VECTOR VALUED MODULAR FORMS

VECTOR VALUED MODULAR FORMS OF DIMENSION 3

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Abstract

In this thesis we describe and relate various representations of 3-dimensional vector valued modular forms. In particular, we give algebraic formulas for families of 3-dimensional vector valued modular forms on $\Gamma_0(2)$, a subgroup of the modular group $\Gamma = SL_2(\mathbb{Z})$. These formulas enable us to compute CM values of the 3-dimensional vector valued modular forms at CM points in the upper half plane.

We also define families of Eisenstein series corresponding to one-dimensional representation, χ, on $\Gamma_0(2)$. This gives a different description of the algebraic family discussed in the preceding paragraph. For Eisenstein series of weight 4 and 6, we evaluate their Fourier series expansion and compute their Fourier coefficients. The constant term in the Fourier series expansion of Eisenstein series of weight 4 and 6 is then expressed using Bessel function of the first kind and Kloosterman sums.

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- Gagandeep Kaur Virk

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Chapter 1

Introduction

1.1 Motivation

Modular forms hold significant importance in the sphere of Number Theory. Andrew Wiles in [29] proved Fermat's Last Theorem using theories related to the modular forms.

These functions are known to establish profound interconnections not only between different areas of mathematics, but also between mathematics and physics. Gannon in [17] explores these connections and explains how modular forms are closely related to the finite simple groups and vertex operator algebras. He also discusses applications of modular forms to conformal field theory, string theory, etc. in physics. Franc and Mason in [16] conduct thorough examination of connections and applications of modular forms to quantum physics and conformal field theory. The interested reader is referred to their research paper [16] and its references for further exploration of the topic.

In the applications of modular forms in physics, these appear as different types of families. We are looking at one such family of modular forms of dimension 3. We used [10] as the main reference for the background on modular forms in chapter 2.

Franc and Mason in [14] identified the connections between vector valued modular forms of dimension 3 and hypergeometric series. In chapter 3, we are trying to extend this relationship further to polynomials. We give algebraic formulas for families of 3-dimensional vector valued modular forms on $\Gamma_0(2)$, a subgroup in $SL_2(\mathbb{Z})$ using Bailey's classical cubic transformations ([4], [21]) and Vidunas' elementary expressions of hypergeometric functions in [28].

Every vector valued modular form corresponds to a Fourier series. In chapter 4, we define a series related to vector valued modular forms in chapter 3 and show that it behaves similar to Eisenstein series. We compute Fourier coefficients of these analogues of Eisenstein series and express the constant terms in their Fourier series expansion in terms of Bessel function of the first kind and Kloosterman sums.

In the next section, we lay down notations that have not been defined in the later chapters. This is then followed by backgound on vector valued modular forms and ordinary differential equations in chapter 2.

1.2 Notations

$\Re$: Real part of complex number

$\Im$: Imaginary part of complex number

Chapter 2

Background

2.1 Vector valued modular forms (vvmfs)

The *upper half plane*, $\mathcal{H}$ is defined as the set of complex numbers with strictly positive imaginary part.

$$\mathcal{H} = \{\tau \in \mathbb{C} \mid Im(\tau) > 0\}.$$

The *Modular Group*, $\Gamma = SL_2(\mathbb{Z})$ is defined as the group of 2x2 matrices with integer

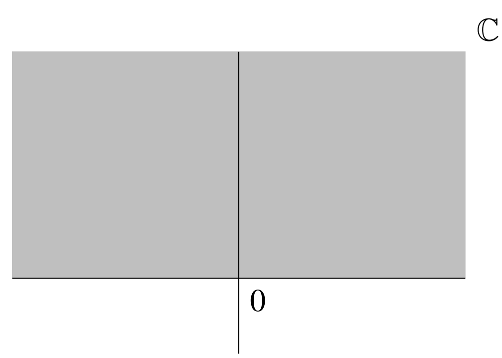

Figure 2.1: Upper half plane.

entries and determinant 1,

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{Z}; ad - bc = 1 \right\}.$$

The group Γ is generated by $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The order of T is infinite because $T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ for $n \in \mathbb{Z}$ and order of S is 4 because $S^2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. To express convenient descriptions of representations of Γ, we define another generator of Γ, $R = ST = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$. The order of $R = 6$ because $R^3 = -I$. So,

$$\Gamma = < R, S \mid R^3 = S^2 = -I, R^6 = S^4 = I > .$$

Next, we define the action of Γ on $\mathcal{H}$. For $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (\tau) = \frac{a\tau + b}{c\tau + d}, \ \forall \tau \in \mathcal{H}.$$

Note that if $\gamma \in \Gamma$ and $\tau \in \mathcal{H}$, then $\gamma(\tau) \in \mathcal{H}$. This happens because

$$Im(\gamma(\tau)) = \frac{Im(\tau)}{|c\tau + d|^2}. \tag{2.1}$$

Definition 2.1. A *representation*, ρ, of Γ is a group homomorphism

$$\rho : \Gamma \to GL_d(\mathbb{C})$$

where d gives the dimension of ρ.

Definition 2.2. A *vector valued modular form* (vvmf) of weight $k \in \mathbb{Z}$ with respect to ρ is a meromorphic function $f : \mathcal{H} \to \mathbb{C}^d$ such that:

- $f = (f_1, f_2, ..., f_d)^t$ is holomorphic on $\mathcal{H}$,
- $f(\gamma(\tau)) = (c\tau + d)^k \rho(\gamma) f(\tau), \ \forall\, \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \tau \in \mathcal{H}$ and,
- f is holomorphic at the "cusp" (∞).

For a detailed discussion of what it means for a vector-valued modular form to be holomorphic at the cusp, see [6].

Example 2.3. For trivial $\rho : \Gamma \to \mathbb{C}^\times$, i.e. $\rho(\gamma) = 1$ for all $\gamma \in \Gamma$, we define modular forms of weight k of level 1 over ρ as follows:

$$f(\gamma(\tau)) = (c\tau + d)^k f(\tau),\ \tau \in \mathcal{H},\ \gamma \in \Gamma = SL_2(\mathbb{Z}).$$

Example 2.4. An example of level 1 modular forms are the Eisenstein series. For even $k > 2$, the Eisenstein series are defined as

$$G_k(\tau) = \sum_{(c,d) \in \mathbb{Z}^2 - (0,0)} \frac{1}{(c\tau + d)^k} \quad \tau \in \mathcal{H}.$$

The Eisenstein series are modular forms of weight k over Γ. Using these, we also define normalized Eisenstein series as follows:

$$E_k(\tau) = \frac{G_k(\tau)}{2\zeta(k)}$$

where $\zeta(k) = \sum_{k=1}^{\infty} \frac{1}{d^k}$ for $Re(k) > 1$ is the Riemann zeta function.

Example 2.5. For dim $\rho = 2$, consider $\rho : \Gamma \to GL_2(\mathbb{C})$ to be the Identity map. Then,

$$f(\tau) = \begin{bmatrix} \tau \\ 1 \end{bmatrix}$$

is a VVMF of weight -1.

Idea for proof: Since $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are generators of Γ, it suffices to check that $f(\gamma(\tau)) = (c\tau + d)^{-1}\rho(\gamma)f(\tau)$ for $\gamma = S,\ T$ for all $\tau \in \mathcal{H}$. These two conditions can be checked directly by a simple finite computation. $\square$

The set of modular forms of weight k over group Γ is denoted by $\mathcal{M}_k(\Gamma)$. Note that for $f_1, f_2 \in \mathcal{M}_k(\Gamma)$, we have $\alpha f_1 + \beta f_2 \in \mathcal{M}_k(\Gamma)$ for all $\alpha, \beta \in \mathbb{C}$. Therefore, $\mathcal{M}_k(\Gamma)$ is a vector space over $\mathbb{C}$.

Also, for $f \in \mathcal{M}_k(\Gamma)$ and $g \in \mathcal{M}_l(\Gamma)$, we have $fg \in \mathcal{M}_{k+l}(\Gamma)$. Hence, we define the space of all modular forms over the group Γ by the sum

$$\mathcal{M}(\Gamma) = \bigoplus_{k \in \mathbb{Z}} \mathcal{M}_k(\Gamma),$$

which forms a graded ring.

We can also denote these spaces with respect to the representation ρ (over Γ). The set of vvmfs of weight $k \in \mathbb{Z}$ for ρ is denoted by $M_k(\rho)$. Similarly, we can denote the set of all vvmfs over ρ by the sum

$$M(\rho) = \bigoplus_{k \in \mathbb{Z}} M_k(\rho).$$

The following theorem informs us about the structure of this sum.

Theorem 2.6 ***[The Free-Module Theorem]*** *([22], [14]) Let ρ denote an n-dimensional complex representation of Γ. Then, $M(\rho)$ is free of rank n as a $\mathbb{C}[E_4, E_6]$-module.*

Next, we will define some important subgroups of $\Gamma = SL_2(\mathbb{Z})$.

Definition 2.7. For $N \in \mathbb{Z}^+$. We define the *principal congruence subgroup* of level N as

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (mod\ N) \right\}.$$

Definition 2.8. A subgroup $X \subset SL_2(\mathbb{Z})$ is called a *congruence subgroup* of level N, if $\Gamma(N) \subseteq X$ for some $N \in \mathbb{Z}^+$ and N is the least such positive integer.

An important congruence subgroup that we will be studying in this thesis is $\Gamma_0(N)$.

Example 2.9. We define $\Gamma_0(N)$ as:

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} (mod\ N) \right\}.$$

Note that the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$ is given by $N \prod_{p|N}(1 + \frac{1}{p})$, where the product is taken over the prime divisors of N. ([10], p. 14)

Another important notation that we need to introduce is that of the weight-k operator, also known as the slash operator.

Definition 2.10. Let f be a vvmf of weight k. Then for any $\gamma \in SL_2(\mathbb{Z})$, we define the slash operator $|_\gamma$ as

$$f|_\gamma(\tau) = \frac{f(\gamma(\tau))}{(c\tau + d)^k}.$$

The slash operator is generally denoted with respect to the weight k as $f|_k\gamma$, but for simplicity we will suppress the weight k from the notation and symbolize the operator as $f|_\gamma$. It can be easily observed that the definition of vvmfs can be re-written using the slash operator as

$$f|_\gamma(\tau) = \rho(\gamma)f(\tau),$$

where $\gamma \in SL_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

Following are the definitions of some of the important functions that we will use in calculations for vvmf formulae in Chapter 3.

Definition 2.11. The *modular function* $j : \mathcal{H} \to \mathbb{C}$ is defined as:

$$j(\tau) = \frac{1728(g_2(\tau))^3}{\Delta(\tau)}, \tag{2.2}$$

where $g_2(\tau) = 60G_4(\tau)$, $g_3(\tau) = 140G_6(\tau)$ and $\Delta(\tau) = (g_2(\tau))^3 - 27(g_3(\tau))^2$ is the Discriminant function.

The j-function is also known as the *modular invariant* and plays a crucial role in the theory of modular forms. It should be noted that the j-function is **not** a modular form. Even though it is holomorphic on $\mathcal{H}$ and is Γ-invariant i.e. $j(\gamma(\tau)) = j(\tau)$ for all $\gamma \in \Gamma$ and $\tau \in \mathcal{H}$, but it is not holomorphic at ∞.

Definition 2.12. The *Dedekind eta function*, denoted by η, is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{2\pi i\tau}$ and $\tau \in \mathcal{H}$. This function is a weight $1/2$ modular form of level 1.

Definition 2.13. For a complex number s with positive real part, we define the *gamma function* of s as the integral

$$\Gamma(s) = \int_{t=0}^{\infty} t^{s-1} e^{-t} dt.$$

It can be easily shown that $\Gamma(1) = 1$ and that the Gamma function satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

Therefore, $\Gamma(n) = (n+1)!$ for all $n \in \mathbb{Z}^+$.[1]

Definition 2.14. Let v and t be complex variables with $\Im(t) > 0$. Then, we define the *theta function* for v and t as follows:

$$\theta(v,t) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)\pi i v},$$

[1] We are using the same notation Γ for the modular group $SL_2(\mathbb{Z})$ and the Gamma function. The meaning of the notation will be based on the context in which it is used.

where $q = e^{\pi i t}$. For fixed t, the theta function is a holomorphic function of v on $\mathbb{H}$ ([7], p. 58). Similarly, we can define special cases of theta functions for complex variables v and t as follows:

$$\theta_2(v,t) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)\pi i v},$$

$$\theta_4(v,t) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i v},$$

$$\theta_3(v,t) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i v},$$

where $q = e^{\pi i t}$ and $\Im(t) > 0$. Like the theta function $\theta(v,t)$, the above mentioned functions are also holomorphic functions of v on $\mathbb{H}$ for fixed t ([7], p. 58).

We will now use the above mentioned special theta functions to define the *modular lambda function*.

Definition 2.15. The *modular lambda function*, denoted by λ, is defined with respect to the theta function as follows:

$$\lambda(\tau) = \frac{{\theta_2}^4(0,\tau)}{{\theta_3}^4(0,\tau)}$$

where $\tau \in \mathcal{H}$. The λ-function is a holomorphic function on the upper half plane, $\mathcal{H}$, and its q-expansion is given by:

$$\lambda(\tau) = 16q - 128q^2 + 704q^3 - 3072q^4 + 11488q^5 - 38400q^6 + \cdots$$

[1].

The formulas in chapter 3 enable us to compute the CM values of the 3-dimensional vector valued modular forms at CM points in the upper half plane. So, it is useful to define the terms CM points and CM values.

Definition 2.16. A point $\tau \in \mathcal{H}$ is a *complex multiplication (CM) point* if it belongs to a quadratic field $\mathbb{Q}(\sqrt{d})$ where $d \in \mathbb{Z}^-$. For example, i, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, etc. We define CM values as the values of modular forms at CM points.

In chapter 4, we will use the fact that each component of VVMF f has a Fourier series expansion([10], [13]), so, it is important to lay down some basic definitions from Fourier Analysis.

Definition 2.17. ([27], p.34) Let F be an integrable function given on an interval $[a, b]$ of length L (that is, $b - a = L$), then the n^{th} Fourier coefficient of F is defined by

$$\hat{F}(n) = \frac{1}{L} \int_a^b F(x) e^{-2\pi i n x/L} dx, \quad n \in \mathbb{Z}.$$

Note that if the function F is periodic on $\mathbb{R}$ of length $b - a$, then it is determined by its values on the interval $[a, b]$.

The Fourier series of F is given by

$$F(x) = \sum_{n=-\infty}^{\infty} \hat{F}(n) e^{2\pi i n x/L}.$$

Each component of VVMF f has Fourier series expansion of the form:

$$f_j(\tau) = q^{m_j} \sum_{l=0}^{\infty} t_{n_l} q^{n_l}$$

where $q = e^{2\pi i \tau}$ and $\tau \in \mathcal{H}$ ([13]). The t_{n_l} are the Fourier coefficients.

In the expansion of Fourier series of certain families of vvmfs in chapter 4, we will observe that the constant term can be expressed using Kloosterman sums, Bessel functions of the first kind and Modified Bessel functions of the first kind. So, it is important that we define these functions. We will discuss Bessel functions in section 2.2 after introducing differential equations. For now, we define Kloosterman sums.

Definition 2.18. ([11]. [18], [19]) Let u, v, n be natural numbers. Then the Kloosterman sums are defined by the formula

$$K(u, v, n) = \sum_{\substack{d=1 \\ gcd(d,n)=1}}^{n-1} e^{\frac{2\pi i}{n}(ud + v\bar{d})}$$

where $\overline{d}$ is the multiplicative inverse of $d \; modulo \; n$.

2.2 Ordinary Differential Equations

A field, K, is called a *differential field* if there exists a map $\partial : K \longrightarrow K$, such that:

- $\partial(a+b) = \partial a + \partial b$ for all $a, b \in K$,
- $\partial(ab) = a\partial b + b\partial a$ for all $a, b \in K$.

Definition 2.19. An *ordinary differential equation* of order n over differential field K is an equation of the form

$$\partial^n y + p_1 \partial^{n-1} y + p_2 \partial^{n-2} y + \cdots + p_{n-1}\partial y + p_n y = 0,$$

where $p_1, p_2, \cdots p_n \in K$.

Example 2.20. For $n = 2$ and $K = \mathbb{C}(z)$, the field of rational functions, we define the hypergeometric differential equation as

$$(z)(z-1)\frac{dy^2}{dz^2} + ((a+b+1)z - c)\frac{dy}{dz} + aby = 0, \tag{2.3}$$

where $a, b, c \in \mathbb{C}$.

Definition 2.21. Consider the linear differential equation of order n,

$$\frac{dy^n}{dz^n} + p_1(z)\frac{dy^{n-1}}{dz^{n-1}} + \cdots + p_{n-1}(z)\frac{dy}{dz} + p_n(z)y = 0 \tag{2.4}$$

where $p_i(z) \in \mathbb{C}(z)$. Assume that the coefficients $p_i(z)$ share no common factor $(z-p)$ for $p \in \mathbb{C}$.

We say that a point $p \in \mathbb{C}$ is a *singular point* of equation 2.4, if it is a pole of some coefficient $p_i(z)$, otherwise we call it a *regular point*. The point $p \in \mathbb{C}$ is called a *regular singularity* of equation 2.4, if the coefficient $p_i(z)$ has a pole of order at most i at p for $i = 1, 2, \cdots, n$. *The point* ∞ *is said to be a regular singularity* of equation 2.4, if we can get 0 as a regular singularity of the equation obtained by writing 2.4 in terms of $w = 1/z$.

Example 2.22. Equation 2.3 in example 2.20 has three regular singular points at $z = 0, 1$ and ∞.

We are mainly interested in solutions of equation 2.3. When c is non-integral, the hypergeometric differential equation has two linearly independent solutions given by

$$y_1(z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}$$

and

$$y_2(z) = z^{1-c} \sum_{k=0}^{\infty} \frac{(a+1-c)_k (b+1-c)_k z^k}{(2-c)_k k!}$$

([5], p.19).

The above two equations lead us to the next definition of this section, which is that of the Gauss Hypergeometric functions.

Definition 2.23. Let $a, b, c \in \mathbb{C}$ and $c \notin \mathbb{Z}_{\leq 0}$. Then, the *Gauss hypergeometric function* is given by the following sum

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!},$$

where $(x)_k$ is the Pochhammer symbol defined by

$$(x)_k = \begin{cases} 1, & k = 0 \\ (x)(x+1)(x+2)\cdots(x+k-1), & k \in \mathbb{Z}^+ \end{cases}$$

for all $x \in \mathbb{C}$.

