Completeness of squared eigenfunctions of the Zakharov-Shabat spectral problem

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Abstract

The completeness of eigenfunctions for linearized equations is critical for many applications, such as the study of stability of solitary waves. In this thesis, we work with the Nonlinear Schrödinger (NLS) equation, associated with the Zakharov-Shabat spectral problem. Firstly, we construct a complete set of eigenfunctions for the spectral problem. Our method involves working with an adjoint spectral problem and deriving completeness and orthogonality relations between eigenfunctions and adjoint eigenfunctions. Furthermore, we prove completeness of squared eigenfunctions, which are used to represent solutions of the linearized NLS equation. For this, we find relations between the variation of potential and the variation of scattering data. Moreover, we show the connection between the squared eigenfunctions of the Zakharov-Shabat spectral problem and solutions of the linearized NLS equation.

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Thanks to my parents and close friends for believing in me.

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Declaration of authorship

I, **Al-Tarazi Assaubay**, declare that this thesis titled, “Completeness of squared eigenfunctions of the Zakharov-Shabat spectral problem”, and the work presented in it are my own, based on reading of the book [1] and consultations with the thesis advisor.

Chapter 1

Introduction

This thesis is devoted to the question of completeness of eigenfunctions for linearized equations associated with integrable nonlinear PDEs. We consider one particular integrable nonlinear PDE given by the Nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} + 2|u|^2 u = 0, \qquad u(x,t) : \mathbb{R}^2 \to \mathbb{C}. \tag{1.1}$$

The key property which leads to the integrability of (1.1) is the existence of a pair of linear equations for $v(x,t) : \mathbb{R}^2 \to \mathbb{C}^2$ with a potential $u(x,t) : \mathbb{R}^2 \to \mathbb{C}$ and spectral parameter $k \in \mathbb{C}$:

$$v_x = \begin{bmatrix} -ik & u \\ -\overline{u} & ik \end{bmatrix} v, \tag{1.2}$$

$$v_t = \begin{bmatrix} -2ik^2 + i|u|^2 & iu_x + 2ku \\ i\overline{u}_x - 2k\overline{u} & 2ik^2 - i|u|^2 \end{bmatrix} v. \tag{1.3}$$

The first equation is a spectral problem and the second one defines the evolution of eigenfunctions. Compatibility condition $v_{xt} = v_{tx}$ is satisfied if $u = u(x,t)$ is a solution to the NLS equation (1.1). This pair of linear equations was discovered by Zakharov and Shabat in [2], and we refer to (1.2) as to the Zakharov-Shabat spectral problem.

1.1 Applications of NLS

Since waves in the real world are nonlinear, it makes sense to model them with nonlinear partial differential equations. To these days, most of nonlinear PDEs cannot be solved analytically, with the exception of integrable nonlinear PDEs, to which NLS belongs. NLS equation has many exact solutions due to its integrability. One of the basic solutions on the zero background ($u \to 0$ as $x \to \pm\infty$) is the solitary wave given by

$$u(x,t) = u_0 \operatorname{sech}\left[u_0(x - 2p_0 t)\right] e^{i\left[p_0 x + (u_0^2 - p_0^2)t\right]},$$

where u_0 is constant amplitude and p_0 is a shift of carrier-wave wave number. This solution can be found by separation of variables and integration of ODEs. Another solution on the nonzero background is the rogue wave discovered by D. Peregrine [3] and given by:

$$u(x,t) = \left[-1 + \frac{4(1+it)}{1+4x^2+t^2}\right] e^{it/2}.$$

Other solutions to (1.1) on nonzero background are Akhmediev breather: [4]

$$u(x,t) = \left[-1 + \frac{2k^2 \cosh(\lambda k t) + 2i\lambda k \sinh(\lambda k t)}{\cosh(\lambda k t) - \lambda \cos(2kx)}\right] e^{it/2},$$

where $k = \sqrt{1-\lambda^2}$ and $\lambda \in (0,1)$ is a free parameter, and Kuznetsov-Ma breather: [5]

$$u(x,t) = \left[-1 + \frac{2\beta^2 \cos(\lambda\beta t) + 2i\lambda\beta \sin(\lambda\beta t)}{\lambda \cosh(2\beta x) - \cos(\lambda\beta t)}\right] e^{it/2},$$

where $\beta = \sqrt{\lambda^2 - 1}$ and $\lambda \in (1, +\infty)$ is a free parameter.

NLS has a wide range of applications such as modeling of the light propagation in optical fibers [6], [7], wave propagation in oceans and seas [8], and atomic Bose-Einstein condensate [9].

