DERIVATION OF MOMENTS DURING A BUSY PERIOD

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By

ROCKY YUK-KEUNG FAN, B.Sc.

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AUTHOR: Rocky Yuk-keung Fan, B.Sc. (McMaster University)

Supervisor: Dr. S.G. Mohanty

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ABSTRACT

The purpose of this project is to derive the first two moments of two random variables, that is, the number served during and the length of a busy period. Two single-server models are discussed in this project, namely, the $M_b^c/E_a/1$ model, and the $M_b/M_b/1$ model. Moreover, in the development, standard methods such as the moment generating function technique are used, application of a computer system will also be introduced.

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CHAPTER 1: PRELIMINARIES

1.1 Introduction

Queueing theory is the mathematical study of queues. It attempts to formulate and interpret different mathematical models for the purpose of better understanding of queueing systems. Its ultimate goal, as S. F. Hillier and G. J. Lieberman [3] mentioned, is to achieve an economic balance between the cost of service and the cost associated with waiting for that service.

The first major study on the queueing theory started in 1908 by A. K. Erlang [2], a Danish engineer who worked for the Copenhagen Telephone Company. His work on the queueing theory concerned mainly with problems of telephone operations. Nowadays, the queueing theory is a well developed branch of the modern applied probability, applied in numerous varied fields. Literature on this subject continues to grow rapidly.

As the word "queue" is mentioned, a picture of a group of people (customers) lining up to be served by another group of people (servers) will usually come to our minds. However, in reality, customers and servers do not necessarily need to be people, or that there is a visible existence of queues. For example, a computer system (server) might execute different computer programmes (customers) which queue up in an invisible form. Therefore, in the queueing theory, customers and servers refer to units demanding services and facilities providing services respectively.

The basic mechanism of a simple queueing process is as follows: (1) Customers arrive from some input sources according to a certain probabilistic law, for instance, a Poisson process. (2) The customers wait in the system, and are served by servers following a given queue discipline, for example, on a first-come-first-serve basis. (3) The required time for each individual service is governed by a certain probabilistic law, for example, exponentially distributed service time. After being served, the customers depart.

Once the queueing model is specified, it may be possible to analyze the model mathematically, in order to predict some aspects of the behaviour of the system. In particular, customers and servers will be most concerned with the duration of the waiting time and of the busy period respectively; management will probably pay more attention to the size of the queue.

Generally, as a system begins to operate, it is greatly affected by the initial conditions and the amount of time elapsed. The system is considered to be in a transient state. However, after sufficient time being elapsed, it may become essentially independent from the initial state and time elapsed. If this happens, it is said to be in a steady state. As a result, the study of queueing theory involves two major branches: non-steady state (transient) behaviour and steady state (equilibrium) behaviour.

A queueing process under some general conditions tends toward equilibrium irrespective of the initial state and its study is done through a set of so-called balance equations. In contrast with the study of equilibrium behaviour, that of the transient behaviour is rather more complicated. However, the investigation of the transient behaviour of queueing processes are also important, not only from the theory point of view, but also in the applications, for example, to determine the distribution of the length of a busy period. By the term "busy period", we mean a time interval during which all servers of the queueing system are occupied continuously without a break.

In studying the behaviour of a busy period, different methods have been introduced, namely, the spectral theory developed by W. Ledermann and G. E. H. Reuter [6], the standard generating function techniques by N. T. J. Bailey [1], and the combinatorial methods by L. Takâcs [10]. However, though analytical formulae can be found in a vast amount of literature, most of them exist in rather complicated forms which, as a result, cannot provide a clear picture of the distribution of a busy period. Furthermore, in designing a queueing system, one will generally find that it is sufficient to know just the mean and variance of the distribution.

The objective of this project is therefore to derive, through the adoption of the combinatorial method, the first two moments of the distribution of two random variables, the number served during and the length of a busy period for two single-server queueing models. These are, the $M_b^c/E_a/1$ model (Poisson input and Erlangian service time distribution with a stages, arrivals in batches (in short "b") from c independent sources. For a complete description of notations, see Appendix A), and the $M_b/M_b/1$ model (Poisson input and exponentially distributed service time, arrivals and services are both in batches). Note that the $M/E_a/1$ model (Poisson input and Erlangian service time distribution with a stages) is a special case of the $M_b^c/E_a/1$ model. Two different methods will be used for finding the moments. The generating function technique will be used for the first model, and an iteration technique will be introduced for the second model.

This chapter is concluded by stating some known mathematical results needed for our purpose. In Chapter 2, the moments for the two models will be derived and presented, in particular, the iteration technique will be explained in detail. Finally, three appendices, one describing the symbols and notations, the others being the demonstrative computer programmes and selected tables to support the iteration method, will be attached at the end.

1.2 Some Mathematical Results

In the early sixties, L. Takács demonstrated in his book [9] and many of his papers [10] [11] [12], his pioneering work in deriving the distribution of the busy period through the application of combinatorial methods.

Within a few years, J. L. Jain and S. G. Mohanty [4] [5] extended Takács' work to the $M_b^c/G/1$ model, in which, (1) customers arrive from c independent input sources, arrival from the i^{th} source is of fixed batch size b_i according to a Poisson process of mean rate L_i, i = 1, ..., c, and are served singly by a single server, (2) the service times are identically and independently distributed, positive random variables with distribution function F(t), and are also independent of the arrival instants.

During a busy period which has initially h ($h \geq 1$) customers, let M_i be the number of batches arriving from the i^{th} input source, i = 1, ..., c, and let T be the length of the busy period. Then the probability $P(M_1=m_1,\ldots,M_c=m_c;T\le t)$ is given by the following proposition [5]:

Proposition 1.1

$$P(M_1=m_1,\ldots,M_c=m_c;T\le t) =$$

$$\frac{h}{n}\int_0^t \prod_{i=1}^{c} [e^{-L_i x}\frac{(L_i x)^{m_i}}{m_i!}]\, dF_n(x) \;;$$

$m_i = 0, 1, \ldots$ for $i = 1, \ldots, c$; $0 \le t < \infty$

where $n = h + \sum_{i=1}^{c} m_i b_i$ and

$F_n(t)$ denotes the n^{th} iterated covolution of $F(t)$.

Note that the $M_b^c/E_a/1$ model which is considered in this project is a special case of the $M_b^c/G/1$ model, Proposition 1.1 will be of interest in the development of Chapter 2. In addition to the Proposition 1.1, a generalized Lagrange's inversion formula, as given in Mohanty [7], will be used for deriving the moments for the model $M_b^c/E_a/1$ and is stated as follows:

Proposition 1.2

$$Z^A = \sum_{m_1=0}^{\infty}\cdots\sum_{m_c=0}^{\infty}\frac{A}{A+\Sigma_i m_i B_i}\binom{A+\Sigma_i m_i B_i}{m_1,\ldots,m_c}\prod_{i=1}^{c} W_i^{m_i}$$

where Σ_i stands for the summation over i from 1 to c,

if and only if Z satisfies the following relation:

$$Z\,(1 - \sum_{i=1}^{C} W_i Z^{B_i - 1}) = 1 \;.$$

Finally, in the $M_b/M_b/1$ model, evaluation of

$$\tau(a,b,t) = \int_0^t e^{-bx}\, x^{a-1}\, b^a\, dx \;;$$

$$a = 1,\, 2,\, \ldots \;; \quad 0 \le b < \infty \;; \quad 0 \le t < \infty$$

is needed. With the use of a computer system, it will be found useful to write the definite integral as the following finite sum, which is obtained by repeatedly applying the technique of integration by parts.

Proposition 1.3

$$\tau(a,b,t) = (a - 1)!\,[1 - \sum_{i=0}^{a-1} \frac{e^{-bt}\,(bt)^i}{i!}] \;.$$

CHAPTER 2: DERIVATION OF MOMENTS

2.1 Introduction

The purpose of this chapter is to derive the first two moments of two single-server models, namely, $M_b^c/E_a/1$ which includes $M/E_a/1$, and $M_b/M_b/1$. Although the higher moments can be found in a similar manner, it is sufficient for our purpose to restrict to the first two moments.

For explanation of notations, one may refer to Appendix A.

2.2 The $M_b^c/E_a/1$ model

The model is a special case of the $M_b^c/G/1$ model (see Proposition 1.1) in which the service time has Erlangian distribution $G(a,U)$, $U/a > \Sigma_i L_i b_i$.

Since $F(t) \sim G(a,U)$ implies that $F_n(t) \sim G(na,U)$, it follows from Proposition 1.1 that

$P(M_1=m_1,\ldots,M_c=m_c;T\le t) =$

$$\frac{h}{n}\int_0^t \pi_i \left[\frac{e^{-L_i x}(L_i x)^{m_i}}{m_i!}\right]\left[\frac{e^{-Ux}\,U^{na}\,x^{na-1}}{(na-1)!}\,dx\right] =$$

$$\frac{h}{n}\;\frac{U^{na}\,\pi_i\,L_i^{m_i}}{(na-1)!\;\pi_i\,m_i!}\int_0^t e^{-(\Sigma_i L_i+U)x}\,x^{\Sigma_i m_i+na-1}\,dx$$

where π_i stands for the product over i from 1 to c. Differentiating it with respect to t and using the Fundamental Theorem of Calculus, we obtain the joint probability density function $g(m_1,\ldots,m_c,t)$ of M_1, ..., M_c, and T as

$g(m_1,\ldots,m_c,t) =$

$$\frac{h}{n}\;\frac{U^{na}\,\pi_i\,L_i^{m_i}}{(na-1)!\;\pi_i\,m_i!}\,e^{-(\Sigma_i L_i+U)t}\,t^{\Sigma_i m_i+na-1}\;; \tag{2.1}$$

$m_i = 0, 1, \ldots$ for $i = 1, \ldots, c$; $0 \le t < \infty$

where

$n = h + \Sigma_i\, m_i b_i$.

To obtain the moments, we define

$H(s_1,\ldots,s_c,r) =$

$$\sum_{m_1=0}^{\infty}\cdots\sum_{m_c=0}^{\infty} s_1^{m_1}\cdots s_c^{m_c}\int_0^\infty e^{rt}\,g(m_1,\ldots,m_c,t)\,dt. \tag{2.2}$$

For simplicity, we will denote $H(s_1,\ldots,s_c,r)$ by H. H may be seen as the joint probability and moment generating function.

