INVENTORY AND PRICING MANAGEMENT OF PERISHABLE PRODUCTS WITH FIXED AND RANDOM SHELF LIFE

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BY

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Abstract

In this dissertation, we study inventory and revenue management problems for perishable products with customer choice considerations. This dissertation is composed of six chapters. In Chapter 1, we provide an overview and the motivation of problems. Subsequently, in Chapter 2, we propose a joint inventory and pricing problem for a perishable product with two freshness levels. After a stochastic time, a fresh item turns into a non-fresh item, which will expire after another random duration. Under an (r, Q) ordering policy and a markdown pricing strategy for non-fresh items, we formulate a model that maximizes the long-run average profit rate. We then reduce the model to a mixed-integer bilinear program (MIBLP), which can be solved efficiently by state-of-the-art commercial solvers. We also investigate the value of using a markdown strategy by establishing bounds on it under limiting regimes of some parameters such as large market demand. Further, we consider an Economic Order Quantity (EOQ)-type heuristic and bound the optimality gap asymptotically. Our results reveal that although the clearance strategy is always beneficial for the retailer, it may hurt customers who are willing to buy fresh products.

In Chapter 3, we extend this model to the dynamic setting with multiple freshness levels of perishable products. Due to the complexity of the problem, we study the structural properties of value function and characterize the structure of the optimal policies by using the concept of anti-multimodularity. The structural analysis enables us to devise three

novel and efficient heuristic policies. We further extend the model by considering donation policy and replenishment system. Our results imply that freshness-dependent pricing and dynamic pricing are two substitute strategies, while freshness-dependent pricing and donation strategy are two complement strategies for matching supply with demand. Also, high variability in product quality under dynamic pricing benefits the firm, but it may result in significant losses with a static pricing strategy.

In Chapter 4, we study a joint inventory-pricing model for perishable items with fixed shelf lives to examine the effectiveness of different markdown policies, including single-stage, multiple-stage, and dynamic markdown policies both theoretically and numerically. We show that the value of multiple-stage markdown policies over single-stage ones asymptotically vanishes as the shelf life, market demand, or customers' maximum willingness-to-pay increase.

In chapter 5, with a focus on blood products, we optimize blood supply chain structure along with the operations optimization. Specifically, we study collection, production, replenishment, issuing, inventory, wastage, and substitution decisions under three different blood supply chain channel structures, i.e., the decentralized, centralized, and coordinated. We propose a bi-level optimization program to model the decentralized system and use the Karush–Kuhn–Tucker (KKT) optimality conditions to solve that. Although centralized systems result in a higher performance than decentralized systems, it is challenging to implement them. Thus, we design a novel coordination mechanism to motivate hospitals to operate in a centralized system. We also extend the model to the case with demand uncertainty and compare different issuing and replenishment policies. Analysis of a realistic case-study indicates that integration can significantly improve the performance of the system. Finally, Chapter 6 concludes this dissertation and proposes future research directions.

To my beloved parents and wife

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Contents

List of Figures

List of Tables

Declaration of Academic Achievement

This dissertation has resulted in submissions currently under review in different peer-reviewed journals. Specifically, the research paper in Chapter 2 is currently under review at the Production and Operations Management (POM) journal. The research paper based on Chapter 3 will be soon submitted to the Manufacturing & Service Operations Management (M&SOM) journal. Chapter 4 has resulted in a research paper which is currently under review at the Omega journal. Finally, a research paper derived from the deterministic model in Chapter 5 is currently under review at the International Transactions in Operational Research (ITOR) journal, and a research paper based on the stochastic model in Chapter 5 is under preparation for submission to a peer-reviewed journal.

Chapter 1

Introduction

1.1 Overview and Motivations

Perishable products encompass a wide variety of items, ranging from common retail store goods such as food, pharmaceuticals, and cut flowers to perishable medical products, including blood components. These products are considerably important to the global economy. In particular, the retail industry relies heavily on perishable products as an important source of revenue. Recent data on fresh produce sales in the United States indicates remarkable growth, setting a record of \$75.1 billion in 2022 (Riemenschneider 2023). Despite their profitability, perishable products are susceptible to wastage because they have limited shelf life. Such wastages pose a significant threat to both profitability and sustainability. For instance, Refed, an organization working to reduce food waste in the United States, reports that supermarkets discard around 43 billion pounds of food each year, accounting for 10% of all U.S. food waste. This contributes to an estimated yearly cost of \$18.2 billion, resulting in substantial revenue loss (Grocery Dive 2021).

Managing inventory and pricing are two crucial controls for firms to reduce wastage and enhance profitability. However, the primary challenge in an effective management system often lies in matching the perishable supply with uncertain demand. To address this challenge, different firms adopt different strategies including freshness-dependent decision-making, dynamic decision-making, and integrated decision-making, which are elaborated below.

1.1.1 Freshness-dependent Decision-making

Perishable products lose their value and utility over time. In response, retailers typically implement markdown strategies to clear inventory nearing expiration. These strategies involve price reductions to increase profits while minimizing wastage. According to a report by Smith School of Business, approximately one-third of total unit sales are generated through the implementation of markdowns (Morantz 2016). Therefore, effective markdown policy management is critical for firms dealing with perishable products. On the one hand, markdown pricing appeals to customers who are willing to trade quality for lower prices, which in turn increases sales and reduces wastages. On the other hand, products of different freshness levels create an assortment with a dynamically changing composition, leading to cannibalization of each other's demand. Thus, precise management of both their inventory levels and prices is critical to strike a balance between reducing wastage and mitigating cannibalization. Thus, to examine this balance, in Chapter 2 and 3 of this dissertation, we study a joint inventory-pricing problem with customer choice for a perishable product that has deteriorating quality levels. We specifically analyze an assortment of perishable items, including products of different freshness levels, that has its composition dynamically changing.

Firms have the option to adopt various markdown approaches, including single-stage, multiple-stage, and dynamic markdown policies. A single-stage markdown policy entails a single price reduction before the expiration date, offering simplicity but risking lost sales or waste due to improper timing. A multiple-stage markdown policy involves several price reductions at various intervals before expiration, potentially yielding higher revenue compared to single-stage policies. A dynamic markdown strategy utilizes real-time data to adjust prices, optimizing revenue and waste reduction. However, determining optimal prices in real-time for perishable items with fixed shelf-lives poses a significant challenge. Among different markdown policies, the optimal policy is not always clear-cut, as there is a trade-off between the simplicity/complexity of the policy and the revenue generated. While simple markdown policies may be easier to implement, they may not lead to optimal revenue generation. Thus, in Chapter 4 of this dissertation, we examine the effectiveness of different markdown policies given their complexities under different scenarios. To that end, we explore joint inventory and pricing decisions within different markdown models and conduct a comparative analytical analysis of results.

1.1.2 Dynamic Decision-making

Another approach commonly employed to achieve an improved alignment between supply and demand is dynamic decision-making. Recent technological advancements, including digital marketing, electronic shelf labeling, and Radio Frequency Identification (RFID) sensing, have enabled the implementation of dynamic pricing strategies in various industries, including grocery retailing. This trend is evident in Amazon's operations, particularly with AmazonFresh, where prices are adjusted roughly every 10 minutes on average, a frequency significantly higher than that of Walmart (Business Insider 2018). Similarly, E.Leclerc, a French hypermarket chain, has implemented electronic shelf tags in numerous stores, leading to over 5,000 weekly price changes (RW3 2016).

Dynamic pricing can offer significant advantages, particularly for perishable inventory like fresh produce, which must navigate shifts in demand and shelf life. This strategy allows retailers to dynamically adjust the price for perishable items approaching their expiration, attracting price-conscious consumers willing to trade quality for a reduced price, while also managing cannibalization effects. Despite these benefits, dynamic optimization of pricing and production decisions for perishable products remains challenging, especially when an assortment includes items of varying freshness levels and customer choices evolve over time. To address this problem, in Chapter 3 of this dissertation, we study structural properties of dynamic inventory and pricing decisions for a perishable product with multiple freshness levels. Subsequently, we utilize the structure of optimal decisions to devise three efficient heuristic polices.

1.1.3 Integrated Decision-making

The integration of decisions across different stages of the supply chain is a key factor in achieving a better match between supply and demand. This aspect is particularly critical in the management of blood products, where uncertain supply and demand pose significant challenges. In many countries, including Canada, the decentralized structure of the blood supply chain hinders effective matching due to independent decisions about production quantities and inventory levels by hospitals and blood centers. This lack of information sharing can create a bullwhip effect, leading to overcollection, overproduction, and consequently high wastage rate (Li et al. 2021, Motamedi et al. 2021). For instance, Canadian Blood Services (CBS) reported discard rates of 6% for whole blood and 10% for platelets in 2018 (Emadi 2020). Conversely, there are cases of stock-outs due to hospitals inflating orders to ensure sufficient supplies, even by carrying excess inventory, placing emergency orders, or purchasing components externally. This inefficiency is highlighted by instances such as Canada's \$512 million purchase of blood components from the United States in 2016 (Dinerstein 2018) and CBS's urgent call for at least 22K blood donations in response to shortages in 2018 (Global News 2018). The coexistence of wastage and shortage underscores the need for an integrated decision-making process that can more accurately match supply and demand. Thus, in Chapter 5 of this dissertation, we optimize the blood supply chain structure along with the operations optimization. To that end, we investigate the optimal operational decisions under three different supply chain structures: decentralized, centralized, and coordinated. A benchmark centralized structure is established to evaluate the performance of the existing decentralized system, while a coordination mechanism is introduced to facilitate the implementation of integrated decision-making process.

1.2 Summary of Main Contributions

In this section, we provide a summary of main results and contributions of this dissertation:

Joint static inventory and pricing problem with customer choice: in Chapter 2, we study a joint inventory-pricing problem for dynamic assortments of a single perishable item, consisting of fresh and non-fresh items. We reformulate the problem to obtain global optimal solutions. We further discuss some theoretical results, based on which an asymptotic optimal heuristic is developed. More specifically:

1. We present a pioneering study on the assortment pricing and inventory problem with customer choice for perishable products under uncertainty. In contrast to the previous works, our model considers a dynamic assortment composition of a single perishable item, including fresh and non-fresh products.

2. we examine the ordering policy that considers the non-fresh inventory and the policy that does not. The results suggest that integrating non-fresh items into the ordering policy can be beneficial when demand or shelf life is low. Additionally, the results imply that incorporating non-fresh items into the ordering policy is advantageous when the value of non-fresh products or order arrival rate is high.

3. Recognizing the intricate nature of the nonlinear queueing model, we reframe the problem as a Mixed-Integer Bi-Linear Programming model (MIBLP). Through this approach, we attain a globally optimal solution for the proposed model.

4. We introduce upper and lower bound models to establish theoretical bounds on clearance value under various parameter regimes. Our findings indicate that the value of clearance diminishes when market demand or mean time between expirations becomes very large. This insight is extended to encompass general renewal demand and expiration processes.

5. Given the complexities of large-scale scenarios, we propose an approximation method based on the classical Economic Order Quantity (EOQ) model and demonstrate the asymptotic optimality of this method.

Joint dynamic inventory and pricing problem for a perishable product with multiple freshness levels: in Chapter 3, we extend the problem in Chapter 2 to the dynamic setting with multiple freshness levels. Specifically, we study joint dynamic production and pricing decisions for a dynamically changing assortment consisting of a single product with multiple freshness levels. We further analyze the structural properties of optimal decisions and develop three novel heuristic models to overcome the complexity of the problem. Specifically:

1. We make the first attempt to study a dynamic inventory-pricing model for a single perishable item, considering varying freshness levels and dynamically changing assortment compositions. Unlike previous works focusing on static settings, our model considers dynamic decisions.

2. We analyze the structural properties of the value function and optimal production and pricing policies. The results suggest that the optimal production decision follows a threshold-based policy.

3. To tackle the model's intricacy, we introduce three new heuristics, with the third one considering the optimal policy structures to enhance efficiency.

4. We extend our model to encompass donation options, replenishment systems, and multi-phase quality transformations.

Joint inventory and markdown policies for perishable products with fixed shelf life: Chapter 4 focuses on perishable products with fixed shelf life, distinguishing it from Chapters 2 and 3, which centered around items with random shelf life. In Chapter 4, we study joint inventory and pricing problems for perishable products under various markdown policies, including single-stage markdown, multiple-stage markdown, and dynamic markdown policies. Then, we compare the performance of different markdown policies both theoretically and numerically. More specifically:

1. This research is the first that both theoretically and numerically compares the efficiency of different markdown strategies for perishable products with multi-period

shelf life. We prove and numerically show that the value of markdown policies vanishes asymptotically as the shelf life, market demand, or maximum willingness-to-pay grow large, while the benefits of markdown policies increase as per-unit expiration, shortage, and purchasing costs increase.

2. We derive the long-run average profit rate in a steady-state for various markdown policy models. Additionally, we perform an asymptotic analysis of the optimal inventory and pricing choices under different parameter settings, including market demand, shelf life, and maximum WTP.

3. To overcome the complexity of dynamic markdown policies, we introduce an approximation method. This approach involves approximating the connection between remaining shelf life and price using mapping functions.

4. We conduct computational experiments on two real-world case studies: a fresh produce supply chain, a farm in Canada, and a bakery chain in France. Our findings indicate that adopting single markdown policies can significantly enhance profits and reduce wastage. However, applying multiple-stage markdown policies might not always outweigh their complexities.

5. We extend the base model to the case of Last In, First Out (LIFO) issuing policy and freshness-dependent demand.

Coordinating a decentralized blood supply chain with interactions between supply-side and demand-side operational decisions: in contrast to Chapters 2, 3, and 4 which mainly focus on the perishable products in general, Chapter 5 specifically centers around the special case of blood products. This chapter studies a joint optimization of collection, production, distribution, demand satisfaction, substitution, and transshipment decisions in the multi-product setting under different Blood Supply Chain (BSC) structures, i.e., decentralized, centralized, and coordinated systems.

1. This study pioneers an integrated optimization model that addresses the coordination of both supply- and demand-related operations within a blood supply chain, encompassing different blood products and blood groups.

2. Although the interaction between the blood center and hospitals in blood supply chain plays a pivotal role in supply and demand matching, it has been overlooked in the literature. To fill this gap, we make the first attempt to optimize both the structure and operations of the blood supply chain on both supply stage (collection and production) and demand stage (inventory and issuing). To that end, we study and compare three different supply chain structures: decentralized, centralized, and coordinated.

3. We develop a bi-level optimization program to formulate the decentralized problem and apply Karush–Kuhn–Tucker (KKT) to solve the problem. Additionally, we establish a centralized framework that serves as a benchmark model for assessing the performance of decentralized model. Then, we introduce a coordination mechanism that aims to facilitate the implementation of centralized model.

4. Given the inherent uncertainty in patient demand, we extend our model to the case with demand uncertainty. Then, we employ a robust optimization method to address this uncertainty. In contrast to earlier works that conducted one-dimensional analysis for selecting robustness parameters, our research adopts a more comprehensive two-dimensional analysis.

5. This study pioneers the comparison of various issuing and replenishment policies

in a BSC. Specifically, we compare and analyze the performance of FIFO, LIFO, and threshold-based issuing policies in terms of total costs and freshness level of transfused items. Further, we assess the performance of two commonly used ordering policies, namely (s,S) and (R,T), and compare their performance against the optimal replenishment policy.

6. We apply the proposed models to a real-world problem in Hamilton, Canada. Our results imply that integration of operations can benefit the blood supply chain by reducing the mean gap between production and consumption by and total costs. We also perform comprehensive assessment of the proposed models and provide managerial insights for blood centers.

1.3 Overview of Dissertation

This dissertation is organized as follows. In Chapter 2, we study a continuous-review inventory-pricing model for a perishable product with two freshness levels, fresh and non-fresh, whose assortment is dynamically changing. We propose a reformulation of the model to solve the problem to the global optimal solution, and then we present an EOQ-type heuristic model. In Chapter 3, we analyze a problem similar to the one in Chapter 2. However, this time, we optimize dynamic inventory and pricing decisions for perishable products with multiple freshness levels. We analyze the structural properties of the optimal solutions and develop three effective heuristic policies. Chapter 4 presents a comparative analysis of various markdown policies, including single-stage, multiple-stage, and dynamic markdown policies. We also theoretically discuss the value of markdown policies under different parameter regimes and use two case studies to discuss the results numerically.

In Chapter 5, we focus on blood products as a special case of perishable products. This chapter investigates the integrated optimization of collection, production, distribution, and demand fulfillment decisions within a multi-product framework, considering various BSC structures: decentralized, centralized, and coordinated. Finally, Chapter 6 concludes the dissertation, and outlines directions for future works.

In closing, we note that this dissertation is represented in a sandwich format and hence different notations have been used across different chapters. Moreover, in each chapter, we independently explore the existing literature, identify gaps, and subsequently position our contribution to the existing body of literature.

Chapter 2

Mark Down to Waste Less: A Joint Inventory and Pricing Problem with Two Freshness Levels

2.1 Introduction

Perishable products with limited shelf-lives, such as foodstuffs, pharmaceuticals, and cut flowers constitute the majority of the retail sector. Wastage of such products is among the top threats to profitability and sustainability. For example, a survey conducted by the National Supermarket Research Group shows that a 300-store grocery chain lost around \$34 million a year due to spoilage of grocery products. 20% (or 11 million tons) of all the agri-food produced in Canada annually becomes food loss or waste (Value Chain Management International and Secord Harvest 2019). Inventory and pricing decisions are two key controls for a firm to reduce wastage. In particular, perishable products lose their value over time, and it is common for retailers to use markdown pricing strategies to clear the inventory of items close to the expiration. On the one hand, markdown pricing attracts customers who are willing to trade quality for lower price, thereby increasing sales and reducing wastage. On the other hand, items of different quality levels create an assortment with a dynamically changing composition and may cannibalize each other's demand. Thus, both their inventory levels and prices need to be carefully chosen to strike a balance between reducing wastage and mitigating cannibalization.

In this research, we consider a continuous-review inventory system and study the joint inventory replenishment and pricing decisions. We consider items of two quality levels, namely, fresh items and non-fresh items. Fresh items turn into non-fresh ones, and non-fresh items expire, both of which happen after a random period of time. Customer demand arrives according to a Poisson process, and each customer purchases a fresh item, a non-fresh item, or walks away without buying anything, based on whichever option maximizes their surplus. We formulate the joint inventory management and pricing problem as a mixed integer programming (MIP) model, and then explore exact and approximate methods to solve for the optimal inventory and pricing decisions. Our model applies not only to perishable products but also to products with deteriorating quality during storage. For examples, being handled by customers while displayed in store, products may show signs of tear and wear or have the package damaged.

Our research contributes to the literature in the following ways. First, to the best of the authors' knowledge, our work makes the first attempt to study the assortment pricing problem with customer choice for perishable products under uncertainty. A remarkable feature of our model is that the assortment, which consists of fresh and non-fresh items, has its composition dynamically changing as fresh items becomes non-fresh and non-fresh items expire. Further, we consider the ordering policy that considers the non-fresh inventory and the policy that does not. The results suggest that incorporating non-fresh items in the ordering policy can be advantageous when demand or shelf life is low, or the value of non-fresh products or order arrival rate is high. Second, given the inherent complexity of the highly non-linear queueing model, we reformulate the problem as a Mixed-Integer Bi-linear Programing model (MIBLP) and obtain a global optimal solution for the proposed model. Third, we introduce upper and lower bound models to theoretically establish bounds on the value of clearance across various parameter regimes. Our results imply that when market demand or mean time between two expirations become very large, the value of clearance vanishes. We further generalize the results to the case of general renewal demand and expiration processes. Finally, given the complexity of the large-scale problems, we propose an approximation method based on the classical economic order quantity (EOQ) model and show that it is asymptotically optimal.

The remainder of this chapter is organized as follows. In Section 2.2, we review the relevant literature. Section 2.3 presents the problem and the its formulation. Section 2.4 presents bounds on the value of clearance, and section 2.5 shows that under certain conditions, managing perishable inventory becomes equivalent to non-perishable one. In section 2.6, we introduce an EOQ-type heuristic model. We present a realistic case study and present computational results in section 2.7. We present managerial insights in section 2.8 and conclude this chapter in section 2.9.

2.2 Literature Review

The literature on the inventory management and pricing decisions of perishable products is extensive. Our research is related to three streams of literature, namely, inventory management of perishable products, joint inventory management and pricing of perishable products, and assortment pricing problems with customer choice.

2.2.1 Perishable Inventory Management

Given that our work studies a continuous-review inventory system, in what follows we focus on papers under the continuous-review setting. Ravichandran (1995) studied the (s,S) continuous-review inventory model with random order lead-times, constant lifetime of the product, Poisson demand, and lost-sales. They obtained closed-form expressions for the optimal ordering policy. Baron et al. (2010) studied an (s,S) inventory model of perishable products arriving in batches. They derived the optimal inventory policy in closed-form for some special cases and developed efficient heuristics based on the fluid approximation. Olsson (2014) studied the base-stock ordering policy for a model with a combination of backorders and lost-sale in case of a stockout. Kouki et al. (2018) studied the value of dual sourcing in a perishable inventory system. Berk and Gürler (2008) examined the (r,Q) replenishment policy for a product with a constant lifetime and order lead-time. With lost-sales, they modeled the inventory dynamics as an embedded Markov process and showed that this policy performs well. In a different paper, the same authors extended the model to the case that allows multiple outstanding orders (Berk et al. 2020). Kouki et al. (2015) investigated continuous-review can-order (s,c,S) replenishment policy for coordination of ordering multiple perishable products with fixed lead-time and exponential lead-time. They presented an iterative algorithm to address their extended Markov process model.

Some papers in the literature have studied perishable inventory models with random shelf life. Kalpakam and Shanthi (2006) and Liu and Shi (1999) studied the optimal (s,S) replenishment policy for a problem with exponentially distributed product shelf-life and order lead-time, Poisson demand process, and lost-sales. Baron et al. (2020) and Barron (2019) considered batch arrivals of customer demand. Gürler and Özkaya (2008) considered items with generally distributed shelf-life, renewal arrival process, and zero lead-time. They showed that the loss resulting from ignoring the randomness of the lifetime can be extreme, and that the distribution of lifetime considerably affects the total cost and should be estimated accurately. Barron and Baron (2020) considered general distributions for product shelf-life and replenishment lead-time. They applied the queueing and Markov chain decomposition (QMCD) approach to enable analysis. Kouki et al. (2020) studied an inventory system with a base-stock replenishment policy, uncertain product shelf-life and order lead-times, and lost-sales. They showed some monotonicity properties of cost function and applied the queueing network approach to overcome the complexity of their problem. The papers reviewed above assumed price as an exogenous parameter, while in this research, we optimize prices along with replenishment decisions and propose a clearance strategy to improve profitability.

2.2.2 Inventory-Pricing Problems for Perishable Products

Abad (1996) studied a joint pricing and inventory optimization model where demand can be partially backordered. Li et al. (2009) characterized the structure of the optimal policy for a product with a two-period lifetime and developed a base-stock/list-price heuristic model for products with multi-period lifetime, when demand is backlogged and the planning horizon is finite. Li et al. (2012) further studied the model for the case of lost sales over an infinite horizon. Under both backorders and lost-sales, Chen et al. (2014) obtained structural properties of the optimal inventory and pricing decisions and developed an effective heuristic policy. Fang et al. (2021) considered a model with multiple substitutable products, for which they provide some analytic properties of the optimal decisions and a numerical scheme for computation.

The above papers focused on joint inventory and pricing decisions while overlooking customers' choices over fresh and non-fresh items. Specifically, these works considered a uniform quality and price for perishable products within a single assortment, neglecting the varying preferences customers may have based on freshness. To address this gap, the papers reviewed in the following section have examined inventory-pricing problems while taking into account the influence of customer choice.

2.2.3 Inventory-Pricing Problems with Customer Choice

Existing papers have studied pricing and/or inventory decisions for a given assortment. Mahajan and Van Ryzin (2001) and Netessine and Rudi (2003) optimized inventory levels of products in an assortment with given prices by considering stockout-based substitution. Aydin and Porteus (2008) studied joint inventory and pricing model of a given assortment in a single period model for general demand models. They showed that the optimal price for each product is unique and optimum though profit function is not even quasi-concave in prices. For the multi-period case, Song et al. (2021) and Dong et al. (2009) presented joint inventory and pricing models for substitutable products. They showed that expected revenue function is a concave function of market shares of the products characterized the structure of the optimal policy. Ceryan et al. (2013) developed a joint pricing and flexible capacity planning for a given assortment to manage mismatch between supply and demand. They considered a problem with two products and characterized the optimal production and pricing decisions. They showed that the presence of flexible resources helps the firm to keep stable price differences across different products over time.

Some papers in the literature studied joint optimization of assortment planning and pricing. Anderson et al. (1992) optimality of offering the same markup, i.e., selling price minus cost, for all products. However, when there are different price sensitivity coefficients, their result may no longer be optimal (Wang 2012, Gallego and Wang 2014). Maddah and Bish (2004) developed a joint, inventory, pricing, and assortment optimization problem in a single-period setting. They assumed customers arrive according to Position process and chose products based on and multinomial logit choice model. To mitigate the complexity, they approximated the expected profit.

In the literature of revenue management, many papers studied the markdown pricing strategy with cannibalization between regular and markdown-priced items. Early work such as Barnhart and Talluri (1997) and McGill and Van Ryzin (1999) assumed no cannibalization between regular-priced items and those on clearance-sales. Talluri and Van Ryzin (2004) relaxed this assumption. Taking into account product availability on customers' choice, they studied the capacity allocation of products sold at different prices. Research on network revenue management has studied markdown pricing problems for multiple (i.e., more than two) resources; see, e.g., Gallego et al. 2004 and Liu and Van Ryzin 2008. Li et al. (2016) considered a joint dynamic replenishment and clearance problem. Hu et al. (2016) studied dynamic inventory and markdown policies in a model where each period consists of clearance and regular-sales phases. Their results suggest that the firm should put all left-over inventory on clearance if it is higher than a threshold. Ferguson and Koenigsberg (2007) optimized price and inventory decisions of new and old products in a two-period setting. They assumed that there is no uncertainty in the second period and the retailer realizes demand before the second period. Their results show the benefit of selling old items in the second period pales the detrimental effect of cannibalization between new and old items. Li et al. (2012) considered a similar problem in the context of grocery products, but they assumed that new and old items cannot be sold at the same time. Gallego et al. (2008) developed a two-period pricing problem in which retailers can set different prices in both periods. Similar to Li et al. (2012), they assumed that regular and markdown-priced items cannot be sold together. They showed that with a mixture of myopic and strategic customers, the markdown policy may be optimal; if all customers are strategic, however, the single-price policy is optimal. Sainathan (2013) studied the joint pricing and ordering decisions for products with a two-period lifetime. Their results show that selling old products can be profitable under demand uncertainty. den Boer et al. (2022) considered a single perishable product with a multi-period shelf life. The authors considered both the fixed price policy and the markdown policy that applies a price discount to the items in their last period of shelf life. They showed that in the scaling limit model, the markdown policy cannot improve the profit over the fixed price policy and the optimal solution reduces to a single price in a single assortment in contrast to our research. Under deterministic conditions, the optimal approach is setting a single price for a single assortment, regardless of the product's shelf life. This finding contrasts with the results presented in our work, which considers the effects of uncertainty on the optimal policy.

Compared to the extant literature, we study a joint inventory-pricing problem for a perishable product that has deteriorating quality levels. To the best of our knowledge, this is the first study that considers pricing for an assortment with dynamically changing composition (due to quality deterioration), along with inventory replenishment decisions under uncertain conditions. To assess the value of pricing items of different quality levels distinctly, we analytically compare the performance of systems with and without such a pricing mechanism. Additionally, we are the first to analyze the optimal inclusion of expiring products within the (r, Q) ordering policy.

As the underlying model suffers from a high degree of nonlinearity, we reformulate the problem as a mixed-integer bi-linear programming model (MIBLP) and solve it to attain a globally optimal solution. For large-scale problems, we propose an EOQ-type heuristic model and show that it is asymptotically optimal. Furthermore, we introduce upper and lower bound models to establish theoretical bounds on the value of clearance across different parameter regimes. Then, we generalize our results to the case of general renewal demand and expiration processes.

2.3 Problem Statement and Model Formulation

We consider a continuous-review inventory system of a product with two quality levels, namely, fresh and non-fresh. Fresh products turn into non-fresh after an exponentially distributed period of time with the rate θ_1. Non-fresh products becomes spoiled and needs to disposed of after an exponentially distributed period of time with rate θ_2. All deterioration times are independent of each other. Thus, when there are i fresh products in the inventory, the rate of transformation from fresh to non-fresh products is represented by $i\theta_1$; In other words, the time until the next fresh item becomes non-fresh is exponentially distributed with the rate $i\theta_1$. Similarly, when there are j non-fresh products in the inventory, the expiration rate can be expressed as $j\theta_2$. Once a non-fresh unit perishes, that unit is disposed of. The firm makes both the pricing decisions for the two types of items and inventory replenishment decisions.

This research adopts exponential phase transformation rates for tractability and to achieve globally optimal solutions. While this model is applicable for perishable items such as strawberries and tomatoes with fluctuating freshness, Chapter 3 extends its applicability to products with less variable shelf life. The third extension in Chapter 3 considers phase transformation rates as sum of exponential phases. In cases where mean shelf life times are nearly deterministic, a practical approach involves using a large number of phases, treating each stage as a deterministic time bucket.

The notation in Table 2.1 is used for formulation throughout this research.

Table 2.1: Notations

Notation	**Meaning**
Sets	
i	Set of fresh inventory levels. k and l are indicators of i
j	Set of non-fresh inventory levels.
Parameters	
λ_1	Rate of demand arrival for fresh products
λ_2	Rate of demand arrival for non-fresh products
θ_1	Rate of changing fresh products to non-fresh products
$\nu_1 = 1/\theta_1$	Mean time for a fresh item to become non-fresh
θ_2	Rate of perishing non-fresh products
$\nu_2 = 1/\theta_2$	Mean time for a non-fresh item to expire
γ	Rate of order replenishment arrival
	(lead-time is exponentially distributed with a mean of $1/\gamma$)
I_{Max}	Upper bound of fresh inventory level
J_{Max}	Upper bound of non-fresh inventory level
A	Fixed ordering cost
C_p	Unit purchase cost
C_h^1	Unit holding cost for fresh products
C_h^2	Unit holding cost for non-fresh products
C_e	Unit expiration cost
C_d	Unit discarding cost of non-fresh products due to the lack of space
C_{cap}	Unit cost for maintaining the total storage capacity
M	A very large number
Decision Variables	
r	Reorder point (integer variable)
Q	Order quantity (integer variable)

S	The capacity of non-fresh products (integer variable)
$\phi_1(\varphi_1)$	Fraction of customers buying fresh products when there are (are not) non-fresh products in the inventory
$\phi_2(\varphi_2)$	Fraction of customers buying non-fresh products when there are (are not) fresh products in the inventory
$\pi_{i,j}$	Steady-state probability when inventory levels of fresh and non-fresh products are i and j, respectively
β	1 when non-fresh products are included in the ordering policy, 0 otherwise
Auxiliary Variables	
$x^1_{i,j}$	The total rate out of the state (i,j) due to demand arrival or expiration of fresh and non-fresh products
$x^2_{i,j}$	The total rate out of the state (i,j) due to order arrival
$y^1_{i,j}$	The total rate into the state (i,j) due to demand arrival or expiration of non-fresh products
$y^2_{i,j}$	The total rate into the state (i,j) due to demand arrival for fresh items
$y^3_{i,j}$	The total rate into the state (i,j) due to changing fresh to non-fresh products when $j \in [0, S-1]$
$y^4_{i,j}$	The total rate into the state (i,j) due to changing fresh to non-fresh products when $j = S$
$y^5_{i,k,j}$	The total rate from state (k,j) into state (i,j) due to order arrival
Z^1_i	1 when $i \leq r+Q$, 0 otherwise
Z^2_j	1 when $j \leq S$, 0 otherwise
$Z^3_{i,j}$	1 when $i+\beta j \leq r$, 0 otherwise
Z^4_j	1 when $j \geq S$, 0 otherwise
$Z^5_{i,k}$	1 when $k \geq i-Q$, 0 otherwise
$Z^6_{i,k}$	1 when $k \leq i-Q$, 0 otherwise

2.3.1 The Pricing Decisions

The firm sets the prices statically for both fresh and non-fresh items. Let us denote the prices for fresh and non-fresh items as P_1 and P_2, respectively, and define the price vector $\boldsymbol{P} := \{P_1, P_2\}$. Demand for fresh and non-fresh items follow Poisson processes with rate $\lambda_1(\boldsymbol{P})$ and $\lambda_2(\boldsymbol{P})$, respectively. Potential customers who intend to buy the product arrive at the rate Λ, representing the total market size of the product. Let V be a consumer's willing-to-pay (WTP) for fresh items with density function $f(.)$ and distribution $F(.)$.

Consumers have a lower valuation of non-fresh items than for fresh items. We assume that the value of non-fresh items is $\psi(V) \in [0, V]$, which is a decreasing function of V. One example can be $\psi(V) = \delta V$, where $\delta \in [0,1]$. Given introduced notation, the surplus of a customer from buying a fresh (resp., non-fresh) item is $U_1 = V - P_1$ (resp., $U_2 = \psi(V) - P_2$). We derive the probability of a customer buying a fresh (resp., non-fresh) item when both products are available denoted by ϕ_1 (resp., ϕ_2) as follows.

$$\phi_1(P_1, P_2) = P(V - p_1 \geq \psi(V) - P_2, V - P_1 \geq 0), \tag{2.1}$$

$$\phi_2(P_1, P_2) = P(\psi(V) - P_2 \geq V - P_1, \psi(V) - P_2 \geq 0). \tag{2.2}$$

The probability of no purchase can be obtained as $1 - \phi_1 - \phi_2 \geq 0$. The probability for a customer to buy a fresh (resp., non-fresh) item when non-fresh (resp., fresh) items are out of stock, denoted by φ_1 (resp., φ_2), can be expressed as follows.

$$\varphi_1(P_1) = P(V - P_1 \geq 0) = 1 - F(P_1), \tag{2.3}$$

$$\varphi_2(P_2) = P(\psi(V) - P_2 \geq 0) = 1 - F\left(\psi^{-1}(P_2)\right). \tag{2.4}$$

Therefore, when both items are available, the demand rates for fresh and non-fresh items are given by $\lambda_1(\boldsymbol{P}) = \phi_1(\boldsymbol{P})\Lambda$ and $\lambda_2(\boldsymbol{P}) = \phi_2(\boldsymbol{P})\Lambda$, respectively. Further, $\bar{\lambda}_1(P_1) = \varphi_1(P_1)\Lambda$ is the demand rate for fresh products when non-fresh items are not available and $\bar{\lambda}_2(P_2) = \varphi_2(P_2)\Lambda$ is the demand rate for non-fresh products when fresh items are out of stock. It is assumed that customers who arrive during stock-out situation are lost. Following Yan et al. (2017), it is equivalent to use the fractions of customers purchasing fresh and non-fresh items when both items are in stock, i.e., $\phi_1 = \phi_1(\mathbf{P})$ and $\phi_2 = \phi_2(\mathbf{P})$, as decisions. Once ϕ_1 and ϕ_2 are determined, one may easily express the corresponding prices as functions of ϕ_1 and ϕ_2 by inverting (2.1) and (2.2) as follows.

$$P_1(\phi_1, \phi_2) = \psi\left(F^{-1}(1-\phi_1-\phi_2)\right) + F^{-1}(1-\phi_1) - \psi\left(F^{-1}(1-\phi_1)\right), \tag{2.5}$$

$$P_2(\phi_1, \phi_2) = \psi\left(F^{-1}(1-\phi_1-\phi_2)\right), \tag{2.6}$$

where $(\phi_1, \phi_2) \in \Omega := \{(\phi_1, \phi_2) : 0 \le \phi_1 \le 1,\ 0 \le \phi_2 \le 1,\ 0 \le \phi_1 + \phi_2 \le 1\}$. By writing the demand rates as $\lambda_1 = \phi_1\Lambda$, $\lambda_2 = \phi_2\Lambda$, $\bar{\lambda}_1 = \varphi_1(P_1(\phi_1, \phi_2))\Lambda$, and $\bar{\lambda}_2 = \varphi_2(P_2(\phi_1, \phi_2))\Lambda$, we can express the revenues from selling fresh and non-fresh items as functions of ϕ_1 and ϕ_2 as follows.

Revenue from selling fresh items when both items are in stock:

$$G_1(\phi_1, \phi_2) := P_1(\phi_1, \phi_2) \times \phi_1\Lambda, \tag{2.7}$$

Revenue from selling non-fresh items when both items are in stock:

$$G_2(\phi_1, \phi_2) := P_2(\phi_1, \phi_2) \times \phi_2\Lambda, \tag{2.8}$$

Revenue from selling fresh items when only fresh items are in stock:

$$\bar{G}_1(\phi_1, \phi_2) := P_1(\phi_1, \phi_2) \times \varphi_1(P_1(\phi_1, \phi_2))\Lambda, \tag{2.9}$$

Revenue from selling non-fresh items when only fresh items are in stock:

$$\bar{G}_2(\phi_1, \phi_2) := P_2(\phi_1, \phi_2) \times \varphi_2(P_2(\phi_1, \phi_2))\Lambda. \tag{2.10}$$

For the sake of simplicity, hereafter we assume that customers' utility uniformly distributed over $[0, \Gamma]$, where Γ is the maximum willingness-to-pay for fresh items. Further, we assume that the value of non-fresh items is a fraction of the value of fresh products, i.e., $\psi(V) = \delta V$ for some $\delta \in [0, 1]$. Under these assumptions, the demand rates λ_1, λ_2, $\bar{\lambda}_1$, $\bar{\lambda}_2$ can be simplified as follows.

$$\lambda_1(\phi_1) = \phi_1\Lambda, \;\; \lambda_2(\phi_2) = \phi_2\Lambda, \;\; \bar{\lambda}_1(\phi_1, \phi_2) = \Lambda(\phi_1 + \delta\phi_2), \text{ and } \bar{\lambda}_2(\phi_1, \phi_2) = \Lambda(\phi_1 + \phi_2). \tag{2.11}$$

The revenue rates G_1, G_2, $\bar{G}_1$, $\bar{G}_2$ can be written as follows.

$$G_1(\phi_1, \phi_2) = \Lambda\Gamma\phi_1(1 - \phi_1) - \delta\Lambda\Gamma\phi_1\phi_2, \tag{2.12}$$

$$G_2(\phi_1, \phi_2) = \delta\Lambda\Gamma\phi_2(1 - \phi_1 - \phi_2), \tag{2.13}$$

$$\bar{G}_1(\phi_1, \phi_2) = \Lambda\Gamma(\phi_1 + \delta\phi_2)(1 - \phi_1 - \delta\phi_2), \tag{2.14}$$

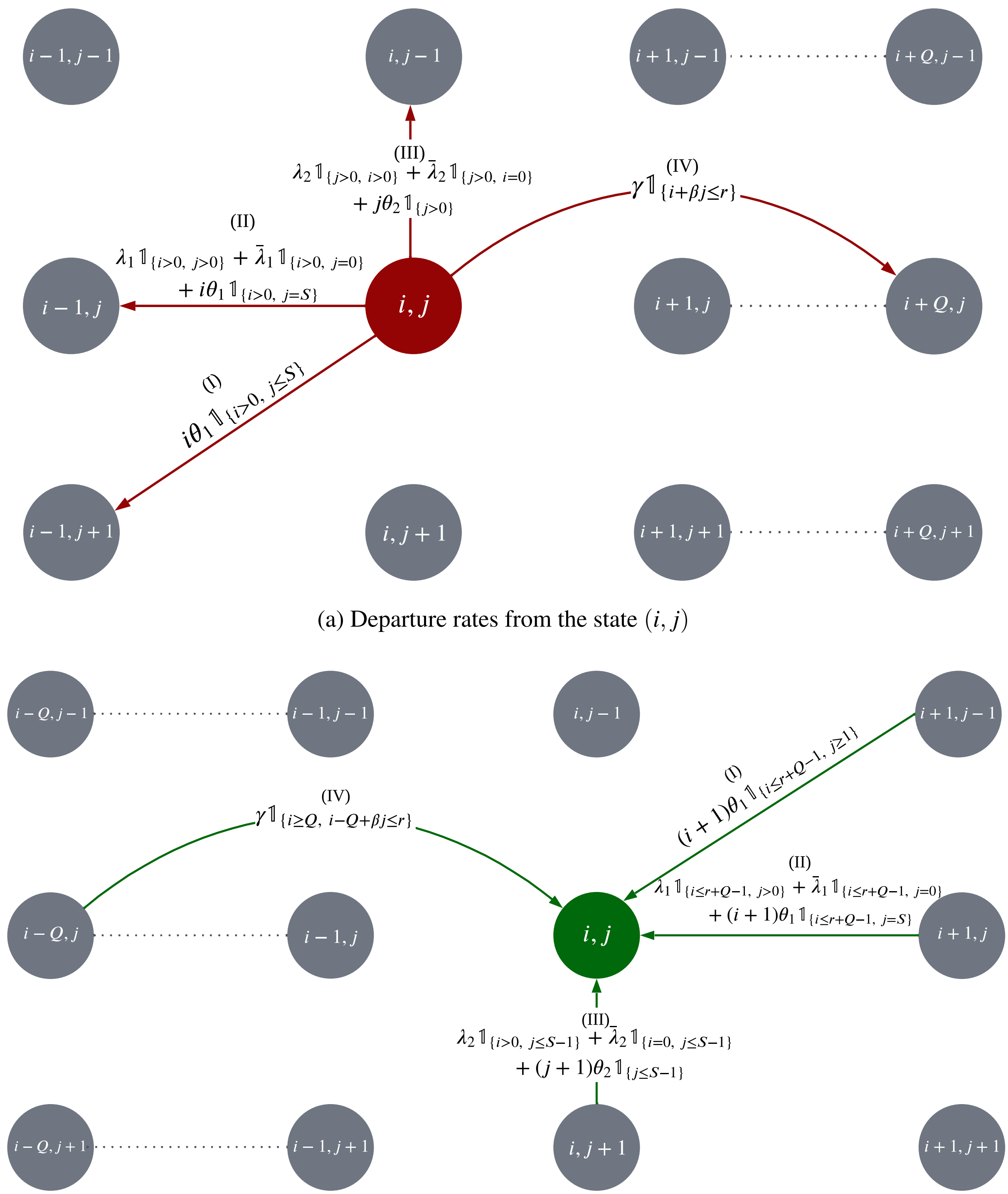
$$\bar{G}_2(\phi_1, \phi_2) = \delta\Lambda\Gamma(\phi_1 + \phi_2)(1 - \phi_1 - \phi_2). \tag{2.15}$$

2.3.2 The Replenishment Decisions and Inventory Dynamics

The firm makes inventory replenishment decisions according to a modified (r, Q) policy. Let $I(t)$ and $J(t)$ denote the numbers of fresh and non-fresh items in stock, respectively. Under the modified (r, Q) policy, the firm places an order for Q fresh items when $\alpha I(t) + \beta J(t)$ reaches or falls below r for some $\alpha > 0$ and $\beta \geq 0$. We set $\alpha = 1$ without loss of generality and consider $\beta \in \{0, 1\}$; i.e., we optimize β while considering two cases: either ignoring the inventory of non-fresh products ($\beta = 0$) or including it ($\beta = 1$) in making replenishment decisions. We make the assumption that lead-time follows an exponential distribution with a mean of $1/\gamma$. Considering lead times in our models as deterministic is not only intractable computationally but also prohibits gaining structural results. While this assumption is made for the sake of tractability and analytical simplicity, we analyze the scenario of zero lead-time in Appendix A.3.

Under a given modified (r, Q) policy and given demand rates (i.e., λ_1 and λ_2 when both products are available, or $\bar{\lambda}_1$ and $\bar{\lambda}_2$ when fresh/non-fresh products are unavailable), the random vector $(I(t), J(t))$ of inventory quantities evolve as a two-dimension Markov chain (MC). It is easy to see that the inventory quantity $I(t)$ of fresh items would never exceed $r + Q$ if the initial inventory $I(0)$ does not exceed $r + Q$. Therefore, the maximum quantity allowed for fresh items is $r + Q$. Also, for the inventory of non-fresh items, we assume that a maximum quantity of S units is allowed; if there are already S units of non-fresh items in stock and one more fresh item becomes non-fresh, one unit of the non-fresh items has to be disposed of so that the total number of non-fresh items remains S. Therefore, we will use $\mathscr{S} := \{(i, j) \mid i = 0, 1, \ldots, r + Q, j = 0, 1, \ldots, S\}$ as the state space of the MC.

We are interested in the steady-state probabilities $\pi_{i,j} := \lim_{t\to\infty} \mathbb{P}(I(t) = i, J(t) = j)$ for $i = 0, 1, \ldots, r + Q$ and $j = 0, \ldots, S$, which is necessary for deriving the long-run average profit rate of the firm. To that end, we derive the transition rates between different inventory states; see Figures 2.1 (a) and 2.1 (b) for a summary and illustration of the transition rates departing and entering a state (i, j), respectively. First, let us consider departure rates from the state (i, j). Departure from a given state may occur due to demand arrival or expiration of fresh or non-fresh products and order arrival, which are elaborated below.



(a) Departure rates from the state (i, j)

(b) Arrival rates into the state (i, j)

Figure 2.1: Transition rates into and out of a given state (i, j)

I Outgoing rate (I) indicates transition rate from state (i, j) into the state $(i-1, j+1)$. Fresh products change into non-fresh products at the rate $i\theta$, when there are $i > 0$ fresh products in the inventory and the capacity of non-fresh items is not full, i.e., $j \leq S-1$. Therefore, transition rate from state (i, j) into state $(i-1, j+1)$ is $i\theta_1$ when $i > 0$ and $j \leq S-1$.

II Outgoing rate (II) implies transition from state (i, j) into state $(i-1, j)$. Fresh products are sold to the customers at the rate λ_1 when there are some non-fresh products on the selves ($j > 0$) and at the rate $\bar{\lambda}_1$ when non-fresh products are not available ($j = 0$). Further, fresh products are expired and one of non-fresh products is discarded at the rate $i\theta_1$ when the capacity of on-fresh products is full. Therefore, transition rate from state (i, j) into state $(i-1, j)$ can be expressed as $\lambda_1 \mathbb{1}_{\{i>0,\ j>0\}} + \bar{\lambda}_1 \mathbb{1}_{\{i>0,\ j=0\}} + i\theta_1 \mathbb{1}_{\{i>0,\ j=S\}}$.

III Transition from state (i, j) into state $(i, j-1)$ may occur due to demand arrival or expiration of non-fresh items and is captured by outgoing rate (III). When there are some fresh products in the inventory ($i > 0$), non-fresh items are sold at the rate λ_2, while the selling rate is $\bar{\lambda}_2$ when there is no fresh product in the inventory ($i = 0$). Also, non-fresh products get expired at the rate $j\theta_2$. Thus, the rate of transition from state (i, j) into state $(i, j-1)$ is $\lambda_2 \mathbb{1}_{\{j>0,\ i>0\}} + \bar{\lambda}_2 \mathbb{1}_{\{j>0,\ i=0\}} + j\theta_2 \mathbb{1}_{\{j>0\}}$.

IV Outgoing rate (IV) indicates transition rate due to order arrival. When $i + \beta j \leq r$, order arrives at the rate γ and adds Q fresh products to the inventory. Thus, the state (i, j) changes to the state $(i+Q, j)$ at the rate $\gamma \mathbb{1}_{\{i+\beta j \leq r\}}$.

On the other hand, state (i, j) can be reached from states $(i+1, j-1)$, $(i+1, j)$, $(i, j+1)$, and $(i, j+1)$ as depicted in Figure 2.1 (b). These four transitions are elaborated as

follows.

I Incoming rate (I) indicates the rate of changing fresh products into non-fresh products when the capacity of non-fresh products is not full, i.e., $j-1 \leq S-1$.

II State (i,j) can be reached from state $(i+1,j)$ due to selling or discarding fresh products only when $i \leq r+Q-1$. Fresh products are sold at the rate λ_1 when non-fresh items are available in stock and at rate $\bar{\lambda}_1$ when there is no non-fresh item available for sale. Further, state (i,j) can be reached at the fresh products expiration rate $(i+1)\theta_1$ when the capacity of non-fresh items is full $j=S$, and one of non-fresh products must be discarded. Thus, the total incoming rate (II) can be expressed as $\lambda_1 \mathbb{1}_{\{i \leq r+Q-1,\, j>0\}} + \bar{\lambda}_1 \mathbb{1}_{\{i \leq r+Q-1,\, j=0\}} + (i+1)\theta_1 \mathbb{1}_{\{i \leq r+Q-1,\, j=S\}}$.

III Incoming rate (III) implies transition from state $(i,j+1)$ to state (i,j) which can occur only when $j \leq S-1$. Non-fresh items are sold at the rate λ_2 when fresh items are available and at the rate $\bar{\lambda}_2$ when they are not available. Further, non-fresh products are expired at rate $(j+1)\theta_2$, therefore the total incoming rate (III) can be expressed as $\lambda_2 \mathbb{1}_{\{i>0,\, j \leq S-1\}} + \bar{\lambda}_2 \mathbb{1}_{\{i=0,\, j \leq S-1\}} + (j+1)\theta_2 \mathbb{1}_{\{j \leq S-1\}}$.

IV State (i,j) can be reached from state $(i-Q,j)$ due to the order arrival. When $i-Q+\beta j \leq r$ and $i \geq Q$ orders may arrive at rate γ and transition the state from $(i-Q,j)$ into (i,j). Therefore, incoming rate (IV) can be written as $\gamma \mathbb{1}_{\{i \geq Q,\, i-Q+\beta j \leq r\}}$.

The total transition rates departing and entering state (i, j) are given as follows.

$$\text{Rate\_In}_{(i,j)} = \begin{array}{l} \underbrace{(i+1)\theta_1 \mathbb{1}_{\{i\leq r+Q-1,\ j\geq 1\}}}_{\text{Change from fresh to non-fresh item}} + \underbrace{\lambda_1 \mathbb{1}_{\{i\leq r+Q-1,\ j>0\}} + \bar{\lambda}_1 \mathbb{1}_{\{i\leq r+Q-1,\ j=0\}}}_{\text{Demand for fresh items}} + \\ \underbrace{\lambda_2 \mathbb{1}_{\{i>0,\ j\leq S-1\}} + \bar{\lambda}_2 \mathbb{1}_{\{i=0,\ j\leq S-1\}}}_{\text{Demand for non-fresh items}} + \underbrace{(j+1)\theta_2 \mathbb{1}_{\{i>0,\ j\leq S-1\}}}_{\text{Expiration of a non-fresh item}} + \underbrace{\gamma \mathbb{1}_{\{i\geq Q,\ i-Q+\beta j\leq r\}}}_{\text{Order arrival}} \end{array} \tag{2.16}$$

$$\text{Rate\_Out}_{(i,j)} = \begin{array}{l} \underbrace{i\theta_1 \mathbb{1}_{\{i>0,\ j\leq S\}}}_{\text{Change from fresh to non-fresh item}} + \underbrace{\lambda_1 \mathbb{1}_{\{i>0,\ j>0\}} + \bar{\lambda}_1 \mathbb{1}_{\{i>0,\ j=0\}}}_{\text{Demand for fresh items}} + \\ \underbrace{\lambda_2 \mathbb{1}_{\{j>0,\ i>0\}} + \bar{\lambda}_2 \mathbb{1}_{\{j>0,\ i=0\}}}_{\text{Demand for non-fresh items}} + \underbrace{j\theta_2 \mathbb{1}_{\{j>0\}}}_{\text{Expiration of a non-fresh item}} + \underbrace{\gamma \mathbb{1}_{\{i+\beta j\leq r\}}}_{\text{Order arrival}} \end{array} \tag{2.17}$$

The steady-state probabilities can be obtained by solving the balance equations (i.e., in steady-state, the expected total rate entering a state equals the expected total rate departing the state) and an equation requiring that the probabilities sum up to one. We refer the readers to Appendix A.1 for the balance equations.

2.3.3 The Joint Inventory-Pricing Problem

Combining the pricing decisions in section 2.3.1 and inventory decisions in section 2.3.2, we illustrate the conceptual modeling framework of the underlying problem depicted in Figure 2.2. This figure displays the interplay between pricing and inventory decisions. Pricing decisions directly influence market demand, which, in turn, impacts inventory depletion and subsequent ordering decisions. On the other hand, inventory decisions influence the availability of fresh and non-fresh items in the assortment, affecting purchasing probabilities and pricing decisions.

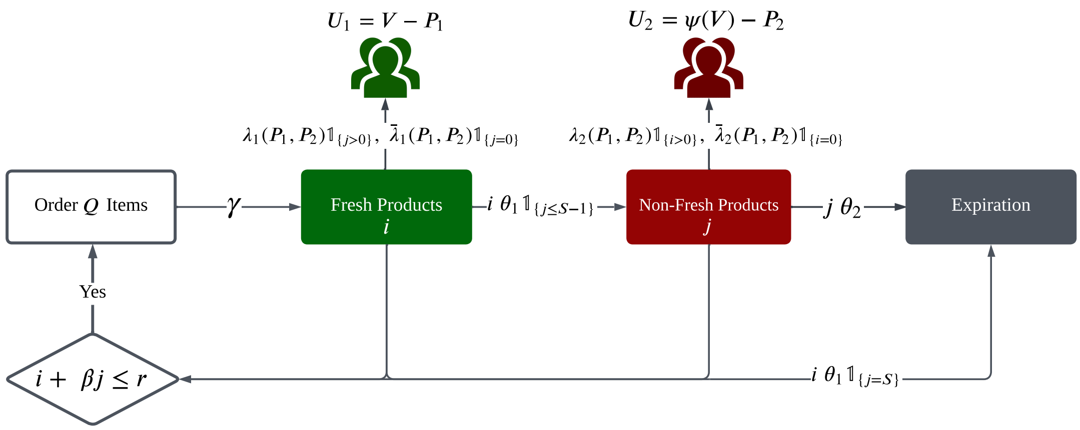


Figure 2.2: Conceptual figure of the problem

We now formulate the joint inventory-pricing problem with the objective of maximizing the average profit rate in steady-state. We consider the following cost components: ordering cost (A per order), purchase cost (C_p per unit ordered), holding cost for fresh products (C_h^1 per unit of inventory per time unit), cost of maintaining total storage capacity(C_{cap} per unit of capacity), holding cost for non-fresh products (C_h^2 per unit of inventory per time unit), expiration cost of non-fresh items (C_e per unit), and disposal cost of non-fresh products when their capacity is full (C_d per unit). Let $\boldsymbol{\Pi} = (\pi_{i,j})_{i=0,\ldots,r+Q;j=0,\ldots,S}$ be the matrix form of the steady-state probabilities. The average profit rate in steady-state can be expressed as

follows.

$$
\begin{aligned}
&TP(r,Q,S,\beta,\phi_1,\phi_2,\boldsymbol{\Pi}) \\
&= [G_1(\phi_1,\phi_2)+G_2(\phi_1,\phi_2)]\sum_{i=1}^{r+Q}\sum_{j=1}^{S}\pi_{i,j}+\bar{G}_1(\phi_1,\phi_2)\sum_{i=1}^{r+Q}\pi_{i,0}+\bar{G}_2(\phi_1,\phi_2)\sum_{j=1}^{S}\pi_{0,j} \\
&\quad -C_{cap}\,(S+r+Q)-C_h^1\sum_{i=0}^{r+Q}\sum_{j=0}^{S} i\pi_{i,j}-C_h^2\sum_{i=0}^{r+Q}\sum_{j=0}^{S} j\pi_{i,j}-(A+C_pQ)\,\gamma\sum_{i+\beta j\le r}\pi_{i,j} \\
&\quad -C_e\sum_{i=0}^{r+Q} j\theta_2\pi_{i,j}-C_d\sum_{i=0}^{r+Q} i\theta_1\pi_{i,S}. \qquad (2.18)
\end{aligned}
$$

In the above expression, the first term is the total revenue when both fresh and non-fresh products are available. The second term represents the revenue rate from selling fresh products when non-fresh products are not-available. The third expression is the total revenue rate from selling non-fresh products when fresh products are not available. The fourth term is cost rate for maintaining the total storage capacity. The fifth and the sixth terms are the total holding cost rates for fresh and non-fresh products, respectively. The seventh term is the total purchasing and ordering cost rate of fresh items. The eighth term gives the total expiration cost rate, and finally the last term is the cost rate of discarding non-fresh products due to lack of space.

Then, we have the following model for maximizing the average profit rate.

$$\max\ TP(r,Q,S,\beta,\phi_1,\phi_2,\boldsymbol{\Pi}) \qquad (2.19)$$

$s.t.$ The steady-state probabilities satisfy the balance equations; see (A.1.1)–(A.1.19) (2.20)

$$\sum_{i=0}^{r+Q}\sum_{j=0}^{S}\pi_{i,j}=1. \qquad (2.21)$$

This model, however, has a high degree of non-linearity. Particularly, in both the objective function and the constraints, the ranges of the summations over i and j involves the decision variables r and Q. To linearize the model, we choose sufficiently large numbers $I_{\max}$ and $J_{\max}$ that are greater than the optimal $r+Q$ and S, respectively. We extend the definition of $\pi_{i,j}$ to $i = 0,\ldots,I_{\max}$ and $j = 0,\ldots,J_{\max}$. In our reformulated model (see Proposition 2.1), any feasible solution is guaranteed to yield $\pi_{i,j} = 0$ for all $i > r+Q$ and for all $j > S$ through constraints.

Proposition 2.1 *The joint inventory-pricing problem (2.19)–(2.21) is equivalent to the following problem (2.22)–(2.48).*

$$
\begin{aligned}
\max \quad & (G_1+G_2)\sum_{i=1}^{I_{Max}}\sum_{j=1}^{J_{Max}}\pi_{i,j}+\bar{G}_1\sum_{i=1}^{I_{Max}}\pi_{i,0}+\bar{G}_2\sum_{j=1}^{J_{Max}}\pi_{0,j}-C_{cap}\left(S+r+Q\right)\\
& -(A+C_pQ)\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}}x_{i,j}^2-C_h^1\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}}i\pi_{i,j} \qquad (2.22)\\
& -C_h^2\sum_{i=0}^{I_{Max}}\sum_{i=0}^{J_{Max}}j\pi_{i,j}-C_e\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}}j\theta_2\pi_{i,j}-C_d\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}}y_{i,j}^4
\end{aligned}
$$

$$s.t.\ x_{i,j}^1+x_{i,j}^2=y_{i,j}^1+y_{i,j}^2+y_{i,j}^3+y_{i,j}^4+\sum_{k=1}^{I_{\max}}y_{i,k,j}^5,\forall i\in[0,I_{\max}],\ j\in[0,J_{\max}] \qquad (2.23)$$

$$r+Q-i+1\le MZ_i^1\le M-i+r+Q,\quad \forall i\in[0,I_{\max}] \qquad (2.24)$$

$$S-j+1\le MZ_j^2\le M-j+S,\quad \forall j\in[0,J_{\max}] \qquad (2.25)$$

$$\pi_{i,j}\le Z_i^1 \text{ and } \pi_{i,j}\le Z_j^2,\quad \forall i\in[0,I_{\max}],\ j\in[0,J_{\max}] \qquad (2.26)$$

$$x_{i,j}^1=\begin{cases}(\lambda_1+i\theta_1+\lambda_2+j\theta_2)\,\pi_{i,j}, & \forall i\in[1,I_{\max}],\ j\in[1,J_{\max}]\\ (\lambda_1+i\theta_1+\lambda_2+j\theta_2)\,\pi_{i,j}, & \forall i\in[1,I_{\max}],\ j\in[1,J_{\max}]\\ \left(\bar{\lambda}_2+j\theta_2\right)\pi_{i,j}, & \forall i=0,\ j\in[1,J_{\max}]\\ \left(\bar{\lambda}_1+i\theta_1\right)\pi_{i,j}, & \forall i\in[1,I_{\max}],\ j=0\\ 0 & \forall i=0,\ j=0\end{cases} \qquad (2.27)$$

$$\gamma\pi_{i,j}+Z_{i,j}^3-1\le x_{i,j}^2\le\gamma\pi_{i,j}-Z_{i,j}^3+1,\quad \forall i\in[0,I_{\max}],\ j\in[0,J_{\max}] \qquad (2.28)$$

$$x_{i,j}^2\le MZ_{i,j}^3,\forall i\in[0,I_{\max}],\quad j\in[0,J_{\max}] \qquad (2.29)$$

$$r-i-\beta j+1\le MZ_{i,j}^3\le M-i-\beta j+r,\quad \forall i\in[0,I_{\max}],\ j\in[0,J_{\max}] \qquad (2.30)$$

$$y_{i,j}^1=\begin{cases}(\lambda_2+(j+1)\theta_2)\,\pi_{i,j+1}, & \forall i\in[1,I_{\max}],\ j\in[0,J_{\max}-1]\\ \left(\bar{\lambda}_2+(j+1)\theta_2\right)\pi_{i,j+1} & \forall i=0,\ j\in[0,J_{\max}-1]\end{cases} \qquad (2.31)$$

$$y_{i,j}^2=\begin{cases}\lambda_1\pi_{i+1,j}, & \forall i\in[0,I_{\max}-1],\ j\in[1,J_{\max}]\\ \bar{\lambda}_1\pi_{i+1,j}, & \forall i\in[0,I_{\max}-1],\ j=0\end{cases} \qquad (2.32)$$

$$\begin{cases} y^3_{i,j} \geq (i+1)\theta_1\pi_{i+1,j-1} - 1 + Z^2_j \\ y^3_{i,j} \leq (i+1)\theta_1\pi_{i+1,j-1} + 1 - Z^2_j \end{cases} \quad \forall i \in [0, I_{\max}-1],\, j \in [1, J_{\max}] \tag{2.33}$$

$$y^3_{i,j} \leq MZ^2_j, \forall i \in [0, I_{\max}],\ j \in [0, J_{\max}] \tag{2.34}$$

$$\begin{cases} y^4_{i,j} \geq (i+1)\theta_1\pi_{i+1,j} - 2 - Z^2_j - Z^4_j \\ y^4_{i,j} \leq (i+1)\theta_1\pi_{i+1,j} + 2 - Z^2_j - Z^4_j, \end{cases} \quad \forall i \in [0, I_{\max}-1], \forall j \in [0, J_{\max}] \tag{2.35}$$

$$y^4_{i,j} \leq MZ^2_j,\ y^4_{i,j} \leq MZ^4_j, \quad \forall i \in [0, I_{\max}],\ j \in [0, J_{\max}] \tag{2.36}$$

$$-S + j + 1 \leq MZ^4_j \leq M - S + j, \quad \forall j \in [0, J_{\max}] \tag{2.37}$$

$$\begin{cases} y^5_{i,k,j} \geq \gamma\pi_{k,j} - 5 + Z^1_i + Z^2_j + Z^3_{k,j} + Z^5_{i,k} + Z^6_{i,k} \\ y^5_{i,k,j} \leq \gamma\pi_{k,j} + 5 - Z^1_i - Z^2_j - Z^3_{k,j} - Z^5_{i,k} - Z^6_{i,k} \end{cases} \quad \forall i \in [0, I_{\max}],\ j \in [0, J_{\max}] \tag{2.38}$$

$$\begin{matrix} y^5_{i,j} \leq MZ^1_i,\ y^5_{i,j} \leq MZ^2_j, y^5_{i,k,j} \leq MZ^3_{k,j}, \\ y^5_{i,k,j} \leq MZ^5_{i,k},\ \text{and}\ y^5_{i,k,j} \leq MZ^6_{i,k}, \end{matrix} \quad \forall i,k \in [0, I_{\max}],\ j \in [0, J_{\max}] \tag{2.39}$$

$$k - i + Q + 1 \leq MZ^5_{i,k} \leq M + k - i + Q, \quad \forall i,k \in [0, I_{\max}], j \in [0, J_{\max}] \tag{2.40}$$

$$i - k - Q + 1 \leq MZ^6_{i,k} \leq M + i - k - Q, \quad \forall i,k \in [0, I_{\max}], j \in [0, J_{\max}] \tag{2.41}$$

$$\sum_{i=0}^{I_{\max}} \sum_{j=0}^{J_{\max}} \pi_{i,j} = 1 \tag{2.42}$$

$$r + Q \leq I_{\max} \tag{2.43}$$

$$S \leq J_{\max} \tag{2.44}$$

$$(2.11) - (2.15)$$

r, Q, S are nonnegative integers (2.45)

$$\lambda_1,\ \lambda_2,\ \bar{\lambda}_1,\ \bar{\lambda}_2,\ G_1,\ G_2,\ \bar{G}_1,\ \bar{G}_2 \geq 0 \tag{2.46}$$

$$\pi_{i,j},\ x^1_{i,j},\ x^2_{i,j},\ y^1_{i,j},\ y^2_{i,j},\ y^3_{i,j},\ y^4_{i,j},\ y^5_{i,k,j} \geq 0 \qquad \forall i,k \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{2.47}$$

$$\beta, Z^1_i, Z^2_j, Z^3_{i,j}, Z^4_j, Z^5_{i,k}, Z^6_{i,k} \in \{0,1\} \qquad \forall i,k \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{2.48}$$

Proposition 2.1 addresses the challenge posed by the original model, which suffers from a high degree of non-linearity. This non-linearity arises from the involvement of decision variables r and Q in the summation ranges of both the objective function and constraints, rendering the problem intractable. The reformulated model in Proposition 2.1 linearizes all terms involving decision variables within the summations over i and j. This reformulation involves the introduction of several auxiliary variables, including both continuous and binary variables, defined in Table 2.1 and detailed in Appendix A.2.

In the reformulated model, the balance equations presented in the Appendix A.1 are condensed into a single equation (2.23), i.e., $x^1_{i,j}+x^2_{i,j}=y^1_{i,j}+y^2_{i,j}+y^3_{i,j}+y^4_{i,j}+\sum_{k=1}^{I_{Max}} y^5_{i,k,j}\ \forall i \in [0,I_{Max}],\ j \in [0,J_{Max}]$. In this equation, $x^1_{i,j}+x^2_{i,j}$ represents the total outflow from the state, while $y^1_{i,j}+y^2_{i,j}+y^3_{i,j}+y^4_{i,j}+\sum_{k=1}^{I_{Max}} y^5_{i,k,j}$ represents the total inflow into the state, as depicted in Figure 2.3.

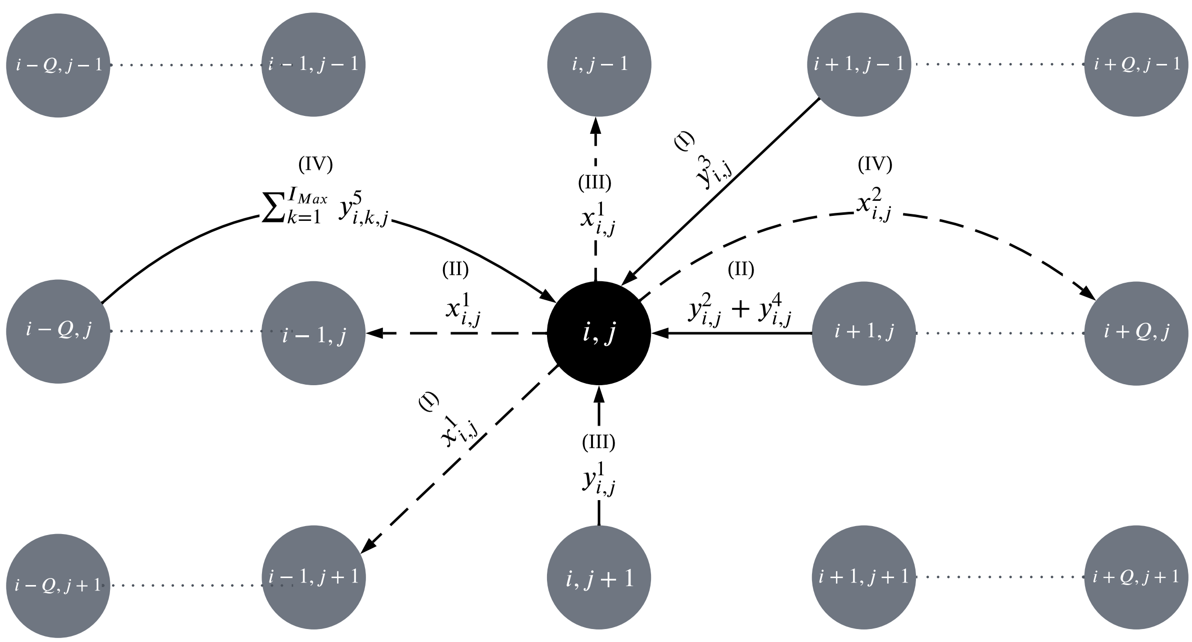


Figure 2.3: Rates into and out of a given state (i, j) defined using non-negative variables

The variables $x^1_{i,j}$, $x^2_{i,j}$, $y^1_{i,j}$, $y^2_{i,j}$, $y^3_{i,j}$, $y^4_{i,j}$, and $y^5_{i,k,j}$ are defined using indicators that involves decision variables (see Figures 2.1a, 2.1b, and 2.3). This results in highly non-linear expressions. Therefore, we introduce binary variables Z^1_i, Z^2_j, $Z^3_{i,j}$, Z^4_j, $Z^5_{i,k}$, and $Z^6_{i,k}$ to linearly model these indicators. A more detailed discussion of the reformulation can be found in Appendix A.2.

The reformulated problem comprises both linear and bi-linear constraints, resulting in a bi-linear mixed-integer programming model that is solvable, in contrast to the original problem. To solve this model, we employ the Gurobi solver, which guarantees an exact global optimal solution for a Mixed-Integer Bi-linear Programing (MIBLP) model (Gurobi Optimization 2023).

2.4 The Value of Clearance Under Different Parameter Regimes

In this section, we study the value of the clearance strategy by establishing bounds on it under several parameter regimes, including large market demand, high maximum willingness-to-pay (WTP), and high expiration rate. To that end, we introduce two distinct inventory-pricing policies: the "clearance policy" and the "no-clearance policy." In both policies, customers perceive non-fresh products as having lower quality than fresh products. Under the clearance policy (a system with clearance), fresh products are sold at a reduced price, denoted as P_2, compared to the price of fresh products, denoted as P_1. This serves as the base model studied in this chapter. Conversely, under the no-clearance policy (a system without clearance), both fresh and non-fresh products are offered at the same price, denoted as $P = P_1 = P_2$.

To determine the value of the clearance policy compared to the no-clearance policy, in what follows, we introduce equivalent upper and lower bound models. Through theoretical analysis, we demonstrate that the gap between the upper and lower bound models vanishes under certain parameter regimes. Note that in Appendix A.3, we examine the case with zero lead-time, while below, we delve into analysis of the case involving positive lead-time.

Subsequently, we denote the decision variables and objective function associated with the upper bound, lower bound, no clearance, and with clearance models using superscripts U, L, NC, and OPT, respectively.

2.4.1 An Upper Bound Model (U)

To establish an upper bound on the optimal solution, we adopt the best-case scenario where fresh items never expire or transition into non-fresh products (i.e., we assume the minimum wastage rate). Essentially, we examine a system where all fresh items are sold before transition or expiration, which is analogous to an (r,Q) system with non-perishable items. The state space reduces to a single dimension i indicating the inventory level of perishable products. We define $\mathscr{S}^U := \left\{i \mid i = 0, 1, \dots, r^U + Q^U\right\}$ as the state space of the MC and $\pi_i^U := \lim_{t\to\infty} \mathbb{P}(I(t) = i)$ for $i = 0, 1, \dots, r^U + Q^U$ as the steady-state probabilities. To obtain these probabilities, we solve the balance equations, which ensure that in steady-state, the expected total rate entering a state equals the expected total rate departing the state. Additionally, we include a normalization equation that enforces the sum of probabilities to equal one. For the sake of brevity, we omit the detailed calculation procedure here and write the steady-state probabilities π_i^U as follows.

$$\pi_i^U = \begin{cases} \frac{\gamma\left(\gamma+\phi_1^U\Lambda\right)^{i-1}}{\left(\phi_1^U\Lambda\right)^{i}}\pi_0^U & i=1,\ldots,r^U+1 \\ \frac{\gamma\left(\gamma+\phi_1^U\Lambda\right)^{r^U}}{\left(\phi_1^U\Lambda\right)^{r^U+1}}\pi_0^U & i=r^U+2,\ldots,Q^U \\ \frac{\gamma\left(\gamma+\phi_1^U\Lambda\right)^{r^U}}{\left(\phi_1^U\Lambda\right)^{r^U+1}}\left[1-\left(\frac{\phi_1^U\Lambda+\gamma}{\phi_1^U\Lambda}\right)^{i-r^U-Q^U-1}\right]\pi_0^U & i=Q^U+1,\ldots,r^U+Q^U \end{cases} \tag{2.49}$$

where $\pi_0^U = \xi^U$, which represents the stock-out probability, is given as follows.

$$\xi^U = \frac{\left(\phi_1^U\Lambda\right)^{r^U+1}}{\left(\phi_1^U\Lambda\right)^{r^U+1}+Q^U\gamma\left(\gamma+\phi_1^U\Lambda\right)^{r^U}} \tag{2.50}$$

The ordering rate can be obtained as follows.

$$\eta^U = \frac{\phi_1^U\Lambda}{Q^U}\left(1-\xi^U\right) \tag{2.51}$$

Also, the average inventory level $\sum_{i=0}^{r^U+Q^U} i\pi_i^U$ can be written as follows.

$$\bar{I}^U = \left[\left(\frac{\gamma}{\phi_1^U\Lambda}\right)\left(\frac{\gamma}{\phi_1^U\Lambda}\right)^{r^U}\left(Q^U r^U+\frac{Q^U\left(Q^U+1\right)}{2}\right)-Q^U\left(1+\frac{\gamma}{\phi_1^U\Lambda}\right)^{r^U}+Q^U\right]\xi^U \tag{2.52}$$

Hence, using the above definitions, we can write the long-run average profit for the upper bound model, i.e., TP^U, as follows.

$$TP^U = \max_{r^U,Q^U,\phi_1^U}\left\{\Gamma\left(1-\xi^U\right)\left(1-\phi_1^U\right)\phi_1^U\Lambda-\left(A+C_pQ^U\right)\eta^U-C_h^1\bar{I}^U-C_{cap}\left(r^U+Q^U\right)\right\} \tag{2.53}$$

where the first term shows the average revenue, the second term indicates the fixed and per-unit ordering cost, the third term denotes holding costs, and the last term shows per-unit capacity cost.

2.4.2 A Lower Bound Model (L)

To establish a lower bound on the optimal total profit, we adopt the worst-case scenario where all the fresh products changing into non-fresh products are expired (maximum wastage rate). In fact, we examine a system where all fresh items are discarded after expiration, analogous to an (r,Q) system with perishable items that are expired with rate θ_1. In this case, the state spaces are simplified to a single dimension denoted by i, representing the inventory level of perishable products. We define the state space of the MC as $\mathscr{S}^L := \left\{ i \mid i = 0,1,\ldots,r^L + Q^L \right\}$. We aim to determine the steady-state probabilities $\pi_i^L := \lim_{t\to\infty} \mathbb{P}(I(t) = i)$ for $i = 0,1,\ldots,r^L + Q^L$, as they are necessary for computing the long-run average profit of the firm. To derive these probabilities, we solve a set of balance equations coupled with the normalization condition and write the steady-state probabilities π_i^L as follows.

$$\pi_i^L = \begin{cases} \frac{\gamma}{\phi_1^L \Lambda + i\theta_1} \Upsilon_{i-1} \pi_0^L & i = 1,\ldots,r^L \\ \frac{\gamma}{\phi_1^L \Lambda + i\theta_1} \Upsilon_{r^L} \pi_0^L & i = r^L + 1,\ldots,Q^L \\ \frac{\gamma}{\phi_1^L \Lambda + i\theta_1} \Upsilon_{r^L} \pi_0^L - \frac{\gamma}{\phi_1^L \Lambda + i\theta_1} \Upsilon_{i-Q^L-1} \pi_0^L & i = Q^L + 1,\ldots,r^L + Q^L \end{cases} \tag{2.54}$$

where

$$\Upsilon_k = \begin{cases} \prod_{n=1}^{k} \left(\frac{\gamma}{\phi_1^L \Lambda + n\theta_1} + 1 \right) & k \geq 1 \\ 1 & k = 0 \end{cases} \tag{2.55}$$

Also, the shortage probability $\xi^L = \pi_0^L$, the ordering rate, and the average inventory level can be written as follows.

$$\xi^L = \left(1 + \Upsilon_{r^L} \sum_{k=r+1}^{r^L+Q^L} \frac{\gamma}{\phi_1^L \Lambda + k\theta_1} + \sum_{k=1}^{r^L} \frac{Q^L \gamma \theta_1}{\left(\phi_1^L \Lambda + k\theta_1\right)\left(\phi_1 \Lambda + \left(k + Q^L\right)\theta_1\right)} \Upsilon_{k-1} \right)^{-1} \tag{2.56}$$

$$\eta^L = \gamma \Upsilon_{r^L} \xi^L \tag{2.57}$$

$$\bar{I}^L = \Upsilon_{r^L} \sum_{k=r+1}^{r^L+Q^L} \frac{k\gamma}{\phi_1^L \Lambda + k\theta_1} \xi^L - \sum_{k=1}^{r^L} \frac{Q^L \gamma \phi_1^L \Lambda \theta_1}{\left(\phi_1^L \Lambda + k\theta_1\right)\left(\phi_1 \Lambda + \left(k + Q^L\right)\theta_1\right)} \Upsilon_{k-1} \xi^L \tag{2.58}$$

Based on the definitions provided above, we can express the long-run average profit for the lower bound model as follows.

$$TP^L = \max_{r^L, Q^L, \phi_1^L} \left\{ \Gamma \left(1 - \xi^L\right)\left(1 - \phi_1^L\right) \phi_1^L \Lambda - \left(A + C_p Q^L\right) \eta^L - C_h^1 \bar{I}^L - C_{cap}\left(r^L + Q^L\right) - C_d \theta_1 \bar{I}^L \right\} \tag{2.59}$$

Remark 1. When $\theta_2 \to \infty$ or $C_d \to \infty$, the base model in Section 2.3 converges to the lower bound model.

