field of positive characteristic, then I is compatibly split with respect to a Frobenius splitting of R.

Proof. The statements (v), (vi), and (vii) are exactly Lemmas 2.2.8, 2.2.15, and 2.3.4, respectively. In particular, the splitting for (vii) is exactly the one constructed in Lemma 2.3.4. Since $\mathrm{in}_<(I)$ is a complete intersection because it is an ideal of indeterminates, so I is a complete intersection as well by [Gro67, Corollary 19.3.8].

We will show that I is prime by showing that R/I is an integral domain, and in particular, that it is a polynomial ring in $(n-r)$-many variables. This argument suffices to show (iv) as well [Har77, Chapter I, Corollary 3.7].

Assume without loss of generality that $\{f_1, \dots, f_r\}$ forms a *reduced* Gröbner basis for I with respect to $<$. That is, after relabelling the variables, we may write $f_i = x_i - q_i$ for each $i = 1, \dots, r$, where no x_i divides any term of q_j for all i and j. It is then clear that

$$R/I \cong \mathbb{K}[q_1, \dots, q_r, x_{r+1}, \dots, x_n] = \mathbb{K}[x_{r+1}, \dots, x_n],$$

so I is prime and $\mathbb{V}(I) \cong \mathbb{A}^{n-r}_{\mathbb{K}}$. Lastly, we use the formula [Har77, Chapter I, Theorem 1.8A] to compute the height of I as follows:

$$\mathrm{ht}(I) = \mathrm{ht}(R) - \mathrm{ht}(R/I) = n - (n-r) = r,$$

where we use the fact that the height of a polynomial ring over a field is given by the number of variables (see, for instance, the corollary in [Mat86, p. 35]). ■

2.4 Subvarieties of the flag variety

We conclude Chapter 2 by setting up the main objects that we will discuss in the next chapters. We introduce Hessenberg varieties as subvarieties of the flag variety and discuss the approach, introduced by Insko and Yong [IY12], of studying Hessenberg varieties via their defining equations. Throughout this section, and for much of the later chapters, we take $\mathbb{K}$ to be an algebraically closed field of arbitrary characteristic, while $R = \mathbb{K}[\mathbf{x}]$ is a polynomial ring in finitely-many variables $\mathbf{x}$. We expect that the reader is familiar with algebraic geometry—affine and projective varieties, the Nullstellensatz, and so on—to the level of a first course following the first few sections of, for instance, [CLO15; Sha13].

The structure of this section is as follows. In the first subsection, we introduce flag varieties and the Plücker embedding. In Section 2.4.2, we discuss Hessenberg varieties and their local defining equations which will form the main objects of study in Chapters 3 and 4. Lastly in Section 2.4.4, we discuss Schubert varieties and Schubert cells, the latter of which we will intersect with Hessenberg varieties in Chapter 3.

2.4.1 Flag varieties

Our main reference for this subsection will be [MS05]. Classically, the Plücker embedding embeds the Grassmannian into projective space. We can use the same approach to embed flag varieties into projective space as they can be viewed as subvarieties of products of Grassmannians.

Definition 2.4.1 ([Har92, Example 6.6])**.** The **Grassmannian** $\mathrm{Gr}(d, n)$ is the collection of d-dimensional vector subspaces of $\mathbb{K}^n$.

Notice that we must have $d \leq n$. The **Plücker embedding** [MS05, Chapter 14.1] gives rise to coordinates defining an element of the Grassmannian in projective space in the following way. An element g of $\mathrm{Gr}(d, n)$ can be represented as an $n \times d$ matrix G with entries in $\mathbb{K}$ whose d columns form a basis for g. For any subset $\sigma \subseteq [d]$ with $|\sigma| = r$, denote by G_σ the $r \times r$ submatrix of G consisting of the columns $1, \ldots, r$ and rows $\sigma_1, \ldots, \sigma_r$. The **Plücker coordinates** of g are the minors $\det G_\sigma$ where σ ranges over all subsets of $[d]$. Hence the Grassmannian $\mathrm{Gr}(d, n)$ is parametrized by the collection of minors $\{\det(\mathbf{x})\}_{\sigma \subseteq [d]}$ for a $d \times n$ matrix of indeterminates $[\mathbf{x}]_{i,j} = x_{i,j}$.

Definition 2.4.2 ([MS05, Chapter 14.1])**.** The **(full) flag variety** in $\mathbb{K}^n$ is the collection of vector subspaces of $\mathbb{K}^n$

$$\mathrm{Flags}(\mathbb{K}^n) = \big\{ V_\bullet = (V_0 \subset V_1 \subset \cdots \subset V_n) \;\big|\; \dim_{\mathbb{K}} V_i = i \text{ for all } i \big\}.$$

Each element $V_\bullet$ is called a **flag**.

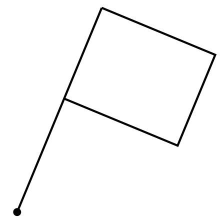

Figure 2.2: A flag.

Figure 2.2 provides justification by example for the nomenclature. It illustrates a flag $V_\bullet = (V_0 \subset V_1 \subset V_2)$ where V_0 is the origin, V_1 is the line, and V_2 is the plane.[1]

It is not necessarily clear from the definition that Flags($\mathbb{K}^n$) is indeed a variety, but this follows from the Plücker embedding as follows. Fix a flag $V_\bullet$ and a basis for $\mathbb{K}^n$, then construct a matrix $G \in \mathbf{GL}_n(\mathbb{K})$ so that V_i is the span of the first i columns of G. The Plücker coordinates of $V_\bullet$ are the 1×1 minors of the first column of G, the 2×2 minors of the first two columns of G, and so on.

We end this subsection by making explicit the relation between Flags($\mathbb{K}^n$) and $\mathbf{GL}_n(\mathbb{K})$. Notice that the process through which we associated a matrix in $\mathbf{GL}_n(\mathbb{K})$ to a flag is not unique. Indeed, should we change the basis for some V_i, then the matrix representative changes too. In particular, right-multiplication of $G \in \mathbf{GL}_n(\mathbb{K})$ by any invertible lower triangular matrix U preserves the flag represented by G. That is, G and GU represent the same flag. Said differently, if we denote by B the subgroup of invertible upper triangular matrices,[2] we have the following.

Proposition 2.4.3. Flags($\mathbb{K}^n$) $\cong \mathbf{GL}_n(\mathbb{K})/B$.

Fix a permutation $w \in \mathfrak{S}_n$. We will abuse notation and denote both the permutation and its corresponding permutation matrix by w. Let U^- be the subgroup of $\mathbf{GL}_n(\mathbb{K})$ consisting of lower triangular matrices with 1's along the diagonal. We denote by $\mathcal{N}_w := wU^-B$ an **open cell**—the coordinate chart—in $\mathbf{GL}_n(\mathbb{K})/B$ containing w. Specifically, let $M \in U^-$ be the matrix

$$M := \begin{bmatrix} 1 & & & & \\ \star & 1 & & & \\ \star & \star & 1 & & \\ \vdots & & \ddots & \ddots & \\ \star & \star & \cdots & \star & 1 \end{bmatrix}, \tag{2.4.4}$$

where each $\star$ is in $\mathbb{K}$. As there are $n(n-1)/2$ such $\star$'s, it is clear that $U^- \cong \mathbb{A}_{\mathbb{K}}^{n(n-1)/2}$. Each permutation w thus gives rise to an embedding $M \mapsto wMB$ in $\mathbf{GL}_n(\mathbb{K})/B$ and

[1] Technically, the line and plane both stretch off to infinity, but our illustration must conform to the finite amount of paper reserved for this thesis.

[2] We use the letter B as it is a *Borel* subgroup of $\mathbf{GL}_n(\mathbb{K})$.

hence $\mathcal{N}_w \cong \mathbb{A}^{n(n-1)/2}$. A point in $\mathcal{N}_w$ is thus identified uniquely with a matrix wM whose (i,j)-th entry is given by

$$[wM]_{i,j} = \begin{cases} 1 & \text{if } j = w^{-1}(i), \\ 0 & \text{if } j > w^{-1}(i), \\ x_{i,j} & \text{otherwise.} \end{cases} \tag{2.4.5}$$

Here, each $x_{i,j}$ is an indeterminate. For each $w \in \mathfrak{S}_n$, we denote by $\mathbf{x}_w$ the collection of these indeterminates $x_{i,j}$. From the isomorphism $\mathcal{N}_w \cong \mathbb{A}^{n(n-1)/2}$ it follows that the coordinate ring of $\mathcal{N}_w$ is isomorphic to the polynomial ring in $n(n-1)/2$ indeterminates over $\mathbb{K}$, specifically, the ring $\mathbb{K}[\mathbf{x}_w]$. We thus have the following.

Fact 2.4.6. The set of all cells $\mathcal{N}_w$ form an open cover of $\mathrm{Flags}(\mathbb{K}^n)$.

2.4.2 Hessenberg varieties

In the late 1980s, De Mari, Procesi, and Shayman introduced Hessenberg varieties, a family of subvarieties of the full flag variety [DPS92; DS88]. These varieties generalize the notion of Hessenberg matrices, which formed a computationally convenient class of matrices in the field of numerical linear algebra [SB80]. Our main reference for this section is [ADGH18]. We note that although the literature for Hessenberg varieties typically works over $\mathbb{C}$, both the algebraic and geometric arguments required for this thesis can be relaxed to an arbitrary algebraically closed field $\mathbb{K}$.

For a positive integer n, we denote $[n] \coloneqq \{1, \ldots, n\}$. A **Hessenberg function** $h : [n] \to [n]$ is a nondecreasing map satisfying $h(i) \geq i$ for all i. We say that a Hessenberg function is **indecomposable** if it satisfies $h(i) > i$ for all $i \in [n-1]$. In either case, since Hessenberg functions are nondecreasing, we always have that $h(n) = n$. We will write a Hessenberg function as a tuple $h = (h(1), h(2), \ldots, h(n))$.

Denote by $\mathfrak{gl}_n(\mathbb{K})$ the Lie algebra of $\mathbf{GL}_n(\mathbb{K})$, which consists of $n \times n$ matrices with entries in $\mathbb{K}$. To a Hessenberg function $h : [n] \to [n]$ we can associate the following (vector) subspace of $\mathfrak{gl}_n(\mathbb{K})$.

Definition 2.4.7. The **Hessenberg space** associated to a Hessenberg function h is the vector subspace of $\mathfrak{gl}_n(\mathbb{K})$ whose (i,j)-th entry vanishes for all $i > h(j)$.

Visually, the Hessenberg space specifies a set of entries (or "boxes") in the lower-left corner that must be zero. Figure 2.3 illustrates the Hessenberg space corresponding to $h = (2, 3, 3, 5, 5)$, where the filled-in boxes denote the matrix entries that must be zero. The reader familiar with Dyck paths may note that the vanishing condition for the Hessenberg space associated to h is exactly the set of boxes that lie beneath the Dyck path from $(0, n)$ to $(n, 0)$ induced from h.

Fix a linear operator $\mathsf{A} \in \mathfrak{gl}_n(\mathbb{K})$ and a Hessenberg function $h : [n] \to [n]$. Classically, and in type A, the Hessenberg variety associated to A and h is the subvariety of $\mathrm{Flags}(\mathbb{K}^n)$ consisting of flags $V_\bullet = (V_0 \subset \cdots \subset V_n)$ that satisfy $\mathsf{A}V_i \subseteq V_{h(i)}$ for all

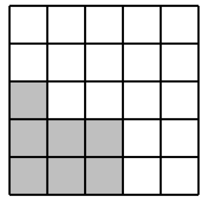

Figure 2.3: The Hessenberg space corresponding to $h = (2, 3, 3, 5, 5)$.

$i = 1, \ldots, n$. Using the identification from Proposition 2.4.3, we give the following equivalent definition of Hess(A, h). We will justify this equivalence following the definition. Recall that B denotes the subgroup of $\mathbf{GL}_n(\mathbb{K})$ consisting of invertible upper triangular matrices.

Definition 2.4.8. View a linear operator $\mathsf{A} \in \mathfrak{gl}_n(\mathbb{K})$ as an $n \times n$ matrix and fix a Hessenberg function h. The **Hessenberg variety** associated to A and h is the set of cosets

$$\mathrm{Hess}(\mathsf{A}, h) := \left\{ MB \in \mathbf{GL}_n(\mathbb{K})/B \mid M^{-1}\mathsf{A}M \in H(h) \right\},$$

where $H(h)$ is the Hessenberg space from Definition 2.4.7.

The above is well-defined because the Hessenberg space is invariant under conjugation by invertible upper-triangular matrices. Indeed, left- and right-multiplication on $\mathfrak{gl}_n(\mathbb{K})$ by elements of B correspond, respectively, to rightward column operations and upward row operations. It follows that if $X \in B$ and $Y \in H(h)$, then $X^{-1}YX \in H(h)$ as well.

So suppose that $MB = \widetilde{M}B$ and $M^{-1}\mathsf{A}M \in H(h)$ for some $\mathsf{A} \in \mathfrak{gl}_n(\mathbb{K})$ and some Hessenberg function h. Since $MB = \widetilde{M}B$, write $\widetilde{M} = MX$ for some $X \in B$. We then have that

$$\widetilde{M}^{-1}\mathsf{A}M = (MX)^{-1}\mathsf{A}(MX) = X^{-1}(M^{-1}\mathsf{A}M)X \in H(h).$$

The following example demonstrates the equivalence between these two equations via local defining equation. Afterwards, we provide a general argument.

Example 2.4.9. Fix a basis on $\mathbb{K}^3$ and let $\mathsf{A} = \mathrm{diag}(1, 2, 3)$. Pick the (indecomposable) Hessenberg function $h = (2, 3, 3)$. Then, the coset represented by the matrix

$$M := \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix},$$

where a, b, and c are indeterminates, is in the Hessenberg variety if and only if

$$M^{-1}\mathsf{A}M = \begin{bmatrix} 1 & 0 & 0 \\ a & 2 & 0 \\ 2b - ac & c & 3 \end{bmatrix} \in H(h),$$

or equivalently, if and only if $2b - ac = 0$. On the other hand, denote by v_i the i-th column of M and the flag $V_\bullet = (V_0 \subset V_1 \subset V_2 \subset V_3)$ where

$$V_1 = \text{span}\{v_1\}, \; V_2 = \text{span}\{v_1, v_2\}, \text{ and } V_3 = \text{span}\{v_1, v_2, v_3\}.$$

We are required to establish conditions guaranteeing that $\mathsf{A}V_i \subseteq V_{h(i)}$ for all i. Since $h = (2, 3, 3)$, this condition is equivalent to ensuring that $\mathsf{A}V_1 \subseteq V_2$. Said differently, this holds if and only if the matrix

$$\begin{bmatrix} | & | & | \\ \mathsf{A}v_1 & v_1 & v_2 \\ | & | & | \end{bmatrix}$$

has rank 2, or if and only if,[3] the above matrix has determinant 0. (Of course, its 2×2 minors do not vanish since the second and third columns are linearly independent.) So we compute,

$$\begin{vmatrix} 1 & 1 & 0 \\ 2a & a & 1 \\ 3b & b & c \end{vmatrix} = ac + 3b - b - 2ac = 2b - ac,$$

which agrees with the first computation. $\Diamond$

More generally, fix a flag $V_\bullet = (V_0 \subset V_1 \subset \cdots \subset V_n) \in \text{Flags}(\mathbb{K}^n)$. Pick a basis $\{v_1, \ldots, v_n\}$ for $\mathbb{K}^n$ such that $V_i = \text{span}\{v_1, \ldots, v_i\}$ and construct a matrix

$$M := \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}.$$

Now for any linear operator $\mathsf{A} : \mathbb{K}^n \to \mathbb{K}^n$ and Hessenberg function $h : [n] \to [n]$, we have that $V_\bullet \in \text{Hess}(\mathsf{A}, h)$ if and only if $\mathsf{A}V_i \subseteq V_{h(i)}$ for all i. Equivalently,

$$\text{rank}\begin{bmatrix} | & & | & | & & | \\ \mathsf{A}v_1 & \cdots & \mathsf{A}v_i & v_1 & \cdots & v_{h(i)} \\ | & & | & | & & | \end{bmatrix} = h(i),$$

which, because M is invertible, is equivalent to

$$\text{rank}\begin{bmatrix} | & & | & | & & | \\ M^{-1}\mathsf{A}v_1 & \cdots & M^{-1}\mathsf{A}v_i & e_1 & \cdots & e_{h(i)} \\ | & & | & | & & | \end{bmatrix} = h(i),$$

where e_j is the j-th elementary basis vector of $\mathbb{K}^n$. Colloquially, this says that the first i columns of $M^{-1}\mathsf{A}M$ must be 0 in rows $h(i)+1, \ldots, n$. In particular, the collection of

[3] A matrix has rank at most r if and only if all of its $(r+1)\times(r+1)$ minors vanish.

rank conditions as above as we range over i is thus equivalent to $M^{-1}\mathsf{A}M \in H(h)$. We conclude that the two definitions of Hessenberg variety set-theoretically agree.

It is also clear from Definition 2.4.8 that, for any invertible $g : \mathbb{K}^n \to \mathbb{K}^n$, there exists an isomorphism $\mathrm{Hess}(\mathsf{A}, h) \cong \mathrm{Hess}(g\mathsf{A}g^{-1}, h)$ for any linear operator A and Hessenberg function h. For a proof, see [Tym06, Proposition 2.7]. As a result, we can assume without loss of generality that (the matrix representing) A is in Jordan canonical form. We recall the following classes of linear operators.

Definition 2.4.10. A linear operator $\mathsf{A} : \mathbb{K}^n \to \mathbb{K}^n$ is

- **nilpotent** if $\mathsf{A}^k = 0$ for some positive integer k,
- **regular** if its eigenvalues are pairwise distinct,
- **semisimple** if it is diagonalizable.

A **nilpotent Hessenberg variety** is one associated to a nilpotent operator (and any Hessenberg function), and similarly for other classes of operators. For instance, in Chapter 3, we will focus on the case of an operator that is both regular and nilpotent.

The vanishing condition arising from the Hessenberg space, as given in Definition 2.4.8, gives rise to local defining equations. By the result of Fact 2.4.6, to study flag varieties locally, it suffices to study the open cells $\mathcal{N}_w$. Then to study Hessenberg varieties locally, we recall from Definition 2.4.8 that $\mathrm{Hess}(\mathsf{A}, h) = \{MB \in \mathbf{GL}_n(\mathbb{K})/B \mid M^{-1}\mathsf{A}M \in H(h)\}$ and make the following definition.

Definition 2.4.11. Fix a coordinate chart $\mathcal{N}_w \cap \mathrm{Hess}(\mathsf{A}, h)$ and denote by wM the matrix as in Equation (2.4.5). To each index $(k, \ell) \in [n]^2$, denote by $f^w_{k,\ell}$ the polynomial

$$f^w_{k,\ell}(\mathsf{A}) := \left[(wM)^{-1}\mathsf{A}(wM)\right]_{k,\ell} \subseteq \mathbb{K}[\mathbf{x}_w].$$

In this setting, the **Hessenberg patch ideal** is the ideal $I^{\mathsf{A}}_{w,h}$ generated by the polynomials $f^w_{k,\ell}(\mathsf{A})$ for each $k > h(\ell)$. When the linear operator A is clear from context, we will write $f^w_{k,\ell}$ and $I_{w,h}$ for $f^w_{k,\ell}(\mathsf{A})$ and $I^{\mathsf{A}}_{k,h}$, respectively.

