A VIRTUAL FINITE ELEMENT METHOD FOR CONTACT PROBLEMS

A VITRUAL FINITE ELEMENT METHOD FOR CONTACT PROBLEMS

by

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ABSTRACT

An algorithm is presented for the solution of mechanical contact problems using the displacement based Finite Element Method. The corrections are applied as forces at the global level, together with any corrections for other nonlinearities, without having to nominate either body as target or contactor. The technique requires statically reducing the global stiffness matrices to each degree of freedom involved in contact. Nodal concentrated force are redistributed as continuous tractions. These tractions are re-integrated over the element domains of the opposing body. This creates a set of virtual elements which are assembled to provide a convenient mesh of the properties of the opposing body no matter what its actual discretizaton into elements. Virtual nodal quantities are used to calculate corrective forces that are optimal to first order.

The work also presents a derivation of referential strain tensors. This sheds new light on the updated Lagrangian formulation, gives a complete and correct incremental form for the Lagrangian strain tensor and illustrates the role of the reference configuration and what occurs when it is changed.

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NOMENCLATURE

Symbols based on Latin letters

$\bar{b}$	body force
c	compression of a spring
C^0	zeroth order Sobolev space
C^{-1}	negative first Sobolev space
D	quadratic form of deformation
E_F	equilibrium error
E	elastic part of friction - slippage constitutive relation
$\bar{\bar{E}}$	Lagrange pseudo-strain tensor
$\bar{f}$	force density
ℓ	redistributed contact traction
f	virtual nodal contact force
F	nodal force vector
F_P	penetration force vector
$\bar{\bar{F}}$	deformation gradient
$\overset{n+1}{g}$	arbitrary material function
g	gap function
$\bar{g}_a$	local tangent basis vectors of space

G	Lagrange coefficient matrix
h	slippage function
$\overset{n}{h}$	arbitrary material function
I	Identity matrix
$\bar{j}_a$	basis vectors
$\bar{j}^a$	inverse basis vectors
J	Jacobian
$\bar{\bar{J}}$	surface Jacobian
k	redistributed contact stiffness density
k	virtual nodal contact stiffness
K	stiffness matrix
K_P	penetratation stiffness matrix
K_T	tangent stiffness matrix
$\mathcal{K}$	kinetic energy density
K	kinetic energy
N_{el}	element ditribution matrix
N	global distribution matrix
$\overset{n}{p}$	arbitrary material function
p	penetration function
P	nodal penetration vector
$\overset{n}{q}$	arbitrary material function
q	arbitrary distributed (scalar) function
q	nodal values of q
Q_{el}	elemental contribution to global Q
Q	global nodal values of concentrated property corresponfing to q
R	nodal penetration gradient matrix

$\bar{\bar{S}}$ Second Piola - Kirchhoff Pseudo-stress tensor

$\bar{\bar{T}}$ First Piola - Kirchhoff Pseudo-stress tensor

$\bar{T}$ (pseudo-) surface traction

$\mathcal{T}$ surface work density

$\bar{\bar{u}}$ displacement gradient

$\bar{u}$ displacement

$\mathcal{U}$ strain energy density

U strain energy

U nodal displacement vector

$\bar{v}$ velocity

$\mathcal{V}$ volume work density

W work

$\bar{x}$ position

X body-point

Y body-point

Y yield function of friction - slippage elasto-plastic constitutive relation

$\bar{x}$ position

Symbols based on Greek letters

α	penalty matrix, penalty number
α_L	augmented Lagrange penalty matrix
β	agent property of body force
δ	a small number
δ_a^b	Kronecker delta
$\bar{\bar{\varepsilon}}$	Euler strain tensor
ε	strain
λ	Lagrange multiplier vector
μ	Coulomb coefficient of friction, friction matrix
μ_d	dynamic coeficient of friction
μ_s	static coefficient of friction
Π	potential energy
Π^*	modified potential energy
Π_C	contact potential energy
Π_F	frictional potential energy
Π_L	Lagrange potential energy
Π_P	proximity potential energy
Π_P	penetration potential energy
Π_S	separation potential energy
ρ	mass density
$\bar{\bar{\sigma}}$	true (Cauchy) stress tensor
ϕ	vector of shape functions
ζ	associative slippage parameter for elasto - plastic friction
ξ	slip potential in elasto - plastic friction

Ω	domain of a body or an element
$\partial\Omega$	surface of a body or an element

Other symbols

$\bar{\bar{1}}$	Identity tensor
$\hat{\circ}$	unit vector
$\circ$	not subject to differentiation
$\overset{\cdot}{\circ}^{sym}$	symmetric part
$\circ^{A}$	pertaining to body A
$\circ^{T}$	transpose
$\circ_{n}$	pertaining to the direction normal to contact
$\circ_{t}$	pertaining to a direction tangential to contact
$\circ^{-1}$	inverse
$\Delta\circ$	incremental change
$\dot{\circ}$	time rate
${}^{A}\circ$	pertains to body in configuration, C_A
$\dfrac{\partial \circ}{\partial \bar{x}}$	gradient
${}^{A,B}\circ$	gradient

$\frac{\delta \circ}{\delta \bar{x}}$	configurational derivative
$^{A;B}{}_{\circ}$	configurational derivative
$\sum$	assembly operator
$\overset{m}{\cdot}$	m^{th} order dot product

CHAPTER ONE

INTRODUCTION

1.1 INTRODUCTION

This thesis presents an adaptation of the popular finite element method for the purpose of solving contact problems. This immediately requires three explanations. One is about what is novel about this. Another is about the finite element method, how and why it is used. The other is about contact. What is contact? Why is it interesting? Where does it occur? Why is it useful or interesting or difficult to solve contact problems? Contact problems are those in which physical bodies touch each other. These are so common that it is surprising that they are so little understood. Examples include stones resting on another, every bolt, screw, nail or rivet holding parts together, the bearing of skeletal joints, the cutting or grinding of tools on workpieces, and many more than can be listed.

For practical purposes it is often not necessary to know the details of motions and stresses in the region of contact. It suffices to have some basic description or to use some assumptions to provide enough information to allow the problem at hand to be solved. For many

cases it is easy enough to carry out some experiment to discover whatever information was needed. In either case there was no need to do detailed calculations. But this is not always true. There are many times when there are advantages to a calculational approach to finding needed information. These include situations where an experiment is not desirable. This could happen for reasons of expense. Perhaps the materials or facilities needed are too costly. This could happen for reasons of safety. The experiment could be so dangerous that it could not be attempted without prior analysis to explore possible outcomes. The experimental approach may simply not be possible. The facilities to do the experiment may not exist. Nonetheless detailed knowledge of conditions at a contact interface may be desired. Detailed knowledge of stress and motion in an arthritic hip may lead to better understanding of the condition. Detailed knowledge of stresses and bearing areas in a cutting tools may lead to better designs to improve cutting technique and extend tool life. Detailed knowledge of contact conditions could be used to numerically simulate crash conditions and so allow 'experiments' to be done with complete safety. So, there is plenty of motivation to have a way to simulate contact.

Is this so difficult? The answer depends on the quality of the results you desire. Results that are exact to the extent of the mathematical theories of deformable bodies are almost unknown for contact problems. For most situations this defeats the skill of even the best of mathematicians. Approximate results are all that we can currently provide. Even these can be found in more or less accuracy and

detail. For problems where the geometry of the bodies involved is not extremely simple the only success at approximation has been through numerical techniques. There are many of these in use. Different ones have been developed to suit different classes of problems. Here we consider only the contact of two or more solid bodies. For studying the displacements and stresses of solid bodies the displacement based finite element method has often been very effective. This thesis presents a way to extend this usefulness and release some restrictions that exist in earlier works on contact problems.

1.2 REVIEW OF CLASSICAL WORK OF HERTZ

Timoshenko and Goodier [1] and Johnson [2] review the classical studies of Heinrich Hertz. The work of Hertz in the nineteenth century remains one of the fundamental reference points in the study of contact problems. It provides some analytical solutions to the problem of linearized contact over very small regions without friction for elastic materials. In particular, he approximated any surface by an ellipsoid with the same principal curvatures at the point of contact as shown in Figure 1.1. The case of contact at a concavity is represented by one or both radii of curvature being negative. For techniques to solve more complicated problems this is usually considered a simplified version and so later works attempt to demonstrate their validity and accuracy by

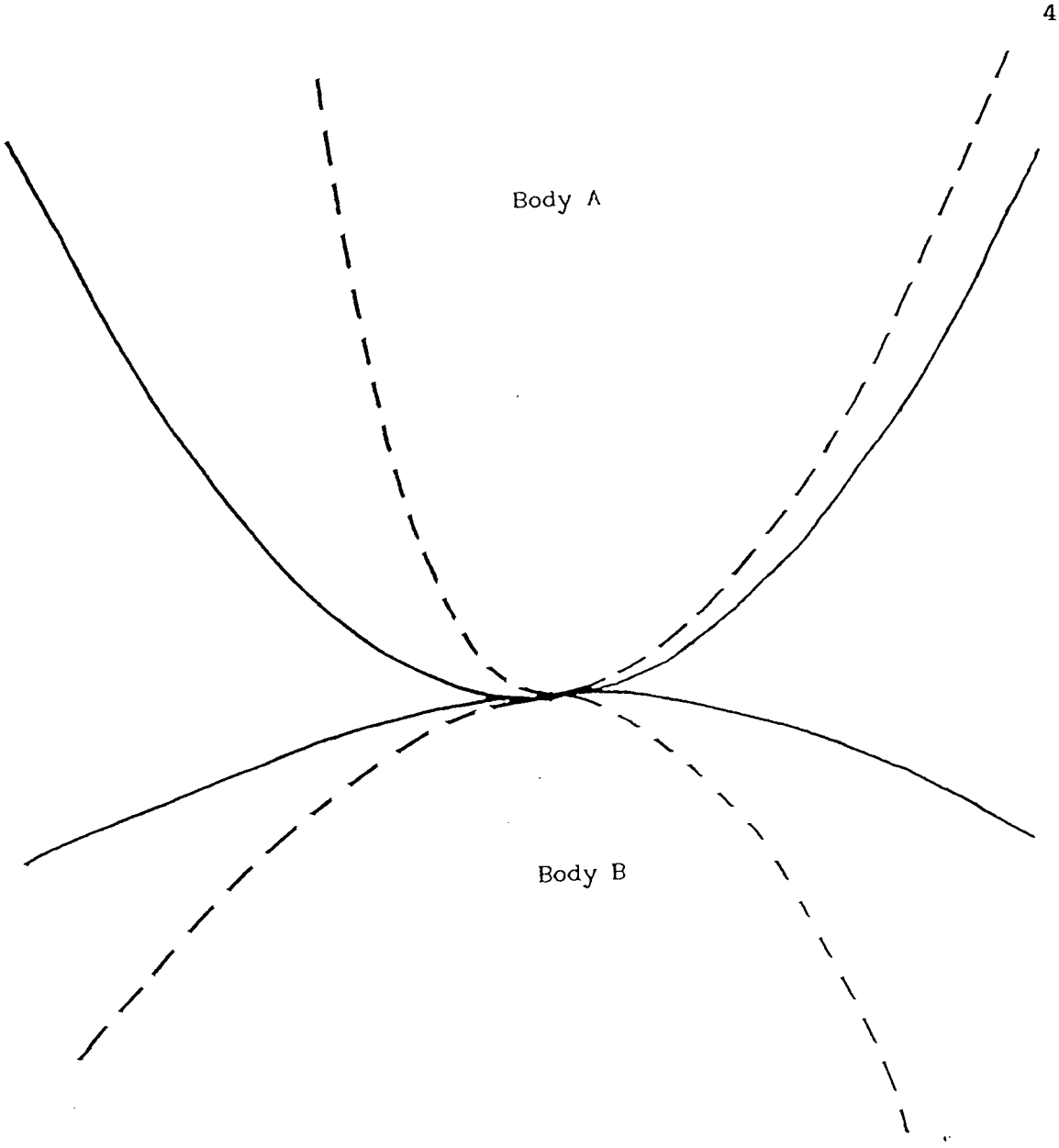


Figure 1.1 Hertz contact

— — — — Principal curves

presenting solutions to similar problems.

He showed that the contact region was an ellipse lying in a plane and that the pressure distribution was an ellipsoid. The total pressure integrated over the contact ellipse equilibrates the forces applied to the distant parts of the body that cause such motion as to result in contact. The peak pressure is 3/2 the average. The distribution of stresses can be described in closed form. The highest compressive stresses appear at the center of the contact. Maximum tensile stress appears at the edge of the contact ellipse in the outward direction. The maximum shear appears on the normal axis through the center of contact. The details of magnitudes and distribution of pressure and stress vary with the relative magnitudes of the radii of curvature and the angle between the normal planes containing the principal curvatures. Many tables have been constructed to provide calculational assistance to the analyst who would use this approach. Indeed this was almost the only approach until after the advent of the high speed digital computer.

This work is restricted in many ways. The semi-axes of the elliptical contact region must be small compared to the radii of curvature. The bodies must be linearly elastic. There can be no friction. Displacements as small and only in the direction normal to the plane in which the contact surface lies. All of these restrictions still leave a method that can be applied to a wide variety of practical problems where these conditions are satisfied. There are many other cases where these conditions are either not known to be satisfied or are

known not to be satisfied. These include any problem involving friction, any problem involving nonlinear material characteristics, any problem where the displacements are not small enough and all those cases where the contact is not over a small elliptical area.

1.3 INTRODUCE THE FINITE ELEMENT METHOD

The finite element method was developed to solve problems where similar difficulties arise. There are many problems of the motions and loads of continua or structures that were so complicated by geometric or material considerations that workers required numerical techniques to find adequate approximations. This is the case for almost all contact problems. A quick scan of the literature showed that contact had been investigated by the finite element method for a long list of applications.

The obvious favourites are punch problems (Chen and Tsai [3], Haber and Harandria [4], Shyu et al. [5], Chang et al. [6], Simo et al. [7], Jing and Liao [8], Ostachowitz [9], Chen and Yeh [10], Zhong and Sun [11], Gallego and Anza [12], Sachdeva and Ramakrishnan [13], Tsai and Chen [14], Bohm [15], Tsiang and Mandrell [16] and Fredriksson et al. [17]). Rolling has also drawn attention (Wong [18], Kulkarni et al. [19], Bhargava et al. [20 and 21]). There are problems of fasteners. Rivets (Torstenfelt [22] and Bertholf et al. [23]), pin joints (Rahman et al. [24] and

Mangalgiri et al. [25]), dove tail joints (Sachdeva et al. [26]) and similarly pin to piston rod problems (Hung and de Saxcé [27]) have all been investigated. Tires are the subject of much work (Padovan et al. [28], Tabbador [29], Chen and Yeh [30], Nakajima and Padovan [31], Purushothaman et al. [32], Zeid and Padovan [33] and Rothert et al. [34]). Some workers have investigated contact with foundations. The contact of beams with rigid curved foundations (Gu [35]) and plates with elastic or rigid foundations or edge supports (Mohr [36], Heinisuo and Miettinen[37], Chandrasekaran et al. [38] and Talaslidis and Panagiotopoulos [39]) have been looked at several ways. The regular mechanical engineering interference problems have been tried too: bearings (Cheng et al. [40] and Torstenfelt [22]), gears (Vijayakar et al. [41]), compound cylinders (Hsu and Bertels [42]) and the fitting of shafts to sleeves and hubs (Okamato and Nakazawa [43], Francis [44], and Wilson and Parsons [45]). Civil engineering problems of the behaviour of pilings (Herrmann [46]) and reinforcing steel in concrete (Mehlhorn et al. [47] and Hsu and Bertels [42]) have benefitted from numerical simulation of contact aspects. Military engineers have used such algorithms to simulate a wide variety of contact, penetration and perforation problems (Chan and Tsai [3], Chen and Yeh [10], Asano [48 and 49], Doi and Naoki [50], Bertholf et al. [23], Garnet and Armen [51] and Johnson et al. [52 and 53]).

Various special examples appear in the literature: Wong [18] investigates paper handling, pipeline closures are the subject of Liu and Huston [54], Mazurkiewicz and Ostachowitz [55] study one disk

pressed flat face to flat face on another, agricultural produce handling is the application of Rumsey and Fridley [56], reflector assembly in a fast breeder reactor changing shape due to operating conditions is the special interest of Zolti [57], Stadter and Weiss [58] investigate a radioisotope thermoelectric generator , strength of brittle shells is examined in the work of Senyushenkov [59] and the flow of visco-plastic materials is elucidated by Bohatier and Chenot [60].

Sometimes the purpose in studying contact is to allow the study of something else. Tribology testing has been simulated by Ihara et al. [61 and 62] and several workers in the problems of fracture mechanics have used contact to simulate frictional load support across cracks (Mehlhorn et al. [47], Komvopoulos [63], Yagawa and Hirayama [64], Bastias et al. [65], Tsiang and Mandell [16] and Fredriksson et al. [17]). The validity and usefulness of proposed friction laws is another use (Oden and Pires [66], Klarbring [67], Plesha et al. [68], Tseng and Olsen [69] and Francis [70]). Finally, many such algorithms claim to be for 'general purposes' and frequently use Hertzian contact to demonstrate their capabilities (Osmont [71], Oden and Pires [66], Pascoe and Mottershead [72], Chang et al. [6], Jing and Liao [8], Okamoto and Nakazawa [43], Endahl [73], Rahman et al. [24], Torstenfelt [22], Hung and de Sacxé [27], Stadter and Weiss [58], Rumsey and Fridley [56], Chen and Yeh [10], Chandrasekaran et al. [38], Bhargava et al. [20 and 21], Sachdeva et al. [26] and Gu [74]).

1.4 INTRODUCE DIFFICULTIES

To explain the difficulties in solving this class of problems it is best to begin with a review of some aspects of finite elements as used to study solids and structures. Usually we pose the problem as the minimization of potential energy, Π, over some region. The information needed to do this is divided into two categories: interior and exterior.

The exterior information applies at all points on the boundary of the region and at any other points where it is known. Later we will see that a priori knowledge of exterior information about a point effectively makes that an exterior point. Exterior information is the a priori knowledge of the stress or the displacement. These are incompatible quantities. It is possible to predetermine only one or the other at any one point. In practice it is often the case that the majority of the exterior information is of the stress type. This is so for cases where it is known that there is no stress acting over much of the boundary. So the exterior information there is that the stress is predetermined to be zero and the displacements are unknowns to be determined.

Nonetheless, we wish to find both kinds of exterior information, and possibly other quantities, at all points in the domain of our problem. For this we need the interior information. This allows us to relate the pointwise quantities all over the structure and to relate the two kinds of exterior information to each other. The interior information for the analysis of solids and structures is the

constitutive properties of the material (including inertial properties for dynamic analysis). This gives a relation between the stresses, and so the loads, and strains, and so the displacements. This is allows us to solve for the exterior information that we sought but could not predetermine.

The minimization problem for potential energy is very difficult if we insist on solving it exactly. It involves integrals over the domain of derivatives of the exterior information. These integrals may be very difficult to evaluate because of the shape of the domain. In the case of analyses that involve large displacements the final shape of the domain may not even be known at the outset. The derivatives may also be difficult to evaluate given extremely complex behaviour of some materials. On the other hand, we are usually more interested in getting a solution at all than we are in insisting that the solution be exact. An adequate approximation will always serve engineering purposes since it is impossible to build anything exactly and no material is truly represented by the idealizations that our equations represent. Having acknowledged the need and acceptability of approximate solution methods, it is still necessary to design one. One of the most popular is the finite element method. This is because of the ease with which it can be used in analyses involving complex geometry and the straight forward way in which it handles most problems. One reason given for using the finite element method: If both the interior and exterior information are constants fixed throughout the analysis, then the usual displacement based finite element method converts the the minimization problem into a

linear algebraic one. An algebraic problem may still be large and difficult, but there are well developed numerical techniques for handling them. Thus the problem has been rendered tractable.

There were two conditions laid down for linearity. If the interior information changes during analysis, then we have a problem that we call materially nonlinear. The commonest and most important example of which is the analysis of plastic deformations. This can be illustrated by considering a bilinearly elasto-plastic material loaded cyclically past its elastic limit. Each time the stress at any point in the material passes the (current) elastic limit the constitutive behaviour in that neighbourhood changed. Each time the differential change in stress became negative the constitutive response changed again. These changes in the stress - strain relation cause changes in the linear equations that describe the converted minimization problem. So, the conversion has to be repeated to represent the actual conditions at that load. There is a rich literature dealing with materially nonlinear problems. It is not our purpose to do more than give a brief description here.

The other condition for the minimization problem to become a linear algebraic one is that the exterior information must not change during the analysis. There are several ways in which such changes can occur. It may happen in some analyses that the exterior information is a function of load or displacement or time. In that case it is usually not possible to make a linear analysis. In many cases it is possible to use a sequence of linear analyses, each of which represents some portion

of the problem to be analysed. In the case of large displacement problems the shape of the solution domain is one of the variables to be found. This requires a truly nonlinear analysis. Like many nonlinear problems this can often be handled by converting the differential form of the problem to an incremental form. This has received wide attention and is now commonly done with various Lagrangian formulations.

One of the cases where the external information changes is the contact problem. Here the exterior information can change abruptly part way through the analysis of a problem. This leads to strong nonlinearities in the solution. Further there is a change in the nature of the exterior information. We no longer necessarily have foreknowledge of either the stresses or the displacements. The style of information on which we have relied to establish our substitute algebraic problem is simply not available. Instead of having one or the other of the two kinds of information that we use outside the areas of contact, we have a combination of two pieces of information that are appropriate inside the region of contact. These are based on physical arguments: the Principle of Impenetrability. Any point in space may be outside of both bodies, in one body or the other, or on the contact surface where the bodies touch, but never in the interior of more than one body. The Principle of Equal and Opposite Reaction: At any point on the contact surface the stress exerted on the surface of one body is equal and opposite to the stress exerted on the surface of the touching body. Clearly these two principles are not independent. The first affects displacements and thereby the strains. This brings about change in the

stresses through the constitution of the material. So it interferes with the free operation of the second principal. A similar chain of logic shows how the principle of equal and opposite reaction interferes with the free operation of the first principle. These two principles contain all of the information needed to solve the contact problem. It remains to find a way to express them mathematically and to put that to use.

1.5 INTRODUCE NUMERICAL APPROACHES

All attempts to express these principles and use them can be lumped into three main branches. These are those based on the use of Lagrange multipliers, those based on the penalty methods and the direct methods. The minimization problem can be stated as:

> Find the value of the nodal displacement, U, such that the potential energy is minimized while not violating the Principles of Impenetrability or Equal and Opposite Reaction.

The first part is easily expressed as

$$\text{Find} \quad \text{Min } \Pi(U), \quad \Pi = \frac{1}{2} U^T K U - U^T F \qquad (1.5.1)$$

The second part is more difficult. How does one express the Principle of impenetrability? There are two common approaches to this. In one the separation of the bodies is measured at some time and it is required that the displacements not exceed this gap function. The gap function may be established at the outset or re-evaluated at every iteration. The other common approach is to detect penetration by inspection and then do something about it. The first approach is very commonly used in the solution attempts using Lagrange multipliers.

1.6 LAGRANGE MULTIPLIERS

To show the flavour of this approach consider first the case of frictionless contact between a linearly elastic deformable body and a fixed, rigid barrier. The initial distance between the elastic body and the barrier is described by a gap function, g. As any point in the body approaches the barrier the value of g for that point decreases and reaches zero as the body-point just touches the barrier. If through some arithmetic violation of the laws of physics the body continued forward and penetrated the barrier, then the gap function would become negative. So, it is required that the gap function remain non-negative for all body-points in all displacements. That is,

$$g(\bar{x}) \geq 0 \qquad (1.6.1)$$

where g is defined only for $\bar{x}$ lying on the surface of the deformable body. To enforce this condition multiply g by a free variable, λ, the Lagrange multiplier. Add this to the original potential. The modified potential, Π^*, is

$$\Pi^* = \Pi + \Pi_L \tag{1.6.2}$$

where

$$\Pi_L = g^T\lambda \tag{1.6.3}$$

$$\Pi^* = \frac{1}{2} U^T K U - U^T F + g^T\lambda \tag{1.6.4}$$

It is also convenient to express g as a Taylor series in U, up to linear terms as shown in eqn(1.6.5).

$$g(U) \simeq g_o + \frac{\partial g}{\partial U} U \tag{1.6.5}$$

or

$$g(U) \simeq g_o + GU \tag{1.6.6}$$

where G is the derivative of the gap function with respect to the nodal displacements. Now minimization of Π^* with respect to both the nodal displacements, U, and the Lagrange multipliers, λ, gives two equations.

$$\Pi^*(U,\lambda) = \frac{1}{2} U^T K U - U^T F + (g_0 + GU)^T \lambda \qquad (1.6.7)$$

$$\frac{\partial \Pi^*}{\partial U} = K U - F + G^T \lambda = 0 \qquad (1.6.8)$$

$$\frac{\partial \Pi^*}{\partial \lambda} = G U + g_0 = 0 \qquad (1.6.9)$$

Equations (1.6.8) and (1.6.9) can be expressed in matrix form as

$$\begin{bmatrix} K & G^T \\ G & 0 \end{bmatrix} \begin{bmatrix} U \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ -g_0 \end{bmatrix} \qquad (1.6.10)$$

Satisfaction of this equation will automatically impose the needed displacement constraints. That is the effect of the second line in the above equation. At the same time it will not disturb the energy of the actual solution, since such a term only need be added for those nodes which would actually make contact. For such a node the final value of the gap is zero, so the satisfaction of these constraints adds zero to the original potential.

Clearly there are some disadvantages to this style of imposing constraints. Now for every node to be constrained there is an extra variable to find in passing through the linear equation solver. There is also a block of zeroes on the main diagonal, which will disturb many

available programs.

What is to be gained by solving for all these extra variables? As is often the case in Lagrange multiplier techniques the multiplier is itself a quantity that can be interpreted physically. In this case it is the contact pressure. We have gained the value of the contact pressure at every node so constrained. This gives us further insight into how to reform the above for the case of two deformable bodies in contact. We can define a contact potential, Π_c, which represents the work done by the contact pressures. The gap function is zero everywhere the surfaces mate. The contact pressure is zero everywhere that the surfaces are separate. Physically no work can be done by such loads, so the original potential remains numerically undisturbed.

Once again start with the gap function, g, to represent the separation of the two bodies which will move into contact. Express g as a Taylor series to linear terms in the nodal displacements of each body in contact.

$$g(U^A, U^B) \simeq g_o + \frac{\partial g}{\partial U^A} U^A + \frac{\partial g}{\partial U^B} U^B \qquad (1.6.11)$$

$$g(U^A, U^B) \simeq g_o + G^A U^A + G^B U^B \qquad (1.6.12)$$

Post multiply g by λ and call this the contact potential, Π_c

$$\Pi_c = g^T(U^A, U^B)\,\lambda \qquad (1.6.13)$$

since the gap function is only zero where the surfaces mate. The potential to be minimized is the sum of the mechanical and contact potential. This gives

$$\frac{\partial \Pi^*}{\partial U^A} = K^A U^A - F^A + G^{AT}\lambda = 0 \qquad (1.6.14)$$

$$\frac{\partial \Pi^*}{\partial U^B} = K^B U^B - F^B + G^{BT}\lambda = 0 \qquad (1.6.15)$$

$$\frac{\partial \Pi^*}{\partial \lambda} = G^A U^A + G^B U^B + g_o = 0 \qquad (1.6.16)$$

Equations (1.6.14), (1.6.15) and (1.6.16) can be combined as

$$\begin{bmatrix} K^A & 0 & G^{AT} \\ 0 & K^B & G^{BT} \\ G^A & G^B & 0 \end{bmatrix} \begin{bmatrix} U^A \\ U^B \\ \lambda \end{bmatrix} = \begin{bmatrix} F^A \\ F^B \\ -g_o \end{bmatrix} \qquad (1.6.17)$$

There are many variations on this. As presented so far, this is only suitable for analyses involving small displacements and linear elastic materials. If the displacements become large, or if there are material nonlinearities or if there is friction involved it becomes

necessary to change to an incremental approach. The basic idea is much the same. The subject of friction is a complex one and will be discussed later.

Incremental approaches are frequently required even if the displacements are small and the materials are linear. This is because of strong nonlinearity in the gap function. Especially for cases where there is little curvature of the surfaces in the contact region, the distribution of nodes free, sliding or sticking can vary dramatically even within iterative corrections to a small load step.

Among the Lagrange multiplier methods there can be a lumping into those which concentrate on the gap function and derive values for G by geometric arguments and those which concentrate on the contact potential and derive G as an integral over the contact area of derivatives of Π_c with respect to the nodal displacements.

Among those that use the geometric approach the discussions range mostly on the topic of the difficulties associated with sliding motion with friction. Ordinarily this leads to extra terms being added to the contact forces to represent the effects of friction. The result is that the system matrix becomes asymmetric. This is unfortunate since this doubles the computer storage requirements. Okamoto and Nakazawa [43] ameliorate this somewhat by a node matching scheme. Each node in contact in one body is explicitly identified with a node in the other body. In that case only one contact force needs to be calculated to guarantee impenetrability and equilibrium. The restriction that this imposes is that such a technique can only be used for situations where

relative motion of matched nodes is very small. If the matched nodes were to separate by a distance comparable to the size of the mesh then the whole problem would need to be rediscretized, reintegrated and the new nodes matched. The utility of this approach is restricted in much the same way as the Hertz solution. The advantages are that it is not necessary to restrict the contact to small ellipsoid shapes and that friction is included. Examples of punching at small displacement are provided. Pascoe and Mottershead [72] address the problem of asymmetry of storage. The advantage in reducing the requirement for computer storage would be important for many users. It is proposed that for small incremental steps it is acceptable to replace the displacement constraints with the transpose of the force constraints. Since the force constraints include extra terms for friction the contact forces are not normal to the contact surface. Likewise this attempts to make the nodes slide along a surface that is not tangent the to physical surface. The tangent of the angle between the two surfaces is the coefficient of friction. It is pointed out that for surfaces with little curvature this is easily corrected in later iterations and for surfaces with great curvature the steps have to be so small that the errors of this method are not a problem.

The group of techniques that integrate the contact potential explicitly have much the same experience. The integrals are usually taken over the faces of the elements in contact rather than over some independent discretization but may use different shape functions. These are often referred to as contact elements and often called mixed because

of the explicit representation of stresses. The results tend to oscillate about the values found by calculation for Hertzian problems. For other examples where the contact stress distributions are not such simple shapes the oscillations grow worse as the curvature of the stress increases. Chen and Tsai [3 and 14] start from the beginning and form a Hamiltonian for dynamic problems with contact and friction. This is done with two sets of Lagrange multipliers: one for impenetrability and one for sliding friction. The resulting matrix equations are asymmetric. The examples provided are for very simple punch problems of small displacement.

1.7 PENALTY METHODS

The penalty methods take a different approach. Essentially these are attempts to change the problem from kinematically nonlinear to materially nonlinear. Instead of trying to build the equations so that only admissible solutions will appear, they attempt to detect and correct errors as they occur. This is done by associating a large penalty of energy with committing the error. The penalty is set up in such a way that nodal motions toward an admissible solution decrease the penalty to be paid and so minimizing the energy of the problem tends to lead to acceptable results. Unlike the Lagrange multipliers the energy of the problem is disturbed. The more closely the approximate solution

matches the true constrained solution the less energy is caught in the penalty terms. Opposing this is the tendency of these methods to require large energy penalties to work, so some price in energy is paid even for small errors. For contact there are two main approaches to applying these penalties: the filled gap and the penetration approaches. They differ in where they allow compatibility errors to be committed. The filled gap techniques fill the space between the bodies with a fictitious material. After displacement a thin sheet of this material remains between the bodies. The penetration detection methods apply corrective forces proportional to the depth of penetration. After displacement only small penetrations are left because large forces prevent greater violation of the compatibility constraints.

The filled gap methods rely on a proximity potential which is modelled as the strain energy of fictitious material filling the space between bodies. Different version of the filled gap approach are based on different ways to discretize this material into elements or different choices for material properties. In the simplest version of the filled gap method, Osmont [71], watch is kept for pairs of nodes about to touch. When the nodes are very close together a short stiff spring is inserted between them. This changes the stiffness matrix of further iterations so it will have to be factored again to continue with the solution. Also, the pair of nodes are explicitly coupled. So it is no longer possible to keep the stiffness matrices of the two bodies separate. It is assumed that neither of these are too great a difficulty. In other versions the gap material can be quite

sophisticated. Zolti [57] points out that while the mechanical properties are fictitious the chemical, thermal and other properties may represent real conditions in the gap.

The constitutive properties in the direction across the gap may be highly nonlinear. At a strain of -1 the element has collapsed and the real elements on either side are touching. The normal stress strain relation is such that until the strain approaches -1 the material offers essentially no resistance. At strains very close to -1 the modulus of the material suddenly becomes very large. The range of this property may be several orders of magnitude. The more sudden is the change the more closely the true kinematic constraints are modelled. On the other hand, a more gentle transition of normal modulus will require less iterations in the nonlinear equation solver. An acceptable stress strain curve is still a matter of trial and error.

Constitutive properties in the directions tangent to contact can be arranged to imposed sticking or sliding friction, or no friction at all. There are many ideas on how this should be done.

As an example consider the bodies in Figure 1.2. The gap between them is filled with a fictitious material that has no shear strength and the elastic modulus in across the gap direction is shown in Figure 1.3. The proximity potential, Π_p, is added to the mechanical potential, Π, and the modified potential, Π^*, is minimized to give the tangent incremental equations. Group the nodal displacements as free, marked with subscript f, or contact, marked with subscript c. The tangent incremental equations are

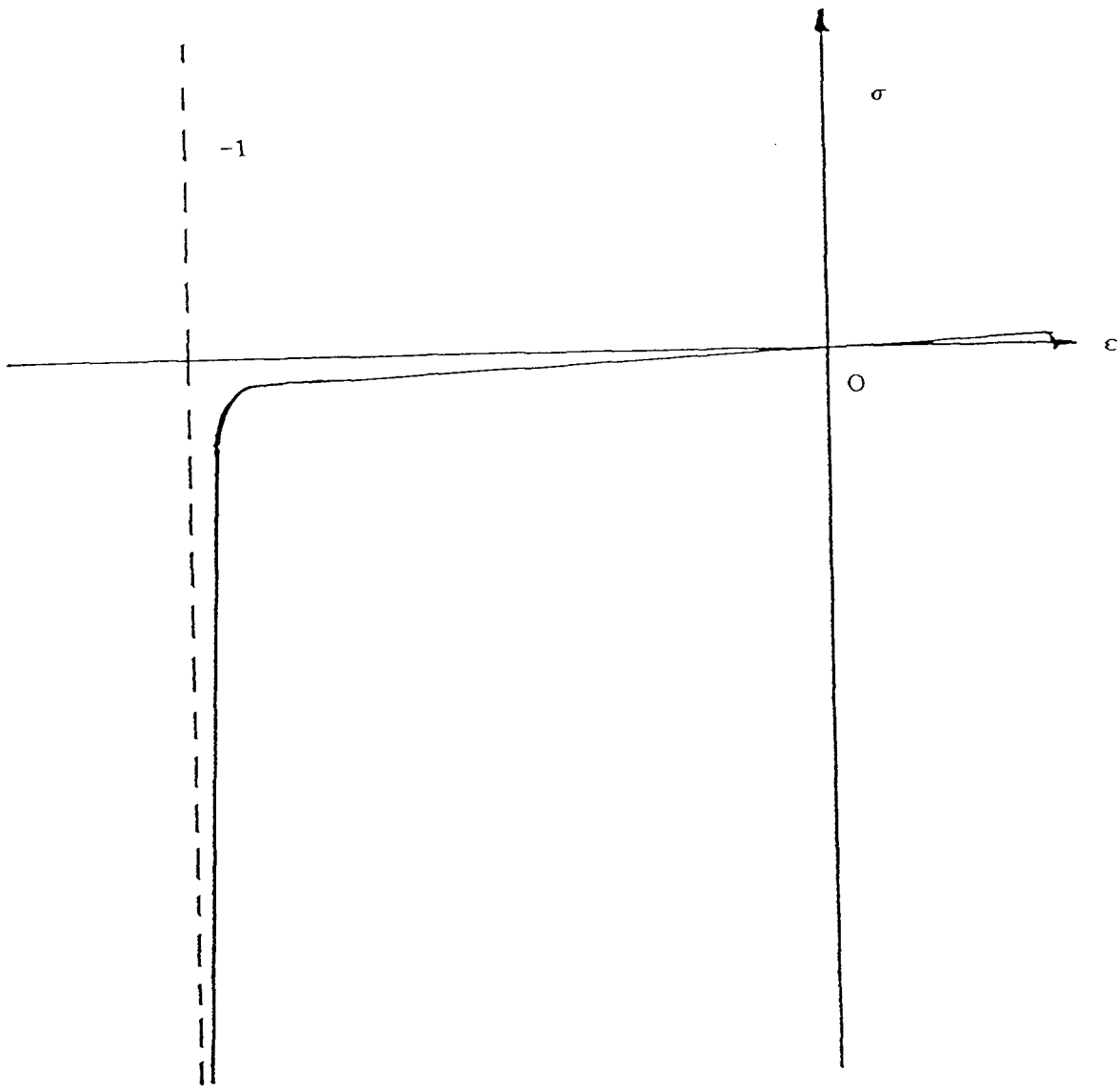


Figure 1.3 Constitutive properties of gap filling material in the direction normal to contact

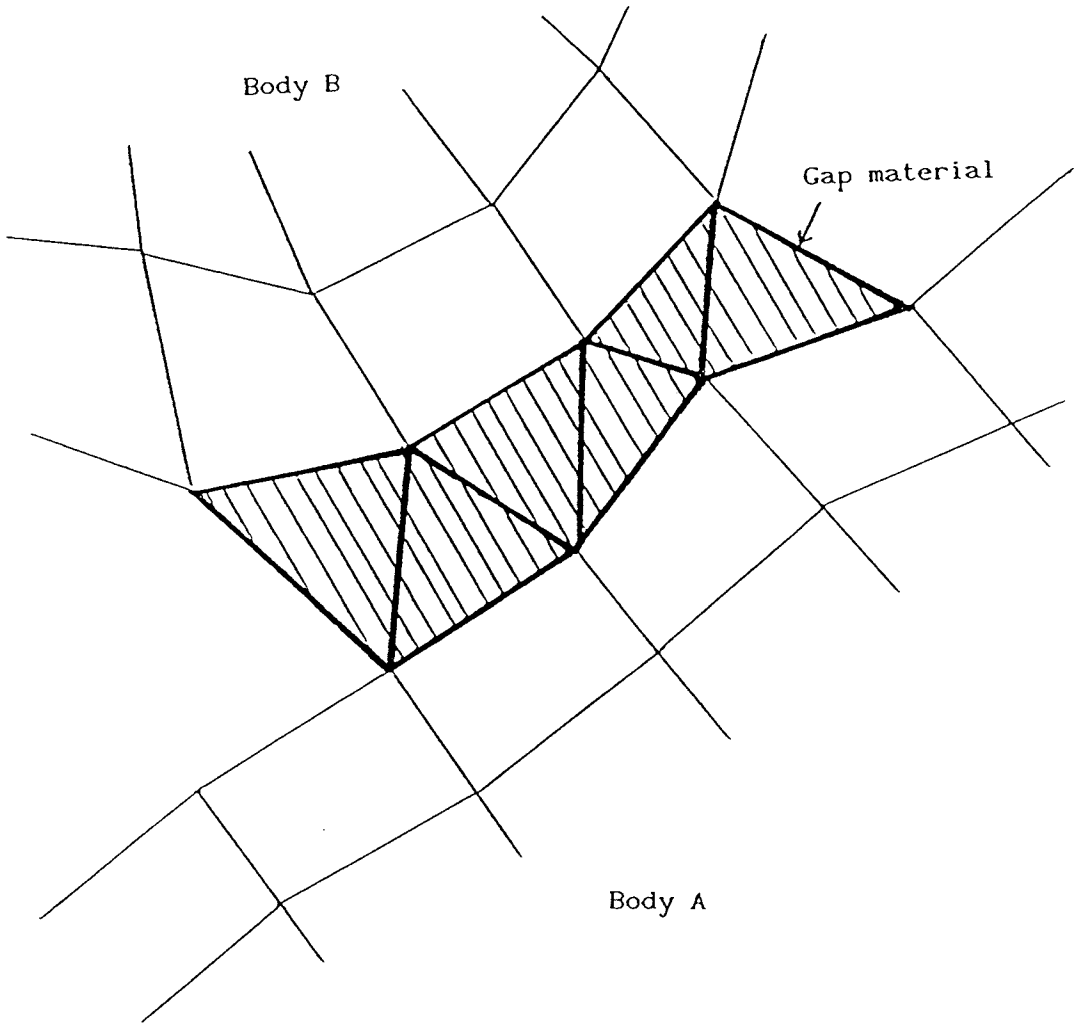


Figure 1.2 Use of gap filling material and gap filling elements

$$\begin{bmatrix} K^{A}_{ff} & K^{A}_{fc} & & 0 & & 0 \\ K^{A}_{cf} & K^{A}_{cc} & + K^{AA}_{TP} & K^{AB}_{TP} & & 0 \\ 0 & K^{BA}_{TP} & & K^{B}_{cc} & + K^{BB}_{TP} & K^{B}_{cf} \\ 0 & 0 & & K^{B}_{fc} & & K^{B}_{ff} \end{bmatrix} \begin{bmatrix} \Delta U^{A}_{f} \\ \Delta U^{A}_{c} \\ \Delta U^{B}_{c} \\ \Delta U^{B}_{f} \end{bmatrix} = \begin{bmatrix} \Delta F^{A}_{f} \\ \Delta F^{A}_{c} \\ \Delta F^{B}_{c} \\ \Delta F^{B}_{f} \end{bmatrix}$$

(1.7.1)

This method of coupling is only suitable for small relative displacement without rediscretizing the gap material into new elements that reflect the current arrangement of the nodes and elements of the real bodies.