NLS in the form (1.1) is only the simplest model among NLS-type equations used in natural sciences. In [10] M. Gedalin, Y. Band and T. Scott used higher order

NLS as a model for propagation of short light pulses in an optical fiber. They were able to find bright solitary wave solutions which exists for certain constraints on the parameters. Similar results were obtained in [11], where authors found dark solitary wave solutions in higher order NLS equation. Other solutions with nonzero boundary conditions were found in [12].

Just like in optical fibers, wave propagation in oceans and seas can be modelled by NLS and its higher-order extensions. One of the applications of NLS is to describe modulation instability and emergence of rogue waves. Modulational instability was also analyzed in [13] for dissipative NLS, which is standard NLS modified by linear dissipation. It was found that dissipation bounds perturbations even in the presence of strong nonlinearity. Authors of [14] considered modulational instability for standard NLS without dissipation, however, their main result is that Penrose stability analysis (analysis performed on an equation after the Wigner Transform) recovers same result with a better method, since it can be applied for incoherent waves. The same conclusion is also supported in [15] and [16], with further applications of Penrose stability analysis. Particularly, in [16] the higher-order NLS equation with time-dependent linear damping (damped Hirota equation) was studied to discuss extreme wave events, such as emergence of rogue waves.

1.2 Motivations

The main motivation for this thesis is to use the integrability scheme and to express solutions of the linearized NLS equation in terms of the squared eigenfunctions of the linear system (1.2) and (1.3). The linearized equations are important in understanding the linear stability of some particular solutions of the NLS in the time evolution [17], [18].

Let us show how the linearized NLS can be obtained. Suppose $u_0(x,t)$ is a particular solution of NLS (1.1). Now, by adding a small perturbation $v(x,t)$ we write a solution to NLS (1.1) in the following form

$$u(x,t) = u_0(x,t) + v(x,t). \tag{1.4}$$

Since both $u(x,t)$ and $u_0(x,t)$ are solutions to NLS, we shall find an equation that $v(x,t)$ satisfies. By substituting (1.4) into (1.1) we obtain

$$\begin{aligned}
0 &= i(u_0+v)_t + (u_0+v)_{xx} + 2(u_0+v)(\overline{u}_0+\overline{v})(u_0+v) \\
&= iu_{0t} + u_{0xx} + iv_t + v_{xx} + 2(|u_0|^2 + u_0\overline{v} + \overline{u}_0 v + |v|^2)(u_0+v) \\
&= iu_{0t} + u_{0xx} + iv_t + v_{xx} + 2(|u_0|^2u_0 + u_0^2\overline{v} + 2|u_0|^2 v + 2u_0|v|^2 + \overline{u}_0 v^2 + |v|^2 v) \\
&= \underbrace{iu_{0t} + u_{0xx} + 2|u_0|^2u_0}_{=0} + iv_t + v_{xx} + 2u_0^2\overline{v} + 4|u_0|^2 v + 4u_0|v|^2 + 2\overline{u}_0 v^2 + 2|v|^2 v \\
&= iv_t + v_{xx} + 2u_0^2\overline{v} + 4|u_0|^2 v + 4u_0|v|^2 + 2\overline{u}_0 v^2 + 2|v|^2 v.
\end{aligned}$$

Neglecting quadratic and cubic terms in v we obtain the linearized NLS equation

$$iv_t + v_{xx} + 2u_0^2\overline{v} + 4|u_0|^2 v = 0. \tag{1.5}$$

It is remarkable property of integrability that solutions $v(x,t)$ of (1.5) can be obtained from squared eigenfunctions of (1.2) and (1.3) associated with the particular solution of NLS given by $u_0(x,t)$. The questions of completeness of eigenfunctions is the one which we would like to address in this thesis.

1.3 Previous studies

It was realized in [19] that the method of inverse scattering transform (IST) is analogous to Fourier transform widely used in solving linear PDEs. Particularly, in [19, section 4] the authors considered direct and inverse scattering problem of the spectral problem (1.2). Moreover, in appendix 6, they derived the completeness relation for eigenfunctions of the linear system (1.2) and (1.3).

D. Kaup extended the theory of completeness to squared eigenfunctions of the spectral problem (1.2) in [20] and derives closure relation, which was used to prove completeness (with respect to L^2) of a set of squared eigenfunctions. Here completeness means that any L^2 function can be expressed as a unique linear combination of squared eigenfunctions from the complete set. To prove completeness of squared eigenfunctions, D. Kaup used Marchenko equations. Squared eigenfunctions play an important role in understanding the effect of variation of scattering data due to perturbations of the potential. In [21] D. Kaup studied application of the method to one soliton solution of the NLS equation (1.1). Particularly, he derived a completeness relation for a linearized NLS operator, the eigenfunctions of this operator were related to the squared eigenfunctions of (1.2) and (1.3). This completeness relation was used to study propagation of Raman pumped soliton in an optical fiber.