Definition 2.24. A differential equation over $\mathbb{C}(z)$ is called Fuchsian if all points on $\mathbb{P}^1$ are either regular or a regular singularity.

Example 2.25. Let $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_n \in \mathbb{C}$ with $\beta_i \notin \mathbb{Z}_{\leq 0}$. Also, define *$\theta$-operator* as

$\theta = z\frac{d}{dz}$. Then, the generalized hypergeometric equation in one variable

$$z(\theta+\alpha_1)\cdots(\theta+\alpha_n)y = (\theta+\beta_1-1)\cdots(\theta+\beta_n-1)y \tag{2.5}$$

is a Fuchsian equation of order n with singularities at $0, 1$ and ∞.

We are interested in the special type of functions which show up in solutions of equation 2.5. We define these functions in the following definition.

Definition 2.26. Let $\alpha_1, \cdots, \alpha_n, \beta_1, \cdots, \beta_{n-1} \in \mathbb{C}$ with $\beta_i \notin \mathbb{Z}_{\leq 0}$. Then, the *generalized hypergeometric function* is given by:

$$_nF_{n-1}(\alpha_1, \cdots, \alpha_n; \beta_1, \cdots, \beta_{n-1}; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_n)_k z^k}{(\beta_1)_k \cdots (\beta_{n-1})_k k!} \tag{2.6}$$

where $(x)_k$ is the Pochhammer symbol (as defined in Definition 2.23) for all $x \in \mathbb{C}$.

Recall from section 2.1 that $M_k(\rho)$ is the set of vector valued modular forms of weight $k \in \mathbb{Z}$ for a representation ρ. Let $F \in M_k(\rho)$. Then, by Definition 2.2, we have that

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \rho(\gamma) F(\tau),$$

$\forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma, \tau \in \mathcal{H}$. Differentiating both sides of the above equation with respect to τ, we get:

$$F'\left(\frac{a\tau+b}{c\tau+d}\right) = kc(c\tau+d)^{k+1}\rho(\gamma)F(\tau) + (c\tau+d)^{k+2}\rho(\gamma)F'(\tau)$$

Clearly, F' does not satisfy the definition of vvmfs unless $k = 0$. The question that arises with this differentiation is that what happens when $k \neq 0$? How do we adjust F' so that we get a modular form? This question leads us to the notion of Modular Derivative.

Definition 2.27. Let $F \in M_k(\rho)$, then the Modular Derivative of F is defined as follows:

$$DF = \frac{1}{2\pi i}\frac{dF}{d\tau} - \frac{k}{12}E_2F$$

Note that the modular derivative $DF \in M_{k+2}(\rho)$ and increases the weight of modular forms by 2.

Combining the modular derivative with Theorem 2.6, Franc and Mason in [14] worked out the basis for vvmfs of dimension 3 in terms of hypergeometric functions. In their work, they consider $\rho : \Gamma \to GL_3(\mathbb{C})$ to be an irreducible representation such that $\rho(T)$ has a finite order. By these assumptions and the Free Module Theorem, they show that $\rho(T)$ must have distinct eigenvalues and that these eigenvalues are each distinct root of unity.

Definition 2.28. Let $\rho : \Gamma \to GL_3(\mathbb{C})$ be an irreducible representation with diagonalized $\rho(T)$ as discussed in [14]. Then, $\rho(T)$ is conjugate to the matrix

$$\begin{pmatrix} e^{2\pi i r_1} & 0 & 0 \\ 0 & e^{2\pi i r_2} & 0 \\ 0 & 0 & e^{2\pi i r_3} \end{pmatrix}$$

where $0 \leq r_i < 1$ for all $1 \leq i \leq 3$. The r_i's here are called exponents of eigenvalues of $\rho(T)$.

Let r_1, r_2, r_3 be the exponents of eigenvalues of $\rho(T)$. Then the coordinates of a non-zero VVMF of lowest weight,

$$k = 4(r_1 + r_2 + r_3) - 2 \tag{2.7}$$

over ρ are given by

$$\eta^{2k} K^{\frac{a_l+1}{6}} {}_3F_2\left(\frac{a_l+1}{6}, \frac{a_l+3}{6}, \frac{a_l+5}{6}; r_l - r_m + 1, r_l - r_m + 1; K\right) \tag{2.8}$$

for $l = 1, 2, 3$, where $a_l = 4r_l - 2r_m - 2r_n$ for $\{l, m, n\} = \{1, 2, 3\}$ and $K = \frac{1728}{j}$ ([14], p.

23).

We will use equations 2.7 and 2.8 throughout chapter 3 to compute our algebraic formulas for vvmfs of dimension 3.

As mentioned earlier in section 2.1, the constant term in the Fourier series expansion of certain families of vvmfs in chapter 4 can be expressed using different types of Bessel functions. So, we conclude this section with a brief discussion around the Bessel functions of the first kind and the Modified Bessel functions of the first kind.

Definition 2.29. ([2], p.358, 360 (Definition 9.1.10)) The Bessel functions of the first kind are solutions to the differential equations of the form

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - v^2)w = 0.$$

These functions are given by the sum:

$$J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+v+1)} \left(\frac{z}{2}\right)^{2k+v}, \tag{2.9}$$

where $\Gamma(z)$ is the Gamma function, as defined in Definition 2.13.

It should be noted that in Equation (2.9) that if v is not an integer then none of the coefficients vanish. For integer values of v, we have

$$\frac{1}{\Gamma(k+v+1)} = 0 \quad \text{for} \;\; k+v+1 \leq 0,$$

and

$$\frac{1}{\Gamma(k+v+1)} = \frac{1}{(k+v)!} \quad \text{for} \;\; k+v+1 > 0$$

([9], p.484).

Definition 2.30. ([2], p.374, 375 (Definition 9.6.10)) The Modified Bessel functions of the first kind are solutions to the differential equations of the form

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} - (z^2 + v^2)w = 0.$$

These functions are given by the sum:

$$I_v(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+v+1)} \left(\frac{z}{2}\right)^{2k+v} \tag{2.10}$$

where $\Gamma(z)$ is the Gamma function, as defined in Definition 2.13.

We can easily deduce from equations 2.9 and 2.10 that

$$I_v(z) = i^{-v} J_v(iz) \tag{2.11}$$

The modified Bessel functions of the first kind can also be expressed in terms of the hypergeometric functions as follows ([2], p. 377 (Result 9.6.47)):

$$I_v(z) = \frac{\left(\frac{z}{2}\right)^v}{\Gamma(v+1)} {}_0F_1\left(; v+1, \frac{z^2}{4}\right). \tag{2.12}$$

We will use equations 2.11 and 2.12 to derive results in chapter 4.

Chapter 3

Formulas for families of vvmfs on $\Gamma_0(2)$

In this chapter, we are looking at vvmfs evaluated at CM points in the upper half plane. We focus on the family that has already been looked at by Franc and Mason in [14] and [15]. The new ingredient that will allow us to compute these values are algebraic formulas for vvmfs.

The group $\Gamma_0(2)$ is the smallest index subgroup in Γ that has infinitely many 1-dimensional representations ([15]). We will study two families of unitary character χ for two reasons. One, we can get all 1-dimensional representations on $\Gamma_0(2)$ using these two families; and two, 1-dimensional representation χ on $\Gamma_0(2)$ helps us to get 3-dimensional representation ρ on Γ. Then using this ρ, we will give algebraic expressions for vvmfs on Γ.

Define

$$U = ST^{-1}S^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
$$V = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = TU^{-2} = TST^2S^{-1},$$

It is well known that $\Gamma_0(2)$ is generated by $U^2 = ST^{-2}S^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $V = TST^2S^{-1}$ ([15]).

Define a unitary character $\chi\colon \Gamma_0(2) \to \mathbb{C}^\times$ by setting

$$\chi \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = e^{2\pi i\alpha}, \qquad \chi \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = \epsilon,$$

where $\alpha \in [0, 1)$ and $\epsilon = \pm 1$. Let $M(\chi)$ denote the space of holomorphic forms transforming under χ, i.e. $M(\chi) = \bigoplus_{k\in\mathbb{Z}} M_k(\chi)$ where for each k, $M_k(\chi)$ represents the set of vvmfs of weight k over $\Gamma_0(2)$ holomorphic at 0 and ∞.

Note that if $\alpha \in \mathbb{Q}$, then χ is of finite order; When χ is of finite order, for all but finitely many α one can show that $\ker \chi$ is noncongruence ([15], Theorem 5).[1]

Since the formulae for $\epsilon = \pm 1$ will be different, we will treat these two cases differently.

[1]

Remark *In this thesis, we will restrict our computations to the case $\alpha \in (0, 1)$. This helps us keep the exponents of the eigenvalues of the induced representation on Γ in the range of $(0, 1)$, which further ensures that the induced representation on Γ is irreducible. Changing this range of exponents would make the induced representation reducible and this would entail adjusting dimension formulas for the corresponding spaces of modular forms. Also, the forms themselves would now possibly vanish at cusp or have a pole there. In addition to this, $\alpha \neq \frac{1}{3}, \frac{2}{3}$ and $\alpha \neq \frac{1}{2}$ by Lemma 4 and 18 respectively in [15]. For values of $\alpha = \frac{1}{n}$ where $n|3$, the induced representation ρ becomes reducible. Since our formulas make use of irreducible representations, we let $\alpha \in (0, 1)$ and $\alpha \neq \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$ to simplify this discussion.*

3.1 The case $\epsilon = 1$

Since $\epsilon = 1$ and $V = TU^{-2}$, we have that $\chi(U^2) = \chi(T)$. Given $f \in M(\chi)$, consider the vector valued function

$$\phi(f) = \begin{pmatrix} f \\ f|_S \\ f|_U \end{pmatrix}.$$

Notice that since $ST = T^{-1}U$ and $UT = VS$, we have

$$\phi(f)|_T = \begin{pmatrix} f|_T \\ f|_{ST} \\ f|_{UT} \end{pmatrix} = \begin{pmatrix} f|_T \\ f|_{T^{-1}U} \\ f|_{VS} \end{pmatrix} = \begin{pmatrix} \chi(T)f \\ \chi(T^{-1})f|_U \\ f|_S \end{pmatrix} = \begin{pmatrix} \chi(T) & 0 & 0 \\ 0 & 0 & \chi(T^{-1}) \\ 0 & 1 & 0 \end{pmatrix} \phi(f)$$

Similarly, since $\chi(V) = 1$, $V^2 = S^2 = -1$, and $US = V^{-1}U$, we have

$$\phi(f)|_S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \phi(f).$$

Below we will see that it is possible to diagonalize the action of T without changing the first coordinate f of the corresponding vvmf on Γ. This is important because our ${}_3F_2$ formulas in (2.8) have a diagonalized T. So, to apply this result, we need to diagonalize the action of T. To this end, define

$$\psi(f) = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & \sqrt{2} & \sqrt{2\chi(T)^{-1}} \\ 0 & -\sqrt{2} & \sqrt{2\chi(T)^{-1}} \end{pmatrix} \phi(f) = \frac{1}{2} \begin{pmatrix} 2f \\ \sqrt{2}f|_S + \sqrt{2\chi(T^{-1})}f|_U \\ -\sqrt{2}f|_S + \sqrt{2\chi(T^{-1})}f|_U \end{pmatrix}.$$

Then one finds that

$$\psi(f)|_T = \begin{pmatrix} \chi(T) & 0 & 0 \\ 0 & \chi(T)^{-1/2} & 0 \\ 0 & 0 & -\chi(T)^{-1/2} \end{pmatrix} \psi(f),$$

$$\psi(f)|_S = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \\ -\sqrt{2} & 1 & 1 \end{pmatrix} \psi(f)$$

Call this representation $\rho\colon \Gamma \to GL_3(\mathbb{C})$, i.e.,

$$\rho(T) = \begin{pmatrix} \chi(T) & 0 & 0 \\ 0 & \chi(T)^{-1/2} & 0 \\ 0 & 0 & -\chi(T)^{-1/2} \end{pmatrix},$$

$$\rho(S) = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \\ -\sqrt{2} & 1 & 1 \end{pmatrix}.$$

The exponents (as defined for equation 2.7) for $\rho(T)$ are α, $(2-\alpha)/2$ and $(1-\alpha)/2$. Notice that if α is contained in the interval $(0,1)$, then all three exponents are contained in the interval $(0,1)$. If ρ is irreducible then the minimal weight where $M_{k_0}(\rho) \neq 0$, and hence the minimal weight where $M_{k_0}(\chi) \neq 0$, is given by (2.7), which in this case is

$$k_0 = 4(\text{sum of exponents}) - 2 = 4(\alpha + \tfrac{2-\alpha}{2} + \tfrac{1-\alpha}{2}) - 2 = 4 + 2 - 2 = 4.$$

Using Theorem 2.6, we compute the dimensions of the modular spaces for different k. This is documented in Table 3.1.

k	$\dim M_k(\chi)$
< 4	0
4	1
6	1
8	2
10	2
$k \geq 12$	$\dim M_{k-12}(\chi) + 3$

Table 3.1: Dimensions of modular spaces for k.

Let a non-zero vvmf of weight 4 be denoted by F_χ. Then, F_χ is contained in $M_4(\chi)$ and by theorem 2.6, we have that a free basis for $M(\chi)$ as an $\mathbb{C}[E_4, E_6]$-module is F_χ, DF_χ and D^2F_χ where D is the modular derivative ([14]). In particular, F_χ spans $M_4(\chi, \mathbb{Q})$ and DF_χ spans $M_6(\chi, \mathbb{Q})$ where $M_k(\chi, \mathbb{Q})$ denotes the space of weight k vvmfs over $\Gamma_0(2)$ that transform under χ. Therefore, the Eisenstein series of weight 4 and 6 for χ are unique up to scaling by a complex number, and in fact we can write down an analytic family for them varying with α. Using (2.8), we define

$$F_\chi := \eta^8 j^{\frac{1-3\alpha}{3}} {}_3F_2\left(\frac{3\alpha-1}{3}, \alpha, \frac{3\alpha+1}{3}; \frac{3\alpha}{2}, \frac{3\alpha+1}{2}; \frac{1728}{j}\right). \tag{3.1}$$

Similarly, using (2.8) we can define 2nd and 3rd co-ordinate of vvmf with respect to ρ and Γ, namely, F_2 and F_3, respectively, as follows:

$$F_2 := \eta^8 j^{\frac{3\alpha-4}{6}} {}_3F_2\left(\frac{4-3\alpha}{6}, \frac{2-\alpha}{2}, \frac{8-3\alpha}{6}; \frac{3}{2}, \frac{4-3\alpha}{2}; \frac{1728}{j}\right), \tag{3.2}$$

$$F_3 := \eta^8 j^{\frac{3\alpha-1}{6}} {}_3F_2\left(\frac{1-3\alpha}{6}, \frac{1-\alpha}{2}, \frac{5-3\alpha}{6}; \frac{3-3\alpha}{2}, \frac{1}{2}; \frac{1728}{j}\right). \tag{3.3}$$

Then (F_χ, F_2, F_3) is a vvmf for ρ. The following theorem describes the coordinates of these vvmfs as algebraic expressions that are useful for evaluating these vvmfs at quadratic imaginary points of the upper half plane. It is the first of our main results. Our formulas reduce the calculations to evaluating the j-function at these points and the eta function at these points, which is a classical result using Chowla-Selberg Formula.

[2]

[2]

Theorem 3.1 *Let $x \in \overline{\mathbb{Q}(j)}$ denote a solution to the equation $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$. Then*

1. $F_\chi = \eta^8 \left(\frac{x^2}{512(1+\sqrt{1-x})^3}\right)^{\frac{3\alpha-1}{3}}$,

2. $F_2 = \eta^8 \left(\frac{64}{x}\right)^{\frac{3\alpha-4}{6}} \left(\frac{4}{(1-3\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha-1} - 1\right)\right)$,

3. $F_3 = \eta^8 \frac{4^{3\alpha}x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2} - 1\right) - \frac{x^{\frac{7-3\alpha}{3}}}{2\alpha(3\alpha+2)} \times$
$\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{1}{2-3\alpha}x + \frac{2(1-x)x}{2-3\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha}\right)\right) + \frac{2}{x^3} - \frac{3\alpha+2}{4x^2}\right]$.

