CRL INTERNAL REPORT SERIES

No. CRL-5

Network Optimization Computer Program Package

J.W. Bandler and V.K. Jha

COMMUNICATIONS RESEARCH LABORATORY

FACULTY OF ENGINEERING

McMASTER UNIVERSITY

HAMILTON, ONTARIO, CANADA

CRL INTERNAL REPORT SERIES

No. CRL-5

Network Optimization Computer Program Package

J.W. Bandler and V.K. Jha

NETWORK OPTIMIZATION

COMPUTER PROGRAM PACKAGE

by

JOHN W. BANDLER and VIRENDRA K. JHA
Communications Research Laboratory
McMaster University
Hamilton, Ontario, Canada

November 1972

ACKNOWLEDGEMENTS

Mrs. J. Rizoniko Popović provided an important group of subprograms devoted to least pth approximation techniques.* C. Charalambous, some of whose recent work is embodied in the package, is gratefully acknowledged. Continued interaction with C.M. Kudsia of RCA Limited in Montreal helped shape this package of programs.

---

*These programs are fully described in the M.Eng. thesis "General programs for least pth and near minimax approximation", McMaster University, October 1972.

CHAPTER ONE

INTRODUCTION

Objectives

The main objective of this work is to provide RCA Limited with a comprehensive, user-oriented computer program package that will analyze and optimize certain electrical networks. The organization of the package is such that the optimization of microwave filters including allpass networks is readily facilitated. Another important objective which has been fulfilled is to organize the programs on a modular approach so that future deletions or additions can be readily implemented by a user.

Main Features of the Package

The package has been originally developed on a CDC 6400 digital computer at McMaster University, Hamilton, using batch processing. It is written in FORTRAN IV. It has been tested and run from a Univac DCT 500 terminal at RCA Limited at St. Anne de Bellevue using the CDC 6000 series computer at Sir George Williams University, Montreal, on the Kronos time-sharing system.

The package features some of the latest and most efficient methods of computer-aided design currently available. At the user's command, either the well-known and highly respected Fletcher-Powell

(1963) method of minimizing unconstrained functions of many variables may be used, or the more recent, and generally more efficient, method by Fletcher (1970).

The package has been designed to incorporate the adjoint network method of sensitivity evaluation to produce accurate first derivatives needed by these efficient gradient minimization methods. Many formulas published by Bandler and Sevoria (1970) have been built into the package. Considerable savings in computer effort are usually to be gained by proper use of the adjoint network method, not only in computing sensitivities but also because efficient optimization methods often rely for efficiency on the availability of very accurate first derivative information.

It was agreed to use perturbation techniques to calculate group delay and group delay sensitivities with respect to variables since the small savings in computing time realized by using the adjoint network method did not appear to be worth the additional programming complexity.

State-of-the-art techniques in least pth approximation generalized for such tasks as filter design as proposed by Bandler and Charalambous (1971 and 1972) are incorporated. Thus, a variety of upper and lower response specifications as well as simple upper and lower desired bounds for variable parameters are catered for. Low values of p, e.g., 2, intermediately large values of p, e.g., 10 to 1,000, as well as extremely large values of p, e.g., 1,000,000 are optional to the user depending on how close to a minimax (Chebyshev, equal-ripple) solution he wants to come.

The package has been developed on an interactive basis between RCA and CRL with the aims of making RCA engineers aware of progress, inevitable limitations in the capabilities of the package due to time constraints and of possibilities of expansion of the package by RCA at a later date.

Program Capabilities

As it stands at present, the package is capable of analyzing and optimizing certain linear, time-invariant, lumped and distributed networks in the frequency domain subject to the following specifications.

The network is assumed to be a cascade of two-port building blocks terminated in a unit normalized frequency-independent resistance at the source and a user-specified frequency-independent resistance at the load (taken as unity when allpass networks are present).

Resistors, inductors, capacitors, lossless short-circuited and open-circuited transmission-line stubs, and series and parallel RLC resonant circuits can be called upon by the user and connected as series or shunt elements, in any order. Lossless transmission-lines as well as microwave allpass C- and D-sections can also be added.

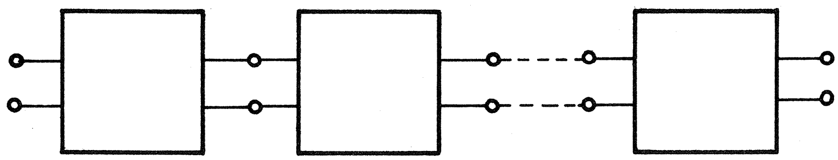
Gradients are automatically checked before optimization.

Responses before and after optimization are printed out. Much other useful information which can be used to check on the progress of the optimization process and to diagnose errors is printed out at the user's discretion.

CHAPTER TWO

CIRCUIT CONFIGURATION

AND BUILDING BLOCKS



CIRCUIT CONFIGURATION

Possibilities

1. A cascade connection of two-port circuit blocks consisting of any of the elements depicted on the following pages in any order, and as many as required.
2. As many C- and D-sections as required.
3. Modification of program to accommodate new blocks is readily effected. See the last page in this chapter.

Implementation*

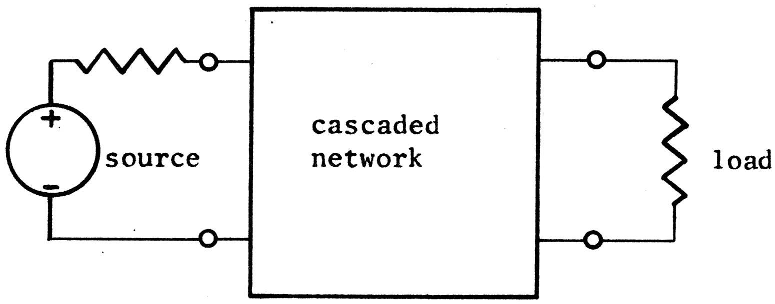
1. All blocks are numbered sequentially from left to right.
2. Each block has a code number associated with it defining the element it contains.

*Except allpass networks.

Parameters Required

Other than the parameters listed and defined together with the individual blocks, the following values must be supplied.

1. The total number of blocks (not including C- and D-sections).
2. The total number of parameters in these blocks.
3. The number of C-sections.
4. The number of D-sections.
5. The center frequency (e.g., in MHz, for normalization).
6. The cutoff frequency for C- and D-sections (e.g., in MHz).
7. The d-level for allpass networks (see Kudsia (1970)). This parameter is treated like any other circuit parameter. It is the very last variable to be entered.



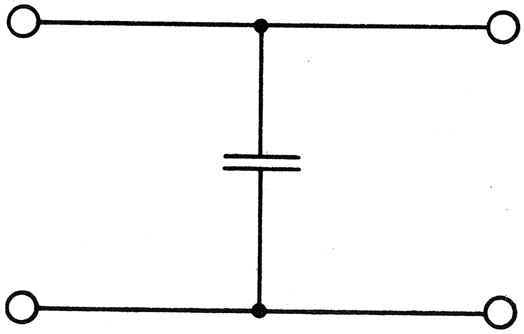
SOURCE
AND
LOAD
CONFIGURATION

Possibilities

1. Complex (but constant) load impedance; will, therefore, usually be a resistance.
2. Modification of program needed to have frequency dependent source and load impedances (source is assumed to be unity).

Parameters Required

1. Load impedance.



SHUNT
CAPACITOR

Code

1

Parameters

1 2 3

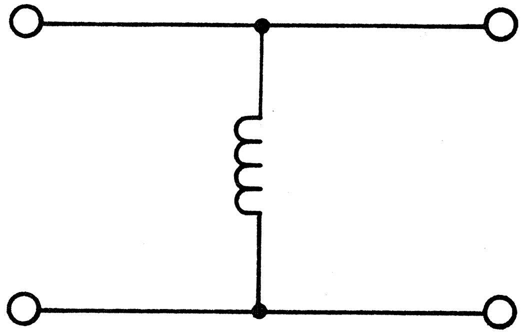
C

Parameter Definition

C = capacitance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SHUNT
INDUCTOR

Code

2

Parameters

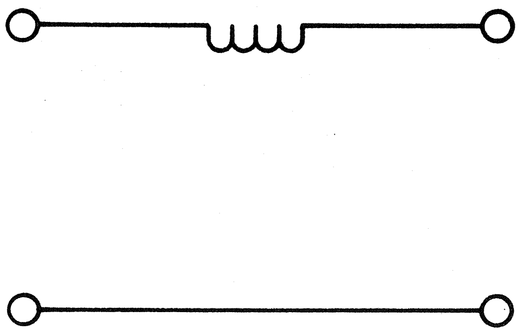
1	2	3
L		

Parameter Definition

L = inductance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SERIES
INDUCTOR

Code

3

Parameters

1 2 3

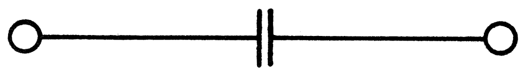
L

Parameter Definition

L = inductance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SERIES
CAPACITOR

Code

4

Parameters

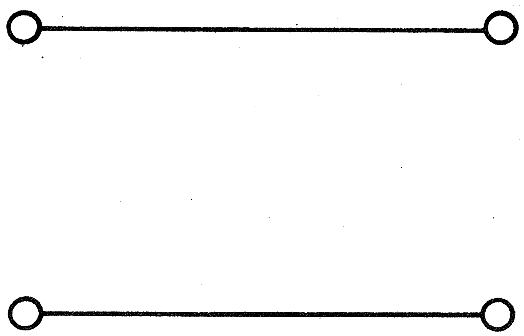
1	2	3
C		

Parameter Definition

C = capacitance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



LOSSLESS
TRANSMISSION
LINE

Code

5

Parameters

1	2	3
ℓ	Z_0	

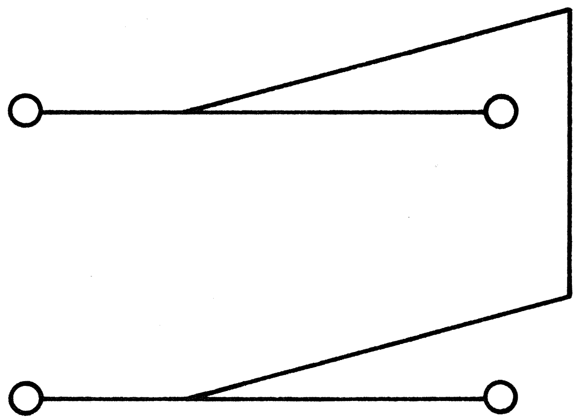
Parameter Definition

ℓ = length (normalized)

Z_0 = characteristic impedance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SHUNT
SHORTED
LOSSLESS
TRANSMISSION
LINE

Code

6

Parameters

1	2	3
ℓ	Z_0	

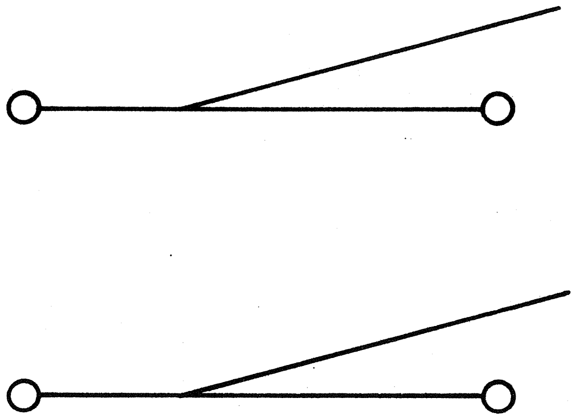
Parameter Definition

ℓ = length (normalized)

Z_0 = characteristic impedance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SHUNT
OPEN
LOSSLESS
TRANSMISSION
LINE

Code	Parameters		
	1	2	3
7	ℓ	Z_0	

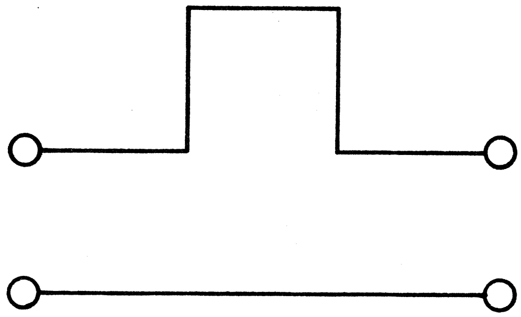
Parameter Definition

ℓ = length (normalized)

Z_0 = characteristic impedance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SERIES
SHORTED
LOSSLESS
TRANSMISSION
LINE

Code

8

Parameters

1	2	3
ℓ	Z_0	

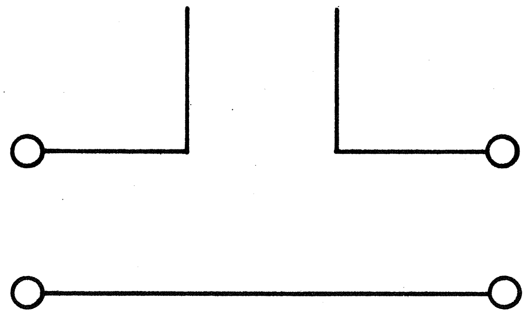
Parameter Definition

ℓ = length (normalized)

Z_0 = characteristic impedance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SERIES
OPEN
LOSSLESS
TRANSMISSION
LINE

Code

9

Parameters

1	2	3
ℓ	Z_0	

Parameter Definition

ℓ = length (normalized)

Z_0 = characteristic impedance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.

SERIES
RESONANT
CIRCUIT

Code

10

Parameters

1	2	3
ω_R	Q	X'

Parameter Definition

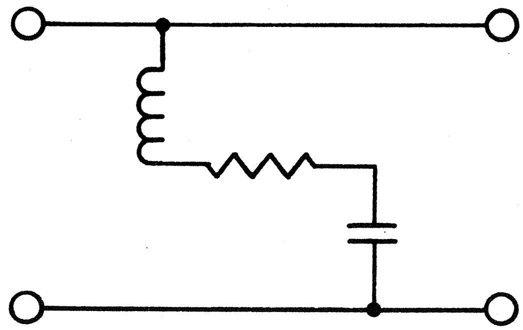
ω_R = resonant frequency (normalized)

Q = quality factor

X' = slope of reactance at resonance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SHUNT
RESONANT
CIRCUIT

Code

11

Parameters

1	2	3
ω_R	Q	X'

Parameter Definition

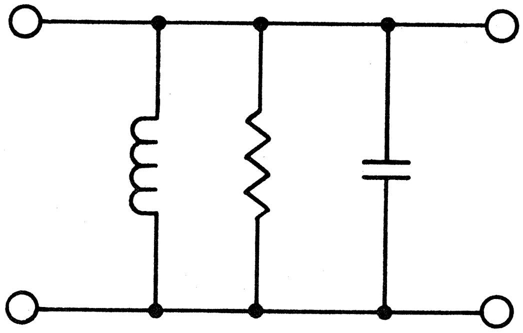
ω_R = resonant frequency (normalized)

Q = quality factor

X' = slope of reactance at resonance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SHUNT
ANTIRESONANT
CIRCUIT

Code

12

Parameters

1	2	3
ω_R	Q	B'

Parameter Definition

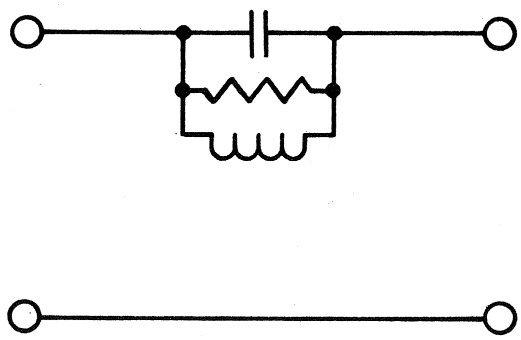
ω_R = antiresonant frequency (normalized)

Q = quality factor

B' = slope of susceptance at antiresonance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SERIES
ANTIRESONANT
CIRCUIT

Code

13

Parameters

1	2	3
ω_R	Q	B'

Parameter Definition

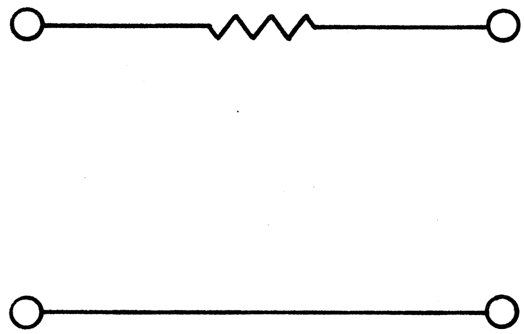
ω_R = antiresonant frequency (normalized)

Q = quality factor

B' = slope of susceptance at antiresonance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SERIES
RESISTOR

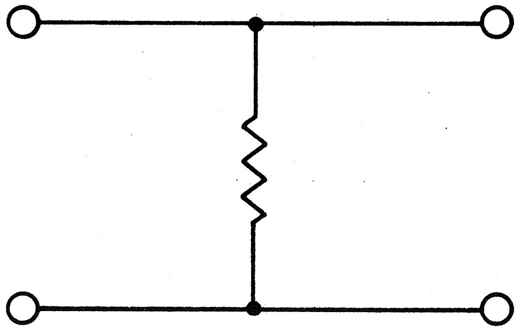
Code	Parameters		
	1	2	3
14	R		

Parameter Definition

R = resistance (normalized)

Comments

Upper and lower bounds or fixed values can be accommodated.



SHUNT
RESISTOR

<u>Code</u>

15

<u>Parameters</u>

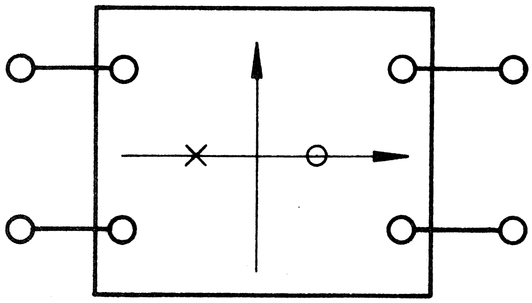
1	2	3
R		

<u>Parameter Definition</u>

R = resistance (normalized)

<u>Comments</u>

Upper and lower bounds or fixed values can be accommodated.



ALLPASS
C-SECTIONS
(Total number n_c)

Code

16 (not used)

Parameters

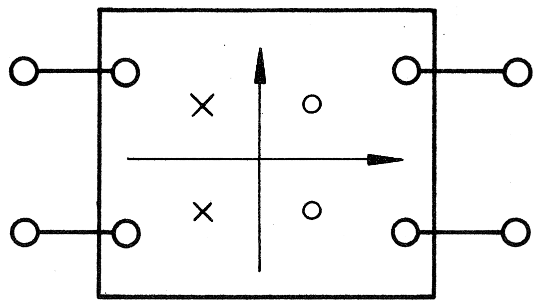
1	2	3 ...	n_c
σ_1	σ_2	σ_3...	σ_{n_c}

Parameter Definition

σ_i = location of ith real zero

Comments

1. The user specifies the number of C-sections required.
2. One cutoff frequency (fixed) and one d-level (variable) must be specified whenever any C- or D-section is used.
3. The user should consult theoretical concepts reviewed by Kudsia (1970).
4. C- and D-section parameters are either all fixed or all variable.



ALLPASS
D-SECTIONS
(Total number n_d)

Code

17 (not used)

Parameters

1	2	3 ...	n_{d+1}	n_{d+2}	n_{d+3}	...
σ_1	σ_2	σ_3...	ω_1	ω_2	ω_3	...

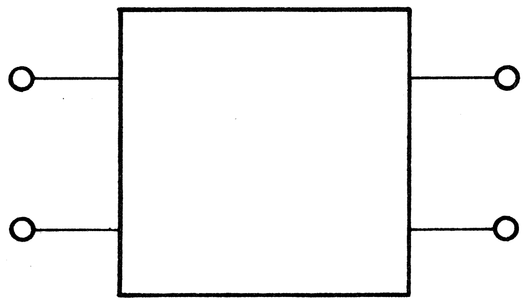
Parameter Definition

σ_i = location of real part of ith zero

ω_i = location of imaginary part of ith zero

Comments

1. The user specifies the number of D-sections required.
2. One cutoff frequency (fixed) and one d-level (variable) must be specified whenever any C- or D-section is used.
3. The user should consult theoretical concepts reviewed by Kudsia (1970).
4. C- and D-section parameters are either all fixed or all variable.



TWO-PORT
SECTION

Possibilities

Addition of various new blocks is possible because of the modular approach which has been used in the development of the package. The following basic procedure has to be carried out.

Implementation

An analysis subroutine must be written to calculate input voltage and current given the output voltage and current (ABCD matrix analysis). The subroutine is called exactly as any other analysis subroutine in the package is called and sensitivity formulas obtained by the adjoint network method (see Bandler and Seviora (1970), a copy of which is included in this report) if the parameters of the two-port are to be varied.

Comments

A wide variety of other two-ports can be added, e.g., distributed RC lines, transistor amplifier stages, operational amplifier stages, etc.

CHAPTER THREE

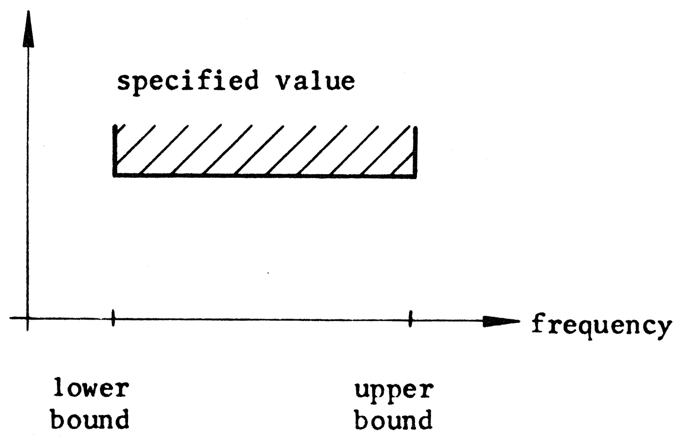
SPECIFICATIONS AND CONSTRAINTS

Possibilities

1. As many upper and lower specifications on reflection coefficient, insertion loss and relative group delay as the user desires can be accommodated.
2. Upper and lower bounds on all variables can be specified.

Parameters Required

1. Total number of frequency intervals including necessary ones to define parameter constraints.
2. When asked for, +1.0 to denote upper and -1.0 to denote lower specifications.
3. When asked for: 0 denotes parameter constraints
   1 denotes reflection coefficient
   2 denotes insertion loss
   3 denotes relative group delay



UPPER RESPONSE SPECIFICATION

Defining Parameters

1. Lower bound (frequency point)

2. Upper bound (frequency point)

3. Number of subintervals (equals sample points minus one)

4. Specified value

Associated Quantities

1. Weighting factor (positive). If in doubt use 1.

2. Upper specification may be

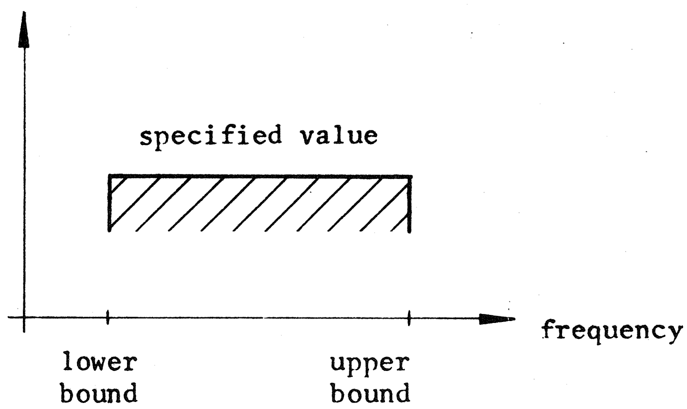
   (i) reflection coefficient
   (ii) insertion loss (dB)
   (iii) relative group delay (nsec)
   (iv) upper constraint bound on parameter

Comments

1. For single point specification, let the number of subintervals be zero and set upper bound equal to lower bound.

2. The user should consult theoretical concepts reviewed by Bandler (1969).

3. Continuous specifications require program modification.



LOWER
RESPONSE
SPECIFICATION

Defining Parameters

1. Lower bound (frequency point)

2. Upper bound (frequency point)

3. Number of subintervals (equals sample points minus one)

4. Specified value.

Associated Quantities

1. Weighting factor (positive). If in doubt use 1.

2. Lower specification may be

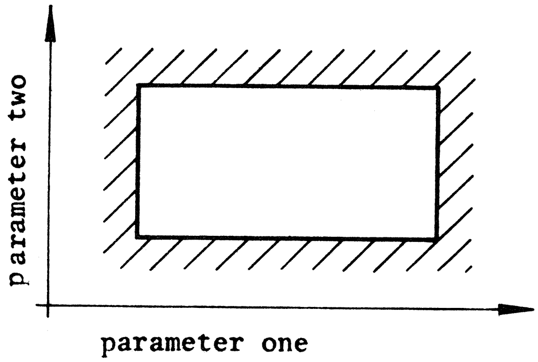
   (i) reflection coefficient
   (ii) insertion loss (dB)
   (iii) relative group delay (nsec)
   (iv) lower constraint bound on parameter

Comments

1. For single point specification, let the number of subintervals be zero and set upper bound equal to lower bound.

2. The user should consult theoretical concepts reviewed by Bandler (1969).

3. Continuous specifications require program modification.



PARAMETER CONSTRAINTS

Possibilities

1. Any circuit parameter may be fixed or varied as specified by the user.
2. If variable parameters are to be constrained, then each must have an associated lower and upper desired bound supplied by the user.

Implementation

For upper and lower parameter constraints, fictitious frequency points of value 1, 2, 3,... etc. are associated with each variable parameter in correct sequence.

Comments

The constraints are treated exactly like single point specifications.

CHAPTER FOUR

OPTIMIZATION METHODS

Possibilities

1. Use of Fletcher-Powell (1963) or Fletcher (1970) methods. The Fletcher method is generally the faster method.
2. Any finite value of p greater than 1. Low values of p will generally allow quicker optimization to nonequal ripple solutions. Larger values of p may slow down optimization but better near equal ripple solutions will be obtained. Recommendation: start with 2, increase to 10 then to 100, etc., as needed.

Parameters Required

1. When asked for, 1 denotes Fletcher, 2 denotes Fletcher-Powell.
2. Maximum number of iterations (e.g., 100).
3. Integer denoting how many iterations should be executed before printout of intermediate output.
4. Value of p (positive integer, greater than one).
5. Starting values of variables in correct sequence.
6. Small test quantities used in the optimization methods to test for convergence (e.g., 10^{-4}).
7. Estimate of lower bound on function to be minimized. Supply realistic underestimate.

8. Artificial margin (see Bandler and Charalambous (1972), included in this report). Set to 0 if unsure.
9. Difference between objective function values in successive optimizations for termination. Set to 0 if unsure.
10. The number of complete optimizations desired. Each optimization starts from the previous optimum obtained.