In [22] D. Kaup and T. Lakoba studied squared eigenfunctions for the massive Thirring model (MTM) instead of NLS. The difference in the approach is that they do not find an equation for squared eigenfunctions, and instead start with a general Wronskian relation between original and adjoint eigenvalue problems. Very similar procedure was also done for Benjamin-Ono (BO) equation in [23] by D. Kaup, T. Lakoba and Y. Matsuno. Significant difference here is that unlike in the case of NLS and MTM, both potential and its variation are assumed to be real-valued functions. Key tool in establishing completeness of squared eigenfunctions is using Green's function for the Lax operator of BO equation. In 2009 D. Kaup and J. Yang revisited the question on completeness of squared eigenfunctions by studying Sasa-Satsuma equation in [24]. To derive squared eigenfunctions they followed the same procedure: computing variation of potential through variation of scattering data with the use of Riemann-Hilbert problem, and then computing variation of scattering data through variation of potential.

The above results on completeness have been obtained for zero boundary conditions. In 2019, D. Bilman and P. Miller considered a Cauchy initial-value problem for NLS (1.1) with nonzero constant boundary conditions [25]. They constructed squared eigenfunctions as solutions to the linearized NLS equation at the Peregrine's rogue wave in the space of decaying data. P. Grinevich and P. Santini in [26] considered periodic boundary conditions and provided solutions to linearized NLS at Akhmediev breather. They also sketched a proof of completeness of squared eigenfunctions in the periodic case, leaving detailed proof in the future plans. M. Haragus and D. Pelinovsky in [18] constructed all possible solutions to linearized NLS equations at Akhmediev and Kusnetsov-Ma breathers and raised an open question on completeness of squared eigenfunctions for solutions of the NLS with nonzero (constant or periodic) boundary conditions at infinity.

1.4 Main Results of this study

Motivated by open questions on completeness of squared eigenfunctions for the potential with nonzero boundary conditions, I have reviewed the proof of completeness of squared eigenfunctions for the potentials with zero boundary conditions at infinity. These proofs has been summarized by J. Yang in the book [1]. I have followed his proofs with some modifications. Particularly, I changed notations and kept them consistent throughout my thesis for better understanding. I also provided proofs that were skipped, one of which is the orthogonality relations between eigenfunctions and adjoint eigenfunctions.

Outcomes of this thesis are presented in the following three main results. Firstly, we prove completeness of single eigenfunctions, which is presented in Theorem 1. Secondly, by obtaining relations between variation of potential and variation of scattering data we construct and prove completeness of squared eigenfunctions in Theorem 2. Both findings mean that we can express any square integrable function as a unique linear combination of eigenfunctions (squared eigenfunctions) from the complete set, where coefficients are calculated from adjoint eigenfunctions (squared adjoint eigenfunctions). Lastly, we show that squared eigenfunctions are solutions to the linearized NLS equation, and adjoint squared eigenfunctions are solutions to the adjoint linearized NLS equation, which are stated in Theorems 3, 4.

1.5 Structure of the thesis

The thesis is structured as follows. In Chapter 2, we formulate a regular Riemann-Hilbert problem for the spectral system (1.2). We introduce eigenfunctions as a product of Jost functions and exponential functions. To understand analyticity and properties of eigenfunctions we derive Jost equations (2.4), (2.5), solutions of which are matrices of vectors $M, \widehat{M}, \widehat{N}, N$. Then, in Lemma 2.1 we figure out that solutions M, N are analytic in upper half-plane, while $\widehat{M}, \widehat{N}$ are analytic in the lower half-plane. These solutions are linearly dependent on $\mathbb{R}$, where they both exist. We relate them by scattering data (2.22). The next step is proceed in a similar way with adjoint Jost equation (2.23). One of the key properties of an adjoint equation is that its solution is an inverse to a solution of spectral problem (2.4). Analyticity of solutions to adjoint problem is proved in Lemma 2.8, thus, vectors M^*, N^* are analytic in the lower half-plane, while $\widehat{M}^*, \widehat{N}^*$ are analytic in the upper half-plane. Combining solutions of spectral and adjoint equations we formulate a Riemann-Hilbert problem (2.42) and solve it with Plemelj formula (2.46).

In Chapter 3, we prove completeness of a set of eigenfunctions $\{\phi, \widehat{\phi}\}$. We introduce functions $\mathcal{R}^+, \mathcal{R}^-$ in (3.3), (3.4), which involve $\phi, \widehat{\phi}, \psi, \widehat{\psi}$. After integrating the functions $\mathcal{R}^+$ and $\mathcal{R}^-$ via contour integration, we obtain closure relation (3.9). It is important to note that we assume scattering data $a, \overline{a}$ to be non-zero. In Lemma 3.3 inner products between eigenfunctions and adjoint eigenfunctions are computed, which help to prove our first main result in Theorem 1.