The very simple approach of Osmont [71] is perhaps the most basic version of such a method. The method has been improved by Wong [18] to allow different stiffnesses in the springs in compression and tension. This reflects the behaviour of paper handling in that some tensile forces can be supported once contact has been made. Padovan et al. [28] also use node to node springs in its simulation of steady state rolling contact. These springs have variable stiffness in the normal direction and a rule to allow them to simulate Coulomb friction in the tangential direction. This improves the dynamic behaviour of the solution. In the work of Zolti [57] there is a simple gap material that has two values for its normal modulus. The modulus in open conditions is negligible. The modulus in overlap conditions is enormous. In this it is different from most filled gap methods and resembles the penetration methods. The way in which overlap is detected

is to check the Jacobian of the gap element. Where ever it is negative overlap has occurred. On the next iteration that integration point will use the closed values for material properties. A more sophisticated gap material is used by Stadter and Weiss [58] to simulate 2D contact problems. In this analysis the modulus in the normal direction was adjusted after each iteration. The normal strain is kept within a tolerance, δ, of -1 (complete collapse). For normal strains representing less compression than the allowed tolerance the modulus was decreased. For larger strains that represent unacceptable penetration the modulus was increased. When all contact strains were within the tolerance or essentially zero the step was considered to have converged. Numerical experiment showed that tighter tolerance on the allowed strains lead to improved performance in the simulation of Hertz problems. A value of 1% for δ leads to quite acceptable results of much of the contact area. Unfortunately the strains and stresses tend to oscillate just at the edge of the contact region. This particular analysis represents an interesting variation in that penetration errors have been allowed as well as gap errors. The only commentary available on the accuracy, stability or other computational aspects is that smaller values of δ require more iterations.

The penetration methods add a penetration potential, Π_p, to the mechanical potential, Π. This penetration potential is proportional to the square of the penetration depth, P. The proportionality constant is the penalty factors, α. The penetration depth is treated similarly to the gap used in the Lagrange multipliers. It is useful to express it as

a function of the nodal displacements. So

$$p(U^A, U^B) \simeq p_0 + \frac{\partial p}{\partial U^A} U^A + \frac{\partial p}{\partial U^B} U^B \qquad (1.7.2)$$

$$p \simeq p_0 + R^A U^A + R^B U^B \qquad (1.7.3)$$

where R is the derivative of the penetration depth with respect to the nodal displacements. The penetration potential is Π_p where

$$\Pi_p = \frac{1}{2} p^T \alpha p \qquad (1.7.4)$$

$$\Pi_p = \frac{1}{2} [p_0 + U^A R^A + U^B R^B]^T \alpha [p_0 + R^A U^A + R^B U^B] \qquad (1.7.5)$$

Now differentiation of the total modified potential gives

$$\frac{\partial \Pi^*}{\partial U^A} = K^A U^A - F^A + F_p^A + K_p^{AA} U^A + K_p^{AB} U^B = 0 \qquad (1.7.6)$$

$$\frac{\partial \Pi^*}{\partial U^B} = K^B U^B - F^B + F_P^B + K_P^{BA} U^A + K_P^{BB} U^B = 0 \qquad (1.7.7)$$

where
$$F_P^A = p_0^{AT} \alpha R^A \quad , \quad F_P^B = p_0^T \alpha R^B$$
$$K_P^{AA} = R^{AT} \alpha R^A \quad , \quad K_P^{BA} = R^{BT} \alpha R^A \qquad (1.7.8)$$
$$K_P^{AB} = R^{AT} \alpha R^B \quad , \quad K_P^{BB} = R^{BT} \alpha R^B$$

The penalty matrix, α, is diagonal with each node's penalty as the diagonal entry if the penetration is positive for that node. That is

$$\alpha = \text{diag}(\alpha_i) \qquad (1.7.9)$$

where
$$\alpha_i = \begin{bmatrix} \alpha_n & 0 & 0 \\ 0 & \alpha_n & 0 \\ 0 & 0 & \alpha_n \end{bmatrix} \quad \text{if } p > 0, \quad 0 \text{ otherwise} \qquad (1.7.10)$$

So the system of equations to be solved is

$$\begin{bmatrix} K^A + K_P^{AA} & K_P^{AB} \\ K_P^{BA} & K^B + K_P^{BB} \end{bmatrix} \begin{bmatrix} U^A \\ U^B \end{bmatrix} = \begin{bmatrix} F^A - F_P^A \\ F^B - F_P^B \end{bmatrix} \qquad (1.7.11)$$

The penalty terms get activated for any node in contact and are zero for

the rest. Because the penalty terms are large the system matrix has similar mathematical difficulties as problems that involve nonlinear materials with rapidly varying constitutive relations. Such analyses also occasionally have problems of ill conditioning in the system matrix if the penalty terms are too large.

For the simpler schemes of Cheng et al. [40] the penetration function is found by a node matching method much like that used in the simple filled gap methods. So long as the matrices R^A and R^B can be found from the geometry of the bodies, there is no need to actually integrate the penetration potential. In the literature this is universally the case. Yagawa and Hirayama [64] fill the space between the bodies with eight noded brick elements. These are not filled with contact material, but are used to establish the value and derivatives of p.

The contact forces acting in directions tangential to the surface can be treated by penetration methods too. Ostachowicz [9], Zhong and Sun [11] and Plesha et al. [68] each include tangential terms to represent friction. For Ostachowicz [9] the friction is simply another spring like relation between nodal forces. The normal springs are very stiff for penetration and zero for open gaps. The tangential springs are multipliers of the normals springs. Within the limits of sticking friction they are linear and they are constants beyond that. Plesha et al. [68] use micrography to determine frictional properties from empirical data. Zhong and Sun [11] use a friction material similar to the springs used in Padovan et al. [28].

1.8 AUGMENTED LAGRANGE MULTIPLIER METHOD

There is another approach that is a combination of the Lagrange multipliers and the penalty method. These hybrids are called the augmented Lagrangian methods. Among the disadvantages of the penalty method is that satisfaction of the constraints is only guaranteed in the limit as the penalty numbers, α, approach infinity. In theory this could be approximated by repeated solutions with larger and larger penalty numbers. In practice there is a limit beyond which the modified stiffness matrix becomes so ill conditioned that no improvement can be achieved for an acceptable cost. Usually the distribution of energy in the problem has been badly disturbed at a much lower value for α. So the contact condition can only approximately be satisfied. One popular possibility is to try to restrict the possible solutions to acceptable solutions by including the Lagrange multipliers. This has the computational cost of having to solve for extra variables but converges for much lower (and more acceptable) values of the penalty function. So, the potentials included are the mechanical potential, Π, the penetration potential, Π_P, and the the contact potential, Π_C. The modified potential is

$$\Pi^* = \Pi + \Pi_P + \Pi_C \qquad (1.8.1)$$

$$\Pi^* = \frac{1}{2} U^{AT} K^A U^A - U^{AT} F^A \qquad (1.8.2)$$

$$+ \frac{1}{2} U^{BT} K^B U^B - U^{BT} F^B$$

$$+ [\, p_0 + U^A R^A + U^B R^B \,]^T \alpha [\, p_0 + R^A U^A + R^B U^B \,]$$

$$+ [\, g_0 + G^A U^A + G^B U^B \,]^T \lambda$$

This is differentiated and the derivatives set to zero to give the global system of equations.

$$\frac{\partial \Pi^*}{\partial U^A} = K^A U^A - F^A \qquad (1.8.3)$$

$$+ F_P^A + K_P^{AA} U^A + K_P^{AB} U^B + G^{AT} \lambda = 0$$

$$\frac{\partial \Pi^*}{\partial U} = K^B U^B - F^B \qquad (1.8.4)$$

$$+ F_P^B + K_P^{BA} U^A + K_P^{BB} U^B + G^{BT} \lambda = 0$$

$$\frac{\partial \Pi^*}{\partial \lambda} = G^A U^A + G^B U^B = 0 \qquad (1.8.5)$$

The definitions of F_P^A, F_P^B, K_P^{AA}, K_P^{AB}, K_P^{BA} and K_P^{BB} are the same as in the

section on penalty methods. These three equations assembled are

$$\begin{bmatrix} K^{A} + K_{P}^{AA} & K_{P}^{AB} & G^{AT} \\ K_{P}^{BA} & K^{B} + K_{P}^{BB} & G^{BT} \\ G^{A} & G^{B} & 0 \end{bmatrix} \begin{bmatrix} U^{A} \\ U^{B} \\ \lambda \end{bmatrix} = \begin{bmatrix} F^{A} - F_{P}^{A} \\ F^{B} - F_{P}^{B} \\ -g_{0} \end{bmatrix} \tag{1.8.6}$$

Another mixture, also called the augmented Lagrangian method, attempts to improve on the shortcomings of the method of Lagrange multipliers. It is not always the case that the multipliers calculated represent compressive stresses. This may be due to calculational error or because the gap function has not been explicitly evaluated to make sure that only touching nodes have been given multipliers that are free to not be zero. The multipliers have their own inequality constraint. The contact stress must be negative. To enforce this one could use yet another set of multipliers but the new set would have the same restriction. So, an alternative that has seen good practical use is to use a penalty method on the Lagrange multipliers themselves. The penalty is applied whenever a multiplier goes positive. This helps overcome numerical imprecision in the Lagrange multiplier method and helps damp the oscillatory behaviour that often occurs when these methods are used for contact. Include a separation potential, Π_s, and minimize as before.

$$\Pi^* = \Pi + \Pi_c + \Pi_s \tag{1.8.7}$$

$$\begin{aligned}\Pi^* = & \frac{1}{2} U^{AT} K^A U^A - U^{AT} F^A \\ & + \frac{1}{2} U^{BT} K^B U^B - U^{BT} F^B \\ & + (g_0 + G^A U^A + G^B U^B)^T \lambda \\ & + \alpha_L \lambda^T \lambda\end{aligned} \tag{1.8.9}$$

$$\frac{\partial \Pi^*}{\partial U^A} = K^A U^A - F^A + G^{AT} \lambda = 0 \tag{1.8.10}$$

$$\frac{\partial \Pi^*}{\partial U^B} = K^B U^B - F^B + G^{BT} \lambda = 0 \tag{1.8.11}$$

$$\frac{\partial \Pi^*}{\partial \lambda} = g_0 + G^A U^A + G^B U^B + \alpha_L \lambda = 0 \tag{1.8.12}$$

In assembled form this is

$$\begin{bmatrix} K^A & 0 & G^{AT} \\ 0 & K^B & G^{BT} \\ G^A & G^B & \alpha_L \end{bmatrix} \begin{bmatrix} U^A \\ U^B \\ \lambda \end{bmatrix} = \begin{bmatrix} F^A \\ F^B \\ -g_0 \end{bmatrix} \qquad (1.8.13)$$

The augmented Lagrange penalty matrix, α_L, is diagonal. Each entry is zero for those multipliers that are negative and a large number for those that are positive. This is the form most often used in the literature on contact. It is possible to combine both forms of augmented Lagrangian methods. This would improve the performance of the penalty method by restricting the solutions and tend to keep the Lagrange multipliers negative. No examples of such an approach were found in the literature for contact problems.

Friction laws have been as much the subject of study as the contact constraint for those that have used the augmented Lagrangian to improve the performance of the Lagrange multiplier method. Some are discussed in Shyu et al. [5], Simo et al. [7] and Gallego and Anza [12]. Various 'non-local' laws are investigated. The distribution of Lagrange multipliers is also a point of interest. In Simo et al. [7] and Gallego and Anza [12] the Lagrange multipliers represent average pressure over an element face. This leaves the pressure discontinuous between elements. It also makes it impossible to enforce compatibility exactly at nodes. Instead, integral compatibility is offered in the sense that the gap averaged over an element is zero. As the discretization is refined the compatibility constraint is satisfied in the limit.

Herrmann [46] uses the augmented Lagrangian is used to improve a penalty method. Here the subject of consideration is the bonding between concrete and reinforcing metal and the friction between to concrete surfaces due to roughness.

1.9 DIRECT OPTIMIZATION METHODS

There remains the possibility of looking at the contact constraint by concentrating on the contact forces instead of the compatibility problem. The method of Lagrange multipliers, the penalty method and the augmented Lagrangian method all concentrate on compatibility errors. There is a whole class of methods which work the other way. These are called various names such as quadratic programming methods, or parametric programming methods, or direct methods or flexibility methods. They attempt to solve the contact problem by finding appropriate increments to the forces acting on the contact surfaces. It is assumed that the modelling of forces has been inadequate because of errors in the terms for forces due to contact. Correction of these forces will give correction to displacements and reach the minimum potential energy.

In the simpler versions of Francavilla and Zienkiewicz [75], Sachdeva and Ramakrishnan [13] and Sachdeva et al. [26], the problem involves linear elastic materials, no friction and is restricted to

small displacements. It is assumed that the problem is linear except for the contact boundary conditions. The mechanical potential is minimized in the ordinary fashion.

$$\Pi = \frac{1}{2} U^T K U - U^T F \qquad (1.9.1)$$

$$\frac{\partial \Pi}{\partial U} = K U^T - F = 0 \qquad (1.9.2)$$

$$U = K^{-1} F \qquad (1.9.3)$$

Of course the stiffness matrix is not usually ever inverted. An error in the displacements is calculated as the depth of penetration. The error in forces due to missing contact terms is found as the force required to cause a displacement increment to correct the displacement error.

$$\Delta F = K \Delta U \qquad (1.9.4)$$

In fact many of these have set the equations up in terms of the flexibility matrix, K^{-1}. In that case the value of ΔF must be found by solving a set of linear equations.

$$K^{-1} \Delta F = \Delta U \qquad (1.9.5)$$

The contact pressures are then explicitly available as a separate term for use by the analyst. It is assumed that at the node pair at the edge of the contact region the contact pressure will just reach zero. If the pressure at the edge of contact is appreciably different from zero, then another displacement increment is added to allow the pressure to better approach zero.

The inclusion of friction and the manner of choosing ΔF lead to the differences in the various direct optimization methods. The majority of methods explicitly identify node pairs. This has the advantage of reducing the number of equations to solve and a built in guarantee of nodal equilibrium between bodies. However, such methods are inherently restricted to problems where relative displacement of the nodes is small. If two nodes identified as a pair slide apart by a distance comparable to the size of an element, then the problem must be halted and rediscretized. It also has the disadvantage that the bodies in contact must be kept in storage simultaneously.

The simplest way to choose the displacement increment is that of the analysis of static contact between deformable and rigid bodies. In that case the rigid body need only exist as displacement constraints. Rahman et al. [24] explored this for frictionless contact between elastic orthotropic materials with application to the analysis of joints between wooden structural members and steel pins. Gu [74] extended this, for isotropic materials, to the case of sticking friction in an investigation of the forces loading an automobile tire in steady state rolling. The case of sliding contact with Coulomb friction is presented

in the work of Zeid and Padovan [33].

For contact between two deformable bodies pioneering work was done by Francavilla and Zienkiewicz [75] for the case of small displacements and no friction. This approach is extended to sliding with Coulomb friction by Sachdeva and Ramakrishnan [13]. For them the friction law is applied as a function of nodal pressures. Jing and Liao [8] use Coulomb friction taken as a quantity distributed over the element, that is to be lumped to nodal values by integration. This is extended to a more general approach in the work by Torstenfelt [22]. Dynamic motions are considered by Chen and Yeh [10] who formulated the direct optimization approach without the numerical convenience of node matching. Appropriate surface quantities are integrated to lumped nodal averages or sums. The solution to dynamic problems of Smith [76] and Johnson [52 and 53] also do not require node matching. The contact conditions are imposed by finding a force to impose an impulse that will bring the incremental velocity to an acceptable value. In these analyses the materials used have constitutions that are sensitive to strain rates, so accurate determination of velocity is very important to their success.

1.10 FRICTION LAWS

The effects of friction are of great importance in some contact calculations. The simplest friction law is none at all. This is also a common condition to use in the development of analyses that do use friction. It can be used to check the qualitative behaviour of such an analysis. It is also used for quantitative checks. The only closed form solution for contact problems is that due to Hertz (see Timoshenko and Goodier [1] or Johnson [2]). This is a case of contact without friction. So, demonstrations that an analysis tool works well by comparing the results to Hertzian contact requires the ability to simulate frictionless conditions.

The next simplest friction law is that no sliding should occur at all; all contact is sticking contact. Once bodies have touched they are not slip with respect to each other until they pull apart again. This can be easily implemented since under these conditions the nodal displacements of one body can be eliminated in terms of the nodal displacements of the other. For implementation by a Lagrange multiplier method these are both easily done. The matrices that arise are symmetric.

Sliding friction is a source of a great deal of effort in numerical simulation of contact. It raises a number of issues which have to be settled. It is assumed that sticking friction is still possible. So now there are two different friction regimes to consider. The stress conditions may be quite different in each. It is also

necessary to explicitly state what kind of friction law is to be used. Almost without exception the law proposed by de Coulomb is used throughout the literature. According to this law, for small enough tangential forces, F_t, no slipping occurs. Should the tangential forces exceed some fraction, μ, of the normal forces, F_n, then slipping will occur. This slippage is resisted by a force in the direction opposite to the relative motion and of magnitude μF_n. Actually two values are observed for μ. The value at which motion is imminent, the static coefficient of friction, μ_s, is slightly higher than the maximum fraction of the normal force, the dynamic coefficient of friction, μ_d. This can be observed by the common occurrence of an object that is being pushed suddenly starting to move as the static friction is overcome and the resistance to motion decreases to the dynamic value. This was originally meant as a phenomenological description of the averaged properties of the sliding of blocks of material with macroscopically flat surfaces.

Coulomb friction has been used in many calculations as if it were an accurate description of the microscopic conditions at a point. Or, to use the words of the study of continuum mechanics, this is used as if it were true for every differentially small neighbourhood. This is fortunate in that some description of friction is needed and this is a relatively simple one. On the other hand, this has led to many efforts to overcome the mathematical inconveniences involved. It leads to the use of various terms at the contact surface that are not differentiable or continuous. For terms involving friction the

mathematical inconvenience is similar to problems that involve phase change boundaries. The positions of the boundaries between sticking and sliding friction are not necessarily at the boundaries of the elements. This has lead to efforts by some workers to arrange that over any particular element only one friction regime is present. The most successful and elegant are those done with the moving finite element methods of Haber [77], of Haber and Harandria [4] and of Gu [35].

Others argue over the merits of local and non-local version of the Coulomb friction law. This arises in theory as an argument of the validity of trying to use a macroscopic average where the microscopic geometry of the surface may not be smooth and perhaps some law using a smoothing function is required. Oden and Pires [66] refer to such a function as a mollifier. In practice this arises in how the force terms tangential to the contact surface should be calculated. The proponents of 'local' friction laws integrate the continuous frictional stresses over the elements. Proponents of 'non-local' friction laws have several approaches. The simplest and commonest way is to concern themselves only with the resultant forces at the nodes. These forces are necessarily some kind of average of the frictional conditions between the nodes. Another way is used by Shyu et al. [5], Chang et al. [6] and Simo et al. [7]. These workers have used an augmented Lagrangian method based on the Lagrange multipliers to restrict the set of possible solutions to the set of admissible solutions. Instead of using nodal Lagrange multipliers they use elemental ones. This gives the averaged contact pressure over the entire element. It has interesting side

effects on convergence and accuracy. Contact equilibrium and compatibility are satisfied exactly only in the limit of fine meshes. In practice it may not be necessary for the mesh to be extremely fine. Also the contact stress interpolation is only C^{-1} continuous. The stress is discontinuous between elements. This allows very easy and efficient integration of the stresses.

There are other inconveniences that arise in the modelling of sliding friction. Many models give rise to matrices that are not symmetric. The potential due to the friction forces must be added in with the other energies. Let the friction potential be Π_F. Then

$$\Pi_F = U^{AT} F_t^A + U^{BT} F_t^B = U^{AT} \mu F_n^A + U^{BT} \mu F_n^B \qquad (1.10.1)$$

The total potential is (in a Lagrange multiplier technique)

$$\Pi^* = \Pi + \Pi_C + \Pi_F \qquad (1.10.2)$$

This is differentiated to give the equilibrium conditions

$$\frac{\partial \Pi^*}{\partial U^A} = K^A U^A - F^A + G^{AT} \lambda + \mu F_n^A = 0 \qquad (1.10.3)$$

$$\frac{\partial \Pi^*}{\partial U^B} = K^B U^B - F^B + G^{BT} \lambda + \mu F_n^B = 0 \qquad (1.10.4)$$

$$\frac{\partial \Pi^*}{\partial \lambda} = g_o + G^A U^A + G^B U^B = 0 \qquad (1.10.5)$$

where μ is diagonal and affects only appropriate entries in λ. These may be put in matrix form as

$$\begin{bmatrix} K^A & 0 & G^{AT} + \mu \\ 0 & K^B & G^{BT} + \mu \\ G^A & G^B & 0 \end{bmatrix} \begin{bmatrix} U^A \\ U^B \\ \lambda \end{bmatrix} = \begin{bmatrix} F^A \\ F^B \\ -g_o \end{bmatrix} \qquad (1.10.6)$$

There remains one of the most interesting versions of the Coulomb friction law. It can be observed that in many ways this law behaves in a very much like the constitutive relation of an elastic - perfectly plastic material. The work done by slippage against friction cannot be recovered. The slippage only occurs if a certain criterion of stress is met. The frictional stresses never exceed a particular bound. Padovan et al. [28], Zhong and Sun [11] and Plesha et al. [68] use an elasto-plastic friction - slippage constitutive relation like the one outlined below. Define slippage as

$$h = u^A - u^B \qquad (1.10.7)$$

Let quantities be resolved in three directions. The direction normal to the contact surface is denoted by subscript n. The two tangent directions are denoted by subscripts r and s. It is assumed that the

slippage occurs in elastic and plastic parts. The stresses are supported by the elastic part. The constitutive relation is given by

$$\begin{bmatrix} \dot{\sigma}_n \\ \dot{\sigma}_r \\ \dot{\sigma}_s \end{bmatrix} = \begin{bmatrix} & E & \end{bmatrix} \begin{bmatrix} \dot{h}_n \\ \dot{h}_r \\ \dot{h}_s \end{bmatrix} \qquad (1.10.8)$$

Plesha et al. [68] claim that there is experimental evidence for E to be diagonal. It has also been noted that the normal direction component can be employed in a penalty fashion to prevent penetration. The additivity postulate is

$$\begin{bmatrix} \dot{h}_n \\ \dot{h}_r \\ \dot{h}_s \end{bmatrix} = \begin{bmatrix} \dot{h}^e_n \\ \dot{h}^e_r \\ \dot{h}^e_s \end{bmatrix} + \begin{bmatrix} \dot{h}^p_n \\ \dot{h}^p_r \\ \dot{h}^p_s \end{bmatrix} \qquad (1.10.9)$$

The consistency condition is

$$\dot{Y} = \left[\frac{\partial Y}{\partial \sigma_n} \quad \frac{\partial Y}{\partial \sigma_r} \quad \frac{\partial Y}{\partial \sigma_s} \right] \begin{bmatrix} \dot{\sigma}_n \\ \dot{\sigma}_r \\ \dot{\sigma}_s \end{bmatrix} = \left[\frac{\partial Y}{\partial \sigma} \right] \dot{\sigma} = 0 \qquad (1.10.10)$$

where Y is the 'yield function'. The slip rule is

$$\begin{bmatrix} \dot{h}^p_n \\ \dot{h}^p_r \\ \dot{h}^p_s \end{bmatrix} = \dot{\zeta} \begin{bmatrix} \frac{\partial \xi}{\partial \sigma_n} \\ \frac{\partial \xi}{\partial \sigma_r} \\ \frac{\partial \xi}{\partial \sigma_s} \end{bmatrix} = \dot{\zeta} \left[\frac{\partial \xi}{\partial \dot{\sigma}} \right] \qquad (1.10.11)$$

where ξ is the slip potential and ζ is the associative flow parameter. These may be combined in the same fashion as is used in perfect plasticity calculations to give an elasto-plastic slip constitutive relation, E^{ep}. For instance when the stresses are on the yield surface and the differential stress is outward (slippage is occurring).

$$E^{ep} = E\left[I - \frac{\left[\frac{\partial \xi}{\partial \dot{\sigma}}\right]\left[\frac{\partial Y}{\partial \dot{\sigma}}\right]^T}{\left[\frac{\partial Y}{\partial \dot{\sigma}}\right]^T E \left[\frac{\partial \xi}{\partial \dot{\sigma}}\right]} E \right] \qquad (1.10.12)$$

The matrix, E^{ep}, is asymmetric. This must be since the direction of slippage is not affected by the pressure. So the slip potential cannot be the same as the yield function. This corresponds to non-associated plastic flow. The work done in a small neighbourhood by such slippage is

$$dW_F = h^T \sigma = h^T E^{ep} h \qquad (1.10.13)$$

This may be integrated over an area to give an elasto-plastic friction element. The slippage is interpolated from nodal values, H, by the usual shape functions, Φ. So the work done over such an element during slip is

$$W_F = H^T \int \Phi^T E^{ep} \Phi \; dA \; H \qquad (1.10.14)$$

These nodal values must be rotated to the global coordinates and replaced by the nodal displacement degrees of freedom from which they

come. The asymmetry can be removed by further manipulations. The explicit coupling between contacting bodies of nodal displacements remains.

1.11 OUTLINE OF THE PRESENT WORK

The method developed in this thesis is of the class of direct optimization methods. That is, the calculation of corrective force increments is based on the current displacements. It is not obviated by the use of Lagrange multipliers nor built into the stiffness matrix by penalty terms. The only nodal degrees of freedom are the displacements. The stiffness matrices are those that represent actual material characteristics. There are no extra terms to represent attempts to arithmetically coerce the behaviour of the solution. So, this can be added to any displacement based finite element program if the coding is adequately modular.

The method is symmetric in both senses. The matrices involved are symmetric. The treatment that each body receives is exactly the same as every other. There is no nomination of one body as target or contactor or master or slave. There is no explicit coupling between the displacement degrees of freedom of the bodies. This allows the information for each body to be stored separately, instead of all in one huge matrix. This alleviates the restrictions inherent in some node

pairing schemes and allows any amount of sliding of the touching surfaces without requiring rediscretization.

The algorithm is objective. All corrections are applied as force terms rather than displacement increments. These do not have to be applied before other nonlinearities but are applied at the global level. So, it does not have to be nested inside other nonlinear coding.

There are two kinds of errors that can occur in contact problems: compatibility errors and equilibrium errors. Each kind of error is treated separately. To the first order, the corrective force increment for compatibility errors does not affect equilibrium. Likewise, to the first order, the corrective force increment for equilibrium errors does not affect compatibility. This method is different from other techniques that require a single force increment to correct both kinds of errors.

To accomplish all this each body must be able to find, on the contact surface, the properties of the other body it is touching. The geometrical difficulties are alleviated by the use of virtual nodes and virtual elements. These map the properties of the opposing body onto the same discretization as the body which is touching it. These virtual nodes and elements have equivalent properties to the real nodes and elements. This allows a body, say body P, to have access to the nodal properties of the body it is touching, say body Q, and finding them apparently distributed with exactly the same pattern of elements and nodes as body P has itself. So, each node is effectively paired to a virtual node. The properties of the virtual node can be updated as

often as necessary. The two properties of chief interest are stiffness and traction. The details of how these are mapped across the contact surface are discussed in Chapter 4. Exemplary results are presented in Chapter 5.

Any friction law can be implemented together with this algorithm. The technique is modular in that respect. Various different friction laws could be used in the different contact areas just as different material constitutions can be used in different parts of the volume of the bodies by telling the program that this is desired. This is a matter of programming rather than modelling. This algorithm requires only that certain information about friction resultants be available at the global level. In general, the topic of friction is one to which much effort could be devoted. Since contact per se, rather than friction, is the subject of this work, a simple approach has been taken. The surfaces of bodies used in the examples of Chapter 5 are frictionless.

In the course of this research much effort was spent on the subject of referential descriptions of motion. This exploration gave new insight into strain tensors in particular. Those efforts are explicated in Chapter 2. In Chapter 3 they are used to illuminate the referential form of the Linear Momentum Principle and the Principle of Static Equilibrium.

Finally in Chapter 6 some conclusions are drawn.

CHAPTER TWO

REFERENTIAL MECHANICS

2.1 SUMMARY

This chapter describes some aspects of the theory involved in the current work. It treats the fundamental concepts: how quantities can be treated by reference configurations, some derivatives and integrals and the notation required to keep track of all of these and how to change reference configuration for gradients. All of this is an exact treatment for any body in any motion. So, it can be used to investigate motions that involve large displacements and large distortions. Both of these often occur in contact problems. The Lagrangian strain tensor is then derived with some care. This derivation includes incremental forms and how to change from one configuration of reference to another. This is ideally suited for use in the incremental calculations such as will be used to follow the details of motion during contact. This chapter ends with a review of the concepts involved in changing configuration of reference for physical principles in integral form.

2.2 LAGRANGIAN MECHANICS

2.2.1 INTRODUCTION

In Lagrangian, or referential, mechanics attention is directed at the body as it moves through space. A body is represented as a gross distribution of mass over its configuration, where the configuration is the region of Euclidean space occupied by the body. Mass is considered an appropriate scalar-valued measure of matter. All quantities pertaining to the body and the body-points themselves are labelled by the position of the body-points when the body is in some reference configuration or placement. Commonly, but not necessarily, this is an initial position or an unstrained configuration. In most calculations it is chosen as having the useful property that any quantity of interest, such as stresses, strains or temperature, are known a priori. There is no restriction to use only one reference configuration. It is often useful or convenient to use several different placements as reference configurations. The configuration in which a quantity is measured has no inherent theoretical relationship to the configuration in which all body-points are labelled. So, the value of measured quantities is invariant with respect to choice of reference configuration. Throughout the changes and motions a body undergoes, it is always possible to identify a body-point by where it would be were the body in a reference configuration. For example, particles in a vibrating body are often labelled by their position in an undisturbed,

equilibrium configuration. The choice of reference configuration is quite arbitrary. Any possible configuration would serve the theoretical purpose of having a way to label body-points. In incremental calculations the choice of which configuration to use as a reference configuration is commonly a recently calculated one and the choice may be changed from time to time. Such a practice leads to the so-called Updated Lagrangian formulation. In theory, this is all equally well, but, in practice this can lead to complications.

It is necessary to consider some configurations of a body (consult Figure 2.1):

C_U Unstrained configuration. The only important feature of this configuration is that it is unstrained. The body is relaxed and has no stresses or forces acting on it. The location and orientation of this configuration is unimportant. It may be an initial configuration from which other configurations are reached. The position of a body-point, X, in this configuration is given by the Cartesian vector ${}^{U}\bar{x}(X)$.

C_0 Initial configuration. This is the region of space occupied by the body at the beginning of the period of consideration. All other configurations that the bodies actually occupies at any point in its motion form a sequence that starts with this configuration. The position of a body-point, X, in this configuration is given by the Cartesian vector ${}^{0}\bar{x}(X)$.

C_1 Present configuration. This configuration may be displaced and/or deformed from C_0. It may be any possible configuration for the body. The position of a body-point, X, in this configuration is given by the Cartesian vector ${}^1\bar{x}(X)$.

C_2 Next configuration. This is any possible configuration that the body may occupy through motion (or transplacement). Ordinarily, this may be a configuration occupied by the body at some stage in its motion later in sequence than C_1 possibly incrementally close to C_1. The position of a body-point, X, in this configuration is given by the Cartesian vector ${}^2\bar{x}(X)$.

C_R Reference configurations. These represent any possible configuration of the body in any location and having any orientation. They may be deformed. They are not necessarily in the sequence of configurations that the body actually occupies. The position of a body-point, X, in this configuration is given by the Cartesian vector ${}^R\bar{x}(X)$.

All calculations are always carried out in a reference configuration.

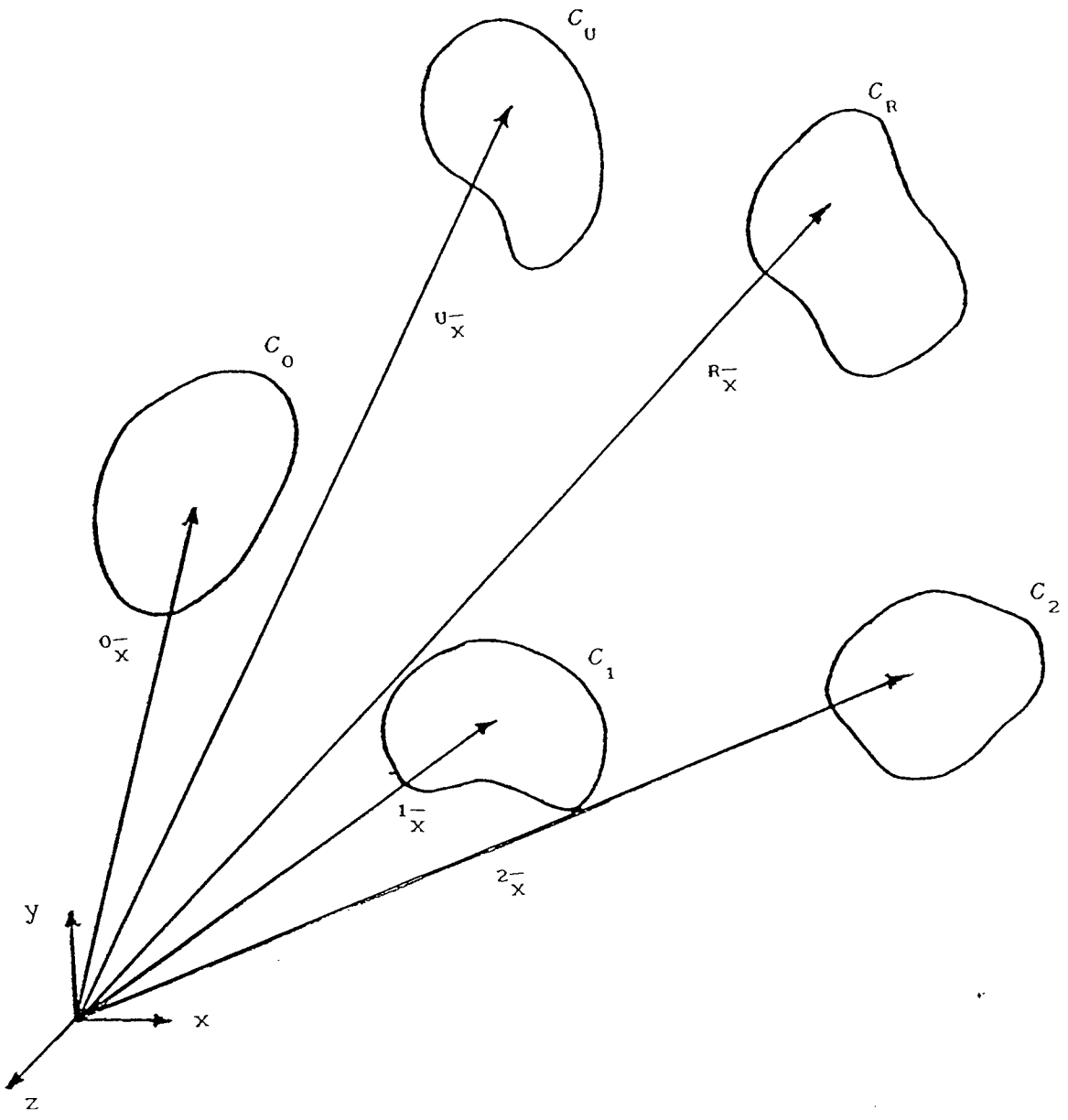


Figure 2.1 Positions of body-point X in various configurations

2.2.2 NOTATION

Overscores

Overscores indicate the tensor order of a quantity. A scalar has no overscores. A vector has one overscore. A dyad or dyadic, being a second order tensor, has two overscores and so on. An alternative notation may be used where a single overscore has a number written above it. In this case the number indicates the tensor order of the quantity underneath. This is mostly used for high order quantities, such as Hookean tensors (fourth order), where the use of a large number of overscores would be inconvenient.

Left superscripts

A single left superscript denotes the configuration in which a quantity is measured For example, ${}^{2}\overline{x}(X)$ is the position vector of body-point X when the body is in configuration C_2.

Multiple left superscripts

For displacements and other quantities which require more than one configuration for their definition, there are more than one left superscripts. To define a displacement requires two configurations. So, the first (leftmost) superscript is used for the "to" configuration.

The second (rightmost) superscript is used for the "from" configuration. For example, the displacement of X from configuration C_A to configuration C_B is ${}^{AB}\bar{u}(X)$. Note that for displacements ${}^{AA}\bar{u} = \bar{0}$.

Left Subscript

The left subscript denotes the configuration of reference. Lack of a left subscript indicates the use of a so-called material formulation. So,

$$ {}^{AB}\bar{u}(X) = {}_{R}{}^{BB}\bar{u}({}^{R}\bar{x}) \qquad (2.2.1) $$

The configuration of reference can be any configuration and this has no effect on the value of the measured quantity. For a body property, T, when the body is in configuration C_A, it must be that ${}^{A}T$, ${}_{R}{}^{A}T$ and ${}_{S}{}^{A}T$ all describe the same quantity and must have the same value, though the formulae may look different. Though not necessary, it is common practice to keep all terms on each side of an equation referred to a single configuration.

2.2.3 LOCATION OF BODY-POINTS

In any configuration, it is required that a unique invertible relationship exist between body-points and Cartesian position vectors locating them. So,

$$^{R}\bar{x} = {}^{R}\bar{x}(X) \tag{2.2.2}$$

and $$X = X(^{R}\bar{x}) \tag{2.2.3}$$

Moreover a differentiable structure is assigned to the body manifold. Hence, if X and Y are two distinct body-points and Y is in a small neighbourhood of X, we can say

$$^{R}\bar{x}(Y) = {}^{R}\bar{x}(X) + {}^{R}d\bar{x}(Y,X) \tag{2.2.4}$$

It is noted that position is a body function. The function that describes the position of body-points in C_A referred to C_B, , ${}^{A}_{B}\bar{x}$, may be any admissible vector-valued function. The function that describes ${}^{R}_{R}\bar{x}$ is just the identity function.

2.3 SPATIAL DERIVATIVES

The meaning of the calculus is clear so long as we are differentiating with respect to a well understood scalar, say time. For differentiation with respect to position there are at least two quite different derivatives that can arise. Let us review the differential calculus with respect to a tensor. Suppose that $\overset{n}{P}$, a tensor of order n, is a function of the m^{th} order tensor $\overset{m}{F}$. That is

$$\overset{n}{P} = \overset{n}{P}(\overset{m}{F}) \tag{2.3.1}$$

Then $d\overset{m}{F}$, a small change in $\overset{m}{F}$, produces $d\overset{n}{P}$, a small change in $\overset{n}{P}$. The function that relates these small changes satisfies

$$d\overset{n}{P} = d\overset{m}{F} \overset{m}{\cdot} \frac{\partial \overset{n}{P}}{\partial \overset{m}{F}} \tag{2.3.2}$$

where $\overset{m}{\cdot}$ indicates that m dot products are to be taken. The small change in $\overset{n}{P}$ is a function of the parametric values of $\overset{m}{F}$. So we know that we can express $d\overset{n}{P}$ as

$$d\overset{n}{P} = d\overset{n}{P}(\overset{m}{F}) = dF^{abc\ldots m} \frac{\partial \overset{n}{P}}{\partial F^{abc\ldots m}} \tag{2.3.3}$$

where repeated indices are summed. The other quantity we know in eqn(2.3.2) is $d\overset{m}{F}$. It is

$$d\overset{m}{F} = dF^{a'b'c'\ldots m'}\ \bar{j}_{a'}\bar{j}_{b'}\bar{j}_{c'}\cdots\bar{j}_{m'} \qquad (2.3.4)$$

where $\bar{j}_p$ p = 1, 2, 3, are the local basis vectors of $\overset{m}{F}$ and the primes indicate that the indices of eqn(2.3.4) are independent of those of eqn(2.3.3)

It is possible to isolate $dF^{a'b'c'\ldots m'}$ as expressed in eqn(2.3.4) by repetitively taking dot products with $\bar{j}^r$, the inverse vectors to the $\bar{j}_p$. Let us illustrate this process by explicitly taking the first such product.

$$d\overset{m}{F} \cdot \bar{j}^m = dF^{a'b'c'\ldots l'm'}\bar{j}_{a'}\bar{j}_{b'}\bar{j}_{c'}\cdots\bar{j}_{l'}\bar{j}_{m'} \cdot \bar{j}^m \qquad (2.3.5)$$

$$= dF^{a'b'c'\ldots l'm'}\bar{j}_{a'}\bar{j}_{b'}\bar{j}_{c'}\quad \bar{j}_{l'}\delta^m_{m'} \qquad (2.3.6)$$

$$= dF^{a'b'c'\ldots l'm}\bar{j}_{a'}\bar{j}_{b'}\bar{j}_{c'}\quad \bar{j}_{l'} \qquad (2.3.7)$$

Notice in eqns (2.3.5), (2.3.6) and (2.3.7) that the tensor order has been reduced by one. Also, in eqn(2.3.7) the final index of $d\overset{n}{P}$, m, is not primed. This relates it to the indices in eqn(2.3.3). After m such products have been taken we find

$$d\underline{\underline{F}}^{\underline{\underline{m}}} \cdot \underline{j}^{m} \underline{j}^{l} \ldots \underline{j}^{c} \underline{j}^{b} \underline{j}^{a} = dF^{abc\ldots lm} \qquad (2.3.8)$$

This can be substituted in to eqn(2.3.3) to give

$$d\underline{\underline{P}}^{\underline{\underline{n}}} = d\underline{\underline{F}}^{\underline{\underline{m}}} \cdot \underline{j}^{m} \underline{j}^{l} \ldots \underline{j}^{c} \underline{j}^{b} \underline{j}^{a} \frac{\partial \underline{\underline{P}}^{\underline{\underline{n}}}}{\partial F^{abc\ldots lm}} \qquad (2.3.9)$$

Now two expressions are available for $d\underline{\underline{P}}^{\underline{\underline{n}}}$. These can be subtracted one from the other and with the resulting identity we can give meaning to the unknown derivative expression in eqn(2.3.2). So,

$$d\underline{\underline{F}}^{\underline{\underline{n}}} - d\underline{\underline{F}}^{\underline{\underline{n}}} = \underline{\underline{0}}^{\underline{\underline{n}}} \qquad (2.3.10)$$

$$\underline{\underline{0}}^{\underline{\underline{n}}} = \left(d\underline{\underline{F}}^{\underline{\underline{m}}} \cdot \underline{j}^{m} \underline{j}^{l} \ldots \underline{j}^{c} \underline{j}^{b} \underline{j}^{a} \frac{\partial \underline{\underline{P}}^{\underline{\underline{n}}}}{\partial F^{abc\ldots lm}} - d\underline{\underline{F}}^{\underline{\underline{m}}} \cdot \frac{\partial \underline{\underline{P}}^{\underline{\underline{n}}}}{\partial \underline{\underline{F}}^{\underline{\underline{m}}}} \right) \qquad (2.3.11)$$

Take out the common factor

$$\underline{\underline{0}}^{\underline{\underline{n}}} = d\underline{\underline{F}}^{\underline{\underline{m}}} \cdot \left(\underline{j}^{m} \underline{j}^{l} \ldots \underline{j}^{c} \underline{j}^{b} \underline{j}^{a} \frac{\partial \underline{\underline{P}}^{\underline{\underline{n}}}}{\partial F^{abc\ldots lm}} - \frac{\partial \underline{\underline{P}}^{\underline{\underline{n}}}}{\partial \underline{\underline{F}}^{\underline{\underline{m}}}} \right) \qquad (2.3.12)$$

Since we have presumed only that $d\overline{\overline{F}}^{m}$ is not $\overline{\overline{0}}^{m}$ then we cannot rely on it lying in some convenient orientation such that the m dot products in eqn(2.3.12) all give zeroes. So, it must be that the quantity in parentheses is identically zero. This can only be if both terms are equal. Now we have the information that was missing in eqn(2.3.2), the meaning of the derivative. Notice that a derivative with respect to an m^{th} order tensor is m tensor orders higher than the original quantity. Notice also that the added vector spaces are the reciprocals of those that defined the tensor with respect to which the derivative was taken and in the reverse order. So, we have

$$\frac{\partial \overline{\overline{P}}^{n}}{\partial \overline{\overline{F}}^{m}} = \bar{j}_m \bar{j}_l \ldots \bar{j}_c \bar{j}_b \bar{j}_a \frac{\partial \overline{\overline{P}}^{n}}{\partial F^{abc\ldots lm}} \qquad (2.3.13)$$

So far this has been developed completely without any restriction.