Theorem 2.1

H satisfies

$$H^{\frac{1}{ha}} - \Sigma_i\, p_i(r)\, H^{\frac{b_i a+1}{ha}} = p_o(r) \qquad (2.3)$$

where

$$p_o(r) = \frac{U}{\Sigma_j L_j + U - r} ;$$

$$p_i(r) = \frac{L_i s_i}{\Sigma_j L_j + U - r} \text{ for } i = 1, \ldots, c .$$

Proof

From (2.1) and (2.2), we get

$$H =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_c=0}^{\infty} s_1^{m_1} \cdots s_c^{m_c} *$$

$$\int_o^{\infty} e^{rt} \left[\frac{h}{n} \frac{U^{na}\, \pi_i\, L_i^{m_i}}{(na-1)!\, \pi_i\, m_i!} e^{-(\Sigma_i L_i+U)t}\, t^{\Sigma_i m_i+na-1}\right] dt =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_c=0}^{\infty} s_1^{m_1} \cdots s_c^{m_c} *$$

$$\frac{h}{n} \frac{U^{na}\, \pi_i\, L_i^{m_i}}{(na-1)!\, \pi_i\, m_i!} \frac{(\Sigma_i m_i + na - 1)!}{(\Sigma_i L_i + U - r)^{\Sigma_i m_i+na}} =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_c=0}^{\infty} \frac{h}{h + \Sigma_i b_i m_i} \frac{U^{(h+\Sigma_i b_i m_i)a}\, \pi_i\, (L_i s_i)^{m_i}}{[(h + \Sigma_i b_i m_i)a - 1]!\, \pi_i\, m_i!} *$$

$$\frac{[ha + \Sigma_i (b_i a + 1) m_i - 1]!}{(\Sigma_i L_i + U - r)^{ha + \Sigma_i (b_i a + 1) m_i}} =$$

$$\sum_{m_1=0}^{\infty} \cdots \sum_{m_c=0}^{\infty} \frac{ha}{ha + \Sigma_i (b_i a + 1) m_i} *$$

$$\binom{ha + \Sigma_i (b_i a + 1) m_i}{m_1, \ldots, m_c} [p_o(r)]^{ha+\Sigma_j b_j m_j a} \pi_i [p_i(r)]^{m_i} . \quad (2.4)$$

Put $A = ha$, $B_i = b_i a + 1$, and $W_i = [p_o(r)]^{b_i a} p_i(r)$ in Proposition 1.2 and compare it with (2.4), we obtain the following

$$[\frac{1}{p_o(r)} H^{\frac{1}{ha}}] \left[1 - \Sigma_i \{[p_o(r)]^{b_i a} p_i(r) [\frac{1}{p_o(r)} H^{\frac{1}{ha}}]^{b_i a}\} \right] = 1. \quad (2.5)$$

Simplify (2.5) and the result follows.

Theorem 2.2

$$E(N) = \frac{hU}{U - a \Sigma_i L_i b_i} .$$

Proof

Differentiating (2.3) with respect to s_u yields

$$\frac{1}{ha} H^{\frac{1}{ha}-1} H_{S_u} - \Sigma_i [p_i(r) \frac{b_i a + 1}{ha} H^{\frac{b_i a+1}{ha}-1} H_{S_u}] -$$

$$p_{u_{S_u}}(r) H^{\frac{b_u a+1}{ha}} = 0 \quad (2.6)$$

where X_{S_u} is the derivative of X with respect to s_u.

Note that $H(1,\ldots,1,0) = 1$ and $H_{S_u}(1,\ldots,1,0) = E(M_u)$.

Therefore at $(s_1,\ldots,s_c,r) = (1,\ldots,1,0)$ in (2.6), we have

$$\frac{1}{ha} E(M_u) - \Sigma_i [p_i \frac{b_i a + 1}{ha} E(M_u)] - p_{u_{S_u}} = 0 \quad (2.7)$$

where

$$p_i = p_{iS_i} = \frac{L_i}{\Sigma_j L_j + U} \; ; \; i = 1, \ldots, c \; .$$

Simplify (2.7) gives

$$E(M_u) = \frac{L_u ha}{U - a \, \Sigma_i L_i b_i} \; .$$

Since $N = h + \Sigma_i M_i b_i$, therefore

$$E(N) = h + \Sigma_i E(M_i) \, b_i \; . \tag{2.8}$$

Simplify (2.8) and the result follows.

Theorem 2.3

$$V(N) =$$

$$\frac{ha}{U - a \, \Sigma_i L_i b_i} \{\Sigma_i L_i b_i^2 + \frac{1}{U - a \, \Sigma_i L_i b_i} *$$

$$[2a \, (\Sigma_i L_i b_i)(\Sigma_i L_i b_i^2) + \frac{(U + a^2 \, \Sigma_i L_i b_i^2)(\Sigma_i L_i b_i)^2}{U - a \, \Sigma_i L_i b_i}]\} .$$

Proof

Differentiating (2.6) with respect to S_v gives

$$\frac{1}{ha} H^{\frac{1}{ha}-2} \; [(\frac{1}{ha} - 1) \, H_{S_u} H_{S_v} + H H_{S_u S_v}] - \Sigma_i \{p_i(r) \; *$$

$$\frac{b_i a + 1}{ha} H^{\frac{b_i a+1}{ha}-2} \; [(\frac{b_i a + 1}{ha} - 1) \, H_{S_u} H_{S_v} + H H_{S_u S_v}]\} -$$

$$p_{vS_v}(r) \, \frac{b_v a + 1}{ha} \, H^{\frac{b_v a+1}{ha}-1} \; H_{S_u} -$$

$$p_{uS_u}(r)\ \frac{b_u a + 1}{ha}\ H^{\frac{b_u a+1}{ha}-1}\ H_{S_v} = 0 . \qquad (2.9)$$

Put $s_i = 1$ for $i = 1, \ldots, c$, and $r = 0$ in (2.9), we have

$$\frac{1}{ha}\ [(\frac{1}{ha} - 1)\ E(M_u)E(M_v) + H_{S_uS_v}(1,\ldots,1,0)] - \Sigma_i\ \{p_i\ *$$

$$\frac{b_i a + 1}{ha}\ [(\frac{b_i a + 1}{ha} - 1)E(M_u)E(M_v) + H_{S_uS_v}(1,\ldots,1,0)]\} -$$

$$p_{vS_v}\ \frac{b_v a + 1}{ha}\ E(M_u) - p_{uS_u}\ \frac{b_u a + 1}{ha}\ E(M_v) = 0$$

which implies that

$$H_{S_uS_v}(1,\ldots,1,0) =$$

$$\frac{L_u L_v ha}{(U - a\ \Sigma_i\ L_i b_i)^2}\ [a\ (h + b_u + b_v) + \frac{U + a^2\ \Sigma_i\ L_i b_i^2}{U - a\ \Sigma_i\ L_i b_i}].$$

When $v = u$, $H_{S_uS_u}(1,\ldots,1,0) = E[M_u(M_u-1)]$, therefore

$$V(M_u) =$$

$$H_{S_uS_u}(1,\ldots,1,0) + E(M_u) - [E(M_u)]^2 =$$

$$\frac{L_u ha}{U - a\ \Sigma_i L_i b_i}\ [1 + \frac{L_u}{U - a\ \Sigma_i L_i b_i}\ (2b_u a + \frac{U + a^2\ \Sigma_i L_i b_i^2}{U - a\ \Sigma_i L_i b_i})].$$

When v does not equal to u, then

$$H_{S_uS_v}(1,\ldots,1,0) = E(M_u M_v), \text{ therefore}$$

$$Cov(M_u, M_v) =$$

$$H_{S_uS_v}(1,\ldots,1,0) - E(M_u)E(M_v) =$$

$$\frac{L_u L_v ha}{(U - a\ \Sigma_i L_i b_i)^2}\ [a\ (b_u + b_v) + \frac{U + a^2\ \Sigma_i L_i b_i^2}{U - a\ \Sigma_i L_i b_i}] .$$

Therefore

$$V(N) = V(h+\Sigma_i M_i b_i)$$

$$= \sum_{i=1}^{c} V(M_i)\, b_i^2 + 2 \sum_{i=1}^{c} \sum_{j=i+1}^{c} Cov(M_i, M_j)\, b_i b_j \,. \qquad (2.10)$$

Simplify (2.10) and the result follows.

Theorem 2.4

$$E(T) = \frac{ha}{U - a\, \Sigma_i\, L_i b_i} \,.$$

Proof

Differentiating (2.3) with respect to r gives

$$\frac{1}{ha}\, H^{\frac{1}{ha}-1}\, H_R - \Sigma_i\, \{H^{\frac{b_i a+1}{ha}-1} \,*$$

$$[p_{iR}(r)\, H + p_i(r)\, (\frac{b_i a + 1}{ha})\, H_R]\} = p_{OR}(r) \,. \qquad (2.11)$$

Since $H_R(1,\ldots,1,0) = E(T)$, therefore

$$\frac{1}{ha}\, E(T) - \Sigma_i\, [p_{iR} + p_i\, \frac{b_i a + 1}{ha}\, E(T)] = p_{OR} \qquad (2.12)$$

where

$$p_{OR} = \frac{U}{(\Sigma_i\, L_i + U)^2} \,; \quad p_{iR} = \frac{L_i}{(\Sigma_j\, L_j + U)^2} \text{ for } i=1,\ldots,c.$$

Simplify (2.12) and the result follows.

Theorem 2.5

$$V(T) = \frac{ha\, (U + a^2\, \Sigma_i\, L_i b_i^2)}{(U - a\, \Sigma_i\, L_i b_i)^3} \,.$$

Proof

Differentiating (2.11) with respect to r gives

$$\frac{1}{ha} H^{\frac{1}{ha}-2} [(\frac{1}{ha} - 1) H_R^2 + HH_{RR}] - \Sigma_i \{(\frac{b_i a + 1}{ha} - 1) *$$

$$H^{\frac{b_i a+1}{ha}-2} H_R [p_{iR}(r) H + p_i(r) \frac{b_i a + 1}{ha} H_R]\} -$$

$$\Sigma_i \left[H^{\frac{b_i a+1}{ha}-1} \{p_{iRR}(r) H + p_{iR}(r) H_R + \frac{b_i a + 1}{ha} * \right.$$

$$\left. [p_{iR}(r) H_R + p_i(r) H_{RR}]\}\right] = p_{ORR}(r) .$$

Realize that $H_{RR}(1,...,1,0) = E(T^2)$, therefore

$$\frac{1}{ha} \{(\frac{1}{ha} - 1) [E(T)]^2 + E(T^2)\} - \Sigma_i \{(\frac{b_i a + 1}{ha} - 1) *$$

$$E(T) [p_{iR} + p_i \frac{b_i a + 1}{ha} E(T)]\} - \Sigma_i \{p_{iRR} + p_{iR} E(T) +$$

$$\frac{b_i a + 1}{ha} [p_{iR} E(T) + p_i E(T^2)]\} = p_{ORR} \qquad (2.13)$$

where

$$p_{ORR} = \frac{2U}{(\Sigma_i L_i + U)^3}; \quad p_{iRR} = \frac{2L_i}{(\Sigma_j L_j + U)^3}, \; i = 1,...,c.$$

Simplify (2.13) gives

$$E(T^2) = \frac{ha}{(U - a \Sigma_i L_i b_i)^2} (ha + \frac{U + a^2 \Sigma_i L_i b_i^2}{U - a \Sigma_i L_i b_i}) .$$

Since $V(T) = E(T^2) - [E(T)]^2$, the result follows.