In Chapter 4, we construct a complete set of squared eigenfunctions $\{Z^-, Z^+\}$. To do so, we choose squared eigenfunctions by considering a perturbation of potential u in the spectral problem (1.2). By expressing variation of scattering data (4.8) in terms of variation of potential we construct the adjoint squared eigenfunctions $\{\Omega^+, \Omega^-\}$ in (4.19). Then, to construct squared eigenfunctions (4.43) we introduce new matrices $F^{\pm}$ that are linearly related to solutions to the Riemann-Hilbert problem, considering asymptotics of $F^{\pm}$ and proving Lemma 4.10 we construct squared eigenfunctions in Lemma 4.11. We obtain the closure relation in Lemma 4.12 and the inner products between $\Omega^{\pm}$ and $Z^{\pm}$ are computed in Lemma 4.13. These Lemmas help us to prove the second main result of this thesis, which is stated as Theorem 2.

In Chapter 5, we show that the squared eigenfunctions are solutions to the linearized NLS, and the adjoint squared eigenfunctions are solutions to the adjoint linearized NLS equation. This is the third main result of this thesis, stated as Theorem 3 and 4.

In Chapter 6, we make concluding remarks by restating main results and methods. Moreover, we discuss possible future extensions of this research.

Chapter 2

Riemann-Hilbert Problem

2.1 Lax equations

Lax pair (1.2), (1.3) can be rewritten as

$$v_x = -ik\sigma_3 v + Q(u)v, \tag{2.1}$$

and

$$v_t = -2ik^2\sigma_3 v + R(u)v, \tag{2.2}$$

where

$$Q(u) = \begin{bmatrix} 0 & u \\ -\overline{u} & 0 \end{bmatrix}, \qquad R(u) = \begin{bmatrix} i|u|^2 & 2ku + iu_x \\ -2k\overline{u} + i\overline{u}_x & -i|u|^2 \end{bmatrix}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Two linearly independent solutions to the system (2.1)-(2.2) exist for $u = 0$:

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ikx-2ik^2t},$$
$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ikx+2ik^2t}.$$

If $u \neq 0$, then two linearly independent solutions can be represented in the matrix form

$$V(x,t) = J(x,t)e^{(-ikx-2ik^2t)\sigma_3}, \tag{2.3}$$

where $J(x,t)$ is supposed to be a non-singular matrix. Substituting (2.3) into (2.1) and (2.2) leads to the following equations

$$J_x = -ik[\sigma_3, J] + Q(u)J, \tag{2.4}$$

and

$$J_t = -2ik^2[\sigma_3, J] + R(u)J, \tag{2.5}$$

where

$$[\sigma_3, J] = \sigma_3 J - J\sigma_3.$$

2.2 Jost functions at $t = 0$.

Here we write integral expressions for elements of matrix Jost solutions $J_\pm$, which satisty the following asymptotics:

$$J_\pm(x) \to I, \quad x \to \pm\infty. \tag{2.6}$$

Without loss of generality, we set $t = 0$ and drop t from the list of arguments. All solutions depend on the spectral parameter k. Following Abel's identity, for a fundamental matrix $V(x)$ we have that

$$\det V(x) = \det V(x_0),$$

because $-ik\sigma_3 + Q(u)$ has zero trace. Applying this to (2.3) we find that $\det J(x)$ is a constant for all x. Due to large x asymptotics (2.6) we have that

$$\det J_\pm(x) = 1. \tag{2.7}$$

Define $J_\pm$ as follows

$$J_- := \begin{bmatrix} M & \widehat{M} \end{bmatrix}, \qquad J_+ := \begin{bmatrix} \widehat{N} & N \end{bmatrix}$$

where

$$M := \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}, \quad \widehat{M} := \begin{bmatrix} \widehat{M}_1 \\ \widehat{M}_2 \end{bmatrix}, \quad \widehat{N} := \begin{bmatrix} \widehat{N}_1 \\ \widehat{N}_2 \end{bmatrix}, \quad N := \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}.$$

