Comp Eng 3KB3

Solutions

Dr. Bandler

20783

2/1 pg 5			
2/10 pg 11			
3/12 pg 22			
3/15 pg 26			
3/22 pg 42			
3/26 pg 63			
4/1 pg 67			
4/4 pg 78			

[redacted] D

[redacted] COPRESS

[redacted] W 25070 J
25071 N
25072 B
E 25073 R
25074 C
25075 V
25078 R
25079 R

[redacted] ANADA INC.
DALE, ONTARIO

LED
FIBRE

*SIGNIFIE 75 % FIBRES RECYCLÉES, 25 % DÉCHETS DE CONSOMMATION

F PRODUCTS
LED

AUTRES PRODUITS: 25 % FIBRES RECYCLÉES

TA 2010 - Co Eng

3KB3

Class Test # 1

Jan. 30, 1996

(Duration of Test: 30 minutes)

1) Expand $f(\underset{\sim}{x})$ around $\underset{\sim}{x}^o$ into the Taylor series.

2) State the formulas for

ⓐ the first order change of $f(\underset{\sim}{x})$.

ⓑ the second order change of $f(\underset{\sim}{x})$.

3) Given a quadratic function

$$U(\underset{\sim}{\varphi}) = \varphi_1^2 + 2\varphi_1\varphi_2 + 2\varphi_2^2 - 3\varphi_1$$

let $\underset{\sim}{\varphi}^o = [1 \quad 1]^T$

ⓐ Calculate $U(\underset{\sim}{\varphi}^o)$

ⓑ Evaluate $\underset{\sim}{\nabla} U$ at $\underset{\sim}{\varphi}^o$

ⓒ Evaluate the Hessian matrix $\underset{\sim}{H}$ at $\underset{\sim}{\varphi}^o$

ⓓ Using the results obtained in ⓐ ⓑ ⓒ estimate the value of U at $\varphi^1 = [1.2 \quad 0.8]^T$

ⓔ How good is the estimate obtained in ⓓ

ⓕ formulate a first order function that approximates U in the region surrounding $\underset{\sim}{\varphi}^o$.

ⓖ What is the extreme value of U? is it the maximum or minimum

ⓗ Sketch a few contours of U.

ⓘ Apply the method of Lagrange multipliers to solve the minimization problem:

minimize $U(\underset{\sim}{\varphi})$

subject to $\varphi_1 = 2$

3KB3 Test #1 Jan. 30, 1996.

Solution:

1) $f(\underset{\sim}{x}) = f(\underset{\sim}{x}^0) + \Delta \underset{\sim}{x}^T \underset{\sim}{\nabla} f(\underset{\sim}{x}^0) + \frac{1}{2} \Delta \underset{\sim}{x}^T \underset{\sim}{H} \Delta \underset{\sim}{x} + \cdots$

$\hookrightarrow$ H refer to $\underset{\sim}{H}(\underset{\sim}{x}_0)$

2) (a) $\Delta \underset{\sim}{x}^T \underset{\sim}{\nabla} f(\underset{\sim}{x}^0)$ (b) $\frac{1}{2} \Delta \underset{\sim}{x}^T \underset{\sim}{H} \Delta \underset{\sim}{x}$

3) (a) $U(\underset{\sim}{\varphi}^0) = 1^2 + 2\cdot 1\cdot 1 + 2\cdot 1^2 - 3\cdot 1 = 2$

(b) $\underset{\sim}{\nabla} U = \begin{bmatrix} \frac{\partial U}{\partial \varphi_1} \\ \frac{\partial U}{\partial \varphi_2} \end{bmatrix} = \begin{bmatrix} 2\varphi_1 + 2\varphi_2 - 3 \\ 2\varphi_1 + 4\varphi_2 \end{bmatrix}$

$\underset{\sim}{\nabla} U(\underset{\sim}{\varphi}^0) = \begin{bmatrix} 2+2-3 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

(c) $\underset{\sim}{H} = \begin{bmatrix} \frac{\partial^2 U}{\partial \varphi_1^2} & \frac{\partial^2 U}{\partial \varphi_1 \partial \varphi_2} \\ \frac{\partial^2 U}{\partial \varphi_1 \partial \varphi_2} & \frac{\partial^2 U}{\partial \varphi_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$

(d) $U(\underset{\sim}{\varphi}') = U(\underset{\sim}{\varphi}^0 + \Delta \underset{\sim}{\varphi})$ $\quad \Delta \underset{\sim}{\varphi} = [0.2 \ \ -0.2]^T$

$= U(\underset{\sim}{\varphi}^0) + \Delta \underset{\sim}{\varphi}^T \underset{\sim}{\nabla} U(\underset{\sim}{\varphi}^0) + \frac{1}{2} \Delta \underset{\sim}{\varphi}^T \underset{\sim}{H} \Delta \underset{\sim}{\varphi}$

$= 2 + [0.2 \ \ -0.2] \begin{bmatrix} 1 \\ 6 \end{bmatrix} + \frac{1}{2} [0.2 \ \ -0.2] \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}$

$= 2 + (-1) + \frac{1}{2} \cdot (0.08) = 1.04$

(e) Because U is a second order function, the estimate in (d) is exact.

(f) Assume the function is $V(\underset{\sim}{\varphi})$ $\quad \underset{\sim}{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$

$$V(\underset{\sim}{\varphi}) = U(\underset{\sim}{\varphi}^0) + (\underset{\sim}{\varphi} - \underset{\sim}{\varphi}^0)^T \underset{\sim}{\nabla} U(\underset{\sim}{\varphi}^0)$$

$$= 2 + \left(\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)^T \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$= 2 + \begin{bmatrix} \varphi_1 - 1 & \varphi_2 - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$= 2 + \varphi_1 - 1 + 6\varphi_2 - 6 = \varphi_1 + 6\varphi_2 - 5$$

(g) At extreme

$$\underset{\sim}{\nabla} U(\underset{\sim}{\varphi}) = \underset{\sim}{0}$$

$$\begin{bmatrix} 2\varphi_1 + 2\varphi_2 - 3 \\ 2\varphi_1 + 4\varphi_2 \end{bmatrix} = \underset{\sim}{0}$$

$$\varphi_{extreme} = \begin{bmatrix} 3 \\ -1.5 \end{bmatrix}$$

$$\because \underset{\sim}{H} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 8 - 4 = 4 > 0$$

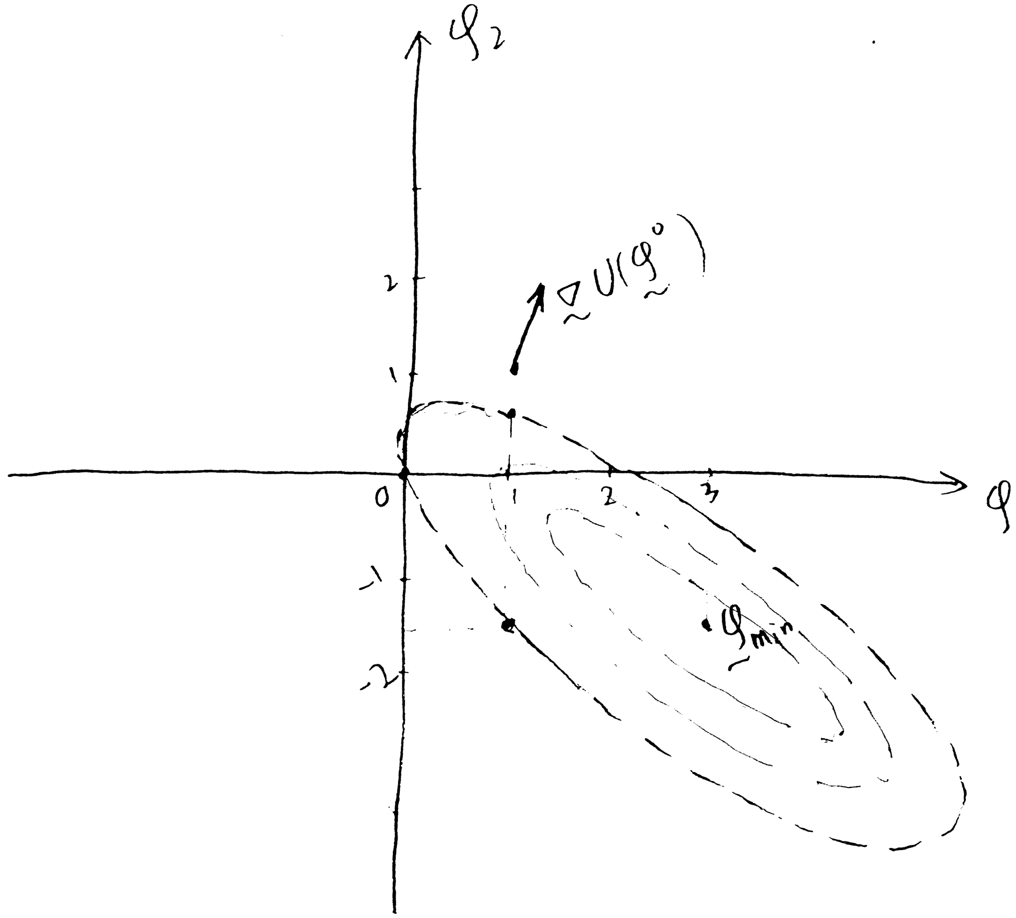
$\underset{\sim}{H}$ is positive definite at $\varphi_{extreme}$.

So $\varphi_{extreme}$ is a minimum.

(h)

For the $U=0$ contour, we first locate some points on the Contour of $U=0$

$$(0,0) \quad \left(1, \frac{-1+\sqrt{5}}{2}\right) \quad \left(1 \quad \frac{-1-\sqrt{5}}{2}\right)$$



ⓘ The constraint $\phi_1 = 2$

can be ~~first~~ rewritten in the form $h = 0$

$$h = \phi_1 - 2 = 0$$

$$L(\phi, \lambda) = U(\phi) + \lambda h(\phi)$$

$$= \varphi_1^2 + 2\varphi_1\varphi_2 + 2\varphi_2^2 - 3\varphi_1 + \lambda\varphi_1 - 2\lambda$$

$$\nabla L(\varphi, \lambda) = \begin{bmatrix} \nabla U(\varphi) + \lambda \nabla h(\varphi) \\ \dfrac{\partial L(\varphi, \lambda)}{\partial \lambda} \end{bmatrix}$$

At the minimum

$$\underset{\sim}{\nabla} L(\underset{\sim}{\varphi}, \lambda) = 0$$

i.e.
$$\underset{\sim}{\nabla} U(\underset{\sim}{\varphi}) + \lambda \underset{\sim}{\nabla} h(\underset{\sim}{\varphi}) = 0$$
$$h(\underset{\sim}{\varphi}) = 0$$

$$\begin{cases} \begin{bmatrix} 2\varphi_1 + 2\varphi_2 - 3 \\ 2\varphi_1 + 4\varphi_2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \\ \phi_1 = 2 \end{cases}$$

$$\begin{cases} 2\varphi_1 + 2\varphi_2 - 3 + \lambda = 0 \\ 2\varphi_1 + 4\varphi_2 = 0 \\ \phi_1 = 2 \end{cases}$$

$$\Downarrow$$

Solution $\begin{cases} \varphi_1 = 2 \\ \varphi_2 = -1 \end{cases}$

$\therefore$ the minimum of $U(\varphi)$ s.t. $\phi_1 = 2$ is $[2, -1]^T$.

COMPUTER ENGINEERING 3KB3

Class Test #2 Feb. 16, 1996

(Duration of Test : 30 min)

Problem #1

Consider three functions below

$$f_1(\phi_1, \phi_2) = \phi_2 - 3$$

$$f_2(\phi_1, \phi_2) = -\phi_1 + 2$$

$$f_3(\phi_1, \phi_2) = \phi_1 - \phi_2 - 1$$

① Sketch two or three minimax contours of $f_1(\phi_1, \phi_2)$, $f_2(\phi_1, \phi_2)$ and $f_3(\phi_1, \phi_2)$.

② Calculate the first order derivatives of f_1, f_2 and f_3.

③ Find the active functions at the point $[0 \;\; 1]^T$.

④ Verify that $[0 \;\; 1]^T$ is not the minimax solution.

Problem #2

Consider the following constrained minimization problem:

Minimize w.r.t. ϕ

$$U = \phi_1 + 2\phi_2$$

s.t.

$$g_1 = \phi_1 + \phi_2 - 2 \geq 0$$

$$g_2 = -\phi_1 + 2\phi_2 + 2 \geq 0$$

① sketch two or three contours of the objective function, plot the constraint boundaries and indicate the feasible region.

② Invoke the Kuhn-Tucker necessary conditions to test the point $[2 \;\; 0]^T$.

Solution to Classroom Test # 2

Problem # 1

For contour of $M=0$

1. Let $f_1 = 0$, $\phi_2 - 3 = 0$

$$\Rightarrow \phi_2 = 3.$$

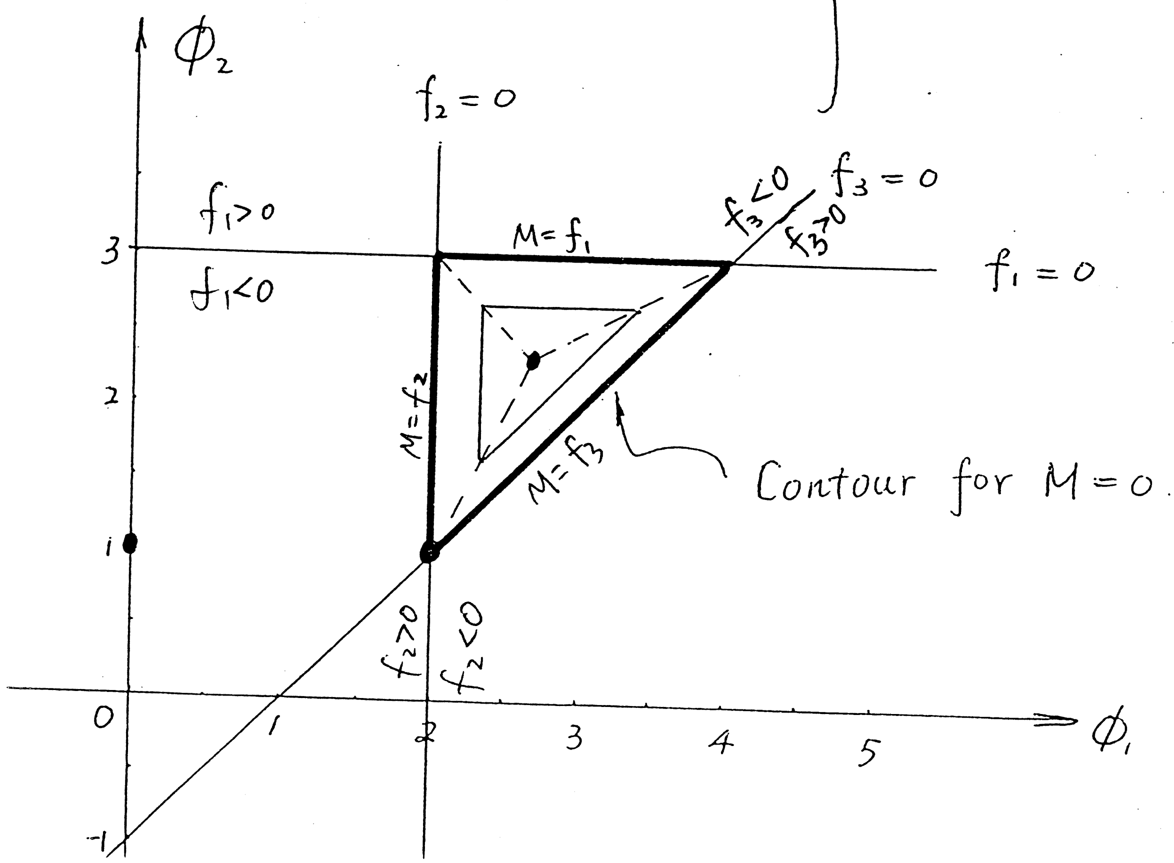
Let $f_2 = 0$, $-\phi_1 + 2 = 0$

$$\Rightarrow \phi_1 = 2.$$

Let $f_3 = 0$, $\phi_1 - \phi_2 - 1 = 0$.

$$\Rightarrow \phi_2 = \phi_1 - 1.$$

$M = \max\{f_1, f_2, f_3\}$



2. $\underset{\sim}{\nabla} f_1 = \begin{bmatrix} \frac{\partial f_1}{\partial \phi_1} \\ \frac{\partial f_1}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\underset{\sim}{\nabla} f_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\underset{\sim}{\nabla} f_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3. At $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$f_1 = 1 - 3 = -2$, $f_2 = -0 + 2 = 2$

$f_3 = 0 - 1 - 1 = -2$.

$M = 2$

$\therefore$ f_2 is the active function.

4. At minimax optimum, the following conditions must hold:

$$\begin{cases} \sum_{i=1}^{3} u_i \underset{\sim}{\nabla} f_i = \underset{\sim}{0} & (1) \\ \sum_{i=1}^{3} u_i = 1 & (2) \\ u_i (M - f_i) = 0 & (3) \\ u_i \geq 0 & (4) \end{cases}$$

At $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$M = 2$.

According to condition (3).

$u_1 = u_3 = 0$.

Substitute this into (2), we have $u_2 = 1$.