As a next step take the case of derivatives with respect to position. The independent tensor is position, $\bar{x}$. A differential change in position is

$$d\bar{x} = dx^r \bar{g}_r \qquad (2.3.14)$$

The parametric values are the spatial coordinates. The basis vectors are the local tangent basis of space. So, the derivative of some quantity with respect to position in space is

$$\frac{\partial \overset{n}{P}}{\partial \overline{x}} = \overline{g}^{r} \frac{\partial \overset{n}{P}}{\partial x^{r}} \qquad (2.3.15)$$

So far we have defined what we mean by a derivative with respect to position. We have not put this into the context of Lagrangian continuum mechanics. There are at least two different reasons why we might wish to examine values to be found at nearby points in space.

One possibility is that the two locations under consideration are two distinct body-points within a single configuration. This is the gradient calculus. Another possibility is that we are considering the same body-point as the body occupies two distinct configurations. This is an entirely different comparison from the gradient calculus. Call this configurational calculus. This is frequently used in the calculus of variations. In an effort to avoid any possible confusion between the two types of derivatives we will use

$$\frac{\partial \overset{n}{P}}{\partial \overline{x}}$$

for gradients and

$$\frac{\delta \overset{n}{P}}{\delta \overline{x}}$$

for configurational derivatives. Where there is any possible confusion

the type of derivative will be specified.

There is still the matter of configurations of measurement and reference. First look at the gradient calculus. Since the above arguments are true for tensors of any order we will simplify the notation here by considering the example of the dependent tensor being always of second order. Any order can be used. There are several possible configurations of which to keep track. Let $\overset{\equiv}{Q}$ be the gradient of $\overset{\equiv}{P}$. Then

$$ {}^{ABCD}_{R}\overset{\equiv}{Q} = \frac{\partial\, {}^{A}_{B}\overset{\equiv}{P}}{\partial\, {}^{C}_{D}\overline{x}} \qquad (2.3.16) $$

The material property, $\overset{\equiv}{P}$, can be measured in any configuration, say C_A. The configuration of reference of ${}^{A}\overset{\equiv}{P}$ could be C_B. That is, when measuring the value of ${}^{A}\overset{\equiv}{P}$ we named the body-points by the position they would have, if the body were in configuration C_B. The alternative, less formal, way to think of this is that when we measured the value of ${}^{A}\overset{\equiv}{P}$, we wrote the value down on a fictional copy of the body in configuration C_B. In this case we have ${}^{A}_{B}\overset{\equiv}{P}$. This is a function of position in C_B. That is

$$ {}^{A}_{B}\overset{\equiv}{P} = {}^{A}_{B}\overset{\equiv}{P}({}^{B}\overline{x}) \qquad (2.3.17) $$

So, it can be differentiated with respect to ${}^{B}\overline{x}$. Differentiation is with respect to position measured in configuration C_C. So, the

configuration of measurement for position in the gradient must be the same as the configuration of reference for the property. Then C_B and C_C must be the same. This position may itself be referred to any configuration. The reference configuration for the position vector is the configuration of reference for the gradient. So in eqn(2.3.16) C_D is the same as C_R. With this clearly understood we may now make a convention for the notation of gradients. Where $\overset{\equiv\equiv}{Q}$ is the gradient of $\overset{\equiv}{P}$, $\overset{\equiv\equiv}{Q}$ has the same left superscripts as $\overset{\equiv}{P}$ plus one more left superscript. This extra left superscript is preceded by a comma and indicates the configuration of measurement for the position vector with respect to which $\overset{\equiv}{P}$ has been differentiated. The left subscript of $\overset{\equiv\equiv}{Q}$ denotes the reference configuration for the gradient, which is the reference configuration of the position vector with respect to which $\overset{\equiv}{P}$ has been differentiated. So, for gradients we have

$$ {}^{A,C}_{R}\overset{\equiv\equiv}{Q} = \frac{\partial\, {}^{A}_{C}\overset{\equiv}{P}}{\partial\, {}^{C}_{R}\overline{x}} \qquad (2.3.18) $$

In a similar examination of the configurational calculus, we start with five configurations. Let $\overset{\equiv\equiv}{Q}$ be the configurational derivative of $\overset{\equiv}{P}$, then

$$ {}^{ABCD}_{R}\overset{\equiv\equiv}{Q} = \frac{\delta\, {}^{A}_{B}\overset{\equiv}{P}}{\delta\, {}^{C}_{D}\overline{x}} \qquad (2.3.19) $$

$\overline{\overline{P}}$ is measured in C_A and referred to C_B. $\overline{x}$ is measured in C_C and referred to C_D. This shows that what is being varied is C_C. Only one body-point is being watched as we make different choices of configuration in which to measure position. We observe changes in the value of ${}^{A}\overline{\overline{P}}$ as various configurations near C_C are chosen for $\overline{x}$. Since this is the position with respect to which we wish to differentiate $\overline{\overline{P}}$, $\overline{\overline{P}}$ must be a function of this position. That is $\overline{\overline{P}}$ must be referred to the same configuration as the one in which position is measured. So C_B must be the same configuration as C_C. The choice of C_D, the reference configuration for the position with respect to which we differentiate $\overline{\overline{P}}$, is independent. So, a notation like that of the gradient calculus can be used. That is

$$ {}^{A;C}_{R}\overline{\overline{Q}} = \frac{\delta\, {}^{A}_{C}\overline{\overline{P}}}{\delta\, {}^{C}_{R}\overline{x}} \qquad (2.3.20) $$

is also useful for the configurational calculus. There is a difference in that the punctuation among the left superscripts to denote the configuration of measurement of the position with respect to which $\overline{\overline{P}}$ is differentiated is changed from a comma to a semicolon.

2.4 CHANGE OF REFERENCE CONFIGURATION

At any time it may be convenient or necessary for an analyst to change the reference configuration. In principle this is a simple operation. The referring of a quantity to a configuration can be thought of as writing the measured value of the quantity on a fictional copy of the configuration chosen for reference. This allows calculations to be carried out in a consistent fashion. Access to all body-points and their properties is easily achieved through the location of that body-point in the reference configuration. However, in the case of spatial derivatives this is not always the most appropriate method. It is useful in many calculations to transform the derivative so that the differentiation is with respect to position in the reference configuration.

Consider the transformation for gradients. To do this use three configurations, C_A, C_B and C_C and two distinct but nearby body-points, X and Y. Then some material property, $\overline{\overline{P}}$, is measured in configuration C_C. So it is ${}^{C}\overline{\overline{P}}$. The values of this property at body-points X and Y are:

$$ {}^{C}\overline{\overline{P}}(X) = {}_{C}^{C}\overline{\overline{P}}({}^{C}\overline{x}) \qquad (2.4.1) $$

and

$$ {}^{C}\overline{\overline{P}}(Y) = {}_{C}^{C}\overline{\overline{P}}({}^{C}\overline{x} + {}^{C}\overline{dx}) \qquad (2.4.2) $$

Eqn (2.4.2) can be expressed as

$$ {}^{c}\bar{\bar{P}}(Y) = {}^{c}_{c}\bar{\bar{P}}({}^{c}\bar{x}) + {}^{c}_{c}d\bar{x} \cdot \frac{\partial {}^{c}_{c}\bar{\bar{P}}}{\partial {}^{c}_{c}\bar{x}} \qquad (2.4.3) $$

Since body-points X and Y are very close to each other the difference in ${}^{c}\bar{\bar{P}}$, ${}^{c}d\bar{\bar{P}}$, can be expressed as

$$ {}^{c}\bar{\bar{P}}(Y) = {}^{c}\bar{\bar{P}}(X) + {}^{c}d\bar{\bar{P}}(X,Y) \qquad (2.4.4) $$

Through the rest of this argument differentials refer to differential changes in values of properties between X and Y. So further references to, for example, ${}^{c}d\bar{\bar{P}}(X)$, are taken to mean ${}^{c}d\bar{\bar{P}}(X,Y)$. Take advantage of the invariance of ${}^{c}\bar{\bar{P}}(Y)$ to compare eqns (2.4.3) and (2.4.4). The difference of these gives the usual form for a gradient

$$ {}^{c}d\bar{\bar{P}} = {}^{c}_{c}d\bar{\bar{P}} = {}^{c}_{c}d\bar{x} \cdot \frac{\partial {}^{c}_{c}\bar{\bar{P}}}{\partial {}^{c}_{c}\bar{x}} \qquad (2.4.5) $$

The value of ${}^{c}\bar{\bar{P}}$ is invariant with respect to configuration of reference and can be referred to C_R.

$$ {}^{c}d\bar{\bar{P}} = {}^{c}_{R}d\bar{\bar{P}} = {}^{c}_{R}d\bar{x} \cdot \frac{\partial {}^{c}_{c}\bar{\bar{P}}}{\partial {}^{c}_{R}\bar{x}} \qquad (2.4.6) $$

As an alternative, we could refer ${}^{c}\bar{\bar{P}}$ to C_A at the beginning of

the argument. We find

$$ {}^{C}\bar{\bar{P}}(X) = {}^{C}_{C}\bar{\bar{P}}({}^{C}\bar{x}) = {}^{C}_{A}\bar{\bar{P}}({}^{A}\bar{x}) \tag{2.4.7} $$

$$ {}^{C}\bar{\bar{P}}(Y) = {}^{C}_{A}\bar{\bar{P}}({}^{A}\bar{x} + {}^{A}d\bar{x}) \tag{2.4.8} $$

$$ {}^{C}\bar{\bar{P}}(Y) = {}^{C}_{A}\bar{\bar{P}} + {}^{A}_{A}d\bar{x} \cdot \frac{\partial {}^{C}_{A}\bar{\bar{P}}}{\partial {}^{A}_{A}\bar{x}} \tag{2.4.9} $$

Using the same approach as for eqn (2.4.5) one finds

$$ {}^{C}d\bar{\bar{P}} = {}^{C}_{A}d\bar{\bar{P}} = {}^{A}_{A}d\bar{x} \cdot \frac{\partial {}^{C}_{A}\bar{\bar{P}}}{\partial {}^{A}_{A}\bar{x}} \tag{2.4.10} $$

or

$$ {}^{C}d\bar{\bar{P}} = {}^{C}_{R}d\bar{\bar{P}} = {}^{A}_{R}d\bar{x} \cdot \frac{\partial {}^{C}_{A}\bar{\bar{P}}}{\partial {}^{A}_{R}\bar{x}} \tag{2.4.11} $$

The differential ${}^{A}_{R}d\bar{x}$ should be noted. As shown in Figure 2.2, ${}^{A}d\bar{x}$ locates body-point Y with respect to body-point X when the body occupies C_A but not when the body occupies any other configuration.

This same process can be repeated to refer ${}^{C}\bar{\bar{P}}$ to C_B. The conclusion must be reached that

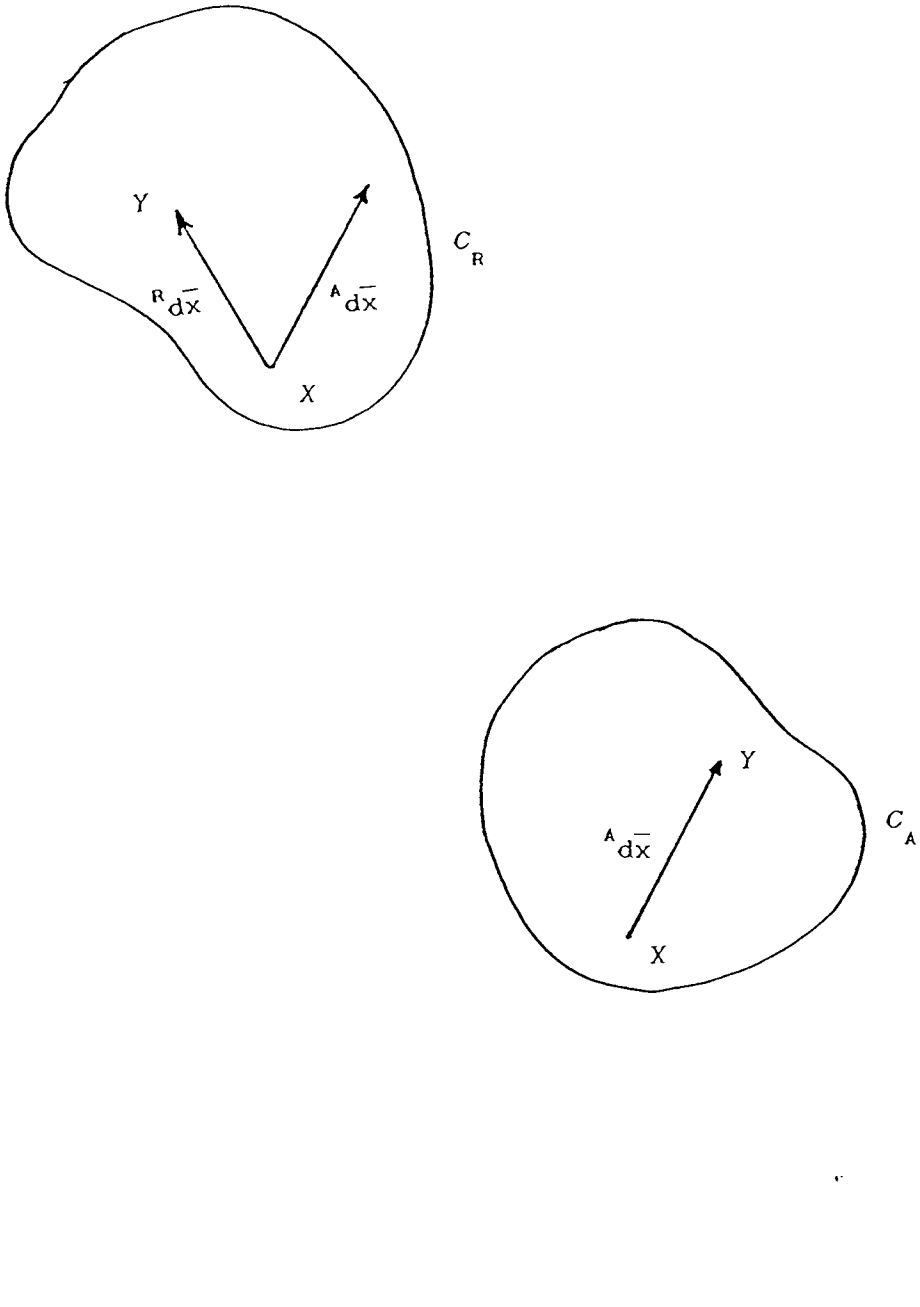


Figure 2.2 Differential position vectors $^A d\bar{x}$ and $^R d\bar{x}$

$$ {}^{C}d\bar{\bar{P}} = {}^{C}_{B}d\bar{\bar{P}} = {}^{B}_{B}d\bar{x} \cdot \frac{\partial^{C}_{B}\bar{\bar{P}}}{\partial^{B}_{B}\bar{x}} \qquad (2.4.12) $$

or

$$ {}^{C}d\bar{\bar{P}} = {}^{C}_{R}d\bar{\bar{P}} = {}^{B}_{R}d\bar{x} \cdot \frac{\partial^{C}_{B}\bar{\bar{P}}}{\partial^{B}_{R}\bar{x}} \qquad (2.4.13) $$

The differential position vector ${}^{B}d\bar{x}$ locates body-point Y with respect to body-point X, only when the body occupies configuration C_B. The argument so far can be applied immediately. Replace ${}^{C}d\bar{\bar{P}}$ with ${}^{A}d\bar{x}$. Then eqn(2.4.13) becomes

$$ {}^{A}d\bar{x} = {}^{A}_{R}d\bar{x} = {}^{B}_{R}d\bar{x} \cdot \frac{\partial^{A}_{B}\bar{x}}{\partial^{B}_{R}\bar{x}} \qquad (2.4.14) $$

This in turn may be substituted into eqn(2.4.11) to give

$$ {}^{C}_{R}d\bar{\bar{P}} = {}^{B}_{R}d\bar{x} \cdot \frac{\partial^{A}_{B}\bar{x}}{\partial^{B}_{R}\bar{x}} \cdot \frac{\partial^{C}_{A}\bar{\bar{P}}}{\partial^{A}_{R}\bar{x}} \qquad (2.4.15) $$

Now we have two expressions for ${}^{C}d\bar{\bar{P}}$ involving ${}^{B}d\bar{x}$. We may subtract these one from the other and extract the common factor of ${}^{B}_{R}d\bar{x}$ to get $\bar{\bar{0}}$ as

$$ {}^{C}_{R}d\bar{\bar{P}} - {}^{C}_{R}d\bar{\bar{P}} = {}^{B}_{R}d\bar{x} \cdot \left[\frac{\partial^{C}_{B}\bar{\bar{P}}}{\partial^{B}_{R}\bar{x}} - \frac{\partial^{A}_{B}\bar{x}}{\partial^{B}_{R}\bar{x}} \cdot \frac{\partial^{C}_{A}\bar{\bar{P}}}{\partial^{A}_{R}\bar{x}} \right] = \bar{\bar{0}} \qquad (2.4.16) $$

Since the relative positions of X and Y were never specified we cannot use any special orientation of ${}^{B}d\bar{x}$ to guarantee a normality condition. So it must be that the term in brackets is identically $\bar{\bar{0}}$. This can only be so if

$$ \frac{\partial^{C}_{B}\bar{\bar{P}}}{\partial^{B}\bar{x}} = \frac{\partial^{A}_{B}\bar{x}}{\partial^{B}\bar{x}} \cdot \frac{\partial^{C}_{A}\bar{\bar{P}}}{\partial^{A}\bar{x}} \qquad (2.4.17) $$

Now we have a useful transformation. At our convenience we may leave the gradient untransformed and merely copy values during a change of configuration of reference or we may take the gradient with respect to the new reference configuration.

This is applied in several important cases. Among the most famous are the First and Second Piola-Kirchhoff pseudo-stress tensors. The term "pseudo-" is very descriptive. These tensors are not identical to the Cauchy stress except under special conditions. Instead some transformation has been applied and a different quantity, a pseudo-stress, has been found. These can be very useful. Another very important "pseudo-" quantity is the Lagrange pseudo-strain tensor. The next section is a detailed treatment of the Lagrangian true and pseudo strains.

2.5 STRAIN TENSORS

Strain tensors are key elements in the finite element equations. A careful derivation of the Lagrangian (referential) strain tensor and of incremental forms of the Lagrangian strain tensor is useful here. It will clarify the later use of these important measures of material distortion. This derivation follows from the classical argument concerning the separation of two distinct body-points in various configurations. The separation of two body-points may change from an unstrained configuration to a deformed configuration. Consider the quadratic form for deformation:

$$ {}^{1}D = {}^{1}d\bar{x} \cdot {}^{1}d\bar{x} - {}^{U}d\bar{x} \cdot {}^{U}d\bar{x} \qquad (2.5.1) $$

Note that ${}^{1}D$ as a symbol does not mention C_U. This is because any unstrained configuration will serve. Now, refer this to the deformed configuration, C_1. The differential ${}^{U}d\bar{x}$ can be replaced by

$$ {}^{U}_{1}d\bar{x} = {}^{1}_{1}d\bar{x} \cdot \frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{1}\bar{x}} \qquad (2.5.2) $$

We can maintain generality and facilitate the discussion of the use of an arbitrary reference configuration if we keep the differential ${}^{1}d\bar{x}$ in the form

$$ {}^1_1d\bar{x} \;=\; {}^1_1d\bar{x} \cdot \frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}} \qquad (2.5.3) $$

Introduce these forms to find

$$ {}^1_1D \;=\; {}^1_1d\bar{x}\cdot\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot {}^1_1d\bar{x}\cdot\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right) \;-\; {}^1_1d\bar{x}\cdot\left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot {}^1_1d\bar{x}\cdot\left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right) \qquad (2.5.4) $$

$$ {}^1_1D \;=\; {}^1_1d\bar{x}\cdot\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right)^T\cdot {}^1_1d\bar{x} \;-\; {}^1_1d\bar{x}\cdot\left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot\left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right)^T\cdot {}^1_1d\bar{x} \qquad (2.5.5) $$

$$ {}^1_1D \;=\; {}^1_1d\bar{x}\cdot\left[\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right)^T - \left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot\left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right)^T\right]\cdot {}^1_1d\bar{x} \qquad (2.5.6) $$

$$ {}^1_1D \;=\; {}^1_1d\bar{x}\;{}^1_1d\bar{x} : \left[\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot\left(\frac{\partial {}^1_1\bar{x}}{\partial {}^1_1\bar{x}}\right)^T - \left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right)\cdot\left(\frac{\partial {}^U_1\bar{x}}{\partial {}^1_1\bar{x}}\right)^T\right] \qquad (2.5.7) $$

$$ {}^1_1D \;=\; {}^1_1d\bar{x}\;{}^1_1d\bar{x} : 2\;{}^1_1\bar{\bar{\varepsilon}}({}^1\bar{x}) \qquad (2.5.8) $$

where ${}^{1}\bar{\bar{\varepsilon}}$ is the true strain tensor of the body in configuration C_1, a colon represents the double dot product and right superscript T denotes the transpose of a tensor.

${}^{1}D$ can be referred to any configuration, say C_A, as

$$ {}^{1}_{A}D = {}^{1}_{A}d\bar{x}\ {}^{1}_{A}d\bar{x} : 2\ {}^{1}_{A}\bar{\bar{\varepsilon}}({}^{A}\bar{x}) \tag{2.5.9} $$

$$ {}^{1}_{A}D = {}^{1}_{A}d\bar{x}\ {}^{1}_{A}d\bar{x} : \left[\left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{A}\bar{x}}\right)\cdot\left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{A}\bar{x}}\right)^{T} - \left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{A}\bar{x}}\right)\cdot\left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{A}\bar{x}}\right)^{T}\right] \tag{2.5.10} $$

In calculation the use of these differentials and gradients can be extremely inconvenient or impossible unless the chosen reference configuration is the present configuration. For any other case the analyst may prefer to transform the quantities so that the differentials are of, and derivatives are taken with respect to, the position in the reference configuration. To do this retrace the steps in deriving ${}^{1}\bar{\bar{\varepsilon}}$ starting with

$$ {}^{1}_{A}d\bar{x} = {}^{A}_{A}d\bar{x}\cdot\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}} \tag{2.5.11} $$

and

$$ {}^{U}_{A}d\bar{x} = {}^{A}_{A}d\bar{x} \cdot \frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}} \qquad (2.5.12) $$

Thence

$$ {}^{1}_{A}D = {}^{A}_{A}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot {}^{A}_{A}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) - {}^{A}_{A}d\bar{x} \cdot \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot {}^{A}_{A}d\bar{x} \cdot \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) $$

(2.5.13)

The same sequence is

$$ {}^{1}_{A}D = {}^{A}_{A}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T} \cdot {}^{A}_{A}d\bar{x} - {}^{A}_{A}d\bar{x} \cdot \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T} \cdot {}^{A}_{A}d\bar{x} $$

(2.5.14)

$$ {}^{1}_{A}D = {}^{A}_{A}d\bar{x} \cdot \left[\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T} - \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T}\right] \cdot {}^{A}_{A}d\bar{x} $$

(2.5.15)

$$ {}^{1}_{A}D = {}^{A}_{A}d\bar{x}\ {}^{A}_{A}d\bar{x} : \left[\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T} - \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right) \cdot \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{A}\bar{x}}\right)^{T}\right] $$

(2.5.16)

$$ {}^{1}_{A}D \;=\; {}^{A}_{A}d\bar{x}\;{}^{A}_{A}d\bar{x} \;:\; 2\;{}^{1A}_{A}\bar{\bar{E}}({}^{A}\bar{x}) \qquad (2.5.17) $$

where ${}^{1A}_{A}\bar{\bar{E}}$ is the Lagrange pseudo-strain of a body in C_1 transformed to C_A and referred to C_A. It is possible to refer this to yet another configuration, C_R. One finds

$$ {}^{1}_{R}D \;=\; {}^{A}_{R}d\bar{x}\;{}^{A}_{R}d\bar{x} \;:\; \left[\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)^{T} - \left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{U}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}}\right)^{T}\right] \qquad (2.5.18) $$

$$ {}^{1}_{R}D \;=\; {}^{A}_{R}d\bar{x}\;{}^{A}_{R}d\bar{x} \;:\; 2\;{}^{1A}_{R}\bar{\bar{E}}({}^{R}\bar{x}) \qquad (2.5.19) $$

This shows how each configuration is used in the pseudo-strain. The use of different configurations for C_A and C_R is uncommon. This is because of the implied use of differentials of form ${}^{A}_{R}d\bar{x}$, where the measurement and reference configurations are not the same. This immediately leads to difficulties as illustrated in Figure 2.2.

The transformation between ${}^{1}\bar{\bar{\varepsilon}}$ and ${}^{1A}\bar{\bar{E}}$ can be found by referring eqn(2.5.4) to C_R.

$$ {}^{1}_{R}D = {}^{1}_{R}d\bar{x}\cdot\left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}}\right)\cdot{}^{1}_{R}d\bar{x}\cdot\left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}}\right) - {}^{1}_{R}d\bar{x}\cdot\left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}}\right)\cdot{}^{1}_{R}d\bar{x}\cdot\left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}}\right) \qquad (2.5.20) $$

We can transform ${}^{1}_{R}d\bar{x}$ to

$$ {}^{1}_{R}d\bar{x} = {}^{A}_{R}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right) \tag{2.5.21} $$

So

$$ \begin{aligned} {}^{1}_{R}D = {}^{A}_{R}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right) \cdot {}^{A}_{R}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right) \\ - {}^{A}_{R}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right) \cdot {}^{A}_{R}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right) \end{aligned} \tag{2.5.22} $$

$$ \begin{aligned} {}^{1}_{R}D = {}^{A}_{R}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{1}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right)^{T} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right)^{T} \cdot {}^{A}_{R}d\bar{x} \\ - {}^{A}_{R}d\bar{x} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial^{U}_{1}\bar{x}}{\partial^{1}_{R}\bar{x}} \right)^{T} \cdot \left(\frac{\partial^{1}_{A}\bar{x}}{\partial^{A}_{R}\bar{x}} \right)^{T} \cdot {}^{A}_{R}d\bar{x} \end{aligned} \tag{2.5.23} $$

$$
{}^{1}_{R}D = {}^{A}_{R}d\bar{x} \cdot \left[\left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right)^{T} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right)^{T} \right.
$$

$$
\left. - \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right)^{T} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right)^{T} \right] \cdot {}^{A}_{R}d\bar{x} \qquad (2.5.24)
$$

$$
{}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right)^{T} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right)^{T} \right.
$$

$$
\left. - \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right)^{T} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right)^{T} \right] \qquad (2.5.25)
$$

$$
{}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right) \cdot \left\{ \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right)^{T} \right. \right.
$$

$$
\left. \left. - \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}} \right)^{T} \right\} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right)^{T} \right] \qquad (2.5.26)
$$

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}}\right) \cdot \left\{ \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}}\right) \cdot \left(\frac{\partial {}^{1}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}}\right)^{T} - \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}}\right) \cdot \left(\frac{\partial {}^{U}_{1}\bar{x}}{\partial {}^{1}_{R}\bar{x}}\right)^{T} \right\} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}}\right)^{T} \right] \tag{2.5.26} $$

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2 \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}}\right) \cdot {}^{1}\bar{\bar{\varepsilon}} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}}\right)^{T} \tag{2.5.27} $$

Also, by the definition of ${}^{1A}\bar{\bar{E}}$

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\ {}^{1A}_{R}\bar{\bar{E}} \tag{2.5.28} $$

Now we may subtract eqn(2.5.28) from eqn(2.5.27) to get 0 as

$$ 0 = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2 \left[\left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}}\right) \cdot {}^{1}\bar{\bar{\varepsilon}} \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}}\right)^{T} - {}^{1A}_{R}\bar{\bar{E}} \right] \tag{2.5.29} $$

Since no conditions were assumed for ${}^{A}d\bar{x}$ it must be that the quantity in brackets in eqn(2.5.29) is $\bar{\bar{0}}$. This can only be true iff

$$\left(\frac{\partial_A^{1}\bar{x}}{\partial_R^{A}\bar{x}}\right) \cdot {}^{1}\bar{\bar{\varepsilon}} \cdot \left(\frac{\partial_A^{1}\bar{x}}{\partial_R^{A}\bar{x}}\right)^{T} = {}_{R}^{1A}\bar{\bar{E}} \qquad (2.5.30)$$

Now we can transform true strain to pseudo-strain and vice versa. This allows us to use either form in further development knowing that which ever form is more convenient can be obtained at need. This also shows the condition required for the pseudo-strain, ${}^{1A}\bar{\bar{E}}$, to be the same as the true strain, ${}^{1}\bar{\bar{\varepsilon}}$. The deformation gradient, ${}^{1,A}\bar{\bar{F}}$, must be an identity tensor which is satisfied only when C_1 and C_A are differ by a rigid body translation. We also note here that this is the same transformation as is required for the First or Second Piola-Kirchhoff pseudo-stress tensors to be identical with the Cauchy true stress tensor.

2.6 DISPLACEMENT FORMS FOR THE STRAIN TENSOR

In most work it is the Lagrange pseudo-strain that is used. So, further investigation of displacement-based forms is developed for that tensor. In practice it is usually the case that the transformation configuration, C_A, is the same as the reference configuration, C_R. However, for the sake of clarity we will assume that C_A is not necessarily the same as C_R.

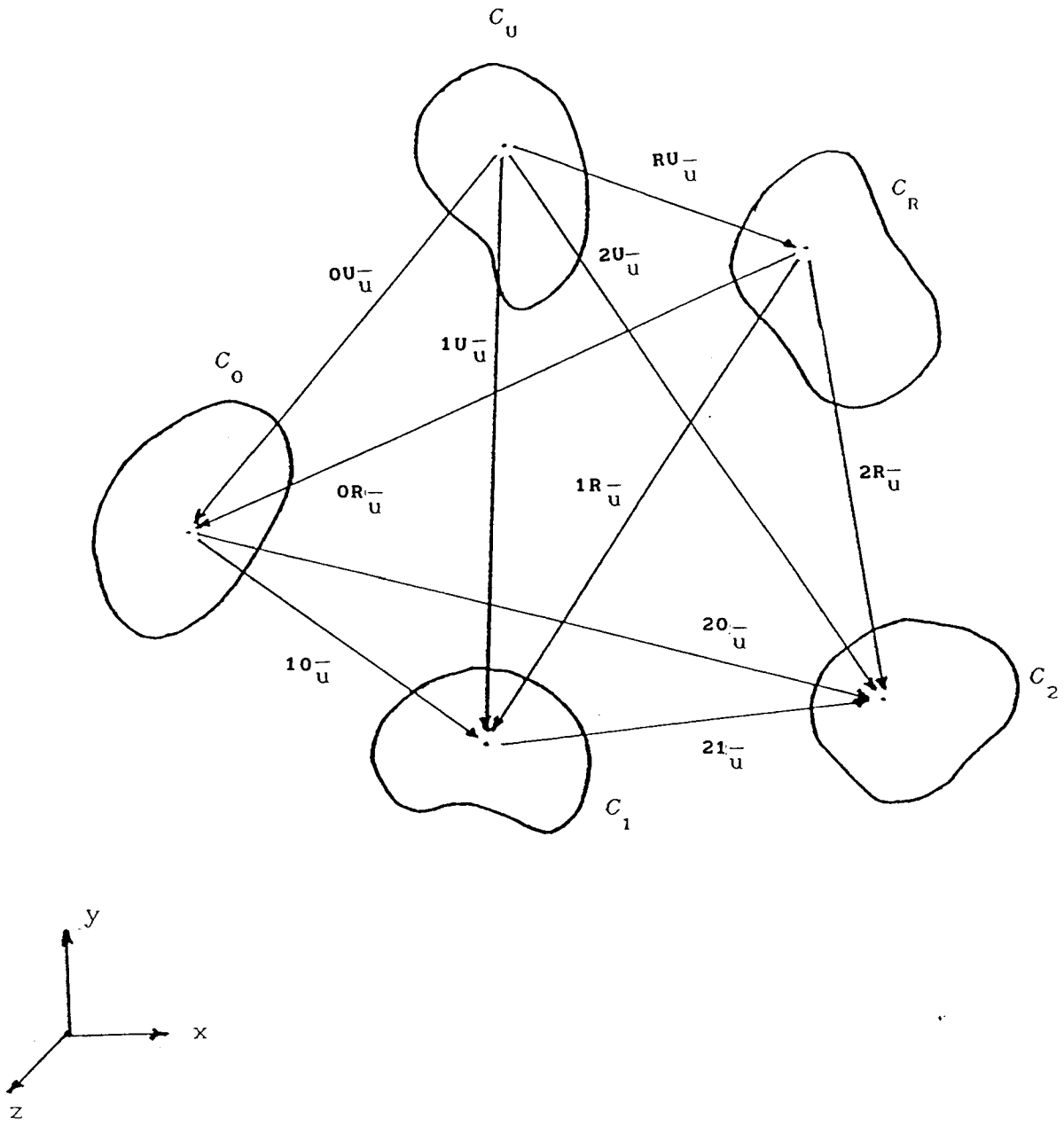


Figure 2.3 Configurations of a body and displacements of body-point X

We may express $^{U}\bar{x}$ and $^{1}\bar{x}$ as displaced from $^{A}\bar{x}$. From Figure 2.3 it can be seen that

$$^{U}\bar{x} = {}^{A}\bar{x} + {}^{UA}\bar{u}, \tag{2.6.1}$$

Or, since

$$^{UA}\bar{u} = - {}^{AU}\bar{u} \tag{2.6.2}$$

$$^{U}\bar{x} = {}^{A}\bar{x} - {}^{AU}\bar{u}. \tag{2.6.3}$$

The gradient

$$\frac{\partial_A^{U}\bar{x}(^{A}\bar{x})}{\partial_R^{A}\bar{x}} = \frac{\partial_A^{A}\bar{x}(^{A}\bar{x})}{\partial_R^{A}\bar{x}} - \frac{\partial_A^{AU}u(^{A}\bar{x})}{\partial_R^{A}\bar{x}} \tag{2.6.4}$$

This can be written as

$$\frac{\partial_A^{U}\bar{x}}{\partial_R^{A}\bar{x}} = \bar{\bar{1}} - {}^{AU,A}_{R}\bar{\bar{u}} \tag{2.6.5}$$

where $\bar{\bar{1}}$ represents the identity tensor and

$$ {}^{AU,A}_{R}\bar{\bar{u}} = \frac{\partial {}^{AU}_{A}\bar{u}}{\partial {}^{A}_{R}\bar{x}} , \qquad (2.6.6) $$

is the displacement gradient. Notice that in eqn(2.6.4) ${}^{A}_{A}\bar{x}$ is a function of itself and is differentiated with respect to itself. Thus its gradient is the identity tensor, even though the gradient is referred to C_R. Similarly,

$$ {}^{1}\bar{x} = {}^{A}\bar{x} + {}^{1A}\bar{u}, \qquad (2.6.7) $$

$$ \frac{\partial {}^{1}_{A}\bar{x}({}^{A}\bar{x})}{\partial {}^{A}_{R}\bar{x}} = \frac{\partial {}^{A}_{A}\bar{x}({}^{A}\bar{x})}{\partial {}^{A}_{R}\bar{x}} + \frac{\partial {}^{1A}_{A}u({}^{A}\bar{x})}{\partial {}^{A}_{R}\bar{x}} \qquad (2.6.8) $$

$$ \frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} = \bar{\bar{1}} + {}^{1A,A}_{R}\bar{\bar{u}} \qquad (2.6.9) $$

Now these expressions are substituted into ${}^{1}_{R}D$.

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{1}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right)^{T} - \left(\frac{\partial {}^{U}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right) \cdot \left(\frac{\partial {}^{U}_{A}\bar{x}}{\partial {}^{A}_{R}\bar{x}} \right)^{T} \right] \qquad (2.6.10) $$

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\left(\bar{\bar{1}} + {}^{1A,A}_{R}\bar{\bar{u}} \right) \cdot \left(\bar{\bar{1}} + {}^{1A,A}_{R}\bar{\bar{u}} \right)^{T} - \left(\bar{\bar{1}} + {}^{AU,A}_{R}\bar{\bar{u}} \right) \cdot \left(\bar{\bar{1}} + {}^{AU,A}_{R}\bar{\bar{u}} \right)^{T} \right] $$

(2.6.11)

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\left(\bar{\bar{1}} \cdot \bar{\bar{1}}^{T} \right) + \left(\bar{\bar{1}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^{T} \right) + \left({}^{1A,A}_{R}\bar{\bar{u}} \cdot \bar{\bar{1}}^{T} \right) + \left({}^{1A,A}_{R}\bar{\bar{u}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^{T} \right) \right. $$

$$ \left. - \left(\bar{\bar{1}} \cdot \bar{\bar{1}}^{T} \right) + \left(\bar{\bar{1}} \cdot {}^{AU,A}_{R}\bar{\bar{u}}^{T} \right) + \left({}^{AU,A}_{R}\bar{\bar{u}} \cdot \bar{\bar{1}}^{T} \right) - \left({}^{AU,A}_{R}\bar{\bar{u}} \cdot {}^{AU,A}_{R}\bar{\bar{u}}^{T} \right) \right] $$

(2.6.12)

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\begin{array}{l} {}^{1A,A}_{R}\bar{\bar{u}}^{T} + {}^{1A,A}_{R}\bar{\bar{u}} + \left({}^{1A,A}_{R}\bar{\bar{u}} \cdot {}^{1A,A}_{R}\bar{\bar{u}}^{T} \right) \\ \quad + {}^{AU,A}_{R}\bar{\bar{u}}^{T} + {}^{AU,A}_{R}\bar{\bar{u}} - \left({}^{AU,A}_{R}\bar{\bar{u}} \cdot {}^{AU,A}_{R}\bar{\bar{u}}^{T} \right) \end{array} \right] $$

(2.6.13)

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\ {}^{1A}_{R}\bar{\bar{E}} \qquad (2.6.14) $$

where the pseudo-strain tensor

$$ {}^{1A}_{R}\overline{\overline{E}} = \frac{1}{2}\left[\left\{{}^{1A,A}_{R}\overline{\overline{u}}^{T} + {}^{1A,A}_{R}\overline{\overline{u}} + \left({}^{1A,A}_{R}\overline{\overline{u}} \cdot {}^{1A,A}_{R}\overline{\overline{u}}^{T}\right)\right\}\right. \qquad (2.6.15) $$

$$ \left. + \left\{{}^{AU,A}_{R}\overline{\overline{u}}^{T} + {}^{AU,A}_{R}\overline{\overline{u}} - \left({}^{AU,A}_{R}\overline{\overline{u}} \cdot {}^{AU,A}_{R}\overline{\overline{u}}^{T}\right)\right\}\right] $$

The braces in eqn(2.6.15) are unnecessary but serve to emphasize a grouping of tensors in the final expression in terms of the original two gradients as their source. It can be seen that the first term in braces depends on C_1 and on C_A but not on C_U. So a different choice of C_U will not affect this term. The second term in braces depends on C_A and on C_U. So a change in C_1 will not affect this term. This property reappears later. In general all six terms are nonzero and all six must be evaluated.

A few special cases should be considered.

If the transformation configuration is an unstrained configuration, then

$$ {}^{A}d\overline{x} = {}^{U}d\overline{x}, \quad {}^{1A,A}_{R}\overline{\overline{u}} = {}^{1U,U}_{U}\overline{\overline{u}} \quad \text{and} \quad {}^{AU,A}_{R}\overline{\overline{u}} = {}^{UU,\,U}_{U}\overline{\overline{u}} = \overline{\overline{0}} \qquad (2.6.16) $$

This shows the quadratic measure of deformation:

$$ {}^{1}_{R}D = {}^{U}_{R}d\bar{x}\ {}^{U}_{R}d\bar{x} : \left[{}^{1U,U}_{R}\bar{\bar{u}} + {}^{1U,U}_{R}\bar{\bar{u}}^{T} + {}^{1U,U}_{R}\bar{\bar{u}} \cdot {}^{1U,U}_{R}\bar{\bar{u}}^{T} \right] \tag{2.6.17} $$

$$ {}^{1}_{R}D = {}^{U}_{R}d\bar{x}\ {}^{U}_{R}d\bar{x} : 2\ {}^{1U}_{R}\bar{\bar{E}}({}^{R}\bar{x}) \tag{2.6.18} $$

where ${}^{1U}_{R}\bar{\bar{E}}$ is recognized as the Green - Lagrange (pseudo-) strain tensor.