2.4.3 A Tighter Lower Bound Model (NC)

The model without clearance provides a more stringent lower bound on the total average profit than the lower bound model (L). In contrast to the lower bound model in which we assumed that all the fresh products are expired, in the no-clearance model, fresh and non-fresh products are offered at the same price (i.e., $P_1 = P_2$). Because customers have lower utility for non-fresh items, they always select fresh items if they are available ($\phi_2 = 0$); otherwise, they buy non-fresh items with probability $\bar{\phi}_2$. The solution obtained from this model yields higher sales and less wastage compared to the lower bound model, and therefore has a higher average total profit.

2.4.4 Bounding The Value of Clearance

In this section, we compare the proposed models (i.e., the lower bound model, the models with and without clearance, and the upper bound model). In the following proposition, we compare the total profit in different models.

Proposition 2.2

$$TP^{L} \leq TP^{NC} \leq TP^{OPT} \leq TP^{U}$$

Proposition 2.2 shows that the total profit of the model without clearance (NC) and the optimal policy (OPT) in the base model lies within the range defined by the total profit of the lower bound model (L) and the upper bound model (U).

Proposition 2.3 *Optimal order quantity in the upper bound model, Q^{U^*}, provides an upper bound for the optimal order quantity under policy f, i.e., $Q^{f^*} \leq Q^{U^*}$, and optimal order* *quantity in the lower bound model, Q^{L^*}, provides a lower bound for the optimal order quantity under policy f, i.e., $Q^{f^*} \geq Q^{L^*}$.*

Proposition 2.3 provides upper and lower bounds for the optimal order quantity under any policy f. Intuitively, this proposition implies that as the degree of perishability increases, the order quantity decreases. In other words, when a product has a high degree of perishability, it means that it has a shorter shelf life or expiration period. In such cases, there is a higher risk of the product becoming unsellable or losing value if it is not sold within a certain time frame. To mitigate this risk, it is preferable to order smaller quantities of the product to ensure that it can be sold before it expires.

In the following theorem, we provide bounds of the gap between the total profit of the upper bound and lower bound models in the order of Λ and θ_1 (ϑ_1).

Theorem 2.1 *The following relations indicate the order of the relative value of clearance in market demand and changing rate from fresh to non-fresh products.*

I

$$\frac{TP^{OPT} - TP^{NC}}{TP^{NC}} = \mathscr{O}(\frac{1}{\sqrt{\Lambda}}) \text{ as } \Lambda \to \infty$$

II

$$\frac{TP^{OPT} - TP^{NC}}{TP^{NC}} = \mathscr{O}(\theta_1) = \mathscr{O}(\frac{1}{\vartheta_1}) \text{ as } \theta_1 \to 0 \quad \text{or } \vartheta_1 \to \infty$$

As expected, the value of clearance diminishes as demand or expiration time grows. Theorem 2.1 quantifies the rate at which the value of clearance diminishes. In Appendix A.6 where we consider the special case of zero leadtime, we show that the value of clearance vanishes with an order of $\mathscr{O}(\frac{1}{\Gamma})$. This implies that in a system with zero lead-time, the value of clearance vanishes in maximum willingness-to-pay vanishes at a faster rate compared to market demand.

2.5 Managing Perishable Inventory as Non-Perishable Inventory

Building upon the results presented in section 2.4, we conclude that when demand or mean time for a fresh item to become non-fresh approach infinity, a system with perishable products becomes equivalent to a system without perishability and managing perishable products inventory become as easy as non-perishable products inventory. We formalize this result in the following proposition.

Proposition 2.4 *The following results hold.*

I

$$\lim_{\Lambda\to\infty}\frac{TP^{OPT}_{\{\beta=1\}}-TP^{OPT}_{\{\beta=0\}}}{TP^{OPT}_{\{\beta=0\}}}=\lim_{\theta_1\to 0}\frac{TP^{OPT}_{\{\beta=1\}}-TP^{OPT}_{\{\beta=0\}}}{TP^{OPT}_{\{\beta=0\}}}=0$$

II

$$\lim_{\Lambda\to\infty}\frac{TP^{OPT}_{\{S=0\}}-(TP^{OPT}_{\{S>0\}}-C_{cap}S)}{TP^{OPT}_{\{S>0\}}-C_{cap}S}=\lim_{\theta_1\to 0}\frac{TP^{OPT}_{\{S=0\}}-(TP^{OPT}_{\{S>0\}}-C_{cap}S)}{TP^{OPT}_{\{S>0\}}-C_{cap}S}=0$$

Proposition 2.4 highlights the convergence between the management of perishable inventory and non-perishable inventory when certain parameters approach asymptotic values. Specifically, Part (I) suggests that as market demand or mean time for a fresh item to become non-fresh grow very large, the gap between total profits in a system incorporating non-fresh items in the ordering policy ($\beta = 1$) and a system excluding non-fresh items from the ordering policy ($\beta = 0$) vanishes. Also, Part (II) indicates that under the same scenarios, the gap in total profits between a system that allocates capacity to non-fresh items ($S > 0$) and one that does not ($S = 0$) vanishes when ignoring the storage costs. This proposition forms the foundation for the heuristic we introduce in the next section.

We further generalize the results to the case of general inter-arrival and expiration distributions.

Proposition 2.5 *The results in Theorem 2.1 and Proposition 2.4 hold for the case where the time for a fresh item to become non-fresh, the expiration of fresh products, or the occurrences of demand follow independent and identically distributed (i.i.d) general distributions.*

Considering general distributions for time between two successive expiration/demand arrival makes the problem more complicated theoretically, therefore, the significance of the above Proposition lies in expanding the understanding of the complex problems with general renewal demand and expiration.

2.6 A Heuristic Model: EOQ-Type Approximation Model (EOQ)

It is well-known in the literature that in large-scale systems demand arrival and expiration events become more predictable and asymptotically equivalent to their deterministic analogy. In this section, as an approximation, we study the deterministic analogy of the original model, with the exception of considering stochastic lead-time. This deterministic model with uncertain lead-time represents a variant of the classic Economic Order Quantity (EOQ) problems. EOQ models are one of the most fundamental models in the inventory management literature introduced by Harris in 1915, and later mathematically formalized by Wilson in 1934. In this model, the objective is to obtain the optimal order quantity to minimize the total ordering and holding costs. The key assumption underlying the EOQ model is deterministic demand, where the demand for the item is considered to be fixed and certain. Although the EOQ model may not directly account for the inherent uncertainties in the system for small-scale problems, it can produce near-optimal solutions in large-scale systems. This is because, as demand and capacity grow significantly, the uncertainties related to demand and expiration processes diminish, and a system with perishable products becomes equivalent to a system with non-perishable products. In what follows, we mathematically argue why the EOQ-type model is a good approximation for our problem.

To analyze the behavior of the arrival process when the system grows large, we consider a sequence of systems. In the n^{th} system, potential demand arrives according to a Poisson process with rate n. As market demand size approaches to infinity ,i.e., $n \to \infty$, the coefficient of variation (c.v. $\frac{1}{\sqrt{n}}$) vanishes in market demand size and market demand approaches to a constant value.

The following Lemma allows us to further reduce the complexity of this problem as market demand approaches to a very large number.

Lemma 2.1 *Given a zero lead-time, as demand approaches infinity, limiting distribution of inventory level for items deteriorating at constant rate* θ_1 *converges to that of classical EOQ model, i.e., Uniform distribution between* 0 *and* Q *(for details readers may refer to Brill (2023)).*

Lemma 2.1 implies that as demand for fresh items λ_1 grows large, an EOQ model with deteriorating items with rate θ_1 becomes equivalent to a classical EOQ model without deterioration with a constant slope. This is consistent with the theoretical results we obtained in the previous section that in the presence of a very large demand, the effect of perishability fades. Therefore, in a large scale system, we can approximate the depletion rate of inventory as $\lambda_1 + \nu\theta_1$ where ν can take a value between 0 and Q. We call this heuristic policy EOQ_ν with the total profit of TP^{EOQ_ν}. Since the supply for non-fresh inventory has a steady rate of $\nu\theta_1$, the retailer can exactly match supply with demand for non-fresh items by adjusting the price. Therefore, at optimiality, we have $\phi_2^{EOQ_\nu}\Lambda = \nu\theta_1$. Also, because for non-fresh products, the supply rate exactly equals to the demand rate, the optimal capacity for non-fresh items is zero, i.e., $S = 0$.

To obtain the steady state probabilities of inventory level, in the following lemma, we apply Level Crossing Theory which indicates that in the steady state, total rates of upcrossings must equal the total rates of downcrossings, see Brill et al. (2008).

Lemma 2.2 *In an (r,Q) system with constant demand rate λ and exponential lead-time γ, the steady-state inventory level distribution, $f(x)$, can be obtained as follows.*

$$f(x) = \begin{cases} \frac{\lambda}{\lambda+\gamma Q e^{\frac{\gamma r}{\lambda}}} & x = 0 \\ \frac{\gamma}{\lambda} e^{\frac{\gamma x}{\lambda}} f(0) & x \in (0,r] \\ \frac{\gamma}{\lambda} e^{\frac{\gamma r}{\lambda}} f(0) & x \in (r,Q] \\ \frac{\gamma}{\lambda}\left(e^{\frac{\gamma r}{\lambda}} - e^{\frac{\gamma(x-Q)}{\lambda}}\right) f(0) & x \in (Q, r+Q] \end{cases} \tag{2.60}$$

Based on Lemma 2.2, the total average profit can be written as follows.

$$TP^{EOQ_\nu} = \max_{\phi_1,\phi_2,r,Q} \left\{ \begin{array}{l} G_1(\phi_1,\phi_2)\left(1-\xi^{EOQ_\nu}\right) + G_2(\phi_1,\phi_2)\left(1-\xi^{EOQ_\nu}\right) \\ -(A + C_p Q)\,\eta^{EOQ_\nu} - C_h^1 \bar{I}^{EOQ_\nu} - C_{cap}(r+Q) \end{array} \right\} \tag{2.61}$$

Subject to

$$\phi_2 = \frac{\nu\theta_1}{\Lambda} \tag{2.62}$$

$$\xi^{EOQ_\nu} = \frac{\phi_1\Lambda + \theta_1\nu}{\phi_1\Lambda + \theta_1\nu + \gamma Q e^{\frac{\gamma r}{\phi_1\Lambda+\theta_1\nu}}} \tag{2.63}$$

$$\eta^{EOQ_\nu} = \gamma e^{\frac{\gamma r}{\phi_1\Lambda+\theta_1\nu}} \xi^{EOQ_\nu} \tag{2.64}$$

$$\bar{I}^{EOQ_\nu} = \frac{Q\left[\gamma(Q+2r)\,e^{\frac{\gamma r}{\phi_1\Lambda+\nu\theta_1}} - 2\,(\phi_1\Lambda+\nu\theta_1)\left(e^{\frac{\gamma r}{\phi_1\Lambda+\nu\theta_1}}-1\right)\right]}{2\left(\gamma Q e^{\frac{\gamma r}{\phi_1\Lambda+\nu\theta_1}} + \phi_1\Lambda + \nu\theta_1\right)} \tag{2.65}$$

The above optimization problem can be easily solved in GUROBI or other commercial solvers. To show the performance of the proposed heuristic model, below we obtain some bounds on the optimality gap of EOQ-type approximation model.

Theorem 2.2 *The following relations indicate the optimality gap EOQ heuristic model with respect to the market demand and expiration rate.*

I

$$\frac{TP^{OPT}\left(\phi_1^*,\phi_2^*,r^*,Q^*\right) - TP^{OPT}\left(\phi_1^{EOQ_\nu},\phi_2^{EOQ_\nu},r^{EOQ_\nu},Q^{EOQ_\nu}\right)}{TP^{OPT}\left(\phi_1^*,\phi_2^*,r^*,Q^*\right)} = \mathscr{O}\left(\frac{1}{\sqrt{\Lambda}}\right)$$
$$\forall \nu \in [0, Q^{EOQ_\nu}]$$

II

$$\frac{TP^{OPT}\left(\phi_1^*,\phi_2^*,r^*,Q^*\right) - TP^{OPT}\left(\phi_1^{EOQ_\nu},\phi_2^{EOQ_\nu},r^{EOQ_\nu},Q^{EOQ_\nu}\right)}{TP^{OPT}\left(\phi_1^*,\phi_2^*,r^*,Q^*\right)} = \mathscr{O}(\theta_1) = \mathscr{O}\left(\frac{1}{\vartheta_1}\right)$$
$$\forall \nu \in [0, Q^{EOQ_\nu}]$$

Theorem 2.2 provides performance guarantee of proposed heuristic model. Specifically, theoretical bound on the optimality gap asymptotically vanish in the order of $\mathscr{O}(\frac{1}{\sqrt{\Lambda}})$ or $\mathscr{O}(\frac{1}{\vartheta_1})$. This theorem specifies that EOQ-type approximation model works the best under large demand or long shelf life scenarios and becomes asymptotically optimal.

2.7 Computational Results: A Real Case of Strawberry Supply Chain

Strawberries are one of the highly perishable fruits with five to seven days lifetime (do Nascimento Nunes 2009). This fruit is made up of 92% water and is characterized with large cells and thin walls, making it susceptible to perishability (Kader 1991). There are different factors such as flavor, size, and appearance determining strawberries' quality. Figure 2.4 shows different quality scores for strawberries. Strawberries scoring less than three are categorized as unmarketable. Accordingly, we classify strawberries with good and excellent quality levels as fresh products, while those with an acceptable quality level are considered as non-fresh products.

During the storage period of strawberries and ripening process, some biochemical changes happen, accounting for the change in the color of strawberries (Moing et al. 2001). This change is highly dependent on the storage temperature. The average temperature for strawberry storage is considered as $15°C$. At this temperature, strawberries start to lose their firmness after about 3 days and red color starts to develop on white areas after 4 days. Decay will increase to 75% of the strawberry after 5 days and to whole strawberry after 8 days. Given the pattern of spoilage, the mean fresh and non-fresh products lifetimes can be estimated as 3 and 2 days, respectively. We consider a medium-sized retail store with an

Figure 2.4: Quality scores for strawberry (do Nascimento Nunes 2015)

Scores	Description	Color, Firmness, and Shriveling	Appearance
1	**Very poor**	Very dark purplish red; extremely overripe or senescent, extremely soft and deteriorated; extremely wilted and dry	
2	**Poor**	overripe; very dark red; soft and leaky; fruit is shriveled, and calyx is wilted and dry	
3	**Acceptable**	fully red; minor signs of softness; fruit and calyx show evident signs of moisture loss	
4	**Good**	Fully light red; firm but less turgid; minor signs of shriveling, calyx slightly wilted	
5	**Excellent**	three-quarter to fully light red; very firm and turgid; field-fresh, fruit and calyx appear very fresh and turgid	

average of 20 demands for strawberry punnets per day arriving according to Poisson distribution. Also, the time between two order arrivals is considered as 4 hours. Cost parameters are estimated in Table 2.2.

Table 2.2: Cost parameters in strawberry supply chain

Description	**Value**
Holding cost for fresh and non-fresh products	\$0.1 per unit per unit of time
Purchasing cost of fresh products	\$1 per unit
Fixed ordering cost	\$2 per order
Expiration cost	\$0.5 per unit

Given the base parameters, in the upcoming sections we solve and analyze the problem by changing different parameters. We first discuss the benefits of clearance strategy and analyze the optimal ordering strategy, followed by the effect of allowance for multiple outstanding orders and detailed sensitivity analysis with respect to different input parameters.

2.7.1 The Value of Clearance

To examine the value of clearance policy, we compare the performance of the system with and without clearance policy. To measure the relative added value of clearance policy, we introduce a measure $\Delta TP_1\%$ as follows:

$$\Delta TP_1\% = \frac{TP^{OPT} - TP^{NC}}{TP^{NC}} \times 100$$

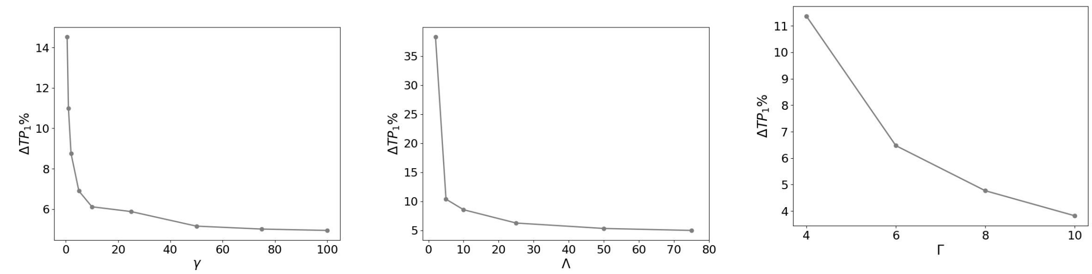
Where TP^{OPT} and TP^{NC} indicate total profits in a system with and without clearance, respectively.

2.7.1.1 The value of clearance policy for the retailers

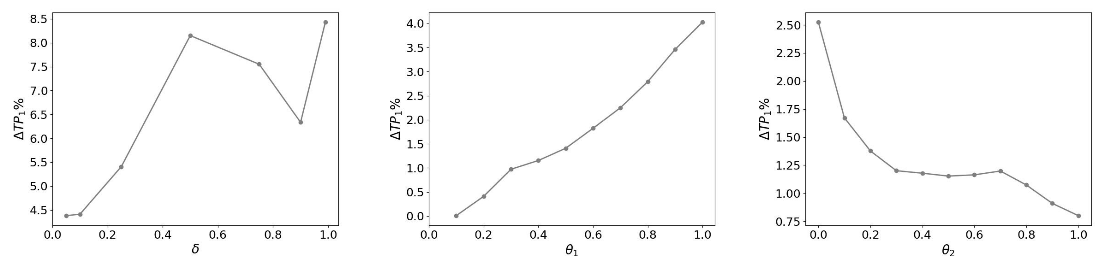
In a retail setting where both fresh and non-fresh products are sold together on the same shelf, a uniform price P is set for both types of products ($P_1 = P_2 = P$). Consequently, customers consistently derive higher utility from purchasing fresh products compared to non-fresh items. From customer's perspective, the optimal issuing policy is Last in First Out (LIFO), favoring fresh products over non-fresh ones. However, when a clearance strategy is implemented, the constraint $P_1 = P_2 = P$ is relaxed, allowing P_2 to take on values greater than zero. Consequently, the system's performance with clearance is guaranteed to be at least as favorable as that without clearance as shown in Figure 2.5. Further, Table 2.3 depicts the number of sales, expired units, and total profit in a system with and without clearance. These results show that a system with clearance always results in a higher sales and lower wastage rates. Although a model with clearance outperforms a model without clearance, the value of clearance strategy varies with the change in different parameters. Figure 2.5 and Table 2.3 illustrate the value of implementing clearance strategy with change in market demand, maximum WTP, order arrival rate, and non-fresh products value.

In accordance with the findings of Theorem 2.1, the relative value of clearance ($\Delta TP_1\%$) diminishes asymptotically with an increase in market demand or a decrease in the expiration rate. Intuitively, when market demand becomes larger or the products' lifetime extends significantly, perishable products approaches the characteristics of non-perishable ones, and consequently, the necessity for clearance diminishes. The results further reveal that the benefit derived from the clearance strategy increases as the mean lead-time extends. The reason is that longer lead-times result in a higher level of uncertainty in order arrival which in turn amplifies the advantages of implementing a clearance strategy.

Figure 2.5 (c) indicates that an increase in maximum WTP results in decreasing relative value of clearance. As the maximum WTP increases, customers are more willing to pay higher prices for products, even for non-fresh items.



(a) Order arrival rate (b) Market demand (c) Maximum willingness-to-pay



(d) Coefficient of non-fresh products value (e) Changing rate from fresh to non-fresh (f) Expiration rate of non-fresh products

Figure 2.5: Comparison between a model with and without clearance

Based on Figure 2.5 (d), when non-fresh products are perceived as medium-value or very high-value (almost the same value as fresh items) products, the relative value of clearance strategy is maximized. However, in the case of low-valued non-fresh items, their limited profitability offers minimal additional profit to the retailer when a clearance model is employed. Specifically, In a system without clearance, if the value of non-fresh products relative to fresh ones is below a threshold, i.e., $\delta \leq \delta_1$, they may be left unsold due to their low value relative to the high price. However, in a system with clearance, these non-fresh items can be sold at lower prices. Consequently, below the threshold $\delta \leq \delta_1$, increasing the value of non-fresh items leads to higher profits in the system with clearance, while the profit remains unchanged in the system without clearance. Thus, below the threshold δ_1, the value of clearance increases as the value of non-fresh products increases.

However, when the value of non-fresh items surpasses the threshold δ_1 (i.e., $\delta > \delta_1$), these products are sold in both systems. In the range of medium perceived value, $\delta_1 \leq \delta \leq \delta_2$, an increase in the value of non-fresh items decreases the value of clearance. Surprisingly, when $\delta > \delta_2$, meaning that non-fresh items have high value close to fresh items, the value of clearance increases as the value of non-fresh products increases. This finding contradicts the common belief that highly valued non-fresh items should not be put on clearance. The counterintuitive result suggests that a clearance strategy can yield substantial profits when the value of non-fresh items is perceived as very high. The main takeaway here is that the efficient clearance strategy involves the perceived value of non-fresh items in addition to their reduced prices. Notably, the clearance strategy yields maximal benefits for the firm when non-fresh items are perceived by customers as either medium-valued or highly-valued products. Conversely, if non-fresh items lack a positive perception in customers' eyes, the clearance strategy results in minimal benefits.

Figures 2.5 (d) and 2.5 (f) suggest that when the transformation rate from fresh to non-fresh items increases, the value of the clearance policy also increases. This is in line with the results obtained in Theorem 2.1. Conversely, a higher expiration rates of non-fresh products imply lower value of clearance, especially when the transformation rate from fresh to non-fresh is high. Intuitively, as expiration rate of non-fresh products tends to infinity, the model becomes equivalent to the lower bound model, and the value of clearance converges to zero.

Table 2.3: Comparison between performance measures in a model with clearance and without clearance

Parameter	Value	Without Clearance			With Clearance			$\Delta TP_1\%$
		Sales	Expirations	TP	Sales	Expirations	TP	
γ	0.5	4.448	1.149	7.855	5.230	0.460	9.190	17.000
	1	5.728	1.241	10.719	6.551	0.479	12.043	12.355
	5	7.459	0.825	14.496	7.818	0.326	15.571	7.409
	10	7.696	0.652	15.146	8.107	0.316	16.133	6.514
	25	7.992	0.657	15.554	8.125	0.250	16.526	6.245
	50	7.909	0.351	15.783	8.069	0.220	16.639	5.422
	100	8.024	0.411	15.954	8.164	0.222	16.784	5.202
δ	0.05	7.177	1.666	13.712	8.780	0.235	14.341	4.581
	0.1	7.176	1.667	13.712	8.635	0.329	14.345	4.614
	0.25	7.179	1.666	13.712	8.411	0.389	14.496	5.717
	0.5	7.177	1.666	13.712	8.103	0.408	14.929	8.873
	0.75	7.612	0.893	14.392	7.783	0.308	15.567	8.166
	0.9	7.471	0.753	15.173	7.915	0.264	16.199	6.763
	0.99	7.524	0.791	15.445	7.840	0.230	16.867	9.208
Λ	2	0.654	0.219	0.250	0.727	0.198	0.406	62.083
	5	1.746	0.286	2.428	1.843	0.223	2.708	11.542
	10	3.753	0.541	6.305	3.887	0.266	6.893	9.326
	25	9.540	1.030	19.015	9.845	0.321	20.280	6.653
	50	19.081	1.586	41.062	20.152	0.428	43.362	5.600
	75	28.698	2.238	63.435	30.124	0.469	66.749	5.225
Γ	2	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	4	6.437	0.551	6.176	6.646	0.207	6.968	12.819
	6	7.553	0.781	14.719	7.994	0.343	15.738	6.920
	8	8.120	1.022	23.784	8.450	0.394	24.975	5.007
	10	8.494	1.308	33.095	8.882	0.509	34.410	3.975

2.7.1.2 The value of clearance policy for the customers

In this subsection, we discuss the benefit of clearance policy from customers' perspective. The results reveal that despite the benefit of markdown policy to the retailer, it might hurt a portion of customers who are willing to buy fresh products. In fact, when fresh products are available, they are usually cheaper in a system without clearance than a system with clearance. This indicates that customers who are willing to buy fresh products may prefer a system without clearance. However, for customers who intend to buy cheaper products, a system without clearance offers them non-fresh items at lower price. Comparisons of prices for fresh and non-fresh products in systems with and without clearance with the change in different parameters are presented in Figures 2.8 (c), 2.9 (c), 2.10 (c), and 2.11 (c).

2.7.2 Discussion on The Ordering Strategy

In this section, we analyze the optimal incluion or exclusion of non-fresh inventory in the ordering policy. To measure the relative added value of including non-fresh items, we introduce a measure $\Delta TP_2\%$ as follows:

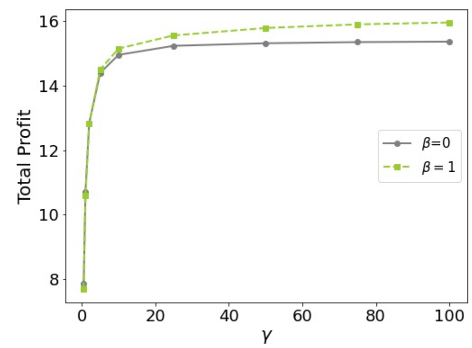
$$\Delta TP_2\% = \frac{TP^{OPT}_{\{\beta=1\}} - TP^{OPT}_{\{\beta=0\}}}{TP^{OPT}_{\{\beta=0\}}} \times 100$$

According to Figure 2.6, including non-fresh products can benefit the system most when demand or shelf life is low because in that case many non-fresh items remain on the shelf and get expired. In accordance with Proposition 2.4, as market demand or mean time between two expirations increases asymptotically, perishable inventory system resembles non-perishable inventory system, and the inclusion or exclusion of non-fresh items in the ordering strategy has no impact on the total profit.

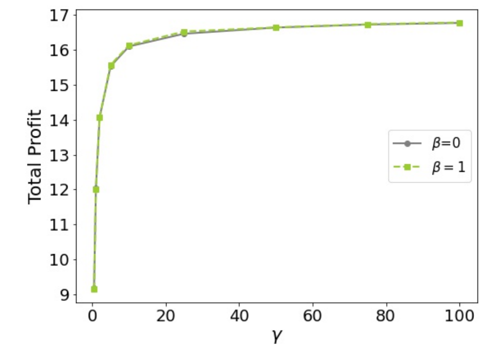
The uncertainty surrounding order arrival and the value attributed to non-fresh items significantly influence the optimal ordering strategy. According to Figures 2.6 (a) and 2.6 (b), when the rate of order arrival is low, it is better to exclude non-fresh items from the ordering policy in that non-fresh products may expire by the time the corresponding order arrives. However, when order arrival rate is high, it is optimal to include non-fresh items in the ordering policy.

Additionally, Figures 2.6 (c) and 2.6 (d) suggest that it is optimal to incorporate non-fresh products into the ordering policy when their value exceeds a certain threshold. However, in a system wherein non-fresh products are perceived as low value items, it is better to ignore non-fresh items in the ordering policy. Intuitively, the higher the value attributed to non-fresh items by customers, the more crucial it becomes to consider their presence in the ordering strategy.

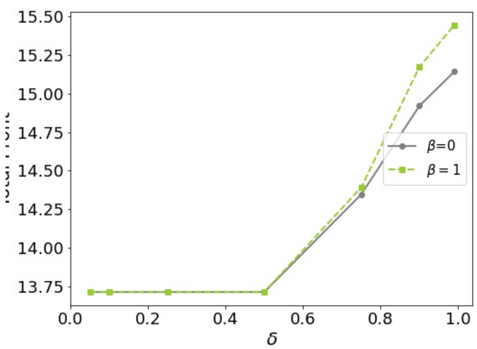
Figures 2.6 (a) and 2.6 (b) show that a system without clearance is more sensitive than a system with clearance to inclusion of non-fresh products in the ordering policy. The reason is that in a system without clearance non-fresh products remain on the shelf to the expiration, and therefore the need for inclusion of non-fresh items in replenishment decisions is higher.



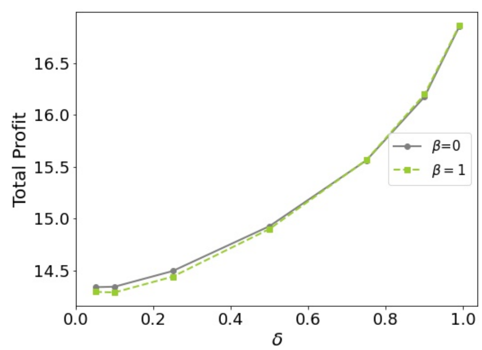
(a) Without clearance



(b) With clearance

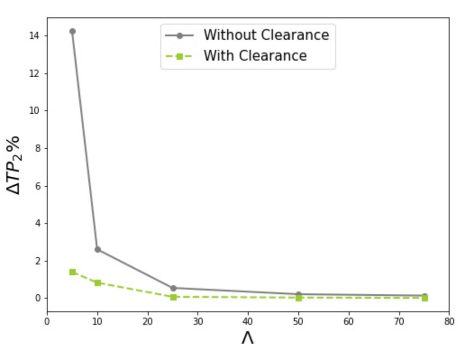


(c) Without clearance

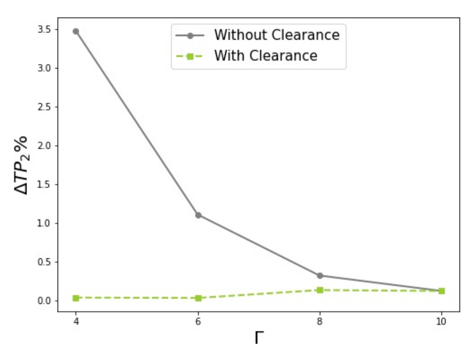


(d) With clearance

Figure 2.6: The effect of non-fresh products on the ordering strategy by changing γ (order arrival rate) and δ (Coefficient of non-fresh products value)



(a)



(b)

Figure 2.7: Sensitivity of including non-fresh products in the ordering strategy to Λ (market demand) and Γ (maximum willingness-to-pay)

Table 2.4: The effect of allowing for multiple outstanding orders by changing A and γ

Model	Parameters		Decisions variables and performance measures										
			β	r	Q	S	P_1	P_2	TP	$Sales$	Exp	m	$\Delta TP_3\%$
MO	A	0.25	1	4	3	3	3.559	2.664	18.141	8.288	0.208	2	2.962
		0.5	1	4	4	3	3.588	2.683	17.527	8.191	0.234	2	1.691
		1	1	3	8	3	3.631	2.700	16.668	7.947	0.286	1	0.000
		2	1	3	10	3	3.646	2.696	15.738	8.002	0.342	1	0.000
		5	1	2	15	4	3.709	2.731	13.804	7.852	0.409	1	0.000
SO	A	0.25	1	4	5	3	3.603	2.686	17.619	8.022	0.254	1	0.000
		0.5	1	4	6	3	3.602	2.681	17.236	8.087	0.284	1	0.000
		1	1	3	8	3	3.631	2.700	16.668	7.947	0.286	1	0.000
		2	1	3	10	3	3.646	2.696	15.738	8.002	0.342	1	0.000
		5	1	2	15	4	3.709	2.731	13.804	7.852	0.409	1	0.000
MO	γ	0.5	1	15	7	4	3.811	2.797	12.775	6.917	0.362	3	39.012
		1	1	10	9	4	3.726	2.736	13.801	7.424	0.402	2	14.596
		2	1	7	7	4	3.705	2.742	14.595	7.713	0.338	2	3.779
		5	1	3	10	3	3.671	2.714	15.571	7.818	0.326	1	0.000
		10	1	2	10	3	3.629	2.693	16.133	8.107	0.316	1	0.000
		25	1	1	10	3	3.626	2.693	16.526	8.125	0.250	1	0.000
		50	1	0	10	3	3.620	2.689	16.639	8.069	0.220	1	0.000
SO	γ	0.5	0	10	19	5	3.990	2.900	9.190	5.230	0.460	1	0.000
		1	0	8	15	5	3.838	2.802	12.043	6.551	0.479	1	0.000
		2	1	6	12	4	3.739	2.747	14.064	7.339	0.405	1	0.000
		5	1	3	10	3	3.671	2.714	15.571	7.818	0.326	1	0.000
		10	1	2	10	3	3.629	2.693	16.133	8.107	0.316	1	0.000
		25	1	1	10	3	3.626	2.693	16.526	8.125	0.250	1	0.000
		50	1	0	10	3	3.620	2.689	16.639	8.069	0.220	1	0.000

2.7.3 The Effect of Multiple Outstanding Orders

In this section, we explore the impact of allowing multiple outstanding orders. The mathematical model that incorporates multiple outstanding orders is presented in Appendix A.8.

The presence of multiple outstanding orders adds complexity to the problem, but it can provide advantages for retailers under certain conditions. By placing an additional order, retailers incur additional fixed and variable ordering costs while simultaneously reducing the order arrival time. This creates a trade-off between the cost of ordering and the time of arrival. The findings of the study demonstrate that the optimal number of outstanding orders is highly influenced by the specific values of the ordering cost and lead-time.

In order to quantify the relative benefit of allowing multiple outstanding orders, we introduce a measure denoted as $\Delta TP_3\%$.

$$\Delta TP_3\% = \frac{TP_{\text{MO}}^{OPT} - TP_{\text{SO}}^{OPT}}{TP_{\text{SO}}^{OPT}} \times 100$$

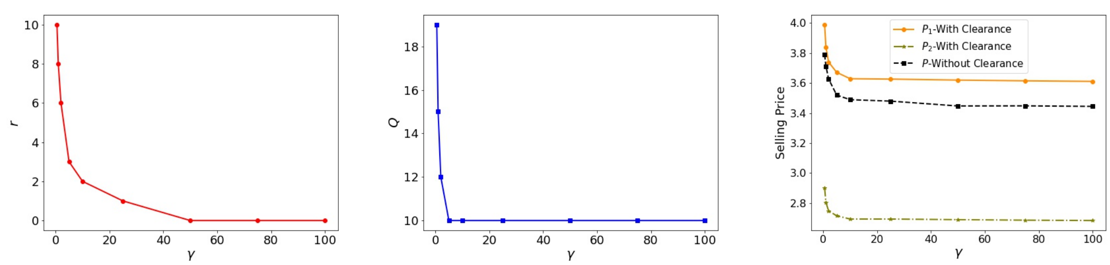
where SO and MO stand for for single outstanding and multiple outstanding order(s), respectively.

Table 2.4 shows the percentage of improvements by allowing for multiple outstanding orders with changes in fixed ordering cost and lead-time. Based on Table 2.4, when lead-time is high, the retailer prefers to allow for more than one outstanding order. Clearly, by increasing order arrival rate and consequently decreasing the uncertainty of order arrival, the maximum number of outstanding orders decreases. In cases of high order arrival rates, it is sufficient to have a maximum of one outstanding order. Moreover, as the fixed ordering cost decreases, the cost incurred by having multiple outstanding orders becomes relatively smaller compared to the overall cost of ordering. This, in turn, increases the benefits of allowing for multiple outstanding orders as indicated by the results in Table 2.4.

2.7.4 Sensitivity Analysis

Order arrival rate: As the order arrival rate increases, the coefficient of variation of lead-time decreases, and the retailer would be more certain about the order arrival time. The results indicates that by increasing the rate of order arrival, reorder point and the capacity of non-fresh products decreases, indicating a higher certainty of the retailer about order arrival. Also, the maximum inventory level for fresh products ($r+Q$) decreases which implies that for a fixed r, order quantity is decreasing as shown in Figures 2.8 (a)-2.8 (c),. It is obvious that the increased certainty of order arrival results in a higher profit as shown in Figure 2.5 (a).

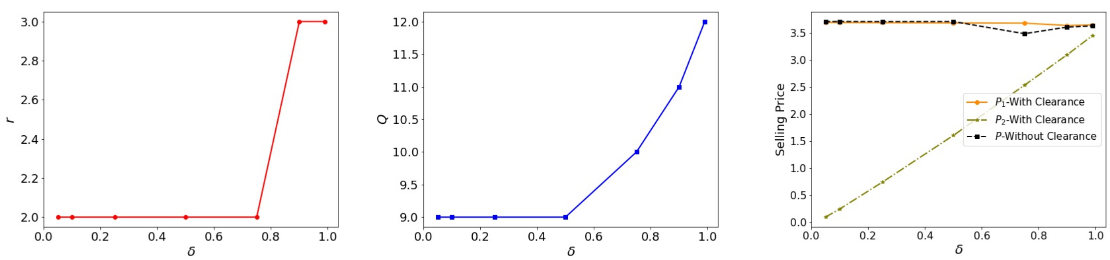
The results further imply that an increase in the order arrival rate decreases the prices of fresh and non-fresh products to an extent and the prices reach stability as the order arrival rate tends to infinity.



(a) (b) (c)

Figure 2.8: Sensitivity of decision variables to the order arrival rate

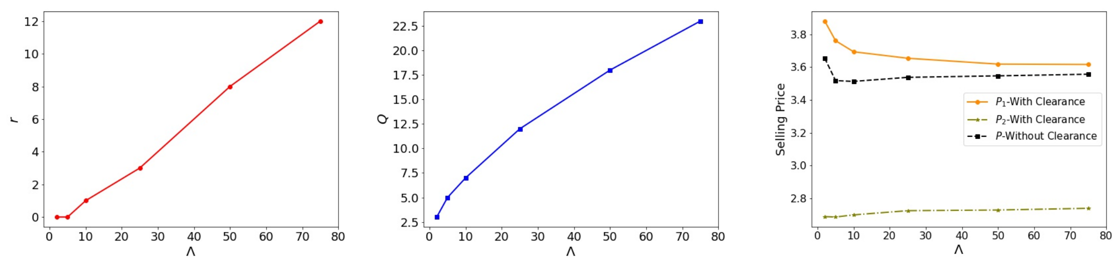
Value of non-fresh products (relative to fresh products): Figures 2.9 (a)-2.9 (c) show that an increase in the value of non-fresh products increases the reorder point, and the order quantity. Intuitively, when the perceived value of non-fresh products is high, carrying a higher inventory level would end in higher profit. Thus, the retailer prefers to increase the reorder point, the order quantity, and the capacity of non-fresh items to increase the gained profit from non-fresh products and the total profit as depicted in Figure 2.5 (d). Moreover, clearly, by increasing the value of non-fresh products, the price of non-fresh products considerably increases, while the price of fresh products slightly decreases.



(a) (b) (c)

Figure 2.9: Sensitivity of decision variables to the value of non-fresh products

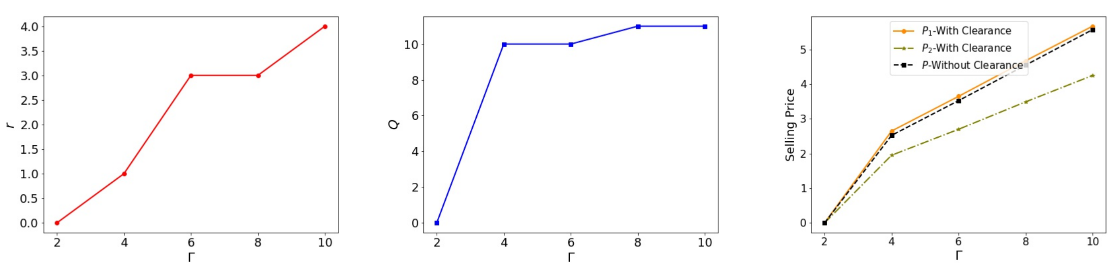
Market demand: As a result of increasing market demand, the retailer increases the reorder point, the order quantity, and the non-fresh items capacity to increase inventory level and be able to satisfy demand. The increase in market demand brings higher sales and in turn higher profit to the retailer. Figures 2.10 (a)-2.10 (c) imply that in a system with clearance, the increase of market demand monotonously decreases the selling price for fresh products. However, the selling price of non-fresh products and the selling price of a system without clearance is not a monotone function of the market demand. By increasing the market demand below a threshold, these prices decrease, while after a threshold the increase in the market demand results in increasing the price of non-fresh items. The reason is that when the market demand is high, the retailer tries to set prices to gravitate a higher proportion of customers a toward fresh items which have higher profit for them.



(a) (b) (c)

Figure 2.10: Sensitivity of decision variables to the market demand

Maximum willingness-to-pay: Based on Figures 2.11 (a)-2.11 (c), when customers have a higher willingness-to-pay, the retailer would be more inclined to increase the reorder point, the order quantity, and the capacity of non-fresh products which indicates keeping a higher inventory level on the shelves. It is obvious that the maximum willingness-to-pay has a positive correlation with selling prices, as depicted in Figure 2.11 (c), and the total profit, as shown in Figure 2.5 (c).



(a) (b) (c)

Figure 2.11: Sensitivity of decision variables to the maximum willingness-to-pay

Expiration rates: Figures 2.12 (a)-2.12 (f) indicate that by increasing the rate of transformation from fresh to non-fresh products and the expiration rate, the firm decreases the reorder point and the order quantity. Doing so, the retailer can reduce the expiration costs by lowering the inventory level and rising the order frequency.

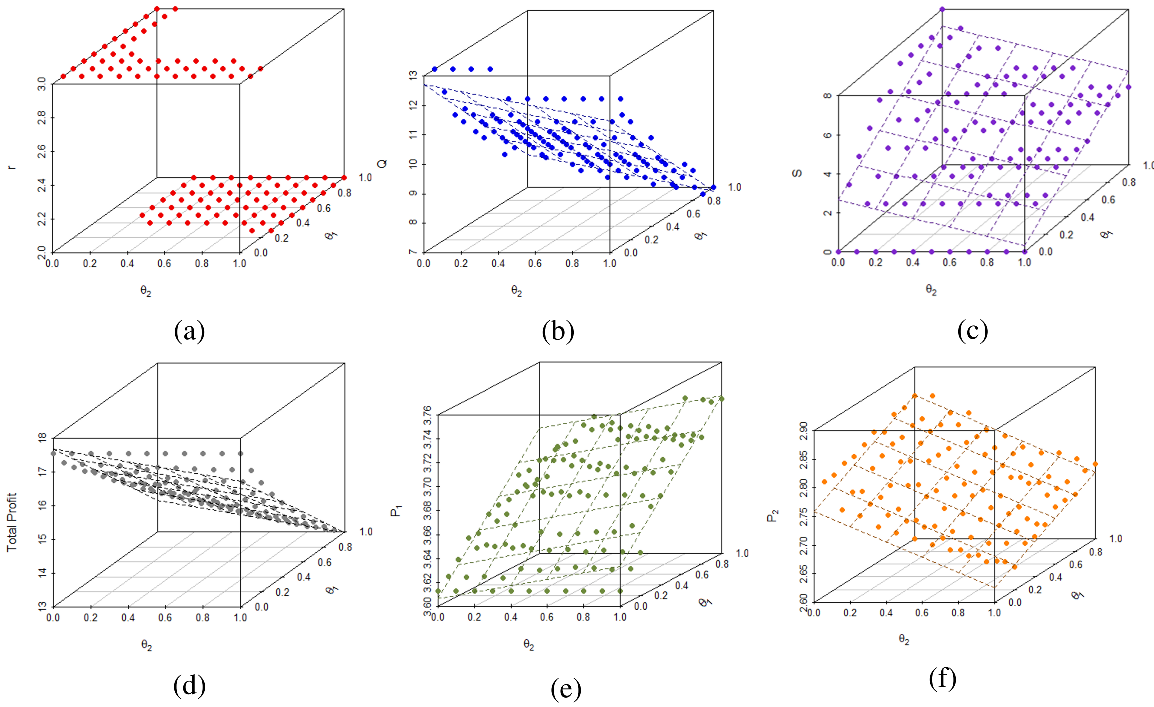


Figure 2.12: Sensitivity of decision variables and total profit to the expiration rates

Moreover, a higher transformation rate or lower expiration rate prompts the retailer to allocate greater capacity to non-fresh items due to increased supply and reduced expiration. Interestingly, the findings suggest that higher expiration rates lead to elevated prices for fresh products, while this effect does not apply to non-fresh items. Instead, the price of non-fresh items exhibits an increasing trend with the transformation rate from fresh to non-fresh but decreases with the expiration rate. Intuitively, the results imply that the fresh products with shorter lifetimes have a higher price compared to those with longer lifetimes, while non-fresh products with longer lifetimes are more expensive than those with shorter lifetimes.

Table 2.5: Sensitivity of the model to various cost parameters

Parameters	Value	Decisions variables and performance measures								
		β	r	Q	S	P_1	P_2	TP	*Sales*	*Expirations*
A	0.25	1	4	5	3	3.603	2.686	17.619	8.022	0.254
	0.5	1	4	6	3	3.602	2.681	17.236	8.087	0.284
	1	1	3	8	3	3.631	2.698	16.668	7.948	0.284
	2	1	3	10	3	3.648	2.698	15.738	7.994	0.343
	5	1	2	15	4	3.708	2.732	13.804	7.853	0.411
C_p	0.25	1	4	12	3	3.260	2.421	22.508	9.360	0.588
	0.5	1	3	12	3	3.396	2.516	20.121	8.821	0.459
	1	1	3	10	3	3.648	2.698	15.738	7.994	0.342
	2	1	1	8	3	4.192	3.114	8.560	5.890	0.167
	5	0/1	0	0	0	0.000	0.000	0.000	0.000	0.000
$C_e = C_d$	0.25	1	3	10	3	3.650	2.722	15.827	7.974	0.379
	0.5	1	3	10	3	3.648	2.698	15.738	7.994	0.342
	1	1	2	10	3	3.675	2.690	15.601	7.746	0.247
	2	1	2	10	3	3.672	2.630	15.379	7.774	0.201
	5	1	2	9	3	3.684	2.522	14.922	7.701	0.126
$C_h^1 = C_h^2$	0.1	1	3	10	3	3.647	2.698	15.738	7.997	0.343
	0.25	1	2	9	3	3.687	2.739	14.820	7.638	0.254
	0.5	1	2	7	3	3.724	2.774	13.480	7.394	0.195
	0.75	1	1	7	2	3.756	2.783	12.392	6.971	0.169
	1	1	1	6	2	3.784	2.808	11.478	6.776	0.138

Sensitivity to cost parameters: In this section, we analyze the sensitivity of decision variables and performance measures to the cost functions. According to Table 2.5, when each of the cost parameters increases, the total sales and the total profit of the system decrease. Based on the results, an increase in purchasing costs, expiration costs, or holding costs leads to a decrease in expiration level, while increasing the fixed ordering cost results in a higher expiration rate due to a higher mean inventory level.

- **Fixed ordering cost:** The results in Table 2.5 indicate that by increasing the fixed ordering cost, the retailer would decrease the reorder point and increase the order quantity to decrease the order frequency and increase the mean inventory on-hand which in turn results in a higher capacity for non-fresh products and a higher expiration rate. Also, selling prices of both fresh and non-fresh items decrease to make up for the increase in the ordering cost.

- **Purchasing cost:** When purchasing cost of products is higher, the retailer decreases the reorder point and order quantity to decrease the order frequency and the purchase amount. Also, the prices of fresh and non-fresh products increase as the purchasing cost of the products is higher for the retailer. As a result, the sales, the expiration, and the total profit would be lower.

- **Expiration cost:** When expiration incurs more costs on the retailer, he/she lowers the reorder point and order quantity to decrease the mean carrying level. Further, results in Table 2.5 show that as a result of the increment in expiration cost, the retailer sells fresh products more expensive while non-fresh products are cheaper. Intuitively, the retailer encourages the customers to buy non-fresh items to decrease the wastage level.

- **Holding costs:** Table 2.5 indicates when holding cost increases, the reorder point and the order quantity decrease, resulting in a lower mean inventory level. Further, the retailer increases the prices of both fresh and non-fresh items to compensate for a high holding cost.

2.8 Managerial Insights

In this section, we summarize the main results derived in the previous sections and provide some managerial remarks.

First of all, as market demand increases, maximum willingness-to-pay (WTP) rises, or expiration rates decrease, perishable inventory systems tend to resemble non-perishable systems. In such cases, retailers can simplify their inventory management by treating it as a non-perishable inventory system, which is significantly simpler. Additionally, in large-scale systems, the inherent uncertainty can be approximated by deterministic equivalents, thereby allowing for the use of EOQ-type approximation heuristic models that we prove to be asymptotically optimal. Therefore, in large-scale retail stores this approximation model can be effectively employed to yield the near optimal solution.

Second, the results indicate that a system with clearance is always better than a system without clearance for the retailer because it increases the number of sales and decreases the number of wastage. However, clearance may hurt some customers who are willing to buy fresh products because the selling price of fresh items is higher in a system with clearance than in a system without clearance. Depending on the value of different parameters, the relative added value of clearance can be variable. In particular, when non-fresh products are medium-valued or very high-valued, market demand is low, lead-time is high, and willingness-to-pay is low, the clearance strategy brings the highest benefit. Intuitively, the higher the uncertainty of the system, the higher the benefit of the clearance strategy (i.e., $\Delta TP_1\%$).

Third, in a system without clearance, the decision to include or exclude non-fresh items in the ordering strategy has a significant impact. This is due to the high rate of expiration faced by the firm in such a system. However, in a system with clearance, the effect of non-fresh items on the ordering policy is generally less pronounced, although it depends on various parameters. Specifically, a higher value assigned to non-fresh items and a shorter lead-time indicate a greater value in including non-fresh items in the ordering policy. It is important to note that the uncertainty of order arrival diminishes the benefit of including non-fresh products in the ordering strategy, as an increase in lead-time raises the probability of expiration. Moreover, as market demand or mean time between two expirations becomes asymptotically large, the perishable inventory system converges towards a non-perishable system, rendering the inclusion or exclusion of non-fresh items in the ordering policy inconsequential.

Finally, as the lifetime of fresh products becomes shorter, the retailer decreases the re-order point and the order quantity to increase the order frequency and increases the capacity of non-fresh items due to the high supply amount for non-fresh items. Further, non-fresh products with shorter lifetimes make the retailer decrease the capacity of non-fresh items due to a lower expiration amount. Interestingly, the results show that fresh products with shorter lifetimes are more valuable than those with longer lifetimes, while it is the reverse for non-fresh products.

2.9 Conclusion

Inventory and revenue management of perishables are challenging due to quality deterioration over time. Non-fresh products are often sold at the same price as fresh items, although they have a lower quality which can hurt the customer perception of the store image. Moreover, there are some customers who cannot afford to buy even non-fresh items. In this research, we propose a markdown policy according to which non-fresh products are sold at a lower price than fresh products. A joint (r,Q) replenishment and pricing problem is modeled as a two-dimensional Markov Chain wherein demands, lifetimes, and lead-time are exponentially distributed. Given the complexity of the extended model, innovatively, we propose an equivalent Mixed-Integer programming model that is solvable exactly. We introduce upper and lower bounds to obtain some theoretical bounds on the value of clearance under different parameter regimes and generalize our results to the case of general renewal demand and expiration processes. Given the complexity of the model, we develop an EOQ-type approximation model and prove that it is asymptotically optimal.

Our computational experiments for the realistic case of the strawberry supply chain suggest the following insights: *First*, a system with clearance always yields a higher profit for the retailer, while might hurt a portion of customers. The results imply that a higher uncertainty of the system implies a higher benefit of the clearance strategy. *Second*, when non-fresh items are more valuable or lead-time is low, indicating a high certainty of order arrival, it is better to include non-fresh items in the ordering strategy. Moreover, a system without clearance is more sensitive than one with clearance to including non-fresh items in the ordering strategy. *Third*, an increase in the value of non-fresh items, market demand, or maximum willingness-to-pay implies a higher profit per unit for the retailer, making the retailer increase the inventory levels of fresh and non-fresh items, while an increase in the order arrival rate makes the retailer decrease reorder point, order quantity, and capacity of non-fresh products due to high certainty of order arrival. *Fourth*, the fresh products with shorter lifetimes are more expensive than those with longer lifetimes, while non-fresh products with shorter lifetimes are cheaper than those with longer lifetime. Moreover, when the expiration rate of a given product increases, the retailer tries to decrease its mean inventory level, while a high supply rate for a product results in assigning a higher capacity for that item. *Fifth*, when ordering a fresh product is more costly, the retailer decreases the reorder point and increases order quantity to decrease order frequency, while by increasing the purchasing cost, the retailer decreases the reorder point and order quantity to decrease the purchase amount. When expiring a product incurs a higher cost on the retailer, he/she attempts to carry fewer products and encourage customers to buy more non-fresh products. Further, a higher holding cost makes the retailer carry fewer products and increases the selling prices of both fresh and non-fresh items. *Finally*, the results imply that in a system with zero lead-time, theoretical bound on the value of clearance vanishes as market demand, mean time between two successive expirations, and maximum willingness-to-pay become large, while in a system with positive lead-time, that vanishes as market demand and mean time between two successive expirations tend to infinity. Also, our proposed EOQ-type heuristic model is asymptotically optimal.

This work can be possibly extended in several directions. *First*, this research considers static pricing and ordering policies which can be improved by considering state-dependent pricing and replenishment policies. *Second*, as a possible extension, one can consider general demand, lead-time, or lifetime distributions. *Finally*, considering the competition between several retailers can also be an interesting research direction.

Chapter 3

Dynamic Inventory and Pricing Control of Perishable Products with Multiple Shelf Life Phases

3.1 Introduction

Perishable products with limited shelf-life constitute a significant portion of the retail sector. However, high wastage rate poses a significant threat to the profitability and sustainability of businesses dealing with such products. Empirical evidence underscores this issue. For instance, in U.S. supermarkets dispose of approximately 43 billion pounds of food annually, constituting 10% of the total U.S. food waste. This translates to an estimated annual cost of $47 billion attributed to food waste, leading to significant revenue depletion (Buzby and Hyman 2012). In response, the firms adopt various strategies to minimize wastage and enhance profitability.

Recent technological advances, such as digital marketing, electronic shelf labeling, and

RFID sensing have enabled dynamic pricing strategies in various industries, including grocery retailing. For instance, AmazonFresh, adjusts product prices approximately every 10 minutes on average, a frequency 50 times higher than Walmart's (Business Insider 2018). Dynamic pricing offers numerous benefits, especially for perishable inventory like fresh produce, whose shelf-life and demand are uncertain. By adjusting prices dynamically, retailers can effectively clear inventory of perishable items approaching their expiration. This dynamic pricing strategy appeals to price-sensitive customers who are willing to compromise on product quality for a discounted price. Consequently, dynamic pricing stimulates demand, aligning it more closely with supply and thus increasing overall profitability. Furthermore, this approach contributes to waste reduction by minimizing the likelihood of unsold perishable products remaining on the shelves.

In spite of these benefits, optimizing pricing and production decisions for perishable products remains challenging, as items of different freshness levels constitute a dynamically varying assortment. Effective inventory and pricing strategies is required to reduce wastage and mitigate demand cannibalization between different freshness levels.

In this research, we consider a joint dynamic inventory-pricing model for perishable products of multiple freshness levels. Our model also considers customers' choices for items of different freshness levels. We contribute to the literature in the following ways.

First, to our knowledge, this work is the first to consider perishable inventory of heterogeneous freshness levels as a dynamic assortment and study both production and (freshness-dependent) pricing decisions of such an assortment.

Second, we characterize the structure of the optimal production and pricing decisions. The results indicate that the optimal production policy is a state-dependent threshold policy and is more sensitive to the inventory of fresher products. The optimal price of a freshness level is decreasing in inventory levels of all freshness levels, with a higher sensitivity to those of closer freshness levels. Moreover, we show that the results obtained under the discounted-profit criterion also hold for the long-run average profit criterion.

Third, we propose and compare three heuristics. The first heuristic employs a dimension reduction approach, while the other two heuristics build on developing static decision-making rules. In particular, the third heuristic considers the structural properties of the optimal policy to improve efficiency.

Lastly, we extend the base model to the case with a donation option, a replenishment system (instead of a production system), and a system with multiple phases of freshness deterioration.

The remainder of this chapter is organized as follows. In Section 2, we review the relevant literature. Section 3 presents the problem and its formulation. In Section 4, we characterize the structure of the optimal policy. Section 5 presents three extensions of the base model. In section 6, we propose three heuristic models, and section 7 presents computational results. We conclude the chapter in section 8.

3.2 Literature Review

Our research is closely related to three streams of literature: inventory management of perishable products, joint inventory-pricing models for perishable products, and structural properties and heuristic policies for managing perishable inventory.

3.2.1 Inventory Management of Perishable Products

Extensive research has been conducted on the inventory management of perishable products. The literature on inventory management problems classifies them into two main categories: period-review and continuous-review systems. Within the domain of continuous-review, many studies have examined optimal ordering policies for perishable items with fixed shelf life, e.g., Ravichandran (1995), Baron et al. (2010), Olsson (2014), Berk and Gürler (2008), and Kouki et al. (2015). Moreover, some papers have investigated inventory management of perishable products with random shelf life. Kalpakam and Shanthi (2006) and Liu and Shi (1999) examined the (s,S) policy with exponentially distributed shelf-life and lead-time, Poisson demand process, and lost-sales. Baron et al. (2020) and Barron (2019) addressed scenarios involving batch arrivals of customer demand. Gürler and Özkaya (2008) explored items with generally distributed shelf-life, renewal arrival process, and zero lead-time, emphasizing the significance of accurately estimating lifetime distribution to minimize losses. Barron and Baron (2020) adopted the queueing and Markov chain decomposition (QMCD) approach to analyze systems with general distributions for product shelf-life and replenishment lead-time. Recently, Kouki et al. (2020) investigated an inventory system with uncertain product shelf-life and lead-times, applying a queueing network approach.

While the literature on inventory management problems in continuous-review systems is extensive, it predominantly focuses on static decisions rather than dynamic ones. In contrast, within the periodic-review context, several papers have explored dynamic inventory decisions. Nahmias and Pierskalla (1973) investigated dynamic inventory control for perishable products, considering a two-period shelf life and demand uncertainty. Later, Nahmias (1975) and Fries (1975) extended their model to the case of multiperiod shelf life and further analyzed the structure of the optimal policy. Chen et al. (2021) and Abouee-Mehrizi et al. (2019) proposed dynamic programming models to investigate optimal issuing, production, and disposal decisions in a system with age-differentiated demand. Then, they both proposed heuristic methods and demonstrated their close-to-optimal performance. Fu et al. (2019) investigated the optimal manufacturing and remanufacturing quantities for perishable products with two period shelf life. Recently, Zhang et al. (2020) studied inventory management problems for perishable products with clearance. Due to the complexity of their problem, they proposed two simple heuristic models and presented some theoretical bounds to show the performance of their heuristics asymptotically.

3.2.2 Joint Inventory-Pricing Management of Perishable Products

The papers reviewed above considered price as an external factor. In contrast, the the second research stream in the literature investigates joint inventory and pricing decisions for perishable products. These pertinent works are reviewed below. Abad (1996) investigated a joint pricing and inventory optimization model that allowed partial backorders. Li et al. (2009) characterized the optimal policy structure for products with a two-period lifetime, and additionally devised a base-stock heuristic model to solve the case with multi-period lifetime. Chen et al. (2014) studied joint inventory and pricing decisions under both backorders and lost-sales scenarios. They derived structural properties of optimal decisions and devised an effective heuristic policy. They further developed their work by considering nonparametric demand learning. Considering a changing environment, Keskin et al. (2022) proposed data-driven dynamic inventory and pricing strategies for perishable items. They presented two distinctive policies and evaluated their associated regret. The papers discussed above primarily concentrated on the integrated inventory and pricing decisions.

However, they overlooked the aspect of customer preferences concerning items with different freshness levels and associated prices. In order to bridge this gap, a few papers have explored the impact of cannibalization between different freshness levels of a single product. Ferguson and Koenigsberg (2007) optimized price and inventory decisions for new and old products in a two-period setting, assuming no uncertainty in the second period and retailer realization of demand before that period. Sainathan (2013) examined pricing and ordering decisions for products with a two-period lifetime, where items with a two-period shelf life are considered new, while those with only one remaining shelf life are categorized as old. Recently, den Boer et al. (2022) investigated inventory and pricing decisions for a single perishable product with two-period shelf life. They compared fixed price and markdown policies under the scaling limit model, wherein parameters are deterministic. Özbilge et al. (2024) investigated inventory, pricing, and donation decisions in a quality-dependent newsvendor problem.

3.2.3 Structural Properties and Heuristic Policies

Finally, this research also relates to the stream of research on structural properties and heuristic policies for perishable products. The state-dependent nature of optimal ordering and pricing decisions for perishable products poses a considerable challenge in obtaining the exact optimal solutions. To address this challenge, some researchers have characterized the structure of the optimal policies for perishable products, while some other papers have utilized these properties to develop efficient heuristics as an alternative for the optimal solutions. The origins of this approach can be traced back to early works by Nahmias and Pierskalla (1973), Nahmias (1975), Fries (1975), and Cohen (1976), who investigated the structure of optimal ordering policies for perishable products. Akçay et al. (2010) examined the structure of the optimal pricing decisions for substitutable perishable products that are horizontally or vertically differentiated. Chen et al. (2014) addressed joint inventory and pricing decisions for perishable items in both backordering and lost sales systems. They established the $L^\natural$-concavity property for transformed cost function and developed a heuristic method by approximating the outdating cost. Zhang et al. (2022) explored an inventory management issue involving platelet products across a two-hospital system practicing transshipment. They harnessed the concept of $L^\natural$-convexity to establish the monotonicity of optimal replenishment and transshipment policies, introducing a myopic transshipment policy that is optimal for some cases and serves as a lower bound for more general cases. In 1985, Hajek introduced the notion of Multimodularity to analyze the structural properties of optimal policies in discrete-time, discrete-state systems (Hajek 1985). In recent years, several have investigated the application of multimodularity and anti-multimodularity in perishable inventory management. Li and Yu (2014) utilized the anti-multimodularity concept to show the structure of optimal ordering and clearance policy, considering exogenous selling prices and a single freshness level of perishable products. They used dimension reduction techniques to develop two heuristic polices and compared their performance. Building upon the structural properties on the optimal ordering and clearance strategies, recently, Zhang et al. (2020) developed two heuristic polices, newsvendor-type model and fluid approximation model, and established theoretical bounds on the performance of both heuristic models. The above paper focused on the structural properties for inventory problem involving a single quality level, without considering the customers' choices over products with different freshness levels. To address this gap, in this research, we made the first attempt to characterize the structural properties for both inventory and pricing decisions for perishable products with customer choice. Further, we develop three heuristic policies, utilizing dimension reduction technique as well as the structural properties of value function.

3.3 Problem Statement and Model Formulation

We consider an inventory-pricing system for a product with multiple phases of freshness levels. The firm incurs a per-unit production cost of c_p for producing fresh items. Each item has n distinct freshness levels and is assigned a specific quality index. For simplicity, we will refer to items of freshness level k as product k (hereafter, we will use the two terms freshness level and product interchangeably). Denote $\mathscr{N} = \{1,2,\ldots,n\}$ as the set of all products where product k has a quality index of q_k (uniform across all consumers). Without loss of generality, we assume that the product quality can be ranked in descending order as $q_1 > q_2 > \ldots > q_n > 0$. The production time is exponentially distributed with mean $1/\mu$. Additionally, an item of freshness level k (referred to product k hereafter) transitions into the next freshness level, denoted as product $k+1$, after an exponentially distributed period of time with mean $1/\theta_k$. Inventory is discarded upon reaching phase n, at a disposal cost φ per unit. Assuming independent of deterioration times of all items, the rate at which product k transition to phase $k+1$ is $i\theta_k$ when there are i units of product k in stock. Holding a unit of product k incurs a holding cost of h_k.

The state of the system at time t is defined as $x(t) := (x_1, x_2, \ldots, x_n)$ for all $t \geq 0$, where $x_k(t) \in \{0,1,\ldots,C\}$ denotes the inventory of product k and C is the total storage capacity of the system, so that $\sum_{k=1}^{n} x_k \leq C$ must hold at all time. Let $\mathscr{S}$ be the set of all possible inventory states.

3.3.1 Customer Choice Model

Potential customers arrive at a rate Λ. Suppose that in the inventory state x, there are $m(x) \leq n$ freshness levels (i.e., products) with positive inventories. Let $A(x)$ be the set of all freshness levels with positive inventory in stock and $\bar{A}(x)$ the set of all freshness levels with zero inventory. For ease of notation, when it does not cause confusion, we will write $A(x)$ and $\bar{A}(x)$ as A and $\bar{A}$, respectively, and $m(x)$ as m. With the set of products $A \subset \mathscr{N}$ in stock, a customer decides whether to buy one of the available products or not. The probability of a customer purchasing product $k \in A$ is denoted as $\alpha_{k|A}$, while the probability of purchasing no product is denoted as $\alpha_{0|A}$. Notably, the probabilities $\left\{\alpha_{k|A} : k \in A \cup 0, A \subset \mathscr{N}\right\}$ are non-negative values that satisfy the condition $\alpha_{0|A} + \sum_{k \in A} \alpha_{k|A} = 1$. The purchasing probabilities of unavailable products are zero, i.e., $\alpha_{k|\bar{A}} = 0$ for all $k \in \bar{A}$. Thus, the vector of purchase probabilities for all products, available or unavailable, can be expressed as follows:

$$\alpha_k = \begin{cases} \alpha_{k|A} & k \in A \\ \alpha_{k|\bar{A}} = 0 & k \in \bar{A} \end{cases}$$

We consider the random utility derived by a customer from purchasing product k at price p_k as $u_k = \zeta q_k - p_k$, where the random coefficient ζ follows a uniform distribution between 0 and 1 and p_k is the price of product k. In this framework, when two products k and j have the same price, a customer will prefer product k over product j if $q_k > q_j$. Additionally, customers have the option of choosing not to purchase. Such an option is represented by a dummy product $n+1$ with $q_{n+1} \equiv p_{n+1} \equiv 0$. Let $\mathbf{P} = \{p_1, p_2, \ldots, p_n\}$ denote the vector of prices for all products. The probability that an arriving customer will purchase product k as $\alpha_k(\mathbf{P}) = \mathbb{P}\left(u_k = \max_{k'}\{u_{k'}, k' = 1, 2, \ldots, n, n+1\}\right)$. We use the approach in Akçay et al.

(2010) to obtain the closed-form customer choice probabilities as follows.

Lemma 3.1 (Akçay et al. (2010)) *Without loss of optimality, one may restrict the prices $p_1, \dots, p_n$ to the set $\mathscr{P}_{\mathbf{X}}$, which is defined as follows:*

$$\mathscr{P}_{\mathbf{X}} = \left\{ (p_1, \dots, p_n) \left| \begin{array}{l} 0 \leq \dfrac{p_n - p_{n+1}}{q_n - q_{n+1}} \leq \dfrac{p_{n-1} - p_n}{q_{n-1} - q_n} \leq \dots \leq \dfrac{p_2 - p_3}{q_2 - q_3} \leq \dfrac{p_1 - p_2}{q_1 - q_2} \leq 1; \\ \dfrac{p_k - p_{k+1}}{q_k - q_{k+1}} = \dfrac{p_{k-1} - p_k}{q_{k-1} - q_k} \quad \text{if } x_k = 0,\ k = 2, 3, \dots, n; \\ \dfrac{p_1 - p_2}{q_1 - q_2} = 1 \quad \text{if } x_1 = 0 \end{array} \right. \right\}$$

Moreover, when $(p_1, \dots, p_n) \in \mathscr{P}_{\mathbf{X}}$, then the probability of a customer choosing a product k is given as follows. For $k \in A$ (i.e., product k is in stock),

$$\alpha_k(\mathbf{P}) = \begin{cases} 1 - \frac{p_1 - p_2}{q_1 - q_2} & k = 1 \\ \frac{p_{k-1} - p_k}{q_{k-1} - q_k} - \frac{p_k - p_{k+1}}{q_k - q_{k+1}} & k = 2, \dots, n \end{cases} \tag{3.1}$$

For $k \in \bar{A}$ (i.e., product k has zero inventory), $\alpha_{k|\bar{A}} = 0$. The probability of no purchase is $\alpha_{n+1} = 1 - \sum_{k=1}^{n} \alpha_k(\mathbf{P}) = \frac{p_n}{q_n}$.

3.3.2 Model Formulation

In this section, we formulate the problem as a continuous-time Markov chain, where state transitions occur due to production, sales, phase changes, or expiration of an item. In each state, the firm must make pricing decisions for all freshness levels and the decision on whether to produce. A control policy, denoted as π, specifies the actions for each state, including whether to produce items and setting the prices in that state. Let use define $\boldsymbol{P}^{\pi}(x) = \left\{ p_1^{\pi}(x), p_2^{\pi}(x), \dots, p_n^{\pi}(x) \right\}$ as the vector of prices and $a^{\pi}(x)$ as the production action under policy π when the system is in state x, where $a^{\pi}(x)$ is a binary variable of the following form.

$$a^{\pi}(x) = \begin{cases} 0 & \text{no production} \\ 1 & \text{production} \end{cases}$$

To evaluate a policy π, we consider the expected discounted profit over an infinite planning horizon with discount rate ψ (later we will extend this analysis to the average profit case in subsection 3.4.3). The expected discounted profit under policy π and a starting state x is denoted as $v^{\pi}(x)$, which is the revenue from product sales less the production, holding, and expiration costs. The objective is to find a policy π that maximizes this expected profit.

To facilitate analysis, we use the uniformization technique (see, e.g., Lippman (1975)) to transform the continuous-time control problem into an equivalent discrete-time control problem, where decisions are made only when there is a transition of state. To that end, we introduce a uniform transition rate given by $\phi = \Lambda + \sum_{k=1}^{n} C\theta_k + \mu$ between every two events. Consequently, the transition times are determined by a sequence of independent and identically distributed exponential random variables, each with a mean of $\frac{1}{\phi}$. As such, the rate ϕ is an upper bound on the actual rate of events (which include arrival of a customer, freshness transition or expiration of an item, and production completion). In fact, we may consider the rate $C\theta_k$ to be consist of the actual transition/expiration rate $x_k\theta_k$ and the dummy transition rate $(C - x_k)\theta_k$ (which does not lead to a change of the state). To simplify the analysis, without loss of generality, we rescale time to let $\psi + \phi = 1$, where ψ indicates the discount factor. The optimal profit function, denoted as $v(x) = \max_{\pi} v^{\pi}(x)$, satisfies the following optimality equation:

$$v(x) = \Lambda \mathscr{R} v(x) + \mu \mathscr{P} v(x) + \sum_{k=1}^{n} \theta_k \mathscr{E}_k v(x) + H(x) \tag{3.2}$$

Where

$$\mathscr{R}v(x) = \max_{\boldsymbol{P}\in\mathscr{P}_{\boldsymbol{X}}} \left\{ \sum_{k}^{n} \alpha_k \left(v(x-\boldsymbol{e}_k) + p_k \right) + \alpha_{n+1} v(x) \right\} \tag{3.3}$$

$$\mathscr{P}v(x) = \max \left\{ v(x+\boldsymbol{e}_1) - c_p, v(x) \right\} \tag{3.4}$$

$$\mathscr{E}_k v(x) = \begin{cases} x_k v(x-\boldsymbol{e}_k+\boldsymbol{e}_{k+1}) + (C-x_k)v(x) & x_k \geq 1 \\ Cv(x) & x_k = 0 \end{cases} \quad \forall k \leq n-1 \tag{3.5}$$

$$\mathscr{E}_n v(x) = \begin{cases} x_n \left[v(x-\boldsymbol{e}_n) - \varphi \right] + (C-x_n)v(x) & x_n \geq 1 \\ Cv(x) & x_n = 0 \end{cases} \tag{3.6}$$

$$H(x) = -\sum_{k=1}^{n} h_k x_k \tag{3.7}$$

The above profit function is recursively expressed using three types of operators: $\mathscr{R}$ denotes the revenue operator concerning sales of different types of products, $\mathscr{P}$ denotes the production operator, and $\mathscr{E}_k$ are transformation operators, showing the product phase transition from phase k to $k+1$. Next, we examine the structural properties of the value function, and then characterize the structure of optimal policies.

3.4 Optimal Inventory-Pricing Policy

3.4.1 Properties of Value Function

In this subsection, we identify and discuss the properties that the optimal value function $\vartheta(x)$ may possess. We then demonstrate these properties are satisfied by the optimal value function v. For the ease of exposition, similar to previous studies in the literature (e.g., Gayon et al. (2009), Benjaafar et al. (2011), Li et al. (2023b)), we define several operators as follows.

Definition 3.1 *For any function $\vartheta(x)$ defined on $\mathbb{Z}^n$, we define the following operators.*

(1) $\Delta_i \vartheta(x) = \vartheta(x + \boldsymbol{e}_i) - \vartheta(x)$

(2) $\Delta_i \Delta_i \vartheta(x) = \Delta_i \vartheta(x + \boldsymbol{e}_i) - \Delta_i \vartheta(x)$

(3) $\Delta_i \Delta_j \vartheta(x) = \Delta_j \vartheta(x + \boldsymbol{e}_i) - \Delta_j \vartheta(x)$

Further, similar to Yang et al. (2022) and Çil et al. (2011), we define the following functional properties.

Definition 3.2 *Define Ω as the set of functions defined on $\mathbb{Z}^n$ that satisfy the following properties.*

(P1) Submodularity: $\Delta_i \Delta_j \vartheta(x) \leq 0 \;\; \forall i, j \neq i \in \mathcal{N}$

(P2) Subconcavity: $\Delta_i \Delta_i \vartheta(x) \leq \Delta_i \Delta_j \vartheta(x) \;\; \forall i, j \neq i \in \mathcal{N}$

(P3) Concavity: $\Delta_i \Delta_i \vartheta(x) \leq 0 \;\; \forall i \in \mathcal{N}$

Property (P1) states that items with different freshness levels are substitutable. This property arises from the condition $\Delta_i \Delta_j \vartheta(x) \leq 0$, indicating that the marginal revenue of item i decreases as the inventory of item j (where $i \neq j$) increases. This implies a competition and cannibalization effect among products of varying freshness levels. Furthermore, Property (P2), the subconcavity property, highlights that the marginal revenue of item i decreases faster in its own on-hand inventory compared to an increase in the on-hand inventory of a different item $j \neq i$. Additionally, the concavity of the value function in all state variables, i.e., Property (P3), signifies that the marginal revenue of item i decreases as its on-hand inventory increases.

Now, in the following theorem, we are ready to show that $v(x) \in \Omega$.

Theorem 3.1 *The optimal value function $v(x) \in \Omega$.*

Next, Lemma 3.2 provides the necessary and sufficient conditions for a two-dimensional function to exhibit anti-multimodularity. On the other hand, Lemma 3.3 establishes that a two-dimensional anti-multimodular function is concave.

Lemma 3.2 (Lemma 1 in Yang et al. (2022)) *A real-valued function ϑ is anti-multimodular if and only if ϑ is submodular and subconcave.*

Lemma 3.3 (Lemma 2 in Yang et al. (2022)) *If a real-valued function ϑ is anti-multimodular, then it is concave function, i.e., $\Delta_i\Delta_i\vartheta(x) \leq 0$.*

Then, the following corollary indicates that the value function $v(x)$ preserves anti-multimodularity property.

Corollary 3.1 *The value function $v(x)$ is anti-multimodular and concave.*

According to the above corollary, the value function exhibits anti-multimodularity, indicating that inventories at different freshness levels are considered economic substitutes. The monotonicity of the optimal inventory levels naturally follows from these anti-multimodularity and concavity properties.

3.4.2 Structure of The Optimal Inventory-Pricing Policy

In this subsection, we use Properties (P1)–(P3) to characterize the optimal production and pricing decisions in Theorems 3.2 and 3.3, respectively.

3.4.2.1 Optimal Production Decisions

Let us begin by analyzing the production decision that solves $\mathscr{P}v(x) = v(x) + \max\left\{\Delta_1 v(x) - c_p, 0)\right\}$. Here, $\Delta_1 v(x)$ represents the marginal revenue obtained by producing one unit of fresh item (item 1) at the inventory level x, while c_p denotes the production cost of a fresh product. From $\Delta_1\Delta_1 v(x) \leq 0$, it is evident that $\Delta_1 v(x)$ decreases as x_1 increases. For a given $x_{-1} = (x_2, x_3, \ldots, x_n) \in \mathbb{Z}_+^{n-1}$, if we can identify the smallest x_1 as $S(x_{-1})$, such that $\Delta_1 v(x) < c_p$, then it follows that $\Delta_1 v(x) < c_p$ for all $x_1 \geq S(x_{-1})$, and $\Delta_1 v(x) \geq c_p$ for all $x_1 < S(x_{-1})$. Thus, it is optimal to produce a fresh item, i.e., item 1, if and only if the current inventory level x satisfies $x_1 < S(x_{-1})$. Thus, the optimal production decision can be fully characterized by $S(x_{-1})$ for $(x_2, x_3, \ldots, x_n) \in \mathbb{Z}_+^{n-1}$, which is formally defined below.

Definition 3.3 *We define state-dependent threshold $S(x_{-1})$ as follows.*

$$S(x_{-1}) = \min\left\{x_1 | \Delta_1 v(x) < c_p\right\},$$

where $x_{-1} = (x_2, x_3, \ldots, x_n)$.

The properties of the value function discussed in the previous section helps us to characterize the structural properties of optimal production decisions in the following theorem.

Theorem 3.2 (Optimal production policy) *Given the inventory level vector x, the optimal production decisions can be determined as follows:*

(1) It is optimal to produce fresh items when $x_1 \leq S(x_{-1})$, and to not produce otherwise, where $x_{-1} = (x_2, x_3, \ldots, x_n)$.

(2) Threshold $S(x_{-1})$ is nonincreasing in each of variables x_i, $i \neq 1$.

(3) $\Delta_j S(x_{-1}) \leq \Delta_i S(x_{-1}) \leq 0$ *for all* $i > j$

The proof of Theorem 3.2 is provided in Appendix B.3. This theorem shed light on how the optimal production decision should be made. By utilizing part (1) of Theorem 3.2, we can generate a set of points $\{(S(x_{-1}), x_{-1})) : x_{-1} \in \mathbb{Z}_+^{n-1}\}$ in the domain of $\mathbb{Z}_+ \times \mathbb{Z}_+^{n-1}$. Connecting these points forms a switching curve, which divides the plane into two distinct regions. The region $\{(x_1, x_{-1}) : x_1 < S(x_{-1})\}$ denotes the optimal production zone for fresh items, whereas the region $\{(x_1, x_{-1}) : x_1 \geq S(x_{-1})\}$ indicates the optimal decision of producing nothing. This encompasses the points lying directly on the curve as well.