Remark 2.4.12. Recall that $\mathbb{K}[\mathrm{x}]$ is the coordinate ring of the coordinate chart $\mathcal{N}_w$. From Definition 2.4.8 it follows that each ideal $I_{w,h}$ set-theoretically cuts out $\mathcal{N}_w \cap \mathrm{Hess}(\mathsf{A}, h)$, however it is not guaranteed that they also scheme-theoretically agree. For that, we must ensure that the ideals are radical, which justifies the term *patch ideal*. We will not delve further into scheme-theoretic ideas beyond verifying that our local defining ideals are radical.

We conclude this subsection with a lemma in the special case of a Hessenberg variety associated to a regular nilpotent operator. In this case, it suffices to only consider those varieties corresponding to indecomposable Hessenberg functions. Recall that a Hessenberg function $h : [n] \to [n]$ is indecomposable if it satisfies $h(i) > i$ for all $i \in [n-1]$.

Lemma 2.4.13 ([Dre15, Theorem 4.5])**.** *Any regular nilpotent Hessenberg variety is the product of indecomposable regular nilpotent Hessenberg varieties.*

2.4.3 Torus actions

In this brief subsection we describe actions of two tori on Hessenberg varieties. As we will see in later chapters, these actions give rise to a grading on the corresponding coordinate ring and moreover, our local defining equations will be homogeneous with respect to this nonstandard grading. We follow [CDHR24, Section 2.3].

Define the following subgroup of $\mathbf{GL}_n(\mathbb{K})$.

$$\mathbf{S} \coloneqq \left\{\operatorname{diag}(s, s^2, \ldots, s^n) \mid s \in \mathbb{K}^*\right\} \cong \mathbb{K}^*$$

As a subgroup of $\mathbf{GL}_n(\mathbb{K})$, $\mathbf{S}$ naturally acts on $\operatorname{Flags}(\mathbb{K}^n)$ by multiplication. Moreover, this action on $\operatorname{Flags}(\mathbb{K}^n)$ preserves regular nilpotent Hessenberg varieties [HT17, Lemma 5.1(3)]. Since $\mathbf{S}$ is a subgroup of the maximal torus

$$\mathbf{T} \coloneqq \{\operatorname{diag}(t_1, t_2, \ldots, t_n) \mid t_i \in \mathbb{K}^* \text{ for each } i \in [n]\} \cong (\mathbb{K}^*)^n$$

and $\mathbf{T}$ preserves coordinate patches $\mathcal{N}_w$, so does $\mathbf{S}$. On each patch, this action gives rise to a grading on the coordinate ring $\mathbb{K}[\mathbf{x}_w]$, which we will now derive. To do so, fix some $\mathbf{t} \in \mathbf{S}$. Explicitly, the action on a flag $[wM] \in \mathbf{GL}_n(\mathbb{K})/B$ is given by

$$\mathbf{t}[wM] = [\mathbf{t}(wM)].$$

We are now required to find a matrix M' such that $\mathbf{t}(wM) = wM'$. The 1's in w occur at $(w(j), j)$ for each $j \in [n]$. Multiplication $\mathbf{t} \cdot w$ sends the 1 in w at $(w(j), j)$ to $t^{w(j)}$. As a result, to return to our original form, we must right-multiply by the diagonal matrix whose j-th diagonal entry is $t^{-w(j)}$. We conclude that the desired M' satisfies

$$[wM']_{i,j} = \begin{cases} 1 & \text{if } j = w^{-1}(i), \\ 0 & \text{if } j > w^{-1}(i), \\ t^{i-w(j)} x_{i,j} & \text{otherwise.} \end{cases} \tag{2.4.14}$$

This gives rise to the following definition.

Definition 2.4.15. Fix a coordinate chart $\mathcal{N}_w$. To the variable $x_{i,j} \in \mathbb{K}[\mathbf{x}_w]$, we assign the degree $w(j) - i$.

In the next subsection, we will introduce the notion of a Schubert cell. An intersection of Hessenberg varieties with a Schubert cell corresponds to setting some variables to zero. These variables $x_{i,j}$ are exactly those that satisfy $w(j) - i < 0$. Upon taking this intersection, the degree of any remaining variable is positive.

2.4.4 Schubert varieties

We conclude this chapter with a short section on Schubert cells and Bruhat order. These tie in nicely to the study of Schubert varieties, subvarieties of the Grassmannian, that are well-studied in algebraic combinatorics and related areas. Our main

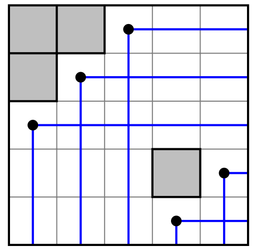

Figure 2.4: The Rothe diagram of $w = 32154$.

references will be [CDHR24] for Schubert cells and [MS05, Chapter 15] for Bruhat order.

Denote by $\mathrm{Mat}_n(\mathbb{K})$ the $\mathbb{K}$-vector space of $n \times n$ matrices with entries in $\mathbb{K}$. The coordinate ring of $\mathrm{Mat}_n(\mathbb{K})$ is the polynomial ring $\mathbb{K}[\mathbf{x}]$, where $\mathbf{x}$ denotes the collection of indeterminates $x_{i,j}$ for $i, j \in [n]$. Abuse notation and denote by w both a permutation in $\mathfrak{S}_n$ and its corresponding $n \times n$ permutation matrix. We take the convention that we write permutation matrices "along columns". That is, if e_i denotes the i-th standard basis vector in $\mathbb{K}^n$, we write the permutation matrix w as follows.

$$w = \begin{bmatrix} | & | & & | \\ e_{w(1)} & e_{w(2)} & \cdots & e_{w(n)} \\ | & | & & | \end{bmatrix}$$

Definition 2.4.16 ([MS05, Definition 15.13])**.** The **(Rothe) diagram** of a permutation $w \in \mathfrak{S}_n$, denoted $D(w)$, is the set of all indices corresponding to entries in the permutation matrix for w neither below nor to the right of a 1. The number of such boxes is called the **length** of w, and is denoted $\ell(w)$.

Example 2.4.17. Figure 2.4 illustrates the diagram for $w = 32145 \in \mathfrak{S}_5$ (written in one-line notation). In the figure, each $\bullet$ represents a 1 in the permutation matrix w, and the boxes corresponding to elements of the diagram are shaded in grey. So $D(w) = \{(1,1), (1,2), (2,1), (4,4)\}$ and the length of w is $\ell(w) = 4$.

The length of a permutation is the dimension of the corresponding Schubert cell.

Definition 2.4.18. The **Schubert cell** of a permutation $w \in \mathfrak{S}_n$ is

$$X_w^\circ := BwB/B \subseteq \mathbf{GL}_n(\mathbb{K})/B.$$

As in the previous subsection, we can parametrize X_w° by viewing its elements as matrices Ω_w given by

$$[\Omega_w]_{i,j} = \begin{cases} 1 & \text{if } j = w^{-1}(i), \\ 0 & \text{if } j > w^{-1}(i) \text{ or } i > w(j), \\ z_{i,j} & \text{otherwise, that is, if } j < w^{-1}(i) \text{ and } i < w(j). \end{cases}$$

As before, we observe that the coordinate ring of X_w° is $\mathbb{K}[\mathbf{z}_w]$ where $\mathbf{z}_w = \{z_{i,j} \mid j < w^{-1}(i) \text{ and } i < w(j)\}$. The conditions $j > w^{-1}(i)$ and $i > w(j)$ correspond to the entries in Ω_w that appear, respectively, to the right and below of 1's. In particular, we have that Ω_w is exactly the matrix wM from Equation (2.4.5) upon setting the variables $x_{i,j}$ to zero for each $(i,j) \notin D(w)$ and then replacing each remaining $x_{i,j}$ by $z_{i,j}$. We will reserve the coordinates $x_{i,j}$ for entries of wM and the coordinates $z_{i,j}$ for the entries of Ω_w.

Bruhat theory guarantees that Schubert cells form a disjoint union of the flag variety Flags$(\mathbb{K}^n) \cong \mathbf{GL}_n(\mathbb{K})/B$. Moreover, each Schubert cell X_w° is isomorphic to the affine space $\mathbb{A}_{\mathbb{K}}^{\ell(w)}$. In Chapter 3 we will discuss the local computational geometry of the intersections of Hessenberg varieties with these Schubert cells. We conclude this chapter with a brief discussion on Bruhat order.

Fix a matrix $Z \in \mathrm{Mat}_n(\mathbb{K})$ and integers $k, \ell \in [n]$. Denote by $Z_{k\times\ell}$ the submatrix consisting of the first k rows and first ℓ columns of Z. Bruhat order is a partial ordering on permutations $\mathfrak{S}_n$.

Definition 2.4.19 ([MS05, Lemma 15.19])**.** Let v and w be permutations in $\mathfrak{S}_n$. We say that $v \leq w$ in **Bruhat order** if, for all $k, \ell \in [n]$, the permutation matrices satisfy $\mathrm{rank}(v_{k\times\ell}) \geq \mathrm{rank}(w_{k\times\ell})$.

Chapter 3

Regular Nilpotent Hessenberg Schubert Cells

The study of Hessenberg varieties via their patch ideals is relatively new. Insko and Yong introduced this approach in 2012 for the case of the Peterson variety: the Hessenberg variety associated to a regular nilpotent operator and Hessenberg function $h = (2, 3, \ldots, n-1, n, n)$ [IY12]. Recall from Remark 2.4.12 that to use this approach, one must first verify in each case that the patch ideals are radical to not only set-theoretically cut out the variety, but also scheme-theoretically agree. In their work, Insko and Yong used the patch ideals to show that Peterson varieties are complete intersections and give a combinatorial description of the singular locus of the Peterson variety. A few years later, Abe, DeDieu, Galetto, and Harada generalized this approach to all regular nilpotent Hessenberg varieties associated to indecomposable Hessenberg functions [ADGH18]. Applications of their work include showing that certain flat families of Hessenberg varieties have reduced fibers and give a formula for the degree of a Hessenberg variety with respect to a Plücker embedding. Other works involving Hessenberg patch ideals includes [IP19; AI22; Ata23; DH23].

In this chapter, we discuss the Gröbner geometry of the intersections of regular nilpotent Hessenberg varieties with Schubert cells. Section 3.1 discusses results from Da Silva and Harada [DH23] regarding the patch ideals corresponding when $w = (n, n-1, \ldots, 2, 1)$. The main result in this section is that the w_0-patch ideals are triangular complete intersections. Sections 3.2 and 3.3 present results from work of the author with Da Silva, Harada, and Rajchgot in [CDHR24]. In Section 3.2, we provide a homomorphism of coordinate rings from the w_0-chart to other charts in the setting of intersections of regular nilpotent Hessenberg varieties with Schubert cells. This allows us to express data such as Gröbner bases for these intersections in terms of the w_0-chart. Importantly, this homomorphism preserves much of the structure of the w_0-patch ideals, included that they are triangular complete intersections. Then in Section 3.3, we discuss applications of our main results to some of the material introduced in Chapter 2: triangular complete intersections and their connections to Hilbert series, geometric vertex decomposability, and Frobenius splitting. We conclude this chapter with Section 3.4 where we discuss some other known results in the regular nilpotent case, as well as what questions and conjectures remain.

Recall that Lemma 2.4.13 guarantees that any regular nilpotent Hessenberg variety is a product of indecomposable regular nilpotent Hessenberg varieties. As a result, throughout this chapter we restrict to indecomposable Hessenberg varieties.

3.1 The w_0-chart

In this section we review results from [DH23] necessary for the following section. We describe here the patch ideals corresponding to a regular nilpotent operator and the permutation $w_0 = (n, n-1, \dots, 2, 1) \in \mathfrak{S}_n$, written in one-line notation. We call this permutation the **longest word permutation** for its Bruhat length satisfies $\ell(w_0) \geq \ell(w)$ for any $w \in \mathfrak{S}_n$. As we remarked in Section 2.4.2, we may freely assume that our linear operator is in Jordan canonical form, so throughout this section we focus on Hess(N, h) where N is as follows:

$$\mathsf{N} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}.$$

Recall from Definition 2.4.11 that we defined $f^w_{k,\ell}(\mathsf{N}) = [(wM)^{-1}\mathsf{N}(wM)]_{k,\ell}$ so that the Hessenberg patch ideal of the coordinate chart $\mathcal{N}_w \cap \text{Hess}(\mathsf{N}, h)$ is given by $I^{\mathsf{N}}_{w,h} = \langle f^w_{k,\ell}(\mathsf{N}) \mid k > h(\ell) \rangle$. Throughout this section, we will write $f^w_{k,\ell}$ and $I_{w,h}$ for $f^w_{k,\ell}(\mathsf{N})$ and $I^{\mathsf{N}}_{w,h}$, respectively. That the patch ideals $I_{w,h}$ are radical and hence scheme-theoretically cut-out $\mathcal{N}_w \cap \text{Hess}(\mathsf{N}, h)$ for any permutation w and Hessenberg function h, per Remark 2.4.12, is guaranteed by [ADGH18, Proposition 3.7]. We next provide an illustrative example that the patch ideals $I_{w_0,h}$ are triangular complete intersections before performing these computations in generality. Part of this example appears in [CDHR24, Example 2.9].

Example 3.1.1. Consider the longest word permutation $w_0 \in \mathfrak{S}_4$. Recall from Equation (2.4.5) that

$$w_0 M = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & 1 \\ x_{2,1} & x_{2,2} & 1 & 0 \\ x_{3,1} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and compute

$$(w_0 M)^{-1}\mathsf{N}(w_0 M) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -x_{2,2} + x_{3,1} & 1 & 0 & 0 \\ -x_{1,2} + x_{1,3}(x_{2,2} - x_{3,1}) + x_{2,1} & -x_{1,3} + x_{2,2} & 1 & 0 \end{bmatrix}.$$

Then, for the indecomposable Hessenberg function $h = (2,3,4,4)$, we have that $I_{w_0,h} = \langle f_{4,1}^{w_0},\, f_{4,2}^{w_0},\, f_{3,1}^{w_0}\rangle$, where

$$\begin{aligned} f_{4,1}^{w_0} &= -x_{1,2} + x_{1,3}(x_{2,2} - x_{3,1}) + x_{2,1}, \\ f_{4,2}^{w_0} &= -x_{1,3} + x_{2,2}, \\ f_{3,1}^{w_0} &= -x_{2,2} + x_{3,1}. \end{aligned}$$

Define a lexicographic monomial order $<$ by

$$x_{1,1} > x_{1,2} > x_{1,3} > x_{2,1} > x_{2,2} > x_{3,1}.$$

The initial terms of the patch ideal generators are thus

$$\mathrm{in}_<(f_{4,1}^{w_0}) = -x_{1,2}, \quad \mathrm{in}_<(f_{4,2}^{w_0}) = -x_{1,3}, \quad \mathrm{in}_<(f_{3,1}^{w_0}) = -x_{2,2}.$$

Upon ordering the generators as in the above list, it is clear from Definition 2.3.5 that the patch ideal $I_{w_0,h}$ is a triangular complete intersection with respect to $<$. $\Diamond$

More generally, note that the $(w_0M)^{-1}$ must be a the form

$$(w_0M)^{-1} = \begin{bmatrix} & & & & 1 \\ & 0 & & 1 & y_{n-1,1} \\ & & \iddots & & \vdots \\ & 1 & \cdots & y_{2,2} & y_{2,1} \\ 1 & y_{1,n-1} & \cdots & y_{1,2} & y_{1,1} \end{bmatrix}, \tag{3.1.2}$$

where each $y_{i,j}$ is a polynomial in $\mathbb{K}[\mathbf{x}_{w_0}]$, that is, in the variables appearing in w_0M. The following lemma characterizes which variables $x_{a,b}$ appear in each $y_{i,j}$.

Lemma 3.1.3 ([DH23, Lemma 2.14]). *As above, let $y_{i,j} = [(w_0M)^{-1}]_{n+1-i,n+1-j}$. Then,*

(i) $y_{i,j} = 1$ *if and only if* $i + j = n + 1$,
(ii) $y_{i,j} = 0$ *if and only if* $i + j > n + 1$,
(iii) if $i + j < n + 1$, *then the polynomial* $y_{i,j}$ *has no constant term,*
(iv) $y_{i,j}$ *depends only on* $x_{a,b}$ *with* $a \geq i$ *and* $b \geq j$.

Sketch of proof. These facts also appear in the proof of [ADGH18, Lemma 3.12]. Items (i) and (ii) follow from the fact that inverse of the lower triangular matrix M must also be lower triangular, each has determinant 1, and $(w_0M)^{-1} = M^{-1}w_0$. Meanwhile, Item (iii) follows by induction and using the equation

$$y_{i,j} = \delta_{n+1-i,j} - \sum_{k=1}^{n-j} y_{i,n+1-k}x_{k,j},$$

where $\delta_{-,-}$ is the Kronecker delta. This equation is derived from the equality

$$[M^{-1}M]_{n+1-i,j} = \delta_{n+1-i,j}.$$

Items (i), (ii), and (iii) are then used in a straightforward inductive argument to show (iv). Moreover, this inductive argument shows the following, which is [ADGH18, Equation (3.7)],

$$y_{i,j} = -x_{i,j} - \sum_{k=i+1}^{n-j} y_{i,n+1-k} x_{k,j}. \tag{3.1.4}$$

In particular, we conclude that the summation depends only on the variables $x_{a',b'}$ with $a' \geq i$, $b' \geq j$, and $(a', b') \neq (i, j)$. □

The variables $x_{a,b}$ appearing in the polynomial $y_{i,j}$ can be visualized as follows [ADGH18]. The polynomial $y_{i,j}$ contains the variable $x_{i,j}$ and all of the variables appearing below and to the right of $x_{i,j}$ in the matrix w_0M.