Therefore,

$$u_2 \cdot \nabla f_2 = \nabla f_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq 0$$

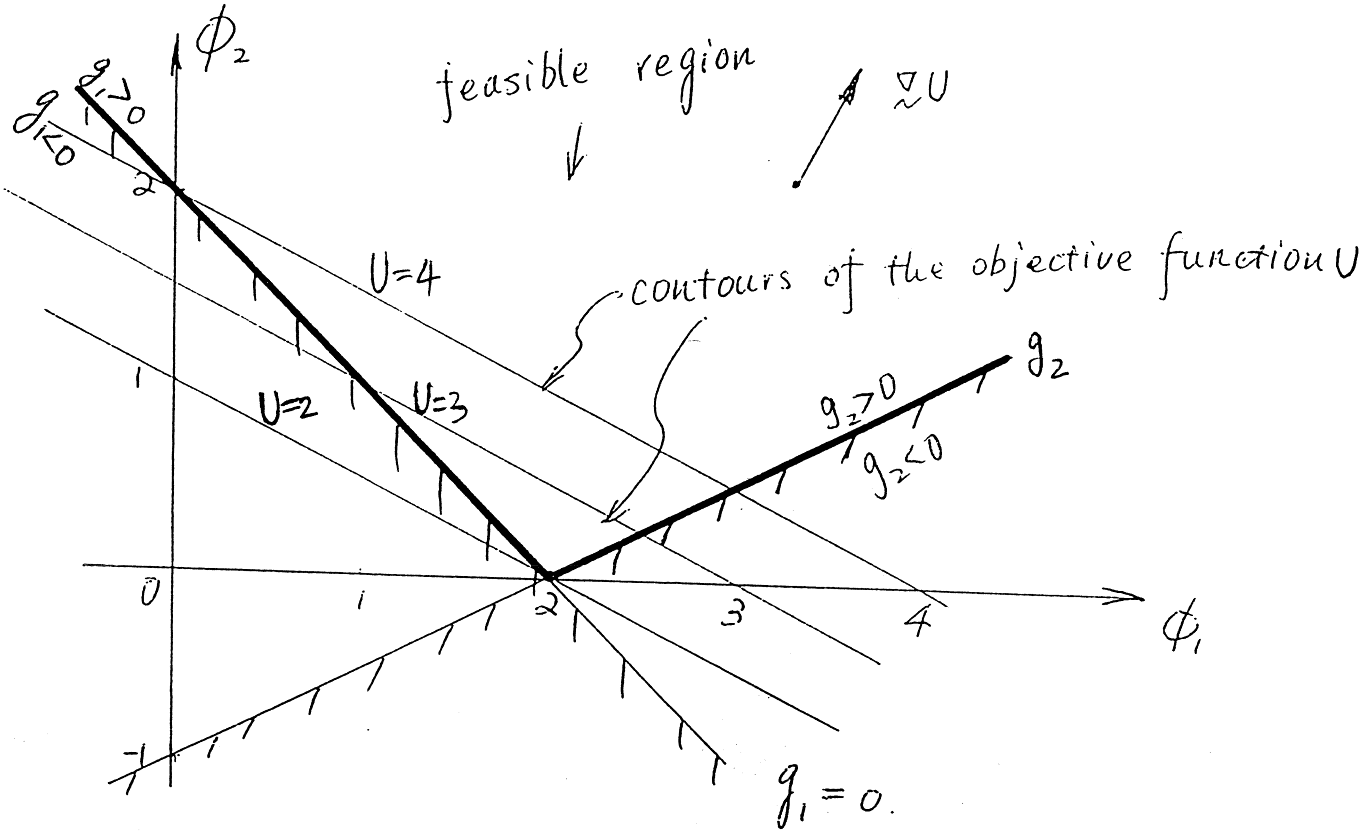
The necessary conditions for minimax optimum are violated. Point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not the minimax solution.

3

9

Problem #2

1.



2. The KT conditions are as follows,

$$\begin{cases} \nabla U = \sum_i u_i \nabla g_i & (1) \\ u_i g_i = 0 & (2) \\ u_i \geq 0 & (3) \end{cases}$$

At point $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\quad \begin{cases} g_1 = 2 + 0 - 2 = 0 \\ g_2 = -2 + 0 + 2 = 0. \end{cases}$

Both constraints are active.

$\underset{\sim}{\nabla} U = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\underset{\sim}{\nabla} g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underset{\sim}{\nabla} g_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Substitute the above into (1).

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 1 = u_1 - u_2 \\ 2 = u_1 + 2u_2 . \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = 4/3 \\ u_2 = 1/3 . \end{cases}$$

Point $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ <u>could be</u> a constrained minimum, since all three <u>necessary</u> conditions are satisfied.

CaEng 3KB3 TUT #3 960209

$$f_1(\phi_1, \phi_2) = \phi_2 - 3$$
$$f_2(\phi_1, \phi_2) = -\phi_1 + 2$$
$$f_3(\phi_1, \phi_2) = \phi_1 - \phi_2 - 1$$

Verify that $[2, 1]^T$ is not the minimax solution.

$$\sum_{i=1}^{3} u_i \nabla f_i = 0 \qquad (1)$$

$$\sum_{i=1}^{3} u_i = 1 \qquad (2)$$

$$u_i (M - f_i) = 0 \qquad (3) \qquad\qquad M = \max_i (f_i)$$

$$u_i \geq 0 \qquad (4)$$

$$\nabla f_1 = \begin{bmatrix} \frac{\partial f_1}{\partial \phi_1} \\ \frac{\partial f_1}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \nabla f_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad \nabla f_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

at $[2\ 1]^T$, $f_1 = -2$, $f_2 = 0$, $f_3 = 0$ $\Rightarrow M = 0$

Since $f_1 = -2$, f_1 is inactive $\Rightarrow u_1 = 0$

$$\therefore \quad u_2 \nabla f_2 + u_3 \nabla f_3 = 0$$

$$\Rightarrow u_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\Rightarrow u_3 = 0 \text{ and } u_2 = 0 \qquad \# \ (2)$$

$\therefore$ $[2\ 1]^T$ is <u>not</u> minimax optimal.

$U = \frac{1}{2}\Phi_1 + 2\Phi_2 + 3$

$g_1 = \Phi_1 + 2\Phi_2 - 3 \geq 0$

$g_2 = 2\Phi_1 + \Phi_2 - 3 \geq 0$

Is $[2\ 1]^T$ a constrained optimum?

KT conditions

$$\boxed{\begin{array}{ll} \nabla U = \sum_{i=1}^{k} u_i \nabla g_i & ① \\ u_i \geq 0, & i=1,\dots,k \\ u_i g_i = 0, & i=1,\dots,k \end{array}}$$

$$\nabla U = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

$$\nabla g_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\nabla g_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

at $[2\ 1]^T$, $g_1 = 1 \geq 0$

$g_2 = 2 \geq 0$

neither g_1 nor g_2 are active at this point

$\Rightarrow u_1$ and $u_2 = 0$ $\quad \not\equiv$ ①

$\therefore [2\ 1]^T$ is not a constrained optimum

$$U = 0.5\phi + 2$$

$$g = \phi \geq 0$$

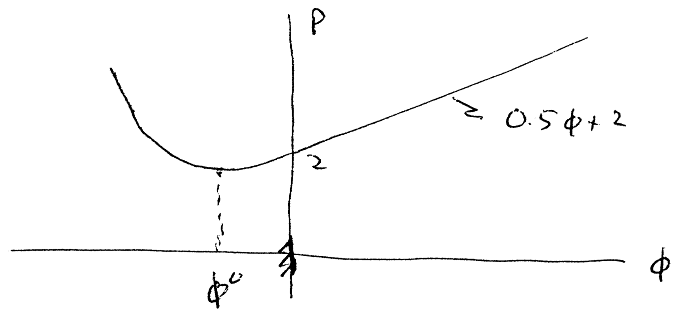
find the location of the minimum by using a penalty function.

$$P = 0.5\phi + 2 + w\phi^2$$

$$\boxed{\begin{array}{l} g_i < 0 \Rightarrow w_i > 0 \\ g_i \geq 0 \Rightarrow w_i = 0 \end{array}}$$

$$\frac{\partial P}{\partial \phi} = 0.5 + 2w\phi \stackrel{\wedge}{=} 0 \Rightarrow \phi^{\circ} = -\frac{1}{4w}$$

$$H = 2w > 0 \Rightarrow \phi^{\circ} \text{ is a minimum}$$



outside feasible region

To improve estimate,

$$\lim_{w \to \infty} \phi^{\circ} = 0 \quad \text{which is clear by the graph.}$$

$\therefore$ The minimum occurs at $\phi^{\circ} = 0$.

$$U = 0.5\phi + 2$$

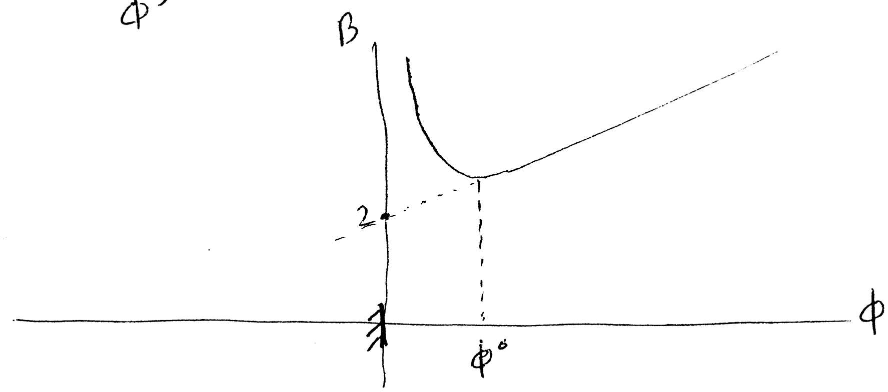
$$g = \phi \geq 0$$

find the location of the minimum by using a barrier fn.

$$B = 0.5\phi + 2 + r\frac{1}{\phi} \quad , \quad r > 0 \quad ; \quad \phi > 0$$

$$\frac{\partial B}{\partial \phi} = 0.5 - r\frac{1}{\phi^2} \stackrel{\wedge}{=} 0 \quad \Rightarrow \quad \phi^o = \sqrt{2r}$$

$$H = 2r\frac{1}{\phi^3} > 0 \quad \therefore \quad \phi^o \text{ is a minimum}$$



inside feasible region.

To improve estimate,

$$\lim_{r \to 0} \phi^o = \lim_{r \to 0} \sqrt{2r} = 0$$ which is clear by graph.

$\therefore$ The minimum occurs at $\phi^o = 0$.

minimize $\quad U = \phi_1^2 + 3\phi_2^2 - 4$

s.t. $\quad \phi_1 - \phi_2 + 1 = 0 \qquad$ by Lagrange Mult.

$$\therefore L = U(\underline{\phi}) + \lambda h(\underline{\phi})$$

$$= \phi_1^2 + 3\phi_2^2 - 4 + \lambda(\phi_1 - \phi_2 + 1)$$

① $\dfrac{\partial L}{\partial \phi_1} = 2\phi_1 + \lambda \triangleq 0$

② $\dfrac{\partial L}{\partial \phi_2} = 6\phi_2 - \lambda \triangleq 0$

③ $\dfrac{\partial L}{\partial \lambda} = \phi_1 - \phi_2 + 1 \triangleq 0$

① = ② $\Rightarrow \phi_1 = -3\phi_2$ ④

④ into ③ $\Rightarrow \phi_2 = \frac{1}{4}$ ⑤

⑤ into ④ $\Rightarrow \phi_1 = -\frac{3}{4}$ ⑥

⑤ ∧ ⑥ into ③ $\Rightarrow \lambda = \frac{3}{2}$ ⑦

$$\therefore \text{Soln } \underline{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$$

Pattern Search : minimize $U = \Phi_1^2 + 2\Phi_2^2 + \Phi_1\Phi_2 + 2\Phi_1 + 1$

wrt Φ starting from $\Phi^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\underline{\delta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\underline{\varepsilon} = \begin{bmatrix} .1 \\ .1 \end{bmatrix}$, $\beta = .25$, $\alpha = 2$

1: $U(2,2) = 21$ explore $\underline{\delta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$U(3,2) = 30$
$U(1,2) = 14$ *
$U(1,3) = 25$
$U(1,1) = 7$ * $\underline{\delta} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

2: $U(1,1) = 7$ pattern move $\underset{\sim}{\Delta} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ $\alpha = 2$ $U(-1,-1) = 3$

explore $\underline{\delta} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$U(-2, -1) = 5$
$U(0, -1) = 3$
$U(-1, -2) = 10$
$U(-1, 0) = 0$ * $\delta = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$

3: $U(-1,0) = 0$ pattern move $\underset{\sim}{\Delta} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ $\alpha = 2$ $U(-5,-2) = 34$

explore $\underline{\delta} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$U(-6, -2) = 45$
$U(-4, -2) = 25$
$U(-5, -3) = 49$
$U(-5, -1) = 23$

NO GOOD.

Going back to $(-1,0)$ $\delta = \begin{bmatrix} +1 \\ +1 \end{bmatrix}$

explore $\underline{\delta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$U(0,0) = 1$
$U(-2,0) = 1$
$U(-1,1) = 1$
$U(-1,-1) = 3$ $\delta = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

NO GOOD

reduce increment $\beta = .25$ $\underline{\delta} = \begin{bmatrix} -.25 \\ -.25 \end{bmatrix}$ $U = (-1.25, 0) = .06$

& explore

$(-.75, 0) = .06$
$(-1, -.25) = .3$
$\rightarrow (-1, .25) = -.1$

$\underline{\delta} = \begin{bmatrix} .25 \\ .25 \end{bmatrix}$

4: $U(-1, .25) = -.125$

etc until $\underline{\varepsilon}$ reached.

Simplex Method

recall centroid = average of (remaining) points $\bar{\Phi}$

reflection : $\phi_r = \bar{\Phi} + \alpha(\bar{\Phi} - \phi_n)$

expansion : $\phi_e = \bar{\Phi} + \gamma(\phi_r - \bar{\Phi})$

contraction : $\phi_c = \bar{\Phi} + \beta(\phi_h - \bar{\Phi})$ $\quad 0 < \beta < 1$

shrinking : $\phi_i \leftarrow 0.5(\phi_i + \phi_\ell)$ $\quad$ all $i \neq \ell$

Set $\alpha = 1$, $\gamma = 2$, $\beta = .5$

minimize $U = \phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_1 + 1$ wrt ϕ

starting from points $(1,1)$, $(1,0)$, $(0,0)$.

1: $U(1,1) = 7$ X

2: $U(1,0) = 4$ X

3: $U(0,0) = 1$

1: largest, centroid : $(.5, 0)$

$\phi_r = (0,-1) \Rightarrow U(0,-1) = 3$

4: $U(0,-1) = 3$ X

2: largest, centroid $(0, -.5)$

$\phi_r = (-1,-1) \Rightarrow U(-1,-1) = 3$

5: $U(-1,-1) = 3$

4: largest, centroid $(-.5, -.5)$

$\phi_r = (-1, 0) \Rightarrow U(-1,0) = 0$

smaller than smallest 3: $\therefore$ expand

$\phi_e = (-1.5, .5) \Rightarrow U(-1.5, .5) = 0$

6: $U(-1.5, .5) = 0$

5: largest, centroid $(-.75, .25)$

$\phi_r = (-.5, 1.5) \Rightarrow U(-.5, 1.5) = 4$

no good $\therefore$ contract

$\phi_c = (-.875, -.375) \Rightarrow U = .625$ OK

7: $U(-.875, -.375) = .625$

etc

Steepest Descent

Minimize $U = \phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_1 + 1$ wrt ϕ

from starting point $\phi^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Do 3 iterations.

recall $\phi^{j+1} = \phi^j + \alpha^j \nabla U^j$, $\alpha^j > 0$

First Iteration

$j = 0 \Rightarrow U^0 = 1$, the starting condition.

$j = 0$, $\phi^1 = \phi^0 - \alpha^0 \nabla U^0 = \begin{bmatrix} -2\alpha^0 \\ 0 \end{bmatrix}$

$\Rightarrow U^1 = (2\alpha^0 - 1)^2$

to minimize wrt α^0, $\alpha^0 = \frac{1}{2}$

$\therefore U^1 = 0$

Second Iteration

$j = 1 \Rightarrow U^1 = 0$

$\alpha^0 = \frac{1}{2} \Rightarrow \phi^1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$j = 1$, $\phi^2 = \phi^1 - \alpha^1 \nabla U^1 = \begin{bmatrix} -1 \\ \alpha \end{bmatrix}$

$\Rightarrow U^2 = 2(\alpha^1 - \frac{1}{4})^2 - \frac{1}{8}$

to minimize wrt α^1, $\alpha^1 = \frac{1}{4}$

$\therefore U^2 = -\frac{1}{8}$

Third Iteration

$j = 2 \Rightarrow u^2 = -\frac{1}{8}$

$\alpha^1 = \frac{1}{4} \Rightarrow \phi^2 = \begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix}$

$j = 2, \quad \phi^3 = \phi^2 - \alpha^2 \nabla u^2 = \begin{bmatrix} -\frac{1}{4}\alpha - 1 \\ \frac{1}{4} \end{bmatrix}$

$\Rightarrow u^3 = \frac{1}{16}\left(\alpha^2 - \frac{1}{2}\right)^2 - \frac{9}{64}$

to minimize wrt α^2, $\alpha^2 = \frac{1}{2}$

$\therefore u^3 = -\frac{9}{64}$ stop now.

finished with $\alpha^2 = \frac{1}{2} \Rightarrow \phi^3 = \begin{bmatrix} -\frac{9}{8} \\ \frac{1}{4} \end{bmatrix}$

Newton Method

Minimize $U = \phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_1 + 1$ wrt ϕ

from starting point $\phi^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, Do 3 iterations.

recall $\phi^{j+1} = \phi^j - \alpha^j H^{-1} \nabla U^j$

First Iteration

$j=0 \Rightarrow U^0 = 1$, the starting condition

$j=0$, $\phi^1 = \phi^0 - \alpha^0 H^{-1} \nabla U^0 = \begin{bmatrix} -\frac{8}{7}\alpha^0 \\ \frac{2}{7}\alpha^0 \end{bmatrix}$

$\Rightarrow U^1 = \frac{8}{7}(\alpha^0 - 1)^2 - \frac{1}{7}$

to minimize wrt α^0, $\alpha^0 = 1$

$\therefore U^1 = -\frac{1}{7}$

Second Iteration

$j=1 \Rightarrow U^1 = -\frac{1}{7}$

$j=1$, $\phi^2 = \phi^1 - \alpha^1 H^{-1} \nabla U^1 = \begin{bmatrix} -\frac{8}{7} \\ \frac{2}{7} \end{bmatrix}$

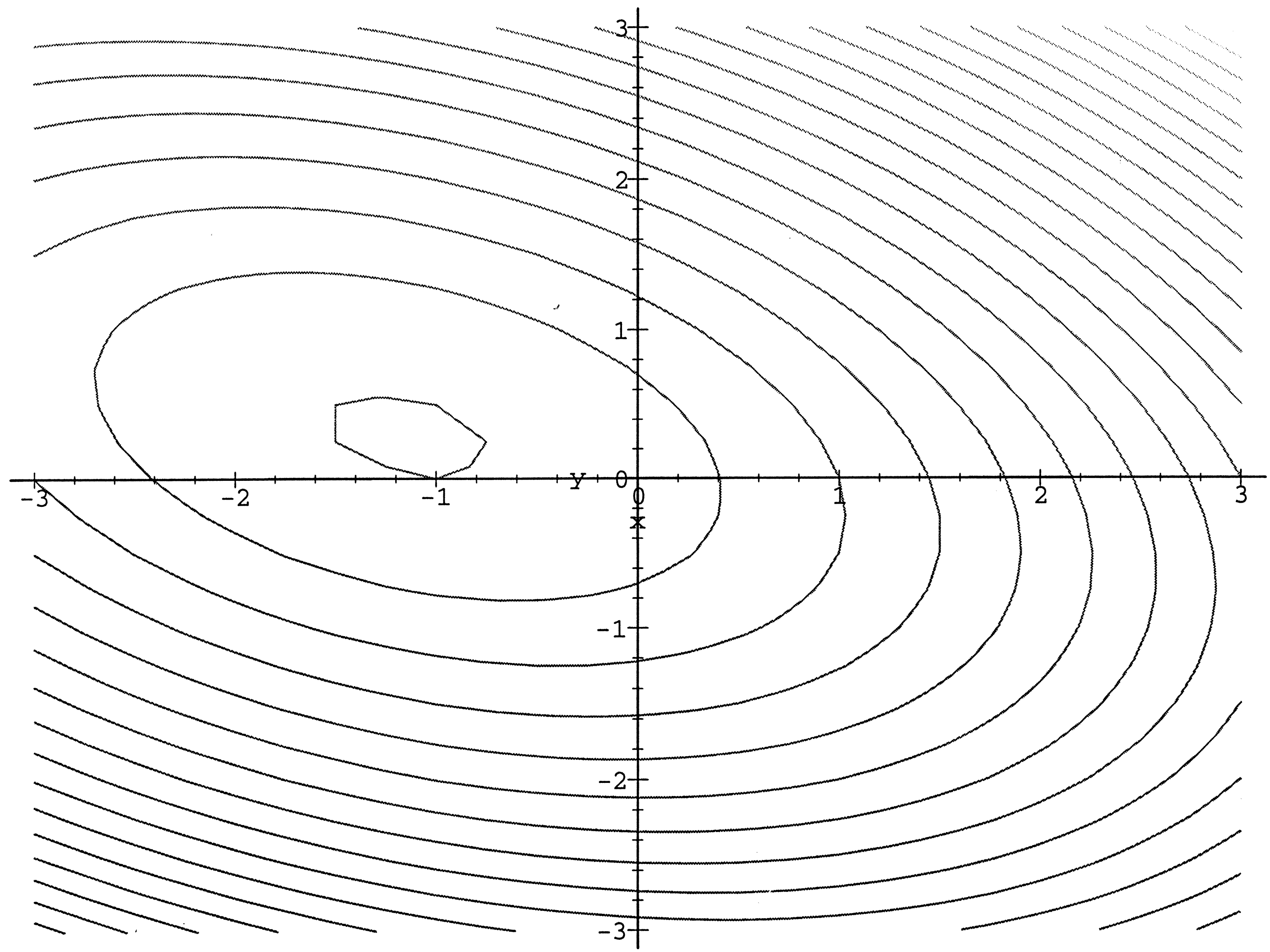
$\Rightarrow U^2 = -\frac{1}{7}$

Note :- no α

- same values of ϕ and U

$\phi^2 = \begin{bmatrix} -\frac{8}{7} \\ \frac{2}{7} \end{bmatrix}$, $U^2 = -\frac{1}{7}$

Newton's Method converges to minimum in one iteration for all quadratic functions. N...