Proof. Before beginning the proof, we will state some results that will be used below. The following cubic transformation for ${}_3F_2$ due to Bailey ([4], [21]):

$$ {}_3F_2\left[\begin{matrix} \frac{a}{3},\ \frac{a}{3}+\frac{1}{3},\ \frac{a}{3}+\frac{2}{3} \\ \frac{3}{4}+\frac{a}{2}+\frac{b}{2},\ \frac{3}{4}+\frac{a}{2}-\frac{b}{2} \end{matrix}; \frac{27x^2}{(4-x)^3}\right] = (1-\frac{x}{4})^a\ {}_3F_2\left[\begin{matrix} a,\ \frac{1}{4}+\frac{a}{2}-\frac{b}{2},\ \frac{1}{4}+\frac{a}{2}+\frac{b}{2} \\ \frac{1}{2}+a+b,\ \frac{1}{2}+a-b \end{matrix}; x\right]. \tag{3.4}$$

We shall also use the following formula from Vidunas' paper [28]:

$$ {}_2F_1\left[\begin{matrix} \frac{a}{2},\ \frac{a+1}{2} \\ a+1 \end{matrix}; x\right] = \left(\frac{1+\sqrt{1-x}}{2}\right)^{-a}. \tag{3.5}$$

The formula for the change of a parameter in denominator of ${}_3F_2$ by 1 [25]:

$$(\theta + b_1 - 1){}_3F_2\left[\begin{matrix} a_1,\ a_2,\ a_3 \\ b_1,\ b_2 \end{matrix}; x\right] = (b_1 - 1){}_3F_2\left[\begin{matrix} a_1,\ a_2,\ a_3 \\ b_1 - 1,\ b_2 \end{matrix}; x\right], \tag{3.6}$$

where θ is the theta operator defined in example 2.25 in chapter 2, and

$$\theta\left({}_3F_2\left[\begin{matrix} a_1,\ a_2,\ a_3 \\ b_1,\ b_2 \end{matrix}; x\right]\right) = x\sum_{k=0}^{\infty} \frac{(a_1)_{k+1}(a_2)_{k+1}(a_3)_{k+1}}{(b_1)_{k+1}(b_2)_{k+1}}\frac{x^k}{k!}. \tag{3.7}$$

Remark *Please note that the expression $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$ in Theorem 3.1 has been well known since the time of Klein to define modular forms on $\Gamma_0(2)$ and its conjugates. This equation arises for us via applications of Bailey's transformations, which are discussed in the proof of the theorem. Also, we are only interested in the solutions of equation $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$ that arise on $\Gamma_0(2)$. We can ignore the other solutions as those arise on conjugates of $\Gamma_0(2)$.*

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We shall also use the formula for change of a parameter in the numerator of ${}_2F_1$ by 1 due to Franc, Gannon and Mason [12]:

$$ {}_2F_1\left[\begin{matrix} a-1,\ b \\ c \end{matrix};x\right] = \left((1-x) - \frac{(a+b-c)x}{c-a} + \frac{1-x}{c-a}\theta\right){}_2F_1\left[\begin{matrix} a,\ b \\ c \end{matrix};x\right]. \tag{3.8} $$

The following fact about rising factorials will also be used in the calculations at some point:

$$ (x)_k = \frac{(x-1)_{k+1}}{x-1}. \tag{3.9} $$

We are now ready to begin the proof. We first examine the coordinate F_χ. By equation (3.1), we have that

$$ \frac{F_\chi}{\eta^8} = j^{\frac{1-3\alpha}{3}}{}_3F_2\left[\begin{matrix} \frac{3\alpha-1}{3},\ \alpha,\ \frac{3\alpha+1}{3} \\ \frac{3\alpha}{2},\ \frac{3\alpha+1}{2} \end{matrix};\frac{1728}{j}\right]. $$

Using Bailey's result in (3.4) and $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$, the above equation can be written as

$$ \frac{F_\chi}{\eta^8} = \left(\frac{x^2}{64(4-x)^3}\right)^{\frac{3\alpha-1}{3}}\left(1-\frac{x}{4}\right)^{3\alpha-1}{}_3F_2\left[\begin{matrix} 3\alpha-1,\ \frac{3\alpha}{2},\ \frac{3\alpha-1}{2} \\ 3\alpha-1,\ 3\alpha \end{matrix};x\right] $$

which simplifies to

$$ \frac{F_\chi}{\eta^8} = \left(\frac{x^2}{4096}\right)^{\frac{3\alpha-1}{3}}{}_2F_1\left[\begin{matrix} \frac{3\alpha}{2},\ \frac{3\alpha-1}{2} \\ 3\alpha \end{matrix};x\right]. $$

Applying formula (3.5), we get the desired result

$$ F_\chi = \eta^8\left(\frac{x^2}{512(1+\sqrt{1-x})^3}\right)^{\frac{3\alpha-1}{3}}. $$

Equation (3.2) gives F_2 as follows:

$$ \frac{F_2}{\eta^8} = j^{\frac{3\alpha-4}{6}}{}_3F_2\left[\begin{matrix} \frac{4-3\alpha}{6},\ \frac{2-\alpha}{2},\ \frac{8-3\alpha}{6} \\ \frac{3}{2},\ \frac{4-3\alpha}{2} \end{matrix};\frac{1728}{j}\right]. $$

Using the result in (3.4) and $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$, the above equation can be written as

$$\frac{F_2}{\eta^8} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{3\alpha-4}{6}} \left(1-\frac{x}{4}\right)^{\frac{4-3\alpha}{2}} {}_3F_2\left[\begin{matrix}\frac{4-3\alpha}{2},\ \frac{3-3\alpha}{2},\ 1\\ 2,\ 3-3\alpha\end{matrix};x\right]$$

which implies

$$\frac{F_2}{\eta^8} = \left(\frac{64}{x}\right)^{\frac{3\alpha-4}{3}} \sum_{k=0}^{\infty}\left(\frac{\left(\frac{4-3\alpha}{2}\right)_k\left(\frac{3-3\alpha}{2}\right)_k x^k}{(3-3\alpha)_k(k+1)!}\right).$$

Using formula (3.9), the above equation simplifies to

$$\frac{F_2}{\eta^8} = \left(\frac{64}{x}\right)^{\frac{3\alpha-4}{3}} \left(\frac{4}{(1-3\alpha)x}\left({}_2F_1\left[\begin{matrix}\frac{2-3\alpha}{2},\ \frac{1-3\alpha}{2}\\ 2-3\alpha\end{matrix};x\right]-1\right)\right).$$

Applying formula (3.5), we get the desired result

$$F_2 = \eta^8\left(\frac{64}{x}\right)^{\frac{3\alpha-4}{3}} \left(\frac{4}{(1-3\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha-1}-1\right)\right).$$

Using the same strategy as above, we have F_3 from equation (3.3) as follows:

$$\frac{F_3}{\eta^8} = j^{\frac{3\alpha-1}{6}}\,{}_3F_2\left[\begin{matrix}\frac{1-3\alpha}{6},\ \frac{1-\alpha}{2},\ \frac{5-3\alpha}{6}\\ \frac{3-3\alpha}{2},\ \frac{1}{2}\end{matrix};\frac{1728}{j}\right].$$

Using $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$ and formulas (3.6) and (3.7), the above equation simplifies to

$$\frac{F_3}{\eta^8} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{3\alpha-1}{6}} \times$$
$$\left({}_3F_2\left[\begin{matrix}\frac{1-3\alpha}{6},\ \frac{1-\alpha}{2},\ \frac{5-3\alpha}{6}\\ \frac{1-3\alpha}{2},\ \frac{1}{2}\end{matrix};\frac{27x^2}{(4-x)^3}\right] - \left(\frac{x^2(5-3\alpha)}{(4-x)^3}\right){}_3F_2\left[\begin{matrix}\frac{7-3\alpha}{6},\ \frac{3-\alpha}{2},\ \frac{11-3\alpha}{6}\\ \frac{5-3\alpha}{2},\ \frac{3}{2}\end{matrix};\frac{27x^2}{(4-x)^3}\right]\right).$$

We use Bailey's transformations from (3.4) for both hypergeometric functions in the right hand side of the above equation, which simplifies the equation to:

$$\frac{F_3}{\eta^8} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{3\alpha-1}{6}} \times \left(\left(1-\frac{x}{4}\right)^{\frac{1-3\alpha}{2}} {}_3F_2\left[\begin{matrix}\frac{1-3\alpha}{2}, \ 1, \ \frac{-3\alpha}{2} \\ 2, \ -3\alpha\end{matrix}; x\right] - \left(\frac{x^2(5-3\alpha)}{(4-x)^3}\right)\left(1-\frac{x}{4}\right)^{\frac{7-3\alpha}{2}} {}_3F_2\left[\begin{matrix}\frac{7-3\alpha}{2}, \ \frac{4-3\alpha}{2}, \ 2 \\ 4-3\alpha, \ 4\end{matrix}; x\right]\right).$$

Using formula (3.9) and (3.5), the above equation simplifies to:

$$\frac{F_3}{\eta^8} = \frac{4^{3\alpha}x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2} - 1\right) - (5-3\alpha)x^{\frac{7-3\alpha}{3}}2^{6\alpha-8}{}_3F_2\left[\begin{matrix}\frac{7-3\alpha}{2}, \ \frac{4-3\alpha}{2}, \ 2 \\ 4-3\alpha, \ 4\end{matrix}; x\right]$$

which implies

$$\frac{F_3}{\eta^8} = \frac{4^{3\alpha}x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2} - 1\right) - (5-3\alpha)x^{\frac{7-3\alpha}{3}}2^{6\alpha-8}\sum_{k=0}^{\infty}\frac{6\left(\frac{7-3\alpha}{2}\right)_k\left(\frac{4-3\alpha}{2}\right)_k x^k}{(4-3\alpha)_k(k+2)(k+3)k!}.$$

Using (3.9), we can further simplify the above equation to:

$$\frac{F_3}{\eta^8} = \frac{4^{3\alpha}x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2} - 1\right) - \frac{x^{\frac{7-3\alpha}{3}}}{2\alpha(3\alpha+2)}\left(\frac{d}{dx}\left(\frac{1}{x^2}{}_2F_1\left[\begin{matrix}\frac{1-3\alpha}{2}, \ \frac{-3\alpha-2}{2} \\ 1-3\alpha\end{matrix}; x\right]\right) + \frac{2}{x^3} - \frac{3\alpha+2}{4x^2}\right).$$

We apply (3.8) formula to the ${}_2F_1$ hypergeometric function, which gives us:

$$\frac{F_3}{\eta^8} = \frac{4^{3\alpha}x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2} - 1\right) - \frac{x^{\frac{7-3\alpha}{3}}}{2\alpha(3\alpha+2)}\times$$
$$\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{1}{2-3\alpha}x + \frac{2(1-x)x}{2-3\alpha}\frac{d}{dx}\right)\left({}_2F_1\left[\begin{matrix}\frac{1-3\alpha}{2}, \frac{-3\alpha}{2} \\ 1-3\alpha\end{matrix}; x\right]\right)\right)\right) + \frac{2}{x^3} - \frac{3\alpha+2}{4x^2}\right].$$

Applying formula (3.5), we get the desired result:

$$\frac{F_3}{\eta^8} = \frac{4^{3\alpha}x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2} - 1\right) - \frac{x^{\frac{7-3\alpha}{3}}}{2\alpha(3\alpha+2)}\times$$
$$\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{1}{2-3\alpha}x + \frac{2(1-x)x}{2-3\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha}\right)\right) + \frac{2}{x^3} - \frac{3\alpha+2}{4x^2}\right].$$

This completes the proof. □

Now, we will demonstrate how we use the above formulae to compute exact formulas for CM values of these vvmfs. We shall evaluate F_χ, F_2 and F_3 at $\tau = i$.

Example 3.2. Compute F_χ, F_2 and F_3 at $\tau = i$.

Proof. Since $j(i) = 1728$ ([10], pg. 7), equation (3.1) simplifies to:

$$F_\chi = \eta(i)^8(1728)^{\frac{1-3\alpha}{3}}\,{}_3F_2\left(\frac{3\alpha-1}{3}, \alpha, \frac{3\alpha+1}{3}; \frac{3\alpha}{2}, \frac{3\alpha+1}{2}; 1\right),$$

Maier computed values of general hypergeometric functions in [20]. The hypergeometric function on the right hand side of the above equation can be computed using Theorem 7.1 in [20]. Building on Maier's work, Milgram in [23] used Maier's index $l = 0$ and computed a particular value of this theorem for ${}_3F_2(\cdots|1)$ as follows:

$${}_3F_2\left(2a, 2a-\frac{1}{3}, 2a+\frac{1}{3}; 3a, 3a+\frac{1}{2}; 1\right) = \left(\frac{3}{2}\right)^{6a-1}. \tag{3.10}$$

Using $a = \frac{\alpha}{2}$ in (3.10), we find that

$$ {}_3F_2\left(\frac{3\alpha-1}{3}, \alpha, \frac{3\alpha+1}{3}; \frac{3\alpha}{2}, \frac{3\alpha+1}{2}; 1\right) = \left(\frac{3}{2}\right)^{3\alpha-1}. $$

Therefore,

$$ F_\chi = \eta(i)^8(1728)^{\frac{1-3\alpha}{3}} \left(\frac{3}{2}\right)^{3\alpha-1} = \eta(i)^8(2)^{3-9\alpha}. $$

We can compute the value of Dedekind eta function for i using Chowla-Selberg formula ([8]). This value is known to be as follows:

$$ \eta(i) = \frac{\Gamma(\frac{1}{4})}{2\pi^{\frac{3}{4}}}. \tag{3.11} $$

Using this value of the eta function for $\tau = i$, we deduce that

$$ F_\chi = \frac{\Gamma\left(\frac{1}{4}\right)^8}{\pi^6 2^{5+9\alpha}}. $$

Next, we work on F_2. Using $j(i) = 1728$, equation (3.2) simplifies to:

$$ F_2 = \eta(i)^8(1728)^{\frac{3\alpha-4}{6}} {}_3F_2\left(\frac{4-3\alpha}{6}, \frac{2-\alpha}{2}, \frac{8-3\alpha}{6}; \frac{3}{2}, \frac{4-3\alpha}{2}; 1\right), $$

The hypergeometric function on the right hand side of the above equation can be computed using Theorem 7.3 in [20]. Particular value of this theorem for ${}_3F_2(\cdots|1)$ is given in [23] as follows:

$$ {}_3F_2\left(a, a+\frac{1}{3}, a+\frac{2}{3}; 3a, \frac{3}{2}; 1\right) = -(3)^{3a}(8^{1-2a}-1)/(12a-6). \tag{3.12} $$

Using $a = \frac{4-3\alpha}{6}$ in (3.12), we find that

$$ {}_3F_2\left(\frac{4-3\alpha}{6}, \frac{2-\alpha}{2}, \frac{8-3\alpha}{6}; \frac{3}{2}, \frac{4-3\alpha}{2}; 1\right) = -(3)^{\frac{4-3\alpha}{2}}(2^{3\alpha-1}-1)/(2-6\alpha). $$

Therefore,

$$F_2 = -\eta(i)^8 (1728)^{\frac{3\alpha-4}{6}} \frac{(3)^{\frac{4-3\alpha}{2}} (2^{3\alpha-1}-1)}{(2-6\alpha)} = -\eta(i)^8 \left(\frac{2^{6\alpha-5}-1}{2-6\alpha} \right).$$

Using the value of $\eta(i)$ from (3.11), we get that F_2 for $\tau = i$ is given by:

$$F_2 = -\frac{\Gamma\left(\frac{1}{4}\right)^8 (2^{6\alpha-5}-1)}{\pi^6 2^9 (1-3\alpha)}.$$

Similarly, we compute F_3. For $j(i) = 1728$, equation (3.3) simplifies to:

$$F_3 = \eta(i)^8 (1728)^{\frac{3\alpha-1}{6}} {}_3F_2 \left(\frac{1-3\alpha}{6}, \frac{1-\alpha}{2}, \frac{5-3\alpha}{6}; \frac{3-3\alpha}{2}, \frac{1}{2}; 1 \right).$$

The hypergeometric function on the right hand side of the above equation can be computed using result 1.2 from [20]. Milgram in [23] computed the value of ${}_3F_2(\cdots|1)$ as follows:

$$ {}_3F_2 \left(a, a+\frac{1}{3}, a+\frac{2}{3}; 3a+1, \frac{1}{2}; 1 \right) = \frac{3^{3a}(1+4^{-3a})}{2}. \tag{3.13}$$

.

Using $a = \frac{1-3\alpha}{6}$ in (3.13), we find that

$$ {}_3F_2 \left(\frac{1-3\alpha}{6}, \frac{1-\alpha}{2}, \frac{5-3\alpha}{6}; \frac{3-3\alpha}{2}, \frac{1}{2}; 1 \right) = \frac{3^{\frac{1-3\alpha}{2}}(1+4^{\frac{3\alpha-1}{2}})}{2}.$$

Therefore,

$$F_3 = \eta(i)^8 (1728)^{\frac{3\alpha-1}{6}} \frac{3^{\frac{1-3\alpha}{2}}(1+4^{\frac{3\alpha-1}{2}})}{2} = \eta(i)^8 2^{3\alpha-2}(1+2^{3\alpha-1}).$$

Using the value of $\eta(i)$ from (3.11), we get that F_3 for $\tau = i$ is given by:

$$F_3 = \frac{\Gamma\left(\frac{1}{4}\right)^8 2^{3\alpha-10}(1+2^{3\alpha-1})}{\pi^6}.$$

□

Combining the results from the above computations, we find that the value of the vvmf over ρ and Γ evaluated at $\tau = i$ is:

$$F = \frac{\Gamma\left(\frac{1}{4}\right)^8}{2^5\pi^6}\left(2^{-9\alpha}, \frac{1-2^{6\alpha-5}}{2^4(1-3\alpha)}, 2^{3\alpha-5}(1+2^{3\alpha-1})\right)^t$$

where t denotes transpose of the matrix.

3.2 The case $\epsilon = -1$

The details for these cases are similar to the first case, but we include them for completeness.

For $\epsilon = -1$, we have $\chi(U^2) = -\chi(T)$ because $V = TU^{-2}$ and $\chi(V) = -1$. Given $f \in M(\chi)$, consider the vector valued function

$$\phi(f) = \begin{pmatrix} f \\ f|_S \\ f|_U \end{pmatrix}.$$

Since $ST = T^{-1}U$ and $UT = VS$, we have

$$\phi(f)|_T = \begin{pmatrix} f|_T \\ f|_{ST} \\ f|_{UT} \end{pmatrix} = \begin{pmatrix} f|_T \\ f|_{T^{-1}U} \\ f|_{VS} \end{pmatrix} = \begin{pmatrix} \chi(T)f \\ \chi(T^{-1})f|_U \\ -f|_S \end{pmatrix} = \begin{pmatrix} \chi(T) & 0 & 0 \\ 0 & 0 & \chi(T^{-1}) \\ 0 & -1 & 0 \end{pmatrix} \phi(f)$$

Similarly, since $\chi(V) = -1$, $V^2 = S^2 = -1$, and $US = V^{-1}U$, we have

$$\phi(f)|_S = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \phi(f)$$

As we did in the first case, we now diagonalize the action of T. We define

$$\psi(f) = \frac{1}{2} \begin{pmatrix} 2\chi(T) & 0 & 0 \\ 0 & i\sqrt{\chi(T^{-1})} & \chi(T^{-1}) \\ 0 & -i\sqrt{\chi(T^{-1})} & \chi(T^{-1}) \end{pmatrix} \phi(f) = \frac{1}{2} \begin{pmatrix} 2\chi(T)f \\ i\sqrt{\chi(T^{-1})}f|_S + \chi(T^{-1})f|_U \\ -i\sqrt{\chi(T^{-1})}f|_S + \chi(T^{-1})f|_U \end{pmatrix}.$$

Then one finds that

$$\psi(f)|_T = \begin{pmatrix} \chi(T) & 0 & 0 \\ 0 & i\chi(T)^{-1/2} & 0 \\ 0 & 0 & -i\chi(T)^{-1/2} \end{pmatrix} \psi(f),$$

$$\psi(f)|_S = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix} \psi(f)$$

Call this representation $\rho\colon \Gamma \to GL_3(\mathbb{C})$ i.e.

$$\rho(T) = \begin{pmatrix} \chi(T) & 0 & 0 \\ 0 & i\chi(T)^{-1/2} & 0 \\ 0 & 0 & -i\chi(T)^{-1/2} \end{pmatrix},$$

$$\rho(S) = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & -1 & 1 \\ \sqrt{2} & 1 & -1 \end{pmatrix}.$$

The exponents (as defined for equation 2.7) for $\rho(T)$ are α, $(5-2\alpha)/4$ and $(3-2\alpha)/4$. Notice that if α is contained in the interval $(0,1)$, then two of the three exponents are contained in the interval $(0,1)$, namely, α and $(3-2\alpha)/4$. To accommodate the third exponent $(5-2\alpha)/4$, we will consider two cases. One, when $\alpha \in (0,1/2)$ and two, when $\alpha \in (1/2,1)$.