With this in hand, we make the following computation for the generators $f^{w_0}_{k,\ell}$. Our eventual goal is to conclude that the patch ideal $I_{w_0,h}$ is a triangular complete intersection so we first show that there exists a monomial order with respect to which, the initial term of $f^{w_0}_{k,\ell}$ is an indeterminate. Let k and ℓ be integers in $[n]$ satisfying $k > \ell$. Then,

$$f^{w_0}_{k,\ell} = \sum_{j=1}^{n} \left[(w_0M)^{-1}\right]_{k,j} \left[\mathsf{N}(w_0M)\right]_{j,\ell} = \sum_{j=n-k+1}^{n} \left[(w_0M)^{-1}\right]_{k,j} \left[\mathsf{N}(w_0M)\right]_{j,\ell}$$

since in row k of $(w_0M)^{-1}$, the first $n-k$ entries are zero, per Equation (3.1.2). Similarly, in column ℓ of $\mathsf{N}(w_0M)$, the last ℓ entries are zero. So,

$$\begin{aligned} f^{w_0}_{k,\ell} &= \sum_{j=n-k+1}^{n-\ell} \left[(w_0M)^{-1}\right]_{k,j} \left[\mathsf{N}(w_0M)\right]_{j,\ell} = \sum_{j=n-k+1}^{n-\ell} y_{n+1-k,n+1-j}\, x_{j+1,\ell} \\ &= y_{n+1-k,\ell+1} x_{n+1-\ell,\ell} + \sum_{j=n-k+1}^{n-\ell} y_{n+1-k,n+1-j} x_{j+1,\ell}. \end{aligned}$$

Moreover, we have $x_{n+1-\ell,\ell} = 1$ by definition and, from Equation (3.1.4),

$$y_{n+1-k,\ell+1} = -x_{n+1-k,\ell+1} - \sum_{j=n-k+2}^{n-\ell-1} y_{n+1-k,n+1-j} x_{j,\ell+1}.$$

To combine the summation from the previous equation with the summation appearing in the equation for $f^{w_0}_{k,\ell}$, we write the latter as follows.

$$\sum_{j=n-k+1}^{n-\ell} y_{n+1-k,n+1-j} x_{j+1,\ell} = y_{n+1-k,k} x_{n-k+2,\ell} + \sum_{j=n-k+2}^{n-\ell-1} y_{n+1-k,n+1-j} x_{j+1,\ell}$$

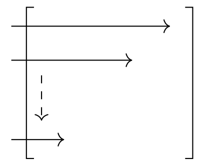

Figure 3.1: Da Silva and Harada's monomial order $<_n$.

Lemma 3.1.3(i) tells us that $y_{n+1-k,k} = 1$. Thus,

$$f^{w_0}_{k,\ell} = -x_{n-k+1,\ell+1} + x_{n-k+2,\ell} + \sum_{j=n-k+2}^{n-\ell-1} y_{n+1-k,n+1-j}\left(x_{j+1,\ell} - x_{j,\ell+1}\right). \tag{3.1.5}$$

With this computation we are able to conclude that the Hessenberg patch ideals are triangular complete intersections with respect to the following monomial order.

Definition 3.1.6 ([DH23, Definition 4.11])**.** Define a lexicographic order $<_n$ on $\mathbb{K}[\mathbf{x}_{w_0}]$ by $x_{i,j} > x_{i',j'}$ if either $i < i'$ or both $i = i'$ and $j < j'$.

Figure 3.1 gives an illustration of this order, where M is the matrix underlying the figure and the arrows order the variables from largest to smallest. Also, we order the polynomials $f^{w_0}_{k,\ell}$ in the following way.

$$f^{w_0}_{n,1},\ f^{w_0}_{n,2},\ \ldots,\ f^{w_0}_{n,n-1},\ f^{w_0}_{n-1,1},\ \ldots,\ f^{w_0}_{n-1,n-2},\ f^{w_0}_{n-2,1},\ \ldots,\ldots,\ f^{w_0}_{3,1}. \tag{3.1.7}$$

We have the following.

Lemma 3.1.8 ([DH23, Lemma 4.13])**.** *Fix indices k and ℓ in $[n]$ with $k > \ell + 1$. Then,*

(i) for $<_n$ as in Definition 3.1.6, the initial term is $\mathrm{in}_{<_n}(f^{w_0}_{k,\ell}) = -x_{n+1-k,\ell+1}$,
(ii) the variable $x_{n+1-k,\ell+1}$ appears exactly once in $f^{w_0}_{k,\ell}$ and all other variables $x_{i,j}$ appearing in $f^{w_0}_{k,\ell}$ satisfy either $i > n+1-k$ or both $i = n+1-k$ and $j > \ell+1$,
(iii) the variable $x_{n+1-k,\ell+1}$ does not appear in any $f^{w_0}_{k',\ell'}$ after $f^{w_0}_{k,\ell}$ in the sequence (3.1.7).

The assumption that $k > \ell + 1$ says that the above results hold for any generator $f^{w_0}_{k,\ell}$ of the patch ideal $I_{w_0,h}$ for any indecomposable Hessenberg function h.

Proof of Lemma 3.1.8. With Equation (3.1.5) and Definition 3.1.6 in hand, (i) follows from (ii), so we show the latter. It suffices to show that each $y_{n+1-k,n+1-j}$ appearing in the summation in (3.1.5) depends only on variables $x_{a,b}$ with $b > \ell + 1$.

From Lemma 3.1.3(iv) it follows that any indeterminate $x_{a,b}$ appearing in the polynomial $y_{n+1-k,n+1-j}$ satisfies, in part, $b \geq n+1-j$. The bounds on the summation in (3.1.5) say that $j \leq n - \ell - 1$, so

$$b \geq n+1-j \geq n+1+(\ell+1-n) > \ell+1,$$

which completes the proof of (i) and (ii). Item (iii) follows from Equation (3.1.5). ∎

Lemma 3.1.8 says exactly that, with respect to the monomial order $<_n$ from Definition 3.1.6 and the ordering Equation (3.1.7), the regular nilpotent Hessenberg w_0-patch ideals are triangular complete intersections.

Theorem 3.1.9 ([DH23, Remark 4.14])**.** *For any indecomposable Hessenberg function h, the w_0-patch ideal $I_{w_0,h}$ is a triangular complete intersection with respect to $<_n$.*

In Chapter 2 we collected several corollaries for the setting of triangular complete intersections. That is, Lemma 2.3.6 immediately implies the following.

Corollary 3.1.10. *Fix an indecomposable Hessenberg function h. Then the w_0-patch ideal is*

(i) a complete intersection,
(ii) geometrically vertex decomposable,
(iii) compatibly split with respect to a Frobenius splitting, when $\mathbb{K}$ is an algebraically closed field of finite characteristic.

Moreover, the generators $\{f^{w_0}_{k,\ell}\}_{k>h(\ell)}$ are a Gröbner basis for $I_{w_0,h}$ with respect to $<_n$.

We end this section with an aside and note that many of these regular nilpotent Hessenberg w_0-patch ideals are toric ideals. Since these ideals are triangular complete intersections, they cut out affine space and hence that they are tori is already known. However, in some cases, there is a natural binomial generating set for these ideals. That is, in these cases, we have a nonstandard torus action on $\mathrm{Hess}(\mathsf{N}, h)\cap\mathcal{N}_{w_0}$ that does not descend from a torus action on the full flag variety.

Example 3.1.11. We saw in Example 3.1.1 that the w_0-patch ideal corresponding to $h = (2, 3, 4, 4)$ has generators

$$\begin{aligned}
f^{w_0}_{4,1} &= -x_{1,2} + x_{1,3}(x_{2,2} - x_{3,1}) + x_{2,1},\\
f^{w_0}_{4,2} &= -x_{1,3} + x_{2,2},\\
f^{w_0}_{3,1} &= -x_{2,2} + x_{3,1}.
\end{aligned}$$

Notice that the parenthetical binomial of $f^{w_0}_{4,1}$ is exactly $-f^{w_0}_{3,1}$, so we may write the patch ideal equivalently as

$$I_{w_0,h} = \langle -x_{1,2} + x_{2,1},\ -x_{1,3} + x_{2,2},\ -x_{2,2} + x_{3,1}\rangle.$$

Lemma 3.1.12. *If $k = \ell + 2$, then $f^{w_0}_{k,\ell}$ is a binomial.*

Proof. Equation (3.1.5) reduces to $f^{w_0}_{\ell+2,\ell} = -x_{n-\ell-1,\ell+1} + x_{n-\ell,\ell}$. ■

This lemma serves as a base case for our argument in the proof of the following theorem.

Theorem 3.1.13. *Let h be a Hessenberg function where $h(i)$ is either $i + 1$ or n for all i. Then $I_{w_0,h}$ has a generating set consisting of differences of variables. In particular, it is a toric ideal.*

Proof. The patch ideal $I_{w_0,h}$ is prime because it is a triangular complete intersection, so we need only show the claim that there exists a generating set consisting of differences of variables.

Rewrite Equation (3.1.5) as follows:

$$f_{k,\ell}^{w_0} = B_{k,\ell} + \sum_{j=n-k+2}^{n-\ell-1} y_{n+1-k,n+1-j}\left(x_{j+1,\ell} - x_{j,\ell+1}\right),$$

so $B_{k,\ell}$ is a binomial of indeterminates. That is, $B_{k,\ell} = -x_{n-k+1,\ell+1} + x_{n-k+2,\ell}$. We will show that $I_{w_0,h} = \langle B_{k,\ell} \mid k > h(\ell)\rangle$. In particular, we will show that every binomial $x_{j+1,\ell} - x_{j,\ell+1}$ appearing in the summation for $f_{k,\ell}^{w_0}$ is exactly $B_{k',\ell}$ for some $k' < k$. Fix some j satisfying $n-k+2 \le j \le n-\ell-1$. Then,

$$B_{n-j+1,\ell} = -x_{n-(n-j+1)+1,\ell} + x_{n-(n-j+1)+2,\ell} = -x_{j,\ell+1} + x_{j+1,\ell}.$$

Moreover, the bounds of the summation guarantee that $n-k+1 < j$, or said differently, $n-j+1 < k$, as desired. We have shown that

$$f_{k,\ell}^{w_0} = -x_{n-k+1,\ell+1} + x_{n-k+2,\ell} + \sum_{j=n-k+2}^{n-\ell-1} y_{n+1-k,n+1-j} \cdot B_{n-j+1,\ell}.$$

The assumption on the Hessenberg function says that each previous $B_{n-j+1,\ell}$ is necessarily in the patch ideal. We thus conclude that $I_{w_0,h}$ is generated by binomials. ∎

3.2 Translating to other charts

We begin this section with the following motivational example that demonstrates that although the w_0-patch ideals have this nice triangular complete intersection structure, the same is certainly not true for other patches. What is true, however, is that the local defining ideals of intersections of regular nilpotent Hessenberg varieties with Schubert cells also have the triangular complete intersection structure. To see this, we construct a homomorphism of coordinate rings that preserves triangular complete intersections as the initial terms do not vanish.

One motivating reason for studying these intersections, which we call **Hessenberg Schubert cells**, is that showing each is isomorphic to affine space is sufficient to conclude that the full Hessenberg variety is paved by affines. We will discuss this more in Section 3.3.

Note first that the intersection of the Hessenberg variety at the w_0-chart with the Schubert cell containing w_0 is exactly the w_0-chart of the Hessenberg variety. That is, $(\mathrm{Hess}(\mathsf{N},h) \cap \mathcal{N}_{w_0}) \cap X_{w_0}^{\circ} = \mathrm{Hess}(\mathsf{N},h) \cap \mathcal{N}_{w_0}$. This is clear from Definition 2.4.18.

3.2.1 The key observation

We begin with a motivational example. The remainder of this section will be spent codifying the following example.

Example 3.2.1. We first show that not every regular nilpotent Hessenberg patch ideal is a triangular complete intersection. We make use of *Macaulay2* and a pair of its packages [CV; GS; Swi]. Pick the indecomposable Hessenberg function $h = (2, 3, 4, 4)$ and fix $w = 3214 \in \mathfrak{S}_4$. Then,

$$wM = \begin{bmatrix} x_{11} & x_{12} & 1 & 0 \\ x_{21} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 1 \end{bmatrix},$$

and $I_{w,h} = \langle f^w_{3,1},\ f^w_{4,1},\ f^w_{4,2} \rangle$ where the polynomials $f^w_{k,\ell}$ are as in Definition 2.4.11. Explicitly, we have

$$\begin{aligned}
f^w_{3,1} &= x_{12}x_{21}x_{41} - x_{11}x_{41} - x_{12} + x_{21}, \\
f^w_{4,1} &= -x_{12}x_{21}x_{41}x_{43} + x_{21}x_{41}x_{42} + x_{11}x_{41}x_{43} - x_{41}^2 + x_{12}x_{43} - x_{21}x_{43} - x_{42}, \\
f^w_{4,2} &= -x_{12}x_{21}x_{42}x_{43} + x_{21}x_{42}^2 + x_{11}x_{42}x_{43} - x_{41}x_{42} - x_{43}.
\end{aligned}$$

Handing off now to *Macaulay2* [GS], we use the `StatePolytope` package [Swi] to compute an exhaustive list of the initial ideals for $I_{w,h}$. In the code, we denote by `I` the patch ideal $I_{w,h}$.

```
i1: inits = initialIdeals I;
i2: any(inits, i -> isGeneratedByIndeterminates ideal i)
o2: false
```

The `GeometricDecomposability` package [CV] then tells us that no initial ideal is generated by indeterminates, so we conclude that $I_{w,h}$ is not a triangular complete intersection. Notice that although $w = 3214$ is a 321-embedding permutation, the patch ideal is quite different to $I_{w_0,h}$.

To contrast, consider the matrix Ω_w representing an element of the Schubert cell containing w, as given in Definition 2.4.18,

$$\Omega_w = \begin{bmatrix} z_{11} & z_{12} & 1 & 0 \\ z_{21} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Following the same procedure for the generators $f^w_{k,\ell}$ of the local defining equations of the Hessenberg patch ideal gives rise to local defining equations for the intersections of Hessenberg varieties with Schubert cells, upon replacing wM by Ω_w. We make this rigorous after this example. For now, we note that

$$\Omega_w^{-1}\mathsf{N}\Omega_w = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -z_{21} \\ -z_{12}+z_{21} & 1 & 0 & z_{12}z_{21}-z_{11} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the local defining ideal of $\mathrm{Hess}(\mathsf{N}, h) \cap X_w^\circ$ is $J_{w,h} = \langle z_{12} - z_{12} \rangle \subseteq \mathbb{K}[\mathbf{z}_w]$. It is clear that this ideal is both radical and a triangular complete intersection. Moreover, we note that Ω_w can be obtained from $w_0 M$ by permuting columns, setting some variables to zero, and relabelling the remaining variables. Explicitly, we have that

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} & 1 \\ x_{21} & x_{22} & 1 & 0 \\ x_{31} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x_{12} & x_{13} & 1 & x_{11} \\ x_{22} & 1 & 0 & x_{21} \\ 1 & 0 & 0 & x_{31} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \Omega_w$$

upon setting x_{11}, x_{21}, and x_{31} to zero and relabelling $x_{12} \mapsto z_{11}$, $x_{13} \mapsto z_{12}$, and $x_{22} \mapsto z_{21}$. Moreover, if we denote by ψ_w this relabelling and setting variables to zero, we can express the (k, ℓ)-th entry of $\Omega_w^{-1}\mathsf{N}\Omega_w$ in terms of the $f_{k',\ell'}^{w_0}$ we computed in Example 3.1.1. Denote by $g_{k,\ell}^w$ the (k, ℓ)-th entry of $\Omega_w^{-1}\mathsf{N}\Omega_w$. Then notice that $g_{3,1}^w = \psi_w(f_{4,2}^{w_0})$, $g_{2,4}^w = \psi_w(f_{3,1}^{w_0})$, and $g_{3,4}^w = \psi_w(f_{3,4}^{w_0})$. ◊

The next portion of this section is devoted to making rigorous the phenomena we observed in Example 3.2.1. A **regular nilpotent Hessenberg Schubert cell**, associated to a Hessenberg function h and permutation $w \in \mathfrak{S}_n$, is the intersection of a Hessenberg variety $\mathrm{Hess}(\mathsf{N}, h)$ with the Schubert cell X_w°. For the rest of this section, we omit "regular nilpotent" and write simply Hessenberg Schubert cell.

We conclude the current subsection with two results on exactly when a Hessenberg Schubert cell is nonempty. Recall that we described in Section 2.4.3 torus actions on the flag variety that descend to actions on Hessenberg varieties. We denote by $\mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$ the set of $w \in \mathfrak{S}_n$ that are fixed points of the $\mathbf{S}$-action described in Section 2.4.3.

Lemma 3.2.2 ([CDHR24, Lemma 3.18])**.** *The Hessenberg Schubert cell associated to a permutation w and Hessenberg function h is nonempty if and only if w is fixed by the* $\mathbf{S}$-*action on* $\mathrm{Hess}(\mathsf{N}, h)$. *That is, if and only if $w \in$* $\mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$.

We provide a sketch of a proof and refer the reader to [CDHR24] for the full proof.

Sketch of proof. We follow the proof of [CDHR24, Lemma 3.18]. If $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$ then $w \in \mathrm{Hess}(\mathsf{N}, h)$ and, by definition, X_w°, so it remains to show the reverse direction.

If the intersection is nonempty, pick a point $gB \in \mathrm{Hess}(\mathsf{N}, h) \cap X_w^\circ$. In Section 2.4.3, we noted that $\mathbf{S}$ preserves regular nilpotent Hessenberg varieties ([HT17, Lemma

5.1(3)]), and since Schubert cells are preserved by the maximal torus $\mathbf{T}$, they are also preserved by $\mathbf{S}$. Hence the Hessenberg Schubert cell too is preserved.

Fix an element $\underline{s} = \mathrm{diag}(t, t^2, \ldots, t^n)$ in $\mathbf{S}$. Following Equation (2.4.14), an entry in $\underline{s} \cdot gB$ is either $t^{i-w(j)}z_{i,j}$, 0, or 1. Definition 2.4.18 tells us that we have $i - w(j) < 0$, so the limit point as $t \to \infty$ is exactly the permutation flag w. Since regular nilpotent Hessenberg varieties are closed, they contain their limit points. □

For our computational purposes, we have the following explicit combinatorial description of the fixed point set given by Abe, Harada, Horiguchi, and Masuda. We make the convention that $w(0) = 0$ for any permutation $w \in \mathfrak{S}_n$.

Lemma 3.2.3 ([AHHM17, Lemma 2.3]). *Let $h : [n] \to [n]$ be any Hessenberg function. Then,*

$$\mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}} = \{w \in \mathfrak{S}_n \mid w^{-1}(w(j) - 1) \leq h(j) \text{ for all } j \in [n]\}.$$

3.2.2 A homomorphism of coordinate rings

We now turn our focus to the local defining equations of Hessenberg Schubert cells. These are defined in the same way as for Hessenberg varieties, in Definition 2.4.11.

Definition 3.2.4 ([CDHR24, Definition 3.12]). Fix a permutation $w \in \mathfrak{S}_n$ and indices $k, \ell \in [n]$. Denote by $g^w_{k,\ell}$ the polynomial in $\mathbb{K}[\mathbf{z}_w]$ at the $(k, \ell)-$th entry of $\Omega_w^{-1}\mathsf{N}\Omega_w$.

Then, the **(regular nilpotent) Hessenberg Schubert cell patch ideal** corresponding to w and a Hessenberg function h is the ideal

$$J_{w,h} := \langle g^w_{k,\ell} \mid k > h(\ell)\rangle \subseteq \mathbb{K}[\mathbf{z}_w].$$

It is clear that these equations set-theoretically cut out the Hessenberg Schubert cells, and we will later show that these ideals are radical. To do so, we will formalize the map ψ_w from Example 3.2.1 as a homomorphism of coordinate rings and argue that it preserves initial terms.

Following the example, for any fixed $w \in \mathfrak{S}_n$, define a permutation $v_w := w_0 w$. This guarantees that the 1's in the matrices $(w_0M)v_w$ and Ω_w appear at the same entries. Moreover, we have that $[(w_0M)v_w]_{i,j} = [w_0M]_{i,v_w(j)}$. For $(w_0M)v_w$ to represent an element of the Schubert cell Ω_w we need only impose the condition from Definition 2.4.18 that $[(w_0M)v_w]_{i,j} = 0$ whenever $w(j) < i$ or $j > w^{-1}(i)$. We next translate this vanishing condition to be in terms of the matrix w_0M.