CHAPTER FIVE

EXAMPLE OF INPUT-OUTPUT

```
 72/11/10. 14.56.11.
KRONOS TIME SHARING SYSTEM - VER. 2.00.
USE:E JBER:  AVYU203
PASSWORD
[redacted]
TERMINAL:      15
SYSDM: DELAY,TTU
READY.

EXEC
OLD, NEW, OR LIB FILE:  OLD
FILE NAME:  COPTA
READY.

RUN,MI=5000

 72/11/10. 14.57.28.
PROGRAM   COPTA

YOU ARE WELCOME TO USE THIS NETWORK OPTIMIZATION
 PACKAGE, PLEASE SUPPLY DATA WHEN ASKED FOR, THANK YOU AND GOOD LUCK
HOW MANY BLOCKS WOULD THERE BE IN THE CIRCUIT
,SUPPLY AN INTEGER NUMBER
? 6
WHAT IS THE TOTAL NUMBER OF PARAMETERS IN ALL BLOCKS
,SUPPLY AN INTEGER NUMBER
? 6
HOW MANY C SECTIONS DO YOU WANT, PRINT 0 IF
 YOU DONT WANT ANY
? 0
HOW MANY D SECTIONS DO YOU WANT ,PRINT 0 IF
 YOU DONT WANT ANY
? 0
WOULD C AND D SECTION PARAMETERS BE VARIABLES
 OR FIXED. TYPE 1 IF VARIABLE AND 0 IF FIXED
? 0
SUPPLY A SEQUENCE OF CODE NUMBERS OF BLOCKS TO
 SPECIFY ORDER IN WHICH BLOCKS WOULD BE CONNECTED
? 1 3 1 3 1 3
SPECIFY VALUES OF EACH PARAMETER IN THE CIRCUIT
 FOLLOWED BY VALUES OF PARAMETERS OF C AND D SECTION AND D LEVEL
? 1 1 1 1 1 1 1
INDICATE WHICH OF THE PARAMETERS IN THE CIRCUIT ARE
 FIXED OR VARIABLES BY PUTTING T FOR VARIABLE AND F FOR FIXED
? T T T T T T
WHAT IS THE TOTAL NUMBER OF INTERVALS INCLUDING
 NECESSARY ONES FOR PARAMETER CONSTRAINTS
? 3
SPECIFY NUMBER OF SUBINTERVALS CORRESPONDING TO EACH
 INTERVAL. TYPE 0 FOR SINGLE POINT SPECIFICATIONS
? 1.E-10 0.9 1.75 1.75 2.5 2.5
ON OR ABOUT LINE NUMBER 01350, ILLEGAL DATA-REENTER REMAINDER.
? 10 0 0
SPECIFY LOWER AND UPPER FREQUENCY BOUNDS
 CORRESPONDING TO EACH INTERVAL. LOWER BOUND EQUALS UPPER BOUND
 FOR SINGLE POINT SPECIFICATIONS.
? 1.E-10 0.9 1.75 1.75 2.5 2.5
 SPECIFY A SEQUENCE OF NUMBERS TO BE USED AS SPECI
```

```
FICATIONS IN EACH INTERVAL INCLUDING PARAMETER CONSTRAINTS
? 0 40 60
INDICATE WHETHER A SPECIFICATION IN ANY GIVEN
 INTERVAL WOULD BE AN UPPER SPECIFICATION OR A LOWER SPEC.
 PUT 1. FOR UPPER AND -1.0 FOR LOWER SPECIFICATION
? 1. -1. -1.
WHAT WILL BE THE WEIGHTING FACTORS ON ERROR
 FUNCTIONS IN EACH INTERVAL. SUPPLY ONE PER INTERVAL
? 5 1 1
WHAT IS THE APPROXIMATING FUNCTION IN EACH INTERVAL
 FOR REFLECTION COEFF. SET=1,FOR INSERTION LOSS SET=2
FOR GROUP DELAY SET=3, FOR PARAMETER CONSTRAINTS SET=0
? 2 2 2
WHAT IS THE LOAD IMPEDANCE
? 1
WHICH OPTIMIZATION TECHNIQUE DO YOU WANT TO USE
 SET EQUAL TO ONE FOR FLETCHER METHOD AND 2 FOR FLETCHER-POWELL
? 1
WHAT IS THE MAXIMUM NUMBER OF ALLOWABLE ITERATIONS
? 100
SPECIFY A REALISTIC UNDER-ESTIMATEOF THE VALUE OF OBJECTIVE FUNCTION
? -10
SUPPLY A SMALL QUANTITY BY WHICH SPECIFICATIONS
WOULD BE SHIFTED ARTIFICIALLY,SET=0 IF NOT SURE
? 0
SUPPLY THE VALUE OF DIFFERENCE BETWEEN
OBJECTIVE FUNCTION VALUES IN SUCCESIVE OPTIMIZATIONSET =0 IF NOT SURE
? 0
WHAT IS THE CENTRE FREQUENCY
? 1
WHAT IS THE CUT OFF FREQUENCY TO BE USED
? 1
INTERMEDIATE OUTPUT WILL BE PRINTED AFTER EVERY
 SPECIFIED NUMBER OF ITERATIONS,SET = 0 IF NO INTERMEDIATE OUTPUTIS DESIRED
? 20
PLEASE SUPPLY THE NUMBER OF VARIABLE PARAMETERS
? 6
WHAT ARE THE STARTING  VALUES FOR THE VARIABLE PARAMETERS
? 1 1 1 1 1 1
SUPPLY AS MANYSMALL QUANTITIES TO BE USED FOR CONVERGENCE
IN FLETCHER METHOD AS THERE ARE VARIABLES
? 1.E-4 1.E-4 1.E-4 1.E-4 1.E-4 1.E-4
HOW MANY DIFFERENT VALUES OF P DO YOU WANT TO USE
? 1
```

RESPONSE AT THE STARTING POINT

FREQUENCY	INSERTION LOSS
.999875737762E-10	-.123433915719E-12
.900000000900E-01	.627325470185E-03
.180000000080E+00	.938744333702E-02
.270000000070E+00	.422855191728E-01
.360000000060E+00	.112363853524E+00
.450000000050E+00	.216134651104E+00
.540000000040E+00	.327204405278E+00
.630000000030E+00	.402323517897E+00
.720000000020E+00	.397961285088E+00
.810000000010E+00	.293526836382E+00
.900000000000E+00	.121617127817E+00

FREQUENCY	INSERTION LOSS
.175000000000E+01	.165996226859E+00

FREQUENCY	INSERTION LOSS
.250000000000E+01	.345393810150E-02

WHAT IS THE VALUE OF P TO BE USED
? 10

0 GRADIENTS CHECKING

GRADIENTS HAVE BEEN CHECKED AT THE FOLLOWING POINT

X(1)= .10000000E+01
X(2)= .10000000E+01
X(3)= .10000000E+01
X(4)= .10000000E+01

```
        X( 5)=    .10000000E+01
        X( 6)=    .10000000E+01

0    ANALYTICAL GRADIENTS    NUMERICAL GRADIENTS    PERCENTAGE ERROR

0      -.47545625E+00          -.47589298E+00          .91770238E-01
0      -.58668560E+01          -.58727177E+01          .99813206E-01
0      -.10887302E+02          -.10903818E+02          .15147304E+00
0      -.10887302E+02          -.10903818E+02          .15147304E+00
0      -.58668560E+01          -.58727177E+01          .99813129E-01
0      -.47545625E+00          -.47589298E+00          .91769283E-01
0

     GRADIENTS ARE O. K.
```

OPTIMIZATION BY [illegible] FLETCHER METHOD

	ITERATION NUMBER	FUNCTION EVALUATIONS	TIME ELAPSED (SECONDS)	OBJECTIVE FUNCTION	VARIABLE VECTOR X(I)	GRADIENT VECTOR G(I)
0	0	1	.15200000E+00	.39879105E+02	.10000000E+01	-.47545625E+00
					.10000000E+01	-.58668560E+01
					.10000000E+01	-.10887302E+02
					.10000000E+01	-.10887302E+02
					.10000000E+01	-.58668560E+01
					.10000000E+01	-.47545625E+00
0	20	32	.40580000E+01	.30073156E+00	.10185255E+01	.70137926E+00
					.16266422E+01	-.25894728E+00
					.19328948E+01	-.12645219E+01
					.18816122E+01	-.16802152E+01
					.16504671E+01	-.52394460E+00
					.10442604E+01	.89735261E+00

0IEXIT= 1CRITERION FOR OPTIMUM (CHANGE IN VECTOR X .LT.EPS) HAS BEEN SATISFIED

FOLLOWING IS THE OPTIMUM SOLUTION

F = .243263O2E+00

X(1)= .10133271E+01
X(2)= .16532968E+01
X(3)= .19154611E+01
X(4)= .19154699E+01
X(5)= .16532863E+01
X(6)= .10133124E+01

0

EXECUTION TIME IN SECONDS= 5.99200

1 P = 10

```
                    FINAL RESPONSE OF THE CIRCUIT
                    -----------------------------

                 FREQUENCY              INSERTION LOSS
               .999875737762E-10     -.123433915719E-12
               .900000000900E-01      .545878878030E-03
               .180000000080E+00      .710161191128E-02
               .270000000070E+00      .247088174402E-01
               .360000000060E+00      .427157551833E-01
               .450000000050E+00      .391416365584E-01
               .540000000040E+00      .120663702819E-01
               .630000000030E+00      .208727079835E-02
               .720000000020E+00      .401122692502E-01
               .810000000010E+00      .418777468811E-01
               .900000000000E+00      .415423199929E-01

                 FREQUENCY              INSERTION LOSS
               .175000000000E+01      .398242384076E+02

                 FREQUENCY              INSERTION LOSS
               .250000000000E+01      .603399576522E+02
ON OR ABOUT LINE NUMBER 00210, EXIT.


CP       7,655 SECS.


RUN COMPLETE.

B_HELLO

AVYU203   LOG OFF.  15.12.47.
AVYU203   CP        7.655 SEC.
```

CHAPTER SIX

EXAMPLES

The purpose of this chapter is to demonstrate the versatility of the package as well as to provide the user with further illustrations on how to enter data and interpret the results.

Example 1

Design of Optimum Group Delay Equalizer

It is desired to use one microwave C-section to optimize a set of group delay specifications over a given band. Table 6-1 shows the given set of frequencies and corresponding group delay.

The number of intervals was taken as 32, i.e., 16 corresponding to upper specifications and 16 to lower specifications. The upper and lower specifications at every frequency point were the same. They were equal to the negative of the given group delay. The weighting was 1 throughout. The Fletcher method was used. The estimate of the lower bound on the objective function was taken as -10. The test quantities used for convergence were 10^{-4} for each parameter. The artificial margin was 0. The number of parameters (unconstrained) is two, namely, σ and d.

Starting and optimized values for the parameters and the corresponding total relative group delay are shown in Table 6-2. Observe that the starting point was the best result obtained by an existing program. Note also that three values of p were used, namely, 2, 10 and 10,000. Optimization for p=10 was started at the optimum for p=2. Extrema in the responses are denoted by *.

Table 6-1

Frequency (MHz)	Group Delay (nsec)
7,976	69.03
7,977	62.61
7,978	58.03
7,979	54.79
7,980	52.52
7,981	50.79
7,982	49.98
7,983	49.49
7,984	49.49
7,985	49.97
7,986	50.95
7,987	52.50
7,988	54.75
7,989	57.99
7,990	62.55
7,991	68.94

Table 6-2

	Parameters			
p	start	2	10	10,000
σ	340	349.05	365.94	368.77
d	86	86.64	87.68	87.75
Frequency (MHz)	Total Relative Group Delay (nsec)			
7,976	4.11	3.53	2.56*	2.49*
7,977	0.30	-0.19	-0.99	-1.04
7,978	-1.48	-1.85	-2.42*	-2.43*
7,979	-1.83	-2.04	-2.33	-2.29
7,980	-1.29	-1.32	-1.29	-1.19
7,981	-0.36	-0.23	0.14	0.29
7,982	0.56	0.83	1.48	1.69
7,983	1.09	1.44	2.26*	2.49*
7,984	1.08	1.44	2.26*	2.49*
7,985	0.54	0.82	1.46	1.67
7,986	-0.38	0.24	0.13	0.29
7,987	-1.32	-1.36	-1.33	-1.23
7,988	-1.89	-2.10	-2.39	-2.35
7,989	-1.54	-1.91	-2.48*	-2.49*
7,990	0.22	-0.27	-1.07	-1.12
7,991	4.01	3.43	2.46*	2.39*
Maximum	4.11	3.53	2.56	2.49
Execution Time (sec)	0	1/2	1-1/4	10

Example 2

Optimization of Two-Section Transmission-Line Transformer

To demonstrate how close to known optimal solutions the package will allow a user to come, a two-section, four-variable lossless transmission-line transformer was optimized from a poor starting point. A relative bandwidth of 100% is assumed and a load to source impedance ratio of 10:1 is taken. The modulus of the reflection coefficient is considered. The problem itself has frequently been used as a test problem for optimization strategies. ℓ_1 and ℓ_2 are the normalized lengths of sections 1 and 2; Z_{01} and Z_{02} are the corresponding normalized characteristic impedances.

One upper specification of 0 reflection coefficient with 20 sub-intervals (21 uniformly distributed sample points) was taken. The Fletcher method was used and every optimization with a different value of p was started at the previous optimum. The test quantities for the Fletcher method were 10^{-6} and the estimate of the lower bound on the objective function was taken as -10. The artificial margin was 0 and the weighting was 1. Table 6-3 shows the results obtained by the package.

Table 6-3

Parameters					
p	Start	2	10	1,000	10,000
ℓ_1	0.8	0.9398	0.9873	0.9999	1.0000
Z_{01}	3.0	1.9897	2.1753	2.2360	2.2361
ℓ_2	0.8	0.9398	0.9873	0.9999	1.0000
Z_{02}	3.5	5.0259	4.5971	4.4722	4.4722
Norm. Freq.	Reflection Coefficient				
.50	.467	.560*	.463*	.4287*	.4286*
.55	.347	.485	.356	.3101	.3099
.60	.205	.398	.235	.1785	.1783
.65	.051	.302	.108	.0438	.0436
.70	.119	.202	.014	.0828	.0830
.75	.266	.103	.124	.1926	.1928
.80	.392	.012	.216	.2811	.2813
.85	.495	.067	.287	.3476	.3478
.90	.574	.131	.337	.3933	.3934
.95	.634	.177	.369	.4198	.4199
1.00	.678	.207	.382*	.4285*	.4286*
1.05	.711	.220*	.378	.4199	.4199
1.10	.734	.217	.356	.3935	.3934
1.15	.749	.196	.315	.3479	.3478
1.20	.758	.159	.255	.2815	.2813
1.25	.760	.105	.174	.1930	.1928
1.30	.758	.035	.073	.0833	.0830
1.35	.749	.050	.044	.0432	.0436
1.40	.734	.145	.170	.1779	.1783
1.45	.711	.246	.295	.3095	.3099
1.50	.678	.345*	.410*	.4282*	.4285*
Maximum	.760	.560	.463	.4287	.4286
Execution Time (sec)	0	4	5	16	5-1/2

Note that to the number of significant figures shown the optimum for p=10,000 is the same as the known Chebyshev solution for the problem. Excellent results are obviously obtainable with p=1,000. Execution times can be cut by roughly 50% if about half the number of sample points are chosen. The maxima in the responses are again denoted by *.

Example 3

Lowpass Lumped LC Filter Design

It is desired to approximate a certain lowpass filter insertion loss specification using a ladder network consisting of lumped lossless inductors and capacitors. The first element is a shunt capacitor, followed by a series inductor and so on, with a total of three capacitors and three inductors. The source and load resistances are each taken as unity.

The response specification is as in Table 6-4. Observe that two problems were to be solved with slightly different specifications.

The Fletcher method was used with the test quantities equal to 10^{-6}, the estimate of the lower bound of the objective function was -10, the artifical margin was 0. Table 6-5 summarizes the results obtained.

Optimization for p=1,000 started from the optimum for p=2. Note that the results for the two problems are identical.

Table 6-4

Problems	Frequency (Hz)	Specification (dB)	Type
1&2	0 - 0.9	0	Upper
1&2	1.75	40	Lower
2	1.75	41	Upper
1&2	2.50	60	Lower

Table 6-5

Parameters					
P	Start	2	1,000	2	1,000
C_1	1.0	1.015	1.011	1.015	1.011
L_1	1.0	1.659	1.654	1.659	1.654
C_2	1.0	1.917	1.915	1.917	1.915
L_2	1.0	1.917	1.915	1.917	1.915
C_3	1.0	1.659	1.654	1.659	1.654
L_3	1.0	1.015	1.011	1.015	1.011

Freq. (Hz)	Weight	Insertion Loss (dB) Start	Problem 1		Problem 2	
.0	5.0	0	0	0	0	0
0.09	5.0	.001	.001	.001	.001	.001
0.18	5.0	.009	.007	.007	.007	.007
0.27	5.0	.042	.024	.024	.024	.024
0.36	5.0	.112	.042	.042	.042	.042
0.45	5.0	.216	.037	.038	.037	.038
0.54	5.0	.327	.011	.012	.012	.012
0.63	5.0	.402	.003	.002	.003	.002
0.72	5.0	.398	.044	.041	.044	.041
0.81	5.0	.294	.044	.042	.044	.042
0.90	5.0	.122	.045	.042	.045	.042
1.75	1.0	.166	39.9	39.8	39.9	39.8
2.50	1.0	34.5	60.5	60.3	60.5	60.3
Execution Time (sec)		0	3-1/2	7-1/2	4	8-1/2

Example 4

High Power Output Filter for CTS

It is desired to meet or exceed the specifications shown in Table 6-6 for a six element filter consisting of blocks 10 and 12 alternating, with unity terminations.

One problem that was tried took the center frequency equal to the resonant frequency f_R = 11,885.5 MHz. The quality factors were to be 6,000. The normalized slope parameters were to be varied between 42 and 2,100.

The Fletcher method was used with relevant optimization parameters as for Example 3. The results are shown in Table 6-7. Weighting factors were unity throughout.

Table 6-6

Frequency (MHz)	Specification (dB)	Type
11,700	66	Lower
11,843 - 11,928	0*	Upper
12,038	31	Lower
12,080	41	Lower

*.85 dB was acutally wanted but 0 was used for convenience in the program.

Table 6-7

		Parameters	
p	Start	2	1,000
B_1'	240	183.5	192.59
X_1'	420	300.2	280.87
B_2'	570	321.6	315.17
X_2'	460	318.9	315.17
B_3'	450	295.9	280.87
X_3'	210	179.7	192.59
Frequency (MHz)		Insertion Loss (dB)	
11,700	85.1	65.8	65.31
11,843	2.81	0.74	0.689
11,847	1.52	0.68	0.675
11,852	1.21	0.65	0.684
11,856	1.09	0.65	0.688
11,860	1.02	0.64	0.674
11,864	0.97	0.62	0.647
11,869	0.94	0.61	0.616
11,873	0.90	0.59	0.592
11,877	0.88	0.58	0.578
11,881	0.86	0.58	0.572
11,886	0.85	0.58	0.571
11,890	0.86	0.58	0.572
11,894	0.88	0.58	0.578
11,898	0.91	0.59	0.592
11,903	0.94	0.61	0.616
11,907	0.97	0.63	0.646
11,911	1.01	0.64	0.674
11,915	1.08	0.65	0.688
11,920	1.21	0.65	0.684
11,924	1.50	0.68	0.675
11,928	2.73	0.74	0.688
12,038	73.9	54.1	53.49
12,080	86.8	67.5	67.05
Execution Time (min)	0	2/3	3

REFERENCES

[1] R. Fletcher and M.J.D. Powell, "A rapidly convergent descent method for minimization", Computer J., vol. 6, pp. 163-168, June 1963.

*[2] J.W. Bandler, "Optimization methods for computer-aided design", IEEE Trans. Microwave Theory and Techniques, vol. MTT-17, pp. 533-552, August 1969.

[3] R. Fletcher, "A new approach to variable metric algorithms", Computer J., vol. 13, pp. 317-322, August 1970.

**[4] J.W. Bandler and R.E. Seviora, "Current trends in network optimization", IEEE Trans. Microwave Theory and Techniques, vol. MTT-18, pp. 1159-1170, December 1970.

[5] C.M.Kudsia, "Synthesis of optimum reflection type microwave equalizers," RCA Review, vol. 31, pp. 571-595, September 1970.

[6] J.W. Bandler and C. Charalambous, "Theory of generalized least pth approximation", IEEE Trans. Circuit Theory, vol. CT-19, pp. 287-289, May 1972.

***[7] J.W. Bandler and C. Charalambous, "Practical least pth optimization of networks", IEEE Trans. Microwave Theory and Techniques, vol. MTT-20, 1972.

* Appendix 2

** Appendix 3

*** Appendix 4

APPENDIX 1

PROGRAM LISTING

The complete listing of the program package including all subprograms needed is supplied in this appendix.

Flow Diagram

```
FMLPO  →  EPSNP   →  (1)
       →  INPUT   (supressed)
       →  WRITE1
       →  FINAL
       →  ERRO    →  (2)
       →  GRDCHK  →  (3)
       →  VMMO1 }    { WRITE2
                  →  {
       →  FMFP  }    { (3)
(1)    →  FCTAPP  →  (4)
       →  FCT     →  FUNCS
       →  FUNCS
       →  W
(2)    →  EPSNP   →  (1)
(3)    →  FUNCT   →  W
                  →  FCTAPP  →  (4)
                  →  ERRO    →  (2)
(4)    →  SUB1
       →  SUB2
            ⋮
       →  SUB17
```

The Subprograms

FMLPO	Supplies all the data for the optimization process and coordinates the other subprograms in the package.
EPSNP	Calculates error functions with respect to the upper and lower specifications.
INPUT	(Suppressed) Prints the input data for the optimization process.
WRITE1, WRITE2	Print intermediate results.
FINAL	Prints the optimum solution.
ERRO	Selects the current weighted error functions relevant to the objective function.
GRDCHK	Checks the gradients of the objective function with respect to the variable parameters at the starting point.
VMM01	Minimizes the objective function using the Fletcher method.
FMFP	Minimizes the objective function using the Fletcher-Powell method.
FCTAPP	Defines an approximating function and its gradients with respect to variable parameters. Specifically, the subprogram does the network analysis and adjoint network analysis. Reflection coefficient, insertion loss, group delay and parameter constraints are evaluated here.

FCT	Calculates the artificial upper and lower specifications.
FUNCS	Defines upper and lower specifications
W	Defines the weighting functions.
SUB1 to	
SUB17	Analysis subprograms for the two-port circuit building blocks corresponding to codes 1 to 17, respectively.

Listing has been omitted for brevity.

Copies are available if required from

J.W. Bandler.

APPENDIX 2

OPTIMIZATION METHODS FOR COMPUTER-AIDED DESIGN

This appendix contains a survey of optimization methods. It deals with response specifications, scaling, constraints, and gradient methods among other topics.

Reprinted from IEEE TRANSACTIONS
ON *MICROWAVE THEORY AND TECHNIQUES*
Volume MTT-17, Number 8, August, 1969
pp. 533-552

Optimization Methods for Computer-Aided Design

JOHN W. BANDLER, MEMBER, IEEE

Invited Paper

Abstract—**This paper surveys recent automatic optimization methods which either have found or should find useful application in the optimal design of microwave networks by digital computer. Emphasis is given to formulations and methods which can be implemented in situations when the classical synthesis approach (analytic or numerical) is inappropriate. Objectives for network optimization are formulated including minimax and least *p*th. Detailed consideration is given to methods of dealing with parameter and response constraints by means of transformations or penalties. In particular, the formulation of problems in terms of inequality constraints and their solution by sequential unconstrained minimization is discussed. Several one-dimensional and multidimensional minimization strategies are summarized in a tutorial manner. Included are Fibonacci and Golden Section search, interpolation methods, pattern search, Rosenbrock's method, Powell's method, simplex methods, and the Newton-Raphson, Fletcher-Powell, and least squares methods. Relevant examples of interest to microwave circuit designers illustrating the application of computer-aided optimization techniques are presented. The paper also includes a classified list of references.**

Manuscript received February 18, 1969; revised May 1, 1969. This work was carried out at the University of Manitoba with financial assistance from the Faculty of Graduate Studies of the University of Manitoba and from the National Research Council of Canada.

The author was with the Numerical Applications Group, Department of Electrical Engineering, University of Manitoba, Winnipeg, Canada. He is now with the Department of Electrical Engineering, McMaster University, Hamilton, Ont., Canada.

I. INTRODUCTION

FULLY AUTOMATED design and optimization is surely one of the ultimate goals of computer-aided design. The amount of human intervention required to produce an acceptable design, even though this is often unavoidable, should, therefore, be regarded as a measure of our ignorance of the problem, our inability to specify our objectives in a meaningful way to the computer, or our failure to anticipate and make provisions for dealing with the possible hazards which could be encountered in the solution of the problem.

An on-line facility which permits the user to propose a circuit configuration, analyze it, and display the results may well be an invaluable educational and research tool providing the user with insight into certain aspects of his design problem. But even with the fastest analysis program it would be misleading to suggest that this method can be efficiently applied to the design and optimization of networks involving more than a few variables and anything other than the simplest of parameter and response constraints. For a fairly complex network optimization problem the number of effective response evaluations can easily run into the thousands even with the most efficient currently available automatic optimization methods before a local optimum is reached—and then only for that predetermined configuration.

Fully automated network design and optimization is still some way off. In the meantime, very effective use of the computer can be made by allowing the computer to optimize a network of predetermined allowable configuration automatically. If the results are unsatisfactory in some way, one could change the objective function, impose or relax constraints, try another strategy, alter the configuration, etc., whichever course of action is appropriate, and try again. Obviously, this can be executed either by batch processing or from an on-line terminal. There is no reason why the on-line designer should not avail himself of an efficient optimization program as well as an analysis program.

With the objective, therefore, of encouraging more effective use of computers, this paper surveys recent automatic optimization methods which either have found or should find useful application in computer-aided network design. Emphasis is given to formulations and methods which can be implemented in practical situations when the classical synthesis approach (analytic or numerical) is inappropriate. Objectives for network optimization including minimax and least pth are formulated and discussed.

Detailed consideration is given to methods of dealing with parameter constraints by means of transformations or penalties. This is rather important for microwave networks where the practical ranges of parameter values can be quite narrow, e.g., characteristic impedance values for transmission lines extend from about 15 to 150 ohms. The configuration, the overall size, the suppression of unwanted modes of propagation, considerations for parasitic discontinuity effects, the stabilization of an active device can all result in constraints on the parameters. Response constraints, which are less easy to deal with than parameter constraints, are also considered in some detail. In particular, the formulation of problems in terms of inequality constraints and their solution by sequential unconstrained minimization is discussed.

Several one-dimensional and multidimensional minimization strategies are summarized in a tutorial manner. Included are Fibonacci and Golden Section search, interpolation methods, pattern search and some variations including Rosenbrock's method, Powell's method, simplex methods, and the Newton-Raphson, Fletcher-Powell and least squares methods. Slightly more emphasis has been accorded to direct search methods than to gradient methods because they appear to date to have been more frequently employed in microwave network optimization. It is probably not widely appreciated that most direct search methods are superior, in general, to the classical steepest descent method and compare rather favorably with other gradient methods as far as efficiency and reliability are concerned. It is generally only near the minimum that differences in efficiency begin to manifest themselves between quadratically convergent and nonquadratically convergent methods—but quadratic convergence is not the prerogative of gradient methods as classified in this paper.

Section II introduces fundamental concepts and definitions. Section III formulates objectives for network optimization. Section IV deals with constraints. Section V describes one-dimensional optimization strategies, followed by Section VI which describes multidimensional direct search strategies and Section VII which describes multidimensional gradient strategies. Section VIII reviews some recent papers which report the application of various methods to network optimization. Finally, the references are divided into broad classifications: references of general interest [1]–[22], references recommended for direct search methods [23]–[55] and gradient methods [56]–[83], references dealing with applications to network design [84]–[119], and some miscellaneous references [120]–[126].

Inevitably, the material presented in this paper tends to reflect some of the author's current interests. Conspicuous omissions include Chebyshev polynomial and rational function approximation techniques using the Remez method or its generalizations [17], [115], [125], and a discussion of optimization by hybrid computer in which the system is simulated on an analog computer while the optimization strategy is controlled by the digital computer [120], [122]. The author apologizes in advance to all those researchers to whose contributions he may not have done full justice. He hopes, however, that the references adequately represent the state of the art of automatic optimization methods for computer-aided design. The use of such automatic computer-aided methods in microwave network design is not so well established as the use of computers in the numerical solution of electromagnetic field problems [126]. For this reason, there are not yet many microwave references from which to choose to illustrate the optimization techniques.

II. Fundamental Concepts and Definitions

The problem is to minimize U where

$$U = U(\boldsymbol{\phi}) \tag{1}$$

and where

$$\boldsymbol{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \cdot \\ \phi_k \end{bmatrix}. \tag{2}$$

U is called the *objective function* and the vector $\boldsymbol{\phi}$ represents a set of independent parameters. Minimizing a function is the same as maximizing the negative of the function, so there is no loss of generality.

In general, there will be *constraints* that must be satisfied either during optimization or by the optimum solution. Each parameter might be constrained explicitly by an upper and lower bound as follows:

$$\phi_{li} \leq \phi_i \leq \phi_{ui} \qquad i = 1, 2, \cdots, k \tag{3}$$

where ϕ_l and ϕ_u are lower and upper bounds, respectively. Furthermore, the problem could be constrained by a set of h implicit functions

$$c_j(\boldsymbol{\phi}) \geq 0 \qquad j = 1, 2, \cdots, h. \tag{4}$$

Any vector ϕ which satisfies the constraints is termed *feasible*. It lies in a *feasible region* R (*closed* if equalities are admissible as in (3) or (4), *open* otherwise) as expressed by $\phi \in R$. It is assumed that $U(\phi)$ can be obtained for any $\phi \in R$ either by calculation or by measurement.

Fig. 1 shows a 2-dimensional *contour* sketch which illustrates some features encountered in optimization problems. A *hypercontour*, described by the relation

$$U(\phi) = U_{\text{const.}}, \tag{5}$$

is the multidimensional generalization of a contour. The feasible region in Fig. 1 is determined by fixed upper and lower bounds on ϕ. The feasible region is seen to contain one *global minimum*, one *local minimum* and one *saddle point*. A minimum may be located by a point $\check{\phi}$ on the *response hypersurface* generated by $U(\phi)$ such that

$$\check{U} = U(\check{\phi}) < U(\phi) \tag{6}$$

for any ϕ in the immediate feasible neighborhood of $\check{\phi}$. (Since methods which guarantee convergence to a global minimum are not available, the discussion must restrict itself to consideration of local minima.) A saddle point is distinguished by the fact that it can appear to be a maximum or a minimum depending upon the direction being investigated. A more formal definition of a minimum follows.

The first three terms of the multidimensional Taylor series are given by

$$U(\phi + \Delta\phi) = U(\phi) + \nabla U^T \Delta\phi + \tfrac{1}{2}\Delta\phi^T H \Delta\phi + \cdots \tag{7}$$

where

$$\Delta\phi = \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \vdots \\ \Delta\phi_k \end{bmatrix} \tag{8}$$

represents the parameter increments,

$$\nabla U = \begin{bmatrix} \dfrac{\partial U}{\partial \phi_1} \\ \dfrac{\partial U}{\partial \phi_2} \\ \vdots \\ \dfrac{\partial U}{\partial \phi_k} \end{bmatrix} \tag{9}$$

is the gradient vector containing the first partial derivatives, and

$$H = \begin{bmatrix} \dfrac{\partial^2 U}{\partial \phi_1^2} & \dfrac{\partial^2 U}{\partial \phi_1 \partial \phi_2} & \cdots & \dfrac{\partial^2 U}{\partial \phi_1 \partial \phi_k} \\ \dfrac{\partial^2 U}{\partial \phi_2 \partial \phi_1} & \cdot & & \vdots \\ \vdots & & \cdot & \vdots \\ \dfrac{\partial^2 U}{\partial \phi_k \partial \phi_1} & & \cdots & \dfrac{\partial^2 U}{\partial \phi_k^2} \end{bmatrix} \tag{10}$$

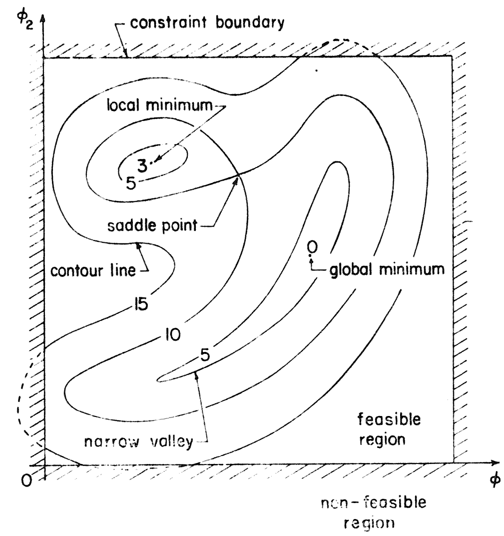


Fig. 1. Two-dimensional contour sketch illustrating some features encountered in optimization problems.

is the matrix of second partial derivatives, the *Hessian* matrix. Assuming the first and second derivatives exist, a point is a minimum if the gradient vector is zero and the Hessian matrix is positive definite at that point.