3
2
1
0
-1
-2
-3
y
x
-3
-2
-1
0
1
2
3

Dr. Bandler March 15, 1996

CoE 3KB3 Class Test # 3 (Duration : 30 minutes)

Use only McMaster standard calculators.

Question 1.
For the function $U(\varphi_1,\varphi_2) = \varphi_1^2 + \varphi_2^2$,
from the current base point $\varphi^0 = [6,2]^T$, execute the pattern search algorithm to find three more base points or to termination.
Be sure to include the point $[\varphi_1,\varphi_2]^T$ and function U value for each base point.

Use the following parameter values:
reduction factor for exploratory increments $\beta = .5$
acceleration factor for pattern move $\alpha = 2$
vector containing exploratory increments $\delta = [2,1]^T$
minimum required exploratory increments:

$e_1 = 2$
$e_2 = 1$

Question 2.
Consider the function

$$U(\varphi_1,\varphi_2) = \begin{cases} 5, & \varphi_1 < 0 \\ 1, & 0 <= \varphi_1 < 5 \\ 2, & 5 <= \varphi_1 < 8 \\ 3, & 8 <= \varphi_1 < 10 \\ 4, & 10 <= \varphi_1 < 12 \\ 7, & 12 <= \varphi_1 < 15 \\ 6, & 15 <= \varphi_1 \end{cases}$$

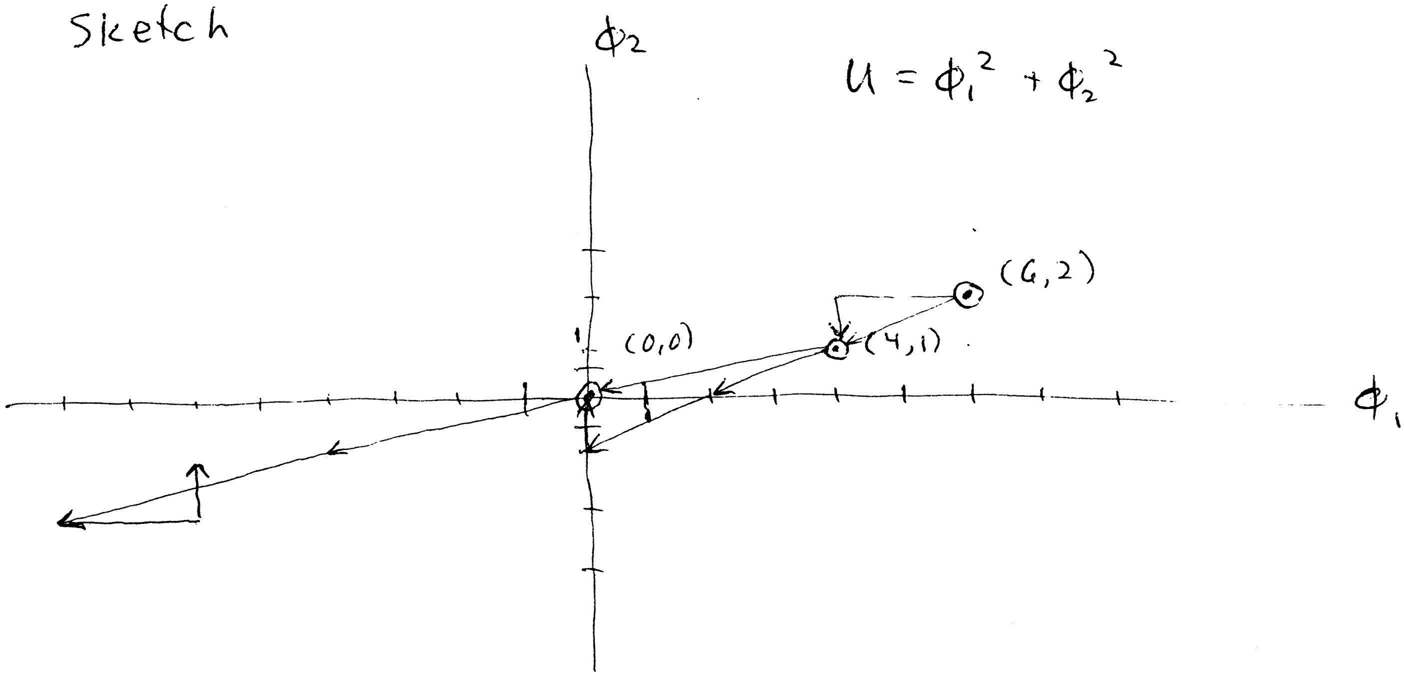
which is composed of constant vertical strips.
Execute the simplex method on U, starting from the simplex defined by the points (9,0), (13,4) and (17,0).
Stop when four more vertex points have been generated.
Be sure to include the points $[\varphi_1,\varphi_2]^T$ and function U values for each vertex.

Use the following parameter values:
The reflection coefficient α is 1.
The ~~extension~~ coefficient γ is 3. extension → expansion
The contraction coefficient β is .5 .
The shrinking coefficient is .5 .

Question 3.
For the function $U(\varphi_1,\varphi_2) = \varphi_1^2 + 2(\varphi_2^2)$ execute 1 iteration of the steepest descent algorithm, including the one-dimensional minimization. Start at the point (2,2).
Calculations need not be exact.
Be sure to include the points $[\varphi_1,\varphi_2]^T$ and function U values.

CoE Class Test #3 Solutions

#1 Sketch



Base Points : Values :

(6,2) 40

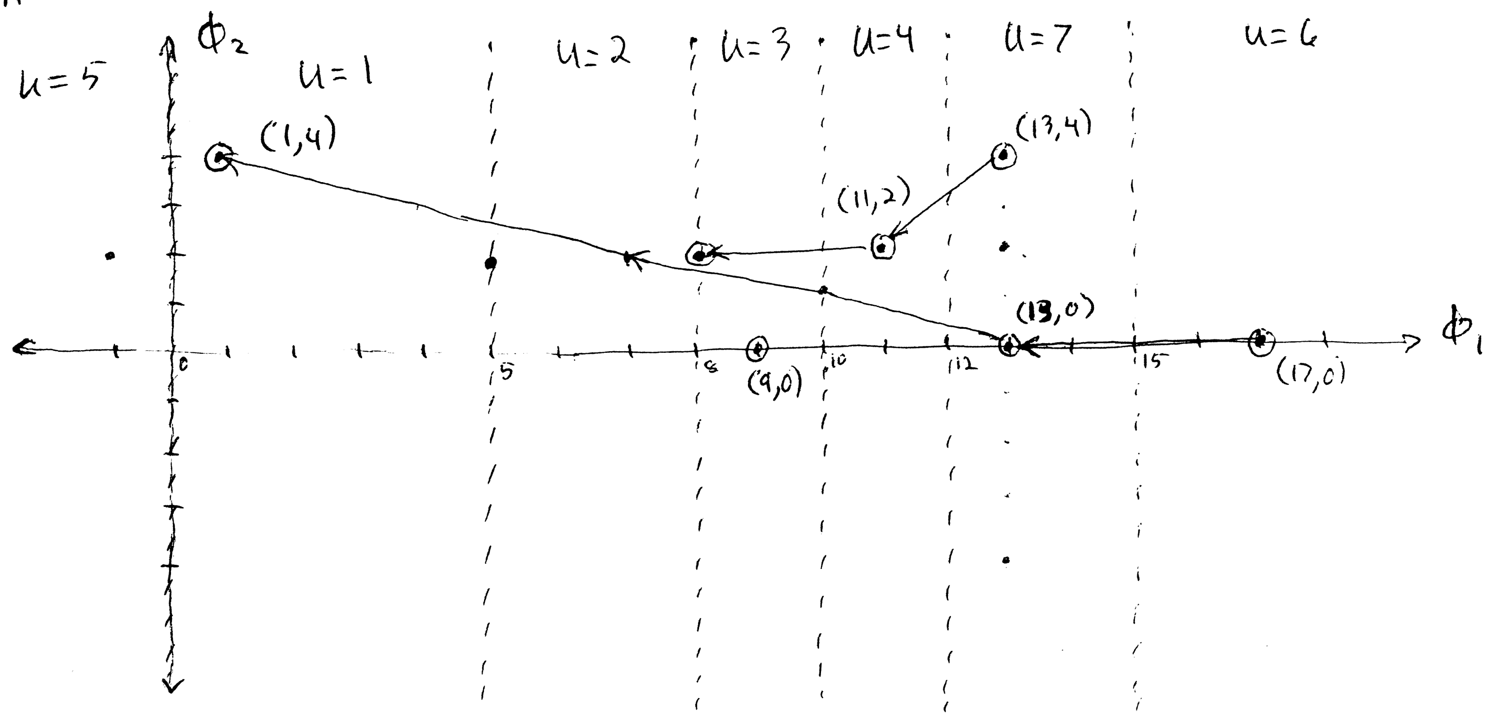
(4,1) . . . . . . 17 ⟵ by exploration

(0,0) . . . . . . 0 ⟵ by pattern move and exploration

termination

$\delta_1 < \varepsilon_1$
$\delta_2 < \varepsilon_2$ ⟵ by reducing increment below minimum required.

#2 Sketch



INITIAL SIMPLEX

(9,0) u=3

(13,4) u=7 ⟶ (11,2) u=4

(17,6) u=6 ⟶ (13,0) u=7 ⟶ (1,4) u=1 ⟶ (11,2) u=3

moved to by shrinking

moved to by expansion

moved to by contraction

four points generated

⇒ Stop.

#3 $\quad U = \phi_1^2 + 2\phi_2^2 \qquad \nabla U = \begin{bmatrix} 2\phi_1 \\ 4\phi_2 \end{bmatrix}$

$j = 0, \quad \phi^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad U^0 = 12$

$$\phi^1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \alpha \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 2-4\alpha \\ 2-8\alpha \end{bmatrix}$$

$$\begin{aligned}
U^1 &= (2-4\alpha)^2 + 2(2-8\alpha)^2 \\
&= 4 - 16\alpha + 16\alpha^2 + 8 - 64\alpha + 128\alpha^2 \\
&= 144\alpha^2 - 80\alpha + 12 \\
&= 144\left(\alpha^2 - \frac{5}{9}\alpha + \frac{1}{12}\right) \\
&= 144\left(\left(\alpha - \frac{5}{18}\right)^2 + \frac{1}{162}\right) \\
&= 144\left(\alpha - \frac{5}{18}\right)^2 + \frac{8}{9}
\end{aligned}$$

$\therefore$ minimum U^1 occurs at $\alpha = \frac{5}{18}$

for $\alpha = \frac{5}{18}$

$$\phi^1 = \begin{bmatrix} \frac{8}{9} \\ -\frac{2}{9} \end{bmatrix}$$

$$U^1 = \frac{8}{9}$$

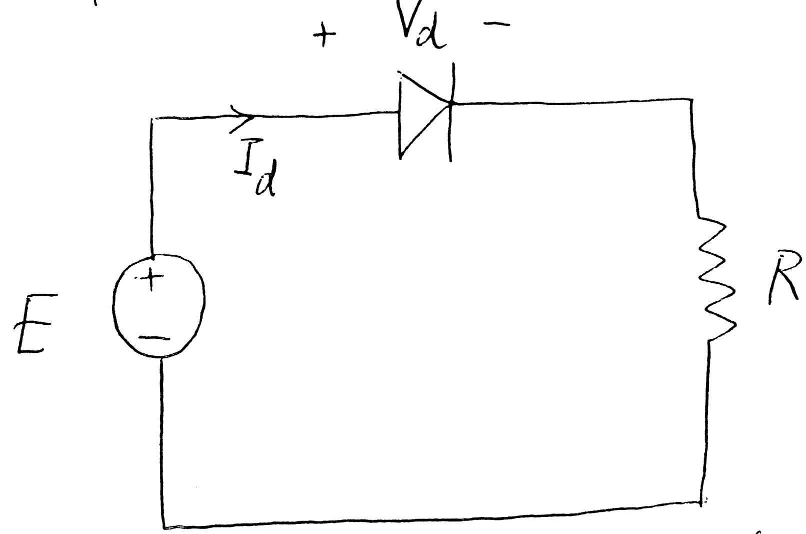
3KB3 tutorial #6

Mar. 20. 1996.

Simple Resistor Diode Circuit

For the circuit below, find the operating point

operating point : V_d



Resistor-diode network.

given : $E = 10\,V$

$R = 1.0\,K\Omega$

$I_s = 10^{-12}\,mA$

$\lambda = 38.7\,V^{-1}$

$$i_d = I_s\left(e^{\lambda V_d} - 1\right)$$

Answer : $V_d = 0.771404$

Circuit equation :

$$f(V_d) = E - V_d - I_d R = 0$$

$$f(V_d) = E - V_d - \underbrace{R I_s (e^{\lambda V_d} - 1)} = 0$$

nonlinear equation

Here, we will apply 3 techniques to solve the nonlinear equation.

Ⅰ Newton - Raphson Method for solving nonlinear ~~equations~~ circuit

Taylor series expansion of $f(V_d)$

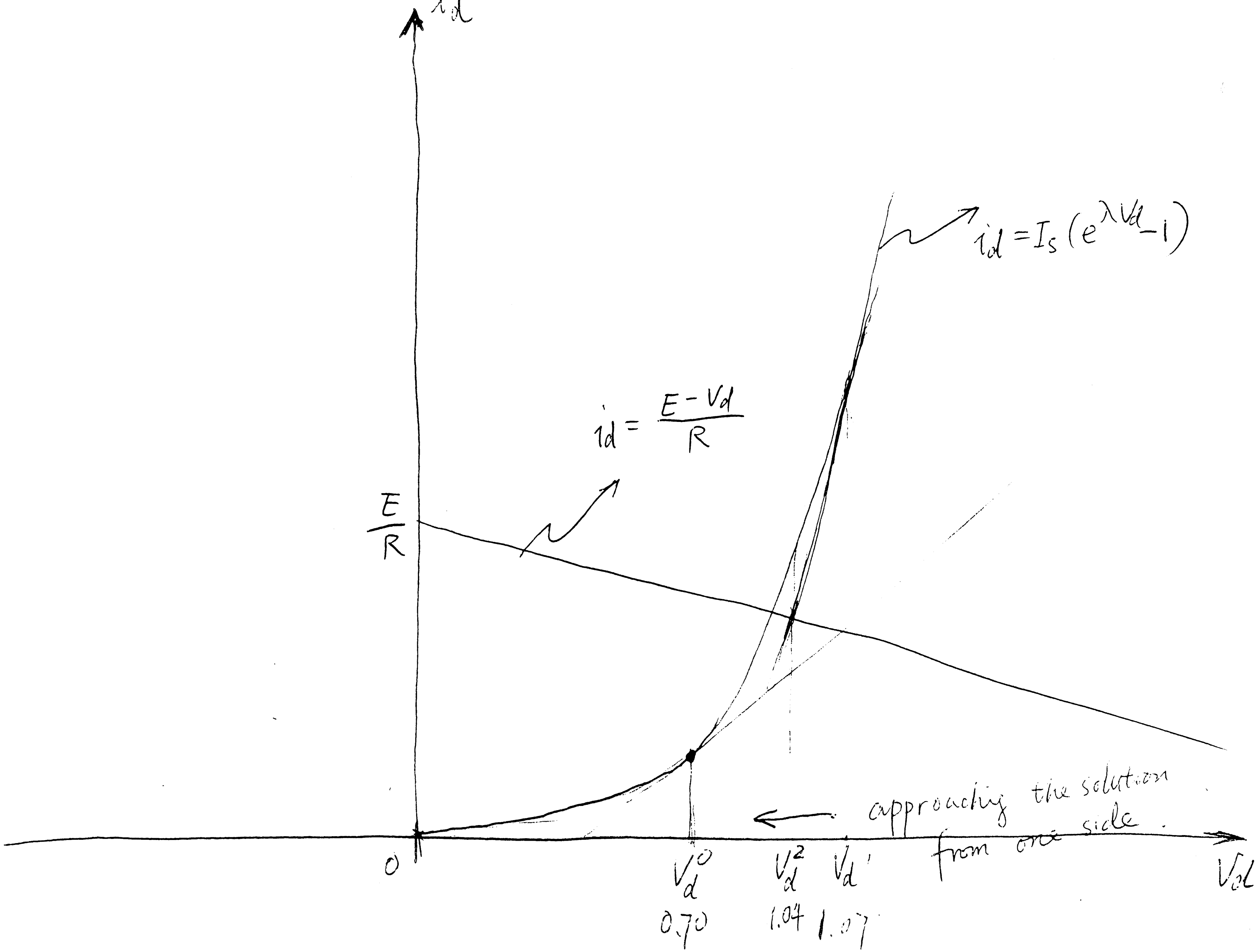
$$0 = f(V_d) \doteq f(V_d^0) + \left.\frac{df(V_d)}{dV_d}\right|^0 (V_d - V_d^0)$$

at the jth iteration,

$$0 = f(V_d^{j+1}) = f(V_d^j) + \left.\frac{df(V_d)}{dV_d}\right|^j (V_d^{j+1} - V_d^j)$$

$$V_d^{j+1} = V_d^j - \frac{f(V_d^j)}{\left.\frac{df(V_d)}{dV_d}\right|^j}$$

$$V_d^{j+1} = V_d^j + \frac{E - V_d^j - R I_S (e^{\lambda V_d^j} - 1)}{1 + R I_S \lambda e^{\lambda V_d^j}}$$



i_d
$\frac{E}{R}$
$i_d = \frac{E-V_d}{R}$
$i_d = I_s(e^{\lambda V_d}-1)$
approaching the solution from one side.
0
V_d^0
0.70
V_d^2
1.04
V_d'
1.07
V_d