Denote by $x_{a,b}$ the (a, b)-th entry of w_0M, where we require $a + b \leq n$, so that $x_{a,b}$ is an indeterminate in $\mathbf{x}_{w_0}$. From the previous paragraph, for $(w_0M)v_w$ to lie in X°_w we must have that $x_{a,b} = 0$ when either $w(v_w^{-1}(b)) < a$ or $v_w^{-1}(b) > w^{-1}(a)$. The former inequality, upon replacing $v_w^{-1} = w^{-1}w_0$ and $w_0(b) = n + 1 - b$, is equivalent to $n + 1 - b < a$. Since we assumed that $a + b \leq n$, this never occurs, so we need only treat the second condition. We denote by Z_w the collection of indices corresponding indeterminates $x_{a,b} \in \mathbb{K}[\mathbf{x}_{w_0}]$ that must be set to zero. Explicitly, we define

$$Z_w := \{x_{i,j} \in \mathbf{x}_{w_0} \mid i + j \leq n \text{ and } v_w^{-1}(j) > w^{-1}(i)\} \subseteq \mathbb{K}[\mathbf{x}_{w_0}]. \tag{3.2.5}$$

With this in hand, we define the ring map ψ_w which, following the previous discussion, relabels the variables $x_{i,j} \in \mathbb{K}[\mathbf{x}_{w_0}]$ for which $x_{i,j} \notin Z_w$.

Definition 3.2.6 ([CDHR24, Definition 3.8])**.** Construct a homomorphism of rings $\psi_w : \mathbb{K}[\mathbf{x}_{w_0}] \to \mathbb{K}[\mathbf{z}_w]$ by

$$\psi_w(x_{i,j}) = \begin{cases} 0 & \text{if } x_{i,j} \in Z_w, \\ z_{i,v_w^{-1}(j)} & \text{if } x_{i,j} \notin Z_w. \end{cases}$$

Following [CDHR24, Remark 3.9], we note that ψ_w corresponds to embedding X_w° into $\mathcal{N}_{w_0}$ by right-multiplying an element of the Schubert cell by v_w^{-1}.

Fix a permutation $w \in \mathfrak{S}_n$. Applying ψ_w entry-by-entry to the matrix $(w_0 M)v_w$ results in a matrix with entries in $\mathbb{K}[\mathbf{z}_w]$. We denote by $\psi_w((w_0 M)v_w)$ this matrix. By construction, we have shown the following.

Remark 3.2.7 ([CDHR24, Remark 3.10])**.** For any permutation $w \in \mathfrak{S}_n$, we have that $\Omega_w = \psi_w\left((w_0 M)v_w\right)$.

In general, if A is a matrix with entries in $\mathbb{K}[\mathbf{x}_{w_0}]$, then $\psi_w(A)$ denotes the matrix with entries in $\mathbb{K}[\mathbf{z}_w]$ obtained by applying ψ_w entry-by-entry. Moreover, we have that ψ_w respects matrix inverses.

Lemma 3.2.8 ([CDHR24, Lemma 3.11])**.** *Fix a permutation $w \in \mathfrak{S}_n$. Then, we have that $\big(\psi_w((w_0 M)v_w)\big)^{-1} = \psi_w\big(((w_0 M)v_w)^{-1}\big)$.*

Proof. Since $\det((w_0 M)v_w) = \operatorname{sgn}(w) = \pm 1$, the inverse of $(w_0 M)v_w$ is a polynomial in its entries and not a rational function. The result is then immediate upon applying ψ_w entry-by-entry. ■

It is now a straightforward computation to express the Hessenberg Schubert cell generators $g_{k,\ell}^w$ in terms of the Hessenberg w_0-patch generators $f_{k',\ell'}^{w_0}$. Recall that we defined the polynomials $g_{k,\ell}^w$ in Definition 3.2.4 and the polynomials $f_{k',\ell'}^{w_0}$ in Definition 2.4.11.

Lemma 3.2.9 ([CDHR24, Lemma 3.20])**.** *For any $w \in \mathfrak{S}_n$ and indices $k, \ell \in [n]$, we have that $g_{k,\ell}^w = \psi_w\big(f_{v_w(k),v_w(\ell)}^{w_0}\big)$.*

Proof. By definition and Lemma 3.2.8 we can compute that

$$\begin{aligned} g_{k,\ell}^w &= \left[\Omega_w^{-1} \mathsf{N} \Omega_w\right]_{k,\ell} = \left[\left(\psi_w((w_0 M)v_w)\right)^{-1} \mathsf{N} \left(\psi_w((w_0 M)v_w)\right)\right]_{k,\ell} \\ &= \left[\psi_w\left(((w_0 M)v_w)^{-1} \mathsf{N} \, (w_0 M)v_w\right)\right]_{k,\ell}. \end{aligned}$$

Then, since ψ_w is a homomorphism of rings applied entry-by-entry, we have that $\mathsf{N} = \psi_w(\mathsf{N})$, and moreover,

$$\begin{aligned} g^w_{k,\ell} &= \psi_w\left(\left[((w_0M)v_w)^{-1}\,\mathsf{N}\,(w_0M)v_w\right]_{k,\ell}\right) \\ &= \psi_w\left(\left[v_w^{-1}(w_0M)\,\mathsf{N}\,(w_0M)v_w\right]_{k,\ell}\right). \end{aligned}$$

The result now follows from the facts that left-multiplication by v_w^{-1} permutes rows and right-multiplication by v_w permutes columns. ∎

The previous lemma holds for all indices k and ℓ. In the case that $\mathrm{Hess}(\mathsf{N}, h) \cap X^\circ_w$ is empty, then the generators $g^w_{k,\ell}$ will be zero for all $k > h(\ell)$, and so moving forward, to avoid this trivial case, we will only consider nonempty intersections. By the result of Lemma 3.2.2, this is equivalent to requiring that $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$.

3.2.3 Preserving triangular complete intersections

In the last subsection, we described the Hessenberg Schubert cell generator $g^w_{k,\ell}$ in terms of the generators $f^{w_0}_{k',\ell'}$ for the w_0-patch ideal, via a homomorphism of rings $\mathbb{K}[\mathbf{x}_{w_0}] \to \mathbb{K}[\mathbf{z}_w]$. Our goal for this subsection is to conclude that, like in the case of the regular nilpotent Hessenberg w_0-patch ideals, the (nonempty) Hessenberg Schubert cells are triangular complete intersections. In doing so, we will tie up one remaining loose end: showing that the Hessenberg Schubert cell ideals are radical, and hence scheme-theoretically cut-out the variety, analogously to Remark 2.4.12.

We begin by defining a monomial order on $\mathbb{K}[\mathbf{z}_w]$ in terms of Da Silva and Harada's monomial order $<_n$ from Definition 3.1.6.

Definition 3.2.10 ([CDHR24, Definition 4.1])**.** Fix a permutation $w \in \mathfrak{S}_n$. We define a lexicographic order $<^w_n$ on $\mathbb{K}[\mathbf{z}_w]$ by $z_{i,j} >^w_n z_{i',j'}$ if and only if $\psi_w^{-1}(z_{i,j}) >_n \psi_w^{-1}(z_{i',j'})$.

Said differently, we have $z_{i,j} >^w_n z_{i',j'}$ if and only if $i < i'$ or both $i = i'$ and $v_w(j) < v_w(j')$. The equivalence of these definitions is guaranteed by definition of ψ_w. So, it is clear that this order is well-defined. Moreover, notice that when $w = w_0$, we recover Da Silva and Harada's monomial order $<_n$, since in this case, we have that v_w is the identity.

The following lemma is the key to showing the titular result of this subsection: that we may use ψ_w to argue that Hessenberg Schubert cell ideals are triangular complete intersections. Recall from Lemma 3.2.2 that requiring that $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$ is equivalent to requiring that $\mathrm{Hess}(\mathsf{N}, h) \cap X^\circ_w$ is nonempty. Moreover, and as discussed in Lemma 3.1.8, a polynomial $f^{w_0}_{a,b}$ is nonconstant whenever $a > b + 1$. So when discussing the polynomial $g^w_{k,\ell} = \psi_w(f^{w_0}_{v_w(k),v_w(\ell)})$, we may freely assume that $v_w(k) > v_w(\ell) + 1$.

Lemma 3.2.11 ([CDHR24, Lemmas 4.3 and 4.4])**.** *Let h be an indecomposable Hessenberg function and $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$. Suppose that $k, \ell \in [n]$ satisfy $k \geq h(\ell)$ and $v_w(k) > v_w(\ell) + 1$. Then,*

(i) *the initial term of* $f^{w_0}_{v_w(k),v_w(\ell)}$ *with respect to* $<_n$ *does not get mapped to zero under* ψ_w*,*
(ii) *upon restricting its domain to* $\mathbb{K}[\mathbf{x}_{w_0} \setminus Z_w]$*, the map* ψ_w *is injective,*
(iii) ψ_w *respects our term orders, that is,* $\mathrm{in}_{<^w_n}(\psi_w(f^{w_0}_{v_w(k),v_w(\ell)})) = \psi_w(\mathrm{in}_{<_n}(f^{w_0}_{v_w(k),v_w(\ell)}))$.

Proof. Lemma 3.1.8 tells us that $\mathrm{in}_{<_n}(f^{w_0}_{v_w(k),v_w(\ell)}) = -x_{n+1-v_w(k),v_w(\ell)+1}$. So by definition of Z_w, as given in Equation (3.2.5), we are required to show that

- $n+1-v_w(k)+v_w(\ell)+1 \le n$, and,
- $v_w^{-1}(v_w(\ell)+1) \le w^{-1}(n+1-v_w(k))$.

The first condition guarantees that $\mathrm{in}_{<_n}(f^{w_0}_{v_w(k),v_w(\ell)})$ is nonconstant. From the discussion preceding the statement of the lemma, we assumed that $v_w(\ell)+1 < v_w(k)$, which is exactly the first condition, after simplifying and rearranging terms. So we verify the second condition to ensure that the initial monomial is not in Z_w.

Since $n+1-v_w(k) = w_0(v_w(k))$ and $v_w = w_0 w$, we have that the right-hand side of the second inequality simplifies to $w^{-1}(w(k)) = k$. For the left-hand side, we use the definition $v_w = w_0 w$ and that $w_0(a) = n+1-a$ to compute

$$v_w^{-1}(v_w(\ell)+1) = w^{-1}w_0(w_0 w(\ell)+1) = w^{-1}(w(\ell)-1).$$

So we have rewritten the second inequality as $w^{-1}(w(\ell)-1) \le k$. Since we assumed that $w \in \mathrm{Hess}(\mathsf{N},h)^{\mathbf{S}}$, Lemma 3.2.3 tells us that $w^{-1}(w(\ell)-1) \le h(\ell)$. Our assumption that $h(\ell) \le k$ then completes the proof of (i).

For (ii) notice that if $x_{i,j}$ and $x_{i',j'}$ are distinct and not in Z_w, then ψ_w permutes indices, so $\psi_w(x_{i,j})$ and $\psi_w(x_{i',j'})$ are distinct too. Item (iii) is clear by Definition 3.2.10 of $<^w_n$. ■

We can now conclude that the Hessenberg Schubert cell ideals, as given in Definition 3.2.4, are triangular complete intersections. Da Silva and Harada showed that the generators $f^{w_0}_{k,\ell}$ form a minimal generating set for the w_0-patch ideal [DH23, Remark 4.14]. As we discussed in the paragraph preceding Lemma 3.2.11, and as we computed in Example 3.2.1, some of the regular nilpotent Hessenberg Schubert cell ideal generators $g^w_{k,\ell}$ may be zero. However, it follows that those that are nonzero form a minimal generating set for $J_{w,h}$. Denote by $\Lambda_{w,h}$ the number of nonzero generators. Explicitly, we have that

$$\Lambda_{w,h} \coloneqq \#\big\{(k,\ell) \in [n]^2 \;\big|\; k > h(\ell) \text{ and } v_w(k) > v_w(\ell)+1\big\}. \tag{3.2.12}$$

Theorem 3.2.13 ([CDHR24, Theorem 5.3])**.** *Let* h *an indecomposable Hessenberg function and* $w \in \mathrm{Hess}(\mathsf{N},h)^{\mathbf{S}}$*. The regular nilpotent Hessenberg Schubert cell ideal at* w *and* h *is a triangular complete intersection of height* $\Lambda_{w,h}$ *with respect to* $<^w_n$*.*

Proof. Consider the list of generators $\{g^w_{k_i,\ell_i}\}_{i=1}^r$ for the ideal $J_{w,h}$. Lemma 3.2.11 guarantees that the list $\{\mathrm{in}_{<^w_n}(g^w_{i_i,\ell_i})\}_{i=1}^r$ is a list of unique indeterminates. To satisfy the remainder of Definition 2.3.5 and conclude that $J_{w,h}$ is a triangular complete intersection, we are required to order the generators so that the initial term of the j-th generator does not appear later in the ordered list.

From Lemma 3.2.9, the set $\{g^w_{k_1,\ell_1}, \ldots, g^w_{k_r,\ell_r}\}$ is equal to the set

$$\left\{\psi_w\left(f^{w_0}_{v_w(k_1),v_w(\ell_1)}\right),\ \psi_w\left(f^{w_0}_{v_w(k_2),v_w(\ell_2)}\right),\ \ldots,\ \psi_w\left(f^{w_0}_{v_w(k_r),v_w(\ell_r)}\right)\right\}.$$

Since the regular nilpotent Hessenberg w_0-patch ideals are triangular complete intersections, there exists an ordering of the $f^{w_0}_{a,b}$ so that the initial term of j-th polynomial does not appear in any polynomial occurring to the right of $f^{w_0}_{a,b}$. After relabelling our indices, write this list as follows:

$$f^{w_0}_{v_w(k_1),v_w(\ell_1)},\ f^{w_0}_{v_w(k_2),v_w(\ell_2)},\ \ldots,\ f^{w_0}_{v_w(k_r),v_w(\ell_r)}.$$

By definition of $<^w_n$, and by Lemma 3.2.11, it follows that the ordered list

$$\psi_w\left(f^{w_0}_{v_w(k_1),v_w(\ell_1)}\right),\ \psi_w\left(f^{w_0}_{v_w(k_2),v_w(\ell_2)}\right),\ \ldots,\ \psi_w\left(f^{w_0}_{v_w(k_r),v_w(\ell_r)}\right)$$

also satisfies the triangular complete intersection condition. ■

The initial ideal of a Hessenberg Schubert cell ideal $J_{w,h}$ is an ideal of indeterminates, which is necessarily radical. This is sufficient to conclude that $J_{w,h}$ is radical as well [CLO15, Chapter 4.2, Exercise 16]. We include a proof for completeness.

Corollary 3.2.14 ([CDHR24, Corollary 4.7])**.** *Let h be an indecomposable Hessenberg function and $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathsf{S}}$. Then the Hessenberg Schubert cell ideal $J_{w,h}$ is radical.*

Proof. Since $\mathrm{in}_{<^w_n}(J_{w,h})$ is an ideal of indeterminates, it is radical. Consider a polynomial $q \in \sqrt{J_{w,h}}$. There exists a positive integer t such that $q^t \in J_{w,h}$, so in particular, $(\mathrm{in}_{<^w_n}(q))^t = \mathrm{in}_{<^w_n}(q^t) \in \mathrm{in}_{<^w_n}(J_{w,h})$. Since the initial ideal is radical, we have $\mathrm{in}_{<^w_n}(q)$ is in the initial ideal too. As a result, there exists some $\widetilde{q} \in J_{w,h}$ with $\mathrm{in}_{<^w_n}\widetilde{q} = \mathrm{in}_{<^w_n} q$. Because $q - \widetilde{q}$ remains in $\sqrt{J_{w,h}}$, and $q - \widetilde{q} < q$, we are done by induction. ■

We conclude this section with the applications of triangular complete intersections, as discussed in Lemma 2.3.6. The following appeared as [CDHR24, Theorem 4.6, Proposition 5.14, and Theorem 5.15].

Corollary 3.2.15 ([CDHR24])**.** *Let h be indecomposable and $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathsf{S}}$. Then,*

(i) the generators $\{g^w_{k,\ell}\}_{k>h(\ell)}$ form a Gröbner basis for the regular nilpotent Hessenberg Schubert cell ideal $J_{w,h}$ with respect to $<^w_n$,
(ii) the ideal $J_{w,h}$ is geometrically vertex decomposable,
(iii) when $\mathbb{K}$ is an algebraically closed field of finite characteristic, the Frobenius splitting constructed in Lemma 2.3.4 compatibly splits $J_{w,h}$.

3.3 Applications

In the previous section we showed that regular nilpotent Hessenberg Schubert cell ideals are the local defining equations of the intersections of regular nilpotent Hessenberg varieties and Schubert cells. Moreover, we showed that these ideals are triangular complete intersections and discussed its immediate corollaries. In this section, we discuss two additional results.

In the first subsection, we recover Tymoczko's result—in type A—that regular nilpotent Hessenberg varieties are paved by affines [Tym07]. To do so, we note that each Hessenberg Schubert cell ideal defines an affine variety, and compatibly "glue" them together to pave the full regular nilpotent Hessenberg variety.

Then, in the second subsection, we find an explicit formula for the Hilbert series of regular nilpotent Hessenberg Schubert cell ideals. This Hilbert series is computed with respect to a nonstandard grading, which arises from the S-action discussed Section 2.4.3. We will see that, with respect to this grading, the generators $g^w_{k,\ell}$ are homogeneous.

3.3.1 Paving by affines

We begin this subsection by briefly introduction the theory of paving by affines and touching on some applications. Our main references for this section will be [Tym07] and [CDHR24, Section 5.2]. Tymoczko showed that regular nilpotent Hessenberg varieties are paved by affines in any Lie type [Tym07]. This generalized earlier work of de Concini, Lusztig, and Procesi [dLP88] and Kostant [Kos96]. Our computational approach provides a new proof of this affine paving, in Lie type A.

Let X be an algebraic variety. A **paving** of X is an ordered partition of disjoint varieties $X_0, X_1, X_2, \ldots$, so that each finite union $\bigcup_{i=0}^{j} X_i$ is closed in X. Each X_i is a called a cell of the paving. We say that a paving is a **paving by affines** when each X_i is homeomorphic to affine space. Although it is beyond the scope of this thesis, one nice application is that it is trivial to compute the cohomology groups of a variety that is paved by affines (see, for instance, [Tym07, Lemma 2.3]).

We first conclude that regular nilpotent Hessenberg Schubert cells are affine spaces. Recall from Equation (3.2.12) that $\Lambda_{w,h}$ denotes the number of minimal generators for the corresponding Hessenberg Schubert cell ideal $J_{w,h}$. Also, recall that the length of a permutation $\ell(w)$ is the size of its diagram, per Definition 2.4.16. From Definition 2.4.18 it is clear that the dimension of the Schubert cell X°_w is exactly $\ell(w)$.