A *unimodal* function may be defined in the present context as one which has a unique optimum in the feasible region. The presence of discontinuities in the function or its derivatives need not affect its unimodality. Fig. 1 has two minima so it is called *bimodal*. A *strictly convex* function is one which can never be underestimated by a linear interpolation between any two points on its surface. Similarly, a *strictly concave* function is one whose negative is strictly convex. Examples of unimodal, convex and concave functions of one variable are illustrated in Fig. 2. (The word "strictly" is omitted if equality of the function and a linear interpolation can occur.)

If the first and second derivatives of a function exist then strict convexity, for example, implies that the Hessian matrix is positive definite and vice versa. Consider the *narrow curved valley* shown in Fig. 3(a). It is possible to underestimate U by a linear interpolation along a contour, for example, which indicates that the function is nonconvex. Contours of this type do present some difficulties to optimization strategies. Ideally, one would like contours to be in the form of concentric *hyperspheres*, and one should attempt to scale the parameters to this end, where possible.

Fig. 3 shows contours of other two-dimensional optimization problems which present difficulties in practice. In Fig. 3(b), the minimum lies on a path of discontinuous derivatives; the constraint boundaries in Fig. 3(c) define a nonconvex feasible region (a feasible region is convex if the straight line joining any two points lies entirely within the region); in Fig. 3(d) the minimum lies at a discontinuity in the function. Theorems which invoke the classical properties of optima or such concepts as convexity may not be so readily applicable to the problems illustrated in Fig. 3(b)–(d), and yet the minima involved are quite unambiguously defined.

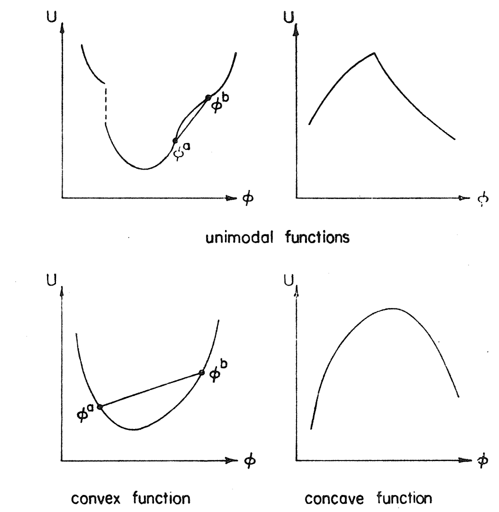


Fig. 2. Examples of unimodal, convex, and concave functions of one variable.

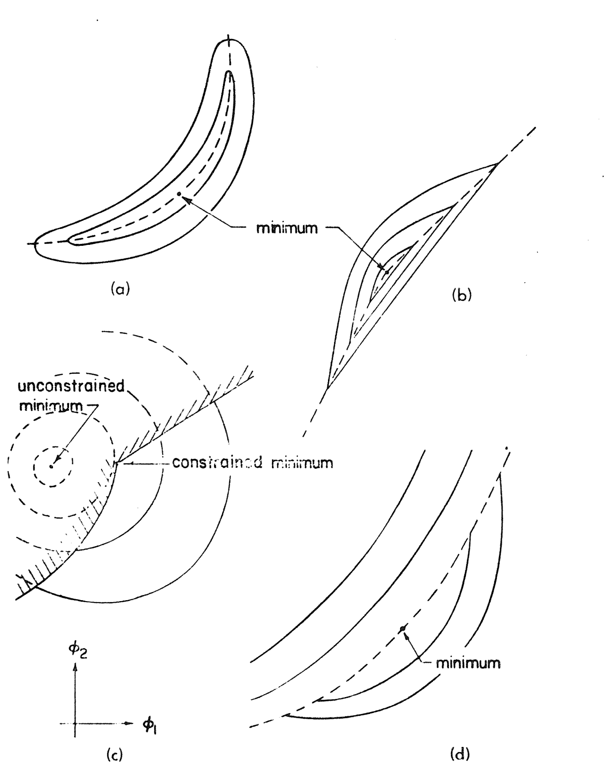


Fig. 3. Contours which present difficulties to optimization strategies. (a) A narrow curved valley. (b) A narrow valley along which a path of discontinuous derivatives lies. (c) A nonconvex feasible region. (d) A discontinuous function.

A number of the general references [1], [17], [19], [20] give good introductions to the fundamental concepts and definitions used in the literature generally. Unfortunately, because of the diverse background of the authors concerned, there exists a profusion of different nomenclature. (The present author has probably added to this confusion.)

III. Objectives for Network Optimization

In this section some objective function formulations for network optimization will be presented and discussed. The emphasis is on formulations which can allow explicit and implicit constraints, e.g., on the network parameters and responses, to be taken into account. This is felt to be particularly important in microwave network optimization where the range of permissible parameter values is often fairly narrow, the choice of physical configurations may be limited and parasitic effects can be acute. Thus, formulations which remain close to physical reality and aim towards physical and practical realizability are preferred, at least by this author.

Direct Minimax Formulation

An ideal objective for network optimization is to minimize U where

$$U = U(\phi, \psi) = \max_{[\psi_l, \psi_u]} \left[w_u(\psi)(F(\phi, \psi) - S_u(\psi)), \; - w_l(\psi)(F(\phi, \psi) - S_l(\psi)) \right] \tag{11}$$

where

$F(\phi, \psi)$ is the response function
ϕ represents the network parameters
ψ is an independent variable, e.g., frequency or time
$S_u(\psi)$ is a desired upper response specification
$S_l(\psi)$ is a desired lower response specification
$w_u(\psi)$ is a weighting factor for $S_u(\psi)$
$w_l(\psi)$ is a weighting factor for $S_l(\psi)$
ψ_u is the upper bound on ψ
ψ_l is the lower bound on ψ.

This formulation is illustrated by Fig. 4. Fig. 4(a) shows a response function satisfying arbitrary specifications; Fig. 4(b) shows a response function failing to satisfy a bandpass filter specification; Fig. 4(c) shows a response function just satisfying a possible amplifier specification. $F(\phi, \psi)$ will often be expressible as a continuous function of ϕ and ψ. But $S_l(\psi)$, $S_u(\psi)$, $w_l(\psi)$, and $w_u(\psi)$ are likely to be discontinuous.

The following restrictions are imposed:

$$S_u(\psi) \geq S_l(\psi) \tag{12}$$

$$w_u(\psi) > 0 \tag{13}$$

$$w_l(\psi) > 0. \tag{14}$$

Under these conditions $w_u(\psi)(F(\phi, \psi) - S_u(\psi))$ and $-w_l(\psi)(F(\phi, \psi) - S_l(\psi))$ are both positive when the specifications are not met; they are zero when the specifications are just met; and they are negative when the specifications are exceeded. The objective is, therefore, to minimize the maximum (weighted) amount by which the network response fails to meet the specifications, or to maximize the minimum amount by which the network response exceeds the specifications. Note the special case when

$$S_u(\psi) = S_l(\psi) = S(\psi) \tag{15}$$

and

$$w_u(\psi) = w_l(\psi) = w(\psi) \tag{16}$$

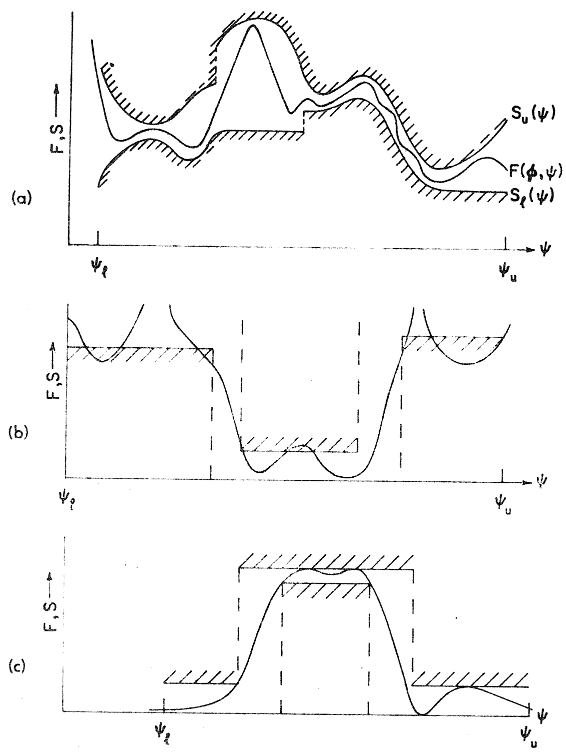


Fig. 4. (a) A response function satisfying arbitrary specifications. (b) A response function failing to satisfy a bandpass filter specification. (c) A response function just satisfying an amplifier specification.

which reduces (11) to

$$U = \max_{[\psi_l, \psi_u]} [\,| w(\psi)(F(\phi, \psi) - S(\psi)) |\,]. \quad (17)$$

This form may be recognized as the more conventional Chebyshev type of objective.

The direct minimax formulation, the optimum of which represents the best possible attempt at satisfying the design specifications within the constraints of the particular problem, appears to have received little attention in the literature on network optimization. This is chiefly due to the fact that discontinuous derivatives are generated in the response hypersurface when the maximum deviation jumps abruptly from one point on the ψ axis to another, and that multidimensional optimization methods which deal effectively with such problems are rather scarce [89], [100].

In spite of these difficulties, some success with objectives in the form of (17) has been reported [23], [88]. But it is felt that considerable research into methods for dealing with objectives in the form of (11) remains to be done.

Formulation in Terms of Inequality Constraints

A less direct formulation than the previous one, but one which seems to have provided considerable success, is the formulation in terms of inequality constraints on the network response described by Waren *et al.* [18]. Their formulation will be slightly adjusted to fit in with the present notation.

The problem is to minimize U subject to

$$U \geq w_{ui}(F_i(\phi) - S_{ui}) \qquad i \in I_u \quad (18)$$

$$U \geq -\, w_{li}(F_i(\phi) - S_{li}) \qquad i \in I_l \quad (19)$$

and other constraints, e.g., as in (3) where U is now an *additional independent variable* and where the subscript i refers to quantities (already defined) evaluated at discrete values of ψ which form the set $\{\psi_i\}$ in the interval $[\psi_l, \psi_u]$. The index sets I_u and I_l, which are not necessarily disjoint, contain those values of i which refer to the upper and lower specifications, respectively. Thus, in the case of Fig. 4(a), the index sets I_u and I_l could be identical. For Fig. 4(b), the set I_u would refer to the passband and the set I_l to the stopbands. In Fig. 4(c), there might be an intersection between I_u and I_l.

At a minimum, at least one of the constraints (18) or (19) must be an equality, otherwise U could be further reduced without any violation of the constraints. If $\check{U}<0$ then the minimum amount by which the network response exceeds the specifications has been maximized. If $\check{U}>0$ then the maximum amount by which the network response fails to meet the specifications has been minimized. It is clear that both this and the previous formulations have ultimately similar objectives. Indeed, if the sets I_u and I_l are infinite then the optimum solutions given by both formulations may be identical. Not surprisingly such a problem may be described as one which has an infinite number of constraints. However, with finite I_u and I_l the present formulation can be used in an optimization process which avoids the generation of discontinuous derivatives within the feasible region, as will be seen in Section IV.

A special case again arises when

$$S_{ui} = S_{li} = S_i \quad (20)$$

$$w_{ui} = w_{li} = w_i \quad (21)$$

$$I_u = I_l = I \quad (22)$$

which reduces (18) and (19) to

$$U \geq w_i(F_i(\phi) - S_i) \quad (23)$$

$$U \geq -\, w_i(F_i(\phi) - S_i) \quad (24)$$

$$i \in I.$$

This formulation, which is an approximation to (17), has been successfully used by Ishizaki and Watanabe [102], [103] (see Section VIII).

Weighting Factors

A discussion of the weighting factors is appropriate at this stage. Essentially, their task is to emphasize or deemphasize various parts of the response to suit the designer's requirements. For example, if one of the factors is unity and the other very much greater than unity, then if the specifications are not satisfied, a great deal of effort will be devoted to forcing the response associated with the large weighting factor to meeting the specifications at the expense of the rest of the response. Once the specifications are satisfied, then effort is quickly switched to the rest of the response while the response associated with the large weighting factor is virtually left alone. In this way, once certain parts of the network response reach acceptable levels they are effectively maintained at those levels while further effort is spent on improving other parts.

Least p*th Approximation*

A frequently employed class of objective functions may be written in the generalized form

$$U = U(\phi, \psi) = \sum_{i=1}^{n} | w_i(F_i(\phi) - S_i) |^p = \sum_{i=1}^{n} | e_i(\phi) |^p \tag{25}$$

where

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \cdot \\ \cdot \\ \psi_n \end{bmatrix} \tag{26}$$

and where the subscript i refers to quantities evaluated at the sample point ψ_i. Thus, the objective is essentially to minimize the sum of the magnitudes raised to some power p of the weighted deviations $e_i(\phi)$ of the network response from a desired response over a set of sample points $\{\psi_i\}$. p may be any positive integer.

The sample points are commonly spaced uniformly along the ψ axis in the interval $[\psi_l, \psi_u]$. If the objective is effectively to minimize the area under a curve then sufficient sample points must be used to ensure that (25) is a good approximation to the area. However, it should be remembered that function evaluations are often by far the most time consuming parts of an optimization process. So the number of sample points should be carefully chosen for the particular problem under consideration. These arguments apply, of course, to any formulation which involves sampling.

With $p=1$, (25) represents the area under the deviation magnitude curve if sufficient sample points are used. With $p=2$ we have a least squares type of formulation. Obviously, the higher value of p the more emphasis will be given to those deviations which are largest. So if the requirement is to concentrate more on minimizing the maximum deviation a sufficiently large value of p must be chosen [17], [79], [100]. The basis of such a formulation is the fact that

$$\max_{[\psi_l, \psi_u]} [| e(\phi, \psi) |] = \lim_{p \to \infty} \left[\frac{1}{\psi_u - \psi_l} \int_{\psi_l}^{\psi_u} | e(\phi, \psi) |^p d\psi \right]^{1/p} \tag{27}$$

when $|e(\phi, \psi)|$ is defined in the interval $[\psi_l, \psi_u]$. In terms of a sampled response deviation the corresponding statement is

$$\max_i [| e_i(\phi) |] = \lim_{p \to \infty} \left[\sum_i | e_i(\phi) |^p \right]^{1/p}. \tag{28}$$

In practice, values of p from 4 to 10 may provide an adequate approximation for engineering purposes to the ideal objective. A good choice of the weighting factors w_i will also assist in emphasizing or deemphasizing parts of the response deviation. It may also be found advantageous to switch objective functions, number of sample points, or weighting factors after any complete optimization if the optimum is unsatisfactory. For example, one may optimize with the weighting factors set to unity and with $p=2$. If the maximum deviation is larger than desired, one could select appropriate scale factors and/or a higher value of p and try again from the previous "optimum."

Combined Objectives

The objective function can consist of several objectives. Indeed, the form of (11) and (25) suggest such a possibility. For example, we could have a linear combination

$$U = \alpha_1 U_1 + \alpha_2 U_2 + \cdots \tag{29}$$

where $U_1, U_2, \cdots$ could take the form of (25). For an amplifier a compromise might have to be reached between gain and noise figure [93]; another example is the problem of approximating the input resistance and reactance of a model to experimental data [100]. The factors $\alpha_1, \alpha_2, \cdots$ would then be given values commensurate with the importance of $U_1, U_2, \cdots$, respectively. If, however, these objectives can be represented instead as inequality constraints, alternative approaches are possible (Section IV).

IV. Constraints

Discussions on how to handle constraints in optimization invariably follow discussions on unconstrained optimization methods in most publications. This is unfortunate because the nature of the constraints and the way they enter into the problem can be deciding factors in the selection of an optimization strategy. And it is rare to find a network design problem which is unconstrained.

This section deals in particular with methods of reducing a constrained problem into an essentially unconstrained one. This can be accomplished by transforming the parameters and leaving the objective function unaltered, or by modifying the objective function by introducing some kind of penalty.

Transformations for Parameter Constraints

Probably the most frequently occurring constraint on the parameter values are upper and lower bounds as indicated by (3). These can be handled by defining ϕ_i' such that [3]

$$\phi_i = \phi_{li} + (\phi_{ui} - \phi_{li}) \sin^2 \phi_i'. \tag{30}$$

If the periodicity caused by this transformation is undesirable and the constraints are in the form

$$\phi_{li} < \phi_i < \phi_{ui} \tag{31}$$

which defines an open feasible region, one could try [86]

$$\phi_i = \phi_{li} + \frac{1}{\pi} (\phi_{ui} - \phi_{li}) \cot^{-1} \phi_i' \tag{32}$$

where $-\infty < \phi_i' < \infty$ but where only solutions within the range

$$0 < \cot^{-1} \phi_i' < \pi \tag{33}$$

are allowed. This transformation has a penalizing effect upon the parameters in the vicinity of the upper and lower bounds.

So if the optimum values are expected to lie away from the bounds this transformation may also introduce a favorable parameter scaling [86].

When the constraints are in the form

$$\phi_i \geq \phi_{li} \tag{34}$$

one can use

$$\phi_i = \phi_{li} + \phi_i'^2. \tag{35}$$

For

$$\phi_i > 0 \tag{36}$$

one can use

$$\phi_i = e^{\phi_i'}. \tag{37}$$

Other transformations of variables can be found [3]. Well chosen transformations may not only reduce an essentially constrained optimization problem to an unconstrained one but might also improve parameter scaling.

Consider the constraint

$$l_{ij} \leq \phi_i/\phi_j \leq u_{ij} \tag{38}$$

which restricts the ratio of two parameters to be within a permissible range $[l_{ij}, u_{ij}]$. This type of constraint can occur when parasitic effects need to be taken into account [17], [88]. Suppose we consider the example

$$l \leq \phi_2/\phi_1 \leq u \tag{39}$$

$$\phi_1 > 0 \tag{40}$$

$$\phi_2 > 0 \tag{41}$$

where $l>0$ and $u>0$. It may be verified that the transformations

$$\phi_1 = e^{z_1} \cos(\theta_l + (\theta_u - \theta_l) \sin^2 z_2) \tag{42}$$

and

$$\phi_2 = e^{z_1} \sin(\theta_l + (\theta_u - \theta_l) \sin^2 z_2) \tag{43}$$

where

$$0 < \theta_l = \tan^{-1} l < \theta_u = \tan^{-1} u < \pi/2 \tag{44}$$

ensure that for any z_1 and z_2 the constraints (39) to (41) are always satisfied.

Inequality Constraints in General

Unfortunately, one cannot always conveniently transform the parameters to incorporate constraints. With implicit constraints of the form of (4) transformations may be out of the question. ψ-dependent constraints in network optimization may, without loss of generality, be written as

$$c_j(\boldsymbol{\phi}, \psi) \geq 0 \qquad j = 1, 2, \cdots, h \tag{45}$$

in the interval $[\psi_l, \psi_u]$ or, at particular points ψ_i

$$c_j(\boldsymbol{\phi}, \psi_i) \geq 0 \qquad \begin{cases} i = 1, 2, \cdots, n \\ j = 1, 2, \cdots, h. \end{cases} \tag{46}$$

A microwave problem having constraints of this form has been described by Bandler [87]. It concerns a stabilizing network of a tunnel-diode amplifier where the objective was to minimize the square of the input reactance of the network at selected frequencies while maintaining certain specifications on the input resistance and reactance at different frequencies (see Fig. 15).

If one is lucky, of course, one might be able to rely on the constraints not being violated. If, for example, a certain parameter must be positive but it is clear from the network configuration that as the parameter tends to zero the response deteriorates anyway then it may not be necessary to constrain the parameter. However, one can not always rely on good fortune so various methods for dealing with inequality constraints in general need to be discussed.

Let all the inequality constraints in a particular problem including the ψ-dependent ones be contained in the vector of m functions

$$\mathbf{g}(\boldsymbol{\phi}) = \begin{bmatrix} g_1(\boldsymbol{\phi}) \\ g_2(\boldsymbol{\phi}) \\ \vdots \\ g_m(\boldsymbol{\phi}) \end{bmatrix} \tag{47}$$

where the feasible region is defined by[1]

$$\mathbf{g}(\boldsymbol{\phi}) \geq \mathbf{0}. \tag{48}$$

For example, constraints in the form of (3) may be written

$$\begin{aligned} \phi_i - \phi_{li} &\geq 0 \\ \phi_{ui} - \phi_i &\geq 0. \end{aligned} \tag{49}$$

Finding a Feasible Point

Finding a feasible point to serve as the initial point in the constrained optimization process may not be easy. It may be found by trial and error [87] or by unconstrained optimization as follows.

Minimize

$$-\sum_{i=1}^{m} w_i g_i(\boldsymbol{\phi}) \qquad w_i \begin{cases} = 0 & g_i(\boldsymbol{\phi}) \geq 0 \\ > 0 & g_i(\boldsymbol{\phi}) < 0. \end{cases} \tag{50}$$

A minimum of zero indicates that a feasible point has been found.

Penalties for Nonfeasible Points

Assuming that the initial solution is feasible, the simplest way of disallowing a constraint violation is by rejecting any set of parameter values which produces a nonfeasible solution. This may be achieved in direct search methods during optimization either by freezing the violating parameter(s) temporarily or by imposing a sufficiently large penalty on the objective function when any violation occurs. Thus, we may add the term

$$\sum_{i=1}^{m} w_i g_i^2(\boldsymbol{\phi}) \qquad w_i \begin{cases} = 0 & g_i(\boldsymbol{\phi}) \geq 0 \\ > 0 & g_i(\boldsymbol{\phi}) < 0 \end{cases} \tag{51}$$

[1] It is hoped that the reader will not be too upset by $\mathbf{g}(\boldsymbol{\phi}) \geq \mathbf{0}$ which is used for $g_i(\boldsymbol{\phi}) \geq 0$, $i = i, 2, \cdots m$.

to the objective function. As long as the constraints are satisfied the objective function is not penalized. However, nonfeasible points can be obtained with this formulation. An alternative which can prevent this is simply to set the objective function to its most unattractive value when $g_i(\phi)<0$. In practice such a value may be easy to determine on physical grounds.

There are disadvantages inherent in this simple approach to dealing with constraints. Depending on the type of penalty used, the objective function may be discontinuous or have steep valleys at the boundaries of the feasible region, and its first or second derivatives may be discontinuous.

Any method which does not modify the objective function in the feasible region and simply causes nonfeasible points to be rejected can run into the following difficulty. Consider the point A on the constraint boundary in Fig. 5. Clearly any exploration along a coordinate direction from A will result either in a nonfeasible point or in an increase in the objective function. Similarly, any excursion along the path of steepest descent (see Section VII) results in a nonfeasible point. This problem does not occur at B, however. Note that direct search methods (Section VI) in particular those good at following narrow curved valleys, might be able to make reasonable progress once a feasible direction is found. A rotation of coordinates might also alleviate the problem to some extent.

The Created Response Surface Technique[2]

This approach originally suggested by Carroll [60] and developed further by Fiacco and McCormick [63], [64] involves the transformation of the constrained objective into a penalized unconstrained objective of the form

$$P(\phi, r) = U(\phi) + r \sum_{i=1}^{m} \frac{1}{g_i(\phi)} \tag{52}$$

where $r>0$.

Define the interior of the region R of feasible points as[1]

$$R^\circ = \{\phi \mid g(\phi) > 0\} \tag{53}$$

where

$$R = \{\phi \mid g(\phi) \geq 0\}. \tag{54}$$

Starting with a point ϕ and a value of r, initially r_1, such that $\phi \in R^\circ$ and $r_1>0$ minimize the unconstrained function $P(\phi, r_1)$. The form of (52) leads one to expect that a minimum will lie in R°, since as any $g_i(\phi)\to 0$, $P\to\infty$. The location of the minimum will depend on the value of r_1 and is denoted $\check{\phi}(r_1)$.

This procedure is repeated for a strictly monotonic decreasing sequence of r values, i.e.,

$$r_1 > r_2 > \cdots r_j > 0, \tag{55}$$

each minimization being started at the previous minimum. For example, the minimization of $P(\phi, r_2)$ would be started

[2] References pertinent to this subsection have been included under gradient methods because of their association with gradient methods of minimization.

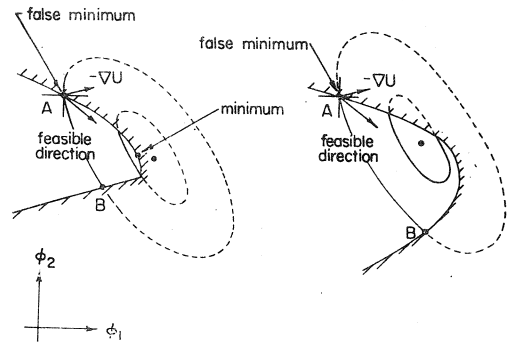


Fig. 5. Pitfalls in constrained minimization when nonfeasible points are simply rejected.

at $\check{\phi}(r_1)$. Every time r is reduced, the effect of the penalty is reduced, so one would expect in the limit as $j\to\infty$ and $r_j\to 0$ that $\check{\phi}(r_j)\to\check{\phi}$ and, consequently, that $U\to\check{U}$, the constrained minimum.

During minimization, should a nonfeasible point be encountered in some current search direction it can simply be rejected since a minimum can always be found in R° by interpolation. If an interior feasible point is not initially available, an attempt to find one can be made either as indicated previously, or by repeated application of the present method [63]. In the latter case, the objective function in (52) is replaced by the negative of any violating constraint function and the satisfied constraints are included as the penalty term. When the constraint is satisfied, the minimization process is stopped and the procedure is repeated for another violating constraint.

Conditions which guarantee convergence have been proved by Fiacco and McCormick. They invoke the requirements that $U(\phi)$ be convex and the $g_i(\phi)$ be concave (see Section II) so that $P(\phi, r)$ is convex. However, it is not unlikely that this method will work successfully on problems for which convergence cannot be readily proved.

To apply the created response surface technique to the formulation in terms of inequality constraints used by Waren *et al.* [18] and introduced in Section III, (52) may be rewritten as

$$P(\phi, U, r) = U + r \sum_{i=1}^{m} \frac{1}{g_i(\phi, U)} \cdot \tag{56}$$

This brings out explicitly the fact that U is both the objective to be minimized and an independent parameter. The constraints $g(\phi)$ are from (18) and (19)

$$U - w_{ui}(F_i(\phi) - S_{ui}) \geq 0 \qquad i \in I_u \tag{57}$$

$$U + w_{li}(F_i(\phi) - S_{li}) \geq 0 \qquad i \in I_l \tag{58}$$

and, for example, (49). Waren *et al.* [18] describe a method for allowing for parameter constraints to be initially violated so that a "reasonably good" initial design can be found. However, the method does not seem to guarantee that these constraints will be ultimately satisfied.

As might be expected, a bad initial value of r will slow down convergence onto each response surface minimum (as indeed a bad initial ϕ will). Too large a value of r_1 will cause

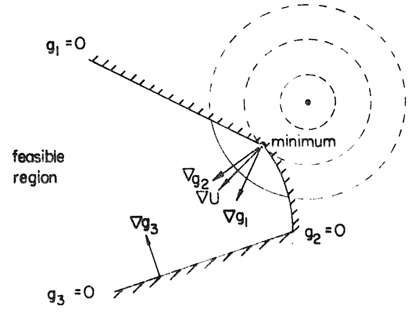


Fig. 6. An interpretation of the Kuhn–Tucker relations, $u_1>0$, $u_2>0$, $u_3=0$.

the first few minima of P to be relatively independent of U, whereas too small a value will render the penalty term ineffective, except near the constraint boundaries where the surface rises very steeply. Once the process is started, however, a constant reduction factor of 10 can be used for successive r values. Another disadvantage of this *sequential unconstrained minimization technique* (SUMT) is that second-order minimization methods are generally required for reasonably fast convergence to the constrained minimum.

Discussions and extensions of these techniques abound in the literature [63], [64], [69], [72], [77], [83]. A book on applications of SUMT is also available [4].

Sufficient Conditions for a Constrained Minimum

Assuming $g(\phi)$ to be concave and differentiable and $U(\phi)$ to be convex and differentiable, a constrained minimum at $\phi=\check{\phi}$ will satisfy

$$\nabla U(\check{\phi}) = \sum_{i=1}^{m} u_i \nabla g_i(\check{\phi}) \tag{59}$$

$$u^T g(\check{\phi}) = 0 \tag{60}$$

where u is a column vector of nonnegative constants and ∇g_i is the gradient vector of the ith constraint function. These are the Kuhn–Tucker relations [123]. They state that $\nabla U(\check{\phi})$ is a nonnegative linear combination of the gradients $\nabla g_i(\check{\phi})$ of those constraints which are active at $\check{\phi}$. An interpretation of these ideas is sketched in Fig. 6. Note that these relations are not, for example, applicable to the case of Fig. 3(c), which is a serious drawback.

Other Methods for Handling Constraints

Other methods for handling constraints include, for example, Rosen's gradient projection method [73], [74], Zoutendijk's method of feasible directions [22], and the method of Glass and Cooper [33]. These methods employ changes in strategy when constraint violations occur. They do not require a transformation of parameters or a penalty function. Thus, they can deal with difficulties such as the one illustrated by Fig. 5 and find a feasible direction yielding an improvement in the objective function. Further details may be found in some of the general references [10], [16], [19], [20]. Alternative methods for dealing with constraints are also indicated, where appropriate, in the following sections.

V. One-Dimensional Optimization Strategies

Many multidimensional optimization strategies employ one-dimensional techniques for searching along some feasible direction to find the minimum in that direction. A brief discussion of efficient one-dimensional strategies is, therefore, appropriate at this stage.

The methods can be divided into two classes: 1) the minimax *direct elimination* methods—minimax, because they minimize the maximum interval which could contain the minimum, and 2) the *approximation* methods. The latter are generally effective on smooth functions, but the former can be applied to arbitrary unimodal functions.