Part (2) of Theorem 3.2 indicates that the threshold $S(x_{-1})$ is nonincreasing in the on-hand inventory of all items $j > 1$. That is, when the inventory level of item j is high, it becomes less likely to produce a fresh item, i.e, item 1. The reason is that a higher inventory level of item j signifies greater product availability within the system, thereby reducing the necessity for maintaining high inventory levels of fresh items.

Part (3) of Theorem 3.2 shows that the decrease of the threshold value, resulting from an increase in the on-hand inventory of other products, is more pronounced for fresher products compared to non-fresh items. In other words, when the on-hand inventory of product i increases, the shift in the switching curve is more significant than the change caused by an increase in the on-hand inventory of product j when $i < j$.

There is an alternative way to describe this property using the optimal production action, denoted as $a(x) = \mathbb{1}_{\{x_1 < S(x_{-1})\}}$. Based on the anti-multimodularity property of the value function, the following inequalities hold: $-1 \leq \Delta_1 a(x) \leq \Delta_2 a(x) \leq \ldots \leq \Delta_{n-1} a(x) \leq \Delta_n a(x) \leq 0$. This implies that the optimal production decision is more sensitive to the on-hand inventory levels of newer products compared to older ones. Also, the marginal effect of each state variable on the optimal production decision is bounded between 0 and -1. As such, by increasing each state variable, the optimal production decision either remains unchanged or decreases by 1. Readers may refer to Appendix B.3 for more information and the proof.

Remark 1: Li and Yu (2014) showed that the replenishment quantity is more sensitive to the newer inventory than older inventory in a periodic-review system. However, their research primarily focused on inventory decisions for perishable products with fixed shelf life, without accounting for freshness-dependent demand. Moreover, we take a step further to utilize this property to devise effective heuristic policies.

3.4.2.2 Optimal Pricing Decisions

Next, we characterize the structure of optimal pricing decisions. We can show that the revenue operator $\mathscr{R}$ is a concave function of the price vector $\boldsymbol{P} = \{p_1, p_2, \ldots, p_n\}$. Then, applying the first-order conditions, in Proposition 3.1, we derive the optimal prices for different freshness levels.

Proposition 3.1 (Optimal pricing decisions) *The optimal price of an available product k in the inventory, i.e, $k \in A$, can be obtained as follows.*

$$p_k = \begin{cases} \frac{1}{2}\left(q_k + \Delta_k v(X)\right) & x_k \geq 1 \\ p_2(X) + (q_1 - q_2) & x_k = 0,\ k = 1 \\ \frac{(q_k - q_{k+1})p_{k-1}(X) + (q_{k-1} - q_k)p_{k+1}(X)}{q_{k-1} - q_{k+1}} & x_k = 0,\ k = 2, 3, \ldots, n \end{cases} \tag{3.8}$$

Proposition 3.1 characterizes the optimal prices for all the products. According to this proposition, the optimal price of a product if available is the average of its corresponding quality index and future marginal value. However, if it is not available, it can be obtained as a function of the prices and quality levels of the adjacent products. Next, the following theorem characterizes the structure of optimal pricing decisions.

Theorem 3.3 (The structure of pricing decisions) *Given the inventory level vector* x*, the following structural results for pricing decisions hold.*

(1) Price of product k*,* p_k*, is decreasing in* x_j *for any* j*.*

(2) Given inventory level vector x*, the following relation holds.*

$$\Delta_k\, p_k \leq \Delta_{k+1} p_k \leq \ldots \leq \Delta_n p_k \quad \forall k = 1,2,\ldots,n$$

$$\Delta_k\, p_k \leq \Delta_{k-1} p_k \leq \ldots \leq \Delta_1 p_k \quad \forall k = 1,2,\ldots,n$$

Proposition 3.1 shows the optimal pricing decision, while Theorem 3.3 explores the structural characteristics of optimal pricing decisions and their relationship with the on-hand inventory. According to Proposition 3.1 the optimal price of an item is determined by its quality index and marginal value. Consequently, all the properties observed in the marginal value of item k ($\Delta_k v(\mathrm{x})$) are preserved in its optimal price. In general, according to Theorem 3.3, the optimal price for item k decreases as its own on-hand inventory and the on-hand inventory of the other item $j \neq k$ increase. This means that when item k has a high inventory level x_k, it is beneficial to set a lower price to reduce the inventory quickly and maximize profits. Additionally, a high inventory level of the other item x_j implies a lower marginal revenue for item k, leading to a lower price for item k as well. Theorem 3.3 further implies that the optimal price of a particular item is more sensitive to the on-hand inventory of adjacent products that have a closer quality index to the item. Intuitively, the price of the least fresh product has a minimal impact on the price of the most fresh product, and vice versa.

3.4.3 Long-Run Average Profit

In this subsection, we demonstrate that the previous findings obtained in the discounted-profit setting hold when considering the long-run average profit setting. In the case of average profit, the optimality equation can be written as follows:

$$v(x)+\eta^* = \Lambda\mathscr{R}v(x)+\mu\mathscr{P}v(x)+\sum_{k=1}^{n}\theta_k\mathscr{E}_k v(x)+H(x) \tag{3.9}$$

Where η^* indicates the optimal average profit rate. The following theorem extends the results to the case of long-run average profit.

Theorem 3.4 *Under the average profit criterion, there is an optimal stationary policy that possesses all the characteristics of the optimal policy under the discounted profit criterion. Moreover, the optimal average profit is finite and remains constant regardless of the initial state, i.e.,* $\eta^*(x)=\eta^*$ *for all states* x.

Later, we make use of Theorem 3.4 to develop our heuristics for solving a problem with average profit criterion. In the next section, we propose some extensions to the base model.

3.5 Extensions

3.5.1 Optimal Control of Inventory-Pricing with Donation

In this section, we assume that in any state, the firm can donate a product to avoid wastage and receive a reward. Firstly, donation contributes to reducing overall waste due to clearing items nearing their expirations which in turn can promote sustainable business practices.

Secondly, by donating an item, the firm gains a tangible reward, fostering positive economic outcomes. Let us denote $\mathscr{D}_k v^D(x)$ as the donation operator corresponding to the decision about whether or not to donate product k. Let δ_k be the donation rate for item k and r_k be the reward obtained after donation. Then, the uniformization rate can be written as $\phi = \Lambda + \sum_{k=1}^{n} C\theta_k + \mu + \sum_{k=1}^{n} \delta_k$ and the value function for the case with donation option can be expressed as follows.

$$v^D(x) = \Lambda \mathscr{R} v^D(x) + \mu \mathscr{P} v^D(x) + \sum_{k=1}^{n} \theta_k \mathscr{E}_k v^D(x) + \sum_{k=1}^{n} \delta_k \mathscr{D}_k v^D(x) + H(x) \tag{3.10}$$

Where

$$\mathscr{R} v^D(x) = \max_{\boldsymbol{P} \in \mathscr{P}_{\boldsymbol{x}}} \left\{ \sum_{k}^{n} \alpha_k \left(v^D(x - \boldsymbol{e}_k) + p_k \right) + \alpha_{n+1} v^D(x) \right\} \tag{3.11}$$

$$\mathscr{P} v^D(x) = \max \left\{ v^D(x + \boldsymbol{e}_1) - c_p, v^D(x) \right\} \tag{3.12}$$

$$\mathscr{E}_k v^D(x) = \begin{cases} x_k v^D(x - \boldsymbol{e}_k + \boldsymbol{e}_{k+1}) + (C - x_k) v^D(x) & x_k \geq 1 \\ C v^D(x) & x_k = 0 \end{cases} \quad \forall k \leq n-1 \tag{3.13}$$

$$\mathscr{E}_n v^D(x) = \begin{cases} x_n \left[v^D(x - \boldsymbol{e}_n) - \varphi \right] + (C - x_n) v^D(x) & x_n \geq 1 \\ C v^D(x) & x_n = 0 \end{cases} \tag{3.14}$$

$$\mathscr{D}_k v^D(x) = \max \left\{ v^D(x - \boldsymbol{e}_k) + r_k, v^D(x) \right\} \quad \forall k \tag{3.15}$$

$$H(x) = -\sum_{k=1}^{n} h_k x_k \tag{3.16}$$

First, in the following theorem, we show preservation of value function properties in a model with donation.

Theorem 3.5 *The optimal value function $v^D(x)$ is an element of Ω, that is $v^D(x) \in \Omega$, and* *it preserves the anti-multimodularity property.*

Hence, all the structural properties presented for optimal production and pricing decisions still hold in a model with donation possibility. Next, we characterize the optimal donation decisions. Let us define x_{-k} as the inventory vercor excluding x_k, then we define state-dependent threshold $S_k^D(x_{-k})$ for any $1 \leq k \leq n$ as follows.

Definition 3.4 *We define state-dependent threshold $S_k^D(x_{-k})$ as follows.*

$$S_k^D(x_{-k}) = \min\left\{x_k | \Delta_k v^D(x) < r_k\right\}$$

Where $x_{-k} = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$.

The properties of the value function discussed in the previous section helps us to characterize the structural properties of optimal donation decisions in the following theorem.

Theorem 3.6 (Optimal donation policy) *Given the inventory level vector x, the optimal donation decisions can be determined as follows:*

(1) It is optimal to donate item k when $x_k \geq S_k^D(x_{-k})$, and to not donate otherwise, where $x_{-k} = \{x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n\}$.

(2) Threshold $S_k^D(x_{-k})$ is nonincreasing in each of variables x_i, $i \neq k$

(3) $\Delta_j S^D(x_{-1}) \leq \Delta_i S^D(x_{-1})$ for all $i > j$

Theorem 3.6 provides information regarding the optimal donation decisions for different products. Using part (1) of Theorem 6, we can create a set of points $\left\{\left(x_{-k}, S_k^D(x_{-k})\right) : x_{-k} \in \mathbb{Z}_+^{n-1}\right\}$ within the domain of $\mathbb{Z}_+ \times \mathbb{Z}_+^{n-1}$. These points can be connected to form a switching curve, partitioning the plane into two distinct regions. The region $\left\{\left(x_{-k}, S_k^D(x_{-k})\right) : x_k \geq S_k^D(x_{-k})\right\}$

indicates the optimal donation zone for product k, while the region $\left\{\left(x_{-k}, S_k^D(x_{-k})\right) : x_k < S_k^D(x_{-k})\right\}$ signifies the optimal choice of retaining item k instead of donating it.

Part (2) of Theorem 3.6 indicates that the threshold $S^D(x_{-k})$ is nonincreasing in the on-hand inventory of all items $j \neq k$. In other words, as the inventory level of item $j \neq k$ increases, the likelihood of retaining item k decreases and the likelihood of donating it increases. The reason is that a higher inventory level of item j implies greater product availability within the system, thereby diminishing the need for maintaining high inventory levels of item k.

Interestingly, part (3) of Theorem 3.6 implies that the reduction in the threshold value due to an increase in the inventory level is greater for products with higher freshness levels compared to those with lower freshness levels. In other words, when the on-hand inventory of product i increases, the shift in the switching curve is more significant than the change caused by an increase in the on-hand inventory of product j when $i < j$. This intuitively suggests that the influence of fresher items on donation decisions is more substantial than that of less fresh items.

3.5.2 Replenishment System

In this section, we consider a replenishment system wherein the firm places orders for fresh items rather than producing them. Each order incurs a per-unit purchasing cost of c_o. The order placement/announcement follows an exponential distribution with a rate of β. Subsequently, the firm faces an exponentially distributed lead time with a rate of γ before receiving the order.

The inventory system's state is denoted as $w = \{x, y\} := (x_1, x_2, \ldots, x_n, y)$, where $x_k \in \{0, 1, \ldots, C\}$ represents the quantity of product k in stock, and $y \in \{0, 1, \ldots, C\}$ denotes the size of an outstanding order. Therefore, the dimension of w is $n+1$. To ensure that the fresh product inventory remains within its capacity, the size of any outstanding order must be less than $C - x_1 - y$ when there are x_1 fresh products in stock. The state space $\mathscr{S}$ is defined as the intersection of all possible inventory levels and order quantities. In the replenishment system, we can write the uniformization rate as $\phi = \Lambda + \sum_{k=1}^{n} C\theta_k + \beta + \gamma$. Then, Bellman equations (readers may refer to Bellman (1966) for more information on Bellman equations) can be written as follows .

$$v^R(w) = \Lambda \mathscr{R} v^R(w) + \sum_{k=1}^{n} \theta_k \mathscr{E}_k v^R(w) + \beta \mathscr{O} v^R(w) + \gamma \mathscr{L} v^R(w) + H(w) \tag{3.17}$$

Where

$$\mathscr{R} v^R(w) = \max_{\boldsymbol{P} \in \mathscr{P}_{\boldsymbol{w}}} \left\{ \sum_{k}^{n} \alpha_k \left(v^R(w - \boldsymbol{e}_k) + p_k \right) + \alpha_{n+1} v^R(w) \right\} \tag{3.18}$$

$$\mathscr{E}_k v^R(w) = \begin{cases} x_k v^R(w - \boldsymbol{e}_k + \boldsymbol{e}_{k+1}) + (C - x_k) v^R(w) & x_k \geq 1 \\ C v^R(w) & x_k = 0 \end{cases} \quad \forall k \leq n-1 \tag{3.19}$$

$$\mathscr{E}_n v^R(w) = \begin{cases} x_n \left[v^R(w - \boldsymbol{e}_n) - \varphi \right] + (C - x_n) v^R(w) & x_n \geq 1 \\ C v^R(w) & x_n = 0 \end{cases} \tag{3.20}$$

$$\mathscr{O} v^R(w) = \max_{Q \in \mathscr{Q}} \left\{ v^R(w + Q\boldsymbol{e}_{n+1}) - c_o Q \right\} \tag{3.21}$$

$$\mathscr{L} v^R(w) = v^R(x, 0) \tag{3.22}$$

$$H(x) = -\sum_{k=1}^{n} h_k x_k \tag{3.23}$$

Where $\mathscr{Q} = \left\{Q : Q \leq C - x_1 - y;\ Q \leq C - \sum_{k=1}^{n} x_k - y\right\}$ denotes the feasible region for ordering decision. Further, $\mathscr{O}v^R(w)$ shows order announcement operator, and $\mathscr{L}v^R(w)$ indicates order arrival operator.

In the following theorem, we show that in replenishment system, the value function preserves properties (P1)-(P3).

Theorem 3.7 *The optimal value function $v^R(x)$ is an element of Ω, i.e., $v^R(x) \in \Omega$. Therefore, it is an anti-multimodular function.*

Thus, all the structural properties presented for optimal pricing decisions still hold in a replenishment system. Next, we characterize the optimal replenishment decisions. To that end, we define state-dependent threshold $S^R(x)$ as follows.

Definition 3.5 *We define state-dependent threshold $S^R(x)$ as follows.*

$$S^R(x) = \min\left\{y | \Delta_y v^R(w) < c_o\right\}$$

Then, the following theorem characterizes the structure of the optimal ordering policy.

Theorem 3.8 (Optimal replenishment policy) *Given the inventory level vector x, the optimal ordering decisions can be determined as follows:*

(1) Given the outstanding order y, when $y \leq S^R - 1$, it is optimal to order $Q^*(x) = S^R(x) - y - 1$ fresh items; otherwise not to order any product, i.e., $Q^*(x) = 0$.

(2) Threshold $S^R(x)$ is nonincreasing in each of variables x_k, $1 \leq k \leq n$

(3) $\Delta_j S^R(x_{-1}) \leq \Delta_i S^R(x_{-1})$ for all $i > j$

Theorem 3.8 characterizes the structure of the optimal ordering policy. Based on part (1) of Theorem 3.8, a set of points $\left\{\left(x, S^R(x)\right) : x \in \mathbb{Z}_+^n\right\}$ can be constructed within the domain of $\mathbb{Z}_+^n \times \mathbb{Z}_+$. These points can be connected to form a switching curve, dividing the plane into two distinct regions. The region $\left\{(x,y) : y \geq S^R(x)\right\}$ represents the zone wherein it is optimal to order nothing. However, the region $\left\{(x,y) : y < S^R(x)\right\}$ indicates the area in which it is optimal to order $Q^* = S^R(x) - y - 1$. Part (2) of Theorem 3.8 indicates as the inventory level of item j increases, signifying greater product availability within the system, the need to order fresh products diminishes. Consequently, this increase in product availability leads to a reduction in the threshold $S^R(x)$. Part (3) of Theorem 3.8 suggests that the reduction in the threshold value due to an increase in the inventory level is more sensitive to the newer items than older items. This observation intuitively indicates that fresher products play more important role than less fresh items in the ordering policy.

3.5.3 Multi-Phase Quality Transformation

In this subsection, we generalize the base model to the case where each product type with a specific quality index may have multiple phases of shelf life before transforming into a product with a different quality index. Let us denote $\boldsymbol{\omega} = \{\omega_1, \omega_2, \ldots, \omega_n\}$ as the set containing a number of phases for each product type. For example, if product type 1 has 2 phases of shelf life, we have $\omega_1 = 2$.

Given that deterioration times are independent, the rate at which product type k in phase j transitions to the phase $j+1$ is equal to $i\theta_{k,j}$ when there are i instances of product k present in phase j, where $j \in [0, \omega_k]$.

For ease of exposition, we define $I_{k,j} = \sum_{i=1}^{k-1} \omega_i + j$, which shows the position of j^{th} phase of product k in the state vector. Next, we uniformize the continuous-time Markov with the uniform rate $\phi = \Lambda + \sum_{k=1}^{n}\sum_{j=1}^{\omega_k} C\theta_{k,j_k} + \mu$ and then write the value function as follows.

$$v^M(x) = \Lambda \mathscr{R} v^M(x) + \mu \mathscr{P} v^M(x) + \sum_{k=1}^{n}\sum_{j=1}^{\omega_k} \theta_{k,j}\mathscr{E}_{k,j} v^M(x) + H(x) \tag{3.24}$$

Where

$$\mathscr{R} v^M(x) = \max_{\boldsymbol{P}\in\mathscr{P}_{\boldsymbol{X}}} \left\{ \sum_{k=1}^{n}\sum_{j=1}^{\omega_k} \alpha_k \rho_{k,j}(x)\left(v^M(x-\boldsymbol{e}_{I_{k,j}}) + p_k\right) + \alpha_{n+1} v^M(x) \right\} \tag{3.25}$$

$$\mathscr{P} v^M(x) = \max\left\{ v^M(x+\boldsymbol{e}_1) - c_p, v^M(x) \right\} \tag{3.26}$$

$$\mathscr{E}_{k,j} v^M(x) = \begin{cases} x_{I_{k,j}} v^M(x - \boldsymbol{e}_{I_{k,j}} + \boldsymbol{e}_{I_{k,j}+1}) + (C - x_{I_{k,j}}) v^M(x) & x_{I_{k,j}} \geq 1 \\ C v^M(x) & x_{I_{k,j}} = 0 \end{cases} \quad \forall k \leq n-1, \;\; j \leq \omega_k \tag{3.27}$$

$$\mathscr{E}_{n,j} v^M(x) = \begin{cases} x_{I_{n,j}} \left[v^M(x - \boldsymbol{e}_{I_{n,j}} + \boldsymbol{e}_{I_{n,j}+1}) - \varphi \right] + (C - x_{I_{n,j}}) v^M(x) & x_{I_{n,j}} \geq 1 \\ C v^M(x) & x_{I_{n,j}} = 0 \end{cases} \quad j \leq \omega_n - 1 \tag{3.28}$$

$$\mathscr{E}_{n,\omega_n} v^M(x) = \begin{cases} x_{I_{n,n}} \left[v^M(x - \boldsymbol{e}_{I_{n,n}}) - \varphi \right] + (C - x_{I_{n,n}}) v^M(x) & x_{I_{n,n}} \geq 1 \\ C v^M(x) & x_{I_{n,n}} = 0 \end{cases} \tag{3.29}$$

$$H(x) = -\sum_{k=1}^{n} h_k x_k \tag{3.30}$$

The following Theorem implies that in a model with multiple phases of quality transformation, the value function preserves properties (P1)-(P3).

Theorem 3.9 *The optimal value function $v^M(x)$ is an element of Ω, that is $v^M(x) \in \Omega$.Therefore,* *it is an anti-multimodular function.*

Therefore, the structural characteristics established for ordering and pricing decisions in the base model remain valid for this extended model.

3.6 Heuristic Policies

In this section, we propose three heuristic models to overcome the complexity of the proposed model, specifically in large-scale systems. The first heuristic model is proposed based on the dimension reduction strategy, while the second and the third heuristics are developed by considering static decisions. In contrast to the second heuristic model, the third model leads to near optimal solutions that satisfy the structural properties we derive in section 5. While these proposed heuristic models offer solutions close to optimality, they do not provide exact optimal solutions due to their static decision-making nature. Therefore, As a prospective avenue, leveraging the identified structural properties allows for a substantial reduction in the search space and development of an exact algorithm that incorporates dynamic decision-making.

3.6.1 Heuristic Policy 1: Pooling Inventories with Different Quality Levels: New and Old Products (H_1)

Dimension reduction is one of the common methods for simplifying problems with state-dependent decisions. In our first heuristic method, denoted as H_1, we cluster different product categories based on their quality indexes into two groups: “new” and “old” inventories. To that end, we partition n products into two sets $\boldsymbol{J} = \{J_1, J_2\}$ such that withing cluster sum of squares (variance) is minimized. Formally, we aim to obtain the optimal $\boldsymbol{J}$

using the following optimization problem.

$$\arg\min_{\boldsymbol{J}} \sum_{i=1}^{2} \sum_{x \in J_i} \|x - \bar{x}_i\|^2 \tag{3.31}$$

Where $\bar{x}_i = \frac{1}{|J_i|} \sum_{x \in J_i} x$.

Let us clarify this approach by assuming that after clustering, products $1, 2, \ldots, m$ with quality indexes $\{q_1, q_2, \ldots, q_m\}$ are categorized as the "new" category and products $m+1, m+2, \ldots, n$ with quality indexes $\{q_{m+1}, q_{m+2}, \ldots, q_n\}$ are categorized as the "old" category. We also consider corresponding transformation rates as $\{\theta_1, \theta_2, \ldots, \theta_m\}$ and $\{\theta_{m+1}, \theta_{m+2} \ldots, \theta_n\}$, respectively. To estimate the rate of transformation from the new inventory to the old inventory, we approximate the average transformation rate from product 1 (the freshest item in the new category) to product $m+1$ (the freshest item in the old category) as $\theta_{\text{new}} = \frac{1}{\sum_{k=1}^{m} \frac{1}{\theta_k}}$. Similarly, we estimate the expiration rate as $\theta_{\text{old}} = \frac{1}{\sum_{k=m+1}^{n} \frac{1}{\theta_k}}$. Let p_{new} and p_{old} represent the prices of new and old products, respectively. Then, we can obtain α_{new} and α_{old} as the purchasing probabilities for new and old products, respectively. Let the state be $(x_{\text{new}}, x_{\text{old}})$, where x_{new} denotes the inventory level of new products and x_{old} shows the inventory level of old products. Then, the model after the dimension reduction under the average profit criterion can then be expressed as follows.

$$\begin{aligned} v^{H_1}(x_{\text{new}}, x_{\text{old}}) + \eta^* =& \Lambda \mathscr{R} v^{H_1}(x_{\text{new}}, x_{\text{old}}) + \mu \mathscr{P} v^{H_1}(x_{\text{new}}, x_{\text{old}}) + \theta_{\text{new}} \mathscr{E}_{\text{new}} v^{H_1}(x_{\text{new}}, x_{\text{old}}) \\ &+ \theta_{\text{old}} \mathscr{E}_{\text{old}} v^{H_1}(x_{\text{new}}, x_{\text{old}}) + H(x_{\text{new}}, x_{\text{old}}) \end{aligned} \tag{3.32}$$

Where

$$\mathscr{R}v^{H_1}(x_{\text{new}},x_{\text{old}}) = \max_{p_{\text{new}},p_{\text{old}}\in\mathscr{P}_{\boldsymbol{x}}} \left\{ \begin{array}{l} \alpha_{\text{new}}\left(v^{H_1}(x_{\text{new}},x_{\text{old}})+p_{\text{new}}\right)+\alpha_{\text{old}}\left(v^{H_1}(x_{\text{new}},x_{\text{old}})+p_{\text{old}}\right) \\ +(1-\alpha_{\text{new}}-\alpha_{\text{old}})v^{H_1}(x_{\text{new}},x_{\text{old}}) \end{array} \right\} \tag{3.33}$$

$$\mathscr{P}v^{H_1}(x_{\text{new}},x_{\text{old}}) = \max\left\{v^{H_1}(x_{\text{new}}+1,x_{\text{old}})-c_p, v^{H_1}(x_{\text{new}},x_{\text{old}})\right\} \tag{3.34}$$

$$\mathscr{E}_{\text{new}}v^{H_1}(x_{\text{new}},x_{\text{old}}) = \begin{cases} x_{\text{new}}v^{H_1}(x_{\text{new}}-1,x_{\text{old}}+1)+(C-x_{\text{new}})v^{H_1}(x_{\text{new}},x_{\text{old}}) & x_{\text{new}}\geq 1 \\ Cv^{H_1}(x_{\text{new}},x_{\text{old}}) & x_{\text{new}}=0 \end{cases} \tag{3.35}$$

$$\mathscr{E}_{\text{old}}v^{H_1}(x_{\text{new}},x_{\text{old}}) = \begin{cases} x_{\text{old}}\left[v^{H_1}(x_{\text{new}},x_{\text{old}}-1)-\varphi\right]+(C-x_{\text{old}})v^{H_1}(x_{\text{new}},x_{\text{old}}) & x_{\text{old}}\geq 1 \\ Cv^{H_1}(x_{\text{new}},x_{\text{old}}) & x_{\text{old}}=0 \end{cases} \tag{3.36}$$

$$H(x_{\text{new}},x_{\text{old}}) = -h_{\text{new}}x_{\text{new}}-h_{\text{old}}x_{\text{old}} \tag{3.37}$$

After obtaining the optimal solutions in the equivalent two-dimensional problem, we map the derived solutions back to the corresponding decisions in the original problem. For this purpose, we consider the following rules:

- **Production decisions:** The optimal decision in state $x_1,\ldots,x_m,x_{m+1},\ldots,x_n$ in the original problem is equivalent to the optimal decision in state $x_1+\ldots+x_m$, $x_{m+1}+\ldots+x_n$ in the reduced problem, where $x_{\text{new}}=x_1+\ldots+x_m$ and $x_{\text{old}}=x_{m+1}+\ldots+x_n$
- **Pricing decisions:** To map the optimal decisions in two dimensions back to the original problem, we consider the price for new and old products as the average price for their respective category, and then we obtain the price for each product

proportional to its quality index. Specifically, the price of product k can be obtained as $p_k = p_{\text{new}} \times \frac{mq_k}{\sum_{k=1}^{m} q_k}$ if k is a new product, and as $p_k = p_{\text{old}} \times \frac{(n-m)q_k}{\sum_{k=n-m}^{n} q_k}$ if k is an old item.

3.6.2 Heuristic Policy 2: Independent Threshold and Pricing Policies (H_2)

The second heuristic involves independent static actions to control production and pricing decisions. For the production control, we introduce a vector $s = \{s_1, s_2, \ldots, s_n\}$ consisting of fixed thresholds for each class of products. When the conditions $\{x_1 \leq s_1, x_2 \leq s_2, \ldots, x_n \leq s_n\}$ are satisfied, the firm produces fresh items; otherwise, it does not.

In addition, pricing decisions are controlled using a vector $\boldsymbol{P} = \{p_1, p_2, \ldots, p_n\}$, representing fixed, independent prices for each product type. Therefore, the static version of the problem under the average profit criterion can be stated as follows.

$$\max_{S,\boldsymbol{P}} \eta^*(s, \boldsymbol{P}) \tag{3.38}$$

$$v^{H_2}(x, s, \boldsymbol{P}) + \eta^*(s, \boldsymbol{P}) = \Lambda \mathscr{R} v^{H_2}(x, s, \boldsymbol{P}) + \mu \mathscr{P} v^{H_2}(x, s, \boldsymbol{P}) + \sum_{k=1}^{n} \theta_k \mathscr{E}_k v^{H_2}(x, s, \boldsymbol{P}) + H(x) \tag{3.39}$$

Where

$$\mathscr{R}v^{H_2}(x,s,\boldsymbol{P}) = \sum_{k}^{n} \alpha_k \left(v^{H_2}(x-\boldsymbol{e}_k,s,\boldsymbol{P}) + p_k\right) + \alpha_{n+1} v^{H_2}(x,s,\boldsymbol{P}) \tag{3.40}$$

$$\mathscr{P}v^{H_2}(x,s,\boldsymbol{P}) = \begin{cases} v^{H_2}(x+\boldsymbol{e}_1,s,\boldsymbol{P}) - c_p & \text{If } x_1 \le s_1, x_2 \le s_2, \ldots, x_n \le s_n \\ v^{H_2}(x,s,\boldsymbol{P}) & \text{otherwise} \end{cases} \tag{3.41}$$

$$\mathscr{E}_k v^{H_2}(x,s,\boldsymbol{P}) = \begin{cases} x_k v^{H_2}(x-\boldsymbol{e}_k+\boldsymbol{e}_{k+1},s,\boldsymbol{P}) + (C-x_k) v^{H_2}(x,s,\boldsymbol{P}) & x_k \ge 1 \\ C v^{H_2}(x,s,\boldsymbol{P}) & x_k = 0 \end{cases} \quad \forall k \le n-1 \tag{3.42}$$

$$\mathscr{E}_n v^{H_2}(x,s,\boldsymbol{P}) = \begin{cases} x_n \left[v^{H_2}(x-\boldsymbol{e}_n,s,\boldsymbol{P}) - \varphi\right] + (C-x_n) v^{H_2}(x,s,\boldsymbol{P}) & x_n \ge 1 \\ C v^{H_2}(x,s,\boldsymbol{P}) & x_n = 0 \end{cases} \tag{3.43}$$

$$H(x) = -\sum_{k=1}^{n} h_k x_k \tag{3.44}$$

In the above optimization problem, the objective is to find the static thresholds and prices to maximize profit.

3.6.3 Heuristic Policy 3: Coordinated Threshold and Pricing Policies (H_3)

Although the first two heuristics reduce the time complexity of solving the problem, they fail to respect the structural properties of the optimal policies obtained in the previous sections. To address this, the third heuristic involves controlling production and pricing decisions by leveraging these structural properties. For production decision, instead of introducing several independent thresholds, a single coordinated threshold line $\xi_1 x_1 + \xi_2 x_2 +$

$\ldots + \xi_n x_n \leq S$ is introduced.

Based on the theoretical results, newer products carry a higher weight than older products in the ordering policy. Thus, we have $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$, and without loss of generality, we assume that $\xi_1 = 1$. To characterize this threshold, we need to determine the optimal S and $\boldsymbol{\xi} = \{\xi_2 \ldots, \xi_n\}$. At a specific state $(x_1, x_2, \ldots, x_n)$, when $x_1 + \xi_2 x_2 + \ldots + \xi_n x_n \leq S$, the firm produces fresh items, and otherwise, it does not produce any item.

In section 3.4, we derive the optimal price of available product k as $p_k = \frac{1}{2}(q_k + \Delta_k v(x))$, which is inversely related to the inventory level of its corresponding product as well as other products. Therefore, in line with the structural properties, we express the price for any product k as $p_k = \frac{1}{2}(q_k + f_k(x))$, where $f_k(x)$ is a decreasing function of all the inventory levels. For simplicity, here we assume that $f_k(x)$ is a linear function of total inventory level, i.e., $f_k(x) = -\sigma^k \sum_{i=1}^n x_i$, where σ^k is the coefficient for product k, and $\boldsymbol{\sigma} = \left\{\sigma^1, \sigma^2, \ldots, \sigma^n\right\}$ is the vector of all the coefficients. Therefore, the optimization problem can be written as follows:

$$\max_{S, \boldsymbol{\xi}, \boldsymbol{\sigma}} \eta^*(S, \boldsymbol{\xi}, \boldsymbol{\sigma}) \tag{3.45}$$

$$\begin{aligned} v^{H_3}(x, S, \boldsymbol{\xi}, \boldsymbol{\sigma}) + \eta^*(S, \boldsymbol{\xi}, \boldsymbol{\sigma}) =& \Lambda \mathscr{R} v^{H_3}(x, S, \boldsymbol{\xi}, \boldsymbol{\sigma}) + \mu \mathscr{P} v^{H_3}(x, S, \boldsymbol{\xi}, \boldsymbol{\sigma}) + \\ & \sum_{k=1}^n \theta_k \mathscr{E}_k v^{H_3}(x, S, \boldsymbol{\xi}, \boldsymbol{\sigma}) + H(x) \end{aligned} \tag{3.46}$$

Where

$$\mathscr{R}v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) = \sum_{k}^{n} \alpha_k \left(v^{H_3}(x-\boldsymbol{e}_k,S,\boldsymbol{\xi},\boldsymbol{\sigma}) + p_k(x,\boldsymbol{\sigma}) \right) + \alpha_{n+1} v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) \tag{3.47}$$

$$\mathscr{P}v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) = \begin{cases} v^{H_3}(x+\boldsymbol{e}_1,S,\boldsymbol{\xi},\boldsymbol{\sigma}) - c_p & \text{If } x_1 + \xi_2 x_2 + \ldots + \xi_n x_n \leq S \\ v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) & \text{otherwise} \end{cases} \tag{3.48}$$

$$\mathscr{E}_k v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) = \begin{cases} x_k v^{H_3}(x-\boldsymbol{e}_k+\boldsymbol{e}_{k+1},S,\boldsymbol{\xi},\boldsymbol{\sigma}) + (C-x_k) v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) & x_k \geq 1 \\ C v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) & x_k = 0 \end{cases} \quad \forall k \leq n-1 \tag{3.49}$$

$$\mathscr{E}_n v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) = \begin{cases} x_n \left[v^{H_3}(x-\boldsymbol{e}_n,S,\boldsymbol{\xi},\boldsymbol{\sigma}) - \varphi \right] + (C-x_n) v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) & x_n \geq 1 \\ C v^{H_3}(x,S,\boldsymbol{\xi},\boldsymbol{\sigma}) & x_n = 0 \end{cases} \tag{3.50}$$

$$H(x) = -\sum_{k=1}^{n} h_k x_k \tag{3.51}$$

$$p_k(x,\boldsymbol{\sigma}) = \frac{1}{2} \left(q_k - \sigma^k \sum_{i=1}^{n} x_i \right) \quad \forall k = 1,\ldots,n \tag{3.52}$$

The objective of the above optimization is to obtain the static threshold line and prices as a function of inventory levels. This problem can be easily solved using grid search or linear programming methods. When the number of products increases, the complexity of this heuristic increases, too. Therefore, one may consider uniform coefficients for different products to simplify the model.

3.7 Computational Results

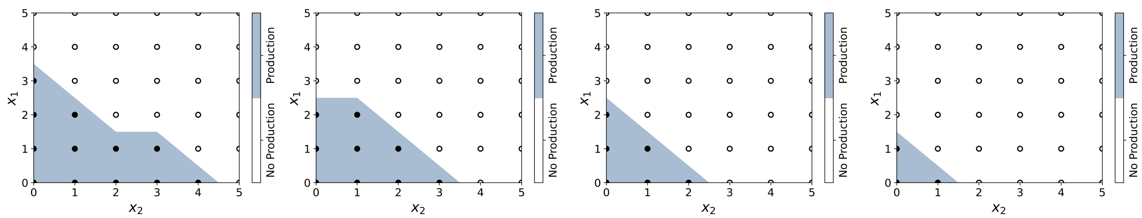
This section aims to demonstrate the robustness of theoretical results, model's sensitivity to various parameters, assess the heuristic models' performance, and evaluate the extended models. We conduct experiments on a small-scale problem with three products with distinct freshness levels indexed by q_1, q_2, and q_3. We consider the following set of parameters. $\mu = 15$, $\Lambda = 10$, $\theta_1 = 1$, $\theta_2 = 1.5$, $\theta_3 = 2$, $q_1 = 6$, $q_2 = 4$, $q_3 = 3$, $c_p = 0.2$, $\phi = 0.1$, $h_1 = 0.01$, $h_2 = 0.01$, $h_3 = 0.01$. To examine the effect of change in the variability in quality index among different products, we further define $\tau = \frac{q_1}{q_2} = \frac{q_2}{q_3}$. Also, for a model with donation, we consider the following additional parameters. $\delta_1 = 2$, $\delta_2 = 2$, $\delta_3 = 2$, $r_1 = 1$, $r_2 = 0.5$, $r_3 = 0.25$.

In what follows, we present sensitivity analysis findings for the average profit criterion, as they are independent of the initial inventory levels and the discount factor.

3.7.1 Behavior of State-Dependent Optimal Production and Pricing Decisions

In this section, we discuss the robustness of theoretical results presented in the previous sections. Figure 3.1 shows the optimal production and Figure 3.2 displays the optimal prices for different state variable values x_1, x_2, and x_3. Note that because the capacity of the system is $C = 5$ and x_1, x_2, and x_3 must satisfy $x_1 + x_2 + x_3 \leq C$. As indicated earlier, according to Figure 3.1, the production threshold in decreasing in the inventory level of all products. Further, the production threshold, and consequently production decision, is more sensitive to fresher products compared to the less fresh items. To illustrate, we examine the values of $S(x_2, x_3)$. According to Figure 3.1, $S(1,0) = 2$, and by increasing the inventory of

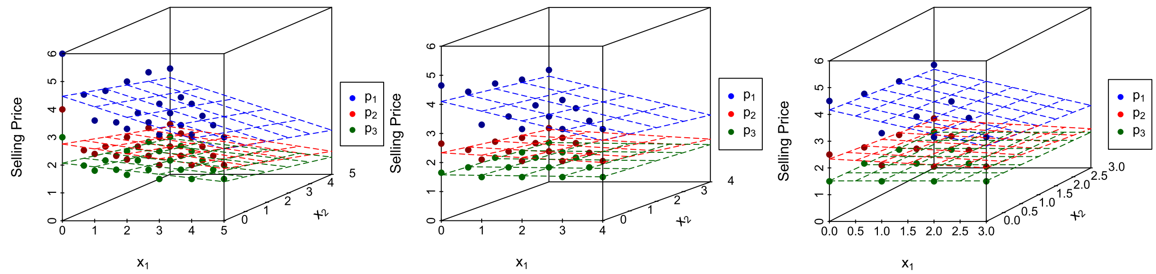
x_2 by one unit, production threshold will decrease by one unit, i.e., $S(2,0) = 1$. However, by increasing the the inventory of x_1 by one unit, production threshold will remain unchanged. This means that the production threshold is more sensitive to fresher products than non-fresh items.



(a) $x_3 = 0$ (b) $x_3 = 1$ (c) $x_3 = 2$ (d) $x_3 = 3$

Figure 3.1: The effect of increasing the inventory levels on the production decisions

According to the Figure 3.2, it is evident that price of a product is proportional to its quality index. This means that items are priced in accordance with their quality index, with the freshest item having the highest price and the non-freshest item having the lowest price. Furthermore, According to Figure 3.2, the optimal price for a given products is decreasing in its own inventory level as well as cross-products inventory level. However, according to the slopes in Figure 3.2, the sensitivity is more to the more adjacent products in terms of quality index. Intuitively, the pricing of fresh items is influenced more by the prices of other fresh items compared to non-fresh items, thereby impacting quality-sensitive customers than those sensitive to prices. Conversely, the pricing of non-fresh items is more responsive to the pricing of non-fresh items with close quality index, thereby affecting customers who are more sensitive to price than quality considerations.



(a) $x_3 = 0$ (b) $x_3 = 1$ (c) $x_3 = 2$

Figure 3.2: The effect of increasing the inventory levels on the pricing decisions

Next, we obtain the optimal profit as well as profits under heuristic methods for different parameter setting to evaluate the performance of heuristic methods.

3.7.2 Optimality Gap of Heuristic Models

In this section, we present the computational results of our heuristic policies and analyze their performance using a small-scale problem. We define the optimality gap of each heuristic policy H_i as the difference between the profit of the solution obtained by the heuristic and the optimal profit for the given problem instance. Formally, the optimality gap is computed as:

$$\Delta v^{\pi}\% = \frac{v^* - v^{\pi}}{v^*}, \ \ \pi \in \{H_1, H_2, H_3\}$$

Where v^{π} shows the maximal profit function under heuristic policy $\pi \in \{H_1, H_2, H_3\}$. To obtain v^*, we use the value iteration method with the terminating condition of reaching the five-digit accuracy. Then, we present the performance of the heuristic policies by varying different parameters in Table 3.1.

Table 3.1: Optimality gap of different heuristic policies

Parameters	**Values**	v^*	$\Delta v^{H_1}\%$	$\Delta v^{H_2}\%$	$\Delta v^{H_3}\%$
μ	5	10.881	15.801	2.491	1.502
	10	12.693	15.680	2.159	1.009
	15	13.041	14.116	2.304	0.432
	20	13.177	17.086	1.373	0.256
	25	13.258	14.636	0.991	0.181
Λ	5	6.368	13.365	1.050	0.272
	10	13.041	11.142	2.304	0.432
	15	19.556	14.628	2.335	0.870
	20	25.658	17.482	2.677	2.115
	25	31.125	16.221	2.854	2.322
τ	1.5	13.031	12.867	2.311	0.397
	2	17.622	13.034	2.174	0.390
	2.5	22.254	14.164	1.973	0.421
	3	26.916	15.388	1.714	0.451
	3.5	31.598	15.700	1.530	0.483
	4	36.293	16.619	1.396	0.467
θ_1	0.5	13.427	9.524	1.133	0.297
	2.5	12.192	21.796	2.302	0.465
	5	11.298	24.400	3.677	0.481
	10	10.308	26.988	3.265	0.395

Table 3.1 shows percentage gap of different heuristic policies. The results indicate that the third heuristic has the best performance, followed by the second and the third heuristics, respectively.

The maximum gap for the heuristic policies H_2 and H_3 across all the instances that we solved are 3.677% and 2.322%, respectively. The results further reveal that heuristic policies H_2 and H_3 work well when supply-to-demand ratio, indicating ample capacity to meet demand. This is because, in such instances, the optimal production threshold structure closely resembles a single threshold line, with minimal staircase patterns. Furthermore, due to the same reasons, these heuristic policies perform effectively when the quality variability among products is high or transformation rate is low. On the other hand, the first heuristic, utilizing dimension reduction techniques, demonstrates better performance when the quality variability or quality transformation rate is lower. This is because the policy classifies all products into two categories, i.e., new and old products. Thus, its effectiveness is enhanced by higher similarity between products within the same category.

3.7.3 The Value of Donation Policy

The relative value of adopting donation policy is denoted by $\Delta v^D\%$, and can be expressed as follows.

$$\Delta v^D\% = \frac{v^D - v}{v} \times 100$$

Where v^D shows the optimal profit in a system with donation policy, and v indicates the optimal profit in the base model without donation policy. Table 3.2 shows the value of donation by varying different parameters. According to this table, the value of donation policy increases as market demand decreases or production rate increases. That is because by decreasing market demand or production rate, the wastage rate increases which in turn underscores the value of donation policy to earn reward rather than incurring expiration cost. Hence, in general, when supply-to-demand ratio increases, the value of donation policy increases, too because of increased likelihood of excess inventory and wastage.

The results of Table 3.2 further indicate that the higher quality index variation, implying a high gap between the quality of products, diminishes the effectiveness of the donation policy. Intuitively, the higher quality items are less susceptible to rapid deterioration. Conversely, lower quality items are more prone to the expiration. Therefore, as the variation in the quality index increases, the value of products wasted becomes comparatively low compared to the fresh products, and consequently, donation policy yields lower benefits in these scenarios.

Table 3.2: Sensitivity of different models to the input parameters

Parameters		Base Model		Donation Model				Single Pricing	
		Profit	**Wastage**	**Profit**	**Wastage**	**Donation**	$\Delta v^D\%$	**Profit**	$\Delta v^F\%$
μ	5	10.881	0.511	11.116	0.305	0.705	2.163	10.655	2.122
	10	12.693	1.108	13.704	0.774	1.723	7.962	12.551	1.138
	15	13.041	1.260	14.353	1.082	1.895	10.067	12.943	0.752
	20	13.177	1.174	14.586	1.282	1.973	10.687	13.098	0.609
	25	13.258	1.137	14.706	1.211	1.987	10.924	13.184	0.560
Λ	5	6.368	1.118	7.777	1.232	1.974	22.121	6.337	0.497
	10	13.041	1.260	14.353	1.082	1.895	10.067	12.943	0.752
	15	19.556	0.988	20.665	0.699	1.808	5.671	19.317	1.237
	20	25.658	0.578	26.457	0.408	1.444	3.112	25.234	1.683
	25	31.125	0.351	31.634	0.229	1.288	1.636	30.537	1.925
τ	1.5	13.031	1.246	14.343	1.087	1.894	10.070	12.932	0.764
	2	17.622	1.210	18.942	0.986	1.899	7.489	17.463	0.912
	2.5	22.254	1.157	23.621	0.930	1.910	6.144	22.014	1.091
	3	26.916	1.070	28.347	0.799	1.913	5.315	26.572	1.293
	3.5	31.598	1.008	33.091	0.758	1.911	4.724	31.136	1.483
	4	36.293	0.934	37.860	0.680	1.839	4.316	35.705	1.648

3.7.4 The Value of Freshness-Dependent Pricing

To evaluate the value of freshness-dependent pricing, we define v^S as the total profit in a system with single pricing policy wherein the selling price is uniform for all products with different freshness levels. Then, the relative value of freshness-dependent pricing (base model in this research) over single-pricing policy, denoted by $\Delta v^F\%$, can be expressed as

follows.

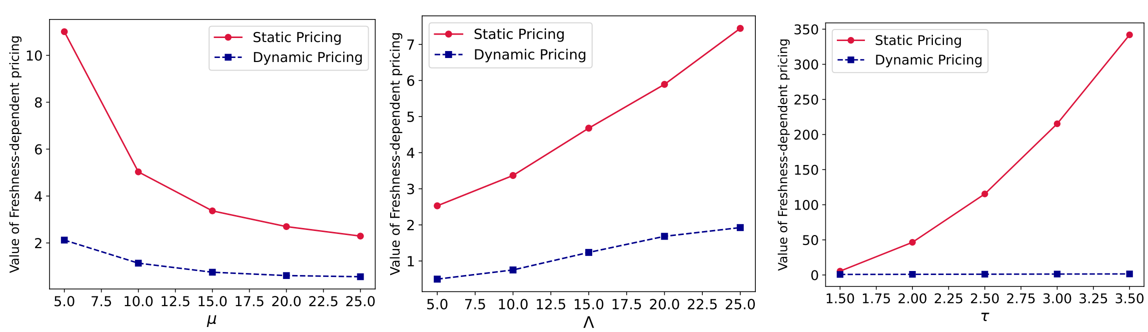
$$\Delta v^F\% = \frac{v - v^S}{v^S} \times 100$$

Where v^S shows the optimal profit in a system under single pricing policy, and v indicates the optimal profit in the base model with freshness-dependent pricing policy. The results in Table 3.2 suggest that as market demand increases, the production rate decreases, and transformation rates decrease, the value of the freshness-dependent pricing policy relative to the single pricing policy increases. Intuitively, under these scenarios, the perishable inventory system becomes closer to a non-perishable inventory system. As a result, the average number of products with lower quality in the inventory decreases, and the benefit of a freshness-dependent pricing policy diminishes. The results in Table 3.2 further imply that by increasing the variation in quality of different products, the need for the freshness-dependent pricing increases, too. The reason is that when there is a high gap between the value of fresh and non-fresh items, a uniform price across all products might result in missed profit opportunities. Adjusting prices to reflect freshness allows retailers to capture additional revenue by selling fresh items at higher prices while still catering to price-conscious customers interested in non-fresh items. Next, we analyze the the interaction among dynamic pricing, the value of freshness-dependent pricing, and donation policy.

3.7.5 Interplay between Dynamic Pricing, Freshness-Dependent Pricing, and Donation Policy

In this subsection, we use numerical examples to analyze the interplay between dynamic pricing, donation policy, and freshness-dependent pricing. It is evident that systems with donation or/and freshness-dependent pricing polices has higher performance than system without those policies. However, they exert different effects under system with static and dynamic pricing schemes. We show the value of freshness-dependent pricing for both static and dynamic systems by changing different parameters in Figure 3.3.

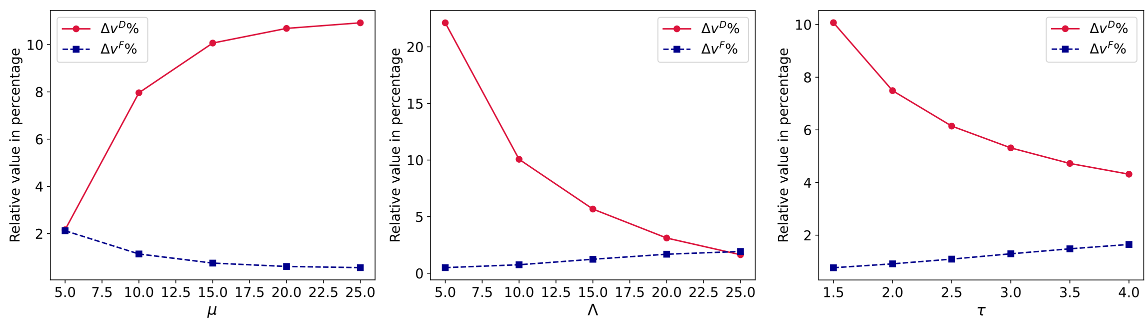
The results in Figure 3.3 reveal interesting insights. The value of freshness-dependent pricing is greater in a system with static pricing policy compared to a system with dynamic pricing policy. That is because freshness-dependent pricing and dynamic pricing can both help matching supply with demand. As a result, in a system with static pricing decisions that highly suffers from supply-demand matching, freshness dependent pricing is more helpful. However, in a system with dynamic prices, although freshness-dependent pricing can help coordinating demand with supply, its effect will be offset by the dynamic pricing. Therefore, the two strategies (dynamic pricing and freshness-dependent pricing) are substitutes.



(a) (b) (c)

Figure 3.3: The value of freshness-dependent pricing under dynamic pricing vs static pricing policies

Figure 3.3 further suggests that freshness-dependent pricing can be most advantageous for the system with high demand, limited supply, or high variability in quality. In other words, when the ratio of supply to demand increases or the variability in quality indices decreases, the advantages of freshness-dependent pricing decrease. Nevertheless, in these cases, implementing a donation policy can greatly assist the system by contributing excess supply and receiving rewards. Consequently, when supply-to-demand ratio is high or quality variability is low, according to Figure 3.4 (c), the donation policy can counterbalance the declining effectiveness of freshness-dependent pricing. Intuitively, donation policy and freshness-dependent policy are two complement policies.

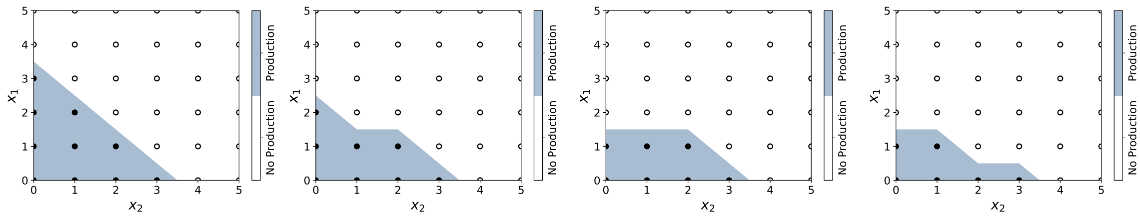


(a) (b) (c)

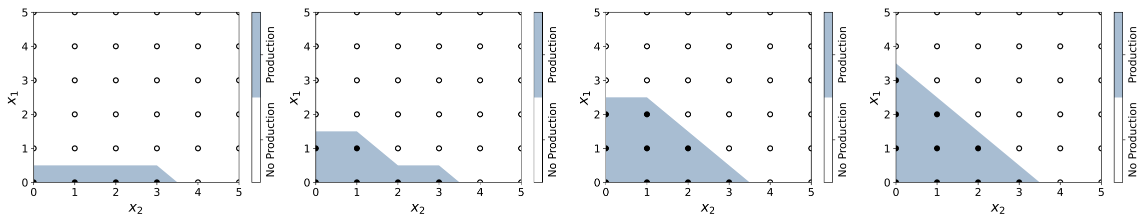
Figure 3.4: The value of freshness-dependent pricing and donation policy by changing different parameters

3.7.6 Impact of Different Parameters on Optimal Decisions

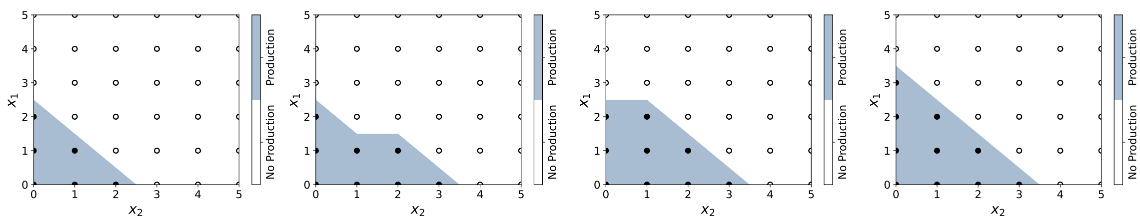
Figure 3.5 shows the impact of market demand, production rate, and quality variability on optimal production and pricing policies. Based on this figure, as market demand increases production threshold increases, meaning that the firm decides to produce more frequently to satisfy market demand. Also, prices of fresh, semi-fresh, and non-fresh items increase as a result of an increase in demand. On the other hand, as the production rate increases, the need for frequent production decreases, and the production threshold shifts to lower values. Also, when the production rate increases, the firm sets lower prices for fresh, semi-fresh, and non-fresh items due to the higher capacity.



(a) $\mu = 10$ (b) $\mu = 20$ (c) $\mu = 30$ (d) $\mu = 50$



(e) $\Lambda = 2$ (f) $\Lambda = 5$ (g) $\Lambda = 10$ (h) $\Lambda = 20$

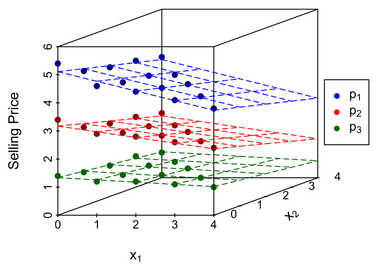


(i) $\tau = 1.05$ (j) $\tau = 1.25$ (k) $\tau = 1.5$ (l) $\tau = 3$

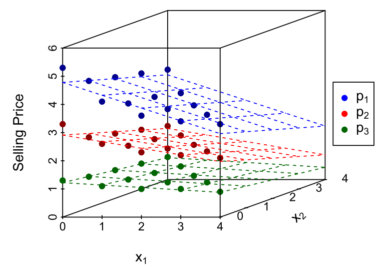
Figure 3.5: Sensitivity of production decisions to the change in different parameters

As depicted in the Figure 3.5, when quality index variability rises, the quality gap between non-fresh and fresh products widens. Consequently, the firm necessitates more frequent production events, leading to an upward shift in the production threshold. Moreover, greater quality variability results in increased price variance among products with distinct quality indices, as illustrated in Figure 3.5.

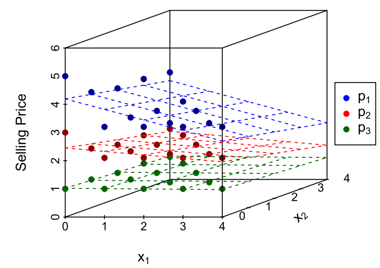
Interestingly, the firm gains benefits from high variability in quality indexes. As shown in Figure 3.6, within a dynamic freshness-dependent pricing model, the company can establish varying prices for distinct products, resulting in substantial profits from items with higher quality indices. However, the findings also indicate that within a static single pricing policy system, high variability in quality indices can severely undermine profitability. This is due to the lack of flexibility for the firm to address quality fluctuations and utilize this diversity.



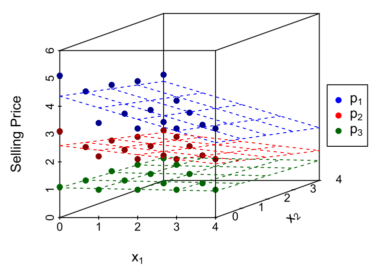
(a) $\mu = 10$



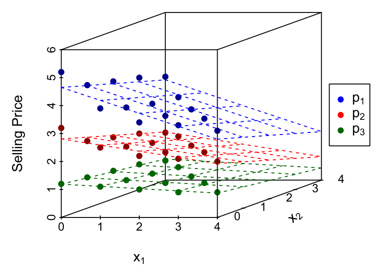
(b) $\mu = 20$



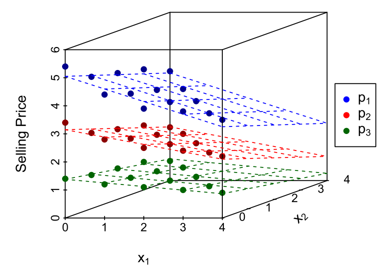
(c) $\mu = 30$



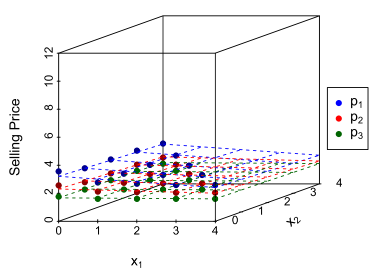
(d) $\Lambda = 5$



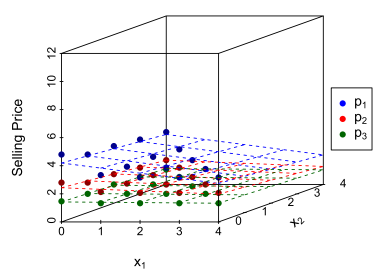
(e) $\Lambda = 10$



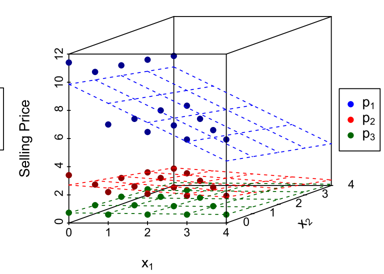
(f) $\Lambda = 20$



(g) $\tau = 1.25$



(h) $\tau = 1.5$



(i) $\tau = 3$

Figure 3.6: Sensitivity of optimal selling prices to the change in different parameters

3.8 Conclusion

In this research, we study a joint inventory-pricing management for perishable products with multiple freshness levels in a dynamic setting. Specifically, each product has an assortment consisting of products of different freshness levels that is dynamically changing. At each state, the firm must decide on whether to produce fresh items and on the prices for different freshness levels of the products given their availabilities. We formulate the problem as a discrete-time Markov decision process, and then we analyze the structural properties of the value function. More specifically, we show the value function preserves anti-multimodularity property, based on which we derive the structural properties of the optimal policies. to overcome the complexity of proposed problem, we make use of dimension reduction technique and structural properties to devise three novel heuristics. Finally, we extend our base model to the case with donation option and replenishment system.

Our extensive theoretical and computational results reveal the following insights. First, the optimal production and donation policies are characterized by state-dependent threshold-based strategies. Notably, the production threshold decreases by increasing the inventory level of the products, with a greater sensitivity to the inventory of fresher products compared to less fresh ones. Second, the results indicate that the prices of the products are monotone in freshness level, meaning that products of higher quality level have higher prices. The optimal price of a product is the average of its quality index and its future marginal value. These prices decrease with higher inventory levels of other products, with greater sensitivity to products of closer quality indexes. Third, computational results imply that freshness-dependent pricing and dynamic pricing are substitute strategies, whereas freshness-dependent pricing and donation policies complement each other in aligning supply with demand. Fourth, our findings imply that the firm can benefit from the quality variation of products in a dynamic system while suffering significant losses in a static system. Fifth, our results demonstrate the efficiency of heuristic policies. Specifically, the first heuristic model performs best when similarity between products is higher, while the second and the third heuristic works best when the firm has enough capacity to satisfy demand (high supply-to-demand ratio).

This research can be possibly extended in a several ways. First, exploring a data-driven dynamic inventory-pricing model for perishable products with multiple freshness levels could be an interesting extension to this research. Second, a future work may explore more general demand or shelf life distributions. Finally, this work can be extended by considering multiple-perishable products and substitutability effect between them.

Chapter 4

Optimal Markdown Policies for Perishable Products with Fixed Shelf Life

4.1 Introduction

Perishable goods lose their value after the "best-before date" or end of the season, and retailers typically try to sell off excess inventory through a markdown strategy. According to Smith and Agrawal (2017), nearly one-third of a retailer's total revenue is generated during the markdown period(s). Hence, an effective management of the markdown strategy is crucial for the firms to deal with perishable products. However, determining the optimal markdown strategy that maximizes expected profits while meeting shelf life constraints remains an open problem.

Despite the crucial role of markdown pricing, many firms still rely on simple markdown policies, that are suboptimal, such as the single-stage markdown policy. There are different types of markdown policies including single-stage markdown, multiple-stage markdown, and dynamic markdown policy. A single-stage markdown policy involves a single price reduction at a time before the expiration date. This policy is relatively simple to implement, but may result in significant lost sales or waste if the price reduction is too low or too late. A multiple-stage markdown policy, in contrast, involves several price reductions at different times before the expiration date. For example, a product with a remaining shelf life of 5 days may be marked down by 20% two days before its expiration date, and then further down by 50% one day before the expiration date. Although determining the optimal number of times and timing of markdown stages can be challenging, it generates higher revenue than single-stage markdown policy. A dynamic markdown policy utilizes real-time data to adjust prices based on factors such as inventory levels and demand patterns. This policy enables retailers to optimize prices based on real-time measures and can lead to significant revenue improvements and waste reduction.

The primary focus of this research is to compare different types of markdown policies. We also provide theoretical bounds on the values of those markdown policies and show that these bounds vanish as the market demand or shelf life increases. Then, we consider several extensions, including the case of LIFO issuing policy and the case with freshness-dependent demand. Lastly, we consider two case studies that are based on real datasets in the farmer's market and bakery industries.

The remainder of this chapter is organized as follows. In Section 4.2, we review the relevant literature. Section 4.3 proposes the problem and Section 4.4 provides its formulation. In section 4.5, we present bounds and asymptotic analysis. Section 4.6 presents two extensions to the base model. We present two case studies in section 4.7 and present computational results in section 4.8. We conclude the chapter in section 4.9. All the appendices are presented in online supplementary materials.

4.2 Related Literature

In this section, we provide an overview of the pertinent literature and establish the position of the current work, which contributes to three interrelated, streams of literature, namely, inventory management of perishable products, joint inventory and pricing optimization of perishable products, and age-dependent pricing model.

4.2.1 Inventory Management of Perishable Products

Numerous studies have studied the management of perishable inventory. Nahmias (1982) and Karaesmen et al. (2011) presented extensive literature reviews. In 1980, Weiss (1980) introduced a continuous review (S, r) policy for a perishable inventory system with fixed self life, Poisson demand, and zero lead-time. Subsequently, researchers extended their model to various settings. For instance, Kalpakam and Sapna (1994), Liu and Shi (1999) extended their model to the case with exponential shelf life and lead time, while Liu and Lian (1999b) and Liu and Lian (1999a) presented a model with a general renewal demand process. Lian and Liu (2001) incorporated a positive lead time to the model and to solve the model, they developed a heuristic algorithm. Ravichandran (1995) proposed a (s,S) model with Poisson demand, random order lead times, fixed product shelf life, and lost-sales. The authors derived closed-form expressions for the optimal ordering policy. Kouki et al. (2015) studied a continuous review (Q,r) replenishment problem in a model with continuous demand distribution, constant shelf life and lead time. In their study, Kouki et al. (2018) explored the benefits of using dual sourcing as a strategy to manage perishable inventory.

The researchers made assumptions of fixed lead time and either fixed or exponential product lifetime. Berk and Gürler (2008) analyzed the (r,Q) replenishment policy for a product with constant lifetime and order lead time. They accounted for lost sales and modeled the system's dynamics under the (r,Q) policy using an embedded Markov process. Using the Queueing and Markov Chain Decomposition methodology, Barron and Baron (2020) investigated a continuous review (S,s) policy. Their analysis included stochastic lead time, perishability, and state-dependent Poisson demand. In a related study, Barron (2019) expanded on the work of Barron and Baron (2020) by incorporating demand uncertainty and stochastic batch demands into the model.

The above papers primarily considered controllable supply. However, some papers in the literature have focused on models with uncertain supply and demand. These stochastic perishable inventory models have several applications in different industries. For instance, this model is applicable to blood banks, where supply refers to donations and demand to transfusions (Nahmias 2011). Other examples can be farmers market or bakery stores. Although these models simplify real-world operations, they still provide valuable insights into practical policy performance (Goh et al. 1993b, Kopach et al. 2008, Sarhangian et al. 2018). Early works in this stream can be traced back to Graves (1978) and Kaspi and Perry (1983) who examined the unsatisfied demand and expiration processes under FIFO policy. Several studies have since explored this problem with variations, such as the renewal supply problem (Kaspi and Perry 1984), batch arrival and demand (Goh et al. 1993a), Obsolescence (Perry and Stadje 2000), quality inspections (Perry 1999), LIFO issuing policy (Keilson and Seidmann 1990, Parlar et al. 2011), Hysteretic control (Perry and Posner 1990), and Outsourcing and urgency classes (Bar-Lev et al. 2005). While most previous research has focused on performance analysis, this study pioneers the optimization of inventory and pricing decisions under a continuous-review system for a perishable product with a fixed shelf life. Additionally, we derive the long-run average profit rate in steady-state for different markdown policies and conduct an asymptotic analysis of the optimal inventory and pricing decisions with respect to several parameters, including the market demand, shelf life and maximum WTP.

4.2.2 Joint Inventory and Pricing Management of Perishable Products

Another research stream looks at coordinating pricing and inventory decisions for these products. Abad (1996) proposed a joint inventory-pricing optimization problem for perishable products where demand can be partially backordered. Under finite horizon setting, Li et al. (2009) discussed the optimal policy structure for a product with a two-period shelf life where demands are backlogged. They further developed a heuristic model for products with multi-period shelf life and backlogged demand. Further, they extended this work to an infinite horizon setting and lost sales (Li et al. 2012). Fang et al. (2021) presented a stochastic dynamic programming model to study joint pricing and inventory decisions for multiple substitutable perishable products. Chen et al. (2014) showed structural properties of optimal inventory and pricing decisions for perishable inventory with both backorders and lost sales and developed heuristics to overcome computational challenges. Ceryan (2019) studied asymmetric inventory and pricing management for seasonal perishable products and regular products that are substitutes. By considering heterogeneity of customers, Herbon (2018), investigated joint inventory and pricing problem in which demand is a function of perceived quality, remaining lifetime, and selling price. While those studies investigated joint inventory-pricing models, they did not consider age-dependent pricing for perishable products. To bridge this gap, our research examines various forms of age-dependent pricing through different markdown policies. This research further provides a theoretical and numerical comparison of these age-dependent pricing strategies with each other and against the fixed-pricing strategy (no-markdown policy).

4.2.3 Age-Dependent Pricing Models for Perishable Products

Age-dependent pricing has been the subject of extensive research in revenue management. Littlewood (2005) conducted pioneering work on capacity allocation to uncertain demands for two fare classes, which has inspired subsequent research on markdown pricing in revenue management problems. Several studies have investigated markdown policies for perishable products reviewed as follows. Some scholars have decomposed the sales period into two distinct periods, including regular sales and markdown sales periods. For instance, Chen (2012) optimized both ordering and promotional decisions, with goods being sold at a regular price, followed by discounted prices during the later phase of the sales period. In a similar vein, cite hu2016joint divided each sales period into two phases, markdown sales, and regular sales, to dispose of unsold inventory from the previous period. Furthermore, Banerjee and Agrawal (2017) partitioned the regular and markdown periods into four segments under the assumption that the products do not deteriorate from the outset. To reduce the wastage rate, some papers studied quality-dependent pricing models potentially with an upper bound on the maximum number of price changes (e.g., (Liu et al. 2015, Qin et al. 2014, Wang and Li 2012, Kayikci et al. 2022, Adenso-Díaz et al. 2017)). The impact of product shelf life and a retailer's pricing strategy on profit and waste has been investigated by several researchers through the use of numerical simulation (e.g., (Buisman et al. 2019, Tekin and Erol 2017, Chung and Li 2014)). Chua et al. (2017) proposed four different models and compared them to investigate the discounting and replenishment policies for short lifetime perishable products with uncertain demand. Qiao et al. (2020) studied the optimization of order quantity, regular price, and markdown price for perishable products that have a limited shelf life of two periods. To solve the single-period optimization problem, they applied the Karush-Kuhn-Tucker conditions and subsequently evaluate the effectiveness of this policy in a multi-period setting numerically. Recently, den Boer et al. (2022) examined the influence of discounts on the wastage rate of perishable products. To overcome the complexity of the system, they considered the equivalent deterministic model and by studying the scaled system, they showed that applying simple pricing rules can result in wastage reduction and profit increment. This research fills a gap in the existing literature by exploring the benefit of age-dependent pricing policies over the fixed pricing policy, both theoretically and numerically. We model various degrees of age-dependent pricing through different markdown policies, including single-stage, multiple-stage, and dynamic markdown pricing, and compare their performance with each other and also against the fixed-pricing policy. In contrast to previous studies that mainly focused on products with a two-period shelf life, in this research we consider a product with a multi-period shelf life.

To summarize, our study makes several contributions to the existing literature on inventory and revenue management of perishable products. First, this work pioneers the study and comparison of the effectiveness of different age-dependent pricing policies for perishable products with multiple-period shelf life, both theoretically and numerically. Specifically, we analyze the benefit of single-stage, multiple-stage, and dynamic markdown policies, providing new insights into the optimal markdown policy for such products. Second, we derive the long-run average profit rate in steady-state for different markdown policy models. We further conduct an asymptotic analysis of the optimal inventory and pricing decisions across different parameter regimes including market demand, shelf life, and maximum WTP. Third, in order to address the intricacies of dynamic markdown policies, we introduce an approximation scheme that involves approximating the relationship between shelf life and price using a mapping function. Fourth, we obtain some theoretical bounds on the benefit of multiple-stage markdown policies over single-stage markdown policies and no-markdown policies and show that the benefits of multiple-stage markdown policies asymptotically vanish in shelf life, market demand, or maximum willingness to pay. Fifth, we extend the main model to the case of LIFO issuing policy and freshness-dependent demand. Finally, we provide numerical evidence of the performance of each markdown policy through two real case studies in a farmers' market and bakery industries.

4.3 Problem Statement and Model Description

In this section, we describe the model and present the performance measures of interest. Consider a profit-maximizing seller of a single perishable item with fixed shelf life θ over a continuous-time system. Perishable products are produced at the firm at a Poisson rate μ, with the age of a newly produced unit being zero. This assumption can be modified for products arriving with a delay. Considering production as a Poisson process is a common assumption in the literature (Li et al. 2023b, Sarhangian et al. 2018, Benjaafar et al. 2011, Gayon et al. 2009). This assumption is particularly applicable to production systems, like the production of agri-food items and fruits, where yield quantities are random, and their arrival can be estimated by a Poisson process, as is observed in our case studies. Moreover, this assumption can be extended to collection processes, such as blood donation or core acquisition for remanufacturing, where the arrival process follows a Poisson distribution.

The unit production cost for the newly produced products is C_p and once their shelf life exceeds θ, they are disposed of at cost C_e. Perishable products can be sold either in the regular stage or during the markdown period elaborated below.

4.3.1 Markdown Policies

The firm may apply different markdown policies wherein the price of products are reduced in a single stage, or multiple stages, or dynamically over time. As a result, the perishable products can be sold either at their regular price or at reduced prices during markdown stages. Let us define $Q \geq 0$ as the number of markdown cutoffs and $\boldsymbol{m}^Q = \left\{m_1^Q, m_2^Q, \ldots, m_Q^Q\right\}$ as the set of all markdown cutoff times. Here, m_i^Q denotes the time point of the i^{th} markdown. To ensure a valid arrangement, the set of markdown cutoff times $\boldsymbol{m^Q}$ must satisfy the condition $\boldsymbol{m^Q} \in \mathscr{A}^Q = \left\{\boldsymbol{m^Q} : 0 \leq m_{i+1}^Q \leq m_i^Q, m_1^Q \leq \theta \;\; \forall i \in [1, Q-1]\right\}$. We refer to a system with Q markdown stages as system Q, which comprises $Q+1$ stages, one regular sale stage (denoted by $j = 0$) and Q markdown stages (denoted by $j = 1, 2, \ldots, Q$).

In system Q, when an item becomes outdated at a stage i, it is transferred to the next stage $i+1$ if $i \leq Q-1$. However, if the item becomes outdated at the last markdown stage $i = Q$, it is disposed of. Each time a product is entitled to a markdown price, the firm will incur a labeling cost of C_l per unit. This cost is associated with the process of applying and displaying the new markdown price on the product. Figure 4.1 provides an illustration of the different types of markdown policies.

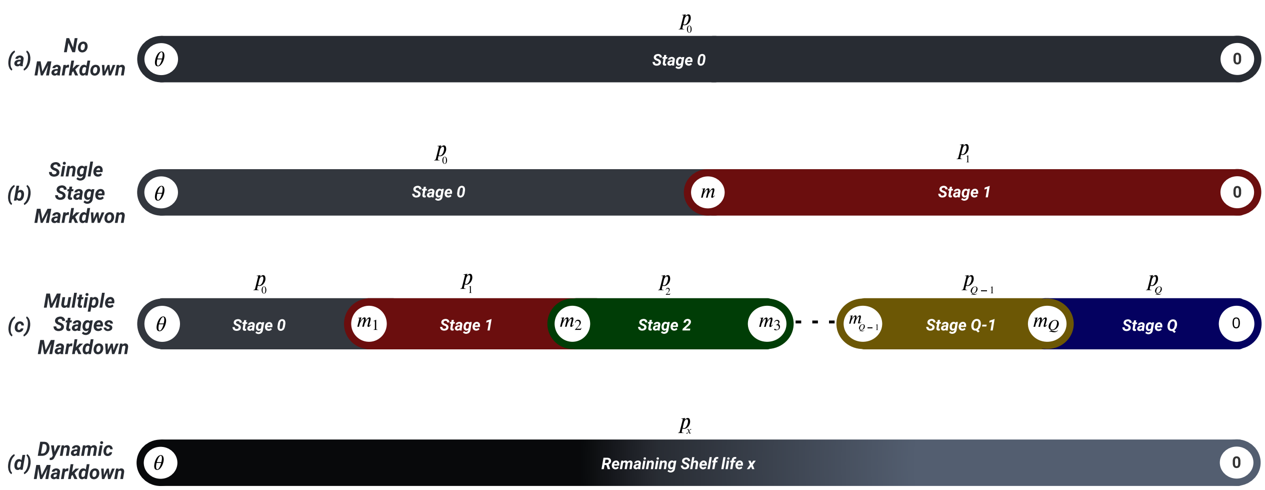


Figure 4.1: Illustration of different markdown models

Remark: If $Q \to \infty$ selling price becomes dynamically changing and is dependent on the remaining shelf life.

4.3.2 Purchasing Behavior

The arrival rate of the potential customers who intend to purchase the product is represented by Λ, indicating the total market size for the product. The consumers' willingness-to-pay for products is defined as V, and is characterized by a density function $g(.)$ and a distribution $G(.)$ with mean ψ and standard deviation σ. For convenience in theoretical analysis, in our subsequent discussions, we assume that WTP follows a uniform distribution between 0 and Γ, where Γ represents the maximum willingness-to-pay. However, in the computational results section, we relax the assumption of a uniform distribution for the WTP and consider a general distribution.

To simplify our theoretical analysis, we assume that products are issued based on a FIFO policy. This means customers' utilities are age-independent and only influenced by the prices. This is a reasonable assumption for products with constant quality over their shelf life such as blood products, bakery items, or certain vegetable products that lack date information and maintain a consistent appearance. Further, issuing policy is influenced by existing items on the shelves. For instance, some firms may only display oldest units to apply FIFO issuing policy. Also, online retailers can employ the FIFO issuing policy as customers are not able to control the product depletion.

The regular and markdown prices are endogenously determined by the firm. In system Q, the prices of the items at stage $j \in [0,Q]$ are denoted by p_j^Q and the utility that a customer can derive from purchasing an item that exists at stage j at price p_j^Q is $V - p_j^Q$. Therefore, when the oldest product exists at stage j, the purchasing probability of that product is represented by $\phi_j^Q = \mathbb{P}\left(V - p_j^Q \geq 0\right), \phi_j^Q \in [0,1]$, and the purchasing probability of products that are younger than the oldest unit is zero. After doing some mathematics, the selling price for a product at stage j, i.e., p_j^Q, can be represented as a function of ϕ_j^Q as follows.

$$p_j^Q = G^{-1}\left(1 - \phi_j^Q\right) = \Gamma\left(1 - \phi_j^Q\right) \tag{4.1}$$

We combine the purchasing probability at different stages in a single vector $\phi^Q = \left\{\phi_0^Q, \ldots, \phi_Q^Q\right\}$, where $\phi^Q \in \mathscr{B}^Q = \left\{\phi^Q : 0 \leq \phi_j \leq 1 \; \forall j \in [0,Q]\right\}$. Therefore, at stage $j \in [0,Q]$, the demand follows Poisson process with rate $\lambda_j^Q(\phi_j^Q) = \Lambda \phi_j^Q$. Lost demand occurs when the inventory level is zero and incurs a per unit cost of C_s on the firm. Next, we characterize the value function of the optimization problem.

