Computer Engineering 3KB3

Tests & Tutorials

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Test #1.

Consider the function

$$U = \varphi_1^2 + 3\varphi_2^2 - 4$$

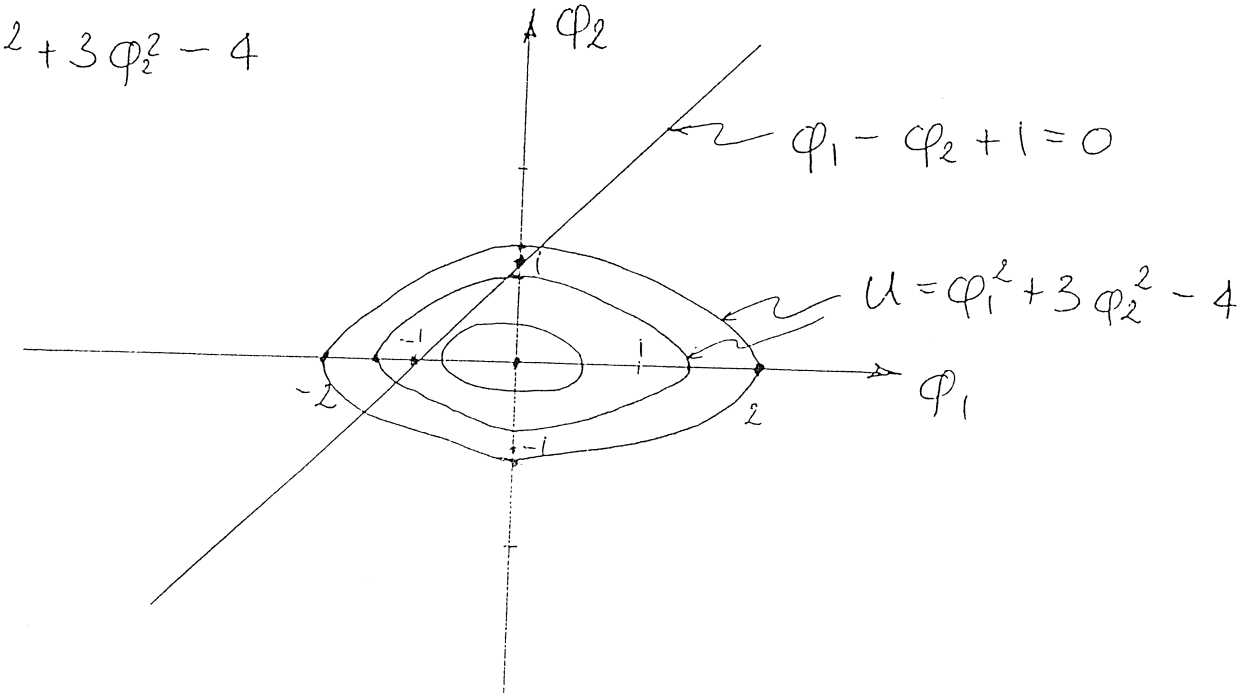
subject to an equality constraint:

$\varphi_1 - \varphi_2 + 1 = 0.$

1. Plot a few contour lines of U.
2. Plot the constraint and indicate the feasible region.
3. Find the unconstrained optimum of U.
4. Show that this optimum is a minimum.
5. Solve the constrained problem by direct substitution.
6. Solve the constrained problem using the Lagrange function.
7. Convert the equality constraint $h = \varphi_1 - \varphi_2 + 1 = 0$ into two inequality constraint

Solution to Test #1 01.02.

(1.) $U = \varphi_1^2 + 3\varphi_2^2 - 4$



(2.) $g = \varphi_1 - \varphi_2 + 1 = 0$. Because g is an equality constraint it defines the feasible region.

(3.) $U = \varphi_1^2 + 3\varphi_2^2 - 4$

$$\frac{\partial U}{\partial \varphi_1} = 2\varphi_1 ; \quad \frac{\partial U}{\partial \varphi_2} = 6\varphi_2 ; \quad \underset{\sim}{\nabla} U = \begin{bmatrix} 2\varphi_1 \\ 6\varphi_2 \end{bmatrix} = \underset{\sim}{0}$$

for $[\varphi_1 \ \varphi_2]^T = \underset{\sim}{0}$

(4.) The Hessian matrix for U is:

$$\frac{\partial^2 U}{\partial \varphi_1^2} = 2 ; \quad \frac{\partial^2 U}{\partial \varphi_2^2} = 6 ; \quad \frac{\partial^2 U}{\partial \varphi_1 \varphi_2} = \frac{\partial^2 U}{\partial \varphi_2 \varphi_1} = 0$$

and $\underset{\sim}{H} = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$ which is positive definite

so $\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a minimum.

(5.) $u = \varphi_1^2 + 3\varphi_2^2 - 4$

$\varphi_1 - \varphi_2 + 1 = 0 \implies \varphi_1 = \varphi_2 - 1$

$u = (\varphi_2 - 1)^2 + 3\varphi_2^2 - 4$

$u = \varphi_2^2 - 2\varphi_2 + 1 + 3\varphi_2^2 - 4 = 4\varphi_2^2 - 2\varphi_2 - 3$

$\frac{\partial u}{\partial \varphi_2} = 8\varphi_2 - 2 = 0 \implies \varphi_2 = \frac{1}{4} = 0.25$

$\varphi_1 = \varphi_2 - 1 = 0.25 - 1 = -0.75$

(6.) $L(\underset{\sim}{\varphi}, \lambda) = u(\underset{\sim}{\varphi}) + \lambda h$

$L(\underset{\sim}{\varphi}, \lambda) = \varphi_1^2 + 3\varphi_2^2 - 4 + \lambda(\varphi_1 - \varphi_2 + 1)$

Then solving

$\frac{\partial L}{\partial \varphi_1} = 2\varphi_1 + \lambda = 0$

$\frac{\partial L}{\partial \varphi_2} = 6\varphi_2 - \lambda = 0$

$\frac{\partial L}{\partial \lambda} = \varphi_1 - \varphi_2 + 1 = 0$

we get $\varphi_1 = -0.75$, $\varphi_2 = 0.25$ and $\lambda = 1.5$

(7.) $h = 0 \implies \begin{cases} h \geq 0 \\ -h \geq 0 \end{cases} \implies \begin{cases} h_1 = \varphi_1 - \varphi_2 + 1 \geq 0 \\ h_2 = -\varphi_1 + \varphi_2 - 1 \geq 0 \end{cases}$

COMPUTER ENGINEERING 3KB3:
COMPUTATIONAL METHODS II

Class Test #1

Duration of Test: 30 minutes

(a) Answer as many questions as you can. Expand $U(\boldsymbol{\phi})$ around $\boldsymbol{\phi}^0$ into the Taylor series.

(b) State the formula for the first order change of $U(\boldsymbol{\phi})$.

(c) State the formula for the second order change of $U(\boldsymbol{\phi})$.

(d) Given a quadratic function $U(\boldsymbol{\phi}) = \phi_1^2 + 2\phi_2^2 - 3\phi_1 - 6$, let $\boldsymbol{\phi}^0 = [\ 0.2 \quad -0.2\ ]^T$.

Calculate $U(\boldsymbol{\phi}^0)$.

(e) Sketch a few contours of U.

(g) Evaluate ∇U at $\boldsymbol{\phi}^0$.

(h) Evaluate the Hessian matrix $\boldsymbol{H}$ at $\boldsymbol{\phi}^0$.

(i) Calculate the first-order change in U at $\boldsymbol{\phi}^0$ for $\Delta\boldsymbol{\phi} = [\ 0.2\ \ -0.2\ ]^T$.

(j) Calculate the second-order change in U at $\boldsymbol{\phi}^0$ for $\Delta\boldsymbol{\phi} = [\ 0.2\ \ -0.2\ ]^T$.

(k) Using the results obtained in (d), (i) and (j), estimate the value of U at $\boldsymbol{\phi}' = [1.2\ 0.8]^T$.

(l) How good is the estimate obtained in (k)?

(m) Find the turning point of U.

(n) Is the turning point found in (m) a maximum or a minimum? Explain your answer.

(o) What is the extreme value of U?

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Solution

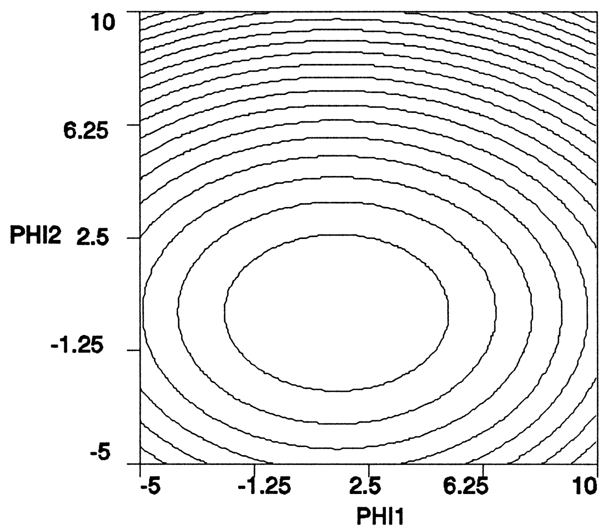
(a) $$U(\boldsymbol{\phi}) = U(\boldsymbol{\phi}^0) + \triangle\boldsymbol{\phi}^T \nabla U(\boldsymbol{\phi}^0) + \frac{1}{2}\triangle\boldsymbol{\phi}^T \boldsymbol{H} \triangle\boldsymbol{\phi} + \cdots$$

(b) $$\triangle\boldsymbol{\phi}^T \nabla U(\boldsymbol{\phi}^0)$$

(c) $$\frac{1}{2}\triangle\boldsymbol{\phi}^T \boldsymbol{H} \triangle\boldsymbol{\phi}$$

(d) $$U(\boldsymbol{\phi}^0) = -6$$

(e)



(g) $$\nabla U(\boldsymbol{\phi}^0) = \begin{bmatrix} 2\phi_1 - 3 \\ 4\phi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

(h) $$\boldsymbol{H}(\boldsymbol{\phi}^0) = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \boldsymbol{H}(\boldsymbol{\phi})$$

(i) $$\triangle\boldsymbol{\phi}^T \nabla U(\boldsymbol{\phi}^0) = [0.2 \quad -0.2] \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -1$$

(j) $$\frac{1}{2}\triangle\boldsymbol{\phi}^T \boldsymbol{H} \triangle\boldsymbol{\phi} = \frac{1}{2}\,[0.2 \quad 0.2] \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} = 0.12$$

(k) Using the Taylor series expansion

$$U(\boldsymbol{\phi}') \approx U(\boldsymbol{\phi}^0) + \Delta\boldsymbol{\phi}^T \nabla U(\boldsymbol{\phi}^0) + \frac{1}{2}\Delta\boldsymbol{\phi}^T \boldsymbol{H} \Delta\boldsymbol{\phi} + \cdots = -6 - 1 + 0.12 = -6.88$$

(l) Because U is a second order function the estimate in (k) is exact.

(m)

$$\nabla U(\boldsymbol{\phi}) = 0 = \begin{bmatrix} 2\phi_1 - 3 \\ 4\phi_2 \end{bmatrix}$$

So

$$\boldsymbol{\phi}^* = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

(n) $\boldsymbol{H}$ is positive definite

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

$$ab > c^2$$

so $\boldsymbol{\phi}^*$ is a minimum.

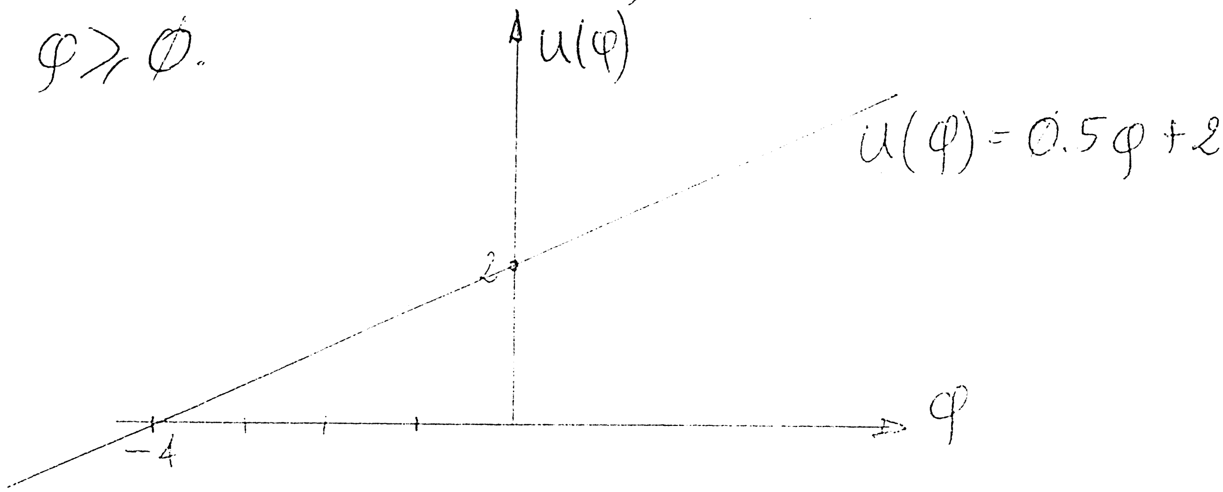
(o)

$$U(\boldsymbol{\phi}^*) = -8.25$$

(2.) Transformation of Variables:

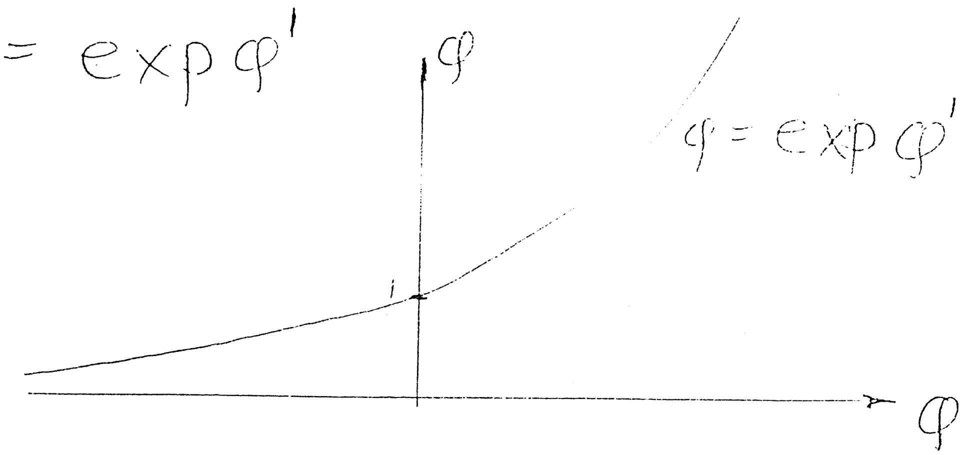
! We do not transform variables if we do not have to do so. !

Transformation of variables may result in badly posed problems. For example, let's minimize $U(\varphi) = 0.5\varphi + 2$ subject to $\varphi \geq 0$.



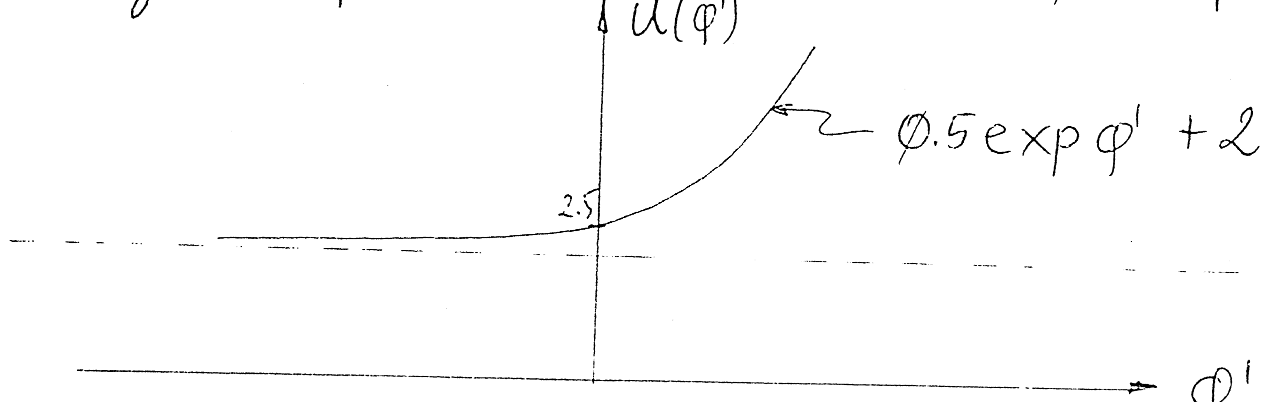
Assuming that our optimizer cannot handle constraints we introduce the following transformation

$$\varphi = \exp \varphi'$$



With this transformation $\varphi > 0$ and $-\infty < \varphi' < \infty$

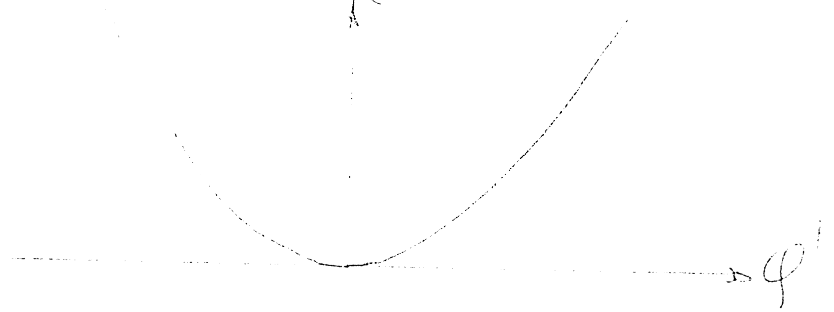
Solving the problem we have: $u = 0.5 \exp \varphi' + 2$



and the minimum accures at $\varphi' \rightarrow -\infty$ and $\varphi \rightarrow 0$. This is a very bad transformation for this example.

Now, let's use another transformation:

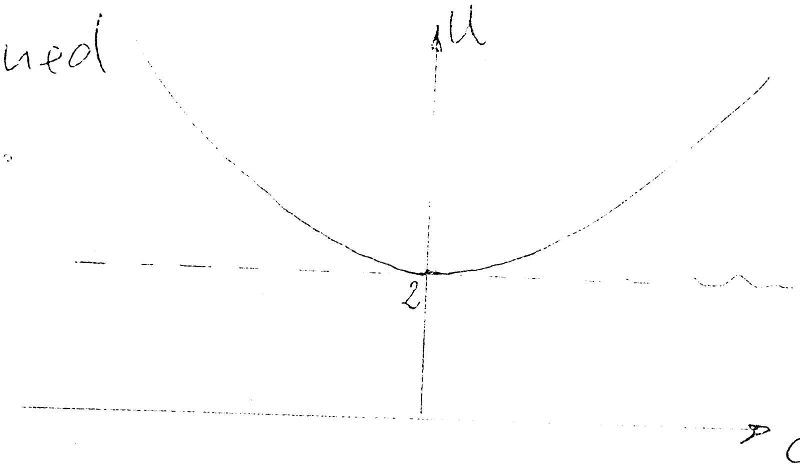
$$\varphi = (\varphi')^2$$



Then $u = 0.5(\varphi')^2 + 2$

and the unconstrained minimum is at 0.

$\varphi'^{m} = \varphi^{m} = 0.$



③ The penalty and barrier functions:

Another way to deal with constraints is the penalty or the barrier function.

- The penalty function:

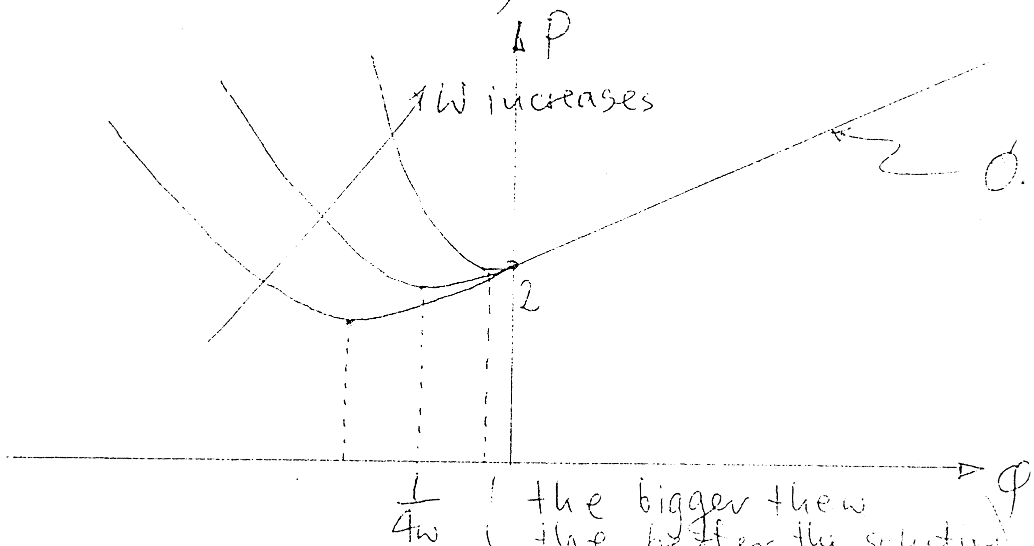
$$P(\underset{\sim}{\varphi}, \underset{\sim}{w}) = U(\underset{\sim}{\varphi}) + \sum_{i=1}^{m} w_i\, g_i^2(\underset{\sim}{\varphi})$$

where $w = \text{const} > 0$ if $g_i < 0$

$w = 0$ if $g_i \geq 0$

For our example $g = \varphi \geq 0$

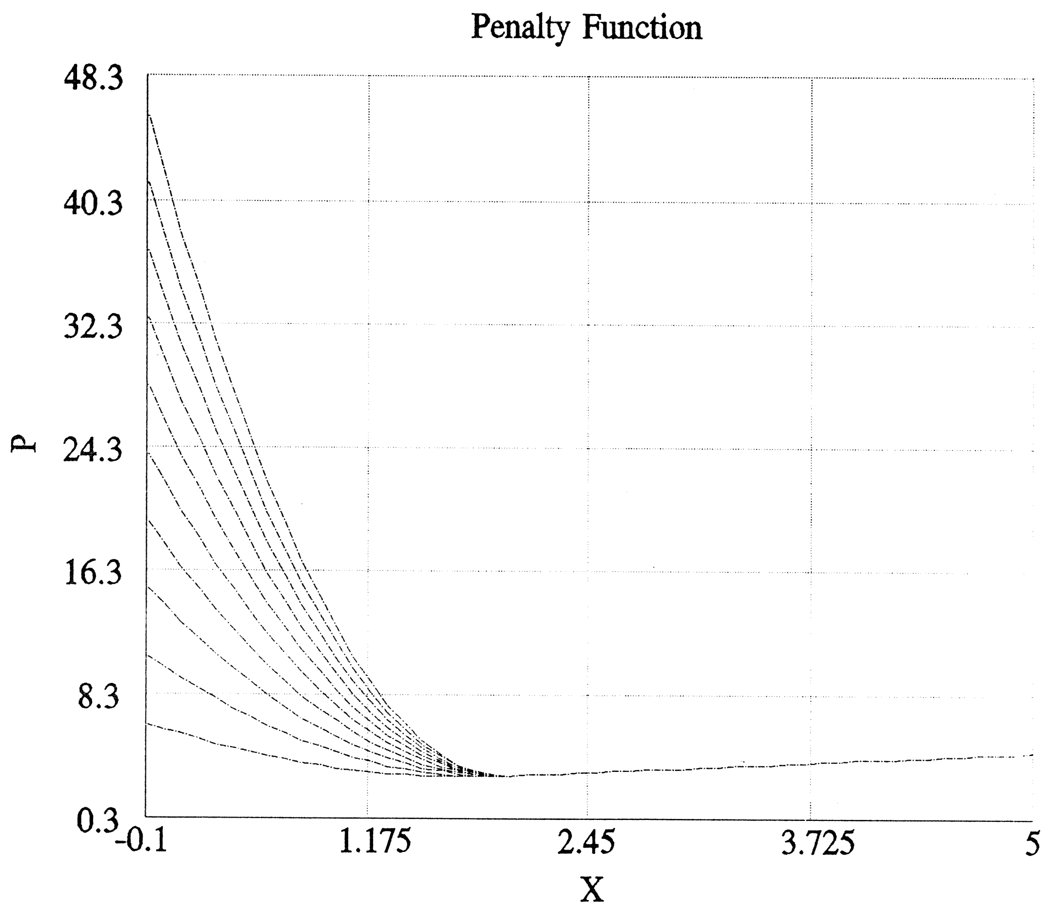
$$P = (0.5\varphi + 2) + w\varphi^2$$ $w > 0$ for $\varphi <$



$$\frac{dP}{d\varphi} = 0.5 + 2w\varphi$$

for $\frac{dP}{d\varphi} = 0$ we have $\varphi^\circ = -\frac{1}{4w}$

We can even look at H. $H = 2w$ and it is positive, so φ° corresponds to a minimum.


Penalty Function
48.3
40.3
32.3
24.3
16.3
8.3
0.3
P
-0.1
1.175
2.45
3.725
5
X

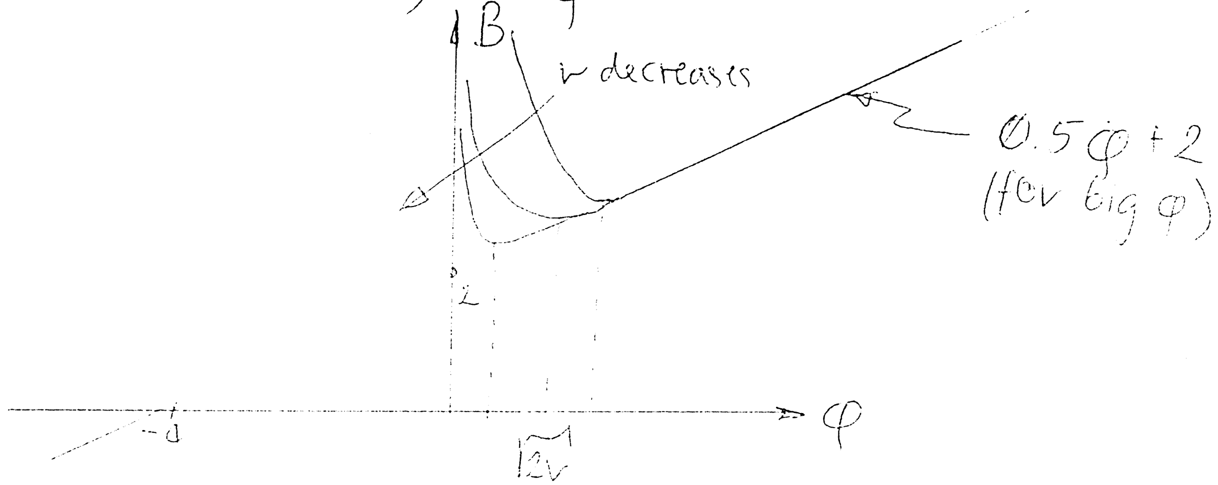
- The barrier function:

$$B(\underset{\sim}{\varphi}, r) = U(\underset{\sim}{\varphi}) + r \sum_{i=1}^{m} \frac{1}{g_i(\underset{\sim}{\varphi})}$$

where $g_i \geq 0$, $r > 0$

For our example $g = \varphi \geq 0$

$$B = (0.5\varphi + 2) + \frac{r}{\varphi} \qquad r > 0,\ \varphi > 0$$

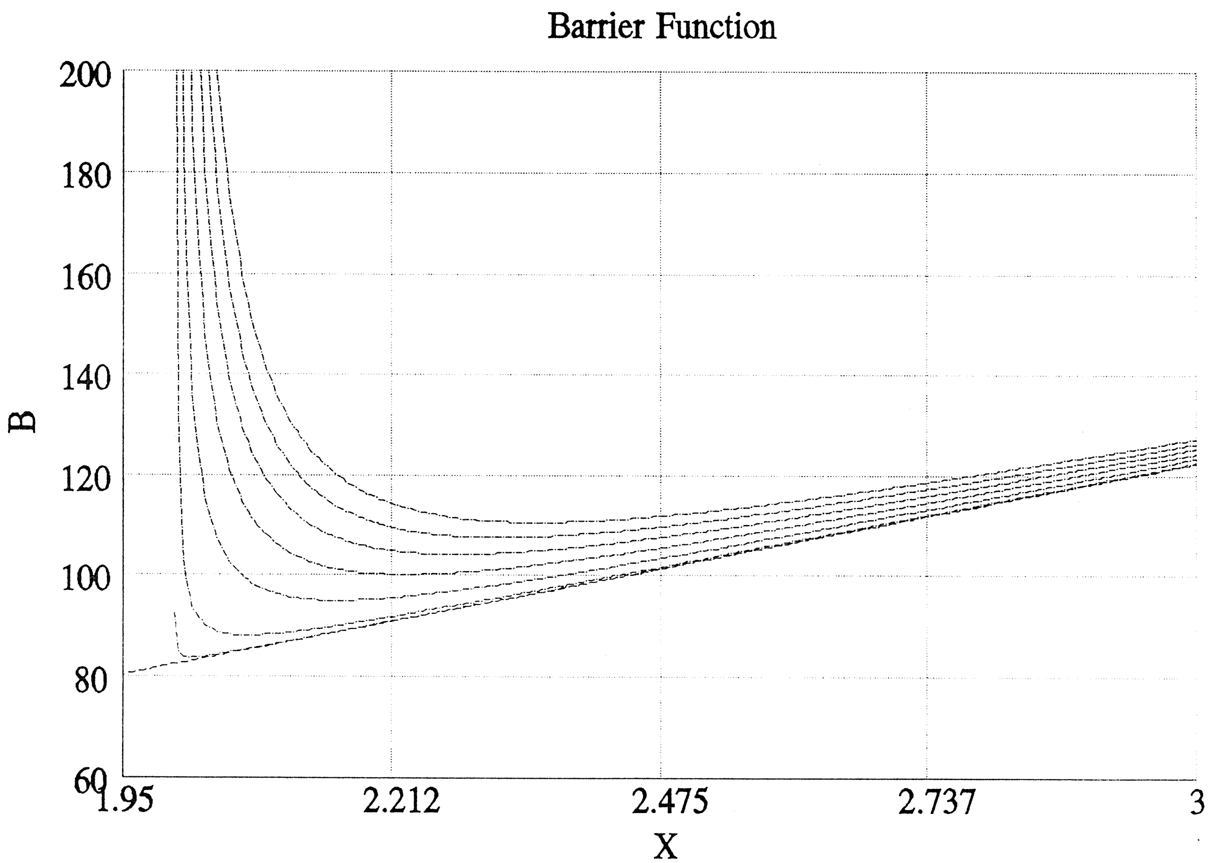


$$\frac{dB}{d\varphi} = 0.5 - \frac{r}{\varphi^2}$$

for $\frac{dB}{d\varphi} = 0$ we have $\varphi^\circ = \sqrt{2r}$

$H = 2r\varphi^{-3}$ which is positive ($\varphi > 0, r > 0$)

so φ° corresponds to a minimum.


Barrier Function
200
180
160
140
120
100
80
60
B
1.95
2.212
2.475
2.737
3
X

The exact-penalty function:

$$V(\underset{\sim}{\phi}, \underset{\sim}{\alpha}) = \max_{1 \le i \le m} [u(\underset{\sim}{\phi}),\ u(\underset{\sim}{\phi}) - \alpha_i g_i(\underset{\sim}{\phi})],$$

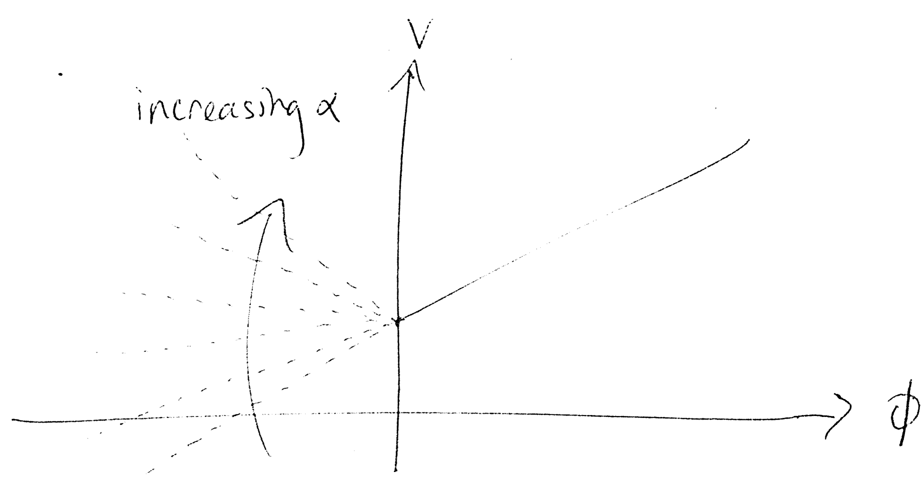
where $\alpha = \text{const.} > 0$

increase α

where $g_i(\hat{\phi}) \geq 0$.

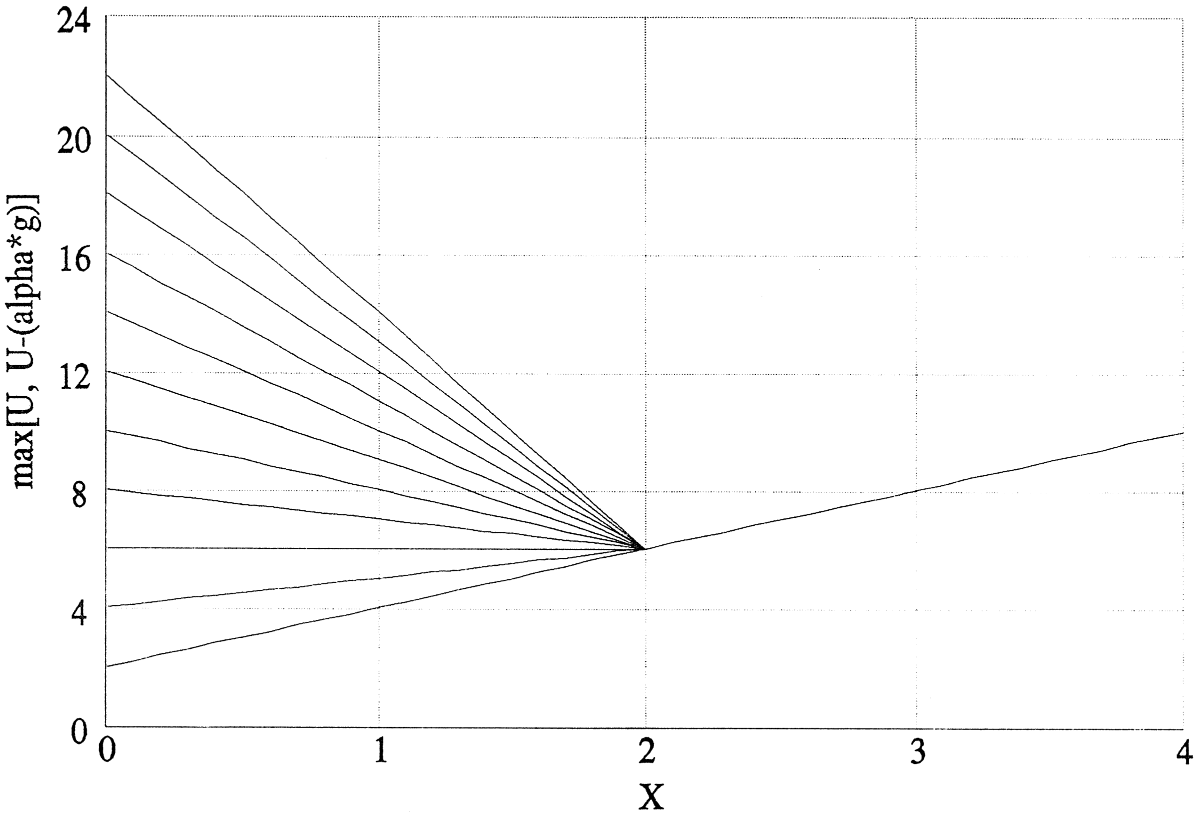
For our example, $g = \phi \geq 0$

$$V(\phi, \alpha) = \max[0.5\phi + 2,\ 0.5\phi + 2 - \alpha\phi]$$



- exact solution
- not differentiable.

Exact-Penalty Function



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! 3KB3 tutorial#3 - Penalty function example

Expression
        X=0; c=1;

        U = 0.5*X + 2;
        g = X - 2;

        w: if (g<0) (c) else (0);

        P = U + w*g^2;
End

Sweep
        c: from 1 to 10 step=1
                X: from -0.1 to 5 step=0.1  P
        { Xsweep Title="Penalty Function" c=all X=X Y=P };
End
```

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```
! 3KB3 tutorial#3 - Barrier function example

Expression
        X=0; r=1;

        U = 40*X + 2;
        g = X - 2;                    ! g = X >= 2

        B: if (X>2) (U + (r / g)) else (NaN);
End

Sweep
        r: 0.01 0.2 1 2 3 4 5
                X: from 1.95 to 3 step=0.003 B U
        { Xsweep Title="Barrier Function" Y_title="B" r=all X=X Y=B & U
          Ymin=70 Ymax=200 LEGEND=OFF };
End
```

```
! 3KB3 tutorial#3 - Exact-penalty function example

Expression
        X = ? 2 ?;
        alpha = 1;

        U = 2*X + 2;
        g = X - 2;
        U1 = U - (alpha*g);

        ! Define exact penalty function (EPF)
        EPF = max(U,U1);

End

Sweep
        alpha: from 0 to 10 step=1
                X: from 0 to 4 step=0.1  EPF
        { Xsweep Title="Exact-Penalty Function" alpha=all LEGEND=OFF
          X=X Xmin=0 Xmax=4
          Y=EPF.green
          Y_Title="max[U, U-(alpha*g)]" };
End

Specification
        EPF;
End
```

COMPUTER ENGINEERING 3KB3:
COMPUTATIONAL METHODS II

Class Test #2

Duration of Test: 30 minutes

PROBLEM #1

Consider three functions below:

$$f_1(\phi_1, \phi_2) = \phi_2 - 3$$
$$f_2(\phi_1, \phi_2) = -\phi_1 + 2$$
$$f_3(\phi_1, \phi_2) = \phi_1 - \phi_2 - 1.$$

1. Sketch two or three minimax contours of $f_1(\phi_1, \phi_2)$, $f_2(\phi_1, \phi_2)$ and $f_3(\phi_1, \phi_2)$.
2. Calculate the first order derivatives of f_1, f_2 and f_3.
3. Find the active functions at the point $[0\ 1]^T$.
4. Verify that $[0\ 1]^T$ is not the minimax solution.

PROBLEM #2

Consider the following constrained minimization problem:

Minimize w.r.t. ϕ

$$U = \phi_1 + 2\phi_2$$

subject to

$$g_1 = \phi_1 + \phi_2 - 2 \geq 0$$
$$g_2 = -\phi_1 + 2\phi_2 + 2 \geq 0.$$

1. Sketch two or three contours of the objective function, plot the constraint boundaries and indicate the feasible region.
2. Invoke the Kuhn-Tucker necessary conditions to test the point $[2\ 0]^T$.