④

Select the starting point $V_d^0 = 0.7$

$$V_d^1 = V_d^0 + \left. \frac{E - V_d^0 - R I_s (e^{\lambda V_d^0} - 1)}{1 + R I_s \lambda e^{\lambda V_d^0}} \right|_{V_d^0 = 0.7} = 1.070510$$

$$V_d^2 = V_d^1 + \cdots\cdots = 1.044670$$

$\vdots$

Iteration	v_d
1	1.070510
2	1.044670
3	1.018831
4	0.992993
5	0.967158
6	0.941331
7	0.915527
8	0.889783
9	0.864206
10	0.839073
11	0.815107
12	0.794018
13	0.778939
14	0.772401
15	0.771423
16	0.771404
17	0.771404

Solution.

termination criterion:

$$| V_d^{j+1} - V_d^j | < 10^{-6}$$

II Modified Newton Method. (still starting from V_d^0)

In the Newton – Raphson Method

the updating scheme is

$$V_d^{j+1} = V_d^j - \frac{f(V_d^j)}{\left.\frac{df(V_d)}{dV_d}\right|^j}$$

difficulties in deriving $f'(V_d)$, when $f(V_d)$ is very complicated

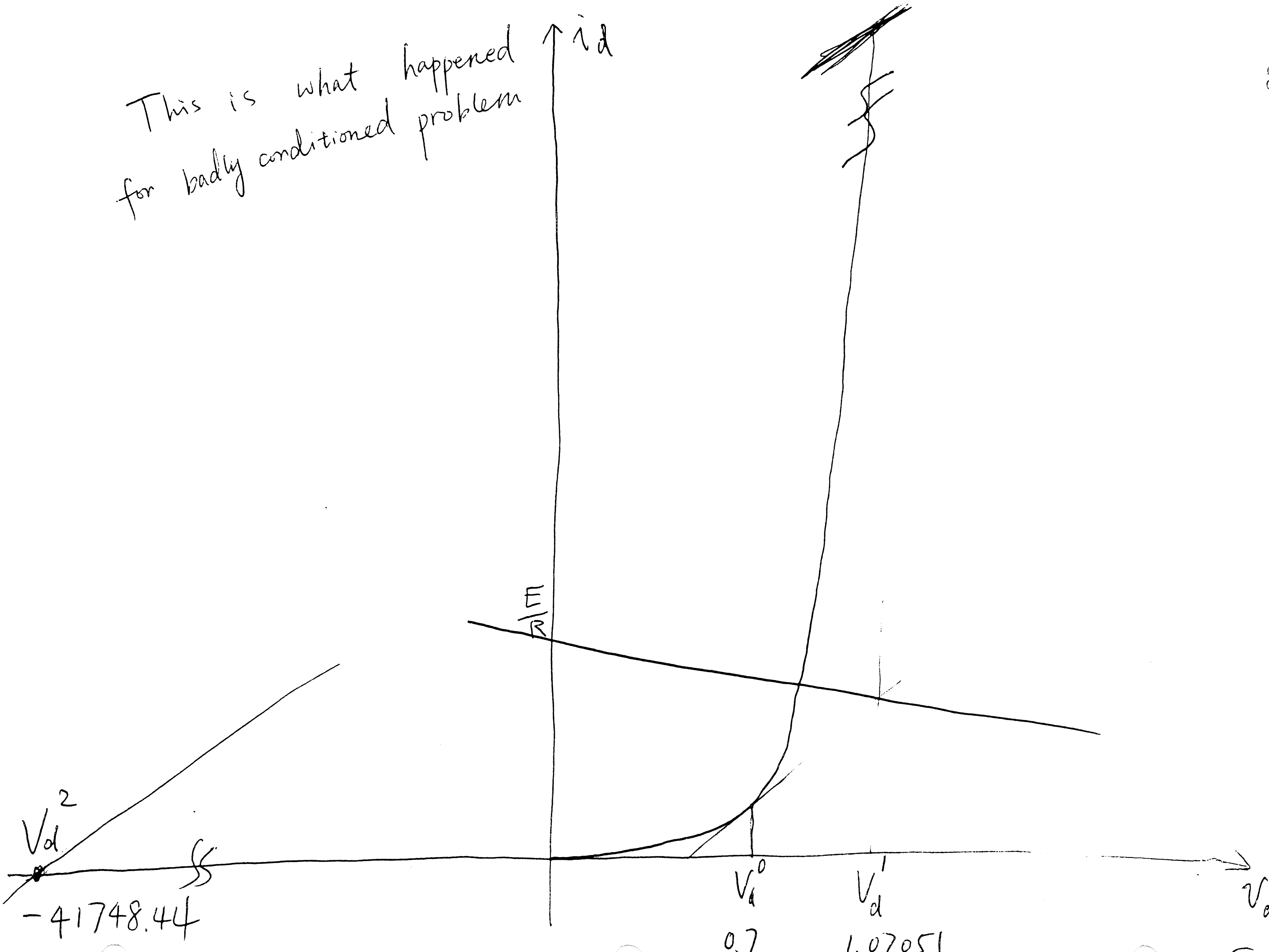
In the modified Newton Method

$$V_d^{j+1} = V_d^j - \frac{f(V_d^j)}{\left.\frac{df(V_d)}{dV_d}\right|^0} = V_d^j - \frac{f(V_d^j)}{m}$$

$$m = \left.\frac{df(V_d)}{dV_d}\right|^0 = 1 + R I_S \lambda e^{\lambda V_d^0} = 23.53$$

it is easy to see that if V_d^0 (0.7) is smaller than the solution, for our problem, we have a badly conditioned problem.

This is what happened for badly conditioned problem



7

We abandon this point, and try ~~another~~ other starting points ① $v_d^0 = 0.85$ V

$$m = \frac{df(v_d^0)}{dv_d^0} = 7479.7$$

$v_d^0 = 0.85$ $m = \underline{7479.7}$

Iteration	v^d
1	0.825387
2	0.816647
⋮	⋮

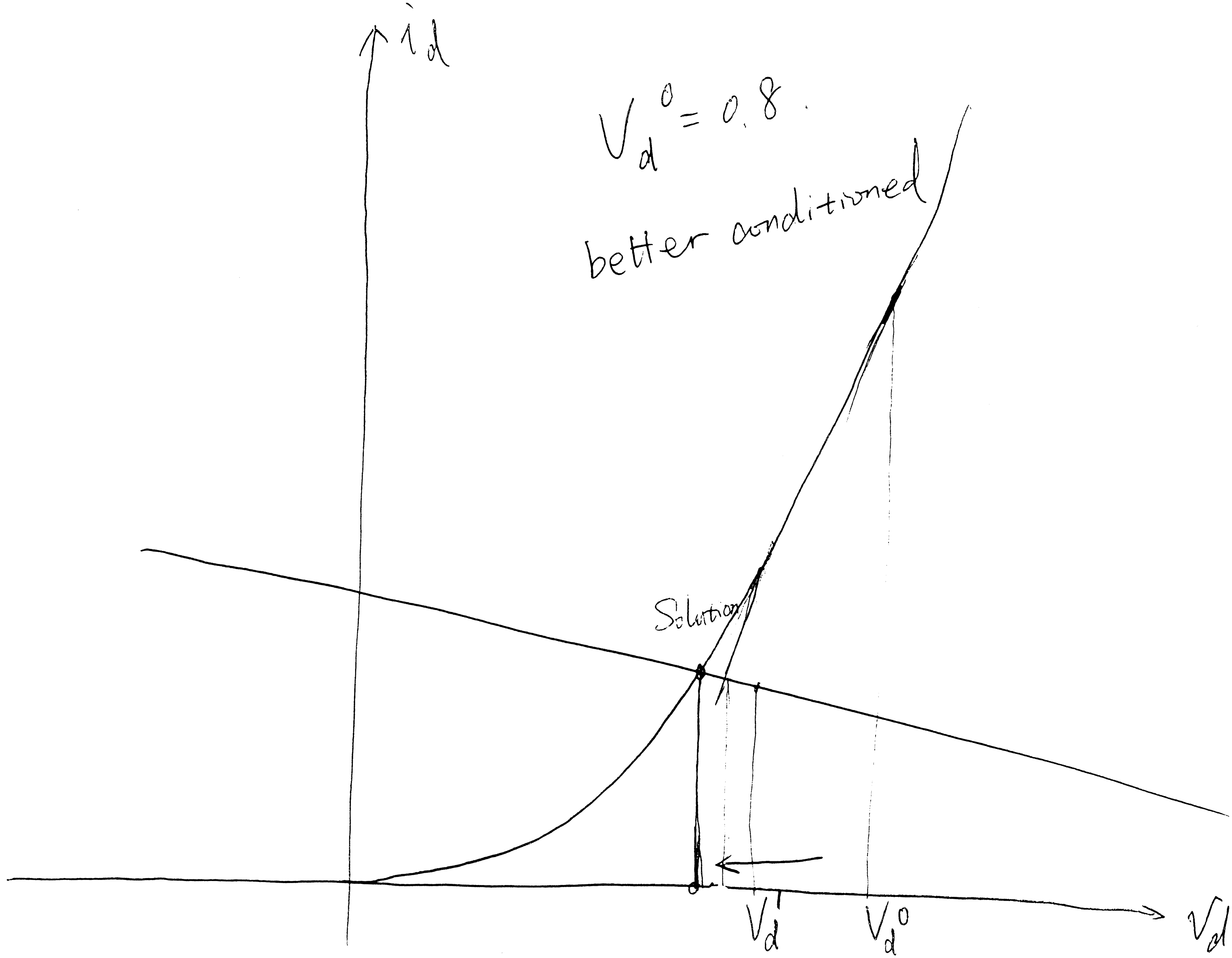
More than 100 iterations

We try another starting point

$v_d^0 = 0.8$ $m = \underline{1081.1}$

Iteration	v^d
1	0.782694
2	0.778006
⋮	⋮
25	0.778006

better conditioned



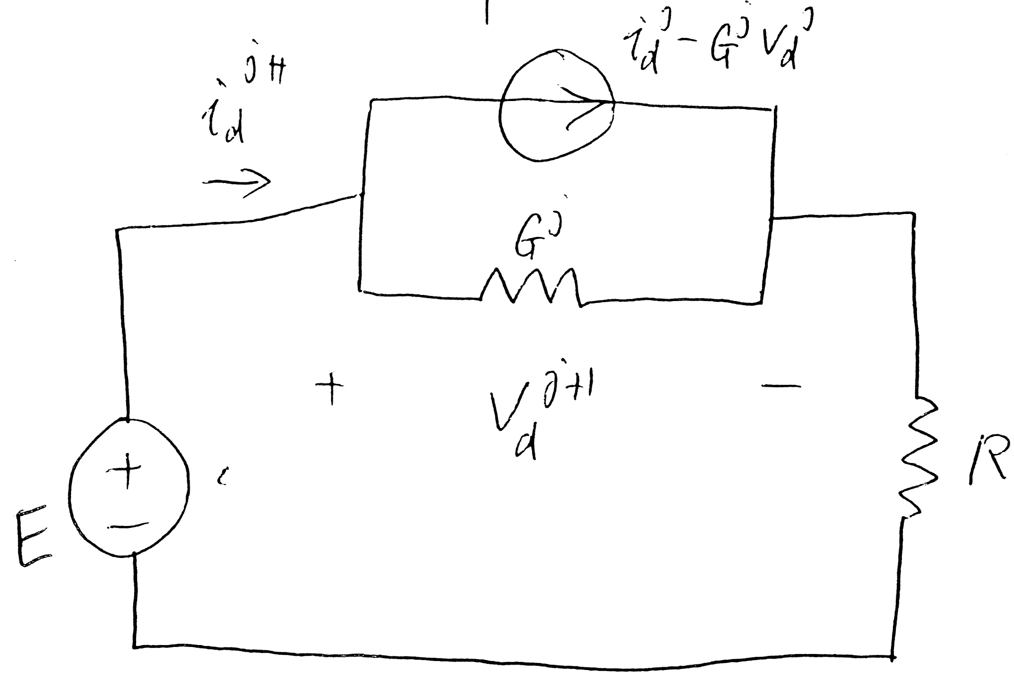
35

(9)

III Companion Network.

$$V_d^0 = 0.7$$

the ckt's companion network.



$$G^j = \left.\frac{d i_d}{d V_d}\right|^j$$

$$i_d^{j+1} = i_d^j + \left.\frac{d i_d}{d V_d}\right|^j (V_d^{j+1} - V_d^j)$$

$$= (i_d^j - G^j V_d^j) + G^j V_d^{j+1}$$

Using KVL :

$$R\, i_d^{j+1} = E - V_d^{j+1}$$

$$R\left[\left(i_d^j - G^j V_d^j\right) + G^j V_d^{j+1}\right] = E - V_d^{j+1}$$

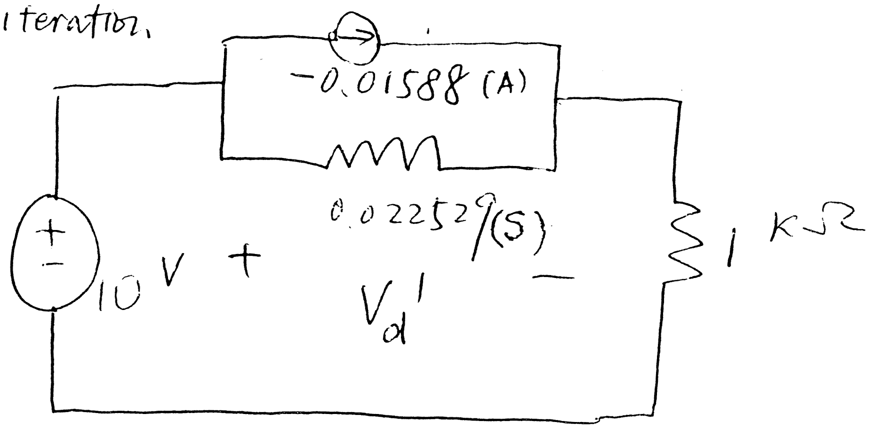
Simplifies to

$$V_d^{j+1} = V_d^j + \frac{E - V_d^j - R I_s \left(e^{\lambda V_d^j} - 1\right)}{1 + R I_s \lambda e^{\lambda V_d^j}}$$

Same updating equation as the Newton-Raphson method.

$$V_d^0 = 0.7$$

0th iteration.



$V_d^1 = 1.070510$

$\vdots$

$V_d^{17} = 0.771404 \longrightarrow$ Solution

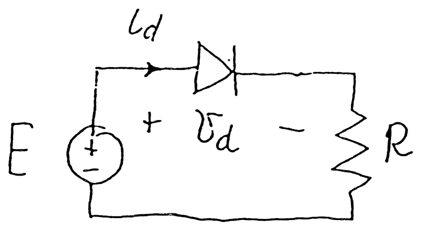
Newton Method:

1. guess the initial solution.
2. set the nonlinear equasion(s) for the system
3. apply a Newton iteration (which implies):
   a) linearization of the nonlinear equation(s)
   b) solving the resulting linear system for a new solution
4. check the termination criterion and either stop or go to point 2

Companion Network: *

1. guess the initial solution
2. for each nonlinear element linearize the nonlinear equation of the element
3. having all elements linear:
   a) set the linear equation(s) for the system
   b) solve the resulting linear system for a new solutio
4. check the termination criterion and either stop or go to point 2.

Example



$$i_d = I_s(e^{\lambda v_d} - 1)$$

given: E, R, I_s, λ

find: v_d

Applying the Algorithms to the Example

Newton Method:

1. e.g. $v_d^0 = 0.7$
2. $f(v_d) = E - v_d - R i_d = 0$

   $f(v_d) = E - v_d - R I_s(e^{\lambda v_d} - 1) = 0$
3. $$v_d^{j+1} = v_d^j - \frac{f(v_d^j)}{f'(v_d^j)}$$

   a), b)
   $$v_d^{j+1} = v_d^j + \frac{E - v_d^j - R I_s(e^{\lambda v_d^j} - 1)}{1 + R I_s \lambda e^{\lambda v_d^j}}$$
4. e.g. $|v_d^{j+1} - v_d^j| <$ SMALL_NUMBER

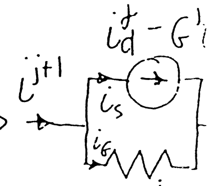
Companion Network:

1. e.g. $v_d^0 = 0.7$
2. $i_d = I_s(e^{\lambda v_d} - 1)$ linearize

   $$i_d^{j+1} = i_d^j + \frac{d i_d^j}{d v_d^j}(v_d^{j+1} - v_d^j) + $$

   or $i_d^{j+1} = i_d^j + G^j(v_d^{j+1} - v_d^j)$

   $$i_d^{j+1} = \underbrace{G^j v_d^{j+1}}_{i_G} + \underbrace{i_d^j - G^j v_d^j}_{i_s} \Rightarrow$$



G^j

$+ \; v^{j+1} \; -$

3. a) $E - v_d^{j+1} - R(G^j v_d^{j+1} + i_d^j - G^j v_d^j) = 0$

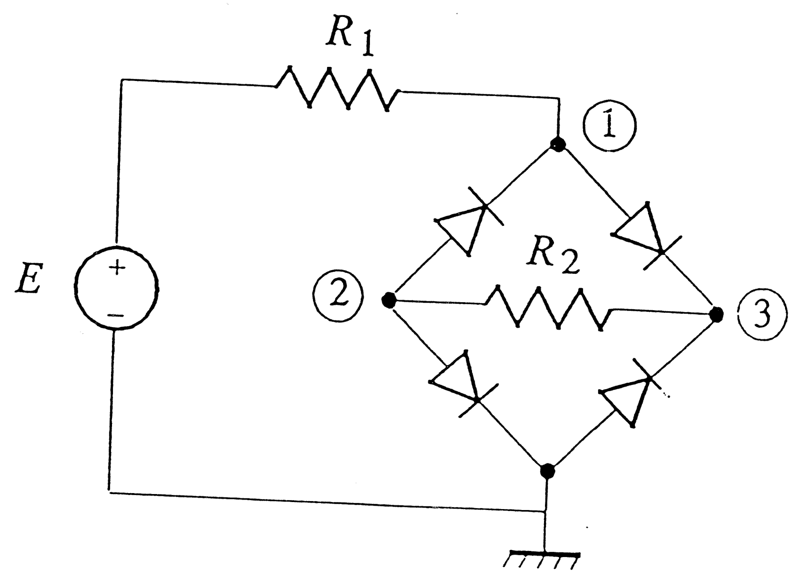
   b)
   $$v_d^{j+1} = v_d^j + \frac{E - v_d^j - R I_s(e^{\lambda v_d^j} - 1)}{1 + R I_s \lambda e^{\lambda v_d^j}}$$
4. e.g. $|v_d^{j+1} - v_d^j| <$ SMALL_NUMBER

Note that the solutions are the same for both methods as expected.