If the transformation configuration is the present configuration, then

$$ {}^{A}d\bar{x} = {}^{1}d\bar{x}, \quad {}^{1A,A}_{R}\bar{\bar{u}} = {}^{11,1}_{R}\bar{\bar{u}} = \bar{\bar{0}} \quad \text{and} \quad {}^{AU,A}_{R}\bar{\bar{u}} = {}^{1U,\ 1}_{R}\bar{\bar{u}} \tag{2.6.19} $$

which yields

$$ {}^{1}_{R}D = {}^{1}_{R}d\bar{x}\ {}^{1}_{R}d\bar{x} : \left[{}^{1U,U}_{R}\bar{\bar{u}} + {}^{1U,U}_{R}\bar{\bar{u}}^{T} - {}^{1U,U}_{R}\bar{\bar{u}} \cdot {}^{1U,U}_{R}\bar{\bar{u}}^{T} \right] \tag{2.6.20} $$

$$ {}^{1}_{R}D = {}^{1}_{R}d\bar{x}\ {}^{1}_{R}d\bar{x} : 2\ {}^{11}_{R}\bar{\bar{E}}({}^{R}\bar{x}) \tag{2.6.21} $$

where ${}^{11}\bar{\bar{E}}$, the Lagrange pseudo-strain tensor, is identical to the Lagrange true strain tensor, ${}^{1}\bar{\bar{\varepsilon}}$. This must be since each time the configuration of transformation matches the configuration of measurement the transformation is pre- and post multiplication by the identity

tensor. ${}^{11}\overline{\overline{E}}$ is recognized as being identical to the Euler strain tensor, ${}^{1}\overline{\overline{\varepsilon}}$.

It should be noted that the entire formulation of the strain tensor has been referential. So the use of C_1 as a reference configuration is an example of an updated Lagrangian formulation. Every time the transformation configuration is updated to coincide with the present configuration, there can be no instantaneous difference from the spatial (Eulerian) formulation. Therefore the last (nonlinear) term is not always added to strain tensors in referential (Lagrangian) formulations, but will always be subtracted unless

$$ {}^{AU}\overline{u} \equiv \overline{0} \tag{2.6.22} $$

If the present configuration is an unstrained configuration, then

$$ {}^{1A,A}_{R}\overline{u} = {}^{UA,A}_{R}\overline{\overline{u}} = -{}^{AU,A}_{R}\overline{\overline{u}} \tag{2.6.23} $$

This choice yields

$$ {}^{1}_{R}D = {}^{U}_{R}D = {}^{A}_{R}\overline{dx}\,{}^{A}_{R}\overline{dx}:\left[\left(-{}^{AU,A}_{R}\overline{\overline{u}}^{T} - {}^{AU,A}_{R}\overline{\overline{u}} + \left({}^{AU,A}_{R}\overline{\overline{u}}\cdot{}^{AU,A}_{R}\overline{\overline{u}}^{T}\right)\right)\right. $$

$$ \left. + \left({}^{AU,A}_{R}\overline{\overline{u}}^{T} + {}^{AU,A}_{R}\overline{\overline{u}} - \left({}^{AU,A}_{R}\overline{\overline{u}}\cdot{}^{AU,A}_{R}\overline{\overline{u}}^{T}\right)\right)\right] \tag{2.6.24} $$

$$ {}^{1}_{R}D = {}^{U}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\ {}^{UA}_{R}\bar{\bar{E}} \qquad (2.6.25) $$

$$ {}^{1}_{R}D = {}^{U}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\ {}_{R}\bar{\bar{0}} = 0 \qquad (2.6.26) $$

The result that ${}^{U}D$ is zero is expected. This could have been arrived at more directly from the definition of ${}^{U}D$ without involving displacements, which gives that ${}^{UA}\bar{\bar{E}}$, ${}^{U}\bar{\bar{\varepsilon}}$ and ${}^{U}\bar{\bar{\varepsilon}}$ are all zero for any unstrained configuration but sheds no light on the nature of being unstrained.

2.7 INCREMENTS OF STRAIN TENSORS

The same quadratic measure of deformation in C_2 is

$$ {}^{2}D = {}^{2}d\bar{x} \cdot {}^{2}d\bar{x} - {}^{U}d\bar{x} \cdot {}^{U}d\bar{x} \qquad (2.7.1) $$

which, when transformed to C_A and referred to C_R, is

$$ {}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : \left[\left\{ {}^{2A,A\bar{\bar{u}}^{T}}_{R} + {}^{2A,A\bar{\bar{u}}}_{R} + \left({}^{2A,A\bar{\bar{u}}}_{R} \cdot {}^{2A,A\bar{\bar{u}}^{T}}_{R} \right) \right\} \right. $$

$$ \left. + \left\{ {}^{AU,A\bar{\bar{u}}^{T}}_{R} + {}^{AU,A\bar{\bar{u}}}_{R} - \left({}^{AU,A\bar{\bar{u}}}_{R} \cdot {}^{AU,A\bar{\bar{u}}^{T}}_{R} \right) \right\} \right] \qquad (2.7.2) $$

$$ {}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\ {}^{2A\bar{\bar{E}}}_{R}(\bar{x}^{R}) \qquad (2.7.3) $$

It should be noted that the terms involving ${}^{AU\bar{\bar{u}}}_{R}$ (the second group in braces) have not changed.

It is possible to think of the displacement from C_A to C_2 as a superposition of a displacement field from C_A to C_1 and an incremental displacement from C_1 to C_2. That is

$$ {}^{2}\bar{x} = {}^{A}\bar{x} + {}^{2A}\bar{u} = {}^{A}\bar{x} + {}^{1A}\bar{u} + {}^{21}\bar{u} \qquad (2.7.4) $$

since

$$ {}^{1A}\bar{u} + {}^{21}\bar{u} = {}^{2A}\bar{u}. \qquad (2.7.5) $$

In that case, the displacement gradient can be expressed as

$$\frac{\partial^{2A}_{A}\bar{u}}{\partial\,{}^{A}_{R}\bar{x}} = {}^{2A,A}_{R}\bar{\bar{u}} = \frac{\partial^{1A}_{A}\bar{u}}{\partial\,{}^{A}_{R}\bar{x}} + \frac{\partial^{21}_{A}\bar{u}}{\partial\,{}^{A}_{R}\bar{u}} = {}^{1A,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}} \tag{2.7.6}$$

Substituting these gradients into the expression for the quadratic measure of deformation gives

$$\begin{aligned} {}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x}:\Bigg[& \Bigg\{ \left({}^{1A,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}} \right) + \left({}^{1A,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}} \right)^{T} \\ & + \left({}^{1A,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}} \right)\cdot\left({}^{1A,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}} \right)^{T} \Bigg\} \\ & + \left\{ {}^{AU,A}_{R}\bar{\bar{u}}^{T} + {}^{AU,A}_{R}\bar{\bar{u}} - \left({}^{AU,A}_{R}\bar{\bar{u}}\cdot{}^{AU,A}_{R}\bar{\bar{u}}^{T} \right) \right\} \Bigg] \end{aligned} \tag{2.7.7}$$

$$\begin{aligned} {}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x}:\Bigg[& \Bigg\{ {}^{1A,A}_{R}\bar{\bar{u}} + {}^{21,A}_{R}\bar{\bar{u}} + {}^{1A,A}_{R}\bar{\bar{u}}^{T} + {}^{21,A}_{R}\bar{\bar{u}}^{T} \\ & + {}^{1A,A}_{R}\bar{\bar{u}}\cdot{}^{1A,A}_{R}\bar{\bar{u}}^{T} + {}^{1A,A}_{R}\bar{\bar{u}}\cdot{}^{21,A}_{R}\bar{\bar{u}}^{T} \\ & + {}^{21,A}_{R}\bar{\bar{u}}\cdot{}^{1A,A}_{R}\bar{\bar{u}}^{T} + {}^{21,A}_{R}\bar{\bar{u}}\cdot{}^{21,A}_{R}\bar{\bar{u}}^{T} \Bigg\} \\ & + \left\{ {}^{AU,A}_{R}\bar{\bar{u}}^{T} + {}^{AU,A}_{R}\bar{\bar{u}} - \left({}^{AU,A}_{R}\bar{\bar{u}}\cdot{}^{AU,A}_{R}\bar{\bar{u}}^{T} \right) \right\} \Bigg] \end{aligned} \tag{2.7.8}$$

The terms can be arranged as:

$$
\begin{aligned}
{}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x}:\Bigg[& \left\{ {}^{1A,A=}_{R}u + {}^{1A,A=T}_{R}u + {}^{1A,A=}_{R}u\,.\,{}^{1A,A=T}_{R}u \right\} \\
& + \left\{ {}^{21,A=}_{R}u + {}^{21,A=T}_{R}u + {}^{1A,A=}_{R}u\,.\,{}^{21,A=T}_{R}u + {}^{21,A=}_{R}u\,.\,{}^{1A,A=T}_{R}u \right\} \\
& + \left\{ {}^{21,A=}_{R}u\,.\,{}^{21,A=T}_{R}u \right\} \\
& + \left\{ {}^{AU,A=T}_{R}u + {}^{AU,A=}_{R}u - \left({}^{AU,A=}_{R}u\,.\,{}^{AU,A=T}_{R}u \right) \right\} \Bigg]
\end{aligned}
\tag{2.7.9}
$$

$$
{}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\ {}^{2A=}_{R}E \tag{2.7.10}
$$

It is convenient to keep these terms in separate groups in ${}^{2A=}_{R}E$ such that

$$
{}^{2A=}_{R}E = {}^{2A=1}_{R}E + {}^{2A=2}_{R}E + {}^{2A=3}_{R}E + {}^{2A=4}_{R}E \tag{2.7.11}
$$

where

$$
{}^{2A=1}_{R}E = \frac{1}{2} \left\{ {}^{1A,A=}_{R}u + {}^{1A,A=T}_{R}u + {}^{1A,A=}_{R}u\,.\,{}^{1A,A=T}_{R}u \right\} \tag{2.7.12}
$$

$$
{}^{2A=2}_{R}E = \frac{1}{2} \left\{ {}^{21,A=}_{R}u + {}^{21,A=T}_{R}u + {}^{1A,A=}_{R}u\,.\,{}^{21,A=T}_{R}u + {}^{21,A=}_{R}u\,.\,{}^{1A,A=T}_{R}u \right\} \tag{2.7.13}
$$

$$ {}^{2A}_{R}\bar{\bar{E}}^{3} = \frac{1}{2}\left\{ {}^{21,A}_{R}\bar{\bar{u}} \cdot {}^{21,A}_{R}\bar{\bar{u}}^{T} \right\} \qquad (2.7.14) $$

$$ {}^{2A}_{R}\bar{\bar{E}}^{4} = \frac{1}{2}\left\{ {}^{AU,A}_{R}\bar{\bar{u}}^{T} + {}^{AU,A}_{R}\bar{\bar{u}} - {}^{AU,A}_{R}\bar{\bar{u}} \cdot {}^{AU,A}_{R}\bar{\bar{u}}^{T} \right\} \qquad (2.7.15) $$

If the body achieves configuration C_2 as a result of an incremental change from C_1, then ${}^{2A}_{R}\bar{\bar{E}}$ can be thought of as incrementally changed from ${}^{1A}_{R}\bar{\bar{E}}$. It is recognized that

$$ {}^{1}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\left({}^{2A}_{R}\bar{\bar{E}}^{1} + {}^{2A}_{R}\bar{\bar{E}}^{4} \right) \qquad (2.7.16) $$

The effect of an increment of motion must appear in ${}^{21}_{R}\bar{u}$ in terms ${}^{2A}_{R}\bar{\bar{E}}^{2}$ and ${}^{2A}_{R}\bar{\bar{E}}^{3}$. ${}^{2A}_{R}\bar{\bar{E}}^{2}$ contains terms linear in ${}^{21}_{R}\bar{u}$ and may be called the linear part of the increment to ${}^{1A}_{R}\bar{\bar{E}}$. ${}^{2A}_{R}\bar{\bar{E}}^{3}$ contains the nonlinear (quadratic) term and may be called the nonlinear part of the increment to ${}^{1A}_{R}\bar{\bar{E}}$. This is mentioned in Underhill et al. (1989), Gadala (1980), Gadala et al. (1982), Abo-Elkhier (1985) and Abo-Elkhier et al. (1985).

Now let some special cases be examined and some results found.

C_A is C_U. If C_R were also C_U, then a total Lagrangian formulation would be in use.

$$ \begin{aligned} {}^{A}_{R}d\bar{x} &= {}^{U}_{R}d\bar{x}, & {}^{1A,A}_{R}\bar{\bar{u}} &= {}^{1U,U}_{R}\bar{\bar{u}}, \\ {}^{21,A}_{R}\bar{\bar{u}} &= {}^{21,U}_{R}\bar{\bar{u}}, & {}^{AU,A}_{R}\bar{\bar{u}} &= {}^{UU,U}_{R}\bar{\bar{u}} = \bar{\bar{0}} \end{aligned} \qquad (2.7.17) $$

and

$$ {}^{2}_{R}D = {}^{U}_{R}\bar{dx}\ {}^{U}_{R}\bar{dx} : 2\ {}^{2U\bar{=}}_{R}\bar{E}({}^{R}_{R}\bar{x}) \tag{2.7.18} $$

We find

$$ {}^{2U\bar{=}1}_{R}\bar{E} = \frac{1}{2}\left\{ {}^{1U,U\bar{=}}_{R}\bar{u} + {}^{1U,U\bar{=}T}_{R}\bar{u} + {}^{1U,U\bar{=}}_{R}\bar{u}\cdot{}^{1U,U\bar{=}T}_{R}\bar{u} \right\} \tag{2.7.19} $$

$$ {}^{2U\bar{=}2}_{R}\bar{E} = \frac{1}{2}\left\{ {}^{21,U\bar{=}}_{R}\bar{u} + {}^{21,U\bar{=}T}_{R}\bar{u} + {}^{1U,U\bar{=}}_{R}\bar{u}\cdot{}^{2U,U\bar{=}T}_{R}\bar{u} + {}^{2U,U\bar{=}}_{R}\bar{u}\cdot{}^{1U,U\bar{=}T}_{R}\bar{u} \right\} \tag{2.7.20} $$

$$ {}^{2U\bar{=}3}_{R}\bar{E} = \frac{1}{2}\left\{ {}^{2U,U\bar{=}}_{R}\bar{u}\cdot{}^{2U,U\bar{=}T}_{R}\bar{u} \right\} \tag{2.7.21} $$

and ${}^{2U\bar{=}4}_{R}\bar{E}$ is $\bar{\bar{0}}$ as always in a total Lagrangian formulation.

If the transformation configuration is the next configuration, C_2, then

$$ {}^{A}_{R}\bar{dx} = {}^{2}_{R}\bar{dx}, \qquad {}^{1A,A\bar{=}}_{R}\bar{u} = {}^{12,2\bar{=}}_{R}\bar{u} = -{}^{21,2\bar{=}}_{R}\bar{u} $$
$$ {}^{21,A\bar{=}}_{R}\bar{u} = {}^{21,2\bar{=}}_{R}\bar{u}, \qquad {}^{AU,A\bar{=}}_{R}\bar{u} = {}^{2U,2\bar{=}}_{R}\bar{u} \tag{2.7.22} $$

and

$$ {}^{2}_{R}D \;=\; {}^{2}_{R}d\bar{x}\;{}^{2}_{R}d\bar{x} \;:\; 2\;{}^{22}_{R}\bar{\bar{E}}({}^{R}_{R}\bar{x}) \qquad (2.7.23) $$

We find

$$ {}^{22}_{R}\bar{\bar{E}}^{1} \;=\; \frac{1}{2}\left\{ -{}^{21,2}_{R}\bar{\bar{u}} - {}^{21,2}_{R}\bar{\bar{u}}^{T} + \left(-{}^{21,2}_{R}\bar{\bar{u}}\right)\cdot\left(-{}^{21,2}_{R}\bar{\bar{u}}^{T}\right) \right\} \qquad (2.7.24) $$

$$ {}^{22}_{R}\bar{\bar{E}}^{2} = \frac{1}{2}\left\{ {}^{21,2}_{R}\bar{\bar{u}} + {}^{21,2}_{R}\bar{\bar{u}}^{T} + \left(-{}^{21,2}_{R}\bar{\bar{u}}\right)\cdot\left({}^{21,2}_{R}\bar{\bar{u}}^{T}\right) + \left({}^{21,2}_{R}\bar{\bar{u}}\right)\cdot\left(-{}^{21,2}_{R}\bar{\bar{u}}^{T}\right)\right\} \qquad (2.7.25) $$

$$ {}^{22}_{R}\bar{\bar{E}}^{3} \;=\; \frac{1}{2}\left\{ {}^{21,2}_{R}\bar{\bar{u}}\cdot{}^{21,2}_{R}\bar{\bar{u}}^{T} \right\} \qquad (2.7.26) $$

$$ {}^{22}_{R}\bar{\bar{E}}^{1} + {}^{22}_{R}\bar{\bar{E}}^{2} + {}^{22}_{R}\bar{\bar{E}}^{3} \;=\; \bar{\bar{0}} \qquad (2.7.27) $$

$$ {}^{22}_{R}\bar{\bar{E}}^{4} \;=\; \frac{1}{2}\left\{ {}^{2U,2}_{R}\bar{\bar{u}} + {}^{2U,2}_{R}\bar{\bar{u}}^{T} - {}^{2U,2}_{R}\bar{\bar{u}}\cdot{}^{2U,2}_{R}\bar{\bar{u}}^{T} \right\} \qquad (2.7.28) $$

This is also as expected. C_A being the same as C_2 is the particular case where referential (Lagrangian) and spatial (Eulerian) formulations coincide.

If C_A is C_1, then this is a case of the referential formulation when the configuration of transformation has been "updated" to C_1. Here

$$
\begin{aligned}
&{}^{A}_{R}d\bar{x} = {}^{1}_{R}d\bar{x}, && {}^{1A,A}_{\ R}\bar{\bar{u}} = {}^{11,1}_{\ R}\bar{\bar{u}} = \bar{\bar{0}} \\
&{}^{21,A}_{\ R}\bar{\bar{u}} = {}^{21,1}_{\ R}\bar{\bar{u}}, && {}^{AU,A}_{\ R}\bar{\bar{u}} = {}^{1U,1}_{\ R}\bar{\bar{u}}
\end{aligned}
\qquad (2.7.29)
$$

and

$$
{}^{2}_{R}D = {}^{A}_{R}d\bar{x}\ {}^{A}_{R}d\bar{x} : 2\ {}^{2A}_{R}\bar{\bar{E}}({}^{R}_{R}\bar{x}) \qquad (2.7.30)
$$

We find

$$
{}^{21}_{1}\bar{\bar{E}}^{1} = \bar{\bar{0}} \qquad (2.7.31)
$$

$$
{}^{21}_{1}\bar{\bar{E}}^{2} = \frac{1}{2}\left\{ {}^{21,1}_{\ 1}\bar{\bar{u}} + {}^{21,1}_{\ 1}\bar{\bar{u}}^{T} \right\} \qquad (2.7.32)
$$

$$
{}^{21}_{1}\bar{\bar{E}}^{3} = \frac{1}{2}\left\{ {}^{21,1}_{\ 1}\bar{\bar{u}} \cdot {}^{21,1}_{\ 1}\bar{\bar{u}}^{T} \right\} \qquad (2.7.33)
$$

$$
{}^{21}_{1}\bar{\bar{E}}^{4} = \frac{1}{2}\left\{ {}^{1U,1}_{\ 1}\bar{\bar{u}} + {}^{1U,1}_{\ 1}\bar{\bar{u}}^{T} - {}^{1U,1}_{\ 1}\bar{\bar{u}} \cdot {}^{1U,1}_{\ 1}\bar{\bar{u}}^{T} \right\} = {}^{11}_{1}\bar{\bar{E}} \qquad (2.7.34)
$$

This shows the strain tensor changing by a linear increment plus a nonlinear increment from ${}^{11}_{1}\bar{\bar{E}}$. As the body passes through configuration C_1 the referential and spatial formulations coincide. As the body proceeds to deform an incremental change in strain occurs. The difference between ${}^{21}_{1}\bar{\bar{E}}^{2}$ and ${}^{21}_{U}\bar{\bar{E}}^{2}$, the linear parts of the increment, should be noticed. Compare the last two terms in each

$$ {}^{2U}_{U}\bar{\bar{E}}^{2} = \frac{1}{2}\left\{ {}^{21,U}_{U}\bar{\bar{u}} + {}^{21,U}_{U}\bar{\bar{u}}^{T} + {}^{1U,U}_{U}\bar{\bar{u}} \cdot {}^{21,U}_{U}\bar{\bar{u}}^{T} + {}^{21,U}_{U}\bar{\bar{u}} \cdot {}^{1U,U}_{U}\bar{\bar{u}}^{T} \right\} \tag{2.7.35} $$

$$ {}^{21}_{1}\bar{\bar{E}}^{2} = \frac{1}{2}\left\{ {}^{21,1}_{1}\bar{\bar{u}} + {}^{21,1}_{1}\bar{\bar{u}}^{T} + {}^{11,1}_{1}\bar{\bar{u}} \cdot {}^{21,1}_{1}\bar{\bar{u}}^{T} + {}^{21,1}_{1}\bar{\bar{u}} \cdot {}^{11,1}_{1}\bar{\bar{u}}^{T} \right\} \tag{2.7.36} $$

Since ${}^{11,1}\bar{\bar{u}} = \bar{\bar{0}}$, the last two terms of ${}^{21}\bar{\bar{E}}^{2}$ vanish identically. This is a distinctive mark of the updated Lagrangian formulation.

2.8 IMPLEMENTATION OF CHANGE OF CONFIGURATION OF TRANSFORMATION

Updated Lagrangian calculations occasionally "update" the configuration of transformation to a recently calculated configuration. This is accomplished easily by the use of the deformation gradient.

This change can be found by the following argument. The Lagrangian pseudo strain in configuration C_A transformed to C_P and referred to C_R is:

$$ {}^{AP}_{R}\overline{\overline{E}} = \frac{1}{2}\left[\left(\frac{\partial^{A}_{P}\bar{x}}{\partial^{P}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{A}_{P}\bar{x}}{\partial^{P}_{R}\bar{x}}\right)^{T} - \left(\frac{\partial^{U}_{P}\bar{x}}{\partial^{P}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{U}_{P}\bar{x}}{\partial^{P}_{R}\bar{x}}\right)^{T}\right] \qquad (2.8.1) $$

The pseudo strain in C_A transformed to C_Q and referred to C_R is:

$$ {}^{AQ}_{R}\overline{\overline{E}} = \frac{1}{2}\left[\left(\frac{\partial^{A}_{Q}\bar{x}}{\partial^{Q}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{A}_{Q}\bar{x}}{\partial^{Q}_{R}\bar{x}}\right)^{T} - \left(\frac{\partial^{U}_{Q}\bar{x}}{\partial^{Q}_{R}\bar{x}}\right)\cdot\left(\frac{\partial^{U}_{Q}\bar{x}}{\partial^{Q}_{R}\bar{x}}\right)^{T}\right] \qquad (2.8.2) $$

The deformation gradient is used to find

$$ \frac{\partial^{A}_{Q}\bar{x}}{\partial^{Q}\bar{x}} = \left(\frac{\partial^{P}_{Q}\bar{x}}{\partial^{Q}\bar{x}}\right)\cdot\left(\frac{\partial^{A}_{P}\bar{x}}{\partial^{P}\bar{x}}\right) \qquad (2.8.3) $$

and

$$ \frac{\partial^{U}_{Q}\bar{x}}{\partial^{Q}\bar{x}} = \left(\frac{\partial^{P}_{Q}\bar{x}}{\partial^{Q}\bar{x}}\right)\cdot\left(\frac{\partial^{U}_{P}\bar{x}}{\partial^{P}\bar{x}}\right) \qquad (2.8.4) $$

so that

$$ {}^{AQ}_{R}\overline{\overline{E}} = \frac{1}{2}\left[\left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)\cdot\left(\frac{\partial {}^{A}_{P}\overline{x}}{\partial {}^{P}_{R}\overline{x}} \right)\cdot\left(\frac{\partial {}^{A}_{P}\overline{x}}{\partial {}^{P}_{R}\overline{x}} \right)^{T}\cdot\left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)^{T} \right. \tag{2.8.5} $$

$$ \left. - \left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)\cdot\left(\frac{\partial {}^{U}_{P}\overline{x}}{\partial {}^{P}_{R}\overline{x}} \right)\cdot\left(\frac{\partial {}^{U}_{P}\overline{x}}{\partial {}^{P}_{R}\overline{x}} \right)^{T}\cdot\left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)^{T} \right] $$

$$ {}^{AQ}_{R}\overline{\overline{E}} = \frac{1}{2}\left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)\cdot\left[\left(\frac{\partial {}^{A}_{P}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)\cdot\left(\frac{\partial {}^{A}_{P}\overline{x}}{\partial {}^{P}_{R}\overline{x}} \right)^{T} \right. \tag{2.8.6} $$

$$ \left. - \left(\frac{\partial {}^{U}_{P}\overline{x}}{\partial {}^{P}_{R}\overline{x}} \right)\cdot\left(\frac{\partial {}^{U}_{P}\overline{x}}{\partial {}^{P}_{R}\overline{x}} \right)^{T} \right]\cdot\left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)^{T} $$

$$ {}^{AQ}_{R}\overline{\overline{E}} = \left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)\cdot {}^{AP}_{R}\overline{\overline{E}}\cdot\left(\frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} \right)^{T} \tag{2.8.7} $$

But, typically the position of a body-point in the new transformation configuration, C_Q, is known as a function of position in the old transformation configuration, C_P, rather than the other way around. So, the deformation gradient

$$ \frac{\partial {}^{P}_{Q}\overline{x}}{\partial {}^{Q}_{R}\overline{x}} $$

is not known explicitly. Fortunately, this difficulty is avoidable since all motions are required to be unique invertible functions of position. It can be shown from the invariance of ${}^{P}d\overline{x}$ that

$$\frac{\partial {}^{P}_{Q}\bar{x}}{\partial {}^{Q}\bar{x}} = \left(\frac{\partial {}^{Q}_{P}\bar{x}}{\partial {}^{P}\bar{x}} \right)^{-1} \tag{2.8.8}$$

The updated pseudo strain tensor becomes

$${}^{AQ}_{R}\bar{\bar{E}} = \left(\frac{\partial {}^{Q}_{P}\bar{x}}{\partial {}^{P}_{R}\bar{x}} \right)^{-1} \cdot {}^{AP}_{R}\bar{\bar{E}} \cdot \left(\frac{\partial {}^{Q}_{P}\bar{x}}{\partial {}^{P}_{R}\bar{x}} \right)^{-T} \tag{2.8.9}$$

As an example, let a calculation follow the motion of a body from an initial, unstrained configuration, C_0. As is common practice, in this example the same configuration will be used for both transformation and the reference. The body moves and deforms and the pseudo strain in C_1 is calculated, with C_0 as the transformation and reference configurations, to be ${}^{10}_{0}\bar{\bar{E}}$. The pseudo strain tensor is updated so that C_1 is the transformation and reference configurations by the use of

$${}^{11}_{1}\bar{\bar{E}} = \left(\frac{\partial {}^{1}_{0}\bar{x}}{\partial {}^{0}_{0}\bar{x}} \right)^{-1} \cdot {}^{10}_{0}\bar{\bar{E}} \cdot \left(\frac{\partial {}^{1}_{0}\bar{x}}{\partial {}^{0}_{0}\bar{x}} \right)^{-T} \tag{2.8.10}$$

Let the calculation continue and the pseudo strain be required in a subsequent configuration, C_2, where the displacement from C_1 to C_2 is known explicitly. Then the pseudo strain in configuration C_2 can be found as incrementally changed from the pseudo strain in configuration C_1.

$$ {}^{21}_{1}\bar{\bar{E}} = {}^{11}_{1}\bar{\bar{E}} + {}^{21}_{1}\bar{\bar{E}}^{2} + {}^{21}_{1}\bar{\bar{E}}^{3} \tag{2.8.11} $$

$$ {}^{21}_{1}\bar{\bar{E}} = {}^{11}_{1}\bar{\bar{E}} + \frac{1}{2}\left({}^{21,1}_{1}\bar{\bar{u}} + {}^{21,1}_{1}\bar{\bar{u}}^{T} + {}^{21,1}_{1}\bar{\bar{u}}\cdot{}^{21,1}_{1}\bar{\bar{u}}^{T} \right) \tag{2.8.12} $$

C_2 can then be made the transformation and the reference configuration by the same kind of transformation as before

$$ {}^{22}_{2}\bar{\bar{E}} = \left(\frac{\partial {}^{2}_{1}\bar{x}}{\partial {}^{1}_{1}\bar{x}} \right)^{-1} \cdot {}^{21}_{1}\bar{\bar{E}} \cdot \left(\frac{\partial {}^{2}_{1}\bar{x}}{\partial {}^{1}_{1}\bar{x}} \right)^{-T} \tag{2.8.13} $$

There is no difficulty in finding the deformation gradient in the calculations based on displacement. Since

$$ {}^{2}\bar{x} = {}^{1}\bar{x} + {}^{21}\bar{u} \tag{2.8.14} $$

then

$$ \frac{\partial {}^{2}_{1}\bar{x}}{\partial {}^{1}_{1}\bar{x}} = \bar{\bar{1}} + {}^{21,1}_{1}\bar{\bar{u}} \, . \tag{2.8.15} $$

2.9 INTEGRALS OF MATERIAL PROPERTIES

Many quantities of interest are not point functions but rather descriptive of a finite region. Such a quantity may be a scalar such as mass or volume, or it could be a tensor of any order. Many vectors in common use describe properties of extensive bodies rather than infinitesimal neighbourhoods. Such is the case when we speak of the weight of a body. However, we consider that properties of bodies are found from the properties of (sets of) infinitesimal neighbourhoods. These extensive properties are usually either averages of local properties, or integrals of local densities. In fact, average quantities are often found as the ratio of the integral of the property to be averaged to the integral of the space (length, area or volume) of the region for which the average is desired. Moreover, it is commonly the case that the extensive property is wanted for some configuration other than the reference configuration. Since, in Lagrangian mechanics, all calculations are carried out in the reference configuration, it is worth a moment to consider the integrals of material properties.

A key idea to keep in mind when considering integrals is that an integral is really a sum. For example, to find the momentum of a body add up the momenta of every particle of the body. The way in which this is usually done is to query each small neighbourhood for the momentum density in that neighbourhood. Multiply this by the size of the neighbourhood. Add up all such contributions. The first way evaluates an extensive property, say $\overset{n}{P}$, as

$$\overset{n}{P} = \int d\overset{n}{P} \tag{2.9.1}$$

Where the integral is taken over the body. The second way is

$$\overset{n}{P} = \int_V \overset{n}{\rho}\, dV \tag{2.9.2}$$

where V is the configuration occupied by the body and
$\overset{n}{\rho}$ is the density of $\overset{n}{P}$. That is

$$\overset{n}{\rho} = \frac{\partial \overset{n}{P}}{\partial V} \tag{2.9.3}$$

Since our access to the properties of the body is usually as a density in a neighbourhood, the second form is commonly more convenient. In the above no explicit mention has been made of the configurations of reference or of measurement. Let $\overset{n}{P}$, $\overset{n}{\rho}$ and dV all be measured in configuration C_A. To begin, let the same configuration also be used as the reference configuration. That is, body-points are named by their position when the body occupies C_A. Then we can write

$$\overset{A}{A}\overset{n}{P} = \int_{A_V} \overset{A}{A}\overset{n}{\rho}\, \overset{A}{A} dV \tag{2.9.4}$$

The same body-points could be named by the position that they would occupy were the body in configuration C_B. This has no effect on the

measured values of ${}^{A}\underline{\underline{P}}$, ${}^{A}\underline{p}$ and ${}^{A}dV$, but the integral is now carried out over the volume of C_B. This is written as

$$ {}^{A}_{B}\underline{\underline{P}} = \int_{B_V} {}^{A}_{B}\underline{P}\ {}^{A}_{B}dV \qquad (2.9.5) $$

One may think of Eqn(2.9.5) as taking all the values used in Eqn(2.9.4) and writing them on a fictitious copy of the body in C_B. Each small neighbourhood of material has contributed exactly the same values: the same density and the same differential volume. So, the value of ${}^{A}\underline{P}$ is unchanged. All that was done was that the adding up was carried out in C_B instead of in C_A. It can be inconvenient, sometimes impossible, to evaluate ${}^{A}dV$. This difficulty arises in calculations involving large displacement. One wishes to use an integral of quantities measured in the newest configuration of the body but does not yet know what that configuration is.

As an example, consider the simplest case: find the volume of the body in C_A. The density is scalar unity. So,

$$ {}^{A}V = \int {}^{A}1\ {}^{A}dV \qquad (2.9.6) $$

It is not necessary to keep the scalar unity explicitly. Refer this to C_A to find

$$ {}^{A}_{A}V = \int_{A_V} {}^{A}_{A}dV \qquad (2.9.7) $$

We may consider that the volume has been built up by accumulating all the small neighbourhoods, ${}^{A}dV$. Moreover, all these small neighbourhoods neatly fit together to give C_A. However, if we referred this to C_B we would get

$$ {}^{A}_{B}V = \int_{B_V} {}^{A}_{B}dV \qquad (2.9.8) $$

Eqn(2.9.8) shows that the volume of C_A can be evaluated in C_B by putting together all the infinitesimal ${}^{A}dV$. These are not necessarily the same volumes as the ${}^{B}dV$ and so do not "fit" together there. This idea of fit is clearer for surface integrals where one finds in different configurations that the differential elements of surface are of different sizes and in different orientations.

It would be very much more convenient to use the ${}^{B}dV$ when integrating over ${}^{B}V$. A transformation can be used to allow this. Such a transformation, called a Jacobian, maintains the value of the original differential volume (or area or length). So, the required behaviour of the Jacobian for volumes is

$$ {}^{A}dV = {}^{AB}J \, {}^{B}dV \qquad (2.9.9) $$

For volumes, the Jacobian is the third scalar invariant of the deformation gradient, ${}^{AB}\overline{\overline{F}}$, where

$$ {}^{AB}\bar{\bar{F}} = \frac{\partial\, {}^{A}_{B}\bar{x}}{\partial\, {}^{B}\bar{x}} \tag{2.9.10} $$

and the third scalar invariant is

$$ {}^{AB}J = \frac{1}{3!}\, {}^{AB}\bar{\bar{F}} \times\!\times {}^{AB}\bar{\bar{F}} : {}^{AB}\bar{\bar{F}} \tag{2.9.11} $$

For surfaces, where the Jacobian is the dyadic ${}^{AB}\bar{\bar{J}}$, the requirement is

$$ {}^{A}d\bar{a} = {}^{B}d\bar{a} \cdot {}^{AB}\bar{\bar{J}} \tag{2.9.12} $$

The Jacobian for surface elements is

$$ {}^{AB}\bar{\bar{J}} = \frac{1}{2!}\, {}^{AB}\bar{\bar{F}} \times\!\times {}^{AB}\bar{\bar{F}} \tag{2.9.13} $$

$$ \frac{1}{2!}\, {}^{AB}\bar{\bar{F}} \times\!\times {}^{AB}\bar{\bar{F}} = \frac{1}{3!} \left({}^{AB}\bar{\bar{F}} \times\!\times {}^{AB}\bar{\bar{F}} : {}^{AB}\bar{\bar{F}} \right) {}^{AB}\bar{\bar{F}}^{-T} \tag{2.9.14} $$

One special case that is frequently important for calculation is where the local tangent basis of space is orthonormal. In that case, which is used through out the calculations presented in Chapter 5, we have some simple relationships in the surface Jacobian. We factor ${}^{A}d\bar{a}$ and ${}^{B}d\bar{a}$ as

$$
\begin{aligned}
{}^{A}d\bar{a} &= {}^{A}dA\ {}^{A}\hat{n} \\
{}^{B}d\bar{a} &= {}^{B}dA\ {}^{B}\hat{n}
\end{aligned}
\tag{2.9.16}
$$

where dA is the area of the differential surface and $\hat{n}$ is a normal unit vector giving the orientation of the surface. Then we can factor ${}^{AB}\bar{\bar{J}}$ as

$$
{}^{AB}\bar{\bar{J}} = {}^{AB}j\ {}^{AB}\hat{j}^{1}\ {}^{AB}\hat{j}^{2}
$$

where

$$
\begin{aligned}
{}^{AB}j &= \frac{{}^{A}dA}{{}^{B}dA} \\
{}^{AB}\hat{j}^{1} &= {}^{B}\hat{n} \\
{}^{AB}\hat{j}^{2} &= {}^{A}\hat{n}
\end{aligned}
\tag{2.9.17}
$$

Also the inverse relationship is simply constructed.

$$
\begin{aligned}
{}^{BA}\bar{\bar{J}} &= {}^{AB}\bar{\bar{J}}^{-1} \\
{}^{BA}\bar{\bar{J}} &= {}^{BA}j\ {}^{BA}\hat{j}^{1}\ {}^{BA}\hat{j}^{2}
\end{aligned}
\tag{2.9.18}
$$

where

$$
\begin{aligned}
{}^{BA}j &= \frac{{}^{B}dA}{{}^{A}dA} = \frac{1}{{}^{AB}j} \\
{}^{BA}\hat{j}^{1} &= {}^{A}\hat{n} = {}^{AB}\hat{j}^{2} \\
{}^{BA}\hat{j}^{2} &= {}^{B}\hat{n} = {}^{AB}\hat{j}^{1}
\end{aligned}
\tag{2.9.19}
$$

2.10 TRANSFORMATION OF PRINCIPLES

We can now apply these forms to investigate the transformation of material principles in a weak form. A divergence term is included so that explicit understanding will be available later. Let ${}^{A}\underline{h}^{n}$ be some function of material properties that represents some material principle. Let it be pointwise true that

$$ {}^{A}\underline{h}^{n} = {}^{A}\underline{p}^{n} + {}^{A}\underline{q}^{n} = \underline{0}^{n} \tag{2.10.1} $$

where

$$ {}^{A}\underline{q}^{n} = \frac{\partial \cdot {}^{A}\underline{q}^{n+1}_{A}}{\partial\, {}^{A}\overline{x}} \tag{2.10.2} $$

So

$$ {}^{A}\underline{h}^{n} = {}^{A}\underline{p}^{n} + \frac{\partial \cdot {}^{A}\underline{q}^{n+1}_{A}}{\partial\, {}^{A}\overline{x}} = \underline{0}^{n} \tag{2.10.3} $$

These can all be referred to C_A. Once the principle is expressed in a referential formulation it can be integrated over the body. The integral of any function that is pointwise zero is also zero. So

$$ \int_{A\Omega} {}^{A}\underline{h}^{n}_{A}\, {}^{A}_{A}dV = \int_{A\Omega} \left({}^{A}\underline{p}^{n}_{A} + {}^{A}\underline{q}^{n}_{A} \right) {}^{A}_{A}dV = \underline{0}^{n} \tag{2.10.4} $$

$$\int_{{}_A\Omega}\left({}_A^A\underline{\rho}^{\underline{n}} + \frac{\partial \cdot {}_A^A\underline{q}^{n+1}}{\partial\, {}_A^A\bar{x}} \right) {}_A dV = \underline{0}^{\underline{n}} \qquad (2.10.5)$$

This appears now to be a very weak argument. It would satisfy eqn(2.10.5) if ${}_A^A\underline{h}^{\underline{n}}$ were zero only on average. This is used to advantage in some numerical methods. For theoretical arguments there is no difficulty. The boundaries of the body under consideration are quite arbitrary. So eqn(2.10.5) must hold for any chosen body. For this to be true it must be that the integrand, ${}_A^A\underline{h}^{\underline{n}}$, is indeed zero for all neighbourhoods. The integral can be split into two terms as

$$\int_{{}_A\Omega} {}_A^A\underline{\rho}^{\underline{n}}\, {}_A dV + \int_{{}_A\Omega} \frac{\partial \cdot {}_A^A\underline{q}^{n+1}}{\partial\, {}_A^A\bar{x}}\, {}_A dV = \underline{0}^{\underline{n}} \qquad (2.10.6)$$

Now the argument appears even weaker. The two terms in eqn(2.10.6) could be such that neither of them are zero but rather are each others opposite. The arbitrary body argument is conjured to counter this possibility. The divergence law can be applied to the second term of eqn(2.9.18) to give

$$\int_{{}_A\Omega} {}_A^A\underline{\rho}^{\underline{n}}\, {}_A dV + \int_{{}_A\partial\Omega} {}_A d\bar{a} \cdot {}_A^A\underline{q}^{n+1} = \underline{0}^{\underline{n}} \qquad (2.10.7)$$

Now the enforcement of eqn(2.10.1) looks even weaker since the second

integral is only over the surface rather than the entire domain. The Principle of Continuity gives the necessary assurance that this term serves as well as the second integral in eqn(2.10.6). The whole integral can be referred to another configuration, say C_B. This is essentially a copying of information. The domains of integration are regions of space and so are changed completely in the changes of configuration of reference and have no connection to the previous reference configuration. All other quantities are still as measured in C_A.

$$\int_{B_\Omega} {}^{A}_{B}\underline{\rho}^{n}\, {}^{A}_{B}dV \;+\; \int_{B_{\partial\Omega}} {}^{A}_{B}d\bar{a} \cdot {}^{A}_{B}\underline{q}^{n+1} \;=\; \underline{0}^{n} \qquad (2.10.8)$$

Let the differentials, ${}^{A}_{B}dV$ and ${}^{A}_{B}d\bar{a}$, be transformed to the new reference configuration.

$$\int_{B_\Omega} {}^{A}_{B}\underline{\rho}^{n}\, {}^{AB}_{B}J\, {}^{B}_{B}dV \;+\; \int_{B_{\partial\Omega}} {}^{B}_{B}d\bar{a} \cdot {}^{AB}_{B}\bar{\bar{J}} \cdot {}^{A}_{B}\underline{q}^{n+1} \;=\; \underline{0}^{n} \qquad (2.10.9)$$

It is often useful to define "pseudo-" quantities

$${}^{AB}\underline{\rho}^{n} \;=\; {}^{A}\underline{\rho}^{n}\, {}^{AB}J \qquad (2.10.10)$$

and

$$ {}^{AB}\underline{q}^{n+1} = {}^{AB}\bar{\bar{J}} \cdot {}^{A}\underline{q}^{n+1} \qquad (2.10.11) $$

These pseudo- quantities can be used to change eqn(2.10.9) to resemble eqn(2.10.7) as eqn(2.10.12).

$$ \int_{B_\Omega} {}^{AB}_{B}\rho^{n}\, {}^{B}_{B}dV + \int_{B_{\partial\Omega}} {}^{AB}_{B}d\bar{a} \cdot {}^{AB}_{B}\underline{q}^{n+1} = \underline{0}^{n} \qquad (2.10.12) $$

Now the process runs backwards. Use the divergence theorem to get

$$ \int_{B_\Omega} {}^{AB}_{B}\rho^{n}\, {}^{B}_{B}dV + \int_{B_\Omega} \frac{\partial}{\partial\, {}^{B}_{B}\bar{x}} \cdot {}^{AB}_{B}\underline{q}^{n+1}\, {}^{B}_{B}dV = \underline{0}^{n} \qquad (2.10.13) $$

The two terms may be combined to give

$$ \int_{B_\Omega} \left({}^{AB}_{B}\rho^{n} + \frac{\partial}{\partial\, {}^{B}_{B}\bar{x}} \cdot {}^{AB}_{B}\underline{q}^{n+1} \right) {}^{B}_{B}dV = \underline{0}^{n} \qquad (2.10.14) $$

It is now possible to use the pseudo- quantities ${}^{AB}\underline{q}^{n}$ and ${}^{AB}\underline{h}^{n}$ where

$$ {}^{AB}\underline{q}^{n} = \frac{\partial\, {}^{AB}\underline{q}^{n+1}}{\partial\, {}^{B}\bar{x}} $$

$$ {}^{AB}\underline{h}^{n} = {}^{AB}\rho^{n} + {}^{AB}\underline{q}^{n} \qquad (2.10.15) $$

Finally we invoke the arbitrary body argument. Since eqn(2.10.14) must be true for any arbitrarily chosen set of infinitesimal neighbourhoods, it must be true independently for each neighbourhood. So the integrand itself must be zero. That is

$$ {}^{AB}\underline{h}^{n} = {}^{AB}\underline{p}^{n} + {}^{AB}\underline{q}^{n} = \underline{0}^{n} \tag{2.10.16} $$

$$ {}^{AB}_{R}\underline{h}^{n} = {}^{AB}_{R}\underline{p}^{n} + \frac{\partial}{\partial\, {}^{B}_{R}\underline{X}} \cdot {}^{AB}_{B}\underline{q}^{n+1} = \underline{0}^{n} \tag{2.10.17} $$

It is important to remember that ${}^{AB}\underline{h}^{n}$ is not ${}^{B}\underline{h}^{n}$.