Theorem 2.6

$$\mathrm{Cov}(N,T) = \frac{haU\,(\Sigma_i\, L_i b_i + a\, \Sigma_i\, L_i b_i^2)}{(U - a\, \Sigma_i\, L_i b_i)^3} .$$

Proof

Differentiating (2.11) with respect to s_u yields

$$\frac{1}{ha} H^{\frac{1}{ha}-2} [(\frac{1}{ha} - 1) H_R H_{S_u} + H H_{RS_u}] - \Sigma_i \{(\frac{b_i a + 1}{ha} - 1) *$$

$$H^{\frac{b_i a+1}{ha}-2} H_{S_u} [p_{i_R}(r)\, H + p_i(r) \frac{b_i a + 1}{ha} H_R]\} -$$

$$\Sigma_i \{H^{\frac{b_i a+1}{ha}-1} [p_{i_R}(r)\, H_{S_u} + p_i(r) \frac{b_i a + 1}{ha} H_{RS_u}]\} -$$

$$H^{\frac{b_u a+1}{ha}-1} [p_{u_{RS_u}}(r)\, H + p_{u_{S_u}}(r) \frac{b_u a + 1}{ha} H_R] = 0 .$$

Realize that $H_{RS_u}(1,\ldots,1,0) = E(M_u T)$, therefore

$$\frac{1}{ha} [(\frac{1}{ha} - 1) E(T)E(M_u) + E(M_u T)] - \Sigma_i \{(\frac{b_i a + 1}{ha} - 1) *$$

$$E(M_u) \{p_{i_R} + p_i (\frac{b_i a + 1}{ha}) E(T)]\} - \Sigma_i [p_{i_R} E(M_u) + p_i *$$

$$\frac{b_i a + 1}{ha} E(M_u T)] - [p_{u_{RS_u}} + p_{u_{S_u}} \frac{b_u a + 1}{ha} E(T)] = 0 \qquad (2.14)$$

where

$$p_{i_{RS_i}} = \frac{L_i}{(\Sigma_j\, L_j + U)^2} \text{ for } i = 1, \ldots, c .$$

Simplify (2.14) gives

$$E(M_uT) = \frac{hL_u a}{(U - a\,\Sigma_i\,L_i b_i)^2}\,[a\,(h + b_u) + \frac{U + a^2\,\Sigma_i\,L_i b_i^2}{U - a\,\Sigma_i\,L_i b_i}] .$$

Hence

$Cov(M_u, T)$ =

$E(M_uT) - E(M_u)E(T)$ =

$$\frac{hL_u a}{(U - a\,\Sigma_i\,L_i b_i)^2}\,(ab_u + \frac{U + a^2\,\Sigma_i\,L_i b_i^2}{U - a\,\Sigma_i\,L_i b_i}) .$$

Realize that

$$Cov(N,T) = Cov(h+\Sigma_i M_i b_i, T)$$
$$= \Sigma_i\ Cov(M_i, T)\ b_i . \qquad (2.15)$$

Simplify (2.15) and the result follows.

Since the correlation coefficient of N and T, denoted by Cor(N,T), is defined as

$$Cor(N,T) = \frac{Cov(N,T)}{\sqrt{[V(N)\ V(T)]}} ,$$

it follows from Theorem 2.3, 2.5, and 2.6 that

$$Cor(N,T) = \frac{U\,(\Sigma_i\,L_i b_i + a\,\Sigma_i\,L_i b_i^2)}{\sqrt{\{[(U - a\,\Sigma_i\,L_i b_i)^2\,(\Sigma_i\,L_i b_i^2) + 2a\,(U - a\,\Sigma_i\,L_i b_i)\,(\Sigma_i\,L_i b_i)\,(\Sigma_i\,L_i b_i^2) + (U + a^2\,\Sigma_i\,L_i b_i^2)\,(\Sigma_i\,L_i b_i)^2] * [U + a^2\,\Sigma_i\,L_i b_i^2]\}}}$$

$$= \sqrt{\{1 - \frac{(U - a\,\Sigma_i\,L_i b_i)^2\,\Sigma_i\,L_i b_i^2}{[U\Sigma_i L_i b_i^2 + (\Sigma_i L_i b_i)^2][U + a^2\Sigma_i L_i b_i^2]}\}} . \qquad (2.16)$$

Note that Cor(N,T) does not depend on h at all.

Although the formulae derived in Theorem 2.2 - 2.6 are not simple enough to be interpreted directly, some observations can still be made. Clearly that the factor U-

$a\ \Sigma_i\ L_i b_i$ plays a very important role in affecting the moments. This is expected as $\Sigma_i\ L_i b_i$ and U/a represent the input and service rates respectively. In contrast, it is quite surprising that the initial number of customers, h, affects the moments as multiples.

Realize that the $M/E_a/1$ model is a special case of the $M_b^c/E_a/1$ model by taking $b_1 = c = 1$. Let us put $b_1 = c = 1$ into Theorem 2.2 - 2.6 and replace L_1 by L, we obtain the moments of the $M/E_a/1$ model as follows:

$$E(N) = \frac{hU}{U - La} \; ; \qquad (2.17)$$

$$V(N) = \frac{ha}{U - La} \{L + \frac{1}{U - La} [2L^2 a + \frac{(U + La^2)\ L^2}{U - La}]\}$$

$$= \frac{hLaU\ (L + U)}{(U - La)^3} \; ; \qquad (2.18)$$

$$E(T) = \frac{ha}{U - La} \; ; \qquad (2.19)$$

$$V(T) = \frac{ha\ (U + La^2)}{(U - La)^3} \; ; \qquad (2.20)$$

$$Cov(N,T) = \frac{hLaU\ (a + 1)}{(U - La)^3} \; ; \qquad (2.21)$$

$$Cor(N,T) = \sqrt{\ } [1 - \frac{(U - La)^2}{(L + U)\ (U + La^2)}] \; . \qquad (2.22)$$

To conclude this section, let us consider two simple but important models, namely, M/M/1 (Poisson input and exponentially distributed service time), and M/D/1 (Poisson input and constant service time).

Denote by exp(U) the exponential distribution with parameter U and by D(w) the deterministic distribution with parameter w. Since exp(U) $\equiv$ G(1,U) and D(w) $\equiv \lim_{U\to\infty}$ G(wU,U), therefore from (2.17) to (2.22), it is clear that

Model	M/M/1	M/D/1
E(N)	$\frac{hU}{U - L}$	$\frac{h}{1 - Lw}$
V(N)	$\frac{hLU (L + U)}{(U - L)^3}$	$\frac{hLw}{(1 - Lw)^3}$
E(T)	$\frac{h}{U - L}$	$\frac{hw}{1 - Lw}$
V(T)	$\frac{h (U + L)}{(U - L)^3}$	$\frac{hLw^3}{(1 - Lw)^3}$
Cov(N,T)	$\frac{2hLU}{(U - L)^3}$	$\frac{hLw^2}{(1 - Lw)^3}$
Cor(N,T)	$\sqrt{[1 - (\frac{U - L}{L + U})^2]}$	1

2.3 The $M_b/M_b/1$ model

This is a model in which arrivals / services are in batches of size b_1 / b_2. Arrivals are in according with a Poisson process of mean rate L. Service times have independent exponential distributions with parameter U, $Ub_2 > Lb_1$, which are also independent of the arrivals.

In order to derive the moments of M (the number of arriving batches during a busy period) and T, one must first of all find out the probability $P(M=m;T\leq t)$. In fact, an explicit formula for the joint distribution has been derived by Mohanty [8]. However, since the formula involves not only determinants of very high order, but also very large binomial coefficients, therefore, the formula is not handy enough for computation. Nevertheless, it shows that the exact distribution can be derived through the application of lattice path counting technique, which will be utilized in the sequel.

Consider a two-dimensional lattice path which starting from the origin moves at any stage either a horizontal unit or a vertical unit. Denote by $P_{(i,j)}$ the probability of the path reaching to the point (i,j), $p_{0(i,j)}$ the probability of moving from (i,j) to (i+1,j), and $p_{1(i,j)}$ the probability of moving from (i,j) to (i,j+1). Then the distribution of M, when there are initially h ($h \geq b_2$) customers, is given in the following theorem.

Theorem 2.7

$$P(M=m) = P_{(n,m)} \; ; \quad m = 0,1,\ldots; \quad n = \left[\frac{h + mb_1}{b_2}\right]^- ; \quad 0 \leq t < \infty \qquad (2.23)$$

where

$[i]^-$ stands for the largest integer $\leq i$;

$$P_{(o,o)} = 1 \; ; \qquad (2.24)$$

$$P_{(i,j)} = p_{o(i-1,j)} P_{(i-1,j)} + p_{1(i,j-1)} P_{(i,j-1)}$$

with

$$p_{o(i,j)} = \begin{cases} \dfrac{U}{L + U} \; ; & i = 0,\ldots,\left[\dfrac{h+jb_1}{b_2}\right]^- - 1 \; ; \; j = 0,1,\ldots \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$p_{1(i,j)} = \begin{cases} \dfrac{L}{L + U} \; ; & i = 0,\ldots,\left[\dfrac{h+jb_1}{b_2}\right]^- - 1 \; ; \; j = 0,1,\ldots \\ 0 \; ; & \text{otherwise .} \end{cases}$$

Proof

Since arrivals and service completion instants are independent Poisson processes of mean rate L and U respectively, every event in the combined process independent of others is either an arrival with probability L/(L+U) or a departure with probability U/(L+U). Moreover, for any instance, if there had been j arrivals, then the number of departing batches, i, should not exceed $[(h+jb_1)/b_2]^-$, where the busy period stops when $i = [(h+jb_1)/b_2]^-$. Represent an arrival by a vertical step and a departure by a horizontal step. Thus the sequence of arrivals and departures during a busy period consisting of m arriving batches is represented by a lattice path from the origin to (n,m), $n = [(h+mb_1)/b_2]^-$. This establishes (2.23). Furthermore, a path to (i,j) can be reached from either (i-1,j) with probability $p_{0(i-1,j)}$ or (i,j-1) with probability $p_{1(i,j-1)}$. Moreover, during the busy period, the path cannot touch the boundary $x = [(b_1y + h)/b_2]^-$ except at the end. Thus we get (2.24). This completes the proof.