4.3.3 Value Function

Let q^π be the probability of outdating and l^π be the probability of a demand being lost under policy π. As long as the steady-state distribution exists, the following **conservation**

law holds.

$$\mu^{\pi}(1-q^{\pi}) = \lambda^{\pi}(1-l^{\pi})$$

Where the left-hand-side shows the rate of product allocation and the right-hand-side shows the rate of demand satisfaction. This system can be viewed as an $M/M/1+D$ queue, where products are the arrivals, and each service completion corresponds to a customer purchasing a product. Hence, the queue has an arrival rate of μ and service rate of λ_j^Q at stage j of system Q. Every time the firm decides to change the price label of a product, it will be subject to a labeling cost. Additionally, products have a fixed patience time until the end of service, represented by θ, meaning they are disposed of if their sojourn time in inventory exceeds θ. Demand is lost when the inventory level is zero. Otherwise, a product is allocated to the demand with the oldest unit. Therefore, the value function under policy π, denoted by V^{π}, consists of revenue, production cost, expiration cost, shortage cost, and labeling cost and can be written as follows.

$$V^{\pi} = \max\left\{\mathscr{R}^{\pi} - \mathscr{S}^{\pi} - \mathscr{W}^{\pi} - \mathscr{C}^{\pi} - \mathscr{L}^{\pi}\right\} \tag{4.2}$$

where $\mathscr{R}^{\pi}, \mathscr{S}^{\pi}, \mathscr{W}^{\pi}, \mathscr{C}^{\pi}$, and $\mathscr{L}^{\pi}$ denote total long-run average revenue, shortage cost, wastage cost, production cost, and labeling cost, respectively. We aim to obtain the optimal production, markdown timing and pricing to maximize the total profit.

4.4 Model Formulation

In this section, we formally formulate different markdown policies including no-markdown policy (N), single-stage markdown policy (S), multiple-stage markdown policy (M), and dynamic markdown policy (D).

4.4.1 Base Model - No-Markdown Policy (N)

The problem of managing perishable inventory with fixed shelf life under continuous-review policy was initially explored by Graves (1978) and Kaspi and Perry (1983). Then, it has been revisited by Parlar et al. (2011). The analysis is based on the Virtual Outdating Process (VOP) $W \equiv \{W(t); t \geq 0\}$, which provides information on the remaining time until the next expiration if no new demands occur. A sample path of process W is shown in Figure 4.2. This process is useful because it is a strong Markov process and contains vital information about the system's state. For example, at any time t, the age of the oldest unit in inventory is $\theta - W(t)$, and if $W(t) > \theta$, there is no inventory at time t, and if $W(t^-) = 0$, a unit was expired at time t. The process W has upward jumps in its sample paths, which occur when the oldest unit is allocated to a demand or expires. The rate of selling products to the customers is $\lambda_0 = \Lambda \phi_0$ and the rate of expiration is the rate of W reaching zero, i.e., $f(0)$. The jump sizes are the inter-arrival time of units to the inventory, which is exponentially distributed with rate μ. Kaspi and Perry (1983) have shown that W has the same distribution as the virtual waiting time process of an $M/M/1+D$ queue, where idle periods are deleted and customers do not join the system if they have to wait more than θ before starting service. This observation allows for the calculation of W 's steady-state distribution, which is denoted as f. To characterize the steady-state distribution f, we apply Level-Crossing Theory according to which for any level x, the total rate of upward jumps is equal to the total rate of downward jumps (see Brill et al. (2008)). Therefore, the balance equation can be written as follows.

$$f(x) = \lambda_0 \int_0^{x \wedge \theta} e^{-\mu(x-w)} f(w) dw + f(0) e^{-\mu x} \quad \forall x \geq 0 \tag{4.3}$$

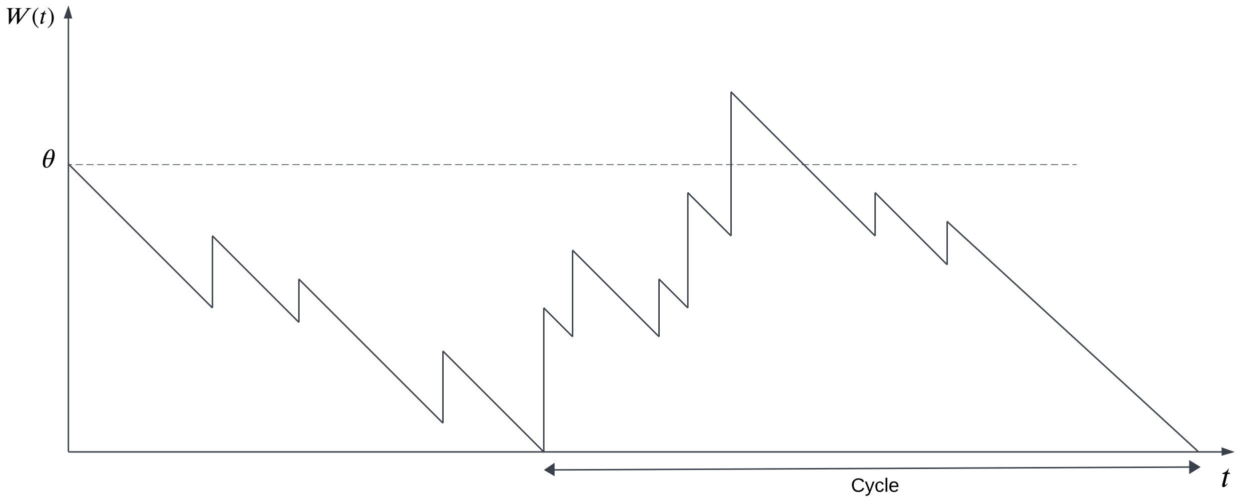


Figure 4.2: Sample path of process W under policy N (adapted from Parlar et al. (2011))

Therefore, the long-term average number of times a level, x, is crossed from above, i.e., $f(x)$, is equal to the long-term average number of times x is up crossed, is represented by the right-hand side of Equation (3). The arrival rate of upward jumps is λ_0 and the probability of jumping above x when starting from $w \in (0, x \wedge \theta]$ is $e^{-\mu(x-w)}$. Using the PASTA principle, the steady-state probability of crossing level x by a jump is given by the integral of $e^{-\mu(x-w)} f(w)dw$ from 0 to $x \wedge \theta$. Additionally, level x can also be crossed at the end of a cycle from level 0, with a rate of cycle endings given by $f(0)$ and a probability of $e^{-\mu x}$ to cross level x from 0. Solving Equation (3) with respect to f yields the steady-state probabilities as follows.

$$f(x) = \begin{cases} f(0)e^{-(\mu-\lambda_0)x}, & 0 \leq x \leq \theta \\ f(0)e^{\lambda_0\theta-\mu x} & x > \theta \end{cases} \tag{4.4}$$

Also, using normalization condition $\int_0^\infty f(w)dw = 1$, we can obtain $f(0)$ as follows.

$$f(0) = \begin{cases} \frac{\mu(\mu-\lambda_0)}{\mu-\lambda e^{-(\lambda_0-\mu)\theta}}, & \mu \neq \lambda_0 \\ \frac{\mu}{1+\theta\mu} & \mu = \lambda_0 \end{cases} \tag{4.5}$$

Proposition 4.1 *The outdating probability q^N when $\lambda_0 \neq \mu$ can be expressed as follows.*

$$q^N = \frac{(\mu-\lambda_0)}{\mu-\lambda e^{-(\lambda_0-\mu)\theta}} \tag{4.6}$$

Also, when $\lambda_0 = \mu$, we have $q^N = \frac{1}{1+\theta\mu}$. Furthermore, using conservation law, one can obtain the probability of demand being lost as $l^N = 1 - \frac{\lambda_0}{\mu}\left(1-q^N\right)$.

Hence, the long-run average profit under policy N can be written as follows.

$$V^{N*} = \max_{\mu \geq 0, 0 \leq \phi_0 \leq 1} p_0\lambda_0\left(1-l^N\right) - C_s\lambda_0 l^N - C_e\mu q^N - C_p\mu, \tag{4.7}$$

where q^N and l^N are given above and also we have $\lambda_0 = \phi_0\Lambda$ and $p_0 = \Gamma(1-\phi_0)$. In the above optimization problem, the first term indicates the average revenue obtained per unit time, the second, third and fourth terms express the average shortage, expiration, and purchase/production costs per unit time. Note that in a system without clearance, the labeling cost associated with changing the price is zero.

Let us define ρ, as the difference between production and sales rate, i.e., $\rho = \mu - \phi_0\Lambda$. We can rewrite the optimization model in terms of ρ and eliminate μ. Then, in the following proposition, we characterize the structure of the optimal policy based on maximum WTP values.

Proposition 4.2 ***(Optimal production rate and selling price)*** *When the maximum willingness-to-pay of customers is equal to or exceeds a threshold value Γ_0^*, the optimal production rate μ^* is greater than or equal to the sales rate $\lambda_0^* = \phi_0^* \Lambda$. On the other hand, when the maximum willingness-to-pay is below Γ_0^*, the optimal production rate is less than the sales rate. Mathematically, we have:*

$$\mu^* = \phi_0^* \Lambda + \rho^* \quad \textit{where} \begin{cases} \rho^* \geq 0, & \Gamma \geq \Gamma_0^* \\ \rho^* < 0 & \Gamma < \Gamma_0^* \end{cases} \tag{4.8}$$

In which Γ_0^* is given by

$$\Gamma_0^* = \frac{2C_p + C_e}{\left(1 - \phi_0^*\right)} + \frac{2\left(C_p + C_e\right)}{\Lambda \theta \phi_0^* \left(2 + \Lambda \theta \phi_0^*\right)\left(1 - \phi_0^*\right)} \tag{4.9}$$

and ϕ_0^* and ρ^* can be obtained by solving $\frac{\partial V^N}{\partial \phi_0} = 0$ and $\frac{\partial V^N}{\partial \rho} = 0$, respectively.

Proposition 4.2 provides insights into the optimal policy structure based on maximum willingness-to-pay values. Specifically, if the maximum WTP is less than the threshold Γ_0^*, the optimal production rate is lower than the average number of customers who purchase the product. Conversely, if the maximum WTP is greater than Γ_0^*, the optimal production rate is higher than the average number of customers who purchase the product. Intuitively, when customers have a higher WTP, it is optimal to produce more products than the average sales rate. This result suggests that the optimal production rate is closely linked to the maximum WTP value, and provides a useful framework for making production decisions based on customer demand.

4.4.2 Single-Stage Markdown Policy (S)

In this section, we formulate the problem with a single markdown stage. Under this policy, the firm offers fresher products at regular price p_0 and products that are older than a threshold m_1 at markdown price p_1. Given selling prices, probability of buying products at regular and markdown stages are denoted by ϕ_0 and ϕ_1, respectively. Then, we can derive regular and markdown prices as functions of ϕ_0 and ϕ_1 as $p_0 = (1-\phi_0)\Gamma$ and $p_1 = (1-\phi_1)\Gamma$. Thus, customer demand at regular sales and markdown stages are $\lambda_0 = \phi_0 \Lambda$ and $\lambda_1 = \phi_1 \Lambda$. Under this policy, a sample path of process W is shown in Figure 4.3. Using level-crossing

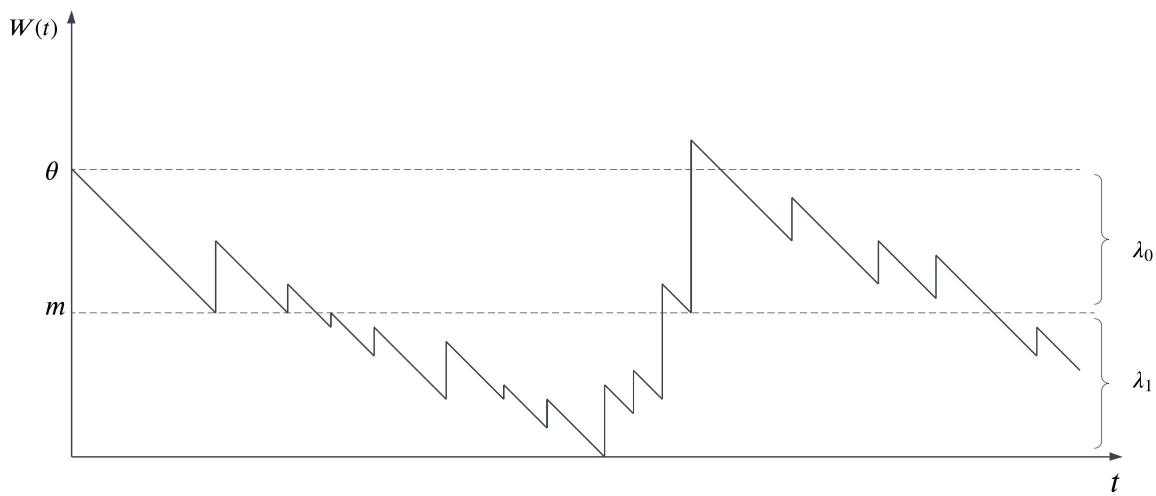


Figure 4.3: Sample path of process W under policy S

theory, we can write the balance equation as follows.

$$f(x) = \lambda_1 \int_0^{x \wedge m_1} e^{-\mu(x-w)} f(w)dw + \lambda_0 \int_{x \wedge m_1}^{x \wedge \theta} e^{-\mu(x-w)} f(w)dw + f(0)e^{-\mu x} \;\; \forall x \in [0, \theta] \tag{4.10}$$

Then, solving the above balance equation for f yields the following steady-state pdf of f.

$$f(x) = \begin{cases} f(0)e^{-(\mu-\lambda_1)x}, & 0 \leq x \leq m_1 \\ f(0)e^{-(\mu-\lambda_0)x+(\lambda_1-\lambda_0)m_1} & m_1 < x \leq \theta \\ f(0)e^{\lambda_0\theta-\mu x+(\lambda_1-\lambda_0)m_1} & x > \theta \end{cases} \tag{4.11}$$

Then using normalizing and continuity conditions, we can obtain $f(0) = q^S\mu$ as follows.

$$f(0) = \left[\frac{1}{\mu-\lambda_1} + e^{-(\mu-\lambda_1)m_1}\left(\frac{1}{\mu-\lambda_0} - \frac{1}{\mu-\lambda_1}\right) + e^{-(\mu-\lambda_0)\theta+(\lambda_1-\lambda_0)m_1}\left(\frac{1}{\mu\theta} - \frac{1}{\mu-\lambda_0}\right)\right]^{-1} \tag{4.12}$$

Let us denote f_0 and f_1 as the fractions of time the system is operating at a demand rate λ_0 and λ_1, respectively. Then, we have:

$$f_0 = f(0)\int_{m_1}^{\theta} e^{-(\mu-\lambda_0)x+(\lambda_1-\lambda_0)m_1} \tag{4.13}$$

$$f_1 = f(0)\int_{0}^{m_1} e^{-(\mu-\lambda_1)x} \tag{4.14}$$

Under policy S, the long-run average profit V^S consisting of revenues from regular and markdown sales, wastage cost, shortage cost, and labeling cost can be expressed as follows.

$$V^S(\mu,\, m_1,\, \phi_0,\, \phi_1) = p_0\lambda_0 f_0 + p_1\lambda_1 f_1 - C_e\mu q^S - C_p\mu - C_l\left(\lambda_1 f_1 + \mu q^S\right) \tag{4.15}$$

We are interested in determining the optimal markdown time, production rate, and regular and markdown prices. Thus, the optimization problem under policy S can be written as follows.

$$V^{S^*} = \max_{\mu\geq 0, 0\leq m_1\leq\theta, 0\leq\phi_0,\phi_1\leq 1} V^S(\mu, m_1, \phi_0, \phi_1) \tag{4.16}$$

While proving the concavity of the optimization problem with respect to all variables is a complex task, our numerical results indicate that the problem is indeed concave. It is straightforward to show the concavity of the above optimization problem in μ. Moreover, since ϕ_0 and ϕ_1 are bounded between 0 and 1, we can employ a grid search method to iteratively solve the problem for optimal μ. This approach allows us to efficiently explore the solution space and identify the optimal values of the variables. Using non-linear optimization packages in MATLAB and Python is also another way of dealing with this problem.

4.4.3 Multiple-Stage Markdown Policy (M)

This policy assumes that products can be sold in the regular stage (stage 0) or during multiple markdown stages. Q represents the number of markdown cutoffs, and m_i^Q indicates the time of the i^{th} markdown when there are Q markdown stages, where $m_i^Q \in \mathscr{A}^Q$.

Let q^Q and l^Q be the probability of outdating and the probability of a demand being lost with a total of Q markdown stages, respectively. Similar to the case with no markdown policy, as long as steady-state limits exist, under policy M with a total of Q markdown stages, the following conservation law holds.

$$\mu^Q\left(1-q^Q\right)=\lambda_0^Q\left(1-l^Q\right)$$

Under multiple-stage markdown policy, a sample path of process W is demonstrated in Figure 4.4. The next Theorem presents steady-state density of process W.

Theorem 4.1 ***(Steady-state density of W for policy M)*** *Let us define f_j^Q as the steady-state density of VOP when the system is operating at stage j in a system with Q markdown*

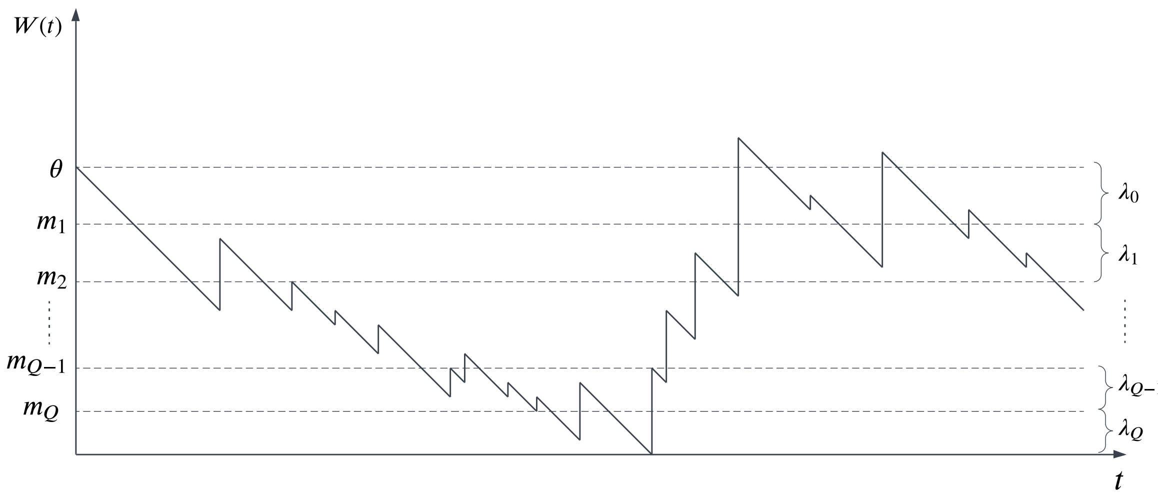


Figure 4.4: Sample path of process W under policy M

cutoffs. Using level-crossing theory, we can write the balance equation for any level x as a single equation as follows.

$$f^Q(x) = \sum_{j=0}^{Q} \lambda_j \int_{x\wedge m_{j+1}}^{x\wedge m_j} e^{-\mu(x-w)} f^Q(w)dw + f^Q(0)e^{-\mu x} \;\; \forall x \in [0,\theta], \tag{4.17}$$

where we assume that $m_0 = \theta$ *and* $m_{Q+1} = 0$*. Solving balance equations couple with normalization condition, for any state* $j \in [0,Q]$*, we can obtain the steady-state density* f_j^Q *as follows.*

$$f_j^Q(x) = f^Q(0)\, e^{\left(\lambda_j-\mu\right)x+\sum_{i=j}^{Q-1}(\lambda_{i+1}-\lambda_i)m_{i+1}} \;\; \forall x \in [m_{j+1}, m_j],\, \forall j \in [0,Q], \tag{4.18}$$

where $m_0 = \theta$ *and* $m_{Q+1} = 0$*. Also,* $f^Q(0)$ *can be written as follows.*

$$f^Q(0) = \left[\sum_{j=0}^{Q} \int_{x\wedge m_{j+1}}^{x\wedge m_j} e^{\left(\lambda_j-\mu\right)x+\sum_{i=j}^{Q-1}(\lambda_{i+1}-\lambda_i)m_{i+1}} + e^{\sum_{i=0}^{Q}\lambda_i(m_{i+1}-m_i)}\frac{e^{-\mu\theta}}{\mu}\right]^{-1}, \tag{4.19}$$

where $f^Q(0)$ denotes the outdating rate.

Theorem 4.1 presents the steady-state density of remaining shelf life of the oldest item at stage j. Then, the long-run average revenue obtained from stage j can be written as $p_j^Q \lambda_j^Q f_j^Q$. Next, we can write $V^{M,Q}$, the total long-run average profit under policy M when there are Q markdown cutoffs, as follows.

$$V^{M,Q}\left(\mu, m^Q, \phi^Q\right) = \sum_{j=0}^{Q} p_j^Q \lambda_j^Q f_j^Q - C_s \lambda_0^Q l^Q - C_e f^Q(0) - C_p \mu - C_l \left(\sum_{j=0}^{Q} j \lambda_j^Q f_j^Q + Q f^Q(0)\right), \tag{4.20}$$

where $\lambda_j^Q(\phi_j^Q) = \Lambda \phi_j^Q$ and $p_j^Q = \Gamma\left(1-\phi_j^Q\right) \quad \forall j \in [0,Q]$. Also, using the conservation law, the shortage rate can be obtained as $\lambda_0^Q l^Q = \lambda_0^Q - \mu + f^Q(0)$. In the above problem, the first term denotes revenues obtained from regular and markdown sales, the second term expresses the shortage cost, the third term shows the expiration cost, the fourth term indicates the production cost, and the last term implies the labeling cost. We are interested in determining the optimal number of markdown cutoffs, optimal markdown points of time, production rate, and regular and markdown prices. Thus, the optimization problem under policy M can be written as follows.

$$V^{M*} = \max_{Q \geq 0} V^{M,Q^*}\left(\mu, \boldsymbol{m}^Q, \phi^Q\right) \tag{4.21}$$

Where

$$V^{M,Q^*} = \max_{\mu \geq 0,\, \boldsymbol{m}^Q \in \mathscr{A}^Q,\, \phi^Q \in \mathscr{B}^Q} V^{M,Q}\left(\mu, \boldsymbol{m}^Q, \phi^Q\right) \tag{4.22}$$

Where $\mathscr{A}^Q = \left\{\boldsymbol{m^Q} : 0 \leq m_{i+1}^Q \leq m_i^Q, m_1^Q \leq \theta \;\; \forall i \in [1, Q-1]\right\}$ and $\mathscr{B}^Q = \left\{\phi^Q : 0 \leq \phi_j \leq 1 \;\; \forall j \in [0,Q]\right\}$. Increasing the number of markdown cutoffs intensifies the complexity of inventory and pricing problem and the loss of revenue due to lowered prices, while not considering a markdown cutoff may result in wastage, so finding the right number of markdown cutoffs is important yet demanding.

In this problem, there is one discrete variable Q, while the rest of the variables are continuous. To determine the optimal decision variables and their corresponding long-run average profit, we propose the following algorithm. We iteratively increase the number of markdown cutoffs until no further improvement is observed, and the profit starts to decline.

Algorithm 1 An iterative algorithm for policy M

Require: Product shelf life, cost parameters, potential market demand size, density function of willingness-to-pay, and V^{N^*}, V^{S^*}, and ε.

Ensure: Determine the optimal markdown policy, selling prices, and production rate.

$Q \leftarrow 0$

$V^{M,0^*} \leftarrow V^{N^*}$

$V^{M,1^*} \leftarrow V^{S^*}$

while $V^{M,Q+1^*} - V^{M,Q^*} > 0$ **do**

  $Q \leftarrow Q+1$

  Solve the problem for the optimal $\mu \geq 0$, $\boldsymbol{m^Q} \in \mathscr{A}^Q$, $\boldsymbol{\phi}^Q \in \mathscr{B}^Q$.

  $V^{M^*} \leftarrow V^{M,Q}\left(\mu^*, \boldsymbol{m}^{Q^*}, \boldsymbol{\phi}^{Q^*}\right)$.

  $Q^* \leftarrow Q$.

end while

4.4.4 Dynamic Markdown Policy (D)

In this section, we propose a state-dependent markdown policy in which markdown price depends on the remaining shelf life and is dynamically changing. Under this policy, purchase probability defined as $\phi(x)$ is state-dependent, where $x \in [0, \theta]$ indicates the remaining shelf life.

Given price $p(x)$, one can obtain the utility of a product with remaining shelf life x as $V - p(x)$ and the probability of purchasing a product with remaining shelf life x can be represented as $\phi(x) = \mathbb{P}(V - p(x) \geq 0)$. Also, selling price can be expressed as $p(x) =$

$\Gamma(1-\phi(x))$. Customer demand for the product is Markovian process with rate $\lambda(x) = \phi(x)\Lambda$ for an item with remaining shelf life x. For calculating steady-state density of W, we set $\lambda(x) = 0$ when $x > \theta$ because a demand arriving at time t where $W(t) > \theta$ is unsatisfied and therefore does not affect the VOP. Additionally, W is regenerative with expiration times as cycle beginnings. This problem can be modeled as a special type of $M/M/1+D$ queuing system in which service rate $\lambda(x)$ is state-dependent. Applying level-crossing theory, we set total downcrossing rate equal to total upcrossing rate and write the balance equation as follows.

$$f(x) = \int_0^{x\wedge\theta} \lambda(w)e^{-\mu(x-w)}f(w)dw + f(0)e^{-\mu x} \tag{4.23}$$

Let $L(x) = \int_0^x \lambda(x)dx$, then, the steady-state density of VOP can be obtained as follows.

$$f(x) = \begin{cases} f(0)e^{-\mu x+L(x)}, & 0 \leq x \leq \theta \\ f(0)e^{L(\theta)-\mu x} & x > \theta \end{cases} \tag{4.24}$$

Using normalization condition $\int_0^\infty f(w)dw = 1$, we can obtain $f(0)$, the outdating rate, as follows.

$$f(0) = \left[\int_0^\theta e^{-\mu x+L(x)}dx + e^{L(\theta)}\int_\theta^\infty e^{-\mu x}dx\right]^{-1} \tag{4.25}$$

Then, we can write the long-run average profit as follows.

$$\begin{aligned} V^{D*} = \max_{\mu\geq 0,\, 0\leq\phi(x)\leq 1} & \int_0^\theta p(x)\lambda(x)f(0)e^{-\mu x+L(x)}dx - \int_\theta^\infty C_s\lambda(x)f(0)e^{L(\theta)}e^{-\mu x} \\ & -C_e\mu q^D - C_p\mu \end{aligned} \tag{4.26}$$

The above control problem is very complex to solve and it is not possible to obtain the closed form solutions for production rate, state-dependent price, and markdown decisions. Therefore, in what follows, we develop a heuristic method to approximate the relationship between state (remaining shelf life) and the selling price.

4.4.4.1 Approximation Framework

We consider mapping function $\phi(x) = H(x)$ indicating the price of perishable product with remaining shelf life x. In this research, we do not constrain H to any specific form but conduct an extensive experiments for different forms of H. Substituting mapping function $H(x)$ in $f(0)$ we can obtain either closed-form functions or numerically evaluable functions of outdating rate, shortage rate, total revenue, and consequently long-run profit.

Examples: We provide several examples of function $H(x)$ that can be used for the approximation. (1)linear function ($H(x) = ax+b$), (2) general Polynomial Function ($H(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$), (3) exponential functions ($H(x) = a + be^{cx}$), etc.

4.5 Bounds and Asymptotic Analysis

In this section, we conduct asymptotic analysis in different parameter regimes including market demand, shelf life, and maximum WTP for the base model. The results can be extended to other models consisting of markdown stages. Then, we propose some theoretical bounds on the value of different markdown policies. More specifically, we characterize the speed at which the value of the markdown policy vanishes in product shelf life, market demand, and WTP. The following propositions show asymptotic optimal purchasing probability and production rate in shelf life, market demand, and WTP regimes.

Proposition 4.3 ***(Asymptotic optimal solutions in shelf life regime)*** *As the shelf life becomes very large, i.e., $\theta \to \infty$, production level μ converges to the average sales rate λ_0,* *and the selling price converges to $\frac{1}{2}\Gamma + \frac{C_p}{2}$, which is greater than the mean WTP.*

Proposition 4.4 ***(Asymptotic optimal solutions in market demand regime)*** *As the market demand grows very large, i.e., $\Lambda \to \infty$, due to the lower uncertainty, the difference between optimal production and sales rates, $\mu - \lambda_0$, converges to a constant ρ^* which is positive when $\Gamma > \Gamma_0^*$ and negative when $\Gamma < \Gamma_0^*$. Further, as $\Lambda \to \infty$, selling price converges to $\frac{1}{2}\Gamma + \frac{C_p}{2}$, which is greater than the mean WTP.*

Proposition 4.5 ***(Asymptotic optimal solutions in WTP regime)*** *When maximum WTP becomes very large, i.e., $\Gamma \to \infty$, selling price converges to the mean WTP, i.e., $p_0^* = \frac{1}{2}\Gamma$, and the optimal production rate tends to infinity with the order of $\log \Gamma$, i.e, $\mu^* = \mathcal{O}(\log \Gamma)$.*

According to Propositions 4.3 and 4.4 , as shelf life and market demand tend to infinity, the optimal selling price tends to converge to $p_0^* \to \frac{1}{2}\Gamma + \frac{C_p}{2}$. This converged price value surpasses the mean maximum willingness-to-pay and is determined by the interplay between the maximum WTP and production cost. Further, the optimal production rate in the shelf life regime converges to the average number of customers buying the product, while in market demand regime, the optimal production rate converges to the average number of customers buying the product , augmented by a constant term ρ, i.e., $\mu^* \to \Lambda\phi_0^* + \rho^*$, where according to Proposition 2, ρ^* may have a positive or a negative value based on the value of Γ.

Proposition 4.4 offers some insights into the asymptotic capacity planning for perishable products. It highlights the behavior of capacity requirements as market demand approaches infinity. When the WTP surpasses the threshold Γ_0^*, the term ρ^* gets a positive value, meaning that additional capacity beyond the base sales rate is required to meet the high demand effectively. Conversely, when WTP falls below the threshold Γ_0^*, the term

ρ^* takes on a negative value and the capacity required in the asymptotic regime is actually lower than the sales rate. In fact, a higher willingness-to-pay implies an increased demand and revenue per product. Consequently, the firm allocates extra capacity beyond the base sales rate to avoid significant revenue loss in the event of shortages. Conversely, when willingness-to-pay is low, the firm allocates a capacity lower than the base sales rate. This is because in the case of shortages, it does not lose considerable revenue.

Proposition 4.5 indicates that as the maximum willingness-to-pay becomes very large, the selling price asymptotically converges to mean willingness-to-pay, i.e., $p_0^* = \frac{1}{2}\Gamma$. Further, the optimal production rate slowly tends to infinity with the order of $\mathscr{O}(\log\Gamma)$. Intuitively, when the maximum willingness-to-pay of customers increases, it becomes advantageous for the retailer to produce a larger quantity of products. The reason is that as the maximum WTP rises, the number of customers willing to purchase the product also experiences a slight increase. Consequently, the retailer can benefit from higher profits resulting from the high prices, which can offset the potential impact of a higher wastage rate.

In what follows, we derive bounds on the benefit of markdown policies and show that the benefit vanishes when some parameters become very large. Let us denote $\Delta V^{N,S}\%, \Delta V^{N,M}\%$, and $\Delta V^{N,D}\%$ as the benefit of single-stage, multiple-stage, and dynamic markdown policies over no markdown policy, respectively, then we have:

$$\Delta V^{N,S}\% = \frac{V^{S^*} - V^{N^*}}{V^{N^*}} \times 100,\ \ \Delta V^{N,M}\% = \frac{V^{M^*} - V^{N^*}}{V^{N^*}} \times 100, \ \ \text{and } \Delta V^{N,D}\% = \frac{V^{D^*} - V^{N^*}}{V^{N^*}} \times 100$$

Further, we define the benefit of multi-stage markdown policy over a single-stage markdown policy as $\Delta V^{S,M}\%$ as follows.

$$\Delta V^{S,M}\% = \frac{V^{M^*} - V^{S^*}}{V^{S^*}} \times 100$$

Hereafter, we refer to $\Delta V^{S,M}$ as the marginal benefit of multi-stage markdown policy. To set up the proofs, in the following lemma, we indicate the order of shelf life in the optimal production rate.

Lemma 4.1 ***(Bound on the optimal*** ρ^****)*** *The optimal* ρ^* *is in the order of* $\mathscr{O}(\frac{1}{\theta})$ *and* $\mathscr{O}(\log \Gamma)$.

The following theorem provides bounds on the benefit of markdown policy M over markdown policy S in shelf life, market demand, and maximum WTP regimes.

Theorem 4.2 ***(Bounds on the marginal benefit of multiple-stage markdown policy)*** *The value of multiple-stage markdown policy (M) over single-stage markdown policy (S), i.e.,* $\Delta V^{S,M} = \frac{V^{M^*} - V^{S^*}}{V^{S^*}}$*, asymptotically vanishes in market demand with the order of* $\mathscr{O}(\frac{1}{\Lambda})$ *as* $\Lambda \to \infty$*, in shelf life with the order of* $\mathscr{O}(\frac{1}{\theta})$ *as* $\Lambda \to \infty$*, and in maximum WTP with the order of* $\mathscr{O}(\frac{1}{\log \Gamma})$ *as* $\Gamma \to \infty$.

Theorem 4.2 shows the speed at which marginal benefit of multi-stage markdown policy vanish in shelf life, market demand, and maximum WTP. Based on the above theorem, as shelf life, market demand, or maximum WTP gets very large, the marginal benefit of policy M over policy S vanishes in the order of $\mathscr{O}\left(\frac{1}{\theta}\right)$, $\mathscr{O}\left(\frac{1}{\Lambda}\right)$, or $\mathscr{O}(\frac{1}{\log \Gamma})$, respectively. These results intuitively show the significance of markdown strategies for products with low demand, shelf life, and willingness-to-pay. Further, the findings imply that the marginal benefit of multiple-stage markdown policy vanishes in market demand and shelf life at a faster rate compared to maximum WTP. Next, utilizing the outcomes from Theorem 4.2, we can establish bounds on the benefits of multi-stage and dynamic markdown policies over the no-markdown policy, denoted as $\Delta V^{N,M}$.

Corollary 4.1 ***(Bounds on $\Delta V^{N,M}$)*** *The benefit of multi-stage markdown policy (M) over no-markdown policy (N), i.e., $\Delta V^{N,M} = \frac{V^{M*} - V^{N*}}{V^{N*}}$, asymptotically vanishes in market demand with the order of $\mathscr{O}(\frac{1}{\Lambda})$ as $\Lambda \to \infty$, in shelf life with the order of $\mathscr{O}(\frac{1}{\theta})$ as $\Lambda \to \infty$, and in maximum WTP with the order of $\mathscr{O}(\frac{1}{\log \Gamma})$ as $\Gamma \to \infty$.*

Corollary 4.1 shows that the value of multi-stage policy over no-markdown policy quickly vanish in market demand and shelf-life, while slowly vanishes in maximum WTP. Intuitively, this indicates that for products with high demand, long shelf life, or high maximum WTP, multiple-stage markdown pricing does not yield significant benefit over a fixed-pricing strategy.

Corollary 4.2 ***(Convergence of the optimal number of markdown stages)*** *As market demand, shelf life, or maximum WTP approach to infinity, the optimal number of markdown stages approach to zero, i.e., $Q^* \to 0$.*

The results in corollary 4.2 can be directly deduced from Corollary 4.1. As market demand, shelf life or maximum WTP asymptotically increase towards infinity, the gap between multi-stage markdown policy and no-markdown policy vanishes. Consequently, the optimal number of markdown stages, denoted as Q^*, decreases and approaches zero, corresponding to the no-markdown policy.

Up to this point, our primary focus has been on developing inventory-pricing models considering various markdown policies under FIFO issuing policies. While some retailers can enforce FIFO policies by displaying the oldest items, certain retail industries operate differently, where consumers observe expiration dates and manage product depletion. Consequently, in these cases, the system operates under LIFO policy. Furthermore, the deteriorating quality of products over time can influence customer's utility, making them not only price-conscious but also sensitive to product freshness. Notably, inventory management systems operating under a LIFO issuing policy are known to be more challenging than those operating under FIFO issuing policy. Thus, to encompass more realistic, yet intricate, inventory-pricing models, the subsequent section extends our base model (no markdown policy) to incorporate LIFO issuing policies and account for customers' sensitivity to product freshness.

4.6 Extensions

4.6.1 Inventory-Pricing Model under LIFO Issuing Policy (L Policy)

Keilson and Seidmann (1990) and Parlar et al. (2011) conducted a study to analyze the LIFO policy under continuous-review system. Under the LIFO policy, the sojourn time of a new unit entering the inventory is solely dependent on the future demand and unit arrivals. Additionally, if a unit has a sojourn time less than θ, it implies that all units that arrived during its sojourn time also have sojourn times below θ and therefore are not expired. In other words, we have $S^L = \min(\tilde{S}^L, \theta)$, where $\tilde{S}^L$ represents the random variable indicating the sojourn time of units with infinite shelf-life. Parlar et al. (2011) demonstrated that the distribution of $\tilde{S}^L$ is equivalent to the distribution of a busy period duration in an $M/M/1$ queue with an arrival rate of μ and a service rate of λ. Then they obtained LT of the truncated sojourn time S^L as follows.

$$s^*(\alpha) = \frac{\mu + \lambda_0 + \alpha - \sqrt{(\mu + \lambda_0 + \alpha)^2 - 4\mu\lambda_0}}{2\mu} \tag{4.27}$$

By inverting the LT in Equation (4.27), the density of sojourn time S^L can be expressed in terms of the modified Bessel function $I_1(z) = \sum_{k=0}^{\infty} \frac{1}{k!(k+1)!} \left(\frac{z}{2}\right)^{2k+1}$ as follows.

$$s(x) = \sqrt{\frac{\lambda_0}{\mu}} x^{-1} e^{-(\lambda_0+\mu)x} I_1(2\sqrt{\lambda_0 \mu x}) \tag{4.28}$$

Then, the cumulative distribution of sojourn time can be written as $S(x) = \int_0^x s(y)dy$.

Proposition 4.6 *Under LIFO policy, outdate rate can be expressed as μq^L, where outdate probability q^L can be expressed as follows.*

$$q^L = 1 - S(\theta) \tag{4.29}$$

Additionally, shortage rate is $\lambda_0 l^L$, where shortage probability l^L can be written as follows.

$$l^L = 1 - \frac{\mu}{\lambda_0} S(\theta) \tag{4.30}$$

Hence, the long-run average profit under LIFO policy L can be written as follows.

$$V^{L*} = \max_{\mu \geq 0, 0 \leq \phi_0 \leq 1} p_0 \lambda_0 \left(1 - l^L\right) - C_s \lambda_0 l^L - C_e \mu q^L - C_p \mu \tag{4.31}$$

Where q^L and l^L are given in Proposition 4.6.

4.6.2 Inventory-Pricing Model with Customer Choice (FL Policy)

In this section, we present a model that incorporates the impact of product quality on customers' willingness-to-pay. Because of quality consideration, products, priced equally, are allocated based on LIFO policy. To simplify the analysis, we introduce a threshold T for remaining shelf life, beyond which fresh products transition into non-fresh items. Prior to reaching this threshold, customers perceive the products as fresh and of high quality. As a result, they are willing to pay a premium price v based on this favorable perception. However, once the threshold T is surpassed, the quality of the products deteriorates to the extent that customers no longer perceive them as fresh. This decline in perceived quality is reflected in a reduced willingness-to-pay, denoted as δv, where δ takes a value between 0 and 1.

In this system, to accommodate the changing quality and customer perception, we allow different prices for fresh and non-fresh products. We denote the price of fresh and non-fresh products as p_0 and p_1, respectively. Then, we can represent the prices of fresh and non-fresh products in terms of the purchase probabilities of fresh and non-fresh items using the following expressions.

$$p_0(\phi_0,\phi_1) = \delta G^{-1}(1-\phi_0-\phi_1) + G^{-1}(1-\phi_0) - \delta G^{-1}(1-\phi_0), \tag{4.32}$$

$$p_1(\phi_0,\phi_1) = \delta G^{-1}(1-\phi_0-\phi_1), \tag{4.33}$$

where $(\phi_0,\phi_1) \in \Omega := \{(\phi_0,\phi_1) : 0 \le \phi_0 \le 1,\ 0 \le \phi_1 \le 1,\ 0 \le \phi_0+\phi_1 \le 1\}$. Then, we can write the demand rates for fresh and non-fresh products as $\lambda_0 = \phi_0\Lambda$ and $\lambda_1 = \phi_1\Lambda$, respectively. In this system, we deal with a queuing network with two tandem $M/M/1+D$ queues operating under LIFO policy. The arrival rate for the fresh products queue is the production/arrival rate of units with Poisson rate μ. Once items pass the remaining shelf life T (outdated in the first queue), they enter the second inventory queue. Thus, the arrival rate of non-fresh products queue is equivalent to the outdate rate of the fresh inventory queue. Because the underlying system is quite complex, as it is common in the papers in the literature, we make an approximation by assuming that the time between successive expirations follows an exponential distribution with rate being equal to the expiration rate of fresh products. Therefore, the arrival rate of fresh and non-fresh queues are denoted by μ and $\mu\left(1-S(T)\right)$, respectively. Additionally, demand rates for the fresh and the non-fresh queues are denoted by λ_0 and λ_1, respectively. Given these parameters, the following proposition presents the expiration and shortage rates in the approximated system.

Proposition 4.7 *Under freshness-dependent LIFO policy, outdate probability can be expressed as follows.*

$$q^{FL} = \left(1-S(T)\right)\left(1-S(\theta-T)\right) \tag{4.34}$$

Additionally, shortage probabilities for fresh and non-fresh products denoted by l_0^{FL} and l_1^{FL} can be written as follows, respectively.

$$l_0^{FL} = 1-\frac{\mu}{\lambda_0}S(T) \tag{4.35}$$

$$l_1^{FL} = 1-\frac{\mu\left(1-S(T)\right)}{\lambda_1}S(\theta-T) \tag{4.36}$$

Thus, the value function under freshness-dependent LIFO policy, can be written as follows.

$$\begin{aligned} V^{FL*} = \max_{\mu\geq 0,\phi_0,\phi_1\in\Omega} & p_0\lambda_0\left(1-l_0^{FL}\right)+p_1\lambda_1\left(1-l_1^{FL}\right)-C_s\left(\lambda_0 l_0^{FL}+\lambda_1 l_1^{FL}\right) \\ & -C_e\mu q^{FL}-C_p\mu \end{aligned} \tag{4.37}$$

4.7 Computational Results

In this section, to illustrate the applicability of the proposed models, we consider two case studies. In the first case study, We consider a farm in British Columbia, Canada, while in the second case-study we consider a local bakery in France to show sensitivity of the model to different parameters and the robustness of the theoretical results numerically.

Next, we present different performance measures utilized in our analysis. Our primary focus is to compare multiple-stage (M) and dynamic (D) markdown pricing policies with single-stage (S) markdown policy and fixed-price policy (N), which serves as the benchmark model. In the dynamic markdown policy, we approximate the relationship between price and remaining shelf life using a simple yet effective 3-degree polynomial mapping function. To evaluate the effectiveness of markdown pricing, we can determine the markdown value by calculating the relative increase in expected profits that occur when a seller switches from a fixed-price policy to a specific markdown policy.

4.7.1 Case Study 1: Fresh Produce Supply Chain

In this section, we investigate the application of the models presented earlier in fresh produce supply chain. For our analysis, we utilize two datasets that comprise records of sales and harvests carried out during the years 2019 to 2020 in a farm located in British Columbia, Canada. The sales records provide information on the revenue generated per product through its main sales channels: markets, community-supported agriculture programs, and wholesale. The harvest records provide information on the quantity of each crop harvested, as well as the date and location of the harvest. The datasets include information on 228 distinct products, and each entry in the sales (harvest) dataset represents a single sales (harvest) event for a specific crop, including information on the weight or the quantity sold (harvested). For computational results, we specifically focus on the summer squash, which is also commonly referred to as zucchini, with the expected shelf life of 7 days under a controlled temperature condition. Figure 4.5 shows the number of wastage for zucchini products during the years 2019 to 2020. We assume that the daily aggregate harvest (supply) and sales (demand) quantity follow a Poisson distribution. The parameters of the Poisson distribution are determined based on the average daily supply and demand observed in the dataset, specifically 32.258 for supply and 26.273 for demand.

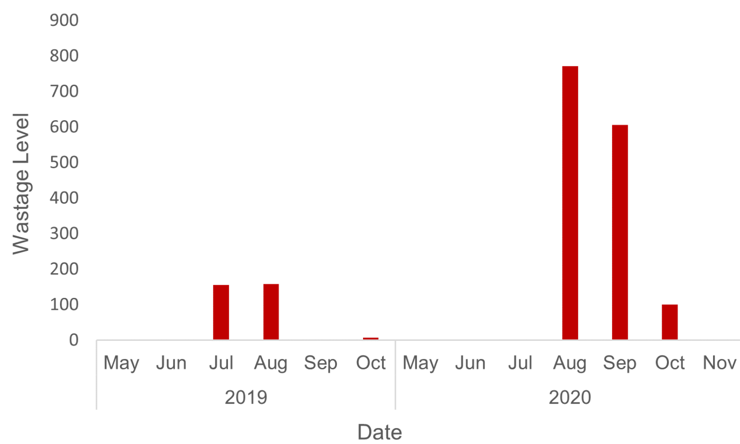


Figure 4.5: Wastage level for zucchini products

Zucchinis are sold at the price of $p = \$2.5$ per pound. According to Curtis et al. (2014), we estimate the willingness-to-pay for summer squash to follow a Normal distribution with mean \$2.925 and standard deviation \$0.383. As such, the potential market demand for zucchini can be obtained as \$0.383. According to the cost analysis carried out by Afeworki et al. (2015), the average production cost of zucchini is \$1.032 per pound. We let the unit lost-sales penalty C_s as the difference between the sales price 2.5 and the unit production cost \$1.032,i.e., $C_s = \$2.5 - \$1.032 = \$1.468$. Also, because disposal cost is 10% of unit sales price, the total expiration cost is set to be the lost profit plus disposal cost, i.e., $C_e = \$0.25 + \$1.468 = \$1.718$. We also consider the cost of adding an additional markdown stage as $C_l = \$0.01$. By utilizing the parameter values introduced earlier and applying the proposed models, we present the results in Table 4.1.

In order to obtain profit function in the current system, we substitute the production rate and selling price of the current system, i.e, $\mu^E = 32.258$ and $p^E = \$2.5$, into the profit function V^{E^*} defined below.

$$V^{E^*} = p^E \lambda^E \left(1 - l^E\right) - C_e \mu^E q^E - C_p \mu^E \tag{4.38}$$

Consequently, we directly derive the total profit in the current system as $V^{E^*} = 22.097$.

Table 4.1: Numerical results for fresh produce supply chain case

Model	**Production Rate**	**Wastage Level**	**Shortage Level**	**Total Profit** ($)
Current System (E)	32.258	5.990	0.000	22.097
No Markdown (N)	26.109	0.150	0.135	37.763
Single-Stage Markdown (S)	25.897	0.018	0.031	38.203
Multiple-Stage Markdown (M)	25.889	0.011	0.028	38.217
Dynamic Markdown (D)	25.961	0.006	0.003	38.444

The findings presented in Table 4.1 indicate that the existing real-world system yields an average profit of 22.097. Adopting optimal production and pricing decisions can substantially enhance profitability, yielding an average profit of 37.763. Introducing a single-stage markdown policy further enhances profitability to 38.203. However, the gains become negligible when employing multiple-stage and dynamic markdown policies, where profits reach 38.217 and 38.444, respectively.

4.7.2 Case Study 2: Bakery Products Chain

The "French bakery daily sales" dataset is a collection of daily sales records from a bakery in France (Gimbert, 2022). The dataset contains information about the date, time, and type of product sold, as well as the quantity and price of each item from 2021-01-01 to 2022-09-30. The dataset has 234005 records and includes a total of 136,000 transactions. In our numerical example, we investigate the value of presented markdown policies for croissants. Selling price of croissants is €1.1 per item. Willingness-to-pay for croissants in Europe is estimated using analysis carried out in Anastasiou et al. (2017). According to this paper, WTP distribution can be estimated as a Uniform distribution between €1.1 and €1.867. We assume that the daily sales rate is Poisson distributed with parameter equal to the average sales quantity per day derived from data, i.e., 30.315. Having WTP distribution and average sales rate, the daily arrival rate of potential market demand can be obtained as 65.749. Similar to the previous case, lost sale cost is set to be the difference between the sales price €1.1 and the unit production cost €0.649,i.e., $C_s = €1.1 - €0.649 = €0.451$, and expiration cost is set to be the lost profit plus disposal cost (10% of selling price), i.e., $C_e = €0.451 + €0.11 = €0.561$. We also consider a €0.01 labelling cost per croissant. Considering the introduced parameter values, we solve the proposed optimization problems and present the results in Table 4.2.

Table 4.2: Numerical results for bakery supply chain case

Model	Production Rate	Wastage Level	Shortage Level	Total Profit (€)
Current System (E)	30.179	0.429	0.561	15.947
No Markdown (N)	28.293	0.442	0.544	16.026
Single-Stage Markdown (S)	28.055	0.053	0.126	16.651
Multiple-Stage Markdown (M)	28.015	0.017	0.105	16.686
Dynamic Markdown (D)	28.135	0.013	0.003	0.011

Similar to the results in Table 4.1, findings in Table 4.2 suggest that the current real-world system generates an average profit of 15.947, but optimizing production and pricing decisions boosts profitability to 16.026. Implementing a single-stage markdown policy further increases profit to 16.651, while multiple-stage and dynamic markdown policies offer only marginal improvements, resulting in profits of 16.686 and 16.951, respectively.

4.8 Sensitivity Analysis

In this section, we aim to demonstrate the robustness of our theoretical results by conducting a comprehensive analysis. Firstly, we investigate the effectiveness of different markdown policies, including no markdown, single-stage, multiple-stage, and dynamic markdown strategies. Subsequently, we perform a sensitivity analysis by varying different parameters, leading to some managerial insights. Lastly, we compare the performance of FIFO and LIFO policies, and examine the impact of customer sensitivity towards freshness in the base model. In what follows, we present the computational results for the first case-study (zucchinis) and present the second case-study (croissants) results in Appendix C.8.

4.8.1 The Value of Different Markdown Policies

The results highlight a trade-off between the added value offered by different markdown policies and the level of complexity associated with their implementation. Assuming no cost is incurred for changing product prices, our analysis reveals that the dynamic markdown policy yields the highest profitability, followed by the multiple-stage and single-stage markdown policies. However, it is important to consider that the inclusion of additional markdown stages leads to an increase in the complexity of inventory-pricing problem. Figures 4.6 and C.8.1 demonstrate that implementing a single-stage markdown policy yields significant improvement over a no-markdown policy, especially when market demand, shelf life, or maximum WTP is low (e.g., around 7% for zucchinis and 35% for croissants with a 95% drop in demand). Conversely, the incremental value offered by multiple-stage and dynamic markdown policies over a single-stage markdown policy is found to be negligible, even when considering scenarios with low demand, short shelf life, and low mean WTP. Specifically, Figures 4.6 and C.7.1 demonstrate when the demand, shelf life, and mean WTP are low, the marginal benefit of a multiple-stage markdown policy over a single-stage policy amounts to approximately 2 percent, which is not significant when compared to the relative benefit of a single-stage markdown policy over a no-markdown policy.

While it is true that the benefits of markdown policies diminish as market demand, shelf life, or maximum willingness-to-pay increase, it is important to note that these policies still hold value by effectively reducing wastage. Interestingly, as depicted in Figure 4.8, there is a surprising trend observed whereby the level of wastage increases in the absence of markdown policies when the market demand grows. In contrast, the implementation of multiple-stage and single-stage markdown policies leads to a decrease in wastage.

According to Tables 4.3 and 4.4, the regular selling price under a multiple-stage markdown policy is higher than that under a single-stage markdown policy, and greater compared to a no-markdown policy. For example, when the market demand for zucchini products is 26.272, the prices for the freshest item are as follows: \$2.510 under no markdown, \$2.552 under single-stage markdown, and \$2.554 under multiple-stage markdown. Intuitively, adding markdown stages may negatively affect a portion of customers buying young items.

4.8.2 The Effect of Market Demand

The results suggest that the benefit of the markdown policies is highest when the market demand is low. The reason is that when market demand is low, the firm can use markdown policies to increase demand for the product and improve profitability. As the market demand increases, the uncertainty of the arrival process decreases, and consequently, the relative benefit of markdown policies would vanish as depicted in Figure 4.6 (a).

As the demand rate rises, production rate increases across all models to maintain the equilibrium between supply and demand. Also, according to Tables 4.3 and 4.4, as a result of increase in demand, the selling price under a no-markdown policy tends to rise, while under markdown policies, regular and clearance selling prices experience a decrease, accompanied by a reduction in the duration of markdown sales. This means with an increase in demand, the selling price under a no-markdown policy gets closer to that under the markdown policies.

In dynamic markdown policy, as the demand rate increases, the level of certainty and stability in selling prices increases, too. This is achieved by minimizing the variability in selling prices, ensuring that customers perceive consistent pricing patterns. Intuitively, as market demand increases, the dynamic markdown policy approaches the fixed pricing policy.

4.8.3 The Effect of Shelf Life

As the shelf life of the product increases, the product gets closer to a non-perishable item with a lower wastage rate. Therefore, markdown policies offer a lower gain for products with longer shelf life as shown in Figure 4.6 (b). The results further imply that the marginal benefits of multi-stage markdown and dynamic markdown policies vanish as shelf life becomes longer due to the lower wastage rate.

The findings of the study indicate that an increase in shelf life exhibits a similar effect to that of an increase in demand. Specifically, as the shelf life of a product becomes longer, the production level increases and the selling price under a no-markdown policy tends to approach the pricing observed in markdown policies.

Table 4.3: Optimal no-markdown and single-stage markdown policies with changes in market demand and shelf life

Parameter		***N***			***S***		
		Markdown Policy	Production Rate	Total Profit	Markdown Policy	Production Rate	Total Profit
	-0.75%	2.507	6.55	8.85	2.445, 2.672	6.36	9.16
	-0.50%	2.509	13.07	18.48	2.445, 2.596	12.86	18.87
	-0.25%	2.510	19.59	28.12	2.444, 2.567	19.38	28.54
Λ	0%	2.510	26.11	37.76	2.444, 2.552	25.90	38.20
	+0.25%	2.510	32.63	47.40	2.443, 2.543	32.42	47.86
	+0.50%	2.511	39.15	57.05	2.443, 2.537	38.94	57.51
	+0.75%	2.511	45.66	66.69	2.443, 2.533	45.46	67.16
	-0.75%	2.507	26.21	35.39	2.445, 2.672	25.45	36.63
	-0.50%	2.509	26.14	36.96	2.445, 2.596	25.73	37.73
	-0.25%	2.510	26.12	37.50	2.444, 2.567	25.84	38.05
θ	0%	2.510	26.11	37.76	2.444, 2.552	25.90	38.20
	+0.25%	2.510	26.10	37.92	2.443, 2.543	25.93	38.29
	+0.50%	2.511	26.10	38.03	2.443, 2.537	25.96	38.34
	+0.75%	2.511	26.09	38.11	2.443, 2.533	25.97	38.38

Table 4.4: Optimal multiple-stage markdown policy with changes in market demand and shelf life

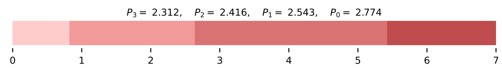
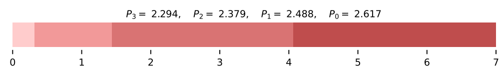
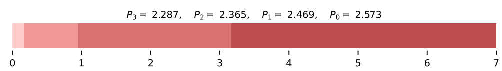
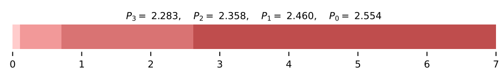
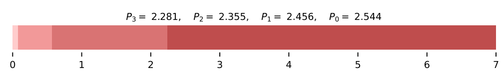
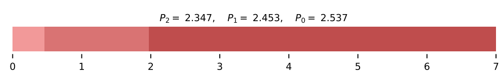
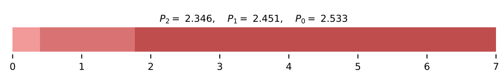
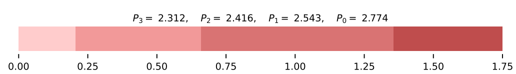
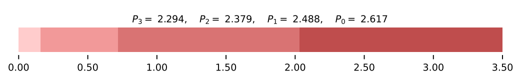
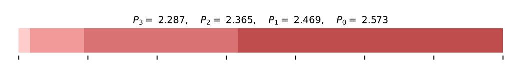
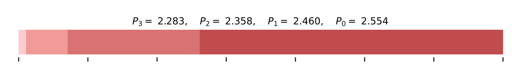
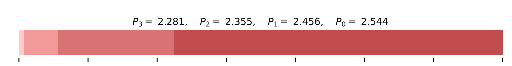
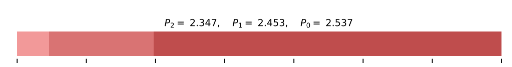
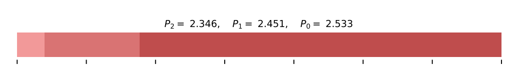
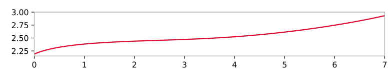
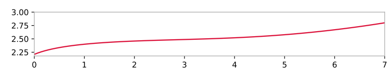
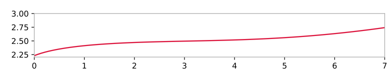
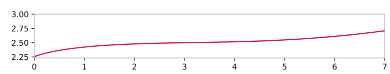
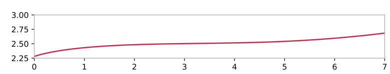
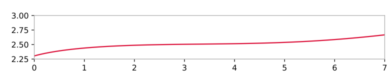
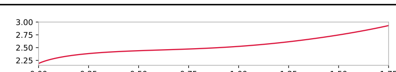
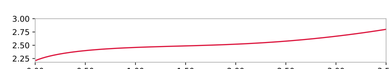
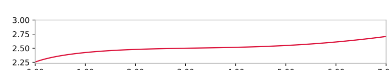
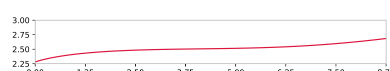
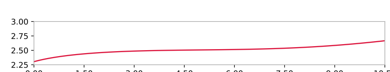
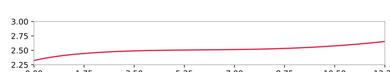
Parameter		***M***			
		Markdown Policy	# of Stages	Production Rate	Total Profit
	-0.75%		3	6.33	9.20
	-0.50%		3	12.84	18.89
	-0.25%		3	19.37	28.56
Λ	0%		3	25.89	38.22
	+0.25%		3	32.41	47.87
	+0.50%		2	38.93	57.52
	+0.75%		2	45.45	67.17
	-0.75%		3	25.33	36.79
	-0.50%		3	25.69	37.78
	-0.25%		3	25.82	38.08
θ	0%		3	25.89	38.22
	+0.25%		3	25.93	38.30
	+0.50%		2	25.95	38.35
	+0.75%		2	25.97	38.38

Table 4.5: Optimal dynamic markdown policy with changes in market demand and shelf life

Parameter		***D***		
		Markdown Policy	Production Rate	Total Profit
	-0.75%		6.348	9.321
	-0.50%		12.883	19.070
	-0.25%		19.424	28.768
Λ	0%		25.961	38.444
	+0.25%		32.495	48.108
	+0.50%		39.025	57.766
	+0.75%		45.553	67.421
	-0.75%		25.391	37.282
	-0.50%		25.765	38.140
	-0.25%		25.897	38.357
θ	0%		25.961	38.444
	+0.25%		25.996	38.486
	+0.50%		26.017	38.511
	+0.75%		26.030	38.526

4.8.4 The Effect of Mean Willingness-To-Pay

As willingness-to-pay rises, customers are willing to pay higher prices for products, resulting in higher selling prices (see Tables 4.6, 4.7, and 4.8), demand, and consequently revenue generated for the firm. This substantial boost in revenue overshadows the relatively smaller wastage costs. Consequently, the impact of a markdown policy, which aims to reduce wastage costs, becomes less influential in the revenue generation process.

Moreover, Figures 4.7 (c) and Table 4.7 indicate a decrease in the number of optimal markdown stages in response to an increase in the mean willingness-to-pay. That is because when customers are willing to pay higher prices, they exhibit a stronger preference for purchasing products at their full price, minimizing the need for markdowns to drive sales.

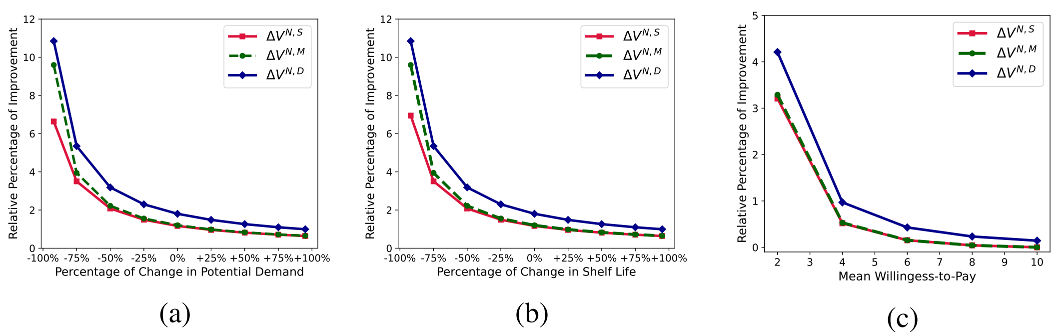
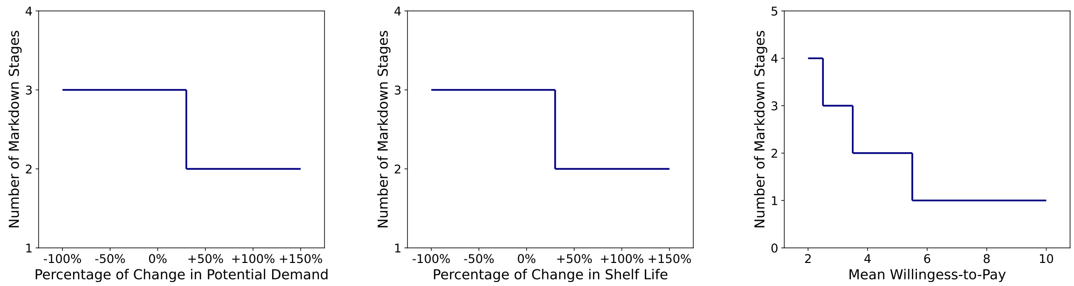
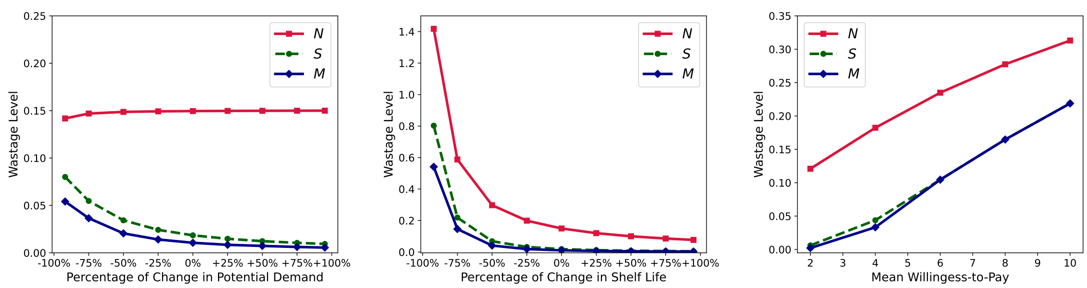


Figure 4.6: Sensitivity of markdown benefits to market demand, shelf life, and mean WTP for zucchini products



(a) (b) (c)

Figure 4.7: Sensitivity of the optimal number of markdown stages to market demand, shelf life, and mean WTP for zucchini products



(a) (b) (c)

Figure 4.8: Sensitivity of wastage level to market demand, shelf life, and mean WTP for zucchini products

4.8.5 The Effect of Standard Deviation of Willingness-To-Pay Distribution

The results indicate that a higher variability in willingness-to-pay among customers results in a greater proportion of products going unsold or being wasted. To mitigate the impact of wastage and enhance profitability, the optimal strategy shifts to implementing a higher number of markdown stages. By doing so, firms can adjust the pricing of their products at different stages to cater to the varying willingness-to-pay levels of customer. As a result, according to Tables 4.6, 4.7, and 4.8, an increase in standard deviation of WTP distribution leads to an increase in the variability of the optimal price and relative benefits of single-stage, multi-stage, and dynamic markdown policies over no markdown policy as shown in Figure 4.9.

Table 4.6: Optimal no-markdown and single-stage markdown policies with changes in mean and standard deviation of WTP

Parameter		***N***			***S***		
		Markdown Policy	Production Rate	Total Profit	Markdown Policy	Production Rate	Total Profit
ψ	2	1.799	21.20	15.58	1.724, 1.829	21.04	16.08
	4	3.456	28.04	66.86	3.400, 3.509	27.82	67.20
	6	5.328	29.28	123.99	5.287, 5.394	29.08	124.18
	8	7.254	29.76	182.57	7.222, 7.324	29.60	182.65
	10	9.204	30.01	241.78	9.176, 9.268	29.89	241.78
σ	0.25	2.574	27.92	42.19	2.527, 2.611	27.68	42.55
	0.5	2.493	24.46	34.91	2.411, 2.538	24.27	35.39
	0.75	2.532	21.24	31.02	2.426, 2.588	21.08	31.54
	1	2.632	18.68	29.02	2.505, 2.701	18.54	29.57
	2	3.237	13.34	28.39	3.050, 3.377	13.20	28.97

Table 4.7: Optimal multiple-stage markdown policy with changes in mean and standard deviation of WTP

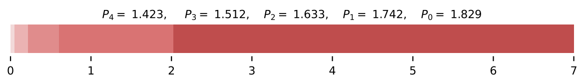
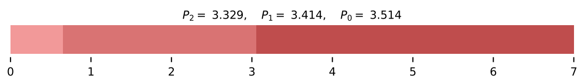
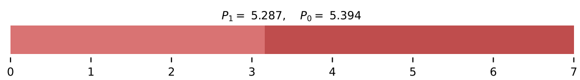
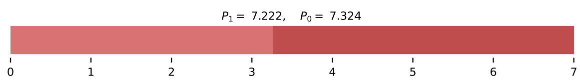
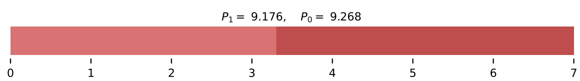
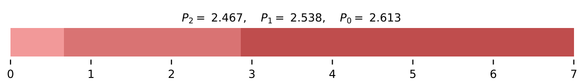
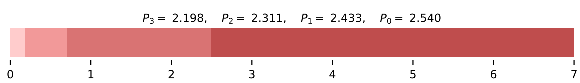
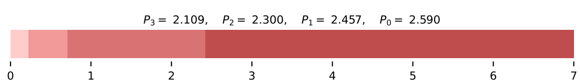
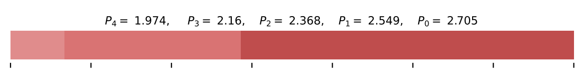
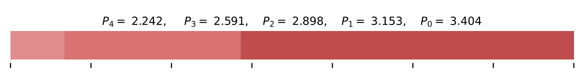
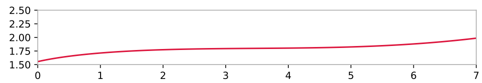
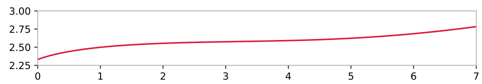
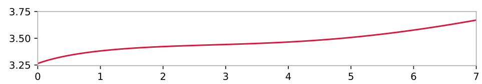
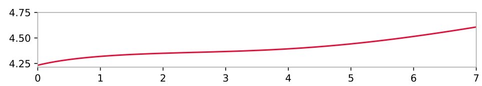
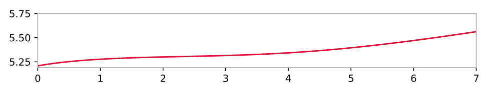
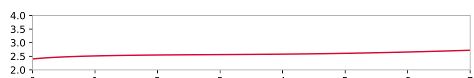
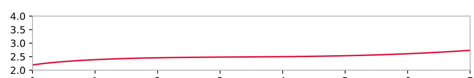
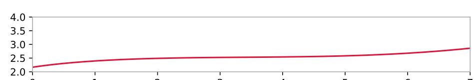
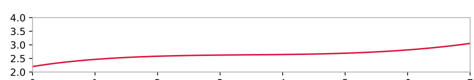
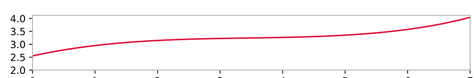
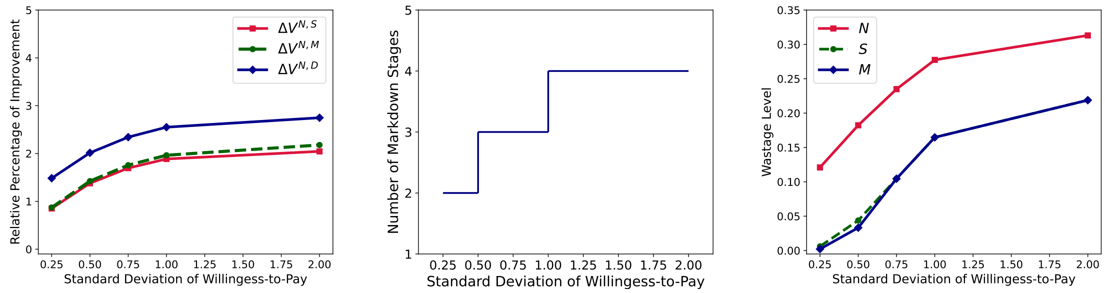
Parameter		*M*			
		Markdown Policy	# of Stages	Production Rate	Total Profit
ψ	2		4	21.04	16.09
	4		2	27.80	67.21
	6		1	29.08	124.18
	8		1	29.60	182.65
	10		1	29.89	241.78
σ	0.25		2	27.66	42.56
	0.5		3	24.27	35.40
	0.75		3	21.07	31.56
	1		4	18.53	29.59
	2		4	13.19	29.01

Table 4.8: Optimal dynamic markdown policy with changes in mean and standard deviation of WTP

Parameter		***D***		
		Markdown Policy	Production Rate	Total Profit
ψ	0		21.134	16.236
	2.5		27.833	67.506
	5		29.060	124.525
	10		29.569	182.999
	50		29.855	242.121
σ	0		27.725	42.814
	2.5		24.337	35.610
	5		24.337	31.746
	10		18.576	29.764
	50		13.214	29.170



(a) (b) (c)

Figure 4.9: Sensitivity of markdown benefits, number of markdown stages, and wastage level to standard deviation of WTP distribution for zucchini products

4.8.6 The Effect of Cost Parameters

In this section, first, we analyze the effect of different cost parameters on the value of markdown policies. Subsequently, we investigate the impact of changing these cost parameters on the optimal solutions. According to Figure 4.10, as per-unit expiration and shortage costs rise, the benefits of markdown policies experience a slight increase, while a significant improvement is observed with an increase in per-unit production costs. That is because the production costs are more directly linked to the profitability of the retailer. Further, Figure 4.11 reveals high sensitivity of the optimal number of markdown stages to expiration and production costs, but lower sensitivity to shortage costs. According to Tables 4.10, as the per unit expiration cost increases,implementing more markdown stages becomes optimal to reduce losses from expiration of unsold products. As purchase costs rise, the firm faces higher expenses when acquiring inventory. To maximize profitability, it becomes more advantageous to distribute the purchase costs over a greater number of sales rather than allowing expensive products to expire, which can be achieved by implementing multiple markdown stages. As a result of increasing the per unit shortage cost, the firm produces more products to avoid shortage situation. In response, as shown in Figure 4.12, the wastage level increases, making the firm to implement multiple markdown stages to avoid high expiration rate.

Table 4.9: Optimal no-markdown and single-stage markdown policies with changes in per unit expiration and shortage costs

Parameter		***N*** Markdown Policy	***N*** Production Rate	***N*** Total Profit	***S*** Markdown Policy	***S*** Production Rate	***S*** Total Profit
	0	2.511	26.30	38.11	2.455, 2.545	25.98	38.25
	0.5	2.511	26.23	37.98	2.450, 2.548	25.94	38.23
C_e	5	2.509	25.96	37.41	2.433, 2.555	25.85	38.16
	10	2.508	25.85	37.11	2.425, 2.557	25.81	38.13
	25	2.507	25.70	36.67	2.414, 2.559	25.76	38.09
	50	2.507	25.59	36.33	2.414, 2.559	25.72	38.05
	0	2.509	25.98	38.01	2.433, 2.538	25.90	38.27
	0.5	2.510	26.03	37.91	2.438, 2.544	25.90	38.24
C_s	5	2.511	26.27	37.42	2.454, 2.568	25.88	38.14
	10	2.511	26.38	37.16	2.461, 2.580	25.87	38.10
	25	2.511	26.54	36.76	2.469, 2.598	25.85	38.05
	50	2.511	26.67	36.44	2.469, 2.598	25.85	38.02

Table 4.10: Optimal multiple-stage markdown policy with changes in per unit expiration and shortage costs

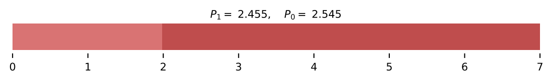
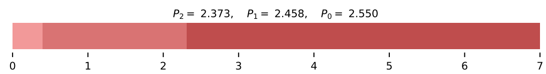
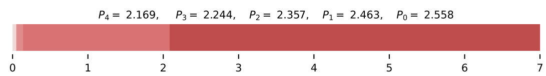
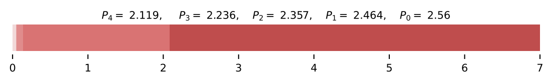
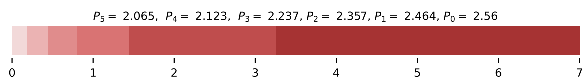
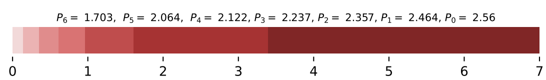
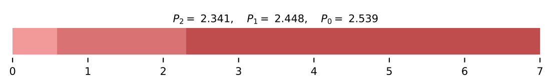
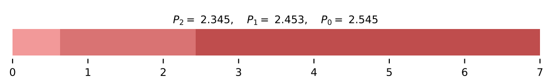
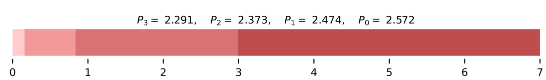
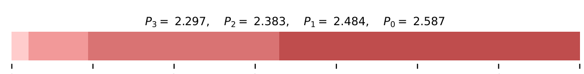
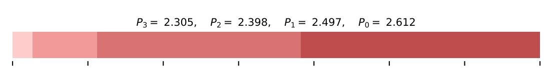
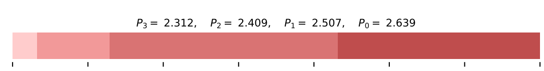
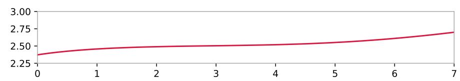
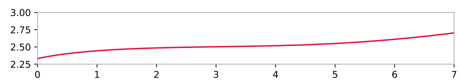
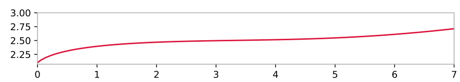
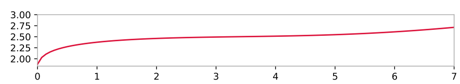
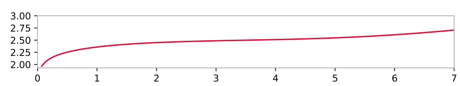
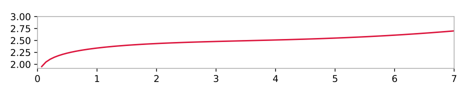
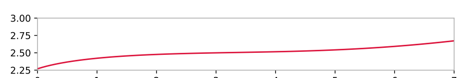
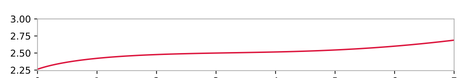
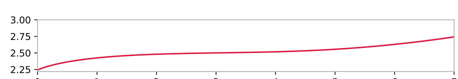
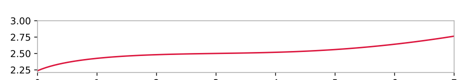
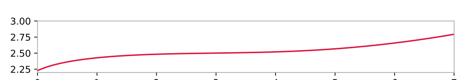
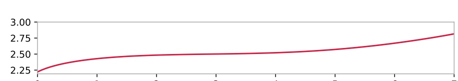
Parameter		***M***			
		Markdown Policy	# of Stages	Production Rate	Total Profit
C_e	0		1	25.98	38.25
	0.5		2	25.93	38.24
	5		3	25.85	38.20
	10		4	25.83	38.18
	25		5	25.81	38.17
	50		6	25.79	38.15
C_s	0		2	25.90	38.28
	0.5		2	25.90	38.25
	5		3	25.87	38.16
	10		3	25.86	38.13
	25		3	25.85	38.09
	50		3	25.85	38.07

Table 4.11: Optimal dynamic markdown policy with changes in expiration and shortage costs

Parameter		***D***		
		Markdown Policy	Production Rate	Total Profit
C_e	0		26.005	38.462
	0.5		25.984	38.454
	5		25.939	38.432
	10		25.923	38.424
	25		25.903	38.412
	50		25.884	38.400
C_s	0		25.964	38.450
	0.5		25.962	38.447
	5		25.957	38.438
	10		25.952	38.434
	25		25.947	38.429
	50		25.944	38.426

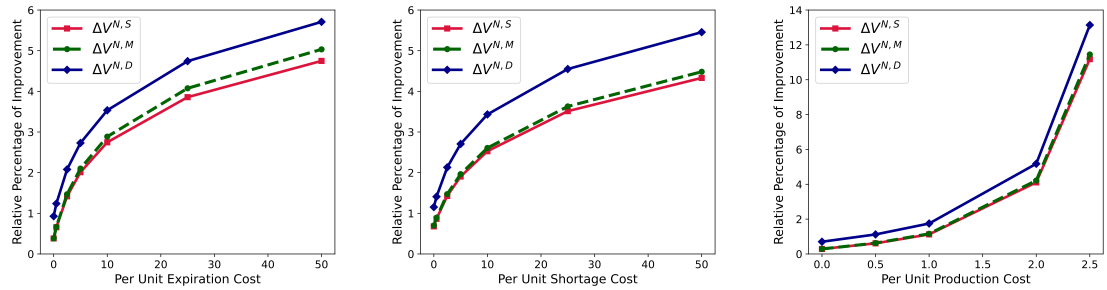
Next, we investigate how changes in various cost parameters impact the optimal solutions. In Tables 4.9-4.11, we provide the optimal prices and markdown policies for different markdown models by changing per-unit expiration and shortage costs. As can be seen, by increasing per unit expiration and shortage costs, the number of markdown stage and selling prices increase. Further, the following properties demonstrate the sensitivity of the optimal solutions to the change in different cost parameters.