In this fashion we can transform the material principle ${}^{A}\underline{h}^{n}$ to a pseudo-principle, ${}^{AB}\underline{h}^{n}$, that may be more convenient for calculation. The particular utility of this is in incremental large displacement calculations where C_A is a newly occupied configuration, as yet unknown, and C_B is a known configuration. Approximate C_A by a sequence of configurations starting with C_B. Find all needed properties of C_A, including its position. Using this corrected approximation to C_A repeat the calculation. Do this as many times as is necessary to find C_A with sufficient accuracy. This discussion will not address the convergence of any such sequence of operations except to note that we assume that every such sequence is convergent if C_B is close enough to C_A and the body is stable.

Some commentary should be made about the differentials and the Jacobians of transformation. Remember that these are integrals of material properties.

The integrals of sections 2.9 and 2.10 are of material functions over regions of space. dV is the size (volume) of a small part of this region. We make an association between material and the region it occupies. So, dV is also the volume of the region of space occupied by a small amount of material. As a material property dV can be measured in, or referred to, any configuration.

We use the mass as a scalar measure of the amount of material. So the mass of the small neighbourhood is dm. dm is not shown as measured in any configuration because it is invariant with respect to configuration. The same amount of matter is always there. When the body is referred to C_A the differential of volume became ${}^{A}_{A}dV$. The amount of matter in a small neighbourhood can then be expressed as

$$dm = {}^{A}_{A}\rho \; {}^{A}_{A}dV \qquad (2.10.18)$$

where ρ is the mass density. This can also be referred to C_B. This changes the name of the differential volume to ${}^{A}_{B}dV$ from ${}^{A}_{A}dV$. This does not change the value (size) of ${}^{A}dV$ nor with which material it is associated.

$$dm = {}^{A}_{B}\rho \; {}^{A}_{B}dV \qquad (2.10.19)$$

Since dm is invariant it can be measured in C_B as well as having calculations referred there.

$$dm = {}^{B}_{B}\rho \; {}^{B}_{B}dV \tag{2.10.20}$$

As discussed above, ${}^{A}_{B}dV$ can be transformed to ${}^{B}_{B}dV$. Use this to find

$$dm = {}^{A}_{B}\rho \; {}^{AB}_{B}J \; {}^{B}_{B}dV \tag{2.10.21}$$

One may subtract eqn(2.10.21) from eqn(2.10.20) to obtain 0. From the remaining relation it may be concluded that

$${}^{AB}_{R}J = \frac{\partial \, {}^{A}_{B}V}{\partial \, {}^{B}_{R}V} = \frac{{}^{A}_{R}\rho}{{}^{B}_{R}\rho} = \frac{1}{3!} \, {}^{AB}_{R}\overline{\overline{F}} \times\times {}^{AB}_{R}\overline{\overline{F}} : {}^{AB}_{R}\overline{\overline{F}} \tag{2.10.22}$$

This expresses the dilation of each small neighbourhood between C_A and C_B.

The transformation of the surface differential, ${}^{A}d\bar{a}$, to ${}^{B}d\bar{a}$ is a little more complicated and interesting because it is a vector quantity. Whatever the integrand that follows the orientation and magnitude must be preserved. As an illustration consider the possibility that ${}^{A}\overline{\overline{g}}^{n+1}$ is the dyad ${}^{A}\overline{\overline{g}}$ and that the local tangent basis set is orthonormal. We express ${}^{A}\overline{\overline{g}}$ as

$${}^{A}\overline{\overline{g}} = {}^{A}g \; {}^{A}\hat{g}^{1} \; {}^{A}\hat{g}^{2} \tag{2.10.23}$$

The invariant differential vector ${}^{A}d\bar{g}$ is

$$ {}^{A}d\bar{g} = {}^{A}d\bar{a} \cdot {}^{A}\bar{\bar{g}} \qquad (2.10.24) $$

We can re-express this as

$$ \begin{aligned} {}^{A}d\bar{g} &= {}^{A}dA\ {}^{A}\hat{n} \cdot {}^{A}g\ {}^{A}\hat{g}^{1}\ {}^{A}\hat{g}^{2} \\ {}^{A}d\bar{g} &= {}^{A}dA\ {}^{A}\mathcal{G}\ {}^{A}\hat{g}^{2} \end{aligned} \qquad (2.10.25) $$

where ${}^{A}\mathcal{G}$ is the density of ${}^{A}\bar{\bar{g}}$ on the differential surface in C_A. The orientation of ${}^{A}d\bar{a}$ and the norm of ${}^{A}\bar{\bar{g}}$ contribute to this measure as

$$ {}^{A}\mathcal{G} = {}^{A}\hat{n} \cdot {}^{A}\hat{g}^{1}\ {}^{A}g \qquad (2.10.26) $$

The same steps can be followed on the differential surface in C_B. From the behaviour of the pseudo- quantity ${}^{AB}\bar{\bar{g}}$

$$ {}^{A}d\bar{g} = {}^{B}d\bar{a} \cdot {}^{AB}\bar{\bar{g}} \qquad (2.10.27) $$

we have

$$ \begin{aligned} {}^{A}d\bar{g} &= {}^{B}dA\ {}^{B}\hat{n} \cdot {}^{AB}g\ {}^{AB}\hat{g}^{1}\ {}^{AB}\hat{g}^{2} \\ {}^{A}dg &= {}^{B}dA\ {}^{AB}\mathcal{G}\ {}^{AB}\hat{g}^{2} \end{aligned} \qquad (2.10.28) $$

By comparing eqn(2.10.28) with eqn(2.10.26), we can see that

$$ \begin{aligned} {}^{AB}\hat{g}^{2} &= {}^{A}\hat{g}^{2} \\ {}^{AB}\mathcal{G} &= \frac{{}^{A}dA}{{}^{B}dA}\ {}^{A}\mathcal{G} = {}^{B}\hat{n} \cdot {}^{AB}\hat{g}^{1}\ {}^{AB}g \end{aligned} \qquad (2.10.29) $$

To find the still unknown factors of the pseudo- quantity ${}^{AB}\overline{\overline{g}}$ we can use

$$ {}^{AB}\overline{\overline{g}} = {}^{AB}\overline{\overline{J}} \cdot {}^{A}\overline{\overline{g}} \qquad (2.10.30) $$

Expand to get

$$ {}^{AB}g \; {}^{AB}\hat{g}^{1} \; {}^{AB}\hat{g}^{2} = \frac{{}^{A}dA}{{}^{B}dA} \; {}^{B}\hat{n} \; {}^{A}\hat{n} \cdot {}^{A}g \; {}^{A}\hat{g}^{1} \; {}^{A}\hat{g}^{2} \qquad (2.10.31) $$

So

$$ {}^{AB}\hat{g}^{1} = {}^{B}\hat{n} \qquad (2.10.32) $$

and

$$ {}^{AB}g = \frac{{}^{A}dA}{{}^{B}dA} \; {}^{A}\hat{n} \cdot {}^{A}\hat{g}^{1} = \frac{{}^{A}dA}{{}^{B}dA} \; {}^{A}\mathcal{g} $$

So

$$ {}^{AB}\mathcal{g} = {}^{B}\hat{n} \cdot {}^{B}\hat{n} \; {}^{AB}g = {}^{AB}g \qquad (2.10.33) $$

We see that the effects of the Jacobian are: to scale the density as seen in eqn(2.10.33) and to compensate for the relative orientation of the two differential surfaces as seen in eqn(2.10.32)

CHAPTER THREE

EQUILIBRIUM PRINCIPLES

3.1 THE LINEAR MOMENTUM PRINCIPLE

The starting point for the next discussion is the Linear Momentum Principle. This we present without further proof or testing and accept as valid.

$$-{}^{A}_{R}\rho \frac{d\,{}^{A}_{R}\bar{v}}{d\,t} + \frac{\partial}{\partial\,{}^{A}_{R}\bar{x}} \cdot {}^{A}_{A}\bar{\bar{\sigma}} + {}^{A}_{R}\bar{f} = \bar{0} \qquad (3.1.1)$$

where ${}^{A}\rho$ is the mass density,

${}^{A}\bar{v}$ is the velocity,

${}^{A}\bar{\bar{\sigma}}$ is the Cauchy (true) stress tensor and

${}^{A}\bar{f}$ is the force density.

All quantities in eqn(3.1.1) are measured in C_A, as the body passes through C_A, and is referred to C_R. This principle we assume to reflect adequately one of the basic universal laws and to be capable of describing the actions within a small neighbourhood of material or any set of small neighbourhoods (body). With this much accepted we may

proceed to use this relationship to study the motions and deformations of bodies.

Unfortunately, the Linear Momentum Principle, eqn(3.1.1), is not always suitable for calculation. A scalar equation would be more convenient. Such is the motivation behind the popular energy methods. These start by forming some kind of scalar from eqn(3.1.1) and work with that scalar. A rather beautiful way to do this is the Rate of Work Principle. This is found by taking the dot product of eqn(3.1.1) with the velocity, ${}^{A}\bar{v}$. One of the advantages of this product is that it keeps all quantities measured in a single configuration, C_A. Another is that we will be able to find principles to describe both dynamic and static displacements from it.

3.2 THE RATE OF WORK PRINCIPLE

Start by simplifying eqn(3.1.1) by assuming that the configuration of measurement is the same as the configuration of reference, so that only one configuration is involved. Then multiply both sides by the velocity, ${}^{A}_{A}\bar{v}$. This gives

$$\left(-{}^{A}_{A}\rho \frac{d\, {}^{A}_{A}\bar{v}}{d\, t} + \frac{\partial \cdot {}^{A}_{A}\bar{\bar{\sigma}}}{\partial\, {}^{A}_{A}\bar{x}} + {}^{A}_{A}\bar{f} \right) \cdot {}^{A}_{A}\bar{v} = \bar{0} \cdot {}^{A}_{A}\bar{v} \qquad (3.2.1)$$

Expand the product to get

$$-{}_{A}\rho \frac{d\,{}_{A}\bar{v}}{d\,t} \cdot {}_{A}\bar{v} + \frac{\partial \cdot {}_{A}\bar{\bar{\sigma}}}{\partial\,{}_{A}\bar{x}} \cdot {}_{A}\bar{v} + {}_{A}\bar{f} \cdot {}_{A}\bar{v} = \bar{0} \qquad (3.2.2)$$

The terms of eqn(3.2.2) can be manipulated separately to yield familiar forms.

3.2.1 KINETIC POWER TERM

A common manipulation improves the symmetry of this term. Consider the rate of $({}_{A}\bar{v} \cdot {}_{A}\bar{v})$

$$\frac{d}{d\,t}({}_{A}\bar{v} \cdot {}_{A}\bar{v}) = \left(\frac{d\,{}_{A}\bar{v}}{d\,t}\right) \cdot {}_{A}\bar{v} + {}_{A}\bar{v} \cdot \left(\frac{d\,{}_{A}\bar{v}}{d\,t}\right) \qquad (3.2.3)$$

$$= \left(\frac{d\,{}_{A}\bar{v}}{d\,t}\right) \cdot {}_{A}\bar{v} + \left(\frac{d\,{}_{A}\bar{v}}{d\,t}\right) \cdot {}_{A}\bar{v} \qquad (3.2.4)$$

Use this in the first term of eqn(3.2.2) to get

$$-{}_{A}\rho \frac{d\,{}_{A}\bar{v}}{d\,t} \cdot {}_{A}\bar{v} = -\frac{1}{2}\,{}_{A}\rho \frac{\partial}{\partial\,t}\left({}_{A}\bar{v} \cdot {}_{A}\bar{v}\right) \qquad (3.3.5)$$

3.2.2 INTERNAL POWER TERM

For this term another re-arrangement is convenient. Here consider the divergence of the product $({}^{A}_{A}\overline{\overline{\sigma}} \cdot {}^{A}_{A}\overline{v})$. Again we examine a combination of a derivative and a product. An underscore indicates that the derivative is not acting.

$$\frac{\partial}{\partial\, {}^{A}_{A}\overline{x}} \cdot \left({}^{A}_{A}\overline{\overline{\sigma}} \cdot {}^{A}_{A}\overline{v} \right) = \left(\frac{\partial}{\partial\, {}^{A}_{A}\overline{x}} \cdot {}^{A}_{A}\overline{\overline{\sigma}} \right) \cdot {}^{A}_{A}\overline{v} + \left(\frac{\partial}{\partial\, {}^{A}_{A}\overline{x}} \cdot {}^{A}_{\underline{A}}\overline{\overline{\sigma}} \cdot {}^{A}_{A}\overline{v} \right) \tag{3.2.6}$$

The last term can be rearranged by the use of parentheses.

$$\frac{\partial}{\partial\, {}^{A}_{A}\overline{x}} \cdot {}^{A}_{\underline{A}}\overline{\overline{\sigma}} \cdot {}^{A}_{A}\overline{v} = \frac{\partial}{\partial\, {}^{A}_{A}\overline{x}} \cdot \left({}^{A}_{A}\overline{v} \cdot {}^{A}_{\underline{A}}\overline{\overline{\sigma}}^{T} \right) \tag{3.2.7}$$

So,

$$\frac{\partial}{\partial\, {}^{A}_{A}\overline{x}} \cdot {}^{A}_{\underline{A}}\overline{\overline{\sigma}} \cdot {}^{A}_{A}\overline{v} = \frac{\partial\, {}^{A}_{A}\overline{v}}{\partial\, {}^{A}_{A}\overline{x}} : {}^{A}_{A}\overline{\overline{\sigma}}^{T} \tag{3.2.8}$$

Since ${}^{A}_{A}\overline{\overline{\sigma}}$ is symmetric the transpose mark can be dropped. Also, only the symmetric part of the velocity gradient participates in eqn(3.2.8), because it is in a double dot product with a symmetric dyadic. Any antisymmetric part will contribute zero after a double dot product with a symmetric dyadic. We now restate eqn(3.2.6) and recognize the first

term on the right hand side as the internal power term of eqn(3.2.2).

$$\frac{\partial}{\partial \overset{A}{\underset{A}{\bar{x}}}} \cdot \left(\overset{A}{\underset{A}{\bar{\bar{\sigma}}}} \cdot \overset{A}{\underset{A}{\bar{v}}} \right) = \left(\frac{\partial \cdot \overset{A}{\underset{A}{\bar{\bar{\sigma}}}}}{\partial \overset{A}{\underset{A}{\bar{x}}}} \right) \cdot \overset{A}{\underset{A}{\bar{v}}} + \frac{\partial \overset{A}{\underset{A}{\bar{v}}}}{\partial \overset{A}{\underset{A}{\bar{x}}}} : \overset{A}{\underset{A}{\bar{\bar{\sigma}}}} \qquad (3.2.9)$$

So the internal power is

$$\left(\frac{\partial \cdot \overset{A}{\underset{A}{\bar{\bar{\sigma}}}}}{\partial \overset{A}{\underset{A}{\bar{x}}}} \right) \cdot \overset{A}{\underset{A}{\bar{v}}} = \frac{\partial}{\partial \overset{A}{\underset{A}{\bar{x}}}} \cdot \left(\overset{A}{\underset{A}{\bar{\bar{\sigma}}}} \cdot \overset{A}{\underset{A}{\bar{v}}} \right) - \frac{\partial \overset{A}{\underset{A}{\bar{v}}}}{\partial \overset{A}{\underset{A}{\bar{x}}}} : \overset{A}{\underset{A}{\bar{\bar{\sigma}}}} \qquad (3.2.10)$$

3.2.3 EXTERNAL POWER TERM

This does not require any modification.

3.3 TRANSFORMATION OF RATE OF WORK PRINCIPLE

Here we follow the steps laid out in section 2.10. First the principle is put into the weak (integral) form.

$$\int_{{}^{A}\Omega} \Bigg[\; - \tfrac{1}{2}\, {}^{A}_{A}\rho \, \frac{d}{d\,t} \left({}^{A}_{A}\bar{v} \cdot {}^{A}_{A}\bar{v} \right) + \frac{\partial}{\partial \, {}^{A}_{A}\bar{x}} \cdot \left({}^{A}_{A}\bar{\bar{\sigma}} \cdot {}^{A}_{A}\bar{v} \right) - \frac{\partial \, {}^{A}_{A}\bar{v}}{\partial \, {}^{A}_{A}\bar{x}} : {}^{A}_{A}\bar{\bar{\sigma}} + {}^{A}_{A}\bar{f} \cdot {}^{A}_{A}\bar{v} \; \Bigg] \, {}^{A}_{A}dV = 0 \tag{3.3.1}$$

Separate the second term and apply the divergence principle.

$$\int_{{}^{A}\Omega} \Bigg[\; - \tfrac{1}{2}\, {}^{A}_{A}\rho \, \frac{d}{d\,t} \left({}^{A}_{A}\bar{v} \cdot {}^{A}_{A}\bar{v} \right) - \frac{\partial \, {}^{A}_{A}\bar{v}}{\partial \, {}^{A}_{A}\bar{x}} : {}^{A}_{A}\bar{\bar{\sigma}} + {}^{A}_{A}\bar{f} \cdot {}^{A}_{A}\bar{v} \; \Bigg] \, {}^{A}_{A}dV + \int_{{}^{A}\partial\Omega} {}^{A}_{A}d\bar{a} \cdot \left({}^{A}_{A}\bar{\bar{\sigma}} \cdot {}^{A}_{A}\bar{v} \right) = 0 \tag{3.3.2}$$

Refer this to C_B.

$$\int_{B_\Omega} \Bigg[\; -\frac{1}{2}\,{}^{A}_{B}\rho\,\frac{d}{dt}\left({}^{A}_{B}\bar{v}\cdot{}^{A}_{B}\bar{v}\right) \qquad (3.3.3)$$

$$-\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}} + {}^{A}_{B}\bar{f}\cdot{}^{A}_{B}\bar{v} \Bigg]\,{}^{A}_{B}dV$$

$$+\int_{B_{\partial\Omega}} {}^{A}_{B}d\bar{a}\cdot\left({}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{A}_{B}\bar{v}\right) = 0$$

Change the differentials, ${}^{A}_{B}dV$ and ${}^{A}_{B}d\bar{a}$, to those measured in C_B. This introduces the Jacobian.

$$\int_{B_\Omega} \Bigg[\; -\frac{1}{2}\,{}^{A}_{B}\rho\,\frac{d}{dt}\left({}^{A}_{B}\bar{v}\cdot{}^{A}_{B}\bar{v}\right) \qquad (3.3.4)$$

$$-\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}} + {}^{A}_{B}\bar{f}\cdot{}^{A}_{B}\bar{v} \Bigg]\,{}^{AB}_{B}J\,{}^{B}_{B}dV$$

$$+\int_{B_{\partial\Omega}} {}^{B}_{B}d\bar{a}\cdot{}^{AB}_{B}\bar{\bar{J}}\cdot\left({}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{A}_{B}\bar{v}\right) = 0$$

Distribute ${}^{AB}_{B}J$ in the volume integral; apply the divergence theorem to the surface integral and then recombine.

$$\int_{A_\Omega} \Bigg[\; - \frac{1}{2} \, {}^{AB}_{B}J \, {}^{A}_{B}\rho \, \frac{d}{d\,t} \left({}^{A}_{B}\bar{v} \cdot {}^{A}_{B}\bar{v} \right) \tag{3.3.5}$$

$$+ \frac{\partial}{\partial \, {}^{B}_{B}\bar{x}} \cdot {}^{AB}_{B}\bar{\bar{J}} \cdot \left({}^{A}_{B}\bar{\bar{\sigma}} \cdot {}^{A}_{B}\bar{v} \right) \; - \; {}^{AB}_{B}J \, \frac{\partial \, {}^{A}_{A}\bar{v}}{\partial \, {}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}}$$

$$+ \; {}^{AB}_{B}J \; {}^{A}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v} \; \Bigg] \, {}^{B}_{B}dV \; = \; 0$$

Invoke the arbitrary body argument to change from weak (integral) back to strong (differential) form. Finally, we have the Rate of Work Principle transformed and referred to C_B.

$$\Bigg[\; - \frac{1}{2} \, {}^{AB}_{B}J \, {}^{A}_{B}\rho \, \frac{d}{d\,t} \left({}^{A}_{B}\bar{v} \cdot {}^{A}_{B}\bar{v} \right) \tag{3.3.6}$$

$$+ \frac{\partial}{\partial \, {}^{B}_{B}\bar{x}} \cdot {}^{AB}_{B}\bar{\bar{J}} \cdot \left({}^{A}_{B}\bar{\bar{\sigma}} \cdot {}^{A}_{B}\bar{v} \right) \; - \; {}^{AB}_{B}J \, \frac{\partial \, {}^{A}_{A}\bar{v}}{\partial \, {}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}}$$

$$+ \; {}^{AB}_{B}J \; {}^{A}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v} \; \Bigg] \; = \; 0$$

It is appropriate to perform some manipulations on the terms of eqn(3.3.6) separately which will put the individual terms into more useful forms.

3.4 KINETIC POWER TERM

$$- \frac{1}{2} \, {}^{AB}_{B}J \, {}^{A}_{B}\rho \, \frac{d}{d\,t} \left({}^{A}_{B}\bar{v} \cdot {}^{A}_{B}\bar{v} \right)$$

First we recognize that Jacobian for volume differentials is the ratio of densities. So, we combine factors.

$$- \frac{1}{2} \, {}^{AB}_{B}J \, {}^{A}_{B}\rho \, \frac{d}{d\,t} \left({}^{A}_{B}\bar{v} \cdot {}^{A}_{B}\bar{v} \right) = - \frac{1}{2} \, {}^{B}_{B}\rho \, \frac{d}{d\,t} \left({}^{A}_{B}\bar{v} \cdot {}^{A}_{B}\bar{v} \right) \qquad (3.4.1)$$

We may define the kinetic energy density, ${}^{A}\mathcal{K}$, as

$${}^{A}_{R}\mathcal{K} = \frac{1}{2} \, {}^{A}_{R}\rho \, {}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \qquad (3.4.2)$$

So, the rate of kinetic energy density is

$$\frac{d}{d\,t} \, {}^{A}_{R}\mathcal{K} = \frac{d}{d\,t} \left(\frac{1}{2} \, {}^{A}_{R}\rho \, {}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \right) \qquad (3.4.3)$$

$$\frac{d \, {}^{A}_{R}\mathcal{K}}{d\,t} = \left(\frac{1}{2} \, \frac{d \, {}^{A}_{R}\rho}{d\,t} \right) {}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} + {}^{A}_{R}\rho \left(\frac{d \, {}^{A}_{R}\bar{v}}{d\,t} \right) \cdot {}^{A}_{R}\bar{v} \qquad (3.4.4)$$

There are two terms: one for changes in the mass density and one for changes in the velocity.

The kinetic energy of a small neighbourhood of material is ${}^{A}_{R}dK$.

$${}^{A}_{R}dK = {}^{A}_{R}\mathcal{K} \, {}^{A}_{R}dV = \frac{1}{2} \, {}^{A}_{R}\rho \, {}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \, {}^{A}_{R}dV \qquad (3.4.5)$$

$$ {}^{A}_{R}dK = \frac{1}{2}\,{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v}\;{}_{R}dm \tag{3.4.6} $$

Find the rate of the kinetic energy of a small neighbourhood to be

$$ \frac{d\,{}^{A}_{R}dK}{d\,t} = \frac{d}{d\,t}\left(\frac{1}{2}\,{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v}\;{}_{R}dm \right) \tag{3.4.7} $$

$$ \frac{d\,{}^{A}_{R}dK}{d\,t} = {}_{R}dm \left(\frac{d\,{}^{A}_{R}\bar{v}}{d\,t} \right) \cdot {}^{A}_{R}\bar{v} \tag{3.4.8} $$

The term for the time derivative of dm has been omitted because it is identically zero. If we expand dm it is possible to see the behaviour of the rates of the Jacobian and the density.

$$ \frac{d\,{}^{A}_{R}dK}{d\,t} = \frac{d}{d\,t}\left(\frac{1}{2}\,{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v}\;\;{}^{A}_{R}\rho\;{}^{A}_{R}dV \right) \tag{3.4.9} $$

$$ \begin{aligned} \frac{d\,{}^{A}_{R}dK}{d\,t} = \Bigg\{ & {}^{A}_{R}\rho\;{}^{A}_{R}dV \left(\frac{d\,{}^{A}_{R}\bar{v}}{d\,t} \right) \cdot {}^{A}_{R}\bar{v} \\ & + \frac{1}{2}\frac{d\,{}^{A}_{R}\rho}{d\,t}\;{}^{A}_{R}dV\;{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \;+\; \frac{1}{2}\frac{d\,{}^{A}_{R}dV}{d\,t}\;{}^{A}_{R}\rho\;\;{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \Bigg\} \end{aligned} \tag{3.4.10} $$

Eqns(3.4.7) and (3.4.10) represent the same quantity and differ only in that in the presence of the final two terms of eqn(3.4.10). So, the last two terms of eqn(3.4.10) must either be equal and opposite or

independently zero. Since the size of a neighbourhood and the density of the material may be changing as the body moves through C_A it cannot be assumed that both of the last two terms are identically zero independently. Therefore, the sum of the last two terms must be zero. That is, the term for the rate of change of the density is equal in magnitude and opposite in sign to the term for the rate of change of the size of the neighbourhood.

In the change of differential from ${}^{A}_{B}dV$ to ${}^{B}_{B}dV$ there is an apparent change in the behaviour of rates. The material density and the volume of a small neighbourhood are both functions of time in eqn(3.4.10). When the body occupies the reference configuration both are constant. Nevertheless, neither the value of the rate of density in C_A nor the value of the rate of kinetic energy density are changed by referring to C_B. The difference is in the transformation of differentials rather than in the change of reference configuration. To see this, look again at eqn(3.4.7). Let the mass of a small neighbourhood be represented as ${}^{AR}_{R}J^{A}_{R}\rho$ instead of as ${}_{R}dm$. We find

$$\frac{d\ {}^{A}_{R}dK}{d\ t} = \frac{d}{d\ t}\left(\frac{1}{2}\ {}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v}\ \ {}^{AR}_{R}J\ {}^{A}_{R}\rho\ {}^{R}_{R}dV \right) \qquad (3.4.11)$$

$$\frac{d\,{}^{A}_{R}dK}{dt} = \left\{ {}^{AR}_{R}J\;{}^{A}_{R}\rho\;{}^{R}_{R}dV \left(\frac{d\,{}^{A}_{R}\bar{v}}{dt} \right) \cdot {}^{A}_{R}\bar{v} \right. \qquad (3.4.12)$$

$$+ \frac{1}{2} \frac{d\,{}^{AR}_{R}J}{dt}\;{}^{A}_{R}\rho\;{}^{R}_{R}dV\;{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \quad + \quad \frac{1}{2}\;{}^{AR}_{R}J \frac{d\,{}^{A}_{R}\rho}{dt}\;{}^{R}_{R}dV\;{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v}$$

$$\left. + \frac{1}{2}\;{}^{AR}_{R}J\;{}^{A}_{R}\rho \frac{d\,{}^{R}_{R}dV}{dt}\;{}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \right\}$$

The last term on the right hand side is the derivative of a constant and so vanishes. Now eqn(3.4.12) may be compared with eqn(3.4.10). Clearly, the last two terms of eqn(3.4.12) must again be equal and opposite. The rate of change of ${}^{A}dV$ is included by the term with the rate of change of ${}^{AR}J$.

So now we understand the absence of a rate term for the density in eqn(3.4.1). Here is one of the peculiarities of Lagrangian calculation. It must be remembered that material quantities in a Lagrangian calculation refer to a constant neighbourhood of material. The size, shape orientation and placement of these neighbourhoods change with time but not their identities. Eqn(3.4.1) contains material quantities and so represents material functions of any predecided small quantity of material. C_B serves here as a reference configuration. The density in the reference configuration is a constant and has no time derivatives. The density in configuration C_A, as the body passes through C_A, is a dynamic quantity and may have non-zero time derivatives. These are exactly compensated by the rate of ${}^{AB}J$. So, in

a Lagrangian calculation, the rate of density in the reference configuration will never contribute.

In an Eulerian calculation the size of a neighbourhood is concerned only with the description of space. As the material moves and density changes the amount of material involved changes. So, for an Eulerian calculation of the rate of kinetic energy density, terms would arise that account for the changing density of material.

The discussion above may be used to define a pseudo kinetic energy density, ${}^{AR}_{R}\mathcal{K}$,

$$ {}^{AR}_{R}\mathcal{K} = \frac{1}{2}\, {}^{AR}_{R}J\, {}^{A}_{R}\rho\, \frac{d}{d\,t} \left({}^{A}_{R}\bar{v} \cdot {}^{A}_{R}\bar{v} \right) \qquad (3.4.13) $$

This gives a concise form for the kinetic power term

$$ -\frac{1}{2}\, {}^{AB}_{B}J\, {}^{A}_{B}\rho\, \frac{d}{d\,t} \left({}^{A}_{B}\bar{v} \cdot {}^{A}_{B}\bar{v} \right) = -\frac{d\, {}^{AB}_{B}\mathcal{K}}{d\,t} \qquad (3.4.14) $$

3.5 STRESS DIVERGENCE TERM

$$\frac{\partial}{\partial\, {}^{B}\bar{X}} \cdot {}^{AB}_{B}\overline{\overline{J}} \cdot \left({}^{A}_{B}\overline{\overline{\sigma}} \cdot {}^{A}_{B}\bar{v} \right)$$

Originally we had one term for internal power and one for external power. After some manipulation the internal power term was divided into two terms. The stress divergence term is one of these. The manipulations to be applied to this terms are particularly simple at this stage. Later, the divergence theorem will be used to effectively change this to an expression for the power of tractions distributed over the surface of the body. For now all that is needed is to recognise the First Piola-Kirchhoff Pseudo-stress tensor in the middle two factors. So, we can write

$$\frac{\partial}{\partial\, {}^{B}\bar{X}} \cdot \left({}^{AB}_{B}\overline{\overline{J}} \cdot {}^{A}_{B}\overline{\overline{\sigma}} \cdot {}^{A}_{B}\bar{v} \right) = \frac{\partial}{\partial\, {}^{B}\bar{X}} \cdot \left({}^{AB}_{B}\overline{\overline{T}} \cdot {}^{A}_{B}\bar{v} \right) \qquad (3.5.1)$$

where ${}^{AB}_{B}\overline{\overline{T}}$ is the First Piola-Kirchhoff Pseudo-stress tensor of C_A transformed to C_B and referred to C_B and

$${}^{AB}_{R}\overline{\overline{T}} = {}^{AB}_{R}\overline{\overline{J}} \cdot {}^{A}_{R}\overline{\overline{\sigma}} \qquad (3.5.2)$$

3.6 VELOCITY GRADIENT TERM

$$ {}^{AB}_{B}J \frac{\partial {}^{A}_{A}\bar{v}}{\partial {}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}} $$

There is sufficient manipulation involved in converting this term to the desired form that it is worth providing a description of what follows:

- Three lemmata will be needed so they are presented first
- The velocity gradient is replaced by a product of deformation gradients and the rate of pseudo-strain
- Arithmetic manipulations re-associate the deformation gradients with the stress tensor

3.6.1 LEMMATA

Lemma 1

Where $\bar{\bar{C}}$ and $\bar{\bar{D}}$ are any two second order tensors

$$ \bar{\bar{C}} : \bar{\bar{D}} = \bar{\bar{D}} : \bar{\bar{C}} $$

Proof

Let the vector spaces of $\overline{\overline{C}}$ be $\overline{P}$ and $\overline{Q}$. Let the vector spaces of $\overline{\overline{D}}$ be $\overline{R}$ and $\overline{S}$. Then

$$\overline{\overline{C}} = \overline{\overline{PQ}} \qquad \text{and} \qquad \overline{\overline{D}} = \overline{\overline{RS}} \tag{3.6.1}$$

So

$$\overline{\overline{C}} : \overline{\overline{D}} = (\overline{\overline{PQ}}) : (\overline{\overline{RS}}) \tag{3.6.2}$$

$$\overline{\overline{C}} : \overline{\overline{D}} = (\overline{P} \cdot \overline{R})(\overline{Q} \cdot \overline{S}) \tag{3.6.3}$$

$$\overline{\overline{C}} : \overline{\overline{D}} = (\overline{R} \cdot \overline{P})(\overline{S} \cdot \overline{Q}) \tag{3.6.4}$$

$$\overline{\overline{C}} : \overline{\overline{D}} = (\overline{\overline{RS}}) : (\overline{\overline{PQ}}) \tag{3.6.5}$$

$$\overline{\overline{C}} : \overline{\overline{D}} = \overline{\overline{D}} : \overline{\overline{C}} \tag{3.6.6}$$

As required

Lemma 2

Where $\overline{\overline{C}}$ and $\overline{\overline{D}}$ are any two second order tensors and $\overline{\overline{1}}$ is the second order identity tensor

$$\overline{\overline{C}} : \overline{\overline{D}} = (\overline{\overline{C}}^T \cdot \overline{\overline{D}}) : \overline{\overline{1}} = (\overline{\overline{C}} \cdot \overline{\overline{D}}^T) : \overline{\overline{1}}$$

Proof

Let the vector spaces of $\overline{\overline{C}}$ be $\overline{P}$ and $\overline{Q}$. Let the vector spaces of $\overline{\overline{D}}$ be $\overline{R}$ and $\overline{S}$. Then

$$\overline{\overline{C}} = \overline{PQ} \qquad \text{and} \qquad \overline{\overline{D}} = \overline{RS} \tag{3.6.7}$$

So

$$(\overline{\overline{C}}^T \cdot \overline{\overline{D}}) : \overline{\overline{1}} = \left[(\overline{QP})^T \cdot (\overline{RS}) \right] : \overline{\overline{1}} \tag{3.6.8}$$

$$(\overline{\overline{C}}^T \cdot \overline{\overline{D}}) : \overline{\overline{1}} = (\overline{P} \cdot \overline{R}) \left[(\overline{QS}) : \overline{\overline{1}} \right] \tag{3.6.9}$$

Refer $\overline{\overline{1}}$ to the vector space $\overline{S}$.

$$(\overline{\overline{C}}^T \cdot \overline{\overline{D}}) : \overline{\overline{1}} = (\overline{P} \cdot \overline{R}) \left[(\overline{Q}_t \overline{S}_t) : (\overline{S}_m \overline{S}^m) \right] \tag{3.6.10}$$

$$(\overline{\overline{C}}^T \cdot \overline{\overline{D}}) : \overline{\overline{1}} = (\overline{P} \cdot \overline{R}) \left[(\overline{Q}_t \cdot \overline{S}_m) (\overline{S}_t \cdot \overline{S}^m) \right] \tag{3.6.11}$$

$$(\bar{\bar{C}}^T \cdot \bar{\bar{D}}) : \bar{\bar{1}} = (\bar{P} \cdot \bar{R}) \left[(\bar{Q}_t \cdot \bar{S}_m) \ \delta_t^{\ m} \right] \tag{3.6.12}$$

where $\delta_t^{\ m}$ is the Kronecker delta

$$(\bar{\bar{C}}^T \cdot \bar{\bar{D}}) : \bar{\bar{1}} = (\bar{P} \cdot \bar{R}) \ (\bar{Q}_m \cdot \bar{S}_m) \tag{3.6.13}$$

$$(\bar{\bar{C}}^T \cdot \bar{\bar{D}}) : \bar{\bar{1}} = (\bar{P} \cdot \bar{R}) \ (\bar{Q} \cdot \bar{S}) \tag{3.6.14}$$

$$(\bar{\bar{C}}^T \cdot \bar{\bar{D}}) : \bar{\bar{1}} = (\overline{PQ}) : (\overline{RS}) \tag{3.6.15}$$

$$(\bar{\bar{C}}^T \cdot \bar{\bar{D}}) : \bar{\bar{1}} = \bar{\bar{C}} : \bar{\bar{D}} \tag{3.6.16}$$

As required. Also

$$(\bar{\bar{C}} \cdot \bar{\bar{D}}^T) : \bar{\bar{1}} = \left[(\overline{PQ})^T \cdot (\overline{SR}) \right] : \bar{\bar{1}} \tag{3.6.17}$$

$$(\bar{\bar{C}} \cdot \bar{\bar{D}}^T) : \bar{\bar{1}} = (\bar{Q} \cdot \bar{R}) \left[(\overline{PR}) : \bar{\bar{1}} \right] \tag{3.6.18}$$

$$(\bar{\bar{C}} \cdot \bar{\bar{D}}^T) : \bar{\bar{1}} = (\bar{Q} \cdot \bar{S}) \left[(\bar{P}_t \bar{R}_t) : (\bar{R}_m \bar{R}^m) \right] \tag{3.6.19}$$

$$(\bar{\bar{C}}\cdot\bar{\bar{D}}^T):\bar{\bar{1}} = (\bar{Q}\cdot\bar{S})\left[(\bar{P}_t\cdot\bar{R}_m)\ (\bar{R}_t\cdot\bar{R}^m)\right] \tag{3.6.20}$$

$$(\bar{\bar{C}}\cdot\bar{\bar{D}}^T):\bar{\bar{1}} = (\bar{Q}\cdot\bar{S})\left[(\bar{P}_t\cdot\bar{R}_m)\ \delta_t{}^m\right] \tag{3.6.21}$$

$$(\bar{\bar{C}}\cdot\bar{\bar{D}}^T):\bar{\bar{1}} = (\bar{Q}\cdot\bar{S})\ (\bar{P}_m\cdot\bar{R}_m) \tag{3.6.22}$$

$$(\bar{\bar{C}}\cdot\bar{\bar{D}}^T):\bar{\bar{1}} = (\bar{Q}\cdot\bar{S})\ (\bar{P}\cdot\bar{R}) \tag{3.6.23}$$

$$(\bar{\bar{C}}\cdot\bar{\bar{D}}^T):\bar{\bar{1}} = (\bar{P}\cdot\bar{R})\ (\bar{Q}\cdot\bar{S}) \tag{3.6.24}$$

$$(\bar{\bar{C}}\cdot\bar{\bar{D}}^T):\bar{\bar{1}} = (\overline{PQ})\ :\ (\overline{RS}) \tag{3.6.25}$$

$$(\bar{\bar{C}}\cdot\bar{\bar{D}}^T):\bar{\bar{1}} = \bar{\bar{C}}\ :\ \bar{\bar{D}} \tag{3.6.26}$$

As required

Lemma 3

$$\frac{\partial \, {}^{A}_{R}d\bar{x}}{\partial \, t} = {}^{A}_{R}d\bar{x} \cdot \frac{\partial \, {}^{A}_{A}\bar{v}}{\partial \, {}^{A}_{R}\bar{x}}$$

Proof

Consider two body-points X and Y. Let Y be differentially close to X, so

$${}^{A}\bar{x}(Y) = {}^{A}\bar{x}(X) + {}^{A}d\bar{x}(X,Y) \qquad (3.6.27)$$

Simplify the notation so that ${}^{A}\bar{x}$ is ${}^{A}\bar{x}(X)$ and ${}^{A}d\bar{x}$ is ${}^{A}d\bar{x}(X,Y)$. Also let the velocity be $\bar{v}$ so

$${}^{A}\bar{v} = \frac{\partial \, {}^{A}\bar{x}}{\partial \, t} \qquad (3.6.28)$$

Then we may write

$${}^{A}\bar{v}(X) = {}^{A}_{A}\bar{v}({}^{A}\bar{x}) \qquad (3.6.29)$$

$${}^{A}\bar{v}(Y) = {}^{A}_{A}\bar{v}({}^{A}\bar{x} + {}^{A}d\bar{x}) \qquad (3.6.30)$$

$$ {}^{A}\bar{v}(Y) = {}^{A}_{A}\bar{v} + {}^{A}_{A}d\bar{x} \cdot \frac{\partial\, {}^{A}_{A}\bar{v}}{\partial\, {}^{A}_{A}\bar{x}} \qquad (3.6.31) $$

So the difference in velocity between the two body-points is

$$ {}^{A}\bar{v}(Y) - {}^{A}\bar{v}(X) = {}^{A}_{A}\bar{v} + {}^{A}_{A}d\bar{x} \cdot \frac{\partial\, {}^{A}_{A}\bar{v}}{\partial\, {}^{A}_{A}\bar{x}} - {}^{A}_{A}\bar{v} \qquad (3.6.32) $$

$$ {}^{A}\bar{v}(Y) - {}^{A}\bar{v}(X) = {}^{A}_{A}d\bar{x} \cdot \frac{\partial\, {}^{A}_{A}\bar{v}}{\partial\, {}^{A}_{A}\bar{x}} \qquad (3.6.33) $$

Also

$$ {}^{A}\bar{v}(Y) = \frac{\partial}{\partial\, t}\left({}^{A}\bar{x} + {}^{A}d\bar{x} \right) = \frac{\partial\, {}^{A}\bar{x}}{\partial\, t} + \frac{\partial\, {}^{A}d\bar{x}}{\partial\, t} \qquad (3.6.34) $$

$$ {}^{A}\bar{v}(Y) = \frac{\partial}{\partial\, t}\left({}^{A}\bar{x} + {}^{A}d\bar{x} \right) = {}^{A}\bar{v} + \frac{\partial\, {}^{A}d\bar{x}}{\partial\, t} \qquad (3.6.35) $$

The difference in the velocities of X and Y using this expression is

$$ {}^{A}\bar{v}(Y) - {}^{A}\bar{v}(X) = {}^{A}\bar{v} + \frac{\partial\, {}^{A}d\bar{x}}{\partial\, t} - {}^{A}\bar{v} = \frac{\partial\, {}^{A}d\bar{x}}{\partial\, t} \qquad (3.6.36) $$

Now it is possible to subtract the two expressions for the difference in velocity

$$\frac{\partial\ {}^{A}d\bar{x}}{\partial\ t} - {}^{A}_{A}d\bar{x} \cdot \frac{\partial\ {}^{A}_{A}\bar{v}}{\partial\ {}^{A}_{A}\bar{x}} = \bar{0} \qquad (3.6.37)$$

Since there are no geometric presuppositions that the vectors on the left hand side of eqn(3.6.37) are themselves zero it must be that the two terms are equal and so the lemma must be true. That is

$$\frac{\partial\ {}^{A}_{R}d\bar{x}}{\partial\ t} = {}^{A}_{R}d\bar{x} \cdot \frac{\partial\ {}^{A}_{A}\bar{v}}{\partial\ {}^{A}_{R}\bar{x}} \qquad (3.6.38)$$

As required

3.6.2 INTRODUCE RATE OF PSEUDO-STRAIN

In the second part we replace the velocity gradient itself with a product of the deformation gradient and the rate of pseudo-strain. To do this we take four steps:

- Find an expression relating the rate of the quadratic measure of deformation to the rate of pseudo-strain,
- Find an expression for the rate of the quadratic measure of deformation involving the velocity gradient,

- Use the first and second parts to identify and expression for the rate of pseudo-strain and
- Isolate the velocity gradient in the expression for the rate of pseudo-strain and replace.