Fibonacci Search

The most effective direct elimination method is the Fibonacci search method [25], [28], [40], [47], [49], [52]. It is so-called because of its association with the Fibonacci sequence of numbers defined by

$$\begin{aligned} F_0 &= F_1 = 1 \\ F_i &= F_{i-1} + F_{i-2} \qquad i = 2, 3, \cdots, \end{aligned} \tag{61}$$

the first six terms, for example, being 1, 1, 2, 3, 5, 8. Assume that we have obtained an initial interval $[\phi_l^1, \phi_u^1]$ over which the objective function is unimodal. At the jth iteration of the Fibonacci search using n function evaluations ($n\geq 2$) we have

$$\phi_a^j = \frac{F_{n-1-j}}{F_{n+1-j}} I^j + \phi_l^j \tag{62}$$

$$\phi_b^j = \frac{F_{n-j}}{F_{n+1-j}} I^j + \phi_l^j \qquad j = 1, 2, \cdots, n-1 \tag{63}$$

where

$$I^j = \phi_u^j - \phi_l^j \tag{64}$$

is the interval of uncertainty at the start of the jth iteration. An example for $n=4$ is illustrated in Fig. 7. Observe that each iteration except the first actually requires only one function evaluation due to symmetry. This fact is summarized by the following relationship.

If $U_a^j > U_b^j$ then

$$\phi_l^{j+1} = \phi_a^j, \quad \phi_a^{j+1} = \phi_b^j, \quad \phi_u^{j+1} = \phi_u^j, \quad U_a^{j+1} = U_b^j; \tag{65a}$$

and if $U_a^j < U_b^j$ then

$$\phi_l^{j+1} = \phi_l^j, \quad \phi_b^{j+1} = \phi_a^j, \quad \phi_u^{j+1} = \phi_b^j, \quad U_b^{j+1} = U_a^j. \tag{65b}$$

Note that the very last function evaluation should, according to this algorithm, be made where the previous one was made. It can, therefore, be omitted if only the minimum value is desired. But to reduce the interval of uncertainty the last function evaluation should be made as close as possible to the previous one, either to the right or to the left.

The interval of uncertainty after j iterations is

$$I^{j+1} = \phi_u^j - \phi_a^j = \phi_b^j - \phi_l^j \tag{66}$$

reducing the interval I^j by a factor

$$\frac{I^j}{I^{j+1}} = \frac{F_{n+1-j}}{F_{n-j}}. \tag{67}$$

After $n-1$ iterations, assuming infinite resolution, the total reduction ratio is

$$\frac{I^1}{I^n} = \frac{F_n}{F_{n-1}} \cdot \frac{F_{n-1}}{F_{n-2}} \cdots \frac{F_2}{F_1} = F_n. \tag{68}$$

For an accuracy of σ the values of n must be such that

$$F_{n-1} < \frac{\phi_u{}^1 - \phi_l{}^1}{\sigma} \leq F_n. \tag{69}$$

In the example of Fig. 7 the initial interval has been reduced by a factor of 5 after 4 function evaluations. Eleven evaluations would have reduced the interval by a factor of 144.

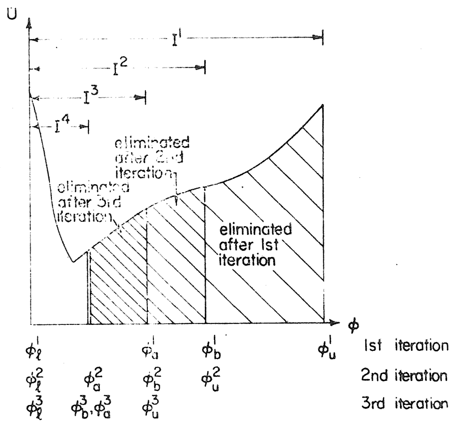


Fig. 7. A Fibonacci search scheme involving three iterations on a unimodal function of one variable.

Search by Golden Section

Almost as effective as Fibonacci search, but with the advantage that n need not be fixed in advance, is the one-dimensional search method using the Golden Section [47], [49], [52].

It is readily shown for Fibonacci search that

$$I^j = I^{j+1} + I^{j+2} \tag{70}$$

as may be verified by the example of Fig. 7. The same relationship between the intervals of uncertainty is true for the present method, with an added restriction that

$$\frac{I^j}{I^{j+1}} = \frac{I^{j+1}}{I^{j+2}} = \tau \tag{71}$$

which leads to

$$\tau^2 = \tau + 1 \tag{72}$$

the solution of interest being $\tau = \frac{1}{2}(1+\sqrt{5}) = 1.6180 \cdots$. The division of a line according to (70) and (71) is called the Golden Section of a line.

The reduction ratio after n function evaluations is

$$\frac{I^1}{I^n} = \tau^{n-1}. \tag{73}$$

It can be shown that for Fibonacci search as $n \rightarrow \infty$

$$\frac{I^1}{I^n} = F_n \approx \frac{\tau^{n+1}}{\sqrt{5}}. \tag{74}$$

The ratio of effectiveness of the Fibonacci search as compared with the Golden Section is, therefore,

$$\frac{F_n}{\tau^{n-1}} \approx \frac{\tau^2}{\sqrt{5}} = 1.1708. \tag{75}$$

Furthermore as $n \rightarrow \infty$

$$\frac{F_n}{F_{n-1}} \approx \tau. \tag{76}$$

Comparing (67) and (71) for $j = 1$ we see that the Fibonacci search and the Golden Section search start at practically the same point, the latter method ultimately providing an interval of uncertainty only some 17 percent greater than the former.

Golden Section search is frequently preferred because the number of function evaluations need not be fixed in advance.

Interpolation Methods

Several methods for finding a minimum have been proposed which repetitively fit a low order polynomial through a number of points until the minimum is obtained to the desired accuracy [28], [41], [47]. The essence of a typical method involving quadratic interpolation may be explained as follows.

At the jth iteration we have a unimodal function over $[\phi_l{}^j, \phi_u{}^j]$ with an interior point $\phi_m{}^j$. Let $a = \phi_l{}^j$, $b = \phi_m{}^j$, and $c = \phi_u{}^j$. Then the minimum of the quadratic through a, b, and c is at

$$d = \frac{1}{2} \frac{(b^2 - c^2)U_a + (c^2 - a^2)U_b + (a^2 - b^2)U_c}{(b - c)U_a + (c - a)U_b + (a - b)U_c}. \tag{77}$$

Then $\phi_l{}^{j+1}$, $\phi_m{}^{j+1}$, and $\phi_u{}^{j+1}$ are obtained as follows:

$$\text{If} \begin{cases} b > d \text{ and } \begin{cases} U_b > U_a & \phi_l{}^{j+1} = a, \ \phi_m{}^{j+1} = d, \ \phi_u{}^{j+1} = b \\ U_b < U_a & \phi_l{}^{j+1} = d, \ \phi_m{}^{j+1} = b, \ \phi_u{}^{j+1} = c \end{cases} \\ b < d \text{ and } \begin{cases} U_b > U_d & \phi_l{}^{j+1} = b, \ \phi_m{}^{j+1} = d, \ \phi_u{}^{j+1} = c \\ U_b < U_d & \phi_l{}^{j+1} = a, \ \phi_m{}^{j+1} = b, \ \phi_u{}^{j+1} = d \end{cases} \end{cases} \tag{78}$$

The procedure may be repeated for greater accuracy, convergence being guaranteed.

This method and certain others like it, are said to have second-order convergence. For this reason they can be more efficient on smooth, well-behaved functions than the Fibonacci search.

Finding Unimodal Intervals

The methods described so far rely on knowing in advance the unimodal interval which contains the desired minimum, otherwise convergence onto it can not be guaranteed. Two situations can arise in practice which require a more cautious strategy.

One is that a given function is expected to be unimodal but the bounds on the unimodal interval are not known in advance. In this case, a quadratic extrapolation method similar to the interpolation method already discussed can be employed repetitively until the minimum is bounded [41], [47]. Alternatively, a sequence of explorations may be performed until such bounds can be established. The second situation is when a given function is expected to be multimodal. In this case, it is advisable to proceed even more cautiously. The function should be evaluated at a sufficient number of uniformly spaced points to determine the unimodal intervals. Once unimodal intervals are established they can be shrunk further by a more efficient method. An example of a multimodal search strategy is the ripple search method [23].

VI. Multidimensional Direct Search Strategies

Methods which do not rely explicitly on evaluation or estimation of partial derivatives of the objective function at any point are usually called *direct search* methods. Broadly speaking, they rely on the sequential examination of trial solutions in which each solution is compared with the best obtained up to that time, with a strategy generally based on past experience for deciding where the next trial solution should be located.

Falling into the category of direct search are: random search; one-at-a-time search [25], [50], [53]; simplex methods [26], [27], [38], [45], [47]; pattern search and its variations [23], [24], [29], [30], [33]–[35], [44], [46], [48], [50], [51], [53], [54]; and some quadratically convergent methods [27], [31], [41], [55]. Multidimensional extensions of Fibonacci search have also been reported [36], [37]. Elimination methods are not as successful, however, as some of the *climbing* methods to be discussed.

One-at-a-Time Search

In this method first one parameter is allowed to vary, generally until no further improvement is obtained, and then the next one, and so on. Fig. 8 illustrates the behavior of this method. It is clear that progress will be slow on narrow valleys which are not oriented in the direction of any coordinate axis.

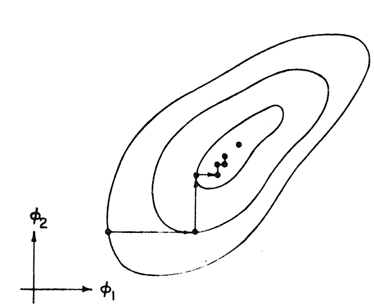


Fig. 8. Minimization by a one-at-a-time method.

Pattern Search

The pattern search strategy of Hooke and Jeeves [34], [50], [53], however, is able to follow along fairly narrow valleys because it attempts to align a search direction along the valley. Fig. 9 shows an example of the pattern search strategy.

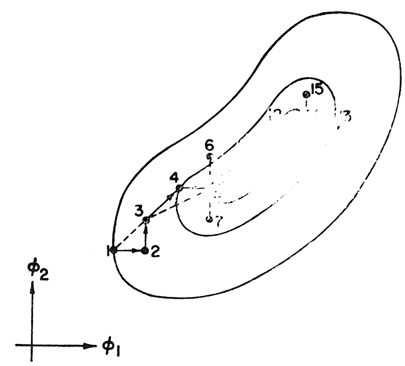


Fig. 9. Following a valley by pattern search.

The starting point $\boldsymbol{\phi}^1$ is the first *base point* $\boldsymbol{b}^1$. In the example the first *exploratory move* from $\boldsymbol{\phi}^1$ begins by incrementing ϕ_1 and resulting in $\boldsymbol{\phi}^2$. Since $U^2 < U^1$, $\boldsymbol{\phi}^2$ is retained and exploration is continued by incrementing ϕ_2. $U^3 < U^2$ so $\boldsymbol{\phi}^3$ is retained in place of $\boldsymbol{\phi}^2$. The first set of exploratory moves being complete, $\boldsymbol{\phi}^3$ becomes the second base point $\boldsymbol{b}^2$. A *pattern move* is now made to $\boldsymbol{\phi}^4 = 2\boldsymbol{b}^2 - \boldsymbol{b}^1$, i.e., in the direction $\boldsymbol{b}^2 - \boldsymbol{b}^1$, in the hope that the previous success will be repeated. U^4 is not immediately compared with U^3. Instead, a set of exploratory moves is first made to try to improve on the pattern direction. The best point found in the present example is $\boldsymbol{\phi}^5$ and, since $U^5 < U^3$, it becomes $\boldsymbol{b}^3$, the third base point. The search continues with a pattern move to $\boldsymbol{\phi}^8 = 2\boldsymbol{b}^3 - \boldsymbol{b}^2$.

When a pattern move and subsequent exploratory moves fail (as around $\boldsymbol{\phi}^{13}$), the strategy is to return to the previous base point. If the exploratory moves about the base point fail (as at $\boldsymbol{\phi}^8$) the pattern is destroyed, the parameter increments are reduced and the whole procedure restarted at that point. The search is terminated when the parameter increments fall below prescribed levels.

Constraints can be taken into account by addition of penalties as described by Weisman and Wood [48], or by the method of Glass and Cooper [33] who describe an alternate strategy for dealing with constraints. Algorithms of pattern search are available in the literature [24], [35].

A variation of pattern search called *spider*, which seems to have enjoyed some success in microwave network optimization [53], has been described by Emery and O'Hagan [30]. The essential difference is that the exploratory moves are made in randomly chosen orthogonal directions. For this reason, there is less likelihood of the search terminating at a false minimum either in a sharp valley or at a constraint boundary as in Fig. 5. Spider can, therefore, be recommended as a useful general purpose direct search method.

Another variation of pattern search called *razor search* [23] has recently been proposed by Bandler and Macdonald to deal with "razor sharp" valleys, i.e., valleys along which a path of discontinuous derivatives lies. Such situations arise in direct minimax response formulations (Section III). An example [23], [89] is shown in Fig. 10. When the

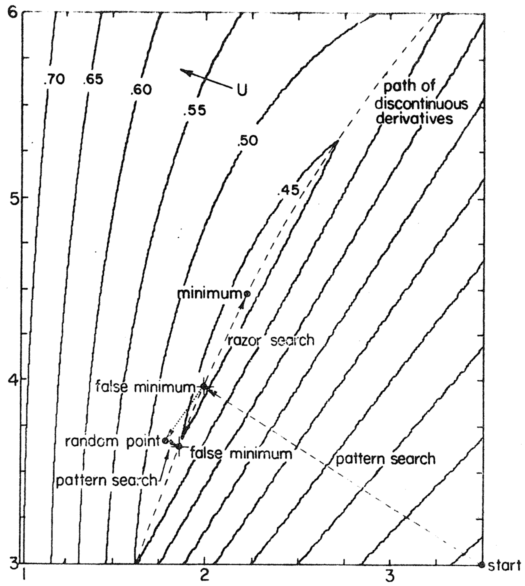


Fig. 10. Following a path of discontinuous derivatives along a narrow valley by razor search. The function is the maximum reflection coefficient over a 100 percent bandwidth of a 2-section 10:1 quarter-wave transmission-line transformer versus characteristic impedances Z_1 and Z_2.

basic pattern search strategy fails it is assumed that a sharp valley whose contours lie entirely within a quadrant of the coordinate axes has been encountered (or for that matter a constraint boundary as in Fig. 5) so a random move is made. When pattern search fails again it is assumed that the same valley (or boundary) is responsible and an attempt to establish a new pattern in the direction of the minimum is tried. The method has been successfully applied to microwave network optimization [23], [88].

Rotating Coordinates

Rosenbrock's strategy [44] is to carry on exploring in directions parallel to the current coordinate axes until one *success* followed by one *failure* has occurred in each direction. Whenever a move is successful (objective function does not become greater than the current best value) the associated increment is multiplied by a factor α; whenever a move fails the increment is multiplied by $-\beta$. When the jth exploratory stage is complete, the coordinates are rotated as described below. First,

$$\begin{aligned} v_k &= d_k u_k^j \\ v_i &= d_i u_i^j + v_{i+1} \qquad i = k-1, \cdots, 1 \end{aligned} \tag{79}$$

where $u_1^j, u_2^j, \cdots u_k^j$ are the orthogonal directions during the jth stage (initially the coordinate directions) and $d_1, d_2, \cdots d_k$ are the distances moved in the respective directions since the previous rotation of the axes. The new set of orthogonal unit vectors $u_1^{j+1}, u_2^{j+1}, \cdots u_k^{j+1}$, the first of which always lies in the direction of total progress made during the jth stage, are obtained from (79) using the Gram–Schmidt procedure:

$$\begin{aligned} w_1 &= v_1 \\ u_1^{j+1} &= \frac{w_1}{\|w_1\|} \\ w_i &= v_i - \sum_{p=1}^{i-1} (v_i^T u_p^{j+1}) u_p^{j+1} \\ u_i^{j+1} &= \frac{w_i}{\|w_i\|} \end{aligned} \Bigg\} \; i = 2, 3, \cdots, k \tag{80}$$

The process is then repeated. The search may be terminated after a predetermined number of function evaluations or when the total progress made during each of several successive exploratory stages becomes smaller than a predetermined value.

Fig. 11 shows a contour plot of Rosenbrock's test function which is frequently used for testing new strategies. Experimentally, Rosenbrock found that $\alpha = 3$, $\beta = -\frac{1}{2}$ gives a good efficiency. Constraints can be taken into account by Rosenbrock's *boundary zone* approach [44], [47].

Swann [46] has described an improvement of Rosenbrock's method which employs linear minimizations once along each direction in turn, after which the coordinates are rotated [27], [31], [47].

More efficient methods of rotating the coordinate directions for Rosenbrock's and Swann's methods have been recently proposed [39], [43].

Powell's Method

An efficient method devised by Powell [41] is based on the properties of conjugate directions defined by a quadratic function, namely

$$U(\phi) = \phi^T A \phi + b^T \phi + c \tag{81}$$

where A is a $k \times k$ constant matrix, b is a constant vector, and c is a constant. The directions u_i and u_j are conjugate with respect to A if

$$u_i^T A u_j = 0 \qquad i \neq j. \tag{82}$$

A two-dimensional example is shown in Fig. 12(a). The consequences of having mutually conjugate directions is that the minimum of a quadratic function can be located by searching for a minimum along each of the directions once.

The jth iteration involves a search for a minimum along k linearly independent directions $u_1^j, u_2^j, \cdots u_k^j$. At the first iteration these are the coordinate directions. Denoting the starting point of the iteration ϕ^0, and the point arrived at after k minimizations ϕ^k, a new direction

$$u = \phi^k - \phi^0 \tag{83}$$

is defined along which another search for a minimum is carried out. u_1^j is then discarded and the linearly independent directions for the $(j+1)$th iteration are defined as

$$[u_1^{j+1}, u_2^{j+1}, \cdots, u_k^{j+1}] = [u_2^j, u_3^j, \cdots, u_k^j, u] \tag{84}$$

and the process is repeated.

If a quadratic is being minimized then after k iterations all the directions are mutually conjugate insuring quadratic

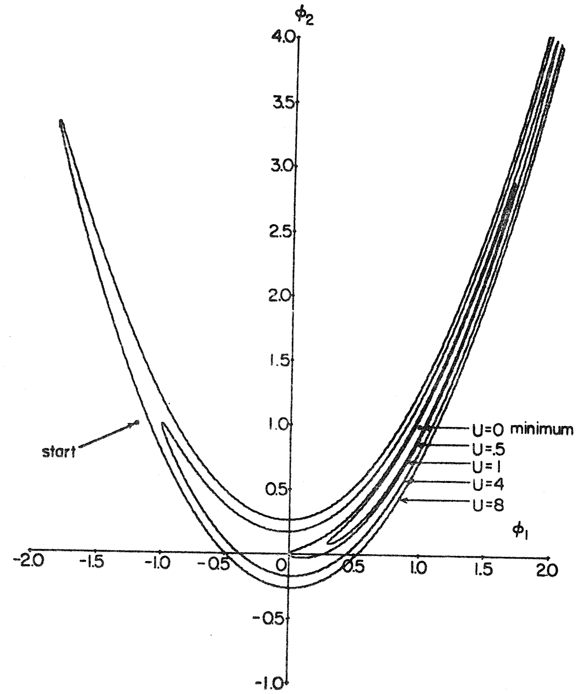


Fig. 11. Contours of a standard test problem: Rosenbrock's function $U=100(\phi_2-\phi_1^2)^2+(1-\phi_1)^2$.

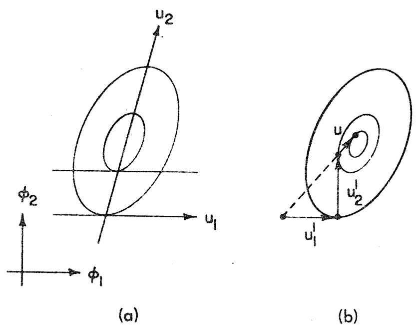


Fig. 12. Illustration of (a) conjugate directions u_1 and u_2, and (b) one iteration of Powell's method (the minimum is found in this problem after two iterations).

convergence. One iteration of Powell's method is represented in Fig. 12(b). In its final form, the method is somewhat more involved than indicated here (see Powell [41] for details, and for the quadratically convergent linear minimization technique). In order to prevent the directions from becoming linearly dependent allowance is made for discarding directions other than u_1^j. Comparisons with other methods are available [27], [31]. Zangwill [55] has simplified Powell's modified method and presented a new one based on Powell's.

Simplex Methods

Simplex methods of nonlinear optimization [26], [27], [38], [45], [47] involve the following operations. A set of $k+1$ points are set up in the k-dimensional ϕ space to form a simplex. The objective function is evaluated at each vertex and an attempt to form a new simplex by replacing

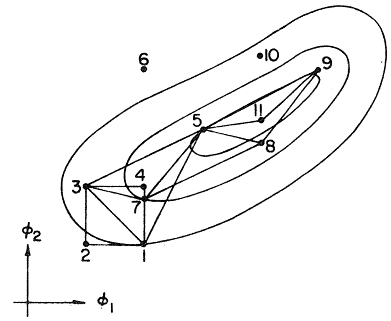


Fig. 13. Following a valley by the simplex method of Nelder and Mead.

the vector with the greatest value of the objective function by another point is made.

An efficient simplex method has been presented by Nelder and Mead [38]. The basic move is to *reflect* the vertex with the greatest value with respect to the centroid of the simplex formed by the remaining vertices. Depending on the outcome, the procedure is repeated or *expansion*, *contraction*, or *shrinking* tactics are employed. Although remarkably efficient for up to four parameters, progress may be slow on problems having more than four parameters [27].

A two-dimensional example of a simplex strategy is given in Fig. 13. Examples of expansion (ϕ^4 to ϕ^5) and contraction (ϕ^6 to ϕ^7 and ϕ^{10} to ϕ^{11}) are shown. Shrinking of the simplex about the vertex having the lowest value follows an unsuccessful attempt at contraction.

A simplex method developed for constrained optimization has been presented by Box [26], [27], [47].

VII. Multidimensional Gradient Strategies

In this section methods are described which utilize partial derivative information to determine the direction of search. The appropriate partial derivatives (which are assumed to exist) may be obtained either by evaluating analytic expressions or by estimation.

The first derivatives can, for example, be estimated from the differences in the objective function produced by small perturbations in the parameter values, say 0.01–1 percent [78]. If the perturbations are too large the estimation will be inaccurate; if they are too small they can still be inaccurate through numerical difficulties. The presence of a narrow curved valley can further confound the issue. Thus, numerical estimation of derivatives must be made somewhat apprehensively.

Steepest Descent

Referring to the multidimensional Taylor series expansion of (7) and neglecting the third term it is clear that a first order change ΔU in the objective function is given by

$$\Delta U = \nabla U^T \Delta \phi. \qquad (85)$$

It is readily shown that maximum change occurs in the direction of the gradient vector ∇U. The steepest *descent* direction is, therefore, given by

$$s = -\frac{\nabla U}{\|\nabla U\|} \tag{86}$$

where the unit vector s is the negative of the *normalized gradient* vector.

At the jth iteration of a simple steepest descent strategy we would have

$$\phi^{j+1} = \phi^j + \alpha^j s^j \tag{87}$$

where α^j is a positive scale factor. It is usual to proceed in the s^j direction until no further improvement is obtained, evaluate s^{j+1}, and continue in this manner.

Fig. 14 illustrates the behavior of this method. Highly dependent on scaling, the method seems to have little advantage over the one at a time method described in Section VI to which it bears a strong resemblance [57], [81], [82].

Parallel Tangents (Partan)

An acceleration technique which makes use of the results of every second iteration to define new search directions can be used rather effectively to speed up the process as should be evident from Fig. 14. A quadratically convergent method, of *parallel tangents* (or *partan*) [75], exploits this basic idea, which can be extended to multidimensional optimization. Excellent discussions of the partan strategy are presented by Wilde [81] and Wilde and Beightler [82].

Generalized Newton–Raphson

Consider the Taylor series expansion of (7) about ϕ in the vicinity of the minimizing point $\check{\phi}$ for a differentiable function such that

$$\check{\phi} = \phi + \Delta\phi. \tag{88}$$

Differentiating (7), and using the fact that $\nabla U(\check{\phi})=0$, we have (neglecting higher order terms)

$$0 \approx \nabla U + H\Delta\phi \tag{89}$$

at ϕ. Hence

$$\Delta\phi \approx -H^{-1}\nabla U \tag{90}$$

where H^{-1} is the inverse of the Hessian matrix. On a quadratic function (90) provides the parameter increments for the minimum to be reached in exactly one step. When U is not quadratic (90) provides the basis of the iterative scheme

$$\phi^{j+1} = \phi^j - H^{-1}\nabla U^j \tag{91}$$

called the *generalized Newton–Raphson* method [72], [77].

Although quadratically convergent, the method has several disadvantages. H must be positive definite, implying that the function must be convex (see Section II), or divergence could occur. To counteract this tendency (91) can be modified to

$$\phi^{j+1} = \phi^j - \alpha^j H^{-1}\nabla U^j \tag{92}$$

where α^j is chosen to minimize U^{j+1} in the direction indicated by $-H^{-1}\nabla U^j$. But even this may be ineffective [72]. Thus, unlike steepest descent, the Newton–Raphson method

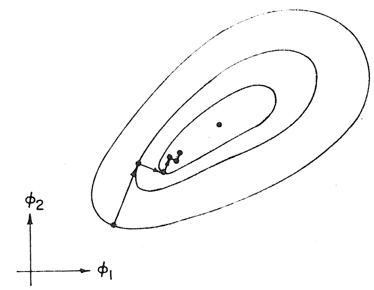


Fig. 14. Minimization by a steepest descent method (see Fig. 8).

may fail to converge from a poor starting point. Furthermore, the computation of H and its inverse are time consuming operations.

Fletcher Powell

Generally acknowledged to be one of the most powerful minimization methods currently available when first derivatives are analytically defined, the *Fletcher Powell* method [66] combines some of the more desirable features of steepest descent and the Newton–Raphson method. It is a development of Davidon's *variable metric* method [61]. A brief discussion of the method follows.

Redefine H as any positive definite matrix. Then at the jth iteration

$$\phi^{j+1} = \phi^j + \alpha^j s^j \tag{93}$$

where

$$s^j = -H^j \nabla U^j. \tag{94}$$

Here, H^j is the jth approximation to the *inverse* of the Hessian matrix. The initial approximation to H is usually the unit matrix. Notice that, in this case, the first iteration is in the direction of steepest descent [cf. (87)]. The α^j are chosen to minimize U^{j+1}. H is continually updated during minimization (hence the name variable metric) such that [72]

$$\phi^{j+1} - \phi^j = H^{j+1}[\nabla U^{j+1} - \nabla U^j]. \tag{95}$$

Thus, only first derivatives are required to update H.

In practice the following procedure is adopted. Let

$$\Delta\phi^j = \alpha^j s^j \tag{96}$$

$$g^j = \nabla U^{j+1} - \nabla U^j. \tag{97}$$

Set

$$H^{j+1} = H^j + A^j + B^j \tag{98}$$

where

$$A^j = \frac{\Delta\phi^j \Delta\phi^{jT}}{\Delta\phi^{jT} g^j} \tag{99}$$

and

$$B^j = \frac{-H^j g^j g^{jT} H^j}{g^{jT} H^j g^j}. \tag{100}$$

The process is repeated from ϕ^{j+1}, replacing j by $j+1$.

Fletcher and Powell prove by induction that if H^j is positive definite then H^{j+1} is also positive definite, since H^0 is taken as positive definite. Fletcher and Powell further prove that on a quadratic function, H^k is the inverse of the Hessian matrix and $\nabla U^k = 0$. However, because of, say, accumulated round-off errors, one extra iteration corresponding to a Newton–Raphson iteration may be required. It is possible for divergence to occur if the α^j are not accurately chosen to minimize the function along s^j. A check for this can be made and H reset to the unit matrix, if necessary.

Algorithms of the Fletcher–Powell method are available [59], [65], [80]. Several comparisons of its performance with other gradient methods have also been published [58], [59], [71], [78]. The reader might also be interested in related methods and extensions which have been proposed [62], [67], [68], [76], in particular, Stewart's modification [76] to accept difference approximations of the derivatives, and Davidon's recent *variance* algorithm [62].

Least Squares

When the objective function can be represented as a sum of squares of a set of functions, special techniques are available [56], [59], [78], [82]. In this case (25) becomes

$$U = \sum_{i=1}^{n} [e_i(\phi)]^2 \tag{101}$$

with $n \geq k$. Define the vector

$$e(\phi) = \begin{bmatrix} e_1(\phi) \\ e_2(\phi) \\ \vdots \\ e_n(\phi) \end{bmatrix}. \tag{102}$$

Then (101) can be written as

$$U = e^T e \tag{103}$$

and

$$\nabla U = 2J^T e \tag{104}$$

where

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial \phi_1} & \frac{\partial e_1}{\partial \phi_2} & \cdots & \frac{\partial e_1}{\partial \phi_k} \\ \frac{\partial e_2}{\partial \phi_1} & \ddots & & \frac{\partial e_2}{\partial \phi_k} \\ \vdots & & \ddots & \vdots \\ \frac{\partial e_n}{\partial \phi_1} & \cdots & & \frac{\partial e_n}{\partial \phi_k} \end{bmatrix} \tag{105}$$

is an $n \times k$ *Jacobian matrix*. Using the first two terms of a Taylor series expansion

$$e(\phi + \Delta\phi) \approx e(\phi) + J\Delta\phi. \tag{106}$$

Assuming J does not change from ϕ to $\phi + \Delta\phi$ we may write [from (104)]

$$\nabla U(\phi + \Delta\phi) \approx 2J^T[e + J\Delta\phi]. \tag{107}$$

The least squares method then is to solve

$$J^T e + J^T J \Delta\phi = 0 \tag{108}$$

for the k components of $\Delta\phi$ causing the gradient at $\phi + \Delta\phi$ to vanish. Note that $J^T J$ is a square matrix of rank k so that

$$\Delta\phi = -[J^T J]^{-1} J^T e. \tag{109}$$

But from (104) $2J^T e = \nabla U$. Now compare (109) with (90). Hence, the term $2J^T J$ corresponds to the Hessian matrix. The least squares method (sometimes called the Gauss method) is, therefore, analogous to the Newton–Raphson method. U is minimized when $[J^T J]^{-1}$ is positive definite which is generally true under the assumptions of the problem.