2

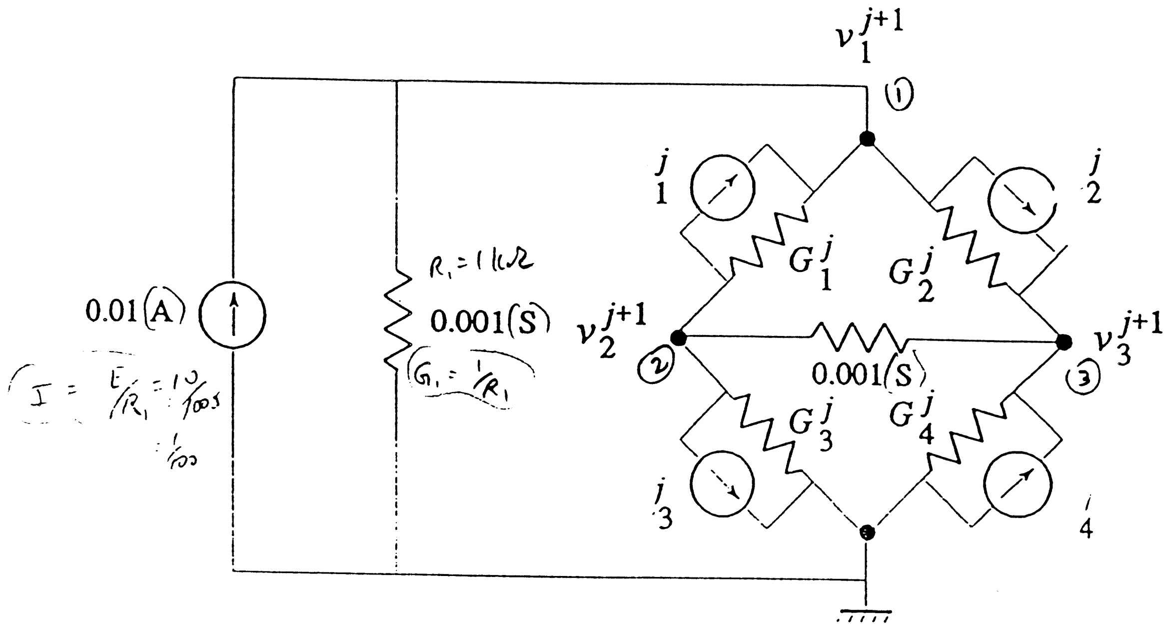
Consider the resistor-diode network shown. Draw the corresponding companion network at the *j*th iteration for its d.c. solution. Write down the nodal equations at this iteration. Where

$$I_d = I_S(e^{\lambda V_d} - 1)$$
$$I_S = 10^{-12} \text{ mA}$$
$$\lambda = \frac{1}{0.026} \; V^{-1}$$
$$E = 10 \text{ V}$$
$$R_1 = R_2 = 1 \text{ k}\Omega$$



A resistor-diode network.

The corresponding companion network at the *j*th iteration is shown below.



$\cancel{i^j \neq I}$

$$G^j = \frac{d I_d^j}{d V_d^j}$$

$$i^j = I_d^j - G^j V_d^j$$

The nodal equations at this iteration can be written as

$$\begin{bmatrix} 0.001 + G_1^j + G_2^j & -G_1^1 & -G_2^j \\ -G_1^j & 0.001 + G_1^j + G_3^j & -0.001 \\ -G_2^j & -0.001 & 0.001 + G_2^j + G_4^j \end{bmatrix} \begin{bmatrix} V_1^{j+1} \\ V_2^{j+1} \\ V_3^{j+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0.01 + (i_{d_1}^j - G_1^j V_{d_1}^j) - (i_{d_2}^j - G_{d_2}^j V_{d_2}^j) \\ -(I_{d_1}^j - G_1^j V_{d_1}^j) - (I_{d_3}^j - G_4^j V_{d_4}^j) \\ (I_{d_2}^j - G_2^j V_{d_2} j) - (i_{d_4}^j - G_4^j V_{d_4}^j) \end{bmatrix} \leftarrow \text{Current in}$$

where

$$V_{d_1}^j = V_2^j - V_1^j, \quad V_{d_2}^j = V_1^j - V_3^j, \quad V_{d_3}^j = V_2^j, \quad V_{d_4}^j = -V_3^j$$

and

$$\left.\begin{aligned} I_{d_k}^j &= I_S(e^{\lambda V_{d_k}^j} - 1) \\ G_k^j &= \lambda I_S \, e^{\lambda V_{d_k}^j} \end{aligned}\right\} \qquad k = 1, 2, 3, 4$$

COMPUTER ENGINEERING 3KB3

DURATION OF TEST: 2 Hours — Wednesday, March 20, 1996

Candidates must attempt Questions

1 or 2   3   4   5   6   7 or 8

Write your name here. NAME: ______________________________

Write your student number here. NO: ______________

Note: (1) All scripts and question papers must be turned in.
(2) Estimated times required to complete the questions are indicated.
(3) Please encircle questions attempted in the following table.
(4) Only McMaster standard calculators (Casio fx-991) are allowed.

Questions Attempted (please encircle)	Weighting	Estimated Time (min.)	Examiner's Use Only
1 or 2	15%	twenty	
3	15%	fifteen	
4	15%	fifteen	
5	15%	twenty	
6	15%	twenty	
7 or 8	25%	thirty	
TOTAL	100%	2 hours	

Al ! these are the solutions of the questions I marked

continued on page 2

Question 1

(a) Given a differentiable function f of many variables x and a corresponding direction vector s,

$$\lim_{\lambda \to 0^+} \frac{f(x + \lambda s) - f(x)}{\lambda} = \ldots\ldots\ldots \text{ (please state) ?}$$

Explain in a few words the meaning of the above expression.

(b) Use the method of Lagrange multipliers to minimize w.r.t. ϕ_1 and ϕ_2 the function

$$U = \phi_1^2 + 2\phi_2^2$$

subject to

$$\phi_1 + \phi_2 = 1$$

Sketch a diagram to illustrate the problem and its solution w.r.t. ϕ_1 and ϕ_2. Verify your answer by substituting the constraint into the function.

Answer (20 min.)

a) see solution question 2011 in Volume 2 of optimization courseware

$$\lim_{\lambda \to 0^+} \frac{f(x + \lambda s) - f(x)}{\lambda} = (\nabla f)^T s$$

This is the definition of gradient of function f at x in the direction of vector s

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

partb) see page 3-15 Volume 2 of optimization courseware question 2c11

continued on page 3

44

Question 2

Examine the points $[-1 \quad 0]^T$ and $[1 \quad 1]^T$ for a minimax problem for which

$$f_1 = \phi_1^4 + \phi_2^2$$

$$f_2 = (2 - \phi_1)^2 + (2 - \phi_2)^2$$

$$f_3 = 2\exp(-\phi_1 + \phi_2)$$

by invoking necessary conditions for a minimax optimum. What is the steepest descent direction, if any, at these two points?

Answer (20 min.)

see solution on page 3-69 for similar question (question 2(6c)).

@ point $[-1, 0]^T$

$f_1 = 1$

$f_2 = 13 \leftarrow$ max

$f_3 = 2e$

∴ only f_2 active

$u_1 = u_3 = 0$ since f_1 & f_3 are inactive

$u_2 \nabla f_2 = 0$

$u_2 \begin{bmatrix} -6 \\ -4 \end{bmatrix} = 0$

if & only if $u_2 = 0$

This contradicts with the condition $u_1 + u_2 + u_3 = 1$

or $u_2 = 1$

Hence, point $[-1, 0]^T$ is not the minimax optimum

The steepest descent direction is $-\dfrac{\nabla f_2}{\|\nabla f_2\|} = \dfrac{1}{\sqrt{52}} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

@ point $[1, 1]^T$

there is no steepest descent direction or it is a zero vector

continued on page 4

45

Question 3

Consider a system of complex linear equations

$$\boldsymbol{Y}\,\boldsymbol{V} \;=\; \boldsymbol{I}$$

where $\boldsymbol{Y}$ is a square nodal admittance matrix of constant, complex coefficients, and $\boldsymbol{I}$ is a specified excitation vector. Set up the appropriate objective function for the <u>least squares solution</u> of this system of equations and derive the gradient vector w.r.t. the real and imaginary parts of the components of $\boldsymbol{V}$.

<u>Answer</u> (15 min.)

Solution on page 3-163, Question 3CEC

in Optimization Volume 2 courseware.

continued on page 5

Question 4

Starting with the interval [0, 6], apply four iterations of the Golden Section search method to the minimization w.r.t. ϕ of a function described by

$$U = -\phi + 5 \qquad \phi \leq 1$$

$$U = 0.5(\phi - 3)^2 + 1 \qquad 1 < \phi \leq 4$$

$$U = 3 - (\phi - 6)^2/3 \qquad \phi > 4$$

What is the solution obtained? By how much has the interval of uncertainty been reduced?

<u>Answer</u> (15 min.)

continued on page 6

Question 4 Midterm CoE 3KB3 March 20 '96

$\tau = \frac{1}{2}(1+\sqrt{5}) \doteq 1.618$, $\phi_a^j = \frac{1}{\tau^2} I^j + \phi_L^j$, $\phi_b^j = \frac{1}{\tau} I^j + \phi_L^j$

Iteration 1 $\phi_L^1 = 0$, $\phi_a^1 \doteq 2.292$, $\phi_b^1 \doteq 3.708$, $\phi_H^1 = 6$

$u_a^1 \doteq 1.251$ $u_b^1 = 1.251$

Iteration 2 $\phi_L^2 = 0$, $\phi_a^2 \doteq 1.416$, $\phi_b^2 \doteq 2.292$, $\phi_H^2 \doteq 3.708$

$u_a^2 \doteq 2.254$ $u_b^2 \doteq 1.251$

Iteration 3 $\phi_L^3 \doteq 1.416$, $\phi_a^3 \doteq 2.292$, $\phi_b^3 \doteq 2.833$, $\phi_H^3 = 3.708$

$u_a^3 \doteq 1.251$, $u_b^3 \doteq 1.014$

Iteration 4 $\phi_L^4 \doteq 2.292$, $\phi_a^4 \doteq 2.833$, $\phi_b^4 \doteq 3.167$, $\phi_H^4 = 3.708$

$u_a^4 \doteq 1.014$ $u_b^4 \doteq 1.014$

At 4 iterations, the solution is contained in

$[2.292, 3.167]$

for an interval of uncertainty $I^4 = .875$

The interval of uncertainty has been reduced by

$\tau^4 \doteq 6.854$

Question 5

Consider the minimization of an objective function $U(\phi)$ w.r.t. ϕ. Starting at $\phi = \phi^0$, one-dimensional search is performed along a given direction s, i.e.

$$\phi = \phi^0 + \alpha s$$

Assume that three uniformly spaced points are achieved on the α axis: a, b, c with $a < b < c$ and $U_a > U_b$, $U_b < U_c$.

(1) Create a quadratic function of α to fit U at a, b and c.
(2) Prove that the minimum of the quadratic function created in (1) is at the point

$$\alpha^* = b + \frac{(b - a)(U_a - U_c)}{2(U_a - 2U_b + U_c)}$$

Note: $b-a = c-b$, $c-a = 2(b-a)$, $c + a = 2b$.

Answer (20 min.)

continued on page 7

49

Question 5

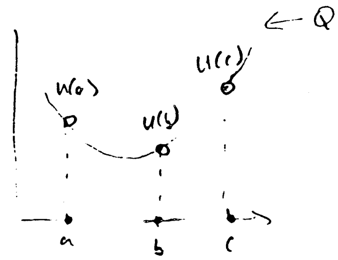
Consider the minimization of an objective function $U(\phi)$ w.r.t. ϕ. Starting at $\phi = \phi^0$, one dimensional search is performed along a given direction s, i.e.

$$\phi = \phi^0 + \alpha s$$

Assume that three uniformly spaced points are achieved on the α axis: a, b, c with $a < b < c$ and $U_a > U_b$, $U_b < U_c$.

(1) Create a quadratic function of α to fit U at a, b and c.
(2) Prove that the minimum of the quadratic function created in (1) is at the point

$$\alpha^* = b + \frac{(b-a)(U_a - U_c)}{2(U_a - 2U_b + U_c)}$$



Note: $b-a = c-b$, $c-a = 2(b-a)$, $c + a = 2b$.

Answer (30 min.)

↳ Let: $Q(\alpha) = A\alpha^2 + B\alpha + C$

where A, B and C are determined from the equations

$$\begin{cases} Aa^2 + Ba + C = U_a & (1) \\ Ab^2 + Bb + C = U_b & (2) \\ Ac^2 + Bc + C = U_c & (3) \end{cases}$$

↳ Solution of these equations gives:

$$A = \frac{(U_a - 2U_b + U_c)}{2(b-a)^2} \qquad (4)$$

$$B = \frac{(U_c - U_a)}{2(b-a)} - \frac{b(U_a - 2U_b + U_c)}{(b-a)^2} \qquad (5)$$

$$C = \frac{bcU_a - 2acU_b + \ldots}{2(b-a)^2}$$

continued on page 7

obtain the minimum of the quadratic function, $Q(\alpha)$, w.r.t. α, we should differentiate Q w.r.t. α and equate to 0:

$$\frac{dQ(\alpha)}{d\alpha} = 2A\alpha + B = 0$$

(notice we don't need C!)

giving the position of the minimum of Q as:

$$\alpha_{min} = \frac{-B}{2A}$$

Using (4) and (5) gives:

$$\alpha_{min} = \frac{-\left\{ \frac{(u_c - u_a)}{2(b-a)} - b\frac{(u_a - 2u_b + u_c)}{(b-a)^2} \right\}}{(2)\left[\frac{u_a - 2u_b + u_c}{2(b-a)^2} \right]}$$

$$= \cancel{\left(\frac{1}{2}\right)}\left[\frac{-(u_c - u_a)}{2(b-a)} + \frac{b(u_a - 2u_b + u_c)}{(b-a)^2} \right] \frac{\cancel{(2)}(b-a)^2}{(u_a - 2u_b + u_c)}$$

$$= \frac{-(u_c - u_a)(b-a)}{(2)(u_a - 2u_b + u_c)} + b$$

$$\therefore \alpha_{min} = b + \frac{(b-a)(u_a - u_c)}{2(\ldots)}$$

Assuming: $b - a = c - b$
$c - a = 2(b-a)$
$c + a = 2b$

51

Question 6

State the conjugate gradient algorithm. Apply the algorithm using a theoretically justified line search for a minimum to the minimization of

$$\phi_1^2 + 3\phi_2^2 + \phi_1\phi_2 + \phi_1 + 2$$

w.r.t. ϕ_1 and ϕ_2 starting at $\phi_1 = 1$ and $\phi_2 = 1$. Show all steps explicitly and comment on the results obtained. Draw an accurate diagram showing the path taken.

Answer (20 min.)

continued on page 8

52

Question 6

State the conjugate gradient algorithm. Apply the algorithm using a theoretically justified line search for a minimum to the minimization of

$$\phi_1^2 + 3\phi_2^2 + \phi_1\phi_2 + \phi_1 + 2$$

w.r.t. ϕ_1 and ϕ_2 starting at $\phi_1 = 1$ and $\phi_2 = 1$. Show all steps explicitly and comment on the results obtained. Draw an accurate diagram showing the path taken.

Answer (20 min.)