3.2.1 Case $\alpha \in (0, 1/2)$

For this case, our exponents are α, $\frac{5-2\alpha}{4} - 1 = \frac{1-2\alpha}{4}$ and $\frac{3-2\alpha}{4}$. Using equation (2.7), we find that the minimal weight where $M_{k_0}(\chi) \neq 0$ in this case is

$$k_0 = 4(\text{sum of exponents}) - 2 = 4(\alpha + \tfrac{1-2\alpha}{4} + \tfrac{3-2\alpha}{4}) - 2 = 4(\tfrac{4}{4}) - 2 = 2.$$

We know by equation (2.8) that the coordinates of non-zero vvmf of lowest weight for ρ are given in terms of generalized hypergeometric series by the following equations:

$$F_\chi := \eta^4 j^{\frac{1-6\alpha}{6}} {}_3F_2\left(\frac{6\alpha-1}{6}, \frac{6\alpha+1}{6}, \alpha+\frac{1}{2}; \frac{6\alpha+3}{4}, \frac{6\alpha+1}{4}; \frac{1728}{j}\right) \tag{3.14}$$

$$F_2 := \eta^4 j^{\frac{6\alpha-1}{12}} {}_3F_2\left(\frac{1-6\alpha}{12}, \frac{5-6\alpha}{12}, \frac{3-2\alpha}{4}; \frac{1}{2}, \frac{5-6\alpha}{4}; \frac{1728}{j}\right) \tag{3.15}$$

$$F_3 := \eta^4 j^{\frac{6\alpha-7}{12}} {}_3F_2\left(\frac{7-6\alpha}{12}, \frac{11-6\alpha}{12}, \frac{5-2\alpha}{4}; \frac{7-6\alpha}{4}, \frac{3}{2}; \frac{1728}{j}\right) \tag{3.16}$$

Then (F_χ, F_2, F_3) is a vvmf for ρ. The following result is useful for evaluating the coordinates of these vvmfs at quadratic imaginary points of the upper half plane. It is the second of our main results.

Theorem 3.3 *Let $x \in \overline{\mathbb{Q}(j)}$ denote a solution to the equation $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$. Then*

1. $F_\chi = \eta^4 \left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-6\alpha}{6}}$,
2. $F_2 = \eta^4 \frac{2^{6\alpha+2} x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)} \left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}} - 1\right) - \frac{x^{\frac{13-6\alpha}{6}} 2^{6\alpha+2} 3}{(1+6\alpha)(6\alpha+5)} \times$
$\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{2}{3-6\alpha}x + \frac{4(1-x)x}{3-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{1+6\alpha}{2}}\right)\right) + \frac{2}{x^3} - \frac{6\alpha+5}{8x^2}\right]$,
3. $F_3 = \eta^4 \left(\frac{64}{x}\right)^{\frac{6\alpha-7}{6}} \left(\frac{8}{(1-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{1-6\alpha}{2}\right)} - 1\right)\right)$.

Proof. To prove this theorem, we will use the formulas mentioned in equations (3.4) - (3.9).

First, we examine the coordinate F_χ. By equation (3.14), we have that

$$\frac{F_\chi}{\eta^4} = j^{\frac{1-6\alpha}{6}} {}_3F_2\left[\begin{matrix} \frac{6\alpha-1}{6}, & \frac{6\alpha+1}{6}, & \alpha+\frac{1}{2} \\ \frac{6\alpha+3}{4}, & \frac{6\alpha+1}{4} & \end{matrix}; \frac{1728}{j}\right].$$

Using Bailey's result in (3.4) and $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$, the above equation can be written as

$$\frac{F_\chi}{\eta^4} = \left(\frac{x^2}{64(4-x)^3}\right)^{\frac{6\alpha-1}{6}} \left(1-\frac{x}{4}\right)^{\frac{6\alpha-1}{2}} {}_3F_2\left[\begin{matrix} \frac{6\alpha-1}{2}, & \frac{6\alpha-1}{4}, & \frac{6\alpha+1}{4} \\ \frac{6\alpha+1}{2}, & \frac{6\alpha-1}{2} & \end{matrix}; x\right]$$

which simplifies to

$$\frac{F_\chi}{\eta^4} = \left(\frac{64}{x}\right)^{1-6\alpha} {}_2F_1\left[\begin{matrix} \frac{6\alpha-1}{4}, & \frac{6\alpha+1}{4} \\ \frac{6\alpha+1}{2} & \end{matrix}; x\right].$$

Applying formula (3.5), we get the desired result

$$F_\chi = \eta^4 \left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-6\alpha}{6}}.$$

Equation (3.15) gives F_2 as follows:

$$\frac{F_2}{\eta^4} = j^{\frac{6\alpha-1}{12}} {}_3F_2\left[\begin{matrix} \frac{1-6\alpha}{12}, & \frac{5-6\alpha}{12}, & \frac{3-2\alpha}{4} \\ \frac{1}{2}, & \frac{5-6\alpha}{4} & \end{matrix}; \frac{1728}{j}\right].$$

Using $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$ and formulas (3.6) and (3.7), the above equation simplifies to

$$\begin{aligned}\frac{F_2}{\eta^4} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{6\alpha-1}{12}} \times \Bigg(& {}_3F_2\left[\begin{matrix} \frac{1-6\alpha}{12}, & \frac{5-6\alpha}{12}, & \frac{3-2\alpha}{4} \\ \frac{1-6\alpha}{4}, & \frac{1}{2} & \end{matrix}; \frac{27x^2}{(4-x)^3}\right] \\ & - \left(\frac{3x^2(3-2\alpha)}{2(4-x)^3}\right) {}_3F_2\left[\begin{matrix} \frac{13-6\alpha}{12}, & \frac{17-6\alpha}{12}, & \frac{7-2\alpha}{4} \\ \frac{9-6\alpha}{4}, & \frac{3}{2} & \end{matrix}; \frac{27x^2}{(4-x)^3}\right]\Bigg).\end{aligned}$$

We use Bailey's transformations from (3.4) for both hypergeometric functions in the right hand side of the above equation, which simplifies the equation to:

$$\frac{F_2}{\eta^4} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{6\alpha-1}{12}} \times \left(\left(1-\frac{x}{4}\right)^{\frac{1-6\alpha}{4}} {}_3F_2\left[\begin{matrix}\frac{1-6\alpha}{4},\ 1,\ \frac{-1-6\alpha}{4}\\ 2,\ \frac{-1-6\alpha}{2}\end{matrix};x\right]\right.$$
$$\left. - \left(\frac{3x^2(3-2\alpha)}{2(4-x)^3}\right)\left(1-\frac{x}{4}\right)^{\frac{13-6\alpha}{4}} {}_3F_2\left[\begin{matrix}\frac{13-6\alpha}{4},\ \frac{7-6\alpha}{4},\ 2\\ \frac{7-6\alpha}{2},\ 4\end{matrix};x\right]\right).$$

Using formula (3.9) and (3.5), the above equation simplifies to:

$$\frac{F_2}{\eta^4} = \frac{2^{6\alpha+2}x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}} - 1\right)$$
$$- 3(3-2\alpha)x^{\frac{13-6\alpha}{6}}2^{6\alpha-8}{}_3F_2\left[\begin{matrix}\frac{13-6\alpha}{4},\ \frac{7-6\alpha}{4},\ 2\\ \frac{7-6\alpha}{2},\ 4\end{matrix};x\right]$$

which implies

$$\frac{F_2}{\eta^4} = \frac{2^{6\alpha+2}x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}} - 1\right)$$
$$- 3(3-2\alpha)x^{\frac{13-6\alpha}{6}}2^{6\alpha-8}\sum_{k=0}^{\infty}\frac{6\left(\frac{13-6\alpha}{4}\right)_k\left(\frac{7-6\alpha}{4}\right)_k x^k}{\left(\frac{7-6\alpha}{2}\right)_k(k+2)(k+3)k!}.$$

Using (3.9), we can further simplify the above equation to:

$$\frac{F_2}{\eta^4} = \frac{2^{6\alpha+2}x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}} - 1\right)$$
$$- \frac{x^{\frac{13-6\alpha}{6}}2^{6\alpha+2}3}{(1+6\alpha)(6\alpha+5)}\left(\frac{d}{dx}\left(\frac{1}{x^2}{}_2F_1\left[\begin{matrix}\frac{1-6\alpha}{4},\ \frac{-6\alpha-5}{4}\\ \frac{1-6\alpha}{2}\end{matrix};x\right]\right) + \frac{2}{x^3} - \frac{6\alpha+5}{8x^2}\right).$$

We apply (3.8) formula to the ${}_2F_1$ hypergeometric function, which gives us:

$$\frac{F_2}{\eta^4} = \frac{2^{6\alpha+2}x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}}-1\right) - \frac{x^{\frac{13-6\alpha}{6}}2^{6\alpha+2}3}{(1+6\alpha)(6\alpha+5)}\times \left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x)+\frac{2}{3-6\alpha}x+\frac{4(1-x)x}{3-6\alpha}\frac{d}{dx}\right)\left({}_2F_1\left[\begin{matrix}\frac{1-6\alpha}{4}, & \frac{-1-6\alpha}{4} \\ \frac{1-6\alpha}{2} & \end{matrix};x\right]\right)\right)\right) + \frac{2}{x^3} - \frac{6\alpha+5}{8x^2}\right].$$

Applying formula (3.5), we get the desired result:

$$\frac{F_2}{\eta^4} = \frac{2^{6\alpha+2}x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}}-1\right) - \frac{x^{\frac{13-6\alpha}{6}}2^{6\alpha+2}3}{(1+6\alpha)(6\alpha+5)}\times \left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x)+\frac{2}{3-6\alpha}x+\frac{4(1-x)x}{3-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{1+6\alpha}{2}}\right)\right) + \frac{2}{x^3} - \frac{6\alpha+5}{8x^2}\right].$$

Using the same strategy as above, we have F_3 from equation (3.16) as follows:

$$\frac{F_3}{\eta^4} = j^{\frac{6\alpha-7}{12}}\,{}_3F_2\left[\begin{matrix}\frac{7-6\alpha}{12}, & \frac{11-6\alpha}{12}, & \frac{5-2\alpha}{4} \\ & \frac{7-6\alpha}{4}, & \frac{3}{2}\end{matrix};\frac{1728}{j}\right].$$

Using the result in (3.4) and $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$, the above equation can be written as

$$\frac{F_3}{\eta^4} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{6\alpha-7}{12}}\left(1-\frac{x}{4}\right)^{\frac{7-6\alpha}{4}}{}_3F_2\left[\begin{matrix}\frac{7-6\alpha}{4}, & 1, & \frac{5-6\alpha}{4} \\ & \frac{5-6\alpha}{2}, & 2\end{matrix};x\right]$$

which implies

$$\frac{F_3}{\eta^4} = \left(\frac{64}{x}\right)^{\frac{6\alpha-7}{6}}\sum_{k=0}^{\infty}\left(\frac{\left(\frac{7-6\alpha}{4}\right)_k\left(\frac{5-6\alpha}{4}\right)_k\ x^k}{\left(\frac{5-6\alpha}{2}\right)_k(k+1)!}\right).$$

Using formula (3.9), the above equation simplifies to

$$\frac{F_3}{\eta^4} = \left(\frac{64}{x}\right)^{\frac{6\alpha-7}{6}} \left(\frac{8}{(1-6\alpha)x}\left({}_2F_1\left[\begin{matrix}\frac{3-6\alpha}{4}, \frac{1-6\alpha}{4} \\ \frac{3-6\alpha}{2}\end{matrix}; x\right] - 1\right)\right).$$

Applying formula (3.5), we get the desired result

$$F_3 = \eta^4 \left(\frac{64}{x}\right)^{\frac{6\alpha-7}{6}} \left(\frac{8}{(1-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{1-6\alpha}{2}\right)} - 1\right)\right).$$

This completes the proof. □

Now, we will demonstrate how we can compute exact formulas for the CM values of these vvmfs. We shall evaluate F_χ, F_2 and F_3 at $\tau = i$.

Example 3.4. Compute F_χ, F_2 and F_3 at $\tau = i$.

Proof. Since $j(i) = 1728$ ([10], pg. 7), equation (3.14) simplifies to:

$$F_\chi = \eta(i)^4 (1728)^{\frac{1-6\alpha}{6}} {}_3F_2\left(\frac{6\alpha-1}{6}, \frac{6\alpha+1}{6}, \alpha+\frac{1}{2}; \frac{6\alpha+3}{4}, \frac{6\alpha+1}{4}; 1\right).$$

We compute the hypergeometric function on the right hand side of the above equation by (3.10). Using $a = \frac{6\alpha+1}{12}$ in equation (3.10) we find that

$${}_3F_2\left(\frac{6\alpha-1}{6}, \frac{6\alpha+1}{6}, \alpha+\frac{1}{2}; \frac{6\alpha+3}{4}, \frac{6\alpha+1}{4}; 1\right) = \left(\frac{3}{2}\right)^{\frac{6\alpha-1}{2}}.$$

Therefore,

$$F_\chi = \eta(i)^4 (1728)^{\frac{1-6\alpha}{6}} \left(\frac{3}{2}\right)^{\frac{6\alpha-1}{2}} = \eta(i)^4 (2)^{\frac{3-18\alpha}{2}}.$$

Using the value of the eta function at i from equation (3.11), we deduce that for $\tau = i$,

$$F_\chi = \frac{\Gamma\left(\frac{1}{4}\right)^4}{\pi^3 2^{\frac{5+18\alpha}{2}}}.$$

Next, we work on F_2. Using $j(i) = 1728$, equation (3.15) simplifies to:

$$F_2 = \eta(i)^4(1728)^{\frac{6\alpha-1}{12}}\,{}_3F_2\left(\frac{1-6\alpha}{12}, \frac{5-6\alpha}{12}, \frac{3-2\alpha}{4}; \frac{1}{2}, \frac{5-6\alpha}{4}; 1\right).$$

We compute the hypergeometric function on the right hand side of the above equation by (3.13). Using $a = \frac{1-6\alpha}{12}$ in equation (3.13), we find that

$${}_3F_2\left(\frac{1-6\alpha}{12}, \frac{5-6\alpha}{12}, \frac{3-2\alpha}{4}; \frac{1}{2}, \frac{5-6\alpha}{4}; 1\right) = \frac{3^{\frac{1-6\alpha}{4}}(1+4^{\frac{6\alpha-1}{4}})}{2}.$$

Therefore,

$$F_2 = \eta(i)^4(1728)^{\frac{6\alpha-1}{12}}\frac{3^{\frac{1-6\alpha}{4}}(1+4^{\frac{6\alpha-1}{4}})}{2} = \eta(i)^4 2^{\frac{6\alpha-3}{2}}\left(1+2^{\frac{6\alpha-1}{2}}\right).$$

Using value of $\eta(i)$ from (3.11), we get that F_2 for $\tau = i$ is given by:

$$F_2 = \frac{\Gamma\left(\frac{1}{4}\right)^4 2^{\frac{6\alpha-11}{2}}(1+2^{\frac{6\alpha-1}{2}})}{\pi^3}.$$

Similarly, we compute F_3. For $j(i) = 1728$, equation (3.19) simplifies to:

$$F_3 = \eta(i)^4(1728)^{\frac{6\alpha-7}{12}}\,{}_3F_2\left(\frac{7-6\alpha}{12}, \frac{11-6\alpha}{12}, \frac{5-2\alpha}{4}; \frac{7-6\alpha}{4}, \frac{3}{2}; 1\right).$$

Hypergeometric functions on the right hand sides of the above equation can be computed by (3.12). Using $a = \frac{7-6\alpha}{12}$ in (3.12), we find that

$${}_3F_2\left(\frac{7-6\alpha}{12}, \frac{11-6\alpha}{12}, \frac{5-2\alpha}{4}; \frac{7-6\alpha}{4}, \frac{3}{2}; 1\right) = \frac{-(3)^{\frac{7-6\alpha}{4}}(8^{\frac{6\alpha-1}{6}}-1)}{1-6\alpha}.$$

Therefore,

$$F_3 = \eta(i)^4(1728)^{\frac{6\alpha-7}{12}}\frac{-(3)^{\frac{7-6\alpha}{4}}(8^{\frac{6\alpha-1}{6}}-1)}{1-6\alpha} = \eta(i)^4 2^{\frac{6\alpha-7}{2}}\frac{(8^{\frac{6\alpha-1}{6}}-1)}{6\alpha-1}.$$

Using value of $\eta(i)$ from (3.11), we get that F_3 for $\tau = i$ is given by:

$$F_3 = \frac{\Gamma\left(\frac{1}{4}\right)^4 2^{\frac{6\alpha-15}{2}}(2^{\frac{6\alpha-1}{2}}-1)}{\pi^3(6\alpha-1)}.$$

□

Combining the results from the above computations, we find that the value of the vvmf over ρ and Γ evaluated at $\tau = i$ is:

$$F = \frac{\Gamma\left(\frac{1}{4}\right)^4}{2^{5/2}\pi^3}\left(2^{-9\alpha}, 2^{3\alpha-3}(1+2^{\frac{6\alpha-1}{2}}), \frac{2^{3\alpha-5}(2^{\frac{6\alpha-1}{2}}-1)}{(6\alpha-1)}\right)^t$$

where t denotes the transpose of the matrix.