Proposition 3.3.1 ([CDHR24, Proposition 5.5])**.** *Let h be an indecomposable Hessenberg function and $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$. Then $\mathrm{Hess}(\mathsf{N}, h) \cap X^\circ_w \cong \mathbb{A}^{\ell(w)-\Lambda_{w,h}}$.*

Proof. Theorem 3.2.13 says that the local defining equations of $\mathrm{Hess}(\mathsf{N}, h) \cap X^\circ_w$ are triangular complete intersections of height $\Lambda_{w,h}$, so apply Lemma 2.3.6(iv). ■

Recall from Lemma 3.2.2 that the regular nilpotent patch $\mathrm{Hess}(\mathsf{N}, h) \cap \mathcal{N}_w$ is nonempty if and only if $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$. Thus to pave the regular nilpotent Hessenberg varieties, we may freely drop any patch with $w \notin \text{Hess}(\mathsf{N}, h)^{\mathbf{S}}$ from the paving. We now have the desired result.

Theorem 3.3.2 ([Tym07]; [CDHR24, Theorem 5.6]). *Indecomposable regular nilpotent Hessenberg varieties are paved by affine spaces given by regular nilpotent Hessenberg Schubert cells.*

Proof. The flag variety Flags($\mathbb{K}^n$) is paved by affines, given by Schubert cells X_w° ordered by any total order that refines Bruhat order (see for instance, [Che91, Étude qualitative des variétés $X(w)$]). Proposition 3.3.1 tells us that each nonempty regular nilpotent Hessenberg Schubert cell is isomorphic to affine space. Since Hessenberg varieties are closed subvarieties of the flag variety, the affine paving of Flags($\mathbb{K}^n$) by Schubert cells descends to an affine paving of Hess(N, h) by regular nilpotent Hessenberg Schubert cells. ■

This result extends naturally to all regular nilpotent Hessenberg varieties, thanks to Lemma 2.4.13.

3.3.2 Hilbert series

Our goal of this subsection is to give an explicit formula for the Hilbert series of regular nilpotent Hessenberg Schubert cell ideals. Since our generators $g_{k,\ell}^w$ are, in general, not homogeneous with respect to the standard grading, we first find a grading of $\mathbb{K}[\mathbf{z}_w]$ with respect to which, the $g_{k,\ell}^w$ are homogeneous.

Recall that in Section 2.4.3, we discussed the action of a torus on Hessenberg varieties, and in particular, in Definition 2.4.15 we defined a $\mathbb{Z}$-grading of $\mathbb{Z}[\mathbf{x}_w]$ by $\deg(x_{i,j}) = w(j) - i$. Da Silva and Harada showed that on the w_0-chart, this is a positive grading. In particular, they showed the following.

Lemma 3.3.3 ([DH23, Lemma 2.18]). *The w_0-patch generators $f_{k,\ell}^{w_0}$ are homogeneous with respect to the nonstandard positive $\mathbb{Z}$-grading of $\mathbb{K}[\mathbf{x}_{w_0}]$ given in Definition 2.4.15.*

The proof of this lemma is a straightforward computation. As with much of Chapter 3, we may use the homomorphism of rings ψ_w to translate this lemma to the setting of regular nilpotent Hessenberg Schubert cells. For instance, notice the similar approach between the following definition and Definition 3.2.10.

Definition 3.3.4. Let $w \in \mathfrak{S}_n$. To an indeterminate $z_{i,j} \in \mathbb{K}[\mathbf{z}_w]$, associate a weight $\deg(z_{i,j}) = \deg(\psi_w^{-1}(z_{i,j}))$, where the weight on the right-hand side is given in Definition 2.4.15 on the w_0-chart.

This definition is well-defined for the same reason that Definition 3.2.10 was well-defined. Explicitly, we have that $\deg(z_{i,j}) = \deg(x_{i,v_w(j)})$ which is $w_0(v_w(j)) - i$. Upon replacing $v_w = w_0 w$, it follows that $\deg(z_{i,j}) = w(j) - i$. This shows that the degree of $z_{i,j}$ in $\mathbb{K}[\mathbf{z}_w]$ agrees with the degree of $x_{i,j}$ in $\mathbb{K}[\mathbf{x}_w]$. However, the restriction to the Schubert cell guarantees that the grading on $\mathbb{K}[\mathbf{z}_w]$ is positive. Moreover, we have the following.

Lemma 3.3.5 ([CDHR24, Lemma 5.10]). *Fix an indecomposable Hessenberg function h and a permutation $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$. Then, the nonzero generators $g^w_{k,\ell}$ of the regular nilpotent Hessenberg Schubert cell ideal are homogeneous with respect to the nonstandard positive $\mathbb{Z}$-grading of $\mathbb{K}[\mathbf{z}_w]$ given in Definition 2.4.15.*

Proof. That the given grading is positive follows from Definition 2.4.18. Since $g^w_{k,\ell}$ is nonzero, write $g^w_{k,\ell} = \psi_w(f^{w_0}_{v_w(k),v_w(\ell)})$ per Lemma 3.2.9. The result now follows from Definition 3.3.4 which preserves the homogeneity of the $f^{w_0}_{k,\ell}$ guaranteed by Lemma 3.3.3. ∎

It is now straightforward to compute the degree of each generator $g^w_{k,\ell}$. Recall that the nonzero generators of a regular nilpotent Hessenberg Schubert cell at w and h satisfy $k > h(\ell)$ and $v_w(k) > v_w(\ell) + 1$.

Lemma 3.3.6. *In the setting of Lemma 3.3.5, we have $\deg(g^w_{k,\ell}) = v_w(k) - v_w(\ell) - 1$.*

Proof. Lemma 3.3.5 says that it suffices to compute the degree of the initial term, so the result follows from Definition 3.3.4, Definition 2.4.15, and Lemma 3.1.8. ∎

We may now compute the Hilbert series for (the quotient by) a regular nilpotent Hessenberg Schubert cell ideal.

Theorem 3.3.7 ([CDHR24, Theorem 5.11]). *Let $J_{w,h}$ be the regular nilpotent Hessenberg Schubert cell ideal associated to an indecomposable Hessenberg function h and the permutation $w \in \mathrm{Hess}(\mathsf{N}, h)^{\mathbf{S}}$. With respect to the nonstandard grading of $R = \mathbb{K}[\mathbf{z}_w]$ from Definition 3.3.4, the Hilbert series of $R/J_{w,h}$ is*

$$HS_{R/J_{w,h}}(t) = \frac{\displaystyle\prod_{\substack{k>h(\ell) \\ v_w(k)>v_w(\ell)+1}} \left(1 - t^{v_w(k)-v_w(\ell)-1}\right)}{\displaystyle\prod_{\substack{i<w(j) \\ j<w^{-1}(i)}} \left(1 - t^{w(j)-i}\right)}.$$

Proof. Per the discussion in the paragraph prior to Lemma 3.2.11, a generator $g^w_{k,\ell}$ with $k > h(\ell)$ is nonzero if and only if $v_w(k) > v_w(\ell) + 1$ and, in this case, has degree $v_w(k) - v_w(\ell) - 1$. The result of Lemma 2.1.11 then guarantees that it remains to show that

$$HS_R(t) = \prod_{\substack{i<w(j) \\ j<w^{-1}(i)}} \left(1 - t^{w(j)-i}\right),$$

which follows from Example 2.1.10 and, for the indices appearing in the product, Definition 2.4.18. ∎

3.4 Regular nilpotent Hessenberg patch ideals

To conclude this chapter, we review some known results on regular nilpotent Hessenberg varieties. A survey paper of Abe and Horiguchi highlights other known results of regular nilpotent and regular semisimple Hessenberg varieties [AH20]. We also discuss here open questions that remain and other potential future research directions.

Insko and Yong in 2012 first used patch ideals in a Hessenberg setting when studying the Peterson variety, the regular nilpotent Hessenberg variety associated to $h = (2, 3, \ldots, n-1, n, n)$. They borrowed this technique from the study of Schubert varieties, and provide a historical background of this case in [IY12, Sections 2 and 3]. In this paper, they gave a combinatorial description of the singular locus of Peterson varieties in type A [IY12, Theorem 4]. They also showed that Peterson patch ideals are complete intersections [IY12, Corollary 7]. This was later generalized to all indecomposable regular nilpotent Hessenberg patch ideals by Abe, DeDieu, Galetto, and Harada [ADGH18, Corollary 3.17]. More recently, Abe and Insko used patch ideals to compute singular points of regular nilpotent Hessenberg varieties and characterize when these varieties are normal [AI22].

These results complement the literature that studies regular nilpotent Hessenberg varieties in arbitrary type using Lie theory. From this side, it is known that the Hessenberg variety $\mathrm{Hess}(\mathsf{N}, h)$ is irreducible of dimension $\sum_{j=1}^{n}(h(j) - j)$ (see [AT10, Lemma 7.1] for the statement in type A and [ST06, Proposition 10.2] for the general statement). Tymoczko was the first to show that regular nilpotent Hessenberg varieties are paved by affines, irrespective of Lie type [Tym07, Corollary 4.3]. She had also previously showed the same result for all Hessenberg varieties in type A in [Tym04, Theorem 23] and [Tym06, Theorem 6.1]. Work of Abe, Harada, Horiguchi, and Masuda gave a description of the cohomology ring of a regular nilpotent Hessenberg variety [AHHM17, Theorem A], as well as connecting it to that of a regular semisimple Hessenberg variety [AHHM17, Theorem B].

We now state some open questions involving regular nilpotent Hessenberg patch ideals. When we concluded that both regular nilpotent w_0-patch ideals are triangular complete intersections, this implied that the given generators form a Gröbner basis. However, we saw in Example 3.2.1 that the same approach cannot be used for an arbitrary regular nilpotent patch ideal. So, we have the following.

Question 3.4.1. Fix a Hessenberg patch ideal $I_{w,h} = \langle f^w_{k,\ell} \mid k > h(\ell)\rangle \subseteq K[\mathbf{x}_w]$. Does there exist a (lexicographic) order on $\mathbb{K}[\mathbf{x}_w]$, with respect to which, the generators $f^w_{k,\ell}$ form a Gröbner basis for $I_{w,h}$. Moreover, on the w_0-patch, does this order agree with Da Silva and Harada's monomial order from Definition 3.1.6?

We also saw in Example 3.2.1 that some patches do not have a squarefree initial ideal. So the result of Lemma 2.3.4 cannot, in general, be used to construct compatible Frobenius splittings of regular nilpotent Hessenberg patch ideals. However, Atar has results in this direction when $h = (n-1, n, \ldots, n)$. That is, when the patch ideal has a single generator.

Theorem 3.4.2 ([Ata23, Theorems 3.2.9 and 3.2.10])**.** *Let $h = (n-1, n, \ldots, n)$. With respect to the lexicographic monomial given in [Ata23, Construction 1], the initial term of $f^w_{n,1}$ is squarefree. Hence there exists a Frobenius splitting that compatibly splits $I_{w,h}$.*

We have computational evidence that every regular nilpotent Hessenberg patch ideal in $n = 4$ is compatibly split. Similar data is also given in [Ata23].

Question 3.4.3. Fix a patch ideal $I_{w,h} \subseteq \mathbb{K}[\mathbf{x}_w]$. Does there exist a Frobenius splitting of $\mathbb{K}[\mathbf{x}_w]$ that compatibly splits $I_{w,h}$?

If so, does there exist a (non-canonical) Frobenius splitting of the flag variety that descends to compatibly split each patch ideal?

We can ask a similar question about geometric vertex decomposition. Again, the methods used in this thesis cannot extend to other patches in general, because of Example 3.2.1.

Question 3.4.4. Which regular nilpotent Hessenberg patch ideals are geometrically vertex decomposable? Which are compatibly geometrically vertex decomposable with respect to a lexicographic order? Does the lex order agree with the order from Question 3.4.1?

Chapter 4

Semisimple Hessenberg Varieties

Regular semisimple Hessenberg varieties were the focus of the paper of De Mari, Procesi, and Shayman that first defined Hessenberg varieties [DPS92]. In their paper, they showed that regular semisimple Hessenberg varieties are smooth [DPS92, Theorem 6] and that a subfamily of these varieties are toric varieties [DPS92, Theorem 11]. Since this paper, regular semisimple Hessenberg varieties have been studied for their connections to, among other areas, the Stanley–Stembridge conjecture [SS93, Conjecture 5.5] (see also [Sta95, Conjecture 5.1]). Shareshian and Wachs conjectured an equivalence of the Stanely–Stembridge conjecture in terms of regular semisimple Hessenberg varieties [SW12, Conjecture 5.3] and this equivalence was proved independently and using different methods by Brosnan and Chow [BC18] and Guay-Paquet [Gua16]. For more on regular semisimple Hessenberg varieties, see [AH20; DPS92].

In contrast, there are relatively few papers that study non-regular semisimple Hessenberg varieties. For instance, a recent preprint of Can, Precup, Shareshian, and Uğurlu gives a condition for semisimple Hessenberg varieties to be irreducible and, in this setting, a formula for its dimension [CPSU23]. Their work restricts to the case of a semisimple operator with exactly two distinct eigenvalues. An analogous result for the regular semisimple was given in De Mari, Procesi, and Shayman's paper [DPS92] and by Sommers and Tymoczko in the regular nilpotent case [ST06]. Insko and Precup showed that, although semisimple Hessenberg varieties are not smooth in general, their irreducible components are smooth. They also gave a explicit description of the intersections of their irreducible components and, hence, of their singular loci [IP19].

An overarching theme of this chapter is this difference between regular semisimple and semisimple Hessenberg varieties. For instance, it is known that regular semisimple Hessenberg varieties are smooth [DPS92, Theorem 6] while arbitrary semisimple varieties are not [IP19, Example 4.7]. We will see that our computational approach similarly agrees that the semisimple case is much more delicate than the regular semisimple case.

4.1 Regular semisimple Hessenberg varieties are local triangular complete intersections

Regular semisimple Hessenberg varieties are known to be smooth over $\mathbb{C}$ [DPS92, Theorem 6] so we should expect that their Gröbner geometry to be correspondingly nice. Indeed, in this section, we will show that the local defining ideals are triangular complete intersections over an arbitrary algebraically closed field $\mathbb{K}$. To the knowledge of the author, this result does not appear in the existing Hessenberg variety literature, but it will not be surprising to experts.

Our set up is as follows. Denote by $\mathbb{K}$ an algebraically closed field of arbitrary characteristic. Recall from Definition 2.4.10 that a regular semisimple matrix is one that is diagonalizable with distinct eigenvalues. Also, since we may freely assume that our linear operator is in Jordan canonical form, throughout this section, we denote by R the diagonal matrix $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where the eigenvalues are pairwise disjoint over $\mathbb{K}$. As derived in Section 2.4.2, the local defining ideal at $w \in \mathfrak{S}_n$ is generated by certain south-west entries of $(wM)^{-1}\mathsf{R}(wM)$. To distinguish the regular semisimple case from our work in previous chapters, denote by $p^w_{k,\ell}$ the (k, ℓ)-th entry of this matrix and denote by $P_{w,h}$ the ideal generated by the $p^w_{k,\ell}$. That is, $P_{w,h}$ is the regular semisimple patch ideal corresponding to $w \in \mathfrak{S}_n$ and Hessenberg function $h : [n] \to [n]$.

We make one final remark before beginning our arguments. In the regular nilpotent case, we freely imposed the condition that our Hessenberg functions be indecomposable for the result of Lemma 2.4.13 said that any regular nilpotent Hessenberg variety is the product of indecomposable regular nilpotent Hessenberg varieties. However, the proof of this lemma, given by Drellich [Dre15, Theorem 4.5], does not immediately adapt to the (regular) semisimple case and the author is not aware of an analogous result in this setting. So throughout Chapter 4 we allow for decomposable Hessenberg functions. That is, our Hessenberg functions throughout this chapter are weakly increasing maps $h : [n] \to [n]$ satisfying $h(i) \geq i$ for all i.

Fix a regular semisimple operator $\mathsf{R} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, so the eigenvalues λ_i are pairwise distinct. As in the previous chapters, denote by M a generic lower triangular matrix with determinant 1, that is, with 1's along the diagonal, 0's above the diagonal, and an $x_{i,j}$ in each (i, j)-th entry with $i > j$. We denote by $\mathbf{x}$ the collection of these $x_{i,j}$ and $\mathbb{K}[\mathbf{x}]$ the polynomial ring in these variables over an algebraically closed field $\mathbb{K}$. For any $w \in \mathfrak{S}_n$, notice that $(wM)^{-1}\mathsf{R}(wM) = M^{-1}(w^{-1}\mathsf{R}w)M$ and the product $w^{-1}\mathsf{R}w$ corresponds to reordering the eigenvalues in R. In particular,

$$w^{-1}\mathsf{R}w = \operatorname{diag}\left(\lambda_{w(1)}, \ldots, \lambda_{w(n)}\right).$$

This allows us to study families of patch ideals, those with a fixed Hessenberg function, simultaneously.

Changing notation from Chapter 3, for the rest of this thesis, we denote by $y_{i,j}$ the (i, j)-th entry of M^{-1}.

Proposition 4.1.1. *For any permutation $w \in \mathfrak{S}_n$ and any $k > \ell$, we have that*

$$p_{k,\ell}^w = \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} + \sum_{j=\ell+1}^{k-1} \left(\lambda_{w(j)} - \lambda_{w(\ell)}\right) y_{k,j} x_{j,\ell}.$$

The assumption that $k > \ell$ says that the above result holds for any generator $p_{k,\ell}^w$ for any $P_{w,h}$ corresponding to any (not necessarily indecomposable) Hessenberg function. After the proof of this proposition, we will construct a lexicographic monomial with respect to which the initial monomial of each $p_{k,\ell}^w$ is $x_{k,\ell}$. This immediately indicates that the assumption for the eigenvalues to be distinct plays a key role in our triangular complete intersection arguments in the present section. Indeed, when eigenvalues are no longer distinct, terms of $p_{k,\ell}^w$ may vanish which, in the next section, causes issues when we attempt to compute initial terms, for example.

We first require the following technical lemma, which is exactly Lemma 3.1.3 and Equation (3.1.4) after changing to the current notation.