A gradient method that exploit the properties of of conjugate directions associated with quadratic functions and do not explicitly evaluate Hessian H or its inverse.

$$\underset{\sim}{S}^0 = -\underset{\sim}{\nabla} U^0$$

$$\underset{\sim}{S}^j = -\underset{\sim}{\nabla} U^j + \beta^j \underset{\sim}{S}^{j-1}, \quad j > 0$$

$$\beta^j = \frac{(\nabla U^j)^T (\nabla U^j)}{(\nabla U^{j-1})^T \nabla U^{j-1}}$$

$$\underset{\sim}{\phi}^{j+1} = \phi^j + \Delta \underset{\sim}{\phi}^j$$

$$U = \phi_1^2 + 3\phi_2^2 + \phi_1\phi_2 + \phi_1 + 2$$

$$\underset{\sim}{\nabla} U = \begin{bmatrix} 2\phi_1 + \phi_2 + 1 \\ 6\phi_2 + \phi_1 \end{bmatrix} \qquad \phi^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{\nabla} U^0 = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$S^0 = -\nabla U^0 = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

Q6-1

continued on page 8

$$\Delta\phi^0 = \alpha S^0 = \begin{bmatrix} \cancel{-4}\alpha \\ -7\alpha \end{bmatrix}$$

$$\phi^1 = \phi^0 + \Delta\phi^0 = \begin{bmatrix} 1-4\alpha \\ 1-7\alpha \end{bmatrix}$$

$$U(\phi^1) = (1-4\alpha)^2 + 3(1-7\alpha)^2 + (1-4\alpha)(1-7\alpha) + (1-4\alpha) + 2$$

at minimum $\dfrac{\partial U(\phi^1)}{\partial \alpha} = 0$

$$-8 + 32\alpha - 42 + 294\alpha - 4 + 28\alpha - 7 + 28\alpha - 4 = 0$$

$$\alpha = 0.170$$

$$\phi^1 = \phi^0 - \alpha \nabla U^0 = \begin{bmatrix} 0.32 \\ -0.19 \end{bmatrix}$$

$$\nabla U(\phi^1) = \begin{bmatrix} 1.45 \\ -0.82 \end{bmatrix} = \nabla U^1$$

$$S^1 = -\nabla U^1 + \frac{(\nabla U^1)^T(\nabla U^1)}{(\nabla U^0)^T(\nabla U^0)} \underset{\sim}{S}^0$$

$$= \begin{bmatrix} -1.45 \\ 0.82 \end{bmatrix} + \frac{\begin{bmatrix} 1.45 & -0.82 \end{bmatrix}\begin{bmatrix} 1.45 \\ -0.82 \end{bmatrix}}{\begin{bmatrix} 4 & 7 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix}} \begin{bmatrix} -4 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -1.621 \\ 0.521 \end{bmatrix}$$

$$\Delta\phi^1 = \alpha S^1 = \begin{bmatrix} -1.621\alpha \\ 0.521\alpha \end{bmatrix}$$

$\boxed{Q6-2}$

$$\phi^2 = \phi^1 + \Delta\phi^1 = \begin{bmatrix} 0.32 - 1.621\alpha \\ -0.19 + 0.521\alpha \end{bmatrix}$$

$$U(\phi^2) = (0.32 - 1.621\alpha)^2 + 3(0.521 - 0.19)^2 + (0.32 - 1.621\alpha)$$

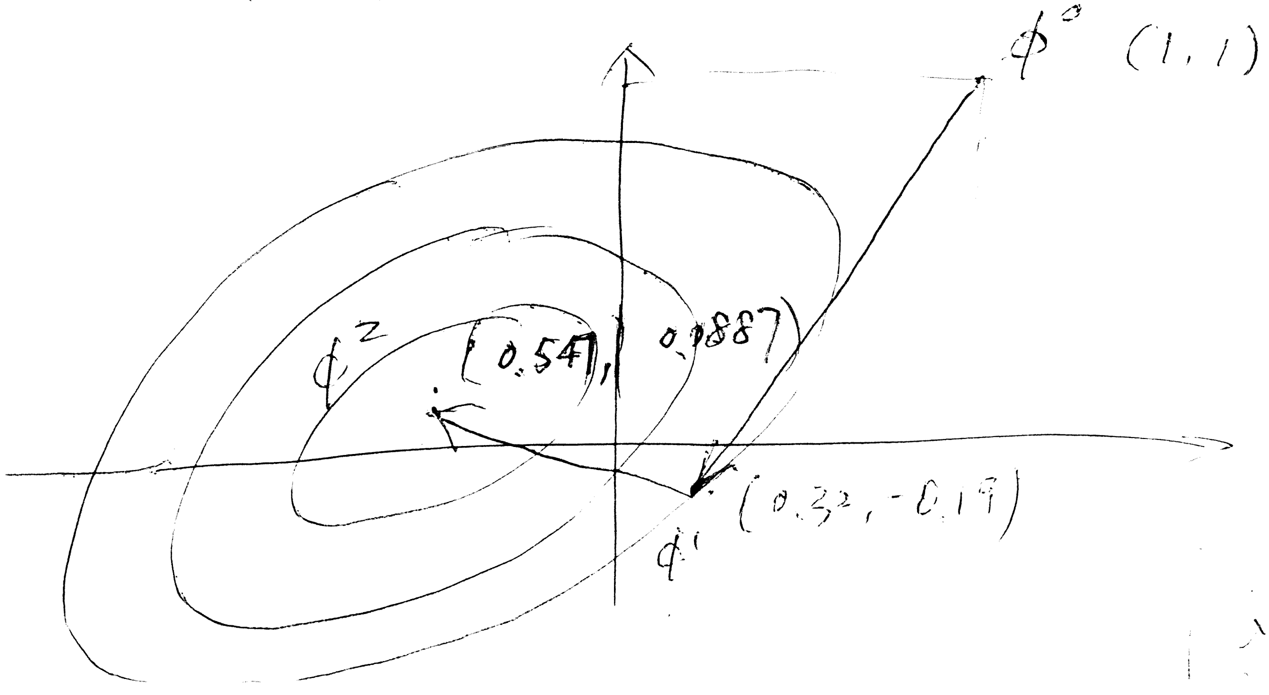
$$(-0.19 + 0.521\alpha) + (0.32 - 1.621\alpha) + 2$$

at minimum $\dfrac{\partial U(\phi^2)}{\partial \alpha} = 0$

$$\alpha = 0.535$$

$$\therefore \phi^2 = \begin{bmatrix} -0.547 \\ 0.0887 \end{bmatrix}$$

For a quadratic function, the minimum is reached at the second iteration



Question 7

Consider the function

$$U = 2\phi_1^2 + \phi_2^2$$

subject to the constraints

$$\phi_2 \geq 1$$

$$\phi_1 \geq \phi_2 .$$

(a) Sketch contours of the objective function U and constraints, indicating the feasible and nonfeasible regions.

(b) Find the unconstrained optimum of U and indicate this point on your sketch.

(c) Form penalty functions P_1 and P_2 using $w_1 = w_2 = 10$ and $w_1 = w_2 = 50$. Find the two minima of P_1 and P_2 and indicate these points on your sketch.

(d) Invoke the Kuhn–Tucker necessary conditions to test the two points from the solutions of the penalty functions.

<u>Answer</u> (30 min.)

continued on page 9

Question 7

a) See graph

b) $\nabla U = \begin{bmatrix} 4\phi_1 \\ 2\phi_2 \end{bmatrix}$, Set $\nabla U = 0 \Rightarrow \underline{\phi} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

c) $P_1 = 2\phi_1^2 + \phi_2^2 + 10(\phi_2 - 1)^2 + 10(\phi_1 - \phi_2)^2$

$\nabla P_1 = 0 \Rightarrow \begin{bmatrix} 24 & -20 \\ -20 & 42 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{608} \begin{bmatrix} 42 & 20 \\ 20 & 24 \end{bmatrix} \begin{bmatrix} 0 \\ 20 \end{bmatrix} \doteq \begin{bmatrix} .6579 \\ .7895 \end{bmatrix}$

$P_2 = 2\phi_1^2 + \phi_2^2 + 50(\phi_2 - 1)^2 + 50(\phi_1 - \phi_2)^2$

$\nabla P_2 = 0 \Rightarrow \begin{bmatrix} 104 & -100 \\ -100 & 202 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{11008} \begin{bmatrix} 202 & 100 \\ 100 & 104 \end{bmatrix} \begin{bmatrix} 0 \\ 100 \end{bmatrix} \doteq \begin{bmatrix} .9084 \\ .9448 \end{bmatrix}$

d) at $\begin{bmatrix} .6579 \\ .7895 \end{bmatrix}$, $g_1 = \phi_2 - 1 = -.2105 \neq 0 \Rightarrow u_1 = 0$

$g_2 = \phi_1 - \phi_2 = -.1316 \neq 0 \Rightarrow u_2 = 0$

$\nabla U(\underline{\phi}) = \begin{bmatrix} 4(.6579) \\ 2(.7895) \end{bmatrix} = \begin{bmatrix} 2.6316 \\ 1.579 \end{bmatrix} \neq 0 = u_1 \nabla g_1 + u_2 \nabla g_2$

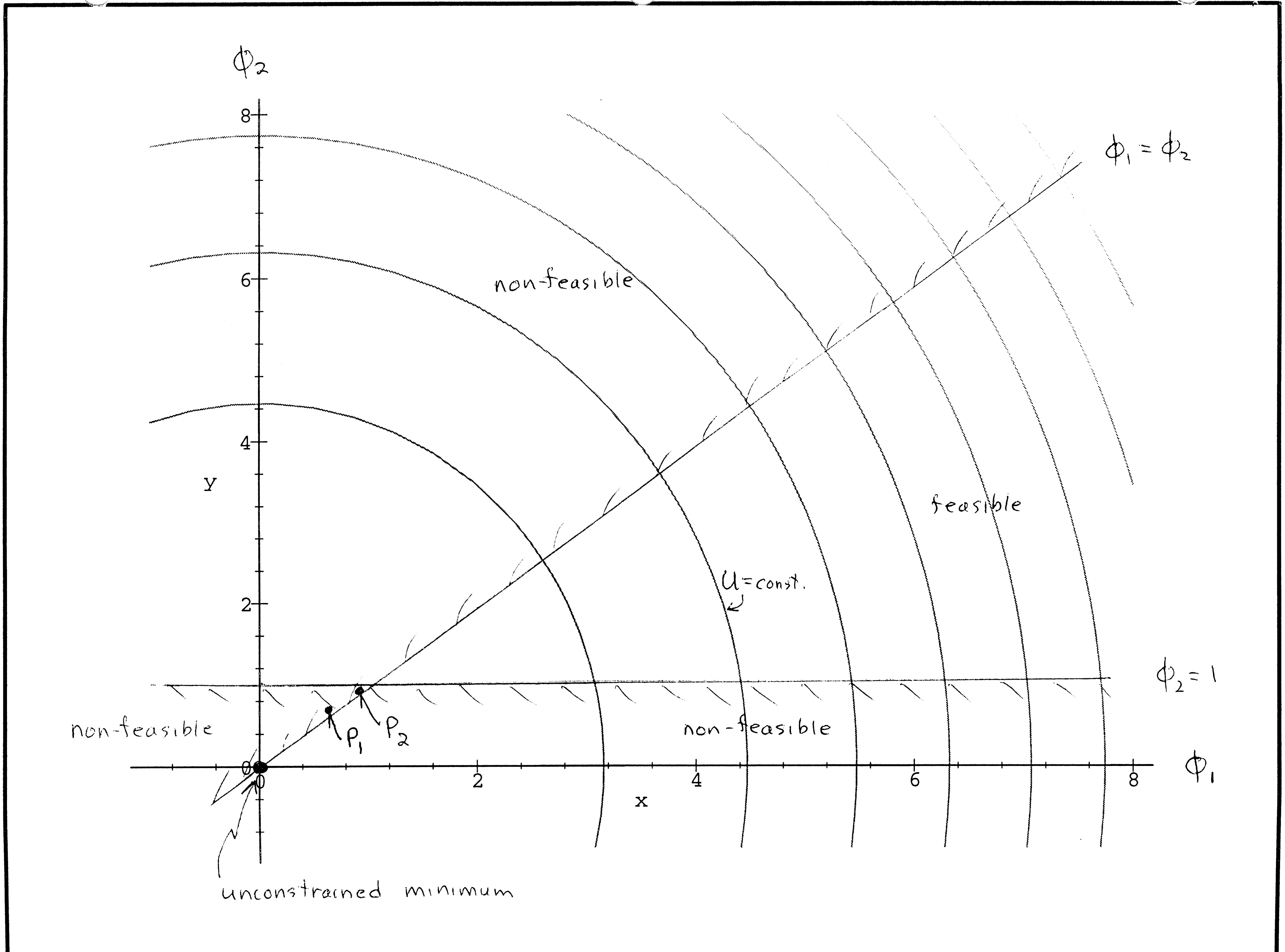
$\therefore$ KT not satisfied

at $\begin{bmatrix} .9084 \\ .9448 \end{bmatrix}$, $g_1 = \phi_2 - 1 = -.0916 \neq 0 \Rightarrow u_1 = 0$

$g_2 = \phi_1 - \phi_2 = -.0364 \neq 0 \Rightarrow u_2 = 0$

$\nabla U(\underline{\phi}) = \begin{bmatrix} 4(.9084) \\ 2(.9448) \end{bmatrix} \neq 0 = u_1 \nabla g_1 + u_2 \nabla g_2$

$\therefore$ KT not satisfied.


ϕ_2
y
$\phi_1 = \phi_2$
non-feasible
feasible
U=const.
$\phi_2 = 1$
non-feasible
non-feasible
P_1
P_2
ϕ_1
x
unconstrained minimum

Question 8

Consider each of the following objective functions

$$(a)\ \ U = 2\,\phi_1 + 2\,\phi_2 - 1$$

$$(b)\ \ U = 0.5\,\phi_1 + 2\,\phi_2 + 3$$

$$(c)\ \ U = 2\,\phi_1 + 0.5\,\phi_2 + 1.5$$

along with the constraints

$$g_1 = \phi_1 + 2\,\phi_2 - 3 \geq 0$$
$$g_2 = 2\,\phi_1 + \phi_2 - 3 \geq 0$$

at each of the three points (i) $[2\ 1]^T$, (ii) $[1\ 1]^T$, (iii) $[3\ 0]^T$. Invoke the Kuhn-Tucker (KT) conditions for each objective function at each of the three points. State the results and comment on them. For any case which does not satisfy the KT conditions, find the steepest feasible downhill direction.

Answer (30 min.)

THE END!

59

Question 8

KT ① $\nabla U = \sum_{i=1}^{K} u_i \nabla g_i$

② $u_i \geq 0$

③ $u_i g_i = 0$

$\Big\}$ all i

$\nabla U_a = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\nabla U_b = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$, $\nabla U_c = \begin{bmatrix} 2 \\ \frac{1}{2} \end{bmatrix}$

$\nabla g_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\nabla g_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

i) at $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $g_1 = 1 \neq 0 \Rightarrow u_1 = 0$

$g_2 = 2 \neq 0 \Rightarrow u_2 = 0$

$\Rightarrow$ R.H.S of ① is 0, but none of the gradients are zero

$\therefore$ KT not satisfied at $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for any of the functions.

iii) at $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $g_1 = 0$

$g_2 = 3 \neq 0 \Rightarrow u_2 = 0$

$\therefore \nabla U = u_1 \nabla g_1$, but none of the gradients are a scalar multiple of ∇g_1

$\therefore$ KT not satisfied at $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ for any of the functions.

ii) at $\begin{bmatrix}1\\1\end{bmatrix}$ $\quad g_1 = 0$

$g_2 = 0$

for (b)

$$\nabla U_b = u_1 \nabla g_1 + u_2 \nabla g_2$$

$$\Rightarrow \begin{bmatrix}\frac{1}{2}\\2\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \Rightarrow \begin{bmatrix}u_1\\u_2\end{bmatrix} = \frac{1}{-3}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{7}{6}\\-\frac{1}{3}\end{bmatrix}$$

\# since $u_2 = -\frac{1}{3} \ngeq 0$

$\therefore$ $\begin{bmatrix}1\\1\end{bmatrix}$ does not satisfy KT for (b)

for (c)

$$\nabla U_c = u_1 \nabla g_1 + u_2 \nabla g_2$$

$$\Rightarrow \begin{bmatrix}2\\\frac{1}{2}\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \Rightarrow \begin{bmatrix}u_1\\u_2\end{bmatrix} = \frac{1}{-3}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}2\\\frac{1}{2}\end{bmatrix}$$

$$= \begin{bmatrix}-\frac{1}{3}\\\frac{7}{6}\end{bmatrix}$$

\# since $u_1 = -\frac{1}{3} \ngeq 0$

$\therefore$ $\begin{bmatrix}1\\1\end{bmatrix}$ does not satisfy KT for (c)

for (a)

$$\nabla U_c = u_1 \nabla g_1 + u_2 \nabla g_2$$

$$\Rightarrow \begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix} \Rightarrow \begin{bmatrix}u_1\\u_2\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}1 & -2\\-2 & 1\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$$

$\therefore$ $\begin{bmatrix}1\\1\end{bmatrix}$ satisfies KT for (a).

Since $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ does not lie on the g_1 or g_2 line

ie $g_1 > 0$ and $g_2 > 0$,

the steepest descent that is feasible lies in the negative gradient direction

$\therefore$ for (a) direction of $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$

(b) " " $\begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$

(c) " " $\begin{bmatrix} -2 \\ -\frac{1}{2} \end{bmatrix}$

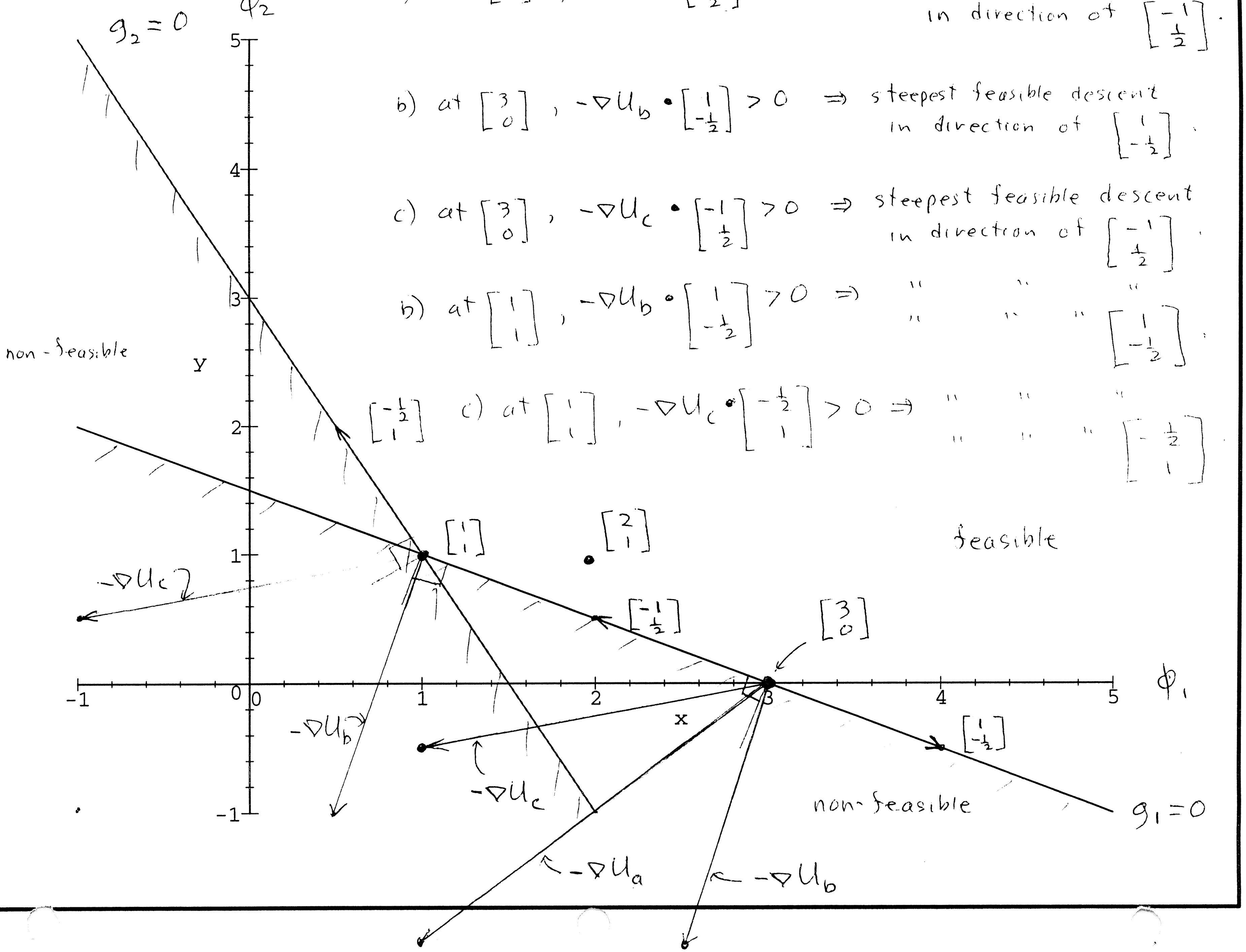
a) at $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $-\nabla U_a \bullet \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} > 0 \Rightarrow$ steepest feasible descent in direction of $\begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$.

b) at $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $-\nabla U_b \bullet \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} > 0 \Rightarrow$ steepest feasible descent in direction of $\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$.

c) at $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $-\nabla U_c \bullet \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} > 0 \Rightarrow$ steepest feasible descent in direction of $\begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$.

b) at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $-\nabla U_b \bullet \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} > 0 \Rightarrow$ " " " $\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$.

c) at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $-\nabla U_c \bullet \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} > 0 \Rightarrow$ " " " $\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$

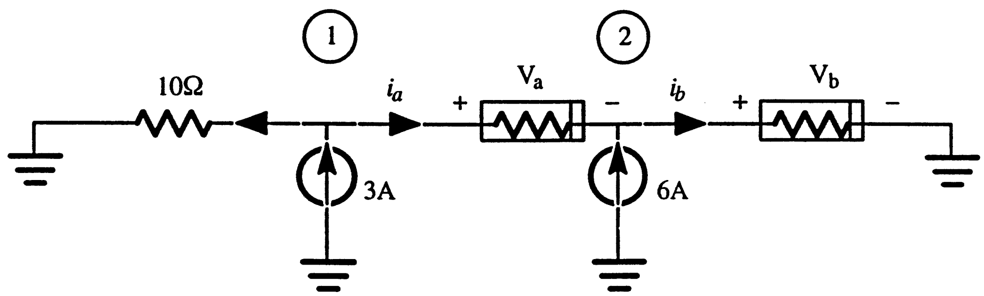


CoE 3KB3 CLASS TEST #4

March 29, 1996 (Duration : 35 minutes)

Q1. Consider the nonlinear circuit shown, where

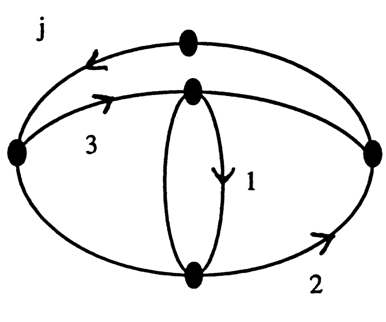
$$i_a = \frac{1}{3}v_a^3 + v_a^2 \quad \text{and} \quad i_b = v_b^3 - 5v_b$$



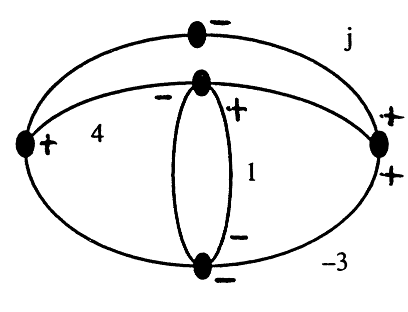
a) Using KCL, state the nonlinear nodal equations. Express the nonlinear equations in linearized form at the jth iteration of the Newton Algorithm (show all the equations, do not solve).

b) Draw the companion network at the jth iteration and state the correspnding nodal equations.

