ON THE RATE-DISTORTION-PERCEPTION TRADEOFF FOR LOSSY COMPRESSION

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Lay Abstract

In image compression, perceptual quality plays a crucial role in the lossy reconstruction of images. Blau & Michaeli [8] introduced a mathematical formulation of perceptual quality and defined the information rate-distortion-perception function, which can be viewed as an extension of the classical rate-distortion function. Built upon the seminal work by Blau & Michaeli, we develop a rate-distortion-perception theory with a special focus on binary and vector Gaussain sources. For binary sources, a closed-form expression of the rate-distortion-perception function in the one-shot setting and a complete characterization of the distortion-perception region for an arbitrary representation are established. We then derive partially tight upper and lower bounds on the minimum rate penalty for universal representations and investigate into the point-wise and set-wise successive refinement. For vector Gaussian sources, we characterize the rate- distortion-perception function, which extends the result for the scalar counterpart in [56], and show that in the high-perceptual-quality regime, each component of the reconstruction is strictly correlated with that of the source, which is in contrast to the traditional water-filling solution. We also consider the notion of universal representation where the encoder is fixed and the decoder is adapted to achieve different distortion-perception pairs. We characterize the achievable distortion-perception region for a fixed representation and demonstrate that the

corresponding distortion-perception tradeoff is approximately optimal.

Abstract

Deep generative models when utilized in lossy image compression tasks can reconstruct realistic looking outputs even at extremely low bit-rates, while traditional compression methods often exhibit noticeable artifacts under similar conditions. As a result, there has been a substantial surge of interest in both the information theoretic aspects and the practical architectures of deep learning based image compression. This thesis makes contributions to the emerging framework of rate-distortion-perception theory. The main results are summarized as follows:

- We investigate the tradeoff among rate, distortion, and perception for binary sources. The distortion considered here is the Hamming distortion and the perception quality is measured by the total variation distance. We first derive a closed-form expression for the rate-distortion-perception tradeoff in the one-shot setting. This is followed by a complete characterization of the achievable distortion-perception region for a general representation. We then consider the universal setting [56] in which the encoder is one-size-fits-all, and derive upper and lower bounds on the minimum rate penalty. Finally, we study successive refinement for both point-wise and set-wise versions of perception-constrained lossy compression. A necessary and sufficient condition for point-wise successive

refinement and a sufficient condition for the successive refinability of universal representations are provided.

- Next, we characterize the expression for the rate-distortion-perception function of vector Gaussian sources, which extends the result in the scalar counterpart in [56], and show that in the high-perceptual-quality regime, each component of the reconstruction (including high-frequency components) is strictly correlated with that of the source, which is in contrast to the traditional water-filling solution. This result is obtained by optimizing over all possible encoder-decoder pairs subject to the distortion and perception constraints. We then consider the notion of universal representation where the encoder is fixed and the decoder is adapted to achieve different distortion-perception pairs. We characterize the achievable distortion-perception region for a fixed representation and demonstrate that the corresponding distortion- perception tradeoff is approximately optimal.

Our findings significantly enrich the nascent rate-distortion-perception theory, establishing a solid foundation for the field of learned image compression.

To my family, for their love and support.

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Abbreviations and Notation

Abbreviations

GANs	Generative Adversarial Networks
RNN	Recurrent Neural Network
TV	Total Variation
RDP	Rate-Distortion-Perception
uRDP	Universal Rate-Distortion-Perception
UB	Upper Bound
LB	Lower Bound
KL	Kullback–Leibler
RV	Random Variable
DL	Deep Learning
MSE	Mean Square Error

Fixed-enc Fixed Encoder

KKT Karush–Kuhn–Tucker

ED Eigenvalue Decomposition

1-shot One-shot Setting

Notation

$\mathbb{E}(\cdot)$ Expectation operator

$(\cdot)^T$ Transpose operator

$(\cdot)^{-1}$ Inverse operator

$\mathbf{tr}(\cdot)$ Trace operator

$\mathbf{det}(\cdot)$ Determinant operator

$\mathbf{diag}^{(L)}(\lambda_1, \ldots, \lambda_L)$

$L \times L$ diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_L$

X^n Abbreviation of $(X_1, \ldots, X_n)$

$\{\omega_\ell\}_{\ell=1}^L$ A set $\{\omega_1, \ldots, \omega_L\}$ with elements from 1 to L

$|\mathcal{X}|$ Cardinality of a set $\mathcal{X}$

Σ_X Covariance matrix of X

$I(\cdot;\cdot)$ Mutual information

$D(\cdot\|\cdot)$	Divergence
$H(\cdot)$	Entropy
$H_b(\cdot)$	Binary entropy
$H_t(\cdot,\cdot)$	Entropy of a ternary random variable
$\lvert x\rvert$	Absolute value of x
$\\|\cdot\\|_1$	ℓ^1 norm
$\\|\cdot\\|^2$	ℓ^2 norm or Mean Square Error
$[x]^+$	Maximum of x and 0
$\arg\min_x f(x)$	The value of x at which $f(x)$ attains its minimum
$cov(U,V)$	Covariance of U and V

Declaration of Academic Achievement

1. Zhang, G., **Qian, J**., Chen, J. and Khisti, A., 2021. Universal rate-distortion-perception representations for lossy compression. Advances in Neural Information Processing Systems, 34, pp.11517-11529.

2. **Qian, J**., Zhang, G., Chen, J. and Khisti, A., 2022, March. A Rate-Distortion-Perception Theory for Binary Sources. In International Zurich Seminar on Information and Communication (IZS 2022). Proceedings (pp. 34-38). ETH Zurich.

3. **Qian, J**., Salehkalaibar, S., Liu, H. and Chen, J., 2023. Rate-Distortion-Perception Tradeoff for Lossy Compression of Vector Gaussian Sources, to be submitted to IEEE Transactions on Information Theory.

4. Zhou, S., Salehkalaibar, S., **Qian, J**., Chen, J., Shi, W., Ge, Y. and Tong, W., 2022. On Distributed Lossy Coding of Symmetrically Correlated Gaussian Sources. IEEE Transactions on Communications, 70(12), pp.7957-7972.

5. Zhou, S., **Qian, J**., and Chen, J., 2023. Symmetric Remote Gaussian Source

Coding with a Centralized Encoder, to be submitted to Entropy.

Chapter 1

Introduction

1.1 Deep Learning based Compression

Compression is a fundamental concept in information theory that aims to efficiently represent data by reducing the number of bits required for its transmission [40]. Compression can be broadly classified into two main categories: lossless compression and lossy compression [11]. Lossy compression algorithms achieve higher compression ratios, while it results in a certain amount of information loss. Lossy compression is widely used in applications such as image and video compression, where small distortions are acceptable. Recently, deep learning models have been employed for lossy compression, which offers a promising approach to efficiently encoding and compressing digital data, such as images, videos, and audio signals. By leveraging the power of deep learning, this approach has the potential to overcome the limitations of traditional compression algorithms, offering higher-quality reconstructions. As a consequence, there has been an upsurge of research on deep learning based compression methods [2–5, 19, 24, 28, 29, 34, 39, 44–46]. Earlier studies focused on evaluating autoencoders and recurrent neural network (RNN) architectures for deep learning based compression primarily based on their rate-distortion performance [2, 4, 5, 19, 24, 43–45]. Subsequent research efforts have integrated Generative Adversarial Networks (GANs) based regularization techniques to enhance perceptual quality in deep learning based compression [3, 17, 21, 33, 39, 46, 50]. This leads to reconstructing realistic looking outputs even at very low bit-rates, when traditional compression methods suffer from significant artifacts.

1.2 Measures in Lossy Compression

The following question arises naturally in lossy compression: How to effectively evaluate the quality of the reconstructions? Extensive research have been dedicated to the design and development of quality measures. At the beginning, the focus was placed on distortion measures, such as mean square error (MSE) [1], Peak Signal-to-Noise Ratio (PSNR), Structural Similarity Index (SSIM) [53], Multi-Scale Structural Similarity Index (MS-SSIM) [52] (which is an extension of SSIM), or deep feature based L_2 distances [18, 30]. Such measures usually rely on a ground truth as reference, against which the reconstruction is compared. Nevertheless, recent works noted that achieving low distortion does not always guarantee high perceptual quality. In other words, minimizing distortion alone does not necessarily imply that the reconstructed output will be visually pleasing, especially at low bit-rates. Deep learning based image compression research has shown that prioritizing higher perceptual quality can lead to an increase in distortion [2, 39, 46]. This naturally leads to a tradeoff between optimizing for perceptual quality and optimizing for distortion, as illustrated

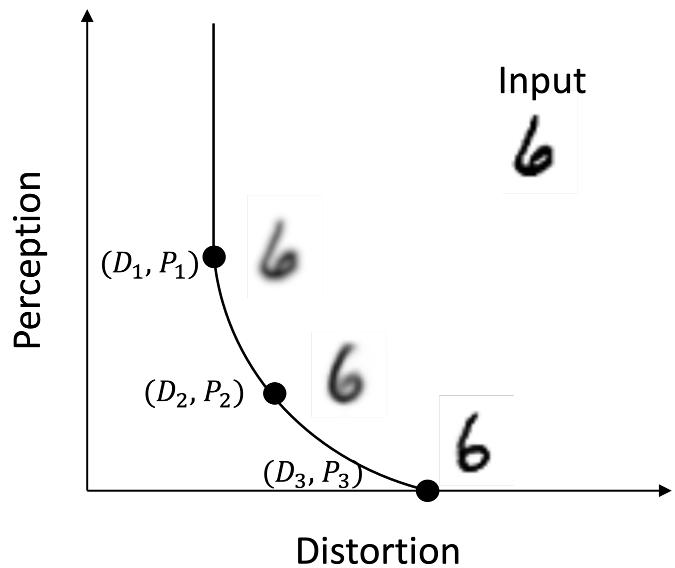

Figure 1.1: A plot of perception-distortion tradeoff curve for a fixed rate. Here, we focus on the low rate regime where this tradeoff can be illustrated clearly. The horizontal axis represents distortion loss, while the vertical axis represents perceptual loss. In both cases, lower values indicate better performance. There are three sets of (D, P) points. (D_1, P_1) represents low distortion and high perception, which is more similar to the input but suffers from blurriness. (D_3, P_3) represents high distortion and perfect perceptual quality, which reduces blurriness at the cost of a less faithful reconstruction of the input (in extreme cases even changing the identity of the digit). And (D_2, P_2) represents the middle point, which produces a median level distortion and perception reconstruction.

in Fig 1.1. Motivated by this, perception-oriented measures have been introduced, aiming to better quantify what human would consider visually pleasing. An image with high perceptual quality is typically characterized by clarity and the absence of visual artifacts. Unlike distortion, the assessment of perceptual quality is considered to be inherently no-reference, meaning that it does not require a reference image for comparison, and only uses statistical features of inputs, specifically measuring the degree to which the reconstruction looks like a natural image. Some commonly used no-reference metrics include Blind/Referenceless Image Spatial Quality Evaluator (BRISQUE), Natural Image Quality Evaluator (NIQE), Perceptual Image Quality Evaluator (PIQE) [30, 47], and Fréchet Inception Distance (FID) [16].

Natural images can be modeled as a space equipped with a prior distribution [31]. Lossy compression is an operation that acts on this space, transforming the prior distribution of input images to a distribution of reconstructed images. To ensure that the reconstructed images look natural, a reasonable way is to require the two probability distributions to be close to each other [23]. From this perspective, the perceptual quality can be mathematically defined as a certain divergence between the distribution $P_{\hat{X}}$ of the reconstruction $\hat{X}$ and the distribution P_X of the input source X, expressed as $d(P_X, P_{\hat{X}})$. In particular, many no-reference image quality measures are based on evaluating the deviations from natural scene statistics [31, 32, 51], which have been shown to correlate well with human opinion scores. In some sense, human visual perception is highly adapted for extracting structural information from a scene [54]. Some works also quantify perceptual quality via real vs. fake user studies, which examine the ability of human observers to tell whether the reconstruction is real or the output of an algorithm [7, 12, 17, 38, 57] (similarly to the idea underlying generative adversarial nets). This line of thinking also boils down to comparing two distributions, one for the input images and the other for the reconstructed ones.

It is obvious that perception loss, as defined above, is not the same as distortion loss. In particular, minimizing the perception loss does not necessarily result in low distortion. For example, if the decoder ignores the source image and simply generates a random sample according to the input distribution, it can achieve perfect perceptual quality but at the expense of very high distortion.

The commonly used divergences include KL divergence [20], Jensen-Shannon divergence [25], Wasserstein distance [48], and TV distance [35]. However, identifying the divergence that is best aligned with human perception remains an ongoing project.

1.3 Rate-Distortion Theory

1.3.1 Rate-Distortion Function

Lossy compression algorithms are typically investigated within the framework of rate-distortion theory, which is an important branch of information theory. Rate-distortion theory [11] offers a computable characterization of an an (operational) objective associated with optimizing an encoder-decoder pair in the presence of a bit interface, as visualized in Fig. 1.2. Specifically, the objective is to minimize the rate of the bit interface subject to a prescribed distortion constraint on the reconstruction. In classical rate-distortion theory, there is no loss of optimality in assuming that both the encoder and decoder are deterministic. The main result of this theory is that the aforementioned fundamental rate-distortion tradeoff is delineated by the rate-distortion function [41] defined as

$$\begin{aligned} R(D) = \inf_{p_{\hat{X}|X}} \; & I(X;\hat{X}) \\ \text{s.t.} \quad & \mathbb{E}[\Delta(X,\hat{X})] \leq D, \end{aligned}$$

where $I(X,\hat{X})$ is the mutual information between the input source X and reconstruction $\hat{X}$, and $\Delta : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ is a distortion function.

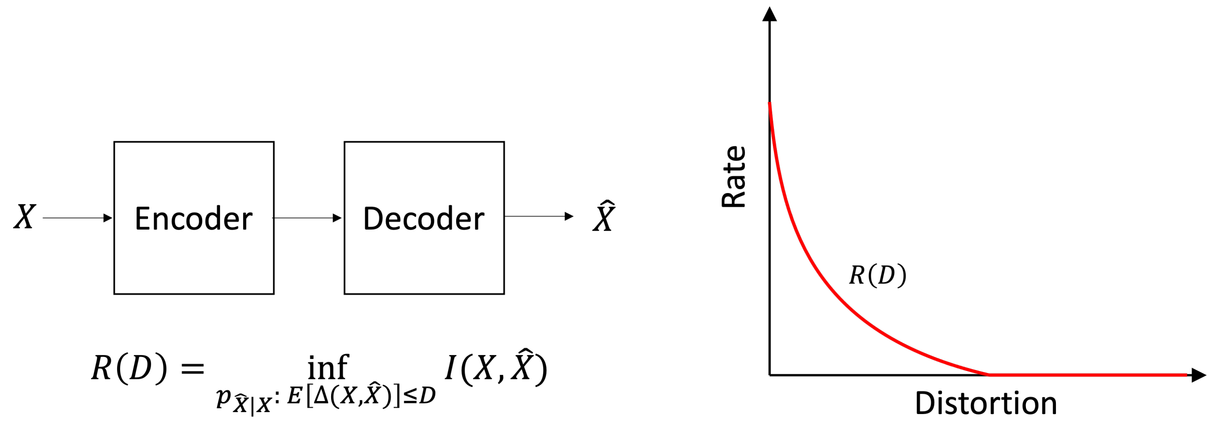

Figure 1.2: An illustration of the rate-distortion function in lossy compression. (i) Assume there is an input source $X \sim P_X$, the encoder describes the input source X by an index, then the decoder represents X by an estimate $\hat{X}$. If the expected distortion is bounded by D, then the lowest achievable rate R is characterized by the information rate-distortion function $R(D)$. (ii) The rate-distortion function is an non-increasing convex function.

1.3.2 Reverse Water-Filling Solution

For a vector Gaussian source with MSE distortion measure, the rate-distortion function is given by the reverse water-filling solution [11]. Specifically, this solution states that to minimize the rate subject to the distortion constraint, one has to evenly allocate the distortion across different eigen-dimensions of the source except for those dimensions that get saturated, as shown in Fig. 1.3. This leads to the following parametric expression of the rate-distortion function [11]

$$R(D) = \sum_{i=1}^{n} \frac{1}{2} \log \frac{\lambda_i}{D_i},$$

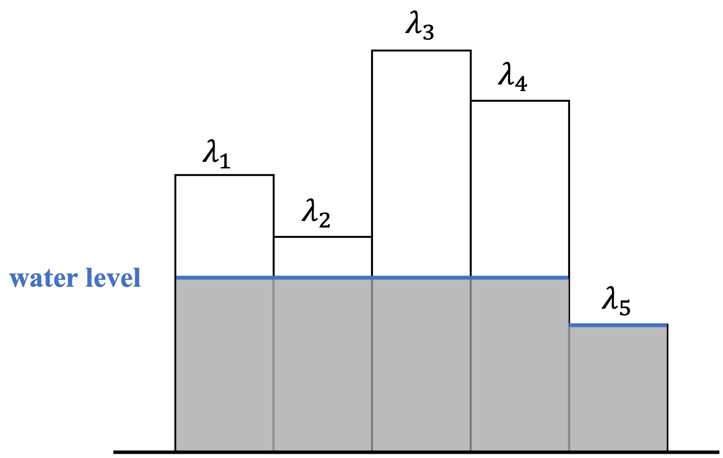

Figure 1.3: An illustration of reverse water-filling solution. Let $X_i \sim N(0, \lambda_i, i = 1, 2, ..., 5)$ be independent Gaussian random variables. Based on the distortion constraint, we choose a water level and only encode those random variables with variances greater than the water level. No bits are allocated to those random variables with variance less than the water level. More generally, the rate-distortion function for a multivariate normal vector can be obtained by reverse water-filling in the eigenspace.

where

$$D_i = \begin{cases} \omega, & \text{if} \quad \omega < \lambda_i, \\ \lambda_i, & \text{if} \quad \omega \geq \lambda_i, \end{cases}$$

with ω chosen so that $\sum_{i=1}^{n} D_i = D$. Parameters $\lambda_i > 0, i = 1, 2, ..., n$ are the eigenvalues of the covariance matrix of the given vector Gaussian source.

For an i.i.d. Gaussian source with MSE distortion measure, the reconstruction given by the reverse water-filling solution is generally of lower power than the source. This means that the dimensions associated with small eigenvalues are often left uncoded. In practice, such dimensions typically correspond to high-frequency components. That is why compressed images tend to look blurry at low bit-rates [49].

1.4 Rate-Distortion-Perception Theory

Considering the importance of perception in image assessment, Blau and Michaeli [8] introduced a mathematical formulation of perceptual quality and made a first attempt to develop a rate-distortion-perception framework for learned image compression. While rate-distortion theory focuses on finding the optimal tradeoff between the rate and distortion, rate-distortion-perception theory considers a three-way tradeoff that involves a third factor: perception, as shown in Fig. 1.4. The rate-distortion-perception function [8] is defined as

$$\begin{aligned} R(D,P) = \inf_{p_{\hat{X}|X}} \ & I(X;\hat{X}) \\ \text{s.t.} \quad & \mathbb{E}[\Delta(X,\hat{X})] \le D \\ \text{s.t.} \quad & d(p_X, p_{\hat{X}}) \le P, \end{aligned}$$

where $d : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+$ is the divergence between two distributions that is convex in its second argument. It has been proved that the rate-distortion-perception function is monotonically non-increasing in D and P and it is a convex function [8].

1.5 Contributions and Thesis Organization

The thesis includes two main topics and consists of three papers that investigate different representation schemes within the rate-distortion-perception framework. The contributions are described in the abstract of each chapter and are summarized below.

- In Chapter 2, we explore the tradeoff among rate, distortion, and perception for

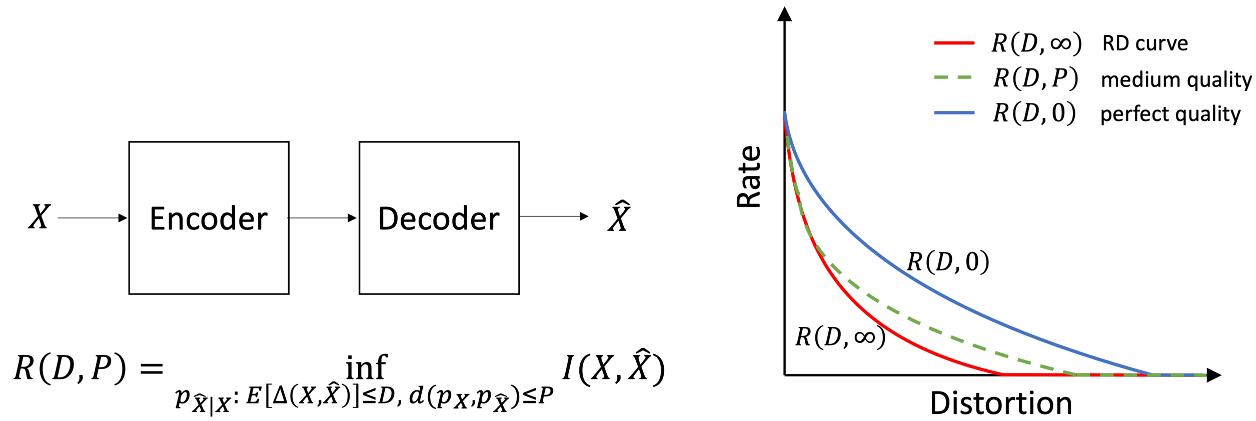

Figure 1.4: An illustration of the rate-distortion-perception function. (i) The perception constraint is added into the optimization problem. (ii) The red curve represents the rate-distortion function, which does not ensure good perceptual quality. When taking perceptual quality into account, the rate-distortion-perception function elevates (green and blue curves). This means that good perceptual quality comes at the expense of either a higher rate, a higher distortion, or both.

binary sources. We first derive a closed-form expression for the rate-distortion-perception tradeoff in the one-shot setting. This is followed by a complete characterization of the achievable distortion-perception region for a general representation. We then consider the universal setting in which the encoder is one-size-fits-all, and derive upper and lower bounds on the minimum rate penalty. Finally, we study successive refinement for both point-wise and set-wise versions of perception-constrained lossy compression. A necessary and sufficient condition for point-wise successive refinement and a sufficient condition for the successive refinability of universal representations are provided.

Qian, J., Zhang, G., Chen, J. and Khisti, A., 2022, March. A Rate-Distortion-Perception Theory for Binary Sources. In International Zurich Seminar on Information and Communication (IZS 2022). Proceedings (pp. 34-38). ETH

Zurich.

- In Chapter 3, we analyze the tradeoff between rate, distortion, and perception for vector Gaussian sources. We adopt MSE as the distortion metric and use either KL divergence or Wassertein-2 distance to measure the perception quality. We first characterize the rate-distortion-perception function of vector Gaussian sources and show that in the high-perceptual-quality regime, each component of the reconstruction is strictly correlated with that of the source which is in contrast to the traditional reverse water-filling solution. We then explore the concept of universal representations and demonstrate that it is possible to construct universal representations that approximately achieve the optimal tradeoff between distortion and perception.

  Qian, J., Salehkalaibar, S., Liu, H. and Chen, J., 2023. Rate-Distortion-Perception Tradeoff for Lossy Compression of Vector Gaussian Sources, to be submitted to IEEE Transactions on Information Theory.

- In Chapter 4, we summarize our findings and provide valuable insights for future research directions.

Chapter 2

A Rate-Distortion-Perception Theory for Binary Sources

2.1 Abstract

Building upon a series of recent works on perception-constrained lossy compression, we develop a rate-distortion-perception theory for binary sources under Hamming distortion and TV perception losses. It includes a closed-form expression of the rate-distortion-perception function in the one-shot setting, a complete characterization of the distortion-perception region for an arbitrary representation, partially tight upper and lower bounds on the minimum rate penalty for universal representations, a necessary and sufficient condition for point-wise successive refinement, and a sufficient condition for the successive refinability of universal representations.

2.2 Introduction

Recently, there has been an upsurge of research on perception-constrained lossy compression for images or videos. Within traditional compression, the well-established rate-distortion formulation is to minimize some notion of distortion under the condition that the given bit rate is not exceeded. In contrast, perceptually-constrained lossy compression takes into account the notion of *perceptual quality*, which turns out to be distinct from the notion of distortion, as well. The motivation for considering both the traditional distortion and the perceptual quality comes from the fact that in many cases, minimizing distortion does not produce visually pleasing results. Such a fact was exemplified by many remarkable deep learning enhanced lossy compression works capable of operating at extremely low rates, such as [39, 6]. In perception-constrained lossy compression, instead, the target is to find the best trade-off among three quantities: rate, distortion and perception. Following the success of deep learning for lossy compression, a mathematical view of this topic was initiated and investigated by Blau and Michaeli [8].

Concretely, perceptual quality aims to quantify the degree of visual satisfaction as measured by the human visual perception system and unlike distortion is taken to be fully no-reference (i.e., not with respect to any *particular* source sample, such as a single image or video). Blau and Michaeli adopt a notion of perceptual quality defined by the divergence (e.g., the KL divergence, the Wasserstein distance, and the TV distance) between the *distribution* of the original source and that of the reconstruction, with the property that perfect perceptual quality is obtained only when the two distributions are identical. By basing this measure on the distributions, we again emphasize that the perceptual quality is in fact a global and inherently no-reference measure of the reconstruction quality. In contrast, the distortion is a local one, expressed in terms of the symbol-by-symbol "distance". As mentioned previously, it may not be possible to attain both low distortion and high perceptual quality at the same time, in the sense that one quality must be sacrificed to improve the other one [6]. Optimizing the tradeoff between distortion and perception, incorporated with distribution-preserving lossy compression [46], is the central idea in studying the rate-distortion-perception tradeoff in the seminal work [8]. However, we note that various versions of distribution-constrained lossy compression have been studied before [8] in information theory literature (e.g., [55, 36, 37]).

In this paper, we investigate the tradeoff among rate, distortion, and perception for binary sources. The distortion considered here is the Hamming distortion and the perception quality is measured by the TV distance. We first derive a closed-form expression for the rate-distortion-perception tradeoff in the one-shot setting. This is followed by a complete characterization of the achievable distortion-perception region for a general representation. We then consider the universal setting [56] in which the encoder is one-size-fits-all, and derive upper and lower bounds on the minimum rate penalty. Finally, we study successive refinement for both point-wise and set-wise versions of perception-constrained lossy compression. A necessary and sufficient condition for point-wise successive refinement and a sufficient condition for the successive refinability of universal representations are provided.

2.3 Problem Definitions and Known Results

2.3.1 Rate-Distortion-Perception Function and Universal Representation

Let $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be a distortion measure and $\omega : \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{X}) \to \mathbb{R}_+$ be a divergence, where $\mathcal{X}$ is the source/reconstruction alphabet and $\mathcal{P}(\mathcal{X})$ denotes the set of distributions defined on $\mathcal{X}$. We assume that ω is convex in its second argument. Let Θ be a non-empty set of (D, P) pairs with each pair being a distortion-perception objective.

Definition 2.3.1 (One-Shot Rate-Distortion-Perception Function) *A rate R is said to be one-shot achievable with respect to Θ for the source variable X if we can find a random seed U (which is independent of X) and an encoder $p_{V|XU}$ with $H(V|U) \leq R$ such that for every $(D, P) \in \Theta$, a decoder $p_{\hat{X}|VU}$ can be constructed to meet the constraints $\mathbb{E}[d(X, \hat{X})] \leq D$ and $\omega(p_X, p_{\hat{X}}) \leq P$, where the joint distribution $p_{XV\hat{X}U}$ is assumed to factor as $p_X p_U p_{V|XU} p_{\hat{X}|VU}$. The infimum of such R is denoted by $R^*(\Theta)$. In the case where Θ consists of a single (D, P) pair, we simply write $R^*(\Theta)$ as $R^*(D, P)$ and refer to it as the one-shot rate-distortion-perception function:*

$$R^*(D, P) := \inf_{p_U, p_{V|XU}, p_{\hat{X}|VU}} H(V|U)$$
$$\text{s.t.} \quad \mathbb{E}[d(X, \hat{X})] \leq D, \quad \omega(p_X, p_{\hat{X}}) \leq P.$$

The random seed U acts as a shared source of randomness, which plays an important role in our formulation, as Fig. 2.1 shows. Note that with U available at both the encoder and decoder, V can be losslessly represented by approximately $H(V|U)$

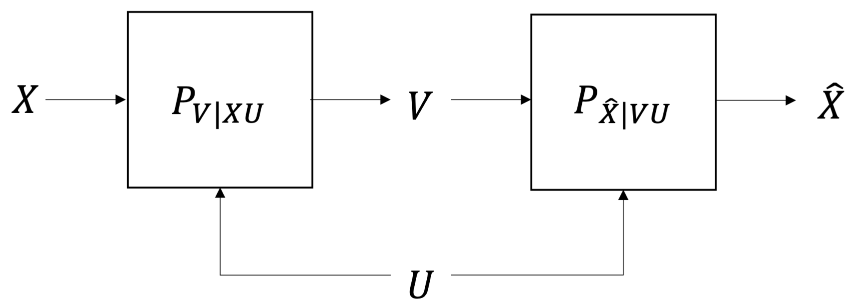

Figure 2.1: One-shot setting with common randomness.

bits using variable-length codes. This provides an operational justification of the rate constraint $H(V|U) \leq R$.

Let $\mathcal{P}_{Z|X}(\Theta)$ denote the set of conditional distributions $p_{Z|X}$ such that for every $(D, P) \in \Theta$, there exists a conditional distribution $p_{\hat{X}|Z}$ satisfying $\mathbb{E}[d(X, \hat{X})] \leq D$ and $\omega(p_X, p_{\hat{X}}) \leq P$, where the joint distribution $p_{XZ\hat{X}}$ is assumed to factor as $p_X p_{Z|X} p_{\hat{X}|Z}$. Define

$$R(\Theta) \triangleq \inf_{p_{Z|X} \in \mathcal{P}_{Z|X}(\Theta)} I(X; Z).$$

Theorem 2.3.1 *It holds that* $R(\Theta) \leq R^*(\Theta) \leq R(\Theta) + \log(R(\Theta) + 1) + 4$*. Moreover,* *in the case where* Θ *consists of a finite number of* (D, P) *pairs,*

$$R^*(\Theta) = \inf_{p_{\hat{X}_\Theta U|X}} H(\hat{X}_\Theta|U) \tag{2.3.1}$$

$$\text{subject to} \quad I(X;U) = 0, \tag{2.3.2}$$

$$H(\hat{X}_\Theta|X, U) = 0, \tag{2.3.3}$$

$$\mathbb{E}[d(X, \hat{X}_{D,P})] \leq D, \quad (D, P) \in \Theta, \tag{2.3.4}$$

$$\omega(p_X, p_{\hat{X}_{D,P}}) \leq P, \quad (D, P) \in \Theta, \tag{2.3.5}$$

where $\hat{X}_\Theta = \{\hat{X}_{D,P}\}_{(D,P)\in\Theta}$.

Proof: The first statement was established in [56] (see also [42] for the special case $\Theta = \{(D, P)\}$) by using the strong functional representation lemma [22]. The second statement follows by showing that there is no loss of optimality in setting $V = \hat{X}_\Theta$ and restricting it to be a deterministic function of (X, U). □

Definition 2.3.2 (Asymptotic Rate-Distortion-Perception Function) *A rate* R *is said to be asymptotically achievable with respect to* Θ *for the i.i.d. source sequence* $\{X(t)\}_{t=1}^{\infty}$ *with each component following the distribution* p_X *if for some positive integer* n*, we can find a random seed* U *and an encoder* $p_{V|X^nU}$ *with* $\frac{1}{n}H(V|U) \leq R$ *such that for every* $(D, P) \in \Theta$*, a decoder* $p_{\hat{X}^n|VU}$ *can be constructed to meet the constraints* $\frac{1}{n}\sum_{t=1}^{n} \mathbb{E}[d(X(t), \hat{X}(t))] \leq D$ *and* $\frac{1}{n}\sum_{t=1}^{n} \omega(p_{X(t)}, p_{\hat{X}(t)}) \leq P$*, where the joint distribution* $p_{X^nV\hat{X}^nU}$ *is assumed to factor as* $p_{X^n}p_Up_{V|X^nU}p_{\hat{X}^n|VU}$*. The infimum of such* R *is denoted by* $R^{(\infty)}(\Theta)$*. In the case where* Θ *consists of a single* (D, P) *pair, we simply write* $R^{(\infty)}(\Theta)$ *as* $R^{(\infty)}(D, P)$ *and refer to it as the asymptotic*

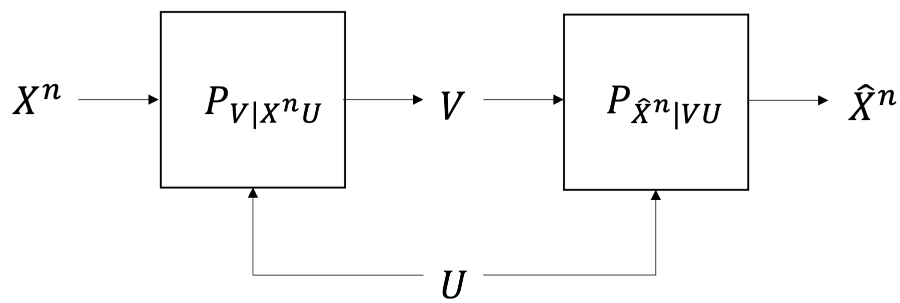

Figure 2.2: Asymptotic setting with common randomness.

rate-distortion-perception function:

$$R^{(\infty)}(D,P) := \inf_{p_U, p_{V|X^nU}, p_{\hat{X}^n|VU}} \frac{1}{n} H(V|U)$$
$$s.t. \quad \frac{1}{n}\sum_{t=1}^{n} \mathbb{E}[d(X(t),\hat{X}(t))] \leq D, \quad \frac{1}{n}\sum_{t=1}^{n} \omega(p_{X(t)}, p_{\hat{X}(t)}) \leq P.$$

Fig. 2.2 depicts the asymptotic setting with common randomness.

As a consequence of Theorem 2.3.1, the following result holds [56] (see also [36, 37, 42] for the special case $\Theta = \{(D,P)\}$).

Theorem 2.3.2 *We have* $R^{(\infty)}(\Theta) = R(\Theta)$.

In view of Theorem 2.3.2, the asymptotic source coding rate is completely characterized by $R(\Theta)$, and such a quantity is expressed in terms of optimization over random variables Z satisfying certain conditions. Hence, we can interpret any random variable Z jointly distributed with X as a *representation* (or reconstruction random variable) of X.

Definition 2.3.3 (Universal Representation) *Given a representation Z of X, its distortion-perception region, denoted by $\Pi(p_{Z|X})$, is the set of all (D,P) pairs for which there exists $p_{\hat{X}|Z}$ satisfying $\mathbb{E}[d(X,\hat{X})] \leq D$ and $\omega(p_X, p_{\hat{X}}) \leq P$, where the* *joint distribution $p_{XZ\hat{X}}$ is assumed to factor as $p_X p_{Z|X} p_{\hat{X}|Z}$. We say that Z is a Θ-universal representation of X if $\Theta \subseteq \Pi(p_{Z|X})$.*

Note that $R^{(\infty)}$ is the minimum rate needed for a fixed encoder to cope with the distortion-perception objectives in Θ. In light of Theorem 2.3.2, it also coincides with the infimum of $I(X;Z)$ over all Θ-universal representations Z of X. On the other hand, $\sup_{(D,P)\in\Theta} R^{(\infty)}(D,P)$ is the rate required to meet the most demanding objective in Θ. As such, $\Delta(\Theta) \triangleq R^{(\infty)}(\Theta) - \sup_{(D,P)\in\Theta} R^{(\infty)}(D,P)$ characterizes the extra rate incurred by meeting all objectives in Θ with the encoder fixed. We can also interpret $\Delta(\Theta)$ equivalently as the minimum rate penalty for using Θ-universal representations as opposed to choosing an optimal representation for each objective in Θ. We are particularly interested in the case $\Theta = \Theta(R)$, where $\Theta(R)$ is the set of distortion-perception objectives achievable with dedicated encoders at rate R, i.e., $\Theta(R) \triangleq \{(D,P) : R^{(\infty)}(D,P) \leq R\}$. It will be seen that for the binary case studied in Section 2.4, $\Delta(\Theta(R))$ is negligible compared to R, namely, objective-agnostic encoders/representations can be (almost) as rate-efficient as objective-aware encoders/representations.