February 1993

Solution to Classroom Test # 2

Problem # 1

1. Let $f_1 = 0$, $\phi_2 - 3 = 0$

$\Rightarrow \phi_2 = 3.$

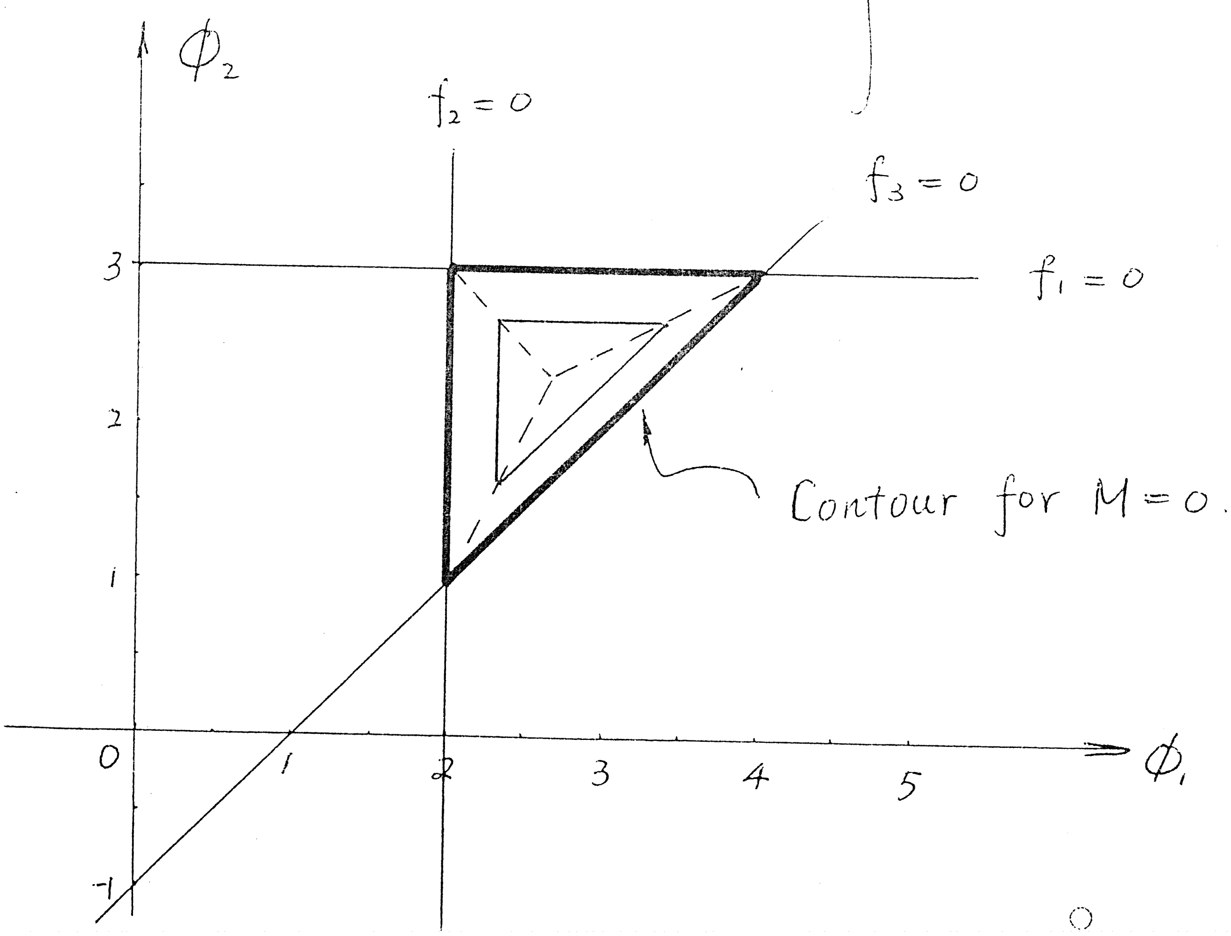
Let $f_2 = 0$, $-\phi_1 + 2 = 0$

$\Rightarrow \phi_1 = 2.$

Let $f_3 = 0$, $\phi_1 - \phi_2 - 1 = 0.$

$\Rightarrow \phi_2 = \phi_1 - 1.$

$M = \max\{f_1, f_2, f_3\}$



2. $$\underset{\sim}{\nabla} f_1 = \begin{bmatrix} \dfrac{\partial f_1}{\partial \phi_1} \\[2ex] \dfrac{\partial f_1}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{\nabla} f_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \underset{\sim}{\nabla} f_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3. At $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$$f_1 = 1 - 3 = -2, \quad f_2 = -0 + 2 = 2$$

$$f_3 = 0 - 1 - 1 = -2.$$

$\therefore$ f_2 is the active function.

4. At minimax optimum, the following conditions must hold:

$$\begin{cases} \displaystyle\sum_{i=1}^{3} u_i \underset{\sim}{\nabla} f_i = \underset{\sim}{0} & (1) \\ \displaystyle\sum_{i=1}^{3} u_i = 1 & (2) \\ u_i (M - f_i) = 0 & (3) \\ u_i \geq 0 & (4) \end{cases}$$

At $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$M = 2$.

According to condition (3).

$u_1 = u_3 = 0$.

Substitute this into (2), we have $u_2 = 1$.

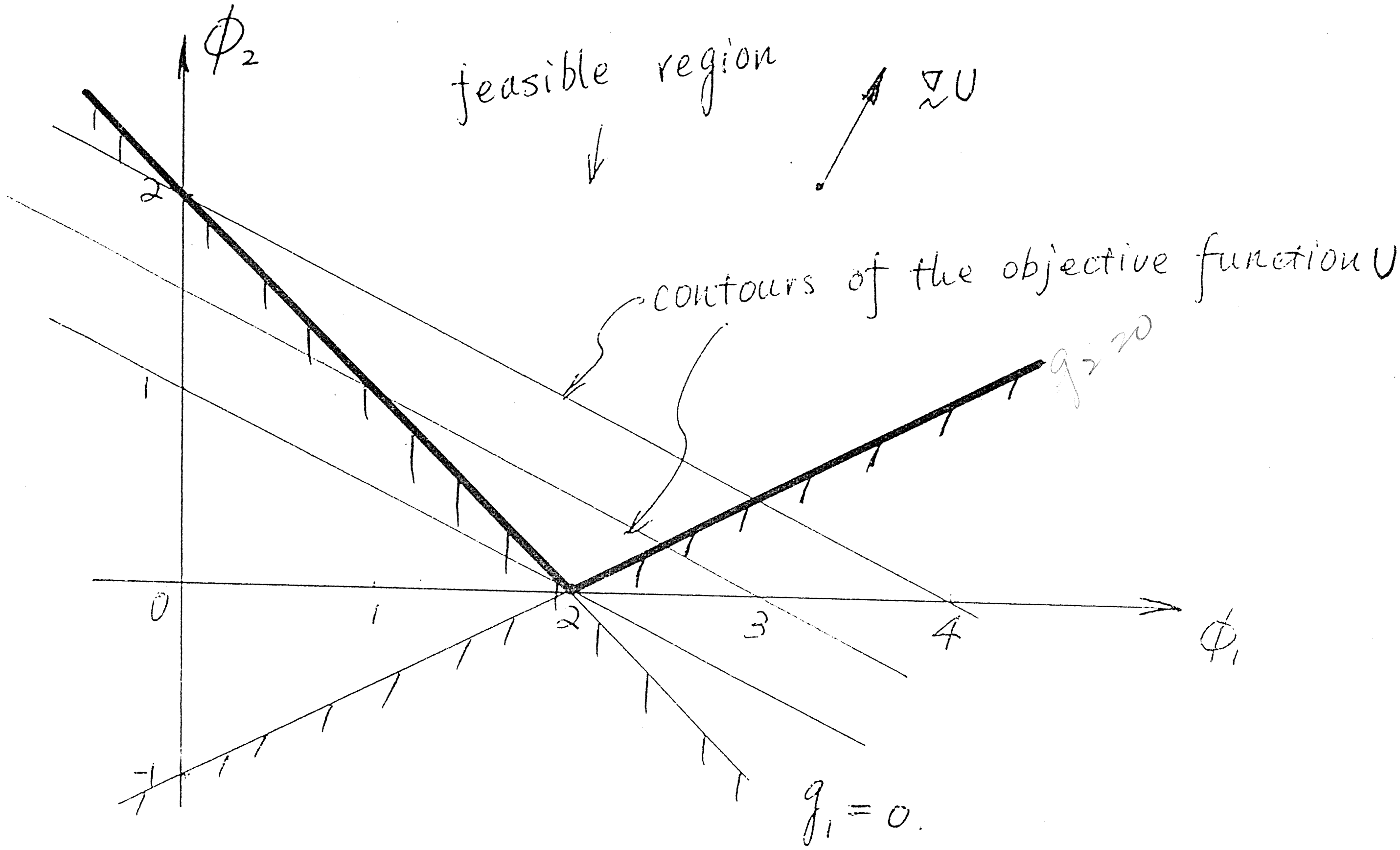
Therefore, the LHS of (1) is

$$u_2 \cdot \nabla f_2 = \nabla f_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \neq \underset{\sim}{0} = RHS.$$

The necessary conditions for minimax optimum are violated. Point $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not the minimax solution.

Problem # 2

1.



2. The KT conditions are as follows,

$$\begin{cases} \nabla U = \sum_i u_i \nabla g_i & (1) \\ u_i g_i = 0 & (2) \\ u_i \geq 0 & (3) \end{cases}$$

At point $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\begin{cases} g_1 = 2 + 0 - 2 = 0 \\ g_2 = -2 + 0 + 2 = 0. \end{cases}$

Both constraints are active.

$$\underset{\sim}{\nabla} U = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \underset{\sim}{\nabla} g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \underset{\sim}{\nabla} g_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Substitute the above into (1).

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 1 = u_1 - u_2 \\ 2 = u_1 + 2u_2 . \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = 4/3 \\ u_2 = 1/3 . \end{cases}$$

Point $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ <u>could be</u> a constrained minimum, since all three <u>necessary</u> conditions are satisfied.

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~~Solution to test~~ Test #2

3KB3

Problem #1

(1.) Let $f_1 = 0 \Rightarrow \phi_2 - 3 = 0$

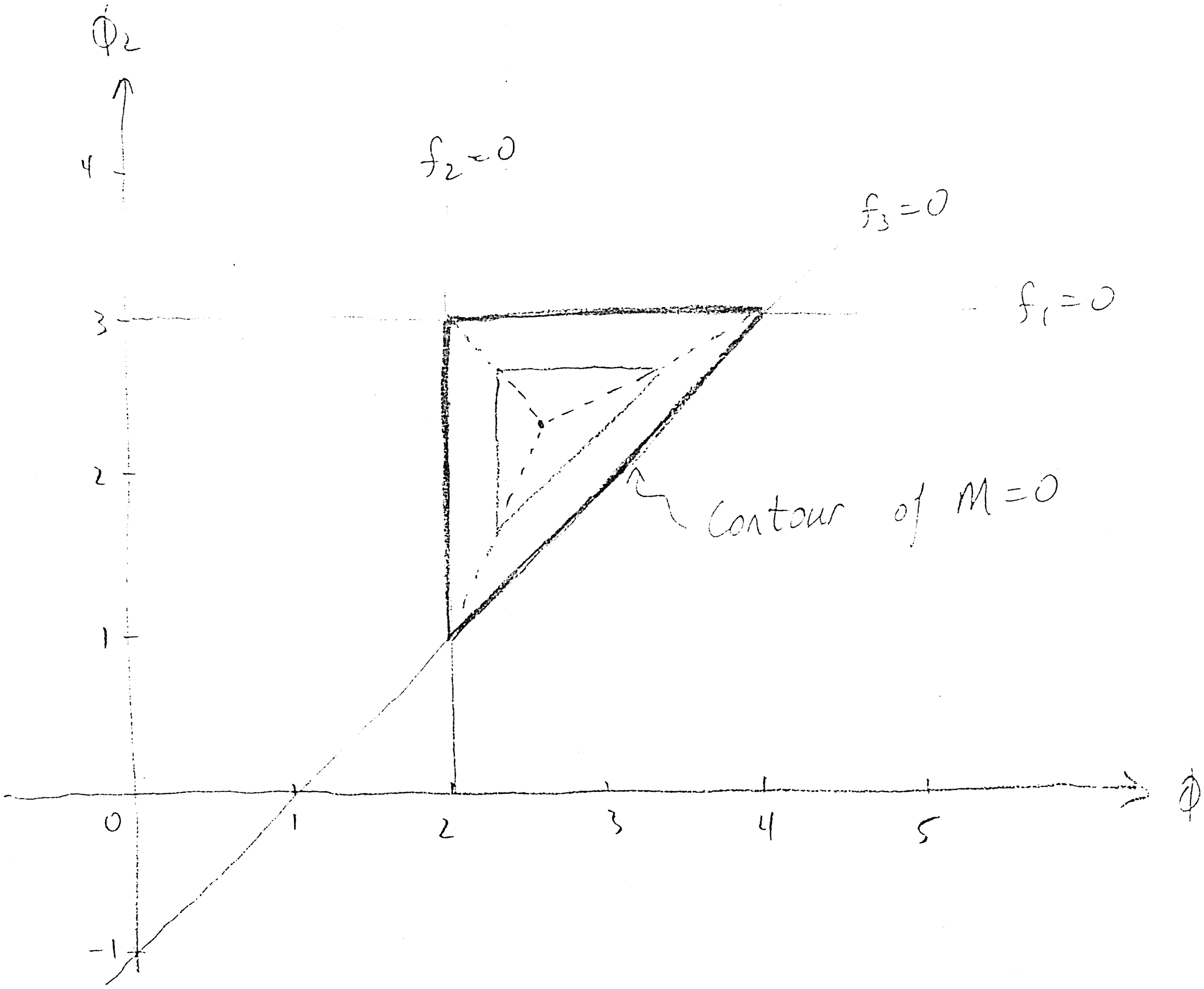
$\therefore \phi_2 = 3$

Let $f_2 = 0 \Rightarrow -\phi_1 + 2 = 0$

$\therefore \phi_1 = 2$

Let $f_3 = 0 \Rightarrow \phi_1 - \phi_2 - 1 = 0$

$\therefore \phi_2 = \phi_1 - 1$



(2). $\nabla f_1 = \begin{bmatrix} \frac{\partial f_1}{\partial \phi_1} \\ \frac{\partial f_1}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\nabla f_3 = \begin{bmatrix} \frac{\partial f_3}{\partial \phi_1} \\ \frac{\partial f_3}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\nabla f_2 = \begin{bmatrix} \frac{\partial f_2}{\partial \phi_1} \\ \frac{\partial f_2}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

3. At $\phi = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\left.\begin{aligned} f_1 &= 1 - 3 = -2 \\ f_2 &= -2 + 2 = 0 \\ f_3 &= 2 - 1 - 1 = 0 \end{aligned}\right\} \therefore \underline{f_2 \text{ and } f_3} \text{ are } \underline{\text{active}} \text{ } \underline{\text{functions}}.$$

(4). At the minimax optimum, the following conditions must hold:

$$\sum_{i=1}^{3} u_i \nabla f_i = 0 \qquad (1)$$

$$\sum_{i=1}^{3} u_i = 1 \qquad (2)$$

$$u_i (M - f_i) = 0 \qquad (3)$$

$$u_i \geq 0 \qquad (4)$$

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$\therefore$ At $\underset{\sim}{\phi} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$m = 0$

According to condition (3)

$u_1 = 0$ $\quad$ ($\because f_1$ is an inactive function)

Substitute this into (1) gives

$u_2 \underset{\sim}{\nabla} f_2 + u_3 \nabla f_3 = 0$

$u_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$

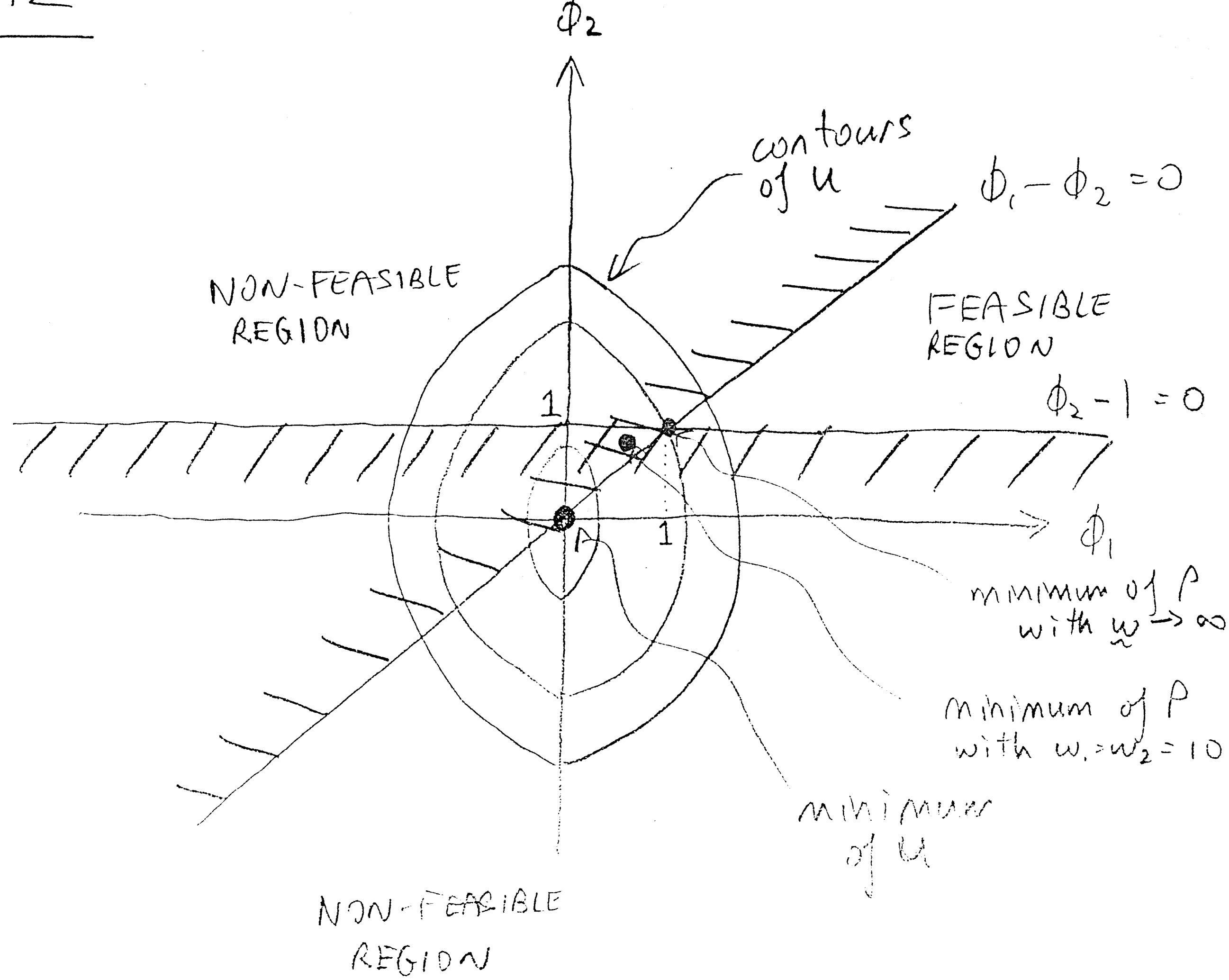
$\begin{cases} -u_2 + u_3 = 0 \\ -u_3 = 0 \end{cases} \Rightarrow \begin{matrix} u_3 = 0 \\ u_2 = 0 \end{matrix}$

which violates condition (2) which says $u_2 + u_3 = 1$.

$\therefore$ The necessary conditions for a minimax optimum are violated. Point $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not the minimax optimum.

'roblem #12

(1)



(2) $u = 2\phi_1^2 + \phi_2^2 + 3$

at minimum (unconstrained) $\nabla u = 0$

$$\left.\begin{aligned}\frac{\partial u}{\partial \phi_1} &= 4\phi_1 = 0\\ \frac{\partial u}{\partial \phi_2} &= 2\phi_2 = 0\end{aligned}\right\} \Rightarrow \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \phi^0$$

ϕ^0 is the <u>unconstrained</u> optimum.

③ In general,

$$P(\underset{\sim}{\phi}, \underset{\sim}{w}) = U(\underset{\sim}{\phi}) + \sum_{i=1}^{m} w_i g_i^2(\underset{\sim}{\phi})$$

In this case,

$$g_1 = \phi_2 - 1 \geq 0$$
$$g_2 = \phi_1 - \phi_2 \geq 0$$

$$\therefore P = (2\phi_1^2 + \phi_2^2 + 3) + w_1(\phi_2 - 1)^2 + w_2(\phi_1 - \phi_2)^2$$

(4) At the minimum of $P \Rightarrow \underset{\sim}{\nabla} P = \underset{\sim}{0}$

$$\frac{\partial P}{\partial \phi_1} = 4\phi_1 + 2w_2(\phi_1 - \phi_2) = 0$$

$$\frac{\partial P}{\partial \phi_2} = 2\phi_2 + 2w_1(\phi_2 - 1) - 2w_2(\phi_1 - \phi_2) = 0$$

$$\begin{cases} (4 + 2w_2)\phi_1 - (2w_2)\phi_2 = 0 \\ (-2w_2)\phi_1 + (2 + 2w_1 + 2w_2)\phi_2 - 2w_1 = 0 \end{cases}$$

$\Rightarrow$ at the minimum of P, both constraints are violated (ie: $g_1 < 0$, $g_2 < 0$) as shown in sketch.

$\therefore w_1 = w_2 = 10$

$$\begin{cases} 24\phi_1 - 20\phi_2 = 0 \\ -20\phi_1 + 42\phi_2 - 20 = 0 \end{cases}$$

(Solve for ϕ_1 and ϕ_2)

$$\begin{bmatrix} 24 & -20 \\ -20 & 42 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$$

$\phi_1 = 0.6579$

$\phi_2 = 0.7895$

$$\therefore \quad \underset{\sim}{\phi_p}^{\circ} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0.6579 \\ 0.7895 \end{bmatrix}$$

and $g_1 = -0.2105$
$g_2 = -0.1316$

CE3KB3

TUTORIAL #5
21.02.94

OSA90/hope's PIPES AND HOW TO USE THEM.

1. Introduction
   a) Definition of a pipe.
   b) Why pipes?
   c) What can pipes do?

2. Main functions of Datapipe.
   a) The parent's (OSA90/hope's) side of a pipe.
   b) The child's side of a pipe.
   c) Datapipe server functions used in a child program.

3. Datapipe protocols in OSA90/hope.

4. Examples.

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1. Introduction

a) <u>Definition of a pipe.</u>

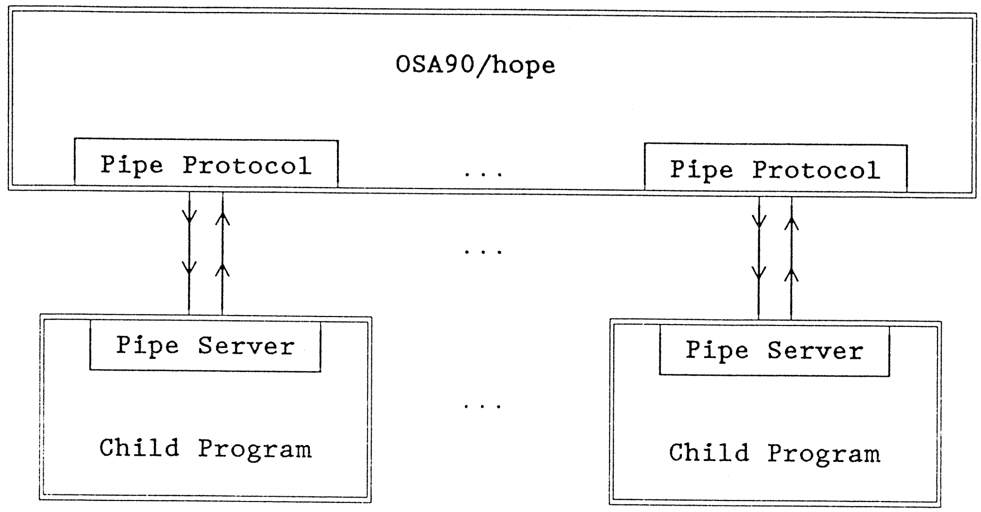
A pipe is an I/O channel intended for use between two cooperating processes: one process writes into the pipe, while the other process reads from the pipe (e.g. standard input, output and error channels of a process).

b) <u>Why pipes?</u>

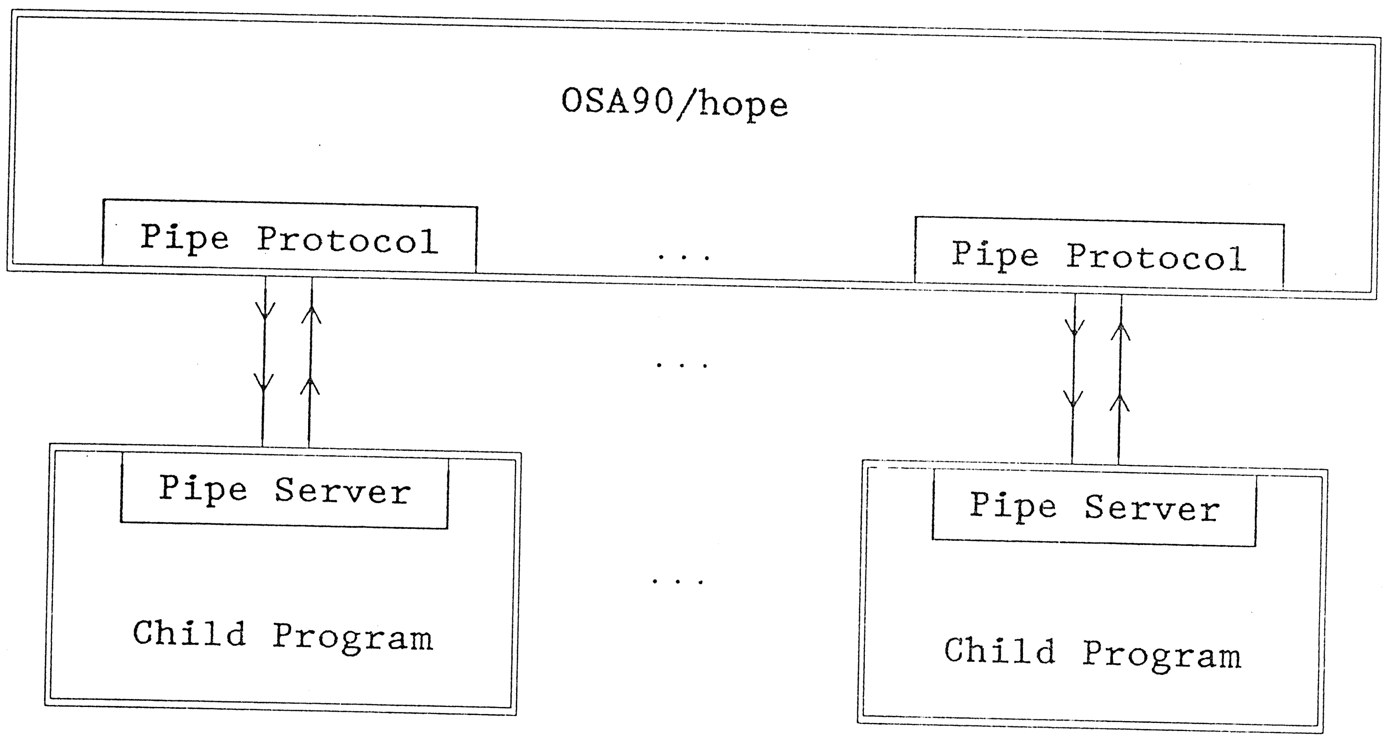
Pipes are: standard features of Unix,
easy to implement,
fast and reliable.

c) <u>What can pipes do?</u>

Pipes can establish a two way communication between independent programs.



Datapipe Schematic



Datapipe Schematic

2. Main functions of Datapipe.

a) <u>The parent's (OSA90/hope's) side of a pipe.</u>

The child program is loaded and activated only once and then it is called from within a loop in the parent program as long as desired. This scheme corresponds to the following sequence of calls to the Datapipe subroutines in the parent program.

```
cid = pipe_open("child_program");            /* open pipe and activate the
                                                child program */
for (i = 0; i < 100; i++) {                  /* iteration in the parent
                                                program, e.g., 100 iter. */
  pipe_initialize(cid);                      /* initialization */
  pipe_write(buffer, size, n_item, cid);     /* send data to child. Upon
                                                receiving all necessary data,
                                                the child program will start
                                                processing */
  pipe_read(buffer, size, n_item, cid);      /* get data from child after
                                                data processing in the child
                                                program is finished */
  /* data processing in parent program here */
}
pipe_close(cid);                             /* tell the child to exit and
                                                close the pipes */
```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b) <u>The child's side of a pipe.</u>

In the child program the user sets up an infinite loop which will perform the desired data processing as long as it is requested to do so by the parent program. The parent program sends to the child an initializing signal in each iteration. The parent will also decide, by sending a different signal to the child, when to terminate the data processing and close the pipe. The general usage of pipe communication in the child program is as follows.

```
for (;;) {                                   /* set up an infinite loop */
  pipe_initialize2();                        /* initialize (synchronize with
                                             the parent) */
  pipe_read2(buffer, size, n_item);          /* get data from parent */

  /* data processing in the child program follows here */

  pipe_write2(buffer, size, n_item);         /* send data to parent */
}
```

c) <u>Datapipe server functions used in a child program.</u>

pipe_initialize2()	synchronizes dialogues
pipe_read2()	reads data from Datapipe
pipe_write2()	writes data to Datapipe

3. **Datapipe protocols in OSA90/hope.**

Datapipes can be define in the Expression and Model blocks of the input file using the following syntax.

Datapipe:	*protocol*	*FILE = "filename"*
	N_INPUT = n	*INPUT = (x1, ..., xn)*
	N_OUTPUT = m	*OUTPUT = (y1, ..., ym)*
	TIMEOUT=k;	

where *protocol* **is a keyword identifying the protocol,** *filename* **is the name of the child program,** *n* **is the number of inputs to the child,** *m* **is the number of outputs from the child and** *k* **is the number of seconds before "time-out".**

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4. Examples.

We will use the *SIM* datapipe protocol to solve an abstract problem in which we want to approximate x^2 by $a_1x+a_2e^x$ over $0 \leq x \leq 2$. Let the initial point be $a_1 = 1$ and $a_2 = 2$. The OSA90/hope's input file for this problem without using pipes could look as follows.

```
Expression
  a1=?-10 1 10?;
  a2=?-10 2 10?;
  x=0;
  y1=x*x;
  y0=a1*x+a2*exp(x);
  err=y0-y1;
End
Sweep
  x: from 0 to 2 step 0.01 y0 y1 err;
End
Specification
  x: from 0 to 2 step 0.01 err=0;
End
```

Using an external program named *match* to calculate the *y0*, *y1* and *err* labels and the *SIM* datapipe protocol we get the following circuit file.

```
Expression
  a1=?-10 1 10?;
  a2=?-10 2 10?;
  x=0;
  Datapipe:      SIM                   FILE="match"
                 N_INPUT=3             N_OUTPUT=3
                 INPUT=(x, a1, a2)     OUTPUT=(y0, y1, err)
                 TIMEOUT=5;
End
Sweep
  x: from 0 to 2 step 0.01 y0 y1 err;
End
Specification
  x: from 0 to 2 step 0.01 err=0;
End
```

In the dialogue between OSA90/hope and the child program information flows from OSA90/hope to the child and then after being processed, back to OSA90/hope. The data sent by OSA90/hope to the child using the *SIM* protocol consists of three fields:

n	integer, number of inputs
m	integer, number of outputs
x	n floats, input values

SIM Protocol Input Data Format

They correspond to *N_INPUT, N_OUTPUT* and *INPUT* keywords in the Datapipe definition respectively. All three fields must be read by the child program. After receiving the input data, the child will proceed to compute the outputs and then to send the results back to OSA90/hope. The output data format is:

0	integer, error flag, 0
y	m floats, output values

SIM Protocol Output Format When Error Free

The first value is an integer error flag (set to 0 if no error) and the second corresponds to the *OUTPUT* keyword in the Datapipe definition.

There are two methods of reporting errors that occurred during the child program execution. The first method sends no message and requires the child program to set the error flag to -1 or 1.

error	integer, error flag, -1 or 1

SIM Protocol Output Format: Error Code Without Message

If a user wants to pass an error message back to OSA90/hope the error flag must be set to the length of the message (including the terminating NULL character) and the format for sending is the following:

error	integer, error message length
message	char string, NULL terminated

SIM Protocol Output Format For Error Messages

After sending a non-zero error flag, with or without a message, the child must not send any other data. It should wait for OSA90/hope to send the terminating signal.

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The source code of the child program for our example:

```
#include <stdio.h>
#include <math.h> /* present here for the exp() function */
#include "ippcv2.h"

main()
{
  int n, m, error = 0;
  float v, x[3], y[3];

  for (;;)
  {
    pipe_initialize2();
    pipe_read2(&n, sizeof(int), 1);
    pipe_read2(&m, sizeof(int), 1);
    pipe_read2(x, sizeof(float), n);
    if(x[0]<0 || x[0]>2)
      error = -1;
else {  y[0] = x[1]*x[0]+x[2]*exp(x[0]);
    y[1] = x[0]*x[0];
    y[2] = y[0]-y[1];
    }  pipe_write2(&error, sizeof(int), 1);
if(!error)  pipe_write2(y, sizeof(float), m);
  }
}
```

Note, that an error is reported if *x* is outside the (0, 2) interval.

Another version of the above program, sending back an error message when *x* is outside the (0, 2) interval.

```
#include <stdio.h>
#include <math.h> /* present here for the exp() function */
#include "ippcv2.h"

main()
{
  int n, m, error = 0;
  char* message;
  float v, x[3], y[3];

  for (;;)
  {
    pipe_initialize2();
    pipe_read2(&n, sizeof(int), 1);
    pipe_read2(&m, sizeof(int), 1);
    pipe_read2(x, sizeof(float), n);

    if(x[0]<0 || x[0]>2)
    {
      message = "x must be in the range 0 ≤ x ≤ 2";
      error =strlen(message) + 1;
    }
    else
    {
      y[0] = x[1]*x[0]+x[2]*exp(x[0]);
      y[1] = x[0]*x[0];
      y[2] = y[0]-y[1];
    }
    pipe_write2(&error, sizeof(int), 1);
    if(error)
      pipe_write2(message, 1, error);
    else
      pipe_write2(y, sizeof(float), m);
  }
}
```

FINAL EXAM 93'

COMPUTER ENGINEERING 3KB3

DAY CLASS | Dr. J.W. Bandler
DURATION OF EXAMINATION: 3 Hours
McMaster University Final Examination | April 1993

This examination paper includes 14 pages and 12 questions. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.

SPECIAL INSTRUCTIONS

Candidates may use slide rules, calculators and log books.

Candidates must not use preprogrammed algorithms, such as those for linear or nonlinear equations, etc.

Candidates must attempt questions 1 2 or 3 4 or 5 6 or 7 8 or 9 10 11 or 12

Write your name here. NAME: ______________________

Write your student number here. NO: ______________________

Date and hour of examination: ______________________

Note: (1) All scripts and question papers must be turned in.
(2) Estimated times required to complete the questions are indicated.
(3) Please encircle questions attempted in the following table.

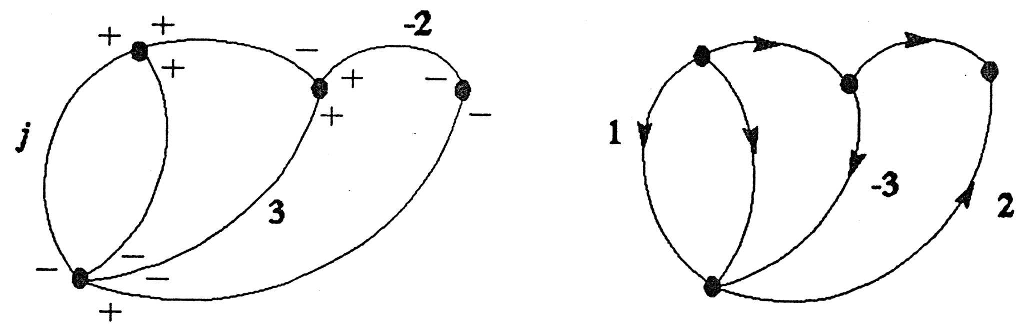
Questions Attempted (please encircle)	Weighting	Estimated Time (min.)	Examiner's Use Only
1	5%	ten	
2 or 3	10%	fifteen	
4 or 5	12.5%	twenty	
6 or 7	12.5%	twenty	
8 or 9	20%	thirty	
10	20%	forty	
11 or 12	20%	forty-five	
TOTAL	100%	3 hours	

continued on page 2

Question 1

(a) V and I contain all corresponding branch voltages and currents, respectively (using associated reference directions) for a circuit and satisfy all the appropriate Kirchhoff laws. $\hat{V}$ and $\hat{I}$ are voltage and current vectors corresponding to V and I but defined for a second topologically equivalent circuit. State Tellegen's Theorem in two forms.

(b) Complete the following diagram and verify Tellegen's theorem.



Answer (10 minutes)

a) Tellegen's Theorem

① $V^T\hat{I} = 0$ so, $V^T\hat{I} - I^T\hat{V} = 0$

$I^T\hat{V} = 0$

② $\Delta V^T\hat{I} = 0$ so; $\Delta V^T\hat{I} - \Delta I^T\hat{V} = 0$

$\Delta I^T\hat{V} = 0$

b) From KCL and KVL we have

$V = [j \ j \ j-3 \ 3 \ -2 \ -5]^T$

$\hat{I} = [1 \ 4 \ -5 \ -3 \ -2 \ 2]^T$

then $V^T\hat{I} = j + j4 - 5j + 15 - 9 + 4 - 10 = 0$

continued on page 3

Question 2

Derive the exact Newton iteration at ϕ^j for minimization of a differentiable multidimensional function $U(\phi)$. Define all terms used. Under what conditions do you expect a locally downhill step from the Newton iteration? Discuss possible pitfalls of the basic Newton method and suggest remedies. What is damping? Illustrate your answers with sketches.

Answer (15 minutes)

continued on page 4

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Question 3

Consider the circuit shown in Fig. Q3, where $G_1 = 1$, $G_2 = 2$, $G_3 = 1$ and $V_{in} = 2V$.