3.2.2 Case $\alpha \in (1/2, 1)$

The exponents (as defined in equation 2.7) for $\rho(T)$ for this case are α, $\frac{5-2\alpha}{4}$ and $\frac{3-2\alpha}{4}$. Using equation (2.7), we find that the minimal weight when $M_{k_0}(\chi) \neq 0$ in this case is

$$k_0 = 4(\text{sum of exponents}) - 2 = 4(\alpha + \tfrac{5-2\alpha}{4} + \tfrac{3-2\alpha}{4}) - 2 = 4(\tfrac{8}{4}) - 2 = 6.$$

As mentioned in the case for $\epsilon = 1$, Eisenstein series for weight 6 are unique up to scaling by a complex number, and we know from equation (2.8) that the coordinates of non-zero vvmf of lowest weight for ρ are given in terms of generalized hypergeometric series by the following equations:

$$F_\chi := \eta^{12} j^{\frac{1-2\alpha}{2}} {}_3F_2\left(\frac{2\alpha-1}{2}, \frac{6\alpha-1}{6}, \frac{6\alpha+1}{6}; \frac{6\alpha-1}{4}, \frac{6\alpha+1}{4}; \frac{1728}{j}\right) \tag{3.17}$$

$$F_2 := \eta^{12} j^{\frac{2\alpha-3}{4}} {}_3F_2\left(\frac{3-2\alpha}{4}, \frac{13-6\alpha}{12}, \frac{17-6\alpha}{12}; \frac{3}{2}, \frac{9-6\alpha}{4}; \frac{1728}{j}\right) \tag{3.18}$$

$$F_3 := \eta^{12} j^{\frac{2\alpha-1}{4}} {}_3F_2\left(\frac{1-2\alpha}{4}, \frac{7-6\alpha}{12}, \frac{11-6\alpha}{12}; \frac{7-6\alpha}{4}, \frac{1}{2}; \frac{1728}{j}\right) \tag{3.19}$$

Then (F_χ, F_2, F_3) is a vvmf for ρ. The following result is useful for evaluating the coordinates of these vvmfs at quadratic imaginary points of the upper half plane. It is the third of our main results.

Theorem 3.5 *Let $x \in \overline{\mathbb{Q}(j)}$ denote a solution to the equation $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$. Then*

1. $F_\chi = \eta^{12}\left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-2\alpha}{2}}$,
2. $F_2 = \eta^{12}\left(\frac{64}{x}\right)^{\frac{2\alpha-3}{2}}\left(\frac{8}{(3-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{3-6\alpha}{2}\right)} - 1\right)\right)$,
3. $F_3 = \eta^{12}\frac{2^{6\alpha}x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}} - 1\right) - \frac{x^{\frac{5-2\alpha}{2}}2^{6\alpha}}{(\alpha-1)(1+2\alpha)}\times$
$\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{2}{5-6\alpha}x + \frac{4(1-x)x}{5-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha-1}{2}}\right)\right) + \frac{2}{x^3} - \frac{6\alpha+3}{8x^2}\right]$.

Proof. To prove this theorem, we will use formulas mentioned in equations (3.4) - (3.9).

First, we examine the coordinate F_χ. By equation (3.17), we have that

$$\frac{F_\chi}{\eta^{12}} = j^{\frac{1-2\alpha}{2}} {}_3F_2\left[\begin{matrix}\frac{2\alpha-1}{2}, \frac{6\alpha-1}{6}, \frac{6\alpha+1}{6} \\ \frac{6\alpha-1}{4}, \frac{6\alpha+1}{4}\end{matrix}; \frac{1728}{j}\right].$$

Using Bailey's result in (3.4) and $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$, the above equation can be written as

$$\frac{F_\chi}{\eta^{12}} = \left(\frac{x^2}{64(4-x)^3}\right)^{\frac{2\alpha-1}{2}}\left(1-\frac{x}{4}\right)^{\frac{6\alpha-3}{2}} {}_3F_2\left[\begin{matrix}\frac{6\alpha-3}{2}, \frac{6\alpha-1}{4}, \frac{6\alpha-3}{4} \\ \frac{6\alpha-3}{2}, \frac{6\alpha-1}{2}\end{matrix}; x\right]$$

which simplifies to

$$\frac{F_\chi}{\eta^{12}} = \left(\frac{64}{x}\right)^{1-2\alpha} {}_2F_1\left[\begin{matrix}\frac{6\alpha-1}{4}, \frac{6\alpha-3}{4} \\ \frac{6\alpha-1}{2}\end{matrix}; x\right].$$

Applying formula (3.5), we get the desired result

$$F_\chi = \eta^{12}\left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-2\alpha}{2}}.$$

Equation (3.18) gives F_2 as follows:

$$\frac{F_2}{\eta^{12}} = j^{\frac{2\alpha-3}{4}}\, {}_3F_2\left[\begin{matrix} \frac{3-2\alpha}{4}, & \frac{13-6\alpha}{12}, & \frac{17-6\alpha}{12} \\ & \frac{3}{2}, & \frac{9-6\alpha}{4} \end{matrix}; \frac{1728}{j}\right].$$

Using the result in (3.4) and $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$, the above equation can be written as

$$\frac{F_2}{\eta^{12}} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{2\alpha-3}{4}} \left(1-\frac{x}{4}\right)^{\frac{9-6\alpha}{4}} {}_3F_2\left[\begin{matrix} \frac{9-6\alpha}{4}, & \frac{7-6\alpha}{4}, & 1 \\ 2, & \frac{7-6\alpha}{2} & \end{matrix}; x\right]$$

which implies

$$\frac{F_2}{\eta^{12}} = \left(\frac{64}{x}\right)^{\frac{2\alpha-3}{2}} \sum_{k=0}^{\infty} \left(\frac{\left(\frac{9-6\alpha}{4}\right)_k \left(\frac{7-6\alpha}{4}\right)_k x^k}{\left(\frac{7-6\alpha}{2}\right)_k (k+1)!}\right).$$

Using formula (3.9), the above equation simplifies to

$$\frac{F_2}{\eta^{12}} = \left(\frac{64}{x}\right)^{\frac{2\alpha-3}{2}} \left(\frac{8}{(3-6\alpha)x}\left({}_2F_1\left[\begin{matrix} \frac{5-6\alpha}{4}, & \frac{3-6\alpha}{4} \\ & \frac{5-6\alpha}{2} \end{matrix}; x\right] - 1\right)\right).$$

Applying formula (3.5), we get the desired result

$$F_2 = \eta^{12}\left(\frac{64}{x}\right)^{\frac{2\alpha-3}{2}} \left(\frac{8}{(3-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{3-6\alpha}{2}\right)} - 1\right)\right).$$

Using the same strategy as above, we have F_3 from equation (3.19) as follows:

$$\frac{F_3}{\eta^{12}} = j^{\frac{2\alpha-1}{4}}\, {}_3F_2\left[\begin{matrix} \frac{1-2\alpha}{4}, & \frac{7-6\alpha}{12}, & \frac{11-6\alpha}{12} \\ & \frac{7-6\alpha}{4}, & \frac{1}{2} \end{matrix}; \frac{1728}{j}\right].$$

Using $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$ and formulas (3.6) and (3.7), the above equation simplifies to

$$\frac{F_3}{\eta^{12}} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{2\alpha-1}{4}} \times \left({}_3F_2\left[\begin{matrix} \frac{1-2\alpha}{4}, & \frac{7-6\alpha}{12}, & \frac{11-6\alpha}{12} \\ & \frac{3-6\alpha}{4}, & \frac{1}{2} \end{matrix}; \frac{27x^2}{(4-x)^3}\right] - \left(\frac{x^2(11-6\alpha)}{2(4-x)^3}\right) {}_3F_2\left[\begin{matrix} \frac{5-2\alpha}{4}, & \frac{19-6\alpha}{12}, & \frac{23-6\alpha}{12} \\ & \frac{11-6\alpha}{4}, & \frac{3}{2} \end{matrix}; \frac{27x^2}{(4-x)^3}\right]\right).$$

We use Bailey's transformations from (3.4) for both hypergeometric functions in the right hand side of the above equation, which simplifies the equation to:

$$\frac{F_3}{\eta^{12}} = \left(\frac{64(4-x)^3}{x^2}\right)^{\frac{2\alpha-1}{4}} \times \left(\left(1-\frac{x}{4}\right)^{\frac{3-6\alpha}{4}} {}_3F_2\left[\begin{matrix} \frac{3-6\alpha}{4}, & 1, & \frac{1-6\alpha}{4} \\ & 2, & \frac{1-6\alpha}{2} \end{matrix}; x\right] - \left(\frac{x^2(11-6\alpha)}{2(4-x)^3}\right)\left(1-\frac{x}{4}\right)^{\frac{15-6\alpha}{4}} {}_3F_2\left[\begin{matrix} \frac{15-6\alpha}{4}, & \frac{9-6\alpha}{4}, & 2 \\ & \frac{9-6\alpha}{2}, & 4 \end{matrix}; x\right]\right).$$

Using formula (3.9) and (3.5), the above equation simplifies to:

$$\frac{F_3}{\eta^{12}} = \frac{2^{6\alpha}x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)}\left(\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}} - 1\right) - (11-6\alpha)x^{\frac{5-2\alpha}{2}}2^{6\alpha-10}\, {}_3F_2\left[\begin{matrix} \frac{15-6\alpha}{4}, & \frac{9-6\alpha}{4}, & 2 \\ & \frac{9-6\alpha}{2}, & 4 \end{matrix}; x\right]\right)$$

which implies

$$\frac{F_3}{\eta^{12}} = \frac{2^{6\alpha}x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)}\left(\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}} - 1\right) - (11-6\alpha)x^{\frac{5-2\alpha}{2}}2^{6\alpha-10}\sum_{k=0}^{\infty}\frac{6\left(\frac{15-6\alpha}{4}\right)_k\left(\frac{9-6\alpha}{4}\right)_k x^k}{\left(\frac{9-6\alpha}{2}\right)_k (k+2)(k+3)k!}\right).$$

Using (3.9), we can further simplify the above equation to:

$$\frac{F_3}{\eta^{12}} = \frac{2^{6\alpha}x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}}-1\right)$$
$$-\frac{x^{\frac{5-2\alpha}{2}}2^{6\alpha}}{(\alpha-1)(1+2\alpha)}\left(\frac{d}{dx}\left(\frac{1}{x^2}{}_2F_1\left[\begin{matrix}\frac{3-6\alpha}{4}, \frac{-3-6\alpha}{4} \\ \frac{3-6\alpha}{2}\end{matrix};x\right]\right)+\frac{2}{x^3}-\frac{6\alpha+3}{8x^2}\right).$$

We apply (3.8) formula to the ${}_2F_1$ hypergeometric function, which gives us:

$$\frac{F_3}{\eta^{12}} = \frac{2^{6\alpha}x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}}-1\right)-\frac{x^{\frac{5-2\alpha}{2}}2^{6\alpha}}{(\alpha-1)(1+2\alpha)}\times$$
$$\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x)+\frac{2}{5-6\alpha}x+\frac{4(1-x)x}{5-6\alpha}\frac{d}{dx}\right)\left({}_2F_1\left[\begin{matrix}\frac{1-6\alpha}{4}, \frac{3-6\alpha}{4} \\ \frac{3-6\alpha}{2}\end{matrix};x\right]\right)\right)\right)\right.$$
$$\left.+\frac{2}{x^3}-\frac{6\alpha+3}{8x^2}\right].$$

Applying formula (3.5), we get the desired result:

$$\frac{F_3}{\eta^{12}} = \frac{2^{6\alpha}x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}}-1\right)-\frac{x^{\frac{5-2\alpha}{2}}2^{6\alpha}}{(\alpha-1)(1+2\alpha)}\times$$
$$\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x)+\frac{2}{5-6\alpha}x+\frac{4(1-x)x}{5-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha-1}{2}}\right)\right)\right.$$
$$\left.+\frac{2}{x^3}-\frac{6\alpha+3}{8x^2}\right].$$

□

Now, we will demonstrate how we can compute exact formulas for CM values of these vvmfs. We shall evaluate F_χ, F_2 and F_3 at $\tau = i$.

Example 3.6. Compute F_χ, F_2 and F_3 at $\tau = i$.

Proof. Since $j(i) = 1728$, equation (3.17) simplifies to:

$$F_\chi = \eta(i)^{12}(1728)^{\frac{1-2\alpha}{2}}{}_3F_2\left(\frac{2\alpha-1}{2}, \frac{6\alpha-1}{6}, \frac{6\alpha+1}{6}; \frac{6\alpha-1}{4}, \frac{6\alpha+1}{4}; 1\right).$$

We compute the hypergeometric function on the right hand side of the above equation by (3.10). Using $a = \frac{6\alpha-1}{12}$ in equation (3.10) we find that

$$ {}_3F_2\left(\frac{2\alpha-1}{2}, \frac{6\alpha-1}{6}, \frac{6\alpha+1}{6}; \frac{6\alpha-1}{4}, \frac{6\alpha+1}{4}; 1\right) = \left(\frac{3}{2}\right)^{\frac{6\alpha-3}{2}}. $$

Therefore,

$$ F_\chi = \eta(i)^{12}(1728)^{\frac{1-2\alpha}{2}}\left(\frac{3}{2}\right)^{\frac{6\alpha-3}{2}} = \eta(i)^{12}(2)^{\frac{9-18\alpha}{2}}. $$

Using value of the eta function at i from equation (3.11), we deduce that for $\tau = i$,

$$ F_\chi = \frac{\Gamma\left(\frac{1}{4}\right)^{12}}{\pi^9 2^{\frac{15+18\alpha}{2}}}. $$

Next, we work on F_2. Using $j(i) = 1728$, equation (3.18) simplifies to:

$$ F_2 = \eta(i)^{12}(1728)^{\frac{2\alpha-3}{4}} {}_3F_2\left(\frac{3-2\alpha}{4}, \frac{13-6\alpha}{12}, \frac{17-6\alpha}{12}; \frac{3}{2}, \frac{9-6\alpha}{4}; 1\right). $$

We compute hypergeometric function on the right hand side of the above equation by (3.12). Using $a = \frac{3-2\alpha}{4}$ in equation (3.12) we find that

$$ {}_3F_2\left(\frac{3-2\alpha}{4}, \frac{13-6\alpha}{12}, \frac{17-6\alpha}{12}; \frac{3}{2}, \frac{9-6\alpha}{4}; 1\right) = -(3)^{\frac{5-6\alpha}{4}}(2^{\frac{2\alpha-1}{2}} - 1)/(1-2\alpha). $$

Therefore,

$$ F_2 = -\eta(i)^{12}(1728)^{\frac{2\alpha-3}{4}}(3)^{\frac{5-6\alpha}{4}}(2^{\frac{2\alpha-1}{2}} - 1)/(1-2\alpha). $$

Using value of $\eta(i)$ from (3.11), we get that F_2 for $\tau = i$ is given by:

$$ F_2 = -\frac{\Gamma\left(\frac{1}{4}\right)^{12} 2^{\frac{6\alpha-33}{2}}(2^{\frac{2\alpha-1}{2}} - 1)}{3\pi^9(1-2\alpha)}. $$

Similarly, we compute F_3. For $j(i) = 1728$, equation (3.19) simplifies to:

$$F_3 = \eta(i)^{12}(1728)^{\frac{2\alpha-1}{4}} {}_3F_2\left(\frac{1-2\alpha}{4}, \frac{7-6\alpha}{12}, \frac{11-6\alpha}{12}; \frac{7-6\alpha}{4}, \frac{1}{2}; 1\right).$$

Hypergeometric functions on the right hand sides of the above equation can be computed by (3.13). Using $a = \frac{1-2\alpha}{4}$ in (3.13), we find that

$${}_3F_2\left(\frac{1-2\alpha}{4}, \frac{7-6\alpha}{12}, \frac{11-6\alpha}{12}; \frac{7-6\alpha}{4}, \frac{1}{2}; 1\right) = \frac{3^{\frac{3-6\alpha}{4}}(1+4^{\frac{6\alpha-3}{4}})}{2}.$$

Therefore,

$$F_3 = \eta(i)^{12}(1728)^{\frac{2\alpha-1}{4}} \frac{3^{\frac{3-6\alpha}{4}}(1+4^{\frac{6\alpha-3}{4}})}{2} = \eta(i)^{12} 2^{\frac{6\alpha-5}{2}}(1+2^{\frac{6\alpha-3}{2}}).$$

Using value of $\eta(i)$ from (3.11), we get that F_3 for $\tau = i$ is given by:

$$F_3 = \frac{\Gamma\left(\frac{1}{4}\right)^{12} 2^{\frac{6\alpha-29}{2}}(1+2^{\frac{6\alpha-3}{2}})}{\pi^9}.$$

□

Combining the results from the above computations, we find that the value of the vvmf over ρ and Γ evaluated at $\tau = i$ is:

$$F = \frac{\Gamma\left(\frac{1}{4}\right)^{12}}{\pi^9}\left(2^{\frac{-15-18\alpha}{2}}, \frac{2^{\frac{6\alpha-33}{2}}(1-2^{\frac{2\alpha-1}{2}})}{3(1-2\alpha)}, 2^{\frac{6\alpha-29}{2}}(1+2^{\frac{6\alpha-3}{2}})\right)^t$$

where t denotes the transpose of the matrix.

Tables 3.2 - 3.4 summarize the results for all the formulas computed in all the cases. Table 3.5 summarizes results for vvmfs for $\tau = i$, and leads us to the following proposition.

Proposition 3.7 *Let F be one of the family of vvmfs discussed above, then the transcendental part of CM value for $\tau = i$ is locally constant in the following sense:*

1. *If $\epsilon = 1$, then $\frac{\pi^6 F_\chi}{\Gamma(1/4)^8} \in \overline{\mathbb{Q}(\sqrt{3})}$ for all $\alpha \in \mathbb{Q} \cap (0,1)$,*

F_i	$\epsilon = 1$
F_χ	$\eta^8\left(\frac{x^2}{512(1+\sqrt{1-x})^3}\right)^{\frac{3\alpha-1}{3}}$
F_2	$\eta^8\left(\frac{64}{x}\right)^{\frac{3\alpha-4}{3}}\left(\frac{4}{(1-3\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha-1}-1\right)\right)$
F_3	$\eta^8\frac{4^{3\alpha}x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2}-1\right)-\frac{x^{\frac{7-3\alpha}{3}}}{2\alpha(3\alpha+2)}\times \left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x)+\frac{1}{2-3\alpha}x+\frac{2(1-x)x}{2-3\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha}\right)\right)+\frac{2}{x^3}-\frac{3\alpha+2}{4x^2}\right]$

Table 3.2: Coordinates of vvmfs for $\epsilon = 1$.