2.3.2 Two-Stage Coding and Successive Refinement

Let Θ_1 and Θ_2 be two non-empty sets of (D,P) pairs.

Definition 2.3.4 (One-Shot Version) *A rate pair (R_1, R_2) is said to be one-shot successively achievable with respect to (Θ_1, Θ_2) for the source variable X if we can find a random seed U and an encoder pair $(p_{V_1|XU}, p_{V_2|XV_1U})$ with U independent of X, $H(V_1|U) \leq R_1$, and $H(V_2|V_1,U) \leq R_2$ such that for every $(D_1,P_1) \in \Theta_1$*

and $(D_2, P_2) \in \Theta_2$*, a decoder pair* $(p_{\hat{X}_1|V_1U}, p_{\hat{X}_1|V_1V_2U})$ *can be constructed to meet the constraints* $\mathbb{E}[d(X, \hat{X}_i)] \leq D_i$ *and* $\omega(p_X, p_{\hat{X}_i}) \leq P_i$*,* $i = 1, 2$*, where the joint distribution* $p_{XV_1V_2\hat{X}_1\hat{X}_2U}$ *is assumed to factor as* $p_X p_U p_{V_1|XU} p_{V_2|XV_1U} p_{\hat{X}_1|V_1U} p_{\hat{X}_2|V_2U}$*. The closure of the set of such* (R_1, R_2) *is denoted by* $\mathcal{R}^*(\Theta_1, \Theta_2)$*.*

Let $\mathcal{P}_{Z_1Z_2|X}(\Theta_1, \Theta_2)$ denote the set of $p_{Z_1Z_2|X}$ such that for every $(D_1, P_1) \in \Theta_1$ and $(D_2, P_2) \in \Theta_2$, there exists $(p_{\hat{X}_1|Z_1}, p_{\hat{X}_2|Z_2})$ satisfying $\mathbb{E}[d(X, \hat{X}_i)] \leq D_i$ and $\omega(p_X, p_{\hat{X}_i}) \leq P_i$, $i = 1, 2$, where the joint distribution $p_{XZ_1Z_2\hat{X}_1\hat{X}_2}$ is assumed to factor as $p_X p_{Z_1Z_2|X} p_{\hat{X}_1|Z_1} p_{\hat{X}_2|Z_2}$. Define

$$\begin{aligned}
\underline{\mathcal{R}}(\Theta_1, \Theta_2) \triangleq & \bigcup_{p_{Z_1Z_2|X} \in \mathcal{P}_{Z_1Z_2|X}(\Theta_1,\Theta_2)} \{(R_1, R_2) \in \mathbb{R}_+^2 : \\
& R_1 \geq I(X; Z_1) + \log(I(X; Z_1) + 1) + 4, \\
& R_1 + R_2 \geq I(X; Z_1, Z_2) + \log(I(X; Z_1) + 1) \\
& \quad + \log(I(X; Z_2|Z_1) + 1) + 8\}, \\
\overline{\mathcal{R}}(\Theta_1, \Theta_2) \triangleq & \bigcup_{p_{Z_1Z_2|X} \in \mathcal{P}_{Z_1Z_2|X}(\Theta_1,\Theta_2)} \{(R_1, R_2) \in \mathbb{R}_+^2 : \\
& R_1 \geq I(X; Z_1), \\
& R_1 + R_2 \geq I(X; Z_1, Z_2)\}.
\end{aligned}$$

Fig. 2.3 illustrates the structure of the two-stage coding. Similarly to Theorem 2.3.1, we have the following theorem [56].

Theorem 2.3.3 *We have* $\mathrm{cl}(\underline{\mathcal{R}}(\Theta_1, \Theta_2)) \subseteq \mathcal{R}^*(\Theta_1, \Theta_2) \subseteq \mathrm{cl}(\overline{\mathcal{R}}(\Theta_1, \Theta_2))$*.*

Definition 2.3.5 (Asymptotic Version) *A rate pair* (R_1, R_2) *is said to be asymptotically successively achievable with respect to* (Θ_1, Θ_2) *for the i.i.d. source sequence*

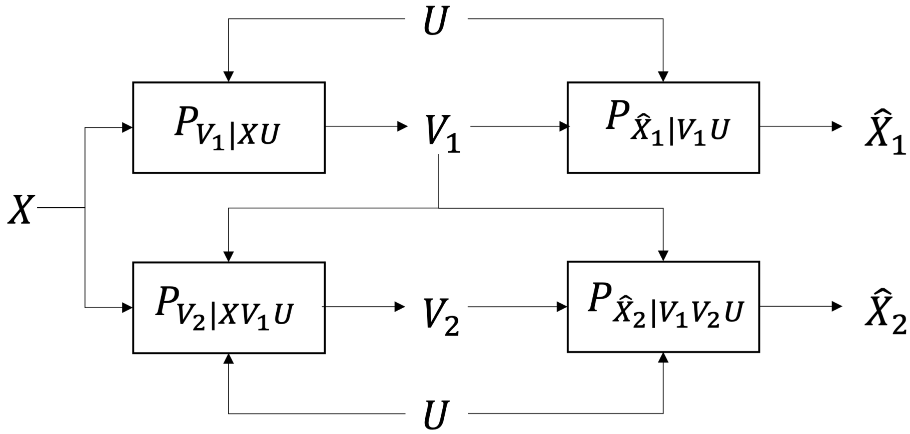

Figure 2.3: Two-stage coding.

$\{X(t)\}_{t=1}^{\infty}$ *if we can find a random seed* U *and an encoder pair* $(p_{V_1|X^nU}, p_{V_2|X^nV_1U})$ *with* $\frac{1}{n}H(V_1|U) \leq R_1$ *and* $\frac{1}{n}H(V_2|V_1,U) \leq R_2$ *such that for every* $(D_1, P_1) \in \Theta_1$ *and* $(D_2, P_2) \in \Theta_2$*, a decoder pair* $(p_{\hat{X}_1^n|V_1U}, p_{\hat{X}_1^n|V_1V_2U})$ *can be constructed to meet the constraints* $\frac{1}{n}\sum_{t=1}^{n}\mathbb{E}[d(X(t),\hat{X}_i(t))] \leq D_i$ *and* $\omega(p_X, \frac{1}{n}\sum_{t=1}^{n} p_{\hat{X}_i(t)}) \leq P_i$, $i = 1, 2$*, where the joint distribution* $p_{X^nV_1V_2\hat{X}_1^n\hat{X}_2^nU}$ *is assumed to factor as* $p_{X^n}p_Up_{V_1|X^nU}p_{V_2|X^nV_1U}p_{\hat{X}_1^n|V_1U}p_{\hat{X}_2^n|V_2U}$*. The closure of the set of such* (R_1, R_2) *is denoted by* $\mathcal{R}^{(\infty)}(\Theta_1,\Theta_2)$*. We say that successive refinement from* Θ_1 *to* Θ_2 *is feasible if* $(R^{(\infty)}(\Theta_1), R^{(\infty)}(\Theta_2) - R^{(\infty)}(\Theta_1)) \in \mathcal{R}^{(\infty)}(\Theta_1,\Theta_2)$*.*

The following result [56] is a corollary of Theorem 2.3.3.

Theorem 2.3.4 *We have* $\mathcal{R}^{(\infty)}(\Theta_1,\Theta_2) = \mathrm{cl}(\overline{\mathcal{R}}(\Theta_1,\Theta_2))$*. Moreover, successive refinement from* Θ_1 *to* Θ_2 *is feasible if and only if* $(R^{(\infty)}(\Theta_1), R^{(\infty)}(\Theta_2) - R^{(\infty)}(\Theta_1)) \in \mathrm{cl}(\overline{\mathcal{R}}(\Theta_1,\Theta_2))$*.*

2.4 Main Results

Throughout this section, we assume $\mathcal{X} = \{0, 1\}$ and $X \sim \text{Bern}(q)$ (i.e., X is a binary source with $p_X(1) = 1 - p_X(0) = q \in (0, \frac{1}{2}]$); moreover, we focus on the Hamming distortion $d(x, \hat{x}) = 1\{x \neq \hat{x}\}$ and the TV distance $w(p_X, p_{\hat{X}}) = \frac{1}{2}\|p_X - p_{\hat{X}}\|_1$.

We first consider the case $\Theta = \{(D, P)\}$ and characterize the one-shot rate-distortion-perception function. Without loss of generality, it is assumed that $P \in [0, q]$.

Theorem 2.4.1 *For a binary source $X \sim$ Bern(q), under Hamming distortion and TV perception losses,*

$$R^*(D, P) = \begin{cases} \frac{q-D}{q} H_b(q) & 0 \leq D \leq P, \\ \frac{(1-q)+P-\frac{D+P}{2q}}{1-q} H_b(q) & P < D \leq D', \\ 0 & \textit{otherwise}, \end{cases}$$

where $H_b(\cdot)$ denotes the binary entropy function and $D' = 2q(1-q) - (1-2q)P$.

Proof: We shall determine the expression of $R^*(D, P)$ by solving the optimization problem in (2.3.1) with $\Theta = \{(D, P)\}$. Without loss of optimality, one can restrict the alphabet size of U to be no more than 4. In fact, $(X, \hat{X})$ takes values in $\{0, 1\}^2$. Hence, without loss of optimality, we can assume that the random seed U takes values in $\{0, 1, 2, 3\}$ and $\hat{X} = X$ if $U = 0$; $\hat{X} = 1 - X$ if $U = 1$; $\hat{X} = 0$ if $U = 2$; and $\hat{X} = 1$ if $U = 3$. (which does not the value of the optimization problem in (2.3.1)). Denoting the probability values $p_U(i) = \epsilon_i, i \in [0 : 3]$, we can rewrite the optimization problem in (2.3.1) as a linear program (with nonnegative variables $(\epsilon_i)_{i \in [0:3]}$ such that $\sum_i \epsilon_i = 1$), since the objective function and the constraints in (2.3.4) and (2.3.5)

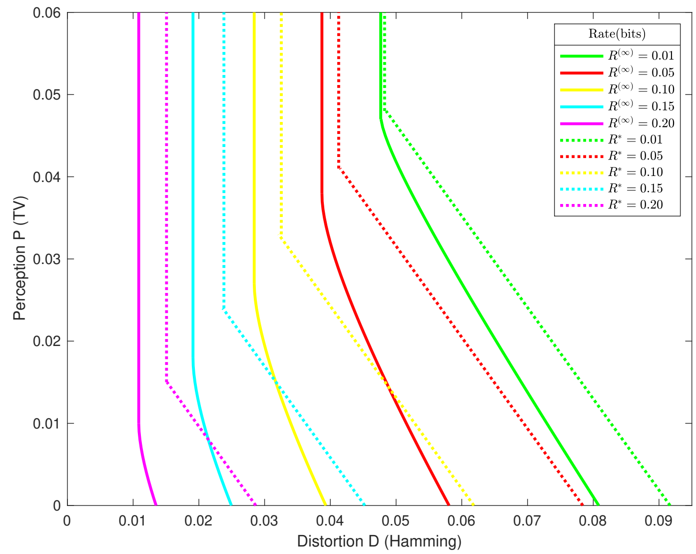

Figure 2.4: Plots of perception-distortion curves for different bit rates, where solid curves denote the asymptotic case while dotted curves denote the one-shot case.

are linear in p_U. Solving this linear program, we obtain the desired result. Detailed proofs are provided in the appendix. □

For comparison, we present the asymptotic rate-distortion-perception function [8] in the following theorem.

Theorem 2.4.2 *For a binary source $X \sim Bern(q)$, under Hamming distortion and TV perception losses,*

$$R^{(\infty)}(D,P) = \begin{cases} H_b(q) - H_b(D), & D \in \mathcal{S}_1, \\ 2H_b(q) + H_b(q-P) & \\ \quad -H_t(\frac{D-P}{2}, q) & \\ \quad -H_t(\frac{D+P}{2}, 1-q), & D \in \mathcal{S}_2, \\ 0, & D \in \mathcal{S}_3. \end{cases}$$

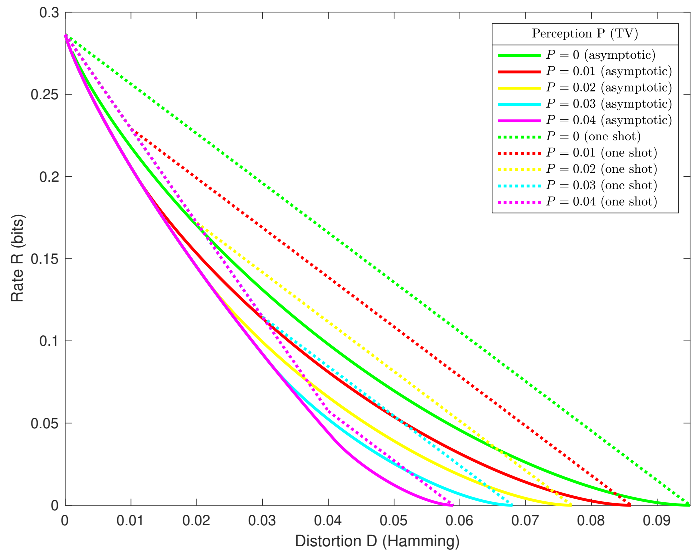

Figure 2.5: Plots of rate-distortion curves for different perception qualities, where solid curves denote the asymptotic case while dotted curves denote the one-shot case.

where $H_t(\alpha, \beta)$ *denotes the entropy of a ternary random variable with probability values* $(\alpha, \beta, 1-\alpha-\beta)$*. Here,* $S_1 = [0, D_1]$*,* $S_2 = (D_1, D_2]$*, and* $S_3 = (D_2, \infty)$ *with* $D_1 = \frac{P}{1-2(q-P)}$ *and* $D_2 = 2q(1-q) - (1-2q)P$*.*

Fig. 2.4 plots perception-distortion curves for different rates, comparing the asymptotic case and one-shot case under the same bit rate. Note that the trade-off curves for the asymptotic case always lie below their counterparts for the one-shot case. A similar phenomenon can be seen from Fig. 2.5, which plots rate-distortion curves for different perception qualities.

Let Z be a representation of a binary source $X \sim \text{Bern}(q)$ with $p_Z(i) = q_i$ and $p_{X|Z}(1|i) = \epsilon_i$, $i \in [n]$, where $\sum_{i=1}^n q_i = 1$ and $\sum_{i=1}^n q_i\epsilon_i = q$. Without loss of generality, we assume that the values of $q_i|1-2\epsilon_i|, i \in [n]$ are in ascending order as i increases. Let j_k with $k \in [m]$ denote the k-th index at which $\epsilon_{j_k} \leq 0.5$. Denote

$k^+ \in [m]$ as the first index such that $j_{k^+} > k$ and $\epsilon_{j_{k^+}} \leq 0.5$, if it exists. Define k^* to be the first positive integer satisfying $\frac{\sum_{i=k^*}^{n} q_i(1-\epsilon_i) - \sum_{i=(k^*)^+}^{m} q_{j_i}}{q_k^*} \leq 1 - \epsilon_{k^*}$.

Theorem 2.4.3 *Let Z be a representation of a binary source $X \sim Bern(q)$ as specified above. Under Hamming distortion and TV perception losses, the lower boundary of $\Pi(p_{Z|X})$ is piecewise linear with k^* turning points $\{(D_k, P_k)\}_{k=1}^{k^*}$ given by*

$$
\begin{aligned}
D_k &= \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k} q_i(2\epsilon_i - 1)(1-\epsilon_i) \\
&\quad + \sum_{i=k^+}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \quad k = 1, \ldots, k^* - 1, \\
P_k &= \left| \sum_{i=k+1}^{n} q_i(1-\epsilon_i) - \sum_{i=k^+}^{m} q_{j_i} \right|, \quad k = 1, \ldots, k^* - 1, \\
D_{k^*} &= \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k^*-1} q_i(2\epsilon_i - 1)(1-\epsilon_i) \\
&\quad + (2\epsilon_{k^*} - 1)\left(\sum_{i=k^*}^{n} q_i(1-\epsilon_i) - \sum_{i=(k^*)^+}^{m} q_{j_i} \right) \\
&\quad + \sum_{i=(k^*)^+}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \\
P_{k^*} &= 0.
\end{aligned}
$$

Proof: We minimize the Hamming distortion $\mathbb{E}[1\{X \neq \hat{X}\}]$ (i.e., $\Pr[X \neq \hat{X}]$) over all the conditional distribution $p_{\hat{X}|Z}$ under the perception constraint $\omega(p_{\hat{X}}, p_X) \leq P$. One can easily observe that this is also a linear program. Solving this linear program, we obtain the desired result. Detailed proofs are provided in the appendix. □

Next consider the case $\Theta = \Theta(R)$. We start by introducing some quantities which are needed for bounding $R^{(\infty)}(\Theta(R))$. Let $D_1 = D_1(R)$ and $D_2 = D_2(R)$ be respectively the solutions of

$$R = H_b(q) - H_b(D_1),$$
$$R = 3H_b(q) - H_t(\frac{D_2}{2}, q) - H_t(\frac{D_2}{2}, 1-q).$$

In fact, D_1 and D_2 correspond to the D_1 and D_2 in Theorem 2.4.2, but here expressed in terms of R, rather than in terms of P. Define

$$\begin{aligned} R_{LB} = (1-q) \sum_{i,j\in\{0,1\}} p_{ij|0} \log \frac{p_{ij|0}}{(1-q)p_{ij|0} + qp_{ij|1}} \\ +q \sum_{i,j\in\{0,1\}} p_{ij|1} \log \frac{p_{ij|1}}{(1-q)p_{ij|0} + qp_{ij|1}}, \end{aligned}$$

where

$$\begin{aligned} p_{00|0} &= 1 - \frac{D_2}{2(1-q)}, \\ p_{01|0} &= \frac{(D_2 - D_1)(2q - 2D_1 + D_2)}{2(1-q)(q - 2D_1 + D_2)}, \\ p_{10|0} &= 0, \\ p_{11|0} &= \frac{(2D_1 - D_2)(q - D_1)}{2(1-q)(q - 2D_1 + D_2)}, \\ p_{00|1} &= \frac{D_2}{2q}, \\ p_{01|1} &= \frac{(D_2 - D_1)(2D_1 - D_2)}{2q(q - 2D_1 + D_2)}, \\ p_{10|1} &= 0, \\ p_{11|1} &= \frac{(q - D_1)(2q - 2D_1 + D_2)}{2q(q - 2D_1 + D_2)}. \end{aligned} \tag{2.4.1}$$

Moreover, define

$$R_{UB} = (1-q)\sum_{i,j\in\{0,1\}} p'_{ij|0}\log\frac{p'_{ij|0}}{(1-q)p'_{ij|0}+qp'_{ij|1}} + q\sum_{i,j\in\{0,1\}} p'_{ij|1}\log\frac{p'_{ij|1}}{(1-q)p'_{ij|0}+qp'_{ij|1}},$$

where

$$\begin{aligned}
p'_{00|0} &= 1-\frac{D_2}{2(1-q)},\\
p'_{01|0} &= \frac{D_2-D_1+P_{UB}}{2(1-q)},\\
p'_{10|0} &= 0,\\
p'_{11|0} &= \frac{D_1-P_{UB}}{2(1-q)},\\
p'_{00|1} &= \frac{D_2}{2q},\\
p'_{01|1} &= \frac{D_1-D_2+P_{UB}}{2q},\\
p'_{10|1} &= 0,\\
p'_{11|1} &= \frac{2q-D_1-P_{UB}}{2q},
\end{aligned} \tag{2.4.2}$$

and

$$P_{UB} = \kappa(D_1-D_2), \tag{2.4.3}$$

$$\kappa = \kappa(D_2) \triangleq \frac{-\log\frac{D_2}{2}+\frac{1}{2}\log\left(1-q-\frac{D_2}{2}\right)+\frac{1}{2}\log\left(q-\frac{D_2}{2}\right)}{\log\frac{q}{1-q}+\frac{1}{2}\log\left(1-q-\frac{D_2}{2}\right)-\frac{1}{2}\log\left(q-\frac{D_2}{2}\right)}. \tag{2.4.4}$$

Theorem 2.4.4 *For a binary source $X \sim Bern(q)$, under Hamming distortion and*

TV perception losses,

$$R_{LB} \leq R^{(\infty)}(\Theta(R)) \leq R_{UB}$$

and consequently

$$R_{LB} - R \leq \Delta(\Theta(R)) \leq R_{UB} - R.$$

Moreover, the upper and lower bounds coincide if and only if

$$\frac{q}{2D_1 - D_2 - q} \geq \kappa. \tag{2.4.5}$$

Fig. 2.6 shows when $R \gtrapprox 0.08$, the dashed lines coincide with the dotted lines, implying that the upper bound meets the lower bound. Fig. 2.7 provides a direct visualization in the rate domain, that is when $R \gtrapprox 0.08$, the two bounds match.

Proof: We first prove the upper bound, i.e., the achievability part. For brevity, denote $\hat{R} := R^{(\infty)}(\Theta(R))$. Observe that the rate-distortion-perception function $R^{(\infty)}(D, P)$ is convex. Hence, the level curve $P(D)$ of $R^{(\infty)}(D, P) = \hat{R}$ is convex as well. Note that the expression of $R^{(\infty)}(D, P)$ is explicitly given in Theorem 2.4.2. By the implicit function theorem, we can compute the derivative $P'(D) = -\frac{\partial R^{(\infty)}(D,P(D))/\partial D}{\partial R^{(\infty)}(D,P(D))/\partial P}$. By the convexity, the curve $P(D)$ is above the line $D \mapsto P'(D_2)(D - D_2)$ where $P'(D_2) = \kappa$ with κ given in (2.4.4). Hence, to ensure that all points in $\Theta(R)$ are achievable, it suffice to require all points on the line segment between $(D_1, P'(D_2)(D_1 - D_2))$ and $(D_2, 0)$ are achievable. In fact, more succinctly, it is only required the two end points $(D_1, P'(D_2)(D_1 - D_2))$ and $(D_2, 0)$ are achievable. This is because, once these two

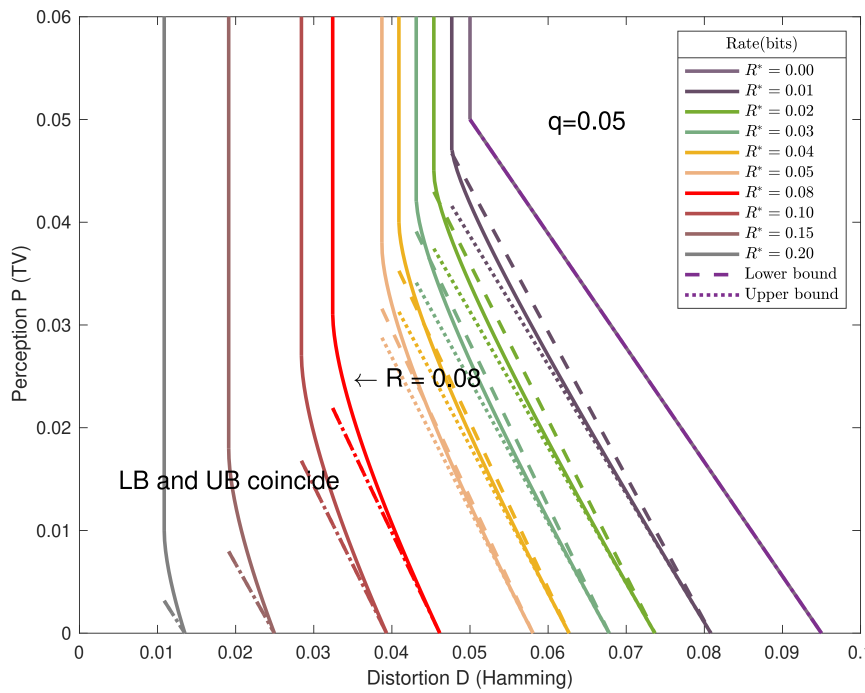

Figure 2.6: Plots of perception-distortion curves for different bit rates under the lower bound and upper bound, where $q = 0.05$. When $R \gtrapprox 0.08$, the upper bound and lower bound coincide. Here, the terms "lower bound" and "upper bound" are with respect to the rate $R^{(\infty)}(\Theta(R))$, not the cross-section curves – which the upper bound curve actually sits below.

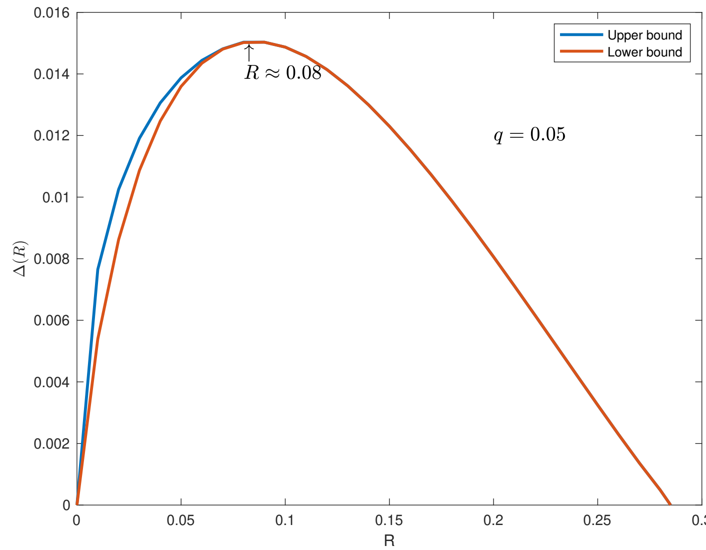

Figure 2.7: Plots of $\Delta(R)$ with R for both lower bound and upper bound, where $q = 0.05$. When $R \gtrapprox 0.08$, the upper bound and lower bound coincide.

end points are achievable, the whole segment is achievable as well, since the set of achievable points is always convex. Substituting $(D_1, P'(D_2)(D_1 - D_2))$ and $(D_2, 0)$ into the RDP function, we obtain the upper bound R_{UB}.

We next prove the lower bound, i.e., the converse part. Since $\Theta(R)$ is achievable and the turning points (D_1, P_1) and $(D_2, 0)$ are on the boundary of this region, obviously, these two points (D_1, P_1) and $(D_2, 0)$ are achievable as well. This implies that the minimum rate $R^{(\infty)}(\Theta(R))$ is at least the value of the following convex program:

$$\begin{aligned} \min_{p_{\hat{X}_1\hat{X}_2|X}} \ & I(X;\hat{X}_1,\hat{X}_2) \\ \text{subject to} \quad & \Pr(X \neq \hat{X}_1) \leq D_1, \\ & \Pr(X \neq \hat{X}_2) \leq D_2, \\ & \omega(p_X, p_{\hat{X}_1}) \leq P_1, \\ & \omega(p_X, p_{\hat{X}_2}) \leq 0. \end{aligned}$$

Since all the constraints are linear and the objective function is convex, a solution is optimal if and only if it satisfies the KKT conditions. Moreover, by substituting the solution $P_{\hat{X}_1\hat{X}_2|X}(i,j|k) = p_{ij|k}$ for $i,j,k \in \{0,1\}$ with $p_{ij|k}$ given in (2.4.1) to the KKT conditions, one can find that the KKT conditions are satisfied, and hence, this solution is optimal. (In fact, one technical obstacle in this proof idea is that the objective function is not differentiable at the boundary points, i.e., at the points with some $p_{ij|k} = 0$. Hence, in the implementation of the proof idea above, we need first to use the log-sum inequality to eliminate the terms involving $p_{10|0}$ or $p_{10|1}$, and then apply the KKT conditions.) This gives the lower bound R_{LB}.

By comparing the upper and lower bounds, one can check that the upper bound and lower bound coincide if and only if $\frac{q}{2D_1-D_2-q} \geq \kappa$.

More detailed proofs are provided in the appendix. □

We next proceed to study successive refinement. Let $\mathcal{A}$ be the regime where both the distortion and perception constraints are active, i.e., $\mathcal{A} \triangleq \{(D,P) : D \in [\frac{P}{1-2(q-P)}, 2q(1-q)-(1-2q)P), P \in [0,q]\}$.

Theorem 2.4.5 *For a binary source $X \sim Bern(q)$, under Hamming distortion and*

TV perception losses, successive refinement from $(D_1, P_1) \in \mathcal{A}$ *to* $(D_2, P_2) \in \mathcal{A}$ *is feasible if and only if*

$$q((D_1 - P_1) - (D_2 - P_2)) \geq D_1P_2 - D_2P_1, \tag{2.4.6}$$

$$(1 - q)((D_1 + P_1) - (D_2 + P_2)) \geq D_2P_1 - D_1P_2. \tag{2.4.7}$$

Proof: For the case $D \in \mathcal{S}_2$, $R^{(\infty)}(D, P)$ is attained by

$$p_{\hat{X}|X}(0|0) = 1 - \frac{D - P}{2(1 - q)},$$
$$p_{\hat{X}|X}(0|1) = \frac{D + P}{2q}.$$

Let $\mathcal{A}$ denote the regime where both D and P are active, i.e., $\mathcal{A} \triangleq \{(D, P) : D \in [\frac{P}{1-2(p-P)}, 2p(1-p)-(1-2p)P), P \in [0, p]\}$. By the standard techniques in information theory [14], successive refinement from $(D_1, P_1) \in \mathcal{A}$ to $(D_2, P_2) \in \mathcal{A}$ is feasible if and only if one can construct a Markov chain $X \leftrightarrow \hat{X}_2 \leftrightarrow \hat{X}_1$ such that

$$p_{\hat{X}_1|X}(0|0) = 1 - \frac{D_1 - P_1}{2(1 - q)},$$
$$p_{\hat{X}_1|X}(0|1) = \frac{D_1 + P_1}{2q},$$
$$p_{\hat{X}_2|X}(0|0) = 1 - \frac{D_2 - P_2}{2(1 - q)},$$
$$p_{\hat{X}_2|X}(0|1) = \frac{D_2 + P_2}{2q}.$$

If we denote $p_{\hat{X}_1|\hat{X}_2}(0|0) = a$ and $p_{\hat{X}_1|\hat{X}_2}(0|1) = b$ in the Markov chain above, then the resultant conditional probability become

$$\begin{aligned} p_{\hat{X}_1|X}(0|0) &= p_{\hat{X}_2|X}(0|0)p_{\hat{X}_1|\hat{X}_2}(0|0) + p_{\hat{X}_2|X}(0|1)p_{\hat{X}_1|\hat{X}_2}(0|1) \\ &= (1 - \frac{D_2 - P_2}{2(1-q)})a + \frac{D_2 - P_2}{2(1-q)}b, \\ p_{\hat{X}_1|X}(0|1) &= p_{\hat{X}_2|X}(0|1)p_{\hat{X}_1|\hat{X}_2}(0|0) + p_{\hat{X}_2|X}(1|1)p_{\hat{X}_1|\hat{X}_2}(0|1) \\ &= \frac{D_2 + P_2}{2q}a + (1 - \frac{D_2 + P_2}{2q})b. \end{aligned}$$

To ensure that the conditional probability are the same as the requirements $p_{\hat{X}_2|X}(0|0) = 1 - \frac{D_2 - P_2}{2(1-q)}, p_{\hat{X}_2|X}(0|1) = \frac{D_2+P_2}{2q}$, the parameters a, b should satisfy

$$\begin{aligned} (1 - \frac{D_2 - P_2}{2(1-q)})a + \frac{D_2 - P_2}{2(1-q)}b = 1 - \frac{D_1 - P_1}{2(1-q)}, \\ \frac{D_2 + P_2}{2q}a + (1 - \frac{D_2 + P_2}{2q})b = \frac{D_1 + P_1}{2q}. \end{aligned}$$

These equations yield

$$\begin{aligned} a &= \frac{(1 - \frac{D_2+P_2}{2q})(1 - \frac{D_1-P_1}{2(1-q)}) - \frac{(D_2-P_2)(D_1+P_1)}{4q(1-q)}}{(1 - \frac{D_2-P_2}{2(1-q)})(1 - \frac{D_2+P_2}{2q}) - \frac{(D_2-P_2)(D_2+P_2)}{4q(1-q)}}, \\ b &= \frac{(1 - \frac{D_2-P_2}{2(1-q)})\frac{D_1+P_1}{2q} - (1 - \frac{D_1-P_1}{2(1-q)})\frac{D_2+P_2}{2q}}{(1 - \frac{D_2-P_2}{2(1-q)})(1 - \frac{D_2+P_2}{2q}) - \frac{(D_2-P_2)(D_2+P_2)}{4q(1-q)}}. \end{aligned}$$

Furthermore, obviously, it should hold that $a \in [0, 1]$ and $b \in [0, 1]$. This is equivalent to the condition given in (2.4.6) and (2.4.7). More detailed proofs are provided in the appendix. □

For $0 < R_1 < R_2$, we denote $D_1 = D_1(R_1), D_2 = D_2(R_1)$, and $D'_1 = D_1(R_2), D'_2 = D_2(R_2)$, where the functions $D_1(R), D_2(R)$ are defined above Theorem 2.4.4.

Theorem 2.4.6 *Let* $0 < R_1 < R_2$*. For a binary source* $X \sim Bern(q)$*, under Hamming distortion and TV perception losses as well as the conditions that*

$$\frac{q}{2D_1 - D_2 - q} \geq \kappa(D_2), \quad \frac{q}{2D_1' - D_2' - q} \geq \kappa(D_2') \tag{2.4.8}$$

with the function κ *given in* (2.4.4)*, successive refinement from* $\Theta(R_1)$ *to* $\Theta(R_2)$ *is feasible if*

$$2D_1 - D_2 \leq 2D_1' - D_2'. \tag{2.4.9}$$

Remark 2.4.1 *Note that the conditions in* (2.4.8) *are the necessary and sufficient conditions for the bounds on* $R^{(\infty)}(\Theta(R))$ *in Theorem 2.4.4 to match at* $R = R_1$ *and* $R = R_2$ *respectively. Hence, the theorem above provides a sufficient condition for the feasibility of set-wise successive refinement, given that* R_1 *and* R_2 *are above the matching threshold.*

Remark 2.4.2 *In fact, we numerically verify that* $2D_1 - D_2 \leq 2D_1' - D_2'$ *automatically holds once the conditions in* (2.4.8) *are satisfied. In other words, successive refinement from* $\Theta(R_1)$ *to* $\Theta(R_2)$ *is feasible if* R_1 *and* R_2 *are above the matching threshold.*

Proof: Under the condition in (2.4.9), set-wise successive refinement from $\Theta(R_1)$ to $\Theta(R_2)$ is feasible if there exists a Markov chain $X \leftrightarrow (\hat{X}_1', \hat{X}_2') \leftrightarrow (\hat{X}_1, \hat{X}_2)$ such that the conditional distributions $P_{\hat{X}_1\hat{X}_2|X}$ and $P_{\hat{X}_1'\hat{X}_2'|X}$ are optimal in attaining $R^{(\infty)}(\Theta(R_1))$ and $R^{(\infty)}(\Theta(R_2))$ respectively. In other words, $P_{\hat{X}_1\hat{X}_2|X}$ corresponds to the distributions given in (2.4.1), and $P_{\hat{X}_1'\hat{X}_2'|X}$ corresponds the same distribution but with D_1, D_2 replaced by D'_1, D'_2. By simple algebraic manipulations, one can find that such a Markov chain exists if and only if $2D_1 - D_2 \leq 2D'_1 - D'_2$ holds. This completes the proof. Detailed proofs are provided in the appendix. □

2.5 Conclusions

In this chapter, we have focused on the advancement of perception-constrained lossy compression for binary sources, specifically addressing Hamming distortion and TV perception losses. We derived a closed-form expression for the rate-distortion-perception tradeoff in the one-shot setting. And we compared the setting with the asymptotic case and resulted that the tradeoff curves for the asymptotic case always lie below their counterparts for the one-shot case. We successfully characterized the achievable distortion-perception region for a general representation. We then established partially tight upper and lower bounds on the minimum rate penalty for the universal setting [56]. Finally, we identified a necessary and sufficient condition for point-wise successive refinement and a sufficient condition for the successive refinability of universal representations.