Realize that the conditional random variable T given M = m has G(m+n,L+U) [8], therefore from Theorem 2.7, we obtain the joint probability density function of M and T as

$$g(m,t) = \frac{e^{-(L+U)t}\,(L+U)^{m+n}\,t^{m+n-1}}{(m+n-1)!}\,P_{(n,m)}\ ; \qquad (2.25)$$

$$m = 0, 1, \ldots\ ; \quad n = \left[\frac{h + mb_1}{b_2}\right]^-\ ; \quad 0 \le t < \infty\ .$$

Theorem 2.8

$$E(M) = \sum_{m=0}^{\infty} m\,P_{(n,m)}\ ; \qquad (2.26)$$

$$V(M) = \sum_{m=0}^{\infty} m^2\,P_{(n,m)} - [E(M)]^2\ ; \qquad (2.27)$$

$$P(T\le t) = \sum_{m=0}^{\infty} P_{(n,m)}\left[1 - \sum_{i=0}^{m+n-1} \frac{e^{-(L+U)t}\,[(L+U)t]^i}{i!}\right]; \qquad (2.28)$$

$$E(T) = \sum_{m=0}^{\infty} \frac{m+n}{L+U} P_{(n,m)} ; \qquad (2.29)$$

$$V(T) = \sum_{m=0}^{\infty} \frac{(m+n)(m+n+1)}{(L+U)^2} P_{(n,m)} - [E(T)]^2 ; \qquad (2.30)$$

$$Cov(M,T) = \sum_{m=0}^{\infty} \frac{m\,(m+n)}{L+U} P_{(n,m)} - E(M)E(T) . \qquad (2.31)$$

Proof

(2.26) and (2.27) are obvious.

For (2.28), we have

$$P(T \le t) = \sum_{m=0}^{\infty} \int_0^t g(m,x)\, dx$$

$$= \sum_{m=0}^{\infty} P_{(n,m)} \int_0^t \frac{e^{-(L+U)x} (L+U)^{m+n} x^{m+n-1}}{(m+n-1)!} dx .$$

By applying Proposition 1.3 , the result follows.

For (2.29), we have

$$E(T) = \sum_{m=0}^{\infty} \int_0^{\infty} t\, g(m,t)\, dt$$

$$= \sum_{m=0}^{\infty} \int_0^{\infty} t \frac{e^{-(L+U)t} (L+U)^{m+n} t^{m+n-1}}{(m+n-1)!} dt\, P_{(n,m)}$$

Simplify the expression and the result follows.

For (2.30), since

$$E(T^2) = \sum_{m=0}^{\infty} \int_0^{\infty} t^2 \frac{e^{-(L+U)t} (L+U)^{m+n} t^{m+n-1}}{(m+n-1)!} dt\, P_{(n,m)}$$

$$= \sum_{m=0}^{\infty} \frac{(m+n)(m+n+1)}{(L+U)^2} P_{(n,m)}$$

and $V(T) = E(T^2) - [E(T)]^2$, the result follows.

For (2.31), since

$$E(MT) = \sum_{m=0}^{\infty} m \int_{0}^{\infty} t \frac{e^{-(L+U)t} (L+U)^{m+n} t^{m+n-1}}{(m + n - 1)!} dt \, P_{(n,m)}$$

$$= \sum_{m=0}^{\infty} \frac{m (m + n)}{L + U} P_{(n,m)}$$

and Cov(M,T) = E(MT) - E(M)E(T), the result follows.

Based on Theorem 2.7 and 2.8, a demonstrative programme is written. Listing of the programme and tables are put in Appendix B. For comparison, parameter sets (L,U,b_1,b_2) are selected such that $Lb_1 : Ub_2 = 1 : 4$. However, interpretation is not the main objective of this project and will not be discussed.

To conclude this section, we will use a slightly different model to demonstrate that the technique introduced in Theorem 2.7 can be extended to more general situation. Consider a model which is similar to the one discussed above, except that it has now a fixed capacity k.

The distribution of M, when there are initially h ($h \geq b_2$) customers and the queue capacity is k, is given by

Theorem 2.9

$$P(M=m) = P_{(n,m)} \; ; \quad m = 0,1,\ldots; \quad n = [\frac{h + mb_1}{b_2}]^-; \quad 0 \leq t < \infty$$

where

$$P_{(0,0)} = 1 \; ;$$

$$P_{(i,j)} = p_{0(i-1,j)} P_{(i-1,j)} + p_{1(i,j-1)} P_{(i,j-1)}$$

with

$$p_{0(i,j)} = \begin{cases} 1 & , \; i = min_j,\ldots,min_{j+1}-1, \; j = 0,1,\ldots, \\ \dfrac{U}{L + U} & , \; i = min_{j+1},\ldots,max_j-1, \; j = 0,1,\ldots, \\ 0 & , \text{ otherwise ;} \end{cases}$$

$$p_{1(i,j)} = \begin{cases} \dfrac{L}{L + U} & , \; i = min_{j+1},\ldots,max_j-1, \; j = 0,1,\ldots, \\ 0 & , \text{ otherwise ;} \end{cases}$$

$$min_j = \max(0,[\frac{h + jb_1 - k}{b_2}]^+) \text{ and } max_j = [\frac{h + jb_1}{b_2}]^- ,$$

$[j]^+$ = the smallest integer which is $\geq j$.

Proof

Arguments are similar to those for Theorem 2.7 except that when a limited capacity k is imposed, the minimum number of departures i when there had been j arrivals is restricted. Explanations are as follows. For any instance, since the number of customers in the system should be $\leq k$, therefore if there had been j arrivals, then we must have $jb_1+h-ib_2 \leq k$. This implies that the corresponding lattice path cannot cross the boundary $x = [(jb_1+h-k)/b_2]^+$. To complete the proof, realize that arrival is allowed only when the capacity left in the system is $\geq b_1$. Therefore the only possible event when $k-(jb_1+h-ib_2) < b_1$ is a departure, which gives $p_{0(i,j)} = 1$ when $i = min_j, \ldots, min_{j+1}-1$.

Base on Theorem 2.9 and the basic definition of moment, a demonstrative programme is written. Listing of the programme and selected tables are put in Appendix C.

APPENDIX A INDEX OF NOTATIONS

MODELS

Notations of the models discussed in this project will be in the form of A/B/1.

$A \in \{M, M_b, M_b^c\}$ represents the input mechanism, where

M = Poisson process, singly from 1 source,
M_b = Poisson process, in batches from 1 source,
M_b^c = Poisson process, in batches from c sources;

$B \in \{D, E_a, G, M, M_b\}$ gives the service time distribution (service times being identically and independently distributed random variables and independent of arrivals).

D = deterministic distribution, singly,
E_a = Erlangian distribution with a stages, singly,
G = general distribution with positive domain,
M = exponential distribution, singly,
M_b = exponential distribution, in batches;

1 at the end represents the number of service channels.

SYMBOLS / NOTATIONS

b_o	batch size (service)
b_i	batch size (input) of the i^{th} source
c	number of independent input sources
Cov(X,Y)	covariance of X and Y
Cor(X,Y)	correlation coefficient of X and Y
D(w)	deterministic distribution with parameter w
E(X)	expectation of X
F(t)	distribution function of T

$F_n(t)$	n^{th} iterated covolution of $F(t)$
$G(a,U)$	Gamma distribution with parameter a and U: $g(x) = \frac{e^{-Ux}\, U^a\, x^{a-1}}{\Gamma(a)}$; $0 \le x < \infty$.
h	number of initial customers
$H(s_1,\ldots,s_c,r)$	moment generating function of $M_1, \ldots, M_c$, T
k	fixed capacity of the system
L (L_i)	input mean rate (of the i^{th} input source)
M (M_i)	no. of input batches (from the i^{th} source)
max(i,j)	the maximum of i and j
N	number served during a busy period
T	length of a busy period
V(X)	variance of X
$F(x) \sim Z(.)$	X has the Z(.) distribution
i * j	i times j
$[i]^+$	the smallest integer $\ge$ i
$[i]^-$	the largest integer $\le$ i
$\sum_{r=i}^{j} z_r$	$z_i + \ldots + z_j$
$\Sigma_i\, z_i$	$z_1 + \ldots + z_c$, c = number of input sources
$\prod_{r=i}^{j} z_r$	$z_i * \ldots * z_j$
$\pi_i\, z_i$	$z_1 * \ldots * z_c$, c = number of input sources
z!	$1 * 2 * \ldots * z$
$X_{Y_1 \ldots Y_i}$	derivative of X with respect to $y_1, \ldots, y_i$
$\binom{u}{v_1,\ldots,v_c}$	$\frac{u!}{(u - \Sigma_i\, v_i)!\ \pi_i\, v_i!}$