F_i	$\epsilon = -1, \alpha \in (0, 1/2)$
F_χ	$\eta^4\left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-6\alpha}{6}}$
F_2	$\eta^4\frac{2^{6\alpha+2}x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}}-1\right)-\frac{x^{\frac{13-6\alpha}{6}}2^{6\alpha+2}3}{(1+6\alpha)(6\alpha+5)}\times \left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x)+\frac{2}{3-6\alpha}x+\frac{4(1-x)x}{3-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{1+6\alpha}{2}}\right)\right)+\frac{2}{x^3}-\frac{6\alpha+5}{8x^2}\right]$
F_3	$\eta^4\left(\frac{64}{x}\right)^{\frac{6\alpha-7}{6}}\left(\frac{8}{(1-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{1-6\alpha}{2}\right)}-1\right)\right)$

Table 3.3: Coordinates of vvmfs for $\epsilon = -1$ and $\alpha \in (0, 1/2)$.

2. *If* $\epsilon = -1$, *then* $\frac{\pi^3 F_\chi}{\Gamma(1/4)^4} \in \overline{\mathbb{Q}(\sqrt{3})}$ *for all* $\alpha \in \mathbb{Q} \cap (0, 1/2)$,

3. *If* $\epsilon = -1$, *then* $\frac{\pi^9 F_\chi}{\Gamma(1/4)^{12}} \in \overline{\mathbb{Q}(\sqrt{3})}$ *for all* $\alpha \in \mathbb{Q} \cap (1/2, 1)$.

In fact, all of these values are algebraically dependent on $\alpha \in (0, 1)$ *except for* $\alpha = 1/2$.

Let λ denote the elliptic modular lambda invariant from chapter 2. Our algebraic formulas for the first coordinate F_χ take a simpler expression if we express them in terms of λ function. Let x_0, x_1, and x_2 denote the solutions of the equation $\frac{27x^2}{(4-x)^3} = \frac{1728}{j}$. Thus we have the minimal polynomial for x as follows:

$$0 = X^3 + (\tfrac{1}{64}j - 12)X^2 + 48X - 64 = (X - x_0)(X - x_1)(X - x_2).$$

Note that the λ is a function on $\Gamma(2)$ and x is a function on $\Gamma_0(2)$. Since $\Gamma(2) \subset \Gamma_0(2)$,

F_i	$\epsilon = -1, \alpha \in (1/2, 1)$
F_χ	$\eta^{12}\left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-2\alpha}{2}}$
F_2	$\eta^{12}\left(\frac{64}{x}\right)^{\frac{2\alpha-3}{2}}\left(\frac{8}{(3-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{3-6\alpha}{2}\right)}-1\right)\right)$
F_3	$\eta^{12}\frac{2^{6\alpha}x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}}-1\right)-\frac{x^{\frac{5-2\alpha}{2}}2^{6\alpha}}{(\alpha-1)(1+2\alpha)}\times$ $\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x)+\frac{2}{5-6\alpha}x+\frac{4(1-x)x}{5-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha-1}{2}}\right)\right)+\frac{2}{x^3}-\frac{6\alpha+3}{8x^2}\right]$

Table 3.4: Coordinates of vvmfs for $\epsilon = -1$ and $\alpha \in (1/2, 1)$.

$Case$	F
$\epsilon = 1$	$\frac{\Gamma\left(\frac{1}{4}\right)^8}{2^5\pi^6}\left(2^{-9\alpha}, \frac{1-2^{6\alpha-5}}{2^4(1-3\alpha)}, 2^{3\alpha-5}(1+2^{3\alpha-1})\right)^t$
$\epsilon = -1, \alpha \in (0, 1/2)$	$\frac{\Gamma\left(\frac{1}{4}\right)^4}{2^{5/2}\pi^3}\left(2^{-9\alpha}, 2^{3\alpha-3}(1+2^{\frac{6\alpha-1}{2}}), \frac{2^{3\alpha-5}(2^{\frac{6\alpha-1}{2}}-1)}{(6\alpha-1)}\right)^t$
$\epsilon = -1, \alpha \in (1/2, 1)$	$\frac{\Gamma\left(\frac{1}{4}\right)^{12}}{\pi^9}\left(2^{\frac{-15-18\alpha}{2}}, \frac{2^{\frac{6\alpha-33}{2}}(1-2^{\frac{2\alpha-1}{2}})}{3(1-2\alpha)}, 2^{\frac{6\alpha-29}{2}}(1+2^{\frac{6\alpha-3}{2}})\right)^t$

Table 3.5: vvmfs for $\tau = i$.

we can write x in terms of λ. We know from [7] and [26] that $j = \frac{256(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$. We substitute this value of the j-function into the the minimal polynomial for x and obtain that (up to permutation) the x_j's are equal to:

$$\frac{4(\lambda-1)}{\lambda^2}, \qquad -\frac{4\lambda}{(\lambda-1)^2}, \qquad -4\lambda(\lambda-1).$$

Let $y = \frac{x^2}{512(1+\sqrt{1-x})^3}$. Then $(F_\chi/\eta^8)(\tau)$ lies in a Kummer extension of $\mathbb{Q}(\tau, j, x, y)$. We now compute the minimal polynomial for y over $\mathbb{Q}(j)$. We have that

$$(1+\sqrt{1-x})^3 = (4-3x) + (4-x)\sqrt{1-x}.$$

Therefore we find that

$$512(4-x)y\sqrt{1-x} = x^2 - 512(4-3x)y.$$

	Γ_0	Γ_1	Γ_2
x_j	$\frac{4(\lambda-1)}{\lambda^2}$	$-4\lambda(\lambda-1)$	$-\frac{4\lambda}{(\lambda-1)^2}$
y_j	$-\frac{\lambda^2}{256(\lambda-1)}$	$\frac{1}{256\lambda(\lambda-1)}$	$\frac{(\lambda-1)^2}{256\lambda}$

Table 3.6: Conjugates of x and y.

A straightforward but somewhat lengthy calculation then shows that y satisfies the equation

$$2^{18}xy^2 + 2^{10}(3x-4)\,y + x^2 = 0.$$

Hence the minimal polynomial of y over $\mathbb{Q}(j)$ divides

$$\prod_{i=0}^{2}\left(2^{18}x_iY^2 + 2^{10}(3x_i-4)\,Y + x_i^2\right) \in \mathbb{Q}(j)[Y].$$

Using a computer one sees that this degree six polynomial is the square of an irreducible cubic. Thus, in this way, one deduces that the minimal polynomial of y over $\mathbb{Q}(j)$ is:

$$P(Y) = Y^3 + \tfrac{3}{256}Y^2 - \left(\frac{1}{2^{24}}j - \frac{3}{2^{16}}\right)Y + \frac{1}{2^{24}}.$$

We substitute the identity $j = \frac{256(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2}$ into the linear term in $P(Y)$ to find that the roots of $P(Y)$ are:

$$-\frac{\lambda^2}{256(\lambda-1)}, \qquad \frac{(\lambda-1)^2}{256\lambda}, \qquad \frac{1}{256\lambda(\lambda-1)}$$

Now let us set some notation so that we can work explicitly with modular forms and their groups. Set $\Gamma_j = R^{-j}\Gamma_0(2)R^j$ where $R = ST$. Hence if f is modular for Γ_i then $f|_{R^j}$ is modular for Γ_{i+j}. With this notation, the roots above are defined on these groups according to Table 3.6 on page 49. In each case x_j and y_j satisfy $64x_jy_j + 1 = 0$.

The next theorem shows that the first coordinates F_χ of vvmfs over ρ and $\Gamma = SL_2(\mathbb{Z})$ for different values of ϵ take a simpler expression if we express them in terms of λ function as claimed above. Since all the coordinates of vvmfs live in the same space, it makes sense to only check their first coordinates.

Theorem 3.8 *With notation as above,*

1. $F_\chi = \eta^8 \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{3\alpha-1}{3}}$ *for* $\epsilon = 1$,

2. $F_\chi = \eta^4 \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{6\alpha-1}{6}}$ *for* $\epsilon = -1$ *and* $\alpha \in (0, 1/2)$,

3. $F_\chi = \eta^{12} \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{2\alpha-1}{2}}$ *for* $\epsilon = -1$ *and* $\alpha \in (1/2, 1)$.

Proof. The Fourier expansions of the y_j for $j \in \{0, 1, 2\}$ are as follows:

$$\begin{aligned} y_0 &= q + 24q^2 + 300q^3 + \cdots , \\ y_1 &= -\frac{1}{4096}q^{-\frac{1}{2}} - \frac{3}{512} - \frac{69}{1024}q^{\frac{1}{2}} + \cdots , \\ y_2 &= \frac{1}{4096}q^{-\frac{1}{2}} - \frac{3}{512} + \frac{69}{1024}q^{\frac{1}{2}} + \cdots . \end{aligned}$$

For $\epsilon = 1$, we have shown that $\frac{F_\chi}{\eta^8} = y_j^{3\alpha-\frac{1}{3}}$ for some $j \in \{0, 1, 2\}$. From equation (3.1), we have that

$$\frac{F_\chi}{\eta^8} = j^{\frac{1-3\alpha}{3}} {}_3F_2\left[\begin{matrix} \frac{3\alpha+1}{3}, \ \alpha, \ \frac{3\alpha+1}{3} \\ \frac{3\alpha}{2}, \ \frac{3\alpha+1}{2} \end{matrix}; \frac{1728}{j}\right] = q^{\frac{3\alpha-1}{3}}(1 + O(q)).$$

Thus, $y_j^{\frac{3\alpha-1}{3}} = q^{\frac{3\alpha-1}{3}}(1 + O(q))$. From the Fourier expansions of the $y_j's$, we can see that the only y_j for which $y_j^{\frac{3\alpha-1}{3}} = q^{\frac{3\alpha-1}{3}}(1 + O(q))$ is y_0. Therefore, we conclude that

$$\frac{F_\chi}{\eta^8} = y_0^{\frac{3\alpha-1}{3}} = \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{3\alpha-1}{3}}.$$

Similarly, for $\epsilon = -1$ and $\alpha \in (0, 1/2)$, we have shown that $\frac{F_\chi}{\eta^4} = y_j^{\frac{6\alpha-1}{6}}$ for some $j \in \{0, 1, 2\}$. From equation (3.14), we have that

$$\frac{F_\chi}{\eta^4} = j^{\frac{1-6\alpha}{6}} {}_3F_2\left[\begin{matrix} \frac{6\alpha-1}{6}, \ \frac{6\alpha+1}{6}, \ \alpha + \frac{1}{2} \\ \frac{6\alpha+3}{4}, \ \frac{6\alpha+1}{4} \end{matrix}; \frac{1728}{j}\right] = q^{\frac{6\alpha-1}{6}}(1 + O(q)).$$

Thus, $y_j^{\frac{6\alpha-1}{6}} = q^{\frac{6\alpha-1}{6}}(1+O(q))$. From the Fourier expansions of the $y_j's$, we can see that the only y_j for which this holds true is y_0. Therefore, we conclude that

$$\frac{F_\chi}{\eta^4} = y_0^{\frac{6\alpha-1}{6}} = \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{6\alpha-1}{6}}.$$

Similar argument follows for the third case as well. For $\epsilon = -1$ and $\alpha \in (1/2, 1)$, we have shown that $\frac{F_\chi}{\eta^{12}} = y_j^{\frac{2\alpha-1}{2}}$ for some $j \in \{0, 1, 2\}$. From equation (3.17), we have that

$$\frac{F_\chi}{\eta^{12}} = j^{\frac{1-2\alpha}{2}} {}_3F_2\left[\begin{matrix} \frac{2\alpha-1}{2}, \frac{6\alpha=1}{6}, \frac{6\alpha+1}{6} \\ \frac{6\alpha-1}{4}, \frac{6\alpha+1}{4} \end{matrix}; \frac{1728}{j}\right] = q^{\frac{2\alpha-1}{2}}(1+O(q)).$$

Thus, $y_j^{\frac{2\alpha-1}{2}} = q^{\frac{2\alpha-1}{2}}(1+O(q))$. From the Fourier expansions of the $y_j's$, we can see that the only y_j for which this holds true is y_0. Therefore, we conclude that

$$\frac{F_\chi}{\eta^{12}} = y_0^{\frac{2\alpha-1}{2}} = \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{2\alpha-1}{2}}.$$

□

In chapter 4 below, we will introduce Eisenstein series for weight 4 and 6 for χ living in the same space. Since these spaces are $1-$dimensional, the Eisenstein series for weight 4 and 6 will be equal up to a scalar. Our goal in the next chapter will be to compute the coefficients of these series.

Chapter 4

Eisenstein families for vvmfs on $\Gamma_0(2)$

In chapter 3, we defined unitary characters, χ, on $\Gamma_0(2)$. In our discussion of Eisenstein series below table 3.1, we had deduced that Eisenstein series of weight 4 and 6 for χ are unique up to scaling by a complex number.

In this chapter, we define $g_{\chi,k}$, an analogue of Eisenstein series ([10]), for varying χ. Since $g_{\chi,4}$ and $g_{\chi,6}$ are contained in 1-dimensional spaces of modular forms, they are unique up to a complex scalar. So, we explore their Fourier series expansions and compute coefficients in these expansions. Discussion in this chapter is concluded by expressing the constant term in the Fourier series expansion of $g_{\chi,4}$ and $g_{\chi,6}$ using Bessel functions of the first kind and Kloosterman sums. This then allows us to compare these families of Eisenstein series to the hypergeometric expressions given previously, since they necessarily differ by this constant term.

For simplicity and to keep the calculations manageable, we will restrict our discussion in this section to the case for $\epsilon = 1$. Similar computations can be done for when $\epsilon = -1$. We plotted results for the $\epsilon = -1$ case and present them in figures 4.3 and 4.4.

Let us now introduce $g_{\chi,k}$.

Recall that in Chapter 3, we observed that $T = VU^2$, where

$$U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, V = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}.$$

So for $\epsilon = 1$, we have that $\chi(T) = e^{2\pi i\alpha}$.

Define

$$g_{\chi,k}(\tau) = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \langle \pm T\rangle\backslash\Gamma_0(2)} \chi\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \frac{e^{2\pi i\alpha\frac{a\tau+b}{c\tau+d}}}{(c\tau+d)^k}.$$

Let us first discuss the elements of $\langle \pm T\rangle\backslash\Gamma_0(2)$. Here, we are looking at the cosets of $\Gamma_0(2)$ in $\langle \pm T\rangle$. So, for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$, the entries a, b, d could be any integers but the c will be an even integer such that $ad - bc = 1$. Therefore, we can say that

$$g_{\chi,k}(\tau) = \sum_{\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in \langle \pm T\rangle\backslash\Gamma_0(2)} \chi\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} \frac{e^{2\pi i\alpha\frac{a\tau+b}{2c\tau+d}}}{(2c\tau+d)^k},$$

where $ad - 2bc = 1$. Also, note that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} = \begin{pmatrix} a+2c & 2c \\ b+d & d \end{pmatrix}.$$

Therefore,

$$\langle \pm T\rangle \begin{pmatrix} a & b \\ 2c & d \end{pmatrix} = \left\{ \begin{pmatrix} \pm a \pm 2nc & \pm b \pm nd \\ \pm 2c & \pm d \end{pmatrix} \in \Gamma_0(2), n \in \mathbb{Z} \right\}.$$

For any given $c, d \in \mathbb{Z}$, we can always choose $a, b \in \mathbb{Z}$ depending on $ad - 2bc = 1$. So, $g_{\chi,k}$ is well-defined. In the following lemma, we discuss the convergence of $g_{\chi,k}(\tau)$ on $\mathcal{H}$.

Lemma 4.1 *Let $k \geq 4$, then $g_{\chi,k}(\tau)$ converges uniformly on compact subsets of $\mathcal{H}$.*

Proof. Since χ is unitary,

$$\left|\chi\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1}\right| = 1. \tag{4.1}$$

Also, by properties of the complex exponential,

$$\left|e^{2\pi i\alpha\Re\left(\frac{a\tau+b}{2c\tau+d}\right)}\right| = 1. \tag{4.2}$$

So,

$$|g_{\chi,k}(\tau)| \leq \sum_{\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}\in\langle\pm T\rangle\backslash\Gamma_0(2)} \frac{e^{-2\pi\alpha\Im\left(\frac{a\tau+b}{2c\tau+d}\right)}}{|2c\tau+d|^k} = \sum_{\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}\in\langle\pm T\rangle\backslash\Gamma_0(2)} \frac{e^{-2\pi\alpha\frac{\Im(\tau)}{|2c\tau+d|^2}}}{|2c\tau+d|^k} \tag{4.3}$$

by equation (2.1). Since $\Im(\tau) > 0$ in $\mathcal{H}$, notice that for $\alpha \geq 0$,

$$e^{-2\pi\alpha\frac{\Im(\tau)}{|2c\tau+d|^2}} \leq 1. \tag{4.4}$$

Hence, combining results from equations (4.1) - (4.4), we have that

$$|g_{\chi,k}(\tau)| \leq \sum_{\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}\in\langle\pm T\rangle\backslash\Gamma_0(2)} \frac{1}{|2c\tau+d|^k}.$$

The sum on the right hand side of the above equation is absolutely convergent for $k \geq 4$ and hence $g_{\chi,k}(\tau)$ converges uniformly on compact subsets of $\mathcal{H}$. Then, by Morera's theorem ([3], p.122), $g_{\chi,k}(\tau)$ is holomorphic on $\mathcal{H}$. □

As we mentioned earlier in Chapter 2, every vvmf has a Fourier q-expansion. We will now compute the coefficients of the Fourier expansion of $g_{\chi,k}$. We will need the following lemma for the computation of the Fourier coefficients.

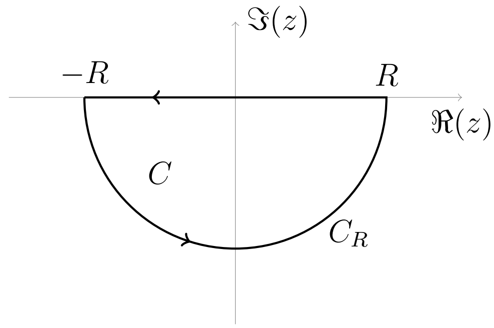

Figure 4.1: Contour C for Lemma 4.2.