2.A Proof of Theorem 2.4.1

For a binary source $X \sim \text{Bern}(q)$, under Hamming distortion and TV perception losses,

$$R^*(D,P) = \begin{cases} \frac{q-D}{q} H_b(q) & 0 \leq D \leq P, \\ \frac{(1-q)+P-\frac{D+P}{2q}}{1-q} H_b(q) & P < D \leq D', \\ 0 & \text{otherwise,} \end{cases}$$

where $H_b(\cdot)$ denotes the binary entropy function and $D' = 2q(1-q) - (1-2q)P$.

Proof: Without loss of optimality, one can restrict the alphabet size of U to be no more than 4. In fact, $(X, \hat{X})$ takes values in $\{0,1\}^2$. Hence, without loss of optimality, we can assume that the random seed U takes values in $\{0,1,2,3\}$ and $\hat{X} = X$ if $U = 0$; $\hat{X} = 1 - X$ if $U = 1$; $\hat{X} = 0$ if $U = 2$; and $\hat{X} = 1$ if $U = 3$. Denoting the probability values $p_U(i) = \epsilon_i, i \in [0:3]$, we seek a conditional distribution $p_{U,\hat{X}|X}$ that solves the one-shot rate-distortion-perception function

$$\begin{aligned} R^*(D,P) = \min_{p_{U,\hat{X}|X}} & I(X;\hat{X}|U) \\ \text{s.t.} \quad & Pr(X \neq \hat{X}) \leq D, \quad \omega_{TV}(p_X, p_{\hat{X}}) \leq P, \\ & I(X;U) = 0, \quad H(\hat{X} \mid X, U) = 0, \end{aligned}$$

where the objective function represented by the mutual information is because

$$\begin{aligned} H(V \mid U) &\geq I(X; V \mid U) \\ &\geq I(X; \hat{X} \mid U). \end{aligned}$$

Here we concentrate on the case where distortion measure is the Hamming distance, and $\omega_{TV}(\cdot,\cdot)$ is the total-variation (TV) divergence. The mutual information term $I(X;\hat{X}|U)$ is given by

$$\begin{aligned} I(X;\hat{X}|U) &= \sum_{i=0}^{3} p_U(i) I(X;\hat{X}|U=i) \\ &= p_U(0)H_b(q) + p_U(1)H_b(q) \\ &= (\epsilon_1 + \epsilon_2)H_b(q). \end{aligned}$$

The Hamming distance term is given by,

$$
\begin{aligned}
Pr(X \neq \hat{X}) &= \sum_{i=0}^{3} p_U(i) P(X \neq \hat{X} | U = i) \\
&= p_U(1) + p_U(2)q + p_U(3)(1-q) \\
&= \epsilon_2 + q\epsilon_3 + (1-q)\epsilon_4.
\end{aligned}
$$

Since U and X are independent, we have

$$
\begin{aligned}
p_{\hat{X}}(1) = &\sum_{z \in \{0,1\}} \sum_{i=0}^{3} P(\hat{X} = 1 | U = i, X = z) p_U(i) p_X(z) \\
&= p_U(0)q + p_U(1)(1-q) + p_U(3) \\
&= q\epsilon_1 + (1-q)\epsilon_2 + \epsilon_4,
\end{aligned}
$$

so the TV divergence term is given by

$$
\begin{aligned}
\omega_{TV}(p_X, p_{\hat{X}}) &= \frac{1}{2} \sum_{z \in \{0,1\}} |p_X(z) - p_{\hat{X}}(z)| \\
&= \frac{1}{2}(|p_X(0) - p_{\hat{X}}(0)| + |p_X(1) - p_{\hat{X}}(1)|) \\
&= |q - q\epsilon_1 - (1-q)\epsilon_2 - \epsilon_4|.
\end{aligned}
$$

Thus, it suffices to solve the linear program

$$
\begin{aligned}
\min_{\epsilon_1,\epsilon_2,\epsilon_3} \quad & (\epsilon_1+\epsilon_2)H_b(q) \\
\text{s.t.} \quad -(1-q)\epsilon_1 + q\epsilon_2 + (2q-1)\epsilon_3 - D + (1-q) &\leq 0, \\
(1-q)\epsilon_1 + q\epsilon_2 + \epsilon_3 - P - (1-q) &\leq 0, \\
-(1-q)\epsilon_1 - q\epsilon_2 - \epsilon_3 - P + (1-q) &\leq 0, \\
-\epsilon_1 &\leq 0, \\
-\epsilon_2 &\leq 0, \\
-\epsilon_3 &\leq 0, \\
\epsilon_1 + \epsilon_2 + \epsilon_2 - 1 &\leq 0.
\end{aligned}
$$

According to the Karush-Kuhn-Tucker conditions, the minimizer can be found if and only if the following conditions are satisfied, and there exist non-negative a, b, c, d,

e, f, g such that

$$
\begin{aligned}
1 - a(1-q) + b(1-q) - c(1-q) - d + e &= 0,\\
1 + aq + bq - cq - f + e &= 0,\\
a(2q-1) + b - c - g + e &= 0,\\
a(-(1-q)\epsilon_1 + q\epsilon_2 + (2q-1)\epsilon_3 - D + (1-q)) &= 0,\\
b((1-q)\epsilon_1 + q\epsilon_2 + \epsilon_3 - P - (1-q)) &= 0,\\
c(-(1-q)\epsilon_1 - q\epsilon_2 - \epsilon_3 - P + (1-q)) &= 0,\\
d\epsilon_1 &= 0,\\
f\epsilon_2 &= 0,\\
g\epsilon_3 &= 0,\\
e(\epsilon_1 + \epsilon_2 + \epsilon_3 - 1) &= 0.
\end{aligned}
$$

Without loss of generality, we assume $0 \leq P \leq q$. It can be verified via algebraic manipulations that when $0 \leq D \leq P$, the optimal solution is given by

$$
\begin{aligned}
\epsilon_1^* &= \frac{q-D}{q},\\
\epsilon_2^* &= 0,\\
\epsilon_3^* &= \frac{D}{q}.
\end{aligned}
$$

In this case, we must have

$$a = \frac{1}{q},$$
$$b, c, d, g = 0,$$
$$e = \frac{1-2q}{q},$$
$$f = 2 + \frac{1-2q}{q}.$$

Since $0 \leq q \leq \frac{1}{2}$, $0 \leq P \leq q$, and $0 \leq D \leq P$, it follows that the original constraints are satisfied and the Lagrange multipliers are all non-negative.

When $P \leq D \leq 2q(1-q) - (1-2q)P$, the optimal solution is given by

$$\epsilon_1^* = \frac{(1-q) + P - \frac{D+P}{2q}}{1-q},$$
$$\epsilon_2^* = 0,$$
$$\epsilon_3^* = \frac{D+P}{2q}.$$

In this case, the Lagrange multipliers are

$$a = \frac{1}{2q(1-q)},$$
$$b = \frac{1-2q}{2q(1-q)},$$
$$c, d, e, g = 0,$$
$$f = 2.$$

Since $0 \leq q \leq \frac{1}{2}$, $0 \leq P \leq q$, and $P \leq D \leq 2q(1-q) - (1-2q)P$, it can be verified that the original constraints are satisfied and the Lagrange multipliers are all non-negative.

When $2q(1-q)-(1-2q)P \leq D$, the optimal solution is given by

$$\epsilon_1^* = 0,$$
$$\epsilon_2^* = 0,$$
$$\max\{\frac{1-q-D}{1-2q}, 1-q-P\} \leq \epsilon_3^* \leq P+(1-q).$$

In this case, the Lagrange multipliers are

$$a, b, c, e, g = 0,$$
$$d = 1,$$
$$f = 1.$$

The Lagrange multipliers are obviously all non-negative. Since $0 \leq q \leq \frac{1}{2}$, $0 \leq P \leq q$, and $q \leq D$, it can be verified that the original constraints are satisfied.

Putting all the pieces together, the overall solution for $P \leq q$ is

$$R^*(D,P) = \begin{cases} \frac{q-D}{q} H_b(q), & 0 \leq D \leq P, \\ \frac{(1-q)+P-\frac{D+P}{2q}}{1-q} H_b(q), & P < D \leq D', \\ 0, & \text{otherwise}, \end{cases}$$

where $H_b(\cdot)$ denotes the binary entropy function and $D' = 2q(1-q)-(1-2q)P$.

For $P > q$, the solution is independent of P and is given by the solution

$$R^*(D) = \begin{cases} \frac{q-D}{q} H_b(q), & 0 \leq D \leq q, \\ 0, & \text{otherwise.} \end{cases}$$

This concludes the proof. □

2.B Proof of Theorem 2.4.3

Let Z be a representation of a binary source $X \sim \text{Bern}(q)$ with $p_Z(i) = q_i$ and $p_{X|Z}(1|i) = \epsilon_i$, $i \in [n]$, where $\sum_{i=1}^n q_i = 1$ and $\sum_{i=1}^n q_i \epsilon_i = q$. Without loss of generality, we assume that the values of $q_i|1-2\epsilon_i|, i \in [n]$ are in ascending order as i increases. Let j_k with $k \in [m]$ denote the k-th index at which $\epsilon_{j_k} \leq 0.5$. Denote $k^+ \in [m]$ as the first index such that $j_{k^+} > k$ and $\epsilon_{j_{k^+}} \leq 0.5$, if it exists. Define k^* to be the first positive integer satisfying $\frac{\sum_{i=k^*}^n q_i(1-\epsilon_i) - \sum_{i=(k^*)^+}^m q_{j_i}}{q_k^*} \leq 1 - \epsilon_{k^*}$. Under Hamming distortion and TV perception losses, the lower boundary of $\Pi(p_{Z|X})$ is piecewise linear with k^* knots $\{(D_k, P_k)\}_{k=1}^{k^*}$ given by

$$\begin{aligned}
D_k &= \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k} q_i(2\epsilon_i - 1)(1-\epsilon_i) \\
&\quad + \sum_{i=k^+}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \quad k = 1, \ldots, k^* - 1 \\
P_k &= \left| \sum_{i=k+1}^{n} q_i(1-\epsilon_i) - \sum_{i=k^+}^{m} q_{j_i} \right|, \quad k = 1, \ldots, k^* - 1, \\
D_{k^*} &= \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k^*-1} q_i(2\epsilon_i - 1)(1-\epsilon_i) \\
&\quad + (2\epsilon_{k^*} - 1)\left(\sum_{i=k^*}^{n} q_i(1-\epsilon_i) - \sum_{i=(k^*)^+}^{m} q_{j_i} \right) \\
&\quad + \sum_{i=(k^*)^+}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \\
P_{k^*} &= 0.
\end{aligned}$$

Proof: Given a binary source X, let Z be any arbitrary representation of X with the known distribution $p_Z(i) = q_i$ and $p_{X|Z}(1|i) = \epsilon_i$, $i \in [n]$, then

$$\begin{aligned}
p_X(0) &= \sum_{i=1}^{n} q_i(1-\epsilon_i), \\
p_X(1) &= 1 - \sum_{i=1}^{n} q_i(1-\epsilon_i).
\end{aligned}$$

There exists some $\hat{X}$ jointly distributed with (X, Z) such that $X \leftrightarrow Z \leftrightarrow \hat{X}$ form a Markov chain. We can parameterize the transition distribution as $p_{\hat{X}|Z}(0|i) = p_i$,

$i \in [n]$, then

$$p_{\hat{X}}(0) = \sum_{i=1}^{n} q_i p_i,$$
$$p_{\hat{X}}(1) = 1 - \sum_{i=1}^{n} q_i p_i.$$

Note that if $1 - \epsilon_i \geq 0.5$, $1 - \epsilon_i \leq p_i \leq 1$; Conversely, if $1 - \epsilon_i < 0.5$, $0 \leq p_i < 1 - \epsilon_i$. We seek a conditional distribution $p_{\hat{X}|Z}$ that solves the distortion-perception problem

$$\begin{aligned} \min_{P_{\hat{X}|Z}} \quad & D = Pr(X \neq \hat{X}) \\ \text{s.t.} \quad & \omega_{TV}(p_X, p_{\hat{X}}) \leq P. \end{aligned}$$

Here we concentrate on the case where distortion measure is the Hamming distance, and $\omega_{TV}(\cdot,\cdot)$ is the total-variation (TV) divergence. The Hamming distance term is given by

$$\begin{aligned} Pr(X \neq \hat{X}) &= p_{X|Z}(0|i)p_Z(i)p_{\hat{X}|Z}(1|i) + p_{X|Z}(1|i)p_Z(i)p_{\hat{X}|Z}(0|i) \\ &= \sum_{i=1}^{n}(q_i(1-\epsilon_i)(1-p_i) + q_i\epsilon_i p_i) \\ &= \sum_{i=1}^{n}(q_i(1-\epsilon_i) + q_i(2\epsilon_i - 1)p_i), \end{aligned}$$

and the TV divergence term is given by

$$\begin{aligned}\omega_{TV}(p_X, p_{\hat{X}}) &= \frac{1}{2}\sum_{z\in\{0,1\}} |p_X(z) - p_{\hat{X}}(z)| \\ &= \frac{1}{2}(|p_X(0) - p_{\hat{X}}(0)| + |p_X(1) - p_{\hat{X}}(1)|) \\ &= |\sum_{i=1}^{n} q_i(1-\epsilon_i) - \sum_{i=1}^{n} q_i p_i|.\end{aligned}$$

Thus, it suffices to solve a piecewise linear program

$$\begin{aligned}\min_{p_1,\ldots,p_n} \quad & \sum_{i=1}^{n}(q_i(1-\epsilon_i) + q_i(2\epsilon_i - 1)p_i) \\ \text{s.t.} \quad & |\textstyle\sum_{i=1}^{n} q_i(1-\epsilon_i) - \sum_{i=1}^{n} q_i p_i| \le P, \\ & 1-\epsilon_i \le p_i \le 1, \quad \text{if} \quad 1-\epsilon_i \ge 0.5, \\ & 0 \le p_i < 1-\epsilon_i, \quad \text{if} \quad 1-\epsilon_i < 0.5.\end{aligned}$$

In order to solve this problem, we sort the distributions in an ascending order by the size of $q_i|1-2\epsilon_i|$, i.e. $q_1|1-2\epsilon_1| \le q_2|1-2\epsilon_2| \le \cdots \le q_n|1-2\epsilon_n|$, and assume $\epsilon_{j_1}, \epsilon_{j_2}, \ldots, \epsilon_{j_m} \le 0.5$ where $q_1|1-2\epsilon_1| \le \cdots \le q_{j_1}|1-2\epsilon_{j_1}| \le \cdots \le q_{j_2}|1-2\epsilon_{j_2}| \le \cdots \le q_{j_m}|1-2\epsilon_{j_m}| \le \cdots \le q_n|1-2\epsilon_n|$ and the left ϵ_i are all greater than 0.5. We also need to define k to be the k^{th} place in the ascending order. If there's no perception constraint ($P = \infty$), we can let $p_{j_1} = \cdots = p_{j_m} = 1$, and set the left p_i to be 0, then we have the minimum distortion which is the first knot in the piecewise distortion-perception line

$$\begin{cases} D_1 = \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=j_1}^{m} q_i(2\epsilon_i - 1), \\ P_1 = |\sum_{i=1}^{n} q_i(1-\epsilon_i) - \sum_{i=1}^{m} q_{j_i}|. \end{cases}$$

In order to minimize the increase in D, we need to change the value of p_i in our previous order until P reaches 0. According to this idea, loops are required to find every knot.

While $1 \leq k < j_1$, the steps in 1^{st} loop are performed as follows.

For $k = 1$, increase p_k until $p_k = 1 - \epsilon_k$, i.e. $p_1 \uparrow, \ldots, p_{j_1-1} \uparrow$, then the $(k+1)^{th}$ knot can be calculated by

$$\begin{cases} D_{k+1} = \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k} q_i(2\epsilon_i - 1)(1-\epsilon_i) + \sum_{i=1}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \\ P_{k+1} = |\sum_{i=k+1}^{n} q_i(1-\epsilon_i) - \sum_{i=1}^{m} q_{j_i}|, \end{cases}$$

if P reaches 0 in the process we increase p_k, end the loop and the finishing knot is

$$\begin{cases} D_{k+1} = \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k-1} q_i(2\epsilon_i - 1)(1-\epsilon_i) \quad + q_k(2\epsilon_k - 1)\frac{\sum_{i=k}^{n} q_i(1-\epsilon_i) - \sum_{i=1}^{m} q_{j_i}}{q_k} \\ \qquad\qquad + \sum_{i=1}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \\ P_{k+1} = 0, \end{cases}$$

else, let $k = k+1$.

After going through this interval, if P still doesn't reach 0, proceed to the next interval.

While $j_1 \leq k < j_2$, the steps in the 2^{nd} loop are performed as follows.

For $k = j_1$, when $k = j_1$, decrease p_k until $p_k = 1 - \epsilon_k$, otherwise increase p_k until $p_k = 1 - \epsilon_k$, i.e. $p_{j_1} \downarrow, p_{j_1+1} \uparrow, \ldots, p_{j_2-1} \uparrow$, then the $(k+1)^{th}$ knot can be calculated by

$$\begin{cases} D_{k+1} = \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k} q_i(2\epsilon_i - 1)(1-\epsilon_i) + \sum_{i=2}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \\ P_{k+1} = |\sum_{i=k+1}^{n} q_i(1-\epsilon_i) - \sum_{i=2}^{m} q_{j_i}|, \end{cases}$$

if P reaches 0 in the process we increase p_k, end the loop and the finishing knot is

$$\begin{cases} D_{k+1} = \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k-1} q_i(2\epsilon_i - 1)(1-\epsilon_i) \quad + q_k(2\epsilon_k - 1)\frac{\sum_{i=k}^{n} q_i(1-\epsilon_i) - \sum_{i=2}^{m} q_{j_i}}{q_k} \\ \qquad\qquad + \sum_{i=2}^{m} q_{j_i}(2\epsilon_{j_i} - 1), \\ P_{k+1} = 0, \end{cases}$$

else, let $k = k+1$.

Specifically, after m loops, it follows by the last loop. While $j_m < k \leq n$, the steps in the last loop are performed as follows.

For $k = j_m + 1$, increase p_k until $p_k = 1 - \epsilon_k$, i.e. $p_{j_m+1} \uparrow, \ldots, p_n \uparrow$, then the $(k+1)^{th}$ knot can be calculated by

$$\begin{cases} D_{k+1} = \sum_{i=1}^{n} q_i(1-\epsilon_i) + \sum_{i=1}^{k} q_i(2\epsilon_i - 1)(1-\epsilon_i), \\ P_{k+1} = |\sum_{i=k+1}^{n} q_i(1-\epsilon_i)|, \end{cases}$$

since in this loop, it's impossible for P to get 0, let $k = k+1$, until $k = n$. This completes the proof.

$\square$

2.C Proof of Theorem 2.4.4

Consider the case $\Theta = \Theta(R)$. We start by introducing some quantities which are needed for bounding $R^{(\infty)}(\Theta(R))$. Let $D_1 = D_1(R)$ and $D_2 = D_2(R)$ be respectively the solutions of

$$R = H_b(q) - H_b(D_1),$$
$$R = 3H_b(q) - H_t(\frac{D_2}{2}, q) - H_t(\frac{D_2}{2}, 1-q).$$

In fact, D_1 and D_2 correspond to the D_1 and D_2 in Theorem 2.4.2, but here expressed in terms of R, rather than in terms of P. Define

$$R_{LB} = (1-q) \sum_{i,j\in\{0,1\}} p_{ij|0} \log \frac{p_{ij|0}}{(1-q)p_{ij|0} + qp_{ij|1}} + q \sum_{i,j\in\{0,1\}} p_{ij|1} \log \frac{p_{ij|1}}{(1-q)p_{ij|0} + qp_{ij|1}},$$

where

$$\begin{aligned}
p_{00|0} &= 1 - \frac{D_2}{2(1-q)}, \\
p_{01|0} &= \frac{(D_2 - D_1)(2q - 2D_1 + D_2)}{2(1-q)(q - 2D_1 + D_2)}, \\
p_{10|0} &= 0, \\
p_{11|0} &= \frac{(2D_1 - D_2)(q - D_1)}{2(1-q)(q - 2D_1 + D_2)}, \\
p_{00|1} &= \frac{D_2}{2q}, \\
p_{01|1} &= \frac{(D_2 - D_1)(2D_1 - D_2)}{2q(q - 2D_1 + D_2)}, \\
p_{10|1} &= 0, \\
p_{11|1} &= \frac{(q - D_1)(2q - 2D_1 + D_2)}{2q(q - 2D_1 + D_2)}.
\end{aligned} \tag{2.C.1}$$

Moreover, define

$$
\begin{aligned}
R_{UB} = (1-q) \sum_{i,j \in \{0,1\}} p'_{ij|0} \log \frac{p'_{ij|0}}{(1-q)p'_{ij|0} + qp'_{ij|1}} \\
+ q \sum_{i,j \in \{0,1\}} p'_{ij|1} \log \frac{p'_{ij|1}}{(1-q)p'_{ij|0} + qp'_{ij|1}},
\end{aligned}
$$

where

$$
\begin{aligned}
p'_{00|0} &= 1 - \frac{D_2}{2(1-q)}, \\
p'_{01|0} &= \frac{D_2 - D_1 + P_{UB}}{2(1-q)}, \\
p'_{10|0} &= 0, \\
p'_{11|0} &= \frac{D_1 - P_{UB}}{2(1-q)}, \\
p'_{00|1} &= \frac{D_2}{2q}, \\
p'_{01|1} &= \frac{D_1 - D_2 + P_{UB}}{2q}, \\
p'_{10|1} &= 0, \\
p'_{11|1} &= \frac{2q - D_1 - P_{UB}}{2q},
\end{aligned}
\tag{2.C.2}
$$

and

$$
P_{UB} = \kappa_{UB}(D_1 - D_2), \tag{2.C.3}
$$

$$
\kappa = \kappa_{UB}(D_2) \triangleq \frac{-\log \frac{D_2}{2} + \frac{1}{2} \log\left(1 - q - \frac{D_2}{2}\right) + \frac{1}{2} \log\left(q - \frac{D_2}{2}\right)}{\log \frac{q}{1-q} + \frac{1}{2} \log\left(1 - q - \frac{D_2}{2}\right) - \frac{1}{2} \log\left(q - \frac{D_2}{2}\right)}. \tag{2.C.4}
$$

For a binary source $X \sim \text{Bern}(q)$, under Hamming distortion and TV perception losses,

$$R_{LB} \leq R^{(\infty)}(\Theta(R)) \leq R_{UB}$$

and consequently

$$R_{LB} - R \leq \Delta(\Theta(R)) \leq R_{UB} - R.$$

Moreover, the upper and lower bounds coincide if and only if $R \geq R^*$, where $R^* = H(q) - H(D_1^*)$ or $3H_b(q) - H_t(\frac{D_2^*}{2}, q) - H_t(\frac{D_2^*}{2}, 1-q)$ and (D_1^*, D_2^*) is the unique solution to

$$\begin{cases} H_b(q) - H_b(D_1^*) = 3H_b(q) - H_t(\frac{D_2^*}{2}, q) - H_t(\frac{D_2^*}{2}, 1-q), \\ \frac{q}{2D_1^* - D_2^* - q} = \frac{-\log \frac{D_2^*}{2} + \frac{1}{2}\log(1-e-\frac{D_2^*}{2}) + \frac{1}{2}\log(e-\frac{D_2^*}{2})}{\log \frac{e}{1-e} + \frac{1}{2}\log(1-e-\frac{D_2^*}{2}) - \frac{1}{2}\log(e-\frac{D_2^*}{2})}. \end{cases}$$

Proof: We first prove the lower bound, i.e., the converse part. Since $\Theta(R)$ is achievable and the turning points (D_1, P_1) and $(D_2, 0)$ are on the boundary of this region, obviously, these two points (D_1, P_1) and $(D_2, 0)$ are achievable as well. This implies that the minimum rate $R^{(\infty)}(\Theta(R))$ is at least the value of the following convex program:

$$\begin{aligned} \min_{p_{\hat{X}_1 \hat{X}_2 | X}} \quad & I(X; \hat{X}_1, \hat{X}_2) \\ \text{s.t.} \quad & Pr(X \neq \hat{X}_1) \leq D_1, \\ & Pr(X \neq \hat{X}_2) \leq D_2, \\ & \omega_{TV}(p_X, p_{\hat{X}_1}) \leq P_1, \\ & \omega_{TV}(p_X, p_{\hat{X}_2}) \leq 0. \end{aligned}$$

The mutual information term $I(X; \hat{X}_1, \hat{X}_2)$ is given by

$$\begin{aligned} I(X; \hat{X}_1, \hat{X}_2) &= \sum_{x,\hat{x}_1,\hat{x}_2 \in \{0,1\}} p(x, \hat{x}_1, \hat{x}_2) \log \frac{p(x, \hat{x}_1, \hat{x}_2)}{p(x)p(\hat{x}_1, \hat{x}_2)} \\ &= (1-q) \sum_{i,j \in \{0,1\}} p_{ij|0} \log \frac{p_{ij|0}}{(1-q)p_{ij|0} + qp_{ij|1}} + q \sum_{i,j \in \{0,1\}} p_{ij|1} \log \frac{p_{ij|1}}{(1-q)p_{ij|0} + qp_{ij|1}}. \end{aligned}$$

The two Hamming distortion terms are given by

$$\begin{aligned} P(X \neq \hat{X}_1) &= p_{\hat{X}_1|X}(1|0)p_X(0) + p_{\hat{X}_1|X}(0|1)p_X(1) \\ &= (1-q)(p_{10|0} + p_{11|0}) + q(p_{00|1} + p_{01|1}), \end{aligned}$$

$$\begin{aligned} Pr(X \neq \hat{X}_2) &= p_{\hat{X}_2|X}(1|0)p_X(0) + p_{\hat{X}_2|X}(0|1)p_X(1) \\ &= (1-q)(p_{01|0} + p_{11|0}) + q(p_{00|1} + p_{10|1}), \end{aligned}$$

respectively. Moreover, the TV divergence terms are given by

$$\begin{aligned} \omega_{TV}(p_X, p_{\hat{X}_1}) &= \frac{1}{2} \sum_{z \in \{0,1\}} |p_X(z) - p_{\hat{X}_1}(z)| \\ &= \frac{1}{2}(|p_X(0) - p_{\hat{X}_1}(0)| + |p_X(1) - p_{\hat{X}_1}(1)|) \\ &= |(1-q)(1 - p_{00|0} - p_{01|0}) - q(p_{00|1} + p_{01|1})|, \end{aligned}$$

$$
\begin{aligned}
\omega_{TV}(p_X, p_{\hat{X}_2}) &= \frac{1}{2}\sum_{z\in\{0,1\}} |p_X(z) - p_{\hat{X}_2}(z)| \\
&= \frac{1}{2}(|p_X(0) - p_{\hat{X}_2}(0)| + |p_X(1) - p_{\hat{X}_2}(1)|) \\
&= |(1-q)(1 - p_{00|0} - p_{10|0}) - q(p_{00|1} + p_{10|1})|,
\end{aligned}
$$

respectively.

In fact, one technical obstacle in this proof idea is that the objective function is not differentiable at the boundary points, i.e., at the points with some $p_{ij|k} = 0$. Hence, in the implementation of the proof idea above, we need first to use the log-sum inequality to eliminate the terms involving $p_{10|0}$ or $p_{10|1}$, and then apply the KKT conditions. According to the log sum inequality [11], we can get the lower bound of the mutual information term $I(X; \hat{X}_1, \hat{X}_2)$, it follows that

$$
\begin{aligned}
& I(X; \hat{X}_1, \hat{X}_2) \\
& \geq (1-q) \\
& \left(p_{00|0} \log \frac{p_{00|0}}{(1-q)p_{00|0} + qp_{00|1}} + p_{01|0} \log \frac{p_{01|0}}{(1-q)p_{01|0} + qp_{01|1}} + p_{11|0} \log \frac{p_{11|0}}{(1-q)p_{11|0} + qp_{11|1}}\right) \\
& + q\left(p_{00|1} \log \frac{p_{00|1}}{(1-q)p_{00|0} + qp_{00|1}} + p_{01|1} \log \frac{p_{01|1}}{(1-q)p_{01|0} + qp_{01|1}} + p_{11|1} \log \frac{p_{11|1}}{(1-q)p_{11|0} + qp_{11|1}}\right) \\
& + \big((1-q)p_{10|0} + qp_{10|1}\big) \log \frac{(1-q)p_{10|0} + qp_{10|1}}{((1-q)p_{10|0} + qp_{10|1})q + ((1-q)p_{10|0} + qp_{10|1})(1-q)} \\
& = (1-q) \\
& \left(p_{00|0} \log \frac{p_{00|0}}{(1-q)p_{00|0} + qp_{00|1}} + p_{01|0} \log \frac{p_{01|0}}{(1-q)p_{01|0} + qp_{01|1}} + p_{11|0} \log \frac{p_{11|0}}{(1-q)p_{11|0} + qp_{11|1}}\right) \\
& + q\left(p_{00|1} \log \frac{p_{00|1}}{(1-q)p_{00|0} + qp_{00|1}} + p_{01|1} \log \frac{p_{01|1}}{(1-q)p_{01|0} + qp_{01|1}} + p_{11|1} \log \frac{p_{11|1}}{(1-q)p_{11|0} + qp_{11|1}}\right) \\
& = I_{LB}(X; \hat{X}_1, \hat{X}_2).
\end{aligned}
$$

Now consider the following convex optimization problem

$$\begin{aligned}
\min_{p_{ij|k}} \quad & f = I_{LB}(X;\hat{X}_1,\hat{X}_2) \\
\text{s.t.} \quad & (1-q)(1-p_{00|0}-p_{01|0})+q(p_{00|1}+p_{01|1}) \leq D_1, \\
& (1-q)(p_{01|0}+p_{11|0})+q(1-p_{01|1}-p_{11|1}) \leq D_2, \\
& |(1-q)(p_{01|0}+p_{11|0})-q(1-p_{01|1}-p_{11|1})| \leq 0, \\
& |(1-q)(1-p_{00|0}-p_{01|0})-q(p_{00|1}+p_{01|1})| \leq P_1, \\
& p_{00|0}+p_{01|0}+p_{11|0} \leq 1, \\
& p_{00|1}+p_{01|1}+p_{11|1} \leq 1, \\
& 0 \leq p_{00|0}, p_{01|0}, p_{11|0} \leq 1, \\
& 0 \leq p_{00|1}, p_{01|1}, p_{11|1} \leq 1.
\end{aligned}$$

According to the Karush-Kuhn-Tucker conditions, $(p^*_{00|0}, p^*_{01|0}, p^*_{11|0}, p^*_{00|1}, p^*_{01|1}, p^*_{11|1})$ is a minimizer of the convex optimization problem if and only if the following conditions are satisfied, and there exist non-negative $v_1, v_2, v_3, v_4, w_1, w_2, \mu_1, \mu_2, \mu_4, \lambda_1, \lambda_2, \lambda_4$ such that

$$
\begin{aligned}
\frac{\partial f}{\partial p_{00|0}} - \mu_1 - (1-q)v_1 - qv_4 + w_1 = 0,\\
\frac{\partial f}{\partial p_{01|0}} - \mu_2 - (1-q)v_1 + (1-q)v_2 + (1-q)v_3 - qv_4 + w_1 = 0,\\
\frac{\partial f}{\partial p_{11|0}} - \mu_4 + (1-q)v_2 + (1-q)v_3 + w_1 = 0,\\
\frac{\partial f}{\partial p_{00|1}} - \lambda_1 + qv_1 - (1-q)v_4 + w_2 = 0,\\
\frac{\partial f}{\partial p_{01|1}} - \lambda_2 + qv_1 - qv_2 + qv_3 - (1-q)v_4 + w_2 = 0,\\
\frac{\partial f}{\partial p_{11|1}} - \lambda_4 - qv_2 + qv_3 + w_2 = 0,\\
\mu_1 p_{00|0} = 0, \mu_2 p_{01|0} = 0, \mu_4 p_{11|0} = 0,\\
\lambda_1 p_{00|1} = 0, \lambda_2 p_{01|1} = 0, \lambda_4 p_{11|1} = 0,\\
w_1(p_{00|0} + p_{01|0} + p_{11|0} - 1) = 0,\\
w_2(p_{00|1} + p_{01|1} + p_{11|1} - 1) = 0,\\
v_1((1-q)(1 - p_{00|0} - p_{01|0}) + q(p_{00|1} + p_{01|1}) - D_1) = 0,\\
v_2((1-q)(p_{01|0} + p_{11|0}) + q(1 - p_{01|1} - p_{11|1}) - D_2) = 0,\\
v_3((1-q)(p_{01|0} + p_{11|0}) - q(1 - p_{01|1} - p_{11|1})) = 0,\\
v_4(|(1-q)(1 - p_{00|0} - p_{01|0}) - q(p_{00|1} + p_{01|1})| - P_{00|0}) = 0,\\
v_1, v_2, v_3, v_4 \geq 0,\\
w_1, w_2 \geq 0,\\
\mu_1, \mu_2, \mu_4 \geq 0,\\
\lambda_1, \lambda_2, \lambda_4 \geq 0.
\end{aligned}
$$

It can be verified via algebraic manipulations that a minimizer $(p^*_{00|0}, p^*_{01|0}, p^*_{11|0}, p^*_{00|1}, p^*_{01|1}, p^*_{11|1})$ is

$$\begin{aligned}
p^*_{00|0} &= 1 - \frac{D_2}{2(1-q)}, \\
p^*_{01|0} &= \frac{(D_2 - D_1)(2q - 2D_1 + D_2)}{2(1-q)(q - 2D_1 + D_2)}, \\
p^*_{11|0} &= \frac{(2D_1 - D_2)(q - D_1)}{2(1-q)(q - 2D_1 + D_2)}, \\
p^*_{00|1} &= \frac{D_2}{2q}, \\
p^*_{01|1} &= \frac{(D_2 - D_1)(2D_1 - D_2)}{2q(q - 2D_1 + D_2)}, \\
p^*_{11|1} &= \frac{(q - D_1)(2q - 2D_1 + D_2)}{2q(q - 2D_1 + D_2)},
\end{aligned}$$

where $2D_1 - D_2 \leq q$, $D_1 \leq q$, $D_2 \leq 2D_1$ in order to ensure these probabilities exist. Under this solution, we can identify the Lagrange multipliers $v_1 = v_2 = v_3 = v_4 = 0, w_1 = w_2 = 0, \mu_1 = \mu_2 = \mu_4 = 0, \lambda_1 = \lambda_2 = \lambda_4 = 0$, which are all nonnegative. And $p^*_{10|0} = 0, p^*_{10|1} = 0$ are calculated based on the solution. Under this solution, we have the new

$$\begin{aligned}
P'_1 &= |(1-q)(1 - p_{00|0} - p_{01|0}) - q(p_{00|1} + p_{01|1})| \\
&= \frac{(D_2 - D_1)q}{q - 2D_1 + D_2},
\end{aligned}$$

then the slope for the lower bound is

$$\kappa_{LB} = \frac{0 - P'_1}{D_2 - D_1} = \frac{q}{2D_1 - D_2 - q}.$$

We next prove the upper bound, i.e., the achievability part. For brevity, denote

$\hat{R} := R^{(\infty)}(\Theta(R))$. Observe that the rate-distortion-perception function $R^{(\infty)}(D,P)$ is convex. Hence, the level curve $P(D)$ of $R^{(\infty)}(D,P) = \hat{R}$ is convex as well. Note that the expression of $R^{(\infty)}(D,P)$ is explicitly given in Theorem 2.4.2. By the implicit function theorem, we can compute the derivative $P'(D) = -\frac{\partial R^{(\infty)}(D,P(D))/\partial D}{\partial R^{(\infty)}(D,P(D))/\partial P}$. By the convexity, the curve $P(D)$ is above the line $D \mapsto P'(D_2)(D-D_2)$ where $P'(D_2) = \kappa$ with κ given in (2.4.4). Hence, to ensure that all points in $\Theta(R)$ are achievable, it suffice to require all points on the line segment between $(D_1, P'(D_2)(D_1-D_2))$ and $(D_2, 0)$ are achievable. In fact, more succinctly, it is only required the two end points $(D_1, P'(D_2)(D_1-D_2))$ and $(D_2, 0)$ are achievable. This is because, once these two end points are achievable, the whole segment is achievable as well, since the set of achievable points is always convex. Substituting $(D_1, P'(D_2)(D_1-D_2))$ and $(D_2, 0)$ into the RDP function, we obtain the upper bound R_{UB}. We already have

$$R^{(\infty)}(D,P) = \begin{cases} H_b(q) - H_b(D), & D \in [0, D_1), \\ 2H_b(q) + H_b(q-P) - H_t(\frac{D-P}{2}, q) - H_t(\frac{D+P}{2}, 1-q), & D \in [D_1, D_2), \\ 0, & D \in [D_2, \infty). \end{cases}$$

In particular, when $D \in [D_1, D_2)$, the slope of the tangent for the upper bound is

$$\kappa_{UB} = \frac{\partial P}{\partial D}\bigg|_{D=D_2, P=0} = -\frac{\frac{\partial R}{\partial D}}{\frac{\partial R}{\partial P}}\bigg|_{D=D_2, P=0} = \frac{-\log\frac{D_2}{2} + \frac{1}{2}\log(1-e-\frac{D_2}{2}) + \frac{1}{2}\log(e-\frac{D_2}{2})}{\log\frac{e}{1-e} + \frac{1}{2}\log(1-e-\frac{D_2}{2}) - \frac{1}{2}\log(e-\frac{D_2}{2})}.$$

By comparing the upper and lower bounds, one can check that the upper bound and lower bound coincide if and only if $\frac{q}{2D_1-D_2-q} \geq \kappa_{UB}$.