So, we begin. The quadratic measure of deformation measured in C_A and transformed and referred to C_R can be expressed using the pseudo-strain as

$$ {}^{A}_{R}D = {}^{R}_{R}d\bar{x}\ {}^{R}_{R}d\bar{x} : 2\,{}^{AR}_{R}\bar{\bar{E}} \tag{3.6.39} $$

The rate of this is

$$ \frac{d\ {}^{A}_{R}D}{d\ t} = \frac{d}{d\ t}\left[\ {}^{R}_{R}d\bar{x}\ {}^{R}_{R}d\bar{x} : 2\,{}^{AR}_{R}\bar{\bar{E}}\ \right] \tag{3.6.40} $$

$$ \frac{d\ {}^{A}_{R}D}{d\ t} = \left(\frac{d\ {}^{R}_{R}d\bar{x}}{d\ t}\right){}^{R}_{R}d\bar{x} : 2\ {}^{AR}_{R}\bar{\bar{E}} \tag{3.6.41} $$

$$ + \ {}^{R}_{R}d\bar{x}\left(\frac{d\ {}^{R}_{R}d\bar{x}}{d\ t}\right) : 2\ {}^{AR}_{R}\bar{\bar{E}} \ + \ {}^{R}_{R}d\bar{x}\ {}^{R}_{R}d\bar{x} : 2\left(\frac{d\ {}^{AR}_{R}\bar{\bar{E}}}{d\ t}\right) $$

The differential position vector in the reference configuration is a constant, so it has a rate of zero. The first two terms on the right hand side vanish. We are left with

$$\frac{d\,{}^{A}_{R}D}{d\,t} = {}^{R}_{R}d\bar{x}\;{}^{R}_{R}d\bar{x} : 2\left(\frac{d\,{}^{AR}_{R}\bar{\bar{E}}}{d\,t}\right) \tag{3.6.42}$$

We can also start from the definition of ${}^{A}D$ and differentiate with respect to time.

$${}^{A}_{R}D = {}^{A}_{R}d\bar{x} \cdot {}^{A}_{R}d\bar{x} - {}^{U}_{R}d\bar{x} \cdot {}^{U}_{R}d\bar{x} \tag{3.6.43}$$

$$\frac{d\,{}^{A}_{R}D}{d\,t} = \frac{d}{d\,t}\left[\,{}^{A}_{R}d\bar{x} \cdot {}^{A}_{R}d\bar{x} - {}^{U}_{R}d\bar{x} \cdot {}^{U}_{R}d\bar{x}\,\right] \tag{3.6.44}$$

$$\frac{d\,{}^{A}_{R}D}{d\,t} = \frac{d\,{}^{A}_{R}d\bar{x}}{d\,t} \cdot {}^{A}_{R}d\bar{x} + {}^{A}_{R}d\bar{x} \cdot \frac{d\,{}^{A}_{R}d\bar{x}}{d\,t} - \frac{d\,{}^{U}_{R}d\bar{x}}{d\,t} \cdot {}^{U}_{R}d\bar{x} - {}^{U}_{R}d\bar{x} \cdot \frac{d\,{}^{U}_{R}d\bar{x}}{d\,t} \tag{3.6.45}$$

The unstrained configuration is a fixed chosen one. Therefore the rate of any properties of this configuration vanishes. So, the terms involving the rate of the differential position vector in the unstrained configuration vanish. So

$$\frac{d\,{}^{A}_{R}D}{d\,t} = \frac{d\,{}^{A}_{R}d\bar{x}}{d\,t} \cdot {}^{A}_{R}d\bar{x} + {}^{A}_{R}d\bar{x} \cdot \frac{d\,{}^{A}_{R}d\bar{x}}{d\,t} \tag{3.6.46}$$

Or

$$\frac{d\,{}^{A}_{R}D}{d\,t} = 2\,{}^{A}_{R}d\bar{x} \cdot \frac{d\,{}^{A}_{R}d\bar{x}}{d\,t} \tag{3.6.47}$$

Invoke Lemma 3 to obtain

$$\frac{d\,{}^{A}_{R}D}{d\,t} = 2\,{}^{A}_{R}d\bar{x} \cdot \left({}^{A}_{R}d\bar{x} \cdot \frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} \right) \tag{3.6.48}$$

$$\frac{d\,{}^{A}_{R}D}{d\,t} = 2\,{}^{A}_{R}d\bar{x}\;{}^{A}_{R}d\bar{x} : \frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} \tag{3.6.49}$$

An effect of the double dot product with the same vector is that any antisymmetric part of the velocity gradient makes no contribution. So,

$$\frac{d\,{}^{A}_{R}D}{d\,t} = 2\,{}^{A}_{R}d\bar{x}\;{}^{A}_{R}d\bar{x} : \left(\frac{1}{2}\left\{ \frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} + \left[\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} \right]^{T} \right\} + \frac{1}{2}\left\{ \frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} - \left[\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} \right]^{T} \right\} \right)$$

$$\frac{d\,{}^{A}_{R}D}{d\,t} = 2\,{}^{A}_{R}d\bar{x}\;{}^{A}_{R}d\bar{x} : \frac{1}{2}\left\{ \frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} + \left[\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} \right]^{T} \right\} \tag{3.6.50}$$

$$\frac{d\,{}^{A}_{R}D}{d\,t} = 2\,{}^{A}_{R}d\bar{x}\;{}^{A}_{R}d\bar{x} : \left(\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{R}\bar{x}} \right)^{sym}$$

Now change the differential position vectors so that they are measured

in C_R. Eqn(3.6.50) becomes

$$\frac{d\ {}^{A}_{R}D}{d\ t} = {}^{R}_{R}d\bar{x}\cdot\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\ \ {}^{R}_{R}d\bar{x}\cdot\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}} : 2\left(\frac{\partial\ {}^{A}_{A}\bar{v}}{\partial\ {}^{A}_{R}\bar{x}}\right)^{sym} \qquad (3.6.51)$$

$$\frac{d\ {}^{A}_{R}D}{d\ t} = {}^{R}_{R}d\bar{x}\cdot\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\cdot 2\left(\frac{\partial\ {}^{A}_{A}\bar{v}}{\partial\ {}^{A}_{R}\bar{x}}\right)^{sym}\cdot\left(\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\right)^{T}\cdot{}^{R}_{R}d\bar{x} \qquad (3.6.52)$$

$$\frac{d\ {}^{A}_{R}D}{d\ t} = {}^{R}_{R}d\bar{x}\ {}^{R}_{R}d\bar{x} : 2\left(\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\ {}^{A}_{A}\bar{v}}{\partial\ {}^{A}_{R}\bar{x}}\right)^{sym}\cdot\left(\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\right)^{T} \qquad (3.6.53)$$

As before the symmetry marked on the velocity gradient is optional since any antisymmetric part vanishes in the double dot product.

The two expressions, eqn(3.6.42) and eqn(3.6.53), for the rate of ${}^{A}_{R}D$ can be compared and it must be concluded that

$$\frac{d\ {}^{AR}_{R}\bar{\bar{E}}}{d\ t} = \left(\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\right)\cdot\left(\frac{\partial\ {}^{A}_{A}\bar{v}}{\partial\ {}^{A}_{R}\bar{x}}\right)^{sym}\cdot\left(\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\right)^{T} \qquad (3.6.54)$$

This we will put to use by isolating the velocity gradient factor as

$$\left(\frac{\partial\ {}^{A}_{A}\bar{v}}{\partial\ {}^{A}_{R}\bar{x}}\right)^{sym} = \left(\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\right)^{-1}\cdot\left(\frac{d\ {}^{AR}_{R}\bar{\bar{E}}}{d\ t}\right)\cdot\left(\frac{\partial\ {}^{A}_{R}\bar{x}}{\partial\ {}^{R}_{R}\bar{x}}\right)^{-T} \qquad (3.6.55)$$

3.6.3 REASSOCIATE FACTORS

The third stage of the conversion of the velocity gradient term can begin. To simplify the notation a little the rate of pseudo-strain will be represented by ${}^{AB}_{B}\bar{\bar{A}}$. Then we have

$$ {}^{AB}_{B}J \frac{\partial {}^{A}_{A}\bar{v}}{\partial {}^{A}_{B}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}} = {}^{AB}_{B}J \left({}^{AB}_{B}\bar{\bar{F}}^{-1} \cdot {}^{AB}_{B}\bar{\bar{A}} \cdot {}^{AB}_{B}\bar{\bar{F}}^{-T} \right) : {}^{A}_{B}\bar{\bar{\sigma}} \qquad (3.6.56) $$

Further notational simplification will be introduced as convenient for the immediate purpose of the following arithmetic. Let

$$ \begin{aligned} \bar{\bar{A}} &= {}^{AB}_{B}\bar{\bar{A}} & \bar{\bar{B}} &= {}^{AB}_{B}\bar{\bar{F}}^{-1} \\ \bar{\bar{G}} &= {}^{AB}_{B}\bar{\bar{F}}^{-T} & \bar{\bar{H}} &= {}^{A}_{B}\bar{\bar{\sigma}} \qquad (3.6.57) \\ \bar{\bar{C}} &= \bar{\bar{B}} \cdot \bar{\bar{A}} \cdot \bar{\bar{G}} & \bar{\bar{D}} &= \bar{\bar{H}} \end{aligned} $$

Then we may apply Lemma 2 in the second form to find

$$ (\bar{\bar{B}} \cdot \bar{\bar{A}} \cdot \bar{\bar{G}}) : \bar{\bar{H}} = [(\bar{\bar{B}} \cdot \bar{\bar{A}} \cdot \bar{\bar{G}}) \cdot \bar{\bar{H}}^{T}] : \bar{\bar{1}} \qquad (3.6.58) $$

Re-associate

$$ [(\bar{\bar{B}} \cdot \bar{\bar{A}} \cdot \bar{\bar{G}}) \cdot \bar{\bar{H}}^{T}] : \bar{\bar{1}} = [\, \bar{\bar{B}} \cdot (\bar{\bar{A}} \cdot \bar{\bar{G}} \cdot \bar{\bar{H}}^{T})] : \bar{\bar{1}} \qquad (3.6.59) $$

Redefine $\bar{\bar{C}}$ and $\bar{\bar{D}}$ so that

$$\bar{\bar{C}} = \bar{\bar{B}} \qquad \bar{\bar{D}} = \bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T \tag{3.6.60}$$

Apply Lemma 2 in the first form

$$[\,\bar{\bar{B}} \cdot (\bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T)\,] : \bar{\bar{1}} = \bar{\bar{B}}^T : (\bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T) \tag{3.6.61}$$

Apply Lemma 1

$$\bar{\bar{B}}^T : (\bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T) = (\bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T) : \bar{\bar{B}}^T \tag{3.6.62}$$

Redefine $\bar{\bar{C}}$ and $\bar{\bar{D}}$ so that

$$\bar{\bar{C}} = \bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T \qquad \bar{\bar{D}} = \bar{\bar{B}}^T \tag{3.6.63}$$

Apply Lemma 2 in the second form

$$(\bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T) : \bar{\bar{B}}^T = [(\bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T) \cdot \bar{\bar{B}}\,] : \bar{\bar{1}} \tag{3.6.64}$$

Re-associate

$$[(\bar{\bar{A}}\cdot\bar{\bar{G}}\cdot\bar{\bar{H}}^T) \cdot \bar{\bar{B}}\,] : \bar{\bar{1}} = [\,\bar{\bar{A}} \cdot (\bar{\bar{G}}\cdot\bar{\bar{H}}^T\cdot\bar{\bar{B}})\,] : \bar{\bar{1}} \tag{3.6.65}$$

Redefine $\bar{\bar{C}}$ and $\bar{\bar{D}}$ so that

$$\bar{\bar{C}} = \bar{\bar{A}} \qquad \bar{\bar{D}} = \bar{\bar{G}} \cdot \bar{\bar{H}}^T \cdot \bar{\bar{B}} \tag{3.6.66}$$

Apply Lemma 2 in the first form

$$[\,\bar{\bar{A}}\cdot(\bar{\bar{G}}\cdot\bar{\bar{H}}^{T}\cdot\bar{\bar{B}})\,]:\bar{\bar{1}}=\bar{\bar{A}}^{T}:(\bar{\bar{G}}\cdot\bar{\bar{H}}^{T}\cdot\bar{\bar{B}}) \tag{3.6.67}$$

Apply Lemma 1

$$\bar{\bar{A}}^{T}:(\bar{\bar{G}}\cdot\bar{\bar{H}}^{T}\cdot\bar{\bar{B}})=(\bar{\bar{G}}\cdot\bar{\bar{H}}^{T}\cdot\bar{\bar{B}}):\bar{\bar{A}}^{T} \tag{3.6.68}$$

Replace the substitutions to find

$$\left({}^{AB}_{B}\bar{\bar{F}}^{-1}\cdot{}^{AB}_{B}\bar{\bar{A}}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-T}\right):{}^{A}_{B}\bar{\bar{\sigma}}=\left({}^{AB}_{B}\bar{\bar{F}}^{-T}\cdot{}^{A}_{B}\bar{\bar{\sigma}}^{T}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-1}\right):{}^{AB}_{B}\bar{\bar{A}}^{T} \tag{3.6.69}$$

Note that both the Cauchy (true) stress and the (pseudo-)strain are symmetric second order tensors. Since the (pseudo-)strain is a symmetric tensor the rate of the (pseudo-)strain must also be symmetric. So the transpose markers on the stress and strain factors may be dropped.

$$\left({}^{AB}_{B}\bar{\bar{F}}^{-1}\cdot{}^{AB}_{B}\bar{\bar{A}}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-T}\right):{}^{A}_{B}\bar{\bar{\sigma}}=\left({}^{AB}_{B}\bar{\bar{F}}^{-T}\cdot{}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-1}\right):{}^{AB}_{B}\bar{\bar{A}} \tag{3.6.70}$$

This can be put into the original term

$${}^{AB}_{B}J\,\frac{\partial\,{}^{A}_{A}\bar{v}}{\partial\,{}^{A}_{B}\bar{x}}:{}^{A}_{B}\bar{\bar{\sigma}}={}^{AB}_{B}J\left({}^{AB}_{B}\bar{\bar{F}}^{-T}\cdot{}^{A}_{B}\bar{\bar{\sigma}}\cdot{}^{AB}_{B}\bar{\bar{F}}^{-1}\right):\frac{d\,{}^{AB}_{B}\bar{\bar{E}}}{d\,t} \tag{3.6.71}$$

The final stage is to recognize all factors on the right hand side

before the double dot product. Taken together these are the Second Piola-Kirchhoff Pseudo-stress tensor, ${}^{AB}_{B}\bar{\bar{S}}$. So

$$ {}^{AB}_{B}J \frac{\partial {}^{A}_{A}\bar{v}}{\partial {}^{A}\bar{x}} : {}^{A}_{B}\bar{\bar{\sigma}} = {}^{AB}_{B}\bar{\bar{S}} : \frac{d {}^{AB}_{B}\bar{\bar{E}}}{d t} \qquad (3.6.72) $$

This is the form sought.

3.7 BODY FORCE TERM

$$ {}^{AB}_{B}J \; {}^{A}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v} $$

It is practical to lump the first two factors together to form a pseudo body force, ${}^{AB}\bar{f}$.

$$ {}^{AB}_{R}\bar{f} = {}^{AB}_{R}J {}^{A}_{R}\bar{f} \qquad (3.7.1) $$

${}^{A}\bar{f}$ is a force density. It is sometimes convenient to express this as acting through some agent property of the material. Examples of this are gravity acting through the mass density and electric forces acting through the charge density. Then we would write

$$ {}^{A}\bar{f} = {}^{A}\beta \, {}^{A}\bar{b} \qquad (3.7.2) $$

where ${}^{A}\beta$ is the agent property of the material and

${}^{A}\bar{b}$ is field that acts through ${}^{A}\beta$.

Exactly the same kind of argument as was used in section(2.10) for mass density may be applied to the agent density here. The conclusion may be drawn that

$$ {}^{AB}J = \frac{{}^{B}\beta}{{}^{A}\beta} \tag{3.7.3} $$

For such forces it may be preferred to keep the transformed equations in terms of an pseudo agent density and an active field. The term pseudo agent density is used since there is no general guaranteed constancy to the agent quantity as there is with mass. The body force term would appear as

$$ {}^{AB}_{B}J \; {}^{A}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v} = {}^{AB}_{B}J \; {}^{A}_{B}\mu \; {}^{A}_{B}\bar{b} \cdot {}^{A}_{B}\bar{v} \tag{3.7.4} $$

$$ {}^{AB}_{B}J \; {}^{A}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v} = {}^{AB}_{B}\beta \; {}^{A}_{B}\bar{b} \cdot {}^{A}_{B}\bar{v} \tag{3.7.5} $$

where the pseudo agent density, ${}^{AB}_{B}\mu$, is

$$ {}^{AB}_{B}\beta = {}^{AB}_{B}J \; {}^{A}_{B}\beta \tag{3.7.6} $$

Alternatively, it may be preferred to keep the agent density unchanged and use a pseudo active field. This second approach is less popular.

3.8 TRANSFORMED RATE OF WORK PRINCIPLE

All terms may now be combined to give the strong form of the Rate of Work Principle transformed to C_A.

$$0 = -\frac{d\ {}^{AB}_{B}\mathcal{K}}{d\ t} + \frac{\partial}{\partial\ {}^{B}_{B}\bar{x}} \cdot \left({}^{AB}_{B}\bar{\bar{T}} \cdot {}^{A}_{B}\bar{v} \right) \qquad (3.8.1)$$

$$- {}^{AB}_{B}\bar{\bar{S}} : \frac{d\ {}^{AB}_{B}\bar{\bar{E}}}{d\ t} + {}^{AB}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v}$$

For calculation this is often kept in weak form with the divergence term expressed as a surface integral.

$$0 = \int_{{}^{B}\Omega} \left\{ -\frac{d\ {}^{AB}_{B}\mathcal{K}}{d\ t} + {}^{AB}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v} \right. \qquad (3.8.2)$$

$$\left. - {}^{AB}_{B}\bar{\bar{S}} : \frac{d\ {}^{AB}_{B}\bar{\bar{E}}}{d\ t} \right\} {}^{B}_{B}dV$$

$$+ \int_{{}^{B}\partial\Omega} {}^{B}_{B}d\bar{a} \cdot \left({}^{AB}_{B}\bar{\bar{T}} \cdot {}^{A}_{B}\bar{v} \right)$$

The differential surface vector, ${}^{B}_{B}d\bar{a}$, can be expanded so that only the area is in differential form.

$$\int_{{}_B\partial\Omega} {}_B^{}d\bar{a} \cdot \left({}_B^{AB}\bar{\bar{T}} \cdot {}_B^{A}\bar{v} \right) = \int_{{}_B\partial\Omega} {}_B^{}da \left({}_B^{}\hat{n} \cdot {}_B^{AB}\bar{\bar{T}} \cdot {}_B^{A}\bar{v} \right) \qquad (3.8.3)$$

Then the unit outward normal is taken in a dot product with the First Piola-Kirchhoff Pseudo-stress tensor to give the (pseudo-) traction on the surface, scaled and rotated to suit C_B, ${}_B^{AB}\bar{T}$.

$$\int_{{}_B\partial\Omega} {}_B^{}da \left({}_B^{}\hat{n} \cdot {}_B^{AB}\bar{\bar{T}} \cdot {}_B^{A}\bar{v} \right) = \int_{{}_B\partial\Omega} {}_B^{}da \; {}_B^{AB}\bar{T} \cdot {}_B^{A}\bar{v} \qquad (3.8.4)$$

This allows the eqn(3.8.1) to be expressed as

$$0 = \int_{{}_B\Omega} \left\{ - \frac{d \; {}_B^{AB}\mathcal{K}}{d\,t} + {}_B^{AB}\bar{f} \cdot {}_B^{A}\bar{v} - {}_B^{AB}\bar{\bar{S}} : \frac{d \; {}_B^{AB}\bar{\bar{E}}}{d\,t} \right\} {}_B^{}dV \qquad (3.8.5)$$

$$+ \int_{{}_B\partial\Omega} {}_B^{}da \; {}_B^{AB}\bar{T} \cdot {}_B^{A}\bar{v}$$

3.9 INCREMENTAL FORM

For the purpose of incremental investigations one could take the velocity based form, eqn(3.8.5), and multiply it by a small time, dt. dt may be very small but must not actually collapse to zero. This gives

$$0 = \int_{B_\Omega} \left\{ - \frac{d\ {}^{AB}_{B}\mathcal{K}}{d\ t} + {}^{AB}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v} - {}^{AB}_{B}\bar{\bar{S}} : \frac{d\ {}^{AB}_{B}\bar{\bar{E}}}{d\ t} \right\} dt\ {}^{B}_{B}dV + \int_{B_{\partial\Omega}} {}^{B}_{B}da\ {}^{AB}_{B}\bar{T} \cdot {}^{A}_{B}\bar{v}\ dt \qquad (3.9.1)$$

The left hand side is still zero. In examining the right hand side it is convenient to distribute the integral as a linear operator and inspect each term separately.

3.9.1 INCREMENT OF KINETIC ENERGY DENSITY

This increment may be expressed as

$$\frac{d\ {}^{AB}_{B}\mathcal{K}}{d\ t}\ dt\ =\ {}^{AB}_{B}d\mathcal{K} \tag{3.9.2}$$

Put this back into the form

$${}^{AB}_{B}d\mathcal{K}\ =\ {}^{B}_{B}\rho\ {}^{A}_{B}\bar{v}\ \cdot\ \left(\frac{d\ {}^{A}_{B}\bar{v}}{d\ t}\right)\ dt \tag{3.9.3}$$

The last two factors combine to give the increment of velocity, ${}^{A}_{B}d\bar{v}$.

$${}^{AB}_{B}d\mathcal{K}\ =\ {}^{B}_{B}\rho\ {}^{A}_{B}\bar{v}\ \cdot\ {}^{A}_{B}d\bar{v} \tag{3.9.4}$$

This may be put into a form involving incremental displacement by using the identity

$${}^{A}\bar{v}\ \cdot\ {}^{A}d\bar{v}\ =\ \frac{d\ {}^{A}\bar{v}}{d\ t}\ \cdot\ {}^{A}d\bar{u} \tag{3.9.5}$$

This follows from the differential of the quadratic measure of velocity, $({}^{A}\bar{v}\ \cdot\ {}^{A}\bar{v})$.

$$d({}^{A}\bar{v}\ \cdot\ {}^{A}\bar{v})\ =\ \frac{d}{d\ t}\ ({}^{A}\bar{v}\ \cdot\ {}^{A}\bar{v})\ dt \tag{3.9.6}$$

$$d({}^{A}\bar{v} \cdot {}^{A}\bar{v}) = \frac{d\,{}^{A}\bar{v}}{d\,t} \cdot {}^{A}\bar{v}\,dt + {}^{A}\bar{v} \cdot \frac{d\,{}^{A}\bar{v}}{d\,t}\,dt \qquad (3.9.7)$$

The two terms on the right hand side are indistinguishable because the dot product is commutative and so they must be equal.

$$\frac{d\,{}^{A}\bar{v}}{d\,t} \cdot {}^{A}\bar{v}\,dt = {}^{A}\bar{v} \cdot \frac{d\,{}^{A}\bar{v}}{d\,t}\,dt \qquad (3.9.8)$$

Now we take the product on either side. In the left hand side, we recognise the rate of velocity as the acceleration and substitute the rate of displacement for velocity to get

$${}^{A}\bar{a} \cdot {}^{A}d\bar{u} = {}^{A}\bar{v} \cdot {}^{A}d\bar{v} \qquad (3.9.9)$$

as required. So eqn(3.9.4) may be rewritten as

$${}^{AB}_{B}dK = {}^{B}_{B}\rho\ {}^{A}_{B}\bar{a} \cdot {}^{A}_{B}d\bar{u} \qquad (3.9.10)$$

3.9.2 INCREMENT OF STRAIN ENERGY DENSITY

Express the third term of eqn(3.9.1) as

$$ {}^{AB}_{B}\bar{\bar{S}} : \frac{d\ {}^{AB}_{B}\bar{\bar{E}}}{d\ t}\, dt \;=\; {}^{AB}_{B}d\mathcal{U} \qquad (3.9.11) $$

Where eqn(3.9.11) defines an incremental change in an energy density. This density is called the strain energy density, ${}^{AB}_{B}\mathcal{U}$. Combine the rate with the time increment to find

$$ {}^{AB}_{B}d\mathcal{U} \;=\; {}^{AB}_{B}\bar{\bar{S}} : {}^{AB}_{B}d\bar{\bar{E}} \qquad (3.9.12) $$

3.9.3 INCREMENT OF VOLUME WORK DENSITY

Express the second term in eqn(3.9.1) as

$$ {}^{AB}_{B}\bar{f} \cdot {}^{A}_{B}\bar{v}\, dt \;=\; {}^{AB}_{B}d\mathcal{V} \qquad (3.9.13) $$

where eqn(3.9.13) defines an incremental change in an energy density. This energy density is called the volume work density, ${}^{AB}_{B}\mathcal{V}$. Replace the velocity by the rate of displacement and combine it with the time increment to get

$$ {}^{AB}_{B}dV = {}^{AB}_{B}\bar{f} \cdot {}^{A}_{B}d\bar{u} \qquad (3.9.14) $$

3.9.4 INCREMENT OF SURFACE WORK DENSITY

Express the last term in eqn(3.9.1) as

$$ {}^{AB}_{B}\bar{T} \cdot {}^{A}_{B}\bar{v}\, dt = {}^{AB}_{B}d\mathcal{T} \qquad (3.9.15) $$

Where eqn(3.9.15) defines an incremental change in a surface energy density. This surface energy density is called the surface work density, ${}^{AB}_{B}\mathcal{T}$. Substitute the rate of displacement for the velocity and combine with the time increment to get

$$ {}^{AB}_{B}d\mathcal{T} = {}^{AB}_{B}\bar{T} \cdot {}^{A}_{B}d\bar{u} \qquad (3.9.16) $$

3.9.5 COMBINED INCREMENTS

These four energy density increments may be combined to express the incremental form of the Rate of Work Principle as

$$0 = \int_{B_\Omega} \left\{ -{}^{AB}_{B}d\mathcal{K} + {}^{AB}_{B}d\mathcal{V} - {}^{AB}_{B}d\mathcal{U} \right\} {}^{B}_{B}dV + \int_{B_{\partial\Omega}} da \; {}^{AB}_{B}d\mathcal{T} \qquad (3.9.17)$$

It is common practice to modify this by lumping all terms. The two work densities are often combined as a single term defining the incremental work, ${}^{AB}_{B}dW$, as

$${}^{AB}_{B}dW = \int_{B_\Omega} {}^{AB}_{B}d\mathcal{V} \; {}^{B}_{B}dV + \int_{B_{\partial\Omega}} {}^{B}_{B}da \; {}^{AB}_{B}d\mathcal{T} \qquad (3.9.18)$$

Eqn(3.9.17) may be expressed as

$$0 = - \int_{B_\Omega} {}^{AB}_{B}d\mathcal{K} \; {}^{B}_{B}dV - \int_{B_\Omega} {}^{AB}_{B}d\mathcal{U} \; {}^{B}_{B}dV + {}^{AB}_{B}dW \qquad (3.9.19)$$

The first two terms define the increment to the kinetic energy, ${}^{AB}_{B}K$, and the increment to the strain energy, ${}^{AB}_{B}U$, respectively. So use

$${}^{AB}_{B}dK = \int_{B_\Omega} {}^{AB}_{B}d\mathcal{K} \; {}^{B}_{B}dV \qquad (3.9.20)$$

to define the increment to the kinetic energy and

$$ {}^{AB}_{B}dU = \int_{B_\Omega} {}^{AB}_{B}d\mathcal{U}\ {}^{B}_{B}dV \qquad (3.9.21) $$

to define the increment to the strain energy. Then the Rate of Work Principle can be stated as

$$ 0 = -{}^{AB}_{B}dK - {}^{AB}_{B}dU + {}^{AB}_{B}dW \qquad (3.9.22) $$

Another popular contraction is to lump together the increments to strain energy and work under the name potential energy. Define this potential energy, ${}^{AB}_{B}\Pi$, as

$$ {}^{AB}_{B}\Pi = {}^{AB}_{B}U - {}^{AB}_{B}W \qquad (3.9.23) $$

So, the Principle becomes

$$ 0 = {}^{AB}_{B}d\Pi - {}^{AB}_{B}dK \qquad (3.9.24) $$

This is a famous form and the starting off point for many analyses.

3.10 STATIC ANALYSES

For static analyses some special conditions should be noted. First the kinetic energy should be examined. The incremental form of the kinetic energy density is

$$ {}^{AB}_{B}d\mathcal{K} = {}^{B}_{B}\rho\ {}^{A}_{B}\bar{a} \cdot {}^{A}_{B}d\bar{u} \qquad (3.10.1) $$

In a static analysis the structure involved is not in active motion. So the displacement is fixed and has no time derivatives of any order and the acceleration vector in eqn(3.10.1) is identically zero. So the kinetic energy density and the kinetic energy are identically zero. Therefore, the Rate of Work Principle becomes

$$ 0 = {}^{AB}_{B}d\Pi \qquad (3.10.2) $$

It is also important to ask at this point what the meaning of the increment of potential energy can be if all is static. Clearly this cannot represent an incremental change as the body moves along some path. Instead the possible changes are choices of configuration. This then allows us to express a principle for static displacements in meaningful terms. That is eqn(3.10.2) should be

$$ d^{AB}_{B}\Pi = 0 \qquad (3.10.3) $$

This is the condition that will allow a choice for C_A among neighbouring

configurations. Eqn(3.10.3) can also be written as

$$d^{AB}_{B}\Pi = d^{A}_{B}\bar{x} \cdot \frac{\delta \, {}^{AB}_{B}\Pi}{\delta \, {}^{A}_{B}\bar{x}} = 0 \qquad (3.10.4)$$

This shows the choice of C_A affecting the value of ${}^{AB}_{B}\Pi$. The position of C_A can also be expressed as a displacement from some other configuration. Common choices for this other configuration are the initial configuration and a nearby configuration. The nearby configuration is one that has already been calculated and that would be occupied were the loading slightly different. The position of a body-point in C_A is ${}^{A}\bar{x}$ and we can write

$${}^{A}\bar{x} = {}^{C}\bar{x} + {}^{AC}\bar{u} \qquad (3.10.5)$$

where ${}^{C}\bar{x}$ is the other configuration mentioned above. A differential change in ${}^{A}\bar{x}$ would give a new position

$${}^{A}\bar{x} + d^{A}\bar{x} = {}^{C}\bar{x} + {}^{AC}\bar{u} + d^{A}\bar{u} \qquad (3.10.6)$$

where $d^{A}\bar{u}$ is the differential change in displacement. In a vectorial sense $d^{A}\bar{u}$ is the displacement from the previously supposed position of X to a putative new position. Two things are clear: (1) $d^{A}\bar{u}$ is not a function of C_C, and (2) $d^{A}\bar{u}$ is identical to $d^{A}\bar{x}$. This leads us to the realization that eqn(3.10.4) can be written as

$$d^{AB}_{B}\Pi = d^{A}_{B}\bar{u} \cdot \frac{\delta\, {}^{AB}_{B}\Pi}{\delta\, {}^{A}_{B}\bar{u}} = 0 \qquad (3.10.7)$$

There are three possibilities for the fulfillment of eqn(3.10.7):

1) $d^{A}_{B}\bar{u}$ is zero

2) $d^{A}_{B}\bar{u}$ is normal to $\dfrac{\delta\, {}^{AB}_{B}\Pi}{\delta\, {}^{A}_{B}\bar{u}}$

3) $\dfrac{\delta\, {}^{AB}_{B}\Pi}{\delta\, {}^{A}_{B}\bar{u}}$ is zero

The first possibility can be discarded because we clearly are considering variations. The configurational derivative of ${}^{AB}_{B}\Pi$ is a vector in a particular direction; whereas $d^{A}_{B}\bar{u}$ could be in any direction. So we cannot rely on a normality condition and must reject the second possibility. This leaves the third possibility. So, the equilibrium condition can be stated as

$$\frac{\delta\, {}^{AB}_{B}\Pi}{\delta\, {}^{A}_{B}\bar{u}} = \bar{0} \qquad (3.10.8)$$

CHAPTER FOUR

CONTACT ALGORITHM

4.1 INTRODUCTION

The proposed algorithm for contact is for an incremental style of solution. Any incremental solver can also solve single step analyses simply by declaring that all boundary conditions are to be met in the first increment. The advantage of incremental solvers is in evolutionary problems where directions and magnitudes of boundary conditions or material behaviour change as the analysis proceeds. Such problems are usually beyond the capabilities of single step solvers.

The first step in discussion of the new algorithm is a brief review of incremental finite element algorithms. Pseudo code for a typical nonlinear algorithm (materially nonlinear, geometrically nonlinear or both) is shown in Figure 4.1. It is assumed that the solution is known up to a certain point. This assumption is safe in that the initial conditions at least are known. A small increment of the known external information is applied and the corresponding increment in the unknown external information is found. Since the internal information may be dependent on external conditions it is

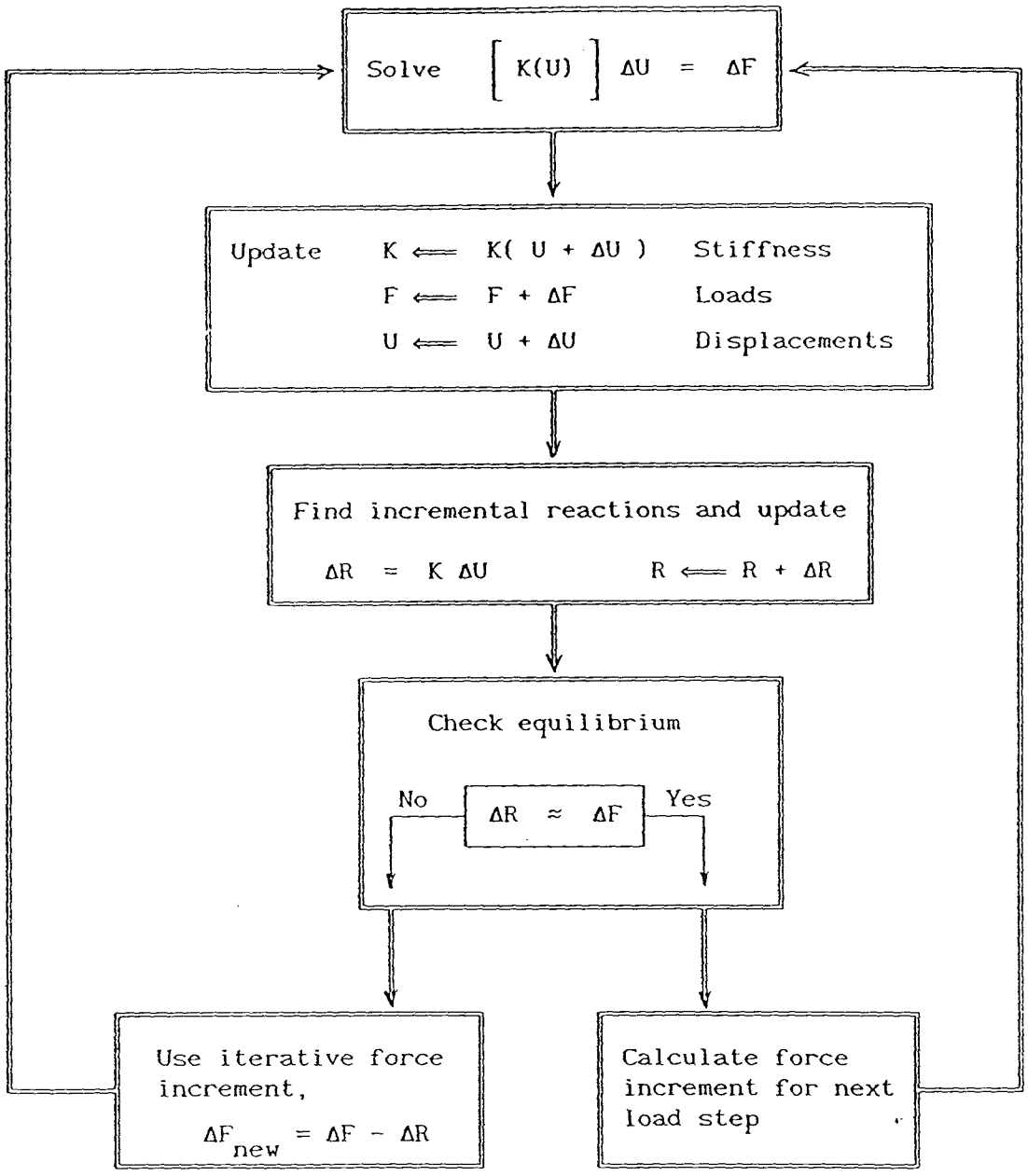


Figure 4.1 Typical incremental nonlinear algorithm

necessary to check whether the internal information has changed. If it is found that it is necessary to update the internal information, that is done. This implies that the external information may no longer be correct, or at least not sufficiently close. Typically, the external information is checked by a balance where the current estimate of one type of external information (usually displacement) is used together with the internal information to generate an estimate of the other type of external information. If it is found that this second estimate is close enough to the values already available, then it is assumed that no correction is necessary and the solution returns to the beginning of the algorithm to start applying another increment of external information. If it is decided that the generated estimate is not close enough to the actual increment, then a corrective iteration is begun. This can take many many forms. A simple and popular method is just to substitute the difference between the generated estimate and the actual increment for the increment of external information and solve again in a recursive fashion. At each recursion the internal information is checked and possibly updated and the external information checked for convergence. It is assumed that such a process will eventually converge. After convergence is achieved the algorithm is started again with a new increment of external information.

For problems of solid mechanics and structures this translates to applying an increment of imposed forces and displacements. Solve for the unknowns. Build a new stiffness matrix to satisfy the new displacement conditions. Multiply the stiffness by the displacements to get the reactions. Check the reactions against the forces. If the

difference is small, then go on to the next increment. If not, then the difference between forces and reactions is applied as a force correction and zeroes are applied as corrections to the imposed displacements. For mechanically stable structures this process will always converge.

The key point is the test for equilibrium. This is done after the imposition of current conditions on all known quantities, usually stiffness, displacement and forces.

Figure 4.2 shows a prototype in pseudo code for a contact algorithm. The key difference between Figure 4.2 and Figure 4.1 is in the nature of the test which triggers another iteration. In the contact algorithm the test is for two possible triggers: compatibility and equilibrium. Different algorithms distinguish themselves in how they detect and generate corrections.

4.2 ONE DIMENSIONAL ANALOGY

In a one dimensional setting the measure of a penetration error is simple; it is the length of overlap. Forces are to be applied to both bodies to correct this situation. So, the compatibility condition on corrective displacements is

$$-\Delta U^{A} + \Delta U^{B} = P \qquad (4.2.1)$$

where P is the penetration error and ΔU is a corrective displacement.