To avoid divergence, however, the jth iteration is often taken as

$$\phi^{j+1} = \phi^j + \alpha^j \Delta\phi^j \tag{110}$$

where α^j, as for the previous methods, may be chosen so as to minimize U^{j+1}. With $\alpha^j < 1$ we have one possible form of *damped least squares*.

Other variations to the least squares method to improve convergence are available [78], [82]. Powell [42] has presented a procedure for least squares which does not require derivatives, these being approximated by differences.

Least pth

Temes and Zai [79] have recently generalized the least squares method to a *least pth method*, where p is any positive even integer. They report improved convergence but also discuss damping techniques similar to those used in least squares. The advantages of using a large value of p as far as reducing the maximum response deviation is concerned are discussed in Section III, so the method should be of considerable interest to network designers. The derivation is analogous to the least squares method which falls out as a special case.

VIII. Application to Network Optimization

A list is appended of selected references [84]–[119] on the application of various methods to the optimal design of networks which should be of interest to microwave engineers. Most of these are briefly discussed and commented upon in this section.

Least pth Objectives

Weighted least squares objectives with the sample points nonuniformly distributed along the frequency axes have been used to design LC ladder filters in the presence of loss [95], [110]. Desoer and Mitra [95] used a steepest descent method, while Murata [110] used a simple direct search method. A comparison of the rather unfavorable results obtained by these formulations with alternative formulations is presented by Temes and Calahan [116].

Sheibe and Huber [112] used a least squares objective function with the created response surface technique (Section IV) to optimize a transistor amplifier subject to various parameter constraints including realistic Q values. Their aim was to fit the gain curve to a desired trapezoidal shape. It turned out that the Q value of one of the tuned circuits was forced to its maximum value, and the response at higher frequencies was rather poor.

An investigation into the design and optimization of LC ladder networks to match arbitrary load immittances to a constant source resistance has been reported by Hatley [100]. After experimentation with several objective functions of the form of (25) on a 6 element resistively terminated LC transformer, $\Sigma_i |\rho_i(\phi)|^4$ was chosen, where ρ is the reflection coefficient, even though max $|\rho|$ was 0.08870 after optimization as compared with the known optimum value of 0.07582. A new minimization technique called the method of quadratic eigenspaces is presented and compared with the Fletcher–Powell method. Examples are presented involving antenna matching, the antennas being characterized by measured data rather than models.

The application of the least pth method developed by Temes and Zai [117] (Section VII) was applied to the optimization of a four-variable RC active equalizer with $p=10$. The maximum deviation from the desired specification for $p=2$ was found to be 33 percent higher. Temes and Zai demonstrated the nonuniqueness of the optimum—they obtained different solutions with different starting points. Indeed, two of the four elements were found to be essentially redundant. The necessity of some experimentation, in general, before accepting an apparently optimal solution (by any numerical optimization procedure) is shown by this example. It is interesting to speculate that since the least pth solution will generally not be the minimax solution, although they could be fairly close, it may be possible to obtain a smaller maximum deviation than given by the least pth solution while still searching for it. The optimization program could check for this possibility.

Inequality Constraints

Two distinct methods of optimizing networks when the objectives are formulated in terms of inequality constraints (Section III) and when minimax solutions are required have been reported.

One of these [102], [103] reduces the nonlinear programming problem to a series of linear programming problems. The constraints are in the form of (23) and (24). The response function $F_i(\phi)$ or the deviation $e_i(\phi)$ is linearized at a particular stage in the optimization process and the linear programming problem thus created can be solved by the simplex method of *linear* programming [9], [20] to reduce U for that stage. Unfortunately, however, because of the linear approximations made, it is possible that the original constraints are violated and that U is not actually minimized. Sufficient *under-relaxing* (or *damping*) may be required to guarantee that $U^{j+1} < U^j$ and that in the limit the process converges to the desired minimax response. A detailed discussion of this method is presented by Temes and Calahan [116]. The paper by Ishizaki and Watanabe [103] presents examples including the design of attenuation equalizers and group delay equalizers. It is felt that their method should have wide application. The reader may also be interested in another recent contribution for nonlinear minimax approximation [124].

The other method which is reviewed by Waren *et al.* [119], uses the sequential unconstrained minimization technique, the advantages and disadvantages of which are discussed in Section IV. They recommend quadratically convergent minimization methods such as the Fletcher–Powell method (Section VII) or Powell's method (Section VI) for rapid convergence to each response surface minimum. Several successful applications have been reported [85], [106], [107], [118], [119]. For example, cascade crystal-realizable lattice filters have been optimized from approximate initial designs, including realistic losses and bounds on the element values [107], [118], [119]. Also of interest to microwave engineers might be the optimization of linear arrays, where allowing additional degrees of freedom can result in improved designs [106], and the more recent extension to planar arrays [119].

Microwave Networks

Several reports of the application of computer-aided optimization methods of varying sophistication to microwave network problems can be found in the literature [84], [86]–[90], [93], [97], [99], [101], [104], [108], [114]. A number of these [88], [90], [104], [108] are found elsewhere in this issue.

One example which demonstrates the effectiveness of computer-aided optimization techniques [87] involved the optimization of the transmission-line network shown in Fig. 15 which was to be used for stabilizing and biasing a tunnel-diode amplifier. The requirements of stability and low noise broad-band amplification in conjunction with the rest of the circuitry (rectangular waveguide components including circulator, matching network and tuning element) imposed nonsymmetrical response restrictions on the input resistance and reactance of the network as shown in Fig. 15. Upper and lower bounds on the final parameter values were also imposed. The objective was to minimize the sum of squares of the input reactance at selected frequencies. A simple direct search method was used, which rejected nonfeasible solutions, an initial feasible solution being found by trial and error. An alternative, and perhaps more elegant, approach would have been the implementation of the sequential unconstrained minimization technique.

Another area in which the computer can be effectively used is the design and optimization of broad-band integrated microwave transistor amplifiers [93], [97], [99]. A block diagram of a two-stage amplifier is shown in Fig. 16. The transistors are usually characterized experimentally at selected frequencies in the band of interest and under the conditions (e.g., operating power level) in which they are to be used. The representation can, for example, be in the form of input and output admittance [97], *ABCD* matrix [93], or scattering matrix [99]. It may also be an advantage to fit the measured data versus frequency to a suitable function in a least-squares sense [93], [97].

The input, output and interstage matching networks usually consist of noncommensurate transmission lines and stubs. The line lengths and characteristic impedances are allowed to vary within upper and lower bounds during the optimization of the amplifier. The spider search method

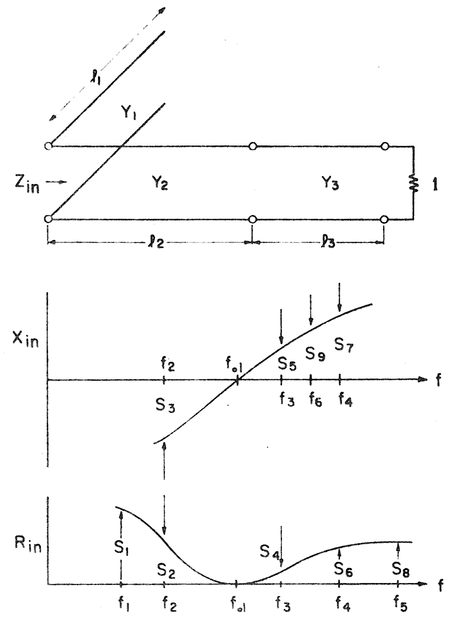


Fig. 15. Noncommensurate stabilizing network for a tunnel-diode amplifier with constraints on input resistance and reactance at certain frequencies.

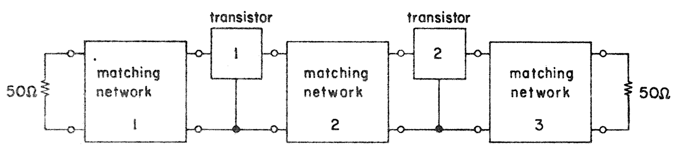


Fig. 16. Block diagram of a two-stage microwave transistor amplifier. The transistors are characterized experimentally. The matching networks usually consist of noncommensurate transmission lines and stubs.

(Section VI) has been applied to the design of such matching networks [97]. The objective functions commonly take the form of (25) with $p=1$ or 2. It is felt, however, that better designs might be achieved by using larger values of p or a minimax objective like (17) to reduce, for example, the maximum deviation of the gain versus frequency from the desired gain. The method of Temes and Zai [117] would be quite appropriate in the former case, while the razor search method [90] could be used in the latter. Since it is difficult to realize component values in integrated circuitry very accurately, the optimal solution should also satisfy appropriate sensitivity constraints.

Multisection inhomogeneous rectangular waveguide impedance transformers (Fig. 17) have been optimized in a minimax equal-ripple sense [88] by the razor search strategy [90] (see Section VI). Suitable parameter constraints—the parameters were the physical dimensions—were imposed to ensure dominant mode propagation and reasonably small junction discontinuity effects which could be taken into account during optimization. Improvements in performance coupled with reduction in size over previous design methods are reported [88].

Automated Design

Approaches to automated network design and optimization which can permit new elements to be "grown" have been suggested by Rohrer [111] and Director and Rohrer [96]. The latter paper, which seems to be a significant contribution, discusses design in the frequency domain of circuits comprising certain types of lumped linear time-invariant elements. A technique is presented whereby the gradient vector of a least squares type of objective function is shown to require only two analyses over the frequency range of interest regardless of the number of variable parameters. And because this gradient depends only on currents and voltages, gradients with respect to nonexistent elements can be calculated. If such a gradient indicates an increase in an element value an appropriate element is grown in the appropriate location. The authors consider an example of broadbanding a transistor amplifier in which they allow for the possibility of growing a number of capacitors. Apparently one has to specify in advance the locations where elements can grow.

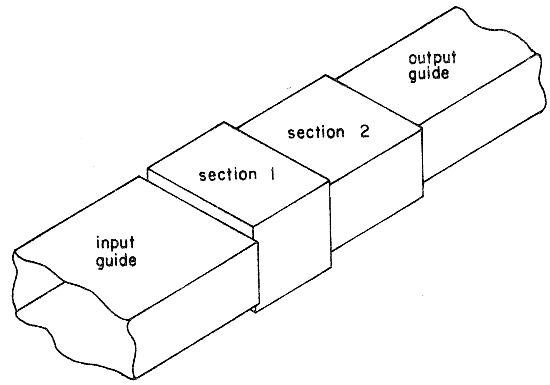


Fig. 17. An inhomogeneous rectangular waveguide impedance transformer. All guides are, in general, noncommensurate.

IX. Conclusions

It is hoped that this paper will not only encourage the use of efficient optimization methods, but will also stimulate the engineer into developing new ones more suited to his design problems. After all, as exemplified by this paper, few optimization strategies have been reported so far which were originally developed with electrical networks in mind. It is also hoped that the present almost instinctive preoccupation with least squares formulations may give way to more attention being paid to minimax objectives and efficient methods of realizing them. Least squares objectives may be flexible and easy to optimize. It is probably their flexibility, however, which is their undoing since any designer who is essentially trying to fit a network response between certain upper and lower levels and is using a least squares objective function may have to employ more human intervention than necessary to achieve an acceptable design. On the other hand, the designer who is employing a minimax objective directly and does not recognize the possible dangers, e.g., of discontinuous derivatives, can easily obtain an equal-ripple response which is still far from the optimum. On-line designers optimizing a network manually with the objective of minimizing the maximum deviation of the network response from a desired response are equally prone to these dangers. An equal-ripple solution need not necessarily be the minimax solution.

Acknowledgment

The author thanks W. J. Getsinger, Guest Editor of this Special Issue on Computer-Oriented Microwave Practices, for the invitation to write this review paper. The author would also like to express his indebtedness to the many individuals with whom he had the pleasure of discussing optimization methods. Much of the material in this paper is drawn from lectures to staff and graduate students at the University of Manitoba during the academic year 1968–1969. The author has benefited considerably from lively feedback from members of the class, particularly Prof. R. A. Johnson. The author is grateful to P. A. Macdonald who checked and criticized the manuscript. The author is also happy to have been associated as a Postdoctorate Fellow with Dr. A. Wexler's group in the Department of Electrical Engineering of the University of Manitoba. Finally, the author must thank Miss L. Milkowski for her careful typing.

References

General

[1] A. H. Boas, "Modern mathematical tools for optimization" (published serially), *Chem. Engrg.*, December 10, 1962, to April 1, 1963.
[2] G. C. Bown, "Optimization methods for circuit design", presented at the IEE Colloquium on Computer Methods in Network Design and Synthesis, London, November 29, 1968.
[3] M. J. Box, "A comparison of several current optimization methods, and the use of transformations in constrained problems," *Computer J.*, vol. 9, pp. 67–77, May 1966.
[4] J. Bracken and G. P. McCormick, *Selected Applications of Nonlinear Programming.* New York: Wiley, 1968.
[5] D. A. Calahan, *Computer-Aided Network Design.* New York: McGraw-Hill, 1968, ch. 4.
[6] A. W. Dickinson, "Nonlinear optimization: Some procedures and examples," *Proc. 19th Natl. Conf. of ACM* (1964), pp. E.1. 2-1 to E.1. 2-8.
[7] P. E. Fleischer, "Optimization techniques in system design," in *System Analysis by Digital Computer*, F. F. Kuo and J. F. Kaiser, Eds. New York: Wiley, 1966.
[8] R. Fletcher, "Function minimization without evaluating derivatives—a review," *Computer J.*, vol. 8, pp. 33–41, April 1965.
[9] H. P. Künzi, H. G. Tzschach, and C. A. Zehnder, *Numerical Methods of Mathematical Optimization with ALGOL and FORTRAN Programs.* New York: Academic Press, 1968.
[10] L. S. Lasdon and A. D. Waren, "Mathematical programming for optimal design," *Electro-Technol.*, vol. 80, pp. 55–70, November 1967.
[11] A. Lavi and T. P. Vogl, Eds., *Recent Advances in Optimization Techniques.* New York: Wiley, 1966.
[12] O. L. Mangasarian, *Nonlinear Programming.* New York: McGraw-Hill, to be published.
[13] M. J. D. Powell, "Minimization of functions of several variables," in *Numerical Analysis: An Introduction*, J. Walsh, Ed. Washington, D.C.: Thompson, 1967.
[14] R. Schinzinger, "Optimization in electromagnetic system design," in *Recent Advances in Optimization Techniques* [11].
[15] H. A. Spang, III, "A review of minimization techniques for nonlinear functions," *SIAM Rev.*, vol. 4, pp. 343–365, October 1962.
[16] W. H. Swann, *Non-linear Optimization Techniques.* London: Oliver and Boyd, to be published.
[17] G. C. Temes and D. A. Calahan, "Computer-aided network optimization the state-of-the-art," *Proc. IEEE*, vol. 55, pp. 1832–1863, November 1967.
[18] A. D. Waren, L. S. Lasdon, and D. F. Suchman, "Optimization in engineering design," *Proc. IEEE*, vol. 55, pp. 1885–1897, November 1967.
[19] D. J. Wilde, *Optimum Seeking Methods.* Englewood Cliffs, N. J.: Prentice Hall, 1964.
[20] D. J. Wilde and C. S. Beightler, *Foundations of Optimization.* Englewood Cliffs, N. J.: Prentice-Hall, 1967.
[21] C. F. Wood, "Review of design optimization techniques," *IEEE Trans. Systems Science and Cybernetics*, vol. SSC-1, pp. 14–20, November 1965. See also E. A. Huber, "Comments on 'Review of design optimization techniques'," *IEEE Trans. Systems Science and Cybernetics*, vol. SSC-2, p. 151, December 1966.
[22] G. Zoutendijk, "Nonlinear programming: a numerical survey," *J. SIAM Control*, vol. 4, pp. 194–210, February 1966.

Direct Search

[23] J. W. Bandler and P. A. Macdonald, "Optimization of microwave networks by razor search," this issue, pp. 552–562.
[24] M. Bell and M. C. Pike, "Remark on Algorithm 178: direct search, "*Commun. ACM*, vol. 9, pp. 684–685, September 1966.
[25] A. H. Boas [1].
[26] M. J. Box, "A new method of constrained optimization and a comparison with other methods," *Computer J.*, vol. 8, pp. 42–52, April 1965.
[27] M. J. Box [3].
[28] D. A. Calahan [5].
[29] A. W. Dickinson [6].
[30] F. E. Emery and M. O'Hagan, "Optimal design of matching networks for microwave transistor amplifiers," *IEEE Trans. Microwave Theory and Techniques* (Correspondence), vol. MTT-14, pp. 696–698, December 1966.
[31] R. Fletcher [8].
[32] I. M. Gelfand and M. L. Tsetlin, "The principle of nonlocal search in automatic optimization systems," *Dokl. Akad. Nauk SSSR*, vol. 137, pp. 295–298, March 1961 (English transl. *Sov. Phys. Dokl.*, vol. 6, pp. 192–194, September 1961.)
[33] H. Glass and L. Cooper, "Sequential search: a method for solving constrained optimization problems," *J. ACM*, vol. 12, pp. 71–82, January 1965.
[34] R. Hooke and T. A. Jeeves, " 'Direct search' solution of numerical and statistical problems," *J. ACM*, vol. 8, pp. 212–229, April 1961.
[35] A. F. Kaupe, Jr., "Algorithm 178: direct search," *Commun. ACM*, vol. 6, pp. 313–314, June 1963.
[36] P. D. Krolak, "Further extensions of Fibonaccian search to nonlinear programming problems," *SIAM J. Control*, vol. 6, pp. 258–265, May 1968.
[37] P. Krolak and L. Cooper, "An extension of Fibonaccian search to several variables," *Commun. ACM*, vol. 6, pp. 639–641, October 1963.
[38] J. A. Nelder and R. Mead, "A simplex method for function minimization," *Computer J.*, vol. 7, pp. 308–313, January 1965.
[39] J. R. Palmer, "An improved procedure for orthogonalising the search vectors in Rosenbrock's and Swann's direct search optimization methods," *Computer J.*, vol. 12, pp. 69–71, February 1969.
[40] M. C. Pike, I. D. Hill, and F. D. James, "Note on Algorithm 2: Fibonacci search," *Computer J.*, vol. 9, pp. 416–417, February 1967.
[41] M. J. D. Powell, "An efficient method for finding the minimum of a function of several variables without calculating derivatives," *Computer J.*, vol. 7, pp. 155–162, July 1964.
[42] M. J. D. Powell, "A method for minimizing a sum of squares of non-linear functions without calculating derivatives," *Computer J.*, vol. 7, pp. 303–307, January 1965.
[43] M. J. D. Powell, "On the calculation of orthogonal vectors," *Computer J.*, vol. 11, pp. 302–304, November 1968.
[44] H. H. Rosenbrock, "An automatic method for finding the greatest or least value of a function," *Computer J.*, vol. 3, pp. 175–184, October 1960.
[45] W. Spendley, G. R. Hext, and F. R. Himsworth, "Sequential application of simplex designs in optimization and evolutionary operations," *Technometrics*, vol. 4, pp. 441–461, November 1962.
[46] W. H. Swann, "Report on the development of a new direct search method of optimization," Central Instrument Laboratory, Imperial Chemical Industries Ltd., England, Research Note 64/3, June 1964.
[47] W. H. Swann [16].
[48] J. Weisman and C. F. Wood, "The use of 'optimal search' for engineering design," in *Recent Advances in Optimization Techniques* [11].
[49] D. J. Wilde [19], ch. 2.
[50] D. J. Wilde [19], ch. 5.
[51] D. J. Wilde, "Direct optimization: a review," *1967 IEEE Internatl. Conv. Rec.*, pt. 3, pp. 53–54.
[52] D. J. Wilde and C. S. Beightler [20], ch. 6.

[53] D. J. Wilde and C. S. Beightler [20], ch. 7.
[54] C. F. Wood [21].
[55] W. I. Zangwill, "Minimizing a function without calculating derivatives," *Computer J.*, vol. 10, pp. 293–296, November 1967.

Gradient Methods

[56] M. R. Aaron, "The use of least squares in system design," *IRE Trans. Circuit Theory*, vol. CT-3, pp. 224–231, December 1956.
[57] A. H. Boas [1].
[58] M. J. Box [3].
[59] D. A. Calahan [5].
[60] C. W. Carroll, "The created response surface technique for optimizing nonlinear, restrained systems," *Operations Res.*, vol. 9, pp. 169–185, 1961.
[61] W. C. Davidon, "Variable metric method for minimization," AEC Research Development Rept. ANL-5990 (Rev.), 1959.
[62] W. C. Davidon, "Variance algorithm for minimization," *Computer J.*, vol. 10, pp. 406–410, February 1968.
[63] A. V. Fiacco and G. P. McCormick, "The sequential unconstrained minimization technique for nonlinear programming, a primal-dual method," *Management Sci.*, vol. 10, pp. 360–366, January 1964.
[64] A. V. Fiacco and G. P. McCormick, "Computational algorithm for the sequential unconstrained minimization technique for nonlinear programming," *Management Sci.*, vol. 10, pp. 601–617, July 1964.
[65] R. Fletcher, "Certification of algorithm 251 [E4] function minimization," *Commun. ACM*, vol. 9, p. 686, September 1966.
[66] R. Fletcher and M. J. D. Powell, "A rapidly convergent descent method for minimization," *Computer J.*, vol. 6, pp. 163–168, June 1963.
[67] R. Fletcher and C. M. Reeves, "Function minimization by conjugate gradients," *Computer J.*, pp. 149–154, July 1964.
[68] D. Goldfarb and L. Lapidus, "Conjugate gradient method for nonlinear programming problems with linear constraints," *I and EC Fundamentals*, vol. 7, pp. 142–151, February 1968.
[69] F. A. Lootsma, "Logarithmic programming: a method of solving nonlinear-programming problems," *Philips Res. Repts.*, vol. 22, pp. 329–344, June 1967.
[70] F. A. Lootsma, "Extrapolation in logarithmic programming," *Philips Res. Repts.*, vol. 23, pp. 108–116, February 1968.
[71] J. D. Pearson, "Variable metric methods of minimization," *Computer J.*, vol. 12, pp. 171–178, May 1969.
[72] M. J. D. Powell [13].
[73] J. B. Rosen, "The gradient projection method for nonlinear programming. Part I. Linear constraints," *J. SIAM*, vol. 8, pp. 181–217, March 1960.
[74] J. B. Rosen, "The gradient projection method for nonlinear programming. Part II. Nonlinear constraints," *J. SIAM*, vol. 9, pp. 514–532, December 1961.
[75] B. V. Shah, R. J. Buehler, and O. Kempthorne, "Some algorithms for minimizing a function of several variables," *J. SIAM*, vol. 12, pp. 74–92, March 1964.
[76] G. W. Stewart, III, "A modification of Davidon's minimization method to accept difference approximations of derivatives," *J. ACM*, vol. 14, pp. 72–83, January 1967.
[77] W. H. Swann [16].
[78] G. C. Temes and D. A. Calahan [17].
[79] G. C. Temes and D. Y. F. Zai, "Least *p*th approximation," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-16, pp. 235–237, May 1969.
[80] M. Wells, "Algorithm 251 function minimization [E4]," *Commun. ACM*, vol. 8, p. 169, March 1965.
[81] D. J. Wilde [19], ch. 5.
[82] D. J. Wilde and C. S. Beightler [20], ch. 7.
[83] G. Zoutendijk [22].

Applications to Network Design

[84] L. Alfredsson, "Microwave equalizer with arbitrary correction characteristic," *Ericsson Technics*, vol. 23, no. 3, 1967.
[85] J. A. Athanassopoulos, J. D. Schoeffler, and A. D. Waren, "Time-domain synthesis by nonlinear programming," *Proc. 4th Allerton Conf. on Circuit and System Theory* (Urbana, Ill), pp. 766–775, October 1966.
[86] J. W. Bandler, "Optimum noncommensurate stepped transmission-line transformers," *Electronics Letters*, vol. 4, pp. 212–213, May 31, 1968.
[87] J. W. Bandler, "Computer optimization of a stabilizing network for a tunnel-diode amplifier," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-16, pp. 326–333, June 1968.
[88] J. W. Bandler, "Computer optimization of inhomogeneous waveguide transformers," this issue, pp. 563–571.
[89] J. W. Bandler and P. A. Macdonald, "Cascaded noncommensurate transmission-line networks as optimization problems," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-16, pp. 391–394, August 1969.
[90] J. W. Bandler and P. A. Macdonald [23].
[91] G. C. Bown [2].
[92] D. A. Calahan [5].
[93] H. F. Cooke and G. J. Policky, "Computer design and optimization of UHF and microwave circuits," *1967 Internatl. Solid-State Circuits Conf., Digest of Tech. Papers*, pp. 106–107.
[94] R. L. Crane, "All-pass network synthesis," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-15, pp. 474–477, December 1968.
[95] C. A. Desoer and S. K. Mitra, "Design of lossy ladder filters by digital computer," *IRE Trans. Circuit Theory*, vol. CT-8, pp. 192–201, September 1961.
[96] S. W. Director and R. A. Rohrer, "Automated network design: the frequency domain case," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 330–337, August 1969.
[97] F. E. Emery and M. O'Hagan [30].
[98] P. E. Fleischer [7].
[99] V. G. Gelnovatch and T. F. Burke, "Computer-aided design of wide-band integrated microwave transistor amplifiers on high dielectric substrates," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-16, pp. 429–439, July 1968.
[100] W. T. Hatley, Jr., "Computer synthesis of wide-band impedance-matching networks," Stanford Electronics Laboratories, Stanford University, Stanford, Calif., Tech. Rept. 6657-2, December 1967.
[101] H. S. Hewitt, "A computer designed, 720 to 1 microwave compression filter," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-15, pp. 687–694, December 1967.
[102] Y. Ishizaki, H. Wakazuki., and H. Watanabe, "Some considerations on min-max approximation problem for network design," *Proc. 3rd Allerton Conf. on Circuit and System Theory* (Urbana, Ill.), pp. 786–795, October 1965.
[103] Y. Ishizaki and H. Watanabe, "An iterative Chebyshev approximation method for network design," *IEEE Trans. Circuit Theory*, vol. CT-15, pp. 326–336, December 1968.
[104] D. W. Kammler, "The design of discrete *N*-section and continuously tapered symmetrical microwave TEM directional couplers," this issue, pp. 577–590.
[105] R. K. Kwan and G. G. Bach, "Simultaneous approximations in filter design," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-16, pp. 117–121, February 1969.
[106] L. S. Lasdon, D. F. Suchman, and A. D. Waren, "Nonlinear programming applied to linear array design," *J. Acoust. Soc. Am.*, vol. 40, pp. 1197–1200, November 1966.
[107] L. S. Lasdon and A. D. Waren, "Optimal design of filters with bounded, lossy elements," *IEEE Trans. Circuit Theory*, vol. CT-13, pp. 175–187, June 1966.
[108] S. Mahdi and A. B. Macnee, "Dominant pole synthesis of transmission line networks," this issue, pp. 591–597.
[109] W. F. McGee, "Numerical approximation technique for filter characteristic functions," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-14, pp. 92–94, March 1967.
[110] T. Murata, "The use of adaptive constrained descent in system design," *1964 IEEE Internatl. Conv. Rec.*, pt. 1, pp. 296–306.
[111] R. A. Rohrer, "Fully automated network design by digital computer: preliminary considerations," *Proc. IEEE*, vol. 55, pp. 1929–1939, November 1967.
[112] P. O. Scheibe and E. A. Huber, "The application of Carroll's optimization technique to network synthesis," *Proc. 3rd Allerton Conf. on Circuit and System Theory* (Urbana, Ill.), pp. 182–191, October 1965.
[113] R. Schinzinger [14].
[114] J. P. Shelton and J. A. Mosko, "Synthesis and design of wide-band equal-ripple TEM directional couplers and fixed phase shifters," *IEEE Trans. Microwave Theory and Techniques*, vol. MTT-14, pp. 462–473, October 1966.
[115] G. C. Temes and J. A. C. Bingham, "Iterative Chebyshev approximation technique for network synthesis," *IEEE Trans. Circuit Theory*, vol. CT-14, pp. 31–37, March 1967.

[116] G. C. Temes and D. A. Calahan [17].
[117] G. C. Temes and D. Y. F. Zai [79].
[118] A. D. Waren and L. S Lasdon, "Practical filter design using mathematical optimization," *Proc. 3rd Allerton Conf. on Circuit and System Theory* (Urbana, Ill.), pp. 677–689, October 1965.
[119] A. D. Waren, L. S. Lasdon, and D. F. Suchman [18].

Miscellaneous

[120] G. A. Bekey and W. J. Karplus, *Hybrid Computation*. New York: Wiley, 1968, ch. 9.
[121] G. C. Broyden, "A new method of solving nonlinear simultaneous equations," *Computer J.*, vol. 12, pp. 94–99, February 1969.
[122] J. Chernak, "Hybrid computer techniques and applications," in *System Analysis by Digital Computer*, F. F. Kuo and J. F. Kaiser, Eds. New York: Wiley, 1966.
[123] H. W. Kuhn and A. W. Tucker, "Non-linear programming," *Proc. 2nd Symp. on Math. Statistics and Probability*. Berkeley, Calif.: University of California Press, 1951, pp. 481–493.
[124] M. R. Osborne and G. A. Watson, "An algorithm for minimax approximation in the nonlinear case," *Computer J.*, vol. 12, pp. 63–68, February 1969.
[125] A. Ralston, "Rational Chebyshev approximation," in *Mathematical Methods for Digital Computers*, vol. 2, A. Ralston and H. S. Wilf, Eds. New York: Wiley, 1967.
[126] A. Wexler, "Computation of electromagnetic fields," this issue, pp. 416–439.

APPENDIX 3

THE ADJOINT NETWORK METHOD

The following paper provides all the sensitivity formulas needed to derive the gradients with respect to parameters as used in the program. It will be helpful in future additions or modifications. It also illustrates how the network analysis is performed.