Property 4.1 ***(The effect of increase in wastage cost per unit)*** *As C_e increases, λ_0^* increases and μ^* decreases. Also, there is a threshold $\bar{C}_e$ for wastage cost per unit beyond which $\mu^* = 0$ and $V^{N^*} = 0$. In other words, when $C_e \geq \bar{C}_e$, it is not economical to produce any product.*

Property 4.2 ***(The effect of increase in shortage cost per unit)*** *As C_s increases, λ_0^* decreases and μ^* increases.*

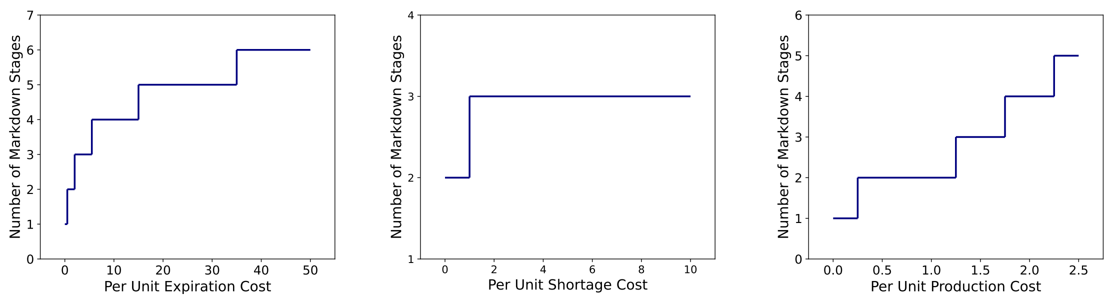
Property 4.3 ***(The effect of increase in production cost per unit)*** *As C_p increases, λ_0^* and μ^* decrease. Also, there is a threshold $\bar{C}_p$ for production cost per unit beyond which $\lambda_0^* = 0$ and $V^{N^*} = 0$. In other words, when $C_p \geq \bar{C}_p$, it is not economical to produce any product.*

Property 4.1 implies that as per unit expiration cost increases, the optimal policy moves toward producing less and selling more. Therefore, optimal arrival/production rate is decreasing in expiration cost per unit and beyond a threshold it becomes zero, meaning that it is optimal to not produce any product. Property 4.2 indicates that by increasing per unit shortage cost, the optimal production rate increases and purchase probability decreases to reduce the risk of stockout situation. Moreover, according to Property 4.3, by increasing per unit production cost, the firm produces and sells fewer products at higher price. Further, when per unit production cost increases beyond a threshold, the optimal production rate becomes zero, and having no system is more economic.



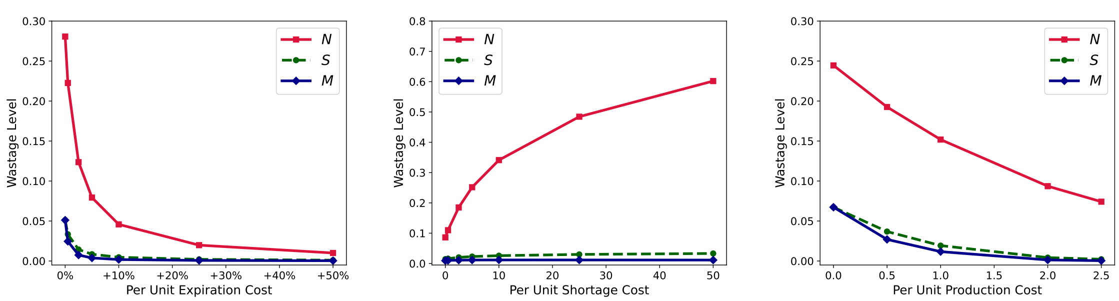
(a) (b) (c)

Figure 4.10: Sensitivity of markdown benefits to the cost parameters for zucchini products



(a) (b) (c)

Figure 4.11: Sensitivity of the optimal number of markdown stages to the cost parameters for zucchini products



(a) (b) (c)

Figure 4.12: Sensitivity of wastage level to the cost parameters for zucchini products

4.8.7 FIFO Policy Versus LIFO Policy

Table 4.12 presents a comparison of the optimal solutions and total profits under two FIFO and LIFO issuing policies. Under the LIFO issuing policy, customers may enjoy the advantage of receiving fresher products. However, this benefit comes at the cost of higher prices compared to the FIFO issuing policy which may negatively impact customer satisfaction. On the other hand, the results, as expected, demonstrate while both the wastage and shortage levels are higher under the LIFO policy, indicating potential inefficiencies in inventory management, the production rate and overall total profit achieved under the FIFO policy is higher compared to the LIFO policy. This suggests that the FIFO policy presents a more favorable outcome for the retailer in terms of profitability. It is noteworthy to mention that under the LIFO policy the effect of changing parameters are similar to that under the FIFO issuing policy.

Table 4.12: Optimal solutions under FIFO and LIFO issuing policies

Parameter		**FIFO Policy**					**LIFO Policy**				
		p_0	λ	WL	SL	V	p_0	λ	WL	SL	V
	-0.75%	2.507	6.55	0.15	0.133	8.85	2.532	6.42	0.54	0.54	6.55
	-0.50%	2.509	13.07	0.15	0.134	18.48	2.526	12.92	0.77	0.76	14.91
	-0.25%	2.510	19.59	0.15	0.135	28.12	2.523	19.43	0.96	0.92	23.56
Λ	0%	2.510	26.11	0.15	0.135	37.76	2.521	25.95	1.11	1.06	32.38
	+0.25%	2.510	32.63	0.15	0.135	47.40	2.520	32.46	1.25	1.18	41.29
	+0.50%	2.511	39.15	0.15	0.135	57.05	2.520	38.97	1.37	1.29	50.27
	+0.75%	2.511	45.66	0.15	0.135	66.69	2.519	45.48	1.48	1.39	59.30
	-0.75%	2.507	26.21	0.59	0.532	35.39	2.532	25.66	2.14	2.17	26.20
	-0.50%	2.509	26.14	0.30	0.268	36.96	2.526	25.85	1.55	1.51	29.81
	-0.25%	2.510	26.12	0.20	0.180	37.50	2.523	25.91	1.28	1.23	31.42
θ	0%	2.510	26.11	0.15	0.135	37.76	2.521	25.95	1.11	1.06	32.38
	+0.25%	2.510	26.10	0.12	0.108	37.92	2.520	25.97	1.00	0.94	33.03
	+0.50%	2.511	26.10	0.10	0.090	38.03	2.520	25.98	0.91	0.86	33.51
	+0.75%	2.511	26.09	0.09	0.077	38.11	2.519	25.99	0.85	0.80	33.89
	2	1.799	21.20	0.12	0.17	15.58	1.827	20.09	0.79	1.15	11.48
	4	3.456	28.04	0.18	0.11	66.86	3.584	26.20	1.14	1.05	34.25
ψ	6	5.328	29.28	0.23	0.08	123.99	5.462	28.45	1.43	0.88	60.43
	8	7.254	29.76	0.28	0.06	182.57	7.385	29.62	1.68	0.74	87.96
	10	9.204	30.01	0.31	0.05	241.78	9.333	30.32	1.88	0.65	116.18
	0.25	2.574	27.92	0.15	0.13	42.19	2.580	27.87	1.17	1.08	36.53
	0.5	2.493	24.46	0.15	0.14	34.91	2.509	24.22	1.07	1.03	29.74
σ	0.75	2.532	21.24	0.15	0.13	31.02	2.558	20.91	1.00	0.95	26.23
	1	2.632	18.68	0.15	0.13	29.02	2.666	18.36	0.96	0.86	24.51
	2	3.237	13.34	0.17	0.11	28.39	3.292	13.25	0.93	0.63	24.36

Table 4.13: Optimal solutions for freshness-dependent willingness-to-pay model

Parameter		First Stage Selling Price	First Stage Selling Price	Production Rate	Wastage Level	Shortage Level	Total Profit
	1	2.629	1.987	26.13	0.71	1.12	31.28
	2	2.651	1.989	26.21	0.66	0.93	32.78
T	3	2.665	1.992	26.23	0.66	0.88	33.32
	4	2.677	1.994	26.22	0.70	0.88	33.46
	5	2.691	1.997	26.21	0.80	0.93	33.27
	6	2.717	2.003	26.17	1.02	1.06	32.53
	0.2	3.305	0.487	26.77	0.69	1.15	32.54
	0.4	3.093	0.984	26.54	0.73	1.08	32.71
δ	0.6	2.887	1.488	26.37	0.76	1.00	32.94
	0.8	2.691	1.997	26.21	0.80	0.93	33.27
	1	2.513	2.513	26.04	0.85	0.85	33.73
	2.5	2.335	1.725	24.44	0.71	0.99	22.95
	5	4.597	3.486	29.45	1.21	0.67	88.91
ψ	10	9.463	7.341	31.47	2.02	0.39	232.22
	20	19.389	15.238	33.46	3.56	0.22	527.16
	50	49.364	39.136	35.99	5.83	0.09	1428.13
	100	99.330	79.065	37.25	7.01	0.04	2938.59
	0.25	2.705	2.05	28.01	0.83	0.94	37.45
	0.5	2.711	1.98	24.55	0.78	0.91	30.57
σ	0.75	2.821	2.01	21.31	0.75	0.86	26.93
	1	2.982	2.09	18.76	0.74	0.80	25.10
	2	3.809	2.57	13.56	0.76	0.63	24.71

As the value of non-fresh items (δ) increases, the selling price of fresh items decreases, while the selling price of non-fresh items rises. Additionally, an increase in the value of non-fresh items results in a higher wastage level, a decrease in production and shortage levels, and an overall increase in total profit. Intuitively, as a result of increased perceived value of non-fresh items, the prices of fresh and non-fresh items converge, leading to an increase in the firm's profit. Furthermore, as the mean WTP increases, both the selling price of fresh and non-fresh items increases, and the retailer produces a greater quantity of products. Consequently, the wastage level increases while the shortage level decreases. The results further indicate that an increase in mean WTP leads to an overall rise in profit, driven by the incremental revenue derived from higher prices. Moreover, higher variability of WTP leads to increased prices for both fresh and non-fresh products, lower production levels, reduced wastage and shortage, and higher total profit. This implies that uncertain customer preferences makes the firm to be more conservative and produce less products. This strategic adjustment aims to mitigate the potential risk of excessive wastage resulting from uncertain demand patterns.

4.9 Conclusion

Retailers often use markdowns to maximize revenues and minimize waste of perishable products due to their limited shelf life and unpredictable demand. Finding the optimal markdown policy can be challenging because of the trade-off between policy complexity and generated revenue, and this issue has not received adequate attention in previous studies. To address this gap, we present a joint inventory and pricing model for perishable items with fixed shelf life and investigate the effectiveness of different markdown policies, including single-stage, multiple-stage, and dynamic markdown policies, both theoretically and numerically.

We do some computational experiments for two realistic cases of fresh produce supply chain and bakery chain. We prove that the benefit of multi-stage markdown policy over single-stag markdown policy vanishes quickly as market demand or shelf life increase, while it vanishes slowly as the customers' maximum willingness-to-pay increase. Further our numerical results indicate that even for products with small market demand, short shelf life, or low maximum WTP, the benefit of multiple-stage markdown policy over single-stage markdown policy is negligible compared to the benefit of single-stage markdown policy over no-markdown policy. Further, our numerical results reveal that for products with small market demand, short shelf life, or low maximum WTP, single-stage markdown policies can significantly benefit the system, while the benefit of multi-stage markdown policy over single-stage markdown policy is negligible and may not always justify their complexities. Also, the results imply that the benefits of different markdown policies increase as the per-unit expiration, shortage, and production costs increase.

This research can be potentially extended in several ways. Considering a randomized issuing policy can be an interesting extension to this work. This work can be further extended by incorporating an online selling channel in addition to the traditional retail store. Another possible direction is to consider a more general settings such as general demand distributions, batch arrival or demand, and so on.

Chapter 5

Coordinating A Decentralized Blood Supply Chain with Interactions between Supply-Side and Demand-Side Operational Decisions

5.1 Introduction

Blood components such as red blood cells (RBC), platelets (PLT), cryoprecipitate, and plasma play a crucial role in the healthcare system. They are delivered from donors to the patients through the blood supply chain that comprises of collection, testing, production and distribution of blood and different components. In Canada, the blood supply chain is coordinated by Canadian Blood Services (CBS), a national organization managing blood supply chain across Canada (except for the province of Québec where it is managed by

Héma-Québec). Figure 5.1 depicts the structure of the blood supply chain network in Canada, which consists of two main echelons, namely, the regional CBS centers and the hospitals. Each regional CBS has its own priorities in fulfilling the demand of the hospitals in its network. However, excess inventory, if any, can be reallocated to other CBS centers and hospitals. We note that each stage of the figure (i.e., donors, CBS, and hospitals) involves many operational decisions; we depict more detailed operations in Figure 5.2.

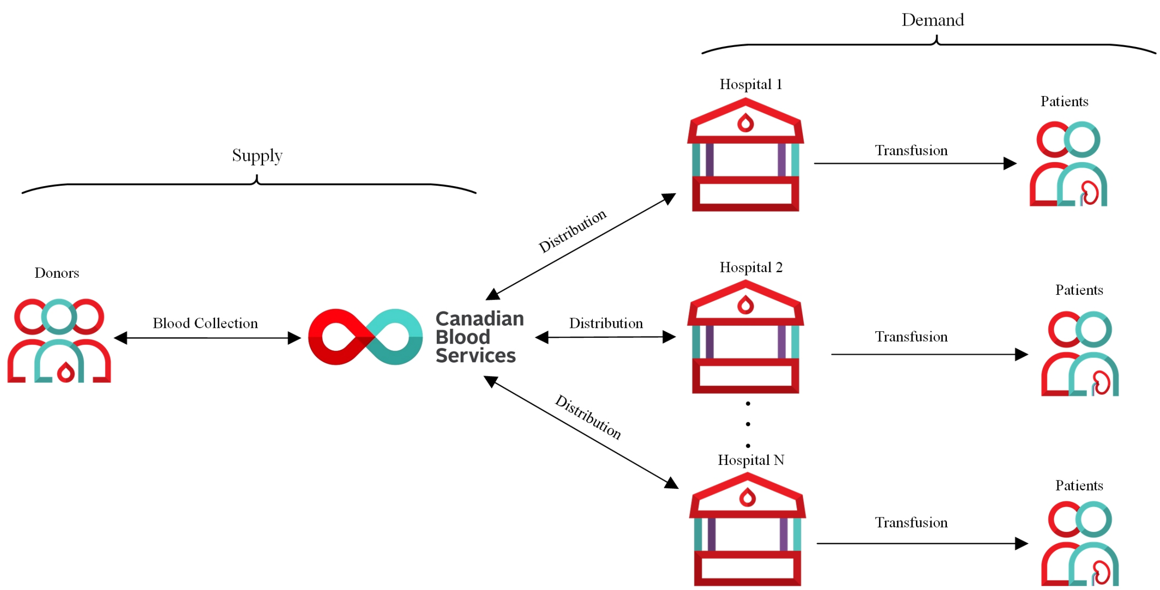


Figure 5.1: Conceptual figure of studied blood supply chain

Managing this supply chain is challenging for at least two reasons. First, unique features of blood products, such as the existence of different blood groups with varying donor levels and limited shelf life of blood components (Osorio et al. 2015), makes it difficult to avoid wastage and carrying large inventory. For example, platelets have a shelf life of 5 days within which it should be consumed or would have to be discarded. It is evident that blood (and components) is needed not just in emergent medical conditions but also to treat severe diseases (and disorders) such as cancer and AIDS, every effort is made to avoid shortage. The latter often implies carrying large inventory, placing emergency orders, and/or purchasing blood components from external sources. Second, the decentralized structure of the blood supply chain in Canada impedes matching because the hospitals and the CBS make independent decisions about production quantity and inventory levels, which leads to bullwhip effect due to lack of information sharing (Li et al. 2021, Motamedi et al. 2021). In general, hospitals place inflated orders that precludes CBS from predicting the right amount to collect and produce, which in turn results in overcollection or overproduction and high wastage rate. For instance, in 2018, CBS reported a 6% discard rate for whole blood and 10% for platelets (Emadi 2020). In another extreme, CBS can experience stock-outs because of inflated ordering by the hospitals which make every effort to ensure enough supplies even if it means carrying large inventory, placing emergency orders, and/or purchasing blood components from external sources. For instance, in 2016, Canada purchased $512 million worth of blood components from the United States (Dinerstein 2018). In another instance, CBS made an urgent request for at least 22K blood donations to compensate for shortage (Global News 2018). Further, there are reports of a 20% drop in donation during the pandemic (The Globe and Mail 2020).

The co-existence of both wastage and shortage in Canada's blood supply chain underscores the inefficiency of the system and calls for the development of a centralized system where supply and demand are more precisely matched. Though a centralized blood supply chain would have a better performance than a decentralized one, it is challenging to implement because hospitals would not be able to order as much as they want, thereby creating a possible conflict with CBS. We postulate that an incentive scheme would encourage hospitals to embrace the centralized structure. Inspired by this real-world problem, we study the decentralized and centralized blood supply chains (BSC). To facilitate implementation of a centralized BSC, we aim to find simple contracting rules to facilitate daily information sharing by hospitals and equitable resource allocation by CBS. More specifically, we intend to investigate whether it is possible for CBS to come up with an equitable coordination contract to facilitate a centralized BSC to replace the existing decentralized structure.

To that end, this research investigates collection, production, replenishment, issuing, inventory and wastage decisions under three different blood supply chain channel structures, i.e., the decentralized, centralized, and coordinated structures. Although this research is motivated by and focuses on the Canadian blood supply chain, the presented problem can be easily applied to other countries' BSC operating under decentralized structure such as United States Paul et al. (2019). This research contributes to the blood supply chain literature in the following ways. First, this work is the first that optimizes blood supply chain structure along with the operations optimization. To that end, we study the optimal operational decisions under three different supply chain structures: decentralized, centralized, and coordinated. Second, we make the very first attempt to propose a bi-level optimization program to model the decentralized blood supply chain system that exists in Canada and many other countries. In addition, a centralized structure is developed as a benchmark to evaluate the performance of the existing decentralized structure, and a coordination mechanism is proposed to facilitate implementation of the centralized structure. Third, in this research, we consider both the supply-side decisions (i.e., those decisions made by the blood center, such as collection, production, and distribution) and demand side decisions (i.e., those decisions made by hospitals, such as inventory management, demand satisfaction, and substitution) in a multi-product setting. Further, the effect of substitution between different blood groups by utilizing a preference table that prioritizes exact substitution. Fourth, the proposed models are used to study a realistic problem instance developed based on publicly available information about the blood supply chain in Hamilton, Canada and develop managerial insights. Our results suggest that integration can benefit the blood supply chain by decreasing the mean gap between production and consumption by 69.84 units and total cost by 93.11%. Further, practicing substitution decreases shortage level and therefore the total cost of the blood system by 14.41%. The centralized system substitutes more products than the decentralized one, leading to lower collection and production levels. Surprisingly, the results suggest that under inappropriate organizational structure, increasing the number of available donors can be detrimental for blood supply chain. Further, our results indicate that coordination contract can effectively facilitate implementation of centralized system. However, offering a subsidy higher than an amount will cause hospitals to make profit, while incurring costs on blood center. Finally, our results also provide insights for the case where integration is not feasible. The remainder of this chapter is organized as follows. Section 5.2 reviews the relevant literature followed by the problem description in Section 5.3. Optimization programs for different channel structures are presented in Section 5.4. A realistic case study is described, studied, and analyzed in Section 5.5. Finally, conclusions and future research directions are outlined in Section 5.6.

5.2 Literature Review

Blood supply chain (BSC) has been an active area of research, and we refer interested readers to Osorio et al. (2015) and Pirabán et al. (2019) for a comprehensive review. Given the focus of this study, we organize the relevant prior works under three streams: supply stage, demand stage, and integrated supply and demand stages.

5.2.1 Supply Stage: Collection and Production Decisions in Blood Supply Chains

The main objective of collection and production stages in blood supply chains is to obtain adequate blood components to satisfy patient demand (Pirabán et al. 2019). To this end, Osorio et al. (2017) combined simulation and optimization techniques to discuss production planning of BSC by considering apheresis and whole blood collection methods. Subsequently, Osorio et al. (2018) developed a multi-objective, multi-product stochastic model to explore joint collection and production decisions in a single-echelon blood supply chain. Özener et al. (2019) quantified the benefits of multi-component apheresis by optimizing donation schedules consisting of donation type and time, while Larimi and Yaghoubi (2019) considered the effects of social announcements and efficiency of collection in a platelet supply chain. Bruno et al. (2019) analyzed the territorial reorganization of blood systems to reduce total management cost without compromising donor attraction goal referred as self-sufficiency goal. They formulated the problem as a facility location model and applied that on a real case-study of Campania Region in Italy. Nagurney and Dutta (2019) developed a novel game theory model to capture the competition between blood service organizations in terms of service quality for collecting blood. Their results imply that increasing competition will result in enhanced service quality. To balance blood production decisions, Yalçındağ et al., (2020) investigated the problem of scheduling blood donations in the face of uncertain donor arrivals.

5.2.2 Demand Stage: Inventory Management Decisions at Hospitals

Inventory management, one of the most critical decisions made at the hospital level in a BSC, has received increased attention over the past decade. Haijema (2014) developed a stochastic dynamic program to investigate ordering, issuance, and disposal policies considering stock level and age of perishable products. They compared their stock-age dependent replenishment policy with base-stock policy and showed that their model works better than classical replenishment policies. Zhou et al. (2011) investigated the optimal order-up-to replenishment policy for platelet products by considering both regular and optional replenishment in a dynamic setting at the hospital level. Kouki et al. (2018) extended an inventory management model for perishable products by considering two supply sources, i.e., ordinary and emergency orders. They applied the base-stock policy and assumed that emergency orders are placed when stock level is lower than a threshold. Duan and Liao (2013) developed a novel age-based replenishment strategy to lower the outdating rate considering a limit for shortage level and indicated that this newly introduced ordering policy works better than classical replenishment models. Rajendran and Ravindran (2017) presented a stochastic integer model to explore ordering policies of platelets aimed at minimizing wastage and shortage and proposed three heuristic rules to compare the model's performance with an order-up-to replenishment policy.

Several researchers have considered different blood groups and substitution. Abdulwahab and Wahab (2014) studied inventory management for platelet supply chain with stochastic supply, demand and substitutable platelets. Duan and Liao (2014) extended their previous work to study substitution and developed three substitution scenarios and minimized the number of outdated units by considering a maximum allowable lost demand.

Dillon et al. (2017) extended a two-stage stochastic inventory management model to explore periodic review policies for RBC products whose demand is uncertain, and assumed substitution based on a priority list. Meneses et al. (2021) studied a two-stage blood supply chain model to minimize total costs as well as shortage and wastage levels. In their paper, they considered minimum service level and ABO-substitutions and designed cut-off policies according to which hospitals can determine their ordering policies. Civelek et al. (2015), Gunpinar and Centeno (2015), and Rajendran and Ravindran (2019) investigated inventory management problem with different types of patient demands requiring blood components of different ages. The papers reviewed above study inventory management problems at the hospital level but did not consider collection and production decisions in the upper stream of the blood supply chain. As blood supply is interwoven with its demand, our research contributes to the above studies by considering interactions between the supply side (i.e., blood center) and the demand side (i.e., the hospitals).

5.2.3 Integrated Models

Simultaneous consideration of supply and demand stages have been attempted in several studies over the past few years. Samani et al. (2019) attempted to incorporate both quantitative and qualitative factors in the BSC network design and aimed to minimize loss of blood unit's freshness and total network cost. Though they considered different collection methods, they did not model heterogeneous blood groups, substitution, or different channel structures. Larimi et al. (2019) discussed lateral transshipment in platelet supply chain where different collection methods are investigated for obtaining typical, irradiated, and washed blood products for age-differentiated patient demands. By considering different types of patients, Ensafian and Yaghoubi (2017) developed a bi-objective robust inventory and production model in a platelet supply chain. Chen et al. (2019) developed a joint collection, production, and disposal model for the platelet supply chain with age-differentiated demand. They modeled their problem as a stochastic dynamic program and addressed their extended model using a lookahead heuristic. Haeri et al. (2020) studied a multi-objective stochastic blood supply chain network design model in an integrated supply and demand system and aimed to incorporate resiliency. Recently, Xu and Szmerekovsky (2022) studied collection, production, and distribution of RBC and PLT products in an integrated BSC with multiple demand types. While they considered the possibility of substitution between compatible blood types, they did not differentiate between exact matching and substitution in their model. In contrast to their work, we take substitution priority into account in our modeling approach. Further, our study focuses on optimizing the structure of the BSC in addition to optimizing its operations. By considering both aspects, we aim to create a more efficient and effective blood supply chain.

Some papers investigated the integrated supply and demand stages models in a BSC under the risk of disruption. Asadpour et al. (2022) presented a recent review on quantitative models under disruption in the blood supply chain. Samani et al. (2020) developed a two-phase disruption scheme for the platelet supply chain in an integrated supply and demand model, where the first phase considers whole blood collection under uncertainty and the second phase reacts to disruption by considering apheresis collection and transshipment between hospitals. Liu et al. (2020) studied the blood distribution problem in China using a vendor-managed inventory routing model wherein the blood center controls the hospital's inventory and fulfills demands at hospitals. Hamdan and Diabat (2020) proposed a robust two-stage programming model to minimize the time and transportation cost under disruption scenarios. They applied Lagrangian relaxation as a solution method to solve large-scale problems. Kamyabniya et al. (2021) considered an integrated network design model in disaster situations where facilities are horizontally and vertically integrated. Further, they considered the matching of injury seriousness with multi-type age-differentiated platelets during the response phase of disasters. Dehghani et al. (2021) presented a proactive transshipment policy to reduce shortages and wastages in a BSC containing several hospitals and a central blood center. They formulated the problem using a two-stage stochastic programming model and used Quasi-Monte Carlo sampling approach to generate different demand scenarios. Most recently, considering uncertainty on both supply and demand sides, Kenan and Diabat (2022) presented a two-stage stochastic programming model to address the inventory management problem in the wake of COVID-19. They considered different blood components and substitutions between blood groups, but different production alternatives were not captured in their model. Most recently, Li et al. (2023a) reviewed the use of novel computational techniques, such as machine learning and optimization, for blood demand forecasting and supply management. They explored the benefits and challenges of data-driven strategies to improve decision-making in demand forecasting and inventory management. Although mentioned works considered integrated operations in a BSC, they mostly focused on operations optimization, rather in this work, along with the operations optimization, we optimize the BSC structure.

5.2.4 Research Gaps and Contributions

By identifying gaps in the current body of knowledge, we can outline the main contributions of our study as follows. First, despite the importance of optimizing the interaction between the blood center and hospitals in BSC, it has been neglected in the literature. All the papers in the literature assumed a single decision maker in BSC, although in most BSCs, blood center and hospitals make individual decisions. To fill this gap, we optimize interaction between BSC facilities as well as operational decisions. To that end, this research investigates and compares three different supply chain structures: decentralized, centralized, and coordinated. To model the decentralized channel structure, we make the first attempt to present a bi-level optimization program and use Karush–Kuhn–Tucker (KKT) to solve the problem. Then, we demonstrate the benefits of a novel coordination contract, under which the decisions by the blood center and the hospitals coincide with those in the centralized BSC. Second, it is evident that the integration of blood operations for different blood products and different blood groups is very critical for supply and demand matching as the inventory management and issuing decisions (demand stage decisions) of different blood products are interwoven with the collection and production decisions (supply stage decisions). However, the extant literature has overlooked all the supply stage (collection and production) and demand stage (inventory and issuing) decisions to manage different blood components considering substitution between blood groups and transshipment between hospitals. In contrast to the existing works in the literature, this research studies a joint optimization of collection, production, distribution, and demand satisfaction, substitution, and transshipment decisions in the multi-product setting under different BSC structures. Third, this research pioneers the analysis and comparison of different issuing policies, including FIFO, LIFO, and Threshold-Based policies in terms of costs and the average freshness level of transfused units. We also formulate two well-known ordering policies, (s,S) and (R,T) policies, and compare their performance with the optimal replenishment policy performance. Fourth, given intrinsic uncertainty in patient demand, a stochastic model is presented, and robust optimization method is applied to cope with the uncertainty. In contrast to previous studies that primarily conducted one-dimensional analyses for selecting robustness parameters, our research adopts a more comprehensive two-dimensional analysis to enhance the performance. Finally, to examine the effectiveness of our proposed models, we apply them to analyze a realistic problem instance of a blood supply chain in Hamilton, Canada and provide some managerial insights.

5.3 Model Description and Formulation

The blood supply chain in Canada comprises of a regional blood center operated by CBS, which first collects blood from donors, processes them and then supplies to the hospitals within the jurisdictional region. CBS, typically, sets up several temporary or permanent collection centers for donors, and where different blood components are extracted either through processing whole blood units or deploying apheresis (i.e., extraction of blood products such as platelets or red blood cells directly from the donor's body, and returning the remaining blood components back to their body). The collected units must be transferred to the blood centers for processing within six hours, where they are fractionated into different blood products using dedicated machines such as centrifuge machines for whole blood units. Different production alternatives are presented in Table 5.1.

Table 5.1: Different collection and production alternatives Osorio et al. (2018)

No	Machine	Process	RBC	PLT
1	Centrifuge Machine	Triplex bag	1	
2	Centrifuge Machine	Quadruple bag	1	1
3	RBC Apheresis Machine	RBC apheresis	2	
4	PLT Apheresis Machine	PLT apheresis		10

The processed blood components are transferred to the hospitals as per the hospital's

Table 5.2: Red blood cell compatibility matrix

	ABo/Rh	Patient							
		A-	A+	AB-	AB+	B-	B+	O-	O+
Donor	A-	✓	✓	✓	✓				
	A+		✓		✓				
	AB-			✓	✓				
	AB+				✓				
	B-			✓	✓	✓	✓		
	B+				✓		✓		
	O-	✓	✓	✓	✓	✓	✓	✓	✓
	O+		✓		✓		✓		✓

order at the beginning of each day and stored in the blood bank to satisfy patient demand. Note that patient demand for different blood groups for all blood components are satisfied based on compatibility rules presented in Tables 5.2, 5.3. For transfusion, exact matches are preferred as much as possible, and substitutes used only in the event of shortages. However, the latter results in mismatching costs Dillon et al. (2017) varying based on matching preferences presented in Tables 5.4 and 5.5. A lower matching preference index leads to more favorable substitutions. It is important that expired blood products incur wastage cost and unsatisfied patient demand result in shortage costs, and hence the objective of such blood supply chain should be to minimize total cost i.e., collection, production, transportation, holding, mismatching, wastage, and shortage.

Table 5.3: Platelet compatibility matrix

	Platelet Type	Patient	
		Rh-	Rh+
Donor	Rh-	✓	✓
	Rh+		✓

Table 5.4: Red blood cell prioritization matrix

	ABo/Rh	Donor							
		A-	A+	AB-	AB+	B-	B+	O-	O+
Patient	A-	0						1	
	A+	1	0					3	2
	AB-	1		0		2			
	AB+	4	2	1	0	5	3	7	6
	B-					0		1	
	B+					1	0	3	2
	O-							0	
	O+							1	0

Table 5.5: Platelet prioritization matrix

	Platelet Type	Patient	
		Rh-	Rh+
Donor	Rh-	0	1
	Rh+		0

In Figure 5.2, we provide a detailed schematic of the Canadian blood supply chain.

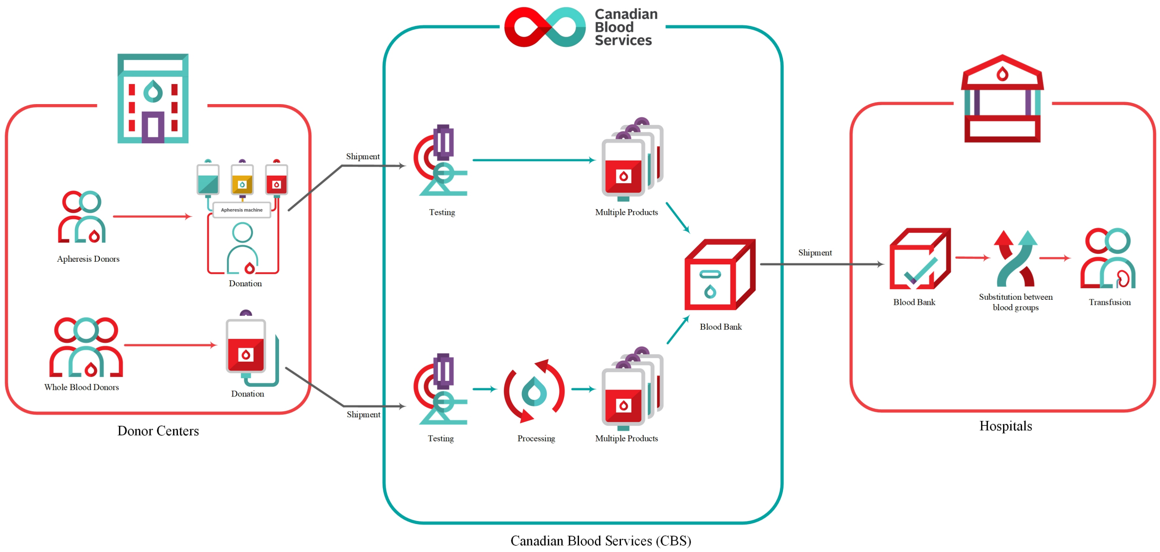


Figure 5.2: A more detailed schematic of the Canadian blood supply chain

The existent blood supply chain structure in Canada is decentralized where the blood center and the hospitals make their own decisions in a sequential manner (Figure 5.3). More specifically, at the beginning of each day, individual hospitals make its ordering decisions, and then regional CBS makes the consequent supply decisions. Since individual hospitals aim at minimizing its shortage and mismatching costs, it usually inflates its orders more than the actual need thereby resulting in overproduction. Motivated by this challenge faced by CBS, we investigate a centralized (integrated) blood supply chain structure, i.e., a central decision maker determines the optimal decisions at both the CBS and affiliated hospitals. Given the loss of control over its decisions, the centralized structure might not be readily acceptable to the hospitals, and hence we also investigate a coordination mechanism to incentivize participation. More precisely, the regional CBS offers a subsidy to each hospital in lieu of the latter agreeing to be a part of the centralized system. Figure 5.3 depicts the three channel structures investigated in this work.

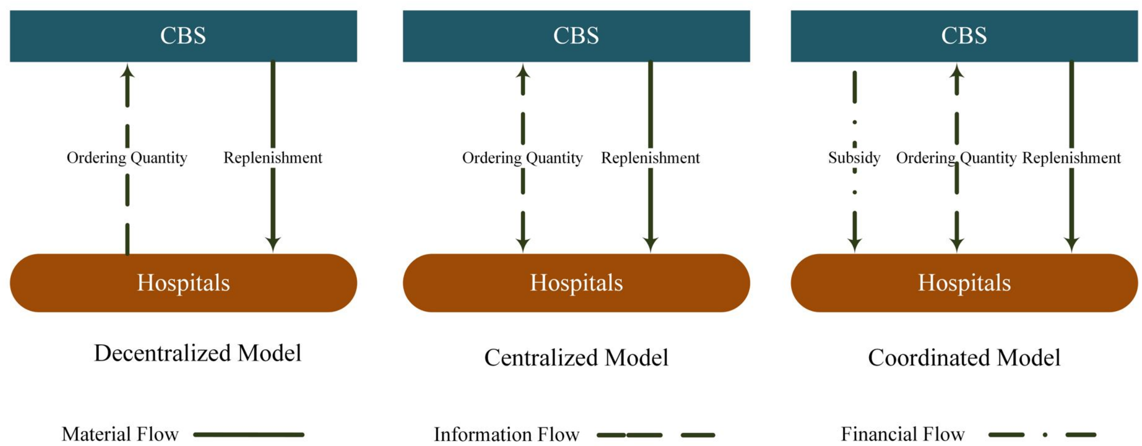


Figure 5.3: Different channel structures of blood supply chain

5.4 Optimization Programs

In this section, we first introduce the notation and then outline the math models for three settings discussed in the previous section, i.e., the decentralized system, the integrated system, and the coordination mechanism.

Table 5.6: Notations

Sets and indices	
T	Set of periods, indexed by t
H	Set of hospitals, indexed by h
J	Set of all distinct blood products consisting of different blood groups (ABO group and Rh +/-) indexed by j (e.g., 8 types of RBC, 2 types of PLT)
G	Set of the blood components (i.e., RBC, PLT, etc.) indexed by g.
L	Set of the blood group types (ABO group and Rh +/-) indexed by l.
K	Set of all processing alternatives consisting both of apheresis and whole blood processing methods indexed by k (presented in Table 5.1).
P	Subset of processing alternatives (K) indexed by p, only consisting of the whole blood processing methods
Q	Subset of processing alternatives (K) indexed by q, only consisting of the apheresis collection methods
R_j	Set of the ages of the product j indexed by r_j
N	Set of machine types for processing blood indexed by n (presented in Table 5.1)
Parameters	
l_1	Collection and production lead-time for the apheresis method
l_2	Collection and production lead-time for the whole blood method
$NW_{l,t}$	Number of available whole blood donors of blood group l in period t
$NA_{l,q,t}$	Number of available donors of blood group l in period t for the apheresis collection method of type q
$R_{k,n}$	Processing time of processing alternative k using a machine of type n, per donated unit

Parameters	
$A_{j,k}$	Quantity of product j derived using alternative k
$d_{j,h,t}$	Patient demand for product j at hospital h in period t
$SO_{j,j'}$	1 if product j is substitutable for product j'; 0 otherwise.
U_g	Maximum lifetime of the product of type g
$M_{j,j'}$	Preference number for satisfying demand for blood component j with blood component j'
CC	Collection and transportation cost of a whole blood unit from collection facilities to blood center
QC_j	Purchasing cost of product j from blood center
PC_k	Processing cost of a blood unit using alternative k
RC_h	Average replenishment and transportation cost of one unit of blood product from blood center to hospital h
$TC_{h,h'}$	Unit transshipment cost between hospitals h and h'
HC_j^B	Holding cost of product j at blood center
$HC_{j,h}^H$	Holding cost of product j at hospital h
WC_j^B	The wastage cost of a unit of product j at the blood center
WC_j^H	The wastage cost of a unit of product j at hospital h
SC_j	Unit shortage cost of product j at hospitals
MC_j	Unit mismatching cost for satisfying demand for product j with the product of the first substitution priority
CB_g	Capacity of blood center for blood product g
$CH_{g,h}$	Capacity of blood product g at hospital h
$CM_{n,t}$	Processing capacity of one machine of type n in period t, expressed as the total time that machine is available in period t
γ_j	The transshipment threshold for product j

Decision Variables	
$D_{j,h,t}$	The order quantity of product j at hospital h in period t
$\alpha^{r_j}_{j,h,t}$	The amount of hospital h's demand for product j in period t which is satisfied with product of age r_j by blood center
$IB^{r_j}_{j,t}$	Inventory level of product j with age r_j at blood center in period t
$IH^{r_j}_{j,h,t}$	Inventory level of product j with age r_j at hospital h in period t
$WB_{j,t}$	Wastage level of product j at the blood center in period t
$WH_{j,h,t}$	Wastage level of product j at hospital h in period t
$SH_{j,h,t}$	Shortage level of product j at hospital h in period t
$W_{l,t}$	Quantity of whole blood of type l transshipped from collection facilities to the blood center in period t
$AC_{l,q,t}$	Number of apheresis collection of type q from donors of blood group l at the blood center in period t
$X_{l,p,t}$	Number of whole blood units of type l processed by alternative p at the blood center in period t
$SD^{r'_j}_{j,j',h,t}$	Quantity of demand for product j of age r_j that is fulfilled by product j' of age r'_j in period t
$TR^{r_j}_{j,h,h',t}$	Quantity of product j with age r_j that is transshipped from hospital h to hospital h' in period t

5.4.1 Decentralized Blood Supply Chain: A Bi-level Model

The decentralized structure of the Canadian blood supply chain entails hospitals placing their orders at the beginning of each day followed by the actions of the blood center to satisfy the demand, which results in a leader-follower structure. In the upper-level problem, hospitals make replenishment and inventory decisions to satisfy patients' demands, and in the lower level, the blood center is obliged to satisfy hospitals' demand by collecting and processing blood units and managing its inventory level. In this system, stock replenishment does not impose any expenses on hospitals, and they can increase the replenishment size without being charged any costs. Therefore, compared with a centralized mode, hospitals are usually willing to use larger order sizes compared to centralized model to lower their costs. In such a system, by acting as the leaders, hospitals have more power than blood center, and decisions are made sequentially, and hence the problem can be formulated as a Stackelberg game with two players (readers may refer to Von Stackelberg (2010) for more information on Stackelberg game). Therefore, in this game, the blood center reacts as best it can to hospitals' decisions, and knowing blood center's reaction, hospitals make their best decisions.

5.4.1.1 Leaders' Problem: Hospitals' Problem

As a leader of blood supply chain, hospitals' problem can be written as follows.

$$\begin{aligned}\operatorname{Min} Z_H^D = & \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} SC_j SH_{j,h,t} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} WC^H_{j,h} WH_{j,h,t} \\ & + \sum_{j\in J}\sum_{j'\in J}\sum_{r_{j'}\in R_{j'}}\sum_{h\in H}\sum_{t\in T} MC_j M_{j,j'} SD^{r_{j'}}_{j,j',h,t} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j} HC^H_{j,h} IH^{r_j}_{j,h,t} \\ & + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} QC_j D_{j,h,t} + \sum_{j\in J}\sum_{h\in H}\sum_{h'\neq h\in H}\sum_{t\in T}\sum_{r_j\in R_j} TC_{h,h'} TR^{r_j}_{j,h,h',t}\end{aligned} \tag{5.1}$$

$S.t$

$$IH^{r_j}_{j,h,t} = IH^{r_j-1}_{j,h,t-1} + \alpha^{r_j}_{j,h,t} - \sum_{j' \in J} SO_{j,j'} SD^{r_j}_{j',j,h,t} + \sum_{h' \neq h \in H} TR^{r_j}_{j,h',h,t} - \sum_{h' \neq h \in H} TR^{r_j}_{j,h,h',t} \qquad \forall\, j,h,t,r_j \tag{5.2}$$

$$\sum_{j' \in J} \sum_{r_{j'} \in R_{j'}} SO_{j,j'} SD^{r_{j'}}_{j,j',h,t} = d_{j,h,t} - SH_{j,h,t} \qquad \forall\, j,h,t \tag{5.3}$$

$$\sum_{j \in G(j)} \sum_{r_j \in R_j} IH^{r_j}_{j,h,t} \leq CH_{g,h} \qquad \forall\, t,g,h \tag{5.4}$$

$$IH^{r_j}_{j,h,t} = WH_{j,h,t} \qquad \forall\, j,h,t,r_j = U_j \tag{5.5}$$

$$TR^{r_j}_{j,h,h',t} = 0 \qquad \forall\, j,h,t,h' \neq h, r_j \leq \gamma_j \tag{5.6}$$

$$D_{j,h,t}, IH^{r_j}_{j,h,t}, WH_{j,h,t}, SH_{j,h,t}, SD^{r'_j}_{j,j',h,t}, TR^{r_j}_{j,h,h',t} \geq 0 \tag{5.7}$$

Objective function (5.1) represents the total costs at the hospitals, which consists of shortage cost, wastage cost, mismatching cost, and holding cost. Constraint (5.2) indicates the inventory balance constraints at hospitals. Equation (5.3) shows the demand satisfaction constraints. The capacity constraints at the hospitals are expressed by constraint (5.4). Constraint (5.5) indicates the wastage level constraints. Constraint (5.6) implies that transshipment cannot occur for blood units that are younger than the transshipment threshold, and finally, constraint (5.7) represents the sign of variables. We remark that the hospitals' decisions are decoupled from each other. As a result, (5.1)-(5.7) can be easily broken down to $|H|$ independent problems, one for each hospital (where $|H|$ is the number of hospitals).

5.4.1.2 Follower's Problem

$$\begin{aligned}\text{Min}\, Z_B^D = &\sum_{l\in L}\sum_{t\in T} CC\, W_{l,t} + \sum_{l\in L}\sum_{p\in P}\sum_{t\in T} PC_p X_{l,p,t} + \sum_{l\in L}\sum_{q\in Q}\sum_{t\in T} PC_q AC_{l,q,t} \\ &+ \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j} RC_h \alpha_{j,h,t}^{r_j} + \sum_{j\in J}\sum_{t\in T} WC_j^B WB_{j,t} + \sum_{j\in J}\sum_{t\in T}\sum_{r_j\in R_j} HC_j^B IB_{j,t}^{r_j}\end{aligned} \tag{5.8}$$

$S.t$

$$WC_{l,t} \leq NW_{l,t} \qquad \forall\, l,t \tag{5.9}$$

$$AC_{l,q,t} \leq NA_{l,q,t} \qquad \forall\, l,q,t \tag{5.10}$$

$$\sum_{l\in L}\sum_{p\in P} X_{l,p,t} R_{p,n} \leq CM_{n,t} \qquad \forall\, n,t \tag{5.11}$$

$$\sum_{l\in L}\sum_{q\in Q} AC_{l,q,t} R_{q,n} \leq CM_{n,t} \qquad \forall\, n,t \tag{5.12}$$

$$W_{l,t-l_2}(1-\varepsilon) \geq \sum_{p\in P} X_{l,p,t} \qquad \forall\, l,t \tag{5.13}$$

$$IB_{j,t}^{r_j} = IB_{j,t-1}^{r_j-1} - \sum_{h\in H} \alpha_{j,h,t}^{r_j} \qquad \forall\, j,t,r_j \neq l_1+1,\; r_j \neq l_2+1 \tag{5.14}$$

$$IB_{j,t}^{r_j} = \sum_{l\in L}\sum_{p\in P} AC_{l,q,t} A_{j,q} + IB_{j,t-1}^{r_j-1} - \sum_{h\in H} \alpha_{j,h,t}^{r_j} \qquad \forall\, j,t,r_j = l_1+1 \tag{5.15}$$

$$IB_{j,t}^{r_j} = \sum_{l\in L}\sum_{p\in P} X_{l,p,t} A_{j,p} + IB_{j,t-1}^{r_j-1} - \sum_{h\in H} \alpha_{j,h,t}^{r_j} \qquad \forall\, j,t,r_j = l_2+1 \tag{5.16}$$

$$\sum_{j\in G(j)}\sum_{r_j\in R_j} IB_{j,t}^{r_j} \leq CB_g \qquad \forall\, t,g \tag{5.17}$$

$$\sum_{r_j\in R_j} \alpha_{j,h,t}^{r_j} = D_{j,h,t} \qquad \forall\, j,h,t \tag{5.18}$$

$$IB_{j,t}^{r_j} = WB_{j,t} \qquad \forall\, j,t,r_j = U_j \tag{5.19}$$

$$\alpha_{j,h,t}^{r_j}, IB_{j,t}^{r_j}, WB_{j,t}, W_{l,t}, AC_{l,q,t}, X_{l,p,t} \;\geq 0 \tag{5.20}$$

Objective function (5.8) minimizes total costs at blood center consisting of collection cost, production costs using the whole blood and apheresis methods, replenishment and transportation costs, wastage cost, and holding cost. Constraint (5.9) indicates that the total whole blood units collected at each collection center must be less than or equal to the number of registered whole blood donors at the collection facilities, respectively. Constraint (5.10) specifies that the total implemented apheresis at the blood center must be no more than the number of available donors registered for apheresis method at blood center. The production capacity limit using the whole blood method and the apheresis method are captured by constraints (5.11) and (5.12), respectively. Constraint (5.13) denotes the total whole blood units allocated to each processing alternatives, where we assume that a given percentage of whole blood donations (ε) does not meet the requirements that has been specified for production and is therefore not usable. These non-conformant blood units cannot be transfused to the patients as they show such abnormal test results as RBC or platelet contamination, underweight, overweight, leakage, and so on (Cobain 2004). Constraints (5.14)-(5.16) show balance equations for the inventory of each blood product with different age groups at the end of each period. Constraint (5.17) indicates the capacity constraint for each blood component at blood center. Constraint (5.18) indicates the amount of blood products of different ages used for satisfying the hospitals' demands. Constraint (5.19) specifies the wastage level at the blood center at the end of each period. Constraint (5.20) is the nonnegativity constraint specifying the sign of decision variables. Let us define N as the vector of all decision variables. N_1 and N_2 as two subsets of N denoting decision variable sets for upper-level and lower-level problems, respectively. Further, assume that upper-level problem has m constraints in total, and lower-level problem has n constraints,

K of which are inequality constraints, and the rest are equality constraints. The upper-level constraints can be reformulated as $C_i(N) \leq 0$ for $i = 1, 2, \ldots, m$. For the lower-level problem, inequality constraints are expressed as $A_k(N) \leq 0$ for $k = 1, 2, \ldots, K$ and equality constraints are shown as $B_l(N) = 0$ for $l = 1, 2, \ldots, n-K$. Then, the bi-level problem has the following structure.

5.4.1.3 Single-Level Reformulation

Solving a bi-level optimization problem is a demanding task (Ben-Ayed et al. 1988). A common technique to deal with a bi-level problem, especially when the lower level is convex, is reformulating the problem as a single-level optimization model. In this research, the lower-level problem is a convex problem, Therefore, the reformulation method is applied by adding Karush–Kuhn–Tucker (KKT) conditions of the lower-level problem, including the stationary and complementary constraints, to the upper-level problem. By considering v_i as dual variable associated with constraint i of lower-level problem, using the KKT optimality conditions, the bi-level problem (P_{BL}) can be reformulated as a single-level problem as follows.

$$P_{BL}: \ \min \ Z_H^D \tag{5.21}$$

$$S.t. \ C_i(N) \leq 0 \qquad i = 1, 2, \ldots, m \tag{5.22}$$

$$\min \ Z_B^D \tag{5.23}$$

$$S.t. \ A_k(N) \leq 0 \qquad k = 1, 2, \ldots, K \tag{5.24}$$

$$B_l(N) = 0 \qquad l = 1, 2, \ldots, n-K \tag{5.25}$$

$$P_{KKTBL}: \ \min Z_h^D \tag{5.26}$$

$$S.t. \ \nabla_{N_2} Z_B^D(N) + \sum_k v_k \nabla_{N_2} A_k(N) + \sum_l v_l \nabla_{N_2} B_l(N) = 0 \qquad \forall N_2 \tag{5.27}$$

$$v_k A_k(N) = 0 \qquad \forall k \tag{5.28}$$

$$C_i(N) \leq 0 \qquad \forall i \tag{5.29}$$

$$A_k(N) \leq 0 \qquad \forall k \tag{5.30}$$

$$B_l(N) = 0 \qquad \forall l \tag{5.31}$$

In the above formulation, constraints (5.27) indicate the stationary constraints and constraints (5.28) are the complementary constraints. To linearize the complementary constraints (5.28), in place of the complementary constraints, the following constraints are added to the single level problem.

$$v_i - M u_i \leq 0 \qquad \forall \, i \tag{5.32}$$

$$A_i(N) + M(1 - u_i) \geq 0 \qquad \forall \, i \tag{5.33}$$

Where M is a very large number and u_i is a binary variable i. The above reformulation is a linear mixed-integer problem which can be solved using commercial solvers such as GAMS, LINGO, and so on. Branch and Bound algorithm in GAMS is used to handle this mixed-integer problem. The following lemma discusses the optimality of single-level reformulated problem.

Proposition 5.1 *When the KKT conditions of lower-level problem are met, P_{BL} is equivalent to P_{KKTBL}, and therefore the optimal solution obtained from reformulated problem using KKT condition (i.e., P_{KKTBL}) is the global minimizer of the original bi-level problem* *(i.e., P_{BL}).*

Proof. See Appendix D.1.

Proposition 5.1 indicates that in our problem, the optimal solution obtained from reformulated single-level problem (P_{KKTBL}) is global minimizer of the objective function, and we can use the reformulated problem for deriving the optimal solutions for bi-level problem.

5.4.2 Centralized (Integrated) Blood Supply Chain Model

In this section, we formulate the centralized model as follows.

$$\begin{aligned}
\text{Min } Z^C = & \sum_{l\in L}\sum_{t\in T} CCW_{l,t} + \sum_{l\in L}\sum_{p\in P}\sum_{t\in T} PC_p X_{l,p,t} + \sum_{l\in L}\sum_{q\in Q}\sum_{t\in T} PC_q AC_{l,q,t} \\
& + \sum_{j\in J}\sum_{t\in T}\sum_{r_j\in R_j} HC_j^B IB_{j,t}^{r_j} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j} HC_{j,h}^H IH_{j,h,t}^{r_j} \\
& + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j} RC_h \alpha_{j,h,t}^{r_j} + \sum_{j\in J}\sum_{t\in T} WC_j^B WB_{j,t} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} SC_j SH_{j,h,t} \\
& + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} WC_{j,h}^H WH_{j,h,t} + \sum_{j\in J}\sum_{j'\in J}\sum_{r_{j'}\in R_{j'}}\sum_{h\in H}\sum_{t\in T} MC_j M_{j,j'} SD_{j,j',h,t}^{r_{j'}} \\
& + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} QC_j D_{j,h,t} + \sum_{j\in J}\sum_{h\in H}\sum_{h'\neq h\in H}\sum_{t\in T}\sum_{r_j\in R_j} TC_{h,h'} TR_{j,h,h',t}^{r_j}
\end{aligned} \tag{5.34}$$

S.t

(5.2) – (5.7) and (5.9) – (5.20)

In the above centralized problem, the objective function is the sum of the upper-level and lower-level objective functions in the decentralized problem discussed in the previous section. Constraints of the centralized problem is the collective constraints of upper-level and lower-level problems in bi-level model. Let us define Z_h^π, Z_H^π, Z_B^π, Z^π as the total costs of hospital h, all hospitals, blood center, and the whole supply chain under model structure $\pi = \{D, C, Co\}$, where D, C, and Co stand for decentralized, centralized, and coordinated systems, respectively (the coordinated system will be introduced and discussed in Section 5.4.3). The following Proposition compares the costs of hospitals, blood center, and the whole supply chain under decentralized model with those under centralized system.

Proposition 5.2 *The costs of hospitals, blood center, and whole supply chain under decentralized and centralized systems satisfy the following relations.*

I. $Z^C \leq Z_H^D + Z_B^D$

II. $Z_H^D \leq Z_H^C$

III. $Z_B^D \geq Z_B^C$

Proof. See Appendix D.2.

Proposition 5.2.I indicates that the total costs under the centralized model is lower than under decentralized system because in a centralized model, total cost of supply chain is minimized with respect to hospitals' and blood center's decisions, while in a bi-level problem, the main objective is minimizing hospitals' costs instead of the total blood supply chain cost.

Proposition 5.2.II implies that the hospitals' total cost under the decentralized model is lower than that under the centralized model. The reason is that in the bi-level model, which is a Stackelberg game, the leader has more power than the follower, and the follower's decisions are made given the feasibility region yielded via the leader's decision. However, in a centralized system hospitals and blood center have the same power and their decisions are made such that the total supply chain cost is minimized.

Furthermore, Proposition 5.2.III shows that integration always benefits the blood center because it benefits the whole supply chain but hurts the hospitals, and hence must benefit the blood center. In fact, in decentralized model, blood center's objective is the lower-level objective of the problem and depends on the hospitals' decisions. However, in a centralized model the blood center's and hospitals' costs are both minimized as the objective of the problem. Proposition 5.2 indicates that hospitals may be reluctant to operate in a centralized system. Thus, in the next section we propose a mechanism to facilitate centralized system implementation.

5.4.3 Coordinated Blood Supply Chain Model

Motivated by the operational challenges that CBS has for integration of blood supply chain, in this section, a coordination mechanism is proposed to motivate hospitals to share their information and operate as if in a centralized system. In this way, the system behaves as if there were a central decision maker who makes optimal centralized decisions for both CBS and hospitals.

As alluded to earlier, in Canada, blood products are free for hospitals, which leads to the undesired effect of order inflation because of the following reasons: first, hospitals want to avoid stock-outs, in which case urgent and costly deliveries need to be made; second, hospitals want to minimize the mismatching cost, associated with blood substitution, which also result in high wastage. CBS is obligated to embark on higher collection and delivery than the actual needs. To resolve this problem, we propose a coordination mechanism to motivate hospitals to operate as they would do in a centralized system, i.e., their optimal decisions will be the same as their optimal decisions in the centralized system.

Such an incentive, however, is viable only if the cost incurred at the hospitals less the incentive is no higher than their costs in the decentralized system. We postulate that CBS provide a subsidy to each hospital for each unit of shortage and mismatching cost such that both stakeholders can benefit from integration. Furthermore, based on the cost at each hospital, CBS can equitably distribute subsidy. It is important that the form of subsidy would depend on the blood supply chain structure. For instance, in the United States, blood components are purchased by the hospitals, and hence subsidy can be interpreted as funding provided by blood collection agencies to motivate hospitals to share their actual demand (Paul et al. 2019). However, in countries such as Canada where blood units are free for hospitals, subsidy can be implemented as free same-day urgent deliveries of blood components to ensure that hospitals place orders based on predicted demand without being concerned about compromising on availability of blood components (Li et al. 2021).

Remark 5.1 *In the coordinated blood supply chain, the optimal decisions by the hospitals and CBS are the same as the optimal decisions in a centralized system. Also, CBS provides each hospital with a subsidy to compensate for additional shortage and mismatching costs incurred by the hospital because of integration. Hence, in a coordinated system, there exists a subsidy level such that both hospitals and CBS have lower costs than in the decentralized model.*

To model this coordination mechanism, we use the notation introduced in the previous section. To create incentive for integration, blood center can derive the range for the total amount of subsidy provided to hospital h, i.e., s_h, as follows. It should be mentioned that subsidy is a fixed parameter that does not depend on decision variables. To coordinate the blood supply chain, it must satisfy the following conditions (5.35)-(5.36).

$$Z_h^{Co} = Z_h^C - s_h \leq Z_h^D \tag{5.35}$$

$$Z_B^{Co} = Z_B^C + \sum_{h \in H} s_h \leq Z_B^D \tag{5.36}$$

Thus, based on inequalities (34) and (35), the lower-bound for total subsidy provided by blood center to hospital h can be obtained as follows.

$$s_h \geq Z_h^C - Z_h^D \tag{5.37}$$

Also, the upper-bound for total subsidies provided to hospitals can be obtained as follows.

$$\sum_{h \in H} s_h \leq Z_B^D - Z_B^C \tag{5.38}$$

For any $s_h \in \left[\underline{s_h}, \overline{s_h}\right]$, where $\underline{s_h}$ and $\overline{s_h}$ can be determined according to relations (5.37) and (5.38) such that $\sum_{h \in H} \overline{s_h} = Z_B^D - Z_B^C$, the hospitals are willing to accept the coordination contract and the blood center is willing to offer it. Therefore, both hospitals and blood center can benefit from this coordination contract, and coordination is accomplished. It should be mentioned that the amount of subsidy needed to coordinate the supply chain may vary for different hospitals and depends on their unit shortage and mismatching costs. In fact, the higher the unit shortage and mismatching costs at a hospital, the higher the subsidy it needs to receive. The detailed discussion on the performance of the proposed coordination mechanism is presented in section 5.5.

A potential challenge associated with the proposed subsidy lies in its real-world applicability. Blood products are considered critical items with clinical implications, making a simple monetary compensation for shortages ethically questionable. It is crucial to acknowledge that the interpretation of the subsidy can vary across countries. In the United States, where hospitals purchase blood and have multiple supplier options, the subsidy could be seen as a monetary value offered by blood centers in exchange for centralization. Conversely, in Canada, where blood products are provided to hospitals for free, the subsidy can be viewed as emergency orders that blood centers are obligated to fulfill in the event of shortages within the coordinated model.

5.4.4 Stochastic Centralized (Integrated) Blood Supply Chain Model

Demand for different blood products can be quite unpredictable. In this research, we model uncertainty by the scenario-based method, where we express uncertain factors as a set of specific scenarios, each with its own probability of happening. To model the uncertain version of the problem, we define a set of scenarios represented as $S = 1, 2, ..., s$, each with an associated probability denoted as P_s. We also use the notation $d^s_{j,h,t}$ to represent the average patient demand for a blood product j at hospital h in period t under scenario s. Furthermore, we introduce the following notations for decision variables that consider the uncertainty.

Decision Variables

$IH^{r_j,s}_{j,h,t}$	Inventory level of product j with age r_j at hospital h in period t under scenario s
$WH_{j,h,t,s}$	Wastage level of product j at hospital h in period t under scenario s
$SH_{j,h,t,s}$	Shortage level of product j at hospital h in period t under scenario s
$SD^{r'_j,s}_{j,j',h,t}$	Quantity of demand for product j of age r_j that is fulfilled by product j' of age r'_j in period t under scenario s
$TR^{r_j,s}_{j,h,h',t}$	Quantity of product j with age r_j that is transshipped from hospital h to hospital h' in period t under scenario s

Using these notations, we formulate the stochastic integrated model, which we refer to as an expected value approach.

$$\begin{aligned}\text{Min}\, Z^S = &\sum_{l\in L}\sum_{t\in T} CCW_{l,t} + \sum_{l\in L}\sum_{p\in P}\sum_{t\in T} PC_p X_{l,p,t} + \sum_{l\in L}\sum_{q\in Q}\sum_{t\in T} PC_q AC_{l,q,t} \\ &+ \sum_{j\in J}\sum_{t\in T}\sum_{r_j\in R_j} HC^B_j IB^{r_j}_{j,t} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j}\sum_{s\in S} HC^H_{j,h} IH^{r_j,s}_{j,h,t} \\ &+ \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j} RC_h \alpha^{r_j}_{j,h,t} + \sum_{j\in J}\sum_{t\in T} WC^B_j WB_{j,t} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{s\in S} SC_j SH^s_{j,h,t} \\ &+ \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{s\in S} WC^H_{j,h} WH^s_{j,h,t} + \sum_{j\in J}\sum_{j'\in J}\sum_{r_{j'}\in R_{j'}}\sum_{h\in H}\sum_{t\in T}\sum_{s\in S} MC_j M_{j,j'} SD^{r_{j'},s}_{j,j',h,t} \\ &+ \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} QC_j D_{j,h,t} + \sum_{j\in J}\sum_{h\in H}\sum_{h'\neq h\in H}\sum_{t\in T}\sum_{s\in S}\sum_{r_j\in R_j} TC_{h,h'} TR^{r_j,s}_{j,h,h',t}\end{aligned} \tag{5.39}$$

$S.t$

$$\begin{aligned} IH^{r_j,s}_{j,h,t} =& IH^{r_j-1,s}_{j,h,t-1} + \alpha^{r_j}_{j,h,t} - \sum_{j' \in J} SO_{j,j'} SD^{r_j,s}_{j',j,h,t} \\ &+ \sum_{h' \neq h \in H} TR^{r_j,s}_{j,h',h,t} - \sum_{h' \neq h \in H} TR^{r_j}_{j,h,h',t,s} \end{aligned} \qquad \forall\, j,h,t,r_j,s \quad (5.40)$$

$$\sum_{j' \in J} \sum_{r_{j'} \in R_{j'}} SO_{j,j'} SD^{r_{j'},s}_{j,j',h,t} = d^s_{j,h,t} - SH^s_{j,h,t} \qquad \forall\, j,h,t,s \quad (5.41)$$

$$\sum_{j \in G(j)} \sum_{r_j \in R_j} IH^{r_j,s}_{j,h,t} \leq CH_{g,h} \qquad \forall\, t,g,h,s \quad (5.42)$$

$$IH^{r_j,s}_{j,h,t} = WH^s_{j,h,t} \qquad \forall\, j,h,t,r_j = U_j,s \quad (5.43)$$

$$TR^{r_j,s}_{j,h,h',t} = 0 \qquad \forall\, j,h,t,h' \neq h, r_j \leq \gamma_j, s \quad (5.44)$$

$$D_{j,h,t}, IH^{r_j,s}_{j,h,t}, WH^s_{j,h,t}, SH^s_{j,h,t}, SD^{r'_j,s}_{j,j',h,t}, TR^{r_j,s}_{j,h,h',t} \geq 0 \qquad (5.45)$$

$$(5.8) - (5.20)$$

Next, we create a robust optimization problem to handle uncertainty in our problem effectively.

5.4.4.1 Solution Approach: Robust Optimization Model

In this section, we adopt and tailor the robust model presented by Aghezzaf et al. (2010) to our setting. The objective function developed by Aghezzaf et al. (2010) is presented as follows.

$$minZ^R = \eta \max_{s \in S} (\xi_s - \xi^\star_s) + \lambda \sum_{s \in S} P_s \, \xi_s \qquad (5.46)$$

where the above objective function minimizes the maximum variability (i.e., $\max_{s \in S} (\xi_s - \xi^\star_s)$) and the expected cost under all scenarios (i.e., $\sum_{s \in S} P_s \, \xi_s$). $\xi^\star_s$ represents the optimal objective value of the deterministic model by considering demand $d^s_{j,h,t}$ under scenario s. In addition, ξ_s is defined as the optimal cost under scenario s in the stochastic problem. Further, let us define $\delta^{s}_{j,h,t}$ as the under-fulfillment of demand for product j at hospital h in period t under scenario s and SC^{Robust}_{j} as the penalty cost for constraint violation or under-fulfillment of demand for product j. The robust optimization model for our stochastic problem can be formulated as follows.

$$\text{Min}\, Z^R = \eta \max_{s\in S} (\xi_s - \xi^{\star}_s) + \lambda \sum_{s\in S} P_s\, \xi_s \tag{5.47}$$

$S.t$

$$\begin{aligned}
\xi_s =& \sum_{l\in L}\sum_{t\in T} CCW_{l,t} + \sum_{l\in L}\sum_{p\in P}\sum_{t\in T} PC_p X_{l,p,t} + \sum_{l\in L}\sum_{q\in Q}\sum_{t\in T} PC_q AC_{l,q,t} \\
&+ \sum_{j\in J}\sum_{t\in T}\sum_{r_j\in R_j} HC^B_j IB^{r_j}_{j,t} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j}\sum_{s\in S} HC^H_{j,h} IH^{r_j,s}_{j,h,t} \\
&+ \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{r_j\in R_j} RC_h \alpha^{r_j}_{j,h,t} + \sum_{j\in J}\sum_{t\in T} WC^B_j WB_{j,t} + \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{s\in S} SC^{Robust}_j SH^s_{j,h,t} \\
&+ \sum_{j\in J}\sum_{h\in H}\sum_{t\in T}\sum_{s\in S} WC^H_{j,h} WH^s_{j,h,t} + \sum_{j\in J}\sum_{j'\in J}\sum_{r_{j'}\in R_{j'}}\sum_{h\in H}\sum_{t\in T}\sum_{s\in S} MC_j M_{j,j'} SD^{r_{j'},s}_{j,j',h,t} \\
&+ \sum_{j\in J}\sum_{h\in H}\sum_{t\in T} QC_j D_{j,h,t} + \sum_{j\in J}\sum_{h\in H}\sum_{h'\neq h\in H}\sum_{t\in T}\sum_{s\in S}\sum_{r_j\in R_j} TC_{h,h'} TR^{r_j,s}_{j,h,h',t}
\end{aligned} \tag{5.48}$$

(5.8) – (5.20) and (5.40) – (5.45)

SC^{Robust}_j is a penalty cost for violating demand fulfillment constraint which helps us to find the optimal solution, referred to solution robustness, and feasible solution, referred to model robustness. This penalty cost is also known as the risk-aversion weight which determines the strategy of decision maker. In robust optimization concept, choosing an appropriate risk aversion weight is important to make a balance between model robustness and solution robustness (Jabbarzadeh et al. 2014). When central blood center is risk-averse, it strongly avoids stock-out situations and prefers a higher degree for under-fulfillment penalty cost. On the other hand, when the blood center takes the risk of stock-out situation, it might be more interested in minimizing the total expected cost and variation between total costs in different scenarios than demand underfulfillment. In this work, because blood components are sensitive products dealing with peoples' lives, decision maker is risk-averse and avoids stock-out situation. We further discuss the choice of robustness parameters in section 5.5.

5.4.4.1.1 Linearization In the objective function of the robust model, there's a non-linear term $\max_{s\in S}(\xi_s - \xi_s^\star)$. To linearize this term, we set $\max_{s\in S}(\xi_s - \xi_s^\star) = \Phi$ and add constraint $\xi_s - \xi_s^\star \leq \Phi$ to the optimization problem. Therefore, the linearized robust optimization problem can be rewritten as follows.

$$\text{Min}\, Z^R = \eta\ \Phi + \lambda \sum_{s\in S} P_s\ \xi_s \tag{5.49}$$

$S.t$

$$\xi_s - \xi_s^\star \leq \Phi \qquad \forall s \tag{5.50}$$

$$(5.8)-(5.20),\ (5.48),\ (5.40)-(5.45), \text{and } \Phi \geq 0$$

5.5 Experimental study

In this section, we first outline the inputs of the case study, and then perform numerical experiments to demonstrate the benefits of a centralized decision-making system, the performance of coordinated system, the value of substitution between blood groups, the value of collaboration between hospitals, comparison between different issuing and replenishment policies, and an analysis of model sensitivity to various input parameters.

5.5.1 Case Study Setting

We apply the proposed optimization models to conduct a case study based on the blood supply chain in the province of Ontario (Canada), with a focus on the interaction between the CBS location in the city of Brampton and the four hospitals in the city of Hamilton (Figure 5.4). The four hospitals are: Hamilton General (HG); Juravinski Hospital (JH); McMaster University Medical Center (MUMC); and, St. Joseph's Healthcare (STJ). Some input data are collected from peer reviewed works and publicly available sources, and appropriate rationale is provided when we make assumptions on some parameters of the models. As indicated earlier, this study only focuses on the two blood components of limited shelf life, i.e., Red Blood Cells (RBC) and Platelets (PLT). The requisite input data can be organized under supply and demand parameters.

5.5.1.1 Supply parameters

Based on the work of Drackley et al. (2012), we estimate about 48,000 units of whole blood donations in the province of Ontario in 2021, which can then be prorated based on population for the city of Hamilton. The parameters associated with collection and processing of blood products that are borrowed from Osorio et al. (2018) are shown in Table 5.1, and we assume that each machine is available for 8 hours each day (i.e., 480 minutes). In line with the extant literature, we assume that the lead time associated with the apheresis method is 0 and that with the whole blood method is 2 days (Ensafian and Yaghoubi 2017, Samani et al. 2020). Finally, transshipment between different hospitals is allowed and transshipment thresholds for RBC and PLT are assumed to be 21 and 4, respectively.

5.5.1.2 Demand parameters

Weekday demand estimations for PLT products at Hamilton hospitals are borrowed from the paper by Abdulwahab and Wahab (2014). Also, we make use of the work by Li et al. (2021) to estimate weekday demand for RBC products at Hamilton hospitals and use the respective hospital capacities to estimate individual demand. In the stochastic model, we explore five distinct scenarios, each representing demand variations. The demand values in each scenario are multiples of the original demand, with factors of 0.25, 0.5, 1, 1.25, and 1.75, occurring with 20% chance.

5.5.1.3 Cost and capacity estimations

We assume transportation cost as \$0.5 (in Canadian dollars) per mile and use distance between CBS and each hospital to estimate their corresponding transportation costs. Similarly, we obtain transshipment cost between two hospitals. Because in Canada, blood products are free for hospitals, purchase costs of blood products from CBS are assumed to be zero. Holding cost for RBC and PLT products is set as \$0.5 per unit. Holding and transportation costs are considered the same for RBC and PLT products, while the wastage costs for RBC and PLT products are proportional to the prices of the corresponding blood components. Price of the PLT components is approximately 1.7 times the price of RBC components; therefore, wastage cost for platelet products is considered as 1.7 times wastage cost for RBC products, which is set to \$20 per unit. The shortage cost for RBC and PLT products are assumed to be \$200 per unit of unsatisfied demand. This assumption is in line with the work of Ensafian and Yaghoubi (2017) that considered shortage cost as ten times the outdating cost. It is worth mentioning that the performance of presented model is investigated with respect to the cost ratios, rather than the absolute values of the costs.

Exact matching incurs no cost on hospitals. However, satisfying demand with inexact blood component imposes some costs on hospitals depending on the matching priorities. To obtain mismatching cost we make use of compatibility and prioritization matrices ($M_{j,j'}$) for RBC and PLT provided in Tables 5.2-5.5. Mismatching cost for the first substitution priority is $\$50/8$ which increases with the increase of priority number. The whole blood production unit cost, which includes the cost of testing and processing, is estimated to be around \$100. The production unit cost of apheresis platelets replies on the product split rate, labor costs, cost of the kits used for collection and the method used for bacterial testing. On average, it is estimated to be \$600 at CBS.

It is assumed CBS has the storage capacity of 5000 blood units and each hospital has the storage capacity of 500 blood units. Capacity of CBS is assumed to be sufficiently large to accommodate blood components at least as much as the aggregated maximum capacity of all hospitals.

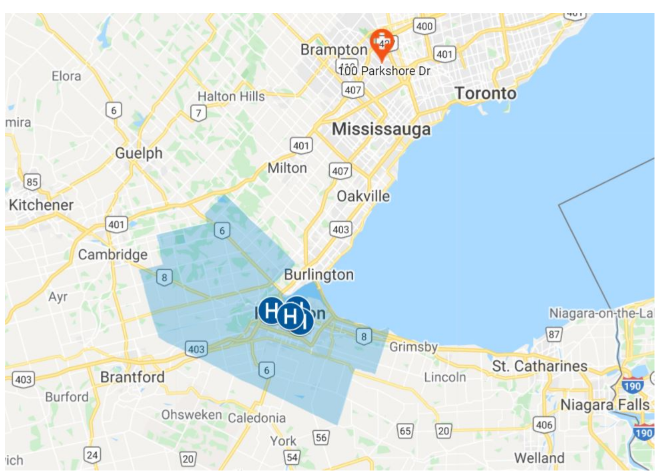


Figure 5.4: Canadian blood services and hospitals in Hamilton region

5.5.2 Performance Measures

To illustrate the applicability of the presented models in real-world, we use the above data to solve the model in GAMS software. To evaluate the performance of the model, we use the following measures: TWL denotes the total wastage level in the blood supply chain; TSL denotes the total shortage level in the blood supply chain; TP indicates the total quantity of blood units produced using whole blood method; $TAPH$ indicates the total quantity of blood units produced using the apheresis method; TR is the total replenishment quantity of blood components at hospitals; and, FL is the average freshness level of transfused blood components. The three cost components of relevance are as follows: ZB is the total cost of CBS; ZH is the total cost of hospitals; and Z is the total cost of the blood supply chain.

Table 5.7: Computational results of different models

Model	**Performance Measures**						
	TWL	TSL	TP	$TAPH$	TR	FL	$Z(\$)$
Decentralized Model	153.720	0	487.476	66	943.398	0.609	95242.79
Centralized Model	31.218	2.455	446.839	0	940.942	0.612	49318.25
Model without Substitution	24.064	28.419	435.855	427.138	951.926	0.587	59027.85

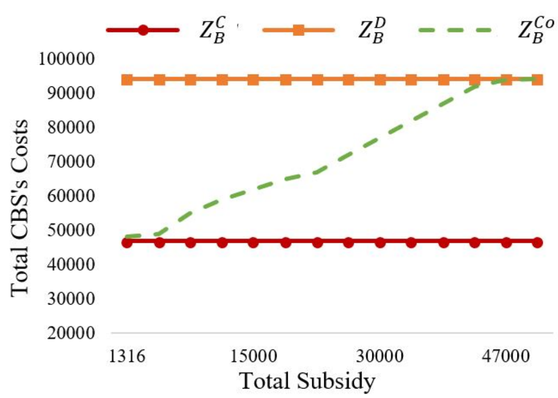
The results obtained for different models, including decentralized model, centralized model with substation, and centralized model without substitution, are presented in Table 5.7. For the model with substitution, we use compatibility matrices presented in Tables 5.2 and 5.3, and for the model without substitution, compatibility matrices for different blood products are considered as identity matrices. Based on the results, the total incurred costs are considerably lower under the centralized system than the decentralized system. Further, because of high demands of hospitals, to be fulfilled by CBS, total collection and production levels are higher under the decentralized system than the centralized system. As a result, wastage level is lower under the centralized system, while depending on the unit shortage cost, the decentralized model may lead to a lower shortage level. Moreover, the results suggest that one can improve the performance of the system by allowing substitution. In next subsections, through extensive sensitivity analysis, some managerial insights are provided.