Using (2.4) we can easily find the following integral expressions, which are available in the literature [27]:

$$\begin{cases} M_1(x) = 1 + \int_{-\infty}^{x} u(y) M_2(y) dy, \\ M_2(x) = -\int_{-\infty}^{x} \overline{u}(y) M_1(y) e^{2ik(x-y)} dy, \end{cases} \tag{2.8}$$

$$\begin{cases} \widehat{M}_1(x) = \int_{-\infty}^{x} u(y) \widehat{M}_2(y) e^{-2ik(x-y)} dy, \\ \widehat{M}_2(x) = 1 - \int_{-\infty}^{x} \overline{u}(y) \widehat{M}_1(y) dy, \end{cases} \tag{2.9}$$

$$\begin{cases} \widehat{N}_1(x) = 1 - \int_{x}^{+\infty} u(y) \widehat{N}_2(y) dy, \\ \widehat{N}_2(x) = \int_{x}^{+\infty} \overline{u}(y) \widehat{N}_1(y) e^{-2ik(y-x)} dy, \end{cases} \tag{2.10}$$

$$\begin{cases} N_1(x) = -\int_{x}^{+\infty} u(y) N_2(y) e^{2ik(y-x)} dy, \\ N_2(x) = 1 + \int_{x}^{+\infty} \overline{u}(y) N_1(y) dy. \end{cases} \tag{2.11}$$

The following lemma has been stated and proven in [27].

Lemma 2.1. *If $u \in L^1(\mathbb{R})$, then for every $k \in \mathbb{R}$ there exist unique bounded solutions $M, \widehat{M}, N, \widehat{N}$. Moreover M, N are analytic functions in k for $Im(k) > 0$ and continuous for $Im(k) \geq 0$, while $\widehat{M}, \widehat{N}$ are analytic functions in k for $Im(k) < 0$, and continuous for $Im(k) \leq 0$.*

Now in the respective planes of analyticity, Jost solutions satisfy $J_{\pm}(x) \to I$ as $|k| \to \infty$, so then

$$M \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \widehat{M} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \widehat{N} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad N \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \text{as } |k| \to \infty. \tag{2.12}$$

2.3 Scattering data

By using $J_- = \begin{bmatrix} M & \widehat{M} \end{bmatrix}$ and $J_+ = \begin{bmatrix} \widehat{N} & N \end{bmatrix}$ we can define two sets of linearly independent solutions of (2.1) as follows:

$$\Phi = \begin{bmatrix} \phi & \widehat{\phi} \end{bmatrix} = \begin{bmatrix} M & \widehat{M} \end{bmatrix} \begin{bmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{bmatrix} = \begin{bmatrix} e^{-ikx} M & e^{ikx} \widehat{M} \end{bmatrix}, \tag{2.13}$$

$$\Psi = \begin{bmatrix} \widehat{\psi} & \psi \end{bmatrix} = \begin{bmatrix} \widehat{N} & N \end{bmatrix} \begin{bmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{bmatrix} = \begin{bmatrix} e^{-ikx} \widehat{N} & e^{ikx} N \end{bmatrix}. \tag{2.14}$$

It follows from (2.6) that $\phi, \widehat{\phi}, \psi, \widehat{\psi}$ satisfy the following boundary conditions:

$$\phi \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ikx}, \quad \widehat{\phi} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ikx}, \qquad \text{as } x \to -\infty, \tag{2.15}$$

$$\psi \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ikx}, \quad \widehat{\psi} \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ikx}, \qquad \text{as } x \to +\infty. \tag{2.16}$$

Since Φ is linearly dependent of Ψ, we introduce a scattering matrix S, entries of which depend on a spectral parameter k,

$$S := \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

from the following scattering relation

$$[\phi, \widehat{\phi}] = [\widehat{\psi}, \psi] \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \qquad \text{or } \Phi = \Psi S. \tag{2.17}$$

Proposition 2.2. *Let* Ψ, Φ *be solutions of* (2.1) *defined in* (2.13)-(2.14). *Then we have the following properties*

$$J_- = J_+ E S E^{-1}, \qquad \det S = 1, \tag{2.18}$$

where

$$E = e^{-ikx\sigma_3} = \begin{bmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{bmatrix}$$

Proof. Substituting (2.13), (2.14) into (2.17) we have the following

$$J_- E = J_+ E S.$$

Since E is invertible, this yields

$$J_- = J_+ E S E^{-1}.$$

Using (2.7) we get

$$1 = \det(J_-) = \det(J_+) \det(E) \det(S) \det(E^{-1}) = \det(S).$$

□

Equation (2.17) can be also rewritten as

$$\begin{cases} \phi = b\psi + a\widehat{\psi} \\ \widehat{\phi} = d\psi + c\widehat{\psi}. \end{cases} \tag{2.19}$$

From (2.13), (2.14) and (2.19) we can obtain relations between $M, \widehat{M}, \widehat{N}, N$:

$$\begin{cases} M = a\widehat{N} + be^{2ikx}N, \\ \widehat{M} = dN + ce^{-2ikx}\widehat{N}, \\ N = a\widehat{M} - ce^{-2ikx}M, \\ \widehat{N} = dM - be^{2ikx}\widehat{M}, \end{cases} \tag{2.20}$$