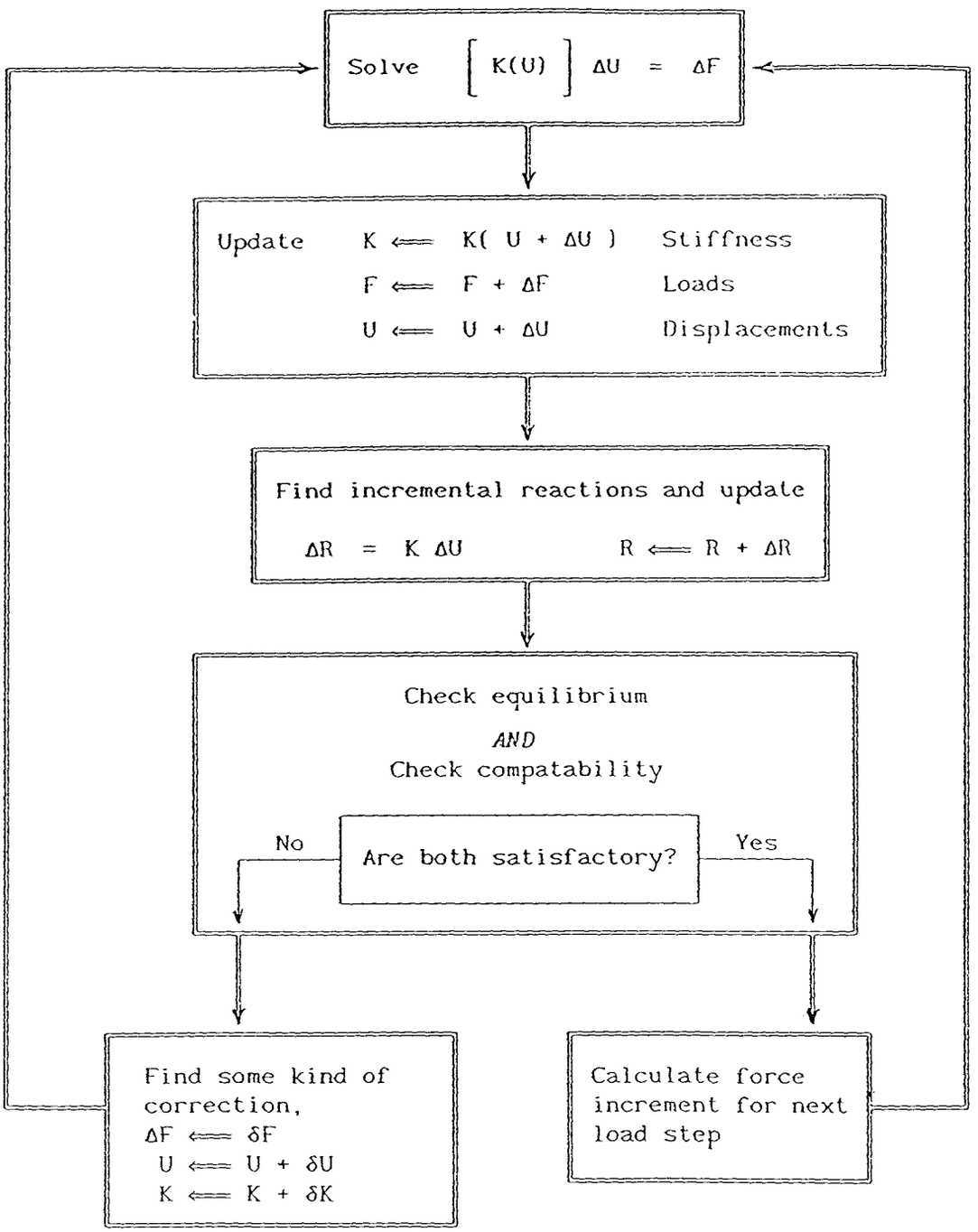


Figure 4.2 Typical incremental contact algorithm

If the displacements are small, then to first order the change in the end reactions must be

$$\begin{aligned} \Delta F^{A} &= K^{A}\Delta U^{A} \\ \Delta F^{B} &= K^{B}\Delta U^{B} \end{aligned} \qquad (4.2.2)$$

where ΔF is a change in end reaction and K is a stiffness. These displacements are to correct compatibility without disturbing equilibrium. So, the condition is put on the reactions that

$$\Delta F^{A} + \Delta F^{B} = 0 \qquad (4.2.3)$$

These conditions can be combined to give the required increments of force to correct a compatibility error without disturbing equilibrium.

$$\begin{aligned} \Delta U^{A} &= -K^{B}P \,/\, (K^{A} + K^{B}) \\ \Delta U^{B} &= K^{A}P \,/\, (K^{A} + K^{B}) \\ \Delta F^{A} &= -K^{A}K^{B}P \,/\, (K^{A} + K^{B}) \\ \Delta F^{B} &= K^{A}K^{B}P \,/\, (K^{A} + K^{B}) \end{aligned} \qquad (4.2.4)$$

An error in equilibrium is simply defined as well. It is just the difference between the net force on the end nodes of each body. The force increments to be applied to correct an equilibrium error are required not to upset compatibility. This requires that

$$\Delta U^{A} = \Delta U^{B} \qquad (4.2.5)$$

The effect of the condition in eqn(4.2.5) is that the end reactions are tied as well through eqn(4.2.2). These force increments are to correct an equilibrium error. So we require of them that

$$\Delta F^{A} + \Delta F^{B} = -E_{F} \tag{4.2.6}$$

where E_F is the equilibrium violation. These conditions may be combined to give

$$\begin{aligned} \Delta F^{A} &= -K^{A}E_{F} / (K^{A} + K^{B}) \\ \Delta F^{B} &= -K^{B}E_{F} / (K^{A} + K^{B}) \end{aligned} \tag{4.2.7}$$

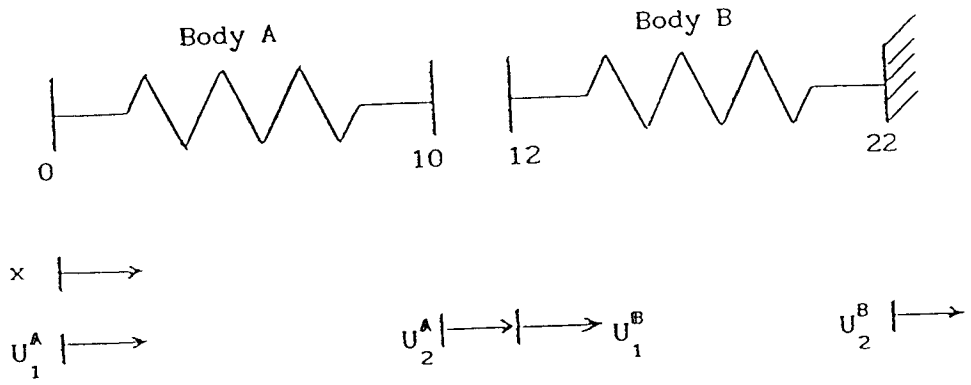
These are applied at the global level along with whatever other force correction terms may be used for the sake of other nonlinearities.

4.2.1 EXAMPLE OF ONE DIMENSIONAL ANALOGY

An example is included here to illustrate the workings in one dimension, where it is easy to see how each term behaves. There are two elements as shown in Figure 4.3. The element on the left, body A, is a nonlinear spring whose stiffness is

$$K^{A} = 1000(1 + 10c\ )\ \text{lb/in} \tag{4.3.1}$$

where c is the fractional compression of the spring. It is initially



$K^A = 1000(1 + 100c) \text{ lb/in}$

$U_1^A = 3.0 \text{ in}$ Prescribed displacement

$K^B = 2000(1 + 200c^2) \text{ lb/in}$

$U_2^B = 0.0 \text{ in}$ Prescribed displacement

Figure 4.3 One dimensional example using nonlinear springs

positioned with its left end at the origin. It is 10 inches long. The second body is a different nonlinear spring, with stiffness

$$K^B = 2000(1 + 20c^2) \text{ lb/in} \qquad (4.3.2)$$

It is initially positioned with its right end at 22 inches to the right. This leaves a two inch gap between springs. The external conditions to be imposed are that the right end of spring B is encastred and the left end of spring A is to move three inches to the right. This will cause the springs to come into contact after spring A has moved two inches. Contact corrections will be necessary for the last inch of motion. Let the motion be applied as half inch increments of the left end of spring A. Calculations are carried out so that forces are found to the nearest pound, stiffnesses to the nearest lb/in and distances to the nearest 0.001 in.

The first three steps bring the right end of spring A to positions 10.5 in, 11.0 in and 11.5 in and no contact is made. The fourth step brings the left spring to just touch the right spring. Since the springs are just touching there is no compatibility error. Since neither spring is compressed the end reaction of each is 0 lb. So, there is no equilibrium error.

In the fifth step the right end of spring A is brought to 12.5 in. There is still no compression of either spring so the equilibrium test does not trigger contact iterations. There is a penetration error of 0.5 in. So, the compatibility condition starts the corrections. The detection of 0.5 in overlap caused a 333 lb force to

be applied to the left on spring A and to the right on spring B. These forces are equal and opposite so as not to upset the equilibrium that is already present. As a result spring A is compressed by 0.333 in and spring B is compressed by 0.167 in. This establishes compatibility, but a check of the end reactions of the springs shows that equilibrium has been lost in the new configuration. This is because the springs are nonlinear and have stiffened when compressed. The end reactions are higher than anticipated by the simple, first order approximation used to attain compatibility. This demonstrates that, in the presence of nonlinearity, there is no guarantee that compatibility or equilibrium will actually be achieved or maintained in any one iteration. These must be checked explicitly. The force that must appear on spring A to maintain this position is 444 lb, while that provided by spring B is only 335 lb. So there is a 109 lb right equilibrium error. This triggers another iteration.

This time only equilibrium corrections are nonzero. The corrective force that must be applied to spring A is 43 lb right and the force on spring B is 66 lb right. In their new positions compatibility is maintained as was the intent in the method of finding force corrections for equilibrium. Equilibrium is still not quite satisfied. However, the error has been reduced to only 8 lb left. This causes the application of 3 lb left to spring A and 5 lb left to spring B. After these forces are applied compatibility is still satisfied and the equilibrium error is only 2 lb. This is close to the precision being used, so we say that the method has converged. The position where the springs are calculated to meet after 2.5 in of motion is at 12.297 in

and the forces on the ends of the springs are 395 lb left and 397 lb right.

The final step has an iteration history similar to the step just completed. It introduces a new 0.5 in compatibility error. The process starts again. A compatibility correction appears in the first iteration, but leaves an equilibrium error of 165 lb right. The second iteration leaves compatibility satisfied but there is still a force imbalance of 33 lb left. The third iteration leaves an equilibrium error of 7 lb right. Finally only a 1 lb left equilibrium error remains after the fourth iteration. Since this is within roundoff errors the process is assumed to have converged.

After the entire 3 inches of motion have been imposed on spring A the two springs are pressed together at 12.431 in with a contact force of 893 ± 1 lb. Spring A is at a stiffness of 1569 lb/in being compressed by 0.569 in. Spring B is at a stiffness of 2074 lb/in being compressed 0.431 in.

In this example, compatibility was established in the first iteration in each case. In the presence of further nonlinearities this may not have happened, nor would compatibility necessarily have been maintained. It is possible to excite both compatibility and equilibrium corrections in a single iteration.

4.3 COMPLICATIONS OF TWO OR THREE DIMENSIONAL CONTACT

There are complications that arise in contact in two or three dimensions. They include:

1) There is more than one direction for compatibility and equilibrium considerations
2) There may be more than one node of each body in each contact area
3) Stiffness is now a matrix quantity rather than a scalar
4) The nodes in contact may not be touching nodes in the other body

Each of these requires some elaboration.

4.3.1 MORE THAN ONE DIRECTION

In two dimensional problems one has to consider equilibrium in directions normal and tangent to the contact surface. In three dimensional contact there are two tangent directions. The chief difficulty is in deciding which is the normal direction to the surface. This is affected by the assumptions which are made on where the true contact surface lies. In this work the difficulty is handled by the simple assumption that the normal to the exterior of the elements in the current configuration is an adequate approximation to the normal to the contact surface. As the elements are deformed during iterations for

contact or any other cause, the estimate of the normal direction is updated. As the iterations converge the element normals improve as estimates of the normal to the contact surface.

It is also necessary to use a depth of penetration. This is simple in one dimension but must be defined in some way for spaces that have more than one dimension. This choice of definition alone could be the subject of much argument. Here a simple choice is made. The depth of penetration is measured in the same direction as the normal forces in the equilibrium check.

4.3.2 MORE THAN ONE NODE IN CONTACT

In the one dimensional case there was no doubt as to which nodes were involved in contact. Only the nodes at the end of a body were ever considered. In two or three dimensions the exterior of a body is no longer just one node at either end. Each node on the exterior is a possible contact node. Of necessity each one must be checked for contact. A node is considered to participate in contact if

1) it is an exterior node AND

2) is embedded in any other body OR

3) has compressive contact forces acting on it.

For purposes of computation there are many approaches to determining which nodes are in contact and which are not. An important feature of calculation that is implicit in the one dimensional case becomes

explicit in the two and three dimensional cases. Any node that is not in contact (all interior nodes and some exterior nodes) may be eliminated statically from the calculation of contact forces. This greatly reduces the number of degrees of freedom involved and so reduces the computational difficulty. The important conceptual step gained is that the entire body has been represented by only the contact surfaces. All deformations due to contact will exhibit the correct amount of strain energy and work.

4.3.3 STIFFNESS IS A MATRIX

In the one dimensional case the stiffness is a scalar. This is coincidental. One may correctly claim that the stiffness is a 1x1 matrix, rather that something that is inherently a scalar. In two and three dimensional problems the stiffness that remains after eliminating the non-contact nodes is a matrix. The number of degrees of freedom is the number of degrees of freedom per node times the number of participating nodes. The stiffness is treated in two stages:

1) Degrees of freedom not associated with contact nodes are statically eliminated.
2) The remaining stiffness matrix is reduced statically to each contact degree of freedom. In this stage of reduction all remaining degrees of freedom are considered as free variables. Any displacement prescriptions are ignored. The

remaining stiffness is a measure of the strain energy required throughout the body for a motion to occur in that dof. The actual calculation of the individual reduced stiffnesses is done recursively to minimize the computational effort.

4.3.4 NODES DO NOT ALIGN

This is an especial problem that many workers have tried to handle by many different techniques. In realistic problems the meshes of opposing bodies will not necessarily match over the region of contact. So each node does not always find an opposing node, the forces and stiffnesses of which can be used to estimate the corrective forces for compatibility and equilibrium. Different quantities are mapped across the contact surface different ways. Stiffness is among those found by interpolation between opposing nodal values. Forces are among the quantities that are redistributed and reintegrated over the mesh of node's own body. This is elucidated in the following sections.

4.4 REDISTRIBUTION OF NODAL QUANTITIES

Some nodal quantities are found as the integrals of distributed quantities. For example, nodal forces can be found as the integral of stress over the surface of the element. In more detail, there are several steps in calculating a nodal force from surface traction. First the traction is weighted by the shape functions. Then the product is integrated over the element to give elemental contributions to the nodal forces. Finally, in the assembly of elements, the individual elemental contributions are accumulated to give the nodal force.

This process may be illustrated here for an unspecified distributed quantity $\mathfrak{q}$. $\mathfrak{q}$ is shown as a scalar. The argument is more long winded if $\mathfrak{q}$ is of higher tensor order, but is essentially the same. It is necessary in that case to expand the calculation to treat each component of $\mathfrak{q}$ separately. First form q as

$$q = \phi\, \mathfrak{q} \tag{4.4.1}$$

where ϕ is a column of shape functions

$\mathfrak{q}$ is the unspecified scalar field

q is a column of the same length as ϕ

We define a global equivalent quantity, Q, concentrated at the nodes. The contribution from an element is calculated as

$$Q_{el} = \int_{\Omega_{el}} q \, dV = \int_{\Omega_{el}} \phi\, \mathfrak{q} \, dV \tag{4.4.2}$$

where Q_{el} is the elemental contribution to the concentrated nodal quantity and

Ω_{el} is the elemental domain

Finally the global concentrated nodal quantity, Q, corresponding to the distributed quantity, q, is found by assembly. That is

$$Q = \sum Q_{el} \tag{4.4.3}$$

where $\sum$ is the assembly operator.

There is a certain amount of information lost in this process. The exact details of the distribution of q are not important to the result, Q. Any other distribution, say q', which has the same integral after weighting by the shape functions will produce the same result. In particular, one may imagine replacing q by some distribution, q', of form

$$q \cong q' = \phi^T q' \tag{4.4.4}$$

where q' is an approximation for q and

q' is a column of pseudo-nodal values

Let Q' be the concentrated nodal value found by using q' instead of q. Q' is then an approximation for Q. If the values of q' are chosen correctly then it may be possible to match Q' to Q exactly.

For the purposes of eqn(4.4.2) it would be allowable for each element to maintain its own vector, q'_{el}, of pseudo nodal values. Many finite element codes use this form for the input for distributed quantities such as pressures and thicknesses. But, to ease our

calculation, let us make the assumption that every element that involves any given node uses the same value for q' at that node. This also gives that the approximation, q', is continuous even though there was no requirement that the original function, q, is continuous. We can now express Q'_{el} as

$$Q'_{el} = \int_{\Omega_{el}} \phi \, \phi^T \, q'_{el} \, dV \qquad (4.4.5)$$

Where q'_{el} are those entries of q' for this element. Since the q' are constants, rather than functions of position they may be taken outside the integral

$$Q'_{el} = \int_{\Omega_{el}} \phi \, \phi^T \, dV \, q'_{el} \qquad (4.4.6)$$

We may define the elemental distribution matrix, N_{el}, for each element as the integral in eqn(4.4.6).

$$N_{el} = \int_{\Omega_{el}} \phi \, \phi^T \, dV \qquad (4.4.7)$$

So eqn(4.4.6) may be written as

$$Q'_{el} = N_{el} \, q'_{el} \qquad (4.4.8)$$

The assembly operation may be applied to the Q'_{el} to give Q'

$$Q' = \sum Q'_{el} = \sum \left(N_{el}\, q'_{el} \right) \qquad (4.4.9)$$

The assembly of the q'_{el} is just the global q'. This leads to a definition of a global distribution matrix, N, as the assembly of elemental distribution matrices, N_{el}. So

$$N = \sum N_{el} \qquad (4.4.10)$$

This allows the global approximation to the concentrated nodal quantity to be expressed concisely as

$$Q' = N\, q' \qquad (4.4.11)$$

The approximation q' can be said to adequately represent the original function q if both give the same global concentrated nodal quantity. Since the only degrees of freedom in q' are the q', all adequacy conditions rest on them. The condition put above can be fulfilled by choosing q for the q', where q is the solution of

$$N\, q = Q \qquad (4.4.12)$$

This requires that the global distribution matrix, N, not be singular. For many situations this will be true. It may be noted that N_{el} has the same form as the consistent mass matrix for many classes of elements. For any such class that has a nonsingular consistent mass matrix the global distribution matrix is also nonsingular.

This manner of approximating q not only loses the local detail of the distribution, but also may lose larger features. For example, suppose that some region (element or set of elements) had q of exactly zero over it. q' would not necessarily also show this but would tend to average the value among those elements and their neighbours. In the interior of such a region the value of q' may be quite close to zero or even exactly zero but near the periphery the results would change smoothly to the surrounding nonzero conditions. Care can be taken to allow for different results. In the above example it would be possible to suppress q' over an element by imposing values of zero for q' for all nodes in the unaffected elements before solving eqn(4.4.12). This would have the effect of forcing the distribution to zero at the edges of adjacent elements as well. An alternative possibility would be to introduce a deliberate discontinuity. This is done by omitting the unaffected elements from the assembly stage to give a modified global distribution matrix N*. Substitute N* into eqn(4.4.12) for N. The pseudo nodal values q' so obtained are not restricted to be zero around the edges of the unaffected elements. It must be remembered in such an approximation that a separate approximation of zero has been made for the omitted region.

In the example above any function that is known a priori can be used. The part of the distribution that is already known need not be zero. The point to be made is that in some circumstances care may be needed to achieve desired results or else a greater than minimal magnitude of approximation may occur.

4.5 VIRTUAL ELEMENTS

The equations for the corrective force increments described in the one dimensional explanation assume that each node in contact is touching a node of the opposing body. This is achieved in a trivial fashion for one dimensional problems. This was achieved in early two dimensional finite element codes by restricting the allowed mesh of nodes and elements so that each node in one body was paired with a node in the touching body. Only very small misalignments of these nodes could be tolerated.

In the present scheme this difficulty is removed by the definition of a mesh of virtual nodes and virtual elements that exactly oppose the actual mesh of either body. Thus each contact node, or element, of body A will find a virtual node, or element, of body B touching it. Likewise the mesh of body B will find the virtual mesh of body A touching it. These virtual nodes provide the comparison for the definition of p, the penetration error, E_F, the equilibrium error, and K, the opposing nodal stiffness. The use of the properties of these virtual nodes and elements gives the analyst freedom in the design of meshes and lifts the restriction that actual nodes must stay close together in pairs. The properties of a virtual node can be calculated at need, including its location.

The needed properties are found in a simple way. For sake of discussion, assume that a node of body A has penetrated into the bulk of body B. For a node embedded in another body the virtual node is located on the surface of the opposing body at a point where an inward normal

would point to the embedded node. Penetration depth is measured as the separation of the real (embedded) node of A and the virtual (surface) node of B as shown in Figure 4.4.

4.5.1 EQUILIBRIUM CORRECTIONS

For this we must find the stiffness and forces of each pair of a node and its opposing virtual node. The forces and stiffness of the actual node are available directly.

To find the stiffness of the virtual B node, first find the coordinates of the virtual B node in the actual B element. Then interpolate between the nodal stiffnesses of the actual B nodes. Since the nodal stiffnesses are available as Castigliano style total stiffness, they represent the resistance of a body to motion in the degrees of freedom. Interpolation serves well to find an estimate of the resistance to motion at a location (virtual node of body B) between these known points (actual nodes in body B).

To find the forces acting on the virtual B node, first the actual forces on the actual B nodes are redistributed. This gives an approximate description of the contact tractions acting on body B. The virtual B element is chosen to occupy the same region and uses the same shape functions as the actual A element, but it is on the surface of body B. The B tractions are integrated over the virtual B element, weighted by the shape functions in the ordinary manner of finding nodal

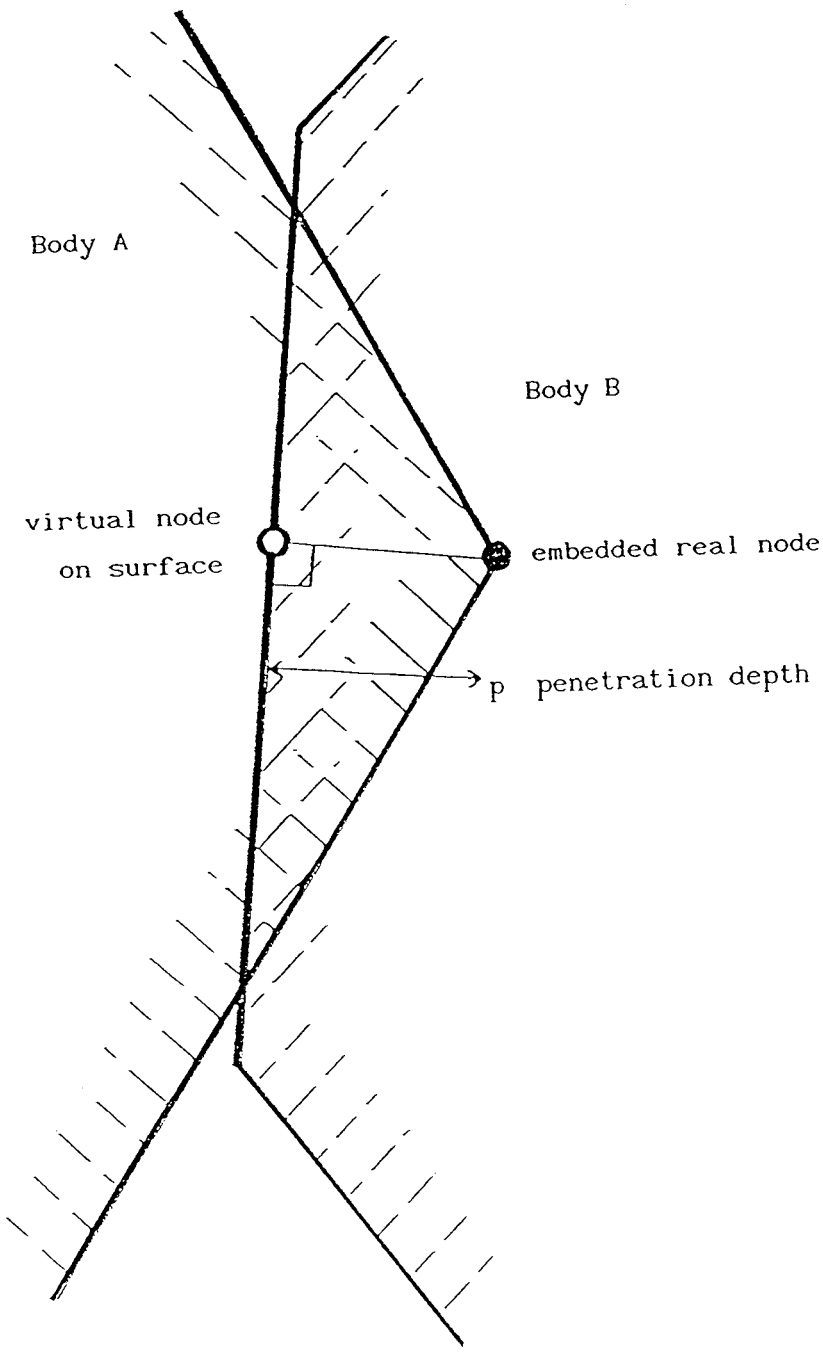


Figure 4.4 Determination of penetration depth

forces from tractions. Each virtual elemental force vector is accumulated so that the complete virtual nodal forces are available when all of the virtual elements have been assembled.

The forces and stiffnesses are re-expressed in a coordinate set with the first coordinate in the normal direction. Equilibrium checks are made in the normal and tangential directions. If a node is found to be in tension in the normal direction it is released. Otherwise corrections are applied to distribute the force imbalance as in eqn(4.2.7).

4.5.2 COMPATIBILITY CORRECTIONS

This is accomplished for each body in two stages. In the first stage the position of the contact surface is estimated and corrective displacements proposed. The penetration depth of an node is the distance between the real node and the virtual node. In the case of a node that has penetrated, there is a region where overlapping volumes must be resolved. In the case of a node that is outside of the opposing body, but has not been released (still has compressive contact forces acting on it), this reveals that the current estimate of contact forces is too high. These forces must be reduced to allow the node to approach the contact surface. In either case, the direction of approach to the contact surface is always along the normal. The distance to approach is given in eqn(4.2.4). The forces given in eqn(4.2.4) could be used, but

these ignore interactions between nodes. It is more effective to use the displacements and convert them to forces by multiplication by the stiffness matrix. That way all the interactions between nodes are preserved in the force increments that correct compatibility errors. The second stage accomplishes this multiplication by using the linear solver routines. The required displacements are given as prescribed displacements. The reactions to these prescribed displacements are the forces needed. Since the stiffness matrix is already factored and available from the linear prediction stage, there is very little cost for this way of converting the compatibility correcting displacements to forces.

CHAPTER FIVE

EXAMPLES

5.1 INTRODUCTION

In this chapter four examples are shown to illustrate the success of the contact algorithm. The first example is a Hertz problem. This is to demonstrate that the results are realistic and accurate. The calculated results are compared to the theoretical solution. The second example is a punch problem. A shallow V shaped wedge is pushed onto a flat plate. The elastic properties of the wedge and the plate are varied to explore the effects of the relative stiffness of the bodies in contact. The third example is two rectangular blocks of different sizes. This shows the behaviour in the presence of sharp corners. The results are robust even in the presence of such challenging geometry. The last example is a weak gasket pushed against a stiff gasket and then retracted. The displacement history shows that the method is reversible.

5.2 HERTZ EXAMPLE

This is a plane strain example. An infinitely long right circular cylinder of isotropic linearly elastic material is pushed against a rigid flat frictionless surface. The cylinder has unit radius; the elastic material has a modulus of 10^6 and a Poisson's ratio of 0.2. The "rigid" surface is modelled as a line of elements, completely fixed on the lower surface, with an elastic modulus of $3x10^9$. That is, the material is $3x10^3$ times as stiff as that of the cylinder. Together with the support conditions, the lower body is effectively rigid. The mesh is shown in the original configuration in Figure 5.1. Since the problem is symmetric about the central plane, only half of it is modelled. At the end of motion a force of 1530 has been applied to the half model. The final configuration of elements near the contact region is shown in Figure 5.2. The normal contact stress is plotted against distance from the central plane in Figure 5.3. The solid quarter circle shows the theoretical results. The horizontal lines show the range of the calculated results. The vertical scale is normal contact stress divided by the maximum normal contact stress. The uncertainties in the data on this scale are negligible. The horizontal axis is horizontal position divided by the radius of contact. As the calculated results can only say that the radius of contact is somewhere in element 207 a range is shown. The left end of each datum (shown with a hollow square) uses the extreme outer edge of element 207 (node 208) as the upper limit of the radius of contact. The right end of each

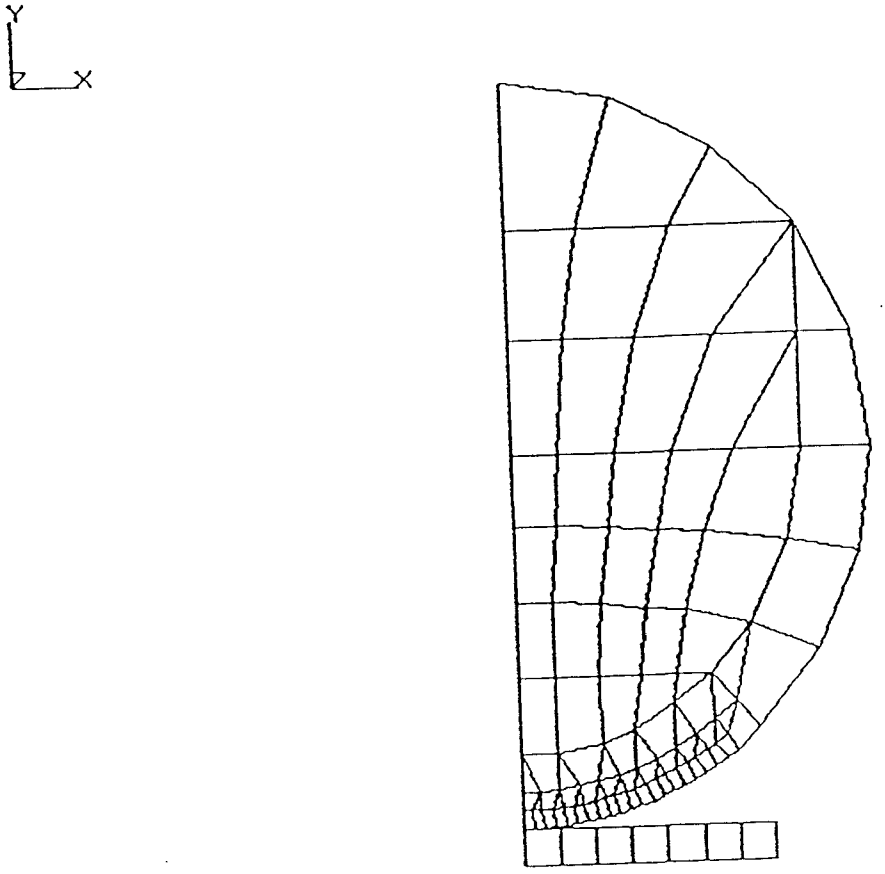


Figure 5.1 Element mesh of Hertz example in the original configuration

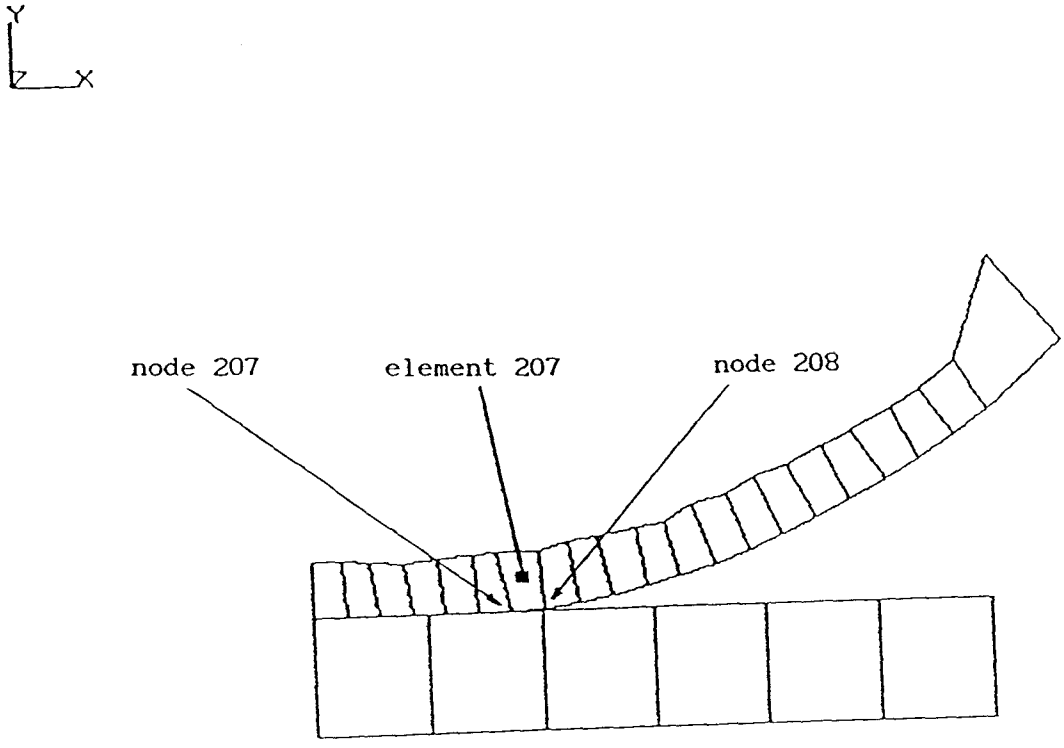


Figure 5.2 Hertz example elements in the contact region in the final configuration

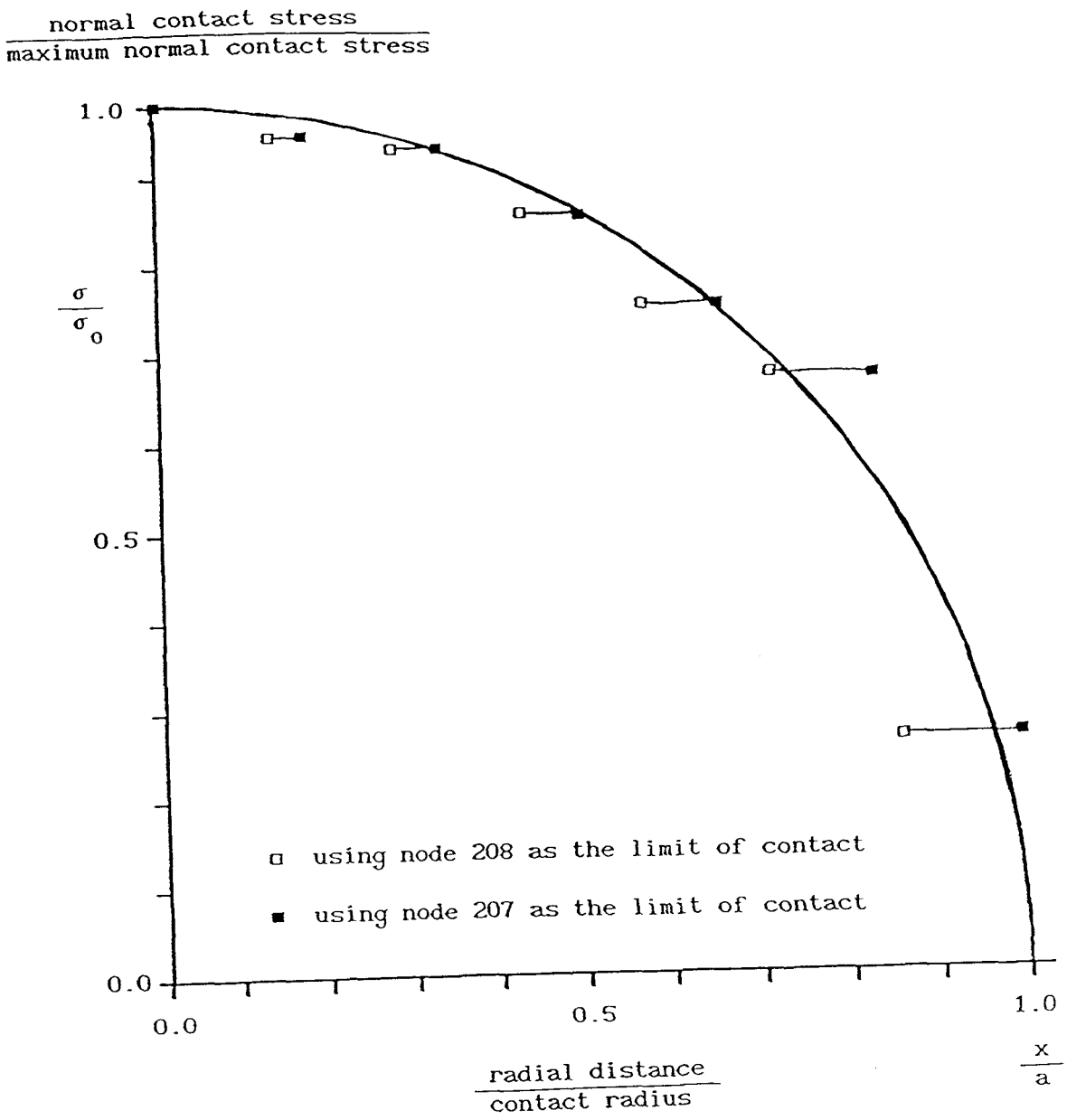


Figure 5.3 Hertz contact stresses

datum (shown with a filled square) uses the extreme inner edge of element 207 (node 207) as the lower limit of the radius of contact.

The data follow the theoretical results quite well. This shows that the method is adequate and accurate for classical problems. This suggests that it should perform well in analyses of practical interest. To decide whether the method is indeed generally useful some further investigation is needed. It is necessary to probe some of the possible conditions that may place demands on the performance of a useful contact algorithm. Each of the other three example look at some particular aspect.

5.3 WEDGE EXAMPLE

In this example a wide shallow wedge is driven slightly into a flat plate. The flat plate is of unit thickness and is rigidly fixed on its lower surface. The wedge varies from unit thickness at the outer edges to a thickness of 1.3 at the center. Initially the point of the wedge is just touching the plate. The central portion of the wedge has an imposed downward motion of 0.3. The problem is symmetric with respect to the central plane, so only the right half is modelled. In five runs the geometry and fixed displacements stay the same while the elastic modulus of the materials is varied. This will show whether there is any practical limit on the relative stiffness of the two materials. The five cases examined are:

1) The wedge is 1000 times stiffer than the plate

2) The wedge is twice as stiff as the plate

3) The wedge and the plate are of equal stiffness

4) The plate is twice as stiff as the wedge

5) The plate is 1000 times as stiff as the wedge

The initial configuration and mesh is shown in Figure 5.4. The linear prediction results in the wedge moving downwards 0.3. This results in overlap as shown in Figure 5.5.

For the case where the wedge material is one thousand times stiffer than the plate material, the final configuration is shown in Figures 5.6, 5.11 and 5.12. This took three iterations. As expected, all of the deflection is in the plate.

For the case where the plate material is one thousand time stiffer than the wedge material, exactly the opposite occurs. The final configuration, shown in Figures 5.7, 5.11 and 5.13 was reached after only two iterations. This time all of the deflection is in the wedge.

For the case where the wedge and the plate are made of identical materials, the final configuration is shown in Figures 5.8 and 5.11 to 5.13. The contact surface is nearly at the center of the overlap produced by the linear prediction. It is actually a little above the center between the two extreme cases. It is expected that the position of the contact surface will be slightly above the middle of the linearly predicted overlap because the plate, being rigidly supported from beneath and thinner than the wedge, is structurally stiffer than the wedge. This case took four iterations to converge.

For the case where material of the wedge is twice as stiff as

that of the plate, the final configuration is shown in Figures 5.9 and 5.12. The contact surface lies between that for the case where the two materials are of equal stiffness and the case where the upper material is much stiffer than the lower material. This took six iterations to converge.

For the case where the plate material is twice the stiffness of the wedge material, the final configuration is shown in Figure 5.10 and 5.13. This shows the final contact surface located between that where the materials are equally stiff and where the plate is rigid by comparison to the wedge. This took four iterations to converge.

For the sake of visual comparison, Figures 5.11 - 5.13 show more than one final position in individual drawings. Figure 5.11 shows the cases 1, 3 and 5. This allows comparison of the extreme cases and the equal stiffness case. Figure 5.12 shows cases 1, 2 and 3. This compares cases where the wedge is as stiff or stiffer than the plate. Figure 5.13 shows cases 3, 4 and 5. These are the cases where the plate is at least as stiff as the wedge.

The cases shown in this example show sensible results. From them it may be inferred that the algorithm behaves well in the presence of a wide range of stiffness ratios.