Lemma 4.2 *Let $m, k \in \mathbb{Z}$ with $k \geq 2$. Let $x, y \in \mathbb{R}$ with $y > 0$, then*

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi imx}dx}{(x+iy)^k} = \begin{cases} 0 & m \leq 0, \\ \frac{m^{k-1}}{(k-1)!}(-2\pi i)^k e^{-2\pi my} & m > 0. \end{cases}$$

Proof. For $m, k \in \mathbb{Z}$ with $k \geq 2$ and $y \in \mathbb{R}$, $y > 0$, consider the function

$$f(z) = \frac{e^{-2\pi imz}}{(z+iy)^k}, \quad z \in \mathbb{C}.$$

For $m > 0$, we will compute integral of this function, i.e. $\oint\limits_C f(z)dz$, over C, where C is a closed curve consisting of real-axis from R to $-R$ and the semi-circle C_R of radius R in the lower half plane. The contour is displayed in figure 4.2.

The function f has one pole of order k at $z = -iy$, which lies inside the contour C. Hence, by the residue theorem, we have that

$$\begin{aligned} \oint\limits_C f(z)dz &= 2\pi i Res_{z=-iy} f(z) \\ &= 2\pi i \frac{1}{(k-1)!} \lim_{z \to -iy} \frac{d^{k-1}}{dz^{k-1}} [(z-(-iy))^k f(z)]_{z=-iy} \\ &= 2\pi i \frac{1}{(k-1)!} \lim_{z \to -iy} \frac{d^{k-1}}{dz^{k-1}} [e^{-2\pi imz}] \\ &= \frac{(2\pi i)^k}{(k-1)!}(-m)^{k-1} e^{-2\pi my}. \end{aligned} \tag{4.5}$$

Also, note that

$$\oint_C f(z)dz = \int_R^{-R} f(z)dz + \oint_{C_R} f(z)dz$$

Since $\Im(z) = 0$ on the real-axis, and by properties of the integrals, we can re-write the above equation as

$$\oint_C f(z)dz = -\int_{-R}^{R} f(x)dx + \oint_{C_R} f(z)dz \tag{4.6}$$

Using properties of complex numbers, we observe that for $R > 0$,

$$\begin{aligned}
\left|\oint_{C_R} f(z)dz\right| = \left|\oint_{C_R} \frac{e^{-2\pi imz}}{(z+iy)^k}dz\right| &\leq \oint_{C_R} \frac{|e^{-2\pi imz}dz|}{|(z+iy)^k|} \\
&= \oint_{C_R} \frac{|e^{-2\pi imz}||dz|}{|(z+iy)|^k} \\
&= \oint_{C_R} \frac{|e^{2\pi my}||dz|}{|(z+iy)|^k} \\
&= \int_0^{\pi} \frac{1.|Rd\theta|}{(\sqrt{R^2+y^2})^k} \quad \text{(since } |e^{2\pi my}| \to 0 \text{ as } y \to 0) \\
&\leq \int_0^{\pi} \frac{Rd\theta}{R^2+y^2} \quad \text{(since } k \geq 2) \\
&= \int_0^{\pi} \frac{1/R^2 d\theta}{1+(y/R)^2}
\end{aligned}$$

Clearly,

$$\left|\oint_{C_R} f(z)dz\right| \to 0 \text{ as } R \to \infty. \tag{4.7}$$

Therefore, by equations 4.5, 4.6 and 4.7, we get that for $m > 0$,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi imx}dx}{(x+iy)^k}dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-2\pi imx}dx}{(x+iy)^k}dx = \frac{m^{k-1}}{(k-1)!}(-2\pi i)^k e^{-2\pi my}.$$

For $m \leq 0$, we can let $p = -m$ and redo the calculations as above for contour C', where

C' is a closed curve consisting of real-axis from $-R$ to R and the semi-circle C'_R of radius R in the upper half plane. As there are no poles of $f(z)$ inside the contour C', the integral, therefore, evaluates to 0.

This completes the proof. $\square$

Notice that we can re-write the expression for $g_{\chi,k}$ as follows:

$$g_{\chi,k}(\tau) = q^{\alpha} + \sum_{c=1}^{\infty} \sum_{\substack{d\in\mathbb{Z} \\ gcd(2c,d)=1}} \chi \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} \frac{e^{2\pi i\alpha \frac{a\tau+b}{2c\tau+d}}}{(2c\tau+d)^k}.$$

The term q^{α} in the above equation corresponds to the identity coset in $\langle \pm T\rangle \backslash \Gamma_0(2)$. Since $gcd(2c, d) = 1$, by Division Algorithm, we have that

$$d = d_0 + 2ct$$

where $t \in \mathbb{Z}$ and $1 \leq d_0 < 2c$. Using the fact that the matrices are coming from $\Gamma_0(2)$, so their determinant has to be 1, we can re-write the definition of $g_{\chi,k}$ as follows:

$$\begin{aligned}
g_{\chi,k}(\tau) &= q^{\alpha} + \sum_{c=1}^{\infty} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \sum_{t\in\mathbb{Z}} \chi \begin{pmatrix} a & b+at \\ 2c & d+2ct \end{pmatrix}^{-1} \frac{e^{2\pi i\alpha \frac{a\tau+b+at}{2c\tau+d+2ct}}}{(2c\tau+d+2ct)^k} \\
&= q^{\alpha} + \sum_{c=1}^{\infty} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \sum_{t\in\mathbb{Z}} \chi \left(\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right)^{-1} \frac{e^{2\pi i\alpha \frac{a\tau+b+at}{2c\tau+d+2ct}}}{(2c\tau+d+2ct)^k} \\
&= q^{\alpha} + \sum_{c=1}^{\infty} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \sum_{t\in\mathbb{Z}} \chi \left(\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} T^t \right)^{-1} \frac{e^{2\pi i\alpha \frac{a\tau+b+at}{2c\tau+d+2ct}}}{(2c\tau+d+2ct)^k} \\
&= q^{\alpha} + \sum_{c=1}^{\infty} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \sum_{t\in\mathbb{Z}} \chi \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} \chi(T^t)^{-1} \frac{e^{2\pi i\alpha \frac{a\tau+b+at}{2c\tau+d+2ct}}}{(2c\tau+d+2ct)^k} \\
&= q^{\alpha} + \sum_{c=1}^{\infty} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \sum_{t\in\mathbb{Z}} \chi \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} \left(\left(e^{2\pi i\alpha}\right)^t \right)^{-1} \frac{e^{2\pi i\alpha \frac{a\tau+b+at}{2c\tau+d+2ct}}}{(2c\tau+d+2ct)^k} \\
&= q^{\alpha} \left(1 + \frac{1}{2^k} \sum_{c=1}^{\infty} \frac{1}{c^k} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \chi \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} \sum_{t\in\mathbb{Z}} \frac{e^{-2\pi i\alpha(\tau+t)} e^{2\pi i\alpha \frac{a(\tau+t)+b}{2c(\tau+t)+d}}}{(\tau+t+\frac{d}{2c})^k} \right)
\end{aligned} \tag{4.8}$$

Notice that the function

$$f(\tau) = \sum_{t\in\mathbb{Z}} \frac{e^{-2\pi i\alpha(\tau+t)} e^{2\pi i\alpha \frac{a(\tau+t)+b}{2c(\tau+t)+d}}}{(\tau+t+\frac{d}{2c})^k}$$

in the equation (4.8) satisfies the translation law $f(\tau+1) = f(\tau)$ because we are summing over $t \in \mathbb{Z}$:

$$f(\tau+1) = \sum_{t\in\mathbb{Z}} \frac{e^{-2\pi i\alpha(\tau+1+t)} e^{2\pi i\alpha \frac{a(\tau+1+t)+b}{2c(\tau+1+t)+d}}}{(\tau+1+t+\frac{d}{2c})^k} = \sum_{t\in\mathbb{Z}} \frac{e^{-2\pi i\alpha(\tau+t)} e^{2\pi i\alpha \frac{a(\tau+t)+b}{2c(\tau+t)+d}}}{(\tau+t+\frac{d}{2c})^k} = f(\tau).$$

So, we will now work on the Fourier expansion of $f(\tau)$. Let $f(\tau) = \sum_{n\in\mathbb{Z}} t_n q^n$. Then, $g_{\chi,k}$ can be re-written as

$$g_{\chi,k} = q^\alpha \left(1 + \sum_{n\in\mathbb{Z}} \frac{1}{2^k} \left(\sum_{c=1}^{\infty} \frac{1}{c^k} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \chi \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} t_n \right) q^n \right). \tag{4.9}$$

Let us work out the formula for all t_n. Recall that $\alpha \in [0,1)$. So, by definition 2.17, we have that

$$\begin{aligned}
t_n &= \int_0^1 f(x+iy) e^{-2\pi i n(x+iy)} dx \\
&= \int_0^1 \sum_{t\in\mathbb{Z}} \frac{e^{-2\pi i\alpha(x+iy+t)} e^{2\pi i\alpha\frac{a(x+iy+t)+b}{2c(x+iy+t)+d}}}{(x+iy+t+\frac{d}{2c})^k} e^{-2\pi i n(x+iy)} dx \\
&= e^{2\pi ny} \int_0^1 \sum_{t\in\mathbb{Z}} \frac{e^{-2\pi i\alpha(x+iy+t)} e^{2\pi i\alpha\frac{a(x+iy+t)+b}{2c(x+iy+t)+d}}}{(x+iy+t+\frac{d}{2c})^k} e^{-2\pi i n x} dx \\
&= e^{2\pi ny} \int_{-\infty}^{\infty} \frac{e^{-2\pi i\alpha(x+iy)} e^{2\pi i\alpha\frac{a(x+iy)+b}{2c(x+iy)+d}}}{(x+iy+\frac{d}{2c})^k} e^{-2\pi i n x} dx \\
&= e^{2\pi(\alpha+n)y} \int_{-\infty}^{\infty} \frac{e^{-2\pi i(\alpha+n)x} e^{2\pi i\alpha\frac{a(x+iy)+b}{2c(x+iy)+d}}}{(x+\frac{d}{2c}+iy)^k} dx
\end{aligned}$$

Let $u = x + \frac{d}{2c}$. Then, $du = dx$; and hence the integral above becomes:

$$\begin{aligned}
t_n &= e^{2\pi(\alpha+n)y} \int_{-\infty}^{\infty} \frac{e^{-2\pi i(\alpha+n)(u-\frac{d}{2c})} e^{2\pi i\alpha\frac{a(u-\frac{d}{2c}+iy)+b}{2c(u-\frac{d}{2c}+iy)+d}}}{(u+iy)^k} du \\
&= e^{2\pi(\alpha+n)y} e^{2\pi i(\alpha+n)\frac{d}{2c}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i(\alpha+n)u} e^{2\pi i\alpha\frac{a(u+iy)-\frac{ad+2bc}{2c}}{2c(u+iy)}}}{(u+iy)^k} du \\
&= e^{2\pi(\alpha+n)y} e^{2\pi i(\alpha+n)\frac{d}{2c}} e^{2\pi i\alpha\frac{a}{2c}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i(\alpha+n)u} e^{2\pi i\alpha\frac{\frac{-1}{2c}}{2c(u+iy)}}}{(u+iy)^k} du \\
&= e^{2\pi(\alpha+n)y} e^{2\pi i(\alpha+n)\frac{d}{2c}} e^{2\pi i\alpha\frac{a}{2c}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i(\alpha+n)u} e^{-2\pi i\alpha\frac{1}{4c^2(u+iy)}}}{(u+iy)^k} du.
\end{aligned} \tag{4.10}$$

Let the integral in equation (4.10) be denoted by $c_k(\alpha, n)$. So,

$$c_k(\alpha, n) = \int_{-\infty}^{\infty} e^{-2\pi i(\alpha+n)u} \frac{e^{-2\pi i\alpha \frac{1}{4c^2(u+iy)}}}{(u+iy)^k} du. \tag{4.11}$$

Let $r \geq 1$. We differentiate the above equation r times with respect to α. We can allow interchange of the differentiation and integration for this step because of the Leibniz integral rule ([24] p. 422, Theorem 1). Differentiating equation (4.11) r times with respect to α gives us

$$\begin{aligned} c_k^{(r)}(\alpha, n) &= \left(-\frac{2\pi i}{4c^2}\right)^r \int_{-\infty}^{\infty} e^{-2\pi i(\alpha+n)u} \frac{e^{-2\pi i\alpha \frac{1}{4c^2(u+iy)}}}{(u+iy)^{k+r}} du \\ &= \left(-\frac{2\pi i}{4c^2}\right)^r c_{k+r}(\alpha, n) \end{aligned} \tag{4.12}$$

So, we can write $c_k(\alpha, n)$ as a power series of the form

$$c_k(\alpha, n) = \sum_{r\geq 0} b_r(\alpha - \zeta)^r$$

where $b_r = \frac{c_k^{(r)}(\zeta,n)}{r!}$. Expanding the above power series about $\zeta = 0$, we get

$$c_k(\alpha, n) = \sum_{r\geq 0} \frac{c_k^{(r)}(0, n)}{r!} \alpha^r \tag{4.13}$$

Substituting the value of $c_k^{(r)}(0, n)$ from equation (4.12) into equation (4.13), we get

$$c_k(\alpha, n) = \sum_{r\geq 0} \frac{1}{r!} \left(\frac{-2\pi i\alpha}{4c^2}\right)^r \int_{-\infty}^{\infty} \frac{e^{-2\pi i(\alpha+n)u}}{(u+iy)^{k+r}} du$$

Using results from lemma 4.2 for the integral in the above equation, we find that

$$c_k(\alpha, n) = \begin{cases} 0 & n \leq -\alpha, \\ \sum_{r\geq 0} \frac{1}{r!} \left(\frac{-2\pi i\alpha}{4c^2}\right)^r \frac{(\alpha+n)^{k+r-1}}{(k+r-1)!} (-2\pi i)^{k+r} e^{-2\pi(\alpha+n)y} & n > \alpha. \end{cases} \tag{4.14}$$

Now, using equations (4.10), (4.11) and (4.14), we find that $t_n = 0$ for $n \leq -\alpha$ and for $n > \alpha$, we have

$$\begin{aligned} t_n &= e^{2\pi(\alpha+n)y} e^{2\pi i(\alpha+n)\frac{d}{2c}} e^{2\pi i\alpha\frac{a}{2c}} \sum_{r\geq 0} \frac{1}{r!}\left(\frac{-2\pi i\alpha}{4c^2}\right)^r \frac{(\alpha+n)^{k+r-1}}{(k+r-1)!}(-2\pi i)^{k+r} e^{-2\pi(\alpha+n)y} \\ &= (-2\pi i)^k (n+\alpha)^{k-1} e^{2\pi i(\alpha+n)\frac{d}{2c}} e^{2\pi i\alpha\frac{a}{2c}} \sum_{r\geq 0} \frac{1}{r!(k+r-1)!}\left(\frac{-4\pi^2\alpha(n+\alpha)}{4c^2}\right)^r. \end{aligned}$$

Recall that $\alpha \in [0, 1)$ for our case, hence we can say that the constant term in the Eisenstein series $g_{\chi,k}$ is as follows:

$$C_0 = 1+\alpha^{k-1}(-\pi i)^k \sum_{c=1}^{\infty} \frac{1}{c^k} \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \chi\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} e^{2\pi i\alpha\frac{a+d}{2c}} \sum_{r\geq 0} \frac{1}{r!(k+r-1)!}\left(\frac{\pi i\alpha}{c}\right)^{2r}. \tag{4.15}$$

Define the Kloosterman sum

$$K(\alpha, c) = \sum_{\substack{d=1 \\ \gcd(2c,d)=1}}^{2c} \chi\begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} e^{2\pi i\alpha\frac{a+d}{2c}} \tag{4.16}$$

and

$$F(z) = \sum_{r\geq 0} \frac{1}{r!(k+r-1)!} z^{2r} \tag{4.17}$$

By equations (4.15) (4.16) and (4.17), we can say that the constant term in $g_{\chi,k}$ is as follows:

$$C_0 = 1 + \alpha^{k-1}(-\pi i)^k \sum_{c=1}^{\infty} \frac{F\left(\frac{\pi i\alpha}{c}\right) K(\alpha, c)}{c^k}. \tag{4.18}$$

Theorem 4.3 *Let $J_\alpha(z)$ denote the Bessel function of the first kind as defined in Definition*

2.29, and let $K(\alpha, c)$ denote the Kloosterman sum. Then, for $\alpha \in [0,1)$, we have:

$$g_{\chi,4} = \left(1 - \pi \sum_{c=1}^{\infty} \frac{J_3\left(\frac{-2\pi\alpha}{c}\right) K(\alpha, c)}{c}\right) F_{\chi}.$$

Proof. We will use equations 2.11 and 2.12 to prove this result.

Note that the function in equation (4.17) can be expressed in terms of the Modified Bessel functions of the first kind using equation (2.12) as follows:

$$\begin{aligned}
F(z) &= \sum_{r \geq 0} \frac{1}{r!(k+r-1)!} z^{2r} \\
&= \frac{1}{(k-1)!} \sum_{r \geq 0} \frac{1}{k(k+1)\cdots(k+r-1)} \frac{z^{2r}}{r!} \\
&= \frac{1}{(k-1)!} {}_0F_1\left(;k, z^2\right) \\
&= \frac{z^{1-k}}{(k-1)!} \Gamma(k) I_{k-1}(2z).
\end{aligned}$$

Therefore,

$$F\left(\frac{\pi i \alpha}{c}\right) = \frac{\left(\frac{\pi i \alpha}{c}\right)^{1-k}}{(k-1)!} \Gamma(k) I_{k-1}\left(\frac{2\pi i \alpha}{c}\right).$$

Using the above result in equation (4.18), we get

$$\begin{aligned}
C_0 &= 1 + \alpha^{k-1}(-\pi i)^k \sum_{c=1}^{\infty} \frac{\left(\frac{\pi i \alpha}{c}\right)^{1-k}}{(k-1)!} \Gamma(k) I_{k-1}\left(\frac{2\pi i \alpha}{c}\right) \frac{K(\alpha, c)}{c^k} \\
&= 1 + (-1)^k \pi i \sum_{c=1}^{\infty} \frac{I_{k-1}\left(\frac{2\pi i \alpha}{c}\right) K(\alpha, c)}{c} \\
&= 1 + (-1)^k \pi i \sum_{c=1}^{\infty} \frac{i^{1-k} J_{k-1}\left(\frac{i 2\pi i \alpha}{c}\right) K(\alpha, c)}{c} \quad \text{(by equation (2.11))} \\
&= 1 + \pi(-1)^k i^{2-k} \sum_{c=1}^{\infty} \frac{J_{k-1}\left(\frac{-2\pi\alpha}{c}\right) K(\alpha, c)}{c}.
\end{aligned}$$

Using $k = 4$ in the above equation, we get that the constant term of $g_{\chi,4}$ is

$$C_0 = \left(1 - \pi \sum_{c=1}^{\infty} \frac{J_3\left(\frac{-2\pi\alpha}{c}\right) K(\alpha, c)}{c}\right)$$

Since F_χ is a one-dimensional modular form with its constant term in its Fourier series expansion as 1. So, $g_{\chi,4}/F_\chi$ gives us the constant term of $g_{\chi,4}$. Hence, we can say that

$$g_{\chi,4}/F_\chi = \left(1 - \pi \sum_{c=1}^{\infty} \frac{J_3\left(\frac{-2\pi\alpha}{c}\right) K(\alpha, c)}{c}\right).$$

This proves the result. □

We can now define a function γ on $(0, 1)$ as $\gamma(\alpha)$ is the constant term in the Fourier expansion of $g_{\chi,k}$. Since F_χ is a one-dimensional modular form with its constant term in its Fourier series expansion as 1, hence,

$$g_{\chi,k} = \gamma(\alpha)F_\chi.$$

This relates our computations with Eisenstein series to our preceding discussion of hypergeometric series formulae. Note that one obtains an analogous formula for $g_{\chi,6}$.