APPENDIX B DEMONSTRATIVE PROGRAMME/TABLES FOR $M_b/M_b/1$

```
(***********************************************************************
*                                                                      *
*  This is a demonstrative programme which computes P(M=m), P(T≤t), and the  *
*  first two moments of M and T for the Mb/Mb/1 model with pre-selected    *
*  parameter sets (h,L,U,b1,b2).  Output is stored in a file and tabulated *
*  output is put following the listing of this programme                   *
*                                                                      *
***********************************************************************)
{$u+}

(* Declaration of the variables to be used ********************************)

const
   last_h = 3;
   last_para = 5;
   last_record = 19;
   last_n = 2000;

type
   one_dim = array [0..last_record] of real;

var
   m, last_m, count_m, count_t, curr_max, prev_max,
   count1, count2, count3, count4                 : integer;
   h, L, U, b1, b2, sum_m, const1, const2, const3,
   curr_p, p_arrival, p_departure                 : real;
   temp_p                                         : array [0..last_n] of real;
   p, sum_p, sum_t, m_val, t_val, const_t,
   para1, para2, para3, poisson, cum_poisson      : one_dim;
   out_file                                       : file of one_dim;
```

```
(* Set up the remaining terms if the probability of the last term is zero ****)

procedure completeM;
begin
   if (count_m <= last_record) then
      for count3 := count_m to last_record do
      begin
         p [count3] := 0;
         sum_p [count3] := sum_m;
      end;
   if (m = last_m) then
   begin
      p [0] := -1;
      sum_p [0] := -1;
      sum_t [0] := -1;
      para2 [count_t] := -1;
      para3 [count_t] := -1;
   end
   else
   begin
      sum_p [0] := sum_p [0] - sqr (p [0]);
      para2 [count_t] := para2 [count_t] - sqr (sum_t [0]);
      para3 [count_t] := para3 [count_t] - p [0] * sum_t [0];
   end;
   write (out_file, p, sum_p, sum_t);
end;

(* Cumulate the probability, mean, and variance ******************************)

procedure update;
begin
   const2 := (m + curr_max + 1) / const1;
   sum_m := sum_m + curr_p;
   if (m = m_val [count_m]) then
   begin
      p [count_m] := curr_p;
      sum_p [count_m] := sum_m;
      count_m := count_m + 1;
   end;
   p [0] := p [0] + m * curr_p;
   sum_p [0] := sum_p [0] + sqr (m) * curr_p;
   sum_t [0] := sum_t [0] + const2 * curr_p;
   const3 := const2 / const1 + sqr (const2);
   para2 [count_t] := para2 [count_t] + curr_p * const3;
   para3 [count_t] := para3 [count_t] + curr_p * const2 * m;
   for count3 := 1 to last_record do
      sum_t [count3] := sum_t [count3] + (1 - cum_poisson [count3]) * curr_p;
end;
```

```
(* Compute the probability of T ************************************************)

procedure computeT;
begin
   for count3 := 1 to last_record do
      for count4 := (m + prev_max) to (m + curr_max) do
      begin
         poisson [count3] := poisson [count3] * const_t [count3] / count4;
         cum_poisson [count3] := cum_poisson [count3] + poisson [count3];
      end;
end;

(* Compute the probability of M ************************************************)

procedure computeM;
begin
   while ((m < last_m) and (curr_p <> 0) and (curr_p <> 1)) do
   begin
      m := m + 1;
      prev_max := curr_max;
      curr_max := trunc ((m * b1 + h) / b2 - 1);
      temp_p [0] := temp_p [0] * p_arrival;
      for count3 := 1 to curr_max + 1 do
      begin
         if (count3 > prev_max) then
            temp_p [count3] := 0
         else
            temp_p [count3] := temp_p [count3] * p_arrival;
         temp_p [count3] := temp_p [count3] + temp_p [count3-1] * p_departure;
      end;
      curr_p := temp_p [curr_max + 1];
      computeT;
      update;
   end;
end;
```

```
(* Compute the probability of M equals to zero ********************************)

procedure compute0;
begin
   m := 0;
   count_m := 1;
   prev_max := 1;
   curr_max := trunc (h / b2) - 1;
   temp_p [0] := 1;
   for count3 := 1 to curr_max + 1 do
      temp_p [count3] := temp_p [count3 - 1] * p_departure;
   curr_p := temp_p [curr_max + 1];
   computeT;
   update;
   computeM;
   if (m = 0) then
      for count3 := 1 to last_record do
         sum_t [count3] := 1;
   completeM;
end;


(* Initialize the variables when h is assigned ********************************)

procedure init_var3;
begin
   count_t := count1 * 3 + count2;
   h := para2 [last_h * last_para + count2];
   last_m := trunc ((last_n * b2 - h) / b1);
   p [0] := 0;
   sum_m := 0;
   sum_p [0] := 0;
   sum_t [0] := 0;
   for count3 := 1 to last_record do
   begin
      sum_p [count3] := 0;
      sum_t [count3] := 0;
      poisson [count3] := exp (- const_t [count3]);
      cum_poisson [count3] := poisson [count3];
   end;
end;
```

```
(* Initialize the variables when L, U, b1, and b2 are assigned ***************)

procedure init_var2;
begin
   L  := para1 [count1 * 4];
   U  := para1 [count1 * 4 + 1];
   b1 := para1 [count1 * 4 + 2];
   b2 := para1 [count1 * 4 + 3];
   const1 := L + U;
   p_arrival := L / const1;
   p_departure := 1 - p_arrival;
   for count2 := 1 to last_record do
      const_t [count2] := (L + U) * t_val [count2];
end;

(* Initialize the common variables to be used for each case *****************)

procedure init_var1;
begin
   assign (out_file, 'b:mbt.out');
   rewrite (out_file);
   for count1 := 0 to last_h * last_para do
   begin
      para2 [count1] := 0;
      para3 [count1] := 0;
   end;
end;

(* Compute the probabilities and moments ***********************************)

procedure compute;
begin
   init_var1;
   for count1 := 0 to last_para - 1 do
   begin
      init_var2;
      for count2 := 1 to last_h do
      begin
         init_var3;
         compute0;
      end;
   end;
   write (out_file, para1, para2, para3, m_val, t_val);
   close (out_file);
end;
```

```
(* Set up the parameters ************************************************************)

procedure init_const;
begin
   para1[ 0] :=    4;  para1[ 1] :=   16;  para1[ 2] :=    1;  para1[ 3] :=    1;
   para1[ 4] :=    2;  para1[ 5] :=   16;  para1[ 6] :=    2;  para1[ 7] :=    1;
   para1[ 8] :=    2;  para1[ 9] :=    4;  para1[10] :=    2;  para1[11] :=    4;
   para1[12] :=    4;  para1[13] :=    4;  para1[14] :=    1;  para1[15] :=    4;
   para1[16] :=    4;  para1[17] :=    2;  para1[18] :=    1;  para1[19] :=    8;
   para2[16] :=    1;  para2[17] :=    4;  para2[18] :=    8;  para2[19] :=    0;
   para3[16] :=    0;  para3[17] :=    0;  para3[18] :=    0;  para3[19] :=    0;
   m_val[ 0] :=    0;  m_val[ 1] :=    0;  m_val[ 2] :=    1;  m_val[ 3] :=    2;
   m_val[ 4] :=    3;  m_val[ 5] :=    4;  m_val[ 6] :=    5;  m_val[ 7] :=    6;
   m_val[ 8] :=    7;  m_val[ 9] :=    8;  m_val[10] :=    9;  m_val[11] :=   10;
   m_val[12] :=   12;  m_val[13] :=   14;  m_val[14] :=   16;  m_val[15] :=   18;
   m_val[16] :=   20;  m_val[17] :=   25;  m_val[18] :=   30;  m_val[19] :=   50;
   t_val[ 0] :=    0;  t_val[ 1] :=    0;  t_val[ 2] :=  0.1;  t_val[ 3] :=  0.2;
   t_val[ 4] :=  0.3;  t_val[ 5] :=  0.4;  t_val[ 6] :=  0.5;  t_val[ 7] :=  0.6;
   t_val[ 8] :=  0.7;  t_val[ 9] :=  0.8;  t_val[10] :=  0.9;  t_val[11] :=    1;
   t_val[12] :=  1.2;  t_val[13] :=  1.4;  t_val[14] :=  1.6;  t_val[15] :=  1.8;
   t_val[16] :=    2;  t_val[17] :=  2.5;  t_val[18] :=    3;  t_val[19] :=  3.5;
end;

(* Main programme *******************************************************************)

begin
   init_const;
   compute;
end.

(* The end of this programme ********************************************************)
```

(L , U , b1 , b2) = (4 ,16 , 1 , 1)

m	h = 1 P(M=m)	h = 1 P(M≤m)	h = 4 P(M=m)	h = 4 P(M≤m)	h = 8 P(M=m)	h = 8 P(M≤m)
0	0.8000	0.8000	0.4096	0.4096	0.1678	0.1678
1	0.1280	0.9280	0.2621	0.6717	0.2147	0.3825
2	0.0410	0.9690	0.1468	0.8185	0.1890	0.5715
3	0.0164	0.9853	0.0805	0.8991	0.1429	0.7144
4	0.0073	0.9927	0.0443	0.9434	0.1001	0.8145
5	0.0035	0.9962	0.0246	0.9679	0.0670	0.8815
6	0.0018	0.9980	0.0138	0.9817	0.0436	0.9251
7	0.0009	0.9989	0.0078	0.9895	0.0279	0.9531
8	0.0005	0.9994	0.0044	0.9939	0.0177	0.9707
9	0.0003	0.9997	0.0025	0.9964	0.0111	0.9818
10	0.0001	0.9998	0.0015	0.9979	0.0069	0.9887
12	0.0000	0.9999	0.0005	0.9993	0.0027	0.9957
14	0.0000	1.0000	0.0002	0.9997	0.0010	0.9984
16	0.0000	1.0000	0.0001	0.9999	0.0004	0.9994
18	0.0000	1.0000	0.0000	1.0000	0.0001	0.9998
20	0.0000	1.0000	0.0000	1.0000	0.0001	0.9999
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	3.333E-01		1.333E+00		2.667E+00	
Var.	7.407E-01		2.963E+00		5.926E+00	

t	h = 1 P(T≤t)	h = 4 P(T≤t)	h = 8 P(T≤t)
0.0	0.0000	0.0000	0.0000
0.1	0.7353	0.0630	0.0002
0.2	0.9001	0.2966	0.0105
0.3	0.9551	0.5383	0.0653
0.4	0.9779	0.7133	0.1796
0.5	0.9885	0.8256	0.3299
0.6	0.9937	0.8946	0.4836
0.7	0.9965	0.9362	0.6191
0.8	0.9980	0.9613	0.7279
0.9	0.9988	0.9764	0.8101
1.0	0.9993	0.9856	0.8698
1.2	0.9998	0.9945	0.9408
1.4	0.9999	0.9979	0.9739
1.6	1.0000	0.9992	0.9887
1.8	1.0000	0.9997	0.9951
2.0	1.0000	0.9999	0.9979
2.5	1.0000	1.0000	0.9998
3.0	1.0000	1.0000	1.0000
3.5	1.0000	1.0000	1.0000
Mean	8.333E-02	3.333E-01	6.667E-01
Var.	1.157E-02	4.630E-02	9.259E-02
Cov.	7.407E-02	2.963E-01	5.926E-01