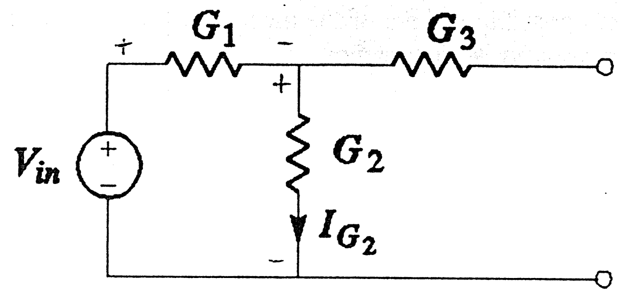
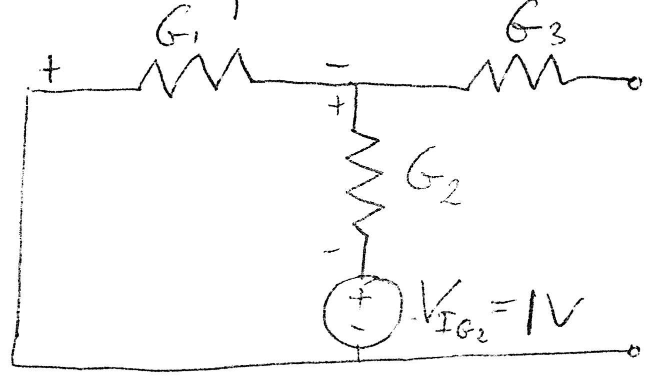


Fig. Q3 A resistive circuit.

Find $\partial I_{G2}/\partial G_1$, $\partial I_{G2}/\partial G_2$ and $\partial I_{G2}/\partial G_3$ by the adjoint circuit method. You may refer to the TABLE at the end of this exam paper. Check your result by direct differentiation.

Answer (15 minutes)

The adjoint circuit is:



$$\frac{\partial I_{G_2}}{\partial G_1} = -V_{G_1}\hat{V}_{G_1} \qquad \frac{\partial I_{G_2}}{\partial G_2} = -V_{G_2}\hat{V}_{G_2} \qquad \frac{\partial I_{G_3}}{\partial G_3} = -V_{G_3}\hat{V}_{G_3}$$

$$\begin{cases} V_{G_1} = V_{in}\cdot\dfrac{G_2}{G_1+G_2} = 2\cdot\dfrac{2}{3} = \dfrac{4}{3} \\ V_{G_2} = V_{in}\cdot\dfrac{G_1}{G_1+G_2} = 2\cdot\dfrac{1}{3} = \dfrac{2}{3} \\ V_{G_3} = 0 \end{cases} \qquad \begin{cases} \hat{V}_{G_1} = -\dfrac{1}{2}V_{G_1} = -\dfrac{2}{3} \\ \hat{V}_{G_2} = -\dfrac{1}{2}V_{G_2} = -\dfrac{1}{3} \\ \hat{V}_{G_3} = 0 \end{cases}$$

continued on page 5

Question 4

Consider the formula

$$G = \sum_{\substack{\text{voltage} \\ \text{sources}}} \hat{V}_i \nabla I_i - \sum_{\substack{\text{current} \\ \text{sources}}} \hat{I}_i \nabla V_i$$

where G is a vector of standard sensitivity expressions, i is the index of the sources and ∇ is the partial derivative operator w.r.t. circuit parameters corresponding to G. Consider a six port network having two constant voltage sources, one constant current source, the remaining ports being terminated by resistors. Use the formula to show how to relate to G the gradient vector of

$$\sum_{\substack{\text{terminating} \\ \text{resistors}}} |V_r|^2/R_r$$

where V_r is the response voltage and R_r is the terminating resistor. Draw the adjoint network and state the proper excitations.

Answer (20 minutes)

continued on page 6

Question 5

Examine the points $[0 \quad 0]^T$ and $[1 \quad 1]^T$ for a minimax problem for which

$$f_1 = \phi_1^4 + \phi_2^2$$

$$f_2 = (2 - \phi_1)^2 + (2 - \phi_2)^2$$

$$f_3 = 2 \exp(-\phi_1 + \phi_2)$$

by invoking necessary conditions for a minimax optimum. What are your conclusions?

Answer (20 minutes)

continued on page 7

Question 6

Consider the voltage divider shown for the response specification and constraint indicated.

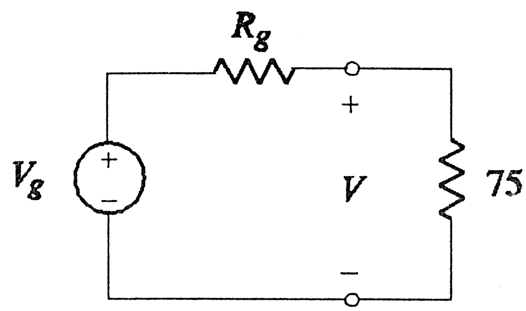


Fig. Q6 A voltage divider.

Design specification: $V \geq 60$
Constraint: $R_g \geq 75$

By testing the Kuhn-Tucker conditions, find V_g and R_g such that the total power dissipated is minimum.

Answer (20 minutes)

continued on page 8

Question 7

Derive from first principles the sensitivity expression and adjoint element corresponding to a voltage controlled current source. Draw circuit diagrams to fully illustrate your results.

Answer (20 minutes)

continued on page 9

Question 8

For the resistor-diode network shown in Fig. Q8, illustrate with the aid of an *I-V* diagram an iterative method of finding V at DC. State Newton's method for solving this problem and derive the network model corresponding to the situation at the jth iteration. What is the significance of this model?

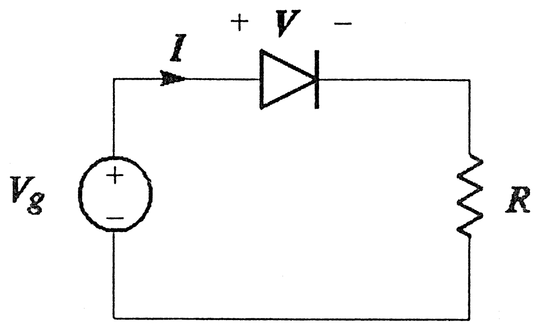


Fig. Q8 Resistor-diode network.

Answer (30 minutes)

continued on page 10

Question 9

Derive from first principles the adjoint element and sensitivity expression for a two-port characterized by

$$\begin{bmatrix} V_p \\ I_p \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_q \\ -I_q \end{bmatrix}$$

Apply the result to the element shown in Fig. Q9 to determine the sensitivity formulas w.r.t. ϕ, where $Y_1 = \phi$ and $Z_2 = 0.5/\phi$.

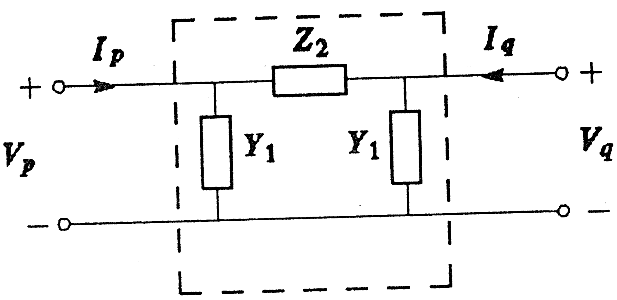


Fig. Q9 A two-port circuit.

Answer (30 minutes)

continued on page 11

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Question 10

Let

$$
\begin{aligned}
h_1 &= -x_1y_1y_2 + 2x_2^2y_2 - 3 = 0\\
h_2 &= 4x_1y_1 - 3y_2 = 0\\
f &= y_2
\end{aligned}
$$

Working directly with these functions, construct the Newton equations to solve the system and formulas to calculate $\partial f/\partial \boldsymbol{x}$ subject to the given constraints. Let $\boldsymbol{x} = [1 \;\; 1.25]^T$, where T denotes transposition. Apply one iteration of the Newton method starting at $y^0 = [1 \;\; 1]^T$. Assuming the solution to be $y = [1.125 \;\; 1.5]^T$ calculate the appropriate $\partial f/\partial \boldsymbol{x}$.

Answer (40 minutes)

continued on page 12

Question 11

Consider the resistive network shown in Fig. Q11, where $G_1 = 1.5$ S, $G_2 = 2.5$ S and $i = 10$ A. Use the adjoint network method to evaluate

$$\frac{\partial i_2}{\partial G_1}, \frac{\partial i_2}{\partial G_2}, \text{ and } \frac{\partial i_2}{\partial i}$$

Check your results by small perturbations.

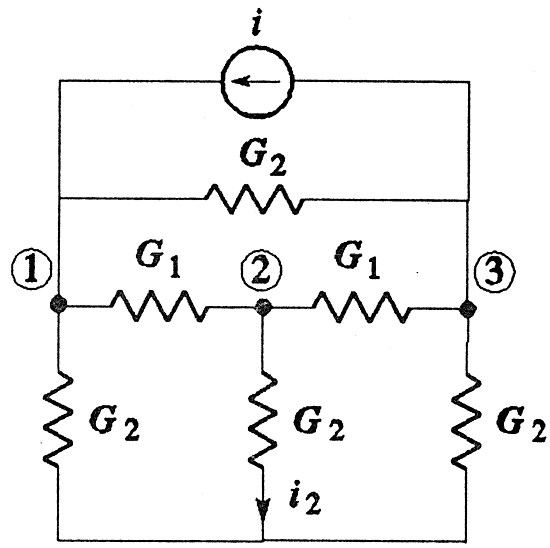

Fig. Q11 A resistive network.

Answer (45 minutes)

continued on page 13

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Question 12

Consider the nonlinear circuit shown in Fig. Q12, where $i_a = 2v_a^3$, $i_b = v_b^3 + 10v_b$.

(a) Express the nodal equations in the linearized form required at the jth iteration of the Newton algorithm.

(b) Apply two iterations of the Newton method, starting at $v_1 = 2$, $v_2 = 1$.

(c) Draw the companion network at the jth iteration and state the corresponding nodal equations.

(d) Continue with two iterations of the companion network method.

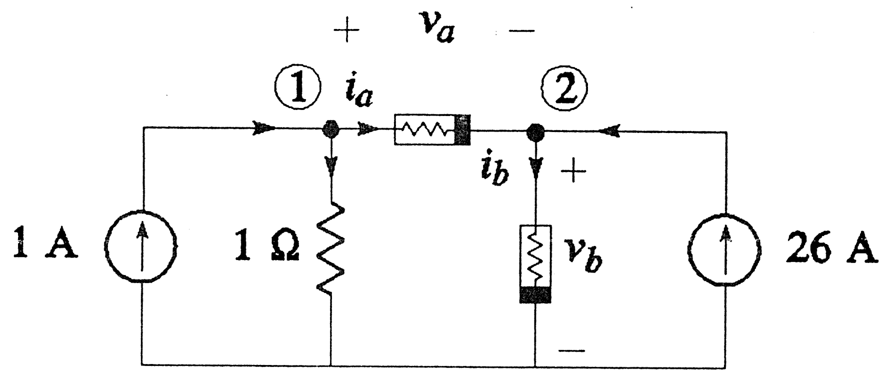

Fig. Q12 Nonlinear circuit example.

Answer (45 minutes)

continued on page 14

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TABLE. SENSITIVITY EXPRESSIONS FOR CERTAIN ELEMENTS

Element	Equation: Original	Equation: Adjoint	Sensitivity	Parameters
Resistor	$V = RI$	$\hat{V} = R\hat{I}$	$I\hat{I}$	R
Capacitor	$I = j\omega CV$	$\hat{I} = j\omega C\hat{V}$	$-j\omega V\hat{V}$	C
Voltage Controlled Voltage Source	$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ I_2 \end{bmatrix}$	$\begin{bmatrix} \hat{I}_1 \\ \hat{V}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\mu \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{V}_1 \\ \hat{I}_2 \end{bmatrix}$	$V_1 \hat{I}_2$	μ

THE END!

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Question 1:

(a) The Tellegen's Theorem can be stated as the following forms:

① $\underset{\sim}{V}^T \underset{\sim}{\hat{I}} = 0$

$\underset{\sim}{I}^T \underset{\sim}{\hat{V}} = 0$

so. $\underset{\sim}{V}^T \underset{\sim}{\hat{I}} - \underset{\sim}{I}^T \underset{\sim}{\hat{V}} = 0$

② $\Delta \underset{\sim}{V}^T \underset{\sim}{\hat{I}} = 0$

$\Delta \underset{\sim}{I}^T \underset{\sim}{\hat{V}} = 0$

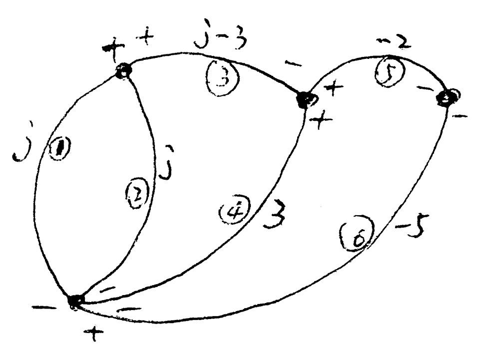
so: $\Delta \underset{\sim}{V}^T \underset{\sim}{\hat{I}} - \Delta \underset{\sim}{I}^T \underset{\sim}{\hat{V}} = 0$

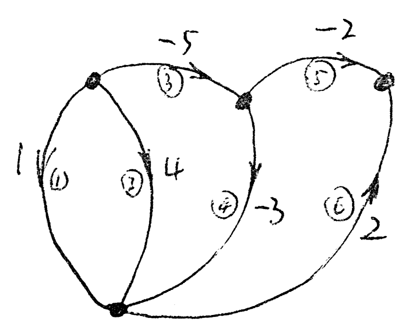
(b) According to the KCL and KVL, we can complete the diagram as shown in the question,

$$\underset{\sim}{V} = [\,j \;\; j \;\; j-3 \;\; 3 \;\; -2 \;\; -5\,]^T$$

$$\underset{\sim}{\hat{I}} = [\,1 \;\; 4 \;\; -5 \;\; -3 \;\; -2 \;\; 2\,]^T$$

then $\underset{\sim}{V}^T \underset{\sim}{\hat{I}} = j + 4j - 5j + 15 - 9 + 4 - 10 = 0$





Question 2 :

① For a differentiable multidimensional function $U(\underset{\sim}{\varphi})$.

the Taylor series expansion of $U(\underset{\sim}{\varphi})$ is given by

$$U(\underset{\sim}{\varphi}+\Delta\underset{\sim}{\varphi}) = U(\underset{\sim}{\varphi}) + \underset{\sim}{\nabla} U^T \Delta\underset{\sim}{\varphi} + \frac{1}{2}\Delta\underset{\sim}{\varphi}^T \underset{\sim}{H} \Delta\underset{\sim}{\varphi} + \cdots$$

where $\underset{\sim}{\varphi} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix}$, $\Delta\underset{\sim}{\varphi} = \begin{bmatrix} \Delta\varphi_1 \\ \Delta\varphi_2 \\ \vdots \\ \Delta\varphi_n \end{bmatrix}$, $\underset{\sim}{\nabla} U = \begin{bmatrix} \frac{\partial U}{\partial \varphi_1} \\ \frac{\partial U}{\partial \varphi_2} \\ \vdots \\ \frac{\partial U}{\partial \varphi_n} \end{bmatrix}$, $\underset{\sim}{H} = \begin{bmatrix} \frac{\partial^2 U}{\partial \varphi_1^2} & \frac{\partial^2 U}{\partial \varphi_1 \partial \varphi_2} & \cdots & \frac{\partial^2 U}{\partial \varphi_1 \partial \varphi_n} \\ \frac{\partial^2 U}{\partial \varphi_2 \partial \varphi_1} & \frac{\partial^2 U}{\partial \varphi_2^2} & \cdots & \frac{\partial^2 U}{\partial \varphi_2 \partial \varphi_n} \\ \vdots & & & \vdots \\ \frac{\partial^2 U}{\partial \varphi_n \partial \varphi_1} & \frac{\partial^2 U}{\partial \varphi_n \partial \varphi_2} & \cdots & \frac{\partial^2 U}{\partial \varphi_n^2} \end{bmatrix}$

then $\underset{\sim}{\nabla} U(\underset{\sim}{\varphi}+\Delta\underset{\sim}{\varphi}) = \underset{\sim}{\nabla} U(\underset{\sim}{\varphi}) + \underset{\sim}{H}\Delta\underset{\sim}{\varphi} + \cdots$ (neglected)

if $\underset{\sim}{\varphi}+\Delta\underset{\sim}{\varphi}$ is the minimizing point $\hat{\varphi}$

then $\underset{\sim}{\nabla} U(\underset{\sim}{\varphi}+\Delta\underset{\sim}{\varphi}) = 0$, i.e. $\Delta\underset{\sim}{\varphi} = -\underset{\sim}{H}^{-1}\underset{\sim}{\nabla} U$

so : $\phi^{i+1} = \varphi^i + [-\underset{\sim}{H}_i^{-1}\underset{\sim}{\nabla} U(\underset{\sim}{\varphi}^i)]$

② If $U(\underset{\sim}{\varphi})$ is a positive definite quadratic function, $\underset{\sim}{H}$ is p.s.d.
a locally downhill step can be expected from this iteration.

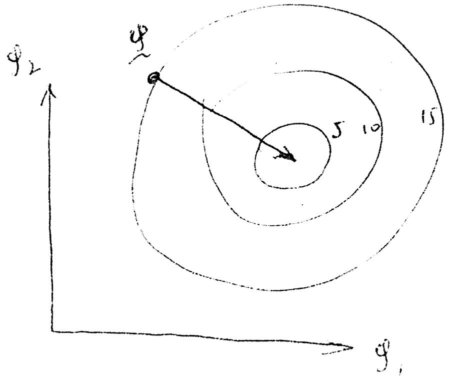
③ If $\underset{\sim}{H}$ is not positive definite, the Newton method ~~can~~ may not converge to an optimum, to avoid this we can use the Newton method with Damping.

④ Damping: $(\underset{\sim}{H}+\lambda\underset{\sim}{I})\Delta\underset{\sim}{\varphi} = -\underset{\sim}{\nabla} U$ $(\lambda \geq 0)$

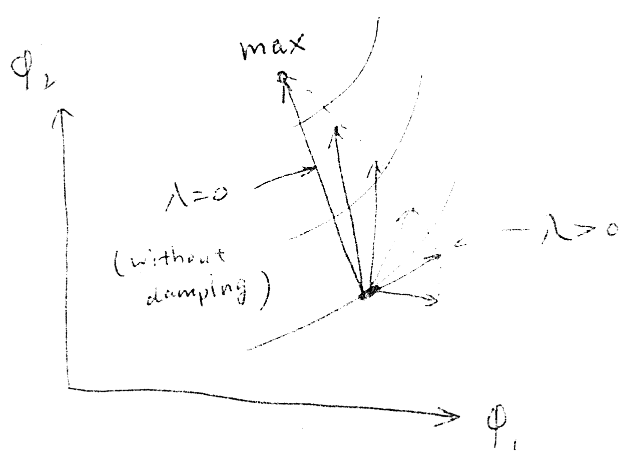
where $\underset{\sim}{I}$ is a unit matrix, λ is a scalar

if $\lambda \to 0$, $\to$ undamped

$\lambda \to \infty$. $\Delta\underset{\sim}{\varphi} \approx -\frac{1}{\lambda}\underset{\sim}{\nabla} U$ the step is in the steepest desent direction.



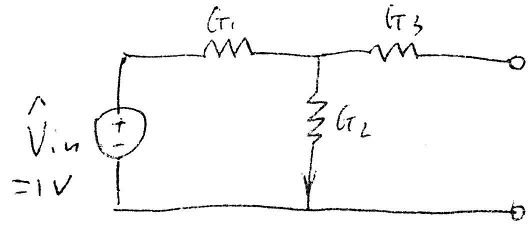
We get the minimum in one step is U is p.s.d. quadratic function.



③

Question 3:

The adjoint circuit is



$$\frac{\partial I_{G_2}}{\partial G_1} = -V_{G_1} \cdot \hat{V}_{G_1}, \quad \frac{\partial I_{G_2}}{\partial G_2} = -V_{G_2} \cdot \hat{V}_{G_2}, \quad \frac{\partial I_{G_3}}{\partial G_3} = -V_{G_3} \cdot \hat{V}_{G_3}$$

$$\begin{cases} V_{G_1} = V_{in} \cdot \dfrac{G_2}{G_1+G_2} = 2 \cdot \dfrac{2}{3} = \dfrac{4}{3} \\ V_{G_2} = V_{in} \cdot \dfrac{G_1}{G_1+G_2} = 2 \cdot \dfrac{1}{3} = \dfrac{2}{3} \\ V_{G_3} = 0 \end{cases} \qquad \begin{cases} \hat{V}_{G_1} = \dfrac{1}{2} V_{G_1} = \dfrac{2}{3} \\ \hat{V}_{G_2} = \dfrac{1}{2} V_{G_2} = \dfrac{1}{3} \\ \hat{V}_{G_3} = 0 \end{cases}$$

$$\therefore \quad \frac{\partial I_{G_2}}{\partial G_1} = -\frac{4}{3} \times \frac{2}{3} = -\frac{8}{9}, \quad \frac{\partial I_{G_2}}{\partial G_2} = -\frac{2}{3} \times \frac{1}{3} = -\frac{2}{9}, \quad \frac{\partial I_{G_2}}{\partial G_3} = 0$$

By direct differentiation:

$$I_{G_2} = \frac{V_{in}}{\left(\frac{1}{G_1} + \frac{1}{G_2}\right)}$$

$$\frac{\partial I_{G_2}}{\partial G_1} = \frac{V_{in}}{\left(\frac{1}{G_1} + \frac{1}{G_2}\right)^2} \cdot \frac{-1}{G_1^2} = \frac{-2}{(1+\frac{1}{2})^2} = -\frac{8}{9}$$

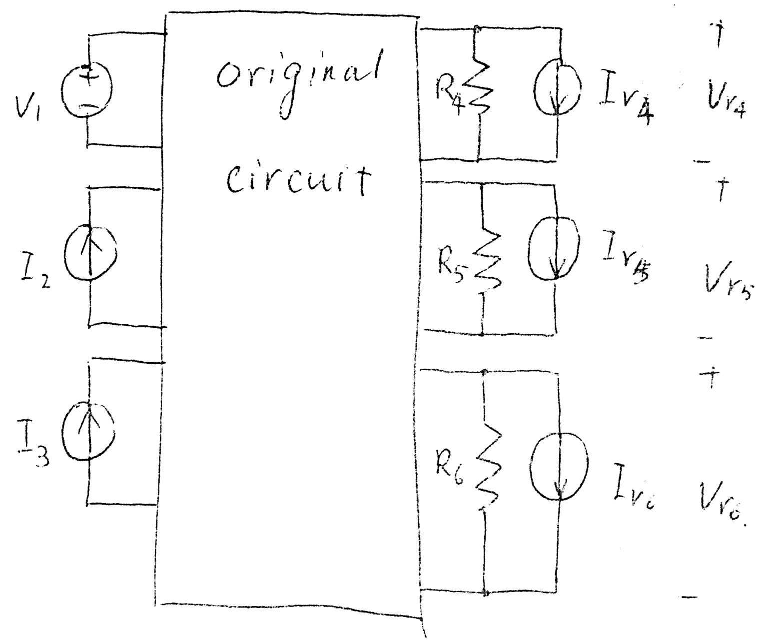
$$\frac{\partial I_{G_2}}{\partial G_2} = \frac{V_{in}}{\left(\frac{1}{G_1} + \frac{1}{G_2}\right)^2} \cdot \frac{1}{G_2^2} = \frac{-2}{(1+\frac{1}{2})^2} \cdot \frac{1}{4} = -\frac{2}{9}$$

$$\frac{\partial I_{G_2}}{\partial G_3} = 0$$

Question 4.

Let $U = \sum_{\mathcal{R}} |V_r|^2 / R_r$

$\mathcal{R}$ stands for terminating resistors.



$$\tilde{\nabla} U = \sum_{\mathcal{R}} 2\,\mathrm{Re}\left\{ \frac{V_r^*}{R_r} \tilde{\nabla} V_r \right\} + \sum_{\mathcal{R}} \frac{|V_r|^2}{R_r^2} \cdot (-\tilde{\nabla} R_r)$$

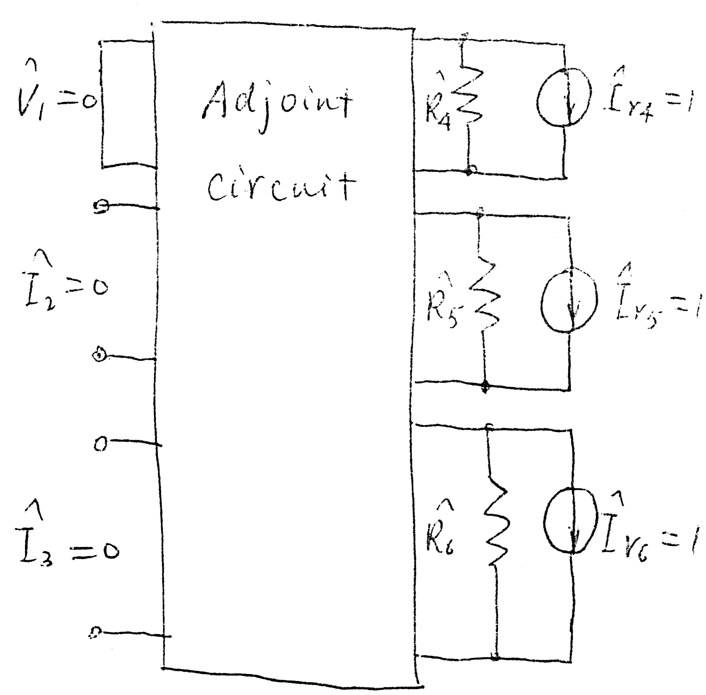
$$= \sum_{\text{Port } 4,5,6} 2\,\mathrm{Re}\left\{ \frac{V_r^*}{R_r} \tilde{\nabla} V_r \right\} - \sum_{\substack{\text{Port} \\ 4,5,6}} \frac{|V_r|^2}{R_r^2} \tilde{\nabla} R_r$$

Define the adjoint network as.

$\hat{V}_1 = 0$.

$\hat{I}_2 = \hat{I}_3 = 0$

$\hat{I}_{r4} = \hat{I}_{r5} = \hat{I}_{r6} = 1$



Therefore,

$$\tilde{G} = \sum_{\mathcal{V}} \hat{V}_i \tilde{\nabla} I_i - \sum_{\mathcal{C}} \hat{I}_i \tilde{\nabla} V_i$$

$$= \hat{V}_1 \tilde{\nabla} I_1 - \hat{I}_2 \tilde{\nabla} V_i - \hat{I}_3 \tilde{\nabla} V_3$$

$$- \hat{I}_{r4} \tilde{\nabla} V_{r4} - \hat{I}_{r5} \tilde{\nabla} V_{r5} - \hat{I}_{r6} \tilde{\nabla} V_{r6}$$

$$\underset{\sim}{G} = - \sum_{\substack{\text{Port} \\ 4,5,6}} \hat{I}_r \nabla V_r .$$

Let $\hat{I}_r = \dfrac{V_r^*}{R_r}$ at Port 4-6. we have

$$\underset{\sim}{\nabla} U = \sum_{\substack{\text{Port} \\ 4.5.6}} 2Re \left\{ \hat{I}_r \underset{\sim}{\nabla} V_r \right\} - \sum_{\substack{\text{Port} \\ 4.5.6}} \frac{|V_r|^2}{R_r} \underset{\sim}{\nabla} R_r$$

$$= 2Re \left\{ \sum_{\substack{\text{Port} \\ 4.5.6}} \hat{I}_r \underset{\sim}{\nabla} V_r \right\} - \sum_{\substack{\text{Port} \\ 4.5.6}} \frac{|V_r|^2}{R_r^2} \underset{\sim}{\nabla} R_r$$

$$= -2Re \left\{ \underset{\sim}{G} \right\} - \sum_{\mathcal{R}} \frac{|V_r|^2}{R_r^2} \underset{\sim}{\nabla} R_r .$$

⑥

Question 5.

Minimax optimality conditions:

$\Sigma u_i \nabla f_i = 0$ .... (1)

$\Sigma u_i = 1$ .... (2)

$u_i \geq 0$ .... (3)

$u_i = 0$ for f_i's which are not active ... (4)

$$\nabla f_1 = \begin{bmatrix} 4\phi_1^3 \\ 2\phi_2 \end{bmatrix} \quad \nabla f_2 = \begin{bmatrix} -2(2-\phi_1) \\ -2(2-\phi_2) \end{bmatrix} \quad \nabla f_3 = \begin{bmatrix} -2\exp(-\phi_1+\phi_2) \\ 2\exp(-\phi_1+\phi_2) \end{bmatrix}$$

at $[0 \ 0]^T$

$f_1 = 0$, $f_2 = 8$, $f_3 = 2$, only f_2 is active, which means $u_1 = u_3 = 0$ (according to (4)).

According to (2), $u_2 = 1$. Substitute u_i's into (1)

$$\Sigma u_i \nabla f_i = u_2 \nabla f_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \neq 0$$

Point $[0 \ 0]^T$ does not satisfy optimality conditions (1-4).

It is not an minimax optimum.

at $[1 \ 1]^T$

$f_1 = f_2 = f_3 = 2$. All functions are active.

According to (1) - (2)

$$\begin{cases} u_1 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + u_2 \begin{bmatrix} -2 \\ -2 \end{bmatrix} + u_3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 0 \\ u_1 + u_2 + u_3 = 1 \end{cases}$$

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Solve the above system of equations.

$$\Rightarrow \begin{cases} u_1 = 1/3 \geq 0 \\ u_2 = 1/2 \geq 0 \\ u_3 = 1/6. \geq 0 \end{cases}$$

All optimality conditions are satisfied.

Point $[1 \ 1]^T$ could be ~~an~~ ~~of~~ a minimax optimum. If this is the case, the value of minimax objective function is 2.

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Question 6.

Let $\Phi_1 = Vg$, $\Phi_2 = Rg$.

Power dissipated $= U(\Phi_1, \Phi_2) = \dfrac{\Phi_1^2}{\Phi_2 + 75}$

two constraints:

$$g_1 = \frac{75\Phi_1}{\Phi_2 + 75} - 60 \geq 0$$

$$g_2 = \Phi_2 - 75 \geq 0$$

$$\underset{\sim}{\nabla} U = \begin{bmatrix} \dfrac{2\Phi_1}{\Phi_2 + 75} \\ -\dfrac{\Phi_1^2}{(\Phi_2+75)^2} \end{bmatrix} \qquad \underset{\sim}{\nabla} g_1 = \begin{bmatrix} \dfrac{75}{\Phi_2 + 75} \\ -\dfrac{75\Phi_1}{(\Phi_2+75)^2} \end{bmatrix} \qquad \underset{\sim}{\nabla} g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Point [120, 75] could be the solution. At this point,

$$g_1 = \frac{75 \times 120}{75 + 75} - 60 = 0 \qquad \text{active}$$

$$g_2 = 75 - 75 = 0 \qquad \text{active.}$$

$$\underset{\sim}{\nabla} U = \begin{bmatrix} \dfrac{2 \times 120}{75+75} \\ -\dfrac{120^2}{(75+75)^2} \end{bmatrix} = \begin{bmatrix} 1.6 \\ -0.64 \end{bmatrix}$$

$$\underset{\sim}{\nabla} g_1 = \begin{bmatrix} \dfrac{75}{75+75} \\ -\dfrac{75 \times 120}{(75+75)^2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.4 \end{bmatrix} \qquad \underset{\sim}{\nabla} g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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solve for the equations.

$$\begin{bmatrix} 1.6 \\ -0.64 \end{bmatrix} = u_1 \begin{bmatrix} 0.5 \\ -0.4 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u_1 = 3.2 \\ u_2 = 0.64 \end{cases}$$

Since $u_i > 0$, all the KT conditions are satisfied at $[120 \quad 75]^T$, it is the solution.

Go back to original problem,

$$\begin{cases} V_g = 120 \\ R_g = 75 \end{cases}$$

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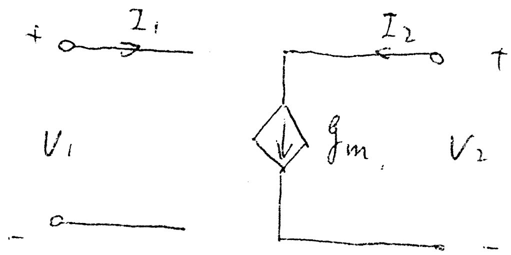
Question 7.

For VCCS.



$$I_2 = g_m V_1.$$

From Tellegen's theorem.

$$\cdots \quad V_1 \hat{I}_1 + V_2 \hat{I}_2 - \hat{V}_1 I_1 - \hat{V}_2 I_2 \quad \cdots$$

Differentiate w.r.t. φ

$$\cdots \quad \frac{\partial V_1}{\partial \varphi} \hat{I}_1 + \frac{\partial V_2}{\partial \varphi} \hat{I}_2 - \frac{\partial I_1}{\partial \varphi} \hat{V}_1 - \frac{\partial I_2}{\partial \varphi} \hat{V}_2 \quad \cdots \qquad (1)$$

From VCCS branch relations and assuming $\varphi \neq g_m$

$$\frac{\partial I_2}{\partial \varphi} = \frac{\partial V_1}{\partial \varphi} g_m + \frac{\partial g_m}{\partial \varphi} V_1 = \frac{\partial V_1}{\partial \varphi} g_m$$

substitute into (1).

$$\cdots \quad \frac{\partial V_1}{\partial \varphi} \hat{I}_1 + \frac{\partial V_2}{\partial \varphi} \hat{I}_2 - \frac{\partial I_1}{\partial \varphi} \hat{V}_1 - \frac{\partial V_1}{\partial \varphi} g_m \hat{V}_2 \quad \cdots \qquad (2)$$

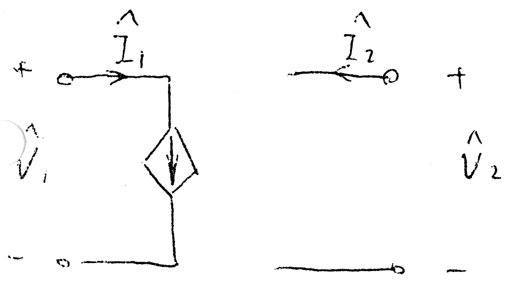
Since $I_1 = 0$. $\frac{\partial I_1}{\partial \varphi} = 0$. (2) becomes

$$\frac{\partial V_1}{\partial \varphi} (\hat{I}_1 - g_m \hat{V}_2) + \frac{\partial V_2}{\partial \varphi} \hat{I}_2 \quad \cdots$$

Define branch relations in adjoint.

$$\hat{I}_1 = g_m \hat{V}_2 \qquad \hat{I}_2 = 0. \qquad (3)$$

this defines



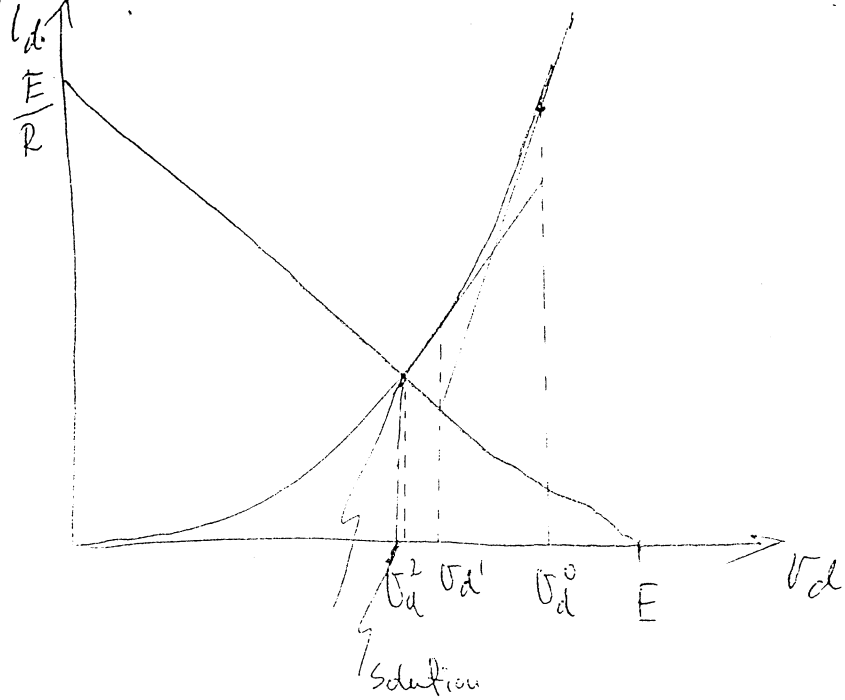
let $\varphi = g_m$ for original circuit

$$\frac{\partial I_2}{\partial \varphi} = \frac{\partial V_1}{\partial \varphi} g_m + V_1$$

substitute into (1). together with (3). yields sensitivity expression,

$$- V_1 \hat{V}_2 \quad //$$

8.1



$$f(v_d) = V_g - v_d^j - R i_d^j = 0 \qquad i_d = i_d(v_d)$$

$$f(v_d) = V_g - v_d^j - R i_d(v_d^j) = 0$$

(*) Newton method

$$v_d^{j+1} = v_d^j - \frac{f(v_d^j)}{f'(v_d^j)}$$

so

$$v_d^{j+1} = v_d^j - \frac{V_g - v_d^j - R i_d(v_d^j)}{-1 - R i_d'(v_d^j)}$$

(*) Network model

we linearize the IV relation for the diode

$$i_d = i_d(v_d)$$

$$i_d^{j+1} = i_d^j + \frac{d i_d(v_d^j)}{d v_d^j}\left(v_d^{j+1} - v_d^j\right) + \cancel{\cdots}$$

or $$i_d^{j+1} = i_d^j + G^j\left(v_d^{j+1} - v_d^j\right)$$

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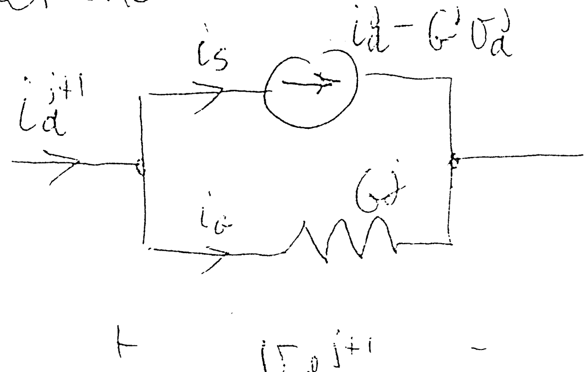
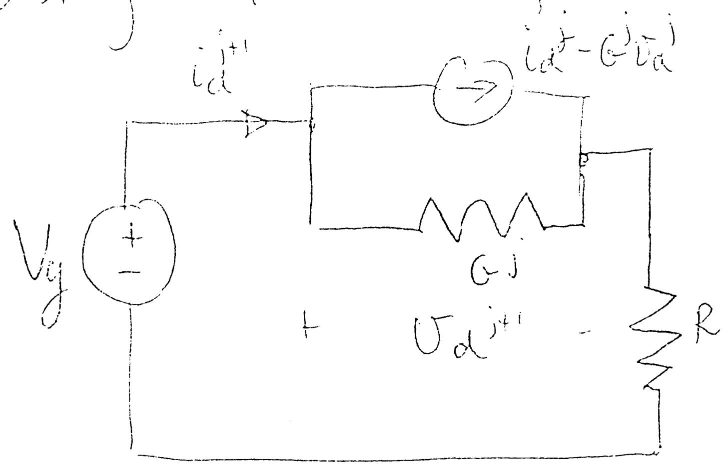
(8₂) $$i_d^{j+1} = \underbrace{G^j v_d^{j+1}}_{i_G} + \underbrace{i_d^j - G^j v_d^j}_{i_s}$$

the network model of i_d^{j+1} is as follows



Using this model for the diode we get



(*) Significance of this model.