Figures 4.2 - 4.4 illustrate values of γ for different values of α for different cases of ϵ.

Remark *Graphs in figures 4.2 - 4.4 were evaluated by sampling* 200 *points for* 60 *coefficients in Fourier series expansion of* $g_{\chi,4}$.

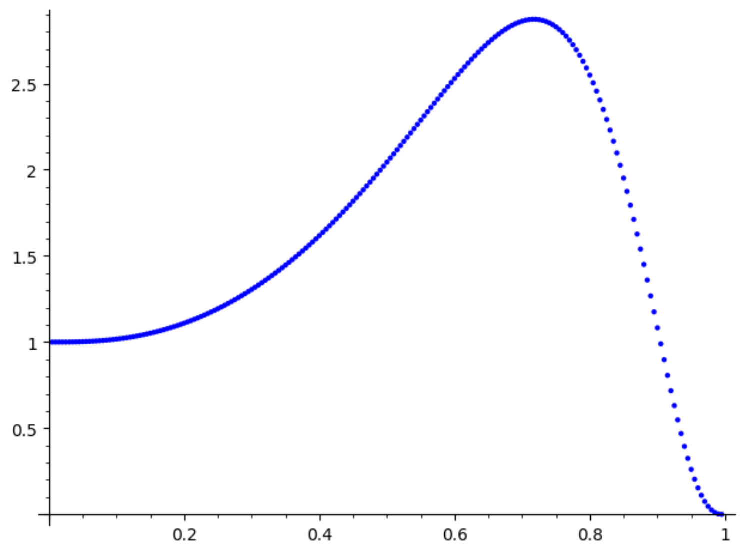

Figure 4.2: Values of γ for $\epsilon = 1$ and $\alpha \in (0, 1)$.

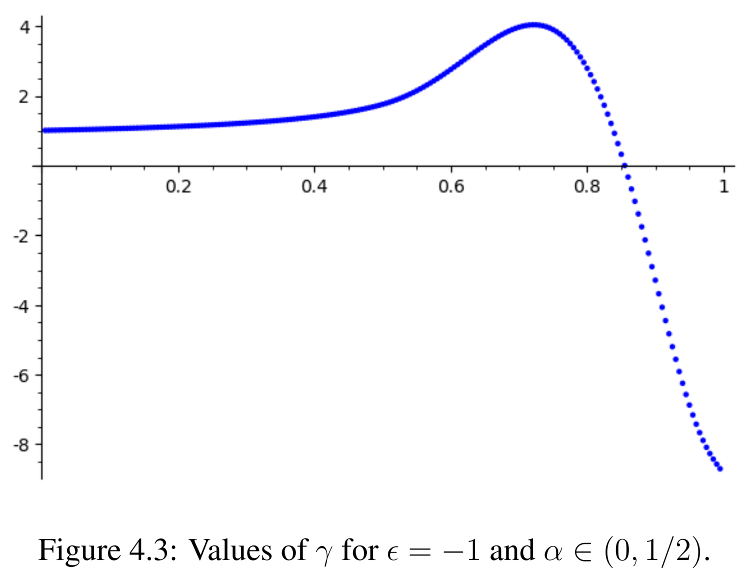

Figure 4.3: Values of γ for $\epsilon = -1$ and $\alpha \in (0, 1/2)$.

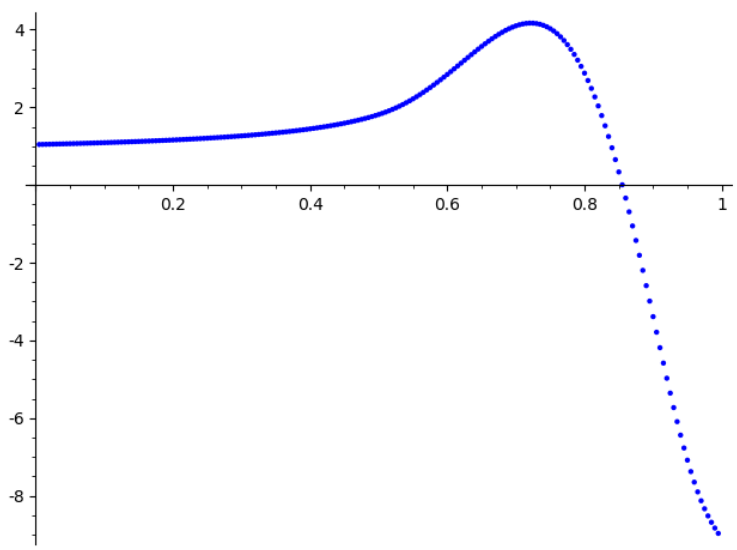

Figure 4.4: Values of γ for $\epsilon = -1$ and $\alpha \in (1/2, 1)$.

Chapter 5

Conclusions

This chapter summarizes the main results presented in this thesis.

5.1 Summary of thesis

In Chapter 3, we computed coordinates of vvmfs over Γ and 3-dimensional ρ for different values of $\alpha \in (0,1)$ and $\epsilon = \pm 1$. Tables 5.1 - 5.3 summarize those results:

F_i	$\epsilon = 1$
F_χ	$\eta^8 \left(\frac{x^2}{512(1+\sqrt{1-x})^3}\right)^{\frac{3\alpha-1}{3}}$
F_2	$\eta^8 \left(\frac{64}{x}\right)^{\frac{3\alpha-4}{3}} \left(\frac{4}{(1-3\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha-1} - 1\right)\right)$
F_3	$\eta^8 \frac{4^{3\alpha} x^{\frac{-3\alpha-2}{3}}}{(-3\alpha-2)} \left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha+2} - 1\right) - \frac{x^{\frac{7-3\alpha}{3}}}{2\alpha(3\alpha+2)} \times \left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{1}{2-3\alpha}x + \frac{2(1-x)x}{2-3\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{3\alpha}\right)\right) + \frac{2}{x^3} - \frac{3\alpha+2}{4x^2}\right]$

Table 5.1: Coordinates of vvmf for $\epsilon = 1$. (Replicated from Table 3.2)

F_i	$\epsilon = -1, \alpha \in (0, 1/2)$
F_χ	$\eta^4 \left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-6\alpha}{6}}$
F_2	$\eta^4 \frac{2^{6\alpha+2} x^{\frac{-6\alpha-5}{6}}}{(-6\alpha-5)} \left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+5}{2}} - 1\right) - \frac{x^{\frac{13-6\alpha}{6}} 2^{6\alpha+2} 3}{(1+6\alpha)(6\alpha+5)} \times$ $\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{2}{3-6\alpha}x + \frac{4(1-x)x}{3-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{1+6\alpha}{2}}\right)\right) + \frac{2}{x^3} - \frac{6\alpha+5}{8x^2}\right]$
F_3	$\eta^4 \left(\frac{64}{x}\right)^{\frac{6\alpha-7}{6}} \left(\frac{8}{(1-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{1-6\alpha}{2}\right)} - 1\right)\right)$

Table 5.2: Coordinates of vvmf for $\epsilon = -1$ and $\alpha \in (0, 1/2)$. (Replicated from Table 3.3)

F_i	$\epsilon = -1, \alpha \in (1/2, 1)$
F_χ	$\eta^{12} \left(\frac{512(1+\sqrt{1-x})^3}{x^2}\right)^{\frac{1-2\alpha}{2}}$
F_2	$\eta^{12} \left(\frac{64}{x}\right)^{\frac{2\alpha-3}{2}} \left(\frac{8}{(3-6\alpha)x}\left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{-\left(\frac{3-6\alpha}{2}\right)} - 1\right)\right)$
F_3	$\eta^{12} \frac{2^{6\alpha} x^{\frac{-2\alpha-1}{2}}}{(-6\alpha-3)} \left(\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha+3}{2}} - 1\right) - \frac{x^{\frac{5-2\alpha}{2}} 2^{6\alpha}}{(\alpha-1)(1+2\alpha)} \times$ $\left[\frac{d}{dx}\left(\frac{1}{x^2}\left(\left((1-x) + \frac{2}{5-6\alpha}x + \frac{4(1-x)x}{5-6\alpha}\frac{d}{dx}\right)\left(\frac{1+\sqrt{1-x}}{2}\right)^{\frac{6\alpha-1}{2}}\right)\right) + \frac{2}{x^3} - \frac{6\alpha+3}{8x^2}\right]$

Table 5.3: Coordinates of vvmf for $\epsilon = -1$ and $\alpha \in (1/2, 1)$. (Replicated from Table 3.4)

$Case$	F
$\epsilon = 1$	$\frac{\Gamma\left(\frac{1}{4}\right)^8}{2^5\pi^6}\left(2^{-9\alpha}, \frac{1-2^{6\alpha-5}}{2^4(1-3\alpha)}, 2^{3\alpha-5}(1+2^{3\alpha-1})\right)^t$
$\epsilon = -1, \alpha \in (0, 1/2)$	$\frac{\Gamma\left(\frac{1}{4}\right)^4}{2^{5/2}\pi^3}\left(2^{-9\alpha}, 2^{3\alpha-3}(1+2^{\frac{6\alpha-1}{2}}), \frac{2^{3\alpha-5}(2^{\frac{6\alpha-1}{2}}-1)}{(6\alpha-1)}\right)^t$
$\epsilon = -1, \alpha \in (1/2, 1)$	$\frac{\Gamma\left(\frac{1}{4}\right)^{12}}{\pi^9}\left(2^{\frac{-15-18\alpha}{2}}, \frac{2^{\frac{6\alpha-33}{2}}(1-2^{\frac{2\alpha-1}{2}})}{3(1-2\alpha)}, 2^{\frac{6\alpha-29}{2}}(1+2^{\frac{6\alpha-3}{2}})\right)^t$

Table 5.4: vvmfs for $\tau = i$. (Replicated from Table 3.5)

We also computed CM values of vvmfs for $\tau = i$, we present these CM values as column vectors in the table 5.4.

In proposition 3.7, we made some observations about the transcendental part of CM value for $\tau = i$. We present these observations below.

Proposition 5.1 *Let F be one of the family of vvmfs discussed above, then the transcendental part of CM value for $\tau = i$ is locally constant in the following sense:*

1. *If $\epsilon = 1$, then $\frac{\pi^6 F_\chi}{\Gamma(1/4)^8} \in \overline{\mathbb{Q}(\sqrt{3})}$ for all $\alpha \in \mathbb{Q} \cap (0, 1)$,*

2. *If $\epsilon = -1$, then $\frac{\pi^3 F_\chi}{\Gamma(1/4)^4} \in \overline{\mathbb{Q}(\sqrt{3})}$ for all $\alpha \in \mathbb{Q} \cap (0, 1/2)$,*

3. *If $\epsilon = -1$, then $\frac{\pi^9 F_\chi}{\Gamma(1/4)^{12}} \in \overline{\mathbb{Q}(\sqrt{3})}$ for all $\alpha \in \mathbb{Q} \cap (1/2, 1)$.*

In fact, all of these values are algebraically dependent on $\alpha \in (0, 1)$ except for $\alpha = 1/2$.

In chapter 3, we also computed the first coordinates F_χ of vvmfs over ρ and Γ in terms of λ, the elliptic modular lambda invariant from chapter 4. We have summarized the results from theorem 3.8 in the table 5.5.

$Case$	F_χ
$\epsilon = 1$	$\eta^8 \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{3\alpha-1}{3}}$
$\epsilon = -1, \alpha \in (0, 1/2)$	$\eta^4 \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{6\alpha-1}{6}}$
$\epsilon = -1, \alpha \in (1/2, 1)$	$\eta^{12} \left(\frac{\lambda^2}{256(1-\lambda)}\right)^{\frac{2\alpha-1}{2}}$

Table 5.5: F_χ for vvmfs over ρ and Γ in terms of modular λ-invariant.

In chapter 4, we defined an analogue of Eisenstein series as follows:

$$g_{\chi,k}(\tau) = \sum_{\begin{pmatrix} a & b \\ 2c & d \end{pmatrix} \in \langle \pm T \rangle \backslash \Gamma_0(2)} \chi \begin{pmatrix} a & b \\ 2c & d \end{pmatrix}^{-1} \frac{e^{2\pi i \alpha \frac{a\tau+b}{2c\tau+d}}}{(2c\tau+d)^k},$$

where $ad - 2bc = 1$. Theorem 4.3 explained a result related to the constant term of this series. We present this theorem here again.

Theorem 5.2 *Let $J_\alpha(z)$ denote the Bessel function of the first kind as defined in Definition 2.29, and let $K(\alpha, c)$ denote the Kloosterman sum. Then, for $\alpha \in [0, 1)$, we have:*

$$g_{\chi,4} = \left(1 - \pi \sum_{c=1}^{\infty} \frac{J_3\left(\frac{-2\pi\alpha}{c}\right) K(\alpha, c)}{c}\right) F_\chi.$$

We also defined a function γ on $(0, 1)$ as $\gamma(\alpha)$ is the constant term in the Fourier expansion of $g_{\chi,k}$. Figures 4.2 - 4.4 illustrated values of γ for different values of α for different cases of ϵ. We present these graphs here in figures 5.1 - 5.3.

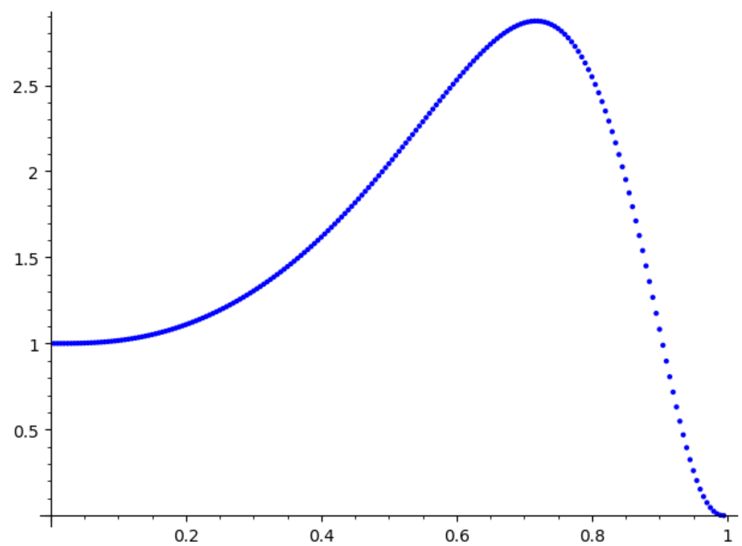

Figure 5.1: Values of γ for $\epsilon = 1$ and $\alpha \in (0, 1)$. (Replicated from Figure 4.2)

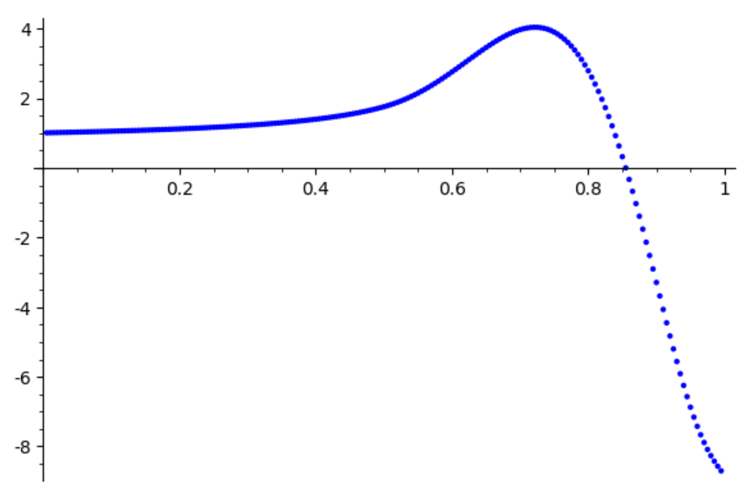

Figure 5.2: Values of γ for $\epsilon = -1$ and $\alpha \in (0, 1/2)$. (Replicated from Figure 4.3)

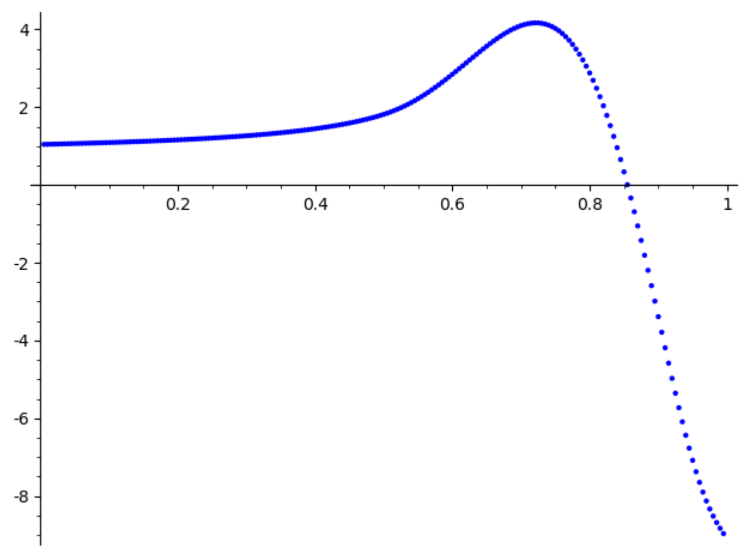

Figure 5.3: Values of γ for $\epsilon = -1$ and $\alpha \in (1/2, 1)$. (Replicated from Figure 4.4)

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