(L , U , b1 , b2) = (2 ,16 , 2 , 1)

	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	0.8889	0.8889	0.6243	0.6243	0.3897	0.3897
1	0.0780	0.9669	0.2192	0.8435	0.2737	0.6635
2	0.0206	0.9875	0.0866	0.9301	0.1562	0.8197
3	0.0072	0.9947	0.0372	0.9673	0.0844	0.9041
4	0.0029	0.9976	0.0169	0.9842	0.0449	0.9489
5	0.0013	0.9989	0.0080	0.9922	0.0238	0.9727
6	0.0006	0.9994	0.0039	0.9960	0.0126	0.9854
7	0.0003	0.9997	0.0019	0.9980	0.0067	0.9921
8	0.0001	0.9999	0.0010	0.9989	0.0036	0.9957
9	0.0001	0.9999	0.0005	0.9994	0.0019	0.9977
10	0.0000	1.0000	0.0003	0.9997	0.0011	0.9987
12	0.0000	1.0000	0.0001	0.9999	0.0003	0.9996
14	0.0000	1.0000	0.0000	1.0000	0.0001	0.9999
16	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
18	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
20	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	1.667E-01		6.667E-01		1.333E+00	
Var.	3.333E-01		1.333E+00		2.667E+00	

t	P(T≤t)	P(T≤t)	P(T≤t)
0.0	0.0000	0.0000	0.0000
0.1	0.7505	0.0684	0.0002
0.2	0.9040	0.3191	0.0124
0.3	0.9528	0.5609	0.0759
0.4	0.9740	0.7242	0.2016
0.5	0.9848	0.8257	0.3565
0.6	0.9908	0.8883	0.5058
0.7	0.9942	0.9275	0.6319
0.8	0.9963	0.9523	0.7309
0.9	0.9976	0.9683	0.8056
1.0	0.9984	0.9788	0.8605
1.2	0.9993	0.9903	0.9288
1.4	0.9997	0.9954	0.9639
1.6	0.9998	0.9978	0.9817
1.8	0.9999	0.9989	0.9907
2.0	1.0000	0.9995	0.9952
2.5	1.0000	0.9999	0.9991
3.0	1.0000	1.0000	0.9998
3.5	1.0000	1.0000	1.0000
Mean	8.333E-02	3.333E-01	6.667E-01
Var.	1.389E-02	5.556E-02	1.111E-01
Cov.	5.556E-02	2.222E-01	4.444E-01

t	P(T≤t)	P(T≤t)	P(T≤t)
0.0	1.0000	0.0000	0.0000
0.1	1.0000	0.3281	0.0611
0.2	1.0000	0.5428	0.1865
0.3	1.0000	0.6824	0.3230
0.4	1.0000	0.7741	0.4472
0.5	1.0000	0.8354	0.5520
0.6	1.0000	0.8774	0.6373
0.7	1.0000	0.9070	0.7059
0.8	1.0000	0.9283	0.7606
0.9	1.0000	0.9440	0.8045
1.0	1.0000	0.9557	0.8397
1.2	1.0000	0.9717	0.8911
1.4	1.0000	0.9814	0.9252
1.6	1.0000	0.9875	0.9480
1.8	1.0000	0.9915	0.9636
2.0	1.0000	0.9942	0.9743
2.5	1.0000	0.9976	0.9890
3.0	1.0000	0.9990	0.9951
3.5	1.0000	0.9995	0.9978
Mean	0.000E+00	2.887E-01	5.981E-01
Var.	0.000E+00	1.240E-01	2.763E-01
Cov.	0.000E+00	3.184E-01	7.224E-01

(L , U , b1 , b2) = (2 , 4 , 2 , 4)

m	h = 1 P(M=m)	h = 1 P(M≤m)	h = 4 P(M=m)	h = 4 P(M≤m)	h = 8 P(M=m)	h = 8 P(M≤m)
0	1.0000	1.0000	0.6667	0.6667	0.4444	0.4444
1	0.0000	1.0000	0.2222	0.8889	0.2963	0.7407
2	0.0000	1.0000	0.0494	0.9383	0.0988	0.8395
3	0.0000	1.0000	0.0329	0.9712	0.0768	0.9163
4	0.0000	1.0000	0.0110	0.9822	0.0293	0.9456
5	0.0000	1.0000	0.0085	0.9907	0.0244	0.9700
6	0.0000	1.0000	0.0033	0.9940	0.0099	0.9799
7	0.0000	1.0000	0.0027	0.9967	0.0086	0.9885
8	0.0000	1.0000	0.0011	0.9978	0.0037	0.9922
9	0.0000	1.0000	0.0010	0.9987	0.0032	0.9954
10	0.0000	1.0000	0.0004	0.9991	0.0014	0.9968
12	0.0000	1.0000	0.0002	0.9996	0.0006	0.9987
14	0.0000	1.0000	0.0001	0.9999	0.0002	0.9994
16	0.0000	1.0000	0.0000	0.9999	0.0001	0.9998
18	0.0000	1.0000	0.0000	1.0000	0.0000	0.9999
20	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		5.774E-01		1.196E+00	
Var.	0.000E+00		1.355E+00		2.981E+00	

(L , U , b1 , b2) = (4 , 4 , 1 , 4)

	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	0.5000	0.5000	0.2500	0.2500
1	0.0000	1.0000	0.2500	0.7500	0.2500	0.5000
2	0.0000	1.0000	0.1250	0.8750	0.1875	0.6875
3	0.0000	1.0000	0.0625	0.9375	0.1250	0.8125
4	0.0000	1.0000	0.0156	0.9531	0.0391	0.8516
5	0.0000	1.0000	0.0156	0.9688	0.0430	0.8945
6	0.0000	1.0000	0.0117	0.9805	0.0352	0.9297
7	0.0000	1.0000	0.0078	0.9883	0.0254	0.9551
8	0.0000	1.0000	0.0024	0.9907	0.0085	0.9636
9	0.0000	1.0000	0.0027	0.9934	0.0098	0.9734
10	0.0000	1.0000	0.0022	0.9956	0.0083	0.9817
12	0.0000	1.0000	0.0005	0.9977	0.0022	0.9901
14	0.0000	1.0000	0.0005	0.9989	0.0022	0.9948
16	0.0000	1.0000	0.0001	0.9994	0.0006	0.9971
18	0.0000	1.0000	0.0001	0.9997	0.0006	0.9985
20	0.0000	1.0000	0.0000	0.9998	0.0002	0.9991
25	0.0000	1.0000	0.0000	1.0000	0.0001	0.9998
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		1.078E+00		2.250E+00	
Var.	0.000E+00		3.033E+00		7.070E+00	

t	P(T≤t)	P(T≤t)	P(T≤t)
0.0	1.0000	0.0000	0.0000
0.1	1.0000	0.3296	0.0615
0.2	1.0000	0.5499	0.1907
0.3	1.0000	0.6959	0.3346
0.4	1.0000	0.7918	0.4669
0.5	1.0000	0.8545	0.5777
0.6	1.0000	0.8958	0.6659
0.7	1.0000	0.9235	0.7346
0.8	1.0000	0.9426	0.7878
0.9	1.0000	0.9562	0.8292
1.0	1.0000	0.9661	0.8617
1.2	1.0000	0.9791	0.9084
1.4	1.0000	0.9868	0.9386
1.6	1.0000	0.9915	0.9586
1.8	1.0000	0.9944	0.9718
2.0	1.0000	0.9963	0.9807
2.5	1.0000	0.9986	0.9923
3.0	1.0000	0.9995	0.9968
3.5	1.0000	0.9998	0.9987
Mean	0.000E+00	2.695E-01	5.624E-01
Var.	0.000E+00	9.835E-02	2.322E-01
Cov.	0.000E+00	4.411E-01	1.067E+00

(L , U , b1 , b2) = (4 , 2 , 1 , 8)

	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	1.0000	1.0000	0.3333	0.3333
1	0.0000	1.0000	0.0000	1.0000	0.2222	0.5556
2	0.0000	1.0000	0.0000	1.0000	0.1481	0.7037
3	0.0000	1.0000	0.0000	1.0000	0.0988	0.8025
4	0.0000	1.0000	0.0000	1.0000	0.0658	0.8683
5	0.0000	1.0000	0.0000	1.0000	0.0439	0.9122
6	0.0000	1.0000	0.0000	1.0000	0.0293	0.9415
7	0.0000	1.0000	0.0000	1.0000	0.0195	0.9610
8	0.0000	1.0000	0.0000	1.0000	0.0043	0.9653
9	0.0000	1.0000	0.0000	1.0000	0.0058	0.9711
10	0.0000	1.0000	0.0000	1.0000	0.0058	0.9769
12	0.0000	1.0000	0.0000	1.0000	0.0043	0.9863
14	0.0000	1.0000	0.0000	1.0000	0.0027	0.9924
16	0.0000	1.0000	0.0000	1.0000	0.0005	0.9949
18	0.0000	1.0000	0.0000	1.0000	0.0008	0.9964
20	0.0000	1.0000	0.0000	1.0000	0.0006	0.9977
25	0.0000	1.0000	0.0000	1.0000	0.0001	0.9992
30	0.0000	1.0000	0.0000	1.0000	0.0001	0.9997
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		0.000E+00		2.092E+00	
Var.	0.000E+00		0.000E+00		8.212E+00	

t	P(T≤t)	P(T≤t)	P(T≤t)
0.0	1.0000	1.0000	0.0000
0.1	1.0000	1.0000	0.1813
0.2	1.0000	1.0000	0.3297
0.3	1.0000	1.0000	0.4512
0.4	1.0000	1.0000	0.5507
0.5	1.0000	1.0000	0.6321
0.6	1.0000	1.0000	0.6986
0.7	1.0000	1.0000	0.7530
0.8	1.0000	1.0000	0.7973
0.9	1.0000	1.0000	0.8333
1.0	1.0000	1.0000	0.8624
1.2	1.0000	1.0000	0.9049
1.4	1.0000	1.0000	0.9324
1.6	1.0000	1.0000	0.9505
1.8	1.0000	1.0000	0.9627
2.0	1.0000	1.0000	0.9713
2.5	1.0000	1.0000	0.9845
3.0	1.0000	1.0000	0.9914
3.5	1.0000	1.0000	0.9951
Mean	0.000E+00	0.000E+00	5.230E-01
Var.	0.000E+00	0.000E+00	3.472E-01
Cov.	0.000E+00	0.000E+00	1.459E+00