The model is a linear approximation of the nonlinear circuit. It is valid only at the jth iteration. Standard methods for solving linear networks can be used to solve the network at the jth iteration.

(9.) From the difference form of the Tellegan theorem:

$$\ldots + [\hat{I}_p \;\; -\hat{V}_p]\begin{bmatrix} V_p \\ I_p \end{bmatrix} + [\hat{I}_q \;\; \hat{V}_q]\begin{bmatrix} V_q \\ -J_q \end{bmatrix} + \ldots = 0$$

Differentiating we get:

$$\ldots + [\hat{I}_p \;\; -\hat{V}_p]\frac{\partial}{\partial\varphi}\begin{bmatrix} V_p \\ I_p \end{bmatrix} + [\hat{I}_q \;\; \hat{V}_q]\frac{\partial}{\partial\varphi}\begin{bmatrix} V_q \\ -J_q \end{bmatrix} + \ldots = 0 \quad (*)$$

From the given branch relation we know that:

$$\frac{\partial}{\partial\varphi}\begin{bmatrix} V_p \\ I_p \end{bmatrix} = \frac{\partial}{\partial\varphi}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_q \\ -I_q \end{bmatrix} + \begin{bmatrix} A & B \\ C & D \end{bmatrix}\frac{\partial}{\partial\varphi}\begin{bmatrix} V_q \\ -I_q \end{bmatrix}$$

Substituting this into $(*)$

$$\ldots + [\hat{I}_p \;\; -\hat{V}_p]\left\{\frac{\partial}{\partial\varphi}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_q \\ -I_q \end{bmatrix} + \begin{bmatrix} A & B \\ C & D \end{bmatrix}\frac{\partial}{\partial\varphi}\begin{bmatrix} V_q \\ -I_q \end{bmatrix}\right\} +$$

$$+ [\hat{I}_q \;\; \hat{V}_q]\frac{\partial}{\partial\varphi}\begin{bmatrix} V_q \\ -J_q \end{bmatrix} + \ldots = 0$$

$$\ldots + [\hat{I}_p \;\; -\hat{V}_p]\frac{\partial}{\partial\varphi}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_q \\ -I_q \end{bmatrix} + [\hat{I}_p \;\; -\hat{V}_p]\begin{bmatrix} A & B \\ C & D \end{bmatrix}\frac{\partial}{\partial\varphi}\begin{bmatrix} V_q \\ -I_q \end{bmatrix} +$$

$$+ [\hat{I}_q \;\; \hat{V}_q]\frac{\partial}{\partial\varphi}\begin{bmatrix} V_q \\ -I_q \end{bmatrix} + \ldots = 0 \quad (**)$$

$$\ldots + [\hat{I}_p \;\; -\hat{V}_p]\frac{\partial}{\partial\varphi}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_q \\ -I_q \end{bmatrix} + \left\{[\hat{I}_p \; -\hat{V}_p]\begin{bmatrix} A & B \\ C & D \end{bmatrix} + [\hat{I}_q \;\; \hat{V}_q]\right\}\frac{\partial}{\partial\varphi}\begin{bmatrix} V_q \\ -I_q \end{bmatrix}$$

$$+ \ldots = 0$$

(q_2) The adjoint element can be defined as:

$$\begin{bmatrix} \hat{I}_p & -\hat{V}_p \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} + \begin{bmatrix} \hat{I}_q & \hat{V}_q \end{bmatrix} = 0$$

Then (**) reduces to

$$\underbrace{\ldots + \begin{bmatrix} \hat{I}_p & -\hat{V}_p \end{bmatrix} \frac{\partial}{\partial \varphi} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_q \\ -I_q \end{bmatrix}}_{\text{This is the sensitivity expression we looked for.}} + \ldots = 0$$

where:

$$V_p = (Y_1 V_q)\left(Z_2 + \frac{1}{Y_1}\right) = \frac{3}{2} V_q \longrightarrow A = \frac{3}{2}$$

$$I_p = Y_1 V_q + Y_1 V_p = Y_1 V_q + Y_1 \frac{3}{2} V_q = \frac{5}{2} \phi V_q \longrightarrow C = \frac{5}{2} \phi$$

$$V_p = -Z_2 I_q = \frac{-I_q}{2\phi} \longrightarrow B = \frac{1}{2\phi}$$

$$I_p = V_p Y_1 - I_q = \frac{-I_q Y_1}{2\phi} - I_q = \left(\frac{Y_1}{2\phi} + 1\right)(-I_q) = \frac{3}{2}(-I_q) \rightarrow D = \frac{3}{2}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2\phi} \\ \frac{5}{2}\phi & \frac{3}{2} \end{bmatrix} \quad \text{so} \quad \frac{\partial}{\partial \varphi} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2\phi^2} \\ \frac{5}{2} & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \hat{I}_p & -\hat{V}_p \end{bmatrix} \frac{\partial}{\partial \varphi} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_q \\ -I_q \end{bmatrix} = -\frac{5}{2} \hat{V}_p V_q + \frac{1}{2\phi^2} \hat{I}_p I_q$$

Question 10 :

Let $h_1(x_1, x_2, y_1, y_2) = -x_1 y_1 y_2 + 2x_2^2 y_2 - 3$

$h_2(x_1, x_2, y_1, y_2) = 4x_1 y_1 - 3y_2$

$$\underset{\sim}{h} = \begin{bmatrix} -x_1 y_1 y_2 + 2x_2^2 y_2 - 3 \\ 4x_1 y_1 - 3y_2 \end{bmatrix}$$

for $\underset{\sim}{x} = [1, 1.25]^T$, we have

$$\underset{\sim}{h} = \begin{bmatrix} -y_1 y_2 + 3.125 y_2 - 3 \\ 4y_1 - 3y_2 \end{bmatrix}$$

The Jacobian $\underset{\sim}{J}$ is given by

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} -y_2 & -y_1 + 3.125 \\ 4 & -3 \end{bmatrix}$$

jth Newton iteration :

$$\underset{\sim}{J}^j \underset{\sim}{y}^{j+1} = \underset{\sim}{J}^j \underset{\sim}{y}^j - \underset{\sim}{h}^j$$

taking $\underset{\sim}{y}^0 = [1 \quad 1]^T$, we have $\underset{\sim}{J}^0 = \begin{bmatrix} -1 & 2.125 \\ 4 & -3 \end{bmatrix}$

$\underset{\sim}{h} = [-0.875 \quad 1]^T$

Substituting these in the Newton iteration :

$$\begin{bmatrix} -1 & 2.125 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} y_1^1 \\ y_2^1 \end{bmatrix} = \begin{bmatrix} -1 & 2.125 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -0.875 \\ 1 \end{bmatrix}$$

$$\begin{cases} -y_1^1 + 2.125\, y_2^1 = 2 \\ 4y_1^1 - 3y_2^1 = 0 \end{cases} \Rightarrow \begin{aligned} y_1^1 &= 1.090908 \\ y_2^1 &= 1.454544 \end{aligned}$$

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Assuming the solution to be $y = [1.125 \quad 1.5]^T$

so. $$\begin{bmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_2}{\partial y_1} \\ \frac{\partial h_1}{\partial y_2} & \frac{\partial h_2}{\partial y_2} \end{bmatrix} \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial y_1} \\ \frac{\partial f}{\partial y_2} \end{bmatrix}$$

i.e. $$\begin{bmatrix} -1.5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow \begin{matrix} \hat{y}_1 = \frac{8}{7} \\ \hat{y}_2 = \frac{3}{7} \end{matrix}$$

Now we may calculate $\partial f / \partial \underset{\sim}{x}$

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = - \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_2}{\partial x_1} \\ \frac{\partial h_1}{\partial x_2} & \frac{\partial h_2}{\partial x_2} \end{bmatrix} \underset{\sim}{\hat{y}} + \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = - \begin{bmatrix} -y_1 y_2 & 4y_1 \\ 4x_2 y_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{8}{7} \\ \frac{3}{7} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1.6875 & 4.5 \\ 7.5 & 0 \end{bmatrix} \begin{bmatrix} \frac{8}{7} \\ \frac{3}{7} \end{bmatrix} = \begin{bmatrix} 0 \\ 8.5714 \end{bmatrix}$$

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Question 11:

$\underset{\sim}{Y}\,\underset{\sim}{V} = \underset{\sim}{I}$ where

$$\underset{\sim}{Y} = \begin{bmatrix} G_1 + 2G_2 & -G_1 & -G_2 \\ -G_1 & 2G_1 + G_2 & -G_1 \\ -G_2 & -G_1 & G_1 + 2G_2 \end{bmatrix} = \begin{bmatrix} 6.5 & -1.5 & -2.5 \\ -1.5 & 5.5 & -1.5 \\ -2.5 & -1.5 & 6.5 \end{bmatrix}$$

$$\underset{\sim}{I} = \begin{bmatrix} 10 \\ 0 \\ -10 \end{bmatrix} = \begin{bmatrix} i \\ 0 \\ -i \end{bmatrix}$$

$$\underset{\sim}{V} = \underset{\sim}{Y}^{-1}\underset{\sim}{I} = \left[\tfrac{10}{9} \quad 0 \quad -\tfrac{10}{9} \right]^T$$

$$\frac{\partial i_2}{\partial q} = \frac{\partial (V_2 G_2)}{\partial q} = \frac{\partial V_2}{\partial q} G_2 + V_2 \frac{\partial G_2}{\partial q}$$

$$\because \frac{\partial \underset{\sim}{I}}{\partial q} = \frac{\partial \underset{\sim}{Y}}{\partial q}\underset{\sim}{V} + \underset{\sim}{Y}\frac{\partial \underset{\sim}{V}}{\partial q} \qquad \therefore \frac{\partial \underset{\sim}{V}}{\partial q} = \underset{\sim}{Y}^{-1}\left(\frac{\partial \underset{\sim}{I}}{\partial q} - \frac{\partial \underset{\sim}{Y}}{\partial q}\underset{\sim}{V}\right)$$

So $$\frac{\partial V_2}{\partial q} = \underset{\sim}{\hat{V}}^T\left(\frac{\partial \underset{\sim}{I}}{\partial q} - \frac{\partial \underset{\sim}{Y}}{\partial q}\underset{\sim}{V}\right)$$

where $\underset{\sim}{\hat{V}}$ is solved from the adjoint network

$$\underset{\sim}{Y}^T \underset{\sim}{\hat{V}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \underset{\sim}{\hat{V}} = \begin{bmatrix} \frac{3}{35} \\ \frac{8}{35} \\ \frac{3}{35} \end{bmatrix}$$

for $q = G_1$, $\frac{\partial \underset{\sim}{I}}{\partial G_1} = 0$, $\frac{\partial V_2}{\partial G_1} = -\underset{\sim}{\hat{V}}^T \frac{\partial \underset{\sim}{Y}}{\partial G_1}\underset{\sim}{V} = -\underset{\sim}{\hat{V}}^T \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \underset{\sim}{V} = 0$

for $q = G_2$, $\frac{\partial \underset{\sim}{I}}{\partial G_2} = 0$, $\frac{\partial V_2}{\partial G_2} = -\underset{\sim}{\hat{V}}^T \frac{\partial \underset{\sim}{Y}}{\partial G_2}\underset{\sim}{V} = -\underset{\sim}{\hat{V}}^T \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} \underset{\sim}{V} = 0$

for $q = i$ $\qquad \frac{\partial \underline{I}}{\partial i} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\frac{\partial V_2}{\partial i} = \hat{\underset{\sim}{V}}^T \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underset{\sim}{V} \right)$$

$$= \hat{\underset{\sim}{V}}^T \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \left[\frac{3}{35} \quad \frac{8}{35} \quad \frac{3}{35} \right] \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$$

So.

$$\frac{\partial i_2}{\partial q} = \frac{\partial V_2}{\partial q} G_2 + V_2 \frac{\partial G_2}{\partial q}$$

$$= 0 \cdot G_2 + 0 \cdot \frac{\partial G_2}{\partial q} = 0$$

$$\therefore \quad \frac{\partial i_2}{\partial G_1} = 0 , \quad \frac{\partial i_2}{\partial G_2} = 0 \quad \frac{\partial i_2}{\partial i} = 0$$

In this circuit, $V_2 \equiv 0$ and it does not change with the change of $G_1 . G_2 . i$, so, $i_2 \equiv 0 = G_2 V_2$ $\quad \frac{\partial i_2}{\partial G_1} = \frac{\partial i_2}{\partial G_2} = \frac{\partial i_2}{\partial i} = 0$

Question 12 :

(a). According to KCL, the nodal equations at node 1 and 2 can be written as:

$$\begin{cases} V_1 + 2(V_2 - V_2)^3 - 1 = 0 \\ V_2^2 + 10V_2 - 2(V_1 - V_2)^3 - 26 = 0 \end{cases}$$

we can define f_1 and f_2 as

$$\begin{cases} f_1 = V_1 + 2(V_1 - V_2)^3 - 1 \\ f_2 = V_2^2 + 10V_2 - 2(V_1 - V_2)^3 - 26 \end{cases} \qquad (1)$$

then the Jacobian matrix at the jth iteration is

$$\underset{\sim}{J}^j = \begin{bmatrix} \frac{\partial f_1}{\partial V_1} & \frac{\partial f_1}{\partial V_2} \\ \frac{\partial f_2}{\partial V_1} & \frac{\partial f_2}{\partial V_2} \end{bmatrix} = \begin{bmatrix} 1 + 6(V_1 - V_2)^2 & -6(V_1 - V_2)^2 \\ -6(V_1 - V_2)^2 & 3V_2^2 + 10 + 6(V_1 - V_2)^2 \end{bmatrix} \qquad (2)$$

The linearized form of nodal equations at the jth iteration of the Newton algorithm can be written as.

$$J^j (V^{j+1} - V^j) = -f^j \qquad (3)$$

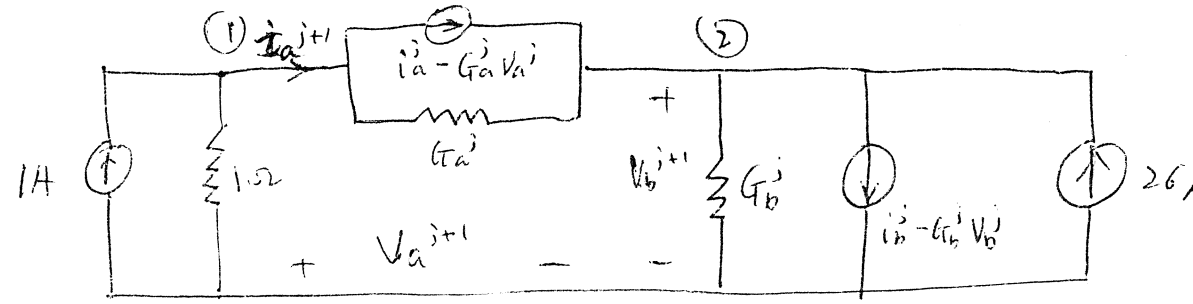
(b) Starting with $\underset{\sim}{V}^0 = [2 \;\; 1]^T$ and using (1), (2), we have

$$\underset{\sim}{f}^0 = \begin{bmatrix} 3 \\ -17 \end{bmatrix}, \quad \underset{\sim}{J}^0 = \begin{bmatrix} 7 & -6 \\ -6 & 19 \end{bmatrix}$$

substituting this into (3), we obtain

$$\underset{\sim}{V}^1 = \begin{bmatrix} 2.46 \\ 2.04 \end{bmatrix}, \quad \text{then} \quad \underset{\sim}{V}^2 = \begin{bmatrix} 1.60 \\ 1.88 \end{bmatrix}$$

(c) the companion network is :



where $i_a^j = 2(V_a^3)^j$ $G_a^j = 6(V_a^2)^j$

$i_b^j = (V_b^3)^j + (10V_b)^j$ $G_b^j = (3V_b^2)^j + 10$

and $V_a = V_1 - V_2$ and $V_b = V_2$

corresponding to nodal equation, we have

$$\begin{bmatrix} 1+G_a^j & -G_a^j \\ -G_a^j & G_a^j + G_b^j \end{bmatrix} \begin{bmatrix} V_1^{j+1} \\ V_2^{j+1} \end{bmatrix} = \begin{bmatrix} 1-(i_a^j - G_a^j V_a^j) \\ (i_a^j - G_a^j V_a^j)-(i_b^j - G_b^j V_b^j)+26 \end{bmatrix} \quad (4)$$

(d). using $\underset{\sim}{V}^0 = [1.60 \quad 1.88]^T$ and the equation (4)

we can obtain

$\underset{\sim}{V}^1 = [1.23 \quad 1.89]^T$, $\underset{\sim}{V}^2 = [1.32 \quad 1.89]^T$

Question 1

Use the method of Lagrange multipliers to minimize w.r.t. ϕ_1 and ϕ_2 the function

$$U = \phi_1^2 + 3\phi_2^2 - 4$$

subject to

$$\phi_1 - \phi_2 + 1 = 0$$

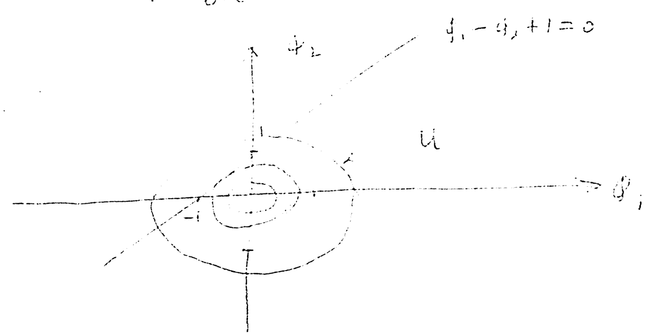
Sketch contours to illustrate this problem w.r.t. ϕ_1 and ϕ_2. Verify the answer by substituting the constraint into the function.

Answer (15 min.)

(1) $L(\Phi, \lambda) = U(\Phi) + \lambda h = \phi_1^2 + 3\phi_2^2 - 4 + \lambda(\phi_1 - \phi_2 + 1)$

Solving $\begin{cases} \dfrac{\partial L}{\partial \phi_1} = 2\phi_1 + \lambda = 0 \\ \dfrac{\partial L}{\partial \phi_2} = 6\phi_2 - \lambda = 0 \\ \dfrac{\partial L}{\partial \lambda} = \phi_1 - \phi_2 + 1 = 0 \end{cases}$ we get $\phi = \begin{bmatrix} -0.75 \\ 0.25 \end{bmatrix}$ $\lambda = 1.5$

(2)



(3) Substituting h into u. $\phi_1 = \phi_2 - 1$

$u = (\phi_2 - 1)^2 + 3\phi_2^2 - 4 = 4\phi_2^2 - 2\phi_2 - 5$

$\dfrac{\partial u}{\partial \phi_2} = 8\phi_2 - 2 = 0$, then $\phi_2 = \frac{1}{4} = 0.25$, $\phi_1 = -0.75$.

$$\phi = \begin{bmatrix} -0.75 \\ 0.25 \end{bmatrix}$$

continued on page 3

Question 2

Apply the Newton method to the minimization of

$$\phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_2 + 1$$

w.r.t. ϕ. Select the two starting points

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}$$

What is the solution?

<u>Answer</u> (15 min.)

$$U = \phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_2 + 1$$

$$\nabla U = \begin{bmatrix} 2\phi_1 + \phi_2 \\ \phi_1 + 4\phi_2 + 2 \end{bmatrix} \qquad H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \qquad H^{-1} = \begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

① $\phi^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \nabla U^0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$\Delta\phi^0 = -H^{-1}\nabla U^0 = -\begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \end{bmatrix}$$

$$\phi^1 = \phi^0 + \Delta\phi^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \end{bmatrix}$$

$$\nabla U^1 = \begin{bmatrix} \frac{2}{7}\times 2 - \frac{4}{7} \\ \frac{2}{7} - \frac{16}{7} + \frac{14}{7} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the solution is $\phi = \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \end{bmatrix}$

② $\phi^0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \nabla U^0 = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

$$\Delta\phi^0 = -H^{-1}\nabla U^0 = -\begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}\begin{bmatrix} 2 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ -\frac{18}{7} \end{bmatrix}$$

$$\phi^1 = \phi^0 + \Delta\phi^0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} \frac{2}{7} \\ -\frac{18}{7} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \end{bmatrix}$$

$$\nabla U^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the solution is $\phi = \begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \end{bmatrix}$

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Question 3

Consider a nodal system of equations as

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Express the solution of this system as a least squares problem. Write down the corresponding Jacobian. Write down the gradient vector of the least squares objective.

Answer (15 min.)

From the equations
$$\begin{cases} g_{11}V_1 + g_{12}V_2 + g_{13}V_3 = 1 \\ g_{21}V_1 + g_{22}V_2 + g_{23}V_3 = 0 \\ g_{31}V_1 + g_{32}V_2 + g_{33}V_3 = 0 \end{cases}$$

so
$$\begin{cases} e_1(\underset{\sim}{V}) = g_{11}V_1 + g_{12}V_2 + g_{13}V_3 - 1 \\ e_2(\underset{\sim}{V}) = g_{21}V_1 + g_{22}V_2 + g_{23}V_3 \\ e_3(\underset{\sim}{V}) = g_{31}V_1 + g_{32}V_2 + g_{33}V_3 \end{cases}$$

the least squares problem is $U = \sum_{i=1}^{3} (e_i(\underset{\sim}{V}))^2$

the Jacobian matrix is

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial e_1}{\partial V_1} & \frac{\partial e_1}{\partial V_2} & \frac{\partial e_1}{\partial V_3} \\ \frac{\partial e_2}{\partial V_1} & \frac{\partial e_2}{\partial V_2} & \frac{\partial e_2}{\partial V_3} \\ \frac{\partial e_3}{\partial V_1} & \frac{\partial e_3}{\partial V_2} & \frac{\partial e_3}{\partial V_3} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

the gradient vector of U is

$$\underset{\sim}{\nabla} U = \sum_{i=1}^{3} 2e_i \underset{\sim}{\nabla} e_i = \begin{bmatrix} 2e_1 \cdot g_{11} \\ 2e_1 g_{12} \\ 2e_1 g_{13} \end{bmatrix} + \begin{bmatrix} 2e_2 g_{21} \\ 2e_2 g_{22} \\ 2e_2 g_{23} \end{bmatrix} + \begin{bmatrix} 2e_3 g_{31} \\ 2e_3 g_{32} \\ 2e_3 g_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 2e_1 g_{11} + 2e_2 g_{21} + 2e_3 g_{31} \\ 2e_1 g_{12} + 2e_2 g_{22} + 2e_3 g_{32} \\ 2e_1 g_{13} + 2e_2 g_{23} + 2e_3 g_{33} \end{bmatrix}$$

continued on page 5

Question 4

Derive the exact Newton iteration at ϕ^j for minimization of a differentiable multidimensional function $U(\phi)$. Define all terms used. Under what conditions do you expect a locally downhill step from the Newton iteration? Discuss possible pitfalls of the basic Newton method and suggest remedies. What is damping? Illustrate your answers with sketches.

Answer (20 min.)

(1) For a differentiable multidimensional function $U(\underline{\phi})$

The Taylor series expansion of $U(\underline{\phi})$ given by

$$U(\underline{\phi}+\Delta\underline{\phi}) = U(\underline{\phi}) + \nabla U^T \Delta\underline{\phi} + \frac{1}{2}\Delta\underline{\phi}^T \underset{\sim}{H} \Delta\underline{\phi} + \cdots$$

where

$$\underline{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}, \quad \Delta\underline{\phi} = \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \vdots \\ \Delta\phi_n \end{bmatrix} \quad \underset{\sim}{\nabla U} = \begin{bmatrix} \frac{\partial U}{\partial \phi_1} \\ \frac{\partial U}{\partial \phi_2} \\ \vdots \\ \frac{\partial U}{\partial \phi_n} \end{bmatrix} \quad \underset{\sim}{H} = \begin{bmatrix} \frac{\partial^2 U}{\partial \phi_1^2} & \frac{\partial^2 U}{\partial \phi_2 \partial \phi_1} & \cdots & \frac{\partial^2 U}{\partial \phi_1 \partial \phi_n} \\ \frac{\partial^2 U}{\partial \phi_1 \partial \phi_2} & \frac{\partial^2 U}{\partial \phi_2^2} & \cdots & \frac{\partial^2 U}{\partial \phi_2 \partial \phi_n} \\ \vdots & & & \\ \frac{\partial^2 U}{\partial \phi_1 \partial \phi_n} & \frac{\partial^2 U}{\partial \phi_2 \partial \phi_n} & \cdots & \frac{\partial^2 U}{\partial \phi_n^2} \end{bmatrix}$$

then $\nabla U(\underline{\phi}+\Delta\underline{\phi}) = \nabla U(\underline{\phi}) + H \Delta\underline{\phi} + \cdots$ (neglected)

if $\underline{\phi}+\Delta\underline{\phi}$ is the minimizing point $\hat{\phi}$,

then $\nabla U(\underline{\phi}+\Delta\underline{\phi}) = 0$ i.e. $\Delta\phi = -H^{-1}\nabla U$ where $H = \nabla(\nabla U)^T$

So $\phi^{j+1} = \phi^j + \left[-H_j^{-1} \nabla U(\underline{\phi}^j)\right]$

② Downhill step condition:

$$\underset{\sim}{H}\,\Delta\underset{\sim}{\varphi} = -\underset{\sim}{\nabla} U(\underset{\sim}{\varphi})$$

$$\Delta\underset{\sim}{\varphi}^T H \Delta\underset{\sim}{\varphi} = -\Delta\underset{\sim}{\varphi}^T \underset{\sim}{\nabla} U(\underset{\sim}{\varphi})$$

if $\underset{\sim}{H}$ is positive definite ($\Delta\underset{\sim}{\varphi}^T H \Delta\underset{\sim}{\varphi} > 0$)

then $-\Delta\underset{\sim}{\varphi}^T \underset{\sim}{\nabla} U(\underset{\sim}{\varphi}) > 0$

and $\Delta\underset{\sim}{\varphi}^T \underset{\sim}{\nabla} U(\underset{\sim}{\varphi}) < 0$ So we go downhill.

③ Problems with Newton method:

(i) the method may diverge, or converge to a ~~ma~~ maximum, rathen than minimum

(ii) $\underset{\sim}{H}$ may be locally singular

(iii) Computations of $\underset{\sim}{H}$ and its inverse are expensive

the suggested remedy is to use damping Newton method.

④ Damped Newton method:

modify $\underset{\sim}{H}$ by $\lambda \underset{\sim}{I}$

i.e. $(\underset{\sim}{H} + \lambda \underset{\sim}{I})\Delta\varphi = -\underset{\sim}{\nabla} U(\underset{\sim}{\varphi})$

When $\lambda \to \infty$, we have $\lambda \underset{\sim}{I} \Delta\underset{\sim}{\varphi} = -\underset{\sim}{\nabla} U(\underset{\sim}{\varphi})$

or $\Delta\underset{\sim}{\varphi} = -\frac{1}{\lambda} \underset{\sim}{\nabla} U(\underset{\sim}{\varphi})$

which is a small ($\frac{1}{\lambda}$) step towards the steepest decent.

Question 5

Examine the points $[0\ \ 0]^T$ and $[1\ \ 1]^T$ for a minimax problem for which

$$f_1 = \phi_1^4 + \phi_2^2$$

$$f_2 = (2 - \phi_1)^2 + (2 - \phi_2)^2$$

$$f_3 = 2\ \exp(-\phi_1 + \phi_2)$$

by invoking necessary conditions for a minimax optimum. What are your conclusions?

<u>Answer</u> (20 min.) At minimax optimum, the following conditions must hold

$$\begin{cases} \sum_{i=1}^{3} u_i \nabla f_i = \underset{\sim}{0} & (1) \\ \sum_{i=1}^{3} u_i = 1 & (2) \\ u_i (M - f_i) = 0 & (3) \\ u_i \geq 0 & (4) \end{cases}$$

for this problem, $\nabla f_1 = \begin{bmatrix} 4\phi_1^3 \\ 2\phi_2 \end{bmatrix}$, $\nabla f_2 = \begin{bmatrix} -2(2-\phi_1) \\ -2(2-\phi_2) \end{bmatrix}$, $\nabla f_3 = \begin{bmatrix} -2e^{(-\phi_1+\phi_2)} \\ 2e^{(-\phi_1+\phi_2)} \end{bmatrix}$

① At $[0, 0]^T$. $f_1 = 0$, $f_2 = 8$, $f_3 = 2$. so f_2 is the active function. $M = 8$

from condition (3) $u_1 = u_3 = 0$.

from condition (2) $u_2 = 1$.

from condition (1) $u_2 \cdot \underset{\sim}{\nabla} f_2 = \begin{bmatrix} -4 \\ -4 \end{bmatrix} \neq 0$,

So the necessary conditions of minimax optimum are violated. $[0, 0]^T$ is not the minimax solution.

② At $[1, 1]^T$. $f_1 = 2$, $f_2 = 2$, $f_3 = 2$. f_1, f_2, f_3 are all active. $M = 2$

from (1) $u_1 \begin{bmatrix} 4 \\ 2 \end{bmatrix} + u_2 \begin{bmatrix} -2 \\ -2 \end{bmatrix} + u_3 \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 0$.

i.e. $4u_1 - 2u_2 - 2u_3 = 0$
$2u_1 - 2u_2 + 2u_3 = 0$.

from (2) $u_1 + u_2 + u_3 = 0$

$\Rightarrow \begin{cases} u_1 = \frac{1}{3} > 0 \\ u_2 = \frac{1}{2} > 0 \\ u_3 = \frac{1}{6} > 0 \end{cases}$

continued on page 7

all the conditions are satisfied, so $[1, 1]^T$ could be the minimax solution

Question 6

Consider the function

$$U = \phi_1 + 2\phi_2$$

along with the constraints

$$g_1 = \phi_1 + \phi_2 - 2 \geq 0$$

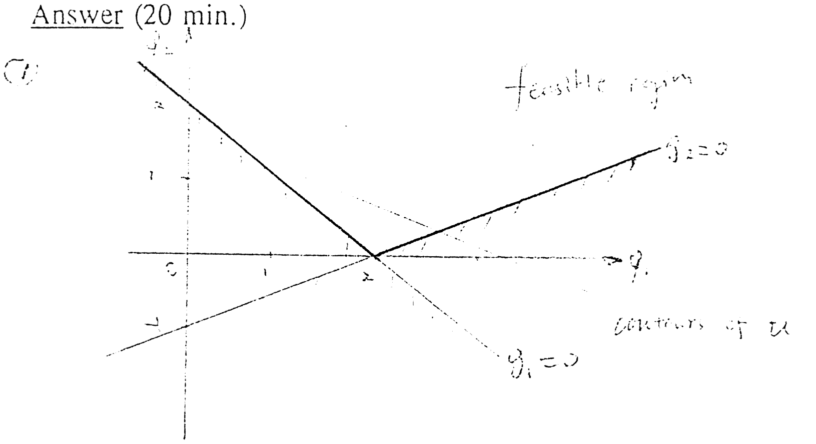
$$g_2 = -\phi_1 + 2\phi_2 + 2 \geq 0$$

Sketch two or three contours of the objective function, plot the constraint boundaries and indicate the feasible region.

Invoke the Kuhn-Tucker (KT) conditions at each of the three points (i) $[2 \;\; 0]^T$; (ii) $[0 \;\; 0]^T$; (iii) $[0 \;\; 2]^T$. State the results and comment on them.

Answer (20 min.)

(1)



(2) KT conditions are

$$\begin{cases} \nabla U = \sum_i u_i \nabla g_i & (1) \\ u_i g_i = 0 & (2) \\ u_i \geq 0 & (3) \end{cases}$$

$$\nabla U = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \nabla g_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \nabla g_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

At $[2 \;\; 0]^T$, $g_1 = 0$, $g_2 = 0$. both g_1 and g_2 are active

from (1) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} u_1 - u_2 = 1 \\ u_1 + 2u_2 = 2 \end{cases} \Rightarrow \begin{cases} u_1 = 4/3 \geq 0 \\ u_2 = 1/3 \geq 0 \end{cases}$

all the conditions are satisfied, so $[2 \;\; 0]^T$ could be a constrained minimum

At $[0 \;\; 0]^T$. $g_1 = -2$, $g_2 = 2$,

from (1) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = u_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} u_1 = \frac{4}{3} \\ u_2 = 1/3 \end{cases}$

substitute $\begin{cases} u_1 = \frac{4}{3} \\ u_2 = \frac{1}{3} \end{cases}$ to (2) $\frac{4}{3}(-2) + \frac{1}{3}(2) \neq 0$. Condition (2) is not satisfied

so $[0, 0]^T$ is not the constrained minimum.

At $[0 \;\; 2]^T$. $g_1 = 0$, $g_2 = 6$.

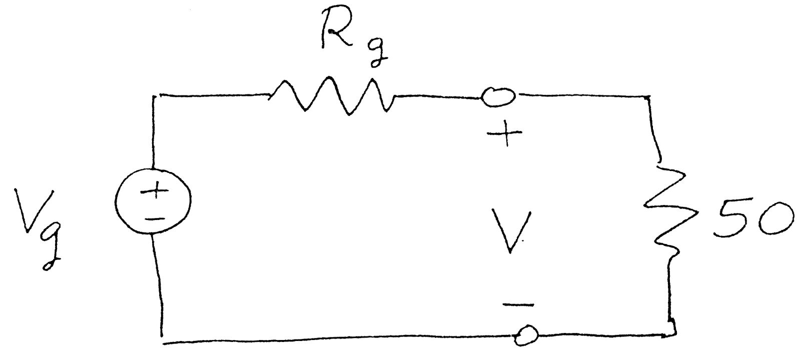
from (1). $u_1 = \frac{4}{3}$, $u_2 = \frac{1}{3}$, subititute $\begin{cases} u_1 = \frac{4}{3} \\ u_2 = \frac{1}{3} \end{cases}$ into (2): we have $\frac{4}{3} \times 0 + \frac{6}{3} \neq 0$

continued on page 8

Condition (2) is violated. so. $[0 \;\; 2]^T$ is not the constrained minimum.

Question 7

Consider the voltage divider shown for the response specification and constraint indicated.



Design specification: $V \geq 60$
Constraint: $R_g \geq 50$

By testing the Kuhn-Tucker conditions, find V_g and R_g such that the total power dissipated is minimum.

Answer (20 min.)

The power dissipated $P = \dfrac{V_g^2}{R_g+50}$

$$V = \frac{50V_g}{R_g+50} \geq 60, \quad R_g \geq 50$$

Consider the following constrained minimization problem:

$$U = \frac{V_g^2}{R_g+50}$$

Constraints: $\begin{cases} g_1 = 5V_g - 6R_g - 300 \geq 0 \\ g_2 = R_g - 50 \geq 0 \end{cases}$

So $\nabla g_1 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$, $\nabla g_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad \nabla U = \begin{bmatrix} \dfrac{2V_g}{R_g+50} \\ \dfrac{-V_g^2}{(R_g+50)^2} \end{bmatrix}$

from KT condition
$$\begin{cases} \nabla U = u_1 \begin{bmatrix} 5 \\ -6 \end{bmatrix} + u_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} & (1) \\ u_1 g_1 + u_2 g_2 = 0 & (2) \\ u_1 \geq 0, \; u_2 \geq 0 & (3) \end{cases}$$

suppose the point $[V_g \;\; R_g]^T = [120, 50]^T$. both g_1 and g_2 are active

from (1). we have $\begin{bmatrix} 2.4 \\ -1.44 \end{bmatrix} = u_1 \begin{bmatrix} 5 \\ -6 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$u_1 = 0.48$. $u_2 = 1.44$ (3) is satisfied.

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So. $V_g = 120$, $R_g = 50$ is the optimal value of V_g. R_g such that P is minimum

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Question 8

Using [0, 5] as the initial interval of uncertainty, apply 3 iterations of the Golden Section search method to the minimization w.r.t. ϕ of a function described by

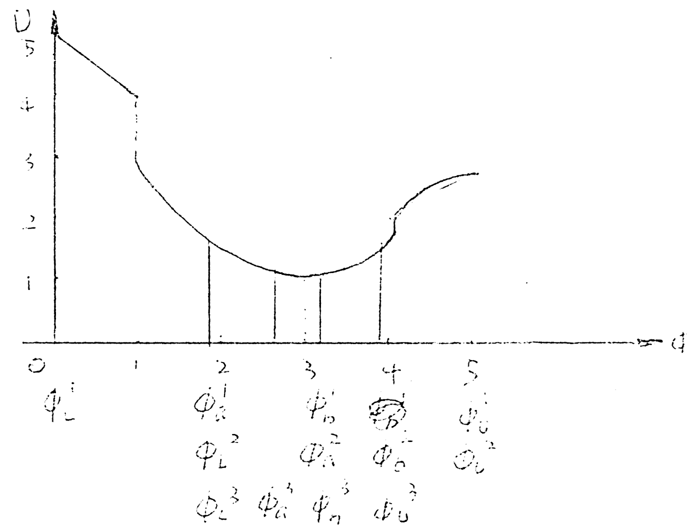
$$U = -\phi + 5 \qquad \phi < 1$$

$$U = 0.5(\phi - 3)^2 + 1 \qquad 1 \le \phi \le 4$$

$$U = 3 - \frac{(\phi - 6)^2}{3} \qquad \phi > 4$$

Show all steps clearly and label a diagram appropriately. State the final interval of uncertainty.

<u>Answer</u> (25 min.)



$\tau = 1.618$

Step 1 $\phi_L^1 = 0$ $\phi_U^1 = 5$. $I^1 = \phi_U^1 - \phi_L^1 = 5 - 0 = 5$, $\phi_a^1 = \frac{1}{\tau^2} I^1 + \phi_L^1 = 1.9098$ $\phi_b^1 = \frac{1}{\tau} I^1 + \phi_L^1 = 3.0902$

$U_a^1 = 0.5(1.9098 - 3)^2 + 1 = 1.5943$. $U_b^1 = 0.5(3.0902 - 3)^2 + 1 = 1.0041$

$U_a^1 > U_b^1$, so $\phi_L^2 = \phi_a^1$, $\phi_a^2 = \phi_b^1$, $\phi_U^2 = \phi_U^1$ see page 5-6 Volume 1

Step 2. $\phi_L^2 = 1.9098$, $\phi_U^2 = 5$, $I^2 = \phi_U^2 - \phi_L^1 = 3.0902$, $\phi_b^2 = \phi_L^2 + \frac{1}{\tau} I^2 = 3.8196$, $\phi_a^2 = 3.0902$

$U_a^2 = U_b^1 = 1.0041$, $U_b^2 = 1.3359$.