Lemma 4.1.2 ([DH23, Lemma 2.14])**.** *Let $y_{k,\ell} = [M^{-1}]_{k,\ell}$. Then,*

(i) $y_{k,\ell} = 1$ *if and only if* $k = \ell$*,*
(ii) $y_{k,\ell} = 0$ *if and only if* $k < \ell$*,*
(iii) $y_{k,\ell} = -x_{k,\ell} - \sum_{j=\ell+1}^{k-1} y_{k,j}\, x_{j,\ell}$ *for any* $k > \ell$*,*
(iv) if $k > \ell$*, then the polynomial* $y_{k,\ell}$ *has no constant term,*
(v) $y_{k,\ell}$ *depends only on* $x_{a,b}$ *with* $a \leq k$ *and* $b \geq \ell$*.*

Proof. Because M is lower triangular it follows that M^{-1} is too. This, together with the fact that $\det M = 1$, completes the proofs of items (i) and (ii). Now, we have that $[M^{-1}M]_{k,\ell} = \delta_{k,\ell}$ and, because M and M^{-1} are lower triangular,

$$\left[M^{-1}M\right]_{k,\ell} = \sum_{j=\ell}^{k} y_{k,j}\, x_{j,\ell}.$$

So, if $k > \ell$, then

$$0 = y_{k,\ell}\, x_{\ell,\ell} + y_{k,k}\, x_{k,\ell} + \sum_{j=\ell+1}^{k-1} y_{k,j}\, x_{j,\ell},$$

which implies that

$$y_{k,\ell} = -x_{k,\ell} - \sum_{j=\ell+1}^{k-1} y_{k,j}\, x_{j,\ell}. \tag{4.1.3}$$

This is item (iii). Items (iv) and (v) follow from (iii) after inducting on $k - \ell$. ∎

Proof of Proposition 4.1.1. We make the following computation:

$$p^w_{k,\ell} = \sum_{j=1}^{n} \left[M^{-1}\right]_{k,j} \left[(w^{-1}\mathsf{R}w)M\right]_{j,\ell} = \sum_{j=\ell}^{k} \left[M^{-1}\right]_{k,j} \left[(w^{-1}\mathsf{R}w)M\right]_{j,\ell},$$

where we use the facts that only the first k entries in row k of M^{-1} are nonzero and the first $\ell - 1$ entries in column ℓ of M (and hence, of $(w^{-1}\mathsf{R}w)M$) are zero. Then, using Lemma 4.1.2(iii),

$$\begin{aligned} p^w_{k,\ell} &= \sum_{j=\ell}^{k} \lambda_{w(j)} y_{k,j} x_{j,\ell} = \lambda_{w(\ell)} y_{k,\ell} x_{\ell,\ell} + \lambda_{w(k)} y_{k,k} x_{k,\ell} + \sum_{j=\ell+1}^{k-1} \lambda_{w(j)} y_{k,j} x_{j,\ell} \\ &= \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} - \lambda_{w(\ell)} \left(\sum_{j=\ell+1}^{k-1} y_{k,j}\, x_{j,\ell} \right) + \sum_{j=\ell+1}^{k-1} \lambda_{w(j)} y_{k,j} x_{j,\ell} \\ &= \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} + \sum_{j=\ell+1}^{k-1} \left(\lambda_{w(j)} - \lambda_{w(\ell)}\right) y_{k,j} x_{j,\ell}. \end{aligned}$$

■

Proposition 4.1.1 above immediately implies the following.

Corollary 4.1.4. *For any $\ell \leq n-1$, the generator $p^w_{\ell+1,\ell}$ is a monomial.*

Proof. In this case, we have that $p^w_{\ell+1,\ell} = \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell}$. ■

We are now only one step away from concluding that each $P_{w,h}$ is a triangular complete intersection. Indeed, recall from Definition 2.3.5 that we are required to find a monomial order with respect to which the initial monomial of each $p^w_{k,\ell}$ is an indeterminate. To that end, we make the following definition.

Definition 4.1.5. Define a lexicographic monomial order $\prec_n$ on $\mathbb{K}[\mathbf{x}]$ by $x_{i,j} \succ_n x_{i',j'}$ whenever $i > i'$ or both $i = i'$ and $j < j'$.

Figure 4.1 provides an illustration of this order. The matrix in the figure is M and the arrows go from largest to smallest with respect to $\prec_n$. Colloquially, this order weights variables more expensive the further south in M they appear and breaks ties (within rows) by preferring variables further west. This is not the unique monomial order that works for our triangular complete intersection argument, see Remark 4.1.10.

Lemma 4.1.6. *For any $w \in \mathfrak{S}_n$ and any $k > \ell$, we have that in$_{\prec_n}(p^w_{k,\ell}) = (\lambda_{w(k)} - \lambda_{w(\ell)}) x_{k,\ell}$.*

Proof. From the equation for $p^w_{k,\ell}$ given in Lemma 4.1.2(iii), it is clear that no $x_{i,j}$ larger than $x_{k,\ell}$ appears in the summation. By applying Lemma 4.1.2(v) to Equation (4.1.3), inducting on $k - \ell$ shows that in$_{\prec_n}(y_{k,\ell}) = -x_{k,\ell}$. This, together with Lemma 4.1.2(iii) and the fact that the eigenvalues are distinct, completes the proof. ■

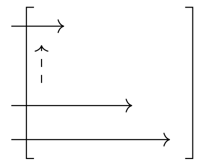

Figure 4.1: Regular semisimple monomial order $\prec_n$.

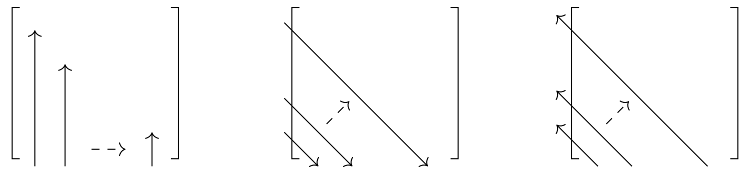

Figure 4.2: Other term orders for which $P_{w,h}$ is a triangular complete intersection.

Moreover, combining items (iii) and (v) in Lemma 4.1.2 yields the following.

Lemma 4.1.7. *Each $p^w_{k,\ell}$ depends only on the variables $x_{a,b}$ with $a \leq k$ and $b \geq \ell$.*

This says that each $p^w_{k,\ell}$ depends only on $x_{k,\ell}$ and the variables appearing weakly north-west of $x_{k,\ell}$ in M. We now may conclude that each regular semisimple patch ideal is a triangular complete intersection. At the same time, this justifies the use of the term "patch ideal", for triangular complete intersections are prime ideals, which are necessarily radical. For the following result, we order the generators $p^w_{k,\ell}$ in the same way that we order the variables $x_{a,b}$ with respect to $\prec_n$. That is, we order them as follows.

$$p^w_{n,1}, p^w_{n,2}, \ldots, p^w_{n,n-1}, p^w_{n-1,1}, p^w_{n-1,2}, \ldots, p^w_{n-1,n-2}, p^w_{n-2}, \ldots, \ldots, p^w_{2,1}. \tag{4.1.8}$$

Theorem 4.1.9. *Regular semisimple patch ideals are triangular complete interesections.*

Proof. Immediate from the ordering in Equation (4.1.8) combined with Lemma 4.1.6 and Lemma 4.1.7, bearing in mind that Lemma 4.1.6 holds because the coefficients of the initial terms are always nonzero in the regular semisimple case. ■

Remark 4.1.10. The above argument works for other term orders. Indeed, what must remain true is the triangular complete intersection condition that states that the initial term of one generator must not divide any term of any generator that occurs later in the list (4.1.8). For instance, the orders illustrated in Figure 4.2 also work. (Recall that M is the matrix underlying each figure.)

We conclude this section with an analogous result to Theorem 3.1.13. That is, we will show that the regular semisimple patch ideal is a monomial ideal for certain Hessenberg functions, and for others, it has a natural binomial generating set. To that end, we will first derive the following recursive result which says that each $p^w_{k,\ell}$ can be written as its initial term plus a polynomial combination of generators $p^w_{k',\ell}$ with $k' < k$. We will also use this result extensively in Section 4.2.

Proposition 4.1.11. *Let $w \in \mathfrak{S}_n$ and $k > \ell$. Then* $q^w_{k,\ell} = \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} - \sum_{j=\ell+1}^{k-1} x_{k,j} p^w_{j,\ell}$.

Proof. We proceed by induction on $k - \ell$. The base case of $k = \ell + 1$ follows immediately from Proposition 4.1.1. Otherwise, the induction hypothesis guarantees that

$$\begin{aligned} p^w_{k-1,\ell} &= \left(\lambda_{w(k-1)} - \lambda_{w(\ell)}\right) x_{k-1,\ell} + \sum_{j=\ell+1}^{k-2} \left(\lambda_{w(j)} - \lambda_{w(\ell)}\right) y_{k-1,j} x_{j,\ell} \\ &= \left(\lambda_{w(k-1)} - \lambda_{w(\ell)}\right) x_{k-1,\ell} - \sum_{j=\ell+1}^{k-2} x_{k-1,j} p^w_{j,\ell}, \end{aligned}$$

so in particular,

$$\sum_{j=\ell+1}^{k-2} \left(\lambda_{w(j)} - \lambda_{w(\ell)}\right) y_{k-1,j} x_{j,\ell} = - \sum_{j=\ell+1}^{k-2} x_{k-1,j} p^w_{j,\ell}.$$

Applying this, and the result of Proposition 4.1.1, says that

$$\begin{aligned} p^w_{k,\ell} &= \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} + \left(\lambda_{w(k-1)} - \lambda_{w(\ell)}\right) y_{k,k-1} x_{k-1,\ell} \\ &\qquad + \sum_{j=\ell+1}^{k-2} \left(\lambda_{w(j)} - \lambda_{w(\ell)}\right) y_{k,j} x_{j,\ell} \\ &= \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} - x_{k,k-1} p^w_{k-1,\ell} - \sum_{j=\ell+1}^{k-2} x_{k,j} p^w_{j,\ell} \\ &= \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} - \sum_{j=\ell+1}^{k-1} x_{k,j} p^w_{j,\ell}, \end{aligned}$$

where the second equality uses the fact from Lemma 4.1.2(iii) that $y_{k,k-1} = -x_{k,k-1}$. ∎

Theorem 4.1.12. *Irrespective of permutation $w \in \mathfrak{S}_n$, the regular semisimple patch ideal $P_{w,h}$ is a monomial ideal whenever $h(i)$ is either i or n for all i. It is generated by binomials whenever $h(i)$ is either $i + 1$ or n for all i.*

Proof. Rewrite the generating set as follows,

$$\begin{aligned} P_{w,h} &= \left\langle \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} - \sum_{j=\ell+1}^{k-1} x_{k,j} p^w_{j,\ell} \;\middle|\; k > h(\ell) \right\rangle \\ &= \left\langle \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} - \sum_{j=\ell+1}^{h(\ell)} x_{k,j} p^w_{j,\ell} \;\middle|\; k > h(\ell) \right\rangle . \end{aligned}$$

If $h(i)$ is always either i or n, then each $p^w_{j,\ell}$ appearing in the summation is itself a generator for the ideal, so we may rewrite the generating set as

$$P_{w,h} = \left\langle \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} \mid k > h(\ell) \right\rangle = \langle x_{k,\ell} \mid k > h(\ell) \rangle,$$

where the second equality holds because the eigenvalues are distinct. On the other hand, if $h(i)$ is always either $i+1$ or n, then the generating set becomes

$$P_{w,h} = \left\langle \left(\lambda_{w(k)} - \lambda_{w(\ell)}\right) x_{k,\ell} - x_{k,\ell+1} p^w_{\ell+1,\ell} \mid k > h(\ell) \right\rangle ,$$

and each $p^w_{\ell+1,\ell}$ is a monomial, per Corollary 4.1.4. ∎

4.2 Semisimple Hessenberg patch ideals

In this section, we discuss the local defining ideals of semisimple Hessenberg varieties. Denote by $\mathsf{S} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ a semisimple linear operator (in Jordan canonical form), so the eigenvalues are not necessarily distinct. Also, denote by $q^w_{k,\ell}$ the (k,ℓ)-th entry of $(wM)^{-1}\mathsf{S}(wM)$ for any $w \in \mathfrak{S}_n$. We maintain that M is the usual generic lower triangular matrix with determinant 1, given in Equation (2.4.4). These polynomials are defined in the same way as the $p^w_{k,\ell}$ from the previous section, but we introduce new notation here to emphasize that some terms may vanish in the semisimple case (see, for instance, Proposition 4.1.1). For any Hessenberg function h, we denote by $K_{w,h}$ the ideal of $\mathbb{K}[\mathbf{x}]$ generated by the $q^w_{k,\ell}$ with $k > h(\ell)$, where $\mathbf{x}$ denotes the finite set of variables in M and $\mathbb{K}$ an arbitrary algebraically closed field.

It is an open question whether these set-theoretic defining ideals $K_{w,h}$ are radical.

Conjecture 4.2.1 ([IP19, Conjecture 5.4]). *Every $K_{w,h}$ is radical.*

As evidence for their conjecture, they sketched a proof for the following special case. Their argument constructed a Gröbner basis for each ideal that has squarefree lead terms.

Proposition 4.2.2 ([IP19, Lemma 5.3]). *The ideal $K_{w,(2,3,\ldots,n,n)}$ is radical for any $w \in \mathfrak{S}_n$.*

Moreover, they noted that their approach is one of two commonly used techniques in the literature to show that local defining ideals are indeed radical:

(i) under additional hypotheses, generically reduced implies reduced [Eis04, Exercise 18.9];
(ii) construct a Gröbner basis with squarefree lead terms [CLO15, Chapter 4.3, Exercise 16].

Their proof sketch used the latter method, while [ADGH18; IY12] both used the former. We devote the present section to giving a positive answer to Conjecture 4.2.1 in some special cases by constructing Gröbner bases with squarefree lead terms. Our arguments will make use of Lemma 2.2.8 which says that a given generating set is a Gröbner basis if the initial terms are relatively prime.

This argument holds over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. If $\mathbb{K}$ is of finite characteristic, then it follows that the initial monomial of the product of the generators is squarefree. Lemma 2.3.4 then tells us how to construct a Frobenius splitting on the corresponding coordinate chart that compatibly splits the patch ideal. We formalize summarize this discussion in the following lemma.

Lemma 4.2.3. *Let $<$ be a monomial order on $\mathbb{K}[\mathbf{x}]$. Suppose that an ideal $I \subseteq \mathbb{K}[\mathbf{x}]$ has a generating set $\{f_1, \ldots, f_r\}$ such that the list of initial terms $\{\mathrm{in}_<(f_1), \ldots, \mathrm{in}_<(f_r)\}$ are squarefree and relatively prime. Then, the given generators form a Gröbner basis with respect to $<$. Moreover, and if $\mathbb{K}$ is finite, there exists a Frobenius splitting on $\mathbb{K}[\mathbf{x}]$ that compatibly splits I.*

We begin with an immediate corollary of Theorem 4.1.9.

Corollary 4.2.4. *Consider an ideal $K_{w,h}$. If, for all indices (k, ℓ) with $k > h(\ell)$, we have that $\lambda_{w(k)} \neq \lambda_{w(\ell)}$, then $K_{w,h}$ is a triangular complete intersection. Hence, $K_{w,h}$ is radical.*

Proof. The assumption says that Lemma 4.1.6 holds in the semisimple case. The remaining arguments of Theorem 4.1.9 are unchanged. ∎

We next treat the case analogous to that of Atar's thesis in the regular nilpotent case [Ata23, Theorem 3.2.9]. We require the following lemma, which we will also use later for the argument in a different case.

Lemma 4.2.5. *Every semisimple generator $q^w_{k,\ell}$ is squarefree.*

Proof. Proceed by induction on $k-\ell$; the base case is Corollary 4.1.4. For the inductive step, from Proposition 4.1.11, it suffices to show that $x_{k,j}$ never divides any term of $q^w_{j,\ell}$ for any $j = \ell+1, \ldots, k-1$. Because we have that $j < k$, Lemma 4.1.7 says that this holds. ∎

Corollary 4.2.6. *The ideal $K_{w,(n-1,n,\ldots,n)}$ is radical.*

Proof. If $q^w_{n,1} = 0$, then there is nothing to show, so assume otherwise. Then since every term of $q^w_{n,1}$ is squarefree, any initial term of $q^w_{n,1}$ is squarefree, so Lemma 4.2.3 trivially holds. ∎

As another straightforward corollary of our work in Section 4.1, we have the following.

Proposition 4.2.7. *Suppose that h is a Hessenberg function for which $h(i)$ is either i or n for all i. Then $K_{w,h}$ is generated by indeterminates, so is radical.*

Proof. Apply Theorem 4.1.12 and discard the generators that are zero. This lands us in the setting of Lemma 4.2.3. ∎

We treat one final case. The case of exactly n eigenvalues is the regular semisimple case. In this final special case, we assume that we have exactly $n-1$ eigenvalues. That is, if $\mathsf{S} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, there exist $i < j$ such that for any $i' < j'$ we have that $\lambda_i' = \lambda_j'$ if and only if $i = i'$ and $j = j'$. In particular, this says that there is at most one generator $q_{k,\ell}^w$ whose initial monomial is not an indeterminate. We first give an illustrative example of the technique we will use in our proof.

Example 4.2.8. Let $\mathsf{S} = \operatorname{diag}(1, 2, 3, 4, 1)$ and consider the corresponding ideal $K_{w,h}$ at the identity chart and $h = (4, 4, 4, 5, 5)$. That is, $K_{w,h} = \left\langle q_{5,1}^w, q_{5,2}^w, q_{5,3}^w \right\rangle$ where

$$\begin{aligned} q_{5,1}^w &= -x_{5,2}q_{2,1}^w - x_{5,3}q_{3,1} - x_{5,4}q_{4,1}, \\ q_{5,2}^w &= (\lambda_5 - \lambda_2)x_{5,2} - x_{5,3}q_{3,2}^w - x_{5,4}q_{4,2}^w, \\ q_{5,3}^w &= (\lambda_5 - \lambda_3)x_{5,3} - x_{5,4}q_{4,3}^w. \end{aligned}$$

As it stands, $\operatorname{in}_{\prec_n}(q_{5,1}^w) = -x_{5,2} \cdot \operatorname{in}_{\prec_n}(q_{2,1}^w)$, which is divisible by the initial term of $q_{5,2}^w$. So to apply Lemma 4.2.3 we replace $q_{5,1}^w$ in the generating set by a polynomial $\widetilde{q}_{k,\ell}^w$ whose initial term is not divisible by $x_{5,2}$. To that end, write

$$\begin{aligned} -q_{5,1}^w = &\frac{q_{2,1}^w}{\lambda_5 - \lambda_2}\left((\lambda_5 - \lambda_2)x_{5,2} - x_{5,3}q_{3,2}^w - x_{5,4}q_{4,2}^w\right) \\ &+ \frac{q_{2,1}^w}{\lambda_5 - \lambda_2}\left(x_{5,3}q_{3,2}^w + x_{5,4}q_{4,2}^w\right) + x_{5,3}q_{3,1}^w + x_{5,4}q_{4,1}^w. \end{aligned}$$

So we may replace $q_{5,1}^w$ in the generating set by

$$\widetilde{q}_{5,1}^w = \frac{q_{2,1}^w}{\lambda_5 - \lambda_2}\left(x_{5,3}q_{3,2}^w + x_{5,4}q_{4,2}^w\right) + x_{5,3}q_{3,1}^w + x_{5,4}q_{4,1}^w.$$

The polynomial $\widetilde{q}_{5,1}^w$ does not have any term divisible by $x_{5,2}$, however, its initial monomial $x_{5,3}x_{31}$ is divisible by the initial monomial of $q_{5,3}^w$. So, we play the same game again, writing

$$\begin{aligned} \widetilde{q}_{5,1}^w &= x_{5,3}\left(\frac{q_{2,1}^w q_{3,2}^w}{\lambda_5 - \lambda_2} + q_{3,1}^w\right) + \frac{x_{5,4}q_{2,1}^w q_{4,2}^w}{\lambda_5 - \lambda_2} + x_{5,4}q_{4,1}^w \\ &= \frac{\frac{q_{2,1}^w q_{3,2}^w}{\lambda_5 - \lambda_2} + q_{3,1}^w}{\lambda_5 - \lambda_3}\left((\lambda_5 - \lambda_3)x_{5,3} - x_{5,4}q_{4,3}^w\right) \\ &\quad + \frac{\frac{q_{2,1}^w q_{3,2}^w}{\lambda_5 - \lambda_2} + q_{3,1}^w}{\lambda_5 - \lambda_3}x_{5,4}q_{4,3}^w + \frac{x_{5,4}q_{2,1}^w q_{4,2}^w}{\lambda_5 - \lambda_2} + x_{5,4}q_{4,1}^w. \end{aligned}$$

Hence we may replace $\widetilde{q}^w_{5,1}$ in the generating set for $K_{w,h}$ by

$$\frac{\frac{q^w_{2,1}q^w_{3,2}}{\lambda_5-\lambda_2}+q^w_{3,1}}{\lambda_5-\lambda_3}x_{5,4}q^w_{4,3}+\frac{x_{5,4}q^w_{2,1}q^w_{4,2}}{\lambda_5-\lambda_2}+x_{5,4}q^w_{4,1},$$

which the reader may verify, using Lemma 4.1.7, is a squarefree polynomial that involves neither $x_{5,2}$ nor $x_{5,3}$. Hence the product of the above generator with $q^w_{5,2}$ and $q^w_{5,3}$ has a squarefree initial term and hence we may apply Lemma 4.2.3 to conclude that $K_{w,h}$ is radical. ♢

We formalize this process in Algorithm 4.1.