5.5.3 Value of Integration

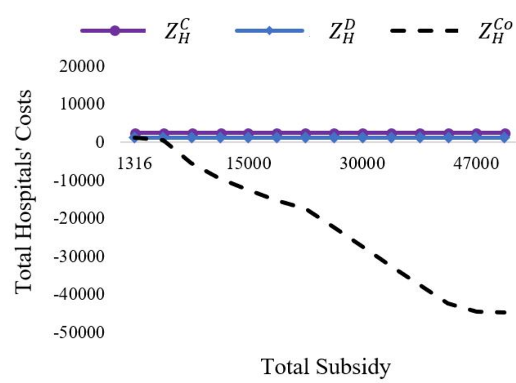
The total cost of the system, under the decentralized setting, is significantly higher than that under the centralized setting. However, given the leader role of the hospitals, the incurred cost is lower under the decentralized structure because hospitals seek to minimize their total cost and do not pay for replenishment or purchase. Unfortunately, the decentralized structure has a negative impact on all pertinent performance measures (Table 5.8). For instance, the increased amount of collection and production in the decentralized model not only results in higher production and replenishment quantities but also in higher wastage costs vis-à-vis the centralized model. However, the shortage level might be higher in the centralized system when shortage cost per unit is low. In addition, the average freshness level is higher in the decentralized system, and both structures converge to similar freshness levels at higher shortage costs. The reason is that in a centralized system, the objective is minimizing the total costs as opposed to prioritizing the minimization of the hospitals' costs as in the decentralized model. Therefore, when shortage cost is low, the optimal shortage level is higher, and the optimal freshness level is lower in the centralized system than decentralized system. Because the health care system in Canada aims to minimize the total costs of the system, the centralized system has a better performance and should be implemented using appropriate information sharing infrastructures between CBS and the hospitals. However, anticipating possible reluctance on the part of the hospitals, we next quantify the amount of subsidy that CBS can offer to the hospitals to incentivize coordination. Figure 5.5 exhibits how the amount of subsidy within the specified range would impact the costs at CBS and the hospitals (i.e., (a) and (b)), and underscores benefits for both parties. Based on Figure 5.5 (a), by increasing the total subsidy between its lower-bound and upper-bound (i.e., $\$1,316$ and $\$47,242$, respectively), CBS's costs ranges between the costs of decentralized and centralized systems, while the gap between total hospitals' costs under decentralized and coordinated model increases. Offering a total subsidy between $\$1,316$ and $\$47,242$ can benefit both blood center and hospitals. However, providing hospitals with an excessively high subsidy may not be necessary as it results in a considerable gap between hospitals' costs under coordinated and decentralized model and increases CBS's cost at the same time. For example, if the amount of subsidy is $\$2,570$, then the total hospitals' costs under centralized model will be equal to that under decentralized model, and CBS will incur $\$49,318$, while by offering a subsidy of $\$45,000$, total hospitals' and CBS's costs will be equal to $\$-42,482$ and $\$91,800$, respectively, which indicates that by offering an unnecessary high subsidy of $\$45,000$, hospitals make profit while CBS incurs a huge costs although the system remains coordinated.

Table 5.8: Comparing decentralized and centralized channel structures

Model	% Change in Shortage Costs	Performance Measure								
		TWL	TSL	TP	$TAPH$	TR	FL	ZB(\$)	ZH(\$)	Z(\$)
Centralized	−75%	31.22	705.07	0.00	0.00	238.32	0.09	813.78	35614.46	36428.24
	−50%	31.22	193.52	255.78	0.00	749.88	0.54	27179.00	22998.79	48389.79
	0%	31.22	2.46	446.84	0.00	940.94	0.61	46800.80	8305.68	49318.25
	+50%	31.22	2.46	446.84	0.00	940.94	0.61	46809.93	12400.96	49563.84
	+75%	31.22	0.00	446.84	1.23	943.40	0.61	47546.41	13595.94	49565.89
Decentralized	−75%	153.72	0.00	487.48	66.00	943.40	0.61	94041.74	1201.05	95242.79
	−50%	153.72	0.00	487.48	66.00	943.40	0.61	94041.74	1201.05	95242.79
	0%	153.72	0.00	487.48	66.00	943.40	0.61	94041.74	1201.05	95242.79
	50%	153.72	0.00	487.48	66.00	943.40	0.61	94041.74	1201.05	95242.79
	75%	153.72	0.00	487.48	66.00	943.40	0.61	94041.74	1201.05	95242.79



(a)



(b)

Figure 5.5: Illustration of CBS and hospitals costs under coordination system between lower-bound and upper-bound for the total subsidy

Another benefit of integration is evident in the gap between the amount of blood components produced (received) and transfused (Figure 5.6). Bars associated with different days of the week (Sat., Fri., Thu., Wed., Tue., Mon., Sun.) are sorted based on their order from up to down. The gap is smaller under the centralized structure thanks to the simultaneous decisions regarding collection, production, and transfusion. Our results show that in decentralized system, mean, standard deviation, and maximum gap between the quantity of production and transfusion are 4.46, 48.48, and 109.52, respectively, while in a centralized system those values are obtained as 74.30, 104.93, and 248.81, respectively.

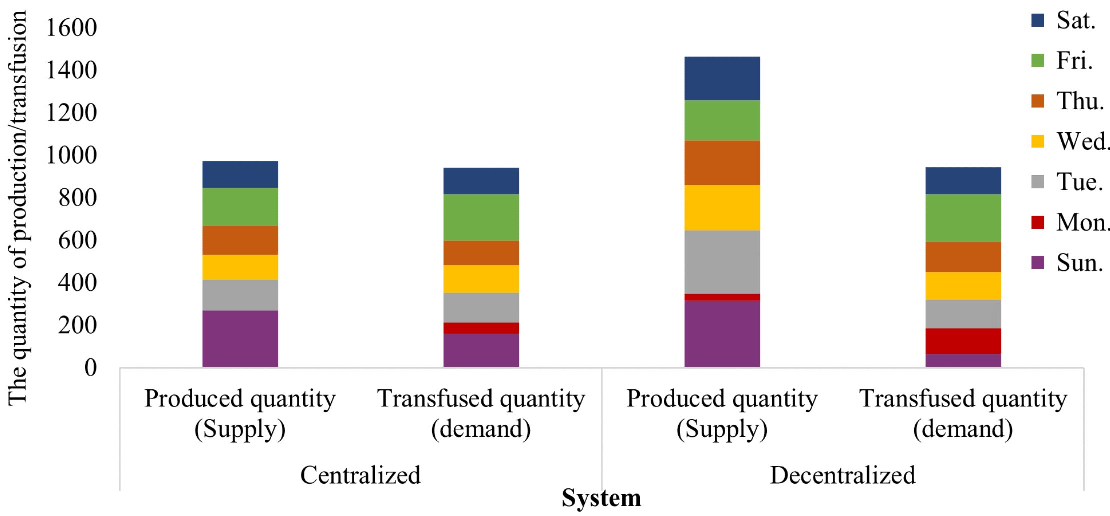


Figure 5.6: Production and transfusion quantities in a week under two channel structures

5.5.4 Substitution between Blood Groups

The effect of substitution between blood groups is depicted in Table 5.9. When substitution is allowed between blood groups, as in the real-world, shortage level for a product can decrease when exact matching for that product is unavailable. Also, to avoid wastage of some blood units that near expiration, inexact substitution can be practiced. Thus, as expected, in the model with substitution, shortage and wastage level decrease. Moreover, by considering substitution between different blood groups, the amount of collection and production and average freshness level slightly increase, while total cost decreases. It is noticeable that when substitution is not allowed and shortage cost is high, the apheresis method, which is more costly than whole blood collection, is applied in addition to the whole blood collection method for satisfying patient demand. The reason is that whole blood donations are not sufficient for satisfying demands. In general, substitution can considerably improve the performance of the blood supply chain.

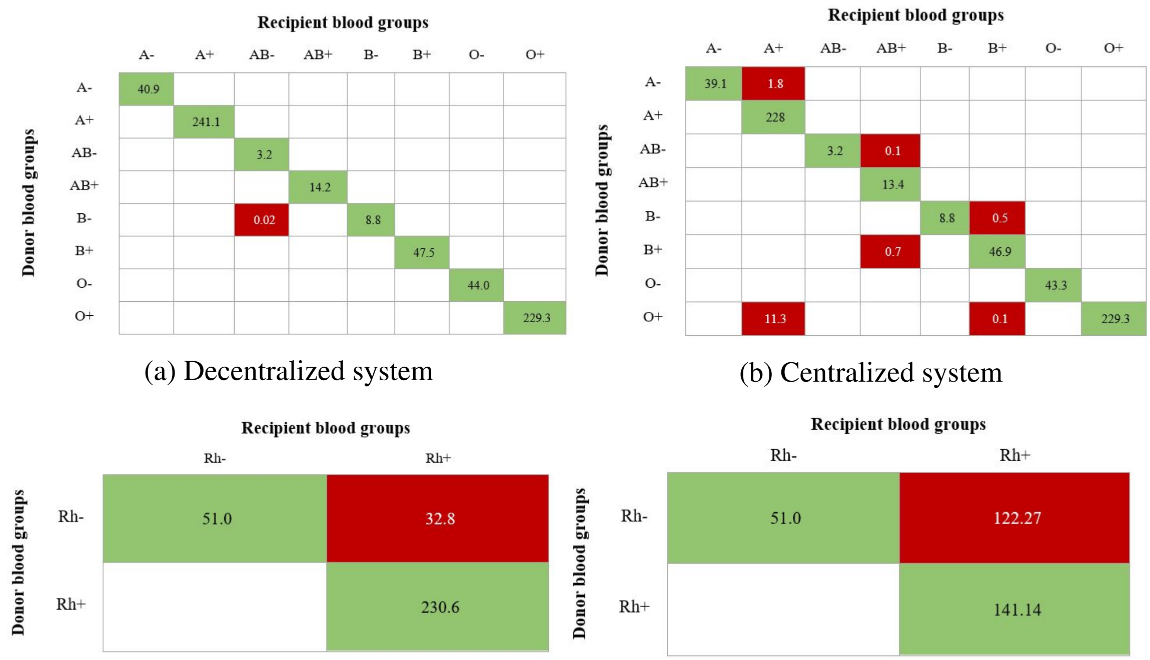
Table 5.9: The effect of substitution on different performance measures

Model	% Change in Shortage Costs	Performance Measure								
		TWL	TSL	TP	$TAPH$	TR	FL	ZB($)	ZH($)	Z($)
With Substitution	−75%	31.218	705.074	0.000	0.000	238.324	0.086	813.781	35614.463	36428.245
	−50%	31.218	193.517	255.778	0.000	749.881	0.535	27178.999	22998.787	48389.791
	0%	31.218	2.456	446.839	0.000	940.942	0.612	46800.796	8305.679	49318.248
	+50%	31.218	2.456	446.839	0.000	940.942	0.612	46809.931	12400.959	49563.845
	+75%	31.218	0.000	446.839	1.228	943.398	0.613	47546.407	13595.936	49565.889
Without Substitution	−75%	31.878	705.734	0.000	0.000	237.664	0.090	833.200	35703.790	36536.990
	−50%	31.878	194.177	255.778	0.000	749.221	0.536	27176.519	21457.925	50599.476
	0%	31.878	13.871	436.084	0.000	929.527	0.610	45692.928	4837.578	56425.605
	50%	31.878	13.871	436.084	0.000	929.527	0.610	45691.103	6226.548	61742.814
	75%	31.878	0.000	436.084	6.936	943.398	0.615	49850.671	2079.220	63720.088

Considering mismatching cost in this setting allows hospitals to use better match for satisfying a demand with a given blood group. Table 5.10 shows the effect of substitution cost on performance measures. If unit mismatching cost increases, the amount of demand satisfied with inexact match decreases, resulting in a higher shortage cost. In the decentralized system, when mismatching cost increases, wastage level increases, too. The reason is that when unit mismatching cost gets higher than shortage cost, hospitals prefer to not satisfy demand over satisfying demand by incurring mismatching cost. Thus, the amount of replenishment decreases and shortage increases. As strictness on substitution between different blood groups directly depends on the mismatching cost it is wise to choose an appropriate mismatching cost to allow some products to be substituted. The acceptable threshold of mismatching cost is affected by hospital policy on the medical and inventory management performance. The results show that total hospitals' cost and supply chain cost increase with an increase in the mismatching cost. Figure 5.7 provides information about the amount of demand satisfied by exact matching and substitution, shaded using light green and dark red colors, respectively, for RBC and PLT. It suggests that the major proportion of demand is satisfied with exact match to avoid mismatching cost. For both RBC products (i.e., (a) and (b)) and PLT products (i.e., (c) and (d)) the centralized model substitutes more products compared to the decentralized model. This is because in the decentralized system, hospitals satisfy more demand from exact matching to reduce the mismatch cost. In contrast, in the centralized system, the mismatch cost carries a smaller weight in the objective function, and therefore products are substituted.

Table 5.10: Sensitivity of performance measures to mismatching cost

Model	Unit Mismatching cost	Performance Measure								
		TWL	TSL	TP	$TAPH$	TR	FL	$ZB(\$)$	$ZH(\$)$	$Z(\$)$
Centralized	**0**	31.218	2.456	446.839	0.000	940.942	0.612	46826.537	562.302	47388.839
	200	31.218	2.456	446.839	0.000	940.942	0.612	46837.967	8268.509	55106.475
	400	31.878	13.761	436.195	0.000	929.637	0.610	45736.863	17021.960	62758.823
	600	31.878	13.761	436.195	9.834	929.637	0.611	51645.492	16835.975	68481.467
	800	31.878	41.795	436.195	9.834	901.603	0.612	51649.718	21462.547	73112.265
	1000	33.928	64.755	441.987	9.834	878.643	0.605	52259.094	22875.581	75134.675
Decentralized	**0**	137.606	0.000	487.476	66.000	943.398	0.633	93669.643	0.000	93669.643
	200	153.720	0.000	487.476	66.000	943.398	0.609	94041.739	4804.212	98845.951
	400	164.631	2.456	487.476	66.000	940.942	0.615	94272.801	9485.625	103758.427
	600	164.631	2.456	487.476	66.000	940.942	0.615	94272.801	13982.842	108255.643
	800	170.082	29.625	487.476	66.000	913.773	0.622	94379.025	18480.058	112859.082
	1000	179.653	53.467	487.476	66.000	889.931	0.628	94583.061	20426.742	115009.803



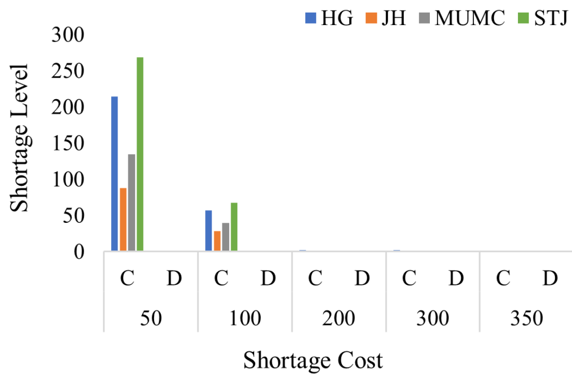
(a) Decentralized system (b) Centralized system

(c) Decentralized system (d) Centralized system

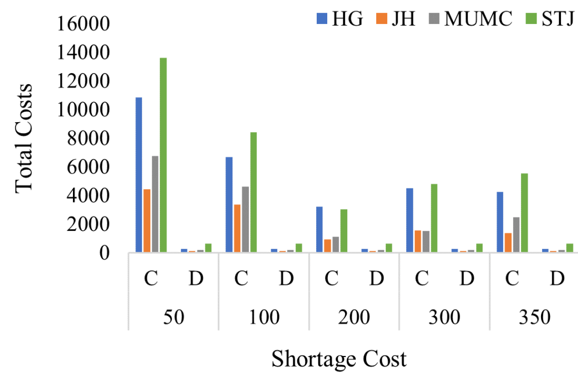
Figure 5.7: Amount of PLT demand satisfied by exact matching and substitution under different structures

5.5.5 Comparing Performance Measures at Different Hospitals

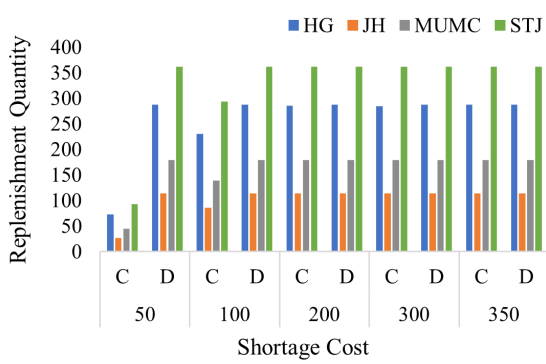
The indicators of interest to the hospitals in Hamilton are exhibited in Figure 5.8, in which bars representing HG, JH, MUMC, STJ are sorted from left to right, respectively. Figure 5.8 (a) and 5.8 (b) depict the shortage level and total costs, while Figure 5.8 (c) and 5.8 (d) highlight the replenishment quantities and lower bound of total subsidy at the four hospitals under centralized and decentralized channel structures denoted by C and D, respectively. Based on results, St. Joseph's Healthcare Hamilton (STJ) has the highest replenishment quantity and lowest shortage level. Also, total subsidy required for coordination is higher at STJ than other hospitals. In contrast, Juravinski Hospital (JH) has the lowest replenishment quantity and highest shortage level. Further, JH requires the lowest amount of subsidy for coordination. These results are in harmony with the capacities of the hospitals. Moreover, when shortage cost per unit increases to 350, no hospital experiences shortage in both channel structures. Also, when shortage cost is high, replenishment quantity is the same under centralized and decentralized systems.



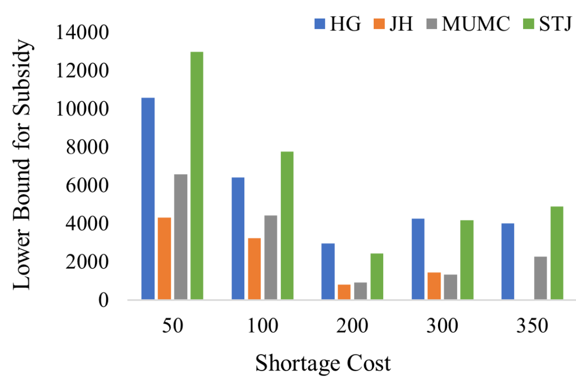
(a) Decentralized system



(b) Centralized system



(c) Decentralized system



(d) Centralized system

Figure 5.8: Shortage level, total costs, replenishment quantity, and lower-bound for the optimal subsidy in different hospitals

5.5.6 Performance of Robust Optimization Model

5.5.6.1 Choice of Robust Optimization Parameters

Selecting the right risk-aversion parameter is of high importance as it determines the strategy of the decision maker. Initially, we explore various values of SC_j^{Robust} to examine the trade-off between solution and model robustness. Figure 5.9 shows the change in solution and model robustness with respect to the underfulfillment cost per unit. As SC_j^{Robust} increases, the blood center becomes more risk averse. Consequently, model robustness, indicating under-fulfillment of demand, decreases. Conversely, solution robustness, representing the total expected costs, increases. This occurs because avoiding risks and satisfying more demands entail additional costs to the system. This observation suggests that when the blood system leans towards risk aversion, it tends to seek feasible solutions across all possible scenarios. In contrast, a blood center willing to take more risks might accept some infeasible solutions in certain scenarios to achieve lower total costs. In our research, as decision makers inclined towards risk aversion, we adopt a strategy that minimizes the risk of shortages. We set SC_j^{Robust} to 3050, which results in zero shortages. Notably, fully satisfying all demands incurs a substantial cost on the blood supply chain, resulting in a total cost of 84864.025. For subsequent calculations, we maintain SC_j^{Robust} at 3050.

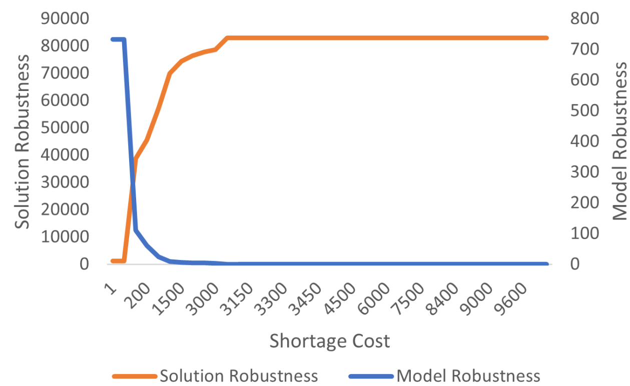
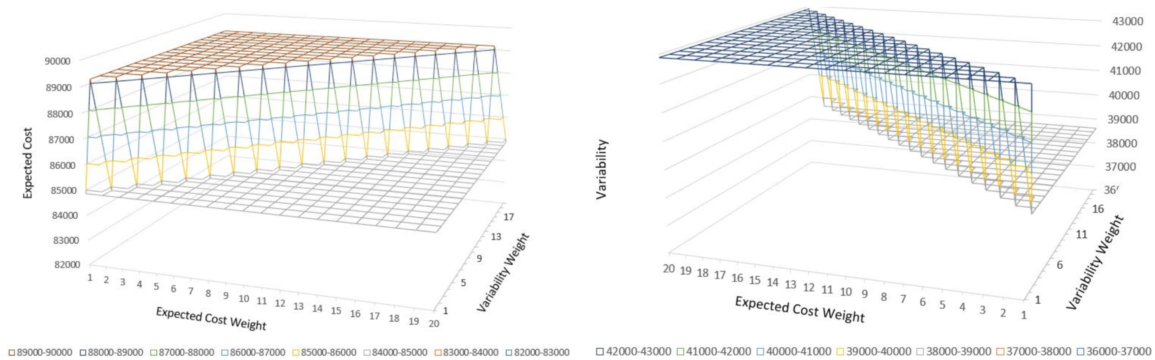


Figure 5.9: Solution robustness and model robustness trade-off

Next, to obtain appropriate variability weight (η) and expected cost wight (λ), we change these two weights and observe the performance of system in terms of expected cost, variability, shortage and wastage level, and freshness level. The sensitivities of expected cost and variability with respect to η and λ are provided in Figures 5.10 (a) and 5.10 (b). It is worth noting that our approach differs from previous studies, such as Ensafian and Yaghoubi (2017) and Jabbarzadeh et al. (2014), which conducted one-dimensional analyses to determine the optimal robustness parameters. In contrast, in our research, we opt for a more precise approach by conducting a two-dimensional analysis, simultaneously adjusting both η and λ as illustrated in Figures 5.10 (a) and 5.10 (b).



(a) (b)

Figure 5.10: Sensitivity of expected cost and variability to expected cost and variability weights

Based on these Figures 5.10 (a) and 5.10 (b), as variability weight increases, the variability decreases and the expected cost increases, while increasing expected cost wight results in increasing variability and decreasing expected cost. In this case-study, we choose $\eta = 17$ and $\lambda = 12$ as it provides a balance between variability and expected cost where both variability and expected cost are low at the same time and the average freshness level is at the highest possible amount.

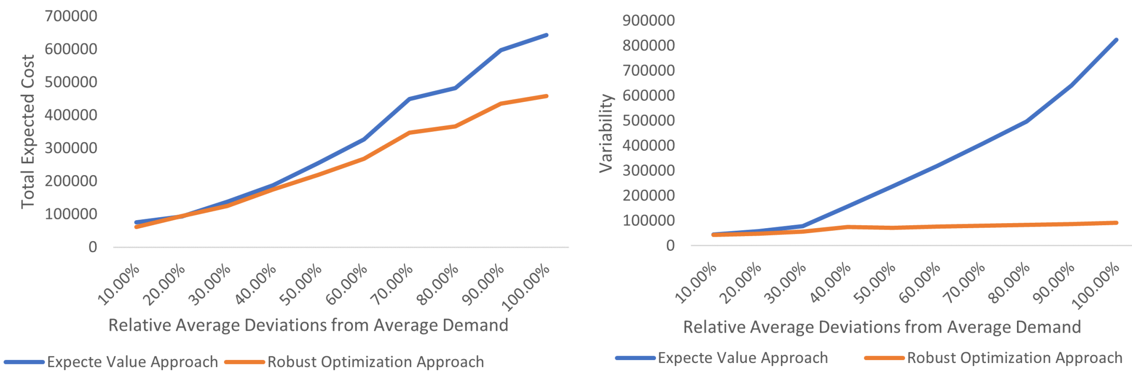
5.5.6.2 Robust Optimization Method Vs. Expected Value Method

In this section, we aim to assess the effectiveness of the robust optimization approach by comparing it to the expected value approach. In the expected value approach, we first obtain the expected values for uncertain parameters, and then use deterministic model in which the values of the random parameters are set equal to their expected values. The difference between the object value under expected value and the objective value under robust stochastic programming yields the value of stochastic solution. To examine this value, we change the relative average deviations from the average demand under different scenarios.

In other words, we generate different demand scenarios categorized by different relative average deviations using the data we detailed in the case study section. Subsequently, we solve both the robust optimization and expected value models for these scenarios and examine the expected cost and variability. For each deviation category, the robust optimization parameters are chosen based on the strategy described above. The outcomes are illustrated in Figure 5.11.

Generally, the robust optimization model exhibits lower expected costs compared to the expected value method. As the relative average deviation from average demand increases, implying greater variation in demand across scenarios, the coefficient of variation in demand also rises. Consequently, the robust optimization model becomes more efficient than the expected value method. In simpler terms, when scenarios differ significantly, the cost difference between the robust optimization and expected value models becomes more pronounced, underscoring the efficiency of the robust optimization approach.

Furthermore, when we compare the maximum variability in the expected value model to that in the robust optimization model, it becomes evident that the robust model consistently demonstrates lower variability, particularly when the relative average deviation from average demand is high. This indicates that the robust model performs significantly better in worst-case scenarios.



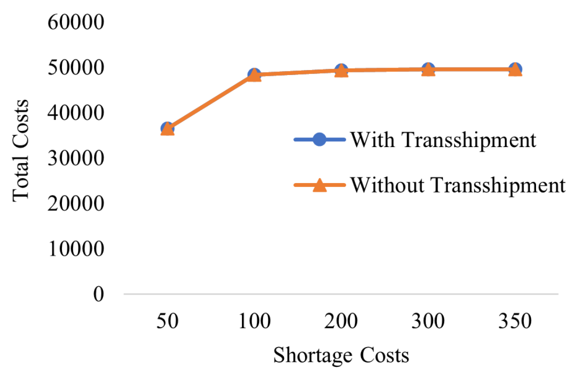
(a) (b)

Figure 5.11: Comparison between expected value approach and robust optimization approach

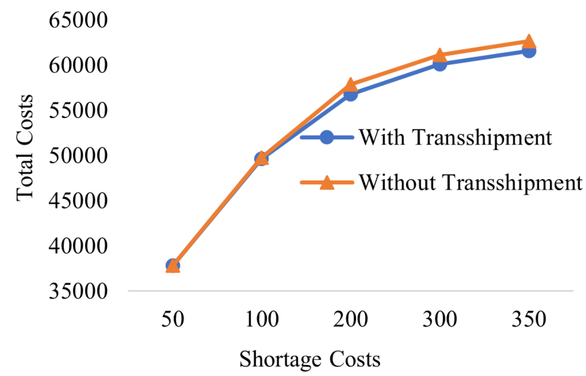
5.5.7 The Value of Collaboration between Hospitals (Transshipment)

Figure 5.12 illustrates the impact of incorporating transshipment practices between hospitals in both deterministic and stochastic models. We employ the robust optimization approach to address the stochastic model. In scenarios with deterministic demand, the introduction of transshipment does not yield any enhancement to the blood supply chain. This is because, in cases where demand is known, blood components are collected and manufactured in proportion to patient needs, and they are directly delivered to the respective hospitals. Thus, no transshipment is needed.

However, when dealing with uncertain demand, collaboration between hospitals can significantly enhance system performance. This collaboration helps reduce shortages at hospitals with inadequate inventory and minimizes wastage at hospitals holding excess inventory. Interestingly, our findings suggest that transshipment can substantially improve the system's performance, particularly when the system operates under LIFO policy and when shortage costs are high. The underlying reason is that, in such scenarios, some hospitals may experience very high wastage, while some others may face significant shortages.



(a) Deterministic model



(b) Stochastic model

Figure 5.12: Effect of transshipment on total costs in deterministic and stochastic models

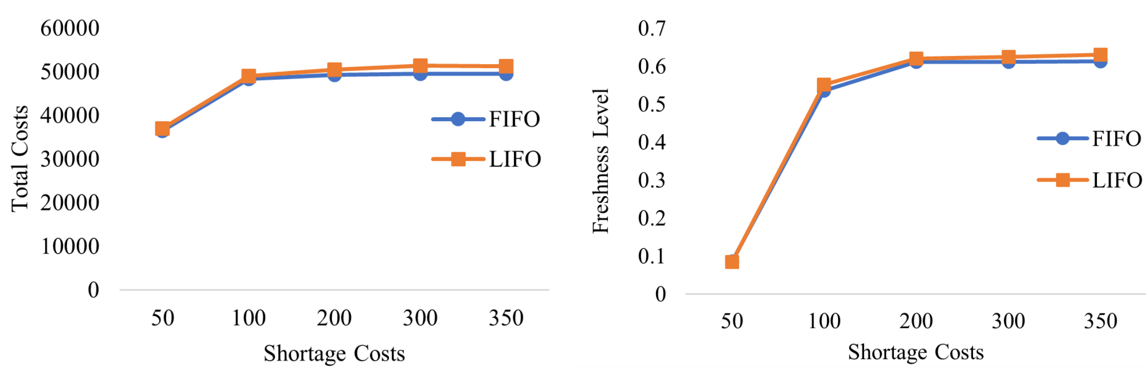
Our results further imply that permitting transshipment between hospitals within a stochastic model brings several positive outcomes: reduced overall wastage and shortage levels, higher average freshness levels, and decreased total costs. The reason is that in a stochastic system, hospitals with surplus inventory can respond to actual demand while redistributing their excess resources to hospitals facing shortages.

5.5.8 Comparison between Different Issuing Policies

FIFO and LIFO issuing policies are well-established in the literature. FIFO policy prioritizes the use of older inventory items before newer ones, while LIFO policy prioritizes the use of newer items. Threshold-based policy operates between two extreme policies, i.e., FIFO and LIFO. The purpose of this allocation policy is to reduce the age of transfused units without compromising its availability. This policy was first introduced by Haijema et al. (2007) for inventory management of platelet, and then applied by Atkinson et al.

(2012) and Sarhangian et al. (2018) to address red blood cell allocation policy. Based on this policy, for a given product j, when there are some units that are younger than a threshold TP_j, those units are allocated based on FIFO policy. When all the units on hand are older than a threshold TP_j and there is no unit younger than TP_j, we allocate them based on LIFO policy.

Appendix D.3 presents the FIFO, LIFO, and Threshold-Based policy models. According to our analysis in Figure 5.13, FIFO policy is the optimal policy, minimizing total costs due to its effective management of wastage and shortage expenses. However, when we consider the average freshness level of transfused blood units in the objective function, the superiority of the FIFO policy diminishes. In contrast, the LIFO policy, while leading to an overall increase in system costs, enhances the freshness of transfused units by prioritizing fresher units for patient use. This policy also results in a slight increase in production and a considerable rise in wastage due to the allocation of fresher units as a top priority.



(a) (b)

Figure 5.13: Total cost and average freshness level in FIFO model vs. LIFO model

Figure 5.14 (a) and 5.14 (b) depict the total costs and freshness levels by changing thresholds for PLT and RBC products within the threshold-based policy. These findings suggest that the threshold-based policy operates as an intermediate between FIFO and LIFO policies. In other words, as the thresholds increase, the total cost and freshness level decrease, aligning the issuing policy more closely with the FIFO approach. Conversely, as thresholds decrease, both total costs and average freshness levels become more similar to those observed in a LIFO policy model, characterized by the highest freshness levels and the highest costs. Hence, we observe a trade-off between the freshness level and total cost. By choosing appropriate thresholds for both products, the blood center can strike a balance, ensuring the desired freshness level while controlling wastage and shortage levels effectively. Intuitively, in a problem with total costs as the objective function, FIFO policy is the optimal policy, while in a problem with average freshness level as the objective function, LIFO policy is the optimal issuing policy. Further, in a bi-objective problem with both total costs and freshness levels as the objective, Threshold-based policy can strike a balance between the two objectives.

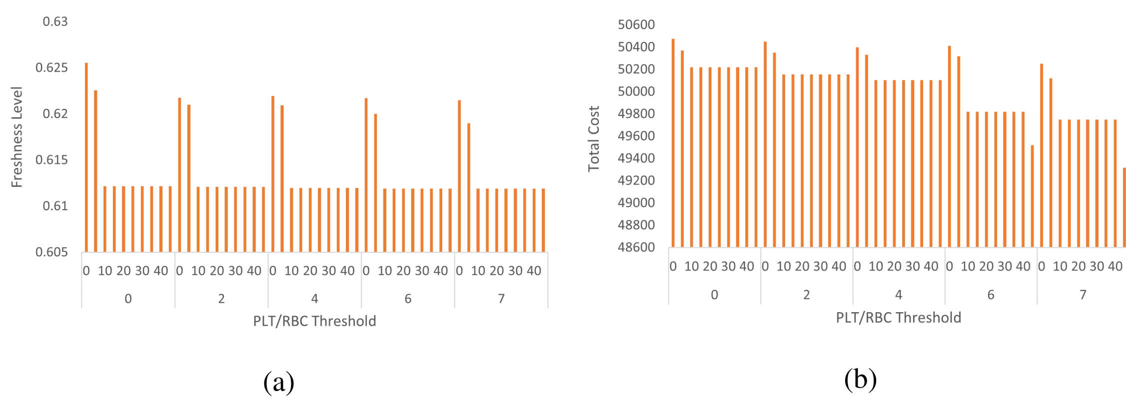


Figure 5.14: Performance of Threshold-Based policy

5.5.9 Comparison between Different Replenishment Policies

In this section, we perform a comparative analysis of two well-established replenishment policies: the (s,S) policy and the (R,T) policy, against an optimal replenishment strategy. Under an (s,S) replenishment policy, when the inventory level at the beginning of a period falls below the reorder point, s, a replenishment order must be initiated to restore the inventory level to S, which represents the order-up-to level. On the other hand, (R,T) replenishment policy places orders at fixed time intervals T and replenishes to a specified level R during each order, regardless of current inventory levels. Detailed mathematical models for (s,S) and (R,T) replenishment policies are available in Appendix D.4. We adopt a review period, i.e., T', of 2 for our analysis.

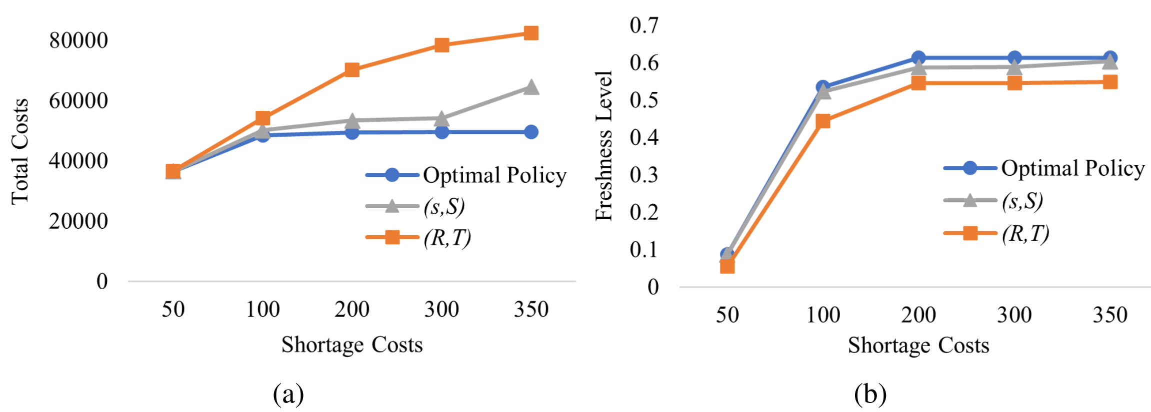


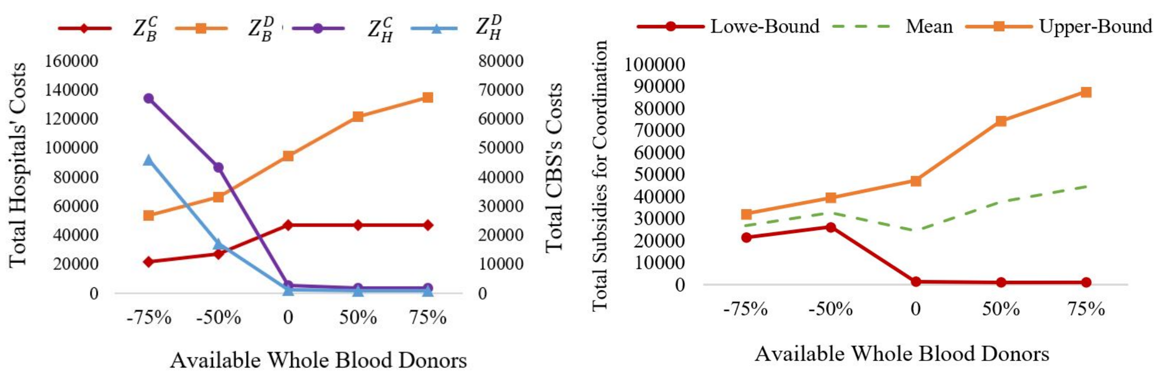
Figure 5.15: Total cost and average freshness level under different replenishment policies

Our findings in Figure 5.15 show that the optimal replenishment policy outperforms both the (s,S) and (R,T) models across various key performance indicators, including shortage level, wastage level, freshness level, and total costs. Notably, the (s,S) policy exhibits better performance in comparison to the (R,T) policy in terms of both total costs and average freshness levels. This is particularly significant in scenarios with high shortage costs, where continuous monitoring is essential to prevent excessive shortages, as depicted in Figure 5.15. In the (R,T) system, due to its lower replenishment frequency, the average age of transfused blood components is higher. However, the (R,T) model compensates for this lower frequency by ordering larger quantities of inventory, as observed in our results, in order to mitigate the impact of reduced replenishment frequency.

5.5.10 Sensitivity Analysis

Increasing the number of available donors: As depicted in Table 5.11, when the number of available whole blood donors increases, total collection and production amount using whole blood method in both decentralized and centralized models increase, too. The reason is that because some constraints (5.9) are binding and have positive shadow price, the increment of supply results in increasing collection and production amount using whole blood method, and therefore, shortage level decreases and total cost decreases in centralized model. However, by increasing the number of available donors in the decentralized model, because of increased collected units and wastage costs, the total cost increases. Counterintuitively, the results indicate that increasing the number of available donors does not always benefit the BSC. When blood system is decentralized, increasing the number of available donors can increase the total cost of the system due to overcollection and overproduction. This is because under the decentralized system, hospitals are leaders of the Stackelberg game and exert influence on CBS's decisions. As a result of increasing the number of available donors, hospitals, upper-level decision maker, try to decrease mismatching cost and transfuse as much compatible units as possible. Therefore, CBS, the lower-level decision maker, collects and produces more blood units to meet hospitals' orders.

In addition, according to Table 5.11, as the number of available whole blood donors increases, the amount of blood components produced through the apheresis method decreases in the centralized model because by increasing supply for some rare blood groups, collecting and producing blood components using the whole blood method seems more reasonable than the apheresis collection method. However, in the decentralized system where minimizing collection and production costs is not the main objective, the amount of blood components produced through the apheresis method remains unchanged while more units are produced using the whole blood method due to the increase in supply. Moreover, as supply increases, the freshness level of transfused units and total replenishment level in both decentralized and centralized models increase, too.



(a) (b)

Figure 5.16: Sensitivity of total hospitals and CBS costs and total subsidy bounds to the number of available whole blood donors

Table 5.11: Sensitivity of centralized (C) and decentralized (D) models to different parameters

Parameter	Model	% Change	Performance Measure								
			TWL	TSL	TP	$TAPH$	TR	FL	$ZB(\$)$	$ZH(\$)$	$Z(\$)$
Number of available donors	**C**	**-75%**	31.218	327.426	121.869	13.391	615.972	0.449	21540.743	67099.554	88640.297
		-50%	31.218	205.592	243.703	1.208	737.806	0.506	26679.036	43143.355	69822.391
		0%	31.218	2.456	446.839	0	940.942	0.612	46800.796	2517.452	49318.248
		50%	31.218	0	449.295	0	943.398	0.62	47028.602	1814.916	48843.519
		75%	31.218	0	449.295	0	943.398	0.62	47004.964	1837.902	48842.866
	D	**-75%**	31.218	222.208	121.869	66	721.19	0.504	53521.472	45918.588	99440.061
		-50%	31.218	76	243.738	66	867.398	0.56	65945.756	17105.047	83050.803
		0%	153.72	0	487.476	66	943.398	0.609	94041.739	1201.053	95242.792
		50%	220.895	0	731.214	66	943.398	0.623	121077.079	897.117	121974.196
		75%	252.343	0	853.083	66	943.398	0.622	134498.04	783.626	135281.666
Production costs	**C**	**-75%**	31.218	0	446.839	1.228	943.398	0.613	13464.326	7824.244	15500.343
		-50%	31.218	0	446.839	1.228	943.398	0.613	24834.086	7809.667	26855.525
		0%	31.218	2.456	446.839	0	940.942	0.612	46800.796	8305.679	49318.248
		50%	31.218	2.456	446.839	0	940.942	0.612	69149.212	8299.233	71660.218
		75%	31.218	2.456	446.839	0	940.942	0.612	80320.197	8299.233	82831.203
	D	**-75%**	153.72	0	487.476	66	943.398	0.609	27781.027	1201.053	28982.08
		-50%	153.72	0	487.476	66	943.398	0.609	49867.931	1201.053	51068.984
		0%	153.72	0	487.476	66	943.398	0.609	94041.739	1201.053	95242.792
		50%	153.72	0	487.476	66	943.398	0.609	138215.547	1201.053	139416.6
		75%	153.72	0	487.476	66	943.398	0.609	160302.451	1201.053	161503.504
Patient demands	**C**	**-75%**	62.887	0	0	4.226	235.85	0.139	4234.827	337.812	4572.639
		-50%	39.097	0	136.946	0	471.699	0.424	15146.11	999.965	16146.08
		0%	31.218	2.456	446.839	0	940.942	0.612	46800.8	2517.452	49318.25
		50%	25.674	280.793	482.984	0	1134.304	0.606	50419.43	59146.925	109566.4
		75%	23.321	433.924	487.101	1.141	1217.022	0.595	51521.61	90307.798	141829.4
	D	**-75%**	302.442	0	487.476	66	235.85	0.676	97561.7	79.678	97641.38
		-50%	251.752	0	487.476	66	471.699	0.627	96361.57	159.357	96520.93
		0%	153.72	0	487.476	66	943.398	0.609	94041.74	1201.053	95242.79
		50%	35.411	150.044	487.476	66	1265.053	0.613	91234.75	32900.62	124135.4
		75%	23.321	304.274	487.476	66	1346.673	0.617	90936.35	64239.44	155175.8

Figure 5.16 exhibits sensitivity of total hospitals and CBS costs in part (a) and optimal

range of subsidies in part (b), to number of available donors. It is notable that increasing the number of available donors widens the range of subsidies within which the coordination mechanism is economically viable. Intuitively, when supply increases, because of increased supply and decreased shortage in the centralized model, the gap between hospitals costs in centralized and decentralized models decreases (Figure 5.16 (a)) and hospitals would likely be more inclined towards integration as the lower-bound decreases significantly. On the other hand, Figure 5.16a shows by increasing the number of available donors the gap between CBS costs in the centralized and decentralized models widens, and CBS would be more willing to offer a higher subsidy to hospitals; therefore, the upper-bound for subsidy increases and coordination is facilitated. According to Figure 5.17 (a), by increasing production costs, the gap between total costs in the centralized and decentralized models increases for both hospitals and CBS. This indicates that the lower bound for subsidy which corresponds to hospitals slightly increases and the upper bound for subsidy pertinent to the CBS considerably increases (Figure 5.17 (b)). Therefore, CBS is more willing to offer a coordination contract with a higher subsidy. This result is quite intuitive and suggests that under the decentralized model, CBS has to collect and produce more units than in the centralized model, and by increasing the production costs, the gap between CBS's cost under the centralized and decentralized structures increases. Thus, CBS is willing to offer higher subsidies to incentivize hospitals to operate in a centralized system and share their information.

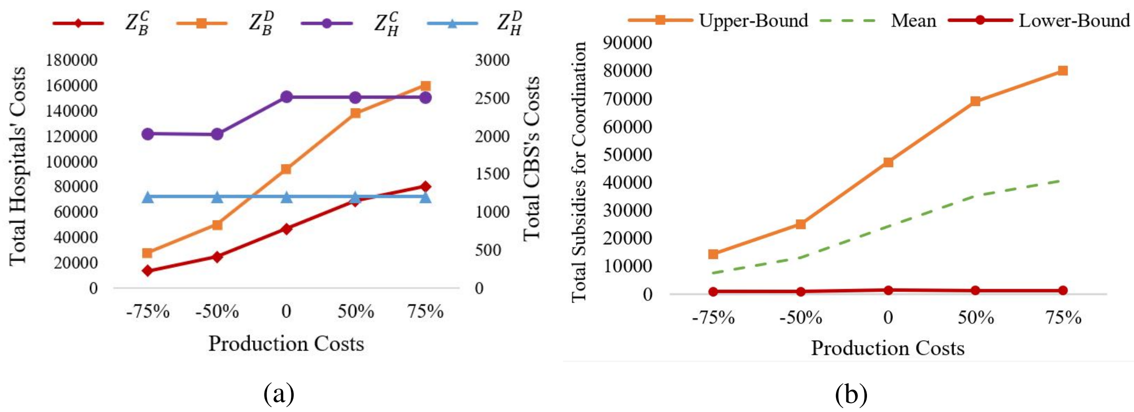
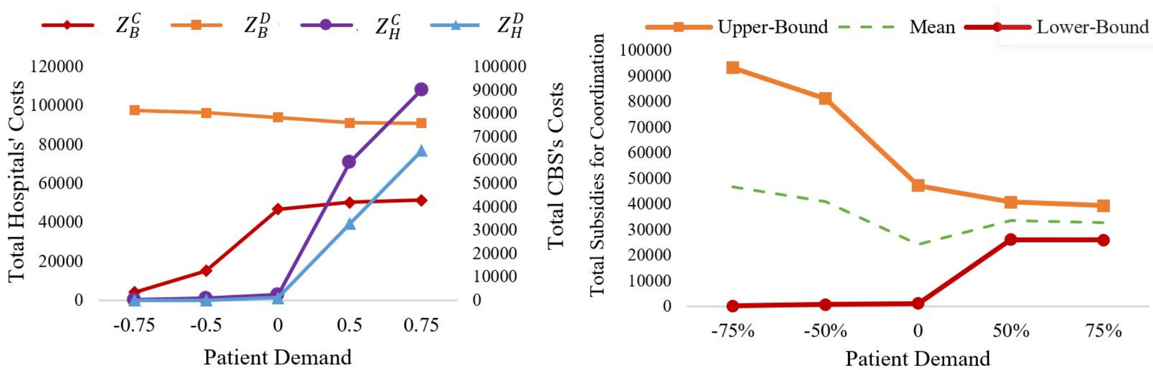


Figure 5.17: Sensitivity of total hospitals and CBS costs and total subsidy bounds to the production costs

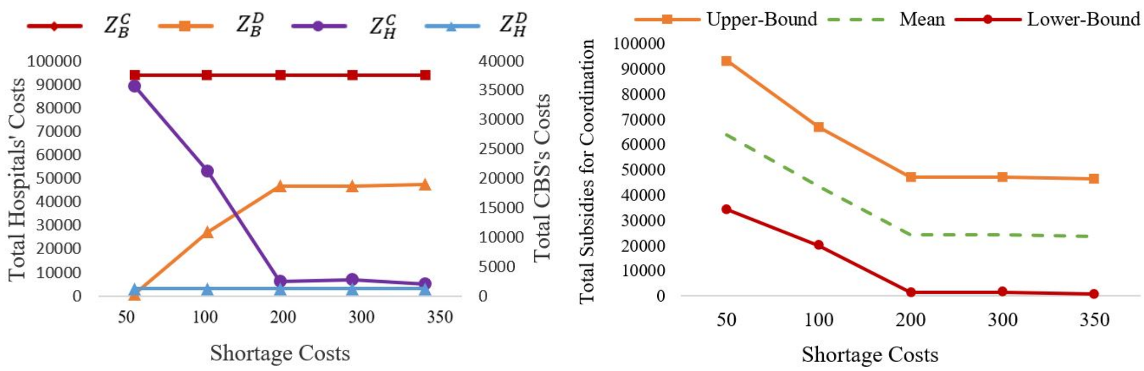
Increasing the patient demands: According to Table 5.11, as demands for both RBC and PLT increase, in both centralized and decentralized models, shortage level increases while wastage level decreases because more units of blood are consumed. Based on the results, it can be inferred when demand is too low under centralized system, apheresis collection method is preferred, while at higher demand rates, whole blood collection method is a better alternative for collection and production because whole blood units can be fractioned into RBC and PLT units and used to satisfy both demands at a lower cost than apheresis method. Further, in both centralized and decentralized models, increasing patient demand results in a higher replenishment quantity to satisfy patient demand. Figure 5.18 (a) suggests as the number of patient demand increases, the gap between total hospitals costs under centralized and decentralized systems widens, while this gap decreases for CBS. Consequently, according to Figure 5.18 (b), increasing the number of patient demands would be likely to make both hospitals and CBS more reluctant towards integration and the lower bound of total subsidy increases while upper bound of total subsidy decreases. Therefore, the feasible subsidy range for a coordination contract narrows and the need for coordination contract is reduced. This result is intuitive in that when implementation of integration is technically not feasible, BSC decision makers can reduce the gap between the performance of centralized and that of the decentralized system, and consequently the need for the integration.



(a) (b)

Figure 5.18: Sensitivity of total hospitals and CBS costs and total subsidy bounds to the patient demands

Increasing shortage costs: The results in Table 5.8 show that an increment in shortage costs decreases shortage level and increases production and replenishment quantity because CBS produces and ships more products to the hospitals to reduce shortage level. Apart from costs, in centralized model, an increase in shortage cost results in an increased average freshness level of transfused units because produced items are consumed quickly. Also, the results suggest that increasing shortage costs increases total costs of supply chain.



(a) (b)

Figure 5.19: Sensitivity of total hospitals and CBS costs and total subsidy bounds to the shortage costs

According to Figure 5.19 (a), with increase in shortage cost, the gap between the total hospitals costs under centralized and decentralized systems decreases due to a lower shortage level, which is more favorable for hospitals and would likely make them more inclined towards integration. In turn, this results in a decrease in the lower bound of subsidy (Figure 5.19 (b)). Also, Figure 5.19 (a) suggests that with an increase in shortage cost, the gap between the total CBS costs under the centralized and decentralized systems decreases due to higher collection and production costs and would likely make CBS more reluctant towards integration, resulting in a decrease in the upper bound of the total subsidy (Figure 5.19 (b)).

5.5.11 Managerial Insights

Detailed analysis of the results enables us to develop the following seven insights. First, the centralized model outperforms the decentralized model in terms of different performance measures (such as wastage level, replenishment level, freshness level, and total cost), and yields a lower gap between production and transfusion resulting from absence of inflated ordering. However, the hospitals' cost is higher under the centralized system, which can be offset by offering subsidy that in turn will motivate hospitals to participate in the centralized system.

Second, the results indicate that depending on the mismatching cost, substitution can considerably lower shortage and wastage levels and total cost. In other words, through substitution, hospitals can use blood groups with high inventory levels and reduce the shortage for blood groups with low inventory levels, enhancing the performance of system. Further, the results show that in the presence of demand uncertainty, lateral transshipment can enhance the performance of system by reducing wastage level at hospitals with excess inventory level and reducing shortages at hospitals facing stock-out situation.

Third, the FIFO issuing policy results in the lowest cost and the freshest transfused units, whereas the LIFO policy leads to the highest cost and freshness level. The threshold-based policy, positioned between these two extremes, offers CBS the flexibility to strike a balance between cost and freshness level based on their priorities. Further, the optimal replenishment policy outperforms both the (s,S) and (R,T) models across key performance indicators, including shortage level, wastage level, freshness level, and total costs. Notably, the (s,S) policy excels in terms of total costs and average freshness levels, making it especially valuable in high shortage cost scenarios where continuous monitoring is crucial. In contrast, the (R,T) system, with its lower replenishment frequency, leads to older transfused blood components but compensates by ordering larger quantities to mitigate the impact of reduced replenishment frequency.

Fourth, our findings from the stochastic model underscore the advantages of the robust optimization model over the expected value method. Specifically, as demand variability increases, the disparity between expected costs and variability widens in both methods. This highlights the efficiency of the robust model even in scenarios characterized by highly variable demand.

Fifth, increase in the number of donors and production costs together with a decrease in patient demand widens the range of total subsidy, which facilitates coordination mechanism because either the cost incurred by the hospitals under a centralized system will decrease or that incurred by the CBS under a decentralized system will increase. Note that this would also provide insights into the cases where integration is not possible.

Sixth, counterintuitively, the results indicate that increasing the number of donors does not necessarily result in a decrease in total cost when the system has an inappropriate organizational structure. For instance, in a decentralized structure, increasing the number of donors might result in an unnecessary increase in collection and production (i.e., overcollection and overproduction), which in turn will increase the total cost. On the other hand, in a centralized system, where resources are produced and allocated optimally, increasing the number of available donors is always beneficial for the system. Further, under decentralized model, increasing the number of patient demand can be beneficial for the blood supply chain, implying inefficient use of resources (e.g., due to the hospital's over ordering), while in centralized model an increase in the number of patient demand results in an increase in the total costs.

Finally, the results show a higher shortage level in blood groups such as A+ and O+ or hospitals such as St. Joseph's Healthcare in Hamilton. To reduce the shortage levels, the blood system can either increase shortage costs to avoid stock-out situations or can procure blood from alternative supply sources. Further, adding apheresis technologies to some large hospitals is helpful for lowering shortage levels. Also, the results indicate that lowering mismatching costs (in order to encourage substitution) and lowering production costs using cost-effective technologies can lower shortage levels and improve system's performance.

5.6 Conclusion

An efficient blood supply chain structure is critical to a successful and sustainable healthcare system. Currently, many countries including Canada are not practicing a centralized decision system in blood supply chain. Rather, they operate as a decentralized system where blood centers and hospitals make their own decisions. We propose a bi-level optimization problem and use Karush–Kuhn–Tucker (KKT) conditions to reformulate bi-level model as a single-level one. In this decentralized system, hospitals tend to place higher orders compared to centralized system and the amount of collection and production are higher, resulting in a high outdating rate. Thus, as a more efficient system, we model a centralized blood supply chain where both the blood centers and hospitals' decisions are made by a central decision maker. The results show that the centralized system outperforms the decentralized one considerably. However, implementing this system in practice is challenging since it increases the hospitals' cost. Motivated by operational challenges in implementing a centralized system, we propose a coordination mechanism, according to which the blood center offers subsidies for shortage and mismatching costs to hospitals. Then the lower bound and upper-bound for the total amount of subsidies offered to hospitals are calculated and analyzed. To further improve system efficiency, substitution between different blood components and transshipment between different hospitals is considered. We further, extend the model to the case of demand uncertainty and compare the optimal issuing and replenishment policies with well-established policies in the literature.

Our extensive computational experiments imply several managerial insights as follows. First, integration will benefit the BSC by decreasing the gap between production and replenishment by 69.84 units and total cost by 93.11%. For higher unit shortage costs, both centralized and decentralized models will have zero shortage levels. Second, substitution between compatible blood groups can reduce the total cost of BSC by 14.41%. In a centralized model, more products are substituted by an inexact match than decentralized model, reducing the total collection and production level. Further, the results indicate that in the presence of demand uncertainty, transshipment between hospitals can significantly improve system performance. Third, the FIFO issuing policy is cost-effective and yields the freshest transfused units, while the LIFO policy is costlier but maintains high freshness levels. The threshold-based policy provides a balance between cost and freshness. The optimal replenishment policy outperforms both (s,S) and (R,T) models across key indicators, particularly in high shortage cost scenarios. The (s,S) policy excels in total costs and freshness levels, while the (R,T) system, with lower replenishment frequency, maintains freshness through larger orders. Fourth, the results indicate that CBS can effectively coordinate BSC by offering a subsidy to hospitals to lower total hospitals' costs at least to their costs under decentralized model. The results indicate that different parameters affect facilitation of integration. For instance, increasing the number of available donors and production costs and decreasing the patient demand will increase the gap between lower-bound and upper-bound for the total subsidies, facilitating the implementation of centralized system. Also, it can provide some insights for the case where integration is not feasible. In that case, we can minimize the gap between the performance of centralized and decentralized systems. Finally, counterintuitively, the results imply that increasing the number of available donors can be detrimental for BSC and/or increasing the number of patient demand can be beneficial for BSC, when organizational structure is not optimal.

We conclude the chapter by mentioning a few future research possibilities. First, the collection strategy can be improved by considering the utility function of donors. Second, as a possible direction, routing decisions can be incorporated in this research. Considering demand and supply disruption is an interesting stream for the future work. Finally, besides the total cost, there are several other objectives such as freshness level, the number of required donors, and so on which can be considered as the objective of blood supply chain system. Modeling different objectives as a multi-objective model can also be considered as an extension to this work.

Chapter 6

Conclusions and Future Work

In this chapter, we summarize the works presented in previous chapters, highlight our major contributions, and propose directions for future research.

In Chapter 1, we presented an overview of the dissertation and motivations of this research. In Chapter 2, we introduced an inventory-pricing model for perishable products with dynamically changing assortment, consisting of fresh and non-fresh products. We also explored the optimal decision of including or excluding non-fresh items in the ordering strategy. To overcome the complexity of the proposed model and attain global optimality, we reformulated the problem as a mixed-integer bilinear program (MIBLP). Then, we proposed theoretical bounds on the value of clearance across various parameter regimes. Our results show that when market demand or mean time between two expirations becomes very large, the clearance value vanishes. Through asymptotic analysis, we propose an EOQ-type heuristic model that is asymptotically optimal. The study highlights the dual impact of clearances, benefitting retailers but potentially hurting customers who prefer fresh items. Surprisingly, the strategy can yield significant profits even for highly valued non-fresh items, contrary to conventional beliefs. This research can possibly be expanded by developing efficient heuristics and offering performance guarantees for these heuristics. For instance, fluid approximation-based or newsvendor-type policies may serve as efficient heuristic models. This work can be further expanded by considering general shelf-life, demand, or lead-time distributions. As such, the validity of the proposed Markovian model is compromised, making the problem more challenging. However, considering phase-type distributions for expiration processes can be used to control the variability of expiration process. Also, simulation methods can be used to examine the robustness of the model against different lead time distribution assumptions. Finally, considering a more general, yet complicated, customer choice models can be considered as another future direction for this work. For instance, multinomial logit choice model was considered in den Boer et al. (2022).

In Chapter 3, we considered a similar problem to the one in Chapter 2. However, in Chapter 4, we considered dynamic decision-making, contrasting with the static decisions studied in Chapter 2. Moreover, the problem in Chapter 3 considers products with multiple freshness levels in a dynamic assortment. We explored structural properties of value function and showed preservation of anti-multimodularity property for value function. We further derived the structure of optimal production and pricing decisions and presented three extensions to the base model. Given the complexity of the proposed model, three novel heuristics developed and compared with the optimal policy. Our findings indicate that the first heuristic performs well in scenarios with low variability in the quality transformation rate, while the second and third heuristics perform well when supply-to-demand ratio is high. Theoretical results show optimal production and donation policies are threshold-based, with the threshold being non-increasing in inventory levels and more sensitive to fresher products. Also, product prices decrease with higher inventory levels and are more sensitive to adjacent products with closer freshness levels. Furthermore, in dynamic settings, quality variability is beneficial, whereas in a static system with a single price, it is significantly detrimental. This work can possibly be extended by considering more general demand and shelf life distributions. The utilization of exponentially-distributed demand and shelf life in this research facilitated the application of the uniformization technique, which would not otherwise be feasible with other distributions. While we introduced the case of multi-phase quality transfusion as an extension to the base model, employing simulation methods can further verify the heuristic models' robustness in scenarios with deterministic shelf life and lead time. In these instances, we handle discrete shelf life and lead time, whereas the review process is continuous, posing significant challenges for the application of analytical methods. Also, it would be interesting to apply a data-driven driven approach to optimize inventory and pricing decisions in the presence of multiple freshness levels. A relevant study with a single freshness level can be found in Keskin et al. (2022). Another possible direction for this work is to obtain performance guarantee bounds for proposed heuristics. Moreover, leveraging the proposed structural results, future studies can develop exact algorithms through the reduction of the search region. Investigating the dynamics of multiple perishable products and their substitutability effects can be another possible direction for future studies. In such a model, competition and cannibalization effects may rise between different products, introducing greater complexity to the model.

In Chapter 4, in contrast to chapters 2 and 3, we focused on the perishable products with fixed shelf life. We studied joint inventory and pricing problems for perishable products under various markdown policies, including single-stage markdown, multiple-stage markdown, and dynamic markdown policies. We theoretically proved and empirically showed that the value of markdown policies vanishes asymptotically as the shelf life, market demand, or maximum willingness-to-pay grows very large. We conducted computational experiments on two real-world case studies: a fresh produce supply chain— a farm in Canada and a bakery chain in France. Our findings show that adopting single markdown policies can significantly enhance profits and reduce wastage. However, applying multiple-stage markdown policies might not always outweigh their complexities. We further extended the base model to the case of LIFO (Last In, First Out) issuing policy and freshness-dependent customers. One of the limitations of this study is considering FIFO policy for different markdown models which can be addressed by assuming a randomized issuing policy. Also, this work can be extended by considering cannibalization effect between different freshness levels of the perishable product. In that way, the problem resembles the proposed model in Chapter 2, albeit with fixed shelf life. Another possible future work direction is studying a system with controlled arrival process (replenishment system) instead of random arrival (production system). Moreover, considering more general supply and demand distributions and the effect of uncertainty on different markdown strategies can be an interesting avenue for future research. Last but not least, this research can be extended by incorporating strategic customers. In such a scenario, intertemporal substitutions between various markdown stages will occur, influenced by the prices at different markdown stages.

Finally, in Chapter 5, we focused on blood products, in contrast to Chapters 2, 3, and 4 that considered general perishable products. In Chapter 5, we studied both the optimal supply and demand stage decisions across three supply chain setups: decentralized, centralized, and coordinated. Specifically, we optimized decisions of collection, production, replenishment, issuing, inventory, wastage, substitution, and transshipment operations across all three organizational structures. Moreover, we analytically compared the performance of the optimal issuing and replenishment policies with commonly-used issuing and replenishment policies in the literature. Then, we studied a problem with demand uncertainty and used robust optimization to deal with the uncertainty. The presented models were applied to a realistic problem instance in Hamilton, Canada. Our findings show that integration improves performance by minimizing the gap between production and consumption, and consequently reducing costs, and increasing the freshness level of transfused units. Interestingly, in an inefficient organizational structure, more donors or reduced patient demand can be detrimental. Substitution significantly enhances system performance, while transshipment is beneficial only with stochastic demand. The optimal issuing policy is FIFO, although LIFO yields higher freshness levels, and a threshold-based policy balances freshness level and cost. This work can be extended by considering disasters and disruptions on demand or supply sides. Considering the freshness of transfused units as an objective function can be an interesting extension to this research. Ensafian and Yaghoubi (2017) considered a similar objective in their paper. Also, considering donation scheduling and uncertainty in supply can be a possible direction for future studies. Furthermore, this study could be expanded by exploring the optimal contract mechanisms between hospitals and blood centers, taking into account various objectives such as equity, fairness, cost, and freshness level.Lastly, integrating new technologies such as blockchain and Artificial Intelligence can be another interesting avenue for future studies. For example, data-driven optimization approaches can be applied to predict both demand and supply, and then optimize the operations in blood supply chain.

Appendix A

Supplement to Chapter 2

A.1 Balance Equations

Below, we present the expanded set of balance equations.

$$\gamma\pi_{i,j} = (\lambda_2 + (j+1)(j+1)\theta_2)\,\pi_{i,j+1} + \lambda_1\pi_{i+1,j} \qquad \forall i = 0,\ j = 0 \quad \text{(A.1.1)}$$

$$(\lambda_1 + i\theta_1 + \gamma)\,\pi_{i,j} = (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} + \lambda_1\pi_{i+1,j} \qquad \forall i \in [1,r],\ j = 0 \quad \text{(A.1.2)}$$

$$(\lambda_1 + i\theta_1)\,\pi_{i,j} = (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} + \lambda_1\pi_{i+1,j} \qquad \forall i \in [r+1, Q-1],\ j = 0 \quad \text{(A.1.3)}$$

$$\begin{aligned}(\lambda_1 + i\theta_1)\,\pi_{i,j} = {} & (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} + \lambda_1\pi_{i+1,j} \\ & + \gamma\pi_{i-Q,j}\end{aligned} \qquad \forall i \in [Q, r+Q-1],\ j = 0 \quad \text{(A.1.4)}$$

$$(\lambda_1 + i\theta_1)\,\pi_{i,j} = (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} + \gamma\pi_{i-Q,j} \qquad \forall i = r+Q,\; j = 0 \tag{A.1.5}$$

$$\begin{aligned}(\lambda_2 + j\theta_2 + \gamma)\,\pi_{i,j} = &(\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} + \lambda_1\pi_{i+1,j} \\ &+ (i+1)\theta_1\pi_{i+1,j-1}\end{aligned} \qquad \forall i = 0,\; j \in [1, S-1] \tag{A.1.6}$$

$$(\lambda_2 + j\theta_2 + \gamma)\,\pi_{i,j} = \lambda_1\pi_{i+1,j} + (i+1)\theta_1\left(\pi_{i+1,j-1} + \pi_{i+1,j}\right) \qquad \forall i = 0,\; j = S \tag{A.1.7}$$

$$\begin{aligned}(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2 + \gamma)\,\pi_{i,j} = &(\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} \\ &+ \lambda_1\pi_{i+1,j} + (i+1)\theta_1\pi_{i+1,j-1}\end{aligned} \qquad \forall i \in [1, r-\beta j],\; j \in [1, S-1] \tag{A.1.8}$$

$$(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2 + \gamma)\,\pi_{i,j} = \lambda_1\pi_{i+1,j} + (i+1)\theta_1\pi_{i+1,j-1} \qquad \forall i \in [1, r-\beta j],\; j = S \tag{A.1.9}$$

$$\begin{aligned}(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} = &(\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} \\ &+ \lambda_1\pi_{i+1,j} \qquad \forall i \in [r-\beta j+1, Q-1],\; j \in [1, S-1] \\ &+ (i+1)\theta_1\pi_{i+1,j-1}\end{aligned} \tag{A.1.10}$$

$$\begin{aligned}(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} =& (i+1)\theta_1\left(\pi_{i+1,j-1} + \pi_{i+1,j}\right)\\ &+ \lambda_1\pi_{i+1,j}\end{aligned} \qquad \forall i \in [r - \beta j + 1, Q-1],\ j = S \tag{A.1.11}$$

$$\begin{aligned}(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} =& (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1}\\ &+ \lambda_1\pi_{i+1,j} + \gamma\pi_{i-Q,j}\\ &+ (i+1)\theta_1\pi_{i+1,j-1}\end{aligned} \qquad \begin{aligned}&\forall i \in [Q, r+Q-1],\ j \in [1, S-1],\\ &i - Q + \beta j \leq r\end{aligned} \tag{A.1.12}$$

$$\begin{aligned}(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} =& (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1}\\ &+ \lambda_1\pi_{i+1,j} + \gamma\pi_{i-Q,j}\\ &+ (i+1)\theta_1\left(\pi_{i+1,j-1} + \pi_{i+1,j}\right)\end{aligned} \qquad \begin{aligned}&\forall i \in [Q, r+Q-1],\ j = S,\\ &i - Q + \beta j \leq r\end{aligned} \tag{A.1.13}$$

$$\begin{aligned}(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} =& (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1}\\ &+ \lambda_1\pi_{i+1,j}\\ &+ (i+1)\theta_1\pi_{i+1,j-1}\end{aligned} \qquad \begin{aligned}&\forall i \in [Q, r+Q-1],\ j \in [1, S-1],\\ &i - Q + \beta j > r\end{aligned} \tag{A.1.14}$$

$$
\begin{aligned}
(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} = {} & (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} \\
& + \lambda_1 \pi_{i+1,j} \\
& + (i+1)\theta_1 \left(\pi_{i+1,j-1} + \pi_{i+1,j}\right)
\end{aligned}
\qquad
\begin{aligned}
& \forall i \in [Q, r+Q-1],\ j = S, \\
& i - Q + \beta j > r
\end{aligned}
\tag{A.1.15}
$$

$$
\begin{aligned}
(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} = {} & (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} \\
& + \gamma \pi_{i-Q,j}
\end{aligned}
\qquad \forall i = r+Q,\ j \in [1, S-1],\ \beta = 0
\tag{A.1.16}
$$

$$
(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} = (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} + \gamma \pi_{i-Q,j} \qquad \forall i = r+Q,\ j = S,\ \beta = 0
\tag{A.1.17}
$$

$$
(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} = (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} \qquad \forall i = r+Q,\ j \in [1, S-1],\ \beta = 1
\tag{A.1.18}
$$

$$
(\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} = (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1} \qquad \forall i = r+Q,\ j = S,\ \beta = 1
\tag{A.1.19}
$$

Also, normalization equation is represented as follows.

$$
\sum_{i=0}^{r+Q} \sum_{j=0}^{S} \pi_{i,j} = 1
\tag{A.1.20}
$$

A.2 Mixed-Integer Programming Model

Objective function (2.18) and constraints (A.1.1)-(A.1.20) are highly nonlinear as the lower bound and upper bound of the ranges for i and j contain decision variables, so it is not straightforward to solve the corresponding optimization problem. To cope with this non-linearity issue, we introduce some new variables and constraints and reformulate the problem as a MIBLP model. In MIBLP model, it is assumed that there are large upper bounds of I_{Max} and J_{Max} for the maximum fresh and non-fresh inventory levels, i.e., $r+Q \leq I_{Max}$ and $S \leq J_{Max}$. Therefore, in the objective function (2.18), upper bounds $r+Q$ and S are replaced by I_{Max} and J_{Max} given that $\pi_{i,j}$ equals to zero when $i > r+Q$ or $j > S$. Using defined variables in Table 2.1, the objective function (2.18) can be rewritten as follows.

$$\begin{aligned} TP^{OPT} = \text{Max } \Bigg\{ & (G_1+G_2)\sum_{i=1}^{I_{Max}}\sum_{j=1}^{J_{Max}}\pi_{i,j} + \bar{G}_1\sum_{i=1}^{I_{Max}}\pi_{i,0} + \bar{G}_2\sum_{j=1}^{J_{Max}}\pi_{0,j} - C_{cap}\,(S+r+Q) \\ & -(A+C_pQ)\sum_{i=1}^{I_{Max}}\sum_{j=1}^{J_{Max}}x_{i,j}^2 - C_h^1\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} i\pi_{i,j} \\ & -C_h^2\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} j\pi_{i,j} - C_e\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} j\theta_2\pi_{i,j} - C_d\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} y_{i,j}^4 \Bigg\} \end{aligned} \tag{A.2.1}$$

For each state (i,j), the total arrival rate to the state equals to the total departure rate out of the state. To linearize the balance equations (A.1.1)-(A.1.20), we introduce several variables to describe the rate into and out of each state.

The departure from or arrival into a given state may occur due to demand for fresh or non-fresh products, changing fresh products to non-fresh products, expiration of non-fresh products, or order arrival. To characterize the rate out of a given state (i, j), we define $x^1_{i,j}$ as the total departure rate from state (i, j) due to demand for fresh or non-fresh products or expiration of fresh or non-fresh products. $x^1_{i,j}$ represents the total outgoing rates $(I)+(II)+(III)$ in Figure 2.1 (a). We also define $x^2_{i,j}$ as the total departure rate from state (i, j) due to order arrival which can occur only if $i+\beta j \leq r$. Outgoing rate (IV) in Figure 2.1 (a) specifies variable $x^2_{i,j}$.

Also, to characterize the arrival into a given state, we introduce $y^1_{i,j}$, $y^2_{i,j}$, $y^3_{i,j}$, $y^4_{i,j}$, and $y^5_{i,k,j}$. $y^1_{i,j}$ denotes the total rate into a given state (i, j) due to demand arrival and expiration of non-fresh products; $y^2_{i,j}$ indicates total incoming rate into state (i, j) due to demand arrival for fresh items; $y^3_{i,j}$ and $y^4_{i,j}$ denote the total rate into state (i, j) due to changing fresh to non-fresh items when $j \in [0, S-1]$ and $j = S$, respectively; and $y^5_{i,k,j}$ describes the total rate from state (k, j) into state (i, j) due to order arrival. Figure A.2.1 illustrate variables $x^1_{i,j}$, $x^2_{i,j}$,$y^1_{i,j}$, $y^2_{i,j}$, $y^3_{i,j}$, $y^4_{i,j}$, and $y^5_{i,k,j}$.

In the following Proposition, all the balance equations (A.1.1)-(A.1.20) are combined into a single equation.

Proposition A.2.1 *Given the above definitions, for any given state $(i, j) \in [(0,0),(I_{Max}, J_{Max})]$, the total rate into the state equals to the total rate out of the state and can be presented as a single balance equation as follows.*

$$x^1_{i,j}+x^2_{i,j} = y^1_{i,j}+y^2_{i,j}+y^3_{i,j}+y^4_{i,j}+\sum_{k=1}^{I_{Max}} y^5_{i,k,j} \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \qquad \text{(A.2.2)}$$

In the above equation, for a given state (i, j), $x^1_{i,j}+x^2_{i,j}$ indicates the total rate out of the state and $y^1_{i,j}+y^2_{i,j}+y^3_{i,j}+y^4_{i,j}+\sum_{k=1}^{I_{Max}} y^5_{i,k,j}$ specifies the total rate into the state.

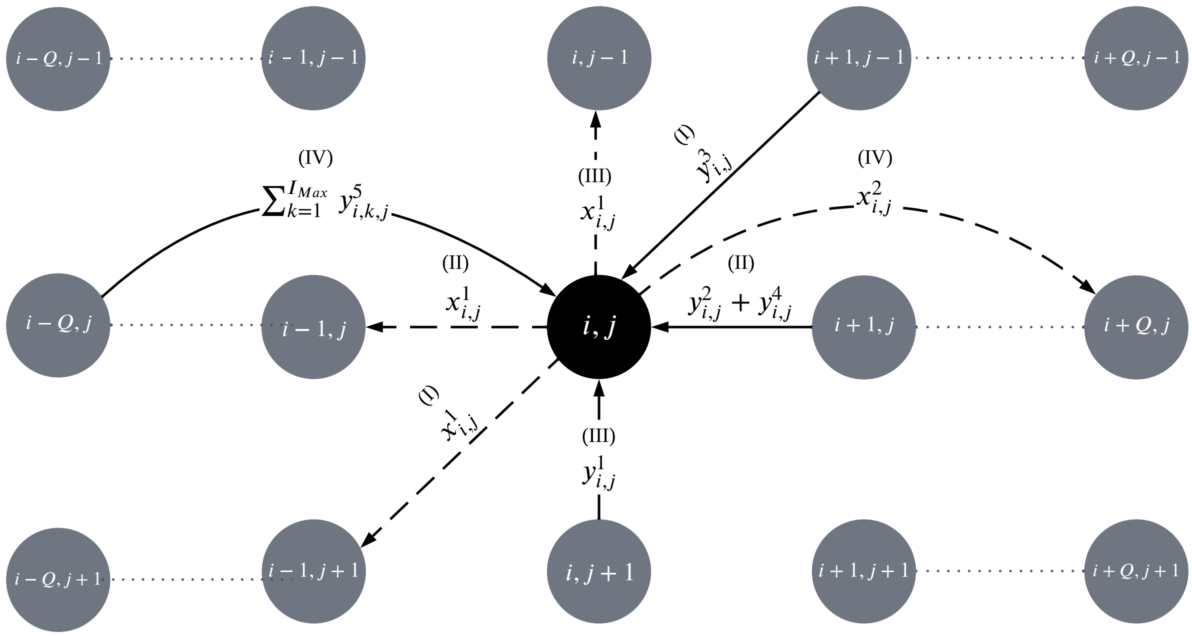


Figure A.2.1: Rates into and out of a given state (i, j) defined using non-negative variables

According to Figures A.2.1 and 2.1 (b), $x^1_{i,j}, x^2_{i,j}, y^1_{i,j}, y^2_{i,j}, y^3_{i,j}, y^4_{i,j}, y^5_{i,k,j}$ are defined using some indicators containing decision variables. Therefore, we introduce binary variables Z^1_i, Z^2_j, $Z^3_{i,j}$, Z^4_j, $Z^5_{i,j}$, and $Z^6_{i,j}$ to model these indicators.