$U_a^2 < U_b^2$, so $\phi_L^3 = \phi_L^2$, $\phi_b^3 = \phi_a^3$, $\phi_U^3 = \phi_b^2$

Step 3. $\phi_L^3 = 1.9098$, $\phi_U^3 = 3.8196$, $I^3 = \phi_U^3 - \phi_L^3 = 1.9098$

$\phi_a^3 = \phi_L^3 + \frac{1}{\tau^2} I^3 = 2.6393$, $\phi_b^3 = 3.0902$. $U_a^3 > U_b^3$

so $\phi_L^4 = \phi_a^3$ $\phi_U^4 = \phi_L^3$

The final interval of uncertainty is [2.6393 3.8196]

continued on page 10

Question 9

Consider the three functions

$$f_1 = \phi_2 - 3$$
$$f_2 = -\phi_1 + 2$$
$$f_3 = \phi_1 - \phi_2 - 1$$

Use a graph to aid your analysis and find the solution to the minimax problem

$$\underset{\phi}{\text{minimize}} \ \underset{i}{\max} \ f_i(\phi)$$

by starting at $[1 \ 0]^T$ and using two iterations of a steepest descent algorithm.

Answer (25 min.)

PP

Question 9:

At the starting point $\varphi^0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$f_1 = \varphi_2 - 3 = -3$, $f_2 = -\varphi_1 + 2 = 1$, $f_3 = \varphi_1 - \varphi_2 - 1 = 0$

only f_2 is active at this point

$$\nabla f_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The steepest descent direction is

$$s^1 = -\frac{\nabla f_2}{\|\nabla f_2\|} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Let $\varphi^1 = \varphi^0 + \gamma_1 s^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \gamma_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma_1 + 1 \\ 0 \end{bmatrix}$ then

$f_1(\varphi^1) = -3$, $f_2(\varphi^1) = -\gamma_1 + 1$, $f_3(\varphi^1) = \gamma_1$

The solution for γ_1 to the minimum of $\max\limits_{i=1,2,3} f_i(\varphi^1)$ is $\gamma_1 = \frac{1}{2}$

$$\varphi^1 = \varphi^0 + \frac{1}{2} s^1 = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}$$

$f_1(\varphi^1) = -3$, $f_2(\varphi^1) = \frac{1}{2}$, $f_3(\varphi^1) = \frac{1}{2}$

f_2 and f_3 are active at this point. For a descent direction s^2,

we must have $\nabla f_2^T s^2 < 0$

$$\nabla f_3^T s^2 < 0$$

Consider $s^2 = -\alpha_2 \nabla f_2 - \alpha_3 \nabla f_3$

$$\alpha_2 + \alpha_3 = 1$$
$$\alpha_2 \geq 0$$
$$\alpha_3 \geq 0$$

Which suggest the linear programming maximize $\alpha \geq 0$

subject to

$$-\nabla f_2^T (\alpha_2 \nabla f_2 + \alpha_3 \nabla f_3) \leq -\alpha \qquad (1)$$

$$-\nabla f_3^T (\alpha_2 \nabla f_2 + \alpha_3 \nabla f_3) \leq -\alpha \qquad (2)$$

$$\alpha_2 + \alpha_3 = 1$$
$$\alpha_2 \geq 0$$
$$\alpha_3 \geq 0$$

substitute $\underset{\sim}{\nabla} f_2 = \begin{bmatrix}-1\\0\end{bmatrix}$, $\underset{\sim}{\nabla} f_3 = \begin{bmatrix}1\\-1\end{bmatrix}$ into (1) (2)

we obtain
$$-\alpha_2 + \alpha_3 \le -\alpha$$
$$\alpha_2 - 2\alpha_3 \le -\alpha$$

let
$$\begin{cases} -\alpha_2 + \alpha_3 = \alpha_2 - 2\alpha_3 \\ \alpha_2 + \alpha_3 = 1 \end{cases} \Longrightarrow \begin{cases} \alpha_2 = \frac{3}{5} \\ \alpha_3 = \frac{2}{5} \end{cases}$$

so
$$s^2 = -\alpha_2 \underset{\sim}{\nabla} f_2 - \alpha_3 \underset{\sim}{\nabla} f_3 = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$\underset{\sim}{\varphi}^2 = \underset{\sim}{\varphi}^1 + \gamma_2 s^2 = \begin{bmatrix} \frac{3}{2} + \frac{1}{5}\gamma_2 \\ \frac{2}{5}\gamma_2 \end{bmatrix}$$

$$f_1(\underset{\sim}{\varphi}) = \frac{2}{5}\gamma_2 - 3, \quad f_2(\underset{\sim}{\varphi}) = -\frac{1}{5}\gamma_2 + \frac{1}{2}, \quad f_3(\underset{\sim}{\varphi}) = -\frac{1}{5}\gamma_2 + \frac{1}{2}$$

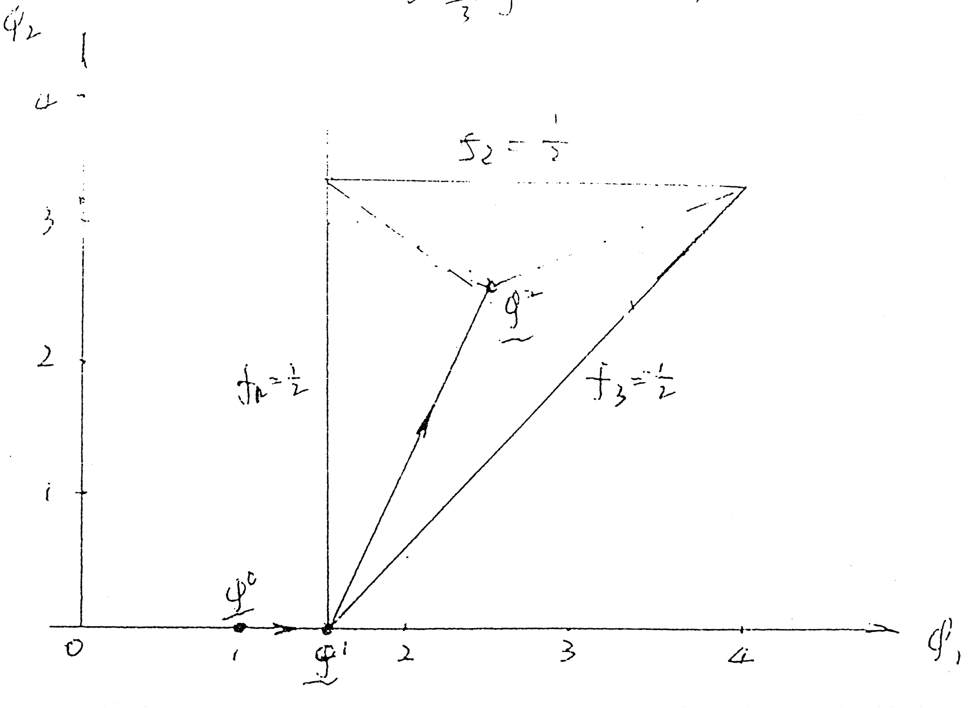
The solution for γ_2 to the minimum value of $\max_{i=1,2,3} f_i(\varphi^2)$

can be obtained by
$$\frac{2}{5}\gamma_2 - 3 = -\frac{1}{5}\gamma_2 + \frac{1}{2}, \quad \gamma_2 = \frac{35}{6}$$

$$\varphi^2 = \begin{bmatrix} \frac{3}{2} + \frac{1}{5}\cdot\frac{35}{6} \\ \frac{2}{5}\cdot\frac{35}{6} \end{bmatrix} = \begin{bmatrix} \frac{8}{3} \\ \frac{7}{3} \end{bmatrix}$$

$$f_1(\underset{\sim}{\varphi}^2) = f_2(\underset{\sim}{\varphi}^2) = f_3(\underset{\sim}{\varphi}^2) = -\frac{2}{3}$$

all function are active $\varphi^2 = \begin{bmatrix} \frac{8}{3} \\ \frac{7}{3} \end{bmatrix}$ is the optimal



Question 10

The updating formula for the Fletcher-Powell-Davidon method is defined by

$$H^0 = 1$$

$$s^j = -H^j \, \nabla U^j, \quad j = 0, 1, 2, \ldots$$

where

$$H^{j+1} = H^j + \frac{\Delta\phi^j \Delta\phi^{j\,T}}{\Delta\phi^{j\,T} g^j} - \frac{H^j g^j g^{j\,T} H^j}{g^{j\,T} H^j g^j}$$

$$\Delta\phi^j \triangleq \alpha^j s^j = \phi^{j+1} - \phi^j$$

$$g^j \triangleq \nabla U^{j+1} - \nabla U^j$$

(a) What is H^j and what is its relationship with the Hessian matrix of a function $U(\phi)$? How is α^j computed in practice?

(b) Apply the algorithm (using a theoretically justified approach to obtain α^j) to the minimization of

$$\phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + \phi_2 + 2$$

w.r.t. ϕ_1 and ϕ_2 starting at $\phi_1 = 0$, $\phi_2 = 2$. Show all steps explicitly and comment on the results obtained. Draw an accurate diagram showing the path taken.

Answer (40 min.)

(a). H^j is the jth positive definite matrix. and is the jth approximation to the inverse of the Hessian matrix.

α^j is computed by minimizing the objective function at ϕ^{j+1} w.r.t α. $\phi^{j+1} = \phi^j + \alpha^j s^j$

Question 10

(b). $\nabla \underset{\sim}{u} = \begin{bmatrix} 2\phi_1 + \phi_2 \\ \phi_1 + 4\phi_2 + 1 \end{bmatrix}$

$\nabla \underset{\sim}{u}(\phi^0) = \begin{bmatrix} 0+2 \\ 0+8+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

$H^0 = \underset{\sim}{I}$

$S^0 = -H^0 \nabla \underset{\sim}{u}^0 = \begin{bmatrix} -2 \\ -9 \end{bmatrix}$ $\qquad \Delta\phi = \alpha S = \begin{bmatrix} -2\alpha \\ -9\alpha \end{bmatrix}$

$\underset{\sim}{\phi}_1' = \underset{\sim}{\phi}^0 + \alpha S = \begin{bmatrix} -2\alpha \\ 2-9\alpha \end{bmatrix}$

$u(\phi_1) = (0-2\alpha)^2 + 2(2-9\alpha)^2 + (0-2\alpha)(2-9\alpha) + (2-9\alpha) + 2$

$= 4\alpha^2 + 2(4 - 36\alpha + 81\alpha^2) + 18\alpha^2 - 4\alpha + 4 - 9\alpha$

$= 184\alpha^2 - 85\alpha + 12$

$\frac{du'}{d\alpha} = 2 \times 184\alpha - 85 = 0$

then $\alpha = \frac{85}{2 \times 184} = 0.2310$

So $\phi_1 = \phi^0 + \alpha S$

$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 9 \end{bmatrix} = \begin{bmatrix} -0.4620 \\ -0.079 \end{bmatrix}$

$\Delta\phi^0 = \alpha S = \begin{bmatrix} -0.4620 \\ -2.079 \end{bmatrix}$

$$\underset{\sim}{Du}^1 = \begin{bmatrix} +2 * (-0.4620) - 0.079 \\ -0.4620 - 4 \times (0.079) + 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1.003 \\ +0.222 \end{bmatrix}$$

$$\underset{\sim}{g}^0 = \underset{\sim}{Du}' - \underset{\sim}{Du}^0 = \begin{bmatrix} -1.003 - 2 \\ 0.222 - 9 \end{bmatrix} = \begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}$$

$$H' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\Delta\Phi^0 \cdot \Delta\Phi^{0T}}{\Delta\Phi^{0T} \cdot g^0} - \frac{H^0 \cdot g^0 \cdot g^{0T} H^0}{g^{0T}\, H^0 \cdot g^0}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\begin{bmatrix} -0.4620 \\ -2.079 \end{bmatrix} \begin{bmatrix} -0.4620 \\ -2.079 \end{bmatrix}^T}{\begin{bmatrix} -0.4260 & -2.079 \end{bmatrix} \begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}} - \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix} \begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} -3.003 & -8.778 \end{bmatrix} \begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.0109 & 0.0489 \\ 0.0489 & 0.2201 \end{bmatrix} - \begin{bmatrix} 0.1048 & 0.3063 \\ 0.3063 & 0.8952 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9061 & -0.2573 \\ -0.2573 & 0.3249 \end{bmatrix}$$

$$S^1 = -H'Du' = -\begin{bmatrix} 0.9061 & -0.2573 \\ -0.2573 & 0.3249 \end{bmatrix} \begin{bmatrix} -1.003 \\ +0.222 \end{bmatrix} = \begin{bmatrix} 0.9659 \\ -0.3302 \end{bmatrix}$$

$$\tilde{\Phi}^2 = \tilde{\Phi}^1 + \Delta\Phi^1 = \begin{bmatrix} -0.4620 + 0.9659\alpha \\ \\ -0.079 - 0.3302\alpha \end{bmatrix}$$

$$u^2 = (-0.4620 + 0.9659\alpha)^2 + 2(-0.079 - 0.3302\alpha)^2$$

$$+ (-0.462 + 0.9659\alpha)(-0.079 - 0.3302\alpha) + (-0.079 - 0.3302\alpha) + 2$$

$$= C_1 + 2 \times (-0.4620) \times (0.9659)\alpha + 0.9659^2\alpha^2$$

$$+ C_2 + 4 \times (0.079) \times 0.3302 \times \alpha + 2 \times 0.3302^2\alpha^2$$

$$+ C_3 + (-0.9659 \times 0.079 + 0.4620 \times 0.3302)\alpha - 0.9659 \times 0.3302\alpha^2$$

$$+ C_4 - 0.3302\,\alpha$$

$$\frac{du^2}{d\alpha} = 2 \times 0.9659^2\alpha + 4 \times 0.3302^2\alpha - 2 \times 0.9659 \times 0.3302\,\alpha$$

$$+ (-2 \times 0.4620 \times 0.9659 + 4 \times 0.079 \times 0.3302 - 0.9659 \times 0.079 + 0.4620 \times 0.3302$$

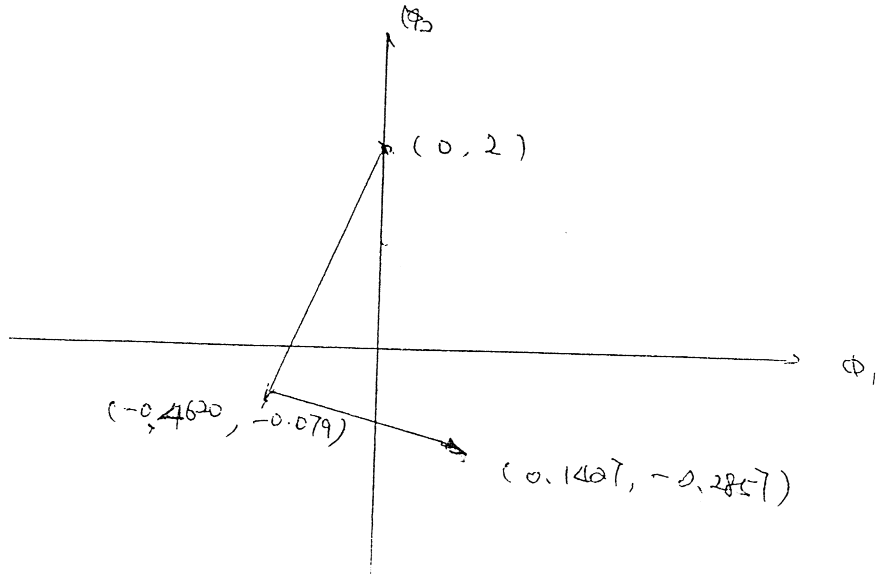
$$- 0.3302)$$

$$\therefore \alpha' \doteq 0.6261$$

$$\tilde{\Phi}^2 = \begin{bmatrix} -0.462 + 0.9659 \times 0.6210 \\ -0.079 + (-0.3302) \times 0.6210 \end{bmatrix} = \begin{bmatrix} 0.1427 \\ -0.2857 \end{bmatrix}$$

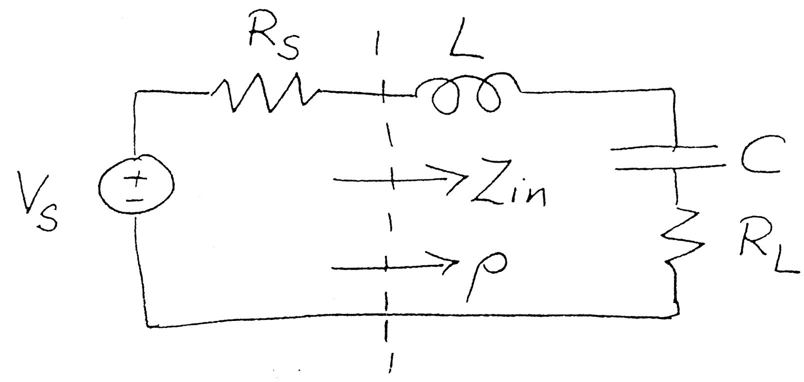
$$\tilde{\nabla} u^2 = \begin{bmatrix} 2\Phi_1 + \Phi_2 \\ \\ \Phi_1 + 4\Phi_2 + 1 \end{bmatrix} \doteq \begin{bmatrix} 0 \\ \\ 0 \end{bmatrix}$$

The Fletcher – Powell Method results in a solution to a quadratic in 2 step.



Question 11

Consider the circuit shown at frequency ω.



R_S fixed

L fixed

V_S fixed

We define a complex coefficient ρ as

$$\rho = \frac{Z_{in} - R_S}{Z_{in} + R_S}$$

Show that $|\rho|^2$ is minimized w.r.t. C for fixed R_L when $\omega^2 LC = 1$. First prove that $\frac{d|z|^2}{dx} = 2Re\left\{z^* \frac{dz}{dx}\right\}$, where z^* is the conjugate of z.

Answer (40 min.)

$$\frac{d|z|^2}{dx} = \frac{d(z \cdot z^*)}{dx} = \frac{dz}{dx} \cdot z^* + \frac{dz^*}{dx} \cdot z.$$

let $z = z_R + j z_i$ $\quad$ so $\frac{d|z|^2}{dx} = \left(\frac{dz_R}{dx} + j\frac{dz_i}{dx}\right)(z_R - jz_i) + \left(\frac{dz_R}{dx} - j\frac{dz_i}{dx}\right)(z_R + jz_i)$

$z^* = z_R - j z_i$

$$= \frac{dz_R}{dx} \cdot z_R - j\frac{dz_R}{dx} \cdot z_i + j\frac{dz_i}{dx} \cdot z_R + \frac{dz_i}{dx} \cdot z_i$$
$$+ \frac{dz_R}{dx} \cdot z_R + j\frac{dz_R}{dx} \cdot z_i - j\frac{dz_i}{dx} \cdot z_R + \frac{dz_i}{dx} \cdot z_i$$
$$= 2\left(\frac{dz_R}{dx} \cdot z_R + \frac{dz_i}{dx} \cdot z_i\right)$$
$$= 2\,Re\left\{z^* \cdot \frac{dz}{dx}\right\}$$

THE END!

(Continue to the back)

For the circuits

we have $Z_{in} = R_L + j\omega L + \frac{1}{j\omega C}$

$= R_L + j(\omega L - \frac{1}{\omega C})$

Then $\rho = \frac{Z_{in} - R_S}{Z_{in} + R_S} = \frac{(R_L - R_S) + j(\omega L - \frac{1}{\omega C})}{(R_L + R_S) + j(\omega L - \frac{1}{\omega C})}$

$\cancel{2Re}\left[\rho^* \frac{d\rho}{dX}\right]$

$\rho^* \frac{d\rho}{dC}$

$$= \frac{(R_L - R_S) - j(\omega L - \frac{1}{\omega C})}{(R_L + R_S) - j(\omega L - \frac{1}{\omega C})} \times \frac{j\frac{1}{\omega C^2}\left[(R_L+R_S) + j(\omega L - \frac{1}{\omega C})\right] - j\frac{1}{\omega C^2}\left[(R_L - R_S) + j(\omega L - \frac{1}{\omega C})\right]}{\left[(R_L + R_S) + j(\omega L - \frac{1}{\omega C})\right]^2}$$

$$= \frac{\left[(R_L - R_S) - j(\omega L - \frac{1}{\omega C})\right] j\frac{1}{\omega C^2}\left[\cancel{R_L} + R_S + \cancel{j(\omega L - \frac{1}{\omega C})} - \cancel{R_L} + R_S - \cancel{j(\omega L - \frac{1}{\omega C})}\right]}{\left[(R_L + R_S)^2 + (\omega L - \frac{1}{\omega C})^2\right]\left[(R_L + R_S) + j(\omega L - \frac{1}{\omega C})\right]}$$

$$= \frac{\left[(R_L - R_S) - j(\omega L - \frac{1}{\omega C})\right] \frac{2jR_S}{\omega C^2} \cdot \left[(R_L + R_S) - j(\omega L - \frac{1}{\omega C})\right]}{\left[(R_L + R_S)^2 + (\omega L - \frac{1}{\omega C})^2\right]^2}$$

$$= \frac{\frac{2jR_S}{\omega C^2} \cdot \left[R_L^2 - R_S^2 - (\omega L - \frac{1}{\omega C})^2 - 2R_L j \cdot (\omega L - \frac{1}{\omega C})\right]}{\left[(R_L + R_S)^2 + (\omega L - \frac{1}{\omega C})^2\right]^2}$$

$$= \frac{2R_S\left\{2R_L(\omega L - \frac{1}{\omega C}) + j\left[R_L^2 - R_S^2 - (\omega L - \frac{1}{\omega C})^2\right]\right\}}{\omega C^2 \left[(R_L + R_S)^2 + (\omega L - \frac{1}{\omega C})^2\right]^2}$$

$$2\,\mathrm{Re}\left\{ \rho^* \frac{d\rho}{dc} \right\}$$

$$= \frac{2 \times 2 \times R_S \left[2R_L \left(\omega L - \frac{1}{\omega C} \right) \right]}{\omega^2 C \left[(R_L + R_S)^2 + \left(\omega L - \frac{1}{\omega C} \right)^2 \right]^2} = 0$$

Then $\omega L - \frac{1}{\omega C} = 0$

that is $\omega^2 L C = 1$

C.E.

~~Test #3~~ Tutorial #6 — 3KB3

Complete the second iteration of the Fletch - Powell - Davidon method to minimize

* — iteratively updates $\underset{\sim}{H}$

* — $\underset{\sim}{\phi}^1 = \underset{\sim}{\phi}^0 + \alpha \underset{\sim}{s}^0$

$$U = \varphi_1^2 + 2\varphi_2^2 + \varphi_1\varphi_2 + 2\varphi_1 + 1$$

w.r.t. $\underset{\sim}{\varphi}$, starting at $\begin{bmatrix}-4\\1\end{bmatrix}$. The first iteration is given as follows

$$\underset{\sim}{\nabla} U = \begin{bmatrix} 2\varphi_1 + \varphi_2 + 2 \\ \varphi_1 + 4\varphi_2 \end{bmatrix} = \begin{pmatrix} \partial U / \partial \phi_1 \\ \partial U / \partial \phi_2 \end{pmatrix}$$

$$\underset{\sim}{\varphi}^0 = \begin{bmatrix}-4\\1\end{bmatrix}$$

$$\underset{\sim}{\nabla} U^0 = \underset{\sim}{\nabla} U(\underset{\sim}{\varphi}^0) = \begin{bmatrix}-5\\0\end{bmatrix}$$

$$\underset{\sim}{H}^0 = \underset{\sim}{1}$$

$$\underset{\sim}{s}^0 = -\underset{\sim}{H}^0 \underset{\sim}{\nabla} U^0 = \begin{bmatrix}5\\0\end{bmatrix}$$

$$\Delta \underset{\sim}{\varphi}^0 = \alpha s^0 = \alpha \begin{bmatrix}5\\0\end{bmatrix} = \begin{bmatrix}5\alpha\\0\end{bmatrix}$$

$$\begin{bmatrix}-4\\1\end{bmatrix} + \begin{bmatrix}5\alpha\\0\end{bmatrix} = \begin{bmatrix}5\alpha-4\\1\end{bmatrix}$$

$$= (5\alpha-4)^2 + 2(1)^2 + (5\alpha-4)(1) + 2(5\alpha-4) + 1$$

$$U^1 = U(\underset{\sim}{\varphi}^0 + \alpha \underset{\sim}{s}^0) = 25\left(\alpha - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}.$$

$\partial U^1 / \partial \alpha = 0$

minimize U^1 w.r.t. α gives $\alpha = \frac{1}{2}$.

Therefore $\underset{\sim}{\varphi}^{1} = \underset{\sim}{\varphi}^{0} + \alpha s^{0} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} + \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -1.5 \\ 1 \end{bmatrix} \quad //$$

End of iter. # 1.

Some useful formulas:

$$\underset{\sim}{\nabla} U = \begin{bmatrix} 2\varphi_1 + \varphi_2 + 2 \\ \varphi_1 + 4\varphi_2 \end{bmatrix}.$$

$$\underset{\sim}{H}^{j+1} = \underset{\sim}{H}^{j} + \frac{\Delta\underset{\sim}{\varphi}^{j}(\Delta\underset{\sim}{\varphi}^{j})^{T}}{(\Delta\underset{\sim}{\varphi}^{j})^{T}\underset{\sim}{g}^{j}} - \frac{\underset{\sim}{H}^{j}\underset{\sim}{g}^{j}(\underset{\sim}{g}^{j})^{T}(\underset{\sim}{H}^{j})^{T}}{(\underset{\sim}{g}^{j})^{T}(\underset{\sim}{H}^{j})^{T}\underset{\sim}{g}^{j}}$$ where $\underset{\sim}{g}^{j} = \underset{\sim}{\nabla} U^{j+1} - \underset{\sim}{\nabla} U^{j}$

$\Delta\underset{\sim}{\varphi}^{j} = \underset{\sim}{\varphi}^{j+1} - \underset{\sim}{\varphi}^{j}$

$$\underset{\sim}{s}^{j+1} = -\underset{\sim}{H}^{j+1}\,\underset{\sim}{\nabla} U^{j+1}$$

Questions:

A). Complete the second iteration following the first one given above

B). Verify that, at the end of 2nd iteration, $\underset{\sim}{\nabla} U^{2} = 0$, the matrix $\underset{\sim}{H}^{(2)}$ is the inverse of Hessian matrix $\underset{\sim}{H}^{-1}$.

Solution to Test #3.

1. The second iteration.

At $\underset{\sim}{\varphi}^1 = \begin{bmatrix} -1.5 \\ 1 \end{bmatrix}$

$$\underset{\sim}{\nabla} U^1 = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$$

$$\underset{\sim}{g}^0 = \underset{\sim}{\nabla} U^1 - \underset{\sim}{\nabla} U^0 = \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} - \begin{bmatrix} -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2.5 \end{bmatrix}$$

$$\underset{\sim}{\Delta\phi}^0 = \underset{\sim}{\phi}^1 - \underset{\sim}{\phi}^0 = \begin{bmatrix} -1.5 \\ 1 \end{bmatrix} - \begin{pmatrix} -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} 25 & 12.5 \\ 12.5 & 6.25 \end{bmatrix}$$

$$\underset{\sim}{H}^1 = \underset{\sim}{H}^0 + \frac{\begin{bmatrix} 2.5 \\ 0 \end{bmatrix} \begin{bmatrix} 2.5 & 0 \end{bmatrix}}{\begin{bmatrix} 2.5 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2.5 \end{bmatrix}} - \frac{\begin{bmatrix} 5 \\ 2.5 \end{bmatrix} \begin{bmatrix} 5 & 2.5 \end{bmatrix}}{\begin{bmatrix} 5 & 2.5 \end{bmatrix} \begin{bmatrix} 5 \\ 2.5 \end{bmatrix}}$$

12.5 31.25

$$= \underset{\sim}{H}^0 + \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -0.8 & -0.4 \\ -0.4 & -0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & -0.4 \\ -0.4 & 0.8 \end{bmatrix}$$

$$\underset{\sim}{S}^1 = -\underset{\sim}{H}^1 \underset{\sim}{\nabla} U^1$$

$$= -\begin{bmatrix} 0.7 & -0.4 \\ -0.4 & 0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 2.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\Delta \underset{\sim}{\varphi}^1 = \alpha S^1 = \begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix}$$

$$\underset{\sim}{\varphi}^{2} = \underset{\sim}{\varphi}^{1} + \Delta\underset{\sim}{\varphi}^{1} = \begin{bmatrix} -1.5 \\ 1 \end{bmatrix} + \begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \alpha - 1.5 \\ 1 - 2\alpha \end{bmatrix}$$

$$U^{2} = U(\underset{\sim}{\varphi}^{2}) = (\alpha - 1.5)^{2} + 2(1-2\alpha)^{2} + (\alpha - 1.5)(1-2\alpha) + 2(\alpha - 1.5) + 1$$

$$= 7\alpha^{2} - 5\alpha + 2.25 + 2 - 1.5 - 3 + 1$$

$$= 7\alpha^{2} - 5\alpha + 0.75$$

$$\frac{dU^{2}}{d\alpha} = 14\alpha - 5 = 0$$

$$\alpha = \frac{5}{14} \quad (= 0.3571)$$

$$\phi^{2} : \phi^{1} + \alpha^{1} S^{1} = \begin{bmatrix} -1.5 \\ 1 \end{bmatrix} + (5/14)\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\underset{\sim}{\varphi}^{2} = \begin{bmatrix} \frac{5}{14} - \frac{3}{2} \\ 1 - \frac{5}{7} \end{bmatrix} = \begin{bmatrix} -\frac{8}{7} \\ \frac{2}{7} \end{bmatrix} \quad // \quad = \begin{bmatrix} -1.14286 \\ 0.28571 \end{bmatrix}$$

B. At $\underset{\sim}{\varphi}^{2} = \begin{bmatrix} -\frac{8}{7} \\ \frac{2}{7} \end{bmatrix}$.

$$\underset{\sim}{\nabla} U = \begin{bmatrix} 2 \times \frac{-8}{7} + \frac{2}{7} + 2 \\ -\frac{8}{7} + 4 \times \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftarrow$$

$$\Delta\underset{\sim}{\varphi}^{1} = \begin{bmatrix} \alpha \\ -2\alpha \end{bmatrix} = \begin{bmatrix} \frac{5}{14} \\ -\frac{5}{7} \end{bmatrix} \qquad \underset{\sim}{g}^{1} = \underset{\sim}{\nabla} U^{2} - \underset{\sim}{\nabla} U^{1}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 2.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2.5 \end{bmatrix}.$$

$$\mathbf{H}^{(2)} = \mathbf{H}^{(1)} + \frac{\Delta\boldsymbol{\varphi}^{1}(\Delta\boldsymbol{\varphi}^{1})^{T}}{(\Delta\boldsymbol{\varphi}^{1})^{T}\,\mathbf{g}^{1}} - \frac{\mathbf{H}^{1}\mathbf{g}^{1}(\mathbf{g}^{1})^{T}(\mathbf{H}^{1})^{T}}{(\mathbf{g}^{1})^{T}(\mathbf{H}^{1})^{T}\,\mathbf{g}^{1}}$$

$$= \begin{bmatrix} 0.7 & -0.4 \\ -0.4 & 0.8 \end{bmatrix} + \frac{\cancel{14}}{25}\cdot\frac{1}{14^2}\begin{bmatrix} 25 & -50 \\ -50 & +100 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & -0.4 \\ -0.4 & 0.8 \end{bmatrix} + \begin{bmatrix} \frac{1}{14} & -\frac{2}{14} \\ -\frac{2}{14} & +\frac{4}{14} \end{bmatrix} - \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & +0.8 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & +\frac{2}{7} \end{bmatrix}$$

$$\mathbf{H}^{(2)}\cdot\mathbf{H} = \begin{bmatrix} \frac{4}{7} & -\frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

$$\left(AA^{-1} = I = A^{-1}A\right)$$

$$= \frac{1}{7}\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

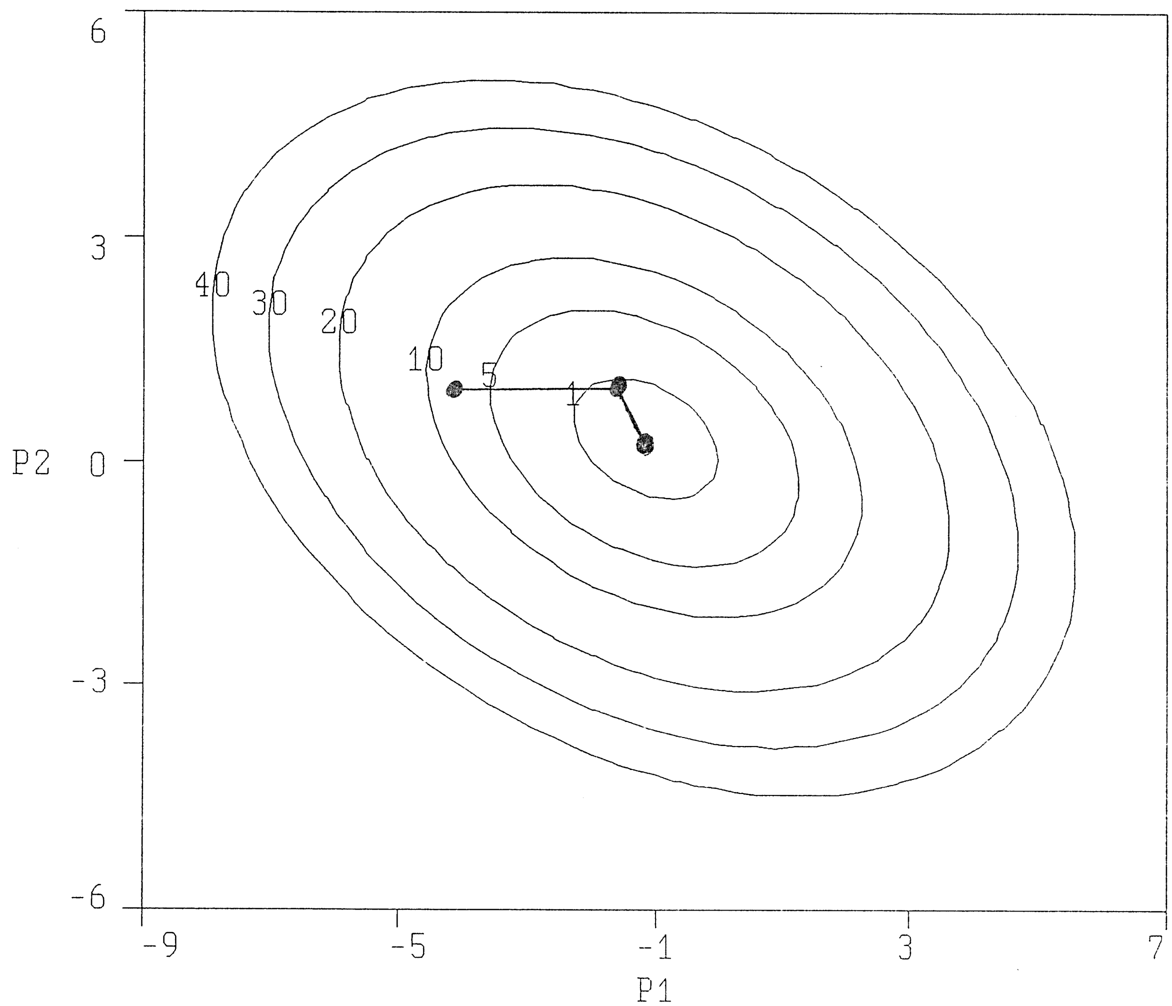
$$= \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\therefore\ \mathbf{H}^{(2)} = \mathbf{H}^{-1}$$

<u>not squared</u>, but $H^{(2)}$ 2nd iteration

Fletcher - Powell - Davidon



CoE 3KB3 Tutorial #5

Mar. 8 / 93.

Question:

Minimize

$$U = \varphi_1^2 + 2\varphi_2^2 + \varphi_1\varphi_2 + 2\varphi_1 + 1$$

w.r.t. $\underset{\sim}{\varphi}$, from the starting point $\underset{\sim}{\varphi}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

A) Use conjugate gradient method.

B). Use steepest descent method.

Check solution analytically.

Set $\underset{\sim}{\nabla} U \equiv \dfrac{\partial U}{\partial \underset{\sim}{\varphi}} = \begin{bmatrix} 2\varphi_1 + \varphi_2 + 2 \\ 4\varphi_2 + \varphi_1 \end{bmatrix} = \underset{\sim}{0}$,

we have $\begin{cases} \varphi_1 = -\dfrac{8}{7} \\ \varphi_2 = \dfrac{2}{7}. \end{cases}$

At the solution,

$$U = \left(-\frac{8}{7}\right)^2 + 2\left(\frac{2}{7}\right)^2 + \left(-\frac{8}{7}\right)\left(\frac{2}{7}\right) + 2\left(-\frac{8}{7}\right) + 1 = -\frac{1}{7}.$$

A). Conjugate Gradient Method.

Algorithm:

Set starting point $\underset{\sim}{\varphi}^{0}$;

Evaluate $\underset{\sim}{s}^{0} = -\underset{\sim}{\nabla} U^{0}$;

$j = 0$;

if not-stop, do

{ $\Delta \underset{\sim}{\varphi}^{j} = \alpha \underset{\sim}{s}^{j}$;

$\underset{\sim}{\varphi}^{j+1} = \underset{\sim}{\varphi}^{j} + \Delta \underset{\sim}{\varphi}^{j}$;

$\underset{\alpha}{\text{minimize}}\ U^{j+1} \equiv U(\underset{\sim}{\varphi}^{j+1})$;

evaluate $\underset{\sim}{\nabla} U^{j+1} = \left.\frac{\partial U}{\partial \underset{\sim}{\varphi}}\right|_{\underset{\sim}{\varphi} = \underset{\sim}{\varphi}^{j+1}}$;

$$\underset{\sim}{s}^{j+1} = -\underset{\sim}{\nabla} U^{j+1} + \frac{(\underset{\sim}{\nabla} U^{j+1})^{T} (\underset{\sim}{\nabla} U^{j+1})}{(\underset{\sim}{\nabla} U^{j})^{T} (\underset{\sim}{\nabla} U^{j})} \underset{\sim}{s}^{j};$$

$j = j + 1$;

}

stop.

Solution:

starting point given $\underset{\sim}{\varphi}^0 = \underset{\sim}{0}$.

evaluate $\underset{\sim}{S}^0 = -\underset{\sim}{\nabla} U^0 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$j = 0$:

$$\Delta \underset{\sim}{\varphi}^0 = \alpha \underset{\sim}{S}^0 = \begin{bmatrix} -2\alpha \\ 0 \end{bmatrix}$$

$$\underset{\sim}{\varphi}^1 = \underset{\sim}{\varphi}^0 + \Delta \underset{\sim}{\varphi}^0 = \begin{bmatrix} -2\alpha \\ 0 \end{bmatrix}$$

$$\begin{aligned} U^1 = U(\underset{\sim}{\varphi}^1) &= (-2\alpha)^2 + 0 + 0 - 4\alpha + 1 \\ &= 4\alpha^2 - 4\alpha + 1 \\ &= (2\alpha - 1)^2 \end{aligned}$$

minimize U^1 w.r.t. α gives $\alpha = \frac{1}{2}$.