Let $g(R) = \frac{q}{2D_1(R)-D_2(R)-q} - \kappa_{UB}$. It can be numerically verified that $g(R)$ is increasing with R when $g(R) \geq 0$. So $\frac{q}{2D_1-D_2-q} \geq \kappa_{UB}$ is equivalent to $R \geq R^*$, where $R^* = H(q) - H(D_1^*)$ or $3H_b(q) - H_t(\frac{D_2^*}{2}, q) - H_t(\frac{D_2^*}{2}, 1-q)$ and (D_1^*, D_2^*) is the unique solution to

$$\begin{cases} H_b(q) - H_b(D_1^*) = 3H_b(q) - H_t(\frac{D_2^*}{2}, q) - H_t(\frac{D_2^*}{2}, 1-q), \\ \frac{q}{2D_1^* - D_2^* - q} = \frac{-\log\frac{D_2^*}{2} + \frac{1}{2}\log(1-e-\frac{D_2^*}{2}) + \frac{1}{2}\log(e-\frac{D_2^*}{2})}{\log\frac{e}{1-e} + \frac{1}{2}\log(1-e-\frac{D_2^*}{2}) - \frac{1}{2}\log(e-\frac{D_2^*}{2})}. \end{cases}$$

This completes the proof. □

2.D Proof of Theorem 2.4.5

For a binary source $X \sim \text{Bern}(q)$, under Hamming distortion and TV perception losses, successive refinement from $(D_1, P_1) \in \mathcal{A}$ to $(D_2, P_2) \in \mathcal{A}$ is feasible if and only if

$$q((D_1 - P_1) - (D_2 - P_2)) \geq D_1 P_2 - D_2 P_1, \tag{2.D.1}$$

$$(1-q)((D_1 + P_1) - (D_2 + P_2)) \geq D_2 P_1 - D_1 P_2. \tag{2.D.2}$$

Proof: Recall that

$$\omega_{TV}(p_X, p_{\hat{X}}) = \frac{1}{2}\sum_{z\in\{0,1\}} |p_X(z) - p_{\hat{X}}(z)|.$$

Without loss of generality, we assume $q \in [0, \frac{1}{2}]$ and $P \in [0, q]$. We have

$$R^{(\infty)}(D,P) = \begin{cases} H_b(q) - H_b(D), & D \in [0, \frac{P}{1-2(q-P)}), \\ 2H_b(q) + H_b(q-P) - H_t(\frac{D-P}{2}, q) - H_t(\frac{D+P}{2}, 1-q), & \\ & D \in [\frac{P}{1-2(q-P)}, 2q(1-q) - (1-2q)P), \\ 0, & D \in [2q(1-q) - (1-2q)P, \infty). \end{cases}$$

In particular, for the second case, $R^{(\infty)}(D,P)$ is attained by

$$p_{\hat{X}|X}(0|0) = 1 - \frac{D-P}{2(1-q)},$$
$$p_{\hat{X}|X}(0|1) = \frac{D+P}{2q}.$$

Let $\mathcal{A}$ denote the regime where both D and P are active, i.e., $\mathcal{A} \triangleq \{(D,P) : D \in [\frac{P}{1-2(q-P)}, 2p(1-q)-(1-2q)P), P \in [0,q]\}$. For $(D_1,P_1) \in A$ and $(D_2,P_2) \in A$ with no conditions, we would like to construct $X \leftrightarrow \hat{X}_2 \leftrightarrow \hat{X}_1$, such that

$$p_{\hat{X}_1|X}(0|0) = 1 - \frac{D_1-P_1}{2(1-q)},$$
$$p_{\hat{X}_1|X}(0|1) = \frac{D_1+P_1}{2q},$$
$$p_{\hat{X}_2|X}(0|0) = 1 - \frac{D_2-P_2}{2(1-q)},$$
$$p_{\hat{X}_2|X}(0|1) = \frac{D_2+P_2}{2q}.$$

The existence of $\hat{X}_2$ is abvious. Now let $p_{\hat{X}_1|\hat{X}_2}(0|0) = a$ and $p_{\hat{X}_1|\hat{X}_2}(0|1) = b$. We have

$$\begin{aligned} p_{\hat{X}_1|X}(0|0) &= p_{\hat{X}_2|X}(0|0)p_{\hat{X}_1|\hat{X}_2}(0|0) + p_{\hat{X}_2|X}(0|1)p_{\hat{X}_1|\hat{X}_2}(0|1) \\ &= (1 - \frac{D_2-P_2}{2(1-q)})a + \frac{D_2-P_2}{2(1-q)}b, \\ p_{\hat{X}_1|X}(0|1) &= p_{\hat{X}_2|X}(0|1)p_{\hat{X}_1|\hat{X}_2}(0|0) + p_{\hat{X}_2|X}(1|1)p_{\hat{X}_1|\hat{X}_2}(0|1) \\ &= \frac{D_2+P_2}{2q}a + (1 - \frac{D_2+P_2}{2q})b. \end{aligned}$$

Setting

$$
\begin{aligned}
(1 - \frac{D_2 - P_2}{2(1-q)})a + \frac{D_2 - P_2}{2(1-q)}b &= 1 - \frac{D_1 - P_1}{2(1-q)},\\
\frac{D_2 + P_2}{2q}a + (1 - \frac{D_2 + P_2}{2q})b &= \frac{D_1 + P_1}{2q},
\end{aligned}
$$

yields

$$
\begin{aligned}
a &= \frac{(1 - \frac{D_2+P_2}{2q})(1 - \frac{D_1-P_1}{2(1-q)}) - \frac{(D_2-P_2)(D_1+P_1)}{4q(1-q)}}{(1 - \frac{D_2-P_2}{2(1-q)})(1 - \frac{D_2+P_2}{2q}) - \frac{(D_2-P_2)(D_2+P_2)}{4q(1-q)}},\\
b &= \frac{(1 - \frac{D_2-P_2}{2(1-q)})\frac{D_1+P_1}{2q} - (1 - \frac{D_1-P_1}{2(1-q)})\frac{D_2+P_2}{2q}}{(1 - \frac{D_2-P_2}{2(1-q)})(1 - \frac{D_2+P_2}{2q}) - \frac{(D_2-P_2)(D_2+P_2)}{4q(1-q)}}.
\end{aligned}
$$

We need to ensure that $a \in [0,1]$ and $b \in [0,1]$. Note that the numerator of a is

$$
\begin{aligned}
&(1 - \frac{D_2 + P_2}{2q})(1 - \frac{D_1 - P_1}{2(1-q)}) - \frac{(D_2 - P_2)(D_1 + P_1)}{4q(1-q)}\\
&= \frac{(2q - (D_2 + P_2))(2(1-q) - (D_1 - P_1)) - (D_2 - P_2)(D_1 + P_1)}{4q(1-q)}
\end{aligned}
$$

We know that $4q(1-q) \geq 0$ always holds. So, we should ensure $(2q-(D_2+P_2))(2(1-q) - (D_1 - P_1)) - (D_2 - P_2)(D_1 + P_1) \geq 0$. Since $D_1 \leq 2q(1-q) - (1-2q)P_1$ and $D_2 \leq 2q(1-q) - (1-2q)P_2$, it follows that

$$
\begin{aligned}
D_1 - P_1 &\leq 2(1-q)(q - P_1),\\
D_2 - P_2 &\leq 2(1-q)(q - P_2),\\
D_1 + P_1 &\leq 2q(1 - q + P_1),\\
D_2 + P_2 &\leq 2q(1 - q + P_2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
2q - (D_2 + P_2) &\geq 2q(q - P_2),\\
2(1-q) - (D_1 - P_1) &\geq 2(1-q)(1-q+P_1),\\
(D_2 - P_2)(D_1 + P_1) &\leq 4q(1-q)(1-q+P_1)(q-P_2).
\end{aligned}
$$

As a consequence,

$$(2q - (D_2 + P_2))(2(1-q) - (D_1 - P_1)) - (D_2 - P_2)(D_1 + P_1) \geq 0.$$

This proves that the numerator of a is non-negative with the conditions $D_1 \leq 2q(1-q) - (1-2q)P_1$ and $D_2 \leq 2q(1-q) - (1-2q)P_2$. Moreover the denominator of a is

$$
\begin{aligned}
&(1 - \frac{D_2 - P_2}{2(1-q)})(1 - \frac{D_2 + P_2}{2q}) - \frac{(D_2 - P_2)(D_2 + P_2)}{4q(1-q)}\\
&= \frac{(2(1-q) - (D_2 - P_2))(2q - (D_2 + P_2)) - (D_2 - P_2)(D_2 + P_2)}{4q(1-q)}.
\end{aligned}
$$

So that

$$
\begin{aligned}
2(1-q) - (D_2 - P_2) &\geq 2(1-q)(1-q+P_2),\\
2q - (D_2 + P_2) &\geq 2q(q - P_2),\\
(D_2 - P_2)(D_2 + P_2) &\leq 2(1-q)(q-P_2)2q(1-q+P_2).
\end{aligned}
$$

As a consequence,

$$(2(1-q) - (D_2 - P_2))(2q - (D_2 + P_2)) - (D_2 - P_2)(D_2 + P_2) \geq 0.$$

This proves that the denominator of a is non-negative with the conditions $D_1 \leq 2q(1-q) - (1-2q)P_1$ and $D_2 \leq 2q(1-q) - (1-2q)P_2$ as well. So we must have $a \geq 0$ when $(D_1, P_1) \in A$ and $(D_2, P_2) \in A$ with no conditions. To ensure $a \leq 1$, we must have

$$\begin{aligned}&(1 - \frac{D_2 + P_2}{2q})(1 - \frac{D_1 - P_1}{2(1-q)}) - \frac{(D_2 - P_2)(D_1 + P_1)}{4q(1-q)} \\ &\leq 1 - \frac{D_2 - P_2}{2(1-q)})(1 - \frac{D_2 + P_2}{2q}) - \frac{(D_2 - P_2)(D_2 + P_2)}{4q(1-q)},\end{aligned}$$

i.e.,

$$(2q - (D_2 + P_2))((D_1 - P_1) - (D_2 - P_2)) - (D_2 - P_2)((D_2 + P_2) - (D_1 + P_1)) \geq 0.$$

Note that

$$\begin{aligned}&(2q - (D_2 + P_2))((D_1 - P_1) - (D_2 - P_2)) - (D_2 - P_2)((D_2 + P_2) - (D_1 + P_1)) \\ &= 2q((D_1 - P_1) - (D_2 - P_2)) - 2(D_1 P_2 - D_2 P_1).\end{aligned}$$

Therefore, we have $a \leq 1$ if

$$q((D_1 - P_1) - (D_2 - P_2)) \geq D_1 P_2 - D_2 P_1.$$

Overall, we must have $a \in [0, 1]$ when $(D_1, P_1) \in A$ and $(D_2, P_2) \in A$ with $q((D_1 - P_1) - (D_2 - P_2)) \geq D_1 P_2 - D_2 P_1$. Now we proceed to show that $b \in [0, 1]$. Note that the numerator of b is

$$
\begin{aligned}
&(1-\frac{D_2-P_2}{2(1-q)})\frac{D_1+P_1}{2q}-(1-\frac{D_1-P_1}{2(1-q)})\frac{D_2+P_2}{2q}\\
&=\frac{(2(1-q)-(D_2-P_2))(D_1+P_1)-(2(1-q)-(D_1-P_1))(D_2+P_2)}{4q(1-q)}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&(2(1-q)-(D_2-P_2))(D_1+P_1)-(2(1-q)-(D_1-P_1))(D_2+P_2)\\
&=2(1-q)((D_1+P_1)-(D_2+P_2))+2(D_1P_2-D_2P_1).
\end{aligned}
$$

Therefore, we have the numerator of $b \geq 0$ if

$$
(1-q)((D_1+P_1)-(D_2+P_2)) \geq D_2P_1-D_1P_2.
$$

Moreover the denominator of b is

$$
\begin{aligned}
&(1-\frac{D_2-P_2}{2(1-q)})(1-\frac{D_2+P_2}{2q})-\frac{(D_2-P_2)(D_2+P_2)}{4q(1-q)}\\
&=\frac{(2(1-q)-(D_2-P_2))(2q-(D_2+P_2))-(D_2-P_2)(D_2+P_2)}{4q(1-q)}.
\end{aligned}
$$

So that

$$
\begin{aligned}
2(1-q)-(D_2-P_2) &\geq 2(1-q)(1-q+P_2),\\
2q-(D_2+P_2) &\geq 2q(q-P_2),\\
(D_2-P_2)(D_2+P_2) &\leq 2(1-q)(q-P_2)2q(1-q+P_2).
\end{aligned}
$$

As a consequence,

$$
(2(1-q)-(D_2-P_2))(2q-(D_2+P_2))-(D_2-P_2)(D_2+P_2) \geq 0.
$$

This proves that the denominator of b is non-negative with the conditions $D_1 \leq 2q(1-q)-(1-2q)P_1$ and $D_2 \leq 2q(1-q)-(1-2q)P_2$ as well. So we must have $b \geq 0$ when $(D_1, P_1) \in A$ and $(D_2, P_2) \in A$ with the condition $(1-q)((D_1+P_1)-(D_2+P_2)) \geq D_2P_1 - D_1P_2$. To ensure $b \leq 1$, it suffices to show

$$\begin{aligned}&(1-\frac{D_2-P_2}{2(1-q)})\frac{D_1+P_1}{2q}-(1-\frac{D_1-P_1}{2(1-q)})\frac{D_2+P_2}{2q}\\ &\leq(1-\frac{D_2-P_2}{2(1-q)})(1-\frac{D_2+P_2}{2q})-\frac{(D_2-P_2)(D_2+P_2)}{4q(1-q)},\end{aligned}$$

i.e.,

$$\begin{aligned}&(2(1-q)-(D_2-P_2))(D_1+P_1)-(2(1-q)-(D_1-P_1))(D_2+P_2)\\ &\leq(2(1-q)-(D_2-P_2))(2q-(D_2+P_2))-(D_2-P_2)(D_2+P_2).\end{aligned}$$

Note that

$$\begin{aligned}&(2(1-q)-(D_2-P_2))(2q-(D_2+P_2))-(D_2-P_2)(D_2+P_2)\\ &-(2(1-q)-(D_2-P_2))(D_1+P_1)-(2(1-q)-(D_1-P_1))(D_2+P_2)\\ &=(1-q-D_2)(q-P_1)-(1-q+P_2)(D_1-q).\end{aligned}$$

Since $D_1 \leq 2q(1-q)-(1-2q)P_1$ and $D_2 \leq 2q(1-q)-(1-2q)P_2$, it follows that

$$\begin{aligned}&(1-q-D_2)(q-P_1)-(1-q+P_2)(D_1-q)\\ &\geq(1-q-(2q(1-q)-(1-2q)P_2))(q-P_1)-(1-q+P_2)((2q(1-q)-(1-2q)P_1)-q)\\ &=(q-P_1)(1-3q+2q^2-1+3q-2q^2)\\ &=0.\end{aligned}$$

This proves that $b \leq 1$ with the conditions $D_1 \leq 2q(1-q) - (1-2q)P_1$ and $D_2 \leq 2q(1-q) - (1-2q)P_2$. So we must have $b \in [0,1]$ when $(D_1, P_1) \in A$ and $(D_2, P_2) \in A$ with $(1-q)((D_1+P_1) - (D_2+P_2)) \geq D_2 P_1 - D_1 P_2$, this ends the proof.

□

2.E Proof of Theorem 2.4.6

Let $0 < R_1 < R_2$. For a binary source $X \sim \text{Bern}(q)$, under Hamming distortion and TV perception losses as well as the conditions that

$$\frac{q}{2D_1 - D_2 - q} \geq \kappa(D_2), \quad \frac{q}{2D_1' - D_2' - q} \geq \kappa(D_2')$$

with the function κ given in (2.4.4), successive refinement from $\Theta(R_1)$ to $\Theta(R_2)$ is feasible if

$$2D_1 - D_2 \leq 2D_1' - D_2'.$$

Proof: Under the condition in (2.4.9), set-wise successive refinement from $\Theta(R_1)$ to $\Theta(R_2)$ is feasible if there exists a Markov chain $X \leftrightarrow (\hat{X}_1', \hat{X}_2') \leftrightarrow (\hat{X}_1, \hat{X}_2)$ such that the conditional distributions $P_{\hat{X}_1\hat{X}_2|X}$ and $P_{\hat{X}_1'\hat{X}_2'|X}$ are optimal in attaining $R^{(\infty)}(\Theta(R_1))$ and $R^{(\infty)}(\Theta(R_2))$ respectively. In other words, $P_{\hat{X}_1\hat{X}_2|X}$ corresponds to the distributions given in (2.4.1), and $P_{\hat{X}_1'\hat{X}_2'|X}$ corresponds the same distribution but with D_1, D_2 replaced by D_1', D_2'. Recall that

$$\omega_{TV}(p_X, p_{\hat{X}}) = \frac{1}{2} \sum_{z \in \{0,1\}} |p_X(z) - p_{\hat{X}}(z)|.$$

Based on theorem 4, the universal rate-distortion-perception function is attained by

$$\begin{aligned}
p_{\hat{X}_1\hat{X}_2|X}(00|0) &= 1 - \frac{D_2}{2(1-q)} = p_1, \\
p_{\hat{X}_1\hat{X}_2|X}(01|0) &= \frac{(D_2 - D_1)(2q - 2D_1 + D_2)}{2(1-q)(q - 2D_1 + D_2)} = p_2, \\
p_{\hat{X}_1\hat{X}_2|X}(11|0) &= \frac{(2D_1 - D_2)(q - D_1)}{2(1-q)(q - 2D_1 + D_2)} = p_4, \\
p_{\hat{X}_1\hat{X}_2|X}(00|1) &= \frac{D_2}{2q} = q_1, \\
p_{\hat{X}_1\hat{X}_2|X}(01|1) &= \frac{(D_2 - D_1)(2D_1 - D_2)}{2q(q - 2D_1 + D_2)} = q_2, \\
p_{\hat{X}_1\hat{X}_2|X}(11|1) &= \frac{(q - D_1)(2q - 2D_1 + D_2)}{2q(q - 2D_1 + D_2)} = q_4.
\end{aligned}$$

For (D_1D_2) and $(D_1'D_2')$ with $D_1 \le D_1'$ and $D_2 \le D_2'$, we would like to construct $X \leftrightarrow (\hat{X}_1', \hat{X}_2') \leftrightarrow (\hat{X}_1, \hat{X}_2)$ such that

$$\begin{aligned}
p_{\hat{X}_1\hat{X}_2|X}(00|0) &= 1 - \frac{D_2}{2(1-q)} = p_1,\\
p_{\hat{X}_1\hat{X}_2|X}(01|0) &= \frac{(D_2-D_1)(2q-2D_1+D_2)}{2(1-q)(q-2D_1+D_2)} = p_2,\\
p_{\hat{X}_1\hat{X}_2|X}(11|0) &= \frac{(2D_1-D_2)(q-D_1)}{2(1-q)(q-2D_1+D_2)} = p_4,\\
p_{\hat{X}_1\hat{X}_2|X}(00|1) &= \frac{D_2}{2q} = q_1,\\
p_{\hat{X}_1\hat{X}_2|X}(01|1) &= \frac{(D_2-D_1)(2D_1-D_2)}{2q(q-2D_1+D_2)} = q_2,\\
p_{\hat{X}_1\hat{X}_2|X}(11|1) &= \frac{(q-D_1)(2q-2D_1+D_2)}{2q(q-2D_1+D_2)} = q_4,\\
p_{\hat{X}_1{}'\hat{X}_2{}'|X}(00|0) &= 1 - \frac{D_2'}{2(1-q)} = p_1',\\
p_{\hat{X}_1{}'\hat{X}_2{}'|X}(01|0) &= \frac{(D_2'-D_1')(2q-2D_1'+D_2')}{2(1-q)(q-2D_1'+D_2')} = p_2',\\
p_{\hat{X}_1{}'\hat{X}_2{}'|X}(11|0) &= \frac{(2D_1'-D_2')(q-D_1')}{2(1-q)(q-2D_1'+D_2')} = p_4',\\
p_{\hat{X}_1{}'\hat{X}_2{}'|X}(00|1) &= \frac{D_2'}{2q} = q_1',\\
p_{\hat{X}_1{}'\hat{X}_2{}'|X}(01|1)2 &= \frac{(D_2'-D_1')(2D_1'-D_2')}{2q(q-2D_1'+D_2')} = q_2',\\
p_{\hat{X}_1{}'\hat{X}_2{}'|X}(11|1) &= \frac{(q-D_1')(2q-2D_1'+D_2')}{2q(q-2D_1'+D_2')} = q_4'.
\end{aligned}$$

The existence of $\hat{X}_1'\hat{X}_2'$ is abvious. Now let

$$p_{\hat{X}_1\hat{X}_2|\hat{X}_1{}'\hat{X}_2{}'}(00|00) = a_1,$$
$$p_{\hat{X}_1\hat{X}_2|\hat{X}_1{}'\hat{X}_2{}'}(00|01) = a_2,$$
$$p_{\hat{X}_1\hat{X}_2|\hat{X}_1{}'\hat{X}_2{}'}(01|00) = b_1,$$
$$p_{\hat{X}_1\hat{X}_2|\hat{X}_1{}'\hat{X}_2{}'}(01|01) = b_2,$$
$$p_{\hat{X}_1\hat{X}_2|\hat{X}_1{}'\hat{X}_2{}'}(11|00) = c_1,$$
$$p_{\hat{X}_1\hat{X}_2|\hat{X}_1{}'\hat{X}_2{}'}(11|01) = c_2.$$

We have

$$P_{\hat{X}_1{}'\hat{X}_2{}'|X} = \sum_{\hat{X}_1\hat{X}_2} P_{\hat{X}_1\hat{X}_2|\hat{X}_1{}'\hat{X}_2{}'} P_{\hat{X}_1\hat{X}_2|X},$$

such that

$$p_{\hat{X}_1{}'\hat{X}_2{}'|X}(00|0) = p_1a_1 + p_2b_1 + p_4c_1,$$
$$p_{\hat{X}_1{}'\hat{X}_2{}'|X}(01|0) = p_1a_2 + p_2b_2 + p_4c_2,$$
$$p_{\hat{X}_1{}'\hat{X}_2{}'|X}(11|0) = p_1(1-a_1-a_2) + p_2(1-b_1-b_2) + p_4(1-c_1-c_2),$$
$$p_{\hat{X}_1{}'\hat{X}_2{}'|X}(00|1) = q_1a_1 + q_2b_1 + q_4c_1,$$
$$p_{\hat{X}_1{}'\hat{X}_2{}'|X}(01|1) = q_1a_2 + q_2b_2 + q_4c_2,$$
$$p_{\hat{X}_1{}'\hat{X}_2{}'|X}(11|1) = q_1(1-a_1-a_2) + q_2(1-b_1-b_2) + q_4(1-c_1-c_2).$$

Setting

$$
\begin{aligned}
p_1 a_1 + p_2 b_1 + p_4 c_1 &= p_1', \\
p_1 a_2 + p_2 b_2 + p_4 c_2 &= p_2', \\
p_1(1 - a_1 - a_2) + p_2(1 - b_1 - b_2) + p_4(1 - c_1 - c_2) &= p_4', \\
q_1 a_1 + q_2 b_1 + q_4 c_1 &= q_1', \\
q_1 a_2 + q_2 b_2 + q_4 c_2 &= q_2', \\
q_1(1 - a_1 - a_2) + q_2(1 - b_1 - b_2) + q_4(1 - c_1 - c_2) &= q_4'.
\end{aligned}
$$

yields

$$
\begin{aligned}
a_1 &= \frac{c_1 p_2 - c_1 q_2 - p_2 q_1' + p_1' q_2 + c_1 p_1 q_2 - c_1 p_2 q_1}{p_1 q_2 - p_2 q_1}, \\
a_2 &= \frac{c_2 p_2 - c_2 q_2 - p_2 q_2' + p_2' q_2 + c_2 p_1 q_2 - c_2 p_2 q_1}{p_1 q_2 - p_2 q_1}, \\
b_1 &= \frac{c_1 q_1 - c_1 p_1 + p_1 q_1' - p_1' q_1 + c_1 p_1 q_2 - c_1 p_2 q_1}{p_1 q_2 - p_2 q_1}, \\
b_2 &= \frac{c_2 q_1 - c_2 p_1 + p_1 q_2' - p_2' q_1 + c_2 p_1 q_2 - c_2 p_2 q_1}{p_1 q_2 - p_2 q_1}.
\end{aligned}
$$

We need to ensure that $a_1, a_2 \in [0,1]$, $b_1, b_2 \in [0,1]$ and $a_1 + a_2 \leq 1$, $b_1 + b_2 \leq 1$. Note that the denominator of all is

$$
p_1 q_2 - p_2 q_1 = \frac{(D_1 - D_2)(D_2 - 2(1-q)D_1)}{2(1-q)q(q - 2D_1 + D_2)}.
$$

Since $D_1 \leq D_2$, $2D_1 - D_2 \leq q$ and $D_2 \leq 2(1-q)D_1$ we must have

$$
p_1 q_2 - p_2 q_1 \geq 0.
$$

In order to make $a_1 \geq 0$, we must have

$$c_1p_2 - c_1q_2 - p_2q_1' + p_1'q_2 + c_1p_1q_2 - c_1p_2q_1 \geq 0,$$
$$(p_2(1-q_1) - q_2(1-p_1))c_1 \geq p_2q_1' - p_1'q_2,$$

Since

$$p_2(1-q_1) - q_2(1-p_1) = \frac{(D_2 - D_1)(q - D_1)}{(1-q)(q - 2D_1 + D_2)} \geq 0,$$

it follows that

$$c_1 \geq \frac{p_2q_1' - p_1'q_2}{p_2(1-q_1) - q_2(1-p_1)}.$$

What's more, it can be proved that

$$q_2 - p_2 = \frac{(D_1 - D_2)(2q^2 - 2D_1 + D_2)}{(2(1-q)q(q - 2D_1 + D_2)} \geq 0,$$

where $D_1 - D_2 \leq 0$, $q - 2D_1 + D_2 \geq 0$ and $2q^2 - 2D_1 + D_2 \leq 0$. Similarly, $q_2' \geq p_2'$. And

$$p_1 - q_1 = \frac{2(1-q)^2 - 2(1-q) + D_2}{2q(q-1)} \geq 0,$$

where $2(1-q)^2 - 2(1-q) + D_2 \leq 0$. Similarly, $p_1' \geq q_1'$. As a consequence, $\frac{p_2q_1' - p_1'q_2}{p_2(1-q_1) - q_2(1-p_1)} \leq 0$, it means $c_1 \geq \frac{p_2q_1' - p_1'q_2}{p_2(1-q_1) - q_2(1-p_1)}$ always exists if $c_1 \geq 0$. To ensure

$a_1 \leq 1$, we must have

$$\begin{aligned} c_1p_2 - c_1q_2 - p_2q_1' + p_1'q_2 + c_1p_1q_2 - c_1p_2q_1 &\leq p_1q_2 - p_2q_1, \\ c_1 &\leq \frac{q_2(p_1 - p_1') - p_2(q_1 - q_1')}{p_2(1-q_1) - q_2(1-p_1)}, \end{aligned}$$

where $p_2(1-q_1) - q_2(1-p_1) \geq 0$. Now we proceed to ensure $b_1 \geq 0$. Note that

$$c_1q_1 - c_1p_1 + p_1q_1' - p_1'q_1 + c_1p_1q_2 - c_1p_2q_1 \geq 0,$$
$$(q_1(1-p_2) - p_1(1-q_2))c_1 \geq p_1'q_1 - p_1q_1'.$$

Since

$$q_1(1-p_2) - p_1(1-q_2) = \frac{(q - D_1)(2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2)}{2(1-q)q(q - 2D_1 + D_2)} \leq 0,$$

where $q - D_1 \geq 0$, $q - 2D_1 + D_2 \geq 0$ and

$$\begin{aligned} &2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2 \\ &\leq 2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + 2(1-q)D_1 \\ &= -2(1-q)q + 2(1-q)(2D_1 - D_2) \\ &= 2(1-q)(2D_1 - D_2 - q) \leq 0, \end{aligned}$$

it follows that

$$c_1 \leq \frac{p_1'q_1 - p_1q_1'}{q_1(1-p_2) - p_1(1-q_2)}.$$

And for $b_1 \leq 1$, we must have

$$\begin{aligned} c_1q_1 - c_1p_1 + p_1q_1' - p_1'q_1 + c_1p_1q_2 - c_1p_2q_1 \leq p_1q_2 - p_2q_1, \\ c_1 \geq \frac{p_1(q_2 - q_1') - q_1(p_2 - p_1')}{q_1(1-p_2) - p_1(1-q_2)}. \end{aligned}$$

where $q_1(1-p_2) - p_1(1-q_2) \leq 0$. As a consequence,

$$\frac{p_1(q_2 - q_1') - q_1(p_2 - p_1')}{q_1(1-p_2) - p_1(1-q_2)} \leq c_1 \leq \min\{\frac{q_2(p_1 - p_1') - p_2(q_1 - q_1')}{p_2(1-q_1) - q_2(1-p_1)}, \frac{p_1'q_1 - p_1q_1'}{q_1(1-p_2) - p_1(1-q_2)}\}$$

Compare the two upper bounds of c_1, we have

$$\begin{aligned} &\frac{q_2(p_1 - p_1') - p_2(q_1 - q_1')}{p_2(1-q_1) - q_2(1-p_1)} - \frac{p_1'q_1 - p_1q_1'}{q_1(1-p_2) - p_1(1-q_2)} \\ &= \frac{D_2' - D_2}{2(q - D_1)} - \frac{(1-q)(D_2' - D_2)(q - 2D_1 + D_2)}{(D_1 - q)(2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2)} \\ &= \frac{(D_2 - D_2')(D_2 - 2(1-q)D_1)}{2(D_1 - q)(2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2)} \geq 0, \\ &\Rightarrow \frac{q_2(p_1 - p_1') - p_2(q_1 - q_1')}{p_2(1-q_1) - q_2(1-p_1)} \geq \frac{p_1'q_1 - p_1q_1'}{q_1(1-p_2) - p_1(1-q_2)}, \end{aligned}$$

where $D_2 - D_2' \leq 0$, $D_2 - 2(1-q)D_1 \leq 0$, $D_1 - q \leq 0$ and $2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2 \leq 0$. Since $p_2q_1 - p_1q_2 \leq 0$ and $q_1(1-p_2) - p_1(1-q_2) \leq 0$, it follows that

$$\begin{aligned} &\frac{p_1'q_1 - p_1q_1'}{q_1(1-p_2) - p_1(1-q_2)} - \frac{p_1(q_2 - q_1') - q_1(p_2 - p_1')}{q_1(1-p_2) - p_1(1-q_2)} \\ &= \frac{p_2q_1 - p_1q_2}{q_1(1-p_2) - p_1(1-q_2)} \geq 0, \end{aligned}$$

which clearly shows that the lower bound is always less than the upper bound of c_1. In other words, we can always find a c_1 in this interval to make sure $a_1 \in [0, 1]$ and $b_1 \in [0, 1]$.