Variables Z^1_i and Z^2_j equal to 1 only if $i \in [0, r+Q]$ and $j \in [0, S]$, respectively; otherwise 0. The fallowing constraints define Z^1_i and Z^2_j.

$$r+Q-i+1 \leq MZ^1_i \leq M-i+r+Q \quad \forall i \in [0, I_{Max}] \tag{A.2.3}$$

$$S-j+1 \leq MZ^2_j \leq M-j+S \quad \forall j \in [0, J_{Max}] \tag{A.2.4}$$

Using these variables, constraint (2.23) ensures that the steady state probabilities can take positive values only when $i \in [0, r+Q]$ and $j \in [0, S]$.

$$\pi_{i,j} \leq Z_i^1 \text{ and } \pi_{i,j} \leq Z_j^2 \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.5}$$

Constraints (A.2.6) define the total rate out of a given state due to demand arrival or expiration of fresh and non-fresh products, i.e., $x^1_{i,j}$, as follows. According to the definition of $x^1_{i,j}$, depending on the availability of fresh or non-fresh products, demand arrival or expiration rates for fresh and non-fresh products differ as can be seen in the following constraints.

$$x^1_{i,j} = \begin{cases} (\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j}, & \forall i \in [1, I_{Max}],\ j \in [1, J_{Max}] \\ (\lambda_1 + i\theta_1 + \lambda_2 + j\theta_2)\,\pi_{i,j} & \forall i \in [1, I_{Max}],\ j \in [1, J_{Max}] \\ \left(\bar{\lambda}_2 + j\theta_2\right)\pi_{i,j} & \forall i = 0,\ j \in [1, J_{Max}] \\ \left(\bar{\lambda}_1 + i\theta_1\right)\pi_{i,j} & \forall i \in [1, I_{Max}],\ j = 0 \\ 0 & \forall i = 0,\ j = 0 \end{cases} \tag{A.2.6}$$

Constraints (A.2.7)-(A.2.9) specify $x^2_{i,j}$, the total rate out of a given state (i, j) due to order arrival. $x^2_{i,j}$ can take a positive value only when $i + \beta j \in [0, r]$. To model this condition, we define $Z^3_{i,j}$ as a binary variable which equals to 1 only if $i + \beta j \in [0, r]$, otherwise 0. Thus, when $Z^3_{i,j} = 1$, $x^2_{i,j}$ can be a positive number, otherwise it must be zero.

$$\gamma\pi_{i,j} + Z^3_{i,j} - 1 \leq x^2_{i,j} \leq \gamma\pi_{i,j} - Z^3_{i,j} + 1 \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.7}$$

$$x^2_{i,j} \leq M Z^3_{i,j} \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.8}$$

$$r - i - \beta j + 1 \leq M Z^3_{i,j} \leq M - i - \beta j + r \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.9}$$

Total rate into a given state due to demand arrival and expiration of non-fresh items is denoted by $y^1_{i,j}$ and is specified using constraint (A.2.10). When there are some available fresh products on the shelves ($i > 0$), non-fresh products are sold at rate λ_2, otherwise, they are sold at rate λ_2 (refer to Figure 2.1 (b)-case (III).

$$y^1_{i,j} = \begin{cases} (\lambda_2 + (j+1)\theta_2)\,\pi_{i,j+1}, & \forall i \in [1, I_{Max}],\ j \in [0, J_{Max}-1] \\ \left(\bar{\lambda}_2 + (j+1)\theta_2\right)\pi_{i,j+1} & \forall i = 0,\ j \in [0, J_{Max}-1] \end{cases} \tag{A.2.10}$$

Constraint (A.2.11) define the total rate into a given state due to demand arrival for fresh items, i.e., $y^2_{i,j}$, as follows. According to Figure 2.1 (b)- case (II), when there is no available non-fresh items in the stock, i.e., $j = 0$, fresh products are sold at rate λ_1, otherwise, i.e., $j > 0$, the selling rate of fresh products is $\bar{\lambda}_1$.

$$y^2_{i,j} = \begin{cases} \lambda_1 \pi_{i+1,j}, & \forall i \in [0, I_{Max}-1],\ j \in [1, J_{Max}] \\ \bar{\lambda}_1 \pi_{i+1,j} & \forall i \in [0, I_{Max}-1],\ j = 0 \end{cases} \tag{A.2.11}$$

According to Figure 2.1 (b)- case (I), When the capacity of non-fresh items is not full in state $(i+1, j-1)$, i.e., $j \leq S$, fresh products add to non-fresh products inventory at rate $(i+1)\theta_1$. Therefore, the total incoming rate into state (i, j) from state $(i+1, j-1)$ due to changing fresh items to non-fresh items is denoted $y^3_{i,j}$ and is defined as follows. In the following constraint, only when $Z^2_j = 1$, $y^3_{i,j}$ can take positive numbers, otherwise it must be zero.

$$(i+1)\theta_1\pi_{i+1,j-1} - 1 + Z_j^2 \leq y_{i,j}^3 \leq (i+1)\theta_1\pi_{i+1,j-1} + 1 - Z_j^2 \ \forall i \in [0, I_{Max}-1], j \in [1, J_{Max}] \tag{A.2.12}$$

$$y_{i,j}^3 \leq MZ_j^2 \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.13}$$

According to Figure 2.1 (b)- case (II), when the capacity of non-fresh items is full in state $(i+1, j)$, i.e., $j = S$, fresh products are discarded at rate $(i+1)$ after transformation to non-fresh. Therefore, when $j = S$, the incoming rate into state (i, j) from state $(i+1, j)$ is specified by $y_{i,j}^4$ and defined using the constraints (A.2.14)-(A.2.16). We use a binary variable Z_j^2 and Z_j^4 to define $y_{i,j}^4$. When $j = S$, $Z_j^2 = Z_j^4 = 1$ and $y_{i,j}^4$ can take positive numbers, otherwise $Z_j^2 = 0$ or/and $Z_j^4 = 0$ and consequently $y_{i,j}^4 = 0$.

$$(i+1)\theta_1\pi_{i+1,j} - 2 - Z_j^2 - Z_j^4 \leq y_{i,j}^4 \leq (i+1)\theta_1\pi_{i+1,j} + 2 - Z_j^2 - Z_j^4 \quad \forall i \in [0, I_{Max}-1], \forall j \in [0, J_{Max}] \tag{A.2.14}$$

$$y_{i,j}^4 \leq MZ_j^2,\ y_{i,j}^4 \leq MZ_j^4 \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.15}$$

$$-S + j + 1 \leq MZ_j^4 \leq M - S + j \quad \forall j \in [0, J_{Max}] \tag{A.2.16}$$

The total rate from state (k, j) into state (i, j) due to order arrival, i.e., $y_{i,k,j}^5$, is defined using constraints (A.2.17)-(A.2.20). To specify $y_{i,k,j}^5$, we introduce variables $Z_{i,k}^5$ and $Z_{i,k}^6$ that are equal to 1 only when $k = i - Q$. According to Figure 2.1 (b)- case (IV), $y_{i,k,j}^5$ can take a positive value only when $k = i - Q$, $i \in [Q, r+Q]$, and $k + \beta j \in [0, r]$.

$$\gamma\pi_{k,j} - 5 + Z_i^1 + Z_j^2 + Z_{k,j}^3 + Z_{i,k}^5 + Z_{i,k}^6 \leq y_{i,k,j}^5 \leq \quad \gamma\pi_{k,j} + 5 - Z_i^1 - Z_j^2 - Z_{k,j}^3 - Z_{i,k}^5 - Z_{i,k}^6$$
$$\forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.17}$$

$$y_{i,j}^5 \leq MZ_i^1,\ y_{i,j}^5 \leq MZ_j^2,\ \ y_{i,k,j}^5 \leq MZ_{k,j}^3,\ y_{i,k,j}^5 \leq MZ_{i,k}^5,\ \text{and}\ y_{i,k,j}^5 \leq MZ_{i,k}^6$$
$$\forall i \in [0, I_{Max}],\ j \in [0, J_{Max}],\ k \in [0, I_{Max}] \tag{A.2.18}$$

$$k - i + Q + 1 \leq MZ_{i,k}^5 \leq M + k - i + Q \quad \forall i \in [0, I_{Max}],\ k \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.19}$$

$$i - k - Q + 1 \leq MZ_{i,k}^6 \leq M + i - k - Q \quad \forall i \in [0, I_{Max}],\ k \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.2.20}$$

Normalization constraint is expressed as follows.

$$\sum_{i=0}^{I_{Max}} \sum_{j=0}^{J_{Max}} \pi_{i,j} = 1 \tag{A.2.21}$$

Demand functions λ_1, λ_2, $\bar{\lambda}_1$, $\bar{\lambda}_2$ are defined in equations (2.11), and revenue functions G_1, G_2, $\bar{G}_1$, $\bar{G}_2$ are defined in equations (2.12)-(2.15). Finally, we add the following constraints to ensure that capacity constraints are met.

$$r + Q \leq I_{Max} \tag{A.2.22}$$

$$S \leq J_{Max} \tag{A.2.23}$$

Thus, MIBLP model can be written as follows.

$$TP^{OPT} = \text{Max } \Bigg\{ (G_1+G_2)\sum_{i=1}^{I_{Max}}\sum_{j=1}^{J_{Max}}\pi_{i,j} + \bar{G}_1\sum_{i=1}^{I_{Max}}\pi_{i,0} + \bar{G}_2\sum_{j=1}^{J_{Max}}\pi_{0,j} - C_{cap}(S+r+Q)$$
$$- (A+C_pQ)\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}}x^2_{i,j} - C_h^1\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} i\pi_{i,j}$$
$$- C_h^2\sum_{i=0}^{I_{Max}}\sum_{i=0}^{J_{Max}} j\pi_{i,j} - C_e\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} j\theta_2\pi_{i,j} - C_d\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} y^4_{i,j}\Bigg\} \tag{A.2.24}$$

$S.t.$

$(2.11)-(2.15)$, and $(A.2.2)-(A.2.23)$

r, Q, S are nonnegative integers (A.2.25)

$\lambda_1,\ \lambda_2,\ \bar{\lambda}_1,\ \bar{\lambda}_2,\ G_1,\ G_2,\ \bar{G}_1,\ \bar{G}_2 \geq 0$ (A.2.26)

$\pi_{i,j},\ x^1_{i,j},\ x^2_{i,j},\ y^1_{i,j},\ y^2_{i,j},\ y^3_{i,j},\ y^4_{i,j},\ y^5_{i,k,j} \geq 0 \qquad \forall i,k \in [0, I_{Max}],\ j \in [0, J_{Max}]$ (A.2.27)

$\beta, Z^1_i, Z^2_j, Z^3_{i,j}, Z^4_j, Z^5_j, Z^6_{i,k} = \{0,1\} \qquad \forall i,k \in [0, I_{Max}],\ j \in [0, J_{Max}]$ (A.2.28)

We use Gurobi solver in Python to exactly solve the above MIBLP model.

A.3 Upper Bound and Lower Bound Models for Zero Lead-Time Case

In this section, we assume that $\gamma \to \infty$, meaning lead-time approaches to zero. Let us define $\widetilde{TP}^f$ as the total profit under policy f when lead-time is zero, i.e., $\gamma \to \infty$. In what follows, we investigate the equivalent upper-bound and lower-bound models for the case of zero lead-time.

A.3.1 The Upper Bound Model (U)

To obtain an upper bound on the optimal solution, we consider the minimum wastage rate and assume that none of the fresh items change into a non-fresh product. In other words, we consider a system in which all the fresh items are sold. This system is equivalent to a system in which items are non-perishable. In this system, demand for a single product forms a renewal process. Let us define $M_1, M_2, \ldots$ times between two consecutive demands that are i.i.d with mean $\lambda_1 = \phi_1 \Lambda$. Therefore, expected cycle time, can be obtained as $E[T] = \sum_{i=1}^{Q} M_i = \frac{Q^U}{\lambda_1}$. Because lead-time is zero, all the demands are satisfied, and the total revenue per cycle can be obtained as $(1-\phi_1)\,Q^U$.

Also, holding cost per cycle can be obtained as $C_h^1 \sum_{i=1}^{Q^U-1} M_i$; ordering and purchasing cost is $A + C_p Q^U$; and the cost of capacity is $C_{cap} Q^U E[T^U]$. Let $E[TPC^U]$ be the total profit in one cycle, then the total average profit under the upper bound policy, i.e., $\widetilde{TP}^U = \frac{E[TPC^U]}{E[T^U]}$, can be written as follows.

$$\widetilde{TP}^U = \max_{\phi_1^U, Q^U} \left\{ \Gamma\left(1-\phi_1^U\right)\phi_1^U \Lambda - \left(A + C_p Q^U\right)\frac{\lambda_1}{Q^U} - C_h^1 \frac{Q^U - 1}{2} - C_{cap} Q^U \right\} \tag{A.3.1}$$

It is easy to check that the average long-run profit is jointly concave in ϕ_1 and Q. Allowing for non-integer order quantity provides, to obtain the optimal order solutions, it suffices that $\widetilde{TP}^U$ satisfies the first order optimality conditions. Taking the first derivative of $\widetilde{TP}$ with respect to ϕ_1 and Q and solving the obtained set of equations, ϕ_1^U* and Q^U* can be obtained as follows.

$$Q^{U*} = \sqrt{\frac{2\phi_1^{U*}\Lambda A}{C_h^1 + 2C_{cap}}} \tag{A.3.2}$$

$$\phi_1^{U*} = \frac{1}{2} - \mathscr{O}(\frac{1}{\Gamma}) - \mathscr{O}(\frac{1}{\Gamma\sqrt{\Lambda}}) \tag{A.3.3}$$

Remark 1. In both the continuous and the discrete EOQ model, the optimal order quantity, Q^*, is in the order of $\sqrt{\Lambda}$.

Remark 2. The above results indicate that as maximum willingness-to-pay become large, i.e., $\Gamma \to \infty$, the optimal fraction of customers buying fresh items tends to $\frac{1}{2}$.

A.3.2 The Lower Bound Model (L)

To obtain a lower bound on the total profit, we consider the maximum wastage rate and assume that all the fresh products changing into non-fresh products are expired. Therefore, to model such a system, we can drop the second dimension, i.e, the inventory level of non-fresh items, and only track the inventory level of fresh products. Thus, for the case when lead-time is zero, using the results of Kalpakam and Arivarignan (1988), we can write the total profit of the lower bound policy as follows.

$$\widetilde{TP}^L = \max_{\phi_1^L, Q^L} \left\{ \Gamma \left(1 - \phi_1^L\right) \phi_1^L \Lambda - \left(A + C_p Q^L\right) \eta^L - C_h^1 \bar{I}^L - C_{cap} Q^L - C_d \theta_1 \bar{I}^L \right\} \tag{A.3.4}$$

In which $\bar{I}^L$ and η^L can be expressed as follows.

$$\bar{I}^L = \frac{\sum_{k=1}^{Q^L} \frac{k}{\phi_1^L \Lambda + k\theta_1}}{\sum_{k=1}^{Q^L} \frac{1}{\phi_1^L \Lambda + k\theta_1}} \tag{A.3.5}$$

$$\eta^L = \frac{1}{\sum_{k=1}^{Q^L} \frac{1}{\phi_1^L \Lambda + k\theta_1}} \tag{A.3.6}$$

We define the optimal order quantity and fraction of customers buying fresh products under lower bound policy as Q^{L^*} and $\phi_1^{L^*}$ respectively.

A.4 Proof of Proposition 2.2

On one hand, changing a fresh product to a non-fresh product may incur some additional costs (expiration and holding costs) and reduce revenue (because $P_2 \leq P_1$), so profit per product for the non-fresh item is lower than the fresh item. Thus, a system in which all fresh items are sold before expiration, i.e., a system without perishability, is considered as an upper bound for the fresh/non-fresh inventory system ($TP^{OPT} \leq TP^U$). On the other hand, keeping non-fresh products in stock and selling them at a lower price P_2 may generate some revenue. In this case , $S^{OPT^*} > 0$, and $TP^{OPT} > TP^L$; otherwise $S^{OPT^*} = 0$, and $TP^{OPT} = TP^L$. Therefore a system in which fresh items are discarded right after changing into non-fresh produces is considered as a lower bound for the fresh/non-fresh inventory system ($TP^{OPT} \geq TP^L$). Next, consider a model in which the retailer keeps non-fresh items without offering clearance. This model is considered as a special case of the original fresh/non-fresh inventory system with the additional constraint that

$P_1 = P_2$. Therefore, because adding a constraint to an optimization problem does not improve the objective function, we have $TP^{OPT} \geq TP^{NC}$. This completes the proof and we have $TP^L \leq TP^{NC} \leq TP^{OPT} \leq TP^U$.

A.5 Proof of Proposition 2.3

For the case with zero lead-time, we consider Q^{U^*} and $\phi_1^{U^*}$ as the optimal solutions for the upper bound problem. In this system expiration cost is zero. Now, let us consider the lower bound problem which is the upper bound problem plus the expiration process. The total profit can be written as follows.

$$\widetilde{TP}^L = \text{Max} \left\{ \Gamma \left(1 - \phi_1^L\right) \phi_1^L \Lambda - \left(A + C_p Q^L\right) \eta^L - C_h^1 \bar{I}^L - C_{cap} Q^L - C_d \theta_1 \bar{I}^L \right\} \qquad \text{(A.5.1)}$$

Let us define expiration cost as $E\left(Q^L, \phi^L\right) = C_d \theta_1 \bar{I}^L$. we have $\widetilde{TP}^L = \widetilde{TP}^U\left(Q^L, \phi^L\right) + E^L\left(Q^L, \phi^L\right)$. The expiration cost is convex in Q^L, i.e., $E''\left(Q^L, \phi^L\right) \geq 0$. Also, it is obvious that the expiration cost is an increasing function of order quantity, i.e., $E'\left(Q^L\right) \geq 0$. Because according to Buchanan and Love (1985), at optimality first order condition is satisfied, we have $\frac{\partial \widetilde{TP}^L}{\partial Q^L} = 0$. Because expiration cost is a convex and increasing function of Q^L, we have $Q^{L^*} \leq Q^{U^*}$. Also, similarly, it can be proved that policy f with lower expiration cost than the extreme lower bound case, i.e., $E^f\left(Q^f, \phi^f\right) = \rho E^L\left(Q^L, \phi^L\right)$ $\rho \in [0, 1]$, we have $Q^{L^*} \leq Q^{f^*}$. T-he same result can be proved for the case with positive lead-time.

A.6 Proof of Theorem 2.1

A.6.1 The Case with Zero Lead-Time

In this part, we prove the results for the case with zero lead-time presented in Appendix A.3. We can propose the results for this case below.

Theorem A.6.1 *Theoretical bound on the gap between the total profit of the upper bound and lower bound model is in order of $\mathscr{O}(\frac{1}{\sqrt{\Lambda}})$, $\mathscr{O}(\frac{1}{\Gamma})$, and $\mathscr{O}(\theta_1)$; in particular we have:*

I

$$\frac{\widetilde{TP}^{OPT}-\widetilde{TP}^{NC}}{\widetilde{TP}^{NC}}=\mathscr{O}(\frac{1}{\sqrt{\Lambda}})$$

II

$$\frac{\widetilde{TP}^{OPT}-\widetilde{TP}^{NC}}{\widetilde{TP}^{NC}}=\mathscr{O}(\frac{1}{\Gamma})$$

III

$$\frac{\widetilde{TP}^{OPT}-\widetilde{TP}^{NC}}{\widetilde{TP}^{NC}}=\mathscr{O}(\theta_1)$$

Proof of part (I): The total average profit in the upper and lower bound models, i.e., $\widetilde{TP}^U$ and $\widetilde{TP}^L$, respectively, can be written as follows.

$$\widetilde{TP}^U=\max_{\phi_1^U,Q^U}\left\{\Gamma\left(1-\phi_1^U\right)\phi_1^U\Lambda-\left(A+C_pQ^U\right)\frac{\lambda_1}{Q^U}-C_h^1\frac{Q^U-1}{2}-C_{cap}Q^U\right\} \tag{A.6.1}$$

$$\widetilde{TP}^L=\max_{\phi_1^L,Q^L}\left\{\Gamma\left(1-\phi_1^L\right)\phi_1^L\Lambda-\left(A+C_pQ^L\right)\eta^L-C_h^1\bar{I}^L-C_{cap}Q^L-C_d\theta_1\bar{I}^L\right\} \tag{A.6.2}$$

In the lower bound model, we can obtain upper bounds for $\bar{I}^L$ and η^L as follows.

$$\bar{I}^L = \frac{\sum_{k=1}^{Q^L} \frac{k}{\phi_1^L \Lambda + k\theta_1}}{\sum_{k=1}^{Q^L} \frac{1}{\phi_1^L \Lambda + k\theta_1}} \le \frac{\sum_{k=1}^{Q^L} \frac{k}{\phi_1^L \Lambda}}{\sum_{k=1}^{Q^L} \frac{1}{\phi_1^L \Lambda}} \le \frac{\frac{Q^L(Q^L-1)}{2}}{Q^L} = \frac{Q^L-1}{2} \tag{A.6.3}$$

$$\eta^L = \frac{1}{\sum_{k=1}^{Q^L} \frac{1}{\phi_1^L \Lambda + k\theta_1}} \le \frac{1}{\frac{Q^L}{\phi_1^L \Lambda + \theta_1 Q^L}} \le \frac{\phi_1^L \Lambda}{Q^L} + \theta_1 \tag{A.6.4}$$

The optimal solutions for upper bound model, $Q^{U^*} = \sqrt{\frac{2\phi_1^{U^*}\Lambda A}{C_h^1 + 2C_{cap}}}$ and $\phi_1^{U^*} = \frac{1}{2} - \mathscr{O}(\frac{1}{\Gamma})$ are feasible solutions for lower bound problem. Let $\widehat{TP}^L$ be the total profit in the lower bound problem evaluated at the optimal solutions of the upper bound problem, i.e., $\widehat{TP}^L = \widetilde{TP}^L(Q^{U^*}, \phi_1^{U^*})$. Based on the results in Proposition 2.2, we have $\widetilde{TP}^{OPT} \le \widetilde{TP}^U$ and $\widetilde{TP}^{NC} \ge \widetilde{TP}^L$. Considering upper bounds for $\bar{I}^L$ and η^L, we can obtain a bound on the gap between $\widetilde{TP}^{OPT}$ and $\widetilde{TP}^{NC}$ as follows.

$$\begin{aligned}
\frac{\widetilde{TP}^{OPT} - \widetilde{TP}^{NC}}{\widetilde{TP}^{NC}} &\le \frac{\widetilde{TP}^U - \widetilde{TP}^L}{\widetilde{TP}^L} \le \frac{\widetilde{TP}^U - \widehat{TP}^L}{\widehat{TP}^L} \\
&\le \frac{\left(A + C_p Q^{U^*}\right)\theta_1 + C_d \theta_1 \frac{Q^{U^*}-1}{2}}{\Gamma\left(1-\phi_1^{U^*}\right)\phi_1^{U^*}\Lambda - \left(A + C_p Q^{U^*}\right)\eta^L - (C_h^1 + C_d\theta_1)\frac{Q^{U^*}-1}{2} - C_{cap}Q^{U^*}} \\
&= \mathscr{O}(\frac{\sqrt{\Lambda}}{\Lambda}) = \mathscr{O}(\frac{1}{\sqrt{\Lambda}})
\end{aligned} \tag{A.6.5}$$

Where the first inequality holds because $\widetilde{TP}^{OPT} \le \widetilde{TP}^U$ and $\widetilde{TP}^L \le \widetilde{TP}^{NC}$, the second inequality is satisfied because $\widehat{TP}^L \le \widetilde{TP}^L$ and the third inequality is satisfied by considering the upper bounds of $\bar{I}^L$ and η^L.

Proof of part (II): Based on the results obtained in part (I), we have:

$$\frac{\widetilde{TP}^{OPT}-\widetilde{TP}^{NC}}{\widetilde{TP}^{NC}} \leq \frac{\left(A+C_pQ^{U*}\right)\theta_1+C_d\theta_1\frac{Q^{U*}-1}{2}}{\Gamma\left(1-\phi_1^{U*}\right)\phi_1^{U*}\Lambda-\left(A+C_pQ^{U*}\right)\eta^L-(C_h^1+C_d\theta_1)\frac{Q^{U*}-1}{2}-C_{cap}Q^{U*}} = \mathscr{O}(\frac{1}{\Gamma}) \tag{A.6.6}$$

Also, the proof of the part (III) directly follows from $\frac{\left(A+C_pQ^{U*}\right)\theta_1+C_d\theta_1\frac{Q^{U*}-1}{2}}{\Gamma\left(1-\phi_1^{U*}\right)\phi_1^{U*}\Lambda-\left(A+C_pQ^{U*}\right)\eta^L-(C_h^1+C_d\theta_1)\frac{Q^{U*}-1}{2}-C_{cap}Q^{U*}}$. Next, we prove the results for the positive lead-time case.

A.6.2 The Case with Positive Lead-Time

The total average profit in the upper and lower bound models, i.e., TP^U and TP^L, respectively, can be written as follows.

$$TP^U = \max_{r^U,Q^U,\phi_1^U}\left\{\Gamma\left(1-\xi^U\right)\left(1-\phi_1^U\right)\phi_1^U\Lambda-\left(A+C_pQ^U\right)\eta^U-C_h^1\bar{I}^U-C_{cap}\left(r^U+Q^U\right)\right\} \tag{A.6.7}$$

$$TP^L = \max_{r^L,Q^L,\phi_1^L}\left\{\Gamma\left(1-\xi^L\right)\left(1-\phi_1^L\right)\phi_1^L\Lambda-\left(A+C_pQ^L\right)\eta^L-C_h^1\bar{I}^L-C_{cap}\left(r^L+Q^L\right) - C_d\theta_1\bar{I}^L\right\} \tag{A.6.8}$$

Where ξ^U, η^U, $\bar{I}^U$, ξ^L, η^L, and $\bar{I}^L$ are given in Equations (2.50-2.52) and (2.56-2.58), respectively. Optimal solutions to the upper bound problem are considered as feasible solutions for the lower bound problem. Let us define $\bar{\xi}^L = \xi^L\left(r^{U*},Q^{U*},\phi_1^{U*},\phi_2^{U*}\right)$ and

$\xi^D = \bar{\xi}^L - \bar{\xi}^U$, then we have:

$$\xi^D = \bar{\xi}^L - \bar{\xi}^U \leq \frac{\left(\phi_1^U \Lambda + r^U \theta_1\right)^{r^U} \left(\phi_1^U \Lambda + \left(r^U + Q^U\right)\theta_1\right)}{\left\{\begin{array}{l}\left(\phi_1^U \Lambda + r^U \theta_1\right)^{r^U} \left(\phi_1^U \Lambda + \left(r^U + Q^U\right)\theta_1\right) \\ + \left(\phi_1^U \Lambda + \gamma + r^U \theta_1\right)^{r^U - 1} \left(\gamma^2 Q^U + 2\gamma\theta_1 r^U Q^U + \phi_1^U \Lambda \gamma Q^U\right)\end{array}\right\}} - \frac{\left(\phi_1^U \Lambda\right)^{r^U+1}}{\left(\phi_1^U \Lambda\right)^{r^U+1} + Q^U \gamma \left(\gamma + \phi_1^U \Lambda\right)^{r^U}} \tag{A.6.9}$$

Therefore, we have:

$$\begin{aligned}\frac{TP^{OPT} - TP^{NC}}{TP^{NC}} \leq & \frac{TP^U - TP^L}{TP^L} \\ & \leq \frac{\Gamma \xi^D \left(1 - \phi_1^{U*}\right) \phi_1^{U*} \Lambda + \left(A + C_p Q^{U*}\right) \left(\frac{\gamma}{\phi_1^{U*}\Lambda}\right) \xi^D + C_d \theta_1 \bar{I}^L \left(r^{U*}, Q^{U*}, \phi^{U*}\right)}{\Gamma \left(1 - \bar{\xi}^L\right) \left(1 - \phi_1^{U*}\right) \phi_1^{U*} \Lambda - \left(A + C_p Q^{U*}\right) \frac{\lambda_1}{Q^{U*}} - \left(C_h^1 + C_d \theta_1\right) \frac{Q^{U*} - 1}{2} - C_{cap} Q^{U*}} \\ & = \mathscr{O}\left(\frac{1}{\sqrt{\Lambda}}\right)\end{aligned} \tag{A.6.10}$$

Where the first inequality holds because $TP^{OPT} \leq TP^U$ and $TP^L \leq TP^{NC}$ and the second inequality is satisfied by putting the optimal solutions of upper bound problem in the total profit of lower bound problem TP^L. Also, the equality is satisfied as the order quantity is in the order of $\mathscr{O}(\sqrt{\Lambda})$, the average inventory level $\bar{I}^L$ is bounded by the maximum inventory level $r^{U*} + Q^{U*}$, and $\xi^D = \mathscr{O}(\frac{1}{\sqrt{\Lambda}})$. Similarly, we can prove that $\frac{TP^{OPT} - TP^{NC}}{TP^{NC}} = \mathscr{O}(\frac{1}{\vartheta_1}) = \mathscr{O}(\theta_1)$ because $\xi^D = \mathscr{O}(\frac{1}{\vartheta_1})$.

A.7 Proofs of Propositions 2.4 and 2.5

A.7.1 Proof of Proposition 2.4

Based on the results presented in Theorem 2.1, it can be deduced that as market demand or mean time for a fresh item to become non-fresh grow very large (tend to infinity), the gap between a model with clearance and a model without clearance vanishes, and the perishable inventory system becomes equivalent to a non-perishable one. Consequently, we deduce the following results:

1. The gap between total profits in a system incorporating non-fresh items within the ordering policy ($\beta = 1$) and a system excluding such non-fresh items from the ordering policy ($\beta = 0$) vanishes when $\Lambda \to \infty$ or $\theta_1 \to 0$ ($\vartheta_1 \to \infty$). Therefore, mathematically, we have

$$\lim_{\Lambda \to \infty} \frac{TP^{OPT}_{\{\beta=1\}} - TP^{OPT}_{\{\beta=0\}}}{TP^{OPT}_{\{\beta=0\}}} = \lim_{\theta_1 \to 0} \frac{TP^{OPT}_{\{\beta=1\}} - TP^{OPT}_{\{\beta=0\}}}{TP^{OPT}_{\{\beta=0\}}} = 0$$

   Which implies the first result in Proposition 2.4.

2. When disregarding storage costs, the gap in total profits between a system that allocates capacity to non-fresh items and one that does not, vanishes as $\Lambda \to \infty$ or $\theta_1 \to 0$ ($\vartheta_1 \to \infty$). Therefore, mathematically, we have:

$$\lim_{\Lambda \to \infty} \frac{TP^{OPT}_{\{S=0\}} - (TP^{OPT}_{\{S>0\}} - C_{cap}S)}{TP^{OPT}_{\{S>0\}} - C_{cap}S} = \lim_{\theta_1 \to 0} \frac{TP^{OPT}_{\{S=0\}} - (TP^{OPT}_{\{S>0\}} - C_{cap}S)}{TP^{OPT}_{\{S>0\}} - C_{cap}S} = 0$$

   Which implies the second result in Proposition 2.4.

A.7.2 Proof of Proposition 2.5

The case with zero lead-time: First, let us consider the case with zero lead-time presented in Appendix A.3. The following Proposition generalizes the results for this case.

Proposition A.7.1 *The results in Theorem A.6.1 holds for the case when time between two successive expiration/demand arrival follow Independent and identically distributed (i.i.d) general distributions.*

When demand arrives according to a renewal process, the optimal Q^{U^*} and $\phi_1^{U^*}$ hold. In the upper bound model, there is no expiration, so general lifetime distribution does not affect upper bound results. Now let us consider the lower bound problem and define $\theta_1 = \frac{1}{\vartheta_1}$ in which ϑ_1 denotes the mean time between two successive expiration of fresh items. The average inventory level is bounded by $\frac{Q-1}{2}$. If we assume the maximum expiration rate for any inventory level, i.e, $\theta_1 Q$, the state transition due to demand or expiration occurs according to a renewal process with the rate $\theta_1 Q + \phi_1 \Lambda$. Therefore, the expected cycle length is bounded by $\frac{Q}{\theta_1 Q + \phi_1 \Lambda}$. Therefore we have:

$$\begin{aligned}\frac{\widetilde{TP}^U - \widetilde{TP}^L}{\widetilde{TP}^L} &\leq \frac{\left(A + C_p Q^{U^*}\right)\theta_1 + C_d \frac{Q^{U^*}-1}{2}}{\Gamma\left(1-\phi_1^{U^*}\right)\phi_1^{U^*}\Lambda - \left(A + C_p Q^{U^*}\right)\frac{\lambda_1}{Q^{U^*}} - \left(C_h^1 + C_d\theta_1\right)\frac{Q^{U^*}-1}{2} - C_{cap}Q^{U^*}} \\ &= \mathscr{O}\left(\frac{1}{\sqrt{\Lambda}}\right)\end{aligned} \tag{A.7.1}$$

According to the above results, bounds with respect to Γ and θ_1 are in the orders of $\mathscr{O}(\frac{1}{\Gamma})$ and $\mathscr{O}(\theta_1)$, respectively.

The case with positive lead-time: Now let us consider the case with positive lead-time.

Similar to the proof for zero lead-time case, it is straightforward to show the results of Theorem 2.1 holds for general renewal demand and expiration processes. Now let us consider the case of general lead-time distribution. Define $g(x)$ as the probability density function of lead-time. Then, the probability of stock out in a system without and with expiration, i.e., ξ^U and ξ^L can be written as follows, respectively.

$$\xi_U = \frac{\int_{\frac{r^U}{\phi_1^U \Lambda}}^{\infty} \left(x - \frac{r^U}{\phi_1^U \Lambda}\right) g(x)dx}{\frac{Q}{\phi_1^U \Lambda 1} + \int_{\frac{r^U}{\phi_1^U \Lambda}}^{\infty} \left(x - \frac{r^U}{\phi_1^U \Lambda}\right) g(x)dx} \tag{A.7.2}$$

$$\xi_L \leq \frac{\int_{\frac{r^L}{\phi_1^L \Lambda + \theta_1 Q^L}}^{\infty} \left(x - \frac{r^L}{\phi_1^L \Lambda + \theta_1 Q^L}\right) g(x)dx}{\frac{Q}{\phi_1^L \Lambda 1 + \theta_1 Q^L} + \int_{\frac{r^L}{\phi_1^L \Lambda + \theta_1 Q^L}}^{\infty} \left(x - \frac{r^L}{\phi_1^L \Lambda + \theta_1 Q^L}\right) g(x)dx} \tag{A.7.3}$$

Define $\xi^D = \xi^L - \xi^U$. Because $\int_0^\infty g(x)dx$ is bounded by 1, after doing some mathematical calculations, we have $\xi^D = \xi^L - \xi^U = \mathscr{O}(\frac{1}{\sqrt{\Lambda}})$. Also, in positive lead-time models, $\bar{I}^L$ vanishes asymptotically in Λ, therefore, according to Appendix A.6, we have:

$$\begin{aligned} \frac{TP^U - TP^L}{TP^L} &\leq \frac{\Gamma \xi^D \left(1 - \phi_1^{U^*}\right) \phi_1^{U^*} \Lambda + \left(A + C_p Q^{U^*}\right) \left(\frac{\gamma}{\phi_1^{U^*} \Lambda}\right) \xi^D + C_d \theta_1 \bar{I}^L \left(r^{U^*}, Q^{U^*}, \phi^{U^*}\right)}{\Gamma \left(1 - \bar{\xi}^L\right) \left(1 - \phi_1^{U^*}\right) \phi_1^{U^*} \Lambda - \left(A + C_p Q^{U^*}\right) \frac{\lambda_1}{Q^{U^*}} - (C_h^1 + C_d \theta_1) \frac{Q^{U^*} - 1}{2} - C_{cap} Q^{U^*}} \\ &= \mathscr{O}(\frac{1}{\sqrt{\Lambda}}) \end{aligned} \tag{A.7.4}$$

Similarly, we can prove that $\frac{TP^U - TP^L}{TP^L} = \mathscr{O}(\theta_1)$.

A.8 Extension: Multiple Outstanding Orders

In this section, we allow for the multiple outstanding orders. As such, the retailer will place an order of size Q whenever inventory position of $i+\beta j$ hits or crosses r from above. To formulate this problem, let us define $m = \lceil \frac{r}{Q} \rceil$ as the maximum number of outstanding orders, where $\lceil x \rceil$ represents the smallest integer greater than x. Thus, we have $(m-1)Q \leq r \leq mQ$. Demand and expiration processes of this system are exactly similar to the base model, while in this system, lead-time depends on the number of outstanding orders and can be expressed as $\gamma_l = l\gamma$ when there are l outstanding orders. To model this problem, we define the following notations.

Table A.8.1: Notations used in the model in which multiple orders are allowed

Notations	**Meanings**
$\tilde{x}^2_{i,j,l}$	the total rate out of the state (i,j) due to order arrival when there are l outstanding orders
$\tilde{y}^5_{i,k,j,l}$	The total rate from state (k,j) into state (i,j) due to order arrival when there are l outstanding orders
$Z^7_{i,j,l}$	1 when $i+\beta j \leq r - l\, Q$, 0 otherwise
$Z^8_{i,j,l}$	1 when $i+\beta j \geq r-(l+1)\, Q+1$, 0 otherwise

For the case that multiple outstanding orders are allowed, balance equation can be rewritten as follows.

$$x^1_{i,j} + \tilde{x}^2_{i,j,l} = y^1_{i,j} + y^2_{i,j} + y^3_{i,j} + y^4_{i,j} + \tilde{y}^5_{i,k,j,l} \quad \forall i \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.1}$$

where $x^1_{i,j}$, $y^1_{i,j}$, $y^2_{i,j}$, $y^3_{i,j}$, and $y^4_{i,j}$ were defined earlier. We define $\tilde{x}^2_{i,j,l}$ as the total rate out of a given state due to order arrival. $\tilde{x}^2_{i,j,l}$ can take a positive value only when $i+\beta j \in [r-(l+1)\,Q+1, r-l\,Q]$ as defined by constraints (A.8.2)-(A.8.5).

$$\gamma\pi_{i,j} + Z^7_{i,j,l} + Z^8_{i,j,l} - 2 \le \tilde{x}^2_{i,j,l} \le \gamma\pi_{i,j} - Z^7_{i,j,l} - Z^8_{i,j,l} + 2 \quad \forall i,l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.2}$$

$$\tilde{x}^2_{i,j,l} \le MZ^7_{i,j,l},\ \tilde{x}^2_{i,j,l} \le MZ^8_{i,j,l} \quad \forall i,l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.3}$$

Where $Z^7_{i,j,l}$ and $Z^8_{i,j,l}$ are defined as follows.

$$r - l\,Q - i - \beta j + 1 \le MZ^7_{i,j,l} \le M - i - \beta j + r - l\,Q \quad \forall i,l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.4}$$

$$-r + (l+1)\,Q + i + \beta j + 2 \le MZ^8_{i,j,l} \le M + i + \beta j - r + (l+1)\,Q \quad \forall i,l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.5}$$

The total rate from state (k,j) into state (i,j) due to order arrival, i.e., $\tilde{y}^5_{i,k,j,l}$, is defined using constraints (A.8.6)-(A.8.9). $\tilde{y}^5_{i,k,j,l}$ can take a positive value only when $k = i - Q$,$i \in$

$[Q, r+Q]$, and $k+\beta j \in [0, r]$.

$$\tilde{y}^5_{i,k,j,l} \geq \gamma\pi_{k,j} - 6 + Z^1_i + Z^2_j + Z^7_{k,j,l} + Z^8_{k,j,l} + Z^5_{i,k} + Z^6_{i,k} \quad \forall i, l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.6}$$

$$\tilde{y}^5_{i,k,j,l} \leq \gamma\pi_{k,j} + 6 - Z^1_i - Z^2_j - Z^7_{k,j,l} - Z^8_{k,j,l} - Z^5_{i,k} - Z^6_{i,k} \quad \forall i, l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.7}$$

$$\begin{aligned} &\tilde{y}^5_{i,k,j,l} \leq MZ^1_i,\ \tilde{y}^5_{i,k,j,l} \leq MZ^2_j,\ \tilde{y}^5_{i,k,j,l} \leq MZ^7_{k,j,l}, \\ &\tilde{y}^5_{i,k,j,l} \leq MZ^8_{k,j,l},\ \tilde{y}^5_{i,k,j,l} \leq MZ^5_{i,k},\ \tilde{y}^5_{i,k,j,l} \leq MZ^6_{i,k} \end{aligned} \tag{A.8.8}$$

$$\begin{aligned} &\forall i, l \in [0, I_{Max}],\ j \in [0, J_{Max}], \\ &\forall\, k \in [0, I_{Max}] \end{aligned} \tag{A.8.9}$$

Therefore, MIBLP model for the case when multiple outstanding orders are allowed can be expressed as follows.

$$\begin{aligned}
TP = \text{Max} \ \Bigg\{ & (G_1+G_2)\sum_{i=1}^{I_{Max}}\sum_{j=1}^{J_{Max}}\pi_{i,j} + \bar{G}_1\sum_{i=1}^{I_{Max}}\pi_{i,0} + \bar{G}_2\sum_{j=1}^{J_{Max}}\pi_{0,j} - C_{cap}\,(S+r+Q) \\
& - (A+C_pQ)\sum_{l=0}^{I_{Max}}\gamma\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}}\tilde{x}^2_{i,j,l} - C_h^1\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} i\pi_{i,j} \\
& - C_h^2\sum_{i=0}^{I_{Max}}\sum_{i=0}^{J_{Max}} j\pi_{i,j} - C_e\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} j\theta_2\pi_{i,j} - C_d\sum_{i=0}^{I_{Max}}\sum_{j=0}^{J_{Max}} y^4_{i,j}\Bigg\}
\end{aligned} \tag{A.8.10}$$

$S.t.$

$(2.11)-(2.15),\ (2.23)-(2.48),$ and $(A.8.2)-(A.8.9)$

$$r, Q, S \ \text{are nonnegative integers} \tag{A.8.11}$$

$$\lambda_1,\ \lambda_2,\ \bar{\lambda}_1,\ \bar{\lambda}_2,\ G_1,\ G_2,\ \bar{G}_1,\ \bar{G}_2 \geq 0 \tag{A.8.12}$$

$$\pi_{i,j},\ x^1_{i,j},\ \tilde{x}^2_{i,j,l},\ y^1_{i,j},\ y^2_{i,j},\ y^3_{i,j},\ y^4_{i,j},\ \tilde{y}^5_{i,k,j,l} \geq 0 \qquad \forall i,k,l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.13}$$

$$\beta, Z^1_i, Z^2_j, Z^3_{i,j}, Z^4_j, Z^5_{i,k}, Z^6_{i,k}, Z^7_{k,j,l}, Z^8_{k,j,l} = \{0,1\} \quad \forall i,k,l \in [0, I_{Max}],\ j \in [0, J_{Max}] \tag{A.8.14}$$

A.9 Proof of Lemma 2.1

First, using the level crossing method, we obtain the steady-state probability of inventory level, denoted by $f(x)$, in a classical EOQ model with deterministic demand rate λ_1. We have the following balance equation.

$$\lambda_1 f(x) = f(0) \qquad \forall x \in [0, Q] \tag{A.9.1}$$

In which the left-hand side shows downcrossing rate and the right-hand side shows the total upcrossing rate. Coupled with normalization condition $\int_0^Q f(x) = 1$, we can obtain $f(x) = \frac{1}{Q}$ $\forall x \in [0,Q]$. Next, again based on the level crossing method, for an EOQ model with deterministic demand rate λ_1 in which fresh items change into non-fresh items at deterministic rate θ_1 we have the following balance equation:

$$(\lambda_1 + \theta_1 x) f(x) = f(0) \qquad \forall x \in [0,Q] \tag{A.9.2}$$

In which the left-hand side shows downcrossing rate and the right-hand side shows the total upcrossing rate. Coupled with normalization condition $\int_0^Q f(x) = 1$, we can obtain steady-state probabilities $f(x)$ as follows.

$$f(x) = \frac{\theta_1}{(\lambda_1 + \theta_1 x)\left(\ln(\lambda_1 + \theta_1 Q) - \ln(\lambda_1)\right)} \qquad \forall x \in [0,Q] \tag{A.9.3}$$

In a large-scale system where market demand grows large to infinity ($\Lambda \to \infty$), the steady-state probability of inventory level in an EOQ model with deteriorating items becomes uniform distribution between 0 and Q, i.e., $f(x) = \frac{1}{Q}$ $\forall x \in [0,Q]$ where $f(x)$ denotes the probability density function of inventory level. In other words, mathematically, we have:

$$\lim_{\lambda_1 \to \infty} f(x) = \lim_{\lambda_1 \to \infty} \frac{\theta_1}{(\lambda_1 + \theta_1 x)\left(\log(\lambda_1 + \theta_1 Q) - \log(\lambda_1)\right)} = \frac{\theta_1}{\log e^{\theta_1 Q}} = \frac{1}{Q} \qquad \forall x \in [0,Q] \tag{A.9.4}$$

This result indicates that in a large system, the EOQ model with deteriorating items becomes equivalent to a classical EOQ model with a constant slope.

A.10 Proof of Lemma 2.2

According to the level crossing method, we can write the following balance equations for different ranges of inventory levels.

$$\lambda f(x) = \gamma \left(\int_0^x f(y)dy + f(0) \right) \qquad x \in (0,r] \tag{A.10.1}$$

$$\lambda f(x) = \gamma \left(\int_0^r f(y)dy + f(0) \right) \qquad x \in (r,Q] \tag{A.10.2}$$

$$\lambda f(x) = \gamma \int_{x-Q}^{r} f(y)dy \qquad x \in (Q,r+Q] \tag{A.10.3}$$

Also, normalization constraint can be written as follows.

$$f(0) + \int_0^r f(y)dy + \int_r^Q f(y)dy + \int_Q^{r+Q} f(y)dy = 1 \tag{A.10.4}$$

Solving the above balance equation coupled with the normalization condition yields the intended results.

$$f(x) = \begin{cases} \frac{\lambda}{\lambda+\gamma Q e^{\frac{\gamma r}{\lambda}}} & x = 0 \\ \frac{\gamma}{\lambda} e^{\frac{\gamma x}{\lambda}} f(0) & x \in (0,r] \\ \frac{\gamma}{\lambda} e^{\frac{\gamma r}{\lambda}} f(0) & x \in (r,Q] \\ \frac{\gamma}{\lambda} \left(e^{\frac{\gamma r}{\lambda}} - e^{\frac{\gamma(x-Q)}{\lambda}} \right) f(0) & x \in (Q,r+Q] \end{cases} \tag{A.10.5}$$

A.11 Proof of Theorem 2.2

We intend to obtain a bound on the optimality gap of proposed EOQ model. To that end, first, we show that $TP^{EOQ} > TP^{OPT}$. Denote TP^{DET} as the total profit in a deterministic equivalent of the fresh/non-fresh problem where all demand and expiration processes are deterministic. Then, we have $TP^{OPT} \leq TP^{DET}$. The total profit in the presented EOQ system (TP^{EOQ}) is higher than deterministic version of the problem (TP^{DET}) due to lower holding and expiration costs. Also, we know that for any set of decision variables, $TP^{L} \leq TP^{OPT}$. Therefore, we have:

$$\frac{TP^{OPT}\left(\phi_1^*,\phi_2^*,r^*,Q^*\right)-TP^{OPT}\left(\phi_1^{EOQ_v{}^*},\phi_2^{EOQ_v{}^*},r^{EOQ_v{}^*},Q^{EOQ_v{}^*}\right)}{TP^{OPT}\left(\phi_1^*,\phi_2^*,r^*,Q^*\right)} \leq \frac{TP^{EOQ_v}\left(\phi_1^{EOQ_v{}^*},\phi_2^{EOQ_v{}^*},r^{EOQ_v{}^*},Q^{EOQ_v{}^*}\right)-TP^{L}\left(\phi_1^{EOQ_v{}^*},\phi_2^{EOQ_v{}^*},r^{EOQ_v{}^*},Q^{EOQ_v{}^*}\right)}{TP^{EOQ_v}\left(\phi_1^{EOQ_v{}^*},\phi_2^{EOQ_v{}^*},r^{EOQ_v{}^*},Q^{EOQ_v{}^*}\right)} \tag{A.11.1}$$

The total average profit in the upper and lower bound models, i.e., TP^{EOQ} and TP^{L}, respectively, can be written as follows.

$$TP^{EOQ_v} = \max_{\phi_1,\phi_2,r,Q}\left\{\begin{array}{l} G_1\left(\phi_1,\phi_2\right)\left(1-\xi^{EOQ_v}\right)+G_2\left(\phi_1,\phi_2\right)\left(1-\xi^{EOQ_v}\right) \\ -\left(A+C_pQ\right)\eta^{EOQ_v}-C_h^1\bar{I}^{EOQ_v}-C_{cap}\left(r+Q\right) \end{array}\right\} \tag{A.11.2}$$

Subject to

$$\phi_2 = \frac{\nu\theta_1}{\Lambda} \tag{A.11.3}$$

$$\bar{I}^{EOQ_v} = \frac{Q\left[\gamma(Q+2r)\,e^{\frac{\gamma r}{\phi_1\Lambda+\nu\theta_1}} - 2\left(\phi_1\Lambda+\nu\theta_1\right)\left(e^{\frac{\gamma r}{\phi_1\Lambda+\nu\theta_1}}-1\right)\right]}{2\left(\gamma Q e^{\frac{\gamma r}{\phi_1\Lambda+\nu\theta_1}}+\phi_1\Lambda+\nu\theta_1\right)} \tag{A.11.4}$$

$$\xi^{EOQ_v} = \frac{\phi_1\Lambda+\theta_1\nu}{\phi_1\Lambda+\theta_1\nu+\gamma Q e^{\frac{\gamma r}{\phi_1\Lambda+\theta_1\nu}}} \tag{A.11.5}$$

$$\eta^{EOQ_v} = \gamma e^{\frac{\gamma r}{\phi_1\Lambda+\theta_1\nu}}\xi^{EOQ_v} \tag{A.11.6}$$

$$TP^L = \max_{r^L,Q^L,\phi_1^L} \left\{\Gamma\left(1-\xi^L\right)\left(1-\phi_1^L\right)\phi_1^L\Lambda - \left(A+C_pQ^L\right)\eta^L - C_h^1\bar{I}^L - C_{cap}\left(r^L+Q^L\right) - C_d\theta_1\bar{I}^L\right\} \tag{A.11.7}$$

Where $\xi^{EOQ_{nu}}$, $\eta^{EOQ_{nu}}$, $\bar{I}^{EOQ_{nu}}$, ξ^L, η^L, and $\bar{I}^L$ are given in Equations (2.63-2.65) and (2.56-2.58), respectively.

The average inventory level for non-perishable items is considered as an upper bound for the average inventory level for perishable items. Therefore, by putting $\theta_1 = 0$ in the lower bound model, we obtain:

$$\begin{aligned}\bar{I}^L &= \Upsilon_{r^L}\sum_{k=r+1}^{r^L+Q^L}\frac{\gamma}{\phi_1^L\Lambda+k\theta_1}\xi^L - \sum_{k=1}^{r^L}\frac{Q^L\gamma\theta_1}{\left(\phi_1^L\Lambda+k\theta_1\right)\left(\phi_1\Lambda+\left(k+Q^L\right)\theta_1\right)}\Upsilon_{k-1}\xi^L \\ &\leq \left(1+\frac{\gamma}{\phi_1^L\Lambda}\right)^{r^L}\left(\frac{Q^L\gamma}{\phi_1^L\Lambda}\right)\end{aligned} \tag{A.11.8}$$

Let us define $\bar{\xi}^L = \xi^L\left(r^{EOQ_v*}, Q^{EOQ_v*}, \phi_1^{EOQ_v*}, \phi_2^{EOQ_v*}\right)$. Further, let us define $\xi^D =$

$\bar{\xi}^L - \xi^{EOQ_v}$, then we have:

$$\xi^D(r,Q,\phi_1) = \bar{\xi}^L - \xi^{EOQ_v} \leq \frac{(\phi_1\Lambda + r\theta_1)^r\,(\phi_1\Lambda + (r+Q)\,\theta_1)}{\left\{\begin{array}{l}(\phi_1\Lambda + r\theta_1)^r\,(\phi_1\Lambda + (r+Q)\,\theta_1)\\ +\,(\phi_1\Lambda + \gamma + r\theta_1)^{r-1}\left(\gamma^2 Q + 2\gamma\theta_1 rQ + \phi_1\Lambda\gamma Q\right)\end{array}\right\}} - \frac{\phi_1\Lambda + \theta_1\nu}{\phi_1\Lambda + \theta_1\nu + \gamma Q e^{\frac{\gamma r}{\phi_1\Lambda + \theta_1\nu}}} \tag{A.11.9}$$

Therefore, we have:

$$\frac{TP^{EOQ_v}\left(\phi_1^{EOQ_v*}, \phi_2^{EOQ_v*}, r^{EOQ_v*}, Q^{EOQ_v*}\right) - TP^L\left(\phi_1^{EOQ_v*}, \phi_2^{EOQ_v*}, r^{EOQ_v*}, Q^{EOQ_v*}\right)}{TP^{EOQ_v}\left(\phi_1^{EOQ_v*}, \phi_2^{EOQ_v*}, r^{EOQ_v*}, Q^{EOQ_v*}\right)} \leq$$

$$\frac{\left\{\begin{array}{l}\Gamma\xi^D\left(1-\phi_1^{EOQ_v*}\right)\phi_1^{EOQ_v*}\Lambda + \left(A + C_pQ^{EOQ_v*}\right)\left(\dfrac{\gamma}{\phi_1^{EOQ_v*}\Lambda}\right)\xi^D + \\ C_d\theta_1\bar{I}^L\left(r^{EOQ_v*}, Q^{EOQ_v*}, \phi_1^{EOQ_v*}\right)\end{array}\right\}}{(G_1+G_2)\left(1-\xi^{EOQ_v*}\right) - \left(A + C_pQ^{EOQ_v*}\right)\eta^{EOQ_v} - C_h^1\bar{I}^{EOQ_v^*} - C_{cap}\left(r^{EOQ_v*} + Q^{EOQ_v*}\right)}$$
$$= \mathscr{O}\left(\frac{1}{\sqrt{\Lambda}}\right) \tag{A.11.10}$$

The above equality is satisfied as order quantity is in the order of $\mathscr{O}(\sqrt{\Lambda})$, the average inventory level $\bar{I}^L\left(r^{EOQ_v*}, Q^{EOQ_v*}, \phi_1^{EOQ_v*}\right)$ is bounded by the maximum inventory level $r^{EOQ_v*} + Q^{EOQ_v*}$, and $\xi^D = \mathscr{O}(\frac{1}{\sqrt{\Lambda}})$. Similarly, the second part of the theorem can be easily proved.

Appendix B

Supplement to Chapter 3

B.1 Preliminaries on Anti-Multimodularity

In this section, we introduce the necessary definitions and properties related to anti-multimodularity. We begin by defining key concepts and then present important lemmas and corollaries that characterize and describe the properties of anti-multimodular functions.

B.1.1 Definitions

Let $e_i \in \mathbb{Z}^m$, for $i = 1, \ldots, m$, denote the vector with elements equal to 1 at the $i^t h$ position and 0 elsewhere. We define $d_i = e_{i-1} - e_i$, for $i = 2, \ldots, m$. Furthermore, we consider the set $F = \{-e_1, d_2, \ldots, d_m, e_m\}$.

Definition B.1.1 *A real-valued function $f : \mathbb{Z}^m \to \mathbb{R}$ is said to be anti-multimodular with respect to F if, for all $x \in \mathbb{Z}^m$ and $a, b \in F$ with $a \neq b$, we have $f(x+a) + f(x+b) \leq f(x) + f(x+a+b)$.*

To further analyze anti-multimodular functions, we define the following difference operators: $\Delta_{e_i} g(x) = g(x+e_i) - g(x)$, $\Delta_{-e_i} g(x) = g(x-e_i) - g(x)$, and $\Delta_{d_i} g(x) = \Delta_{e_{i-1}} g(x) - \Delta_{e_i} g(x) = g(x+e_{i-1}) - g(x+e_i)$. With these definitions, we can establish a characterization of anti-multimodularity using second-order differences.

B.1.2 Characterization of Anti-Multimodularity

Lemma B.1.1 *A function f is anti-multimodular if and only if $\Delta_a \Delta_b f \geq 0$, for all $a, b \in F$ with $a \neq b$.*

Lemma B.1.2 *A function f is anti-multimodular with respect to F if and only if*

$$f(x+a_1+\ldots+a_l) + f(x+a_{l+1}+\ldots+a_k) \leq f(x) + f(x+a_1+\ldots+a_k)$$

holds for any $\{a_1, \ldots, a_l, \ldots, a_k\} \in F$, $0 < l < k$, and $a_i \neq a_j$ for $i \neq j$. Essentially, Lemma B.1.2 provides a more general characterization of anti-multimodularity compared to Lemma B.1.1.

Next, we present some important properties of anti-multimodular functions.

Lemma B.1.3 *For any $a, b \in \overline{F}$ and $p, q \in \mathbb{R}$, we have:*

1. $\Delta_a(pf+qg) = p\Delta_a f + q\Delta_a g$,

2. $\Delta_a \Delta_b f = \Delta_b \Delta_a f$.

Lemma B.1.4 *If f is anti-multimodular with respect to F, then the following properties hold:*

1. $\Delta_{e_i} \Delta_{e_j} f \leq 0$ *for all i, j,*

2. *$\Delta_{e_i}\Delta_{e_i} f \leq \Delta_{e_j}\Delta_{e_i} f$ for all i, j,*

3. *$\Delta_{e_i}\Delta_{d_i} f \geq 0$ for all $i \neq 1$,*

4. *$\Delta_{d_i}\Delta_{d_i} f \leq 0$ for all $i \neq 1$,*

5. *$\Delta_{e_i}\Delta_{e_j} f \leq 0$ if $i < j$ and $\Delta_{e_i}\Delta_{e_j} f \geq 0$ if $i > j$, for $j \neq 1$.*

Corollary B.1.1 *If f is anti-multimodular with respect to F, then it is jointly concave.*

We omit the proofs in this subsection for brevity and refer interested readers to Altman et al. (2000). We utilize these properties later to establish the structure of optimal policies in this research.

B.2 Proof of Theorem 3.1

In order to prove that $v \in \Omega$, we first need to prove if $v \in \Omega$, then $H \in \Omega$, $\mathscr{R}v \in \Omega$, $\mathscr{P}v \in \Omega$, $\mathscr{E}_k v \in \Omega \ \ \forall k = 1, 2, \ldots, n$.

B.2.1 Proof of $H \in \Omega$

Because H is a linear operator, it is obvious that it satisfies submodularity and subconcavity conditions.

B.2.2 Proof of $\mathscr{R}v \in \Omega$

In order to prove submodularity and subconcavity conditions for $\mathscr{R}v$, we define the optimal price vector as $\mathbf{P}^*$. We have $\mathscr{R}v(\mathbf{x}) = \max_{\mathbf{P}} f(\mathbf{x}, \mathbf{P})$. Based on Lemma B.4.1, $f(\mathbf{x}, \mathbf{P})$ is a concave function of $\mathbf{P}$, and according to Proposition 3.1, we can obtain the optimal prices for all the product. Therefore, substituting the optimal prices in $\mathscr{R}v(\mathbf{x})$, we can rewrite the revenue operator as follows.

$$\mathscr{R}v(\mathbf{x}) = \left(q_1 - 2p_1^*(\mathbf{x}) + \sum_{k=1}^{n-1} \frac{\left(p_k^*(\mathbf{x}) - p_{k+1}^*(\mathbf{x})\right)^2}{q_k - q_{k+1}} + \frac{p_n^{2*}}{q_n} \right) + v(\mathbf{x}) \tag{B.2.1}$$

Then we can write $\mathscr{R}v(\mathbf{x})$ as follows.

$$\mathscr{R}v(\mathbf{x}) = f(\mathbf{x}, \mathbf{P}^*) = \tilde{f}(\mathbf{x}, \mathbf{P}^*) + v(\mathbf{x}) \tag{B.2.2}$$

Where $\tilde{f} = (\mathbf{x}, \mathbf{P}^*) \left(q_1 - 2p_1(\mathbf{x}) + \sum_{k=1}^{n-1} \frac{(p_k(\mathbf{X}) - p_{k+1}(\mathbf{X}))^2}{q_k - q_{k+1}} + \frac{p_n^2}{q_n} \right)$. Let us define the optimal price vectors for states $(\mathbf{x})$, $(\mathbf{x}+\mathbf{e}_i)$, $(\mathbf{x}+\mathbf{e}_j)$, $(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)$, $(\mathbf{x}+2\mathbf{e}_i)$, and $(\mathbf{x}+2\mathbf{e}_j)$ as follows:

State	$(\mathbf{x})$	$(\mathbf{x}+\mathbf{e}_i)$	$(\mathbf{x}+2\mathbf{e}_i)$	$(\mathbf{x}+\mathbf{e}_j)$	$(\mathbf{x}+2\mathbf{e}_j)$	$(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)$
Optimal Prices	$\mathbf{P}^*_{0,0}$	$\mathbf{P}^*_{1,0}$	$\mathbf{P}^*_{2,0}$	$\mathbf{P}^*_{0,1}$	$\mathbf{P}^*_{0,2}$	$\mathbf{P}^*_{1,1}$

Next, we prove submodularity and subconcavity conditions for $\mathscr{R}v$.

B.2.2.1 Proof of submodularity

we intend to prove that $\Delta_i\Delta_j\mathscr{R}v(\mathbf{x}) = \mathscr{R}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j) - \mathscr{R}v(\mathbf{x}+\mathbf{e}_i) - \mathscr{R}v(\mathbf{x}+\mathbf{e}_j) + \mathscr{R}v(\mathbf{x}) \leq 0$. Considering the action $\mathbf{P}^*_{1,1}$ for state $(\mathbf{x}+\mathbf{e}_i)$ and $\mathbf{P}^*_{0,0}$ for $(\mathbf{x}+\mathbf{e}_j)$. Then, we have:

$$\begin{aligned}
\Delta_i\Delta_j\mathscr{R}v(\mathbf{x}) =& \mathscr{R}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)-\mathscr{R}v(\mathbf{x}+\mathbf{e}_i)-\mathscr{R}v(\mathbf{x}+\mathbf{e}_j)+\mathscr{R}v(\mathbf{x}) \\
&\leq f(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j,\mathbf{P}^*_{1,1})-f(\mathbf{x}+\mathbf{e}_i,\mathbf{P}^*_{1,1})-f(\mathbf{x}+\mathbf{e}_j,\mathbf{P}^*_{0,0})+f(\mathbf{x},\mathbf{P}^*_{0,0}) \\
&= v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)-v(\mathbf{x}+\mathbf{e}_i)-v(\mathbf{x}+\mathbf{e}_j)+v(\mathbf{x}) \\
&= \Delta_i\Delta_j v(\mathbf{x}) \leq 0
\end{aligned} \tag{B.2.3}$$

Where the first inequality follows from the optimality of $p_{1,0}$ for the state $(\mathbf{x}+\mathbf{e}_i)$ and optimality of $p_{0,1}$ for the state $(\mathbf{x}+\mathbf{e}_j)$, and the second inequality follows from the definition of $\mathscr{R}v(\mathbf{x})$.

B.2.2.2 Proof of subconcavity

In this part, we intend to prove that $\Delta_i\Delta_i\mathscr{R}v(\mathbf{x}) = \Delta_i\Delta_j\mathscr{R}v(\mathbf{x}) \leq 0$, or in other words $\mathscr{R}v(\mathbf{x}+2\mathbf{e}_i)-\mathscr{R}v(\mathbf{x}+\mathbf{e}_i)-\mathscr{R}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)+\mathscr{R}v(\mathbf{x}+\mathbf{e}_j)\leq 0$. Considering the action $\mathbf{P}^*_{2,0}$ for state $(\mathbf{x}+\mathbf{e}_i)$ and $\mathbf{P}^*_{0,1}$ for $(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)$. Then, we have:

$$\begin{aligned}
\Delta_i\Delta_i\mathscr{R}v-\Delta_i\Delta_j\mathscr{R}v =& \mathscr{R}v(\mathbf{x}+2\mathbf{e}_i)-\mathscr{R}v(\mathbf{x}+\mathbf{e}_i)-\mathscr{R}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)+\mathscr{R}v(\mathbf{x}+\mathbf{e}_j) \\
&\leq f(\mathbf{x}+2\mathbf{e}_i,\mathbf{P}^*_{2,0})-f(\mathbf{x}+\mathbf{e}_i,\mathbf{P}^*_{2,0})-f(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j,\mathbf{P}^*_{0,1})+f(\mathbf{x}+\mathbf{e}_j,\mathbf{P}^*_{0,1}) \\
&= v(\mathbf{x}+2\mathbf{e}_i)-v(\mathbf{x}+\mathbf{e}_i)-v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)+v(\mathbf{x}+\mathbf{e}_j) \\
&= \Delta_i\Delta_i v-\Delta_i\Delta_j v \leq 0
\end{aligned} \tag{B.2.4}$$

B.2.3 Proof of $\mathscr{P}v \in \Omega$

In order to proof submodularity and subconcavity of operator $\mathscr{P}v(\mathbf{x}) = \max\left\{v(\mathbf{x}+\mathbf{e}_1) - c_p, v(\mathbf{x})\right\}$, we reformulate this operator as follows.

$$\mathscr{P}v(\mathbf{x}) = \max_{a \in Q(\mathbf{X})} \left\{v(\mathbf{x}+a) - a \times c_p\right\} \tag{B.2.5}$$

where $Q(\mathbf{x}) = \{0 \leq a \leq 1;\ x_1 + a \leq C\}$ is the feasible region for action a. define $g(\mathbf{x}, a)$ as the total value after taking production action a, so we can write $\mathscr{P}v(\mathbf{x}) = \max_{a \in Q(\mathbf{X})} g(\mathbf{x}, a)$. Let us denote the optimal production decision a for states $(\mathbf{x})$, $(\mathbf{x}+\mathbf{e}_i)$, $(\mathbf{x}+\mathbf{e}_j)$, $(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)$, $(\mathbf{x}+2\mathbf{e}_i)$, and $(\mathbf{x}+2\mathbf{e}_j)$ as follows:

State	$(\mathbf{x})$	$(\mathbf{x}+\mathbf{e}_i)$	$(\mathbf{x}+2\mathbf{e}_i)$	$(\mathbf{x}+\mathbf{e}_j)$	$(\mathbf{x}+2\mathbf{e}_j)$	$(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)$
Optimal Production Decisions	$a^*_{0,0}$	$a^*_{1,0}$	$a^*_{2,0}$	$a^*_{0,1}$	$a^*_{0,2}$	$a^*_{1,1}$

Next, we prove the submodularity and subconcavity property for operator $\tilde{\mathscr{P}}v$.

B.2.3.1 Proof of submodularity

From submodularity and subconcavity property of value function v, we conclude that $a^*_{1,1} \leq a^*_{0,0} \leq a^*_{1,1} + 1$. Therefore, because $0 \leq a \leq 1$, we can consider two cases as follows.

B.2.3.1.1 $a^*_{0,0} = a^*_{1,1} = 0$ **or** 1 Consider action $a^*_{1,1}$ for states $(\mathbf{x}+\mathbf{e}_i)$, $(\mathbf{x}+\mathbf{e}_j)$. Then, we have:

$$\begin{aligned}
\Delta_i\Delta_j\mathscr{P}v(\mathbf{x}) =& \mathscr{P}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j) - \mathscr{P}v(\mathbf{x}+\mathbf{e}_i) - \mathscr{P}v(\mathbf{x}+\mathbf{e}_j) + \mathscr{P}v(\mathbf{x}) \\
&\leq g(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j, a^*_{1,1}) - g(\mathbf{x}+\mathbf{e}_i, a^*_{1,1}) - g(\mathbf{x}+\mathbf{e}_j, a^*_{1,1}) + g(\mathbf{x}, a^*_{1,1}) \\
&\leq v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j+a^*_{1,1}\mathbf{e}_1) - v(\mathbf{x}+\mathbf{e}_i+a^*_{1,1}\mathbf{e}_1) - v(\mathbf{x}+\mathbf{e}_j+a^*_{1,1}\mathbf{e}_1) + v(\mathbf{x}+a^*_{1,1}\mathbf{e}_1) \\
&= \Delta_i\Delta_j v(\mathbf{x}+a^*_{1,1}\mathbf{e}_1) \leq 0
\end{aligned} \tag{B.2.6}$$

where the inequality holds because $a^*_{1,1}$ is not necessarily optimal for states $(\mathbf{x}+\mathbf{e}_i)$, $(\mathbf{x}+\mathbf{e}_j)$.

B.2.3.1.2 $a^*_{0,0} = 1$, $a^*_{1,1} = 0$ In this subsection, we consider two cases:

1. $1 \leq i < j$: Consider action $a^*_{1,1} = 0$ for state $(\mathbf{x}+\mathbf{e}_i)$ and action $a^*_{0,0} = 1$ for state $(\mathbf{x}+\mathbf{e}_j)$. Then, we have:

$$\begin{aligned}
\Delta_i\Delta_j\mathscr{P}v(\mathbf{x}) =& \mathscr{P}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j) - \mathscr{P}v(\mathbf{x}+\mathbf{e}_i) - \mathscr{P}v(\mathbf{x}+\mathbf{e}_j) + \mathscr{P}v(\mathbf{x}) \\
&\leq g(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j, 0) - g(\mathbf{x}+\mathbf{e}_i, 0) - g(\mathbf{x}+\mathbf{e}_j, 1) + g(\mathbf{x}, 1) \\
&= v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j) - v(\mathbf{x}+\mathbf{e}_i) - v(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_1) + v(\mathbf{x}+\mathbf{e}_1) \\
&= \Delta_j v(\mathbf{x}+\mathbf{e}_i) - \Delta_j v(\mathbf{x}+\mathbf{e}_1) \leq 0
\end{aligned} \tag{B.2.7}$$

   Where the the last equality is satisfied because when $1 < i < j$, $\Delta_j\Delta_i v(\mathbf{x}) \leq \Delta_j\Delta_1 v(\mathbf{x})$ due to the anti-multimodularity of v.

2. $1 \leq j < i$: Consider action $a^*_{1,1} = 0$ for state $(\mathbf{x}+\mathbf{e}_j)$ and action $a^*_{0,0} = 1$ for state

$(\mathbf{x}+\mathbf{e}_i)$. Then, we have:

$$\begin{aligned}\Delta_i\Delta_j\mathscr{P}v(\mathbf{x}) =& \mathscr{P}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)-\mathscr{P}v(\mathbf{x}+\mathbf{e}_i)-\mathscr{P}v(\mathbf{x}+\mathbf{e}_j)+\mathscr{P}v(\mathbf{x})\\ &\leq g(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j,0)-g(\mathbf{x}+\mathbf{e}_i,1)-g(\mathbf{x}+\mathbf{e}_j,0)+g(\mathbf{x},1)\\ &= v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)-v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_1)-v(\mathbf{x}+\mathbf{e}_j)+v(\mathbf{x}+\mathbf{e}_1)\\ &= \Delta_i v(\mathbf{x}+\mathbf{e}_j)-\Delta_i v(\mathbf{x}+\mathbf{e}_1)\leq 0\end{aligned} \tag{B.2.8}$$

Where the the last equality is satisfied because when $1<j<i$, $\Delta_i\Delta_j v(\mathbf{x})\leq\Delta_i\Delta_1 v(\mathbf{x})$ due to the anti-multimodularity of v.

B.2.3.2 Proof of subconcavity

In this part, we intend to prove that $\Delta_i\Delta_i\mathscr{P}v(\mathbf{x})=\Delta_i\Delta_j\mathscr{P}v(\mathbf{x})\leq 0$, or in other words $\mathscr{P}v(\mathbf{x}+2\mathbf{e}_i)-\mathscr{P}v(\mathbf{x}+\mathbf{e}_i)-\mathscr{P}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)+\mathscr{P}v(\mathbf{x}+\mathbf{e}_j)\leq 0$. To that end, we consider the following cases.

1. $1\leq i<j$:

$$\begin{aligned}&\mathscr{P}v(\mathbf{x}+2\mathbf{e}_i)-\mathscr{P}v(\mathbf{x}+\mathbf{e}_i)-\mathscr{P}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j)+\mathscr{P}v(\mathbf{x}+\mathbf{e}_j)\\ &\leq g(\mathbf{x}+2\mathbf{e}_i,0)-g(\mathbf{x}+\mathbf{e}_i,0)-g(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j,1)+g(\mathbf{x}+\mathbf{e}_j,1)\\ &= v(\mathbf{x}+2\mathbf{e}_i)-v(\mathbf{x}+\mathbf{e}_i)-v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_1)+v(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_1)\\ &= \Delta_i\Delta_i v(\mathbf{x}+\mathbf{e}_1)-\Delta_i\Delta_j v(\mathbf{x}+\mathbf{e}_1)\leq 0\end{aligned} \tag{B.2.9}$$

2. $1 \le j < i$:

$$\begin{aligned}
&\mathscr{P}v(\mathbf{x}+2\mathbf{e}_i) - \mathscr{P}v(\mathbf{x}+\mathbf{e}_i) - \mathscr{P}v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j) + \mathscr{P}v(\mathbf{x}+\mathbf{e}_j) \\
&\le g(\mathbf{x}+2\mathbf{e}_i,0) - g(\mathbf{x}+\mathbf{e}_i,1) - g(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j,0) + g(\mathbf{x}+\mathbf{e}_j,1) \\
&= v(\mathbf{x}+2\mathbf{e}_i) - v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_1) - v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j) + v(\mathbf{x}+\mathbf{e}_j+\mathbf{e}_1) \\
&= \left[\Delta_i\Delta_i v(\mathbf{x}) - \Delta_i\Delta_j v(\mathbf{x})\right] + \left[\Delta_1\Delta_j v(\mathbf{x}) - \Delta_1\Delta_i v(\mathbf{x})\right] \le 0
\end{aligned} \tag{B.2.10}$$

B.2.4 Proof of $\mathscr{E}_k v \in \Omega \ \ \forall k = 1,2,\ldots,n$

B.2.4.1 Proof of submodularity

To prove the submodularity condition for operators $\mathscr{E}_k v \ \ \forall k = 1,2,\ldots,n$, we consider two cases as follows.

B.2.4.1.1 $i \neq j \neq k$ We first consider $\mathscr{E}_k v$ for $k < n$.

$$\Delta_i\Delta_j\mathscr{E}_k v(\mathbf{x}) = \begin{cases} x_k\Delta_i\Delta_j v(\mathbf{x}-\mathbf{e}_k+\mathbf{e}_{k+1}) + (C-x_k)\Delta_i\Delta_j v(\mathbf{x}) \le 0 & x_k \ge 1 \\ C\Delta_i\Delta_j v(\mathbf{x}) \le 0 & x_k = 0 \\ 0 \le 0 & x_k = C \end{cases} \quad \forall k \le n-1 \tag{B.2.11}$$

Then, let us consider $\mathscr{E}_n v$.

$$\Delta_n\Delta_j\mathscr{E}_k v(\mathbf{x}) = \begin{cases} x_n\Delta_i\Delta_j v(\mathbf{x}) + (C-x_n)\Delta_i\Delta_j v(\mathbf{x}) \le 0 & x_n \ge 1 \\ C\Delta_i\Delta_j v(\mathbf{x}) \le 0 & x_n = 0 \end{cases} \tag{B.2.12}$$

B.2.4.1.2 $i = k$ **or** $j = k$ Let us first consider $\mathscr{E}_k v\ \forall k < n$ and $i = k$. When $0 < x_i < C$, we have:

$$\begin{aligned}\Delta_i\Delta_j\mathscr{E}_k v(\mathbf{x}) &= \left[(x_i+1)\,\Delta_j v(\mathbf{x}+\mathbf{e}_{i+1}) + (C-x_i-1)\Delta_j v(\mathbf{x}+\mathbf{e}_i)\right] \\ &\quad - \left[x_i\Delta_j v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_{i+1}) + (C-x_i)\Delta_j v(\mathbf{x})\right] \\ &\leq x_i\Delta_i\Delta_j v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_{i+1}) + (C-x_i-1)\Delta_i\Delta_j v(\mathbf{x}) \leq 0\end{aligned} \tag{B.2.13}$$

Where the first inequality is satisfied because $\Delta_j v(\mathbf{x}+\mathbf{e}_{i+1}) \leq \Delta_j v(\mathbf{x})$ due to the submodularity of v, and the second inequality directly follows from the submodularity of v. Also, when $x_i = 0$, we have $\Delta_i\Delta_j\mathscr{E}_k v(\mathbf{x}) = C\Delta_i\Delta_j v(\mathbf{x}) \leq 0$, and when $x_i = C$, we have $\Delta_i\Delta_j\mathscr{E}_k v(\mathbf{x}) \leq 0$. Now, let us consider the case of $\mathscr{E}_n v$. Assume that $i = n$, then we have:

$$\Delta_n\Delta_j\mathscr{E}_k v(\mathbf{x}) = \begin{cases} x_n\Delta_n\Delta_j v(\mathbf{x}) + (C-x_n-1)\Delta_n\Delta_j v(\mathbf{x}) \leq 0 & x_n \geq 1 \\ C\Delta_n\Delta_j v(\mathbf{x}) \leq 0 & x_n = 0 \end{cases} \tag{B.2.14}$$

B.2.4.2 Proof of subconcavity

We want to show $\Delta_{\tilde{i}}\Delta_{\tilde{j}}\mathscr{E}_k v(\mathbf{x}) = v(\mathbf{x}+2\mathbf{e}_i) - v(\mathbf{x}+\mathbf{e}_i) - v(\mathbf{x}+\mathbf{e}_i+\mathbf{e}_j) + v(\mathbf{x}+\mathbf{e}_j) \leq 0$ for any $k \leq n$. Similar to the proof of submodularity condition, we consider two cases.

B.2.4.3 $i \neq j \neq k$

We first consider $\mathscr{E}_k v$ for $k < n$.

$$\Delta_{\tilde{i}}\Delta_{\tilde{j}}\mathscr{E}_k v(\mathbf{x}) =$$

$$\begin{cases} x_k(\Delta_i\Delta_i v(\mathbf{x}) - \Delta_i\Delta_i v(\mathbf{x}) - \mathbf{e}_k + \mathbf{e}_{k+1}) + (C - x_k)(\Delta_i\Delta_i v(\mathbf{x}) - \Delta_i\Delta_j v(\mathbf{x})) \leq 0 & x_k \geq 1 \\ C(\Delta_i\Delta_i v(\mathbf{x} - \Delta_i\Delta_j v(\mathbf{x})) \leq 0 & x_k = 0 \\ 0 \leq 0 & x_k = C \end{cases} \quad \forall k \leq n-1 \tag{B.2.15}$$

Then, let us consider $\mathscr{E}_n v$.

$$\Delta_n\Delta_n\mathscr{E}_k v(\mathbf{x}) - \Delta_n\Delta_j\mathscr{E}_k v(\mathbf{x}) = \begin{cases} x_n(\Delta_n\Delta_n v(\mathbf{x}) - \Delta_n\Delta_j v(\mathbf{x})) + (C - x_n)(\Delta_n\Delta_n v(\mathbf{x}) - \Delta_n\Delta_j v(\mathbf{x})) \leq 0 & x_n \geq 1 \\ C(\Delta_n\Delta_n v(\mathbf{x}) - \Delta_n\Delta_j v(\mathbf{x})) \leq 0 & x_n = 0 \end{cases} \tag{B.2.16}$$

B.2.4.3.1 $i = k$ **or** $j = k$ Let us first consider $\mathscr{E}_k v$ $\forall k < n$. We consider $i = k$. When $0 < x_i < C$, we have:

$$
\begin{aligned}
\Delta_{\tilde{i}}\Delta_{\tilde{j}}\mathscr{E}_i v(\mathbf{x}) =& x_i\Delta_i\Delta_i v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_{i+1}) + (C-x_i)\Delta_i\Delta_i v(\mathbf{x}) \\
&+ 2\left(\Delta_i v(\mathbf{x}+\mathbf{e}_{i+1} - \Delta_i v(\mathbf{x}+\mathbf{e}_i)\right) \\
&- x_i\Delta_i\Delta_j v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_{i+1}) - (C-x_i)\Delta_i\Delta_j v(\mathbf{x}) \\
&- \left(\Delta_j v(\mathbf{x}+\mathbf{e}_{i+1} - \Delta_j v(\mathbf{x}+\mathbf{e}_i)\right) \\
&\leq x_i\Delta_{\tilde{i}}\Delta_{\tilde{j}} v(\mathbf{x}-\mathbf{e}_i+\mathbf{e}_{i+1}) + (C-x_i-2)\Delta_{\tilde{i}}\Delta_{\tilde{j}} v(\mathbf{x}) \leq 0
\end{aligned}
\tag{B.2.17}
$$

Now, let us consider $\mathscr{E}_n v$.

$$
\Delta_{\tilde{i}}\Delta_{\tilde{j}}\mathscr{E}_n v(\mathbf{x}) =
\begin{cases}
x_n\Delta_{\tilde{i}}\Delta_{\tilde{j}} v(\mathbf{x}-\mathbf{e}_n) + (C-x_n-2)\Delta_{\tilde{i}}\Delta_{\tilde{j}} v(\mathbf{x}) \leq 0 & x_n \geq 1 \\
C\Delta_{\tilde{i}}\Delta_{\tilde{j}} v(\mathbf{x}) \leq 0 & x_n = 0
\end{cases}
\tag{B.2.18}
$$

Also, the subconcavity for the case when $j = k$ can be proved in a similar way.