Lemma 2.3. *Entries of scattering matrix S satisfy the following expressions*

$$\begin{cases} a = 1 + \int_{\mathbb{R}} u(y)M_2(y)dy, \\ b = -\int_{\mathbb{R}} \overline{u}(y)M_1(y)e^{-2iky}dy, \\ c = \int_{\mathbb{R}} u(y)\widehat{M}_2(y)e^{2iky}dy, \\ d = 1 - \int_{\mathbb{R}} \overline{u}(y)\widehat{M}_1(y)dy. \end{cases} . \tag{2.21}$$

Moreover, if $u \in L^1(\mathbb{R})$, then a is analytic in $\mathbb{C}_+$ and d is analytic in $\mathbb{C}_-$.

Proof. Consider the first equation in (2.20) but write it using equations (2.8), (2.9), (2.10), (2.11)

$$\begin{bmatrix} 1 + \int_{-\infty}^{x} u(y)M_2(y)dy \\ -\int_{-\infty}^{x} \overline{u}(y)\widehat{M}_1(y)e^{2ik(x-y)}dy \end{bmatrix} = a \begin{bmatrix} 1 - \int_{x}^{+\infty} u(y)\widehat{N}_2(y)dy \\ \int_{x}^{+\infty} \overline{u}(y)\widehat{N}_1(y)e^{-2ik(y-x)}dy \end{bmatrix} + be^{2ikx} \begin{bmatrix} -\int_{x}^{+\infty} u(y)N_2(y)e^{2ik(y-x)}dy, \\ 1 + \int_{x}^{+\infty} \overline{u}(y)N_1(y)dy \end{bmatrix}$$

Take the limit of $x \to +\infty$ to obtain

$$\begin{bmatrix} 1 + \int_{\mathbb{R}} u(y)M_2(y)dy \\ -\int_{\mathbb{R}} \overline{u}(y)M_1(y)e^{2ik(x-y)}dy \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + be^{2ikx} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

from where we can conclude expressions for a, b as in (2.21). To find c, d we follow the same procedure, i.e. we take the limit of $x \to +\infty$ in the second equation in

(2.20). Now we are left to explain analyticity of scattering data. Consider integral expressions for a, d

$$\begin{cases} a = 1 + \int_{\mathbb{R}} u(y) M_2(y) dy, \\ d = 1 - \int_{\mathbb{R}} \overline{u}(y) \widehat{M}_1(y) dy, \end{cases}$$

where $M_2, \widehat{M}_1$ are analytic in $\mathbb{C}_{\pm}$ respectively by Lemma 2.1. Since $u \in L^1(\mathbb{R})$, a, d are analytic in the domains of analyticity of $M_2, \widehat{M}_1$. □

Remark 2.4. *Consider integral expression for* b

$$b = -\int_{\mathbb{R}} \overline{u}(y) M_1(y) e^{-2iky} dy,$$

where M_1 *is analytic in* $\mathbb{C}_+$*. The exponential term in the integral expression is given by*

$$e^{-2i(Re(k)+iIm(k))y} = e^{-2iRe(k)y} e^{2Im(k)y}.$$

If $Im(k) > 0$*, it diverges when* $y \to +\infty$*, thus,* b *is not analytic in* $\mathbb{C}_+$*. If* $Im(k) < 0$*, then it diverges when* $y \to -\infty$*, then* b *is not analytic in* $\mathbb{C}_-$*. Therefore, the integral expression for* b *is only defined for* $k \in \mathbb{R}$*. Analogously,* c *is not analytic in* $\mathbb{C}$ *and is only defined for* $k \in \mathbb{R}$*.*

Lemma 2.5. *Columns of* Φ, Ψ *satisfy the following symmetry:*

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} \overline{\widehat{\phi}}_2 \\ -\overline{\widehat{\phi}}_1 \end{bmatrix}, \qquad \begin{bmatrix} \widehat{\psi}_1 \\ \widehat{\psi}_2 \end{bmatrix} = \begin{bmatrix} \overline{\psi}_2 \\ -\overline{\psi}_1 \end{bmatrix}, \qquad \text{for every } x \in \mathbb{R}.$$

Moreover, scattering matrix S *can be written as*

$$S = \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix}. \tag{2.22}$$

Proof. From (2.1) we have

$$\begin{cases} \widehat{\phi}_1' + ik\widehat{\phi}_1 = u\phi_2, \\ \widehat{\phi}_2' - ik\widehat{\phi}_2 = -\overline{u}\phi_1. \end{cases}$$