In the following, we proceed to ensure $a_2 \in [0, 1]$ and $b_2 \in [0, 1]$. In order to make $a_2 \geq 0$, we must have

$$\begin{aligned} c_2p_2 - c_2q_2 - p_2q_2' + p_2'q_2 + c_2p_1q_2 - c_2p_2q_1 &\geq 0, \\ (p_2(1-q_1) - q_2(1-p_1))c_2 &\geq p_2q_2' - p_2'q_2, \\ c_2 &\geq \frac{p_2q_2' - p_2'q_2}{p_2(1-q_1) - q_2(1-p_1)}. \end{aligned}$$

where $p_2(1-q_1) - q_2(1-p_1) \geq 0$. To ensure $a_2 \leq 1$, we have

$$\begin{aligned} c_2p_2 - c_2q_2 - p_2q_2' + p_2'q_2 + c_2p_1q_2 - c_2p_2q_1 &\leq p_1q_2 - p_2q_1, \\ (p_2(1-q_1) - q_2(1-p_1))c_2 &\leq q_2(p_1 - p_2') - p_2(q_1 - q_2'), \\ c_2 &\leq \frac{q_2(p_1 - p_2') - p_2(q_1 - q_2')}{p_2(1-q_1) - q_2(1-p_1)}. \end{aligned}$$

where $p_2(1-q_1) - q_2(1-p_1) \geq 0$. Note that

$$\begin{aligned} &(q_2(p_1 - p_2') - p_2(q_1 - q_2')) - (p_2(1-q_1) - q_2(1-p_1)) \\ &\qquad = q_2(1 - p_2') - p_2(1 - q_2') \geq 0, \end{aligned}$$

where the inequality is from $q_2 \geq p_2$ and $1 - p_2' \geq 1 - q_2'$. Therefore, if $c_2 \leq 1$,

$c_2 \leq \frac{q_2(p_1-p_2')-p_2(q_1-q_2')}{p_2(1-q_1)-q_2(1-p_1)}$ always exists. To ensure $b_2 \geq 0$, we have

$$
\begin{aligned}
c_2q_1 - c_2p_1 + p_1q_2' - p_2'q_1 + c_2p_1q_2 - c_2p_2q_1 \geq 0, \\
(q_1(1-p_2) - p_1(1-q_2))c_2 \geq p_2'q_1 - p_1q_2', \\
c_2 \leq \frac{p_2'q_1 - p_1q_2'}{q_1(1-p_2) - p_1(1-q_2)}.
\end{aligned}
$$

where $q_1(1-p_2) - p_1(1-q_2) \leq 0$. For $b_2 \leq 1$, we must have

$$
\begin{aligned}
c_2q_1 - c_2p_1 + p_1q_2' - p_2'q_1 + c_2p_1q_2 - c_2p_2q_1 \leq p_1q_2 - p_2q_1, \\
(q_1(1-p_2) - p_1(1-q_2))c_2 \leq p_1(q_2 - q_2') - q_1(p_2 - p_2'), \\
c_2 \geq \frac{p_1(q_2 - q_2') - q_1(p_2 - p_2')}{q_1(1-p_2) - p_1(1-q_2)}.
\end{aligned}
$$

where $q_1(1-p_2) - p_1(1-q_2) \leq 0$. As a consequence,

$$
\max\{\frac{p_2q_2' - p_2'q_2}{p_2(1-q_1) - q_2(1-p_1)}, \frac{p_1(q_2 - q_2') - q_1(p_2 - p_2')}{q_1(1-p_2) - p_1(1-q_2)}\} \leq c_2 \leq \frac{p_2'q_1 - p_1q_2'}{q_1(1-p_2) - p_1(1-q_2)}.
$$

Compare the two lower bounds of c_2, we have

$$
\begin{aligned}
&\frac{p_1(q_2 - q_2') - q_1(p_2 - p_2')}{q_1(1-p_2) - p_1(1-q_2)} - \frac{p_2q_2' - p_2'q_2}{p_2(1-q_1) - q_2(1-p_1)} \\
&= \frac{(2(1-q)D_1 - D_2)(2q(D_2' - D_1' + D_1 - D_2) + (2D_1' - D_2')(D_1' - D_2') - 2D_1D_1' + 3D_2D_1' - D_2D_2'}{2(D_1 - q)(q - 2D_1' + D_2')(2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2)} \\
&\geq 0,
\end{aligned}
$$

because $2(1-q)D_1 - D_2 \geq 0$, $D_1 - q \leq 0$, $q - 2D_1' + D_2' \geq 0$, $2(1-q)^2 - 2(1-q) +$

$2(1-q)D_1 - 2(1-q)D_2 + D_2) \leq 0$ and if $2D_1' - D_2' \geq 2D_1 - D_2$, we must have

$$\begin{aligned}
&2q(D_2' - D_1' + D_1 - D_2) + (2D_1' - D_2')(D_1' - D_2') - 2D_1D_1' + 3D_2D_1' - D_2D_2' \\
&= 2q(D_2' - D_1' + D_1 - D_2) + (2D_1' - D_2')(D_1' - D_2') + D_2(2D_1' - D_2') - D_1'(2D_1 - D_2) \\
&= 2q(D_2' - D_1' + D_1 - D_2) + (2D_1' - D_2')(D_1' - D_2' + D_2) - D_1'(2D_1 - D_2) \\
&\geq (2D_1' - D_2')(D_2' - D_1' + 2D_1 - D_2) - D_1'(2D_1 - D_2) \\
&= (2D_1' - D_2')(D_2' - D_1') + (2D_1' - D_2')(2D_1 - D_2) - D_1'(2D_1 - D_2) \\
&= (2D_1' - D_2')(D_2' - D_1') - (2D_1 - D_2)(D_2' - D_1') \\
&= (D_2' - D_1')(2D_1' - D_2' - 2D_1 + D_2) \geq 0.
\end{aligned}$$

Therefore,

$$\frac{p_2q_2' - p_2'q_2}{p_2(1-q_1) - q_2(1-p_1)} \leq \frac{p_1(q_2 - q_2') - q_1(p_2 - p_2')}{q_1(1-p_2) - p_1(1-q_2)}.$$

Since $p_1q_2 - p_2q_1 \geq 0$ and $q_1(1-p_2) - p_1(1-q_2) \leq 0$, it follows that

$$\begin{aligned}
&\frac{p_2'q_1 - p_1q_2'}{q_1(1-p_2) - p_1(1-q_2)} - \frac{p_1(q_2 - q_2') - q_1(p_2 - p_2')}{q_1(1-p_2) - p_1(1-q_2)} \\
&= \frac{p_1q_2 - p_2q_1}{q_1(1-p_2) - p_1(1-q_2)} \leq 0.
\end{aligned}$$

which clearly shows that the lower bound is always less than the upper bound of c_2. In other words, we can always find a c_2 in this interval to make sure $a_2 \in [0,1]$ and $b_2 \in [0,1]$.

Next, we need to ensure $a_1+a_2 \leq 1$ and $b_1+b_2 \leq 1$ simultaneously. For $a_1+a_2 \leq 1$, we must have

$$
\begin{aligned}
(p_2(1-q_1) - q_2(1-p_1))(c_1+c_2) &\le p_2q_1' - p_1'q_2 + p_2q_2' - p_2'q_2 + p_1q_2 - p_2q_1,\\
c_1 + c_2 &\le \frac{p_2q_1' - p_1'q_2 + p_2q_2' - p_2'q_2 + p_1q_2 - p_2q_1}{p_2(1-q_1) - q_2(1-p_1)}.
\end{aligned}
$$

where $p_2(1-q_1) - q_2(1-p_1) \ge 0$. And for $b_1 + b_2 \le 1$, we must have

$$
\begin{aligned}
(q_1(1-p_2) - p_1(1-q_2))(c_1+c_2) &\le p_1'q_1 - p_1q_1' + p_2'q_1 - p_1q_2' + p_1q_2 - p_2q_1,\\
c_1 + c_2 &\ge \frac{p_1'q_1 - p_1q_1' + p_2'q_1 - p_1q_2' + p_1q_2 - p_2q_1}{q_1(1-p_2) - p_1(1-q_2)}.
\end{aligned}
$$

where $q_1(1-p_2) - p_1(1-q_2) \le 0$. In order to ensure the existence of $c_1 + c_2$, the lower bound must be smaller than the upper bound, so

$$
\begin{aligned}
&\frac{p_1'q_1 - p_1q_1' + p_2'q_1 - p_1q_2' + p_1q_2 - p_2q_1}{q_1(1-p_2) - p_1(1-q_2)} \le \frac{p_2q_1' - p_1'q_2 + p_2q_2' - p_2'q_2 + p_1q_2 - p_2q_1}{p_2(1-q_1) - q_2(1-p_1)},\\
&\frac{(D_2 - 2(1-q)D_1)(D_1' - q)(2D_1 - D_2 - 2D_1' + D_2')}{2(D_1 - q)(2D_1' - D_2' - q)(2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2)} \ge 0,\\
&\Rightarrow 2D_1 - D_2 \le 2D_1' - D_2'.
\end{aligned}
$$

where $D_2 - 2(1-q)D_1 \le 0$, $D_1' - q \le 0$, $D_1 - q \le 0$, $2D_1' - D_2' - q \le 0$ and $2(1-q)^2 - 2(1-q) + 2(1-q)D_1 - 2(1-q)D_2 + D_2) \le 0$. Therefore, we have $a_1, a_2 \in [0,1]$, $b_1, b_2 \in [0,1]$, $a_1 + a_2 \le 1$ and $b_1 + b_2 \le 1$ if

$$2D_1 - D_2 \le 2D_1' - D_2'.$$

We now argue that $2D_1 - D_2 \le 2D_1' - D_2'$ automatically holds once the conditions in (2.4.8) are satisfied. It is equivalent to prove that when $\frac{q}{2D_1 - D_2 - q} \ge \kappa$ or $R \ge R^*$,

$2D_1 - D_2$ is monotonically decreasing with R. Recall

$$R(D_1) = H_b(q) - H_b(D_1),$$
$$R(D_2) = 3H_b(q) - H_t(\frac{D_2}{2}, q) - H_t(\frac{D_2}{2}, 1-q).$$

Taking the derivatives of $R(D_1)$ and $R(D_2)$ w.r.t D_1 and D_2 respectively, we have

$$\frac{d}{dD_1}R(D_1) = \log\frac{D_1}{1-D_1},$$
$$\frac{d}{dD_2}R(D_2) = \log\frac{D_2}{2} - \frac{1}{2}\log(q - \frac{D_2}{2}) - \frac{1}{2}\log(1-q-\frac{D_2}{2}).$$

So,

$$\frac{d}{dR}D_1(R) = \frac{1}{\log\frac{D_1(R)}{1-D_1(R)}},$$
$$\frac{d}{dR}D_2(R) = \frac{1}{\log\frac{D_2(R)}{2} - \frac{1}{2}\log(q-\frac{D_2(R)}{2}) - \frac{1}{2}\log(1-q-\frac{D_2(R)}{2})}.$$

Let $h(R) = 2D_1(R) - D_2(R)$, the derivative can be expressed as

$$\begin{aligned}\frac{d}{dR}h(R) &= 2\frac{d}{dR}D_1(R) - \frac{d}{dR}D_2(R)\\ &= \frac{2}{\log\frac{D_1(R)}{1-D_1(R)}} - \frac{1}{\log\frac{D_2(R)}{2} - \frac{1}{2}\log(q-\frac{D_2(R)}{2}) - \frac{1}{2}\log(1-q-\frac{D_2(R)}{2})},\end{aligned}$$

which has the sign of

$$S(R) = 2\log\frac{D_2(R)}{2} - \log(q-\frac{D_2(R)}{2}) - \log(1-q-\frac{D_2(R)}{2}) - \log\frac{D_1(R)}{1-D_1(R)}.$$

It can be numerically verified that $S \leq 0$ when $R \geq R^*$. This completes the proof.

□

Chapter 3

Rate-Distortion-Perception Tradeoff for Lossy Compression of Vector Gaussian Sources

3.1 Abstract

In image compression, the perceptual quality plays an important role in lossy reconstruction of images. To that end, the rate-distortion-perception function has been introduced which generalizes the traditional rate-distortion function. We consider lossy compression of a vector source consisting of several (possibly correlated) components. For the image reconstruction, these components can represent frequency elements of the Fourier transform. The traditional reverse water-filling solution which establishes the rate-distortion function of a vector source indicates that some components (mainly high-frequency components) of the reconstruction might be uncorrelated with those of

the source. We characterize the rate-distortion-perception function of a vector Gaussian source, which extends the scalar case proposed in [56], and show that for a high perceptual quality, each component of the reconstruction (including high-frequency components) is strictly correlated with that of the source which is in contrast to the traditional reverse water-filling solution. Our proposed method optimizes over all possible encoder-decoder pairs by jointly optimizing the distortion and perception constraints. We then consider the notion of the universal representation [56] where the encoder is fixed and the decoder is adapted to achieve different distortion-perception pairs. We characterize the achievable distortion-perception region for a fixed representation and discuss that the corresponding distortion-perception tradeoff is approximately optimal.

3.2 Introduction

Deep generative models [15] when applied to lossy image compression tasks can reconstruct realistic looking outputs even at extremely low bit-rates [46], when traditional compression methods suffer from noticeable artifacts. This has led to a growing interest in both the information theoretic aspects, as well as practical architectures of deep learning based image compression [3, 19, 46].

Lossy compression algorithms are typically investigated within the framework of rate-distortion theory, whose goal is to find an optimal tradeoff between the distortion and rate. Nevertheless, it has been noted that achieving low distortion does not always guarantee high perceptual quality. In other words, minimizing distortion alone does not necessarily imply that the reconstructed output will be visually pleasing, especially at low bit-rates. Deep learning-based image compression research has shown that prioritizing higher perceptual quality can lead to an increase in distortion [2, 39, 46]. This suggests a tradeoff where improving one aspect may come at the expense of the other. To tackle this challenge, recent studies by Blau and Michaeli [6, 8] have introduced a mathematical approach to quantify perceptual quality and extended it to the rate-distortion-perception framework. Subsequently, numerous studies emerged that focused on the theory of rate-distortion-perception [10, 26, 27, 43]. Our previous work [56] has also established a rate-distortion-perception theory for scalar Gaussian sources.

The traditional reverse water-filling solution [11] characterized the rate-distortion function for vector Gaussian sources consisting of several components. In the method, some components (mainly high-frequency components) of the reconstruction might be uncorrelated with those of the source when the distortion constraint is loose enough. It is natural to wonder if the same phenomenon holds when extending it to the three-way rate-distortion-perception tradeoff.

Our previous work [56] introduced a concept called universal representation, where the same compressed representation can be used to simultaneously achieve different operating points on the distortion-perception tradeoff. This motivates the study of practical constructions that are approximately universal across the RDP tradeoff, thereby alleviating the need to design a new encoder for each objective.

In this chapter, we analyze the tradeoff between rate, distortion, and perception for vector Gaussian sources. We adopt MSE as the distortion metric and measure perception quality through either KL divergence or Wassertein-2 distance. We first characterize the rate-distortion-perception function of vector Gaussian sources and explain its connection to the classical rate-distortion function given by the reverse water-filling solution. We then explore the concept of universal representations, where the encoder remains fixed while the decoder is adjusted to achieve different distortion-perception combinations. We characterize the achievable distortion-perception region for a fixed representation and demonstrate the existence of universal representations that achieve an approximately optimal tradeoff between distortion and perception.

3.3 Theoretical Framework

The rate-distortion theory aims to express the rate of the bit interface between the encoder and decoder as an *information* objective function which needs to be minimized over the compression constraints. The quality of the compression is quantified by a distortion function. However, from the perception point of view, the choice of the distortion function does not necessarily capture the perceptual quality of the perceived output. To address this issue, [6] introduced an additional constraint called *perception* criterion to match the distributions of the input and output. Recently, [56] proposed a theoretical framework for analyzing the *rate-distortion-perception* characteristics of a compression system for quantizing a single-component source.

Let $X \sim P_X$ be a source which should be compressed. Denote the reconstruction by $\hat{X}$ and let $P_{\hat{X}}$ be the distribution of the reconstruction induced by the encoding and decoding mechanisms. The quality of the compression is measured by the so-called "squared-error" distortion function $d\colon \mathbb{R}^L \times \mathbb{R}^L \to \mathbb{R}_{\geq 0}$ where $d(x, \hat{x}) := \|x - \hat{x}\|^2$. From a perceptual perspective, for a given probability distributions P_X and $P_{\hat{X}}$, let $\phi(P_X, P_{\hat{X}})$ be the perception function capturing the difference between them. Notice that $\phi(P_X, P_{\hat{X}}) = 0$ if and only if $P_X = P_{\hat{X}}$.

Definition 3.3.1 (RDP function) *For a source X, let $\mathcal{P}_{\hat{X}|X}(D,P)$ be the set of all transforms $P_{\hat{X}|X}$ such that for a given (D,P), we have*

$$\mathbb{E}[\|X-\hat{X}\|^2] \leq D, \qquad \phi(P_X, P_{\hat{X}}) \leq P, \tag{3.3.1}$$

where

$$P_{\hat{X}}(\hat{x}) := \int P_{\hat{X}|X}(\hat{x}|x)P_X(x)dx, \qquad \forall \hat{x} \in \mathbb{R}. \tag{3.3.2}$$

Define

$$R(D,P) := \inf_{P_{\hat{X}|X} \in \mathcal{P}_{\hat{X}|X}(D,P)} I(X;\hat{X}), \tag{3.3.3}$$

which is called as the rate-distortion-perception (RDP) function*.*

We distinguish between the rate-distortion-perception function of the *one-shot* setting where only a symbol is compressed at a time, denoted by $R_{\text{1-shot}}(D,P)$, and that of the *asymptotic* setting where n i.i.d. symbols of X are jointly encoded and the behaviour is analyzed as $n \to \infty$, denoted by $R_\infty(D,P)$. Using the functional representation lemma as in [22] and following similar steps to Theorem 2 in Appendix A.2 of [56], one can show that

$$R_\infty(D,P) = R(D,P), \tag{3.3.4}$$

and

$$R(D,P) \leq R_{\text{1-shot}}(D,P) \leq R(D,P) + \log(R(D,P)+1) + 5. \tag{3.3.5}$$

3.3.1 RDP Function for a Vector Gaussian Source

Assume that we have an L-dimensional vector Gaussian source $\boldsymbol{X} \sim P_{\boldsymbol{X}}$ with mean μ_X and covariance matrix $\Sigma_{\boldsymbol{X}} \succ 0$. Consider the eigenvalue decomposition of $\Sigma_{\boldsymbol{X}}$ as follows

$$\Sigma_{\boldsymbol{X}} := \Theta^T \Lambda_{\boldsymbol{Z}} \Theta, \tag{3.3.6}$$

where Θ is an arbitrary unitary matrix and $\Lambda_{\boldsymbol{Z}}$ is a diagonal matrix of size $L \times L$ given as follows:

$$\Lambda_{\boldsymbol{Z}} := \text{diag}^L(\lambda_1, \ldots, \lambda_L), \tag{3.3.7}$$

for some nonnegative $\lambda_1, \ldots, \lambda_L$. We consider two different metrics as the perception function; the KL-divergence and the squared Wasserstein-2 distance. Recall the definitions of the KL-divergence as

$$D(P_{\boldsymbol{X}} \| P_{\hat{\boldsymbol{X}}}) := \int_{\boldsymbol{x}} P_{\boldsymbol{X}}(\boldsymbol{x}) \log \frac{P_{\boldsymbol{X}}(\boldsymbol{x})}{P_{\hat{\boldsymbol{X}}}(\boldsymbol{x})} dx, \tag{3.3.8}$$

and the Wasserstein-2 distance as

$$W_2^2(P_{\boldsymbol{X}}, P_{\hat{\boldsymbol{X}}}) := \inf \mathbb{E}[\|\boldsymbol{X} - \hat{\boldsymbol{X}}\|^2], \tag{3.3.9}$$

where the infimum is taken over all joint distributions of $(\boldsymbol{X}, \hat{\boldsymbol{X}})$ with marginals $P_{\boldsymbol{X}}$ and $P_{\hat{\boldsymbol{X}}}$.

In the following, we will provide the RDP function for a vector Gaussian source. Before stating the result in the following Theorem 3.3.1, we introduce some definitions.

KL-Divergence as the Perception Function

Assume that the perception function is the KL-divergence between the input and reconstruction distributions, i.e., $\phi(P_{\boldsymbol{X}}, P_{\hat{\boldsymbol{X}}}) = D(P_{\hat{\boldsymbol{X}}} \| P_{\boldsymbol{X}})$. Although KL-divergence is asymmetric, it can still be used to measure the distance between two distributions since KL divergence measures how one probability distribution (typically referred to as the ”true” or ”target” distribution) differs from a second probability distribution (often referred to as the ”approximate” or ”estimated” distribution), which is consistent with our goal, in which we measure how the reconstructed distribution differs from the input distribution. Define the functions $\mathsf{D}_\ell \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\mathsf{P}_\ell \colon \mathbb{R}_+ \to \mathbb{R}_+$ as follows.

$$\mathsf{D}_\ell(\alpha, \beta) := \lambda_\ell - 2\sqrt{\beta(\lambda_\ell - \alpha)} + \beta, \tag{3.3.10}$$

$$\mathsf{P}_\ell(\beta) := \frac{1}{2}(\frac{\beta}{\lambda_\ell} - 1 + \log \frac{\lambda_\ell}{\beta}). \tag{3.3.11}$$

The functions $\mathsf{D}_\ell(.,.)$ and $\mathsf{P}_\ell(.)$ denote the distortion and perceptual quality of the ℓ-th component of the reconstruction in terms of two parameters, α and β, which will be discussed later. When the perception function is the Wasserstein-2 distance between the input and reconstruction distributions, the function in (3.3.11) will be replaced by a different definition discussed in the following Section 3.3.1.

Moreover, for a given positive ν_1 and ν_2, define

$$b := 2\lambda_\ell \nu_1 - 4\lambda_\ell^2 \nu_1^2 + \nu_2 - 4\lambda_\ell \nu_1 \nu_2, \tag{3.3.12}$$

and

$$\hat{\lambda}_\ell^*(\nu_1, \nu_2) := \frac{\lambda_\ell(-b + \sqrt{b^2 + 8\lambda_\ell \nu_1 \nu_2 (2\lambda_\ell \nu_1 + \nu_2)} + 2\nu_2^2)}{2(2\lambda_\ell \nu_1 + \nu_2)^2}, \tag{3.3.13}$$

$$\gamma_\ell^*(\nu_1, \nu_2) := \frac{-2\lambda_\ell \nu_1(1 + 2\lambda_\ell \nu_1) - \nu_2 + \sqrt{b^2 + 8\lambda_\ell \nu_1 \nu_2 (2\lambda_\ell \nu_1 + \nu_2)}}{8\lambda_\ell \nu_1^2(-1 + \nu_2)}. \tag{3.3.14}$$

The functions $\hat{\lambda}_\ell^*(.,.)$ and $\gamma_\ell^*(.,.)$ will be used to represent the optimal solutions of the optimization problem (3.3.3) in terms of two parameters ν_1 and ν_2 which will be determined later. For the Wasserstein-2 distance as the perception function, the definitions (3.3.13) and (3.3.14) will be replaced by different representations.

Wasserstein-2 Distance as the Perception Function

Now, assume that the perception function is the Wasserstein-2 distance between input and reconstruction distributions, i.e., $\phi(P_{\boldsymbol{X}}, P_{\hat{\boldsymbol{X}}}) = W_2^2(P_{\boldsymbol{X}}, P_{\hat{\boldsymbol{X}}})$. Let $D_\ell(.,.)$ be defined similar to (3.3.10). However, the perception function in (3.3.11) is replaced by the following:

$$\mathsf{P}_\ell(\beta) := (\sqrt{\lambda_\ell} - \sqrt{\beta})^2. \tag{3.3.15}$$

For a given positive ν_1 and ν_2, define θ_ℓ to be the solution of the following equation:

$$\nu_1 \theta_\ell^3 - 2\nu_1(1 + \lambda_\ell(\nu_1 - \nu_2))(\nu_1 + \nu_2)\theta_\ell^2 + (\nu_1 + 4\lambda_\ell \nu_1^2 + \nu_2)(\nu_1 + \nu_2)\theta_\ell - 2\lambda_\ell \nu_1 (\nu_1 + \nu_2)^2 = 0.$$

and let $\hat{\lambda}_\ell^*(\nu_1, \nu_2)$ and $\gamma_\ell^*(\nu_1, \nu_2)$ in (3.3.13) and (3.3.14) be replaced by the following:

$$\hat{\lambda}_\ell^*(\nu_1, \nu_2) := \frac{\lambda_\ell}{\left(1 + \frac{(1-\theta_\ell)\nu_1}{\nu_2}\right)^2}, \tag{3.3.16}$$

$$\gamma_\ell^*(\nu_1, \nu_2) := \frac{\left(1 + \frac{(1-\theta_\ell)\nu_1}{\nu_2}\right)^2}{8\lambda_\ell \nu_1^2} \cdot \left(-1 + \sqrt{1 + \frac{16\lambda_\ell^2 \nu_1^2}{\left(1 + \frac{(1-\theta_\ell)\nu_1}{\nu_2}\right)^2}}\right). \tag{3.3.17}$$

Functions of Distortion-Perception

The following functions are introduced when the perception function is either the KL-divergence or the Wasserstein-2 distance. Define $\gamma(D)$ be the solution of the following equation:

$$\sum_{\ell=1}^{L} [\lambda_\ell - \gamma(D)]^+ = \left[\sum_{\ell=1}^{L} \lambda_\ell - D\right]^+, \tag{3.3.18}$$

where $[x]^+ := \max\{x, 0\}$. The parameter $\gamma(D)$ represents the constant distortion level, denoted by the *water-level*, that can be achieved using the traditional reverse water-filling solution as in Theorem 10.3 of [11] when there is no perception constraint in the system. It is well-known that the distortion of the ℓ-th component of the reconstruction cannot be larger than the variance of ℓ-th component of the source. Thus, the water-level achieved for the ℓ-th component (using the traditional reverse water-filling solution) is given by the function $\Delta_\ell(D)$ defined as

$$\Delta_\ell(D) := \begin{cases} \lambda_\ell & \text{if } \gamma(D) \geq \lambda_\ell \\ \gamma(D) & \text{if } \gamma(D) < \lambda_\ell \end{cases}. \tag{3.3.19}$$

Moreover, let $\Omega_\ell(D,P)$ be the following function

$$\Omega_\ell(D,P) := \gamma_\ell^*(\nu_1,\nu_2), \tag{3.3.20}$$

where positive ν_1 and ν_2 are chosen such that:

$$\sum_{\ell=1}^{L} \mathsf{D}_\ell(\gamma_\ell^*(\nu_1,\nu_2), \hat{\lambda}_\ell^*(\nu_1,\nu_2)) = D, \tag{3.3.21a}$$

$$\sum_{\ell=1}^{L} \mathsf{P}_\ell(\hat{\lambda}_\ell^*(\nu_1,\nu_2)) = P. \tag{3.3.21b}$$

As will be shown later, the function $\Omega_\ell(D,P)$ determines the optimal rate that can be achieved for the ℓ-th component of the reconstruction when P is small enough.

Theorem 3.3.1 *The rate-distortion-perception function $R(D,P)$ with KL-divergence or Wasserstein-2 distance as the perception function is given by the following optimization problem*

$$R(D,P) = \min_{\{\hat{\lambda}_\ell,\gamma_\ell\}_{\ell=1}^L} \quad \frac{1}{2}\sum_{\ell=1}^{L} \log\frac{\lambda_\ell}{\gamma_\ell} \tag{3.3.22a}$$

$$s.t. \qquad 0 < \gamma_\ell \le \lambda_\ell, \tag{3.3.22b}$$

$$0 \le \hat{\lambda}_\ell \le \lambda_\ell, \tag{3.3.22c}$$

$$\sum_{\ell=1}^{L} \mathsf{D}_\ell(\gamma_\ell, \hat{\lambda}_\ell) \le D, \tag{3.3.22d}$$

$$\sum_{\ell=1}^{L} \mathsf{P}_\ell(\hat{\lambda}_\ell) \le P, \tag{3.3.22e}$$

and the solution to the above program is as follows.

$$R(D,P) = \begin{cases} \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{\lambda_\ell}{\Omega_\ell(D,P)} & \textit{if} \;\; \sum_{\ell=1}^{L}\mathsf{P}_\ell\left(|\lambda_\ell-\gamma(D)|\right) > P \\ \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{\lambda_\ell}{\Delta_\ell(D)} & \textit{if} \;\; \sum_{\ell=1}^{L}\mathsf{P}_\ell\left([\lambda_\ell-\gamma(D)]^+\right) \leq P \\ 0 & \textit{if} \;\; \sum_{\ell=1}^{L}\mathsf{P}_\ell\left(|\lambda_\ell-\gamma(D)|\right) \leq P \\ & \qquad \textit{and} \;\; \sum_{\ell=1}^{L}\mathsf{P}_\ell\left([\lambda_\ell-\gamma(D)]^+\right) > P \end{cases}. \tag{3.3.23}$$

Proof: See Section 3.A. □

In the following, we discuss Theorem 3.3.1 for different values of P. First, assume that P is large enough, i.e., the reconstructions are not visually pleasing. If $\gamma(D) \geq \lambda_\ell$ for some $\ell \in \{1,\ldots,L\}$, then the water-levels are determined by $\Delta_\ell(D)$ under the second clause of (3.3.23). This is indeed the optimal water-filling solution which has been obtained for parallel Gaussian sources in Theorem 10.3 of [11]. According to (3.3.19), if $\gamma(D)$ exceeds λ_ℓ for some ℓ, then the rate allocated to source ℓ would be zero.

On the other hand, for a small enough P (corresponding to high perceptual quality), the following Corollary 3.3.1.1 states that the water-levels $\Delta_\ell(D)$ and $\Omega_\ell(D,P)$ are strictly smaller than λ_ℓ. This is in contrast to the traditional reverse water-filling result for parallel Gaussian sources (without a perception constraint) where the water-level assigned to each source can be equal to λ_ℓ as long as the constant water-level which is assigned to other sources is larger than λ_ℓ, as Fig. 3.1 illustrated.

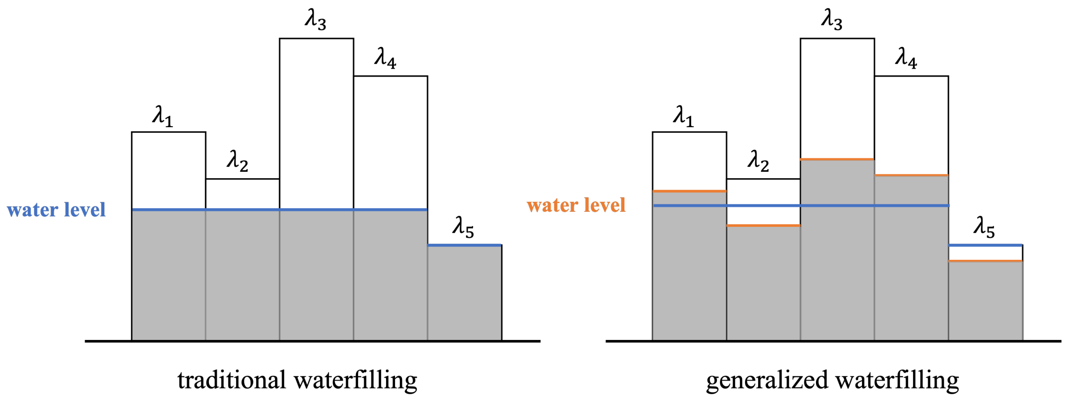

Figure 3.1: Comparison between traditional water-filling and generalized water-filling. Here assume $\ell = 5$, the blue line represents the traditional water level while the orange line represents the generalized water level. It is noticed the orange line doesn't touch the top while the blue line did, which means each component of the reconstruction is strictly correlated with that of the source in generalized setting while some components of the reconstruction might be uncorrelated with those of the source in traditional setting.

Define

$$\{\Lambda_\ell, \Gamma_\ell\}_{\ell=1}^{L} = \arg \min_{\{\hat{\lambda}_\ell, \gamma_\ell\}_{\ell=1}^{L}} \frac{1}{2} \sum_{\ell=1}^{L} \log \frac{\lambda_\ell}{\gamma_\ell}, \tag{3.3.24}$$

such that $\{\hat{\lambda}_\ell, \gamma_\ell\}_{\ell=1}^{L}$ satisfy (3.3.22b)–(3.3.22e).

Corollary 3.3.1.1 *If $P < \infty$ for the KL-divergence metric or $P < \min_\ell \lambda_\ell$ for the Wasserstein-2 distance, then we have*

$$\Lambda_\ell < \lambda_\ell, \qquad \forall \ell \in \{1, \ldots, L\}, \tag{3.3.25}$$

or

$$\Lambda_\ell = \lambda_\ell, \qquad \forall \ell \in \{1, \ldots, L\}, \tag{3.3.26}$$

when $R(D, P) = 0$.

Proof: See Section 3.B. □

Now, we discuss some implications of Corollary 3.3.1.1 on the correlation coefficient of the ℓ-th components of the source and reconstructed images defined as follows

$$\text{corr}(X_\ell, \hat{X}_\ell) := \frac{\mathbb{E}[X_\ell \hat{X}_\ell] - \mathbb{E}[X_\ell]\mathbb{E}[\hat{X}_\ell]}{\sqrt{\mathbb{E}[X_\ell^2]\mathbb{E}[\hat{X}_\ell^2]}}. \tag{3.3.27}$$

Consider the case where $\gamma(D) \geq \lambda_\ell$ for some $\ell \in \{1, \ldots, L\}$. As discussed above, for the traditional reverse water-filling solution solution, the water-level assigned to the ℓ-th component of reconstruction would be saturated by λ_ℓ which implies a zero correlation coefficient between the ℓ-th components of the source and reconstructed images, i.e., $\text{corr}(X_\ell, \hat{X}_\ell) = 0$. However, if a high perceptual quality is expected on the reconstruction, Corollary 3.3.1.1 states that all components of the reconstructed image are correlated with those of the source image (assuming a positive compression rate) and the correlation coefficient is simply given by the following

$$\text{corr}(X_\ell, \hat{X}_\ell) = \sqrt{1 - \frac{\Lambda_\ell}{\lambda_\ell}}. \tag{3.3.28}$$

3.3.2 Universal Representations

The RDP function is defined to be the minimal compression rate where the encoder and decoder are optimized to achieve a given distortion-perception pair (D, P). The universal RDP function is a generalization where the encoder is fixed and the decoder is optimized to meet different distortion-perception pairs $(D, P) \in \Theta_{\mathsf{DP}}$. Of a particular interest is the case where Θ_{DP} is the set of all (D, P) pairs on the RDP function for a fixed rate. Fig 3.2 illustrates the Information-theoretic universal compression, where X_r is the universal representation. The universal RDP function quantifies the additional rate which is needed to achieve these pairs with a fixed encoder. In the following, we discuss that the rate loss due to use of the fixed encoder is not much larger than that of the optimized encoder which achieves a single distortion-perception pair. Before proceeding with the discussion, we present some definitions.

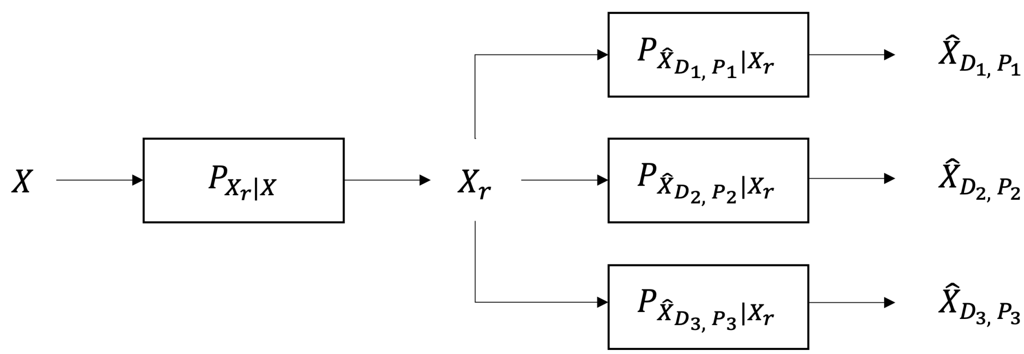

Figure 3.2: Information-theoretic universal compression.

The rate-distortion-perception function defined in Definition 3.3.1, considers all pairs of encoders and decoders which satisfy a given desired threshold pair (D, P). In the following, we propose a generalization where (D, P) can be any threshold pair in a general set Θ_{DP}.

Definition 3.3.2 *Let X_r be a representation of X generated by a random transform $P_{X_r|X}$. Define $\mathcal{P}_{X_r|X}(\Theta_{\mathsf{DP}})$ to be the set of all transforms $P_{X_r|X}$ such that for each $(D, P) \in \Theta_{\mathsf{DP}}$, there exists a transform $P_{\hat{X}|X_r}$ such that $\hat{X} \to X_r \to X$ forms a Markov chain and*

$$\mathbb{E}[\|X - \hat{X}\|^2] \leq D, \qquad \phi(P_X, P_{\hat{X}}) \leq P, \tag{3.3.29}$$

where

$$P_{\hat{X}}(\hat{x}) := \int P_{\hat{X}|X_r}(\hat{x}|x_r)P_{X_r|X}(x_r|x)P_X(x)dxdx_r, \qquad \forall \hat{x} \in \mathbb{R}. \tag{3.3.30}$$

Definition 3.3.3 (Universal Representation) *Given a representation X_r of X, its distortion-perception region, denoted by $\Phi_{\mathsf{DP}}(P_{X_r|X})$, is the set of (D, P) pairs for which there exists $P_{\hat{X}|X_r}$ such that*

$$\mathbb{E}[\|X - \hat{X}\|^2] \leq D, \qquad \phi(P_X, P_{\hat{X}}) \leq P, \tag{3.3.31}$$

where $P_{\hat{X}}$ is defined as in (3.3.30)*. We say that X_r is a Θ_{DP}-universal representation of X if $\Theta_{\mathsf{DP}} \subseteq \Phi_{\mathsf{DP}}(P_{X_r|X})$. Furthermore, define*

$$R(\Theta_{\mathsf{DP}}) := \inf_{P_{X_r|X} \in \mathcal{P}_{X_r|X}(\Theta_{\mathsf{DP}})} I(X; X_r), \tag{3.3.32}$$

which is called as the universal rate-distortion-perception (uRDP) function.