Algorithm 4.1: Semisimple generator replacement.

Data: generator $q^w_{k,\ell}$ with $\lambda_{w(k)} = \lambda_{w(\ell)}$ and Hessenberg function h
Result: $\widetilde{q}^w_{k,\ell}$ with distinct initial term to replace $q^w_{k,\ell}$ in the generating set

1 $i \leftarrow 1$
2 $\widetilde{q}^w_{k,\ell} \leftarrow -q^w_{k,\ell}$
3 **while** $k > h(i+1)$ **do**
4 write $\widetilde{q}^w_{k,\ell} = Qx_{k,\ell+i} + R$ where Q and R satisfy $Q, R \prec_n x_{k,\ell+i}$
5 write $q^w_{k,\ell+i} = Cx_{k,\ell+i} + R'$ for $C \in \mathbb{K}^*$ and $R' \prec_n x_{k,\ell+i}$
6 $\widetilde{q}^w_{k,\ell} \leftarrow R - QR'/C$
7 $i \leftarrow i+1$
8 **end**

In Algorithm 4.1, Line 6 holds because the setup implies that $\widetilde{q}^w_{k,\ell} = (Q/C)q^w_{k,\ell+i} - QR'/C + R$ and $q^w_{k,\ell+i}$ is itself a generator for $K_{w,h}$, as guaranteed by the bounds on the while loop. Hence this algorithm produces a new generating set consisting of the $q^w_{k',\ell'}$ with $(k',\ell') \neq (k,\ell)$ and $\widetilde{q}^w_{k,\ell}$. We will show this generating set satisfies Lemma 4.2.3, for which, we require the following technical lemma.

Lemma 4.2.9. *Denote by $q^w_{k,\ell}(i)$ the polynomial $\widetilde{q}^w_{k,\ell}$ after $i-1$ instances of the while loop in Algorithm 4.1. Then, $q^w_{k,\ell}(i) = Q_i x_{k,\ell+i} + R_i$ where*

$$Q_i := q^w_{\ell+i,\ell} + \sum_{j=1}^{i-1}\frac{Q_j}{C_j}q^w_{\ell+i,\ell+j}, \quad R_i := \sum_{j=\ell+i+1}^{k-1} x_{k,j}\left(q^w_{j,\ell} + \sum_{m=1}^{i-1}\frac{Q_m}{C_m}q^w_{j,\ell+m}\right),$$

and each $C_j := \lambda_{w(k)} - \lambda_{w(\ell+j)}$ is nonzero and is the lead coefficient of $q^w_{k,\ell+j}$. Moreover, each Q_i depends only on the $x_{a,b}$ with $a \leq \ell+i$ and $b \geq \ell$ and hence, every Q_i and R_i is less than $x_{k,\ell+i}$ with respect to the term order from Definition 4.1.5.

We defer the proof of Lemma 4.2.9 until after the proof of our main theorem, which we now state.

Theorem 4.2.10. *Suppose that* S *has* $n-1$ *distinct eigenvalues. Then Algorithm 4.1 produces a Gröbner basis for* $K_{w,h}$ *with relatively prime initial terms.*

Proof. We need only treat the case that $\lambda_{w(k)} = \lambda_{w(\ell)}$ with $k > h(\ell)$, otherwise we are in the setting of Corollary 4.2.4. It then follows from Definition 4.1.5 and Proposition 4.1.11 that $\mathrm{in}_{\prec_n}(q^w_{k,\ell}) = -x_{k,\ell+1} \cdot \mathrm{in}_{\prec_n}(q^w_{\ell+1,\ell})$. If $q^w_{\ell+1,\ell}$ is itself a generator for $K_{w,h}$ and we claim that we are done. Indeed, for this to occur, we must have $h(\ell) = \ell$, in which case, each of the terms in the summand in Proposition 4.1.11 already lie in $K_{w,h}$ and hence we may freely remove $q^w_{k,\ell}$ from the generating set.

So assume that $h(\ell) > \ell$. If we also have $h(\ell+1) \geq k$, then $x_{k,\ell+1}$ does not divide the initial term of any generator of $K_{w,h}$, so the generators will have relatively prime initial terms.

Otherwise, we have that $q^w_{k,\ell+1}$ is a generator of $K_{w,h}$, which has initial monomial $x_{k,\ell+1}$ and we may apply Algorithm 4.1 to replace $q^w_{k,\ell}$ in the generating set by some

$$\widetilde{q}^w_{k,\ell} = Q_i x_{k,\ell+i} + R_i.$$

In particular, the initial term of $\widetilde{q}^w_{k,\ell}$ is a monomial multiple of $x_{k,\ell+i}$. We may freely assume that the monomial does not descend from a product involving other generators of the ideal, for if it did, then we could remove this product. So, since $Q_i \prec_n x_{k,\ell+i}$, it suffices to show that Q_i is squarefree. We proceed by induction; the base case holds because of Lemma 4.2.5.

For the inductive step, we need only show that it is impossible to have some $j \in [i-1]$ for which there is a variable $x_{a,b}$ appearing in both Q_j and $q^w_{j,\ell+j}$. Assume that such an $x_{a,b}$ exists. Lemma 4.1.7 says that $b \geq \ell+j$. Moreover, our variables must always satisfy $a > b$, so we must have that $a > \ell+j$. The condition from Lemma 4.2.9 requires $a \leq \ell+j$, which is a contradiction. Hence no such $x_{a,b}$ exists, so every Q_i is squarefree. ∎

Proof of Lemma 4.2.9. Proceed by induction on i. The base case is a straightforward computation. Indeed, in this case, Proposition 4.1.11 tells us that because $\lambda_{w(k)} = \lambda_{w(\ell)}$,

$$-q^w_{k,\ell} = x_{k,\ell+1} q^w_{\ell+1,\ell} + \sum_{j=\ell+2}^{k-1} x_{k,j} q^w_{j,\ell} = Q_1 x_{k,\ell+1} + R_1.$$

Note that when $i = 1$, both the summation in the definition of Q_i and the nested summation in the definition of R_i are empty.

Now assume that the result holds for i. That is, we have $q^w_{k,\ell}(i) = Q_i x_{k,\ell+i} + R_i$ for Q_i and R_i given above. Algorithm 4.1 instructs us to derive $q^w_{k,\ell}(i+1)$ from $q^w_{k,\ell}(i)$ by using $q^w_{k,\ell+i+1}$. Doing so, and again using Proposition 4.1.11, we have that,

$$\begin{aligned} q^w_{k,\ell}(i) &= Q_i x_{k,\ell+i} + R_i \\ &= \frac{Q_i}{C_i}\left[C_i x_{k,\ell+i} - \sum_{j=\ell+i+1}^{k-1} x_{k,j} q^w_{j,\ell+i}\right] + \frac{Q_i}{C_i} \sum_{j=\ell+i+1}^{k-1} x_{k,j} q^w_{j,\ell+i} + R_i, \end{aligned}$$

so,

$$\begin{aligned}
q^w_{k,\ell}(i+1) &= \frac{Q_i}{C_i}\sum_{j=\ell+i+1}^{k-1} x_{k,j}q^w_{j,\ell+i} + \sum_{j=\ell+i+1}^{k-1} x_{k,j}\left(q^w_{j,\ell} + \sum_{m=1}^{i-1}\frac{Q_m}{C_m}q^w_{j,\ell+m}\right)\\
&= x_{k,\ell+i+1}\left(\frac{Q_i}{C_i}q^w_{\ell+i+1,\ell+i} + q^w_{\ell+i+1,\ell} + \sum_{m=1}^{i-1}\frac{Q_m}{C_m}q^w_{j,\ell+m}\right)\\
&\quad + \frac{Q_i}{C_i}\sum_{j=\ell+i+2}^{k-1} x_{k,j}q^w_{j,\ell+i} + \sum_{j=\ell+i+2}^{k-1} x_{k,j}\left(q^w_{j,\ell} + \sum_{m=1}^{i-1}\frac{Q_m}{C_m}q^w_{j,\ell+m}\right).
\end{aligned}$$

After rearranging by collecting the $x_{k,j}$'s, we have:

$$q^w_{k,\ell}(i+1) = x_{k,\ell+i+1}\left(q^w_{\ell+i+1,\ell} + \sum_{j=1}^{i}\frac{Q_i}{C_i}q^w_{j,\ell+j}\right) + \sum_{j=\ell+i+2}^{k-1} x_{k,j}\left(q^w_{j,\ell} + \sum_{m=1}^{i}\frac{Q_m}{C_m}q^w_{j,\ell+m}\right),$$

which are exactly the desired formulae for Q_{i+1} and R_{i+1}.

To see that Q_{i+1} and R_{i+1} are less than $x_{k,\ell+i+1}$ with respect to the order $\prec_n$ from Definition 4.1.5, notice that the $q^w_{a,b}$ appearing explicitly in the formulae for each must have $a < k$. So we are done by induction after applying Lemma 4.1.7.

We use induction one more time, now showing that the only variables appearing in Q_i are the $x_{a,b}$ with $a \le \ell + i$ and $b \ge \ell$. The base case follows from the definition of Q_1 and the computation in the proof of Corollary 4.1.4. For the inductive step, apply the induction hypothesis and Lemma 4.1.7 to the formula for Q_{i+1} above. ■

We conclude this chapter with a trio of examples that illustrate the delicate nature of the ideals $K_{w,h}$. The first shows that Algorithm 4.1 does not work when S has exactly $n-2$ eigenvalues. The second example shows that the semisimple patch ideals are, in general, worse than those in the regular semisimple and regular nilpotent cases, in a way we make precise later. Our final example demonstrates that there does not exist a global lexicographic order on the coordinate ring of the flag variety with respect to which the $q^w_{k,\ell}$ form a Gröbner basis. Despite this trio of examples, the reader should note that every $K_{w,h}$ seen in the following examples is indeed radical, so despite the issues above, we do not have a counterexample to Insko and Precup's conjecture.

Example 4.2.11. Let S be a semisimple operator with $\lambda_1 = \lambda_4 = \lambda_5$ and consider $K_{w,h}$ with $h = (2,4,4,4)$ and $w = id$. We have that $K_{w,h} = \langle q^{id}_{5,1},\, q^{id}_{4,1}\rangle$ where

$$\begin{aligned}
q^{id}_{5,1} &= (\lambda_1-\lambda_2)x_{5,2}x_{2,1} + (\lambda_1-\lambda_3)x_{5,3}x_{3,1} + (\lambda_2-\lambda_1)x_{5,3}x_{3,2}x_{2,1}\\
&\quad + (\lambda_2-\lambda_1)x_{5,4}x_{4,2}x_{2,1} + (\lambda_3-\lambda_1)x_{5,4}x_{4,3}x_{3,1} + (\lambda_1-\lambda_2)x_{5,4}x_{4,3}x_{3,2}x_{2,1},\\
q^{id}_{4,1} &= (\lambda_1-\lambda_2)x_{4,2}x_{2,1}(\lambda_1-\lambda_3)x_{4,3}x_{3,1} + (\lambda_2-\lambda_1)x_{4,3}x_{3,2}x_{2,1}.
\end{aligned}$$

With respect to $\prec_n$, the initial monomials are $x_{5,2}x_{2,1}$ and $x_{4,2}x_{2,1}$, so Lemma 4.2.3 does not apply. In particular, these generators do not form a Gröbner basis with respect to $\prec_n$. Indeed, the reader can verify that the initial monomial of $x_{5,2}q^{id}_{4,1} - x_{4,2}q^{id}_{5,1}$ is $x_{5,2}x_{4,3}x_{3,2}$ which does not lie in the ideal generated by the initial terms of the $q^{id}_{4,1}$ and $q^{id}_{5,1}$. $\diamondsuit$

We next contrast the semisimple case with the regular nilpotent and regular semisimple cases. Abe, DeDieu, Galetto, and Harada showed that the regular nilpotent patch ideals are indeed radical [ADGH18, Lemma 3.12], as well as the following.

Proposition 4.2.12 ([ADGH18, Lemma 3.11])**.** *Regular nilpotent Hessenberg varieties are local complete intersections.*

However, an analogous result does not hold in the semisimple setting.

Example 4.2.13. Let $h = (3,4,5,5,5)$ and consider the semisimple operator $\mathsf{S} = \text{diag}(1,1,2,1,1)$. Then the ideal $K_{w,h}$ at the identity chart is

$$\begin{aligned} K_{id,h} &= \langle -x_{5,3}x_{3,1} + x_{5,4}x_{4,3}x_{3,1},\ x_{4,3}x_{3,1},\ x_{5,3}x_{3,2} - x_{5,4}x_{4,3}x_{3,2}\rangle \\ &= \langle x_{5,3}x_{3,1},\ x_{4,3}x_{3,1},\ x_{5,3}x_{3,2} - x_{5,4}x_{4,3}x_{3,2}\rangle \,. \end{aligned}$$

The generators given in the first line are (up to sign) the natural generators for $K_{w,h}$ as described in Definition 2.4.11 and Proposition 4.1.11.

Some remarks are in order. First, and most important with regard to Conjecture 4.2.1, is that $K_{id,h}$ is a radical ideal. However, neither Algorithm 4.1 nor a natural modification thereof can be used to produce an alternate set of generators with distinct initial terms. So it is not immediately obvious whether Lemma 4.2.3 can be applied in this example.

Moreover, this ideal is not a complete intersection. Indeed, the reader can verify (for instance, in *Macaulay2* [GS]) that $K_{id,h}$ has height two but the generators above form a minimal generating set. $\diamondsuit$

Recall that the two approaches in the literature to show that patch ideals are radical is to either show that they are generically reduced or construct a Gröbner basis with squarefree lead terms. Our second example demonstrates the necessity of a novel approach in the semisimple case for the natural generators do not form a Gröbner basis with respect to a common lexicographic monomial order. We will make use of *Macualay2* and the `StatePolytope` package [GS; Swi].

Example 4.2.14. First let $h = (4,4,5,5,5)$ and S a semisimple operator with $\lambda_1 = \lambda_2 = \lambda_5$. Then we have that $K_{id,h} = \langle q^{id}_{5,1},\ q^{id}_{5,2}\rangle$ where

$$\begin{aligned} q^{id}_{5,1} &= (\lambda_1 - \lambda_3)x_{5,3}x_{3,1} + (\lambda_1 - \lambda_4)x_{5,4}x_{4,1} + (\lambda_3 - \lambda_1)x_{5,4}x_{4,3}x_{3,1} \\ q^{id}_{5,2} &= (\lambda_1 - \lambda_3)x_{5,3}x_{3,2} + (\lambda_1 - \lambda_4)x_{5,4}x_{4,2} + (\lambda_3 - \lambda_1)x_{5,4}x_{4,3}x_{3,2}. \end{aligned}$$

The `StatePolytope` package for *Macaulay2* says that these generators form a Gröbner basis for $K_{id,h}$ if there exists some monomial order $<$ for which the initial terms, up to scalar multiple, are

$$\text{in}_<(q^{id}_{5,1}) = x_{5,3}x_{3,1} \quad \text{and} \quad \text{in}_<(q^{id}_{5,2}) = x_{5,4}x_{4,2}$$

or an order $<$ for which the initial terms are (up to scalar multiple)

$$\text{in}_<(q^{id}_{5,1}) = x_{5,4}x_{4,1} \quad \text{and} \quad \text{in}_<(q^{id}_{5,2}) = x_{5,3}x_{3,2}.$$

Hence we have the following four options for lexicographic monomial orders.

- $x_{3,1} > x_{4,2} > x_{5,3} >$ all remaining variables
- $x_{4,2} > x_{3,1} > x_{5,3} >$ all remaining variables
- $x_{3,2} > x_{4,1} > x_{5,3} >$ all remaining variables
- $x_{4,1} > x_{3,2} > x_{5,3} >$ all remaining variables

Now consider $K'_{w,h}$ for $h = (3,5,5,5,5)$ and $w = 14325 = (24)$. We then have that $K'_{w,h} = \langle q^w_{4,1},\, q^w_{5,1}\rangle$ where

$$\begin{aligned} q^w_{4,1} &= (\lambda_1 - \lambda_2)x_{4,2}x_{2,1} + (\lambda_1 - \lambda_3)x_{4,3}x_{3,1} + (\lambda_2 - \lambda_1)x_{4,3}x_{3,2}x_{2,1} \\ q^w_{5,1} &= (\lambda_1 - \lambda_2)x_{5,2}x_{2,1} + (\lambda_1 - \lambda_3)x_{5,3}x_{3,1} + (\lambda_2 - \lambda_1)x_{5,3}x_{3,2}x_{2,1} \\ &\qquad + (\lambda_2 - \lambda_1)x_{5,4}x_{4,2}x_{2,1} + (\lambda_3 - \lambda_1)x_{5,4}x_{4,3}x_{3,1} + (\lambda_1 - \lambda_2)x_{5,4}x_{4,3}x_{3,2}x_{2,1}. \end{aligned}$$

The `StatePolytope` package can again be used to compute the possible initial ideals of $K'_{w,h}$ and the reader can verify that none of our candidate orders above yield that $\{q^w_{4,1}, q^w_{5,1}\}$ is a Gröbner basis for $K'_{w,h}$. We conclude that there does not exist a global lexicographic order with respect to which the natural generators $q^w_{k,\ell}$ form a Gröbner basis for every $K_{w,h}$. $\diamondsuit$

Chapter 5

Conclusion

This thesis concerns the local study of Hessenberg varieties in type A using patch ideals. In the setting of Hessenberg varieties, this approach was introduced by Insko and Yong [IY12] but their use dates back to the study of Schubert varieties.

For the case of a regular nilpotent Hessenberg variety, Da Silva and Harada showed that the local defining ideal at the chart containing the longest-word permutation is a triangular complete intersection, a complete intersection for which the generators satisfy additional relations and as a result form a Gröbner basis. In Chapter 3 of this thesis, we discuss the approach from [CDHR24] that shows that the local defining equations of regular nilpotent Hessenberg Schubert cells are also triangular complete intersections. Our method constructs an embedding that preserves triangular complete intersections. As a result, we conclude that regular nilpotent Hessenberg Schubert cells are also geometrically vertex decomposable, compatibly Frobenius split, and compute the Hilbert series of the local defining ideals with respect to a nonstandard grading arising from a torus action.