Q2. Complete the following diagram and verify Tellegen's theorem.



$\underset{\sim}{I}$



$\underset{\sim}{\hat{V}}$

SOLUTIONS : CLASS TEST #4 CoE 3KB3 MARCH 29 '96

Q1. a) According to KCL, the nodal equations at node 1 and 2 can be written as

$$.1\, v_1 + \frac{1}{3}(v_1 - v_2)^3 + (v_1 - v_2)^2 - 3 = 0$$

$$v_2^3 - 5v_2 - \frac{1}{3}(v_1 - v_2)^3 - (v_1 - v_2)^2 - 6 = 0$$

If we let

$$f_1 = .1\, v_1 + \frac{1}{3}(v_1 - v_2)^3 + (v_1 - v_2)^2 - 3$$

$$f_2 = v_2^3 - 5v_2 - \frac{1}{3}(v_1 - v_2)^3 - (v_1 - v_2)^2 - 6$$

Then the Jacobian matrix at the jth iteration is

$$\underline{J}^j = \begin{bmatrix} \frac{\partial f_1}{\partial v_1} & \frac{\partial f_1}{\partial v_2} \\ \frac{\partial f_2}{\partial v_1} & \frac{\partial f_2}{\partial v_2} \end{bmatrix} =$$

$$= \begin{bmatrix} .1 + (v_1 - v_2)^2 + 2(v_1 - v_2) & -(v_1 - v_2)^2 - 2(v_1 - v_2) \\ -(v_1 - v_2)^2 - 2(v_1 - v_2) & 3v_2^2 - 5 + (v_1 - v_2)^2 + 2(v_1 - v_2) \end{bmatrix}$$

and the linearized form of nodal equations at the jth iteration of the Newton algorithm can be written as

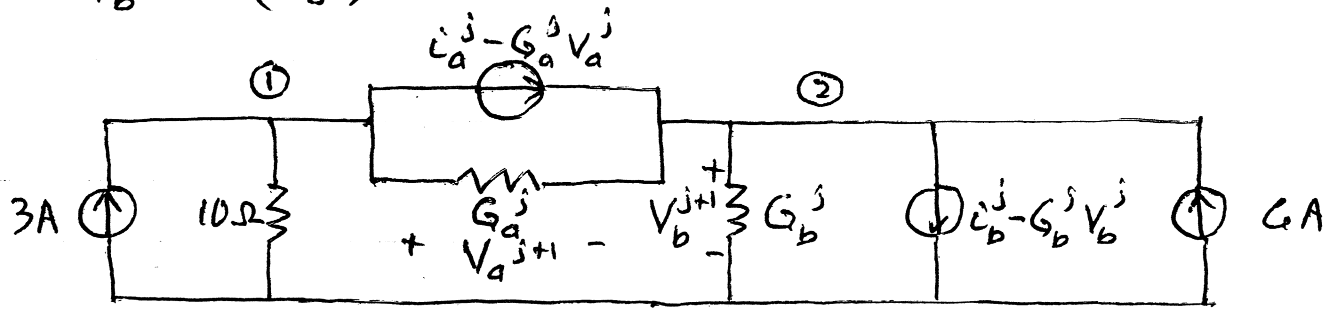
$$\underline{J}^j(\underline{v}^{j+1} - \underline{v}^j) = -\underline{f}^j$$

Q1. b) The companion network at the jth iteration is shown below, where

$$i_a^j = \frac{1}{3}(v_a^3)^j + (v_a^2)^j, \quad i_b^j = (v_b^3)^j - 5(v_b)^j$$

$$G_a^j = (v_a^2)^j + 2(v_a)^j, \quad v_a^j = v_1^j - v_2^j, \quad v_b^j = v_2^j$$

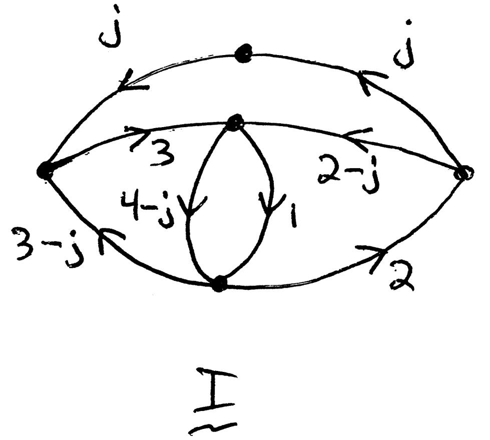
$$G_b^j = 3(v_b^2)^j - 5$$

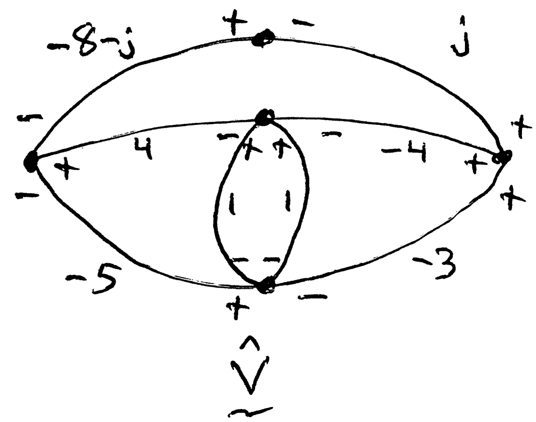


The corresponding nodal equations are

$$\begin{bmatrix} .1 + G_a^j & -G_a^j \\ -G_a^j & G_a^j + G_b^j \end{bmatrix} \begin{bmatrix} v_1^{j+1} \\ v_2^{j+1} \end{bmatrix} = \begin{bmatrix} 3 - (i_a^j - G_a^j V_a^j) \\ 6 + i_a^j - G_a^j V_a^j - (i_b^j - G_b^j V_b^j) \end{bmatrix}$$

Q2





$$\sum_{\text{branches}} vi = (j)(-8-j) + (j)(j) + (3)(4) + (2-j)(-4)$$

$$+ (4-j) + 1 + (-5)(3-j) - (2)(-3)$$

$$= -8j - j^2 + j^2 + 12 - 8 + 4j + 4 - j + 1 - 15 + 5j + 6$$

$$= (-8+4-1+5)j + (12-8+4+1-15+6)$$

$$= 0$$

3KB3 Tutorial #7

Adjoint Network Analysis

(I) Why Adjoint Network

Sensitivity Analysis:

During Circuit Optimization (Gradient search)

$$\underset{\underset{\sim}{\phi}}{\text{minimize}}\ U(\underset{\sim}{\phi})$$

$\underset{\sim}{\phi}$ → Circuit optimizable parameter e.g. R. C, L.

U → Objective function of response e.g. V_{out} I_{out} S_{11} S_{12}

Next Searching point is derived (obtained) from the information of $\frac{\partial U}{\partial \underset{\sim}{\phi}}$ $\left(\frac{\partial U}{\partial \phi_1}, \frac{\partial U}{\partial \phi_2}, \cdots\cdots\right)$.

Computer derivative evaluation method.

① original

$$\frac{\partial U}{\partial \phi_n} \doteq \frac{U(\underset{\sim}{\phi} + \Delta\phi_n) - U(\underset{\sim}{\phi})}{\Delta\phi_n}$$

$$\left(\begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\\ \vdots\end{bmatrix} + \begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \Delta\phi_n\\ \vdots\end{bmatrix}\right)$$

2 network analyses required
- one for evaluating $U(\underset{\sim}{\phi}+\Delta\phi_n)$
- one for $U(\underset{\sim}{\phi})$

For n optimizable parameters, $2n$ network analyses required for each iterative. (time consuming).

② adjoint network method.

No matter how many optimizable parameters are involved, Only 2 network analyses are required for evaluting

$$\frac{\partial U}{\partial \phi_1}, \frac{\partial U}{\partial \phi_2}, \cdots, \frac{\partial U}{\partial \phi_n}, \cdots$$

(II) The idea of adjoint network method.

The adjoint network method is based on Tellegen's Theorem. It is devised for the efficient calculation of $\frac{\partial I}{\partial \phi}, \frac{\partial V}{\partial \phi}$

ϕ: R, C, L, g_m

① Step 1. Construct the adjoint network

Rules:

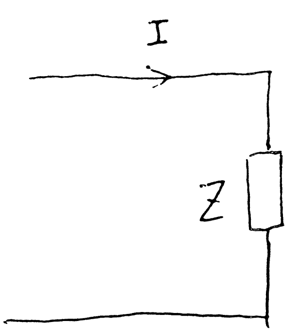
$\frac{\partial I}{\partial \phi}, \frac{\partial V}{\partial \phi}$

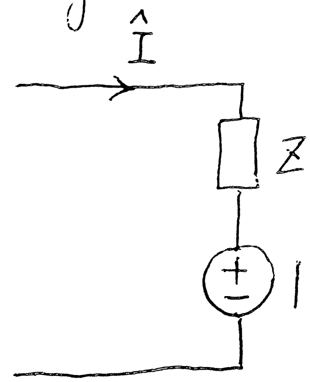
Sensitivity of the Response to be evaluated in the Original branch