APPENDIX C DEMONSTRATIVE PROGRAMME/TABLES FOR $M_b/M_b/1$ WITH FIXED CAPACITY

```
(*******************************************************************************
*                                                                              *
*   This is a demonstative programme which computes the probabilities and the  *
*   first two moments for the Mb/Mb/1 model with fixed capacity k and pre-     *
*   selected parameter sets (h,k,L,U,b1,b2).  Output is stored in a file and   *
*   tabulated output is put following the listing of this programme.           *
*                                                                              *
*******************************************************************************)
{$u+}

(* Declaration of the variables to be used ************************************)

const
   last_h = 3;
   last_k = 2;
   last_para = 5;
   last_n = 2000;
   last_record = 19;

type
   one_dim = array [0..last_record] of real;

var
   m, count_m, last_m, prev_max, curr_max, curr_min, next_min,
   count1, count2, count3, count4         : integer;
   h, k, L, U, b1, b2,
   curr_p, sum_m, p_arrival, p_departure  : real;
   temp_p                                 : array [0..last_n] of real;
   p, sum_p, m_val, para1, para2          : one_dim;
   out_file                               : file of one_dim;

(* A function which gives the maximum of two given numbers *******************)

function max (first_num, second_num : integer) : integer;
begin
   if (first_num > second_num) then
      max := first_num
   else
      max := second_num;
end;
```

```
(* A function which gives the minimum integer which is greater than or equals
   to the given number ***********************************************************)

function min_int (num : real) : integer;
begin
   if (num = trunc (num)) then
      min_int := trunc (num)
   else
      min_int := round (num + 0.5);
end;


(* Cumulate the probability, mean, variance, and store them into an array ****)

procedure store_P;
begin
   sum_m := sum_m + curr_p;
   p [0] := p [0] + m * curr_p;
   sum_p [0] := sum_p [0] + sqr (m) * curr_p;
   if (m = m_val [count_m]) then
   begin
      p [count_m] := curr_p;
      sum_p [count_m] := sum_m;
      count_m := count_m + 1;
   end;
end;


(* Set up the remaining terms if the probability of the last terms is zero ***)

procedure completeM;
begin
   if (count_m <= last_record) then
      for count4 := count_m to last_record do
      begin
         p [count4] := 0;
         sum_p [count4] := sum_m;
      end;
   if (curr_p <> 0) then
   begin
      p [0] := -1;
      sum_p [0] := -1;
   end
   else
      sum_p [0] := sum_p [0] - sqr (p [0]);
end;
```

```
(* Compute the probability of M ************************************************)

procedure computeM;
begin
   if (curr_min > prev_max) then
      curr_p := 0
   else
   begin
      temp_p [curr_min] := temp_p [curr_min] * p_arrival;
      for count4 := curr_min + 1 to curr_max + 1 do
      begin
         if (count4 > prev_max) then
            temp_p [count4] := 0
         else
            temp_p [count4] := temp_p [count4] * p_arrival;
         if (count4 > next_min) then
            temp_p [count4] := temp_p [count4] + temp_p [count4-1] * p_departure
         else
            temp_p [count4] := temp_p [count4] + temp_p [count4-1];
      end;
      curr_p := temp_p [curr_max + 1];
   end;
end;

(* Compute the probability of M equals to zero ********************************)

procedure compute0;
begin
   m := 0;
   count_m := 2;
   while ((m < last_m) and (curr_p <> 0)) do
   begin
      m := m + 1;
      prev_max := curr_max;
      curr_max := trunc ((m * b1 + h) / b2 - 1);
      curr_min := next_min;
      next_min := max (0, min_int (((m+1) * b1 + h - k) / b2));
      computeM;
      if (count_m <= last_record) then
         store_P;
   end;
   completeM;
   write (out_file, p, sum_p);
end;
```

```
(* Initialize the variables to be used *************************************** )

procedure init_compute0;
begin
   h := para2 [count3];
   last_m := trunc ((last_n * b2 - h) / b1);
   curr_max := trunc (h / b2) - 1;
   next_min := max (0, min_int ((b1 + h - k) / b2));
   temp_p [0] := 1;
   for count4 := 1 to curr_max + 1 do
      if (count4 > next_min) then
         temp_p [count4] := temp_p [count4-1] * p_departure
      else
         temp_p [count4] := temp_p [count4-1];
   curr_p := temp_p [curr_max + 1];
   sum_m := curr_p;
   p [0] := 0;
   p [1] := curr_p;
   sum_p [0] := 0;
   sum_p [1] := curr_p;
end;


(* Compute the probabilities and moments of M ******************************* )

procedure compute;
begin
   assign (out_file, 'B:MM1.out');
   rewrite (out_file);
   for count1 := 0 to last_para - 1 do
   begin
      L  := para1 [count1 * 4];
      U  := para1 [count1 * 4 + 1];
      b1 := para1 [count1 * 4 + 2];
      b2 := para1 [count1 * 4 + 3];
      p_arrival := L / (L + U);
      p_departure := 1 - p_arrival;
      for count2 := 1 to last_k do
      begin
         k := para2 [last_h + count2];
         for count3 := 1 to last_h do
         begin
            init_compute0;
            compute0;
         end;
      end;
   end;
   write (out_file, para1, para2, m_val);
   close (out_file);
end;
```

```
(* Set up the parameters *************************************************)

procedure init_const;
begin
   m_val[ 0] :=  0;  m_val[ 1] :=   0;  m_val[ 2] :=  1;  m_val[ 3] :=  2;
   m_val[ 4] :=  3;  m_val[ 5] :=   4;  m_val[ 6] :=  5;  m_val[ 7] :=  6;
   m_val[ 8] :=  7;  m_val[ 9] :=   8;  m_val[10] :=  9;  m_val[11] := 10;
   m_val[12] := 12;  m_val[13] :=  14;  m_val[14] := 16;  m_val[15] := 18;
   m_val[16] := 20;  m_val[17] :=  25;  m_val[18] := 30;  m_val[19] := 50;
   para1[ 0] :=  4;  para1[ 1] :=  16;  para1[ 2] :=  1;  para1[ 3] :=  1;
   para1[ 4] :=  2;  para1[ 5] :=  16;  para1[ 6] :=  2;  para1[ 7] :=  1;
   para1[ 8] :=  2;  para1[ 9] :=   4;  para1[10] :=  2;  para1[11] :=  4;
   para1[12] :=  4;  para1[13] :=   4;  para1[14] :=  1;  para1[15] :=  4;
   para1[16] :=  4;  para1[17] :=   2;  para1[18] :=  1;  para1[19] :=  8;
   para2[ 0] :=  0;  para2[ 1] :=   1;  para2[ 2] :=  4;  para2[ 3] :=  8;
   para2[ 4] := 20;  para2[ 5] := 100;
   for count1 := 6 to 19 do
      para2 [count1] := 0;
end;

(* Main program **********************************************************)

begin
   init_const;
   compute;
end.

(* The end of this programme *********************************************)
```

(L , U , b1 , b2) = (4 ,16 , 1 , 1)

k = 20	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	0.8000	0.8000	0.4096	0.4096	0.1678	0.1678
1	0.1280	0.9280	0.2621	0.6717	0.2147	0.3825
2	0.0410	0.9690	0.1468	0.8185	0.1890	0.5715
3	0.0164	0.9853	0.0805	0.8991	0.1429	0.7144
4	0.0073	0.9927	0.0443	0.9434	0.1001	0.8145
5	0.0035	0.9962	0.0246	0.9679	0.0670	0.8815
6	0.0018	0.9980	0.0138	0.9817	0.0436	0.9251
7	0.0009	0.9989	0.0078	0.9895	0.0279	0.9531
8	0.0005	0.9994	0.0044	0.9939	0.0177	0.9707
9	0.0003	0.9997	0.0025	0.9964	0.0111	0.9818
10	0.0001	0.9998	0.0015	0.9979	0.0069	0.9887
12	0.0000	0.9999	0.0005	0.9993	0.0027	0.9957
14	0.0000	1.0000	0.0002	0.9997	0.0010	0.9984
16	0.0000	1.0000	0.0001	0.9999	0.0004	0.9994
18	0.0000	1.0000	0.0000	1.0000	0.0001	0.9998
20	0.0000	1.0000	0.0000	1.0000	0.0001	0.9999
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	3.333E-01		1.333E+00		2.667E+00	
Var.	7.407E-01		2.963E+00		5.926E+00	

k =100	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	0.8000	0.8000	0.4096	0.4096	0.1678	0.1678
1	0.1280	0.9280	0.2621	0.6717	0.2147	0.3825
2	0.0410	0.9690	0.1468	0.8185	0.1890	0.5715
3	0.0164	0.9853	0.0805	0.8991	0.1429	0.7144
4	0.0073	0.9927	0.0443	0.9434	0.1001	0.8145
5	0.0035	0.9962	0.0246	0.9679	0.0670	0.8815
6	0.0018	0.9980	0.0138	0.9817	0.0436	0.9251
7	0.0009	0.9989	0.0078	0.9895	0.0279	0.9531
8	0.0005	0.9994	0.0044	0.9939	0.0177	0.9707
9	0.0003	0.9997	0.0025	0.9964	0.0111	0.9818
10	0.0001	0.9998	0.0015	0.9979	0.0069	0.9887
12	0.0000	0.9999	0.0005	0.9993	0.0027	0.9957
14	0.0000	1.0000	0.0002	0.9997	0.0010	0.9984
16	0.0000	1.0000	0.0001	0.9999	0.0004	0.9994
18	0.0000	1.0000	0.0000	1.0000	0.0001	0.9998
20	0.0000	1.0000	0.0000	1.0000	0.0001	0.9999
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	3.333E-01		1.333E+00		2.667E+00	
Var.	7.407E-01		2.963E+00		5.926E+00	