Current Trends in Network Optimization

JOHN W. BANDLER, MEMBER, IEEE, AND RUDOLPH E. SEVIORA, STUDENT MEMBER, IEEE

***Abstract*—Some current trends in automated network design optimization which, it is believed, will play a significant role in the computer-aided design of lumped-distributed and microwave networks are reviewed and discussed. In particular, the adjoint network approach due to Director and Rohrer for evaluating the gradient vector of suitable objective functions related to network responses that are to be optimized is presented in a tutorial manner. The advantage of this method is the ease with which the required partial derivatives with respect to variable parameters, such as electrical quantities or geometrical dimensions, can be obtained using at most two network analyses. Least *p*th and minimax approximation in the frequency domain are considered. Networks consisting of linear time-invariant elements are treated, including the conventional lumped elements, transmission lines, RC lines, coaxial lines, rectangular waveguides, and coupled lines. To illustrate the application of the adjoint network method, an example is given concerning the optimization in the least *p*th sense using the Fletcher-Powell method of a transmission-line filter with frequency variable terminations previously considered by Carlin and Gupta.**

I. Introduction

As THE RECENT special issue on Computer-Oriented Microwave Practices of the IEEE TRANSACTIONS ON MICROWAVE THEORY AND TECHNIQUES shows, microwave network optimization is widely carried out using direct search methods, i.e., iterative optimization methods which do not employ evaluation or estimation of derivatives. Murray-Lasso and Kozemchak [1], for example, used pattern search [2] to optimize the parameters of the transmission-line network shown in Fig. 1. The problem was to match the 50-ohm characteristic impedance of a transmission line to the complex input impedance of the transistor specified at a discrete set of frequencies in the band of interest. The ten parameters were the five lengths and five characteristic impedances. A problem studied by Bandler [3] was the optimization of multisection inhomogeneous rectangular waveguide impedance transformers (Fig. 2). The objective was, within certain constraints, to adjust the geometrical dimensions of the sections such that the input and output waveguides were matched over a given frequency band. In general, all waveguides had different cutoff frequencies. Responses which were optimal in the Chebyshev sense, i.e., minimax, were desired. The razor search method [4] was employed to realize them. A modified version of Rosenbrock's method [2] was used more recently by Trick and Vlach [5] to optimize the broad-band amplifier shown in Fig. 3 with, in general, complex frequency-dependent terminations. A weighted least-squares type of objective function was employed to achieve a flat power gain with a reasonable reflection coefficient in the band of interest.

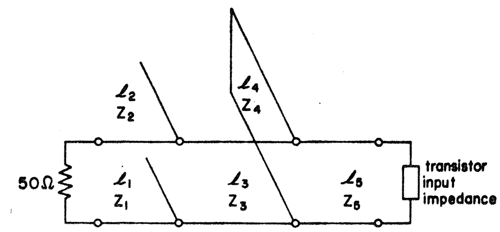


Fig. 1. Matching network optimized by Murray-Lasso and Kozemchak [1].

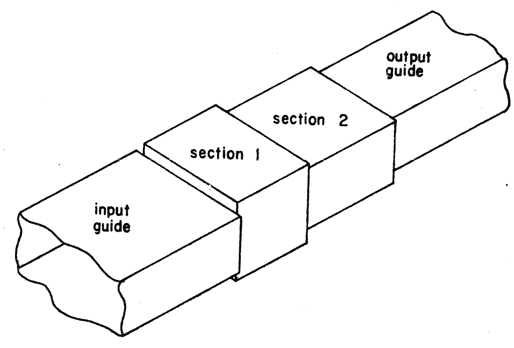


Fig. 2. Inhomogeneous rectangular waveguide impedance transformer optimized by Bandler [3].

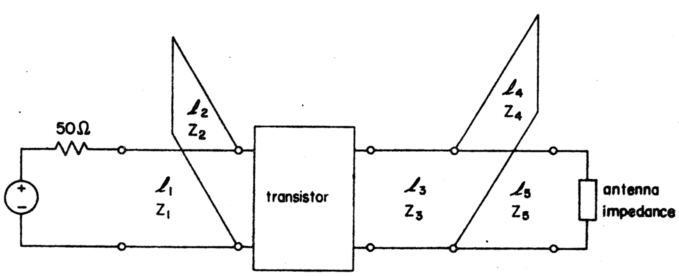


Fig. 3. Broad-band amplifier optimized by Trick and Vlach [5].

These three examples (Figs. 1 to 3) are a good indication of the state of the art in automatic optimization by computer of distributed networks in the microwave region. In the absence of a reasonably simple and efficient method of evaluating derivatives, direct search methods were probably found preferable by the authors instead of gradient methods of minimization. Consider, for example, an *m*-section cascaded network described

Manuscript received August 10, 1970. This work was supported by the National Research Council of Canada, under Grants A7239 and A5277. This paper is based on two papers presented at the 1970 IEEE G-MTT International Microwave Symposium, Newport Beach, Calif., May 11–14.

J. W. Bandler is with the Department of Electrical Engineering, McMaster University, Hamilton, Ont., Canada.

R. E. Seviora is with the Department of Electrical Engineering, University of Toronto, Toronto, Ont., Canada.

by the $ABCD$ matrix. Then

$$\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \prod_{i=1}^{m} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \tag{1}$$

so that, if some jth parameter ϕ_j appears in the nth section [6]

$$\frac{\partial}{\partial \phi_j}\begin{bmatrix} A_t & B_t \\ C_t & D_t \end{bmatrix} = \prod_{i=1}^{n-1} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \begin{bmatrix} \dfrac{\partial A_n}{\partial \phi_j} & \dfrac{\partial B_n}{\partial \phi_j} \\ \dfrac{\partial C_n}{\partial \phi_j} & \dfrac{\partial D_n}{\partial \phi_j} \end{bmatrix} \prod_{i=n+1}^{m} \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}. \tag{2}$$

In general, the functions involved are highly nonlinear, containing transcendental expressions. If care is not exercised to prevent reevaluation of expressions and formulas already evaluated, it may not make much difference in computing time whether analytic expressions are available for the derivatives, the derivatives are estimated numerically by differences produced by small perturbations in the parameter values, or large steps in the parameters are taken as in direct search methods.

The essence of the adjoint network method originally proposed by Director and Rohrer [7], [8] is that all required partial derivatives of the objective function may be obtained from the results of at most two complete analyses of the network regardless of the number of variable parameters and without actually perturbing them. For design of reciprocal networks on the reflection coefficient basis, for example, only one analysis yields all the information needed to compute the derivatives. The procedure is essentially an exact one, so the components could be in analytic or numerical form.

II. Tellegen's Theorem

Tellegen's theorem [9], [10], [11] is invoked to simplify the necessary derivations. Let

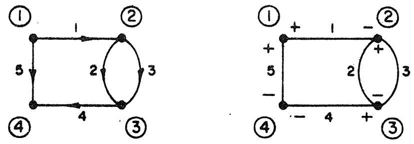
$$v \triangleq \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_b \end{bmatrix} \tag{3}$$

contain all the branch voltages in a network and

$$i \triangleq \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ i_b \end{bmatrix} \tag{4}$$

contain all the corresponding branch currents using associated reference directions [10].[1] Tellegen's theorem

[1] With associated reference directions, the current always enters a branch at the plus sign and leaves at the minus sign.



(a) (b)

Fig. 4. Two networks having the same topology with nodes and branches correspondingly numbered.

states that if v and i satisfy Kirchhoff's voltage law (KVL) and Kirchoff's current law (KCL), respectively,

$$v^T i = 0. \tag{5}$$

The proof is rather straightforward [10, p. 422]. KVL requires that $v = A^T e$, where A is the reduced incidence matrix of the network and e is the node-to-datum voltage vector. So

$$v^T i = (A^T e)^T i = e^T A i.$$

But KCL requires that $Ai = 0$. Therefore,

$$v^T i = 0.$$

As a numerical example of Tellegen's theorem consider Fig. 4, which represents two networks having the same topology. Let.

$$i = [3 \quad -2 \quad 5 \quad 3 \quad -3]^T$$

refer, for example, to Fig. 4(a), and

$$v = [1 \quad 2 \quad 2 \quad 3 \quad 6]^T$$

to Fig. 4(b). Then

$$v^T i = 3 - 4 + 10 + 9 - 18 = 0.$$

Observe that differences in elements or element values between the networks are irrelevant. Thus, i may be essentially arbitrary but subject to KCL and v arbitrary subject to KVL.

III. The Adjoint Network

We need to define an auxiliary network which is topologically the same as the original or given network which is to be optimized. This is called the adjoint network. Let the variables V and I refer to the original network and $\hat{V}$ and $\hat{I}$ refer to the corresponding quantities of the adjoint network. From (5)

$$\begin{aligned} V_B{}^T \hat{I}_B &= 0 \\ I_B{}^T \hat{V}_B &= 0 \end{aligned} \tag{6}$$

where subscript B implies that the associated vectors contain all corresponding complex branch voltages and currents. Perturbing elements in the original network and noting that Kirchoff's laws and hence Tellegen's theorem are applicable to the incremental changes in current and voltage, namely, ΔI_B and ΔV_B, respectively,

$$\begin{aligned} \Delta V_B{}^T \hat{I}_B &= 0 \\ \Delta I_B{}^T \hat{V}_B &= 0 \end{aligned} \tag{7}$$

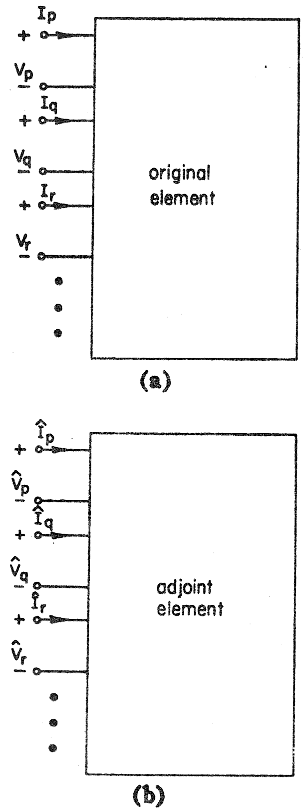


Fig. 5. (a) Multiport original element. (b) Adjoint element.

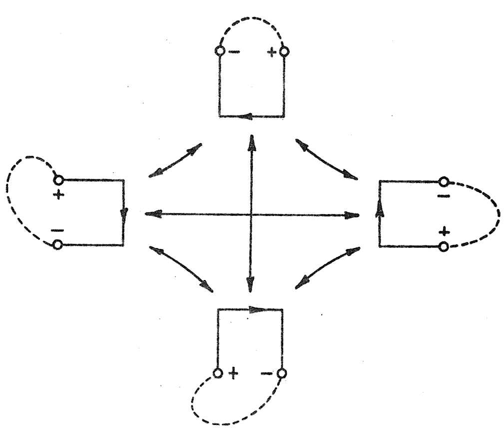


Fig. 6. Representation of multiport element or network for application of Tellegen's theorem.

so that we have the useful form

$$\Delta V_B{}^T \hat{I}_B - \Delta I_B{}^T \hat{V}_B = 0. \tag{8}$$

In general, the network to be optimized will consist of multiport elements (Fig. 5), particularly in the microwave region. To see how Tellegen's theorem may be applied, consider Fig. 6. Obviously, we can still think in terms of network graphs with branch quantities related through some appropriate matrix description. Suppose we take the hybrid matrix description

$$\begin{bmatrix} I_a \\ V_b \end{bmatrix} = \begin{bmatrix} Y & A \\ M & Z \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix} \tag{9}$$

where

$$\begin{bmatrix} I_a \\ I_b \end{bmatrix} = \begin{bmatrix} I_p \\ I_q \\ I_r \\ \cdot \\ \cdot \end{bmatrix}$$

$$\begin{bmatrix} V_a \\ V_b \end{bmatrix} = \begin{bmatrix} V_p \\ V_q \\ V_r \\ \cdot \\ \cdot \end{bmatrix}.$$

Perturbing the parameters of the element and neglecting higher order terms

$$\begin{bmatrix} \Delta I_a \\ \Delta V_b \end{bmatrix} = \begin{bmatrix} Y & A \\ M & Z \end{bmatrix} \begin{bmatrix} \Delta V_a \\ \Delta I_b \end{bmatrix} + \begin{bmatrix} \Delta Y & \Delta A \\ \Delta M & \Delta Z \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}. \tag{10}$$

Substituting (10) in (8), the terms of (8) corresponding to the element are

$$\begin{aligned} &\begin{bmatrix} \Delta V_a \\ \Delta V_b \end{bmatrix}^T \begin{bmatrix} \hat{I}_a \\ \hat{I}_b \end{bmatrix} - \begin{bmatrix} \Delta I_a \\ \Delta I_b \end{bmatrix}^T \begin{bmatrix} \hat{V}_a \\ \hat{V}_b \end{bmatrix} \\ &= \begin{bmatrix} \Delta I_a \\ \Delta V_b \end{bmatrix}^T \begin{bmatrix} -\hat{V}_a \\ \hat{I}_b \end{bmatrix} + \begin{bmatrix} \Delta V_a \\ \Delta I_b \end{bmatrix}^T \begin{bmatrix} \hat{I}_a \\ -\hat{V}_b \end{bmatrix} \\ &= \left(\begin{bmatrix} \Delta V_a \\ \Delta I_b \end{bmatrix}^T \begin{bmatrix} Y & A \\ M & Z \end{bmatrix}^T + \begin{bmatrix} V_a \\ I_b \end{bmatrix}^T \begin{bmatrix} \Delta Y & \Delta A \\ \Delta M & \Delta Z \end{bmatrix}^T \right) \\ &\quad \cdot \begin{bmatrix} -\hat{V}_a \\ \hat{I}_b \end{bmatrix} + \begin{bmatrix} \Delta V_a \\ \Delta I_b \end{bmatrix}^T \begin{bmatrix} \hat{I}_a \\ -\hat{V}_b \end{bmatrix} \end{aligned} \tag{11}$$

which can be reduced to

$$[V_a{}^T \ \ I_b{}^T] \begin{bmatrix} -\Delta Y^T & \Delta M^T \\ -\Delta A^T & \phi Z^T \end{bmatrix} \begin{bmatrix} \hat{V}_a \\ \hat{I}_b \end{bmatrix} \tag{12}$$

if

$$\begin{bmatrix} \hat{I}_a \\ \hat{V}_b \end{bmatrix} \equiv \begin{bmatrix} Y^T & -M^T \\ -A^T & Z^T \end{bmatrix} \begin{bmatrix} \hat{V}_a \\ \hat{I}_b \end{bmatrix} \tag{13}$$

which defines the adjoint element. This definition causes the terms of (8) relating to the element to be expressible only in terms of the unperturbed currents and voltages associated with the original and adjoint elements and incremental changes in the elements of the matrix. Terms containing incremental changes in current and voltage have disappeared.

Table I summarizes these results and results for impedance matrix, admittance matrix, and *ABCD* matrix descriptions. They may be derived independently or as special cases of the derivation for the hybrid matrix. Two important special cases should be noted. The first

TABLE I

Matrix Type	Original Element	Adjoint Element	Expression Yielding Sensitivity
Impedance	$V = ZI$	$\hat{V} = Z^T\hat{I}$	$I^T \Delta Z^T \hat{I}$
Admittance	$I = YV$	$\hat{I} = Y^T\hat{V}$	$-V^T \Delta Y^T \hat{V}$
Hybrid	$\begin{bmatrix} I_a \\ V_b \end{bmatrix} = \begin{bmatrix} Y & A \\ M & Z \end{bmatrix} \begin{bmatrix} V_a \\ I_b \end{bmatrix}$	$\begin{bmatrix} \hat{I}_a \\ \hat{V}_b \end{bmatrix} = \begin{bmatrix} Y^T & -M^T \\ -A^T & Z^T \end{bmatrix} \begin{bmatrix} \hat{V}_a \\ \hat{I}_b \end{bmatrix}$	$[V_a^T \, I_b^T] \begin{bmatrix} -\Delta Y^T & \Delta M^T \\ -\Delta A^T & \Delta Z^T \end{bmatrix} \begin{bmatrix} \hat{V}_a \\ \hat{I}_b \end{bmatrix}$
ABCD	$\begin{bmatrix} V_p \\ I_p \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_q \\ -I_q \end{bmatrix}$	$\begin{bmatrix} \hat{V}_p \\ \hat{I}_p \end{bmatrix} = \frac{1}{AD - BC} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \hat{V}_q \\ -\hat{I}_q \end{bmatrix}$	$[V_q \, I_q] \begin{bmatrix} \Delta A & -\Delta C \\ -\Delta B & \Delta D \end{bmatrix} \begin{bmatrix} \hat{I}_p \\ \hat{V}_p \end{bmatrix}$

is that adjoint of a reciprocal element is identical to the element itself. The second is that a one-port element (resistor, inductor, etc.) is accounted for by Table I.

Suppose the original and adjoint networks are excited by independent sources[2] as indicated by Fig. 7. Let

$$V_V \triangleq [V_1 \, V_2 \cdots V_{n_V}]^T \tag{14}$$

be the n_V-element voltage-excitation vector,

$$I_I \triangleq [I_{n_V+1} \, I_{n_V+2} \cdots I_{n_V+n_I}]^T \tag{15}$$

be the n_I-element current-excitation vector, so that

$$I_V \triangleq [I_1 \, I_2 \cdots I_{n_V}]^T \tag{16}$$

and

$$V_I \triangleq [V_{n_V+1} \, V_{n_V+2} \cdots V_{n_V+n_I}]^T \tag{17}$$

respectively, are the corresponding response vectors. Thus, subscript V refers to voltage-excited ports and subscript I refers to current-excited ports. For the adjoint network, similar definitions from (14) through (17) would be distinguished by ^.

Terms of (8) associated with the port excitations and responses are

$$\Delta V_V{}^T \hat{I}_V - \Delta I_V{}^T \hat{V}_V + \Delta V_I{}^T \hat{I}_I - \Delta I_I{}^T \hat{V}_I. \tag{18}$$

But $\Delta V_V = \Delta I_I = 0$ if the excitations remain constant. Expression (18), therefore, reduces to

$$-\Delta I_V{}^T \hat{V}_V + \Delta V_I{}^T \hat{I}_I. \tag{19}$$

In summary, then, (8) consists of terms of the form of (12) and similar ones as in Table I together with (19) leading, in general, to

$$\Delta I_V{}^T \hat{V}_V - \Delta V_I{}^T \hat{I}_I = G^T \Delta \phi \tag{20}$$

where G is a vector of sensitivities related to the adjustable network parameters contained in ϕ. It is seen that (20) relates changes in the port responses to changes in parameter values, which is usually what we are interested in. The form of the right-hand side of (20) is a direct consequence of the definition of the adjoint network.

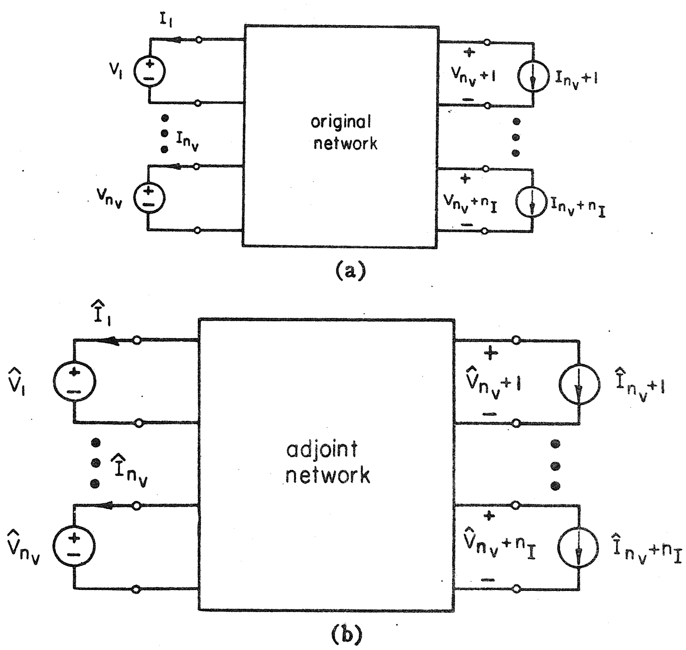


Fig. 7. (a) Excited arbitrary multiport network containing lumped and distributed elements. (b) Topologically equivalent adjoint network with corresponding port excitations.

IV. Derivation of Sensitivities

Table II presents the results of applying the formulas of Table I to a number of commonly used elements. Consider, for example, an inductor. According to the impedance formulas of Table I, the expression yielding the sensitivity is

$$I \Delta Z \hat{I} = (j\omega \, I\hat{I}) \Delta L. \tag{21}$$

Taking the inductance L as the parameter, $j\omega I\hat{I}$ is the sensitivity or component of G and ΔL is the parameter increment.

Now consider a uniformly distributed line as shown in Fig. 8(a). The element is reciprocal, so that

$$Z^T = Z = Z \begin{bmatrix} \coth \theta & \operatorname{csch} \theta \\ \operatorname{csch} \theta & \coth \theta \end{bmatrix} \tag{22}$$

[2] Appropriate zero-valued sources are placed, for convenience, at ports which are not excited.

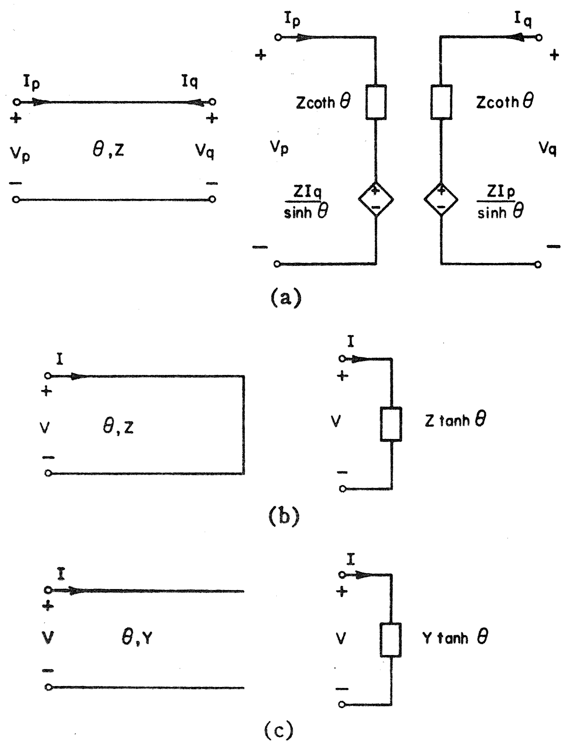


Fig. 8. Uniformly distributed elements with convenient representations. (a) Uniform line. (b) Short-circuited line. (c) Open-circuited line.

where Z is the characteristic impedance. Using the same formula in Table I as for the inductor [12]

$$
\begin{aligned}
I^T \Delta Z^T \hat{I} &= I^T \left(\Delta Z \begin{bmatrix} \coth\theta & \operatorname{csch}\theta \\ \operatorname{csch}\theta & \coth\theta \end{bmatrix} \right. \\
&\quad \left. - \frac{Z\Delta\theta}{\sinh\theta} \begin{bmatrix} \operatorname{csch}\theta & \coth\theta \\ \coth\theta & \operatorname{csch}\theta \end{bmatrix} \right)^T \hat{I} \\
&= \left(\frac{\Delta Z}{Z} \boldsymbol{ZI} - \frac{\Delta\theta}{\sinh\theta} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \boldsymbol{ZI} \right)^T \hat{\boldsymbol{I}} \\
&= \frac{\Delta Z}{Z} \boldsymbol{V}^T \hat{\boldsymbol{I}} - \frac{\Delta\theta}{\sinh\theta} \boldsymbol{V}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{\boldsymbol{I}}. \qquad (23)
\end{aligned}
$$

Corresponding expressions for the lossless transmission line of length l with $\theta = j\beta l$ and the uniform RC line (Fig. 9) with $Z = \sqrt{R/sC}$ and $\theta = \sqrt{sRC}$ are readily obtained [12] and are shown in Table II.

Consider a rectangular waveguide operating in the H_{10} mode, as shown in Fig. 10. The following model may be used if the restrictions outlined by Bandler [3] are observed:

$$Z = b\lambda_g \qquad (24)$$

$$\theta = j\frac{2\pi l}{\lambda_g} = j\beta_g l \qquad (25)$$

where

$$\lambda_g = \frac{\lambda}{\sqrt{1 - (\lambda/2a)^2}} \qquad (26)$$

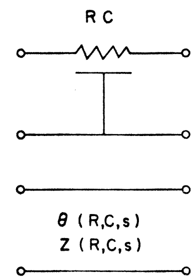


Fig. 9. Uniformly distributed RC line.

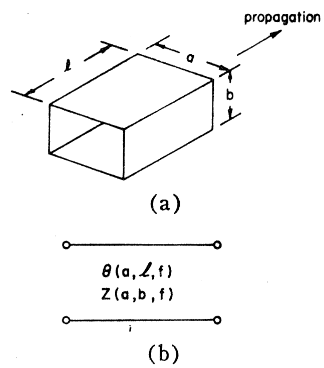


Fig. 10. Rectangular waveguide. (a) Geometrical dimensions. (b) Circuit representation.

where a, b, and l are the width, height, and length, respectively of the waveguide; λ_g is the guide wavelength and $\lambda = c/f$. It is readily shown, neglecting higher order terms, that

$$\Delta Z = \lambda_g \Delta b - \frac{b\lambda_g^3}{4a^3}\Delta a \qquad (27)$$

and

$$\Delta\theta = j\beta_g \Delta l + j\frac{\beta_g l \lambda_g^2}{4a^3}\Delta a. \qquad (28)$$

Expression (23) for the rectangular waveguide then becomes

$$
\begin{aligned}
&-\Delta a \frac{\lambda_g^2}{4a^3}\left(\boldsymbol{V}^T\hat{\boldsymbol{I}} + \frac{\beta_g l}{\sin\beta_g l}\boldsymbol{V}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{\boldsymbol{I}} \right) \\
&\qquad + \frac{\Delta b}{b}\boldsymbol{V}^T\hat{\boldsymbol{I}} - \frac{\beta_g \Delta l}{\sin\beta_g l}\boldsymbol{V}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{\boldsymbol{I}}. \qquad (29)
\end{aligned}
$$

Note that the voltages and currents do not necessarily have to have any physical interpretation, their use is only in being convenient variables for analysis.