Notice the difference between the rate-distortion-perception functions of the one-shot and the asymptotic settings denoted by $R_\infty(\Theta_{\mathsf{DP}})$ and $R_{\text{1-shot}}(\Theta_{\mathsf{DP}})$, respectively. As mentioned after Definition 3.3.1, similar to (3.3.4)-(3.3.5) and as in Theorem 2 of [56], one can show that

$$R_\infty(\Theta_{\mathsf{DP}}) = R(\Theta_{\mathsf{DP}}), \tag{3.3.33}$$

and

$$R(\Theta_{\mathsf{DP}}) \leq R_{\text{1-shot}}(\Theta_{\mathsf{DP}}) \leq R(\Theta_{\mathsf{DP}}) + \log(R(\Theta_{\mathsf{DP}}) + 1) + 5. \tag{3.3.34}$$

The detailed proofs are provided in Appendix A.4 and A.5. The above arguments in (3.3.33) and (3.3.34) were developed when a shared source of stochasticity exists between the encoder and decoder.

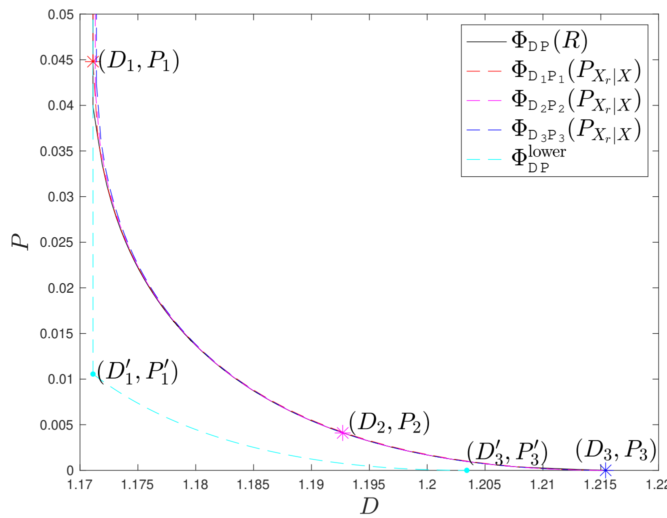

Figure 3.3: Approximate universality for a vector Gaussian source. The boundaries of achievable distortion-perception regions $\Phi_{\mathsf{D_iP_i}}(P_{X_{\mathrm{r}}|X})$ are shown in the figure. Three sets of (D_i, P_i) pairs on $\Phi_{\mathsf{DP}}(R)$ are chosen: for $(D_1, P_1) \in \Phi_{\mathsf{D_1P_1}}(P_{X_{\mathrm{r}}|X})$, P_1 is large enough; for $(D_3, P_3) \in \Phi_{\mathsf{D_3P_3}}(P_{X_{\mathrm{r}}|X})$, we have $P_3 = 0$; and $(D_2, P_2) \in \Phi_{\mathsf{D_2P_2}}(P_{X_{\mathrm{r}}|X})$ is a midpoint on the curve $\Phi_{\mathsf{DP}}(R)$. The curves $\Phi_{\mathsf{D_iP_i}}(P_{X_{\mathrm{r}}|X})$ are close to $\Phi_{\mathsf{DP}}(R)$, i.e., $\Phi_{\mathsf{D_iP_i}}(P_{X_{\mathrm{r}}|X}) \approx \Phi_{\mathsf{DP}}(R)$.

In the following discussion, for simplicity in presentation of results, we proceed with the Wasserstein-2 distance as the perception metric. Define $\Phi_{\mathsf{DP}}(R)$ as follows

$$\Phi_{\mathsf{DP}}(R) := \{(D, P)\colon R(D, P) \leq R\}. \tag{3.3.35}$$

Let X_r be a representation of X which achieves the point (D, P) on the boundary of $\Phi_{\mathsf{DP}}(R)$. For this representation, according to (3.3.22), there exist $\{\bar{\gamma}_\ell, \bar{\lambda}_\ell\}_{\ell=1}^L$ such that

$$R = \frac{1}{2}\sum_{\ell=1}^{L} \log \frac{\lambda_\ell}{\bar{\gamma}_\ell}, \tag{3.3.36a}$$

where

$$0 < \bar{\gamma}_\ell \leq \lambda_\ell, \tag{3.3.36b}$$

$$0 \leq \bar{\lambda}_\ell \leq \lambda_\ell, \tag{3.3.36c}$$

$$\sum_{\ell=1}^{L} \mathsf{D}_\ell(\bar{\gamma}_\ell, \bar{\lambda}_\ell) \leq D, \tag{3.3.36d}$$

$$\sum_{\ell=1}^{L} \mathsf{P}_\ell(\bar{\lambda}_\ell) \leq P. \tag{3.3.36e}$$

Also, define the following set:

$$\Phi_{\mathsf{DP}}^{\text{fixed-enc}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L) := \left\{(D', P') \colon \exists \nu > 0 \text{ s.t. } P' \geq \sum_{\ell=1}^{L} \mathsf{P}_\ell\left(\frac{(\sqrt{\lambda_\ell} + \nu\sqrt{\lambda_\ell - \bar{\gamma}_\ell})^2}{(1+\nu)^2}\right),\right.$$
$$\left. D' \geq \sum_{\ell=1}^{L} \mathsf{D}_\ell\left(\bar{\gamma}_\ell, \frac{(\sqrt{\lambda_\ell} + \nu\sqrt{\lambda_\ell - \bar{\gamma}_\ell})^2}{(1+\nu)^2}\right)\right\}. \tag{3.3.37}$$

The following theorem provides a universal representation of X.

Theorem 3.3.2 *X_r is a $\Phi_{\mathsf{DP}}^{\textit{fixed-enc}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L)$-universal representation of X.*

Proof: See Section 3.C. □

Corollary 3.3.2.1 $\Phi_{\mathsf{DP}}^{\textit{fixed-enc}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L) \approx \Phi_{\mathsf{DP}}(R)$.

Remark 3.3.1 *In the context of a scalar Gaussian source, a significant result has been established in [56], demonstrating that the notion of approximate equality becomes exact equivalence, namely* $\Phi_{\mathsf{DP}}^{\textit{fixed-enc}}(P_{X_r|X}) = \Phi_{\mathsf{DP}}(R)$. *However, for vector Gaussian sources, a rate penalty can be anticipated due to the presence of multiple components. We numerically verify that the corresponding distortion-perception region is approximately optimal, confirming the approximation* $\Phi_{\mathsf{DP}}^{\textit{fixed-enc}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L) \approx \Phi_{\mathsf{DP}}(R)$, *which is shown in Fig. 3.3. Moreover, for an arbitrary scalar source, it is also proved in [56] that* $\Phi_{\mathsf{DP}}^{\textit{fixed-enc}}(P_{X_r|X})$ *is not much smaller than* $\Phi_{\mathsf{DP}}(R)$. *The detailed proof is attached in Appendix A.1. We can extend the theoretical proof into the vector case.*

Consider the following extreme points on $\Phi_{\mathsf{DP}}(R)$ (see Fig. 3.3). The first extreme point corresponding to the highest perceptual quality is given by

$$(D_3, P_3) = (\sum_{\ell=1}^{L} 2\lambda_\ell - 2\sqrt{\lambda_\ell\,(\lambda_\ell - \omega_\ell^0)}, 0), \tag{3.3.38}$$

where ω_ℓ^0 is the water-level assigned to the ℓ-th component of the source defined as

$$\omega_\ell^0 := \frac{2\lambda_\ell}{1 + \sqrt{1 + 16\nu_1^2\lambda_\ell^2}}, \tag{3.3.39}$$

such that $\nu_1 > 0$ satisfies the following

$$R = \frac{1}{2}\sum_{\ell=1}^{L} \log \frac{1 + \sqrt{1 + 16\nu_1^2\lambda_\ell^2}}{2}. \tag{3.3.40}$$

The other extreme point of $\Phi_{\mathsf{DP}}(R)$ corresponding to the lowest perceptual quality is

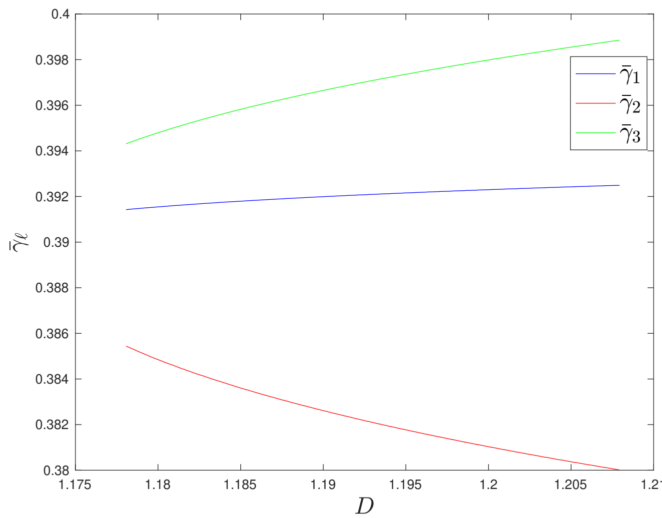

Figure 3.4: Monotonicity of $\bar{\gamma}_\ell$ versus D for $L = 3$ and $R = 3$.

given by the following

$$(D_1, P_1) = (\sum_{\ell=1}^{L} \lambda_\ell - \sum_{\ell=1}^{L}[\lambda_\ell - \delta_\ell]^+, \sum_{\ell=1}^{L} \mathsf{P}_\ell([\lambda_\ell - \delta_\ell]^+)), \tag{3.3.41}$$

where δ_ℓ satisfies the following

$$R = \frac{1}{2}\sum_{\ell=1}^{L}\left[\log\frac{\lambda_\ell}{\delta_\ell}\right]^+. \tag{3.3.42}$$

The water-level at this point is simply given by the traditional reverse water-filling solution solution as follows

$$\omega_\ell^\infty := \begin{cases} \delta_\ell & \text{if } \delta_\ell < \lambda_\ell \\ \lambda_\ell & \text{if } \delta_\ell \geq \lambda_\ell \end{cases}. \tag{3.3.43}$$

Fig. 3.3 shows the boundaries of $\Phi_{\mathsf{D}_1\mathsf{P}_1}(P_{X_{\mathrm{r}}|X})$ and $\Phi_{\mathsf{D}_3\mathsf{P}_3}(P_{X_{\mathrm{r}}|X})$ where $\bar{\gamma}_\ell$ is ω_ℓ^∞ and ω_ℓ^0, respectively. The figure also illustrates the boundary of $\Phi_{\mathsf{D}_2\mathsf{P}_2}(P_{X_{\mathrm{r}}|X})$ where $(D_2, P_2) \in \Phi_{\mathsf{D}_2\mathsf{P}_2}(P_{X_{\mathrm{r}}|X})$ is a midpoint on the curve $\Phi_{\mathsf{DP}}(R)$. For any point on $\Phi_{\mathsf{DP}}(R)$, represented by the set of water-levels $\{\bar{\gamma}_\ell\}_{\ell=1}^L$, the distortion-perception tradeoff is given by $\Phi_{\mathsf{DP}}^{\text{fixed-enc}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L)$ characterized in Theorem 3.3.2. The functions $\mathsf{P}_\ell(.,.)$ and $\mathsf{D}_\ell(.,.)$ in (3.3.37) are monotonically increasing functions of $\bar{\gamma}_\ell$. In order to find a lower bound on the set $\Phi_{\mathsf{DP}}^{\text{fixed-enc}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L)$, we need to find the minimum values of $\{\bar{\gamma}_\ell\}_{\ell=1}^L$ on the tradeoff curve. Through our experiments (see Fig. 3.4), we observed a monotonic behaviour for $\{\bar{\gamma}_\ell\}_{\ell=1}^L$ versus D. This observation implies the following lower bound:

$$\bar{\gamma}_\ell \geq \min(\omega_\ell^0, \omega_\ell^\infty). \tag{3.3.44}$$

Now, define

$$\begin{aligned}\Phi_{\mathsf{DP}}^{\text{lower}} := \quad & \Bigg\{(D', P')\colon \exists \nu > 0 \text{ s.t. } P' \geq \sum_{\ell=1}^L \mathsf{P}_\ell\left(\frac{(\sqrt{\lambda_\ell} + \nu\sqrt{\lambda_\ell - \min(\omega_\ell^0, \omega_\ell^\infty)})^2}{(1+\nu)^2}\right), \\ & D' \geq \sum_{\ell=1}^L \mathsf{D}_\ell\left(\min(\omega_\ell^0, \omega_\ell^\infty), \frac{(\sqrt{\lambda_\ell} + \nu\sqrt{\lambda_\ell - \min(\omega_\ell^0, \omega_\ell^\infty)})^2}{(1+\nu)^2}\right)\Bigg\}.\end{aligned} \tag{3.3.45}$$

Then, we get $\Phi_{\mathsf{DP}}^{\text{fixed-enc}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L) \subseteq \Phi_{\mathsf{DP}}^{\text{lower}}$. The boundary of the set $\Phi_{\mathsf{DP}}^{\text{lower}}$ is shown in Fig. 3.3. The points (D_1, P_1) and (D_3, P_3) are included in the set $\Phi_{\mathsf{DP}}^{\text{lower}}$. The points (D_1', P_1') and (D_3', P_3') on the boundary of $\Phi_{\mathsf{DP}}^{\text{lower}}$ are the extreme points corresponding to a large enough P and $P = 0$, respectively.

3.4 Conclusions

In this chapter, we focused on the crucial role of perceptual quality in image compression, specifically in lossy image reconstruction. We have introduced the rate-distortion-perception function as a generalized approach to considering perceptual metrics into the compression process, going beyond the conventional rate-distortion framework. We addressed the lossy compression of a vector Gaussian source composed of potentially correlated components, such as frequency elements in the Fourier transform of an image. Through the conventional reverse water-filling solution, which determined the rate-distortion function of a vector Gaussian source, we observed that some components of the reconstruction might be uncorrelated with the corresponding components of the source, particularly the high-frequency components. While for vector Gaussian sources, we made a significant finding: for achieving high perceptual quality, each component of the reconstruction, including the high-frequency components, was strictly correlated with its counterpart in the source. This discovery contrasted with the traditional reverse water-filling solution approach. Aiming to jointly optimize the distortion and perception constraints by considering all possible encoder-decoder pairs, we explored the concept of the universal representation, where the encoder remained fixed, and the decoder was adapted to achieve different distortion-perception pairs. We demonstrated that the corresponding distortion-perception tradeoff was approximately optimal.

3.A Proof of Theorem 3.3.1

First, we provide the proof for the case that the perception function is the KL-divergence between input and reconstruction distributions.

We will first need a Lemma from estimation theory. For scalar case, let $\hat{X}$ be a random variable with $\mathbb{E}[\hat{X}] = \mu_{\hat{X}}$, $\mathrm{Var}(\hat{X}) = \sigma^2_{\hat{X}}$ and $\mathrm{Cov}(X, \hat{X}) = \theta$. Let $\hat{X}_G$ be a random variable jointly Gaussian with X with the same first and second order statistics as $\hat{X}$.

Lemma 3.A.1 *Given $\mu_{\hat{X}}, \sigma^2_{\hat{X}}$, and θ, we have that*

$$\mathbb{E}\left[\left(X - \mathbb{E}\left[X \mid \hat{X}_G\right]\right)^2\right] \geq \mathbb{E}\left[(X - \mathbb{E}[X \mid \hat{X}])^2\right]. \tag{3.A.1}$$

The proof of this result can be found in a standard estimation theory reference, e.g. Chapter 3, page 134 of the 6.432 notes by Willsky & Wornell (2004).

We can have

$$\begin{aligned}
\mathbb{E}\left[\left(X - \hat{X}\right)^2\right] &= \mathbb{E}\left[X^2\right] + \mathbb{E}\left[\hat{X}^2\right] - 2\mathbb{E}\left[X\hat{X}\right] \\
&= \sigma^2_X + \mu^2_X + \sigma^2_{\hat{X}} + \mu^2_{\hat{X}} - 2\mu_X \mu_{\hat{X}} - 2\theta \\
&= (\mu_X - \mu_{\hat{X}})^2 + \sigma^2_X + \sigma^2_{\hat{X}} - 2\theta \\
&= \mathbb{E}\left[\left(X - \hat{X}_G\right)^2\right], \qquad (3.A.2)
\end{aligned}$$

where (3.A.2) is because $\hat{X}_G$ has the same first and second order statistics as $\hat{X}$.

For vector case, we assume an L-dimensional random vector $\hat{\boldsymbol{X}}$ with mean $\mu_{\hat{\boldsymbol{X}}}$

and its joint covariance matrix with $\boldsymbol{X}$ is given by

$$\Sigma_{\boldsymbol{X},\hat{\boldsymbol{X}}} \triangleq \begin{pmatrix} \Sigma_{\boldsymbol{X}} & \Theta \\ \Theta^T & \Sigma_{\hat{\boldsymbol{X}}} \end{pmatrix}.$$

Let $\hat{\boldsymbol{X}}_G$ be an L-dimensional random vector jointly Gaussian with $\boldsymbol{X}$ with the same first and second order statistics as $\hat{\boldsymbol{X}}$. Lemma 3.A.1 also applies in vector case.

Lemma 3.A.2 *Given $\mu_{\hat{\boldsymbol{X}}}$, and $\Sigma_{\boldsymbol{X},\hat{\boldsymbol{X}}}$, we have that*

$$\det(\Sigma_{\boldsymbol{X}-\mathbb{E}[\boldsymbol{X}|\hat{\boldsymbol{X}}_G]}) \geq \det(\Sigma_{\boldsymbol{X}-\mathbb{E}[\boldsymbol{X}|\hat{\boldsymbol{X}}]}). \tag{3.A.3}$$

Restriction to Jointly Gaussian Distributions:

We shall show that there is no loss of optimality in assuming that $\hat{\boldsymbol{X}}$ is jointly Gaussian with $\boldsymbol{X}$. It is clear that $\mathbb{E}[\|\boldsymbol{X}-\hat{\boldsymbol{X}}\|^2] = \mathbb{E}[\|\boldsymbol{X}-\hat{\boldsymbol{X}}_G\|^2]$, as the first and second order statistics are all given.

Moreover, notice that for every random variable $\hat{\boldsymbol{X}}$ which has the same mean and covariance matrix as $\hat{\boldsymbol{X}}_G$, we have

$$
\begin{aligned}
I(\boldsymbol{X};\hat{\boldsymbol{X}}) &= h(\boldsymbol{X}) - h(\boldsymbol{X} \mid \hat{\boldsymbol{X}}) && (3.A.4)\\
&\geq h(\boldsymbol{X}) - h(\boldsymbol{X} - \mathbb{E}[\boldsymbol{X} \mid \hat{\boldsymbol{X}}]) && (3.A.5)\\
&\geq h(\boldsymbol{X}) - \frac{1}{2}\log\left((2\pi e)^L \det(\Sigma_{\boldsymbol{X}-\mathbb{E}[\boldsymbol{X}\mid\hat{\boldsymbol{X}}]})\right) && (3.A.6)\\
&\geq h(\boldsymbol{X}) - \frac{1}{2}\log\left((2\pi e)^L \det(\Sigma_{\boldsymbol{X}-\mathbb{E}\left[\boldsymbol{X}\mid\hat{\boldsymbol{X}}_G\right]})\right) && (3.A.7)\\
&= h(\boldsymbol{X}) - h\left(\boldsymbol{X} - \mathbb{E}\left[\boldsymbol{X} \mid \hat{\boldsymbol{X}}_G\right]\right) && (3.A.8)\\
&= h(\boldsymbol{X}) - h\left(\boldsymbol{X} - \mathbb{E}\left[\boldsymbol{X} \mid \hat{\boldsymbol{X}}_G\right] \mid \hat{\boldsymbol{X}}_G\right) && (3.A.9)\\
&= h(\boldsymbol{X}) - h\left(\boldsymbol{X} \mid \hat{\boldsymbol{X}}_G\right) && (3.A.10)\\
&= I\left(\boldsymbol{X};\hat{\boldsymbol{X}}_G\right). && (3.A.11)
\end{aligned}
$$

where (3.A.5) is due to $\mathbb{E}[\boldsymbol{X} \mid \hat{\boldsymbol{X}}]$ is the MMSE of $\hat{\boldsymbol{X}}$, (3.A.6) is because the Gaussian distribution maximizes differential entropy for a given variance, (3.A.7) follows from Lemma 3.A.2 and (3.A.9) is because the estimation error $\boldsymbol{X} - \mathbb{E}\left[\boldsymbol{X} \mid \hat{\boldsymbol{X}}_G\right]$ is independent of $\hat{\boldsymbol{X}}_G$, and (3.A.10) follows because $\mathbb{E}\left[\boldsymbol{X} \mid \hat{\boldsymbol{X}}_G\right]$ is a function of $\hat{\boldsymbol{X}}_G$.

Finally, note that

$$
\begin{aligned}
D(P_{\hat{\boldsymbol{X}}}\|P_{\boldsymbol{X}}) &= \int P_{\hat{\boldsymbol{X}}}(\boldsymbol{x}) \log \frac{P_{\hat{\boldsymbol{X}}}(\boldsymbol{x})}{P_{\boldsymbol{X}}(\boldsymbol{x})} d\boldsymbol{x} \\
&= -h(\hat{\boldsymbol{X}}) - \int P_{\hat{\boldsymbol{X}}}(\boldsymbol{x}) \log P_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \\
&= -h(\hat{\boldsymbol{X}}) - \int P_{\hat{\boldsymbol{X}}}(\boldsymbol{x}) \log \frac{1}{\sqrt{(2\pi)^L \det(\Sigma_{\boldsymbol{X}})}} e^{-\frac{1}{2}(\boldsymbol{x}-\mu_{\boldsymbol{X}})^T \Sigma_{\boldsymbol{X}}^{-1}(\boldsymbol{x}-\mu_{\boldsymbol{X}})} d\boldsymbol{x} \\
&= -h(\hat{\boldsymbol{X}}) - \int P_{\hat{\boldsymbol{X}}}(\boldsymbol{x}) \left(-\frac{1}{2}\log(2\pi)^L \det(\Sigma_{\boldsymbol{X}}) - \frac{1}{2}(\boldsymbol{x}-\mu_{\boldsymbol{X}})^T \Sigma_{\boldsymbol{X}}^{-1}(\boldsymbol{x}-\mu_{\boldsymbol{X}}) \right) d\boldsymbol{x} \\
&= -h(\hat{\boldsymbol{X}}) + \frac{1}{2}\int P_{\hat{\boldsymbol{X}}}(\boldsymbol{x})(\boldsymbol{x}-\mu_{\boldsymbol{X}})^T \Sigma_{\boldsymbol{X}}^{-1}(\boldsymbol{x}-\mu_{\boldsymbol{X}}) d\boldsymbol{x} + \frac{1}{2}\log(2\pi)^L \det(\Sigma_{\boldsymbol{X}}) \\
&= -h(\hat{\boldsymbol{X}}) + \frac{1}{2}\int P_{\hat{\boldsymbol{X}}_G}(\boldsymbol{x})(\boldsymbol{x}-\mu_{\boldsymbol{X}})^T \Sigma_{\boldsymbol{X}}^{-1}(\boldsymbol{x}-\mu_{\boldsymbol{X}}) d\boldsymbol{x} + \frac{1}{2}\log(2\pi)^L \det(\Sigma_{\boldsymbol{X}}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(3.A.12)} \\
&= -h(\hat{\boldsymbol{X}}) - \int P_{\hat{\boldsymbol{X}}_G}(\boldsymbol{x}) \log P_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \qquad \text{(3.A.13)} \\
&\geq -h(\hat{\boldsymbol{X}}_G) - \int P_{\hat{\boldsymbol{X}}_G}(\boldsymbol{x}) \log P_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \qquad \text{(3.A.14)} \\
&= D(P_{\hat{\boldsymbol{X}}_G}\|P_{\boldsymbol{X}}), \qquad \text{(3.A.15)}
\end{aligned}
$$

where (3.A.12) follows because the expression $(\boldsymbol{x}-\mu_{\boldsymbol{X}})^T \Sigma_{\boldsymbol{X}}^{-1}(\boldsymbol{x}-\mu_{\boldsymbol{X}})$ for a vector $\boldsymbol{x} = (x_1, \ldots, x_L)$ contains the terms like x_ℓ^2, x_ℓ and $x_\ell x_{\ell'}$ for $\ell, \ell' \in \{1, \ldots, L\}$, and since the $\hat{\boldsymbol{X}}$ has the same mean and covariance matrix as $\hat{\boldsymbol{X}}_G$, the expected values of these terms with respect to $P_{\hat{\boldsymbol{X}}}$ are equal to those calculated with respect to $P_{\hat{\boldsymbol{X}}_G}$; (3.A.14) follows because the differential entropy is maximized for a Gaussian distribution.

So, we will restrict our search to Gaussian distributions that satisfy both the distortion and perception metrics. Thus, the optimization problem of Definition 3.3.1

reduces to the following:

$$\min_{P_{\hat{\boldsymbol{X}}_G|\boldsymbol{X}}} I(\boldsymbol{X}; \hat{\boldsymbol{X}}_G), \tag{3.A.16}$$

$$\text{s.t.} \qquad \mathbb{E}[\|\boldsymbol{X} - \hat{\boldsymbol{X}}_G\|^2] \leq D, \qquad D(P_{\hat{\boldsymbol{X}}_G}\|P_{\boldsymbol{X}}) \leq P, \tag{3.A.17}$$

We can also restrict our search to $\hat{\boldsymbol{X}}_G$ such that $\mathbb{E}[\hat{\boldsymbol{X}}_G] = \mu_{\hat{\boldsymbol{X}}}$ since this choice minimizes both distortion and perception metrics as can be verified by the following identities:

$$\mathbb{E}[\|\boldsymbol{X} - \hat{\boldsymbol{X}}_G\|^2] = \|\mu_{\boldsymbol{X}} - \mu_{\hat{\boldsymbol{X}}}\|^2 + \mathbb{E}[\|(\boldsymbol{X} - \mu_{\boldsymbol{X}}) - (\hat{\boldsymbol{X}}_G - \mu_{\hat{\boldsymbol{X}}})\|^2], \tag{3.A.18}$$

and

$$D(P_{\hat{\boldsymbol{X}}_G}\|P_{\boldsymbol{X}}) = \frac{1}{2}\left(\operatorname{tr}(\Sigma_{\boldsymbol{X}}^{-1}\Sigma_{\hat{\boldsymbol{X}}}) + (\mu_{\boldsymbol{X}} - \mu_{\hat{\boldsymbol{X}}})^T\Sigma_{\boldsymbol{X}}^{-1}(\mu_{\boldsymbol{X}} - \mu_{\hat{\boldsymbol{X}}}) - L + \log\frac{\det(\Sigma_{\boldsymbol{X}})}{\det(\Sigma_{\hat{\boldsymbol{X}}})}\right). \tag{3.A.19}$$

Therefore, there is no loss of optimality in assuming $\mu_{\boldsymbol{X}} = \mu_{\hat{\boldsymbol{X}}}$. Recall the eigenvalue decomposition of $\Sigma_{\boldsymbol{X}}$ from (3.3.6)–(3.3.7). Furthermore, set $\boldsymbol{Z} = \Theta\boldsymbol{X}$ and define $\hat{\boldsymbol{Z}}_G := \Theta\hat{\boldsymbol{X}}_G$. Let $\Sigma_{\boldsymbol{X}-\mathbb{E}[\boldsymbol{X}|\hat{\boldsymbol{X}}_G]}$ be the covariance matrix of $\boldsymbol{X} - \mathbb{E}[\boldsymbol{X}|\hat{\boldsymbol{X}}_G]$ and $\Lambda_{\boldsymbol{Z}-\mathbb{E}[\boldsymbol{Z}|\hat{\boldsymbol{Z}}_G]}$ be a diagonal matrix whose diagonal entries coincide with those of $\Theta^T\Sigma_{\boldsymbol{X}-\mathbb{E}[\boldsymbol{X}|\hat{\boldsymbol{X}}_G]}\Theta$, i.e.,

$$\Lambda_{\boldsymbol{Z}-\mathbb{E}[\boldsymbol{Z}|\hat{\boldsymbol{Z}}_G]} := \operatorname{diag}^L(\gamma_1, \ldots, \gamma_L), \tag{3.A.20}$$

for some nonnegative $\gamma_1, \ldots, \gamma_L$. Moreover, let $\Sigma_{\hat{\boldsymbol{X}}_G}$ be the covariance matrix of

$\hat{\boldsymbol{X}}_G$ and $\Lambda_{\hat{\boldsymbol{Z}}_G}$ be a diagonal matrix whose diagonal elements coincide with those of $\Theta\Sigma_{\hat{\boldsymbol{X}}_G}\Theta^T$, i.e.,

$$\Lambda_{\hat{\boldsymbol{Z}}_G} := \mathrm{diag}^L(\hat{\lambda}_1, \ldots, \hat{\lambda}_L). \tag{3.A.21}$$

<u>*Simplification of the Optimization Problem for the Gaussian Distribution:*</u> Now, we simplify the optimization problem when the reconstruction random variable is restricted to be jointly Gaussian with desired properties given in the previous subsection. Consider the following set of inequalities for the mutual information:

$$\begin{aligned}
I(\boldsymbol{X}; \hat{\boldsymbol{X}}_G) =& h(\boldsymbol{X}) - h(\boldsymbol{X}|\hat{\boldsymbol{X}}_G) && (3.A.22)\\
=& h(\boldsymbol{X}) - h(\boldsymbol{X} - \mathbb{E}[\boldsymbol{X}|\hat{\boldsymbol{X}}_G]|\hat{\boldsymbol{X}}_G) && (3.A.23)\\
=& h(\boldsymbol{X}) - h(\boldsymbol{X} - \mathbb{E}[\boldsymbol{X}|\hat{\boldsymbol{X}}_G]) && (3.A.24)\\
=& h(\Theta^T\boldsymbol{Z}) - h(\Theta^T\boldsymbol{Z} - \Theta^T\mathbb{E}[\boldsymbol{Z}|\hat{\boldsymbol{Z}}_G]) && (3.A.25)\\
=& I(\boldsymbol{Z}; \boldsymbol{Z} - \mathbb{E}[\boldsymbol{Z}|\hat{\boldsymbol{Z}}_G]) && (3.A.26)\\
=& \sum_{\ell=1}^{L} h(Z_\ell) - h(\boldsymbol{Z} - \mathbb{E}[\boldsymbol{Z}|\hat{\boldsymbol{Z}}_G]) && (3.A.27)\\
\geq& \sum_{\ell=1}^{L} h(Z_\ell) - \sum_{\ell=1}^{L} h(Z_\ell - \mathbb{E}[Z_\ell|\hat{Z}_{G,\ell}]) && (3.A.28)\\
=& \sum_{\ell=1}^{L} \frac{1}{2}\log\left((2\pi e)\lambda_\ell\right) - \sum_{\ell=1}^{L} \frac{1}{2}\log\left((2\pi e)\gamma_\ell\right) && (3.A.29)\\
=& \sum_{\ell=1}^{L} \frac{1}{2}\log\frac{\lambda_\ell}{\gamma_\ell}, && (3.A.30)
\end{aligned}$$

where

- (3.A.26) follows because $h(\Theta^T \boldsymbol{Z}) = h(\boldsymbol{Z}) + \log(|\det(\Theta^T)|)$ which is a property of the differential entropy,
- (3.A.27) follows because $Z_1, \ldots, Z_L$ are independent,
- (3.A.28) follows because conditioning reduces entropy,
- (3.A.29) follows because $\mathbb{E}[Z_\ell^2] = \lambda_\ell$ and $\mathbb{E}[(Z_\ell - \mathbb{E}[Z_\ell|\hat{Z}_{G,\ell}])^2] = \gamma_\ell$.

Next, consider the expected distortion function as follows:

$$\begin{aligned}
\mathbb{E}[\|\boldsymbol{X} - \hat{\boldsymbol{X}}_G\|^2] =& \mathbb{E}[\|\boldsymbol{Z} - \hat{\boldsymbol{Z}}_G\|^2] && (3.A.31)\\
=& \sum_{\ell=1}^{L} \mathbb{E}[(Z_\ell - \hat{Z}_{G,\ell})^2] && (3.A.32)\\
=& \sum_{\ell=1}^{L} \mathbb{E}[Z_\ell^2] - 2\mathbb{E}[Z_\ell \hat{Z}_{G,\ell}] + \mathbb{E}[\hat{Z}_{G,\ell}^2] && (3.A.33)\\
=& \sum_{\ell=1}^{L} \lambda_\ell - 2\mathbb{E}[Z_\ell \hat{Z}_{G,\ell}] + \hat{\lambda}_\ell && (3.A.34)\\
=& \sum_{\ell=1}^{L} \lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell - \gamma_\ell)} + \hat{\lambda}_\ell && (3.A.35)
\end{aligned}$$

where

- (3.A.31) follows because $\boldsymbol{X} = \Theta^T \boldsymbol{Z}$, $\hat{\boldsymbol{X}}_G = \Theta^T \hat{\boldsymbol{Z}}_G$ and Θ is a unitary matrix,
- (3.A.34) follows because $\mathbb{E}[Z_\ell^2] = \lambda_\ell$ and $\mathbb{E}[\hat{Z}_{G,\ell}^2] = \hat{\lambda}_\ell$,
- (3.A.35) follows from the identity $\mathbb{E}[(Z_\ell - \mathbb{E}[Z_\ell|\hat{Z}_{G,\ell}])^2] = \mathbb{E}[Z_\ell^2] - (\mathbb{E}[Z_\ell \hat{Z}_{G,\ell}])^2 (\mathbb{E}[\hat{Z}_{G,\ell}])^{-1}$, and $\mathbb{E}[(Z_\ell - \mathbb{E}[Z_\ell|\hat{Z}_{G,\ell}])^2] = \gamma_\ell$, $\mathbb{E}[Z_\ell^2] = \lambda_\ell$, $\mathbb{E}[\hat{Z}_{G,\ell}^2] = \hat{\lambda}_\ell$ and also because $\mathbb{E}[Z_\ell \hat{Z}_{G,\ell}]$ being positive decreases the distortion function.

Finally, consider the perception function in the following.

$$D(P_{\hat{\boldsymbol{X}}_G}\|P_{\boldsymbol{X}}) = \frac{1}{2}\left(\mathrm{tr}(\Lambda_{\boldsymbol{Z}}^{-1}\Theta\Sigma_{\hat{\boldsymbol{X}}_G}\Theta^T) - L + \log\frac{\det(\Lambda_{\boldsymbol{Z}})}{\det(\Theta\Sigma_{\hat{\boldsymbol{X}}_G}\Theta^T)}\right) \quad (3.A.36)$$

$$= \frac{1}{2}\left(\mathrm{tr}(\Lambda_{\boldsymbol{Z}}^{-1}\Lambda_{\hat{\boldsymbol{Z}}_G}) - L + \log\frac{\det(\Lambda_{\boldsymbol{Z}})}{\det(\Theta\Sigma_{\hat{\boldsymbol{X}}_G}\Theta^T)}\right) \quad (3.A.37)$$

$$\geq \frac{1}{2}\left(\mathrm{tr}(\Lambda_{\boldsymbol{Z}}^{-1}\Lambda_{\hat{\boldsymbol{Z}}_G}) - L + \log\frac{\det(\Lambda_{\boldsymbol{Z}})}{\det(\Lambda_{\hat{\boldsymbol{Z}}_G})}\right) \quad (3.A.38)$$

$$= \frac{1}{2}\sum_{\ell=1}^{L}\left(\frac{\hat{\lambda}_\ell}{\lambda_\ell} - 1 + \log\frac{\lambda_\ell}{\hat{\lambda}_\ell}\right) \quad (3.A.39)$$

where

- (3.A.37) follows because $\Lambda_{\boldsymbol{Z}}^{-1}$ is a diagonal matrix and thus the trace depends on the diagonal elements of $\Theta\Sigma_{\hat{\boldsymbol{X}}_G}\Theta^T$ which are equal to diagonal elements of $\Lambda_{\hat{\boldsymbol{Z}}_G}$,
- (3.A.38) follows from Hadamard's inequality for a positive semidefinite matrix as in Eq. (8.64) of [11].