(L , U , b1 , b2) = (2 ,16 , 2 , 1)

k = 20	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	0.8889	0.8889	0.6243	0.6243	0.3897	0.3897
1	0.0780	0.9669	0.2192	0.8435	0.2737	0.6635
2	0.0206	0.9875	0.0866	0.9301	0.1562	0.8197
3	0.0072	0.9947	0.0372	0.9673	0.0844	0.9041
4	0.0029	0.9976	0.0169	0.9842	0.0449	0.9489
5	0.0013	0.9989	0.0080	0.9922	0.0238	0.9727
6	0.0006	0.9994	0.0039	0.9960	0.0126	0.9854
7	0.0003	0.9997	0.0019	0.9980	0.0067	0.9921
8	0.0001	0.9999	0.0010	0.9989	0.0036	0.9957
9	0.0001	0.9999	0.0005	0.9994	0.0019	0.9977
10	0.0000	1.0000	0.0003	0.9997	0.0011	0.9987
12	0.0000	1.0000	0.0001	0.9999	0.0003	0.9996
14	0.0000	1.0000	0.0000	1.0000	0.0001	0.9999
16	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
18	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
20	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	1.667E-01		6.667E-01		1.333E+00	
Var.	3.333E-01		1.333E+00		2.666E+00	

k =100	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	0.8889	0.8889	0.6243	0.6243	0.3897	0.3897
1	0.0780	0.9669	0.2192	0.8435	0.2737	0.6635
2	0.0206	0.9875	0.0866	0.9301	0.1562	0.8197
3	0.0072	0.9947	0.0372	0.9673	0.0844	0.9041
4	0.0029	0.9976	0.0169	0.9842	0.0449	0.9489
5	0.0013	0.9989	0.0080	0.9922	0.0238	0.9727
6	0.0006	0.9994	0.0039	0.9960	0.0126	0.9854
7	0.0003	0.9997	0.0019	0.9980	0.0067	0.9921
8	0.0001	0.9999	0.0010	0.9989	0.0036	0.9957
9	0.0001	0.9999	0.0005	0.9994	0.0019	0.9977
10	0.0000	1.0000	0.0003	0.9997	0.0011	0.9987
12	0.0000	1.0000	0.0001	0.9999	0.0003	0.9996
14	0.0000	1.0000	0.0000	1.0000	0.0001	0.9999
16	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
18	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
20	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	1.667E-01		6.667E-01		1.333E+00	
Var.	3.333E-01		1.333E+00		2.667E+00	

(L , U , b1 , b2) = (2 , 4 , 2 , 4)

k = 20	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	0.6667	0.6667	0.4444	0.4444
1	0.0000	1.0000	0.2222	0.8889	0.2963	0.7407
2	0.0000	1.0000	0.0494	0.9383	0.0988	0.8395
3	0.0000	1.0000	0.0329	0.9712	0.0768	0.9163
4	0.0000	1.0000	0.0110	0.9822	0.0293	0.9456
5	0.0000	1.0000	0.0085	0.9907	0.0244	0.9700
6	0.0000	1.0000	0.0033	0.9940	0.0100	0.9800
7	0.0000	1.0000	0.0027	0.9967	0.0087	0.9887
8	0.0000	1.0000	0.0011	0.9978	0.0037	0.9924
9	0.0000	1.0000	0.0010	0.9987	0.0033	0.9956
10	0.0000	1.0000	0.0004	0.9992	0.0014	0.9970
12	0.0000	1.0000	0.0002	0.9997	0.0005	0.9988
14	0.0000	1.0000	0.0001	0.9999	0.0002	0.9996
16	0.0000	1.0000	0.0000	1.0000	0.0001	0.9998
18	0.0000	1.0000	0.0000	1.0000	0.0000	0.9999
20	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		5.772E-01		1.194E+00	
Var.	0.000E+00		1.350E+00		2.943E+00	

k =100	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	0.6667	0.6667	0.4444	0.4444
1	0.0000	1.0000	0.2222	0.8889	0.2963	0.7407
2	0.0000	1.0000	0.0494	0.9383	0.0988	0.8395
3	0.0000	1.0000	0.0329	0.9712	0.0768	0.9163
4	0.0000	1.0000	0.0110	0.9822	0.0293	0.9456
5	0.0000	1.0000	0.0085	0.9907	0.0244	0.9700
6	0.0000	1.0000	0.0033	0.9940	0.0099	0.9799
7	0.0000	1.0000	0.0027	0.9967	0.0086	0.9885
8	0.0000	1.0000	0.0011	0.9978	0.0037	0.9922
9	0.0000	1.0000	0.0010	0.9987	0.0032	0.9954
10	0.0000	1.0000	0.0004	0.9991	0.0014	0.9968
12	0.0000	1.0000	0.0002	0.9996	0.0006	0.9987
14	0.0000	1.0000	0.0001	0.9999	0.0002	0.9994
16	0.0000	1.0000	0.0000	0.9999	0.0001	0.9998
18	0.0000	1.0000	0.0000	1.0000	0.0000	0.9999
20	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
25	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		5.774E-01		1.196E+00	
Var.	0.000E+00		1.355E+00		2.981E+00	

(L , U , b1 , b2) = (4 , 4 , 1 , 4)

k = 20	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	0.5000	0.5000	0.2500	0.2500
1	0.0000	1.0000	0.2500	0.7500	0.2500	0.5000
2	0.0000	1.0000	0.1250	0.8750	0.1875	0.6875
3	0.0000	1.0000	0.0625	0.9375	0.1250	0.8125
4	0.0000	1.0000	0.0156	0.9531	0.0391	0.8516
5	0.0000	1.0000	0.0156	0.9688	0.0430	0.8945
6	0.0000	1.0000	0.0117	0.9805	0.0352	0.9297
7	0.0000	1.0000	0.0078	0.9883	0.0254	0.9551
8	0.0000	1.0000	0.0024	0.9907	0.0085	0.9636
9	0.0000	1.0000	0.0027	0.9934	0.0098	0.9734
10	0.0000	1.0000	0.0022	0.9956	0.0083	0.9817
12	0.0000	1.0000	0.0005	0.9977	0.0022	0.9901
14	0.0000	1.0000	0.0005	0.9989	0.0022	0.9949
16	0.0000	1.0000	0.0001	0.9994	0.0006	0.9972
18	0.0000	1.0000	0.0001	0.9997	0.0006	0.9985
20	0.0000	1.0000	0.0000	0.9998	0.0002	0.9992
25	0.0000	1.0000	0.0000	1.0000	0.0001	0.9998
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		1.078E+00		2.249E+00	
Var.	0.000E+00		3.031E+00		7.049E+00	

k =100	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	0.5000	0.5000	0.2500	0.2500
1	0.0000	1.0000	0.2500	0.7500	0.2500	0.5000
2	0.0000	1.0000	0.1250	0.8750	0.1875	0.6875
3	0.0000	1.0000	0.0625	0.9375	0.1250	0.8125
4	0.0000	1.0000	0.0156	0.9531	0.0391	0.8516
5	0.0000	1.0000	0.0156	0.9688	0.0430	0.8945
6	0.0000	1.0000	0.0117	0.9805	0.0352	0.9297
7	0.0000	1.0000	0.0078	0.9883	0.0254	0.9551
8	0.0000	1.0000	0.0024	0.9907	0.0085	0.9636
9	0.0000	1.0000	0.0027	0.9934	0.0098	0.9734
10	0.0000	1.0000	0.0022	0.9956	0.0083	0.9817
12	0.0000	1.0000	0.0005	0.9977	0.0022	0.9901
14	0.0000	1.0000	0.0005	0.9989	0.0022	0.9948
16	0.0000	1.0000	0.0001	0.9994	0.0006	0.9971
18	0.0000	1.0000	0.0001	0.9997	0.0006	0.9985
20	0.0000	1.0000	0.0000	0.9998	0.0002	0.9991
25	0.0000	1.0000	0.0000	1.0000	0.0001	0.9998
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		1.078E+00		2.250E+00	
Var.	0.000E+00		3.033E+00		7.069E+00	

(L , U , b1 , b2) = (4 , 2 , 1 , 8)

k = 20	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	1.0000	1.0000	0.3333	0.3333
1	0.0000	1.0000	0.0000	1.0000	0.2222	0.5556
2	0.0000	1.0000	0.0000	1.0000	0.1481	0.7037
3	0.0000	1.0000	0.0000	1.0000	0.0988	0.8025
4	0.0000	1.0000	0.0000	1.0000	0.0658	0.8683
5	0.0000	1.0000	0.0000	1.0000	0.0439	0.9122
6	0.0000	1.0000	0.0000	1.0000	0.0293	0.9415
7	0.0000	1.0000	0.0000	1.0000	0.0195	0.9610
8	0.0000	1.0000	0.0000	1.0000	0.0043	0.9653
9	0.0000	1.0000	0.0000	1.0000	0.0058	0.9711
10	0.0000	1.0000	0.0000	1.0000	0.0058	0.9769
12	0.0000	1.0000	0.0000	1.0000	0.0060	0.9880
14	0.0000	1.0000	0.0000	1.0000	0.0027	0.9947
16	0.0000	1.0000	0.0000	1.0000	0.0004	0.9968
18	0.0000	1.0000	0.0000	1.0000	0.0005	0.9979
20	0.0000	1.0000	0.0000	1.0000	0.0005	0.9989
25	0.0000	1.0000	0.0000	1.0000	0.0000	0.9998
30	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		0.000E+00		2.069E+00	
Var.	0.000E+00		0.000E+00		7.416E+00	

k =100	h = 1		h = 4		h = 8	
m	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)	P(M=m)	P(M≤m)
0	1.0000	1.0000	1.0000	1.0000	0.3333	0.3333
1	0.0000	1.0000	0.0000	1.0000	0.2222	0.5556
2	0.0000	1.0000	0.0000	1.0000	0.1481	0.7037
3	0.0000	1.0000	0.0000	1.0000	0.0988	0.8025
4	0.0000	1.0000	0.0000	1.0000	0.0658	0.8683
5	0.0000	1.0000	0.0000	1.0000	0.0439	0.9122
6	0.0000	1.0000	0.0000	1.0000	0.0293	0.9415
7	0.0000	1.0000	0.0000	1.0000	0.0195	0.9610
8	0.0000	1.0000	0.0000	1.0000	0.0043	0.9653
9	0.0000	1.0000	0.0000	1.0000	0.0058	0.9711
10	0.0000	1.0000	0.0000	1.0000	0.0058	0.9769
12	0.0000	1.0000	0.0000	1.0000	0.0043	0.9863
14	0.0000	1.0000	0.0000	1.0000	0.0027	0.9924
16	0.0000	1.0000	0.0000	1.0000	0.0005	0.9949
18	0.0000	1.0000	0.0000	1.0000	0.0008	0.9964
20	0.0000	1.0000	0.0000	1.0000	0.0006	0.9977
25	0.0000	1.0000	0.0000	1.0000	0.0001	0.9992
30	0.0000	1.0000	0.0000	1.0000	0.0001	0.9997
50	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
Mean	0.000E+00		0.000E+00		2.091E+00	
Var.	0.000E+00		0.000E+00		8.188E+00	

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