Taking complex conjugation we obtain

$$\begin{cases} \overline{\widehat{\phi}}_2' + ik\overline{\widehat{\phi}}_2 = -u\overline{\widehat{\phi}}_1, \\ \overline{\widehat{\phi}}_1' - ik\overline{\widehat{\phi}}_1 = \overline{u}\overline{\widehat{\phi}}_2, \end{cases}$$

which shows that $(\overline{\widehat{\phi}}_2, -\overline{\widehat{\phi}}_1)$ solves the same problem as (ϕ_1, ϕ_2) and satisfy the same boundary conditions. By Lemma 2.1, the solution (ϕ_1, ϕ_2) is uniquely defined, hence $\phi_1 = \overline{\widehat{\phi}}_2, \phi_2 = -\overline{\widehat{\phi}}_1$. Similarly, we obtain $\psi_1 = -\widehat{\psi}_2, \psi_2 = \widehat{\psi}_1$.

To show relation between entries of scattering data, we combine symmetry of elements of Φ, Ψ with (2.13), (2.14) to obtain the following symmetry between Jost solutions

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} = \begin{bmatrix} \overline{\widehat{M}}_2 \\ -\overline{\widehat{M}}_1 \end{bmatrix}, \qquad \begin{bmatrix} \widehat{N}_1 \\ \widehat{N}_2 \end{bmatrix} = \begin{bmatrix} \overline{N}_2 \\ -\overline{N}_1 \end{bmatrix}.$$

Using the above we can rewrite (2.21) as

$$\begin{cases} a = 1 + \int_{\mathbb{R}} u(y)M_2(y)dy, \\ b = -\int_{\mathbb{R}} \overline{u}(y)M_1(y)e^{-2iky}dy, \\ c = \int_{\mathbb{R}} u(y)\overline{M}_1(y)e^{2iky}dy, \\ d = 1 + \int_{\mathbb{R}} \overline{u}(y)\overline{M}_2(y)dy, \end{cases}$$

from where we can conclude the relation

$$a = \overline{d},$$
$$b = -\overline{c},$$

which yields (2.22). □

2.4 Adjoint spectral problem

Lemma 2.6. *Let J satisfy the spectral problem in* (2.4). *The adjoint spectral problem is given by*

$$K_x = -ik[\sigma_3, K] - KQ(u), \tag{2.23}$$

where the adjoint equation is defined with respect to the inner product (without complex conjugation)

$$f, g \in L^2(\mathbb{R}) : \langle f, g \rangle := \int_{\mathbb{R}} f(x)g(x)dx. \tag{2.24}$$

Proof. Let J be written in the component form:

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}. \tag{2.25}$$

Thus, using (2.25) we can write (2.4) explicitly

$$\begin{bmatrix} J'_{11} & J'_{12} \\ J'_{21} & J'_{22} \end{bmatrix} = \begin{bmatrix} 0 & -2ikJ_{12} \\ 2ikJ_{21} & 0 \end{bmatrix} + \begin{bmatrix} uJ_{21} & uJ_{22} \\ -\overline{u}J_{11} & -\overline{u}J_{12} \end{bmatrix}.$$

This results in following equations:

$$\begin{cases} J'_{11} = uJ_{21}, \\ J'_{21} = 2ikJ_{21} - \overline{u}J_{11}, \\ J'_{12} = -2ikJ_{12} + uJ_{22}, \\ J'_{22} = -\overline{u}J_{12}. \end{cases} \tag{2.26}$$

We write entries of K similar to (2.25):

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}.$$

Then KJ' is the following

$$\begin{bmatrix} K_{11}J'_{11} + K_{12}J'_{21} & K_{11}J'_{12} + K_{12}J'_{22} \\ K_{21}J'_{11} + K_{22}J'_{21} & K_{21}J'_{12} + K_{22}J'_{22} \end{bmatrix} \tag{2.27}$$

Using first two equations of (2.26) in (2.27) and integrating both sides we can see that

$$\int_{\mathbb{R}} \Big(K_{11}J'_{11} + K_{12}J'_{21} \Big) dx = \int_{\mathbb{R}} \Big(uK_{11}J_{21} + 2ikK_{12}J_{21} - \overline{u}K_{12}J_{11} \Big) dx.$$

Using integration by parts we obtain the following

$$-\int_{\mathbb{R}} \Big(K'_{11}J_{11} + K'_{12}J_{21} \Big) dx = \int_{\mathbb{R}} \Big[\Big(2ikK_{21} + uK_{11} \Big) J_{21} - \overline{u}K_{12}J_{11} \Big] dx.$$