Therefore, $\underset{\sim}{\varphi}^1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

evaluate $\underset{\sim}{\nabla} U^1 = \begin{bmatrix} -2+0+2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$$\underset{\sim}{S}^1 = -\underset{\sim}{\nabla} U^1 + \frac{(\underset{\sim}{\nabla} U^1)^T (\underset{\sim}{\nabla} U^1)}{(\underset{\sim}{\nabla} U^0)^T (\underset{\sim}{\nabla} U^0)} \underset{\sim}{S}^0$$

$$= \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

$j = 1$:

$$\Delta \underset{\sim}{\varphi}^1 = \alpha \underset{\sim}{s}^1 = \begin{bmatrix} -\frac{1}{2}\alpha \\ \alpha \end{bmatrix}$$

$$\underset{\sim}{\varphi}^2 = \underset{\sim}{\varphi}^1 + \Delta \underset{\sim}{\varphi}^1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}\alpha \\ \alpha \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}\alpha - 1 \\ \alpha \end{bmatrix}.$$

$$U^2 = U(\underset{\sim}{\varphi}^2) = \left(-\tfrac{1}{2}\alpha - 1\right)^2 + 2\alpha^2 + \alpha\left(-\tfrac{1}{2}\alpha - 1\right) + 2\left(-\tfrac{1}{2}\alpha - 1\right).$$

$$= \frac{7}{4}\alpha^2 - \alpha \;.= \frac{7}{4}\left(\alpha^2 - \frac{4}{7}\alpha\right)$$

$$= \frac{7}{4}\left(\alpha^2 - \frac{4}{7}\alpha + \frac{4}{49} - \frac{4}{49}\right)$$

$$= \frac{7}{4}\left(\alpha - \frac{2}{7}\right)^2 - \frac{7}{4}\cdot\frac{4}{49}$$

$$= \frac{7}{4}\left(\alpha - \frac{2}{7}\right)^2 - \frac{1}{7}$$

minimize U^2 w.r.t α gives $\alpha = \frac{2}{7}$.

Therefore, $\underset{\sim}{\varphi}^2 = \begin{bmatrix} -\frac{8}{7} \\ \frac{2}{7} \end{bmatrix}$.

We find solution after two iterations, because this is a two-dimensional problem.

B. Steepest Descent Method:

Algorithm:

set starting point $\underset{\sim}{\varphi}^{0}$;

$j = 0$;

if not stop, do

$\{$ evaluate $\underset{\sim}{\nabla} U^{j} = \left.\dfrac{\partial U}{\partial \underset{\sim}{\varphi}}\right|_{\underset{\sim}{\varphi} = \underset{\sim}{\varphi}^{j}}$;

$\Delta \underset{\sim}{\varphi}^{j} = -\alpha \underset{\sim}{\nabla} U^{j}$;

$\underset{\sim}{\varphi}^{j+1} = \underset{\sim}{\varphi}^{j} + \Delta \underset{\sim}{\varphi}^{j}$;

$\underset{\alpha}{\text{minimize}}\ U^{j+1} = U(\underset{\sim}{\varphi}^{j+1})$

$j = j+1$;

$\}$

stop.

$\{ \underset{\sim}{\phi}^{j} = \underset{\sim}{\phi}^{0} + \alpha \underset{\sim}{\xi} \}$

Solution:

starting point given $\underset{\sim}{\varphi}^0 = \underset{\sim}{0} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$j = 0$:

evaluate $\underset{\sim}{\nabla} U^0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. $\qquad U^0 = 1$

$$\Delta \underset{\sim}{\varphi}^0 = -\alpha \underset{\sim}{\nabla} U^0 = \begin{bmatrix} -2\alpha \\ 0 \end{bmatrix}$$

$$\underset{\sim}{\varphi}^1 = \underset{\sim}{\varphi}^0 + \Delta \underset{\sim}{\varphi}^0 = \begin{bmatrix} -2\alpha \\ 0 \end{bmatrix}$$

$$U^1 = 4\alpha^2 - 4\alpha + 1$$

$$= (2\alpha - 1)^2$$

minimize U^1 w.r.t. α gives $\alpha = \frac{1}{2}$.

$j = 1$:

$\underset{\sim}{\varphi}^1 = \begin{bmatrix} -2\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\qquad U^1 = 0$

evaluate $\underset{\sim}{\nabla} U^1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\Delta \underset{\sim}{\varphi}^1 = -\alpha \underset{\sim}{\nabla} U^1 = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

$$\underset{\sim}{\varphi}^2 = \underset{\sim}{\varphi}^1 + \Delta \underset{\sim}{\varphi}^1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} = \begin{bmatrix} -1 \\ \alpha \end{bmatrix}$$

$$U^2 = (-1)^2 + 2\alpha^2 + (-1)\alpha + 2(-1) + 1$$

$$= 1 + 2\alpha^2 - \alpha - 2 + 1$$

$$= 2\alpha^2 - \alpha$$

$$= 2\left(\alpha - \frac{1}{4}\right)^2 - \frac{1}{8}.$$

minimize U^2 w.r.t. α gives $\alpha = \frac{1}{4}$.

$$U^2 = -\frac{1}{8}.$$

$j = 2$:

$$\underset{\sim}{\varphi}^2 = \begin{bmatrix} -1 \\ \alpha \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix}$$

evaluate $\underset{\sim}{\nabla} U^2 = \begin{bmatrix} \frac{1}{4} \\ 0 \end{bmatrix}$

$$\Delta \underset{\sim}{\varphi}^3 = -\alpha \underset{\sim}{\nabla} U^2 = \begin{bmatrix} -\frac{1}{4}\alpha \\ 0 \end{bmatrix}$$

$$\underset{\sim}{\varphi}^3 = \underset{\sim}{\varphi}^2 + \Delta \underset{\sim}{\varphi}^2 = \begin{bmatrix} -1 \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} -\frac{1}{4}\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}\alpha - 1 \\ \frac{1}{4} \end{bmatrix}.$$

$$U^3 = \left(-\frac{1}{4}\alpha - 1\right)^2 + 2\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(-\frac{1}{4}\alpha - 1\right) + 2\left(-\frac{1}{4}\alpha - 1\right) + 1$$

$$= \frac{1}{16}\left(\alpha - \frac{1}{2}\right)^2 - \frac{9}{64}.$$

minimize U^3 w.r.t. α give $\alpha = \frac{1}{2}$.

After 3 iterations, we have

$$\underset{\sim}{\varphi}^3 = \begin{bmatrix} -\frac{9}{8} \\ \frac{1}{4} \end{bmatrix}, \text{ at which } U^3 = -\frac{9}{64}.$$

$j = 3$:

$$\underset{\sim}{\varphi}^3 = \begin{bmatrix} -\frac{9}{8} \\ \frac{1}{4} \end{bmatrix}$$

evaluate $\underset{\sim}{\nabla} U^3 = \begin{bmatrix} 2\times(-\frac{9}{8}) + \frac{1}{4} + 2 \\ 4\times(\frac{1}{4}) - \frac{9}{8} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{8} \end{bmatrix}$

$$\underset{\sim}{\varphi}^4 = \underset{\sim}{\varphi}^3 + \Delta\underset{\sim}{\varphi}^3$$

$$= \begin{bmatrix} -\frac{9}{8} \\ \frac{1}{4} \end{bmatrix} + (-\alpha)\cdot\underset{\sim}{\nabla} U^3$$

$$= \begin{bmatrix} -\frac{9}{8} \\ \frac{1}{4} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{8}\alpha \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{9}{8} \\ \frac{1}{4} + \frac{1}{8}\alpha \end{bmatrix}$$

$$U^4 = (-\tfrac{9}{8})^2 + 2(\tfrac{1}{4} + \tfrac{1}{8}\alpha)^2 + (-\tfrac{9}{8})(\tfrac{1}{4} + \tfrac{1}{8}\alpha) + 2(-\tfrac{9}{8}) + \ldots$$

$$= \frac{81}{64} + 2\cdot(\frac{1}{16} + \frac{1}{16}\alpha + \frac{1}{64}\alpha^2) - \frac{9}{8}\cdot\frac{1}{4} - \frac{9}{64}\alpha - \frac{9}{4} + \ldots$$

$$= \frac{1}{32}\alpha^2 - \frac{1}{64}\alpha - \frac{9}{64}.$$

$$= \frac{1}{32}(\alpha - \frac{1}{4})^2 - \frac{73}{512}$$

$$\alpha = \frac{1}{4}.$$

$$\underset{\sim}{\varphi}^4 = \begin{bmatrix} -\frac{9}{8} \\ \frac{9}{32} \end{bmatrix}$$

$$U^4 = -\frac{73}{512}$$

	Conj. Grad.		St. Desc	
iter #	point	U	point	U
0	$[0\ \ 0]^T$	1	$[0\ \ 0]^T$	1
1	$[-1\ \ 0]^T$	0	$[-1\ \ 0]^T$	0
2	$[-\frac{8}{7}\ \ \frac{2}{7}]^T$	-0.1429	$[-1\ \ \frac{1}{4}]^T$	-0.125
3			$[-\frac{9}{8}\ \ \frac{1}{4}]^T$	-0.1406
4			$[-1.125\ \ 0.2812]^T$	-0.1426
⋮				
Final	$[-\frac{8}{7}\ \ \frac{2}{7}]^T$ $\Downarrow$ $[-1.1429\ \ 0.2857]^T$	-0.1429	$[-1.1429\ \ 0.2857]^T$	-0.1429

library copy

COMPUTER ENGINEERING 3KB3: COMPUTATIONAL METHODS II

Class Test #3

Duration of Test: 35 minutes

Minimize

$$U(\phi_1,\phi_2) = 2\phi_1^{\,2} + 2\phi_2^{\,2} + \phi_1\phi_2 + 2\phi_1 + 1$$

w.r.t ϕ_1 and ϕ_2 from the starting point $\phi^{\circ} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Perform

2 iterations using the conjugate gradient method.

The updating formula for s^{j+1} is

$$s^{j+1} = -\nabla U^{j+1} + \frac{(\nabla U^{j+1})^T (\nabla U^{j+1})}{(\nabla U^{j})^T (\nabla U^{j})} s^{j} .$$

Check solution analytically: NOT NECESSARY FOR TEST! ①

Set $\underset{\sim}{\nabla} u \equiv \frac{\partial u}{\partial \underset{\sim}{\phi}} = \begin{bmatrix} 4\phi_1 + \phi_2 + 2 \\ 4\phi_2 + \phi_1 \end{bmatrix} = \underset{\sim}{0}$

$$\begin{cases} 4\phi_1 + \phi_2 + 2 = 0 \\ 4\phi_2 + \phi_1 = 0 \end{cases} \Longrightarrow \phi_1 = -4\phi_2$$

$$4(-4\phi_2) + \phi_2 + 2 = 0$$

$$-15\phi_2 = -2$$

$$\phi_2 = 2/15 = \underline{\underline{0.1333}}$$

$$\phi_1 = -4(0.1333)$$

$$= \underline{\underline{-0.5332}}$$

$$\check{\underset{\sim}{\phi}} = \begin{bmatrix} -0.5332 \\ 0.1333 \end{bmatrix}$$

Solution : Conjugate Gradient Method

$$\underset{\sim}{\nabla U} = \frac{\partial U}{\partial \underset{\sim}{\phi}} = \begin{bmatrix} 4\phi_1 + \phi_2 + 2 \\ 4\phi_2 + \phi_1 \end{bmatrix}$$

Starting from the point given, $\underset{\sim}{\phi}^0 = \underset{\sim}{0}$

$$\underset{\sim}{\nabla U}^0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

evaluate $\underset{\sim}{S}^0 = -\underset{\sim}{\nabla U}^0 = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{j = 0}$

$$\Delta \underset{\sim}{\phi}^0 = \alpha \underset{\sim}{S}^0 = \begin{bmatrix} -2\alpha \\ 0 \end{bmatrix}$$

$$\underset{\sim}{\phi}^1 = \underset{\sim}{\phi}^0 + \Delta \underset{\sim}{\phi}^0 = \begin{bmatrix} -2\alpha \\ 0 \end{bmatrix}$$

$$\begin{aligned} \therefore U^{(1)} = U(\underset{\sim}{\phi}^1) &= 2(-2\alpha)^2 + 0 + 0 + 2(-2\alpha) + 1 \\ &= 2(4\alpha^2) - 4\alpha + 1 \\ &= 8\alpha^2 - 4\alpha + 1 \end{aligned}$$

minimize $U^{(1)}$ w.r.t. α:

$$\frac{\partial U^{(1)}}{\partial \alpha} = 16\alpha - 4 \equiv 0$$

gives $\alpha = 4/16 = \boxed{0.25}$

Therefore:

$$\underset{\sim}{\phi}^1 = \underset{\sim}{\phi}^0 + \Delta\underset{\sim}{\phi}^0$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} (-2)(0.25) \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\underset{\sim}{\phi}^1 = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}}$$

evaluate $\nabla U^1 = \begin{bmatrix} 4(-\frac{1}{2}) + 0 + 2 \\ 0 + (-\frac{1}{2}) \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}$

$$\therefore \underset{\sim}{S}^1 = -\nabla U^1 + \frac{(\nabla U^1)^T (\nabla U^1)}{(\nabla U^0)^T (\nabla U^0)} \underset{\sim}{S}^0$$

$$= -\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} + \frac{\begin{bmatrix} 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix}}{\begin{bmatrix} 2 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -0.125 \\ 0.5 \end{bmatrix}$$

④

$\Rightarrow \boxed{j=1}$ $\quad \underset{\sim}{\Delta\phi}^1 = \alpha \underset{\sim}{S}^1 = \begin{bmatrix} -0.125\alpha \\ 0.5\alpha \end{bmatrix}$

$$\underset{\sim}{\phi}^2 = \underset{\sim}{\phi}^1 + \underset{\sim}{\Delta\phi}^1$$

$$= \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -0.125\alpha \\ 0.5\alpha \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 - 0.125\alpha \\ 0.5\alpha \end{bmatrix}$$

$$\therefore U^{(2)} = U(\underset{\sim}{\phi}^2) = 2(-0.5-0.125\alpha)^2 + 2(0.5\alpha)^2 + (-0.5-0.125\alpha)(0.5\alpha) + 2(-0.5-0.125\alpha) + 1$$

$$= 0.5 - 0.25\alpha + 0.46875\alpha^2$$

minimize $\underline{U^{(2)}}$ w.r.t. α

$$\frac{\partial U^{(2)}}{\partial \alpha} = -0.25 + 0.9375\alpha = 0$$

$$\alpha = \boxed{0.2666}$$

Therefore $\underset{\sim}{\phi}^2 = \underset{\sim}{\phi}^1 + \Delta\underset{\sim}{\phi}^1$

$$= \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} (-0.125)(0.2666) \\ (0.5)(0.2666) \end{bmatrix}$$

$$= \begin{bmatrix} -0.5333 \\ 0.1333 \end{bmatrix}$$

To check if this is the solution:

$$\underline{\nabla U}^{(2)} = \begin{bmatrix} (4)(-0.5333) + (0.1333) + 2 \\ (4)(0.1333) + (-0.5333) \end{bmatrix}$$

$$\approx \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

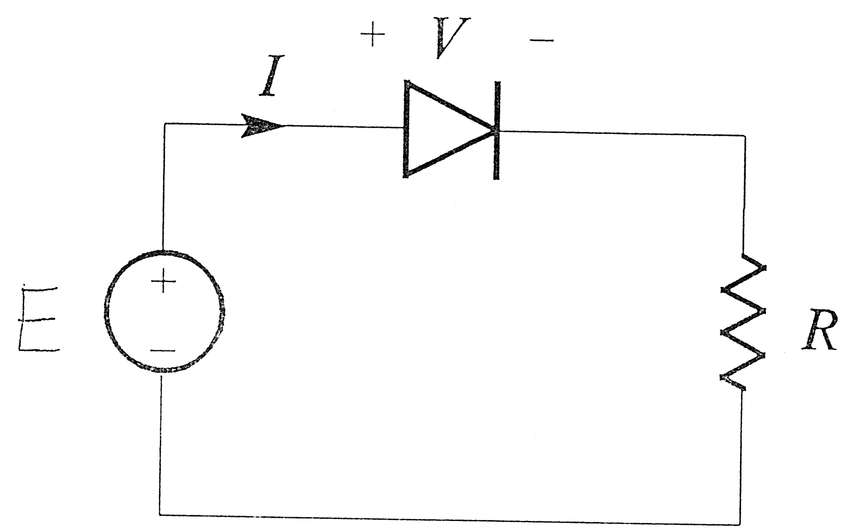
CE 3KB3 Tutorial #7 March 21, 1994

SIMPLE RESISTOR DIODE CIRCUIT

For the circuit below, find the operating point using

a) Techniques for solving nonlinear equations

b) Optimization techniques



Resistor-diode network.

E = 10 V
R = 1 kΩ
I_S = 10^{-12} mA
λ = 38.7 V^{-1}

$$i_d = I_S(e^{\lambda v_d} - 1)$$

Answer: $v_d = 0.771404$

Newton-Raphson Method

$$f(x) = 0 \qquad\qquad x^{j+1} = x^j - \frac{f^j}{\left(\dfrac{\partial f}{\partial x}\right)^j}$$

For our example:

$$f(v_d) = E - v_d - Ri_d = 0$$

$$f(v_d) = E - v_d - RI_S(e^{\lambda v_d} - 1)$$

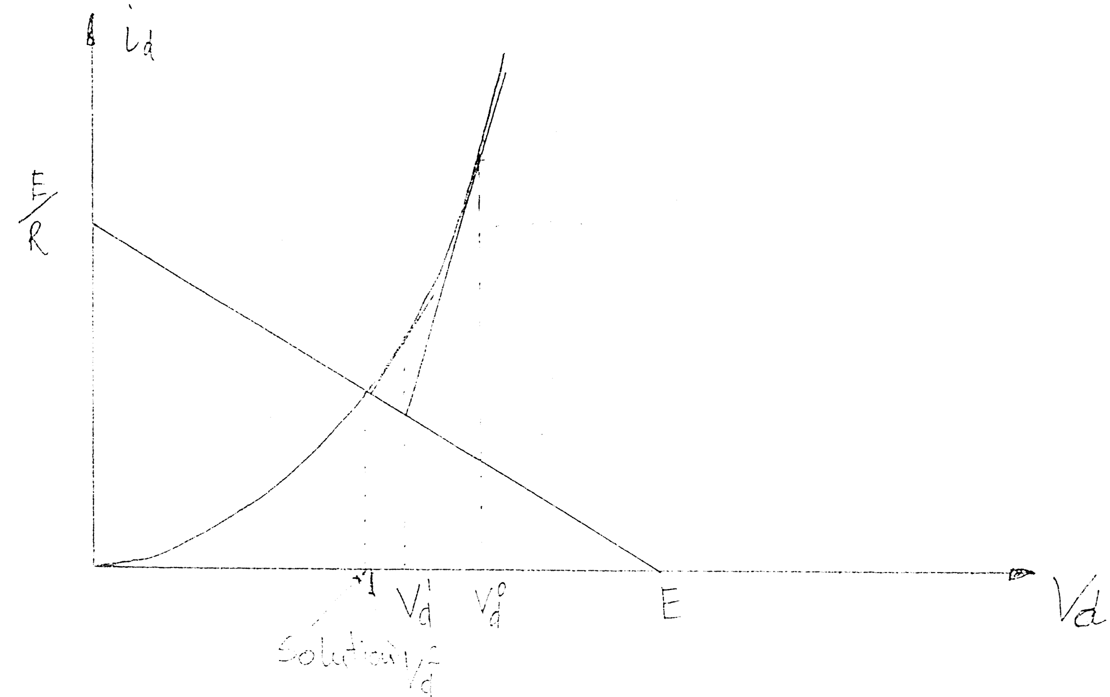
$$v_d^{j+1} = v_d^j + \frac{E - v_d^j - RI_S(e^{\lambda v_d^j} - 1)}{1 + RI_S \lambda e^{\lambda v_d^j}}$$

$f(x) \to 0$

$f(x^{j+1}) = f(x^j) + \dfrac{df(x^j)}{dx}(x^{j+1} - x^j) + \ldots$

$x^{j+1} = x^j - \dfrac{f(x^j)}{\dfrac{df(x^j)}{dx}} = x^j - \dfrac{f^j}{\dfrac{df^j}{dx}}$

Graphical Interpretation

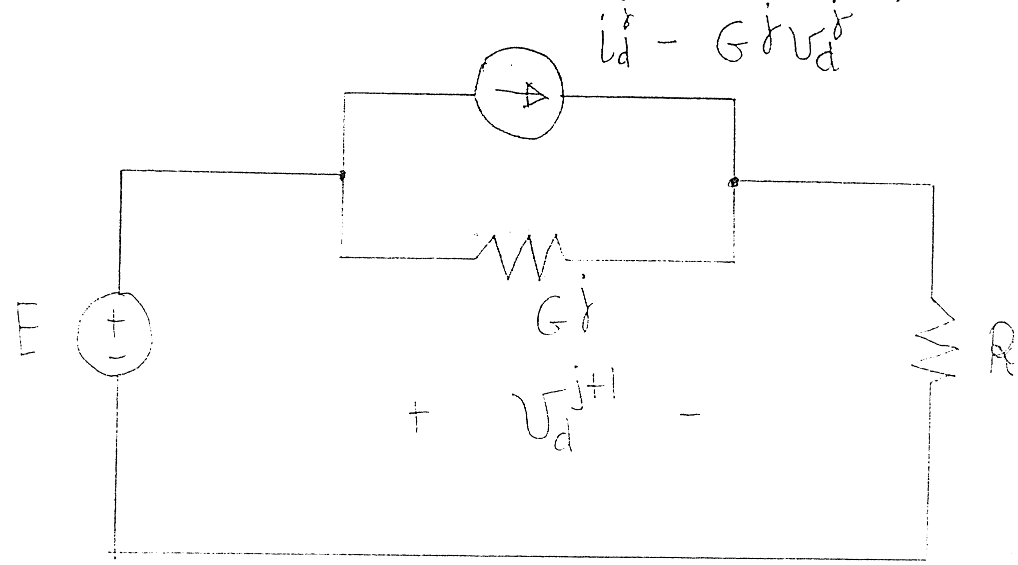


Results for Newton-Raphson (Using HP calculator)

$$v_d^0 = 0.7$$

Iteration	v_d
1	1.070510
2	1.044670
3	1.018831
4	0.992993
5	0.967158
6	0.941331
7	0.915527
8	0.889783
9	0.864206
10	0.839073
11	0.815107
12	0.794018
13	0.778939
14	0.772401
15	0.771423
16	0.771404
17	0.771404

Circuit Interpretation (Companion Network)



$i_d = i(v_d)$

$i_d^{j+1} = i_d^j + \frac{di_d}{dv_d}\Big|^j (v_d^{j+1} - v_d^j) = i_d^j + G^j(v_d^{j+1} - v_d^j) = G^j v_d^{j+1} + i_d^j - G^j v_d^j$

$$i_d^j = I_S(e^{\lambda v_d^j} - 1)$$

$$G^j = \frac{\partial i_d^j}{\partial v_d^j} = \lambda \, I_S \, e^{\lambda v_d^j}$$

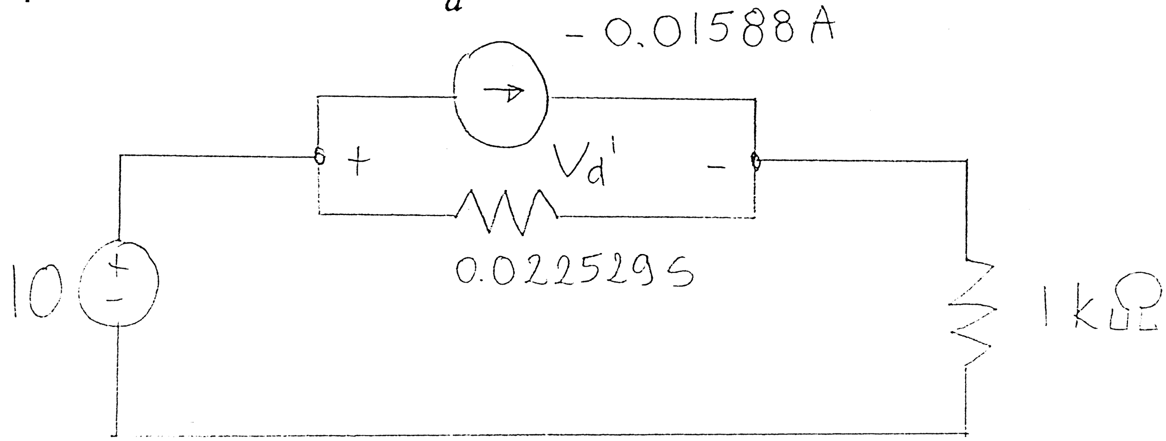
Using KVL

$$R(v_d^{j+1} G^j + i_d^j - G^j v_d^j) = E - v_d^{j+1}$$

Simplifies to

$$v_d^{j+1} = \frac{E - v_d^j - R i_d^j}{RG^j + 1} + v_d^j = v_d^j + \frac{E - v_d^j - RI_s(e^{\lambda v_d^j} - 1)}{1 + RI_S \lambda e^{\lambda v_d^j}}$$

Companion Network: $v_d^0 = 0.7$



$$v_d^1 = 1.070510$$

Modified Newton Method

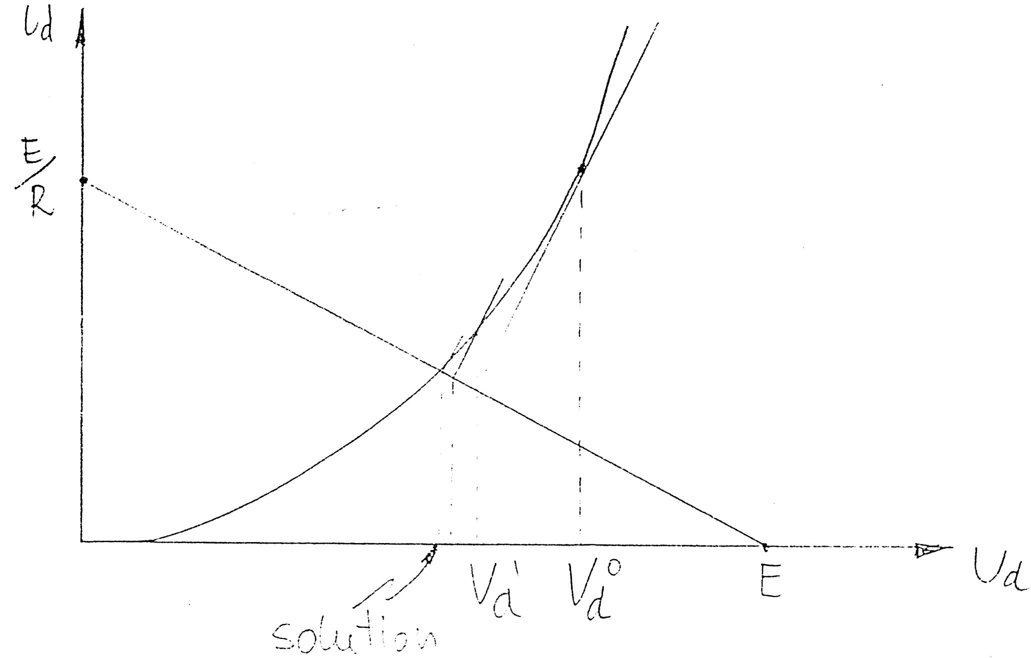
$$f(x) = 0 \qquad x^{j+1} = x^j - \frac{f^j}{m} \qquad m = \left(\frac{\partial f}{\partial x}\right)^0$$

For our example:

$$v_d^{j+1} = v_d^j + \frac{E - v_d^j - RI_S(e^{\lambda v_d^j} - 1)}{m}$$

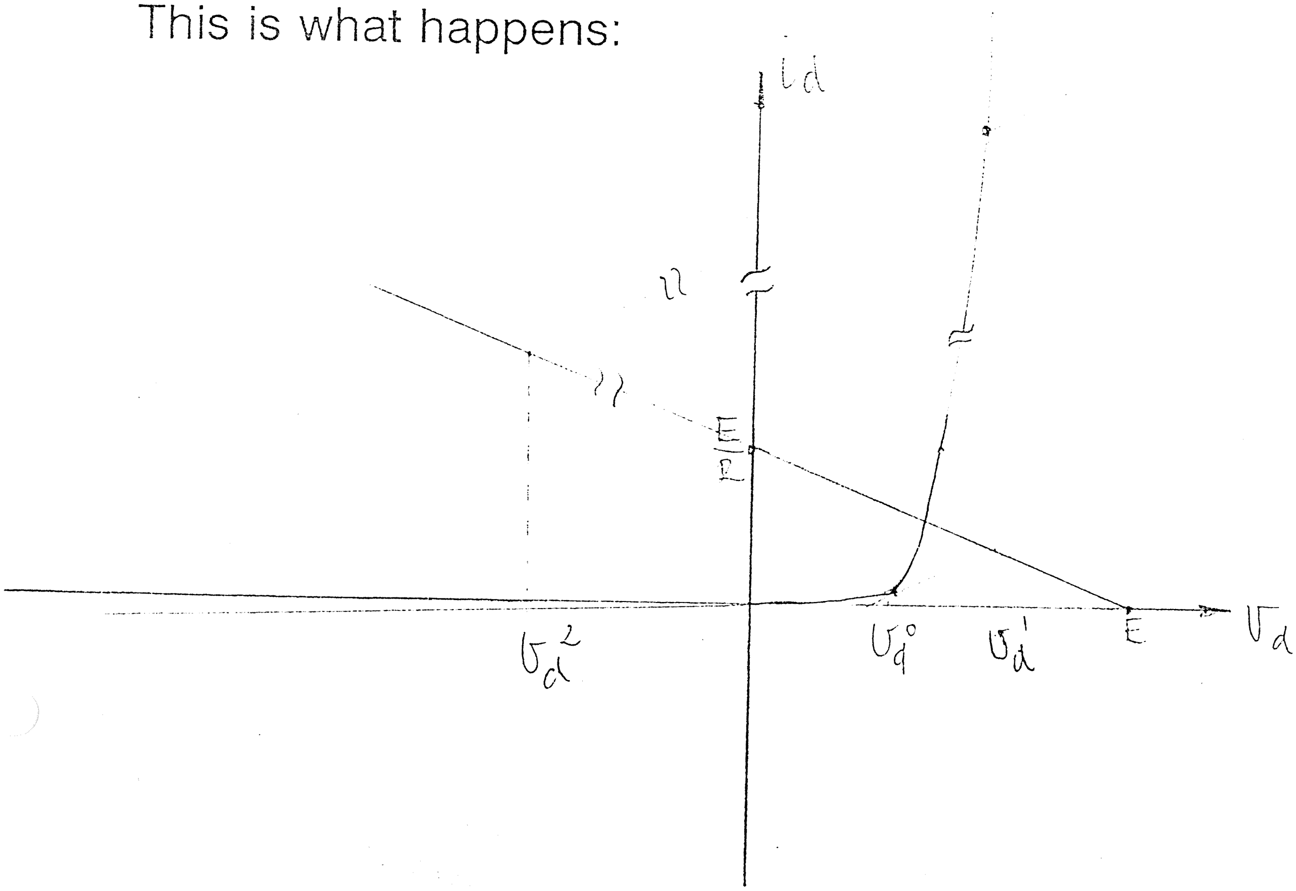
$$m = 1 + RI_S \lambda e^{\lambda v_d^0}$$

Graphical Interpretation



It is easy to see that if v_d^0 is smaller than the solution

for our problem, we have a badly conditioned problem. This is what happens:



Results for Modified Newton Method

$v_d^0 = 0.7$ $\qquad m = 23.53$

Iteration	v^d
1	1.07051
2	-41748.44
⋮	⋮

Badly conditioned

$v_d^0 = 0.85 \qquad m = 7479.7$

Iteration	v^d
1	0.825387
2	0.816647
⋮	⋮

More than 100 iterations

$v_d^0 = 0.8 \qquad m = 1081.1$

Iteration	v^d
1	0.782694
2	0.778006
⋮	⋮
25	0.778006

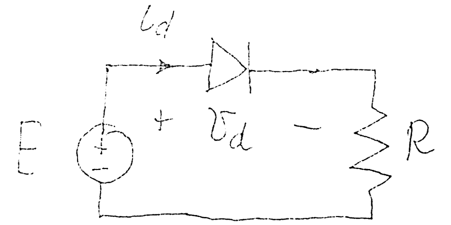
Newton Method:

1. guess the initial solution.
2. set the nonlinear equasion(s) for the system
3. apply a Newton iteration (which implies):
   a) linearization of the nonlinear equation(s)
   b) solving the resulting linear system for a new solution
4. check the termination criterion and either stop or go to point 2

Companion Network:

1. guess the initial solution
2. for each nonlinear element linearize the nonlinear eq tio of the element
3. having all elements linear:
   a) set the linear equation(s) for the system
   b) solve the resulting linear system for a new solution
4. check the termination criterion and either stop or go to point 2.

Example



$$i_d = I_s(e^{\lambda v_d} - 1)$$

given: E, R, I_s, λ

find: v_d

Applying the Algorithms to the Example

Newton Method:

1. e.g. $v_d^0 = 0.7$
2. $f(v_d^j) = E - v_d^j - R i_d^j = 0$

$$f(v_d^j) = E - v_d^j - RI_s(e^{\lambda v_d^j} - 1) = 0$$

3. $$v_d^{j+1} = v_d^j - \frac{f(v_d^j)}{f'(v_d^j)}$$

a), b)

$$v_d^{j+1} = v_d^j + \frac{E - v_d^j - RI_s(e^{\lambda v_d^j} - 1)}{1 + RI_s \lambda e^{\lambda v_d^j}}$$

4. e.g. $|v_d^{j+1} - v_d^j| <$ SMALL_NUMBER

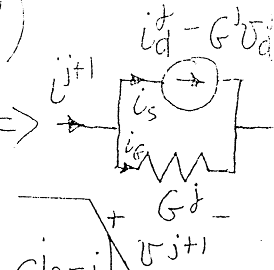
Companion Network:

1. e.g. $v_d^0 = 0.7$
2. $i_d^j = I_s(e^{\lambda v_d} - 1)$ linearization

$$i_d^{j+1} = i_d^j + \frac{d i_d^j}{d v_d^j}(v_d^{j+1} - v_d^j) + \cancel{\dots}$$

or $$i_d^{j+1} = i_d^j + G^j(v_d^{j+1} - v_d^j)$$

$$i_d^{j+1} = \underbrace{G^j v_d^{j+1}}_{i_G} + \underbrace{i_d^j - G^j v_d^j}_{i_s} \Rightarrow$$



3. a) $E - v_d^{j+1} - R(G^j v_d^{j+1} + i_d^j - G^j v_d^j)$

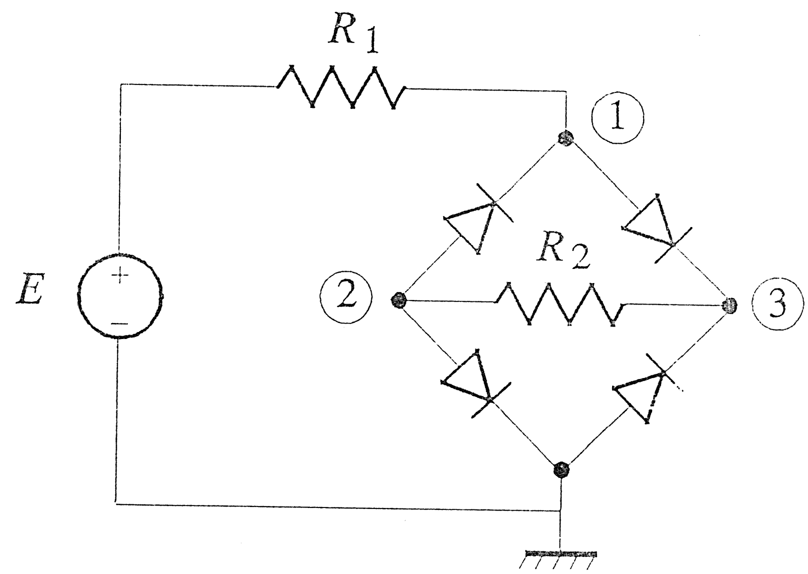
   b) $$v_d^{j+1} = v_d^j + \frac{E - v_d^j - RI_s(e^{\lambda v_d^j} - 1)}{1 + RI_s \lambda e^{\lambda v_d^j}}$$

4. e.g. $|v_d^{j+1} - v_d^j| <$ SMALL_NUMBER

Note that the solutions are the same for both methods as expected.

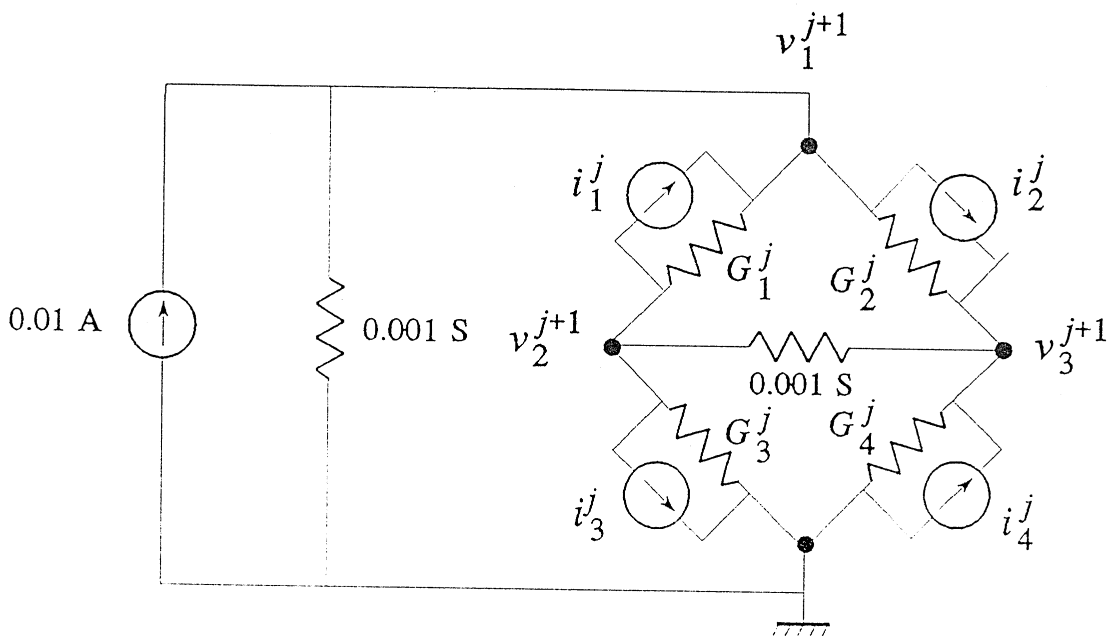
Consider the resistor-diode network shown. Draw the corresponding companion network at the jth iteration for its d.c. solution. Write down the nodal equations at this iteration. Where

$$I_d = I_S(e^{\lambda V_d} - 1)$$
$$I_S = 10^{-12} \text{ mA}$$
$$\lambda = \frac{1}{0.026} \; V^{-1}$$
$$E = 10 \text{ V}$$
$$R_1 = R_2 = 1 \text{ k}\Omega$$



A resistor-diode network.

The corresponding companion network at the *j*th iteration is shown below.