Now consider a uniform lossless coaxial line with

$$Z = \frac{1}{2\pi}\frac{Z_0}{\sqrt{\epsilon_r}}\ln\frac{d_0}{d_i} \qquad (30)$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0}$, ϵ_r is the relative permittivity of the medium, and d_0 and d_i are the outer and inner diameters,

TABLE II

SENSITIVITY EXPRESSIONS FOR SOME LUMPED AND UNIFORMLY DISTRIBUTED ELEMENTS

Element	Equation	Sensitivity (component of **G**)	Increment (component of $\Delta\phi$)
Resistor	$V = RI$	$I\hat{I}$	ΔR
	$I = GV$	$-V\hat{V}$	ΔG
Inductor	$V = j\omega LI$	$j\omega I\hat{I}$	ΔL
	$I = \frac{1}{j\omega}\Gamma V$	$-\frac{1}{j\omega}V\hat{V}$	$\Delta\Gamma$
Capacitor	$V = \frac{1}{j\omega}SI$	$\frac{1}{j\omega}I\hat{I}$	ΔS
	$I = j\omega CV$	$-j\omega V\hat{V}$	ΔC
Transformer	$\begin{bmatrix} V_p \\ I_q \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \begin{bmatrix} I_p \\ V_q \end{bmatrix}$	$V_qI_p + I_p\hat{V}_q$	Δn
Gyrator	$\mathbf{V} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \mathbf{I}$	$I_p\hat{I}_q - I_q\hat{I}_p$	$\Delta\alpha$
Voltage controlled voltage source	$\begin{bmatrix} I_p \\ V_q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \begin{bmatrix} V_p \\ I_q \end{bmatrix}$	$V_p\hat{I}_q$	$\Delta\mu$
Voltage controlled current source	$\mathbf{I} = \begin{bmatrix} 0 & 0 \\ g_m & 0 \end{bmatrix} \mathbf{V}$	$-V_p\hat{V}_q$	Δg_m
Current controlled voltage source	$\mathbf{V} = \begin{bmatrix} 0 & 0 \\ r_m & 0 \end{bmatrix} \mathbf{I}$	I_pI_q	Δr_m
Current controlled current source	$\begin{bmatrix} V_p \\ I_q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix} \begin{bmatrix} I_p \\ V_q \end{bmatrix}$	$-I_p\hat{V}_q$	$\Delta\beta$
Short-circuited uniformly distributed line	$V = Z \tanh\theta\, I$	$\tanh\theta\, I\hat{I}$	ΔZ
		$Z\,\mathrm{sech}^2\theta\, II$	$\Delta\theta$
	$I = Y \coth\theta\, V$	$-\coth\theta\, V\hat{V}$	ΔY
		$Y\,\mathrm{csch}^2\theta\, V\hat{V}$	$\Delta\theta$
Open-circuited uniformly distributed line	$V = Z \coth\theta\, I$	$\coth\theta\, I\hat{I}$	ΔZ
		$-Z\,\mathrm{csch}^2\theta\, I\hat{I}$	$\Delta\theta$
	$I = Y \tanh\theta\, V$	$-\tanh\theta\, V\hat{V}$	ΔY
		$-Y\,\mathrm{sech}^2\theta\, V\hat{V}$	$\Delta\theta$
Uniformly distributed line	$\mathbf{V} = Z\begin{bmatrix} \coth\theta & \mathrm{csch}\,\theta \\ \mathrm{csch}\,\theta & \coth\theta \end{bmatrix} \mathbf{I}$	$\frac{1}{Z}\mathbf{V}^T\hat{\mathbf{I}}$	ΔZ
		$-\frac{1}{\sinh\theta}\mathbf{V}^T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\hat{\mathbf{I}}$	$\Delta\theta$
	$\mathbf{I} = Y\begin{bmatrix} \coth\theta & -\mathrm{csch}\,\theta \\ -\mathrm{csch}\,\theta & \coth\theta \end{bmatrix} \mathbf{V}$	$-\frac{1}{Y}\mathbf{I}^T\hat{\mathbf{V}}$	ΔY
		$-\frac{1}{\sinh\theta}\mathbf{I}^T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\hat{\mathbf{V}}$	$\Delta\theta$
Short-circuited lossless transmission line	$V = jZ \tan\beta l\, I$	$j\tan\beta l\, I\hat{I}$	ΔZ
		$jZ\beta\sec^2\beta l\, II$	Δl
	$I = -jY\cot\beta l\, V$	$j\cot\beta l\, V\hat{V}$	ΔY
		$-jY\beta\csc^2\beta l\, V\hat{V}$	Δl
Open-circuited lossless transmission line	$V = -jZ\cot\beta l\, I$	$-j\cot\beta l\, II$	ΔZ
		$jZ\beta\csc^2\beta l\, II$	Δl
	$I = jY\tan\beta l\, V$	$-j\tan\beta l\, V\hat{V}$	ΔY
		$-jY\beta\sec^2\beta l\, V\hat{V}$	Δl
Lossless transmission line	$\mathbf{V} = -jZ\begin{bmatrix} \cot\beta l & \csc\beta l \\ \csc\beta l & \cot\beta l \end{bmatrix} \mathbf{I}$	$\frac{1}{Z}\mathbf{V}^T\hat{\mathbf{I}}$	ΔZ

TABLE II (*Cont.*)

Element	Equation	Sensitivity (component of G)	Increment (component of $\Delta\phi$)
Lossless transmission line	$I = -jY\begin{bmatrix}\cot\beta l & -\csc\beta l\\ -\csc\beta l & \cot\beta l\end{bmatrix}V$	$-\frac{\beta}{\sin\beta l}V^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{I}$	Δl
		$-\frac{1}{Y}I^T\hat{V}$	ΔY
		$-\frac{\beta}{\sin\beta l}I^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{V}$	Δl
ectangular waveguide operating in H_{10} mode	as for lossless transmission line with $Z = b\lambda_g$, β replaced by $\beta_g = 2\pi/\lambda_g$, where $\lambda_g = \lambda/\sqrt{1-(\lambda/2a)^2}$	$-\frac{\lambda_g^2}{4a^3}V^T\begin{bmatrix}1 & \frac{\beta_g l}{\sin\beta_g l}\\ \frac{\beta_g l}{\sin\beta_g l} & 1\end{bmatrix}\hat{I}$	Δa
		$\frac{1}{b}V^T\hat{I}$	Δb
		$-\frac{\beta_g}{\sin\beta_g l}V^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{I}$	Δl
niform RC line	as for uniformly distributed line with $Z = \sqrt{\frac{R}{sC}}$ and $\theta = \sqrt{sRC}$	$\frac{1}{2R}V^T\begin{bmatrix}1 & -\frac{\theta}{\sinh\theta}\\ -\frac{\theta}{\sinh\theta} & 1\end{bmatrix}\hat{I}$	ΔR
		$-\frac{1}{2C}V^T\begin{bmatrix}1 & \frac{\theta}{\sinh\theta}\\ \frac{\theta}{\sinh\theta} & 1\end{bmatrix}\hat{I}$	ΔC
Uniform coaxial line	as for lossless transmission line with $Z = \frac{1}{2\pi}\frac{Z_0}{\sqrt{\epsilon_r}}\ln\frac{d_0}{d_i}$ and $\beta = \beta_0\sqrt{\epsilon_r}$	$\frac{1}{d_0\ln\frac{d_0}{d_i}}V^T\hat{I}$	Δd_0
		$-\frac{1}{d_i\ln\frac{d_0}{d_i}}V^T\hat{I}$	Δd_i
		$-\frac{\beta_0\sqrt{\epsilon_r}}{\sin\beta_0\sqrt{\epsilon_r}l}V^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{I}$	Δl
		$-\frac{1}{2\epsilon_r}\left(V^T\hat{I}+\frac{\beta_0\sqrt{\epsilon_r}l}{\sin\beta_0\sqrt{\epsilon_r}l}V^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{I}\right)$	$\Delta\epsilon_r$
Coupled lines (1) capacitance matrix description	$I = c\begin{bmatrix}C\coth\theta & -C\operatorname{csch}\theta\\ -C\operatorname{csch}\theta & C\coth\theta\end{bmatrix}V$ where $C \triangleq \begin{bmatrix}C_{01}+C_{12} & -C_{12}\\ -C_{12} & C_{02}+C_{12}\end{bmatrix}$ (see text and Fig. 11)	$-c[V_{1p}\ V_{1q}]\begin{bmatrix}\coth\theta & -\operatorname{csch}\theta\\ -\operatorname{csch}\theta & \coth\theta\end{bmatrix}\begin{bmatrix}\hat{V}_{1p}\\ \hat{V}_{1q}\end{bmatrix}$	ΔC_{01}
		$-c[V_{2p}\ V_{2q}]\begin{bmatrix}\coth\theta & -\operatorname{csch}\theta\\ -\operatorname{csch}\theta & \coth\theta\end{bmatrix}\begin{bmatrix}\hat{V}_{2p}\\ \hat{V}_{2q}\end{bmatrix}$	ΔC_{02}
		$-cV^T\begin{bmatrix}1'\coth\theta & -1'\operatorname{csch}\theta\\ -1'\operatorname{csch}\theta & 1'\coth\theta\end{bmatrix}\hat{V}$	ΔC_{12}
		$-\frac{1}{\sinh\theta}I^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{V}$ (see text for definitions of 1′, 1, and 0)	$\Delta\theta$
Coupled lines (2) even- and odd-mode description for symmetrical arrangement	$I = -j\begin{bmatrix}Y_M\cot\beta l & -Y_M\csc\beta l\\ -Y_M\csc\beta l & Y_M\cot\beta l\end{bmatrix}$ where $Y_M \triangleq \frac{1}{2}\begin{bmatrix}Y_e+Y_o & Y_e-Y_o\\ Y_e-Y_o & Y_e+Y_o\end{bmatrix}$ and where Y_e and Y_o are the even- and odd-mode admittances	$-\frac{1}{2Y_e}V^T\begin{bmatrix}I_{1p}+I_{2p}\\ \hat{I}_{1p}+\hat{I}_{2p}\\ \hat{I}_{1q}+\hat{I}_{2q}\\ \hat{I}_{1q}+\hat{I}_{2q}\end{bmatrix}$	ΔY_e
		$-\frac{1}{2Y_o}V^T\begin{bmatrix}\hat{I}_{1p}-\hat{I}_{2p}\\ \hat{I}_{2p}-I_{1p}\\ I_{1q}-I_{2q}\\ \hat{I}_{2q}-I_{1q}\end{bmatrix}$	ΔY_o
		$-\frac{\beta}{\sin\beta l}I^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{V}$	Δl

respectively, of the line. Here,

$$\theta = j\beta_0\sqrt{\epsilon_r}\, l \tag{31}$$

where β_0 is the free-space phase constant. Thus

$$\Delta Z = Z\left\{\frac{\Delta d_0}{d_0 \ln \frac{d_0}{d_i}} - \frac{\Delta d_i}{d_i \ln \frac{d_0}{d_i}} - \frac{\Delta\epsilon_r}{2\epsilon_r}\right\} \tag{32}$$

and

$$\Delta\theta = j\beta_0\sqrt{\epsilon_r}\,\Delta l + \frac{j\beta_0 l \Delta\epsilon_r}{2\sqrt{\epsilon_r}}. \tag{33}$$

Expression (23) for the coaxial line becomes

$$\left\{\frac{\Delta d_0}{d_0 \ln \frac{d_0}{d_i}} - \frac{\Delta d_i}{d_i \ln \frac{d_0}{d_i}}\right\} V^T\hat{I} - \frac{\beta_0\sqrt{\epsilon_r}\,\Delta l}{\sin \beta_0\sqrt{\epsilon_r} l} V^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{I}$$

$$- \frac{\Delta\epsilon_r}{2\epsilon_r}\left(V^T\hat{I} + \frac{\beta_0\sqrt{\epsilon_r}\, l}{\sin\beta_0\sqrt{\epsilon_r}\, l} V^T\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{I}\right). \tag{34}$$

Finally, consider the admittance matrix formulas of Table I applied to the pair of coupled lines above a ground plane [13], [14] shown in Fig. 11. The admittance matrix description is

$$\begin{bmatrix} I_{1p} \\ I_{2p} \\ I_{1q} \\ I_{2q}\end{bmatrix} = c\begin{bmatrix} C \coth\theta & -C \operatorname{csch}\theta \\ -C \operatorname{csch}\theta & C\coth\theta\end{bmatrix}\begin{bmatrix} V_{1p} \\ V_{2p} \\ V_{1q} \\ V_{2q}\end{bmatrix} \tag{35}$$

where subscript p denotes the two ports formed between each conductor and the ground plane at one end and q the corresponding ports at the other end; subscript 1 refers to one conductor and 2 to the other. The matrix C is given by

$$C \triangleq \begin{bmatrix} C_{01} + C_{12} & -C_{12} \\ -C_{12} & C_{02} + C_{12}\end{bmatrix} \tag{36}$$

the elements of which are defined in Fig. 11. Treating C_{01}, C_{02}, C_{12}, and θ as variables we have

$$-V^T \Delta Y^T \hat{V}$$

$$= -c[V_{1p} V_{1q}]\begin{bmatrix}\coth\theta & -\operatorname{csch}\theta \\ -\operatorname{csch}\theta & \coth\theta\end{bmatrix}\begin{bmatrix}\hat{V}_{1p} \\ \hat{V}_{1q}\end{bmatrix}\Delta C_{01}$$

$$- c[V_{2p} V_{2q}]\begin{bmatrix}\coth\theta & -\operatorname{csch}\theta \\ -\operatorname{csch}\theta & \coth\theta\end{bmatrix}\begin{bmatrix}\hat{V}_{2p} \\ \hat{V}_{2q}\end{bmatrix}\Delta C_{02}$$

$$- cV^T\begin{bmatrix}1' \coth\theta & -1' \operatorname{csch}\theta \\ -1' \operatorname{csch}\theta & 1'\coth\theta\end{bmatrix}\hat{V}\Delta C_{12}$$

$$- \frac{c}{\sinh\theta} V^T\begin{bmatrix} -C \operatorname{csch}\theta & C\coth\theta \\ C\coth\theta & -C\operatorname{csch}\theta\end{bmatrix}\hat{V}\Delta\theta \tag{37}$$

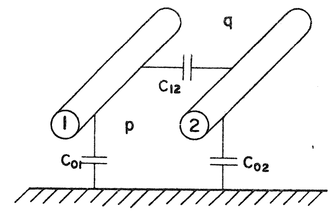


Fig. 11. Coupled lines above a ground plane with static capacitances per unit length.

where

$$1' \triangleq \begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}. \tag{38}$$

The last term may be rewritten as

$$-\frac{1}{\sinh\theta} I^T \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\hat{V}\Delta\theta \tag{39}$$

where

$$0 \triangleq \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \tag{40}$$

and

$$1 \triangleq \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \tag{41}$$

These results are summarized in Table II, along with expressions based on the approach using even- and odd-mode characteristic admittances.

V. Gradient Computations

There are a number of ways in which the adjoint network method can be used effectively in gradient computations.

Consider Figs. 12 and 13. Fig. 12(a) depicts the situation when insertion loss or gain is to be optimized. Here we are interested at some frequency in the partial derivatives of I_L with respect to the parameters and hence ∇I_L. Fig. 13(a) is appropriate for design on the reflection coefficient basis. In this case we are interested at some frequency in ∇I_g. Suppose the adjoint networks are excited as shown in Figs. 12(b) and 13(b). Then, for Fig. 12, (20) can be reduced to

$$\Delta I_L \hat{V}_L = G^T \Delta\phi. \tag{42}$$

Dividing by $\hat{V}_L$ we have

$$\Delta I_L = \nabla I_L{}^T \Delta\phi = \left[\frac{1}{\hat{V}_L} G^T\right]\Delta\phi$$

from which

$$\nabla I_L = \frac{1}{\hat{V}_L} G. \tag{43}$$

For Fig. 13, (20) can be reduced to

$$\Delta I_g \hat{V}_g = G^T \Delta\phi. \tag{44}$$

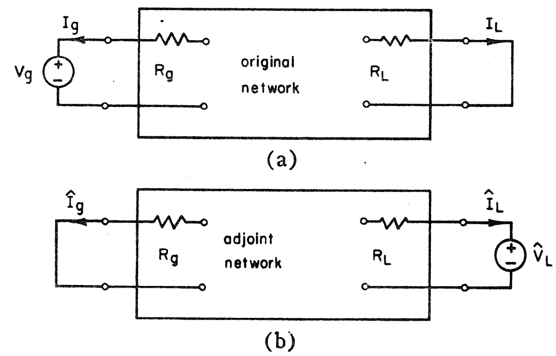


Fig. 12. Special case of Fig. 7 for insertion loss design.

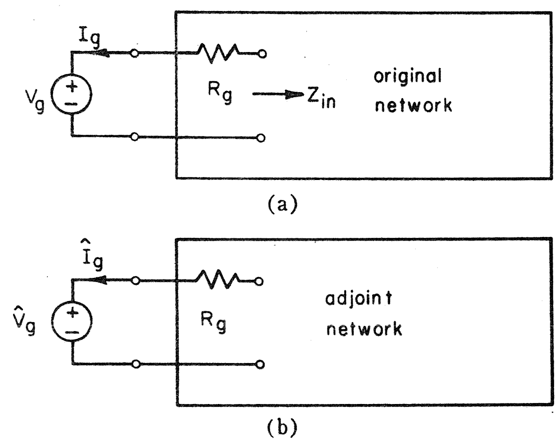


Fig. 13. Special case of Fig. 7 for reflection coefficient design.

Noting that (44) has the same form as (42) we get

$$\nabla I_g = \frac{1}{\hat{V}_g} G. \tag{45}$$

Observe that ∇I_L in (43) and ∇I_g in (45) are evaluated from the currents and voltages present in the unperturbed original and adjoint networks. At most, two network analyses using *any* suitable method will, therefore, yield the information required for the evaluation. Of course, if desired, analytic expressions for the partial derivatives could also be found by this approach.[3] It is interesting to note that for design of reciprocal networks on the reflection coefficient basis we are at liberty to set $\hat{V}_g \equiv V_g$ and use the results of just *one* analysis at each frequency.

To relate ∇I_L or ∇I_g to the gradient vector of suitable least pth or minimax objective functions [2]–[4], [6], [15]–[18] is a straightforward process [12]. In anticipation of the numerical example (Section VI), we will first consider discrete least pth approximation using the reflection coefficient. Let

$$U = \sum_{\Omega_d} \frac{1}{p} \mid \rho(j\omega_d) \mid^p, \qquad \omega_d \in \Omega_d \tag{46}$$

[3] It is debatable, however, whether any computational advantage would, in general, be gained by deriving analytic expressions.

where ρ is the reflection coefficient between R_g and the one-port network, Ω_d is a set of discrete frequencies ω_d, and p is any positive integer. Suppose it is required to minimize U. In this case we are trying to approximate zero reflection coefficient in a least pth sense. For large p we would expect a nearly equal-ripple response to correspond to the minimum of U [2].

$$\rho(j\omega_d) = \frac{Z_{in}(j\omega_d) - R_g}{Z_{in}(j\omega_d) + R_g} = 1 - \frac{2R_g}{Z_{in}(j\omega_d) + R_g} = 1 + \frac{2R_g I_g(j\omega_d)}{V_g(j\omega_d)} \tag{47}$$

so that

$$\nabla U = \sum_{\Omega_d} \operatorname{Re} \left\{ \mid \rho(j\omega_d) \mid^{p-2} \rho^*(j\omega_d) \nabla \rho(j\omega_d) \right\} = \sum_{\Omega_d} \operatorname{Re} \left\{ \frac{2R_g}{V_g(j\omega_d)} \mid \rho(j\omega_d) \mid^{p-2} \rho^*(j\omega_d) \nabla I_g(j\omega_d) \right\}. \tag{48}$$

If, instead of minimizing (46), the problem is to minimize a nonnegative independent variable U, subject to

$$U \geq g(\omega_d) = \tfrac{1}{2} \mid \rho(j\omega_d) \mid^2, \qquad \omega_d \in \Omega_d \tag{49}$$

then we have minimax approximation [2], for which

$$\nabla g(\omega_d) = \operatorname{Re} \left\{ \frac{2R_g}{V_g(j\omega_d)} \rho^*(j\omega_d) \nabla I_g(j\omega_d) \right\}. \tag{50}$$

Finally, let us address ourselves to the approximation problem considered by Director and Rohrer [8], generalizing it to least p [19], [20]. Equation (20) is readily rearranged to give

$$G = \sum_{i=1}^{n_V} \hat{V}_i \nabla I_i - \sum_{i=n_V+1}^{n_V+n_I} \hat{I}_i \nabla V_i \tag{51}$$

since, neglecting higher order terms,

$$\Delta I_i = \nabla I_i^T \Delta \phi$$
$$\Delta V_i = \nabla V_i^T \Delta \phi.$$

Given, for example, the objective function

$$U = \sum_{i=1}^{n_V+n_I} \int_\Omega \frac{1}{p} \mid e_i(\phi, j\omega) \mid^p d\omega \tag{52}$$

where

$$e_i(\phi, j\omega) \triangleq w_i(\omega)(F_i(\phi, j\omega) - S_i(j\omega)) \tag{53}$$

where

$$F_i(\phi, j\omega) \triangleq \begin{cases} I_i(\phi, j\omega) & i = 1, 2, \cdots, n_V \\ V_i(\phi, j\omega) & i = n_V + 1, \cdots, n_V + n_I \end{cases} \tag{54}$$

and Ω defines the frequency range of interest. Here, $S_i(j\omega)$ is a desired complex port response with $w_i(\omega)$ a nonnegative real weighting function. In this case

$$\nabla U = \sum_{i=1}^{n_V+n_I} \int_\Omega \operatorname{Re} \left\{ \mid e_i(\phi, j\omega) \mid^{p-2} w_i(\omega) \cdot e_i^*(\phi, j\omega) \nabla F_i(\phi, j\omega) \right\} d\omega. \tag{55}$$

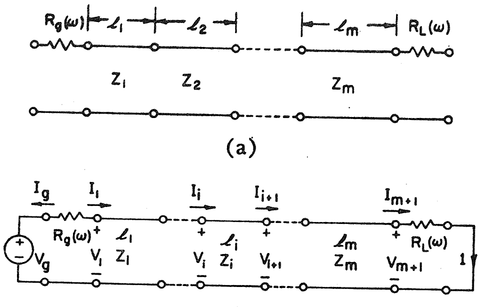


Fig. 14. Cascaded transmission lines terminated in frequency variable resistances.

By comparing (51), (54), and (55) it is seen that by arranging for the adjoint network voltage and current excitations to be given by

$$|e_i(\phi, j\omega)|^{p-2} w_i(\omega) e_i^*(\phi, j\omega) = \begin{cases} \hat{V}_i(j\omega) & i = 1, 2, \cdots, n_V \\ -\hat{I}_i(j\omega) & i = n_V + 1, \cdots, n_V + n_I \end{cases} \tag{56}$$

we obtain

$$\nabla U = \int_\Omega \text{Re}\,\{\boldsymbol{G}\}\, d\omega. \tag{57}$$

If there is no excitation at a particular port the appropriate source is obviously set to zero. If the response at a particular port is not to be controlled the corresponding adjoint excitation should be zero. Elements or parameters not to be varied during optimization do not, of course, contribute to ϕ or $\boldsymbol{G}$.

VI. Example

Carlin and Gupta [21] recently considered the optimal design of filters with lumped–distributed elements or frequency-variable terminations. Although any of their design examples are amenable to computer-oriented optimization techniques, let us discuss the design of the symmetrical seven-section cascaded transmission-line filter shown in Fig. 14(a).

The terminating impedances are real but frequency dependent, specifically

$$R_g(\omega) = R_L(\omega) = 377/\sqrt{1-(f_c/f)^2}$$

where f is the frequency in GHz and

$$f_c = 2.077 \text{ GHz}.$$

Thus, the terminating impedances can be thought of as rectangular waveguides operating in the H_{10} mode with cutoff frequency 2.077 GHz. Carlin and Gupta required a passband insertion loss of less than 0.4 dB over 2.16 to 3 GHz and an edge to the useful band of 5 GHz. They constrained all section lengths to be 1.5 cm so that each section would be quarter wave at 5 GHz and causing the maximum insertion loss to occur at that frequency. The response of their design is shown in Figs. 15 and 16 and the values of characteristic impedance in Table III.

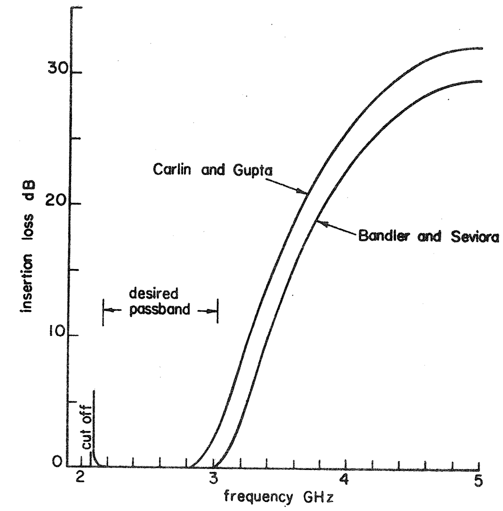


Fig. 15. The response of the seven-section filter whose configuration is shown in Fig. 14. The authors' response was optimized for minimum passband insertion loss.

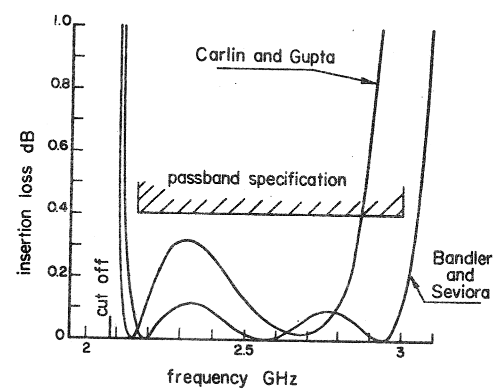


Fig. 16. Details of the passband insertion loss of the seven-section filter.

A question of interest to the present authors is this: how small can the passband insertion loss be made under the constraints of the problem? (Note that the question is trivial if the terminating impedances were frequency independent or if the section lengths were freely variable.)

The least pth objective function of (46) was set up using 51 uniformly spaced points over the range 2.16 to 3 GHz and with $p=10$. Optimization was carried out by the Fletcher–Powell method [22], the required derivatives being evaluated from the results of *one* analysis of the network of Fig. 14(b). To apply the adjoint network method a simple *ABCD* matrix analysis algorithm employing the approach indicated in Fig. 14(b) was written. Instead of fixing V_g, it was found more convenient to assume that $I_L=1$ and to calculate the required currents and voltages including V_g. The appropriate formulas from Table II were used (not forgetting to reverse the currents at the junctions when necessary). The design parameter values of Carlin and Gupta were used as starting values.

TABLE III
COMPARISON OF PARAMETER VALUES FOR THE SEVEN-SECTION FILTER

Characteristic Impedances (normalized)	Carlin and Gupta [21]	Bandler and Seviora
Z_1	1476.5	1469.5
Z_2	733.6	763.2
Z_3	1963.6	1945.1
Z_4	461.8	558.7
Z_5	1963.6	1945.1
Z_6	733.6	763.2
Z_7	1476.5	1469.5

The resulting response is plotted in Figs. 15 and 16 and the final parameter values are given in Table III. Observe the almost equal-ripple behavior of the response with a maximum insertion loss over the passband of about 0.1 dB. It would appear then that under the design constraints imposed by Carlin and Gupta, a much lower maximum passband insertion loss is probably not achieveable. This was verified more recently by applying a minimax approximation algorithm to the same problem. A substantially equal-ripple response was obtained with a maximum insertion loss of 0.086 dB. (The algorithm uses the general philosophy behind the razor search method [4] but relies on gradient information generated by the adjoint network method.)

The reader should note that our design is not optimal in the filtering sense required by Carlin and Gupta; to achieve this one would want to maximize the stopband insertion loss subject to a passband insertion loss less than or equal to 0.4 dB. Allowing the section lengths to vary might also improve the response somewhat.

VII. Discussion

A nonexistent lumped element may be thought of as an appropriate zero-valued element connected between two nodes. Since the gradients depend only on voltages between nodes and currents through branches, they may be evaluated with respect to such nonexistent elements. If an increase in element value is indicated, the element can be grown from a short circuit or open circuit, as appropriate. Thus, changes in topology can be accommodated by this means. The adjoint network method does not seem, however, to provide any clear advantage over other methods as an aid to choosing the best topology except possibly in computation time. A direct search method, for example, can also investigate changes with respect to zero-valued elements.

As the authors have found [12], it is not, in general, obvious what kind of element should be grown, whether lumped or distributed, when distributed elements are also allowed. A knowledge only of the currents and voltages is not really sufficient. Furthermore, a variety of physical, economic, and other practical constraints on circuit configuration will also affect the choice. How would one decide, for example, whether a short-circuited transmission line should be grown rather than a lumped inductor?

Many circuit designers claim to have had success in computer-aided network design using direct search methods, so why should they adopt a gradient method? Well, if it is steepest descent they are thinking of, they are better off using the direct search methods. The Fletcher–Powell method [22], on the other hand, is, at the time of writing, still most widely acknowledged as the most powerful unconstrained minimization method available. Factors affecting the choice of an optimization method undoubtedly include familiarity with a particular program, the presence of constraints, the type of approximation required (whether least pth or minimax), the number of variables, and the available computation system [23]. As far as approximation methods are concerned, algorithms which should benefit considerably from the adjoint network method of evaluating derivatives are the minimax approximation methods of Lasdon and Waren [6], [15], Ishizaki and Watanabe [16], Osborne and Watson [24], and the least pth approximation method of Temes and Zai [17], [18].

Extensions of the adjoint network method to second-order network sensitivities have been presented [25]–[27]. The results may be used with those optimization methods, such as the Newton method [2], which require second derivatives. However, since gradient methods involving only first derivatives are generally considered superior, it seems unlikely that widespread application of these results will be seen in the very near future.

VIII. Conclusions

The ease of implementation of the adjoint network method of evaluating partial derivatives and the immediate savings in computation time for the computer-aided design of circuits make it very attractive. A great deal of the uncertainty and inefficiency inherent in the numerical estimation of partial derivatives can be eliminated.

It is believed that this approach will find very wide application. Another very recent report in this area and of interest to microwave engineers is available [28]. There seems little doubt, from the circuit designer's point of view at any rate, that the introduction of the adjoint network method by Director and Rohrer is a turning point in computer-aided design.

IX. Acknowledgment

The authors would like to thank Dr. M. Sablatash of the Department of Electrical Engineering, University of Toronto, Ont., Canada, for his stimulation of this work, A. Lee-Chan of the McMaster Data Processing and Computing Centre, McMaster University for his programming assistance, and W. J. Butler, of General Electric Company, Schenectady, N. Y., for useful discussions. R. E. Seviora received financial assistance under a Mary H. Beatty Fellowship from the University of Toronto.

References

[1] M. A. Murray-Lasso and E. B. Kozemchak, "Microwave circuit design by digital computer," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 514–526, August 1969.

[2] J. W. Bandler, "Optimization methods for computer-aided design," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 533–552, August 1969.

[3] ——, "Computer optimization of inhomogeneous waveguide transformers," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 563–571, August 1969.

[4] J. W. Bandler and P. A. Macdonald, "Optimization of microwave networks by razor search," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 552–562, August 1969.

[5] T. N. Trick and J. Vlach, "Computer-aided design of broad-band amplifiers with complex loads," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-18, pp. 541–547, September 1970.

[6] L. S. Lasdon and A. D. Waren, "Optimal design of filters with bounded, lossy elements," *IEEE Trans. Circuit Theory*, vol. CT-13, pp. 175–187, June 1966.

[7] S. W. Director and R. A. Rohrer, "The generalized adjoint network and network sensitivities," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 318–323, August 1969.

[8] ——, "Automated network design—the frequency-domain case," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 330–337, August 1969.

[9] B. D. H. Tellegen, "A general network theorem, with applications," *Philips Res. Reps.*, vol. 7, pp. 259–269, August 1952.

[10] C. A. Desoer and E. S. Kuh, *Basic Circuit Theory*. New York: McGraw-Hill, 1969, ch. 9.

[11] P. Penfield, Jr., R. Spence, and S. Duinker, *Tellegen's Theorem and Electrical Networks*. Cambridge, Mass.: MIT Press, 1970.

[12] J. W. Bandler and R. E. Seviora, "Computation of sensitivities for noncommensurate networks" (to be published).

[13] E. M. T. Jones and J. T. Bolljahn, "Coupled-strip-transmission-line filters and directional couplers," *IRE Trans. Microwave Theory Tech.*, vol. MTT-4, pp. 75–81, April 1956.

[14] R. J. Wenzel, "Theoretical and practical applications of capacitance matrix transformations to TEM network design," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-14, pp. 635–647, December 1966.

[15] A. D. Waren, L. S. Lasdon, and D. F. Suchman, "Optimization in engineering design," *Proc. IEEE*, vol. 55, pp. 1885–1897, November 1967.

[16] Y. Ishizaki and H. Watanabe, "An iterative Chebyshev approximation method for network design," *IEEE Trans. Circuit Theory*, vol. CT-15, pp. 326–336, December 1968.

[17] G. C. Temes and D. Y. F. Zai, "Least *p*th approximation," *IEEE Trans. Circuit Theory* (Correspondence), vol. CT-16, pp. 235–237, May 1969.

[18] G. C. Temes, "Optimization methods in circuit design," in *Computer Oriented Circuit Design*, F. F. Kuo and W. G. Magnuson, Jr., Eds. Englewood Cliffs, N. J.: Prentice-Hall, 1969.

[19] S. W. Director, "Network design by mathematical optimization," *WESCON*, San Francisco, Calif., August 1969.

[20] R. Seviora, M. Sablatash, and J. W. Bandler, "Least *p*th and minimax objectives for automated network design," *Electron. Lett.*, vol. 6, pp. 14–15, January 8, 1970.

[21] H. J. Carlin and O. P. Gupta, "Computer design of filters with lumped-distributed elements or frequency variable terminations," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-17, pp. 598–604, August 1969.

[22] R. Fletcher and M. J. D. Powell, "A rapidly convergent descent method for minimization," *Comput. J.*, vol. 6, pp. 163–168, June 1963.