Moving our attention to the case of semisimple Hessenberg varieties, in Chapter 4 we show that regular semisimple Hessenberg varieties also are triangular complete intersections. Then in the semisimple (not necessarily regular case), we discuss a conjecture of Insko and Precup that these set-theoretic defining ideals agree scheme-theoretically, that is, that these ideals are radical [IP19]. Making use of our work in the regular semisimple case, we give a positive answer to the conjecture of Insko and Precup in several special cases including an analogous case to that treated in the thesis of Atar [Ata23]. The main result of Section 4.2 is that their conjecture holds for any patch of any semisimple Hessenberg variety in $\mathrm{Flags}(\mathbb{K}^n)$ when the semisimple operator has exactly $n-1$ eigenvalues.

To conclude this thesis we provide future research directions for the study of Hessenberg varieties using patch ideals. The conjecture of Insko and Precup [IP19] that the set-theoretic ideals in the semisimple case agree scheme-theoretically remains open for the case of a semisimple operator $\mathsf{S} : \mathbb{K}^n \to \mathbb{K}^n$ with between 2 and $n-2$ eigenvalues (inclusive). A related question is to ask whether the natural generators form a Gröbner basis for these ideals with respect to a lexicographic monomial order. Or, to answer the question using Frobenius splitting, on each semisimple patch, does the exist a compatible Frobenius splitting? Can we strengthen this to a (non-canonical) splitting of the flag variety that descends to a compatible splitting on each semisimple patch?

In the case of regular nilpotent Hessenberg Schubert cells, we computed the Hilbert series of these triangular complete intersection ideals. To that end, we found a nonstandard grading of the polynomial ring with respect to which each triangular complete intersection ideal was homogeneous. Can we perform a similar computation in the regular semisimple case, where we also know that the patch ideals are triangular complete intersections? It is known that both the $\mathbf{S}$- and $\mathbf{T}$-actions discussed in Section 2.4.3 preserve regular semisimple Hessenberg varieties (see [AHHM17, Section 2]), so do either of these give rise to a grading for which the patch ideals are homogeneous?

For the setting of regular nilpotent Hessenberg varieties, it is known that the patch ideals at w_0 are triangular complete intersections [DH23]. Moreover, regular nilpotent Hessenberg varieties of codimension 1 in the flag variety have patch ideals that are compatibly Frobenius split [Ata23]. As discussed in Section 3.4, it remains an open problem to construct Gröbner bases for each of the regular nilpotent non-w_0-patches. If there does exist a monomial order with respect to which the natural generators form a Gröbner basis, perhaps depending on the choice of w, does this generalize Da Silva and Harada's monomial order? Similarly, and working now over a field of finite characteristic, can we construct a Frobenius splitting on each chart of the flag variety that compatibly splits the corresponding Hessenberg patch ideal? We can also ask to strengthen this to a (non-canonical) Frobenius splitting on the whole flag variety that descends to a compatible splitting of each patch ideal.

Work of Abe, Fujita, and Zeng [AFZ20] and Insko, Tymoczko, and Woo [ITW20] use patch ideals to study the cohomology and K-theory of regular Hessenberg varieties. We can then ask the same questions about Gröbner bases, Frobenius splitting, and so on in the regular setting. Similar questions can again be asked for other families of Hessenberg varieties, for instance, for nilpotent Hessenberg varieties. This thesis and much of the Hessenberg patch ideal literature (including [IY12; ADGH18; ITW20; DH23; CDHR24] and some of [IP19]) restrict to Lie type A where we make the identification Flags$(\mathbb{K}^n) \cong \mathbf{GL}_n(\mathbb{K})/B$. More generally we may identify the flag variety with G/B for an arbitrary complex semisimple Lie algebra G and its corresponding Borel subgroup B. We may then ask the same question in other Lie types.

Bibliography

[ADGH18] Hiraku Abe, Lauren DeDieu, Federico Galetto, and Megumi Harada. Geometry of Hessenberg varieties with applications to Newton-Okounkov bodies. *Selecta Mathematica, New Series* 24 (July 2018), pp. 2129–2163.
Cited on pages 1, 2, 18, 25–28, 42, 51, 56, 59

[AFZ20] Hiraku Abe, Naoki Fujita, and Haozhi Zeng. Geometry of regular Hessenberg varieties. *Transformation Groups* 25 (Mar. 2020), pp. 305–333.
Cited on pages 1, 2, 59

[AHHM17] Hiraku Abe, Megumi Harada, Tatsuya Horiguchi, and Mikiya Masuda. The cohomology rings of regular nilpotent Hessenberg varieties in Lie type A. *International Mathematics Research Notices* 2019.17 (2017), pp. 5316–5388.
Cited on pages 34, 42, 59

[AH20] Hiraku Abe and Tatsuya Horiguchi. A Survey of Recent Developments on Hessenberg Varieties. In: *Schubert Calculus and Its Applications in Combinatorics and Representation Theory*. Ed. by Jianxun Hu, Changzheng Li, and Leonardo C. Mihalcea. Vol. 332. Springer Proceedings in Mathematics & Statistics. Springer, Singapore, Oct. 2020, pp. 251–279.
Cited on pages 42, 44

[AI22] Hiraku Abe and Erik Insko. On singularity and normality of regular nilpotent Hessenberg varieties. Preprint, available at arxiv.org/abs/2212.13817. Dec. 2022.
Cited on pages 1, 25, 42

[AT10] Dave Anderson and Julianna Tymoczko. Schubert polynomials and classes of Hessenberg varieties. *Journal of Algebra* 323.10 (May 2010), pp. 2605–2623.
Cited on pages 1, 42

[Ata23] Büşra Atar. Hessenberg Patch Ideals of Codimension 1. MSc thesis. McMaster University, Apr. 2023.
Cited on pages 1, 25, 43, 51, 58, 59

[AM69] Michael F. Atiyah and Ian Grant Macdonald. *Introduction to Commutative Algebra*. Addison–Wesley Series in Mathematics. Addison–Wesley Publishing Company, Inc., 1969.
Cited on page 11

[BK05] Michel Brion and Shrawan Kumar. *Frobenius Splitting Methods in Geometry and Representation Theory*. Vol. 231. Progress in Mathematics. Birkhäuser Boston, MA, 2005.
Cited on pages 13, 14

[BC18] Patrick Brosnan and Timothy Y. Chow. Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties. *Advances in Mathematics* 329 (Apr. 2018), pp. 955–1001.
Cited on pages 1, 44

[Buc06] Bruno Buchberger. Bruno Buchberger's PhD thesis 1965: An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. *Journal of Symbolic Computation* 41.3–4 (Mar. 2006), pp. 475–511.
Cited on page 8

[CPSU23] Mahir Bilen Can, Martha Precup, John Shareshian, and Ölzem Uğurlu. K-orbit closures and Hessenberg varieties. Preprint, available at arxiv.org/abs/2309.05770. Sept. 2023.
Cited on page 44

[CDH05] Marta Casanellas, Elena Drozd, and Robin Hartshorne. Gorenstein liaison and ACM sheaves. *Journal für die reine und angewandte Mathematik* 2005.584 (July 2005), pp. 149–171.
Cited on page 12

[Che91] Claude Chevalley. Algebraic Groups and Their Generalizations: Classical Methods. In: ed. by William J. Haboush and Brian J. Parshall. Vol. 56, Part 1. Proceedings of Symposia in Pure Mathematics. American Mathematical Society, July 1991. Chap. Sur les décompositions cellulaires des espaces G/B.
Cited on page 40

[CLO05] David A. Cox, John Little, and Donal O'Shea. *Using Algebraic Geometry*. 2nd edition. Vol. 185. Graduate Texts in Mathematics. Springer New York, NY, 2005.
Cited on page 8

[CLO15] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms*. 4th edition. Undergraduate Texts in Mathematics. Springer Cham, 2015.
Cited on pages 8–10, 13, 16, 38, 51

[CDHR24] Mike Cummings, Sergio Da Silva, Megumi Harada, and Jenna Rajchgot. Gröbner geometry for regular nilpotent Hessenberg Schubert cells. *Journal of Pure and Applied Algebra* 228.7 (July 2024).
Cited on pages 1, 10, 15, 22, 23, 25, 26, 33–41, 58, 59

[CV] Mike Cummings and Adam Van Tuyl. GeometricDecomposability: A package to check whether ideals are geometrically vertex decomposable. Version 1.4. A *Macaulay2* package available at macaulay2.com.
Cited on page 32

[DH23] Sergio Da Silva and Megumi Harada. Geometric vertex decomposition, Gröbner bases, and Frobenius splittings for regular nilpotent Hessenberg varieties. *Transformation Groups* (July 2023).
Cited on pages 1, 12, 14, 15, 25–27, 29, 30, 37, 40, 46, 59

[dLP88] Corrado de Concini, George Lusztig, and Claudio Procesi. Homology of the zero-set of a nilpotent vector field on a flag manifold. *Journal of the American Mathematical Society* 1.1 (1988), pp. 15–34.
Cited on page 39

[DPS92] Filippo De Mari, Claudio Procesi, and Mark A. Shayman. Hessenberg varieties. *Transactions of the American Mathematical Society* 332.2 (Aug. 1992), pp. 529–534.
Cited on pages 1, 2, 18, 44, 45

[DS88] Filippo De Mari and Mark A. Shayman. Generalized Eulerian numbers and the topology of the Hessenberg variety of a matrix. *Acta Applicandae Mathematica* 12 (July 1988), pp. 213–235.
Cited on page 18

[Dre15] Elizabeth J. Drellich. Combinatorics of Equivariant Cohomology: Flags and Regular Nilpotent Hessenberg Varieties. PhD thesis. University of Massachusetts Amherst, Mar. 2015.
Cited on pages 1, 21, 45

[DF04] David S. Dummit and Richard M. Foote. *Abstract Algebra*. 3rd edition. John Wiley & Sons, Inc., 2004.
Cited on page 14

[Eis04] David Eisenbud. *Commutative Algebra with a View Towards Algebraic Geometry*. Vol. 150. Graduate Texts in Mathematics. Springer New York, NY, 2004.
Cited on pages 4–6, 10, 51

[EPS22] Laura Escobar, Martha Precup, and John Shareshian. Hessenberg varieties of codimension one in the flag variety. Preprint, available at arxiv.org/abs/2208.06299. 2022.
Cited on pages 1, 2

[Fun03] Francis Y.C. Fung. On the topology of components of some Springer fibers and their relation to Kazhdan–Lusztig theory. *Advances in Mathematics* 178.2 (Sept. 2003), pp. 244–276.
Cited on page 1

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at macaulay2.com.
Cited on pages 32, 56

[Gro67] Alexander Grothendieck. *Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphisms de schémas, Quatrième partie*. Vol. 32. Publications Mathématiques de l'IHÉS, 1967.
Cited on page 15

[Gua16] Mathieu Guay-Paquet. A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra. Unpublished, avaialble at arxiv.org/abs/1601.05498. Jan. 2016.
Cited on pages 1, 44

[HT11] Megumi Harada and Julianna Tymoczko. A positive Monk formula in the S^1-equivariant cohomology of type A Peterson varieties. *Proceedings of the London Mathematical Society* 103.1 (Jan. 2011), pp. 40–72.
Cited on page 1

[HT17] Megumi Harada and Julianna Tymoczko. Poset pinball, GKM-compatible subspaces, and Hessenberg varieties. *Jounal of the Mathematical Soceity of Japan* 69.3 (July 2017), pp. 945–994.
Cited on pages 22, 33

[Har92] Joe Harris. *Algebraic Geometry: A First Course*. Vol. 133. Graduate Texts in Mathematics. Springer-Verlag, 1992.
Cited on page 16

[Har77] Robin Hartshorne. *Algebraic Geometry*. Vol. 52. Graduate Texts in Mathematics. Springer New York, NY, 1977.
Cited on page 15

[Hun96] Craig L. Huneke. *Tight Closure and its Applications*. Regional Conference Series in Mathematics 88. American Mathematical Society, 1996.
Cited on page 13

[IP19] Erik Insko and Martha Precup. The singular locus of semisimple Hessenberg varieties. *Journal of Algebra* 521 (Mar. 2019), pp. 65–96.
Cited on pages 1, 2, 25, 44, 50, 58, 59

[IT16] Erik Insko and Julianna Tymoczko. Intersection theory of the Peterson variety and certain singularities of Schubert varieties. *Geometriae Dedicata* 180 (2016), pp. 95–116.
Cited on page 1

[ITW20] Erik Insko, Julianna Tymoczko, and Alexander Woo. A formula for the cohomology and K-class of a regular Hessenberg variety. *Journal of Pure and Applied Algebra* 224.5 (May 2020), 14pp.
Cited on pages 1, 2, 59

[IY12] Erik Insko and Alexander Yong. Patch ideals and Peterson varieties. *Transformation Groups* 17 (Sept. 2012), pp. 1011–1036.
Cited on pages 1, 2, 16, 25, 42, 51, 58, 59

[KR21] Patricia Klein and Jenna Rajchgot. Geometric vertex decomposition and liaison. *Forum of Mathematics, Sigma* 9.e70 (Oct. 2021), pp. 1–23.
Cited on pages 11, 12

[KMMNP01] Jan O. Kleppe, Juan C. Migliore, Rosa Miró-Roig, Uwe Nagel, and Chris Peterson. Gorenstein Liaison, Complete Intersection Liaison Invariants and Unobstructedness. *Memoirs of the American Mathematical Society* 154.732 (Nov. 2001), pp. vii+116.
Cited on page 12

[KMY09] Allen Knutson, Ezra Miller, and Alexander Yong. Gröbner geometry of vertex decompositions and of flagged tableaux. *Journal für die reine und angewandte Mathematik* 630 (Mar. 2009), pp. 1–31.
Cited on pages 10–12

[Kos96] Bertram Kostant. Flag manifold quantum cohomology, the Toda lattice, and the representation with highest weight ρ. *Selecta Mathematica, New Series* 2.1 (1996), pp. 43–91.
Cited on page 39

[Lan65] Serge Lang. *Algebra*. Addison–Wesley Series in Mathematics. Addison–Wesley Publishing Company, Inc., 1965.
Cited on page 11

[Mat86] Hideyuki Matsumura. *Commutative ring theory*. Vol. 8. Cambridge studies in advanced mathematics. Cambridge University Press, 1986.
Cited on pages 4, 6, 11, 15

[MR85] Vikram B. Mehta and A. Ramanathan. Frobenius splitting and cohomology vanishing for Schubert varieties. *Annals of Mathematics, Second Series* 122.1 (July 1985), pp. 27–40.
Cited on page 13

[MS05] Ezra Miller and Bernd Sturmfels. *Combinatorial Commutative Algebra*. Vol. 227. Graduate Texts in Mathematics. Springer New York, NY, 2005.
Cited on pages 7, 16, 23, 24

[NRV23] Thái Thàn Nguyễn, Jenna Rajchgot, and Adam Van Tuyl. Three invariants of geometrically vertex decomposable ideals. Preprint, available at arxiv.org/abs/2311.08541. Nov. 2023.
Cited on page 12

[Pee11] Irena Peeva. *Graded Syzygies*. Vol. 14. Algebra and Applications. Springer London, 2011.
Cited on pages 7, 8

[PB80] J. Scott Provan and Louis J. Billera. Decompositions of simplicial complexes related to diameters of convex polyhedra. *Mathematics of Operations Research* 5.4 (Nov. 1980), pp. 576–594.
Cited on page 10

[Sha13] Igor R. Shafarevich. *Basic Algebraic Geometry 1: Varieties in Projective Space*. 3rd edition. Springer Berlin, Heidelberg, 2013.
Cited on page 16

[SW12] John Shareshian and Michelle L. Wachs. Chromatic quasisymmetric functions and Hessenberg varieties. In: *Configuration spaces*. Ed. by A. Bjorner, F. Cohen, C. de Concini, C. Procesi, and M. Salvetti. CRM Series. Edizioni della Normale, Pisa, 2012, pp. 433–460.
Cited on pages 1, 44

[SZ15] Karen E. Smith and Wenliang Zhang. Frobenius splitting in commutative algebra. In: Mathematical Sciences Research Institution Publications 67. Cambridge University Press, 2015, pp. 291–345.
Cited on page 13

[ST06] Eric Sommers and Julianna Tymoczko. Exponents for B-stable ideals. *Transactions of the American Mathematical Society* 358.8 (Aug. 2006), pp. 3493–3509.
Cited on pages 42, 44

[Spr76] Tonny A. Springer. Trigonometric sums, green functions of finite groups and representations of Weyl groups. *Inventiones Mathematicae* 36 (Dec. 1976), pp. 173–207.
Cited on page 1

[Sta75] Richard P. Stanley. The upper bound conjecture and Cohen-Macaulay rings. *Studies in Applied Mathematics* LIV.2 (1975), pp. 135–142.
Cited on page 6

[Sta95] Richard P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Advances in Mathematics* 111.1 (Mar. 1995), pp. 166–194.
Cited on page 44

[SS93] Richard P. Stanley and John R. Stembridge. On immanants of Jacobi–Trudi matrices and permutations with restricted position. *Journal of Combinatorial Theory, Series A* 62.2 (Mar. 1993), pp. 261–279.
Cited on page 44

[SB80] Josef Stoer and Roland Bulirsch. *Introduction to Numerical Analysis*. Springer New York, NY, 1980.
Cited on page 18

[Swi] Dave Swinarski. StatePolytope: the state polytope of an ideal. Version 1.2. A *Macaulay2* package available at macaulay2.com.
Cited on pages 32, 56

[Tym04] Julianna Tymoczko. Decomposing Hessenberg Varieties over Classical Groups. PhD thesis. Princeton University, June 2004.
Cited on page 42

[Tym06] Julianna Tymoczko. Linear conditions imposed on flag varieties. *American Journal of Mathematics* 128.6 (Dec. 2006), pp. 1587–1604.
Cited on pages 21, 42

[Tym07] Julianna Tymoczko. Paving Hessenberg varieties by affines. *Selecta Mathematica, New Series* 13 (Sept. 2007), pp. 353–367.
Cited on pages 2, 39, 40, 42

[Tym08] Julianna Tymoczko. Permutation representations on Schubert varieties. *American Journal of Mathematics* 130.5 (Oct. 2008), pp. 1171–1194.
Cited on page 1

[Wat12] John J. Watkins. Krull Dimension of Polynomial and Power Series Rings. In: *Progress in Commutative Algebra 2: Closures, Finiteness and Factorization*. Ed. by C. Francisco, L. C. Klinger, S. M. Sather-Wagstaff, and J. C. Vassilev. Progress in Mathematics. De Gruyter, 2012, pp. 205–219.
Cited on page 6

[Yun17] Zhiwei Yun. Lectures on Springer theories and orbital integrals. In: *Geometry of Moduli Spaces and Representation Theory*. Ed. by Roman Bezrukavnikov, Alexander Braverman, and Zhiwei Yun. Vol. 24. IAS/Park City Mathematics Series. American Mathematical Society and IAS/Park City Mathematics Institute, 2017.
Cited on page 1