The nodal equations at this iteration can be written as

$$
\begin{bmatrix}
0.001 + G_1^j + G_2^j & -G_1^j & -G_2^j \\
-G_1^j & 0.001 + G_1^j + G_3^j & -0.001 \\
-G_2^j & -0.001 & 0.001 + G_2^j + G_4^j
\end{bmatrix}
\begin{bmatrix}
V_1^{j+1} \\
V_2^{j+1} \\
V_3^{j+1}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0.01 + (i_{d_1}^j - G_1^j V_{d_1}^j) - (i_{d_2}^j - G_{d_2}^j V_{d_2}^j) \\
-(I_{d_1}^j - G_1^j V_{d_1}^j) - (I_{d_3}^j - G_4^j V_{d_4}^j) \\
(I_{d_2}^j - G_2^j V_{d_2} j) - (i_{d_4}^j - G_4^j V_{d_4}^j)
\end{bmatrix}
$$

where

$$V_{d_1}^j = V_2^j - V_1^j, \quad V_{d_2}^j = V_1^j - V_3^j, \quad V_{d_3}^j = V_2^j, \quad V_{d_4}^j = -V_3^j$$

and

$$
\left.
\begin{aligned}
I_{d_k}^j &= I_S(e^{\lambda V_{d_k}^j} - 1) \\
G_k^j &= \lambda I_S \, e^{\lambda V_{d_k}^j}
\end{aligned}
\right\} \qquad k = 1, 2, 3, 4
$$

SOLUTIONS

COMPUTER ENGINEERING 3KB3

DURATION OF TEST: 2 hours

Monday, March 21, 1994

Candidates must attempt Questions

1 or 2 or 3   4 or 5   6 or 7   8 or 9   10 or 11

Write your name here. NAME: ______________________

Write your student number here. NO: ______________________

Note: (1) All scripts and question papers must be turned in.
(2) Estimated times required to complete the questions are indicated.
(3) Please encircle questions attempted in the following table.

Questions Attempted (please encircle)	Weighting	Estimated Time (min.)	Examiner's Use Only
1 or 2 or 3	15%	fifteen	
4 or 5	15%	twenty	
6 or 7	20%	twenty	
8 or 9	25%	twenty-five	
10 or 11	25%	forty	
TOTAL	100%	2 hours	

continued on page 2

Question 1

Use the method of Lagrange multipliers to minimize w.r.t. ϕ_1 and ϕ_2 the function

$$U = \phi_1^2 + 2\phi_2^2$$

subject to

$$\phi_1 + \phi_2 = 1$$

Sketch contours to illustrate this problem w.r.t. ϕ_1 and ϕ_2. Verify the answer by substituting the constraint into the function.

Answer (15 min.)

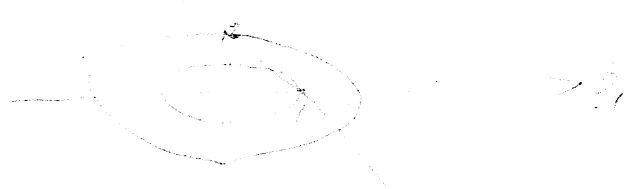
① $L(\underset{\sim}{\phi}, \lambda) = U(\underset{\sim}{\phi}) - \lambda h$ ...

Solving

$$\begin{cases} \frac{\partial L}{\partial \phi_1} = 2\phi_1 - \lambda = 0 \\ \frac{\partial L}{\partial \phi_2} = 4\phi_2 - \lambda = 0 \\ \frac{\partial L}{\partial \lambda} = \phi_1 + \phi_2 - 1 = 0 \end{cases}$$

we get $\underset{\sim}{\phi} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$ $\lambda = -4/3$

②



③ substituting h into U $\phi_1 = -\phi_2 + 1$

$U = (-\phi_2 + 1)^2 + 2\phi_2^2 = 3\phi_2^2 - 2\phi_2 + 1$

$\frac{\partial U}{\partial \phi_2} = 6\phi_2 - 2 = 0$ then $\phi_2 = 1/3$ $\phi_1 = 2/3$

$$\underset{\sim}{\phi} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

continued on page 3

Question 2

Apply the Newton method to the minimization of

$$\phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_2 + 1$$

w.r.t. $\underset{\sim}{\phi}$. Select the starting point

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

What is the solution?

Now apply one iteration of steepest descent with exact line search and compare and comment on the result.

Answer (15 min.)

$U = \phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_2 + 1$ $\quad \nabla U = \begin{bmatrix} 2\phi_1 + \phi_2 \\ \phi_1 + 4\phi_2 + 2 \end{bmatrix}$ $\quad H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ $\quad H^{-1} = \begin{bmatrix} 4/7 & -1/7 \\ -1/7 & 2/7 \end{bmatrix}$

② $\phi^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \nabla U^0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ $\quad \Delta\underset{\sim}{\phi}^0 = -H^{-1}\nabla U^0 = -\begin{bmatrix} 4/7 & -1/7 \\ -1/7 & 2/7 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/7 \\ -4/7 \end{bmatrix}$

So $\phi^1 = \phi^0 + \Delta\phi^0 = \begin{bmatrix} 0 + 2/7 \\ 0 - 4/7 \end{bmatrix} = \begin{bmatrix} 2/7 \\ -4/7 \end{bmatrix}$

$\nabla U^1 = \begin{bmatrix} 2/7 \times 2 - 4/7 \\ 2/7 - 4 \times 4/7 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so the solution is $\underset{\sim}{\phi} = \begin{bmatrix} 2/7 \\ -4/7 \end{bmatrix}$

① $\underset{\sim}{s}^0 = -\dfrac{\nabla U^0}{\|\nabla U^0\|} = \dfrac{-\begin{bmatrix} 0 \\ 2 \end{bmatrix}}{2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\underset{\sim}{\phi}^1 = \underset{\sim}{\phi}^0 + \alpha^0 \underset{\sim}{s}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ -\alpha \end{bmatrix}$

apply $\underset{\sim}{\phi}^1$ to $U = \phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + 2\phi_2 + 1 = 2\alpha^2 - 2\alpha + 1$

minimize $U = 2\alpha^2 - 2\alpha + 1$, we get $\alpha = 1/2$

so $\underset{\sim}{\phi}^1 = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$

continued on page 4

Question 3

Consider a nodal system of equations as

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Express the solution of this system as a least squares problem. Write down the corresponding Jacobian. Write down the gradient vector of the least squares objective.

Answer (15 min.)

From
$$\begin{cases} g_{11}V_1 + g_{12}V_2 + g_{13}V_3 = 0 \\ g_{21}V_1 + g_{22}V_2 + g_{23}V_3 = 0 \\ g_{31}V_1 + g_{32}V_2 + g_{33}V_3 = 0 \end{cases}$$
we get
$$\begin{cases} e_1(V) = g_{11}V_1 + g_{12}V_2 + g_{13}V_3 - 1 \\ e_2(V) = g_{21}V_1 + g_{22}V_2 + g_{23}V_3 \\ e_3(V) = g_{31}V_1 + g_{32}V_2 + g_{33}V_3 \end{cases}$$

the least squares problem is $U = \sum_{i=1}^{3} [e_i(V)]^2$

The Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial e_1}{\partial V_1} & \frac{\partial e_1}{\partial V_2} & \frac{\partial e_1}{\partial V_3} \\ \frac{\partial e_2}{\partial V_1} & \frac{\partial e_2}{\partial V_2} & \frac{\partial e_2}{\partial V_3} \\ \frac{\partial e_3}{\partial V_1} & \frac{\partial e_3}{\partial V_2} & \frac{\partial e_3}{\partial V_3} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

the gradient vector of U is

$$\nabla U = \sum_{i=1}^{3} 2e_i \nabla e_i = \begin{bmatrix} 2e_1 g_{11} + 2e_2 g_{21} + 2e_3 g_{31} \\ 2e_1 g_{12} + 2e_2 g_{22} + 2e_3 g_{32} \\ 2e_1 g_{13} + 2e_2 g_{23} + 2e_3 g_{33} \end{bmatrix}$$

continued on page 5

Question 4

Derive the exact Newton iteration at ϕ^j for minimization of a differentiable multidimensional function $U(\phi)$. Define all terms used. Under what conditions do you expect a locally downhill step from the Newton iteration? Discuss possible pitfalls of the basic Newton method and suggest remedies. What is damping? Illustrate your answers with sketches.

Answer (20 min.)

(1) The Taylor series expension of the differentiable mutidimensiona function $U(\underset{\sim}{\phi})$ is

$$U(\underset{\sim}{\phi}+\Delta\underset{\sim}{\phi}) = U(\underset{\sim}{\phi}) + \nabla U^T(\underset{\sim}{\phi})\,\Delta\underset{\sim}{\phi} + \frac{1}{2}\Delta\underset{\sim}{\phi}^T H(\underset{\sim}{\phi})\,\Delta\underset{\sim}{\phi} + \cdots$$

where

$\underset{\sim}{\phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix}$ is a point in the optimizable parameters' space

$\Delta\underset{\sim}{\phi} = \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \\ \vdots \\ \Delta\phi_n \end{bmatrix}$ is a increment vector in the optimizable parameters' space.

$\underset{\sim}{\nabla} U(\underset{\sim}{\phi}) = \begin{bmatrix} \frac{\partial U}{\partial \phi_1} \\ \frac{\partial U}{\partial \phi_2} \\ \vdots \\ \frac{\partial U}{\partial \phi_n} \end{bmatrix}$ is the gradient of the function $U(\underset{\sim}{\phi})$

$$\underset{\sim}{H}(\underset{\sim}{\phi}) = \begin{bmatrix} \frac{\partial^2 U}{\partial \phi_1^2} & \frac{\partial^2 U}{\partial \phi_1 \partial \phi_2} & \cdots & \frac{\partial^2 U}{\partial \phi_1 \partial \phi_n} \\ \frac{\partial^2 U}{\partial \phi_2 \partial \phi_1} & \frac{\partial^2 U}{\partial \phi_2^2} & \cdots & \frac{\partial^2 U}{\partial \phi_2 \partial \phi_n} \\ \vdots & & & \\ \frac{\partial^2 U}{\partial \phi_n \partial \phi_1} & \frac{\partial^2 U}{\partial \phi_n \partial \phi_2} & \cdots & \frac{\partial^2 U}{\partial \phi_n^2} \end{bmatrix}$$

is the Hessian matrix of the function $U(\underset{\sim}{\phi})$

Question 4

continued on page 6

Differentiating $U(\underset{\sim}{\phi}+\Delta\underset{\sim}{\phi})$ and suppose $\underset{\sim}{\phi}+\Delta\underset{\sim}{\phi}$ is a minimum point, we have

$$\underset{\sim}{\nabla} U(\underset{\sim}{\phi}) + \underset{\sim}{H}(\underset{\sim}{\phi})\Delta\phi = 0$$

(neglecting higher order terms)

$$\Delta\phi = -H^{-1}\nabla U \quad \cdots\cdots\cdots\cdots (1)$$

So the exact Newton iteration at $\underset{\sim}{\phi}^{j}$ for minimization of $U(\underset{\sim}{\phi})$ is

$$\underset{\sim}{\phi}^{j+1} = \underset{\sim}{\phi}^{j} + \left[-H_j^{-1}\nabla U(\phi^{j})\right]$$

(2) Locally down hill condition

from (1) we have

$$\underset{\sim}{H}\Delta\underset{\sim}{\phi} = -\nabla U(\underset{\sim}{\phi})$$

then $\quad \Delta\underset{\sim}{\phi}^{T}\nabla U(\underset{\sim}{\phi}) = -\Delta\underset{\sim}{\phi}^{T}\underset{\sim}{H}\Delta\underset{\sim}{\phi}$

So, the downhill condition is

$\underset{\sim}{H}$ is positive definite (i.e. $\Delta\underset{\sim}{\phi}^{T}\underset{\sim}{H}\Delta\underset{\sim}{\phi} > 0$)

under this condition,

$\Delta\underset{\sim}{\phi}^{T}\nabla U(\underset{\sim}{\phi}) < 0 \quad$ $\underset{\sim}{\phi}$ going downhill

This Newton method has some draw-backs

<i> the method may diverge, or converge to a maximum rather than minimum.

<ii> $\underset{\sim}{H}$ may be locally singular, makes inverse of $\underset{\sim}{H}$ impossible

<iii> computations of $\underset{\sim}{H}$ and its inverse are expensive.

Question 4

The suggested remedy is to use damping Newton method.

Damping Newton method:

modify $\underset{\sim}{H}$ by $\lambda \underset{\sim}{I}$, i.e.

$$(\underset{\sim}{H} + \lambda \underset{\sim}{I}) \Delta \underset{\sim}{\phi} = -\nabla U(\underset{\sim}{\phi})$$

when $\lambda \to \infty$, we have $\lambda \underset{\sim}{I} \Delta \underset{\sim}{\phi} = -\underset{\sim}{\nabla} U(\underset{\sim}{\phi})$

or $\Delta \underset{\sim}{\phi} = -\frac{1}{\lambda} \underset{\sim}{\nabla} U(\underset{\sim}{\phi})$

which is a small $(\frac{1}{\lambda})$ step towards the steepest descent.

Question 4

Question 5

Examine the points $[1 \;\; 2]^T$ and $[1 \;\; 1]^T$ for a minimax problem for which

$$f_1 = \phi_1^4 + \phi_2^2$$

$$f_2 = (2 - \phi_1)^2 + (2 - \phi_2)^2$$

$$f_3 = 2 \exp(-\phi_1 + \phi_2)$$

by invoking necessary conditions for a minimax optimum. What are your conclusions?

Answer (20 min.)

The necessary conditions for this minimax optimum are

$$\begin{cases} \sum_{i=1}^{3} u_i \nabla f = 0 & (1) \\ \sum_{i=1}^{3} u_i = 1 & (2) \\ u_i (M - f_i) = 0 & (3) \\ u_i \geq 0 & (4) \end{cases}$$

where $i = 1, 2, 3$, $M = \mathrm{Max}\{f_1, f_2, f_3\}$

$$\nabla f_1 = \begin{bmatrix} 4\phi_1^3 \\ 2\phi_2 \end{bmatrix} \quad \nabla f_2 = \begin{bmatrix} -2(2-\phi_1) \\ -2(2-\phi_2) \end{bmatrix} \quad \nabla f_3 = \begin{bmatrix} -2\exp(-\phi_1+\phi_2) \\ 2\exp(-\phi_1+\phi_2) \end{bmatrix}$$

(1) If point $[1 \;\; 2]^T$ is a minimax optimum, the necessary conditions must be satisfied.

at this point, $f_1 = 5$ $f_2 = 1$ $f_3 = 2e$ $M = 2e$

from condition (3), we can derive

$u_1 = 0$ $u_2 = 0$

from condition (2), we can derive

$u_3 = 1$

then, for condition (1),

Question 5

continued on page 7

$$\sum_{i=1}^{3} u_i \nabla f_i = u_3 \begin{bmatrix} -2\exp(-\phi_1+\phi_3) \\ 2\exp(-\phi_1+\phi_2) \end{bmatrix} = \begin{bmatrix} 2e \\ 2e \end{bmatrix} \neq \underset{\sim}{0}$$

So the necessary conditions at point $[1\ 2]^T$ are violated, $[1\ 2]^T$ is not a minimax optimum point.

(2) If $[1\ 1]^T$ is a minimax optimum, the necessary conditions must be satisfied,

at this point, $f_1=2$, $f_2=2$, $f_3=2$ $\qquad M=2$

Condition (3) is automatically satisfied.

As for condition (1) (2), if they are satisfied, then

$$\left.\begin{aligned} u_1\begin{bmatrix}4\\2\end{bmatrix} + u_2\begin{bmatrix}-2\\-2\end{bmatrix} + u_3\begin{bmatrix}-2\\2\end{bmatrix} = 0 \\ u_1+u_2+u_3=1 \end{aligned}\right\} \Longrightarrow \begin{cases} u_1=\frac{1}{3} \\ u_2=\frac{1}{2} \\ u_3=\frac{1}{6} \end{cases}$$

This set of u also satisfy condition (4).

So the necessary conditions are satisfied at point $[1\ 1]^T$, point $[1\ 1]^T$ could be a minimax point.

Question 5

Question 6

Consider the function

$$U = 2\phi_1^2 + \phi_2^2 + 3$$

subject to the constraints

$$\phi_2 \geq 1$$

$$\phi_1 \geq \phi_2$$

(1) Sketch the objective function U and constraints, indicating the feasible and nonfeasible regions.

(2) Find the unconstrained optimum of U.

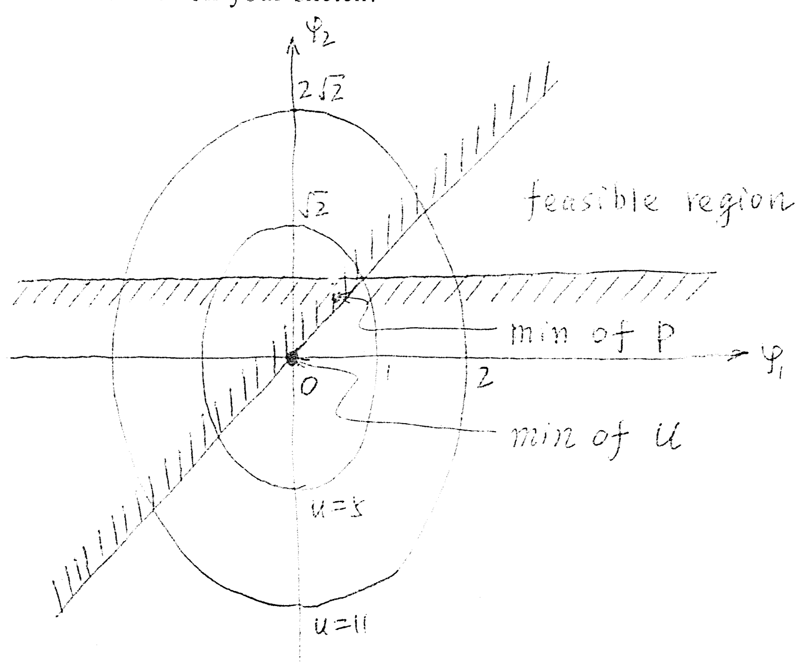
(3) Form a penalty function P with $w_1 = w_2 = 10$.

(4) Find the minimum of P.

(5) Indicate the minimum of U and P on your sketch.

Answer (20 min.)

(1)



(2) $\nabla U(\varphi^{\circ}) = 0$ for the unconstrained minimum φ°

continued on page 8

$$\underset{\sim}{\nabla} u = \begin{pmatrix} 4\varphi_1 \\ 2\varphi_2 \end{pmatrix}$$

Set $\underset{\sim}{\nabla} u = \underset{\sim}{0}$; we get the minimum

$$\underset{\sim}{\varphi}^0 = \begin{pmatrix} \varphi_1^0 \\ \varphi_2^0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(3) In general

$$P(\underset{\sim}{\varphi}, \underset{\sim}{w}) = u(\underset{\sim}{\varphi}) + \sum_{i=1}^{m} w_i^2 g_i^2(\underset{\sim}{\varphi})$$

In this case, $m=2$

$g_1 = \varphi_2 - 1 \geq 0$

$g_2 = \varphi_1 - \varphi_2 \geq 0$

$$P(\underset{\sim}{\varphi}, \underset{\sim}{w}) = (2\varphi_1^2 + \varphi_2^2 + 3) + 10(\varphi_2 - 1)^2 + 10(\varphi_1 - \varphi_2)^2$$

(4) $\underset{\sim}{\nabla} P = \underset{\sim}{0}$ for the minimum of P

$$\underset{\sim}{\nabla} P = \begin{pmatrix} 4\varphi_1 + 20(\varphi_1 - \varphi_2) \\ 2\varphi_2 + 20(\varphi_2 - 1) + 20(\varphi_2 - \varphi_1) \end{pmatrix} = \begin{pmatrix} 24\varphi_1 - 20\varphi_2 \\ 42\varphi_2 - 20\varphi_1 - 20 \end{pmatrix}$$

Set $\underset{\sim}{\nabla} P = \underset{\sim}{0}$, we get the minimum

$$\underset{\sim}{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 25/38 \\ 15/19 \end{pmatrix} = \begin{pmatrix} 0.6579 \\ 0.7895 \end{pmatrix}$$

(5) See the sketch

Question 7

Consider the voltage divider shown for the response specification and constraint indicated.

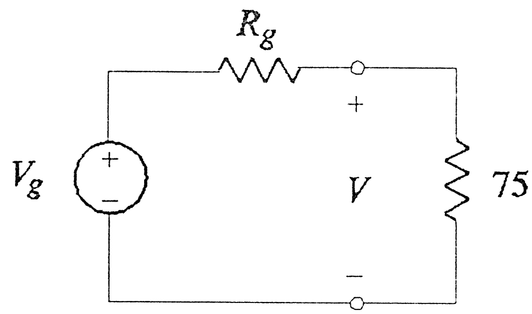


Fig. Q7 A voltage divider.

Design specification: $V \geq 60$
Constraint: $R_g \geq 75$

By testing the Kuhn-Tucker conditions, find V_g and R_g such that the total power dissipated is minimum.

Answer (20 min.)

The power dissipated is

$$P = \frac{V_g^2}{R_g + 75}$$

Design specification

$$V = \frac{75 V_g}{R_g + 75} \geq 60$$

Constraint

$$R_g \geq 75$$

continued on page 9

Considering the following constrained minimization problem

$$u = \frac{Vg^2}{Rg+75}$$

with constraints

$$\begin{cases} g_1 = 5Vg - 4Rg - 300 \geq 0 \\ g_2 = Rg - 75 \geq 0 \end{cases}$$

Kuhn-Tucker conditions

$$\begin{cases} \nabla u = u_1 \nabla g_1 + u_2 \nabla g_2 & (1) \\ u_1 g_1 + u_2 g_2 = 0 & (2) \\ u_1 \geq 0, \quad u_2 \geq 0 & (3) \end{cases}$$

$$\nabla u = \begin{pmatrix} \dfrac{2Vg}{Rg+75} \\ \dfrac{-Vg^2}{(Rg+75)^2} \end{pmatrix}$$

$$\nabla g_1 = \begin{pmatrix} 5 \\ -4 \end{pmatrix} \qquad \nabla g_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Test the point $\begin{pmatrix} Vg \\ Rg \end{pmatrix} = \begin{pmatrix} 120 \\ 75 \end{pmatrix}$

where both g_1 and g_2 are active.

From (1), we have

$$\begin{pmatrix} 1.6 \\ -0.64 \end{pmatrix} = u_1 \begin{pmatrix} 5 \\ -4 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.32 \\ 0.64 \end{pmatrix}$$

(3) is satisfied.

So at $\begin{pmatrix} Vg \\ Rg \end{pmatrix} = \begin{pmatrix} 120 \\ 75 \end{pmatrix}$, the dissipated power is minimum.

Question 8

Using [0, 5] as the initial interval of uncertainty, apply 3 iterations of the Golden Section search method to the minimization w.r.t. ϕ of a function described by

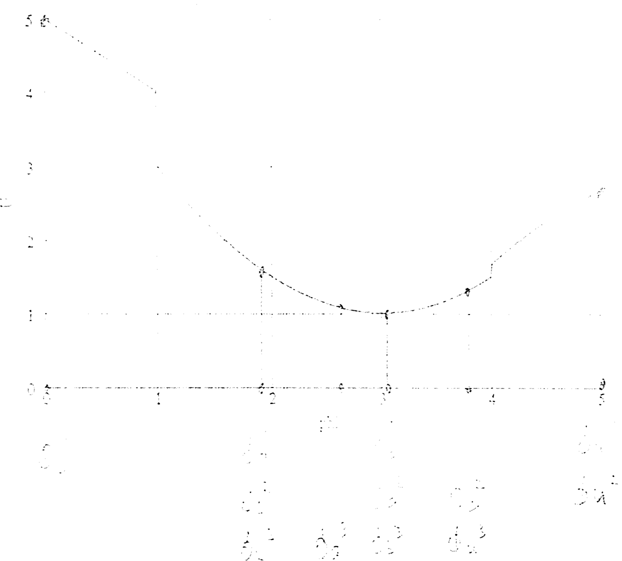
$$U = -\phi + 5 \qquad \phi < 1$$

$$U = 0.5(\phi - 3)^2 + 1 \qquad 1 \le \phi \le 4$$

$$U = 3 - \frac{(\phi - 6)^2}{3} \qquad \phi > 4$$

Show all steps clearly and label a diagram appropriately. State the final interval of uncertainty.

Answer (25 min.)



$\tau = 1.6180$

Step 1 $\phi_L^1 = 0$, $\phi_u^1 = 5$, $I^1 = \phi_u^1 - \phi_L^1 = 5 - 0 = 5$, $\phi_a^1 = \frac{1}{\tau^2} I^1 + \phi_L^1 = 1.9099$, $\phi_b^1 = \frac{1}{\tau} I^1 + \phi_L^1 = 3.0902$

$U_a^1 = 0.5(1.9099 - 3)^2 + 1 = 1.5942$, $U_b^1 = 0.5(3.0902 - 3)^2 + 1 = 1.0041$

$\because U_a^1 > U_b^1 \Rightarrow$ so $\phi_L^2 = \phi_a^1$ $\phi_a^2 = \phi_b^1$ $\phi_u^2 = \phi_u^1$

Step 2 $\phi_L^2 = 1.9099$, $\phi_u^2 = 5$, $I^2 = \phi_u^2 - \phi_L^2 = 3.0901$, $\phi_b^2 = \frac{1}{\tau} I^2 + \phi_L^2 = 3.8197$

$\phi_a^2 = 3.0902$

$U_a^2 = U_b^1 = 1.0041$, $U_b^2 = 1.3359$

$\because U_a^2 < U_b^2 \Rightarrow$ so $\phi_L^3 = \phi_L^2$, $\phi_u^3 = \phi_b^2$, $U_a^3 = \phi_b^2$

Step 3 $\phi_L^3 = 1.9099$, $\phi_u^3 = 3.8197$, $I^3 = \phi_u^3 - \phi_L^3 = 1.9098$

$\phi_a^3 = \frac{1}{\tau^2} I^3 + \phi_L^3 = 2.6394$, $\phi_b^3 = 3.0902$, $U_a^3 = 1.065$, $U_b^3 = 1.004$

$\because U_a^3 > U_b^3 \Rightarrow$ so $\phi_L^4 = \phi_a^3$, $\phi_u^4 = \phi_u^3$

$\therefore$ The final interval of uncertainty is [2.6394 3.8197]

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Question 9

Consider the three functions

$$f_1(\phi_1, \phi_2) = \phi_2 - 3$$

$$f_2(\phi_1, \phi_2) = -\phi_1 + 2$$

$$f_3(\phi_1, \phi_2) = \phi_1 - \phi_2 - 1$$

(1) Sketch two to three minimax contours of $f_1(\phi_1, \phi_2)$, $f_2(\phi_1, \phi_2)$ and $f_3(\phi_1, \phi_2)$.

(2) Calculate the first-order derivatives of f_1, f_2 and f_3.

(3) Find the active functions at the point $[1\ 4]^T$.

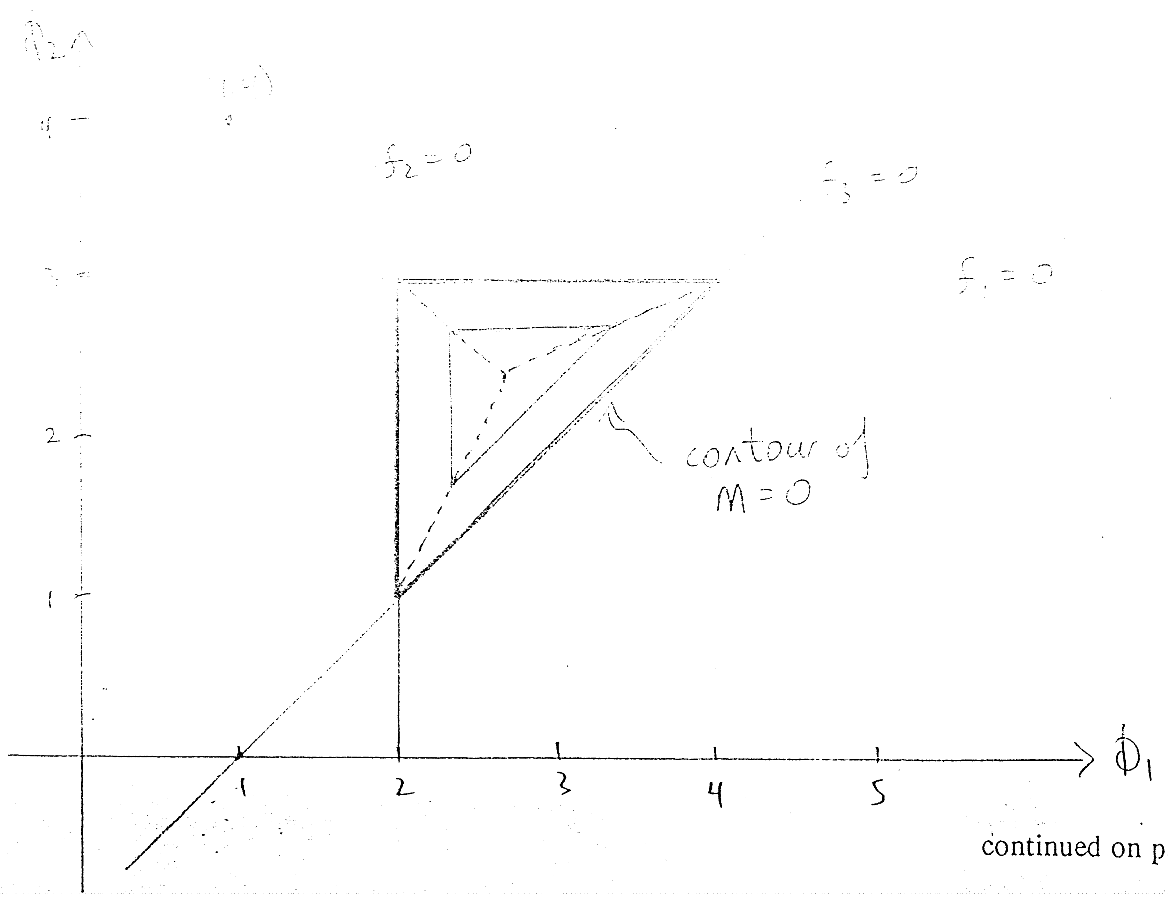
(4) Verify that $[1\ 4]^T$ is not the minimax solution.

Answer (25 min.)

① Let: $f_1 = 0 \longrightarrow \phi_2 - 3 = 0 \Longrightarrow \phi_2 = 3$

$f_2 = 0 \longrightarrow -\phi_1 + 2 = 0 \Longrightarrow \phi_1 = 2$

$f_3 = 0 \longrightarrow \phi_1 - \phi_2 - 1 = 0 \Longrightarrow \phi_2 = \phi_1 - 1$



continued on page 11

(2) $\nabla f_1 = \begin{bmatrix} \dfrac{\partial f_1}{\partial \phi_1} \\[2ex] \dfrac{\partial f_2}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\qquad \nabla f_2 = \begin{bmatrix} \dfrac{\partial f_2}{\partial \phi_1} \\[2ex] \dfrac{\partial f_2}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\nabla f_3 = \begin{bmatrix} \dfrac{\partial f_3}{\partial \phi_1} \\[2ex] \dfrac{\partial f_3}{\partial \phi_2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(3) at $\phi = [1 \quad 4]^T$

$$\left.\begin{aligned} f_1 &= 4 - 3 = 1 \\ f_2 &= -1 + 2 = 1 \end{aligned}\right\} \therefore f_1 \text{ and } f_2 \text{ are the } \underline{\text{active functions}} \text{ at } [1 \quad 4]^T$$

$$f_3 = 1 - 4 - 1 = -4$$

(4) At the minimax optimum, the following conditions must hold:

$$\sum_{i=1}^{3} u_i \nabla f_i = 0 \qquad (1)$$

$$\sum_{i=1}^{3} u_i = 1 \qquad (2)$$

$$u_i (M - f_i) = 0 \qquad (3)$$

$$u_i \geq 0 \qquad (4)$$

$\therefore$ at $\underline{\phi} = [1 \;\; 4]^T \quad \rightarrow M = 1$

$\therefore$ According to condition (3)

$u_3 = 0 \quad (\because f_3$ is an inactive function$)$

$\therefore$ Substitute this into (1) and (2) gives;

$$\begin{cases} u_1 \nabla f_1 + u_2 \nabla f_2 = 0 \\ u_1 + u_2 = 1 \end{cases}$$

$$\begin{cases} u_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = 0 \\ u_1 + u_2 = 1 \end{cases}$$

$-u_2 = 0 \longrightarrow \boxed{u_2 = 0}$

$u_1 = 0$

$u_1 + u_2 = 1 \longrightarrow$ inconsistent.

which violates condition (3)

$\therefore$ The necessary conditions for a minimax optimum are violated. Point $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is NOT the minimax optimum.

Question 10

The updating formula for the Fletcher-Powell-Davidon method is defined by

$$H^0 = \mathbf{1}$$

$$s^j = -H^j \nabla U^j, \quad j = 0, 1, 2, \ldots$$

where

$$H^{j+1} = H^j + \frac{\Delta\phi^j \Delta\phi^{j^T}}{\Delta\phi^{j^T} g^j} - \frac{H^j g^j g^{j^T} H^j}{g^{j^T} H^j g^j}$$

$$\Delta\phi^j \triangleq \alpha^j s^j = \phi^{j+1} - \phi^j$$

$$g^j \triangleq \nabla U^{j+1} - \nabla U^j$$

(a) What is H^j and what is its relationship with the Hessian matrix of a function $U(\phi)$? How is α^j computed in practice?

(b) Apply the algorithm (using a theoretically justified approach to obtain α^j) to the minimization of

$$\phi_1^2 + 2\phi_2^2 + \phi_1\phi_2 + \phi_2 + 2$$

w.r.t. ϕ_1 and ϕ_2 starting at $\phi_1 = 0$, $\phi_2 = 2$. Show all steps explicitly and comment on the results obtained. Draw an accurate diagram showing the path taken.

Answer (40 min.)

a) H^j is the jth approximation of the inverse of the Hessian matrix.

α^j is computed by minimizing the objective function of $\underset{\sim}{\phi}^{j+1}$ w.r.t. α, where $\underset{\sim}{\phi}^{j+1} = \underset{\sim}{\phi}^j + \alpha^j \underset{\sim}{s}$.

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(b). $\nabla u = \begin{bmatrix} 2\phi_1 + \phi_2 \\ \phi_1 + 4\phi_2 + 1 \end{bmatrix}$

$u(\phi^0) = \begin{bmatrix} 0 + 2 \\ 0 + 8 + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

$H^0 = I$

$S^0 = -H^0 \nabla u^0 = \begin{bmatrix} -2 \\ -9 \end{bmatrix}$ $\quad \Delta\phi = \alpha S = \begin{bmatrix} -2\alpha \\ -9\alpha \end{bmatrix}$

$\phi_1 = \phi^0 + \alpha S = \begin{bmatrix} -2\alpha \\ 2 - 9\alpha \end{bmatrix}$

$u(\phi_1) = (0-2\alpha)^2 + 2(2-9\alpha)^2 + (0-2\alpha)(2-9\alpha) + (2-9\alpha) \cdot 2$

$= 4\alpha^2 + 2(4 - 36\alpha + 81\alpha^2) + 18\alpha^2 - 4\alpha + 4 - 9\alpha$

$= 184\alpha^2 - 85\alpha + 12$

$\frac{du}{d\alpha} = 2 \times 184\alpha - 85 = 0$

then $\alpha = \frac{85}{2 \times 184} = 0.2310$

So $\phi_1 = \phi^0 + \alpha S$

$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ -9 \end{bmatrix} = \begin{bmatrix} -0.4620 \\ -0.079 \end{bmatrix}$

$\Delta\phi^0 = \alpha S = \begin{bmatrix} -0.4620 \\ -2.079 \end{bmatrix}$

$$\nabla u^1 = \begin{bmatrix} 2*(-0.4620) - 0.079 \\ -0.4620 - 4\times(0.079) + 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1.003 \\ +0.222 \end{bmatrix}$$

$$g^0 = \nabla u^1 - \nabla u^0 = \begin{bmatrix} -1.003 - 2 \\ 0.222 - 9 \end{bmatrix} = \begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}$$

$$H^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\Delta\Phi^0 \cdot \Delta\Phi^{0T}}{\Delta\Phi^T \cdot g^0} - \frac{H^0 \cdot g^0 \cdot g^{0T} H^0}{g^{0T} H^0 \cdot g^0}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{\begin{bmatrix} -0.4620 \\ -2.079 \end{bmatrix}\begin{bmatrix} -0.4620 \\ -2.079 \end{bmatrix}^T}{\begin{bmatrix} -0.4260 & -2.079 \end{bmatrix}\begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}} - \frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}\begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}^T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{\begin{bmatrix} -3.003 & -8.778 \end{bmatrix}\begin{bmatrix} -3.003 \\ -8.778 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.0109 & 0.0489 \\ 0.0489 & 0.2201 \end{bmatrix} - \begin{bmatrix} 0.1048 & 0.3063 \\ 0.3063 & 0.8952 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9061 & -0.2573 \\ -0.2573 & 0.3249 \end{bmatrix}$$

$$S^1 = -H^1 \nabla u^1 = -\begin{bmatrix} 0.9061 & -0.2573 \\ -0.2573 & 0.3249 \end{bmatrix}\begin{bmatrix} -1.003 \\ +0.222 \end{bmatrix} = \begin{bmatrix} 0.9659 \\ -0.3302 \end{bmatrix}$$

$$\underset{\sim}{\Phi}^2 = \underset{\sim}{\Phi}^1 + \Delta\Phi^1 = \begin{bmatrix} -0.4620 + 0.9659\alpha \\ -0.079 - 0.3302\alpha \end{bmatrix}$$

$$u^2 = (-0.4620 + 0.9659\alpha)^2 + 2(-0.079 - 0.3302\alpha)^2$$

$$+ (-0.462 + 0.9659\alpha)(-0.079 - 0.3302\alpha) + (-0.079 - 0.3302\alpha) + 2$$

$$= C_1 + 2 \times (-0.4620) \times (0.9659)\alpha + 0.9659^2\alpha^2$$

$$+ C_2 + 4 \times (0.079) \times 0.3302 \times \alpha + 2 \times 0.3302^2\alpha^2$$

$$+ C_3 + (-0.9659 \times 0.079 + 0.4620 \times 0.3302)\alpha + - 0.9659 \times 0.3302\alpha^2$$

$$+ C_4 - 0.3302\alpha$$

$$\frac{du^2}{d\alpha} = 2 \times 0.9659^2\alpha + 4 \times 0.3302^2\alpha - 2 \times 0.9659 \times 0.3302\alpha$$

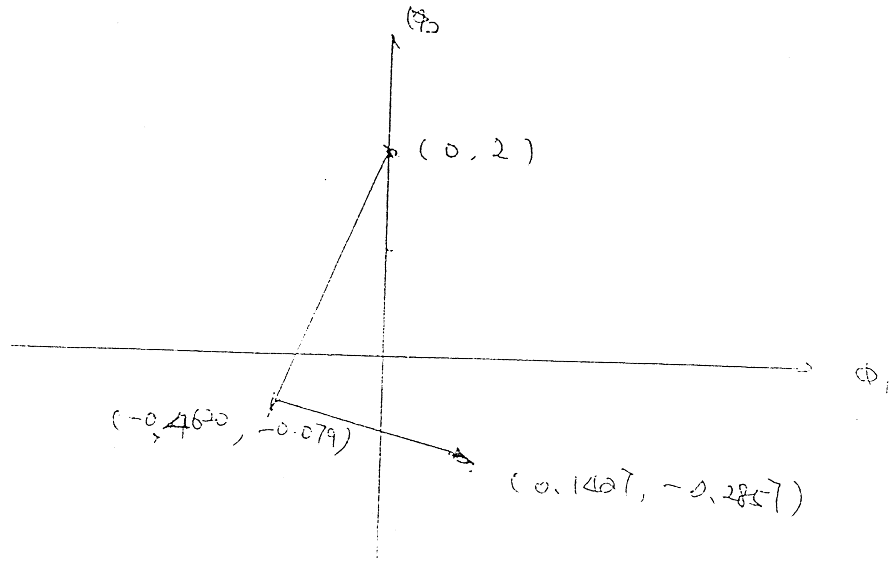
$$+ (-2 \times 0.4620 \times 0.9659 + 4 \times 0.079 \times 0.3302 - 0.9659 \times 0.079 + 0.4620 \times 0.3302$$
$$- 0.3302)$$

$$\therefore \alpha \doteq 0.6261$$

$$\underset{\sim}{\Phi}^2 = \begin{bmatrix} -0.462 + 0.9659 \times 0.6210 \\ -0.079 + (-0.3302) \times 0.6210 \end{bmatrix} = \begin{bmatrix} 0.1427 \\ -0.2857 \end{bmatrix}$$

$$\underset{\sim}{\nabla} u^2 = \begin{bmatrix} 2\Phi_1 + \Phi_2 \\ \Phi_1 + 4\Phi_2 + 1 \end{bmatrix} \doteq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The Fletcher – Powell Method results in a solution to a quadratic in 2 step.