Optimization Problem: The optimization problem reduces to the following:

$$R(D,P) := \min_{\{\hat{\lambda}_\ell,\gamma_\ell\}_{\ell=1}^{L}} \quad \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{\lambda_\ell}{\gamma_\ell} \quad (3.A.40a)$$

$$\text{s.t.} \quad 0 < \gamma_\ell \leq \lambda_\ell, \quad (3.A.40b)$$

$$0 \leq \hat{\lambda}_\ell \leq \lambda_\ell, \quad (3.A.40c)$$

$$\sum_{\ell=1}^{L}\left(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell-\gamma_\ell)} + \hat{\lambda}_\ell\right) \leq D, \quad (3.A.40d)$$

$$\frac{1}{2}\sum_{\ell=1}^{L}\left(\frac{\hat{\lambda}_\ell}{\lambda_\ell} - 1 + \log\frac{\lambda_\ell}{\hat{\lambda}_\ell}\right) \leq P. \quad (3.A.40e)$$

It can be easily verified that the above optimization problem is convex. Thus, the solution to the above program is equal to that of the following dual optimization problem.

$$R(D,P) = \max_{\nu_1,\nu_2} \min_{\substack{\{\gamma_\ell,\hat{\lambda}_\ell\}_{\ell=1}^L:\\ 0<\gamma_\ell\leq\lambda_\ell \\ 0\leq\hat{\lambda}_\ell\leq\lambda_\ell}} \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{\lambda_\ell}{\gamma_\ell} + \nu_1\left(\sum_{\ell=1}^{L}(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell-\gamma_\ell)} + \hat{\lambda}_\ell) - D\right) + \nu_2\left(\frac{1}{2}\sum_{\ell=1}^{L}\left(\frac{\hat{\lambda}_\ell}{\lambda_\ell} - 1 + \log\frac{\lambda_\ell}{\hat{\lambda}_\ell}\right) - P\right) \tag{3.A.41}$$

$$= \max_{\nu_1,\nu_2} \sum_{\ell=1}^{L} \min_{\substack{\{\gamma_\ell,\hat{\lambda}_\ell\}_{\ell=1}^L:\\ 0<\gamma_\ell\leq\lambda_\ell \\ 0\leq\hat{\lambda}_\ell\leq\lambda_\ell}} \left(\frac{1}{2}\log\frac{\lambda_\ell}{\gamma_\ell} + \nu_1\left(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell-\gamma_\ell)} + \hat{\lambda}_\ell - D_\ell\right) + \nu_2\left(\frac{1}{2}\left(\frac{\hat{\lambda}_\ell}{\lambda_\ell} - 1 + \log\frac{\lambda_\ell}{\hat{\lambda}_\ell}\right) - P_\ell\right)\right), \tag{3.A.42}$$

where $\{\nu_1, \nu_2\}$ are nonnegative Lagrange multipliers and positive D_ℓ and nonnegative P_ℓ are chosen such that

$$\nu_1\left(\sum_{\ell=1}^{L} D_\ell - D\right) = 0, \tag{3.A.43}$$

$$\nu_2\left(\sum_{\ell=1}^{L} P_\ell - P\right) = 0. \tag{3.A.44}$$

So, the optimization problem reduces to solving the program for each $\ell \in \{1, \ldots, L\}$

as in (3.A.42). For a given ν_1 and ν_2 and each $\ell \in \{1, \ldots, L\}$, define

$$G_\ell(\nu_1, \nu_2, \gamma_\ell, \hat{\lambda}_\ell) := \frac{1}{2}\log\frac{\lambda_\ell}{\gamma_\ell} + \nu_1\left(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell - \gamma_\ell)} + \hat{\lambda}_\ell - D_\ell\right) + \frac{\nu_2}{2}\left(\frac{\hat{\lambda}_\ell}{\lambda_\ell} - 1 + \log\frac{\lambda_\ell}{\hat{\lambda}_\ell} - P_\ell\right). \tag{3.A.45}$$

In the following, we solve the minimization program

$$\min_{\substack{\{\gamma_\ell, \hat{\lambda}_\ell\}_{\ell=1}^L:\\ 0<\gamma_\ell\le\lambda_\ell\\ 0\le\hat{\lambda}_\ell\le\lambda_\ell}} G_\ell(\nu_1, \nu_2, \gamma_\ell, \hat{\lambda}_\ell). \tag{3.A.46}$$

We setup the KKT conditions [9]. Let $(\gamma_\ell^*, \hat{\lambda}_\ell^*)$ be any optimal solution for the primal problem and $\{\xi_\ell\}_{\ell=1}^L$, $\{\eta_\ell, \eta'_\ell\}_{\ell=1}^L$ be nonnegative Lagrange multipliers. Thus, for $\ell \in \{1, \ldots, L\}$, we have:

$$\frac{1}{2\gamma_\ell^*} - \nu_1\frac{\sqrt{\hat{\lambda}_\ell^*}}{\sqrt{\lambda_\ell - \gamma_\ell^*}} - \xi_\ell = 0, \tag{3.A.47a}$$

$$\nu_1\left(-\sqrt{\frac{\lambda_\ell - \gamma_\ell^*}{\hat{\lambda}_\ell^*}} + 1\right) + \frac{1}{2}\nu_2\left(\frac{1}{\lambda_\ell} - \frac{1}{\hat{\lambda}_\ell^*}\right) + \eta_\ell - \eta'_\ell = 0, \tag{3.A.47b}$$

$$\xi_\ell(\gamma_\ell^* - \lambda_\ell) = 0, \tag{3.A.47c}$$

$$\eta_\ell(\hat{\lambda}_\ell^* - \lambda_\ell) = 0, \tag{3.A.47d}$$

$$\eta'_\ell\hat{\lambda}_\ell^* = 0, \tag{3.A.47e}$$

$$\nu_1\left((\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell^*(\lambda_\ell - \gamma_\ell^*)} + \hat{\lambda}_\ell^*) - D_\ell\right) = 0, \tag{3.A.47f}$$

$$\nu_2\left(\frac{1}{2}\left(\frac{\hat{\lambda}_\ell^*}{\lambda_\ell} - 1 + \log\frac{\lambda_\ell}{\hat{\lambda}_\ell^*}\right) - P_\ell\right) = 0. \tag{3.A.47g}$$

We consider different cases based on the values of ν_1, ν_2, ξ_ℓ, η_ℓ and η'_ℓ.

Case 0) $\eta'_\ell > 0$: In this case, we have $\hat{\lambda}^*_\ell = 0$ and $\gamma^*_\ell = \lambda_\ell$.

Thus, in all of the following cases, we assume that $\eta'_\ell = 0$.

Assumption 1 **(**$\nu_1, \nu_2 > 0$**)**:

Case 1) $\xi_\ell = \eta_\ell = 0$**:** In this case, the condition (3.A.47b) implies that

$$\lambda_\ell < \gamma^*_\ell + \hat{\lambda}^{*1}_\ell, \tag{3.A.48}$$

which together with (3.A.47a) yields the following.

$$\gamma^*_\ell < \frac{1}{2\nu_1}. \tag{3.A.49}$$

Combining (3.A.48) and (3.A.49), we get:

$$\hat{\lambda}^*_\ell > \lambda_\ell - \frac{1}{2\nu_1}. \tag{3.A.50}$$

Notice that combining (3.A.48) with the distortion constraint in (3.A.47f) yields the following:

$$\lambda_\ell - \hat{\lambda}^*_\ell < D_\ell. \tag{3.A.51}$$

Moreover, (3.A.47f) implies that

$$D_\ell < \lambda_\ell + \hat{\lambda}^*_\ell. \tag{3.A.52}$$

Considering (3.A.51) and (3.A.52) yields the following inequality

$$\hat{\lambda}_\ell^* > |\lambda_\ell - D_\ell|. \tag{3.A.53}$$

Assuming that inequalities (3.A.48)–(3.A.50) hold, we can solve the equations (3.A.47a)–(3.A.47b) to get

$$\hat{\lambda}_\ell^* = \frac{\lambda_\ell(-b + \sqrt{b^2 + 8\lambda_\ell\nu_1\nu_2(2\lambda_\ell\nu_1 + \nu_2)} + 2\nu_2^2)}{2(2\lambda_\ell\nu_1 + \nu_2)^2} \left(= \frac{\lambda_\ell - \gamma_\ell^*}{4\gamma_\ell^{*2}\nu_1^2} \right) \tag{3.A.54}$$

$$= \hat{\lambda}_\ell^*(\nu_1, \nu_2), \tag{3.A.55}$$

$$\gamma_\ell^* = \frac{-2\lambda_\ell\nu_1(1 + 2\lambda_\ell\nu_1) - \nu_2 + \sqrt{b^2 + 8\lambda_\ell\nu_1\nu_2(2\lambda_\ell\nu_1 + \nu_2)}}{8\lambda_\ell\nu_1^2(-1 + \nu_2)} \left(= \frac{-1 + \sqrt{1 + 16\lambda_\ell\hat{\lambda}_\ell^*\nu_1^2}}{8\hat{\lambda}_\ell^*\nu_1^2} \right) \tag{3.A.56}$$

$$= \gamma_\ell^*(\nu_1, \nu_2), \tag{3.A.57}$$

where b is defined as in (3.3.12). Moreover, we have:

$$\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell^*(\lambda_\ell - \gamma_\ell^*)} + \hat{\lambda}_\ell^* = D_\ell, \tag{3.A.58a}$$

$$\frac{1}{2}\left(\frac{\hat{\lambda}_\ell^*}{\lambda_\ell} - 1 + \log\frac{\lambda_\ell}{\hat{\lambda}_\ell^*}\right) = P_\ell. \tag{3.A.58b}$$

Case 2) $\xi_\ell > 0$: In this case, we have $\gamma_\ell^* = \lambda_\ell$ which is not a feasible case since

$$\left.\frac{dG_\ell(\nu_1, \nu_2, \gamma_\ell, \hat{\lambda}_\ell)}{d\gamma_\ell}\right|_{\gamma_\ell = \lambda_\ell} = +\infty, \tag{3.A.59}$$

which makes the function $G_\ell(\nu_1, \nu_2, \gamma_\ell = \lambda_\ell, \hat{\lambda}_\ell)$ be a non-decreasing function thanks to $\nu_1 > 0$.

Case 3) $\eta_\ell > 0$ **and** $\xi_\ell = 0$**:** In this case, we have $\hat{\lambda}_\ell^* = \lambda_\ell$, $P_\ell = 0$ and

$$\eta_\ell = \nu_1 \left(\sqrt{\frac{\lambda_\ell - \gamma_\ell^*}{\lambda_\ell}} - 1 \right) < 0, \tag{3.A.60}$$

which is an infeasible case.

Assumption 2 ($\nu_1 > 0, \nu_2 = 0$):

Case 1) $\xi_\ell = 0$**,** $\eta_\ell = 0$: In this case, the KKT conditions simplify to the following.

$$\frac{1}{2\gamma_\ell^*} - \nu_1 \frac{\sqrt{\hat{\lambda}_\ell^*}}{\sqrt{\lambda_\ell - \gamma_\ell^*}} = 0, \tag{3.A.61a}$$

$$\nu_1 \left(-\sqrt{\frac{\lambda_\ell - \gamma_\ell^*}{\hat{\lambda}_\ell^*}} + 1 \right) = 0, \tag{3.A.61b}$$

$$\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell^* (\lambda_\ell - \gamma_\ell^*)} + \hat{\lambda}_\ell^* = D_\ell, \tag{3.A.61c}$$

$$\mathsf{P}_\ell(\hat{\lambda}_\ell^*) \leq P_\ell. \tag{3.A.61d}$$

Thus, condition (3.A.61b) implies that

$$\lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*. \tag{3.A.62}$$

Then, we have $D_\ell = \lambda_\ell - \hat{\lambda}_\ell^* = \frac{1}{2\nu_1}$,

$$\gamma_\ell^* = \frac{1}{2\nu_1}, \tag{3.A.63}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell - \frac{1}{2\nu_1}. \tag{3.A.64}$$

Case 2) $\xi_\ell > 0$**,** $\eta_\ell = 0$: In this case, the KKT conditions imply that

$$\lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*, \tag{3.A.65a}$$

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.65b}$$

$$D_\ell = \lambda_\ell + \hat{\lambda}_\ell^*, \tag{3.A.65c}$$

$$\hat{\lambda}_\ell^* = 0. \tag{3.A.65d}$$

Case 3) $\xi_\ell > 0$**,** $\eta_\ell > 0$: In this case, the KKT conditions imply that

$$\lambda_\ell > \hat{\lambda}_\ell^* + \gamma_\ell^*, \tag{3.A.66}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell, \tag{3.A.67}$$

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.68}$$

which are not feasible conditions together.

Case 4) $\xi_\ell = 0$**,** $\eta_\ell > 0$: In this case, the KKT conditions imply that

$$\lambda_\ell > \gamma_\ell^* + \hat{\lambda}_\ell^*, \tag{3.A.69}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell, \tag{3.A.70}$$

which are not feasible conditions together.

Assumption 3 $(\nu_1 = 0, \nu_2 > 0)$: In this case, the KKT conditions in (3.A.47) imply that

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.71}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell, \tag{3.A.72}$$

$$D_\ell \geq 2\lambda_\ell, \tag{3.A.73}$$

$$P_\ell = \mathsf{P}_\ell(\lambda_\ell) = 0. \tag{3.A.74}$$

Assumption 4 ($\nu_1 = \nu_2 = 0$):

Case 1 ($\xi_\ell > 0$): In this case, the KKT conditions in (3.A.47) imply that

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.75}$$

$$D_\ell \geq \lambda_\ell + \hat{\lambda}_\ell^*, \tag{3.A.76}$$

$$P_\ell \geq \mathsf{P}_\ell(\hat{\lambda}_\ell^*). \tag{3.A.77}$$

Case 2 ($\xi_\ell = 0$): According to (3.A.47a), this is an infeasible case.

Summarizing all of the above cases, we get:

$$R(D, P) = \max_{\substack{\nu_1, \nu_2 \geq 0: \\ \nu_1(\sum_{\ell=1}^L D_\ell - D) = 0 \\ \nu_2(\sum_{\ell=1}^L P_\ell - P) = 0}} \frac{1}{2} \sum_{\ell=1}^L \log \frac{\lambda_\ell}{\gamma_\ell}, \tag{3.A.78}$$

where

$$
(\gamma_\ell, D_\ell, P_\ell) :=
\begin{cases}
(\gamma_\ell^*(\nu_1,\nu_2), \mathsf{D}_\ell(\gamma_\ell^*(\nu_1,\nu_2), \hat{\lambda}_\ell^*(\nu_1,\nu_2)), \mathsf{P}_\ell(\hat{\lambda}_\ell^*(\nu_1,\nu_2))), \\
\qquad \text{if } \nu_1,\nu_2 > 0,\ \hat{\lambda}_\ell^*(\nu_1,\nu_2) > |\lambda_\ell - D_\ell|,\ \lambda_\ell < \gamma_\ell^*(\nu_1,\nu_2) + \hat{\lambda}_\ell^*(\nu_1,\nu_2),\ \gamma_\ell^*(\nu_1,\nu_2) < \frac{1}{2\nu_1}, \\
\qquad\qquad \hat{\lambda}_\ell^*(\nu_1,\nu_2) > \lambda_\ell - \frac{1}{2\nu_1}, \\
(\frac{1}{2\nu_1}, \frac{1}{2\nu_1}, P_\ell), \\
\qquad \text{if } \nu_1 > 0, \nu_2 = 0,\ D_\ell = \lambda_\ell - \hat{\lambda}_\ell^*,\ \ \lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*,\ \gamma_\ell^* = \frac{1}{2\nu_1},\ \hat{\lambda}_\ell^* = \lambda_\ell - \frac{1}{2\nu_1}, \\
\qquad\qquad \mathsf{P}_\ell(\hat{\lambda}_\ell^*) \le P_\ell, \\
(\lambda_\ell, \lambda_\ell, \mathsf{P}_\ell(0)), \\
\qquad \text{if } \nu_1 > 0, \nu_2 = 0,\ D_\ell = \lambda_\ell - \hat{\lambda}_\ell^*,\ \ \lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*,\ \ \gamma_\ell^* = \lambda_\ell,\ \ \hat{\lambda}_\ell^* = 0, \\
(\lambda_\ell, 2\lambda_\ell, \mathsf{P}_\ell(\lambda_\ell)), \\
\qquad \text{if } \nu_1 = 0, \nu_2 > 0,\ D_\ell \ge 2\lambda_\ell,\ \gamma_\ell^* = \lambda_\ell,\ \hat{\lambda}_\ell^* = \lambda_\ell, \\
(\lambda_\ell, D_\ell, P_\ell), \\
\qquad \text{if } \nu_1 = \nu_2 = 0,\ \ D_\ell - \lambda_\ell \ge \hat{\lambda}_\ell^*,\ \ \gamma_\ell^* = \lambda_\ell,\ \ \mathsf{P}_\ell(\hat{\lambda}_\ell^*) \le P_\ell.
\end{cases}
\tag{3.A.79}
$$

and ν_1 and ν_2 are chosen such that

$$\nu_1 \left(\sum_{\ell=1}^{L} D_\ell - D \right) = 0, \tag{3.A.80a}$$

$$\nu_2 \left(\sum_{\ell=1}^{L} P_\ell - P \right) = 0. \tag{3.A.80b}$$

First, we analyze the second and third clauses of (3.A.79) to obtain the value of D_ℓ since in these two clauses, we have $D_\ell = \lambda_\ell - \hat{\lambda}_\ell^*$. Consider the following sum distortion constraint

$$\sum_{\ell=1}^{L} D_\ell = D, \tag{3.A.81}$$

which can be written as follows:

$$\sum_{\ell=1}^{L} (\lambda_\ell - \hat{\lambda}_\ell^*) = D. \tag{3.A.82}$$

Plugging $\hat{\lambda}_\ell^* = [\lambda_\ell - \frac{1}{2\nu_1}]^+$ into the above inequality yields the following:

$$\sum_{\ell=1}^{L} \left[\lambda_\ell - \frac{1}{2\nu_1}\right]^+ = \left[\sum_{\ell=1}^{L} \lambda_\ell - D\right]^+. \tag{3.A.83}$$

Let $\gamma(D) := \frac{1}{2\nu_1}$ which yields the definition in (3.3.18). Moreover, we have

$$D_\ell = \frac{1}{2\nu_1} = \gamma(D). \tag{3.A.84}$$

After deriving the value of D_ℓ, we get back to analysis of different clauses of (3.A.79). Under the first clause, we have

$$R(D, P) = \frac{1}{2} \sum_{\ell=1}^{L} \log \frac{\lambda_\ell}{\gamma_\ell^*(\nu_1, \nu_2)}, \tag{3.A.85}$$

where $\hat{\lambda}_\ell^*(\nu_1, \nu_2)$ and $\gamma_\ell^*(\nu_1, \nu_2)$ are defined in (3.3.13) and (3.3.14), respectively, and satisfy the following equations

$$\sum_{\ell=1}^{L} \mathsf{D}_\ell(\gamma_\ell^*(\nu_1,\nu_2), \hat{\lambda}_\ell^*(\nu_1,\nu_2)) = D, \tag{3.A.86}$$

$$\sum_{\ell=1}^{L} \mathsf{P}_\ell(\hat{\lambda}_\ell^*(\nu_1,\nu_2)) = P. \tag{3.A.87}$$

Moreover, D_ℓ given in (3.A.84) satisfies the following inequality

$$\hat{\lambda}_\ell^*(\nu_1,\nu_2) > |\gamma(D) - \lambda_\ell|. \tag{3.A.88}$$

Combining the above inequality with (3.A.87) yields the following constraint:

$$P < \sum_{\ell=1}^{L} \mathsf{P}_\ell(|\gamma(D) - \lambda_\ell|). \tag{3.A.89}$$

Thus, the first clause of (3.A.79) is active when the above inequality is satisfied. If the above inequality is violated, the other clauses of (3.A.79) would be active.

Now, we analyze the second and third clauses of (3.A.79). Under these two clauses, we have:

$$R(D,P) = \frac{1}{2}\sum_{\ell=1}^{L} \log \frac{\lambda_\ell}{\Delta_\ell(D)}, \tag{3.A.90}$$

where $\Delta_\ell(D)$ is defined in (3.3.19). Notice that these two clauses are active when

$$\hat{\lambda}_\ell^* = [\lambda_\ell - D_\ell]^+, \tag{3.A.91}$$

which gives the following inequality on the perception constraint

$$\sum_{\ell=1}^{L} \mathsf{P}_\ell([\lambda_\ell - \gamma(D)]^+) \leq P. \tag{3.A.92}$$

So, if the above inequality is violated, the fourth and fifth clauses of (3.A.79) would be active. Under the last two clauses of (3.A.79), we have:

$$R(D, P) = 0, \tag{3.A.93}$$

since $\gamma_\ell^* = \lambda_\ell$ for all $\ell \in \{1, \dots, L\}$. Summarizing all of the above cases, we get to the expression in (3.3.23). This concludes the proof of the optimization problem.

Now, consider the Wasserstein-2 distance as the perception metric. The proof follows similar steps to the previous case. We just need to study the perception function which is determined by the Wasserstein-2 distance. Define the following joint distribution

$$P^*_{\boldsymbol{UV}} = \arg \inf_{\substack{\tilde{P}_{\boldsymbol{UV}}: \\ \tilde{P}_{\boldsymbol{U}} = P_{\boldsymbol{X}} \\ \tilde{P}_{\boldsymbol{V}} = P_{\hat{\boldsymbol{X}}}}} \mathbb{E}_{\tilde{P}}[\|\boldsymbol{U} - \boldsymbol{V}\|^2], \tag{3.A.94}$$

and let $P^G_{\boldsymbol{UV}}$ be a jointly Gaussian distribution such that $\mathrm{cov}_{P^G}(\boldsymbol{U}, \boldsymbol{V}) = \mathrm{cov}_{P^*}(\boldsymbol{U}, \boldsymbol{V})$.

Consider the following sets of inequalities:

$$\begin{aligned}
W_2^2(P_{\boldsymbol{X}}, P_{\hat{\boldsymbol{X}}}) =& \mathbb{E}_{P^*}[\|\boldsymbol{U} - \boldsymbol{V}\|^2] && \text{(3.A.95)}\\
=& \mathbb{E}_{P^G}[\|\boldsymbol{U} - \boldsymbol{V}\|^2] && \text{(3.A.96)}\\
\geq& \inf_{\substack{\hat{P}_{\boldsymbol{U}\boldsymbol{V}}:\\ \hat{P}_{\boldsymbol{U}}=P_{\boldsymbol{X}}\\ \hat{P}_{\boldsymbol{V}}=P_{\hat{\boldsymbol{X}}_G}}} \mathbb{E}_{\hat{P}}[\|\boldsymbol{U} - \boldsymbol{V}\|^2] && \text{(3.A.97)}\\
=& W_2^2(P_{\boldsymbol{X}}, P_{\hat{\boldsymbol{X}}_G}), && \text{(3.A.98)}
\end{aligned}$$

So, from the perception point of view, one can restrict to $\hat{\boldsymbol{X}}_G$ jointly Gaussian with $\boldsymbol{X}$, without loss of optimality.

Next, notice that for two Gaussian distributions with zero-mean, the Wasserstein-2 distance simplifies to the following.

$$\begin{aligned}
W_2^2(P_{\boldsymbol{X}}, P_{\hat{\boldsymbol{X}}_G}) =& \mathrm{tr}(\Sigma_{\boldsymbol{X}} + \Sigma_{\hat{\boldsymbol{X}}_G} - 2(\Sigma_{\boldsymbol{X}}^{\frac{1}{2}} \Sigma_{\hat{\boldsymbol{X}}_G} \Sigma_{\boldsymbol{X}}^{\frac{1}{2}})^{\frac{1}{2}}) && \text{(3.A.99)}\\
=& \sum_{\ell=1}^{L} \left(\sqrt{\lambda_\ell} - \sqrt{\hat{\lambda}_\ell}\right)^2. && \text{(3.A.100)}
\end{aligned}$$

Thus, the optimization problem reduces to the following:

$$R(D,P) = \min_{\{\hat{\lambda}_\ell,\gamma_\ell\}_{\ell=1}^L} \ \frac{1}{2}\sum_{\ell=1}^{L} \log\frac{\lambda_\ell}{\gamma_\ell} \tag{3.A.101a}$$

$$\text{s.t.} \qquad 0 < \gamma_\ell \le \lambda_\ell, \tag{3.A.101b}$$

$$0 \le \hat{\lambda}_\ell \le \lambda_\ell, \tag{3.A.101c}$$

$$\sum_{\ell=1}^{L}\left(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell-\gamma_\ell)} + \hat{\lambda}_\ell\right) \le D, \tag{3.A.101d}$$

$$\sum_{\ell=1}^{L}\left(\sqrt{\lambda_\ell} - \sqrt{\hat{\lambda}_\ell}\right)^2 \le P. \tag{3.A.101e}$$

The solution to the above program can be derived following similar steps to the proof of the KL-divergence case. It can be easily verified that the above optimization problem is convex. Thus, the solution to the above program is equal to that of the following dual optimization problem.

$$R(D,P) = \max_{\nu_1,\nu_2} \min_{\substack{\{\gamma_\ell,\hat{\lambda}_\ell\}_{\ell=1}^L:\\ 0<\gamma_\ell\le\lambda_\ell \\ 0\le\hat{\lambda}_\ell\le\lambda_\ell}} \ \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{\lambda_\ell}{\gamma_\ell} + \nu_1\left(\sum_{\ell=1}^{L}(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell-\gamma_\ell)} + \hat{\lambda}_\ell) - D\right)$$
$$+ \nu_2\left(\sum_{\ell=1}^{L}\left(\sqrt{\lambda_\ell}-\sqrt{\hat{\lambda}_\ell}\right)^2 - P\right) \tag{3.A.102}$$

$$= \max_{\nu_1,\nu_2} \sum_{\ell=1}^{L} \min_{\substack{\{\gamma_\ell,\hat{\lambda}_\ell\}_{\ell=1}^L:\\ 0<\gamma_\ell\le\lambda_\ell \\ 0\le\hat{\lambda}_\ell\le\lambda_\ell}}$$
$$\left(\frac{1}{2}\log\frac{\lambda_\ell}{\gamma_\ell} + \nu_1\left(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell-\gamma_\ell)} + \hat{\lambda}_\ell - D_\ell\right) + \nu_2\left(\left(\sqrt{\lambda_\ell}-\sqrt{\hat{\lambda}_\ell}\right)^2 - P_\ell\right)\right), \tag{3.A.103}$$

where $\{\nu_1, \nu_2\}$ are nonnegative Lagrange multipliers and positive D_ℓ and nonnegative P_ℓ are chosen such that

$$\nu_1 \left(\sum_{\ell=1}^{L} D_\ell - D \right) = 0, \tag{3.A.104}$$

$$\nu_2 \left(\sum_{\ell=1}^{L} P_\ell - P \right) = 0. \tag{3.A.105}$$

So, the optimization problem reduces to solving the program for each $\ell \in \{1, \ldots, L\}$ as in (3.A.103). For a given ν_1 and ν_2 and each $\ell \in \{1, \ldots, L\}$, define

$$G_\ell(\nu_1, \nu_2, \gamma_\ell, \hat{\lambda}_\ell) := \frac{1}{2} \log \frac{\lambda_\ell}{\gamma_\ell} + \nu_1 \left(\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell(\lambda_\ell - \gamma_\ell)} + \hat{\lambda}_\ell - D_\ell \right) + \nu_2 \left(\left(\sqrt{\lambda_\ell} - \sqrt{\hat{\lambda}_\ell} \right)^2 - P_\ell \right). \tag{3.A.106}$$

In the following, we solve the minimization program

$$\min_{\substack{\{\gamma_\ell, \hat{\lambda}_\ell\}_{\ell=1}^{L}: \\ 0 < \gamma_\ell \leq \lambda_\ell \\ 0 \leq \hat{\lambda}_\ell \leq \lambda_\ell}} G_\ell(\nu_1, \nu_2, \gamma_\ell, \hat{\lambda}_\ell). \tag{3.A.107}$$

We setup the KKT conditions. Let $(\gamma_\ell^*, \hat{\lambda}_\ell^*)$ be any optimal solution for the primal problem and $\{\xi_\ell\}_{\ell=1}^{L}$, $\{\eta_\ell, \eta_\ell'\}_{\ell=1}^{L}$ be nonnegative Lagrange multipliers. Thus, for $\ell \in$

$\{1, \ldots, L\}$, we have:

$$\frac{1}{2\gamma_\ell^*} - \nu_1 \frac{\sqrt{\hat{\lambda}_\ell^*}}{\sqrt{\lambda_\ell - \gamma_\ell^*}} - \xi_\ell = 0, \tag{3.A.108a}$$

$$\nu_1 \left(-\sqrt{\frac{\lambda_\ell - \gamma_\ell^*}{\hat{\lambda}_\ell^*}} + 1 \right) + \nu_2 \left(1 - \sqrt{\frac{\lambda_\ell}{\hat{\lambda}_\ell^*}} \right) + \eta_\ell - \eta'_\ell = 0, \tag{3.A.108b}$$

$$\xi_\ell(\gamma_\ell^* - \lambda_\ell) = 0, \tag{3.A.108c}$$

$$\eta_\ell(\hat{\lambda}_\ell^* - \lambda_\ell) = 0, \tag{3.A.108d}$$

$$\eta'_\ell \hat{\lambda}_\ell^* = 0, \tag{3.A.108e}$$

$$\nu_1 \left((\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell^*(\lambda_\ell - \gamma_\ell^*)} + \hat{\lambda}_\ell^*) - D_\ell \right) = 0, \tag{3.A.108f}$$

$$\nu_2 \left(\left(\sqrt{\lambda_\ell} - \sqrt{\hat{\lambda}_\ell} \right)^2 - P_\ell \right) = 0. \tag{3.A.108g}$$

We consider different cases based on the values of ν_1, ν_2, ξ_ℓ, η_ℓ and η'_ℓ.

Case 0) $\eta'_\ell > 0$: In this case, we have $\hat{\lambda}_\ell^* = 0$ and $\gamma_\ell^* = \lambda_\ell$.

Thus, in all of the following cases, we assume that $\eta'_\ell = 0$.

Assumption 1 $(\nu_1, \nu_2 > 0)$:

Case 1) $\xi_\ell = \eta_\ell = 0$**:** In this case, the condition (3.A.108b) implies that

$$\lambda_\ell < \gamma_\ell^* + \hat{\lambda}_\ell^{*2}, \tag{3.A.109}$$

which together with (3.A.108a) yields the following.

$$\gamma_\ell^* < \frac{1}{2\nu_1}. \tag{3.A.110}$$

Combining (3.A.109) and (3.A.110), we get:

$$\hat{\lambda}_\ell^* > \lambda_\ell - \frac{1}{2\nu_1}. \tag{3.A.111}$$

Notice that combining (3.A.109) with the distortion constraint in (3.A.108f) yields the following:

$$\lambda_\ell - \hat{\lambda}_\ell^* < D_\ell. \tag{3.A.112}$$

Moreover, (3.A.108f) implies that

$$D_\ell < \lambda_\ell + \hat{\lambda}_\ell^*. \tag{3.A.113}$$

Considering (3.A.112) and (3.A.113) yields the following inequality

$$\hat{\lambda}_\ell^* > |\lambda_\ell - D_\ell|. \tag{3.A.114}$$

Assuming that inequalities (3.A.109)–(3.A.111) hold, we can solve the equations (3.A.108a)–(3.A.108b) to get

$$\hat{\lambda}_\ell^*(\nu_1, \nu_2) := \frac{\lambda_\ell}{\left(1 + \frac{(1-\theta_\ell)\nu_1}{\nu_2}\right)^2}, \tag{3.A.115}$$

$$\gamma_\ell^*(\nu_1, \nu_2) := \frac{\left(1 + \frac{(1-\theta_\ell)\nu_1}{\nu_2}\right)^2}{8\lambda_\ell \nu_1^2} \cdot \left(-1 + \sqrt{1 + \frac{16\lambda_\ell^2 \nu_1^2}{\left(1 + \frac{(1-\theta_\ell)\nu_1}{\nu_2}\right)^2}}\right). \tag{3.A.116}$$

where θ_ℓ is the unique solution of the following equation:

$$\nu_1\theta_\ell^3 - 2\nu_1(1+\lambda_\ell(\nu_1-\nu_2))(\nu_1+\nu_2)\theta_\ell^2 + (\nu_1+4\lambda_\ell\nu_1^2+\nu_2)(\nu_1+\nu_2)\theta_\ell - 2\lambda_\ell\nu_1(\nu_1+\nu_2)^2 = 0. \tag{3.A.117}$$

Case 2) $\xi_\ell > 0$**:** In this case, we have $\gamma_\ell^* = \lambda_\ell$ which is not a feasible case since

$$\frac{dG_\ell(\nu_1,\nu_2,\gamma_\ell,\hat{\lambda}_\ell)}{d\gamma_\ell}\Big|_{\gamma_\ell=\lambda_\ell} = +\infty, \tag{3.A.118}$$

which makes the function $G_\ell(\nu_1,\nu_2,\gamma_\ell=\lambda_\ell,\hat{\lambda}_\ell)$ be a non-decreasing function thanks to $\nu_1 > 0$.

Case 3) $\eta_\ell > 0$ **and** $\xi_\ell = 0$**:** In this case, we have $\hat{\lambda}_\ell^* = \lambda_\ell$, $P_\ell = 0$ and

$$\eta_\ell = \nu_1\left(\sqrt{\frac{\lambda_\ell-\gamma_\ell^*}{\lambda_\ell}} - 1\right) < 0, \tag{3.A.119}$$

which is an infeasible case.

Assumption 2 $(\nu_1 > 0, \nu_2 = 0)$:

Case 1) $\xi_\ell = 0$**,** $\eta_\ell = 0$: In this case, the KKT conditions simplify to the following.

$$\frac{1}{2\gamma_\ell^*} - \nu_1\frac{\sqrt{\hat{\lambda}_\ell^*}}{\sqrt{\lambda_\ell-\gamma_\ell^*}} = 0, \tag{3.A.120a}$$

$$\nu_1\left(-\sqrt{\frac{\lambda_\ell-\gamma_\ell^*}{\hat{\lambda}_\ell^*}} + 1\right) = 0, \tag{3.A.120b}$$

$$\lambda_\ell - 2\sqrt{\hat{\lambda}_\ell^*(\lambda_\ell-\gamma_\ell^*)} + \hat{\lambda}_\ell^* = D_\ell, \tag{3.A.120c}$$

$$\mathsf{P}_\ell(\hat{\lambda}_\ell^*) \le P_\ell. \tag{3.A.120d}$$

Thus, condition (3.A.120b) implies that

$$\lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*. \tag{3.A.121}$$

Then, we have $D_\ell = \lambda_\ell - \hat{\lambda}_\ell^* = \frac{1}{2\nu_1}$,

$$\gamma_\ell^* = \frac{1}{2\nu_1}, \tag{3.A.122}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell - \frac{1}{2\nu_1}. \tag{3.A.123}$$

Case 2) $\xi_\ell > 0$**,** $\eta_\ell = 0$: In this case, the KKT conditions imply that

$$\lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*, \tag{3.A.124a}$$

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.124b}$$

$$D_\ell = \lambda_\ell + \hat{\lambda}_\ell^*, \tag{3.A.124c}$$

$$\hat{\lambda}_\ell^* = 0. \tag{3.A.124d}$$

Case 3) $\xi_\ell > 0$**,** $\eta_\ell > 0$: In this case, the KKT conditions imply that

$$\lambda_\ell > \hat{\lambda}_\ell^* + \gamma_\ell^*, \tag{3.A.125}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell, \tag{3.A.126}$$

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.127}$$

which are not feasible conditions together.

Case 4) $\xi_\ell = 0$, $\eta_\ell > 0$: In this case, the KKT conditions imply that

$$\lambda_\ell > \gamma_\ell^* + \hat{\lambda}_\ell^*, \tag{3.A.128}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell, \tag{3.A.129}$$

which are not feasible conditions together.