[23] J. W. Bandler, "Automatic optimization of engineering designs—possibilities and pitfalls," *NEREM Rec.*, pp. 26–27, November 1969.

[24] M. R. Osborne and G. A. Watson, "An algorithm for minimax approximation in the nonlinear case," *Comput. J.*, vol. 12, pp. 63–68, February 1969.

[25] S. W. Director, "Increased efficiency of network sensitivity computations by means of LU factorization," presented at the Midwest Symposium on Circuit Theory, Austin, Tex., April 1969.

[26] P. J. Goddard and R. Spence, "Efficient method for the calculation of first- and second-order network sensitivities," *Electron. Lett*, vol. 5, pp. 351–352, August 7, 1969.

[27] G. A. Richards, "Second-derivative sensitivity using the concept of the adjoint network," *Electron. Lett.*, vol. 5, pp. 398–399, August 21, 1969.

[28] M. E. Mokari-Bolhassan and T. N. Trick, "Design of microwave integrated amplifiers," presented at the Midwest Symposium on Circuit Theory, Minneapolis, Minn., May 1970.

Reprinted from IEEE TRANSACTIONS
ON *MICROWAVE THEORY AND TECHNIQUES*
Volume MTT-18, Number 12, December, 1970
pp. 1159-1170

APPENDIX 4

PRACTICAL LEAST PTH OPTIMIZATION

On the following pages the formulation of least pth approximation problems as used in the package is described. Much useful insight into scaling, weighting, values of p, the artifical margin, efficiency, satisfied and violated specifications and some examples are provided.

PRACTICAL LEAST *p*th OPTIMIZATION OF NETWORKS

BY

J. W. BANDLER AND C. CHARALAMBOUS

Reprinted from IEEE TRANSACTIONS
ON *MICROWAVE THEORY AND TECHNIQUES*
Volume MTT-20, Number 12, December, 1972
pp. 834-840

Practical Least pth Optimization of Networks

JOHN W. BANDLER AND CHRISTAKIS CHARALAMBOUS

Abstract—**A new and practical approach to computer-aided design optimization is presented. Central to the process is the application of least pth approximation using extremely large values of p, typically 1000 to 1 000 000. It is shown how suitable and reasonably well-conditioned objective functions can be formulated, giving particular emphasis to more general approximation problems as, for example, in filter design. It is demonstrated how easily and efficiently extremely near minimax results can be achieved on a discrete set of sample points. Highly efficient gradient methods can be employed and, in network design problems, the use of the adjoint network approach for evaluating gradients results in greater savings in computer effort. A comparison between the Fletcher–Powell method and the more recent Fletcher method is made on the application of least pth approximation, using a range of values of p up to 1 000 000 000 000 on transmission-line transformer problems for which optimal minimax solutions are known. This is followed by filter design examples subject to certain constraints.**

I. Introduction

LEAST pth approximation with a sufficiently large value of p can, in principle, be used to achieve near minimax approximations for a wide class of circuit- and system-design problems. Early reports [1]–[4] simply suggested that appropriate error functions be raised to a power p. This approach, in practice, can lead to ill-conditioning of the objective function for values of p greater than or equal to about 10. In certain design problems the unwary designer may be led to the conclusion that his problem has many local minima (see, for example, [1] and [2]) in a region of the parameter space where, in fact, a unique minimum exists.

Bandler and Charalambous [5] have shown how to apply least pth approximation to design problems having upper and lower response specifications, e.g., as in filter design. However, the same ill-conditioning could arise in that particular formulation. More recent theoretical work has been published on conditions for optimality in least pth approximation with $p \rightarrow \infty$ [6], from which conditions for a minimax approximation [7] fall out.

It is the purpose of the present paper to present a computationally practical approach to least pth approximation for use in design problems. The important feature of the approach is the use that can be made of efficient gradient minimization techniques, such as the Fletcher–Powell method [8] and the more recent Fletcher method [9], in conjunction with least pth objective functions employing extremely large values of p, typically 1000–1 000 000. It is demonstrated how easily and efficiently extremely near minimax results can be achieved on a discrete set of sample points.

A comparison between the Fletcher–Powell and Fletcher methods is made using a range of values of p up to 1 000 000 000 000 on transmission-line transformer problems for which optimal minimax solutions are known. Filter-design examples with constraints are also provided. In all cases the adjoint network method [4] is used to obtain all the required partial derivatives at a given point in the parameter space from the results of one network analysis.

Manuscript received June 9, 1972; revised August 4, 1972. This work was supported by the National Research Council of Canada under Grants A7239 and C154, and by a Frederick Gardner Cottrell Grant from the Research Corporation. This paper is based on papers presented at the 5th Asilomar Conference on Circuits and Systems, Pacific Grove, Calif., November 1971, and the 1972 IEEE International Microwave Symposium, Chicago, Ill., May 22–24, 1972.

The authors are with the Communications Research Laboratory, Department of Electrical Engineering, McMaster University, Hamilton, Ont., Canada.

II. Theory

Definitions

The notation to be used in this paper largely follows that used previously by the authors [3], [5].

$F(\phi, \psi)$	The approximating function (actual response).
$S_u(\psi)$	An upper specified function (desired response bound).
$S_u'(\psi, \xi)$	An artificial upper specified function.
$S_l(\psi)$	A lower specified function (desired response bound).
$S_l'(\psi, \xi)$	An artificial lower specified function.
$w_u(\psi)$	An upper positive weighting function.
$w_l(\psi)$	A lower positive weighting function.
ϕ	A vector containing the k independent parameters.
ψ	An independent variable (e.g., frequency or time).
ξ	Margin of errors with respect to the artificial and desired specifications.

The introduction of the artificial margin ξ, which is a constant during optimization, allows for certain flexibility in formulating the optimization problem. Its advantage will become evident at a later stage.

Now we can define real error functions related to the upper and lower specifications as

$$e_u(\phi, \psi) \triangleq w_u(\psi)(F(\phi, \psi) - S_u(\psi)) \quad (1)$$

$$e_u'(\phi, \psi, \xi) \triangleq w_u(\psi)(F(\phi, \psi) - S_u'(\psi, \xi)) = e_u(\phi, \psi) - \xi \quad (2)$$

$$e_l(\phi, \psi) \triangleq w_l(\psi)(F(\phi, \psi) - S_l(\psi)) \quad (3)$$

$$e_l'(\phi, \psi, \xi) \triangleq w_l(\psi)(F(\phi, \psi) - S_l'(\psi, \xi)) = e_l(\phi, \psi) + \xi \quad (4)$$

where $S_u'(\psi, \xi)$ and $S_l'(\psi, \xi)$ are taken, respectively, as

$$S_u'(\psi, \xi) = S_u(\psi) + \frac{\xi}{w_u(\psi)} \quad (5)$$

$$S_l'(\psi, \xi) = S_l(\psi) - \frac{\xi}{w_l(\psi)} \cdot \quad (6)$$

In practice, we will evaluate all the functions at a finite discrete set of values of ψ taken from one or more closed intervals. Therefore, we will define the functions

$$e_{ui}'(\phi, \xi) \triangleq e_u'(\phi, \psi_i, \xi), \qquad i \in I_u \quad (7)$$

$$e_{li}'(\phi, \xi) \triangleq e_l'(\phi, \psi_i, \xi), \qquad i \in I_l \quad (8)$$

where it is assumed that a sufficient number of sample points have been chosen so that the discrete approximation problem adequately approximates the continuous problem. I_u and I_l are appropriate index sets. We assume that we can choose ψ_i such that the corresponding $e_{ui}'(\phi, \xi)$ and $e_{li}'(\phi, \xi)$ are continuous with continuous derivatives with respect to ϕ.

The Objective Function

Here we have to consider two separate cases, the first one when the specification is violated and the second one when the specification is satisfied.

In the first case some of the $e_{ui}'(\phi, \xi)$ or $-e_{li}'(\phi, \xi)$ are positive. To meet the artificial specification [same as original if $\xi = 0$ as indicated by (5) and (6)] we might propose the following objective function to be minimized:

$$U(\phi, \xi) = \left(\sum_{i \in J_u(\phi,\xi)} [e_{ui}'(\phi, \xi)]^p + \sum_{i \in J_l(\phi,\xi)} [-e_{li}'(\phi, \xi)]^p \right)^{1/p} \quad (9)$$

where[1]

$$J_u(\phi, \xi) \triangleq \{ i \mid e_{ui}'(\phi, \xi) \geq 0, \qquad i \in I_u \} \quad (10)$$

$$J_l(\phi, \xi) \triangleq \{ i \mid -e_{li}'(\phi, \xi) \geq 0, \qquad i \in I_l \} \quad (11)$$

and

$$p > 1.$$

For larger values of p we would expect the maximum of the functions to be emphasized, since

$$\max_{i,j} [e_{ui}'(\phi, \xi), -e_{lj}'(\phi, \xi)] = \lim_{p \to \infty} U(\phi, \xi), \qquad i \in J_u(\phi, \xi), \; j \in J_l(\phi, \xi). \quad (12)$$

Letting

$$\nabla \triangleq \begin{bmatrix} \frac{\partial}{\partial \phi_1} \\ \frac{\partial}{\partial \phi_2} \\ \vdots \\ \frac{\partial}{\partial \phi_k} \end{bmatrix} \quad (13)$$

we have

$$\nabla U(\phi, \xi) = \left(\sum_{i \in J_u(\phi,\xi)} [e_{ui}'(\phi, \xi)]^p + \sum_{i \in J_l(\phi,\xi)} [-e_{li}'(\phi, \xi)]^p \right)^{(1/p)-1} \cdot \left(\sum_{i \in J_u(\phi,\xi)} [e_{ui}'(\phi, \xi)]^{p-1} \nabla e_{ui}'(\phi, \xi) - \sum_{i \in J_l(\phi,\xi)} [-e_{li}'(\phi, \xi)]^{p-1} \nabla e_{li}'(\phi, \xi) \right). \quad (14)$$

For the second case all the $-e_{ui}'(\phi, \xi)$ and $e_{li}'(\phi, \xi)$ will be positive. To exceed the specification by the greatest amount, we might propose the following objective function to be minimized:

$$U(\phi, \xi) = -\left(\sum_{i \in I_u} [-e_{ui}'(\phi, \xi)]^{-p} + \sum_{i \in I} [e_{li}'(\phi, \xi)]^{-p} \right)^{-(1/p)} \quad (15)$$

for

$$-e_{ui}'(\phi, \xi) > 0, \qquad i \in I_u \quad (16)$$

$$e_{li}'(\phi, \xi) > 0, \qquad i \in I_l \quad (17)$$

and

$$p \geq 1.$$

Again, for larger values of p we would expect the maximum of the functions to be emphasized, since

$$\max_{i,j} [e_{ui}'(\phi, \xi), -e_{lj}'(\phi, \xi)] = \lim_{p \to \infty} U(\phi, \xi), \qquad i \in I_u, \; j \in I_l. \quad (18)$$

[1] It is important to note that the sets J_u and J_l are dependent on ϕ and ξ. Thus temporarily excluded sample points are immediately included when the corresponding errors violate the specification, and temporarily included sample points are immediately excluded when the corresponding errors satisfy the specification.

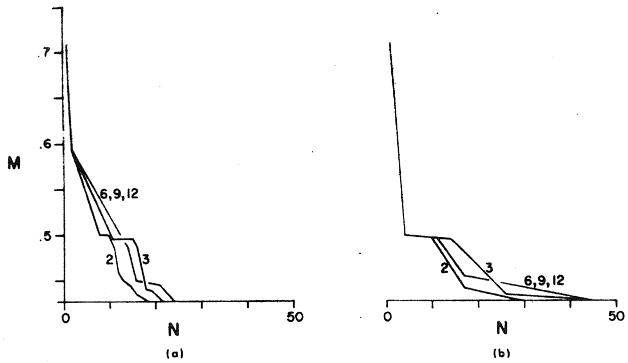


Fig. 5. Optimization from $Z_1=1.0$, $Z_2=3.0$. (a) Fletcher. (b) Fletcher–Powell.

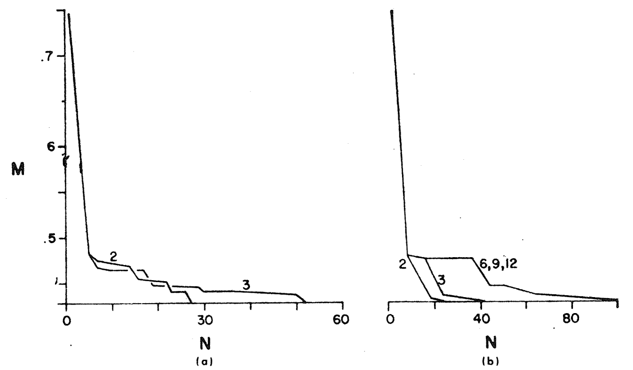


Fig. 6. Optimization from $Z_1=1.0$, $Z_2=6.0$. (a) Fletcher. (b) Fletcher–Powell.

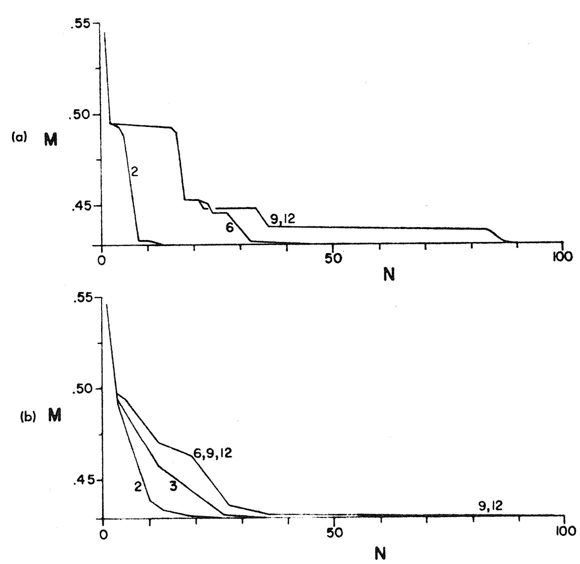


Fig. 7. Optimization from $Z_1=3.5$, $Z_2=6.0$. (a) Fletcher. (b) Fletcher–Powell.

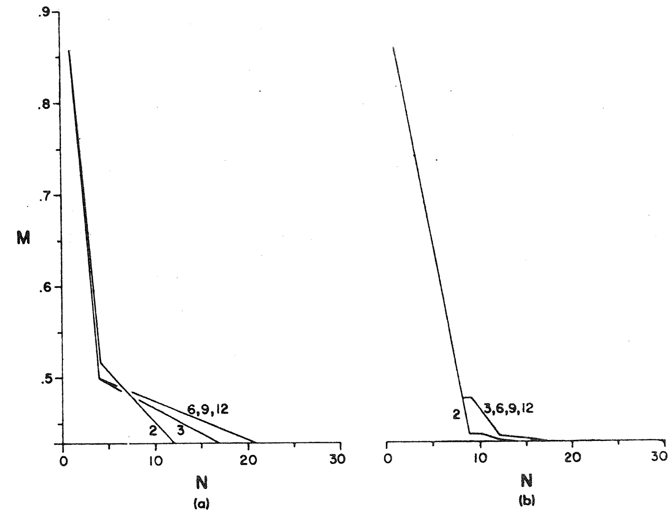


Fig. 8. Optimization from $Z_1=3.5$, $Z_2=3.0$. (a) Fletcher. (b) Fletcher–Powell.

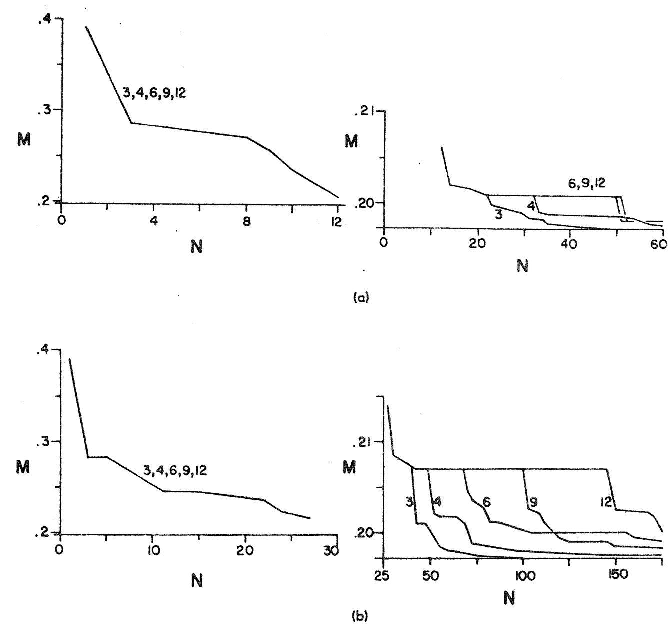


Fig. 9. Optimization from $Z_1=1.5$, $Z_2=3.0$, $Z_3=6.0$, $l_1/l_q=0.8$, $l_2/l_q=1.2$, $l_3/l_q=0.8$. (a) Fletcher. (b) Fletcher–Powell.

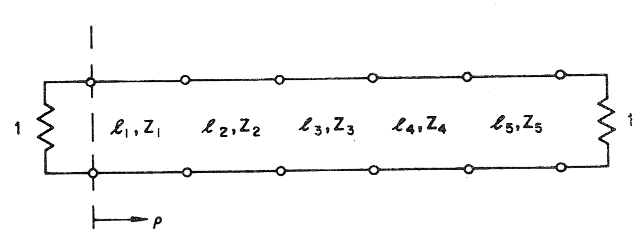


Fig. 10. 5-section transmission-line low-pass filter.

$\xi=0$. Since the response at zero frequency is independent of the parameters, and to avoid numerical difficulties, the frequency point 0.02 replaced 0 in problem 1.

Optimization using the Fletcher method in accordance with the foregoing ideas with $p=1000$ gave the results shown in Table III, where l_q is the quarter-wave value at 1 GHz.

The responses are depicted in Figs. 11–13. The final

TABLE I

OPTIMIZATION OF A 2-SECTION 10:1 QUARTER-WAVE TRANSFORMER OVER 100-PERCENT BANDWIDTH WITH VARIABLE CHARACTERISTIC IMPEDANCES Z_1 AND Z_2

Figure	Starting Point Z_1	Starting Point Z_2	n where $p=10^n$	Number of Function Evaluations N[a] Fletcher [9]	Number of Function Evaluations N[a] Fletcher–Powell [8]
5	1.0	3.0	2	22	31
			3	28	49
			6	33	56
			9	33	56
			12	33	56
6	1.0	6.0	2	30	26
			3	58	50
			6	b	133
			9	b	172
			12	b	198
7	3.5	6.0	2	15	23
			3	b	41
			6	44	101
			9	102	118
			12	102	118
8	3.5	3.0	2	14	16
			3	19	56
			6	21	75
			9	21	85
			12	21	310

[a] The number of N listed are those required to bring M within 0.01 percent of the known optimum value, namely, 0.42857.
[b] Missing entries are due to parameters becoming negative—constraints were not imposed during optimization.

TABLE II

OPTIMIZATION OF A 3-SECTION 10:1 TRANSFORMER OVER 100-PERCENT BANDWIDTH WITH VARIABLE LENGTHS AND CHARACTERISTIC IMPEDANCES

Starting Point: $Z_1=1.5$, $Z_2=3.0$, $Z_3=6.0$, $l_1/l_q=0.8$, $l_2/l_q=1.2$, $l_3/l_q=0.8$, where l_q is the quarter wavelength at center frequency

n where $p=10^n$	Number of Function Evaluations N to Reach the Value of M Shown in Brackets[a]; the Optimum Value of M is 0.19729: Fletcher [9]		Fletcher–Powell [8]	
3	57	(0.19734)	115	(0.19733)
4	86	(0.19730)	378	(0.19729)
6	418	(0.19729)	702	(0.19740)
9	634	(0.19730)	661	(0.19740)
12	668	(0.19736)	645	(0.19851)

[a] A time limit of 64 s/run was imposed, at which time the optimum for large p had still not been reached.

TABLE III

OPTIMIZATION OF THE CIRCUIT SHOWN IN FIG. 10 USING VARIABLE LENGTHS

Parameters	Starting Point	Problem 1	Problem 2
$\frac{l_1}{l_q}=\frac{l_5}{l_q}$	0.07	0.09593	0.09098
$\frac{l_3}{l_q}$	0.15	0.16278	0.18928
$\frac{l_2}{l_q}=\frac{l_4}{l_q}$	0.15	0.19798	0.15821

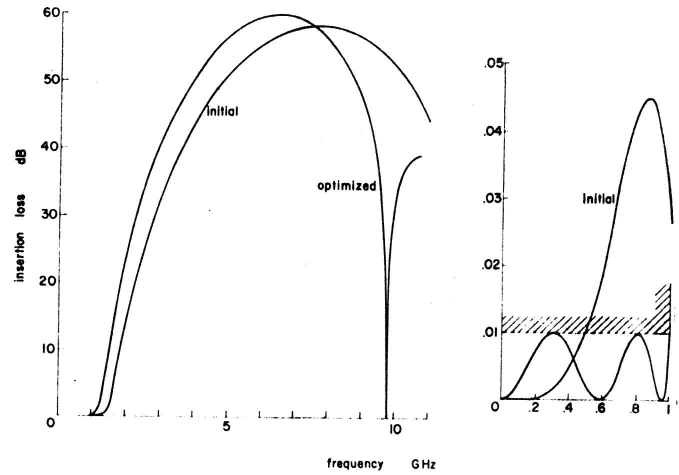


Fig. 11. Optimized response of the circuit of Fig. 10 subject to the constraints imposed for problem 1.

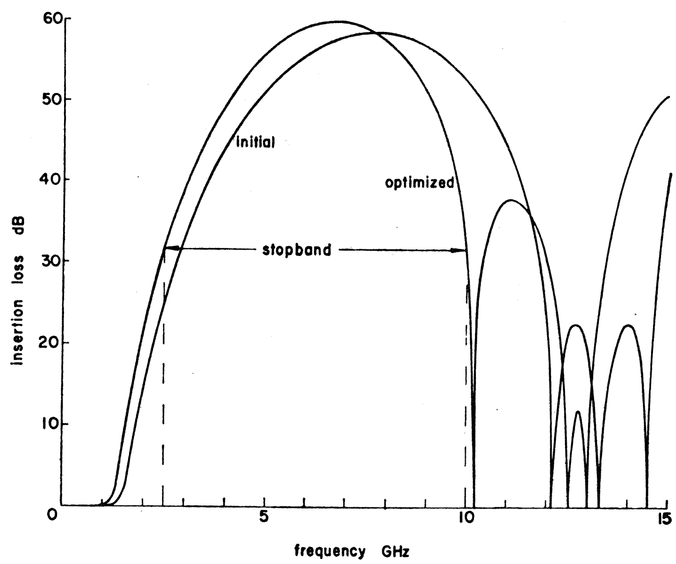


Fig. 12. Optimized response of the circuit of Fig. 10 subject to the constraints imposed for problem 2.

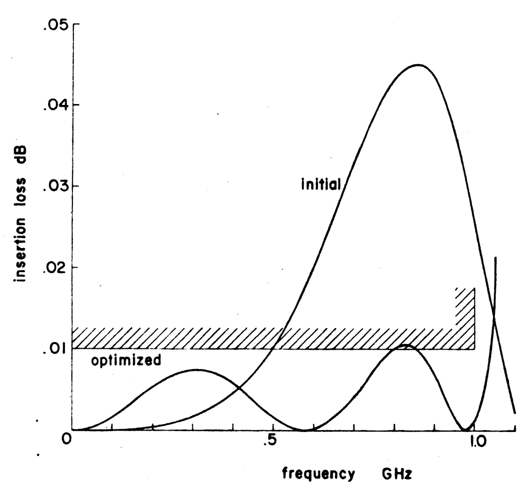


Fig. 13. Passband details of the optimized response shown in Fig. 12.

results are the same whether or not symmetry is assumed. Problem 2, for example, was solved using the Fletcher–Powell method without the symmetry assumption.

Three comments on the responses shown in Figs. 11–13 are in order. The first is that perfectly equal-ripple responses should not be expected in general nonlinear approximation problems, with or without constraints. The second is that, unless interpolation methods [3], [11] are used, actual extrema in the response errors will usually lie between adjacent discrete sample points. The third is that, in the present examples, slight deviations from the passband specification are to be expected, since the stopband specification is unattainable in practice.

IV. Discussion

From a minimax point of view ($p=\infty$), the value of the parameter ξ does not affect the location of the optimum. For finite values of p, however, it can play an extremely important role. It can be chosen, if desired, so that the M of (20) is always positive or, alternatively, always negative. In the first case an economy in gradient computation may be realized since only sample points satisfying the conditions in (10) and (11) are considered. This is a subset of all the possible sample points. In the second case all the sample points would generally have to be considered, but in our experience convergence to a good solution is usually faster. In this case, of course, we avoid the mild possible hazards mentioned in Section II encountered in the transition region when $M=0$.

Theoretically, if ξ is chosen such that $M=0$ is optimal, then a *finite* value of p will yield the minimax solution! In practice, a good estimate for ξ may allow relatively low values of p to yield results much closer to the minimax solution than a bad estimate. As Figs. 5–9 indicate the lower the value of p the faster M is reduced in the early stages of optimization. This is not unexpected since the minimization algorithms used are based on quadratic models. As p increases the objective function will generally deviate further and further from a quadratic form so that the algorithms will progressively slow down.

It is, incidentally, always good practice to monitor the current minimum value of M and the associated ϕ while U is being minimized, since a lower, and hence presumably preferable, value of M may be realized on the way than might prevail at the minimizing point for U.

V. Conclusions

An approach to computer-aided minimax design of microwave circuits employing highly efficient optimization methods has been presented. Typically, less than 1 min of CDC 6400 computer time is sufficient to optimize the type of examples given in this paper to a high degree of accuracy.

Other recent work on least pth approximation using very large p is the work by Fletcher *et al.* [15] on linear approximation problems, and the work by Bandler *et al.* [16] on optimum system modeling problems in the time domain. The latter paper, in particular, compares the grazor search method [12] with the present approach.

No attempts at modifying the minimization methods to improve convergence for extremely large values of p nor a detailed study of other possible effects of numerical ill-conditioning have as yet been carried out. But, if the success we have had is widely repeatable, then far-reaching consequences are foreseen, not only in nonlinear approximation, but in the closely related field of nonlinear programming [17].

Acknowledgment

The authors wish to thank Dr. R. Fletcher, Atomic Energy Research Establishment, Harwell, England, who made available his computer program [9]. They also wish to thank Dr. M. D. Hebden, formerly at McMaster University, now also at the Atomic Energy Research Establishment, Harwell, England, for useful suggestions on least pth approximation.

References

[1] G. C. Temes and D. Y. F. Zai, "Least pth approximation," *IEEE Trans. Circuit Theory* (Corresp.), vol. CT-16, pp. 235–237, May 1969.
[2] G. C. Temes, "Optimization methods in circuit design," in *Computer Oriented Circuit Design*, F. F. Kuo and W. G. Magnuson, Jr., Eds. Englewood Cliffs, N. J.: Prentice-Hall, 1969.
[3] J. W. Bandler, "Optimization methods for computer-aided design," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Computer-Oriented Microwave Practices*), vol. MTT-17, pp. 533–552, Aug. 1969.
[4] J. W. Bandler and R. E. Seviora, "Current trends in network optimization," *IEEE Trans. Microwave Theory Tech.* (*1970 Symposium Issue*), vol. MTT-18, pp. 1159–1170, Dec. 1970.
[5] J. W. Bandler and C. Charalambous, "Theory of generalized least pth approximation," *IEEE Trans. Circuit Theory* (Corresp.), vol. CT-19, pp. 287–289, May 1972.
[6] ——, "On conditions for optimality in least pth approximation with $p\rightarrow\infty$," *J. Optimiz. Theory Appl.*, to be published.
[7] J. W. Bandler, "Conditions for a minimax optimum," *IEEE Trans. Circuit Theory* (Corresp.), vol. CT-18, pp. 476–479, July 1971.
[8] R. Fletcher and M. J. D. Powell, "A rapidly convergent descent method for minimization," *Comput. J.*, vol. 6, pp. 163–168, June 1963.
[9] R. Fletcher, "A new approach to variable metric algorithms," *Comput. J.*, vol. 13, pp. 317–322, Aug. 1970.
[10] J. W. Bandler and P. A. Macdonald, "Cascaded noncommensurate transmission-line networks as optimization problems," *IEEE Trans. Circuit Theory* (Corresp.), vol. CT-16, pp. 391–394, Aug. 1969.
[11] ——, "Optimization of microwave networks by razor search," *IEEE Trans. Microwave Theory Tech.* (*Special Issue on Computer-Oriented Microwave Practices*), vol. MTT-17, pp. 552–562, Aug. 1969.
[12] J. W. Bandler, T. V. Srinivasan, and C. Charalambous, "Minimax optimization of networks by grazor search," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-20, pp. 596–604, Sept. 1972.
[13] G. L. Matthaei, L. Young, and E. M. T. Jones, *Microwave Filters, Impedance Matching Networks and Coupling Structures.* New York: McGraw-Hill, 1964, ch. 6.
[14] C. Brancher, F. Maffioli, and A. Premoli, "Computer optimization of cascaded noncommensurable-line lowpass filters," *Electron. Lett.*, vol. 6, pp. 513–515, Aug. 1970.
[15] R. Fletcher, J. A. Grant, and M. D. Hebden, "Linear minimax approximation as the limit of best L_p approximation," Atomic Energy Res. Estab., Harwell, England, TP470, Dec. 1971.
[16] J. W. Bandler, N. D. Markettos, and T. V. Srinivasan, "A comparison of recent minimax techniques on optimum system modelling," in *Proc. 6th Princeton Conf. Information Sciences and Systems* (Princeton, N. J., Mar. 1972), pp. 540–544.
[17] J. W. Bandler and C. Charalambous, "A new approach to nonlinear programming," in *Proc. 5th Hawaii Int. Conf. System Sciences* (Honolulu, Hawaii, Jan. 1972), pp. 127–129.

ΤΑ ΠΑΝΤΑ ΕΝ ΧΡΙΣΤΩΙ ΣΥΝΕΣΤΗΚΕΝ