Question 11

For the resistor-diode network shown in Fig. Q11, illustrate with the aid of an I-V diagram an iterative method of finding V at DC. State Newton's method for solving this problem and derive the network model corresponding to the situation at the jth iteration. What is the significance of this model?

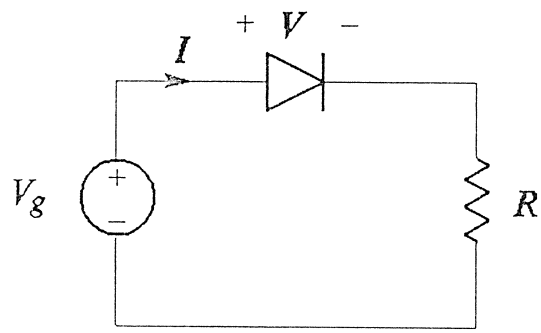


Fig. Q11 Resistor-diode network.

Answer (40 min.)

THE END!

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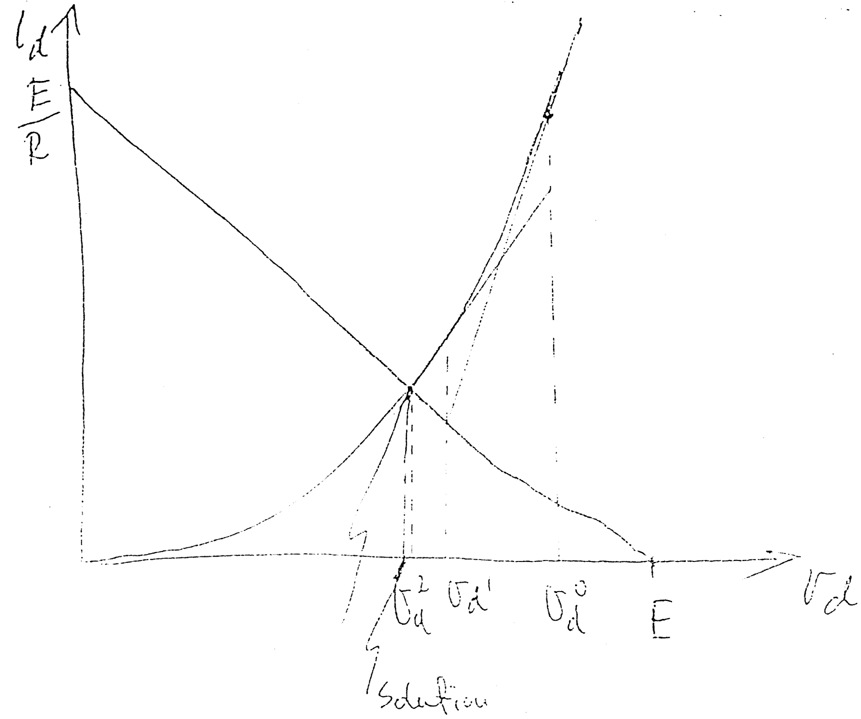


$$f(v_d) = V_g - v_d^j - R i_d^j = 0 \qquad i_d = i_d(v_d)$$

$$f(v_d) = V_g - v_d^j - R i_d(v_d^j) = 0$$

(*) Newton method

$$v_d^{j+1} = v_d^j - \frac{f(v_d^j)}{f'(v_d^j)}$$

so

$$v_d^{j+1} = v_d^j - \frac{V_g - v_d^j - R i_d(v_d^j)}{-1 - R i_d'(v_d^j)}$$

(*) Network model
We linearize the IV relation for the diode

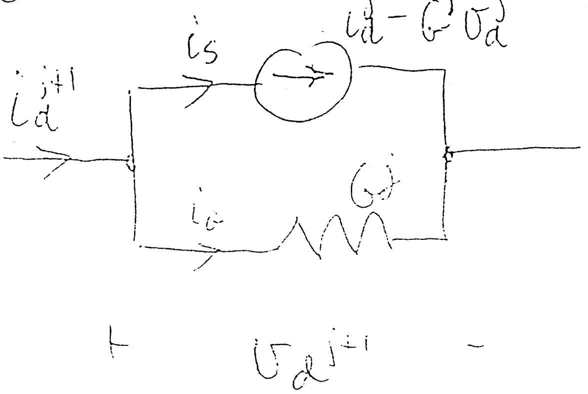
$$i_d = i_d(v_d)$$

$$i_d^{j+1} = i_d^j + \frac{d i_d(v_d^j)}{d v_d^j}(v_d^{j+1} - v_d^j) + \cdots$$

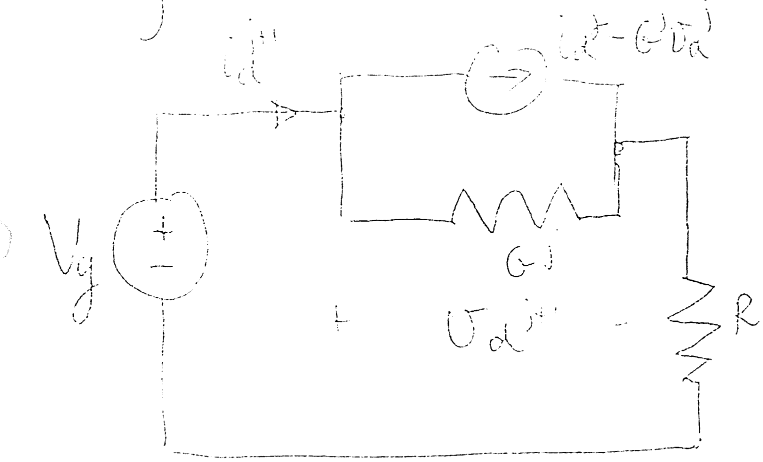
or $$i_d^{j+1} = i_d^j + G^j(v_d^{j+1} - v_d^j)$$

$$i_d^{j+1} = \underbrace{G^j v_d^{j+1}}_{i_G} + \underbrace{i_d^j - G^j v_d^j}_{i_s}$$

the network model of i_d^{j+1} is as follows



Using this model for the diode we get



(*) Significance of this model.

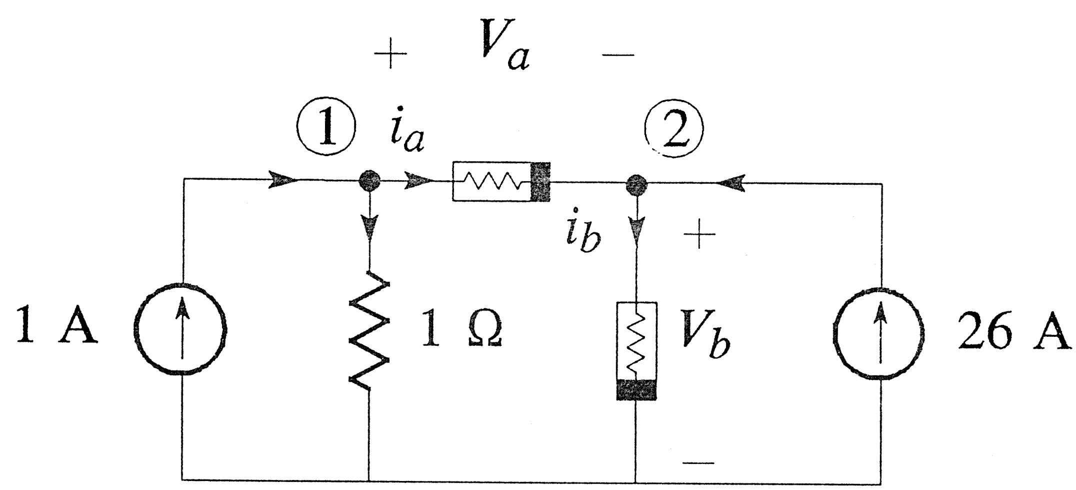
The model is a linear approximation of the nonlinear circuit. It is valid only at the jth iteration. Standard methods for solving linear networks can be used to solve the network at the jth iteration.

COMPUTER ENGINEERING 3KB3:
COMPUTATIONAL METHODS II

Class Test #4 (Duration of Test: 30 minutes)

Answer questions Q1 (a) **or** Q1 (b) **and** Q2.

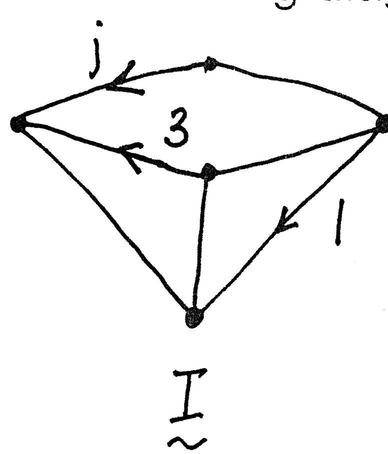
Q1. Consider the nonlinear circuit shown, where $i_a = 2v_a^3$ and $i_b = v_b^3 + 10v_b$.

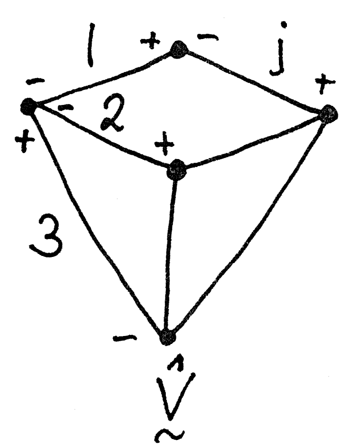


(a) Express the nodal equations in the linearized form required at the *j*th iteration of the Newton algorithm.

(b) Draw the companion network at the *j*th iteration and state the corresponding nodal equations.

Q2. Complete the following diagram and verify Tellegen's theorem.





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Solution

Q1 (a) According to KCL, the nodal equations at node 1 and 2 can be written as

$$v_1 + 2(v_1 - v_2)^3 - 1 = 0$$
$$v_2^3 + 10\,v_2 - 2(v_1 - v_2)^3 - 26 = 0$$

If we let

$$f_1 = v_1 + 2(v_1 - v_2)^3 - 1$$
$$f_2 = v_2^3 + 10\,v_2 - 2(v_1 - v_2)^3 - 26$$

Then the Jacobian matrix at the *j*th iteration is

$$\boldsymbol{J}j = \begin{bmatrix} \dfrac{\partial f_1}{\partial v_1} & \dfrac{\partial f_1}{\partial v_2} \\[2ex] \dfrac{\partial f_2}{\partial v_1} & \dfrac{\partial f_2}{\partial v_2} \end{bmatrix}^j = \begin{bmatrix} 1 + 6(v_1 - v_2)^2 & -6(v_1 - v_2)^2 \\ -6(v_1 - v_2)^2 & 3v_2^2 + 10 + 6(v_1 - v_2)^2 \end{bmatrix}^j$$

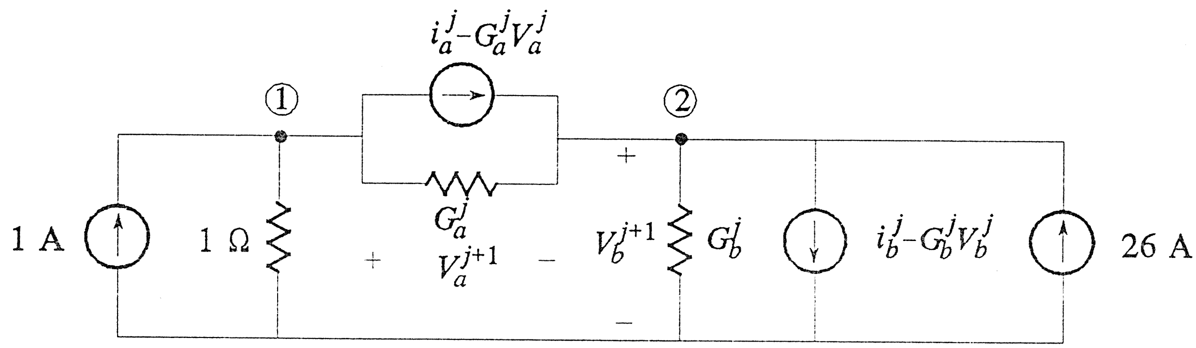
and the linearized form of nodal equations at the *j*th iteration of the Newton algorithm can be written as

$$\boldsymbol{J}^j (\boldsymbol{v}^{j+1} - \boldsymbol{v}^j) = -\boldsymbol{f}^j$$

Q1 (b) The companion network at the jth iteration is shown in below, where

$$i_a^j = 2(v_a^3)^j \quad i_b^j = (v_b^3)^j + (10\ v_b)^j \quad G_a^j = 6(v_a^2)^j$$

$$v_a = v_1 - v_2, \quad v_b = v_2 \quad G_b^j = (3\ v_b^2)^j + 10$$



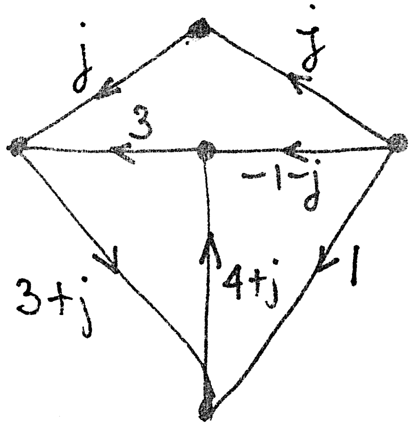
The corresponding nodal equations are

$$\begin{bmatrix} 1 + G_a^j & -G_a^j \\ -G_a^j & G_a^j + G_b^j \end{bmatrix} \begin{bmatrix} v_1^{j+1} \\ v_2^{j+1} \end{bmatrix} = \begin{bmatrix} 1 - (i_a^j - G_a^j\ v_a^j) \\ (i_a - G_a^j\ v_a^j) - (i_b^j - G_b^j\ v_b^j) + 26 \end{bmatrix}$$

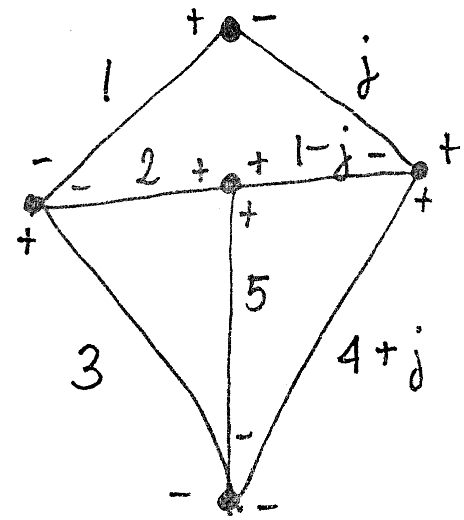
Q2. Tellegen's theorem:

$$\left.\begin{aligned} \underset{\sim}{V}^T \underset{\sim}{\hat{I}} &= \emptyset \\ \underset{\sim}{I}^T \underset{\sim}{\hat{V}} &= \emptyset \end{aligned}\right\} \longrightarrow \underset{\sim}{V}^T \underset{\sim}{\hat{I}} - \underset{\sim}{I}^T \underset{\sim}{\hat{V}} = \emptyset$$

$$\left.\begin{aligned} \Delta \underset{\sim}{V}^T \underset{\sim}{\hat{I}} &= \emptyset \\ \Delta \underset{\sim}{I}^T \underset{\sim}{\hat{V}} &= \emptyset \end{aligned}\right\} \longrightarrow \Delta \underset{\sim}{V}^T \underset{\sim}{\hat{I}} - \Delta \underset{\sim}{I}^T \underset{\sim}{\hat{V}} = \underset{\sim}{\emptyset}$$



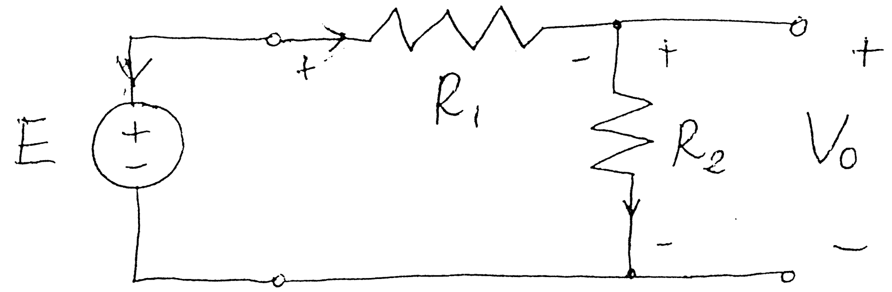
$\underset{\sim}{I}$



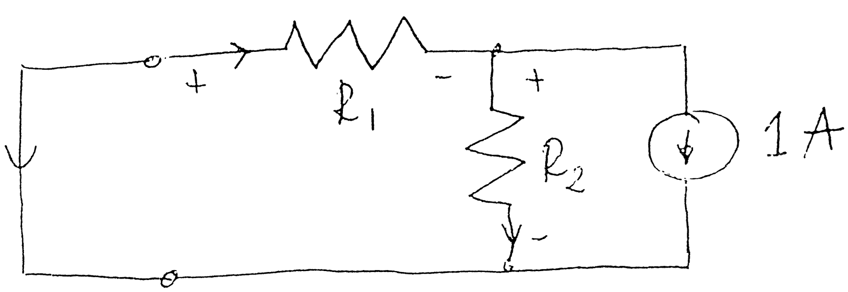
$\underset{\sim}{\hat{V}}$

$$\underset{\sim}{I}^T \underset{\sim}{\hat{V}} = (3+j)(3) + j(1) + (j)(j) + (1)(4+j) + (3)(2)$$
$$- (-1-j)(1-j) - (4+j)(5) = \emptyset.$$

Consider the following circuit



and the adjoint circuit



1. Write down the difference form of the Telligen theorem for this circuit.
2. Develop the derivative of this equation to expose explicit formulas for $\frac{\partial V_0}{\partial R_1}$ and $\frac{\partial V_0}{\partial R_2}$.
3. Calculate $\frac{\partial V_0}{\partial R_1}$ and $\frac{\partial V_0}{\partial R_2}$ for $E=3V$, $R_1=2\Omega$, $R_2=1\Omega$
4. Verify your results analytically.

Solution:

1. $$\left(V_E \hat{I}_E - \hat{V}_E I_E\right) + \left(V_{R_1} \hat{I}_{R_1} - \hat{V}_{R_2} I_{R_2}\right) + \left(V_{R_2} \hat{I}_{R_2} - \hat{V}_{R_2} I_{R_2}\right) +$$
$$+ \left(V_o \hat{I}_o - \hat{V}_o I_o\right) = 0$$

2. From the original and adjoint circuits we know that:

$$V_E = E \qquad \text{and} \qquad I_o = 0$$
$$\hat{V}_E = 0 \qquad\qquad \hat{I}_o = 1\,A$$

Hence

$$E \hat{I}_E + \left(V_{R_1} \hat{I}_{R_1} - \hat{V}_{R_1} I_{R_1}\right) + \left(V_{R_2} \hat{I}_{R_2} - \hat{V}_{R_2} I_{R_2}\right) + V_o = 0$$

or

$$V_o = -E \hat{I}_E - \left(V_{R_1} \hat{I}_{R_1} - \hat{V}_{R_1} I_{R_1}\right) - \left(V_{R_2} \hat{I}_{R_2} - \hat{V}_{R_2} I_{R_2}\right)$$

Differentiating w.r.t. φ, where φ is either R_1 or R_2 we get:

$$\frac{\partial V_o}{\partial \varphi} = \left(\frac{\partial I_{R_1}}{\partial \varphi} \hat{V}_{R_1} - \frac{\partial V_{R_1}}{\partial \varphi} \hat{I}_{R_1}\right) + \left(\frac{\partial I_{R_2}}{\partial \varphi} \hat{V}_{R_2} - \frac{\partial V_{R_2}}{\partial \varphi} \hat{I}_{R_2}\right)$$

using

$$V_{R_1} = I_{R_1} R_1 \qquad\qquad \hat{V}_{R_1} = \hat{I}_{R_1} R_1$$
$$V_{R_2} = I_{R_2} R_2 \qquad \text{and} \qquad \hat{V}_{R_2} = \hat{I}_{R_2} R_2$$

leads to

$$\frac{\partial V_o}{\partial \varphi} = \frac{\partial I_{R_1}}{\partial \varphi} \hat{I}_{R_1} R_1 - \frac{\partial (I_{R_1} R_1)}{\partial \varphi} \hat{I}_{R_1} + $$
$$+ \frac{\partial I_{R_2}}{\partial \varphi} \hat{I}_{R_2} R_2 - \frac{\partial (I_{R_2} R_2)}{\partial \varphi} \hat{I}_{R_2}$$

or

$$\frac{\partial V_o}{\partial \varphi} = \frac{\partial I_{R_1}}{\partial \varphi} \hat{I}_{R_1} R_1 - \frac{\partial I_{R_1}}{\partial \varphi} \hat{I}_{R_1} R_1 - \frac{\partial R_1}{\partial \varphi} I_{R_1} \hat{I}_{R_1} +$$
$$+ \frac{\partial I_{R_2}}{\partial \varphi} \hat{I}_{R_2} R_2 - \frac{\partial I_{R_2}}{\partial \varphi} \hat{I}_{R_2} R_2 - \frac{\partial R_2}{\partial \varphi} I_{R_2} \hat{I}_{R_2}$$

$$\frac{\partial V_o}{\partial \varphi} = - \frac{\partial R_1}{\partial \varphi} I_{R_1} \hat{I}_{R_1} - \frac{\partial R_2}{\partial \varphi} I_{R_2} \hat{I}_{R_2}$$

and finally:

$$\frac{\partial V_o}{\partial R_1} = - I_{R_1} \hat{I}_{R_1} \quad \text{and} \quad \frac{\partial V_o}{\partial R_2} = - I_{R_2} \hat{I}_{R_2}$$

3. Solving the original and adjoint circuits we get

$$I_{R_1} = I_{R_2} = 1$$

and $\hat{I}_{R_1} = \frac{1}{3}$, $\hat{I}_{R_2} = -\frac{2}{3}$

Then $\frac{\partial V_o}{\partial R_1} = -\frac{1}{3}$ and $\frac{\partial V_o}{\partial R_2} = \frac{2}{3}$

4. $$V_0 = \frac{E R_2}{R_1 + R_2}$$

$$\frac{\partial V_0}{\partial R_1} = -\frac{E R_2}{(R_1 + R_2)^2} = -\frac{1}{3}$$

$$\frac{\partial V_0}{\partial R_2} = \frac{E}{R_1 + R_2} - \frac{E R_2}{(R_1 + R_2)^2} = \frac{2}{3}$$

Question 110 Consider the circuit shown in Fig. 18, which is assumed to be in the sinusoidal steady state.

Derive from first principles the adjoint network and sensitivity expressions for all the elements of the circuit. Derive the adjoint excitations appropriate for calculating the first-order sensitivities of V_{C_2} w.r.t. all the parameters.

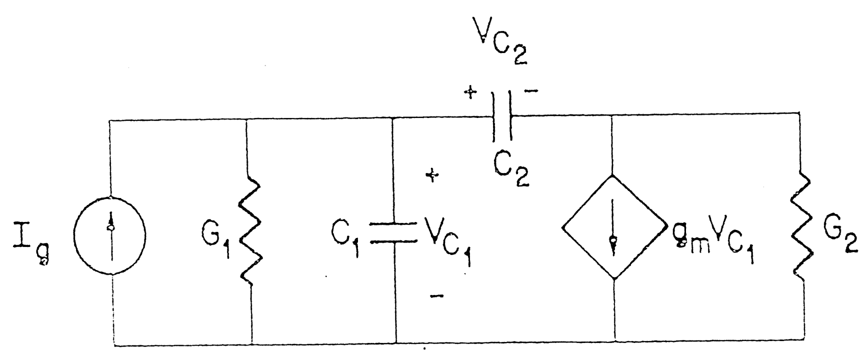
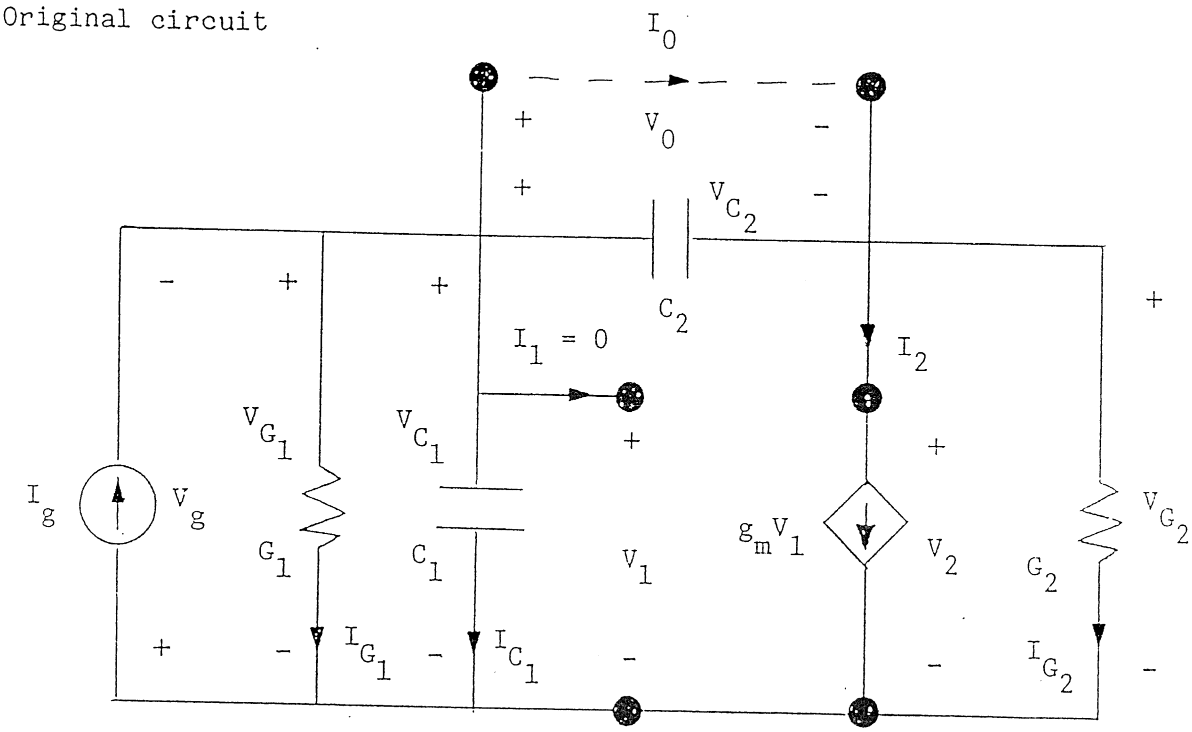


Fig. 18 Active circuit (Question 110).

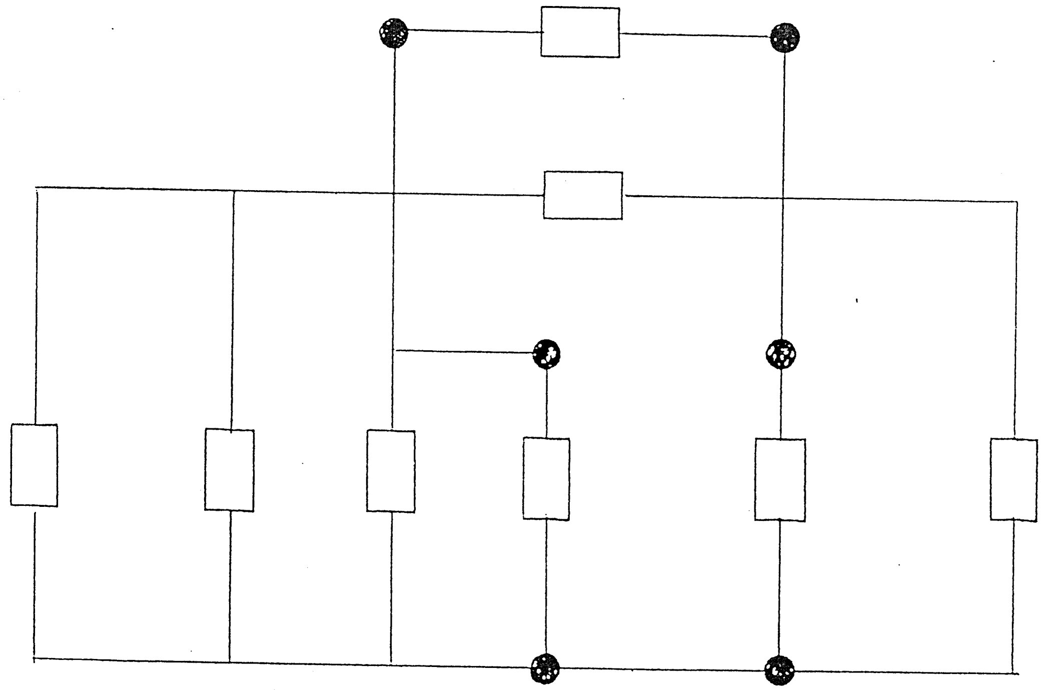
A SOLUTION TO QUESTION 110

Original circuit



Adjoint Circuit

(notation to correspond to original circuit)



Tellegen's Theorem

$$V_g\hat{I}_g + V_{G_1}\hat{I}_{G_1} + V_{C_1}\hat{I}_{C_1} + V_{C_2}\hat{I}_{C_2} + V_1\hat{I}_1 + V_2\hat{I}_2$$

$$+ V_{G_2}\hat{I}_{G_2} + V_0\hat{I}_0 - I_g\hat{V}_g - I_{G_1}\hat{V}_{G_1} - I_{C_1}\hat{V}_{C_1}$$

$$- I_{C_2}\hat{V}_{C_2} - I_1\hat{V}_1 - I_2\hat{V}_2 - I_{G_2}\hat{V}_{G_2} - I_0\hat{V}_0 = 0$$

Differentiate w.r.t. ϕ

$$\frac{\partial V_g}{\partial\phi}\hat{I}_g + \frac{\partial V_{G_1}}{\partial\phi}\hat{I}_{G_1} + \frac{\partial V_{C_1}}{\partial\phi}\hat{I}_{C_1} + \frac{\partial V_{C_2}}{\partial\phi}\hat{I}_{C_2} + \frac{\partial V_1}{\partial\phi}\hat{I}_1$$

$$+ \frac{\partial V_2}{\partial\phi}\hat{I}_2 + \frac{\partial V_{G_2}}{\partial\phi}\hat{I}_{G_2} + \frac{\partial V_0}{\partial\phi}\hat{I}_0 - \frac{\partial I_g}{\partial\phi}\hat{V}_g - \frac{\partial I_{G_1}}{\partial\phi}\hat{V}_{G_1}$$

$$- \frac{\partial I_{C_1}}{\partial\phi}\hat{V}_{C_1} - \frac{\partial I_{C_2}}{\partial\phi}\hat{V}_{C_2} - \frac{\partial I_1}{\partial\phi}\hat{V}_1 - \frac{\partial I_2}{\partial\phi}\hat{V}_2 - \frac{\partial I_{G_2}}{\partial\phi}\hat{V}_{G_2}$$

$$- \frac{\partial I_0}{\partial\phi}\hat{V}_0 = 0$$

Branch relations (original circuit)

$$I_g = \text{constant, hence } \frac{\partial I_g}{\partial\phi} = 0$$

$$I_{G_1} = G_1V_{G_1}\text{, hence } \frac{\partial I_{G_1}}{\partial\phi} = \frac{\partial G_1}{\partial\phi}V_{G_1} + G_1\frac{\partial V_{G_1}}{\partial\phi}$$

$$I_{C_1} = j\omega C_1V_{C_1}\text{, hence } \frac{\partial I_{C_1}}{\partial\phi} = \frac{\partial(j\omega C_1)}{\partial\phi}V_{C_1} + j\omega C_1\frac{\partial V_{C_1}}{\partial\phi}$$

$$I_{C_2} = j\omega C_2V_{C_2}\text{, hence } \frac{\partial I_{C_2}}{\partial\phi} = \frac{\partial(j\omega C_2)}{\partial\phi}V_{C_2} + j\omega C_2\frac{\partial V_{C_2}}{\partial\phi}$$

$$I_1 = 0\text{, hence } \frac{\partial I_1}{\partial\phi} = 0$$

$$I_2 = g_m V_1, \text{ hence } \frac{\partial I_2}{\partial \phi} = \frac{\partial g_m}{\partial \phi} V_1 + g_m \frac{\partial V_1}{\partial \phi}$$

$$I_{G_2} = G_2 V_{G_2}, \text{ hence } \frac{\partial I_{G_2}}{\partial \phi} = \frac{\partial G_2}{\partial \phi} V_{G_2} + G_2 \frac{\partial V_{G_2}}{\partial \phi}$$

$$I_0 = 0, \text{ hence } \frac{\partial I_0}{\partial \phi} = 0$$

Substitute branch relations directly and collect up terms.

$$\frac{\partial V_g}{\partial \phi} \hat{I}_g + \frac{\partial V_{G_1}}{\partial \phi} (\hat{I}_{G_1} - G_1 \hat{V}_{G_1}) + \frac{\partial V_{C_1}}{\partial \phi} (\hat{I}_{C_1} - j\omega C_1 \hat{V}_{C_1})$$

$$+ \frac{\partial V_{C_2}}{\partial \phi} (\hat{I}_{C_2} - j\omega C_2 \hat{V}_{C_2}) + \frac{\partial V_1}{\partial \phi} (\hat{I}_1 - g_m \hat{V}_2)$$

$$+ \frac{\partial V_2}{\partial \phi} \hat{I}_2 + \frac{\partial V_{G_2}}{\partial \phi} (\hat{I}_{G_2} - G_2 \hat{V}_{G_2}) + \frac{\partial V_0}{\partial \phi} \hat{I}_0$$

$$- \frac{\partial G_1}{\partial \phi} V_{G_1} \hat{V}_{G_1} - \frac{\partial (j\omega C_1)}{\partial \phi} V_{C_1} \hat{V}_{C_1} - \frac{\partial (j\omega C_2)}{\partial \phi} V_{C_2} \hat{V}_{C_2}$$

$$- \frac{\partial g_m}{\partial \phi} V_1 \hat{V}_2 - \frac{\partial G_2}{\partial \phi} V_{G_2} \hat{V}_{G_2} = 0$$

Define adjoint circuit branch relations to simplify this equation

$$\hat{I}_g = 0$$

$$\hat{I}_{G_1} = G_1 \hat{V}_{G_1}$$

$$\hat{I}_{C_1} = j\omega C_1 \hat{V}_{C_1}$$

$$\hat{I}_{C_2} = j\omega C_2 \hat{V}_{C_2}$$

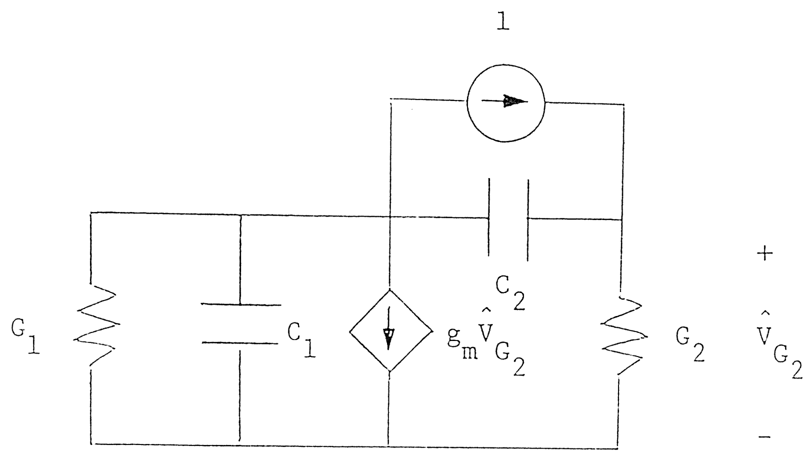
$$\hat{I}_1 = g_m \hat{V}_2$$

$$\hat{I}_2 = 0$$

$\hat{I}_{G_2} = G_2 \hat{V}_{G_2}$

$\hat{I}_0 = 1$

Adjoint Circuit



Hence,

$$\frac{\partial V_0}{\partial \phi} = \frac{\partial G_1}{\partial \phi} V_{G_1} \hat{V}_{G_1} + \frac{\partial (j\omega C_1)}{\partial \phi} V_C \hat{V}_{C_1} + \frac{\partial (j\omega C_2)}{\partial \phi} V_{C_2} \hat{V}_{C_2}$$

$$+ \frac{\partial g_m}{\partial \phi} V_{C_1} \hat{V}_{G_2} + \frac{\partial G_2}{\partial \phi} V_{G_2} \hat{V}_{G_2}$$

Let $\phi = G_1$, hence $\frac{\partial V_0}{\partial G_1} = V_{G_1} \hat{V}_{G_1}$

Let $\phi = C_1$, hence $\frac{\partial V_0}{\partial C_1} = j\omega V_{C_1} \hat{V}_{C_1}$

Let $\phi = C_2$, hence $\frac{\partial V_0}{\partial C_2} = j\omega V_{C_2} \hat{V}_{C_2}$

Let $\phi = g_m$, hence $\dfrac{\partial V_0}{\partial g_m} = V_{C_1} \hat{V}_{G_2}$ [sic.]

Let $\phi = G_2$, hence $\dfrac{\partial V_0}{\partial G_2} = V_{G_2} \hat{V}_{G_2}$

$V_0 = V_{C_2}$, hence $\partial V_0/\partial . \equiv \partial V_{C_2}/\partial .$

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