Original branch

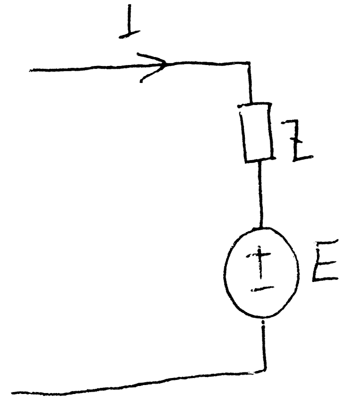
Corresponding adjoint branch

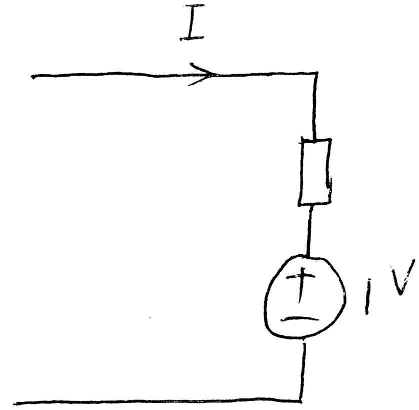
I



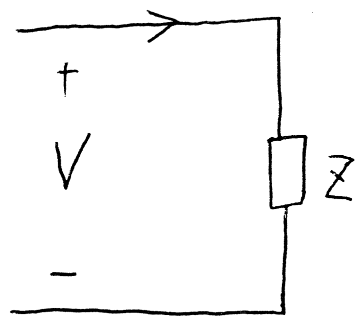


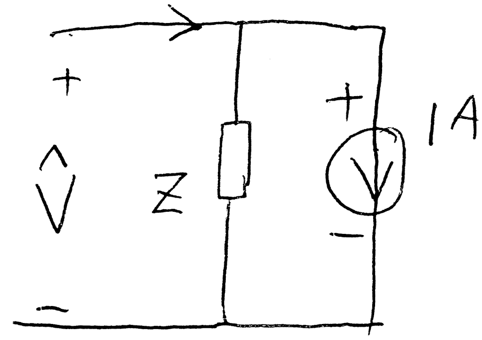
I



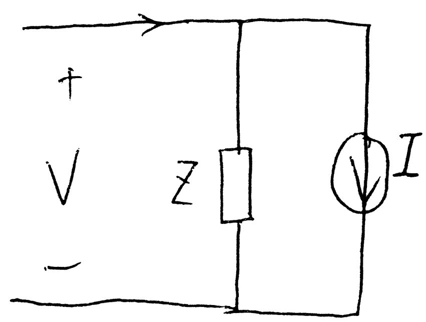


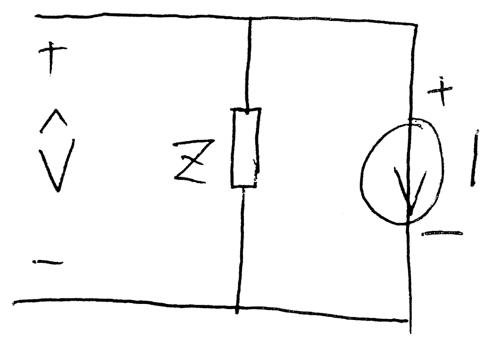
V



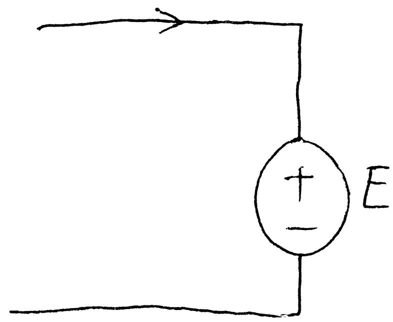


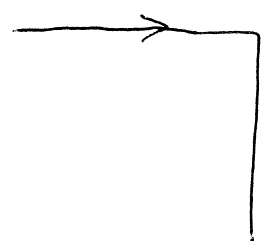
V



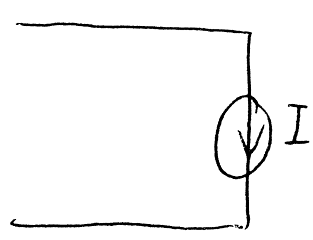


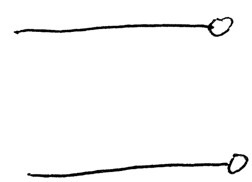
None



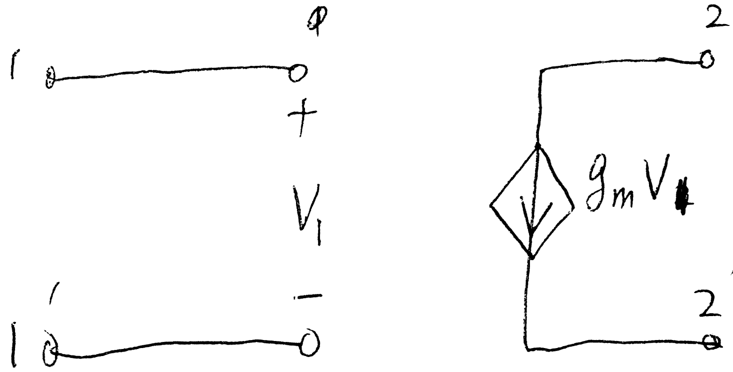


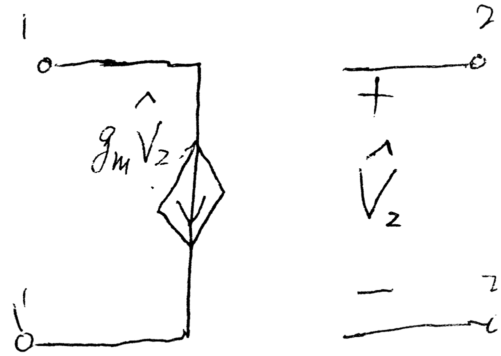
None





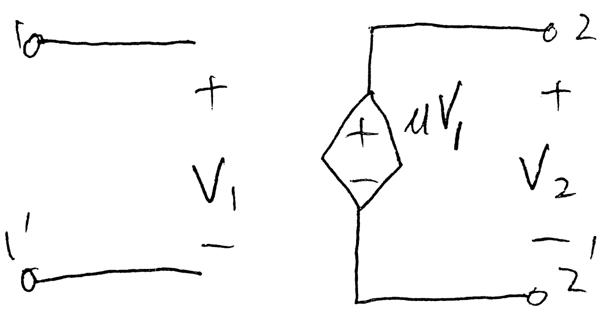
None

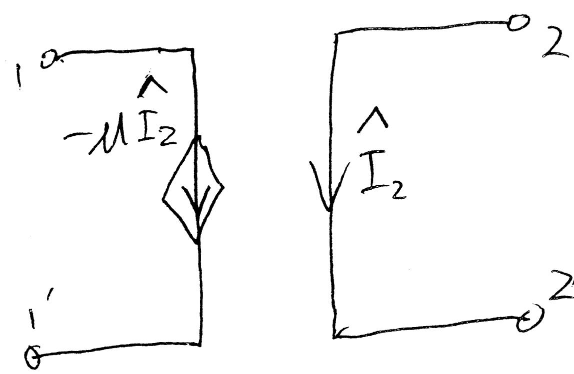




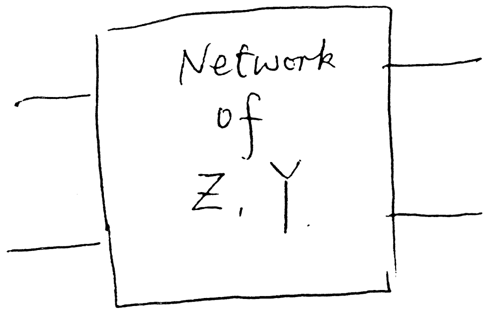
associated directions

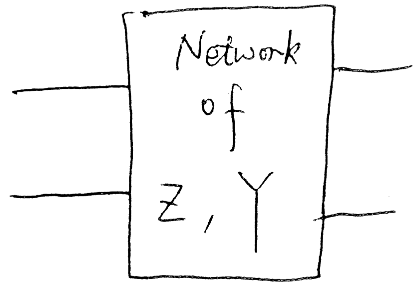
None





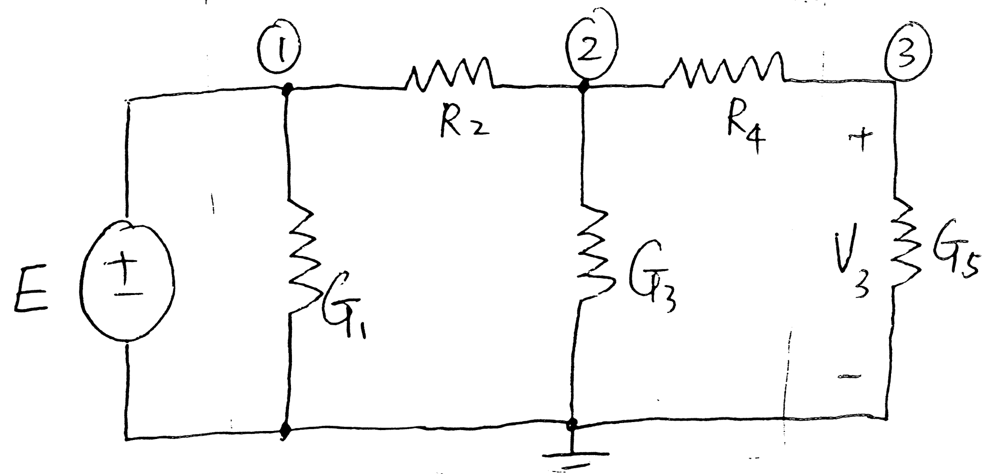
None



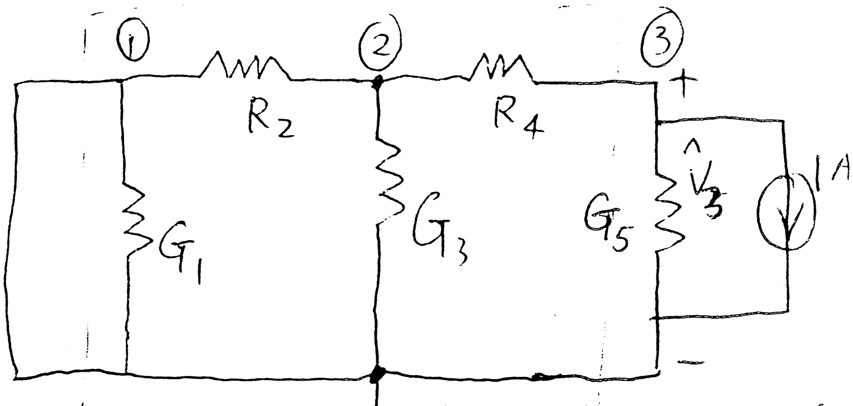


<u>Example 1</u> In the following circuit, the sensitivity of V_3 is of interest, construct its adjoint network.

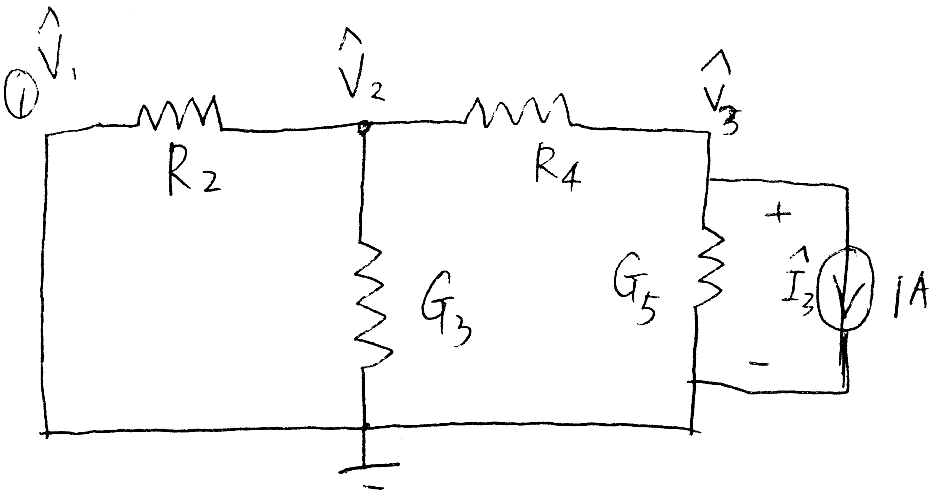
Original network



adjoint network.



The Adjoint network is simplified to



② Step 2
Evaluate the sensitivity according to the first principle :

$$\sum \hat{V} \frac{\partial I}{\partial \phi} - \sum \hat{I} \frac{\partial V}{\partial \phi} = G_\phi \qquad ①$$

corresponding branch

Voltage Sources in the adjoint network

Current Sources in the adjoint network.

ϕ branch

For the evaluation of $\frac{\partial V}{\partial \phi}$

① is expressed as $\sum \hat{I} \frac{\partial V}{\partial \phi} = -G_\phi$

For the evaluation of $\frac{\partial I}{\partial \phi}$

① is expressed as $\sum \hat{V} \frac{\partial I}{\partial \phi} = G_\phi$

$$G_\phi = \begin{bmatrix} \underset{\sim}{V_a} \\ \\ \underset{\sim}{I_b} \end{bmatrix}^T \begin{bmatrix} -\dfrac{\partial Y^T}{\partial \phi} & \dfrac{\partial M^T}{\partial \phi} \\ \\ -\dfrac{\partial A^T}{\partial \phi} & \dfrac{\partial Z^T}{\partial \phi} \end{bmatrix} \begin{bmatrix} \hat{V}_a \\ \\ \hat{I}_b \end{bmatrix}$$

generalized Sensitivity of ϕ

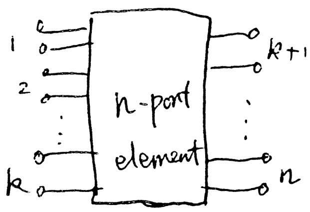
For element R $\longrightarrow$ only 1 port

In generalized form, assume

element X $\longrightarrow$ n ports

original

$$\begin{array}{c} \underset{\sim}{V_a} \\ \\ \underset{\sim}{V_b} \end{array} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \\ V_{k+1} \\ \vdots \\ V_n \end{bmatrix} = Z \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_k \\ I_{k+1} \\ \vdots \\ I_n \end{bmatrix} \begin{array}{c} \underset{\sim}{I_a} \\ \\ \underset{\sim}{I_b} \end{array}$$



adjoint

$$\begin{array}{c} \underset{\sim}{\hat{V}_a} \\ \\ \underset{\sim}{\hat{V}_b} \end{array} \begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \vdots \\ \hat{V}_k \\ \hat{V}_{k+1} \\ \vdots \\ \hat{V}_n \end{bmatrix} = Z \begin{bmatrix} \hat{I}_1 \\ \hat{I}_2 \\ \vdots \\ \hat{I}_k \\ \hat{I}_{k+1} \\ \vdots \\ \hat{I}_n \end{bmatrix} \begin{array}{c} \hat{I}_a \\ \\ \hat{I}_b \end{array}$$

$$\overset{n\times 1}{\underset{\sim}{V}} = [Z] \overset{n\times 1}{\underset{\sim}{I}}$$ Characterization.

$$\underset{\sim}{I} = [Y] \underset{\sim}{V}$$

$$\begin{bmatrix} \underset{\sim}{I_a} \\ \underset{\sim}{V_b} \end{bmatrix} = \begin{bmatrix} Y & A \\ M & Z \end{bmatrix} \begin{bmatrix} \underset{\sim}{V_a} \\ \underset{\sim}{I_b} \end{bmatrix}$$

For resistance $\phi = R$

$$G_\phi = \begin{bmatrix} I \end{bmatrix} \begin{bmatrix} \diagdown & & \\ & \frac{\partial Z^T}{\partial R} & \\ & & \diagdown \end{bmatrix} \begin{bmatrix} \hat{I} \end{bmatrix} \qquad Z = R$$

$$G_R = I \frac{\partial R}{\partial R} \hat{I} = I \hat{I}$$

I — current through R

For inductance

$$G_\phi = \begin{bmatrix} V \end{bmatrix} \begin{bmatrix} -\frac{\partial Y^T}{\partial G} & & \\ & \diagdown & \\ & & \diagdown \end{bmatrix} \begin{bmatrix} \hat{V} \end{bmatrix} \qquad Y = G$$

$$G_G = -V \frac{\partial G}{\partial G} \hat{V} = -V \hat{V}$$

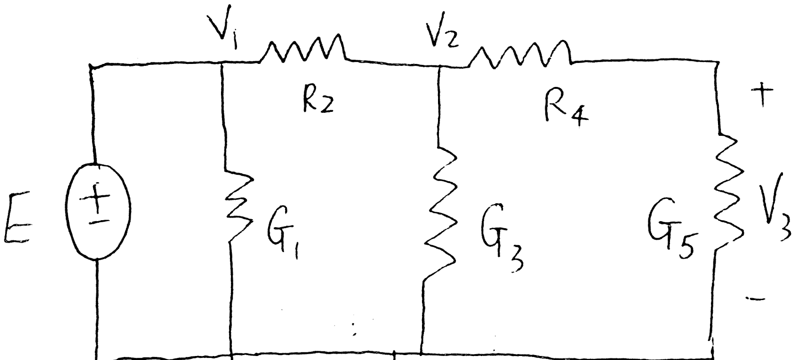
V — voltage across G

For the rest (e.g. VCVS VCCS $\phi = g_m, \mu, r_m \cdots$) see table 7.1

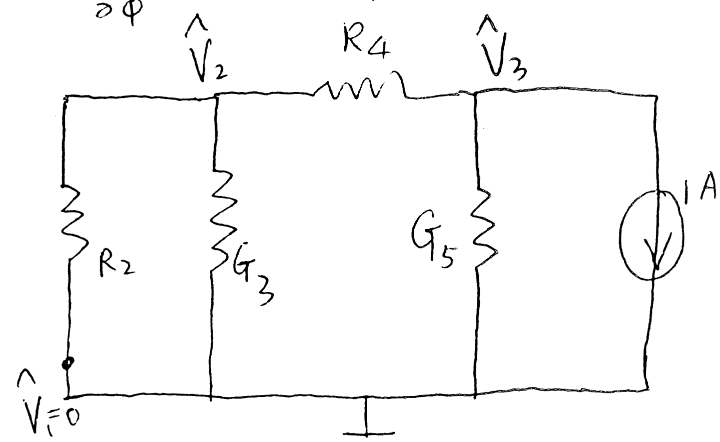
example 2 : For example 1, assume $G_1 = G_3 = G_5 = 1\,S$

$R_2 = R_4 = 0.5\,\Omega$, evaluate $\frac{\partial V_3}{\partial G_1}$ $\frac{\partial V_3}{\partial R_2}$ $\frac{\partial V_3}{\partial G_3}$

(Since the value of E will not affect the $\frac{\partial V}{\partial \phi}$ → results, we set $E = 1$)



Original



adjoint

perform 1st network analysis

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{6}{11} \\ \frac{4}{11} \end{bmatrix}$$

perform 2nd network analysis.

$$\begin{bmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{11} \\ -\frac{5}{11} \end{bmatrix}$$

$$\frac{\partial V_3}{\partial G_1} : \qquad \sum \hat{I}_3 \frac{\partial V_3}{\partial G_1} = -G_{G_1}$$

V_ϕ

$$\Downarrow$$

$Y = G_1$

$$1 \cdot \frac{\partial V_3}{\partial G_1} = -\left(-V_1 \frac{\partial Y}{\partial G_1} \hat{V}_1\right) = +V_1 \hat{V}_1$$

$$\frac{\partial V_3}{\partial G_1} = -1 \cdot 0 = 0.$$

— the change of G_1 will not affect V_3
(by inspecting the network, we find it is true)

$$\frac{\partial V_3}{\partial R_2} : \qquad \sum \hat{I} \frac{\partial V_3}{\partial R_2} = -G_{R_2}$$

$$\Downarrow$$

$Z = R_2$

$$1 \cdot \frac{\partial V_3}{\partial R_2} = -\left(+ I_{R_2} \frac{\partial Z}{\partial R_2} \hat{I}_{R_2}\right) = -I_{R_2} \hat{I}_{R_2}$$

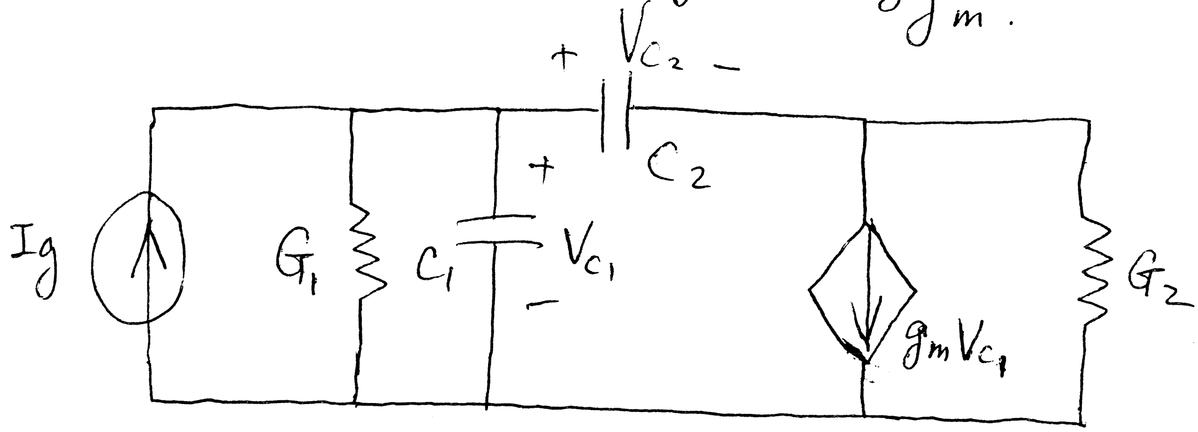
$$\frac{\partial V_3}{\partial R_2} = -\frac{V_1 - V_2}{R_2} \frac{\hat{V}_1 - \hat{V}_2}{R_2} = -\left(1 - \frac{6}{11}\right)\left(0 + \frac{2}{11}\right) / R_2^2$$

$$= -\frac{10}{121} \Big/ \frac{1}{4} = -\frac{40}{121}$$

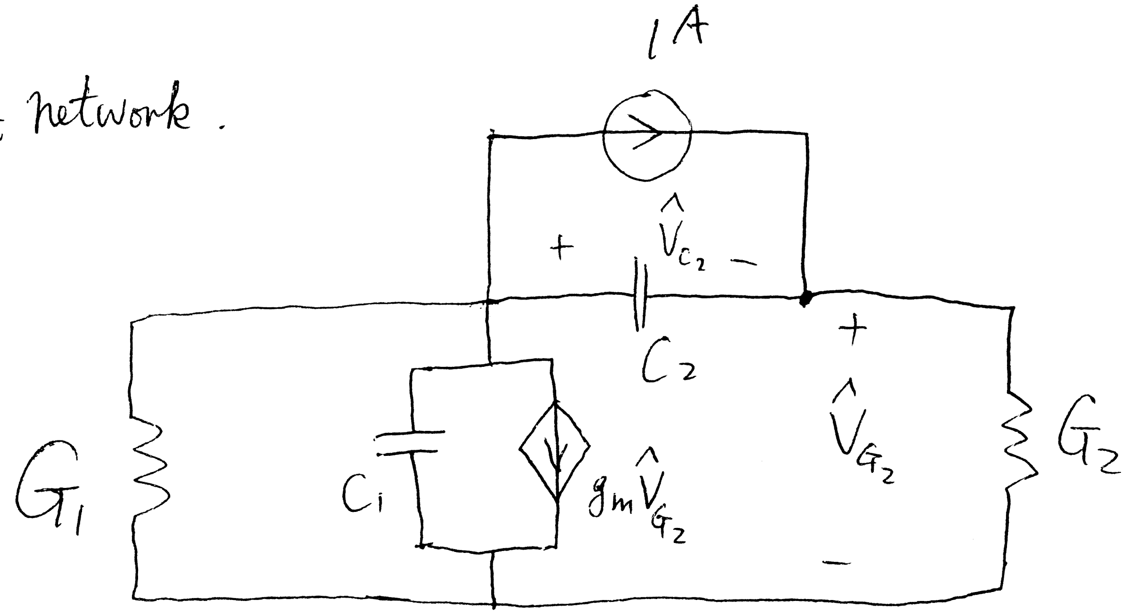
$Y = G_3$

$$\frac{\partial V_3}{\partial G_3} = -\left(-V_2 \frac{\partial Y}{\partial G_3} \hat{V}_2\right) = V_2 \cdot \hat{V}_2 = \frac{6}{11} \cdot \left(-\frac{2}{11}\right) = -\frac{12}{121}$$

Example 3. Consider the circuit, which is assumed to be in the sinusoidal steady state. Derive the adjoint ~~excitations~~ network appropriate for calculating the first-order sensitivity of $\frac{\partial V_{C2}}{\partial g_m}$.



adjoint network.



current source

$$\sum \hat{I} \frac{\partial V}{\partial \phi} = -G_\phi$$

according to table 7.1 $\quad G_\phi = -V_{C1}\hat{V}_{G_2}$

then

$$1 \cdot \frac{\partial V_{c_2}}{\partial g_m} = V_{C_1} \hat{V}_{G_2}$$

20783

2/1 pg 5			
2/10 pg 11			
3/12 pg 22			
3/15 pg 26			
3/22 pg 42			
3/26 pg 63			
4/1 pg 67			
4/4 pg 78			

[redacted] COPRESS

[redacted] W 25070 J
[redacted] 25071 N
[redacted] 25072 B
[redacted] E 25073 R
[redacted] 25074 C
[redacted] 25075 V
[redacted] 25078 R
[redacted] 25079 R

[redacted] ANADA INC.
[redacted] DALE, ONTARIO

[redacted] LED
[redacted] FIBRE

*SIGNIFIE 75 %
FIBRES RECYCLÉES,
25 % DÉCHETS DE
CONSOMMATION

[redacted] F PRODUCTS
[redacted] LED

AUTRES PRODUITS:
25 % FIBRES RECYCLÉES

0 50505 25075 2