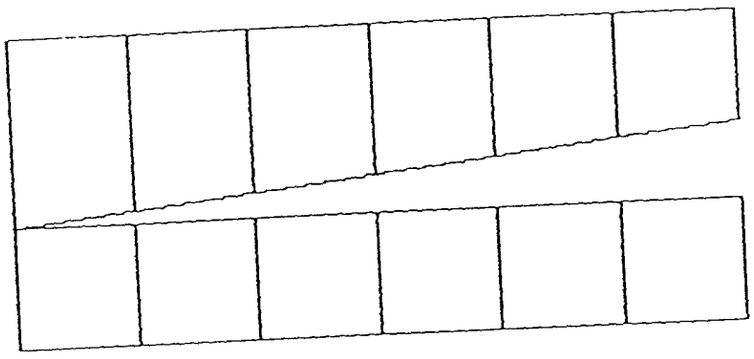


Figure 5.4 Wedge example: Element mesh in the original configuration

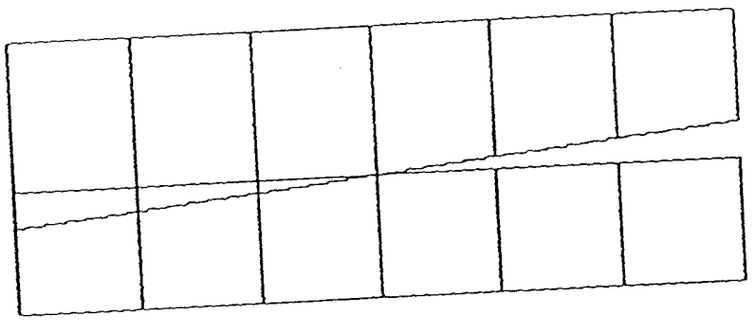


Figure 5.5 Wedge example: Element mesh after the linear prediction

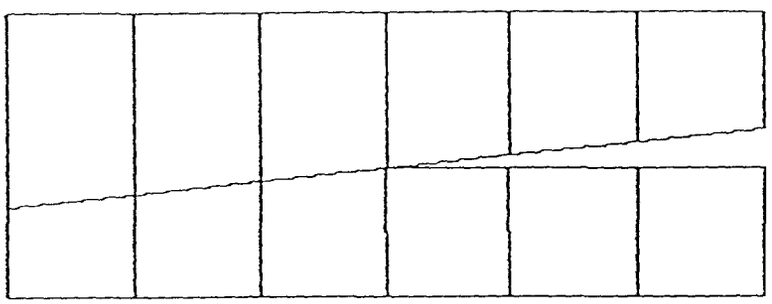


Figure 5.6 Wedge example: Final configuration for case 1; the wedge material is 1000 times as stiff as the plate material

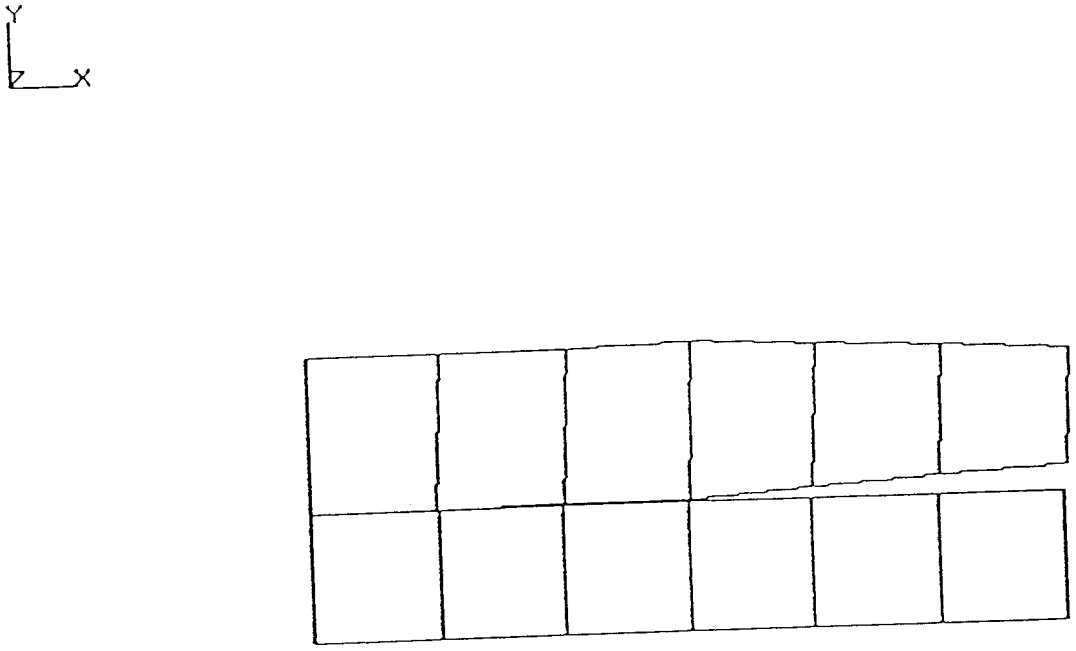


Figure 5.7 Wedge example: Final configuration for case 5; the plate material is 1000 times as stiff as the wedge material

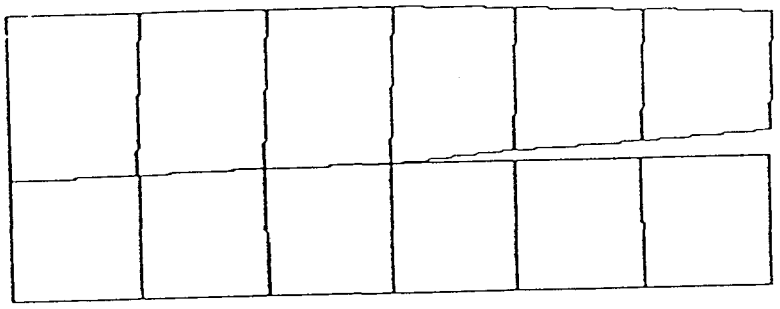


Figure 5.8 Wedge example: Final configuration for case 3; the wedge and the plate are made of the same material

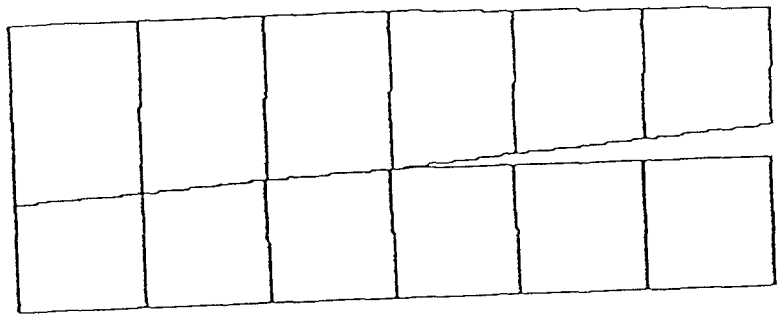


Figure 5.9 Wedge example: Final configuration for case 2; the wedge material is twice as stiff as the plate material

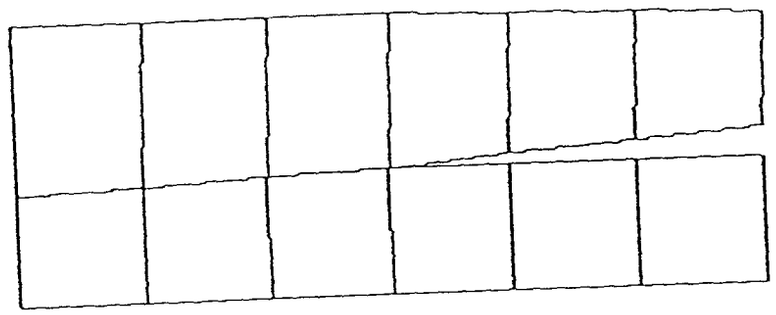
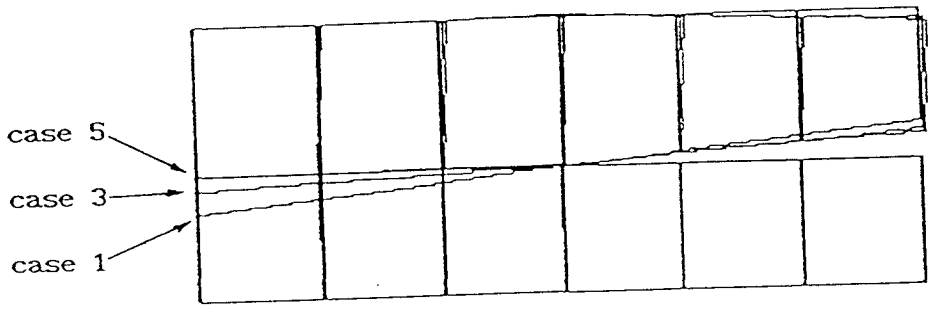


Figure 5.10 Wedge example: Final configuration for case 4; the plate material is twice as stiff as the wedge material

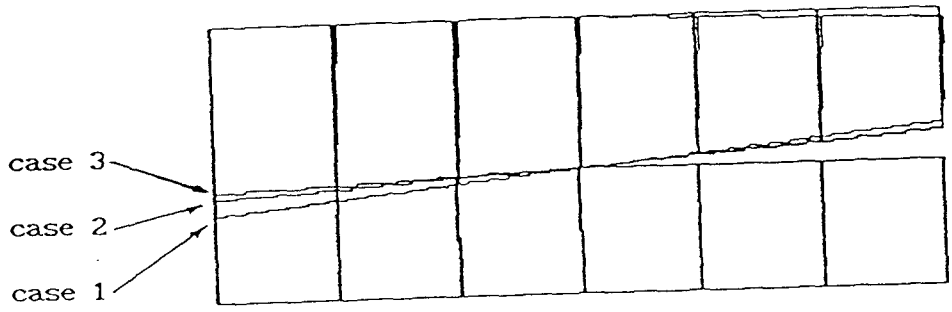


case 1: the wedge material is 1000 times as stiff as the plate material

case 3: the wedge and the plate are made of the same material

case 5: the plate material is 1000 times as stiff as the wedge material

Figure 5.11 Wedge example: Final configurations for cases 1, 3 and 5

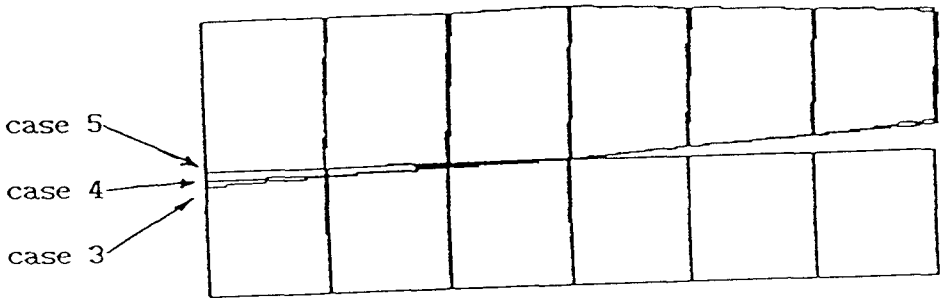


case 1: the wedge material is 1000 times as stiff as the plate material

case 2: the wedge material is twice as stiff as the plate material

case 3: the wedge and the plate are made of the same material.

Figure 5.12 Wedge example: Final configurations for cases 1, 2 and 3



case 3: the wedge and the plate are made of the same material

case 4: the plate material is twice as stiff as the wedge material

case 5: the plate material is 1000 times as stiff as the wedge material

Figure 5.13 Wedge example: Final configurations for cases 3, 4 and 5

5.4 BLOCK EXAMPLE

In the third example a larger block is forced against a smaller, fixed block. The materials in the blocks are identical. The challenge to a contact calculation that this example presents is the sharp corners of the smaller block. How well are the contact conditions met near such an irregular boundary? The initial configuration is shown in Figure 5.14, the linear prediction in Figure 5.15 and the final in Figure 5.16. The model used is very crude. Even so it can be seen that reasonable results are produced in the presence of sudden angular changes in the boundary.

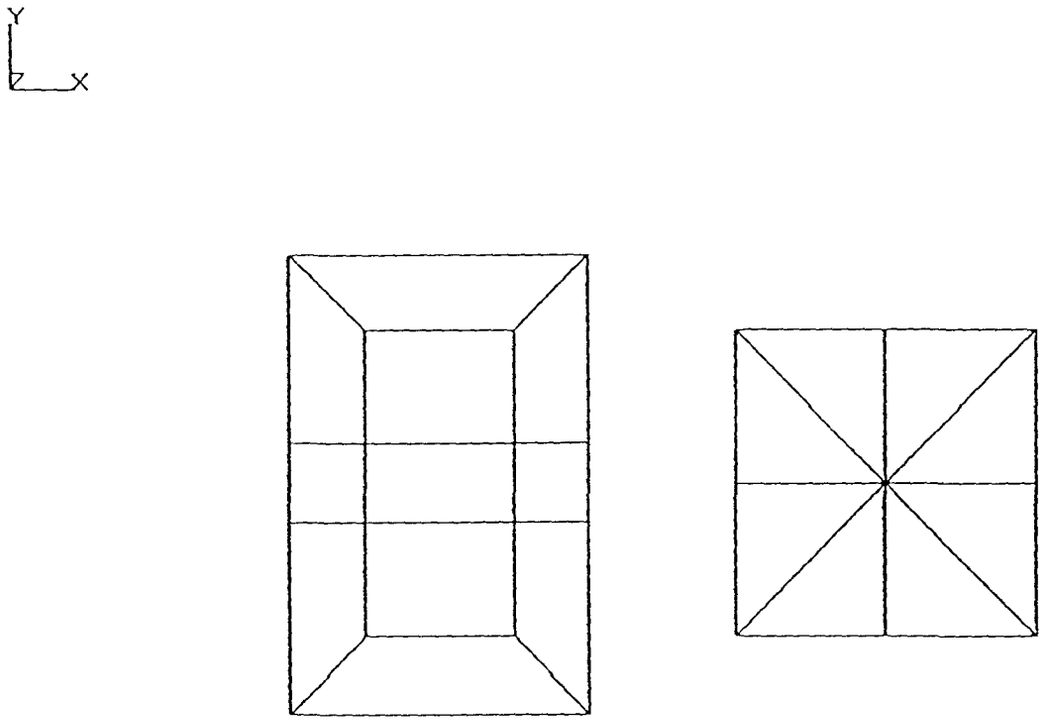


Figure 5.14 Block example: Initial configuration

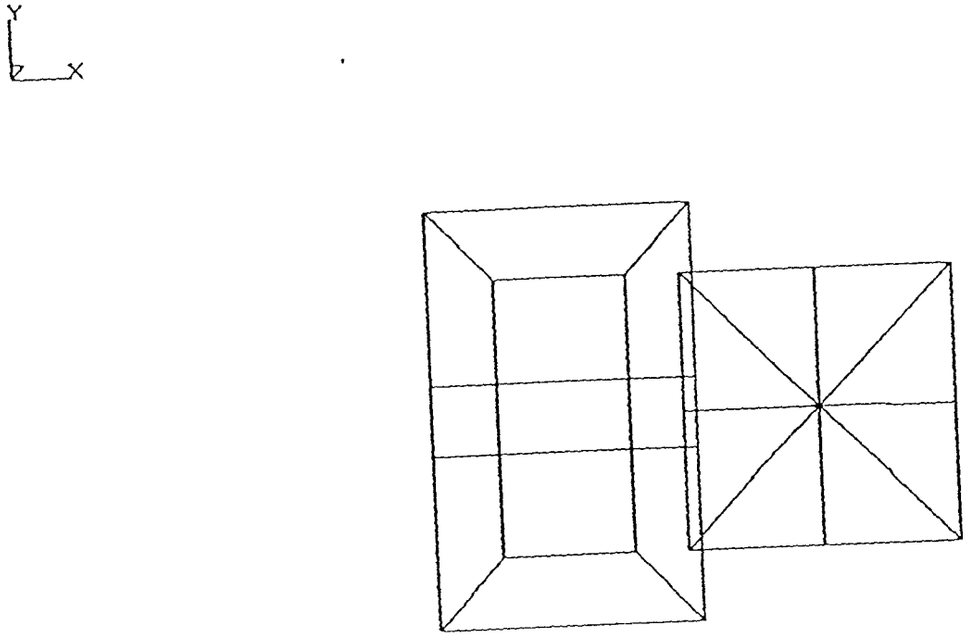


Figure 5.15 Block example: Linear prediction

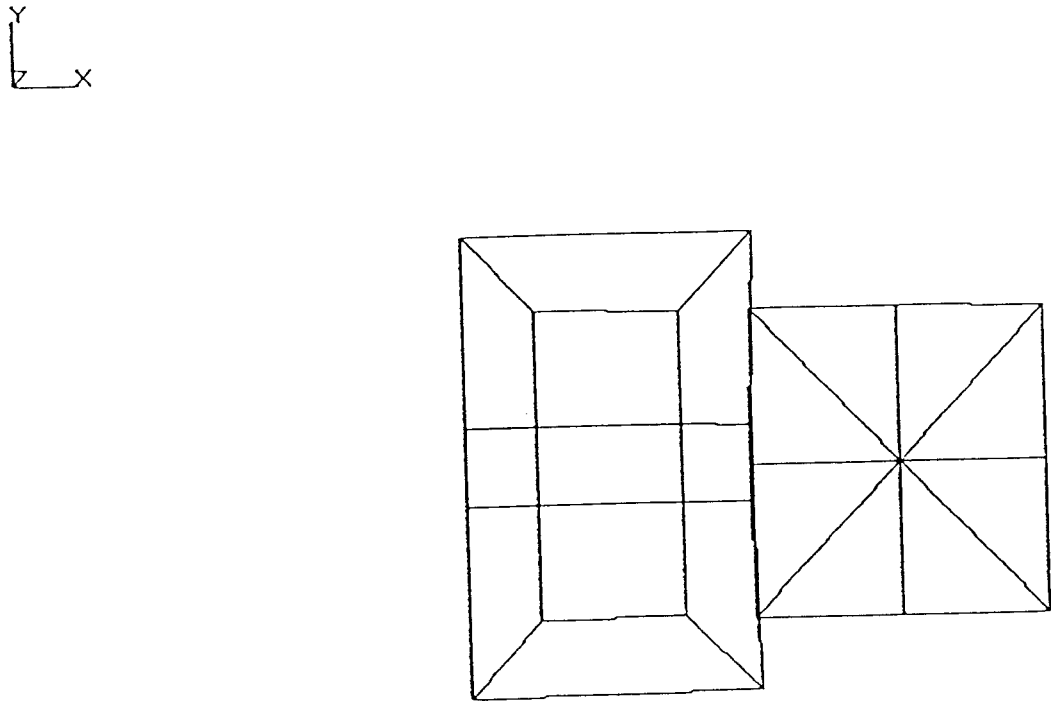


Figure 5.16 Block example: Final configuration

5.5 GASKET EXAMPLE

This example demonstrates that the method behaves reversibly in the absence of other nonlinearities. The bodies in contact are a soft gasket of semicircular section and a similar stiff gasket against which the soft one is pushed. See Figure 5.17. The back side of each gasket has all of its displacements prescribed. Both gaskets have unit radius and a thickness of 0.1. There is a vertical offset of 0.25 between the centers and a horizontal separation of 0.2. They are forced together by a prescribed horizontal motion of their backs such that each moves 0.25. This would cause considerable overlap if contact were ignored. The initial and final positions, ignoring contact, are shown in Figure 5.18. The calculation, including contact, is advanced by small steps. At step 5 overlap occurs for the first time as shown in Figure 5.19. After iterating contact is resolved as shown in Figure 5.20. The maximum deflection occurs in the tenth step. The linear prediction is shown in Figure 5.21 and the resolved positions in Figure 5.22. Then the gaskets start to retreat from each other. The final and initial positions are shown in Figure 5.23. They overlay very closely. This is a good demonstration of the reversibility of the method.

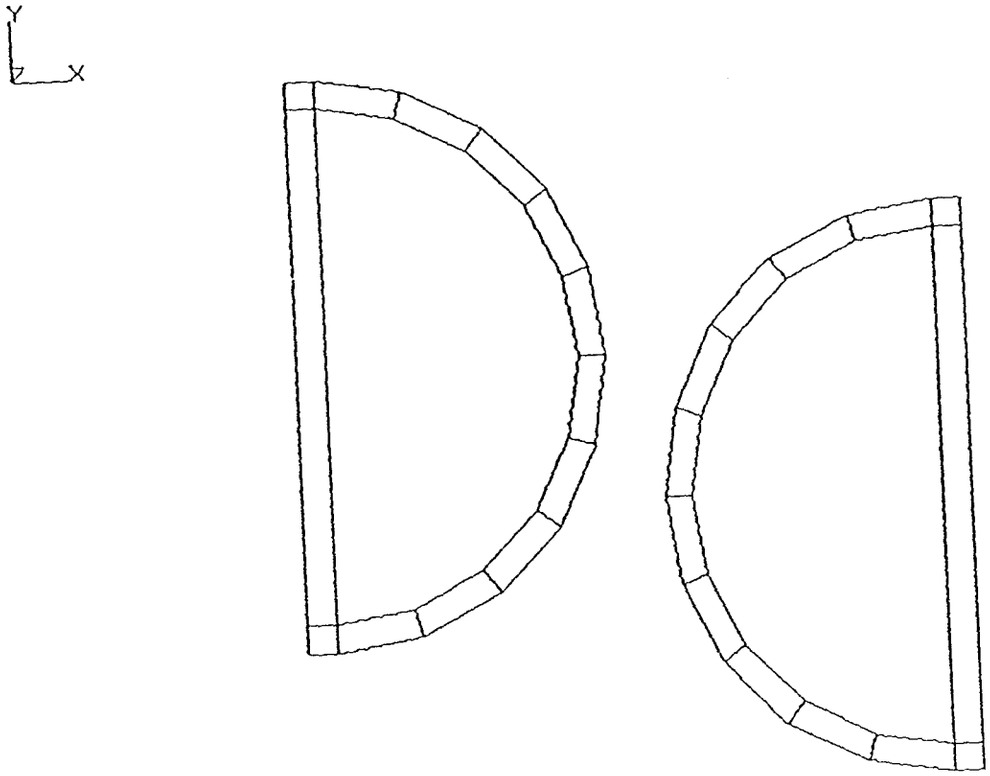


Figure 5.17 Gasket example: Initial configuration

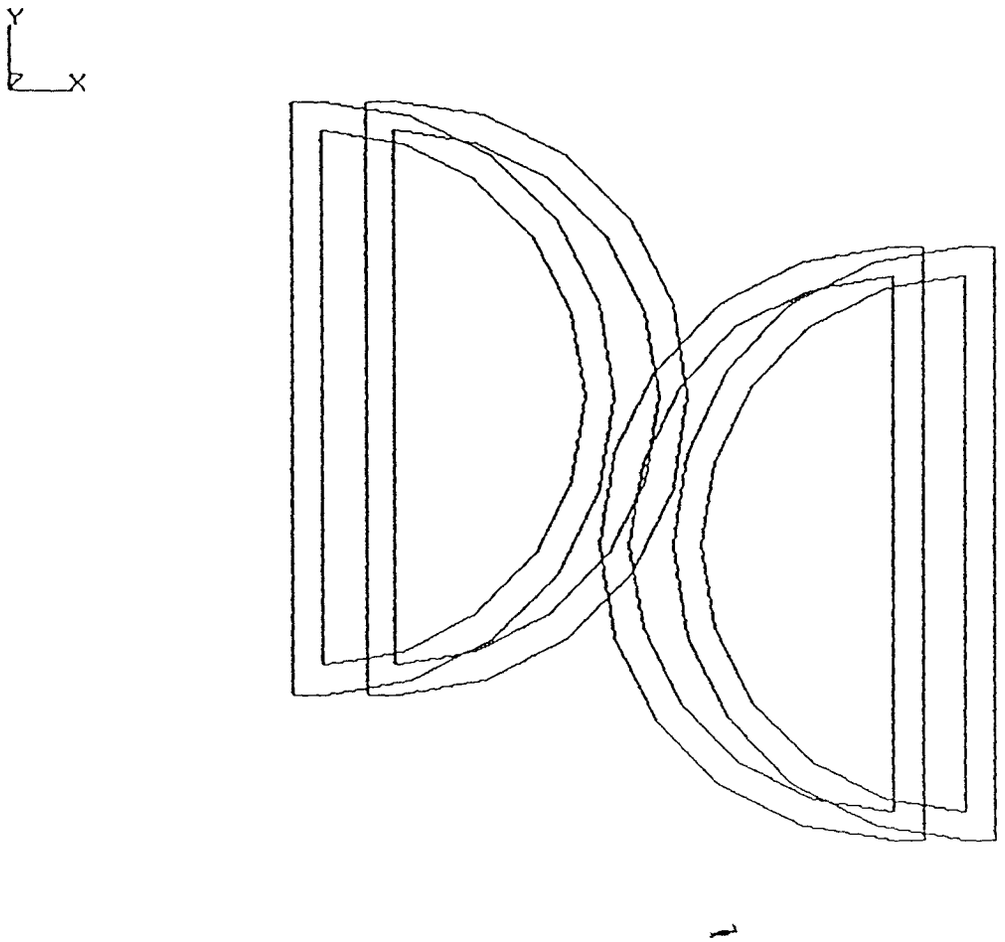


Figure 5.18 Gasket example: Initial and final configurations obtained by ignoring contact

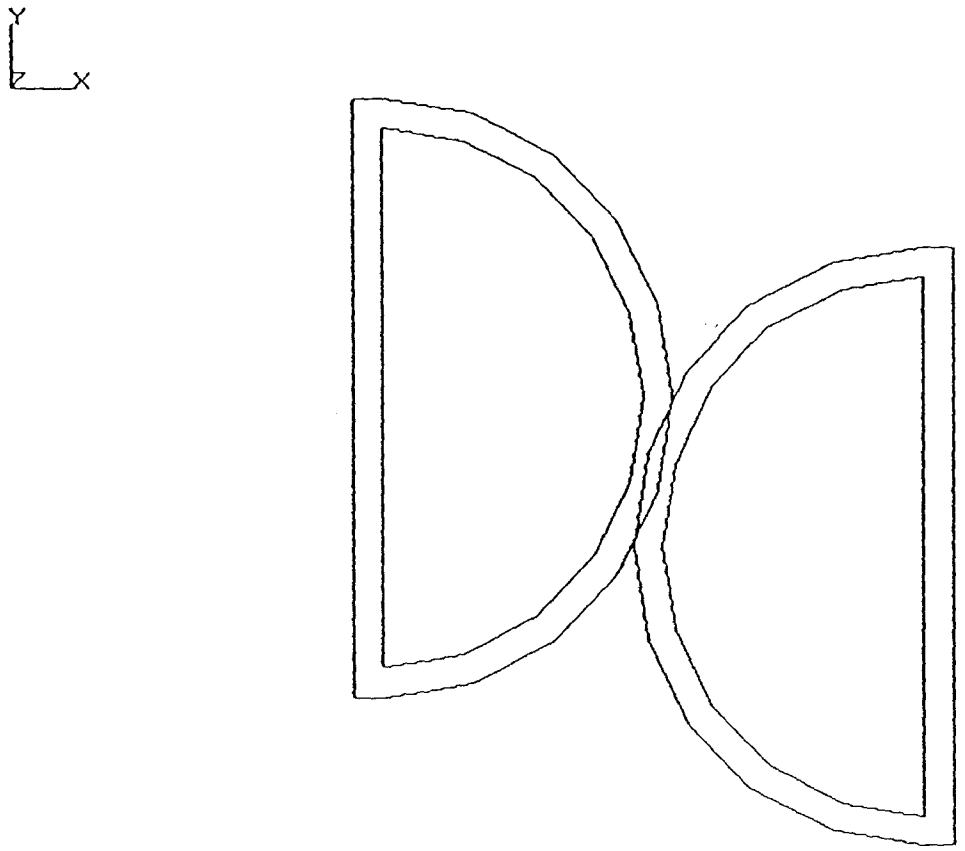


Figure 5.19 Gasket example: Linear prediction for step 5

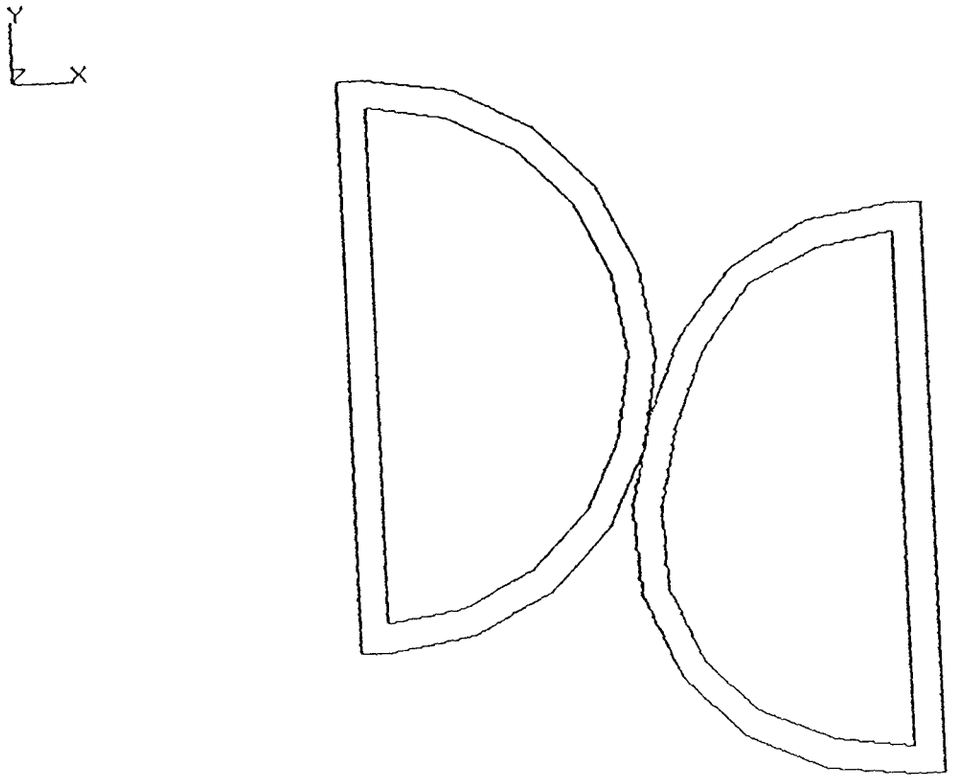


Figure 5.20 Gasket example: Resolved position after iterations for step 5

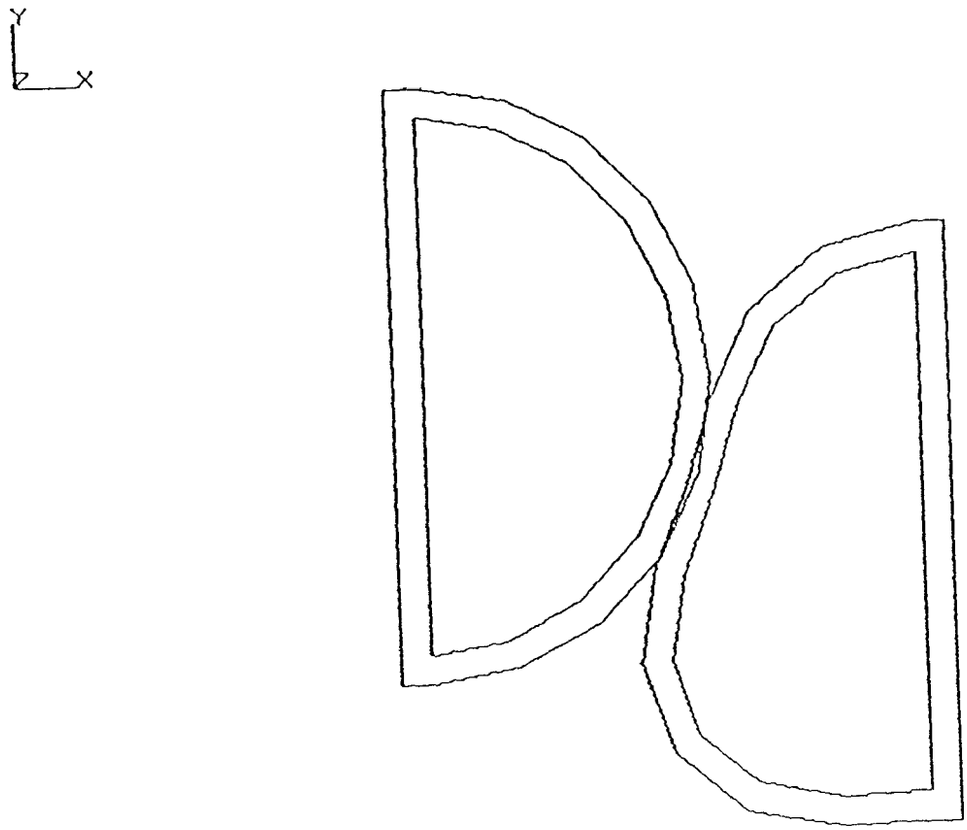


Figure 5.21 Gasket example: Linear prediction for step 10

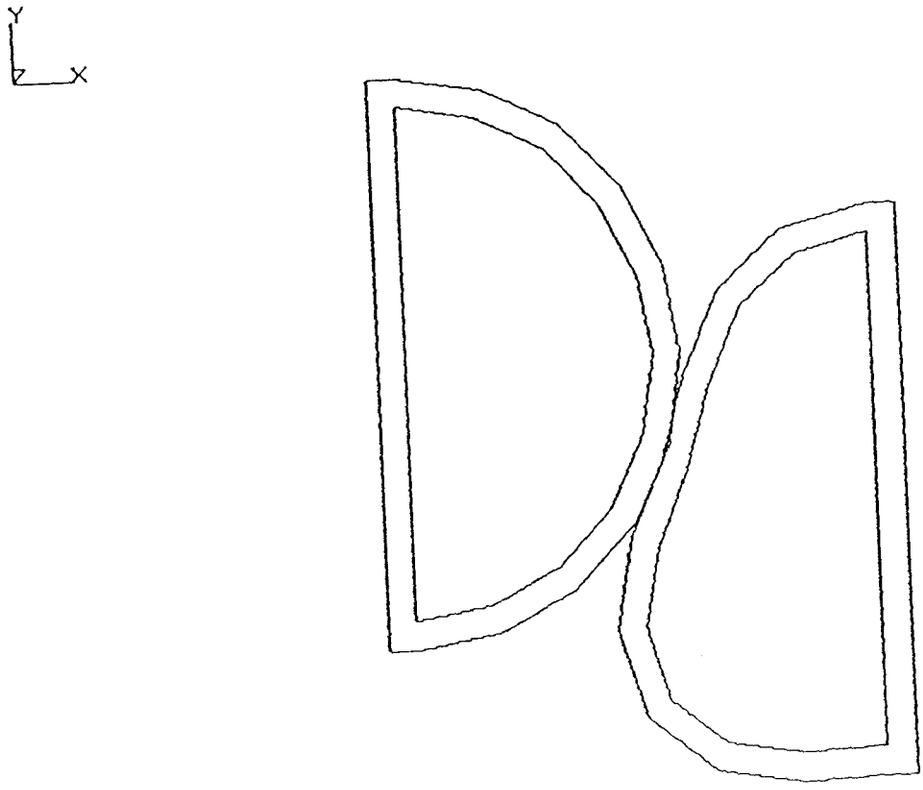


Figure 5.22 Gasket example: Resolved position after iterations for step 10

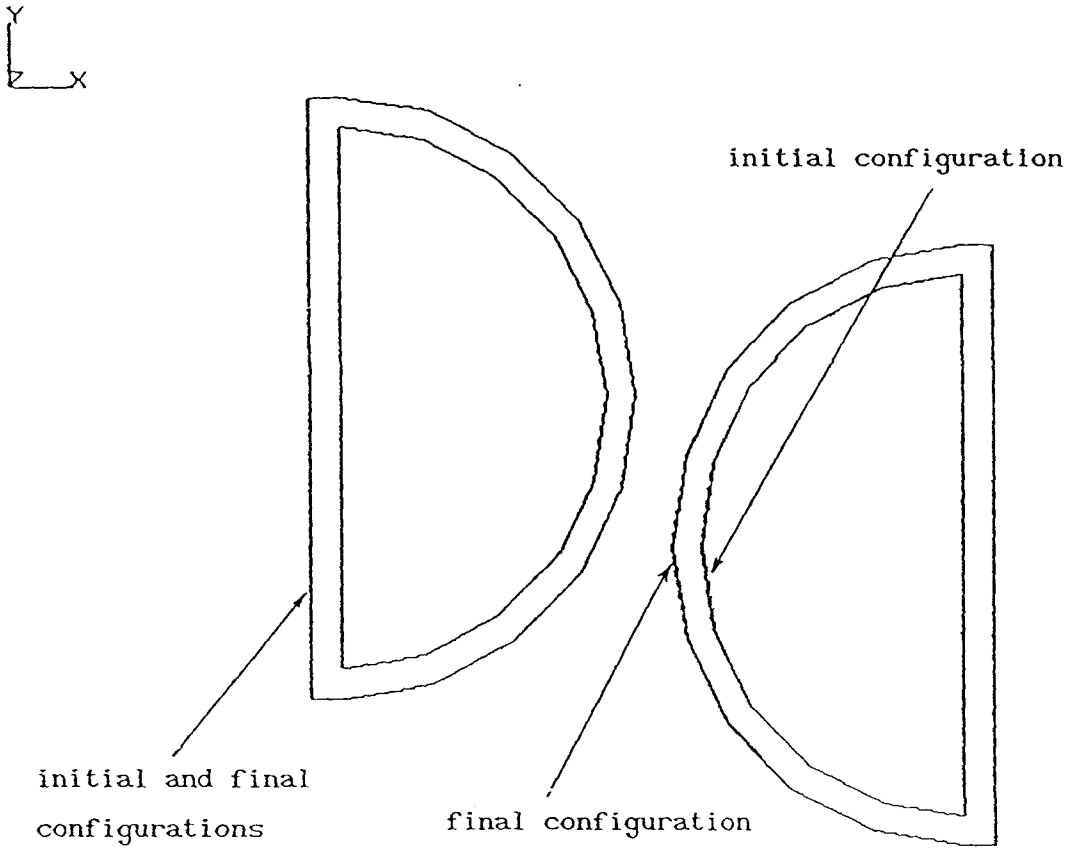


Figure 5.23 Gasket example: Initial and final configurations

5.6 DISCUSSION

There are a few aspects of the method that should be discussed. This is still a nonlinear problem of some severity. There are certain aspects of nonlinear finite element calculations that should be mentioned: step size, mesh refinement, tolerances, convergence and relaxation.

The first let us look at step size. For nonlinear calculations to converge the step size must usually be small. For accurate results, the step size is often strictly limited; otherwise the calculated history of loads and displacements may diverge from the intended problem to an unacceptable extent. Further, there are cost penalties as well as accuracy disadvantages to the use of large steps. It is common for the use of larger steps to require disproportionately more iterations. Given that the method is stable, there is a certain problem dependent limit, within which any step will converge quickly. It is not clear a priori how big this limit is. There is a necessary conceptual limit in these contact calculations in particular. One of the requisites is that the penetration or emergence of nodes must be shallow. This is necessary both to keep the step size small and to keep the direction of the surface normal well defined. The limitation that has been used in the calculations presented here is that no node may penetrate all the way through the layer of elements that form the outer rind of the body. If a node is found embedded in an element that is in the bulk of the body's volume, then an error is declared and calculations are stopped. This puts a limit on the size of step that can be undertaken in the linear

prediction. If the elements on the surface are thin, then the steps must be kept very small. This is of practical importance. For such an analysis, it is possible that the requirement that nodes not penetrate past the surface layer of elements may place a more restrictive upper bound on the size of a step than the limitations imposed by the need to define the outward normal or the desire for rapid convergence.

Mesh refinement is a matter for thought in designing a finite element model. How well any algorithm can follow the details of contact in a small region is limited by the fineness of the mesh. Also, the relative refinement of the mesh on either side of the contact surface is important to this method. If on one side the mesh were very fine and on the other side the mesh were very crude, then there would be implications for the numerical integrations of the virtual elements. According to the smaller elements, there would be very slow variation in the opposing contact traction as compared to the internal coordinates and low order integration would suffice. For the larger elements, the opposing contact tractions may seem to vary very rapidly as compared to their internal coordinate systems. For the large elements, it may be necessary to use an integration scheme of quite high order. Good results are observed to occur when the elements on either side of contact are similar in size. A size ratio of 3:1 is easily tolerated. For larger ratios it is suggested that care be taken that the surface integrals are of sufficiently high order. This is not an important calculational burden, but it does affect performance.

The method uses two tolerances. If a node has penetrated, or emerged, only a very small distance, then this is considered a

negligible error. No compatibility correction is found for it. Instead its displacements are considered free variables in the backsubstitution pass that calculates the forces needed move the body to the contact surface. The other tolerance is on the equilibrium errors at pairs of real and virtual nodes. If this error is sufficiently small, then further corrections are not attempted. Clearly the choices made for these tolerances limit the accuracy that it is possible to achieve. They also limit the cost of an analysis. The engineer who wishes to find the solution to a contact problem must decide how much accuracy can afforded. The behaviour of the method is insensitive to the equilibrium tolerance so long as it is small compared to the nodal loads. The penetration tolerance has more effect on the behaviour of the method. There is an interaction between element size and penetration tolerance. This tolerance must always be small compared to the thickness of an element or else the linear prediction stage will tend to cause nodes to penetrate past the surface layer of elements. If very detailed results are needed, then this tolerance can be set so low as to enforce near perfect compatibility. This may lead to using many iterations.

The criterion for convergence is another matter that affects nonlinear calculations strongly. As used in these calculations, a step is considered to have converged when either of two criteria is met. The first possibility is that the corrective forces for an iteration are less than some specified fraction of the largest for the step. Magnitude is compared by the Euclidean norm of sum of the incremental equilibrium and compatibility forces. The other possibility is that the displacements that resulted from applying the corrective forces are

small when compared to the largest incremental displacement for the step. This means that at least two iterations are required for each step where contact is detected. One to establish initial values for the magnitudes of corrective forces and resultant displacements and one or more further iterations until one of the criteria are met. If the force criterion causes iterations to cease, then that increment of force is not applied. How many iterations are required and how well the contact conditions are met depends on what fraction is specified for convergence. Experience so far suggests that 1% gives good results. It is possible to get results with a less stringent convergence limit, but it is not guaranteed that the contact conditions will be well met. This is particularly true where the bodies involved have greatly different elastic moduli and the stiffer body has sharp irregularities on a scale similar to the size of the elements of the body with the more flexible material. In that case, it is possible that the sharp points of the stiff material will not be completely ejected from the flexible material. For a very approximate analysis this may not matter. For an investigation that requires accuracy in the fine details, it is better to set the limits of convergence to be very fine and to reduce the size of the elements in the flexible material.

Another consideration is the use of relaxation factors. These will accelerate or retard convergence depending on how they are used. The best values to use seem to be problem dependent. Experiments show that for many analyses it is effective to under relax the first few iterations. It is sometimes effective to over relax later iterations.

The most useful application of this arises in the release of

nodes that are found to have tensile contact forces. Recall that the algorithm works by allowing a physical error to occur and then correcting it. It is possible when step sizes are large for the compatibility corrections of the first iteration to apply tensile forces to nodes near the edge of the contact region. For physical reasons these must be released. However, in many cases, if all of the tension is released at once, then there is an unwonted disturbance to the calculations. If, instead, the release of this excess traction is done over two (or more) iterations, then the magnitude of subsequent corrections is much less and the overall convergence is greatly accelerated. Trials to date suggest that releasing half of the force over each of two iterations results in disturbances which are easily tolerated by the calculation.

The final discussion is to assess the utility of this new algorithm. Here the evidence of the examples is helpful. It has been shown the method is accurate, that it behaves well under a wide variety of conditions of relative stiffness, that geometric irregularities do not disturb it unduly and that it is reversible. This is a strong argument that this algorithm is sufficiently robust and rapidly convergent to be very suitable for practical applications.

CHAPTER SIX

CONCLUSIONS

6.1 CONCERNING REFERENTIAL MECHANICS

In Chapter Two a careful study was presented of several basic quantities used in referential mechanics. Care was used to keep track of the use of configurations for various purposes, especially for gradients and for reference. For this work a detailed notation was developed from earlier similar notations. The result was that correct forms were found for incremental displacement-based pseudo strains. In particular, the form of the linear part of the increment of pseudo strain for the updated Lagrangian formulation was corrected from earlier works.

In Chapter Three the repercussions of the forms developed in Chapter 2 are investigated. The detailed form of the Rate of Work Principle for static and dynamic problems is presented.

6.2 CONCERNING VIRTUAL FINITE ELEMENTS FOR CONTACT PROBLEMS

In Chapter Four a new algorithmic procedure is presented for approximate solutions to contact problems using the finite element method. In Chapter Five some examples are presented which demonstrate certain features of this new procedure.

This contact algorithm has two salient new features. It allows independence of the meshes of the bodies involved in contact. It introduces a new concept, the virtual mesh, which facilitates the translation of quantities across the contact surface. Other uses for virtual meshes may be found. It is noted that the algorithm used converges very rapidly. This is attributed to the manner in which the corrections are undertaken. That is, there are separate corrective forces applied for kind of error and these corrections do not interfere with each other to the first order.

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