B.3 Proof of Theorem 3.2

B.3.1 Part 1

The definition of $S(\mathbf{x}_{-1})$ implies that when the current inventory level $\mathbf{x}$ satisfies $x_1 < S(\mathbf{x}_{-1})$, we have $\Delta_1 v(\mathbf{x}) \geq c_p$. This indicates that it is more advantageous to produce one fresh product rather than producing nothing.

We now demonstrate that when the current level $\mathbf{x}$ satisfies $x_1 \geq S(\mathbf{x}_{-1})$, we have $\Delta_1 v(\mathbf{x}) < c_p$. Using induction, we can establish this. Initially, we have $\Delta_1 v(S(\mathbf{x}_{-1}), \mathbf{x}_{-1}) <$

c_p. Assuming that for some $\mathbf{x}$ with $x_1 \geq S(\mathbf{x}_{-1})$, $\Delta_1 v(\mathbf{x}) < c_p$ holds true, we can deduce that $\Delta_1 v(\mathbf{x}+\mathbf{e}_1) \leq \Delta_1 v(\mathbf{x}) < c_p$, where the first inequality is a result of value function structural properties. Thus, $(\mathbf{x}+\mathbf{e}_1)$ also satisfies $\Delta_1 v(\mathbf{x}+\mathbf{e}_1) < c_p$. Consequently, for $\mathbf{x}$ satisfying $x_1 \geq S(\mathbf{x}_{-1})$, we have $\Delta_1 v(\mathbf{x}) < c_p$ which indicates that it is optimal to produce nothing.

B.3.2 Part 2

By utilizing property (P1) and the definition of $S(\mathbf{x}_{-1})$, we find that $\Delta_1 v(S(\mathbf{x}_{-1}), \mathbf{x}_{-1} + \mathbf{e}_j) \leq \Delta_1 v(S(\mathbf{x}_{-1}), \mathbf{x}_{-1}) < c_p \ \forall j \neq 1$. Also, because we have $S(\mathbf{x}_{-1}+\mathbf{e}_j) = \min\left\{x_1 | \Delta_1 v\left(\mathbf{x}+\mathbf{e}_j\right) < c_p\right\}$ and $\Delta_1 v(\mathbf{x}+\mathbf{e}_1) \leq \Delta_1 v(\mathbf{x}) < c_p$, we can conclude that $S(\mathbf{x}_{-1}+\mathbf{e}_j) \leq S(\mathbf{x}_{-1})$.

B.3.3 Part 3

To prove $\Delta_j S(\mathbf{x}_{-1}) \leq \Delta_i S(\mathbf{x}_{-1})$ for all $i > j$, it suffices to show that $S(\mathbf{x}_{-1}+\mathbf{e}_j) \leq S(\mathbf{x}_{-1} + \mathbf{e}_i)$. Based on anti-multimodularity property of value function, we have $\Delta_1 v(S(\mathbf{x}_{-1}), \mathbf{x}_{-1} + \mathbf{e}_j) \leq \Delta_1 v(S(\mathbf{x}_{-1}), \mathbf{x}_{-1}+\mathbf{e}_i) \leq \Delta_1 v(S(\mathbf{x}_{-1}), \mathbf{x}_{-1}) < c_p \ \forall 1 < j < i$. Because $\Delta_1 v(\mathbf{x}+\mathbf{e}_1) \leq \Delta_1 v(\mathbf{x}) < c_p$, we conclude that $\Delta_j S(\mathbf{x}_{-1}) \leq \Delta_i S(\mathbf{x}_{-1})$ for all $i > j$.

Also, using anti-multimodularity, there is an alternative way to express property in part (3) elaborated as follows. In Appendix B.2, we reformulate the production operator $\mathscr{P}v(\mathbf{x})$ as follows.

$$\mathscr{P}v(\mathbf{x}) = \max_{a \in Q(\mathbf{X})} \left\{v(\mathbf{x}+a) - a \times c_p\right\} \tag{B.3.1}$$

Where $a \in \{0,1\}$ denotes the production action. Thus, the production operator can be written as follows.

$$\mathscr{P}v(\mathbf{x}) = \max_{a \in Q(a,\mathbf{X})} \tilde{g}(a,\mathbf{x}) \tag{B.3.2}$$

Where $Q(a,\mathbf{x}) = \{0 \leq a \leq 1;\ x_1 + a \leq C\}$ and

$$\tilde{g}(a,\mathbf{x}) = \left\{v(\mathbf{x}+a) - a \times c_p\right\} \tag{B.3.3}$$

Because $\mathscr{P}v(\mathbf{x})$ is anti-multimodular and $Q(a,\mathbf{x})$ is a polyhedron satisfying necessary condition for anti-multimodularity (See Li and Yu (2014) for more details), then, based on Theorem 1 in Li and Yu (2014), we can conclude:

$$-1 \leq \Delta_1 a(\mathbf{x}) \leq \Delta_2 a(\mathbf{x}) \leq \ldots \leq \Delta_{n-1} a(\mathbf{x}) \leq \Delta_n a(\mathbf{x}) \leq 0 \tag{B.3.4}$$

B.4 Proof of Proposition 3.1

First, we assume that $\mathbf{x} > 0$. We can rewrite operator $\mathscr{R}$ as $\mathscr{R}v(\mathbf{x}) = \max_{\mathbf{P} \in \mathscr{P}} f(\mathbf{x},\mathbf{P})$ where $f = \sum_k^n \alpha_k \left(v(\mathbf{x} - \mathbf{e}_k) + p_k\right) + \alpha_{n+1} v(\mathbf{x})$. Substituting purchase probabilities in operator f, it is easy to confirm that $f(\mathbf{x},\mathbf{P})$ is a quadratic function of $\mathbf{P}$. The following Lemma also indicates that $f(\mathbf{x},\mathbf{P})$ is concave in $\mathbf{P}$.

Lemma B.4.1 *The revenue operator $f(\mathbf{x},\mathbf{P})$ is a concave function of the price vector $\mathbf{P} = \{p_1, p_2, \ldots, p_n\}$.*

Readers may refer to Akçay et al. (2010) for the proof. Then, based on the first-order condition, we can obtain the optimal prices by solving a system of equations $\frac{\partial f(\mathbf{X},\mathbf{P})}{p_k} = 0\ \forall k = 1,2,\ldots,n$ which yields $\frac{1}{2}\left(q_k + \Delta_k v(\mathbf{x})\right)$. Therefore, for any available product k, the optimal price equals $\frac{1}{2}\left(q_k + \Delta_k v(\mathbf{x})\right)$. Now, let us consider products with $x_k = 0$ for some k. Based on the feasible region $\mathscr{P}$, the optimal prices for unavailable products must be set such that the purchase probability for those products be zero. Therefore, using conditions in feasible set $\mathscr{P}$, we can obtain the prices for products with $x_k = 0$ for some k as follows.

$$p_k = \begin{cases} p_2(\mathbf{x}) + (q_1 - q_2) & x_k = 0,\ k = 1 \\ \frac{(q_k - q_{k+1})p_{k-1}(\mathbf{x}) + (q_{k-1} - q_k)p_{k+1}(\mathbf{x})}{q_{k-1} - q_{k+1}} & x_k = 0,\ k = 2, 3, \ldots, n \end{cases} \tag{B.4.1}$$

Therefore, in summary, there are two steps in determination of the optimal prices. First, we ignore the unavailable products, and solve the problem for positive inventory which yields the optimal price of $\frac{1}{2}\left(q_k + \Delta_k v(\mathbf{x})\right)$ for product k. Then, in the second step, for products with zero inventory level, we obtain the price by setting its corresponding purchase probability to 0. Therefore, the optimal prices can be written as follows.

$$p_k = \begin{cases} \frac{1}{2}\left(q_k + \Delta_k v(\mathbf{x})\right) & x_k \geq 1 \\ p_2(\mathbf{x}) + (q_1 - q_2) & x_k = 0,\ k = 1 \\ \frac{(q_k - q_{k+1})p_{k-1}(\mathbf{x}) + (q_{k-1} - q_k)p_{k+1}(\mathbf{x})}{q_{k-1} - q_{k+1}} & x_k = 0,\ k = 2, 3, \ldots, n \end{cases} \tag{B.4.2}$$

B.5 Proof of Theorem 3.3

In Proposition 3.1, we obtained the optimal price for item i as $p_i = \frac{1}{2}\left(q_i + \Delta_i v(\mathbf{x})\right)$, therefore, the properties for $\Delta_i v(\mathbf{x})$ is also preserved by the optimal price.

B.5.1 Part 1

Based on optimal price equation $p_i = \frac{1}{2}\left(q_i + \Delta_i v(\mathbf{x})\right)$, we have:

$$\Delta_i p_i = \Delta_i \Delta_i v(\mathbf{x}) \leq 0 \tag{B.5.1}$$

where the inequality holds because of concavity of value function.

B.5.2 Part 2

Similarly, we can calculate $\Delta_j p_i$ for any $j \neq i$ as follows.

$$\Delta_j p_i = \Delta_j \Delta_i v(\mathbf{x}) \leq 0 \tag{B.5.2}$$

where the inequality holds because of submodularity of value function.

B.5.3 Part 3

Based on anti-multimodularity property we have:

$$\Delta_i \Delta_i v(\mathbf{x}) \leq \Delta_i \Delta_{i+1} v(\mathbf{x}) \leq \ldots \leq \Delta_i \Delta_n v(\mathbf{x})$$

and

$$\Delta_i \Delta_i v(\mathbf{x}) \leq \Delta_i \Delta_{i-1} v(\mathbf{x}) \leq \ldots \leq \Delta_i \Delta_1 v(\mathbf{x})$$

Because there is a one-to-one correspondence between p_i and $\Delta_i v(\mathbf{x})$, we can write

$$\Delta_i\, p_i \leq \Delta_{i+1} p_i \leq \ldots \leq \Delta_n p_i \quad \forall i$$

$$\Delta_i\, p_i \leq \Delta_{i-1} p_i \leq \ldots \leq \Delta_1 p_i \quad \forall i$$

.

B.6 Proof of Theorem 3.4

The existence of an average optimal policy is guaranteed by Theorem 8.4.5a of Puterman (1994), provided that the state and action spaces for each state are finite, the profits are bounded, and the model is unichain. A model is considered unichain if the transition matrix contains a single recurrent class, as well as a set of possibly empty transient states. In this particular model, any state $\mathbf{x}$ can be accessed by every stationary policy from any state $\mathbf{x}'$ in $\mathscr{S}$, i.e., state space, demonstrating that the model is unichain.

B.7 Proof of Theorems 3.5, 3.7, and 3.9

It should be noted that in each extended model, several operators are exactly similar to those in the base model. Therefore, here we only examin the operators that are new to the extended model.

Proof of Theorem 3.5: To prove Theorem 3.5, it suffices to show that operator $\mathscr{D}_k v^D(\mathbf{x})$ is submodular and subconcave. This operator is similar to the production operator and its submodularity and subconcavity properties can be shown in a similar way. Therefor, due to brevity, we omit its proof here.

Proof of Theorem 3.7: To prove Theorem 3.7, it suffices to show that operators

$\mathscr{O}v^R(\mathbf{w})$ and $\mathscr{L}v^R(\mathbf{w})$ are submodular and subconcave. By induction on $v^R(\mathbf{w})$ (assuming submodularity and subconcavity for v^R, it is obvious that $\mathscr{L}v^R(\mathbf{w})$ is submodular an subconcave. Next, let us consider operator $\mathscr{O}v^R(\mathbf{w})$. This operator is similar to the batch demand operators in the work by Yang et al. (2022). In that paper, they proved the submodulaity and subconcavity of the operator. Therefore, in a replenishment system, the value function holds subconcavity and submodularity.

Proof of Theorem 3.9: In a model with multiple phases of quality transformation, the structure of all the operators remain similar to the base mode. Therefore, the submodularity and sibconcavity property remain valid in this extended model.

B.8 Proof of Theorem 3.6

B.8.1 Part 1

In this part, we prove the result for the product k which can be similarly extended to the other items. The definition of $S_k^D(\mathbf{x_{-k}})$ implies that when the current inventory level $\mathbf{x}$ satisfies $x_k < S_k^D(\mathbf{x_{-k}})$, we have $\Delta_k v^D(\mathbf{x}) \geq r_k$. This indicates that it is more advantageous to not donate item k rather than donating it.

We now demonstrate that when the current level $\mathbf{x}$ satisfies $x_k \geq S_k^D(\mathbf{x_{-k}})$, we have $\Delta_k v^D(\mathbf{x}) < r_k$. Using induction, we can establish this. Initially, we have $\Delta_k v^D(x_1, x_2, \ldots, x_{k-1}, S_k^D(\mathbf{x_{-k}}), x_{k+1}, \ldots, x_n) < r_k$. Assuming that for some $\mathbf{x}$ with $x_k \geq S_k^D(\mathbf{x_{-k}})$, $\Delta_k v^D(\mathbf{x}) < r_k$ holds true, we can deduce that $\Delta_k v^D(\mathbf{x}+\mathbf{e}_k) \leq \Delta_k v^D(\mathbf{x}) < r_k$, where the first inequality is a result of value function structural properties. Thus, $(\mathbf{x}+\mathbf{e}_k)$ also satisfies $\Delta_k v^D(\mathbf{x}+\mathbf{e}_k) < r_k$. Consequently, for $\mathbf{x}$ satisfying $x_k \geq S^D(\mathbf{x}_{-k})$, we have $\Delta_k v^D(\mathbf{x}) < r_k$ which indicates that it is optimal to donate product k. Similarly, we can show the result for other products.

B.8.2 Part 2

Let us consider product k and $j \neq k$. Without loss of generality, we assume that $j < k$. By utilizing property (P1) and the definition of $S_k^D(\mathbf{x_{-k}}))$, for all $j \neq k$, we find that:

$$\Delta_k v^D(x_1,\ldots,x_j+1,\ldots,x_{k-1},S_k^D(\mathbf{x_{-k}}),x_{k+1},\ldots,x_n) \leq \Delta_k v^D(x_1,\ldots,x_{k-1},S_k^D(\mathbf{x_{-k}}),x_{k+1},\ldots,x_n) < r_k$$

Also, because we have $S_k^D(\mathbf{x_{-k}}+\mathbf{e}_j) = \min\left\{x_k|\Delta_k v^D\left(\mathbf{x}+\mathbf{e}_j\right) < r_k\right\}$ we can conclude that $S_k^D(\mathbf{x_{-k}}+\mathbf{e}_j) \leq S_k^D(\mathbf{x}_{-k})$.

B.8.3 Part 3

To prove $\Delta_j S_k^D(\mathbf{x}_{-k}) \leq \Delta_i S_k^D(\mathbf{x}_{-k})$ for all $i > j$, it suffices to show that $S_k^D(\mathbf{x}_{-k}+\mathbf{e}_j) \leq S_k^D(\mathbf{x}_{-k}+\mathbf{e}_i)$. Based on anti-multimodularity property of value function, we have:

$$\begin{aligned}
&\Delta_k v^D(x_1,\ldots,x_j+1,\ldots,x_{k-1},S_k^D(\mathbf{x_{-k}}),x_{k+1},\ldots,x_n)\\
&\leq \Delta_k v^D(x_1,\ldots,x_i+1,\ldots,x_{k-1},S_k^D(\mathbf{x_{-k}}),x_{k+1},\ldots,x_n)\\
&\leq \Delta_k v^D(\mathbf{x}) < r_k
\end{aligned} \tag{B.8.1}$$

Because $\Delta_k v^D(\mathbf{x}+\mathbf{e}_k) \leq \Delta_1 v^D(\mathbf{x}) < r_k$, we conclude that $\Delta_j S_k^D(\mathbf{x}_{-k}) \leq \Delta_i S_k^D(\mathbf{x}_{-k})$ for all $i > j$.

B.9 Proof of Theorem 3.8

B.9.1 Part 1

The definition of $S^R(\mathbf{x})$ implies that given the current inventory level $\mathbf{x}$, when $y < S^R(\mathbf{x})$, we have $\Delta_y v^R(\mathbf{w}) \geq c_o$. This indicates that as long as $y < S^R(\mathbf{x})$ it is more beneficial to order one more unit rather than not ordering nothing. Therefore, when the inventory level is $\mathbf{x}$ and the number of units in order is y, the optimal order quantity is $y - (S^R - 1)$ that still satisfies $y < S^R(\mathbf{x})$.

We now demonstrate that when the current level is $\mathbf{x}$, when $y \geq S^R(\mathbf{x})$, we have $\Delta_y v^R(\mathbf{x}) < c_o$. Using induction, we can establish this. Initially, we have $\Delta_y v^R(\mathbf{x}, S^R(\mathbf{x})) < c_o$. Assuming that for some $(\mathbf{x}, y)$ with $y \geq S^R(\mathbf{x})$, $\Delta_y v^R(\mathbf{w}) < c_o$ holds true, we can deduce that $\Delta_y v^R(\mathbf{x}, y+1) \leq \Delta_y v^R(\mathbf{x}, y) < c_o$, where the first inequality is a result of value function structural properties. Thus, $(\mathbf{x}, y+1)$ also satisfies $\Delta_y v^R(\mathbf{x}, y+1) < c_o$. Consequently, for $(\mathbf{x}, y)$ satisfying $y \geq S^R(\mathbf{x}))$, we have $\Delta_y v^R(\mathbf{x}, y) < c_o$ which indicates that it is optimal to not order any fresh product.

B.9.2 Part 2

By utilizing property (P1) and the definition of $S^R(\mathbf{x})$, we find that $\Delta_y v^R(\mathbf{x} + \mathbf{e}_j, S^R(\mathbf{x})) \leq \Delta_y v^R(\mathbf{x}, S^R(\mathbf{x})) < c_o \;\; \forall j \neq y$. Also, because we have $S^R(\mathbf{x}) = \min\left\{y | \Delta_y v\left(\mathbf{w} + \mathbf{e}_j\right) < c_o\right\}$, we can conclude that $S^R(\mathbf{x} + \mathbf{e}_j) \leq S^R(\mathbf{x})$.

B.9.3 Part 3

To prove $\Delta_j S^R(\mathbf{x}) \leq \Delta_i S^R(\mathbf{x})$ for all $i > j$, it suffices to show that $S^R(\mathbf{x}+\mathbf{e}_j) \leq S^R(\mathbf{x}+\mathbf{e}_i)$. Based on anti-multimodularity property of value function, we have $\Delta_y v^R(\mathbf{x}+\mathbf{e}_j, S^R(\mathbf{x})) \leq \Delta_y v^R(\mathbf{x}+\mathbf{e}_i, S^R(\mathbf{x})) \leq \Delta_y v^R(\mathbf{x}, S^R(\mathbf{x})) < c_o \ \forall 1 < j < i$. Because $\Delta_y v^R(\mathbf{x}, y+1) \leq \Delta_y v^R(\mathbf{x}) < c_o$, we conclude that $\Delta_j S^R(\mathbf{x}) \leq \Delta_i S^R(\mathbf{x})$ for all $i > j$.

Appendix C

Supplement to Chapter 4

C.1 Proofs of Propositions 4.1, 4.6, and 4.7

C.1.1 Proof of Proposition 4.1

According to the process W, outdates occur when W hits zero (remaining time until the next outdate is zero). Therefore, the average rate of outdates is equivalent to $f(0)$ provided in Equation 4.5 and the otdating probability can be obtained as $q^N = \frac{f(0)}{\mu}$ which is given by:

$$q^N = \begin{cases} \frac{(\mu-\lambda_0)}{\mu-\lambda e^{-(\lambda_0-\mu)\theta}}, & \mu \neq \lambda_0 \\ \frac{1}{1+\theta\mu} & \mu = \lambda_0 \end{cases} \tag{C.1.1}$$

C.1.2 Proof of Proposition 4.6

Under LIFO policy, the outdating probability, denoted by q^L, is equivalent to the probability that the busy period duration exceeds the shelf life θ. Therefore, using the cumulative function of busy period at time x, i.e., $S(x)$, the outdating probability can be written as $q^L = 1 - S(\theta)$. Further, based on the conservation law, the shortage probability under LIFO issuing policy can be written as $l^L = 1 - \frac{\mu}{\lambda_0} S(\theta)$.

C.1.3 Proof of Proposition 4.7

The probability that an item becomes expired in a freshness-dependent demand is that the item becomes expired in the first system with probability $1 - S(T)$ and becomes expired in the second system with probability of $1 - S(\theta - T)$. Therefore, the probability of expiration can be written as $(1 - S(T))(1 - S(\theta - T))$. Then, based on the conservation law for the first system, the shortage probability for fresh products can be written as $l_0^{FL} = 1 - \frac{\mu}{\lambda} S(T)$, and shortage probability for non-fresh items can be obtained as $l_1^{FL} = 1 - \frac{\mu(1-S(T))}{\lambda_1} S(\theta - T)$.

C.2 Proof of Proposition 4.2

To obtain the desired results, we reformulate the problem by defining $\mu = \phi_0 \Lambda + \rho$. Then, substituting this equation into the original value function, we can reformulate the problem in terms of ϕ_0 and ρ as follows.

$$V^{N*} = \max_{\rho, 0 \leq \phi_0 \leq 1} (\Gamma (1 - \phi_0) + C_e + C_s)(\phi_0 \Lambda + \bar{q}) - C_s \phi_0 \Lambda - (C_e + C_p)(\phi_0 \Lambda + \rho) \tag{C.2.1}$$

where $\bar{q} = \frac{\phi_0 \Lambda \rho}{\phi_0 \Lambda - (\phi_0 \Lambda + \rho) e^{\theta \rho}}$. It is obvious that in the above optimization problem, the second and the third terms are linear functions, and subsequently are concave. To prove the concavity of the first expression in optimization problem, we obtain its corresponding hessian matrix H_1 and use minor principals. Please note that due to the complexity of the formulas involved, we have omitted the complete representation of the Hessian matrix in this context. Instead, we will focus on the first and second minors of the Hessian matrix H_1, denoted by $\Delta_1^{H_1}$ and $\Delta_2^{H_1}$, respectively. To prove the concavity of the first term, we should prove that $\Delta_1^{H_1} \leq 0$ and $\Delta_2^{H_1} \geq 0$. We can express Δ_{H_1} as follows.

$$\begin{aligned} \Delta_1^{H_1} = & -\frac{2\Lambda^2 \rho^2 e^{\theta \rho} \left(1 - e^{\theta \rho}\right) \left(\Gamma(1-\phi_0) + C_e + C_s\right)}{1 - e^{\theta \rho}} \\ & - \frac{2\Lambda \Gamma \left(1 - e^{\theta \rho}\right) \left(\Lambda^2 \phi_0^2 \left(1 - e^{\theta \rho}\right) - \rho^2 e^{\theta \rho} - 2\Lambda \phi_0 \rho e^{\theta \rho}\right)}{\left(\Lambda \phi_0 \left(1 - e^{\theta \rho}\right) - \rho e^{\theta \rho}\right)^2} \leq 0 \end{aligned} \tag{C.2.2}$$

Second minor of Hessian matrix results in a very huge mathematical formulation that makes it very challenging to evaluate. To prove that $\Delta_2^{H_1} \geq 0$ we simulate the values of $\Delta_2^{H_1}$ for a wide range of combinations of parameters. Let us denote ρ_0 as the root to the equation $\Delta_2^{H_1} = 0$. We show that when $\rho \geq \rho_0$, then $\Delta_2^{H_1} \geq 0$. To that end, we evaluate $\Delta_2^{H_1}$ for combinations arises when $C_e, C_s, \Gamma, \Lambda$ are generated from uniform distribution between $[0, 10000]$, and θ, ϕ_0, and ρ are generated from a uniform distribution between $[0, 10]$, $[0, 1]$, and $[\rho_0, \Lambda \phi]$, respectively. We generate 10^6 distinct combinations of parameters and observe the values of $\Delta_2^{H_1}$ as shown in Figure C.2.1.

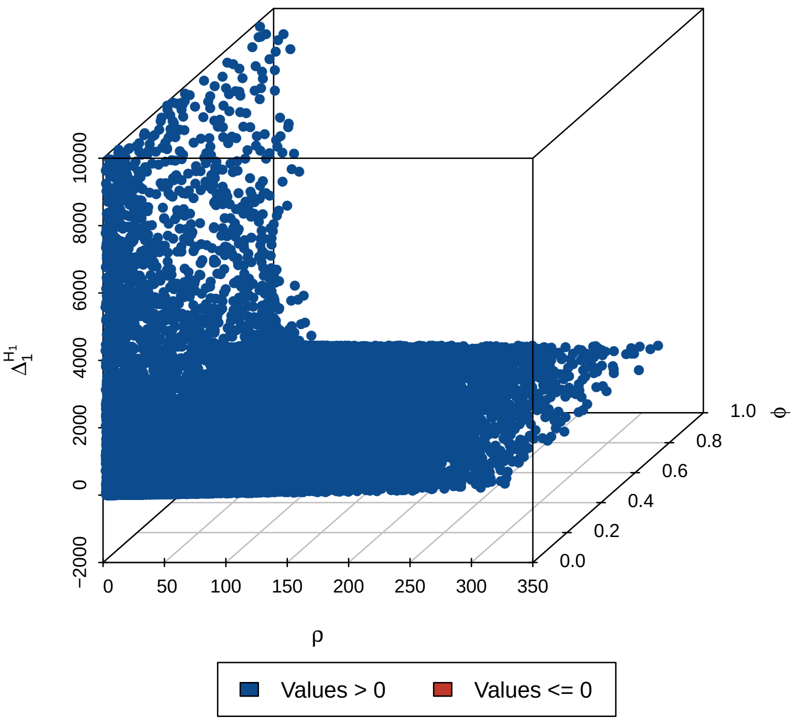


Figure C.2.1: Determinant of Hessian Matrix when $\rho \geq \rho_0$

Figure C.2.1 indicates that the second minor of the Hessian matrix is always positive when $\rho \geq \rho_0$ where $\rho_0 < 0$ and therefore the optimization problem is concave for $\rho \geq \rho_0$. Next, we need to show that the optimal ρ occurs when $\rho \geq \rho_0$. For any ϕ_0, the optimization problem is continuous and increasing in $\rho \leq \rho_0$ as shown in Figure C.2.2, while it is concave in $\rho \geq \rho_0$.

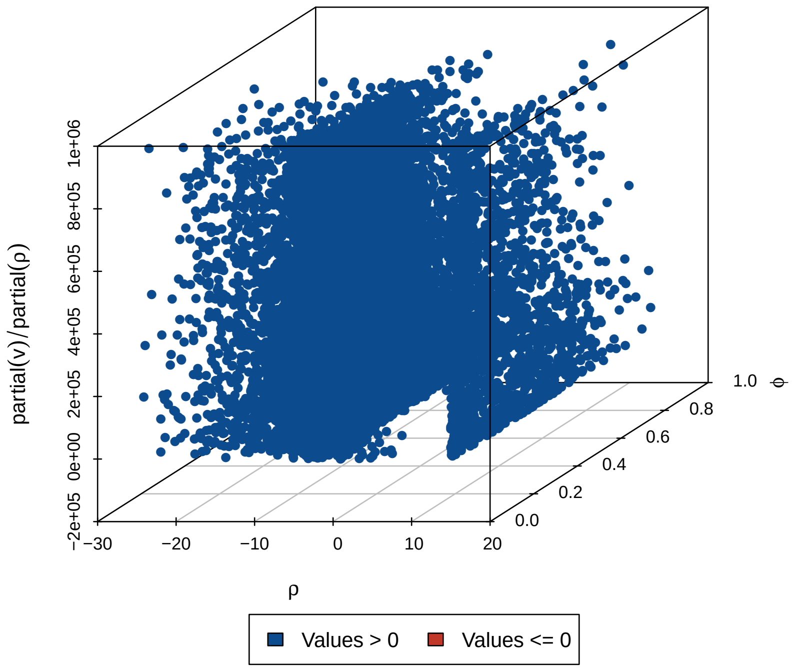


Figure C.2.2: Partial derivative of profit function with respect to ρ when $\rho \leq \rho_0$

Since $\frac{\partial V^N}{\partial \rho} < 0$ for $\rho < \rho_0$, the optimal ρ is greater than $\rho_0 < 0$, or in other words $\exists \rho^* \geq \rho_0 \ : \ \frac{\partial V^N}{\partial \rho^*} = 0$. Based on the above simulation results, the optimal solutions can be obtained by solving $\frac{\partial V^N}{\partial \phi_0} = 0$ and $\frac{\partial V^N}{\partial \rho} = 0$. Then, taking the first derivative from value function with respect to ρ, we have:

$$\frac{2\Lambda\theta\phi_0\left(2\theta\rho^* + \Lambda\theta\phi_0^* + 2\right)\left(\Gamma + C_s + C_e - \Gamma\phi_0^*\right)}{2\theta\rho^* + \theta^2\rho^{*2} + 2\Lambda\theta\phi_0^* + \Lambda\theta^2\phi_0^*\rho^* + 2} - C_e - C_p = 0 \tag{C.2.3}$$

By introducing $X = e^{\theta\rho}\left(\rho + \phi_0\Lambda\right)$, the above equation can be written as a quadratic equation as follows.

$$AX^2 + BX + C = 0 \tag{C.2.4}$$

where

$$A = -\frac{C_e + C_p}{C_e + C_s + \Gamma\left(1-\phi_0^*\right)} \tag{C.2.5}$$

$$B = \Lambda\phi_0^*\rho^*\theta - \Lambda\phi_0^* + 2A\Lambda\phi_0^* \tag{C.2.6}$$

$$C = \Lambda\phi_0^*\rho^* e^{\theta\rho^*} + (1-A)\Lambda^2\phi_0^{*2} \tag{C.2.7}$$

where $0 < A < 1$ Because $X = e^{\theta\rho}\left(\rho + \phi_0\Lambda\right)$ is always positive, the solution to the above equation is $X^* = \frac{-B-\sqrt{B^2-4AC}}{2A}$. Then, after doing some mathematics, we have $\rho^* = \frac{\log\frac{-B-\sqrt{B^2-4AC}}{2A} - \log(\rho+\phi_0\Lambda)}{\theta}$. We can verify that when $A \to 0$, $\rho^* < 0$ and when $A \to 1$, $\rho^* > 0$. Because above solution is decreasing in A, when A is greater than a threshold A_0, then $\frac{-B-\sqrt{B^2+4AC}}{-2A} < (\rho + \phi_0\Lambda)$. Equivalently, when Γ is less than a threshold Γ_0, ρ is negative, otherwise it has a positive value. Thus, solving the above equation for Γ, we obtain Γ_0 as follows.

$$\Gamma_0^* = \frac{2C_p + C_e}{\left(1-\phi_0^*\right)} + \frac{2\left(C_p + C_e\right)}{\Lambda\theta\phi_0^*\left(2+\Lambda\theta\phi_0^*\right)\left(1-\phi_0^*\right)} \tag{C.2.8}$$

where and ϕ_0^* and ρ^* are solutions to the obtained system of equations from $\frac{\partial V^N}{\partial\phi_0} = 0$ and $\frac{\partial V^N}{\partial\rho} = 0$. Also, we have:

$$\mu^* = \phi^*\Lambda + \rho^* \quad \text{where} \begin{cases} \rho^* \geq 0, & \Gamma \geq \Gamma_0^* \\ \rho^* < 0 & \Gamma < \Gamma_0^* \end{cases} \tag{C.2.9}$$

C.3 Proof of Theorem 4.1

In the multiple-stage markdown policy, the arrival rate of products follows a Poisson process with a rate of μ, while the sales rate at each stage j depends on the remaining shelf life of the products and is represented by λ_j. This model assumes a constant production rate but a sales rate that varies depending on the product's state. To analyze this model, we utilize the findings from Boxma et al. (2023), who derived the steady-state density of time until the next product outdate for the case with state-dependent supply and demand. Here, we define μ as the arrival rate and $\lambda(x)$ as the demand rate at state x. Additionally, we introduce $L(x) = \int_0^x \lambda(y)dy$ as the cumulative demand rate. Considering a scenario with constant supply and state-dependent demand, we can formulate the balance equation as follows.

$$f(x) = \int_0^{\theta} \lambda(w)e^{-\mu(x-w)} f(w)dw + f(0)e^{-\mu x} \ \ \forall x \in [0, \theta] \tag{C.3.1}$$

Now we break down shelf life into $Q+1$ stages, where at stage $j \in [0, Q]$ demand is denoted by λ_j. Each stage j starts at the shelf life of $x \wedge m_{j+1}$ and ends at the shelf life of $x \wedge m_j$. Therefore, we can rewrite the balance equation for the case with Q markdown stages as follows.

$$f^Q(x) = \sum_{j=0}^{Q} \lambda_j \int_{x\wedge m_{j+1}}^{x\wedge m_j} e^{-\mu(x-w)} f^Q(w)dw + f^Q(0)e^{-\mu x} \ \ \forall x \in [0, \theta], \tag{C.3.2}$$

where we assume that $m_0 = \theta$ and $m_{Q+1} = 0$. By using normalization constraint $\int_0^{\infty} f^Q(x) = 1$, we can obtain the steady-state density of VOP as follows.

$$f_j^Q(x) = f^Q(0)\, e^{(\lambda_j-\mu)x+\sum_{i=j}^{Q-1}(\lambda_{i+1}-\lambda_i)m_{i+1}} \ \ \forall x \in [m_{j+1}, m_j],\ \forall j \in [0, Q], \tag{C.3.3}$$

where $m_0 = \theta$ and $m_{Q+1} = 0$. Also, $f(0)$ can be written as follows.

$$f^Q(0) = \left[\sum_{j=0}^{Q} \int_{x \wedge m_{j+1}}^{x \wedge m_j} e^{\left(\lambda_j - \mu\right)x + \sum_{i=j}^{Q-1} (\lambda_{i+1} - \lambda_i) m_{i+1}} + e^{\sum_{i=0}^{Q} \lambda_i (m_{i+1} - m_i)} \frac{e^{-\mu\theta}}{\mu} \right]^{-1}, \qquad \text{(C.3.4)}$$

C.4 Proofs of Theorems 4.3, 4.4, and 4.5

C.4.1 Proof of Theorem 4.3

As $\theta \to \infty$, we have the following relations.

$$\lim_{\theta \to \infty} q = \begin{cases} \frac{\mu - \phi_0 \Lambda}{\mu}, & \phi_0 \Lambda \leq \mu \\ 0 & \phi_0 \Lambda \geq \mu \end{cases} \qquad \text{(C.4.1)}$$

Therefore, when $\theta \to \infty$, the value function V^N can be rewritten as follows.

$$V^{N*} = \begin{cases} \max_{\mu \geq 0, 0 \leq \phi_0 \leq 1} (1 - \phi_0) \Gamma \lambda - C_p \mu - C_e (\mu - \lambda), & \phi_0 \Lambda \leq \mu \\ \max_{\mu \geq 0, 0 \leq \phi_0 \leq 1} \left[(1 - \phi_0) \Gamma - C_p \right] \mu - C_s (\lambda - \mu) & \phi_0 \Lambda \geq \mu \end{cases} \qquad \text{(C.4.2)}$$

Next, we use Karush–Kuhn–Tucker (K.K.T) conditions to obtain the optimal ϕ_0 and μ in the above problem.

We introduce a Lagrange multiplier ζ. When $\phi_0 \Lambda \leq \mu$, the Lagrangian function can be written as follows.

$$L_1 = \max_{\mu \geq 0, 0 \leq \phi_0 \leq 1} (1 - \phi_0) \Gamma \lambda - C_p \mu - C_e (\mu - \lambda) - \zeta (\phi_0 \Lambda - \mu) \qquad \text{(C.4.3)}$$

Then we write the K.K.T optimality conditions as follows.

$$\frac{\partial L_1}{\partial \phi_0} = \Gamma\Lambda - 2\phi_0\Gamma\Lambda + C_e\Lambda - \zeta\Lambda = 0 \tag{C.4.4}$$

$$\frac{\partial L_1}{\partial \mu} = -C_p - C_e + \zeta = 0 \tag{C.4.5}$$

$$\zeta\left(\phi_0\Lambda - \mu\right) = 0 \tag{C.4.6}$$

Solving the above equations, the optimal solution can be obtained as $\phi_0^* = \frac{\Gamma - C_p}{2\Gamma}$ and $\mu^* = \phi_0^*\Lambda$. Also, when $\phi_0\Lambda \geq \mu$, the Lagrangian function can be written as follows.

$$L_2 = \max_{\mu \geq 0, 0 \leq \phi_0 \leq 1} \left[(1-\phi_0)\Gamma - C_p\right]\mu - C_s\left(\lambda - \mu\right) - \zeta\left(\mu - \phi_0\Lambda\right) \tag{C.4.7}$$

For this case, we can write K.K.T optimality conditions as follows.

$$\frac{\partial L_2}{\partial \phi_0} = -\Gamma\mu - C_s\Lambda + \zeta\Lambda = 0 \tag{C.4.8}$$

$$\frac{\partial L_2}{\partial \mu} = (1-\phi_0)\Gamma - C_p + C_s - \zeta = 0 \tag{C.4.9}$$

$$\zeta\left(\phi_0\Lambda - \mu\right) = 0 \tag{C.4.10}$$

Solving the above relations, we can obtain $\phi_0^* = \frac{\Gamma - C_p}{2\Gamma}$ and $\mu^* = \phi_0^*\Lambda$. Therefore, considering two cases, when $\theta \to \infty$ the optimal solution tends to $\phi_0^* = \frac{\Gamma - C_p}{2\Gamma}$ and $\mu^* = \phi_0^*\Lambda$.

C.4.2 Proof of Theorem 4.4

In this part, to prove the results, we utilize the modified value function $\tilde{V^N}^*$ represented as follows.

$$\tilde{V^N}^* = \max_{\rho, 0 \le \phi_0 \le 1} \left[(1-\phi_0)\Gamma + C_e + C_s \right](\lambda + \psi) - C_s\lambda - (C_e + C_p)(\lambda + \rho) \tag{C.4.11}$$

As $\Lambda \to \infty$, we have $\psi \to \tilde{\psi} = \frac{-\rho}{e^{-\theta\rho}-1}$. Therefore, the value function can be written as follows.

$$\tilde{V^N}^* = \max_{\rho, 0 \le \phi_0 \le 1} \left[(1-\phi_0)\Gamma + C_e + C_s \right](\phi_0\Lambda + \tilde{\psi}) - C_s\phi_0\Lambda - (C_e + C_p)(\phi_0\Lambda + \rho) \tag{C.4.12}$$

Thus, using first-order condition $\frac{\partial V^N}{\partial \phi_0} = 0$, the optimal ϕ_0 can be obtained as follows.

$$\phi_0^* = \frac{\Gamma - C_p}{2\Gamma} + o(\frac{1}{\Lambda}) \tag{C.4.13}$$

Substituting the optimal ϕ_0 into the value function, we have:

$$\begin{aligned} V^{N*} = \max_{\rho, 0 \le \phi_0 \le 1} & \left[\left(1 - \frac{\Gamma - C_p}{2\Gamma} + o(\frac{1}{\Lambda}) \right) \Gamma + C_e + C_s \right] \left[\left(\frac{\Gamma - C_p}{2\Gamma} + o(\frac{1}{\Lambda}) \right) \Lambda + \tilde{\psi} \right] - C_s\phi_0\Lambda \\ & - (C_e + C_p) \left(\frac{\Gamma - C_p}{2\Gamma} + o(\frac{1}{\Lambda})\Lambda + \rho \right) \end{aligned} \tag{C.4.14}$$

Using the first order condition $\frac{\partial V^N}{\partial \rho} = 0$, the following relation is satisfied.

$$\frac{e^{\theta\rho}\left(e^{\theta\rho} - \theta\rho - 1\right)}{\left(e^{\theta\rho} - 1\right)^2} - \frac{2(C_e + C_p)}{\Gamma + C_p + 2C_e} = 0 \tag{C.4.15}$$

The optimal ρ^* can be obtained by solving the above relation.

C.4.3 Proof of Theorem 4.5

Based on Lemma 4.1, as maximum willingness-to-pay Γ increases, the difference between arrival/production level and sales quantity (ρ) increases with an order of $\mathscr{O}(\log\Gamma)$. Therefore, as $\Gamma \to \infty$, the production level tends to infinity with an order of $\mathscr{O}(\log\Gamma)$, resulting in a zero shortage. Thus, when $\Gamma \to \infty$, the value function can be written as follows.

$$V^{N*} = \max_{\rho, 0 \leq \phi_0 \leq 1} (1-\phi_0)\Gamma\phi_0\Lambda - C_p\mu - C_e(\mu-\lambda) \tag{C.4.16}$$

Using first-order condition it is easy to obtain the optimal ϕ_0 and μ. When $\Gamma \to \infty$, we obtain the optimal decisions as $\phi_0^* \to \lim_{\Gamma\to\infty}\frac{\Gamma+C_e}{2\Gamma} \to \frac{1}{2}$ and $\mu^* \to \frac{1}{2}\Lambda + \rho^*$, where $\rho^* \to \infty$ with an order of $\mathscr{O}(\log\Gamma)$.

C.5 Proof of Lemma 4.1

Based on Proposition 3.1, we obtain the optimal ρ^* as follows.

$$\rho^* = \frac{\log\frac{-B-\sqrt{B^2-4AC}}{2A} - \log(\rho + \phi_0\Lambda)}{\theta} \tag{C.5.1}$$

Where A, B, and C are given as follows.

$$A = -\frac{C_e + C_p}{C_e + C_s + \Gamma\left(1-\phi_0^*\right)} \tag{C.5.2}$$

$$B = \Lambda\phi_0^*\rho^*\theta - \Lambda\phi_0^* + 2A\Lambda\phi_0^* \tag{C.5.3}$$

$$C = \Lambda\phi_0^*\rho^* e^{\theta\rho^*} + (1-A)\Lambda^2\phi_0^{*2} \tag{C.5.4}$$

Based on the definition of big-O notation, $f(x) \in \mathcal{O}(g(x))$ as $x \to \infty$ if there exist positive constants x_0 and c such that $|f(x)| \le cg(x)$ for all $x \ge x_0$.

To prove that $|\rho^*(\theta)| = \mathcal{O}(\frac{1}{\theta})$, it suffices to show that there exist positive θ_0 and c such that $|\rho^*(\theta)| \le c\frac{1}{\theta}$ for all $\theta \ge \theta_0$. In other words, we have $|\rho^*| \le \frac{|\log\frac{-B-\sqrt{B^2-4AC}}{2A}| + |-\log(\rho^*+\phi_0^*\Lambda)|}{\theta} \le c\frac{1}{\theta}$ for positive c and all $\theta \ge \theta_0$.

Replacing ρ^* by $\frac{c}{\theta}$ in the above relation, we can obtain c using the following equation.

$$\frac{|\log\frac{\Lambda\phi_0^*c-\Lambda\phi_0^*+2A\Lambda\phi_0^*+\sqrt{\left(\Lambda\phi_0^*c-\Lambda\phi_0^*+2A\Lambda\phi_0^*\right)^2-4A\left(\Lambda\phi_0^*\frac{c}{\theta}e^c+(1-A)\Lambda^2\phi_0^{*2}\right)}}{-2A}| + |-\log(\frac{c}{\theta}+\phi_0^*\Lambda)|}{\theta} = \frac{c}{\theta} \tag{C.5.5}$$

Putting $\theta_0 = 1$, we can obtain a positive $c < \infty$ by solving the following problem.

$$\max_{0<\phi_0^*<1}\left|\log\frac{\Lambda\phi_0^*c-\Lambda\phi_0^*+2A\Lambda\phi_0^*+\sqrt{\begin{array}{l}(\Lambda\phi_0^*c-\Lambda\phi_0^*+2A\Lambda\phi_0^*)^2\\-4A\left(\Lambda\phi_0^*ce^c+(1-A)\Lambda^2\phi_0^{*2}\right)\end{array}}}{-2A}\right| + \left|-\log(\frac{c}{\theta}+\phi_0^*\Lambda)\right| = c \tag{C.5.6}$$

Similarly, we can prove that $|\rho^*(\theta)| = \mathcal{O}(\log\Gamma)$.

C.6 Proofs of Theorem 4.2

To obtain an upper-bound on the marginal benefit of multiple-stage markdown policy as a function of the shelf life θ, we obtain an upper bound on $\frac{V^{M*}-V^{S*}}{V^{S*}}$ as follows.

$$\frac{V^{M*}-V^{S*}}{V^{S*}} \le \frac{V^{M*}-V^{N*}}{V^{N*}} = \frac{\begin{array}{c}\max\limits_{Q\ge 0,\mu^Q\ge 0,\boldsymbol{m}^Q\in\mathscr{A}^Q,\phi^Q\in\mathscr{B}^Q}\left\{\sum_{j=0}^{Q} p_j^Q\lambda_j^Q f_j^Q - C_s\lambda_0^Q l^Q - C_e f^Q(0) - C_p\mu^Q\right\}\\ -\max\limits_{\mu^N\ge 0,0\le\phi_0^N\le 1}\left\{p_0^N\lambda_0^N\left(1-l^N\right) - C_s\lambda_0^N l^N - C_e\mu q^N - C_p\mu\right\}\end{array}}{\max_{\mu^N\ge 0,0\le\phi_0^N\le 1}\left\{p_0^N\lambda_0^N\left(1-l^N\right) - C_s\lambda_0^N l^N - C_e\mu q^N - C_p\mu\right\}}$$

$$\le \frac{\begin{array}{c}\left[(p_0^{Q*}-C_p)\mu^{Q*} - C_s\phi_0^{Q*}\Lambda + \left(-p_0^{Q*}-C_e+C_s\right)\mu^{Q*}q^{Q*}\right] - \\ \left[(p_0^{Q*}-C_p)\mu^{N*} - C_s\phi_0^{Q*}\Lambda + \left(-p_0^{Q*}-C_e+C_s\right)\mu^{N*}q^{N}{}_{\left\{\mu^{N*},\phi_0^{Q*}\right\}}\right]\end{array}}{p_0^{N*}\mu^{N*} - C_s\phi_0^{N*}\Lambda + \left(-p_0^{N*}-C_e+C_s\right)\mu^{N*}q^{N*} - C_p\mu^{N*}}$$

$$\le \frac{(\Gamma + C_e - C_s)\left(\mu^{N*}q^{N}{}_{\left\{\mu^{N*},\phi_0^{Q*}\right\}} - \mu^{Q*}q^{Q*}\right)}{p_0^{N*}\mu^{N*} - C_s\phi_0^{N*}\Lambda + \left(-p_0^{N*}-C_e+C_s\right)\mu^{N*}q^{N*} - C_p\mu^{N*}}$$

$$\le \frac{(\Gamma + C_e - C_s)\frac{\left(\phi_0^{Q*}\Lambda+\rho^{N*}\right)\rho^{N*}}{\phi_0^{Q*}\Lambda+\rho^{N*}-\phi_0^{Q*}\Lambda e^{-\rho^{N*}\theta}}}{\left[\begin{array}{l}\left(\Gamma(1-\phi_0^{N*})-C_p\right)\left(\phi_0^{N*}\Lambda+\rho^{N*}\right) - C_s\phi_0^{N*}\Lambda\\ +\left(-\Gamma(1-\phi_0^{N*})-C_e+C_s\right)\left(\phi_0^{N*}\Lambda+\rho^{N*}\right)q^{N*}\end{array}\right]} = \mathscr{O}\left(\frac{1}{\theta}\right) \tag{C.6.1}$$

Where the first inequality holds because:

1. Under multiple-stage markdown model, we have $p_0 \ge p_1 \ldots \ge p_Q$

2. ϕ_0^{Q*} is a feasible solution for no-markdown model with value function V^N. Therefore, we have $V^N\left(\phi_0^{Q*},\mu^{N*}\right) \le V^{N*}$

the second inequality is true because $\mu^{Q^*} \leq \mu^{N^*}$, and the last inequality can be obtained by dropping the term $-\mu^{Q^*}q^{Q}$. Finally, the equality holds because $0 < \phi^{Q^*} < 1$ and according to Lemma 4.1, we know that $|\rho^{N^*}| = \mathscr{O}(\frac{1}{\theta})$.

Similarly, to prove the second part of this theorem, we have:

$$\frac{V^{M*}-V^{S*}}{V^{S*}} \leq \frac{(\Gamma + C_e - C_s)\,\frac{\left(\phi_0^{Q^*}\Lambda+\rho^{N^*}\right)\rho^{N^*}}{\phi_0^{Q^*}\Lambda+\rho^{N^*}-\phi_0^{Q^*}\Lambda e^{-\rho^{N^*}\theta}}}{\begin{bmatrix}\left(\Gamma(1-\phi_0^{N^*})-C_p\right)\left(\phi_0^{N^*}\Lambda+\rho^{N^*}\right)-C_s\phi_0^{N^*}\Lambda \\ +\left(-\Gamma(1-\phi_0^{N^*})-C_e+C_s\right)\left(\phi_0^{N^*}\Lambda+\rho^{N^*}\right)q^{N^*}\end{bmatrix}} = \mathscr{O}(\frac{1}{\Lambda}) \tag{C.6.2}$$

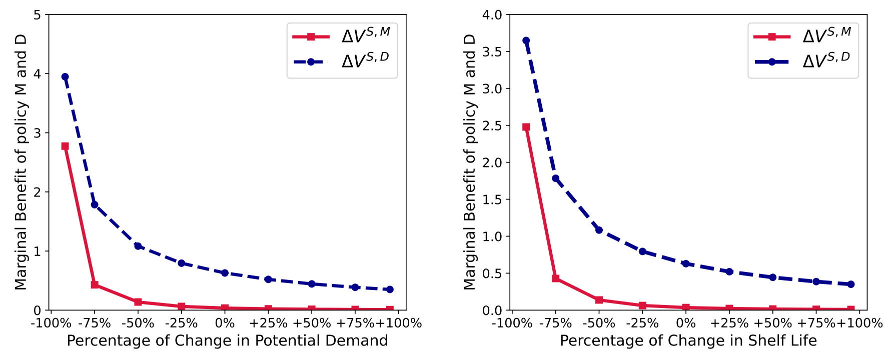
Where the inequality directly follows from first part of theorem,and the equality is true because $0 < \phi^{Q^*} < 1$ and according to Lemma 4.1, we know that ρ^{N^*} converges to a constant value.

Next,to prove the third part of this theorem, we have:

$$\frac{V^{M*}-V^{S*}}{V^{S*}} \leq \frac{(\Gamma + C_e - C_s)\,\frac{\left(\phi_0^{Q^*}\Lambda+\rho^{N^*}\right)\rho^{N^*}}{\phi_0^{Q^*}\Lambda+\rho^{N^*}-\phi_0^{Q^*}\Lambda e^{-\rho^{N^*}\theta}}}{\begin{bmatrix}\left(\Gamma(1-\phi_0^{N^*})-C_p\right)\left(\phi_0^{N^*}\Lambda+\rho^{N^*}\right)-C_s\phi_0^{N^*}\Lambda \\ +\left(-\Gamma(1-\phi_0^{N^*})-C_e+C_s\right)\left(\phi_0^{N^*}\Lambda+\rho^{N^*}\right)q^{N^*}\end{bmatrix}} = \mathscr{O}(\frac{1}{\log\Gamma}) \tag{C.6.3}$$

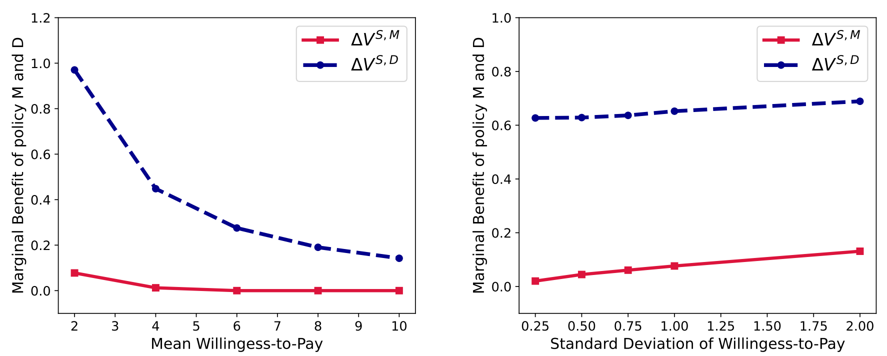
where the equality is true because as $\Gamma \to \infty$, $\phi_0^{N^*}$ converges to $\frac{1}{2}$, and $\rho^{N^*} = \mathscr{O}(\log\Gamma)$. Therefore, $\Gamma \to \infty$ the nominator increases with an order of $\mathscr{O}(\Gamma)$, while denominator increases with an order of $\mathscr{O}(\Gamma\log\Gamma)$, which yields the intended result.

C.7 Marginal Benefit of Markdown Policy M and D for Fresh Produce Case



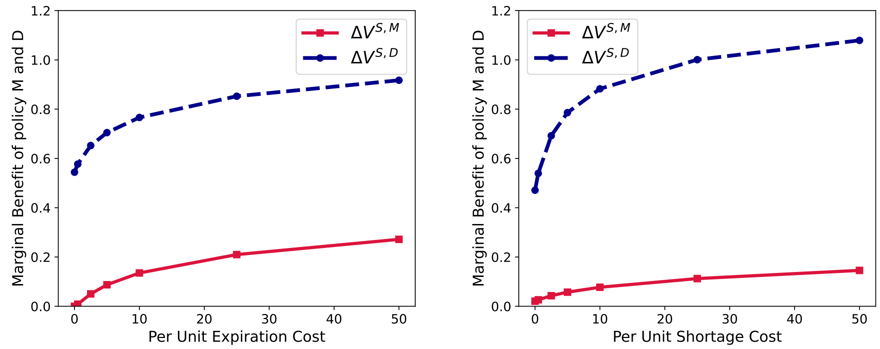
(a) (b)

Figure C.7.1: Sensitivity of marginal markdown benefits to market demand and shelf life for zucchini products



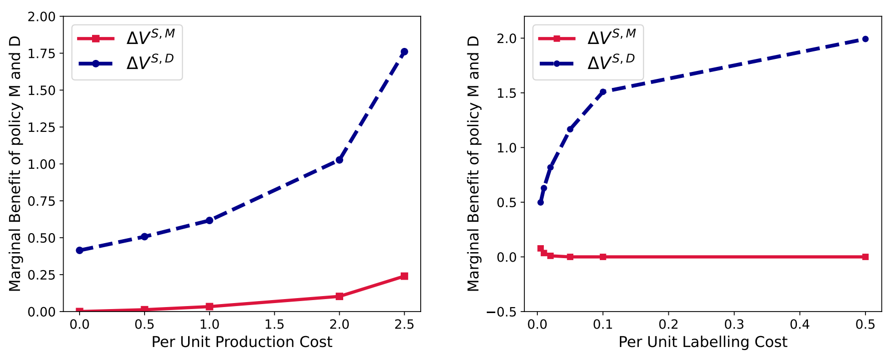
(a) (b)

Figure C.7.2: Sensitivity of marginal markdown benefits to mean and standard deviation of WTP for zucchini products



(a) (b)

Figure C.7.3: Sensitivity of marginal markdown benefits to per unit expiration and shortage costs

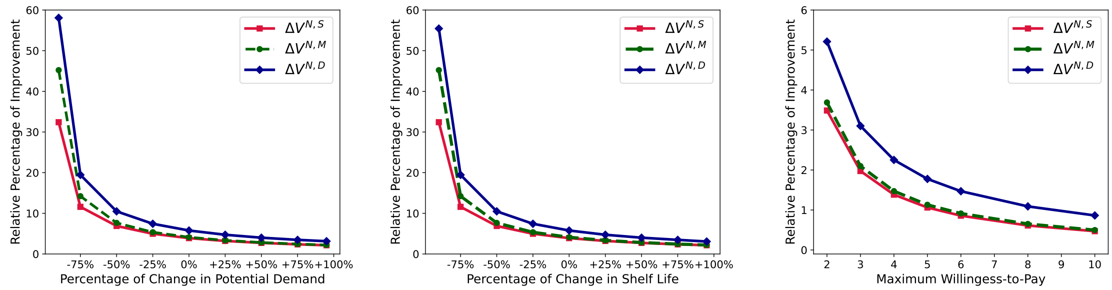


(a) (b)

Figure C.7.4: Sensitivity of marginal markdown benefits to per unit production and labelling costs

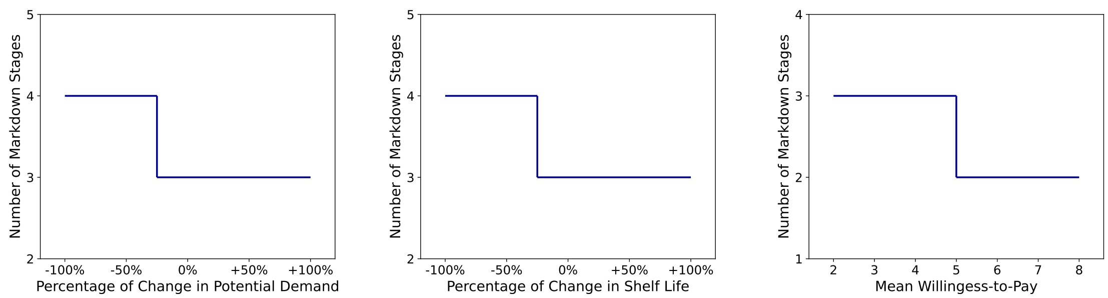
C.8 Sensitivity Analysis for Bakery Chain Case Study

Sensitivity analysis with respect to market demand, shelf life, and maximum WTP:



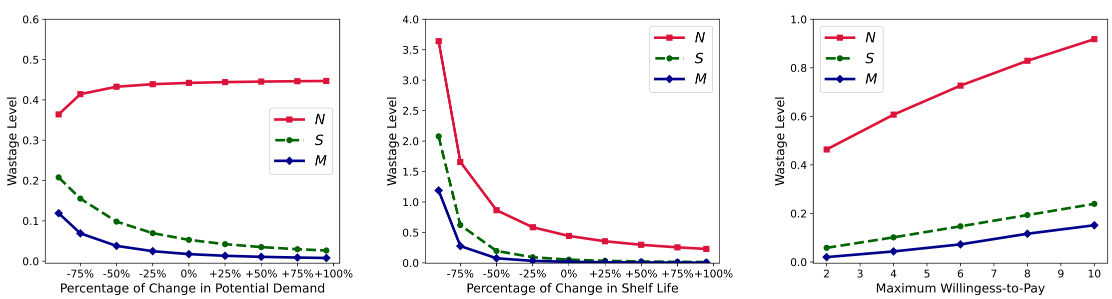
(a) (b) (c)

Figure C.8.1: Sensitivity of markdown benefits to market demand, shelf life, and maximum WTP



(a) (b) (c)

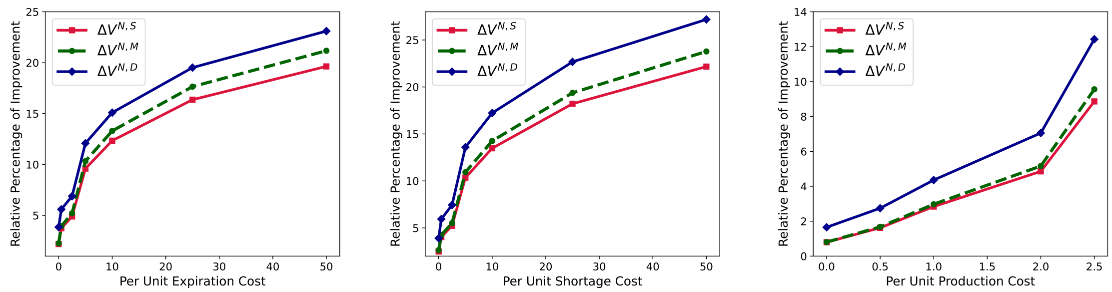
Figure C.8.2: Sensitivity of the optimal number of markdown stages to market demand, shelf life, and maximum WTP



(a) (b) (c)

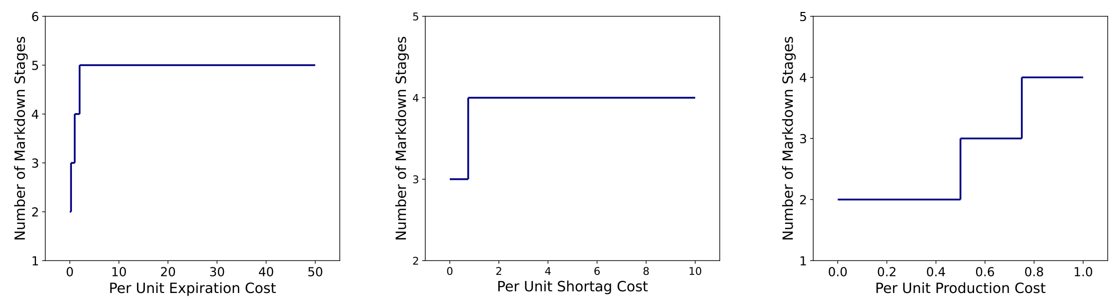
Figure C.8.3: Sensitivity of wastage level to market demand, shelf life, and maximum WTP

Sensitivity analysis with respect to cost parameters:



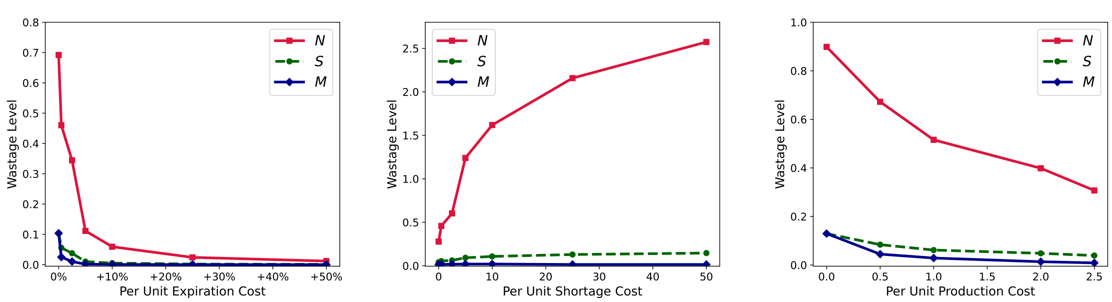
(a) (b) (c)

Figure C.8.4: Sensitivity of markdown benefits to the cost parameters



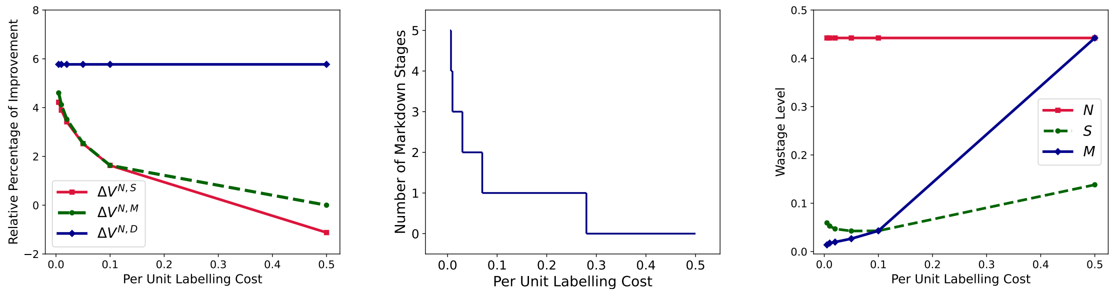
(a) (b) (c)

Figure C.8.5: Sensitivity of the optimal number of markdown stages to the cost parameters



(a) (b) (c)

Figure C.8.6: Sensitivity of wastage level to the cost parameters



(a) (b) (c)

Figure C.8.7: Sensitivity of markdown benefits, number of markdown stages, and wastage level to labelling cost

Appendix D

Supplement to Chapter 5

D.1 Proof of Proposition 5.1

Because both leader's and follower's problems contain only linear objective function and constraints, they are both convex and satisfy constraint qualification. Therefore, based on the Allende and Still (2013), the optimal solution N obtained from reformulated problem using KKT condition (i.e., P_{KKTBL}) is the global minimizer of the original bi-level problem (i.e., P_{BL}) if and only if the KKT conditions of lower-level problem are satisfied. Under this condition, P_{BL} is equivalent to P_{KKTBL}. This completes the proof of Proposition 5.1.

D.2 Proof of Proposition 5.2

Based on the introduced notations, the structure of decentralized and centralized problems can be written as follows, respectively.

$$P_{\text{Decentralized}}: \min Z_H^D \tag{D.2.1}$$

$$S.t.\ \nabla_{N_2} Z_B^D(N) + \sum_k v_k \nabla_{N_2} A_k(N) + \sum_l v_l \nabla_{N_2} B_l(N) = 0 \qquad \forall N_2 \tag{D.2.2}$$

$$v_k A_k(N) = 0 \qquad \forall k \tag{D.2.3}$$

$$C_i^1(N) \leq 0 \qquad \forall i \tag{D.2.4}$$

$$A_k(N) \leq 0 \qquad \forall k \tag{D.2.5}$$

$$B_l(N) = 0 \qquad \forall l \tag{D.2.6}$$

$$P_{\text{Centralized}}: \min Z_H^C + Z_B^C \tag{D.2.7}$$

$$S.t.\ C_i^1(N) \leq 0 \qquad \forall i \tag{D.2.8}$$

$$A_k(N) \leq 0 \qquad \forall k \tag{D.2.9}$$

$$B_l(N) = 0 \qquad \forall l \tag{D.2.10}$$

Let us consider $N_1^{C\star}$ and $N_2^{C\star}$ as the optimal decision variables associated with upper-level and lower-level problems, respectively, under centralized model, and consider $N_1^{D\star}$ and $N_2^{D\star}$ as the optimal solutions under centralized model. Further, the optimal hospitals, blood center, and whole supply chain cost under decentralized model can be written as $Z_H^D(N_1^{D\star}, N_2^{D\star})$, $Z_B^D(N_1^{D\star}, N_2^{D\star})$, and $Z^D(N_1^{D\star}, N_2^{D\star})$, respectively. Similarly, $Z_H^C(N_1^{C\star}, N_2^{C\star})$, $Z_B^C(N_1^{C\star}, N_2^{C\star})$, and $Z^C(N_1^{C\star}, N_2^{C\star})$ denote the optimal hospitals, blood center, and whole supply chain cost under decentralized model, respectively.

Assume $P^{N_2^D}_{\text{Decentralized}}$ as the decentralized problem in which $N_2^D \in \Omega_D$ is fixed, where Ω_D is the feasible region. The optimal objective function obtained from solving relaxed $P^{N_2^D}_{\text{Decentralized}}$ is denoted by $\tilde{Z}^D_{H,N_2^D}(N_1^{D\star})$. Also, in this problem, the optimal blood center's cost can be written as $\tilde{Z}^D_{B,N_2^D}(N_1^{D\star})$.

Similarly, consider $P^{N_1^C}_{\text{Centralized}}$ as the centralized model wherein $N_1^C \in \Omega_C$ is fixed, where Ω_C is the feasible region. The optimal hospital's cost obtained from solving relaxed $P^{N_1^C}_{\text{Centralized}}$ is denoted as $\tilde{Z}^C_{H,N_1^C}(N_2^{C\star})$. Also, in this problem, the optimal blood center's cost can be written as $\tilde{Z}^C_{B,N_1^C}(N_2^{C\star})$.

Next, for blood center, we can conclude the following relations.

$$Z_B^C(N_1^{C\star}, N_2^{C\star}) \leq \tilde{Z}^C_{B,N_1^{D\star}}(N_2^{C\star}) = \tilde{Z}^D_{B,N_1^{D\star}}(N_2^{D\star}) = Z_B^D(N_1^{D\star}, N_2^{D\star}) \tag{D.2.11}$$

The first inequality follows from the fact that all feasible solutions have higher costs than the optimal solution. The second equality shows that by fixing centralized and decentralized models at $N_1^{D\star}$, both problems are equivalent. The above argument completes the proof for relation (iii). Also, for hospitals' objective function, we have:

$$Z_H^D(N_1^{D\star}, N_2^{D\star}) \leq \tilde{Z}^D_{H,N_1^{C\star}}(N_2^{D\star}) = \tilde{Z}^C_{H,N_1^{C\star}}(N_2^{C\star}) = Z_H^C(N_1^{C\star}, N_2^{C\star}) \tag{D.2.12}$$

In the above relation, the first inequality is satisfied because all feasible solutions result in a higher cost than the optimal solution. Also, the first equality implies that when $N_2^{C\star}$ is fixed, centralized and decentralized problems result in the same solution. This completes the proof for relation (ii). Finally, for the total cost of supply chain, we have:

$$Z_H^C(N_1^{C\star}, N_2^{C\star}) + Z_B^C(N_1^{C\star}, N_2^{C\star}) \leq \tilde{Z}^C_{H,N_1^{C\star}}(N_2^{D\star}) + \tilde{Z}^D_{B,N_1^{D\star}}(N_2^{D\star}) \tag{D.2.13}$$

$$= \tilde{Z}^D_{H,N_1^{D\star}}(N_2^{D\star}) + \tilde{Z}^D_{B,N_1^{D\star}}(N_2^{D\star}) \tag{D.2.14}$$

$$= \tilde{Z}^D_{H,N_1^{D\star}}(N_2^{D\star}) + \tilde{Z}^D_{B,N_1^{D\star}}(N_2^{D\star}) \tag{D.2.15}$$

In the above relation, because $Z_H^C(N_1^{C\star}, N_2^{C\star}) \leq \tilde{Z}^C_{H,N_1^{C\star}}(N_2^{D\star})$ and $Z_B^C(N_1^{C\star}, N_2^{C\star}) \leq \tilde{Z}^D_{B,N_1^{D\star}}(N_2^{D\star})$, we can conclude the first inequality. Also, the first equality is satisfied because by fixing $N_2^{C\star}$, centralized and decentralized problems yield the same solution. This completes the proof of Proposition 5.2.

D.3 Different Issuing Policies Models

In this section, two well-known issuing policies, i.e., FIFO and LIFO policies, and Threshold-based policy are extended to the case of integrated model to assess the performance of the optimal issuing policies.

D.3.1 FIFO Issuing Policy Model

In the original model, there are not constrains forcing FIFO and LIFO policies. To evaluate the performance of the optimal issuing policy compared to these policies, two models under FIFO and LIFO policies are extended. To enforce FIFO issuing policy, we add the following constraints to the centralized (integrated) model.

$$\sum_{j\in J} SD^{r_j}_{j',j,h,t} SO_{j,j'} \leq MZ^{r_j}_{j,h,t} \qquad \forall\, j,h,t,r_j \tag{D.3.1}$$

$$Z^{r_j}_{j,h,t} \leq M \sum_{j'\in J} SD^{r_j}_{j',j,h,t} SO_{j,j'} \qquad \forall\, j,h,t,r_j \tag{D.3.2}$$

$$IH^{r_j}_{j,h,t} + \sum_{j'\in J} SD^{r_j}_{j',j,h,t} SO_{j,j'} \leq MY^{r_j}_{j,h,t} \qquad \forall\, j,t,r_j \tag{D.3.3}$$

$$Y^{r_j}_{j,h,t} \leq M \left(IH^{r_j}_{j,h,t} + \sum_{j'\in J} SD^{r_j}_{j',j,h,t} SO_{j,j'} \right) \qquad \forall\, j,h,t,r_j \tag{D.3.4}$$

$$Z^{r_j}_{j,h,t} \leq Z^{r'_j}_{j,h,t} + M\left(1 - Y^{r'_j}_{j,h,t}\right) \qquad \forall\, j,h,t,r_j,r'_j > r_j \tag{D.3.5}$$

$$IH^{r_j}_{j,h,t} \leq M\left(1 - Z^{r'_j}_{j,h,t}\right) \qquad \forall\, j,t,r_j,r'_j < r_j \tag{D.3.6}$$

Constraints (D.3.1) and (D.3.2) associate a binary variable to the demand satisfaction variable, indicating when any patient demand is not satisfied by product j with age r_j at hospital h in period t, its corresponding binary variable $Z^{r_j}_{j,h,t}$ is zero; otherwise, it is 1. Constraints (D.3.3) and (D.3.4) associate a binary variable to the inventory level of each product with different ages in each period, that is when inventory level is 0, its corresponding binary variable $Y^{r_j}_{j,h,t}$ is 0; otherwise, it is 1. Constraints (D.3.5) and (D.3.6) satisfy FIFO policy conditions.

D.3.2 LIFO Issuing Policy Model

In LIFO model, constraints (D.3.5)-(D.3.6) should be replaced by the following constraints.

$$Z^{r_j}_{j,h,t} \leq Z^{r'_j}_{j,h,t} + M\left(1 - Y^{r'_j}_{j,h,t}\right) \qquad \forall\, j, j', t, r_j, r'_j < r_j \tag{D.3.7}$$

$$IH^{r_j}_{j,h,t} \leq M\left(1 - Z^{r'_j}_{j,h,t}\right) \qquad \forall\, j, t, r_j, r'_j > r_j \tag{D.3.8}$$

Where constraints (D.3.7) and (D.3.8) satisfy LIFO policy conditions.

D.3.3 Threshold-Based Policy Model

To model Threshold-based policy, the constraints (D.3.5)-(D.3.6) in FIFO model should be replaced by the following constraints.

$$Z^{r_j}_{j,h,t} \leq Z^{r'_j}_{j,h,t} + M\left(1 - Y^{r'_j}_{j,h,t}\right) \qquad \forall\, j, t, r_j \leq TP_j, r'_j > r_j \tag{D.3.9}$$

$$Z^{r_j}_{j,h,t} \leq Z^{r'_j}_{j,h,t} + M\left(1 - Y^{r'_j}_{j,h,t}\right) \qquad \forall\, j, t, r_j > TP_j, r'_j < r_j \tag{D.3.10}$$

$$IH^{r_j}_{j,h,t} \leq M\left(1 - Z^{r'_j}_{j,h,t}\right) \qquad \forall\, j, t, r_j \leq TP_j, r'_j < r_j \tag{D.3.11}$$

$$IH^{r_j}_{j,h,t} \leq M\left(1 - Z^{r'_j}_{j,h,t}\right) \qquad \forall\, j, t, r_j > TP_j, r'_j > r_j \tag{D.3.12}$$

Where constraints (D.3.9)-(D.3.12) denote threshold-based policy, based on which if there are units older than TP_j, they should be issued based on LIFO policy, and if there is no unit older than TP_j, products younger than TP_j should be issued based on FIFO policy.

D.4 Different Replenishment Policies Models

Given the simplicity of implementation of (S,s) and (R,T) replenishment policies, many health care systems may prefer to use these systems for their replenishment decisions. To assess the performance of the optimal policy, (S,s) and (R,T) replenishment models are formulated and compared with the optimal policy. Next, we present the formulations for (S,s) and (R,T) replenishment models.

D.4.1 (s,S) Replenishment Policy

In this section, the general problem is extended to the case of (S,s) replenishment policy as follows.

$$\sum_{r_j \in R_j} IH_{j,h,t-1}^{r_j} \geq s_{j,h} - M\left(1 - O_{j,h,t}\right) \qquad \forall\, j,h,t \tag{D.4.1}$$

$$\sum_{r_j \in R_j} IH_{j,h,t-1}^{r_j} \leq s_{j,h} + M\,O_{j,h,t} \qquad \forall\, j,h,t \tag{D.4.2}$$

$$D_{j,h,t} \leq M\left(1 - O_{j,h,t}\right) \qquad \forall\, j,h,t \tag{D.4.3}$$

$$D_{j,h,t} \geq -M\left(1 - O_{j,h,t}\right) \qquad \forall\, j,h,t \tag{D.4.4}$$

$$D_{j,h,t} \leq S_{j,h} - \sum_{r_j \in R_j} IH_{j,h,t-1}^{r_j} + M\,O_{j,h,t} \qquad \forall\, j,h,t \tag{D.4.5}$$

$$D_{j,h,t} \geq S_{j,h} - \sum_{r_j \in R_j} IH_{j,h,t-1}^{r_j} - M\,O_{j,h,t} \qquad \forall\, j,h,t \tag{D.4.6}$$

In the above mixed-integer optimization model, binary variable $O_{j,h,t}$ equals 1 when inventory level is less than $s_{j,h}$ for product j at hospital h; otherwise, it is zero. Then, constraints (D.4.1)-(D.4.6) show (s,S) replenishment policy according to which an order is placed to restore the inventory level of product j at hospital h to $S_{j,h}$, when it is below $s_{j,h}$; otherwise, no order is placed for the corresponding product at hospital h.

D.4.2 (R,T) Replenishment policy

To formulate (R,T) replenishment policy, we denote T' as the reviewing period and $R'_{j,h}$ as the order-up-to-level for product j at hospital h. Then, we add the following constraints to the original centralized model to ensure that the system operates under (R,T) replenishment policy.

$$D_{j,h,t} = R'_{j,h} - IH^{r_j}_{j,h,t-1} \qquad \forall\, j,h,r_j,t = i \times T' \;\; i \in (0, \left\lfloor \frac{T}{T'} \right\rfloor) \tag{D.4.7}$$

$$D_{j,h,t} = 0 \qquad \forall\, j,h,r_j,t \neq i \times T' \;\; i \in (0, \left\lfloor \frac{T}{T'} \right\rfloor) \tag{D.4.8}$$

In the above reformulation, constraints (D.4.7) and (D.4.8) imply (R,T) policy in which every T' period (reviewing period), an order is placed to restore the inventory level of product j at hospital h to $R'_{j,h}$.

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