Assumption 3 ($\nu_1 = 0, \nu_2 > 0$): In this case, the KKT conditions in (3.A.108) imply that

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.130}$$

$$\hat{\lambda}_\ell^* = \lambda_\ell, \tag{3.A.131}$$

$$D_\ell \geq 2\lambda_\ell, \tag{3.A.132}$$

$$P_\ell = \mathsf{P}_\ell(\lambda_\ell) = 0. \tag{3.A.133}$$

Assumption 4 ($\nu_1 = \nu_2 = 0$):

Case 1 ($\xi_\ell > 0$): In this case, the KKT conditions in (3.A.108) imply that

$$\gamma_\ell^* = \lambda_\ell, \tag{3.A.134}$$

$$D_\ell \geq \lambda_\ell + \hat{\lambda}_\ell^*, \tag{3.A.135}$$

$$P_\ell \geq \mathsf{P}_\ell(\hat{\lambda}_\ell^*). \tag{3.A.136}$$

Case 2 ($\xi_\ell = 0$): According to (3.A.108a), this is an infeasible case.

Summarizing all of the above cases, we get:

$$R(D,P) = \max_{\substack{\nu_1,\nu_2\geq 0: \\ \nu_1(\sum_{\ell=1}^L D_\ell - D)=0 \\ \nu_2(\sum_{\ell=1}^L P_\ell - P)=0}} \frac{1}{2}\sum_{\ell=1}^{L} \log\frac{\lambda_\ell}{\gamma_\ell}, \tag{3.A.137}$$

where

$$(\gamma_\ell, D_\ell, P_\ell) :=$$
$$\begin{cases} (\gamma_\ell^*(\nu_1,\nu_2), \mathsf{D}_\ell(\gamma_\ell^*(\nu_1,\nu_2), \hat{\lambda}_\ell^*(\nu_1,\nu_2)), \mathsf{P}_\ell(\hat{\lambda}_\ell^*(\nu_1,\nu_2))), \\ \qquad \text{if } \ \nu_1,\nu_2 > 0,\ \hat{\lambda}_\ell^*(\nu_1,\nu_2) > |\lambda_\ell - D_\ell|,\ \lambda_\ell < \gamma_\ell^*(\nu_1,\nu_2) + \hat{\lambda}_\ell^*(\nu_1,\nu_2),\ \gamma_\ell^*(\nu_1,\nu_2) < \frac{1}{2\nu_1}, \\ \qquad\qquad \hat{\lambda}_\ell^*(\nu_1,\nu_2) > \lambda_\ell - \frac{1}{2\nu_1}, \\ (\frac{1}{2\nu_1}, \frac{1}{2\nu_1}, P_\ell), \\ \qquad \text{if } \ \nu_1 > 0, \nu_2 = 0,\ D_\ell = \lambda_\ell - \hat{\lambda}_\ell^*,\ \ \lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*,\ \gamma_\ell^* = \frac{1}{2\nu_1},\ \hat{\lambda}_\ell^* = \lambda_\ell - \frac{1}{2\nu_1}, \\ \qquad\qquad \mathsf{P}_\ell(\hat{\lambda}_\ell^*) \leq P_\ell, \\ (\lambda_\ell, \lambda_\ell, \mathsf{P}_\ell(0)), \\ \qquad \text{if } \ \nu_1 > 0, \nu_2 = 0,\ D_\ell = \lambda_\ell - \hat{\lambda}_\ell^*,\ \ \lambda_\ell = \gamma_\ell^* + \hat{\lambda}_\ell^*,\ \ \gamma_\ell^* = \lambda_\ell,\ \ \hat{\lambda}_\ell^* = 0, \\ (\lambda_\ell, 2\lambda_\ell, \mathsf{P}_\ell(\lambda_\ell)), \\ \qquad \text{if } \ \nu_1 = 0, \nu_2 > 0,\ D_\ell \geq 2\lambda_\ell,\ \gamma_\ell^* = \lambda_\ell,\ \hat{\lambda}_\ell^* = \lambda_\ell, \\ (\lambda_\ell, D_\ell, P_\ell), \\ \qquad \text{if } \nu_1 = \nu_2 = 0,\ \ D_\ell - \lambda_\ell \geq \hat{\lambda}_\ell^*,\ \ \gamma_\ell^* = \lambda_\ell,\ \ \mathsf{P}_\ell(\hat{\lambda}_\ell^*) \leq P_\ell. \end{cases} \tag{3.A.138}$$

and ν_1 and ν_2 are chosen such that

$$\nu_1 \left(\sum_{\ell=1}^{L} D_\ell - D \right) = 0, \tag{3.A.139a}$$

$$\nu_2 \left(\sum_{\ell=1}^{L} P_\ell - P \right) = 0. \tag{3.A.139b}$$

First, we analyze the second and third clauses of (3.A.138) to obtain the value of D_ℓ since in these two clauses, we have $D_\ell = \lambda_\ell - \hat{\lambda}_\ell^*$. Consider the following sum distortion constraint

$$\sum_{\ell=1}^{L} D_\ell = D, \tag{3.A.140}$$

which can be written as follows:

$$\sum_{\ell=1}^{L} (\lambda_\ell - \hat{\lambda}_\ell^*) = D. \tag{3.A.141}$$

Plugging $\hat{\lambda}_\ell^* = [\lambda_\ell - \frac{1}{2\nu_1}]^+$ into the above inequality yields the following:

$$\sum_{\ell=1}^{L} \left[\lambda_\ell - \frac{1}{2\nu_1} \right]^+ = \left[\sum_{\ell=1}^{L} \lambda_\ell - D \right]^+. \tag{3.A.142}$$

Let $\gamma(D) := \frac{1}{2\nu_1}$ which yields the definition in (3.3.18). Moreover, we have

$$D_\ell = \frac{1}{2\nu_1} = \gamma(D). \tag{3.A.143}$$

After deriving the value of D_ℓ, we get back to analysis of different clauses of (3.A.138).

Under the first clause, we have

$$R(D,P) = \frac{1}{2}\sum_{\ell=1}^{L} \log \frac{\lambda_\ell}{\gamma_\ell^*(\nu_1,\nu_2)}, \tag{3.A.144}$$

where $\hat{\lambda}_\ell^*(\nu_1,\nu_2)$ and $\gamma_\ell^*(\nu_1,\nu_2)$ are defined in (3.3.13) and (3.3.14), respectively, and satisfy the following equations

$$\sum_{\ell=1}^{L} \mathsf{D}_\ell(\gamma_\ell^*(\nu_1,\nu_2), \hat{\lambda}_\ell^*(\nu_1,\nu_2)) = D, \tag{3.A.145}$$

$$\sum_{\ell=1}^{L} \mathsf{P}_\ell(\hat{\lambda}_\ell^*(\nu_1,\nu_2)) = P. \tag{3.A.146}$$

Moreover, D_ℓ given in (3.A.143) satisfies the following inequality

$$\hat{\lambda}_\ell^*(\nu_1,\nu_2) > |\gamma(D) - \lambda_\ell|. \tag{3.A.147}$$

Combining the above inequality with (3.A.146) yields the following constraint:

$$P < \sum_{\ell=1}^{L} \mathsf{P}_\ell(|\gamma(D) - \lambda_\ell|). \tag{3.A.148}$$

Thus, the first clause of (3.A.138) is active when the above inequality is satisfied. If the above inequality is violated, the other clauses of (3.A.138) would be active.

Now, we analyze the second and third clauses of (3.A.138). Under these two clauses, we have:

$$R(D,P) = \frac{1}{2}\sum_{\ell=1}^{L} \log \frac{\lambda_\ell}{\Delta_\ell(D)}, \tag{3.A.149}$$

where $\Delta_\ell(D)$ is defined in (3.3.19). Notice that these two clauses are active when

$$\hat{\lambda}_\ell^* = [\lambda_\ell - D_\ell]^+, \tag{3.A.150}$$

which gives the following inequality on the perception constraint

$$\sum_{\ell=1}^{L} \mathsf{P}_\ell([\lambda_\ell - \gamma(D)]^+) \leq P. \tag{3.A.151}$$

So, if the above inequality is violated, the fourth and fifth clauses of (3.A.138) would be active. Under the last two clauses of (3.A.138), we have:

$$R(D, P) = 0, \tag{3.A.152}$$

since $\gamma_\ell^* = \lambda_\ell$ for all $\ell \in \{1, \ldots, L\}$. Summarizing all of the above cases, we get to the expression in (3.3.23). This concludes the proof.

3.B Proof of Corollary 3.3.1.1

Assume that the perception function is the KL-divergence between input and reconstruction distributions. The proof for the Wasserstein-2 distance is similar. Fix a finite P. If P satisfies the first clause of (3.3.23), then the optimal water-levels Γ_ℓ are given by $\Omega_\ell(D, P)$ as follows:

$$\Gamma_\ell = \Omega_\ell(D, P) \tag{3.B.1}$$

$$= \gamma_\ell^*(\nu_1, \nu_2) \tag{3.B.2}$$

$$= \frac{-1 + \sqrt{1 + 16\lambda_\ell \hat{\lambda}_\ell^*(\nu_1, \nu_2)\nu_1^2}}{8\hat{\lambda}_\ell^*(\nu_1, \nu_2)\nu_1^2} \tag{3.B.3}$$

$$< \lambda_\ell. \tag{3.B.4}$$

Now, assume that P satisfies the second clause of (3.3.23). Since P is finite, the second clause of (3.3.23) implies that

$$\gamma(D) < \lambda_\ell, \qquad \forall \ell \in \{1, \ldots, L\}, \tag{3.B.5}$$

or equivalently

$$\Gamma_\ell = \Delta_\ell(D) < \lambda_\ell, \qquad \forall \ell \in \{1, \ldots, L\}. \tag{3.B.6}$$

Finally, assume that P satisfies the third clause of (3.3.23). In this case, since $R(D, P) = 0$, we have:

$$\Gamma_\ell = \lambda_\ell, \qquad \forall \ell \in \{1, \ldots, L\}. \tag{3.B.7}$$

Considering (3.B.4), (3.B.6) and (3.B.7) concludes the proof of corollary.

3.C Proof of Theorem 3.3.2

Recall from (3.3.36) that X_{r} is a representation of X such that there exist $\{\bar{\gamma}_\ell, \bar{\lambda}_\ell\}_{\ell=1}^L$ such that

$$R = \frac{1}{2}\sum_{\ell=1}^{L} \log\frac{\lambda_\ell}{\bar{\gamma}_\ell}, \tag{3.C.1}$$

where

$$0 < \bar{\gamma}_\ell \leq \lambda_\ell, \tag{3.C.2}$$

$$0 \leq \bar{\lambda}_\ell \leq \lambda_\ell, \tag{3.C.3}$$

$$\sum_{\ell=1}^{L}\left(\lambda_\ell - 2\sqrt{\bar{\lambda}_\ell(\lambda_\ell-\bar{\gamma}_\ell)} + \bar{\lambda}_\ell\right) \leq D, \tag{3.C.4}$$

$$\sum_{\ell=1}^{L}\left(\sqrt{\lambda_\ell} - \sqrt{\bar{\lambda}_\ell}\right)^2 \leq P. \tag{3.C.5}$$

Thus, there exists a reconstruction $\hat{X}$ such that $(\bar{\lambda}_1, \ldots, \bar{\lambda}_L)$ are the eigenvalues of its covariance matrix, i.e., $\Sigma_{\hat{X}} = \Theta\Lambda_{\hat{X}}\Theta^T$ where

$$\Lambda_{\hat{X}} := \mathrm{diag}^L(\bar{\lambda}_1, \ldots, \bar{\lambda}_L). \tag{3.C.6}$$

Let

$$\hat{X}_A := A\hat{X}^T, \tag{3.C.7}$$

where A is an L-dimensional vector as $A := (A_1, \ldots, A_L)$ for some positive real numbers $\{A_\ell\}_{\ell=1}^L$. Notice that we have $\Sigma_{\hat{X}_A} = A\Sigma_{\hat{X}}A^T$. If we let $\Sigma_{\hat{X}_A} = \Theta\Lambda_{\hat{X}_A}\Theta^T$

where

$$\Lambda_{\hat{X}_A} := \mathrm{diag}^L(\lambda_{A,1}, \ldots, \lambda_{A,L}), \tag{3.C.8}$$

Then, we get

$$\lambda_{A,\ell} = A_\ell \bar{\lambda}_\ell. \tag{3.C.9}$$

Since the rate on the boundary of $\Phi_{\mathsf{DP}}(R)$ is fixed to be R, then according to (3.3.36a), $\{\bar{\gamma}_\ell\}_{\ell=1}^L$ are also fixed. Thus, according to (3.3.36c)–(3.3.36e), $\Phi_{\mathsf{DP}}(P_{X_{\mathrm{r}}|X})$ includes the following set:

$$\Phi_{\mathsf{DP}}^{\mathrm{in}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L) := \left\{ \begin{array}{l} (D', P') \colon \exists \{A_\ell\}_{\ell=1}^L \text{ s.t.:} \\ \\ 1 \le A_\ell, \\ \sum_{\ell=1}^L \left(\lambda_\ell - 2\sqrt{A_\ell \bar{\lambda}_\ell (\lambda_\ell - \bar{\gamma}_\ell)} + A_\ell \bar{\lambda}_\ell \right) \le D', \\ \sum_{\ell=1}^L \left(\sqrt{\lambda_\ell} - \sqrt{A_\ell \bar{\lambda}_\ell} \right)^2 \le P' \end{array} \right\}. \tag{3.C.10}$$

In order to obtain the boundary of the above region, we solve an optimization problem for each D'. That is, the set of all feasible P' on the boundary of $\Phi_{\mathsf{DP}}^{\mathrm{in}}(\bar{\gamma}_1, \ldots, \bar{\gamma}_L)$ is determined by the following.

$$P' = \min_{\{A_\ell\}_{\ell=1}^L} \sum_{\ell=1}^L \left(\sqrt{\lambda_\ell} - \sqrt{A_\ell \bar{\lambda}_\ell} \right)^2, \tag{3.C.11a}$$

$$\text{s.t.:} \quad 1 \le A_\ell, \tag{3.C.11b}$$

$$\sum_{\ell=1}^L \left(\lambda_\ell - 2\sqrt{A_\ell \bar{\lambda}_\ell (\lambda_\ell - \bar{\gamma}_\ell)} + A_\ell \bar{\lambda}_\ell \right) \le D'. \tag{3.C.11c}$$

The above program is convex, so we can setup the dual Lagrange problem and continue with the KKT conditions. The dual Lagrange function is given as follows.

$$\max_{\nu\geq 0} \min_{\substack{\{A_\ell\}_{\ell=1}^L:\\ 1\leq A_\ell}} \sum_{\ell=1}^{L}\left(\sqrt{\lambda_\ell}-\sqrt{A_\ell\bar{\lambda}_\ell}\right)^2+\nu\sum_{\ell=1}^{L}\left(\lambda_\ell-2\sqrt{A_\ell\bar{\lambda}_\ell(\lambda_\ell-\bar{\gamma}_\ell)}+A_\ell\bar{\lambda}_\ell-D'\right) \tag{3.C.12}$$

$$=\max_{\nu\geq 0}\sum_{\ell=1}^{L}\min_{\substack{\{A_\ell\}_{\ell=1}^L:\\ 1\leq A_\ell}}\left(\left(\sqrt{\lambda_\ell}-\sqrt{A_\ell\bar{\lambda}_\ell}\right)^2+\nu\left(\lambda_\ell-2\sqrt{A_\ell\bar{\lambda}_\ell(\lambda_\ell-\bar{\gamma}_\ell)}+A_\ell\bar{\lambda}_\ell-D_\ell\right)\right), \tag{3.C.13}$$

where nonnegative $\{D_\ell\}_{\ell=1}^L$ are chosen such that $\sum_{\ell=1}^L D_\ell = D'$. In order to solve the minimization problem in (3.C.13), we setup KKT conditions. Thus, there exist nonnegative Lagrange multipliers $\{\xi_\ell\}_{\ell=1}^L$ such that

$$\left(-\sqrt{\frac{\lambda_\ell\bar{\lambda}_\ell}{A_\ell}}+\bar{\lambda}_\ell\right)+\nu\left(\bar{\lambda}_\ell-\sqrt{\frac{\bar{\lambda}_\ell(\lambda_\ell-\bar{\gamma}_\ell)}{A_\ell}}\right)-\xi_\ell=0, \tag{3.C.14}$$

$$\xi_\ell(-A_\ell+1)=0, \tag{3.C.15}$$

$$\nu\left(\lambda_\ell-2\sqrt{A_\ell\bar{\lambda}_\ell(\lambda_\ell-\bar{\gamma}_\ell)}+A_\ell\bar{\lambda}_\ell-D_\ell\right)=0. \tag{3.C.16}$$

We discuss the above program based on different values of ν. If $\nu=0$, then (3.C.14) implies that

$$\xi_\ell>0,\ A_\ell=1. \tag{3.C.17}$$

If $\nu>0$, then we have two cases.

Case 1 ($\xi_\ell = 0$): In this case, we have:

$$A_\ell = \frac{\lambda_\ell}{\bar{\lambda}_\ell}\left(\frac{1+\nu\sqrt{1-\frac{\bar{\gamma}_\ell}{\bar{\lambda}_\ell}}}{1+\nu}\right)^2. \tag{3.C.18}$$

Case 2 ($\xi_\ell > 0$): Here, we have $A_\ell = 1$.

To sum up, the case of $A_\ell = 1$ gives the trivial solution that $\{(D,P)\} \subseteq \Phi_{\mathsf{DP}}(P_{X_\mathrm{r}|X})$. Plugging (3.C.18) into (3.C.11a) and (3.C.11c) yields the set of the pairs (D', P') stated in theorem. This concludes the proof.

3.D Derivation of Extreme Points of $\Phi_{\mathsf{DP}}(R)$

If $P = 0$, we are under the first clause of $R(D,P)$ in (3.3.23). Here, we have:

$$R = \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{\lambda_\ell}{\Omega_\ell(D,P)}. \tag{3.D.1}$$

The equation (3.3.21b) together with $P = 0$ implies that $\hat{\lambda}^*_\ell(\nu_1,\nu_2) = \lambda_\ell$ for every $\ell \in \{1,\dots,L\}$. Also, (3.3.16) and (3.3.17) yield the following:

$$\gamma^*_\ell(\nu_1,\nu_2) = \frac{2\lambda_\ell}{1+\sqrt{1+16\nu_1^2\lambda_\ell^2}} := \omega^0_\ell, \tag{3.D.2}$$

where ν_1 is chosen such that

$$R = \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{\lambda_\ell}{\omega^0_\ell} \tag{3.D.3}$$

$$= \frac{1}{2}\sum_{\ell=1}^{L}\log\frac{1+\sqrt{1+16\nu_1^2\lambda_\ell^2}}{2}. \tag{3.D.4}$$

Moreover, we get

$$D = \sum_{\ell=1}^{L} \left[2\lambda_\ell - 2\sqrt{\lambda_\ell(\lambda_\ell - \omega_\ell^0)} \right] := D_3. \tag{3.D.5}$$

Now, for a large enough P, the second clause of $R(D, P)$ in (3.3.23) holds. Then, we have:

$$R = \frac{1}{2} \sum_{\ell=1}^{L} \log \frac{\lambda_\ell}{\delta_\ell}, \tag{3.D.6}$$

where δ_ℓ satisfies the following equation:

$$\sum_{\ell=1}^{L} [\lambda_\ell - \delta_\ell]^+ = \left[\sum_{\ell=1}^{L} \lambda_\ell - D_1 \right]^+. \tag{3.D.7}$$

This concludes the proof.

3.E Proof of Convexity of Program (3.A.40)

First, consider that the second-order derivative of the objective function (3.A.40a) with respect to γ_ℓ is $\frac{1}{2\gamma_\ell^2}$ which is a positive term. The second-order derivative of the constraint (3.A.40e) with respect to $\hat{\lambda}_\ell$ is $\frac{1}{2\hat{\lambda}_\ell^2}$ which is again a positive term. It just remains to study the constraint (3.A.40d). The Hessian matrix of LHS of this constraint is given as follows.

$$\begin{pmatrix} \frac{\sqrt{\lambda_\ell - \gamma_\ell}}{2\sqrt{\hat{\lambda}_\ell^3}} & \frac{1}{2\sqrt{\hat{\lambda}_\ell(\lambda_\ell - \gamma_\ell)}} \\ \frac{1}{2\sqrt{\hat{\lambda}_\ell(\lambda_\ell - \gamma_\ell)}} & \frac{\sqrt{\hat{\lambda}_\ell}}{2\sqrt{(\lambda_\ell - \gamma_\ell)^3}} \end{pmatrix} \tag{3.E.1}$$

The determinant of the above matrix is zero and thus it is a positive semi-definite matrix which implies the convexity of the underlying function. This concludes the proof.

3.F Proof of Monotonicity

First, notice that the function $\mathsf{P}_\ell\left(\lambda_\ell\left(\frac{1+\nu\sqrt{1-\frac{\bar{\gamma}_\ell}{\lambda_\ell}}}{1+\nu}\right)^2\right)$ is an increasing function of $\bar{\gamma}_\ell$. Now, it remains to study the function $\mathsf{D}_\ell(.,.)$. With some simple calculations, it can be shown that

$$\mathsf{D}_\ell\left(\bar{\gamma}_\ell, \lambda_\ell\left(\frac{1+\nu\sqrt{1-\frac{\bar{\gamma}_\ell}{\lambda_\ell}}}{1+\nu}\right)^2\right) = \bar{\gamma}_\ell + \left(\frac{\sqrt{\lambda_\ell-\bar{\gamma}_\ell}-\sqrt{\lambda_\ell}}{1+\nu}\right)^2. \tag{3.F.1}$$

If we take the derivative of the above term with respect to $\bar{\gamma}_\ell$, we have:

$$\frac{\nu^2+2\nu}{(1+\nu)^2} + \frac{\sqrt{\lambda_\ell}}{(1+\nu)^2\sqrt{\lambda_\ell-\bar{\gamma}_\ell}}, \tag{3.F.2}$$

which is a nonnegative term. Thus, the function $\mathsf{D}_\ell(.,.)$ is an increasing function of $\bar{\gamma}_\ell$.

Chapter 4

Conclusion and Future Work

3.1 Conclusion

In this thesis, we reviewed the notion of perception loss, which emerges naturally during the training of deep learning models. We then considered the rate-distortion-perception (RDP) function, which serves as a benchmark in assessing the performance of image compression systems trained through deep learning. In addition, we reviewed the notion of universal encoded representations, where the same compressed representation could be used to simultaneously achieve different operating points on the distortion-perception tradeoff curve. We endowed the information-theoretic definitions with operational meanings by proving coding theorems in both one-shot and asymptotic settings, then provided thorough analyses for both the binary and Gaussian sources.

For binary sources, we employed the Hamming distance as the distortion measure and the TV distance as the metric to evaluate the perception quality of the reconstructed output. Firstly, we derive an analytical expression for the rate-distortion-perception tradeoff in the one-shot setting. Then, a complete characterization of the achievable distortion-perception region for an arbitrary representation was obtained. Furthermore, we derived upper and lower bounds on the minimum rate penalty in the universal representation setting, where the encoder is designed to be applicable across different scenarios without the need for customization. Lastly, we delved into the concept of successive refinement in the context of perception-constrained lossy compression for both point-wise and set-wise settings and provided a necessary and sufficient condition for point-wise successive refinement and a sufficient condition for the successive refinability of universal representation.

For vector Gaussian sources, we provided a complete characterization of the rate-distortion-perception function, which extends its scalar counterpart in [56]. The distortion considered was the MSE loss and the perception quality was measured by KL divergence or Wasserstein-2 distance. Our finding demonstrated that when aiming for a high perceptual quality, every component of the reconstructed output, including high-frequency components, needs to retain a strong correlation with the corresponding component of the source. This observation should be contrasted with the traditional water-filling solution, which allows some components of the reconstruction to be uncorrelated with those of the source. Furthermore, we explored the concept of universal representation, where the encoder remained fixed while the decoder was adapted to accommodate varying distortion-perception requirements. We characterized the achievable distortion-perception region for a fixed representation and demonstrated that the distortion-perception tradeoff achieved by universal representations can be nearly optimal.

Theoretical discoveries presented in this thesis greatly enrich the emerging rate-distortion-perception theory, laying a strong groundwork for the advancement of lossy image compression techniques.

3.2 Future Work

There are still untapped opportunities for future research to advance and refine the proposed approaches, leading to further improvements.

Some technical issues still remain unresolved. While at some places we have provided numerical verification, it is crucial to supplement it with more rigorous mathematical proofs. This can include addressing the conditions mentioned in Remark 2.4.2, providing a monotonicity proof for $\Phi_{\mathsf{DP}}^{\mathrm{lower}}$ and theoretically proving Corollary 3.3.2.1. It is essential to further refine and strengthen the mathematical aspects of these three parts.

We employed KL divergence as a perceptual metric in our study. However, it is important to note that KL divergence possesses the property of asymmetry, which may limit its suitability as a perception measurement. Despite this limitation, KL divergence remains a valuable measurement in the field of Information Theory and serves a significant role. To address the broader issue, further exploration is needed to determine an appropriate perceptual measure that more accurately approximates human judgment.

It is crucial to supplement our theoretical findings with experimental results on real datasets to validate our claims. Additionally, exploring universal representations for the binary case under a real-valued reconstruction set is an intriguing area of interest. Relaxing the binary alphabet constraint raises the question of whether similar phenomena occur in the general case and whether it leads to the discovery of "free" universal representations. Therefore, conducting further investigations in this direction can yield valuable insights and contribute to the advancement of the field.

Appendix A

Appendices

A.1 Partial Explanation for Corollary 3.3.2.1

For arbitrary distribution with expectation μ_X and variance σ_X^2, assume $P_3 = 0$ (i.e., $P_{\hat{X}_{D_3,P_3}} = P_X$), it follows that

$$D_3 = \mathbb{E}\left[\left\|X - \hat{X}_{D_3,P_3}\right\|^2\right] = 2\sigma_X^2 - 2\mathbb{E}\left[(X-\mu_X)^T\left(\hat{X}_{D_3,P_3} - \mu_X\right)\right]. \qquad \text{(A.1.1)}$$

Note that $I\left(X; \mathbb{E}\left[X \mid \hat{X}_{D_3,P_3}\right]\right) \leq I\left(X; \hat{X}_{D_3,P_3}\right) = R\left(D_1, \infty\right)$, which implies $\mathbb{E}[\|X - \mathbb{E}\left[X \mid \hat{X}_{D_3,P_3}\right]\|^2] \geq D_1$. Let $c = \frac{2\sigma_X^2 - D_3}{2\sigma_X^2}$. We have

$$\begin{aligned}
D_1 &\leq \mathbb{E}\left[\left\|X - \mathbb{E}\left[X \mid \hat{X}_{D_3,P_3}\right]\right\|^2\right] \\
&\leq \mathbb{E}\left[\left\|X - \mu_X - c\left(\hat{X}_{D_3,P_3} - \mu_X\right)\right\|^2\right] \\
&= \left(1 + c^2\right)\sigma_X^2 - 2c\mathbb{E}\left[(X-\mu_X)^T\left(\hat{X}_{D_3,P_3} - \mu_X\right)\right] \\
&\stackrel{\text{(a)}}{=} \frac{4\sigma_X^2 D_3 - D_3^2}{4\sigma_X^2},
\end{aligned}$$

where (a) is due to (A.1.1). So

$$D_3 \geq 2\sigma_X^2 - 2\sigma_X\sqrt{\sigma_X^2 - D_1}$$

which together with the fact that $D^{(b)} \leq 2D_1$ implies

$$\begin{aligned} &D^{(b)} - D_3 \leq 2D_1 - 2\sigma_X^2 + 2\sigma_X\sqrt{\sigma_X^2 - D_1}, \\ &\frac{D^{(b)}}{D_3} \leq \frac{D_1}{\sigma_X^2 - \sigma_X\sqrt{\sigma_X^2 - D_1}}, \end{aligned}$$

where $\left(D^{(a)}, P^{(a)}\right)$ and $\left(D^{(b)}, P^{(b)}\right)$ are the two extreme points, corresponding to upper-left and lower-right. It is easy to verify that

$$\begin{aligned} &\frac{1}{2}\sigma_X^2 \geq 2D_1 - 2\sigma_X^2 + 2\sigma_X\sqrt{\sigma_X^2 - D_1} \overset{D_1 \approx 0 \text{ or } \sigma_X^2}{\approx} 0, \\ &2 \geq \frac{D_1}{\sigma_X^2 - \sigma_X\sqrt{\sigma_X^2 - D_1}} \overset{D_1 \approx \sigma_X^2}{\approx} 1. \end{aligned}$$

A similar argument can be used to bound the gap between (D_1, P_1) and the upper-left extreme point $\left(\tilde{D}^{(a)}, \tilde{P}^{(a)}\right)$. Note that

$$\tilde{D}^{(a)} = \mathbb{E}\left[\left\|X - \mathbb{E}\left[X \mid \hat{X}_{D_3,P_3}\right]\right\|^2\right] \leq \frac{4\sigma_X^2 D_3 - D_3^2}{4\sigma_X^2}$$

which together with the fact that $D_1 \geq \frac{1}{2}D_3$ implies

$$\begin{aligned} &\tilde{D}^{(a)} - D_1 \leq \frac{1}{2}D_3 - \frac{D_3^2}{4\sigma_X^2}, \\ &\frac{\tilde{D}^{(a)}}{D_1} \leq 2 - \frac{D_3}{2\sigma_X^2}. \end{aligned}$$

Finally, this implies

$$\frac{1}{4}\sigma_X^2 \geq \frac{1}{2}D_3 - \frac{D_3^2}{4\sigma_X^2} \overset{D_3\approx 0 \text{ or } 2\sigma_X^2}{\approx} 0,$$
$$2 \geq 2 - \frac{D_3}{2\sigma_X^2} \overset{D_3\approx 2\sigma_X^2}{\approx} 1.$$

A.2 Functional Representation Lemma

Lemma A.2.1 (Functional representation lemma [13]) *For any pair of random variables $(X, \hat{X})$, there exists random variable U, independent of X, such that $\hat{X}$ can be expressed as a deterministic function of (X, U).*

A.3 Strong Functional Representation Lemma

Lemma A.3.1 (Strong Functional Representation Lemma [22]) *For general (X, Z), their exists a U such that $H(Z|U)$ is close to $I(X;Z)$. Specifically, we can strengthen the functional representation lemma to show that for any X and Z, there exists a U independent of X such that Z is a function of X and U, and*

$$H(Z \mid U) \leq I(X;Z) + \log(I(X;Z) + 1) + 4.$$

A.4 Proof of $R(\Theta_{\mathsf{DP}}) \leq R_{\textbf{1-shot}}(\Theta_{\mathsf{DP}}) \leq R(\Theta_{\mathsf{DP}}) + \log(R(\Theta_{\mathsf{DP}}) + 1) + 5$

Proof: First, we prove $R_{\text{1-shot}}(\Theta_{\mathsf{DP}}) \leq R(\Theta_{\mathsf{DP}}) + \log(R(\Theta_{\mathsf{DP}}) + 1) + 5$. Let Z be jointly distributed with X such that for any $(D, P) \in \Theta_{\mathsf{DP}}$, there exists $p_{\hat{X}_{D,P}|Z}$ satisfying $\mathbb{E}[\|X - \hat{X}_{D,P}\|^2] \leq D$ and $\phi(P_X, P_{\hat{X}_{D,P}}) \leq P$. It follows by the strong functional representation lemma A.3.1 that there exist a random variable U, independent of X, and a deterministic function g such that $Z = g(X, U)$ and $H(Z \mid U) \leq I(X; Z) + \log(I(X; Z) + 1) + 4$. So with U available at both the encoder and the decoder, we can use a class of prefix-free binary codes whose expected codeword length no greater than $H(Z \mid U) + 1$ to lossless represent Z. Now it suffices for the decoder to simulate $p_{\hat{X}_{D,P}|Z}$. Specifically, it follows by the functional representation lemma that there exists a random variable $U_{D,P}$, independent of (X, U), and a deterministic function $\psi_{D,P}$ such that $\hat{X}_{D,P} = \psi_{D,P}(Z, U_{D,P})$. Note that U and $U_{D,P}$ can be extracted from random seed.

Then we prove $R(\Theta_{\mathsf{DP}}) \leq R_{\text{1-shot}}$. For any random variable Q, encoding function $f_Q : \mathcal{X} \to \mathcal{C}_Q$, and decoding functions $g_{Q,D,P} : \mathcal{C}_Q \to \hat{\mathcal{X}}, (D, P) \in \Theta_{\mathsf{DP}}$ satisfying $\mathbb{E}[\|X - \hat{X}_{D,P}\|^2] \leq D$ and $\phi(P_X, P_{\hat{X}_{D,P}}) \leq P$, we have

$$\begin{aligned}
\mathbb{E}\left[\ell\left(f_Q(X)\right)\right] &\geq H\left(f_Q(X) \mid Q\right) \\
&\geq I\left(X; f_Q(X) \mid Q\right) \\
&= I\left(X; f_Q(X), Q\right) \\
&\geq R(\Theta_{\mathsf{DP}}),
\end{aligned}$$

where the last inequality follows by defining $(f_Q(X), Q)$ as Z, which satisfies the conditions in the definition of $R(\Theta_{\mathsf{DP}})$. Therefore, we have

$$R(\Theta_{\mathsf{DP}}) \leq R_{\text{1-shot}}(\Theta_{\mathsf{DP}}) \leq R(\Theta_{\mathsf{DP}}) + \log(R(\Theta_{\mathsf{DP}}) + 1) + 5.$$

□

A.5 Proof of $R_\infty(\Theta_{\mathsf{DP}}) = R(\Theta_{\mathsf{DP}})$

Proof: We first show achievability. By assumption, there exists $p_{Z|X}$ and $p_{\hat{X}_{D,P}|Z}$ such that $\mathbb{E}[\|X - \hat{X}_{D,P}\|^2] \leq D$ and $\phi(P_X, P_{\hat{X}_{D,P}}) \leq P$ for each $(D,P) \in \Theta_{\mathsf{DP}}$. Encode according to the product measure

$$p_{Z^n|X^n} = \prod_{i=1}^{n} p_{Z_i|X_i},$$

where $p_{Z_i|X_i} = p_{Z|X}$ for each i. By the strong functional representation lemma, there exists a random variable U independent of X^n and function $\phi^{(n)}(\cdot,\cdot)$ such that $Z^n = \phi^{(n)}(X^n, U)$, with

$$\begin{aligned}
\frac{H(Z^n \mid U)}{n} &\leq \frac{I(X^n; Z^n) + \log(I(X^n; Z^n) + 1) + 4}{n} \\
&= \frac{nI(X;Z) + \log(nI(X;Z) + 1) + 4}{n} \\
&\to I(X;Z) \text{ as } n \to \infty.
\end{aligned}$$

For the converse, we have

$$
\begin{aligned}
H\left(f_U^{(n)}\left(X^n\right) \mid U\right) &\geq I\left(f_U^{(n)}\left(X^n\right); X^n \mid U\right) \\
&= I\left(f_U^{(n)}\left(X^n\right); X^n \mid U\right) + I\left(X^n; U\right) \\
&= I\left(f_U^{(n)}\left(X^n\right), U; X^n\right) \\
&= \sum_{i=1}^{n} I\left(f_U^{(n)}\left(X^n\right), U; X_i \mid X^{i-1}\right) \\
&= \sum_{i=1}^{n} I\left(f_U^{(n)}\left(X^n\right), U; X_i \mid X^{i-1}\right) + I\left(X_i; X^{i-1}\right) \\
&= \sum_{i=1}^{n} I\left(f_U^{(n)}\left(X^n\right), U, X^{i-1}; X_i\right) \\
&\geq \sum_{i=1}^{n} I\left(f_U^{(n)}\left(X^n\right), U; X_i\right) \\
&\geq nR(\Theta_{\mathsf{DP}})
\end{aligned}
$$

where the final inequality follows because each reconstruction $\hat{X}_{D,P,i}$ is a function of $\left(f_U^{(n)}\left(X^n\right), U\right)$ so the data processing inequality can be applied. This establishes the converse. Therefore, we have

$$
R_\infty(\Theta_{\mathsf{DP}}) = R(\Theta_{\mathsf{DP}}).
$$

□

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