From above we can derive equations for K_{11} and K_{12}:

$$\begin{cases} K'_{11} = \overline{u}K_{12}, \\ K'_{12} = -2ikK_{12} - uK_{11}. \end{cases} \tag{2.28}$$

Using last two equations of (2.26) in (2.27) and integrating both sides we can see that

$$\int_{\mathbb{R}} \Big(K_{21}J'_{12} + K_{22}J'_{22} \Big) dx = \int_{\mathbb{R}} \Big[\Big(-2ikK_{21} - \overline{u}K_{22} \Big) J_{12} + uK_{21}J_{22} \Big] dx.$$

Using integration by parts as before we obtain

$$-\int_{\mathbb{R}} \Big(K'_{21}J_{12} + K'_{22}J_{22} \Big) dx = \int_{\mathbb{R}} \Big[\Big(-2ikK_{21} - \overline{u}K_{22} \Big) J_{12} + uK_{21}J_{22} \Big] dx,$$

from which the following equations can be derived:

$$\begin{cases} K_{21}' = 2ikK_{21} + \overline{u}K_{22} \\ K_{22}' = -uK_{21}. \end{cases} \tag{2.29}$$

Now combining (2.28) and (2.29) we can finally write the adjoint equation for K in the form (2.23). □

Corollary 2.7. *Solution to the adjoint spectral problem* (2.23) *can be expressed as* $K = J^{-1}$.

Proof. Using (2.4) and the following property

$$(JJ^{-1})_x = 0$$

we can see that J^{-1} satisfy

$$(J^{-1})_x = -ik[\sigma_3, J^{-1}] - J^{-1}Q, \tag{2.30}$$

hence $K = J^{-1}$ up to constant multiplication. □

Let us define inverse matrices for J_- and J_+ as follows

$$J_-^{-1} := \begin{bmatrix} M_1^* & M_2^* \\ \widehat{M}_1^* & \widehat{M}_2^* \end{bmatrix}, \qquad J_+^{-1} := \begin{bmatrix} \widehat{N}_1^* & \widehat{N}_2^* \\ N_1^* & N_2^* \end{bmatrix}.$$

Using (2.30) we can find integral expressions for entries of J_-^{-1} and J_+^{-1}:

$$\begin{cases} M_1^*(x) = 1 + \int_{-\infty}^{x} \overline{u}(y) M_2^*(y) dy, \\ M_2^*(x) = -\int_{-\infty}^{x} u(y) M_1^*(y) e^{-2ik(x-y)} dy, \end{cases} \tag{2.31}$$

$$\begin{cases} \widehat{M}_1^*(x) = \int_{-\infty}^{x} \overline{u}(y)\widehat{M}_2^* e^{2ik(x-y)}dy, \\ \widehat{M}_2^*(x) = 1 - \int_{-\infty}^{x} u(y)\widehat{M}_1^*(y)dy, \end{cases} \tag{2.32}$$

$$\begin{cases} \widehat{N}_1^* = 1 - \int_{x}^{+\infty} \overline{u}(y)\widehat{N}_2^*(y)dy, \\ \widehat{N}_2^* = \int_{x}^{+\infty} u(y)\widehat{N}_1^*(y)e^{2ik(y-x)}dy, \end{cases} \tag{2.33}$$

$$\begin{cases} N_1^* = -\int_{x}^{+\infty} \overline{u}(y)N_2^*(y)e^{-2ik(y-x)}dy, \\ N_2^* = 1 + \int_{x}^{+\infty} u(y)N_1^*(y)dy. \end{cases} \tag{2.34}$$

Comparing systems of equations (2.8) and (2.31), it is not hard to see that solution of (2.31) is complex conjugate of the solution of (2.8) for $k \in \mathbb{R}$. The same is true for the systems (2.9) and (2.32), (2.10) and (2.33), (2.11) and (2.34). Hence, the asterisk is equivalent to complex conjugation for $M_{1,2}^* = \overline{M}_{1,2}, \widehat{M}_{1,2}^* = \overline{\widehat{M}}_{1,2}, N_{1,2}^* = \overline{N}_{1,2}, \widehat{N}_{1,2}^* = \overline{\widehat{N}}_{1,2}$. Therefore we can say that the asterisk denotes Hermite conjugation.

We denote rows of J_-^{-1} and J_+^{-1} as follows

$$M^* := \begin{bmatrix} M_1^* & M_2^* \end{bmatrix}, \qquad \widehat{M}^* := \begin{bmatrix} \widehat{M}_1^* & \widehat{M}_2^* \end{bmatrix},$$
$$\widehat{N}^* := \begin{bmatrix} \widehat{N}_1^* & \widehat{N}_2^* \end{bmatrix}, \qquad N^* = \begin{bmatrix} N_1^* & N_2^* \end{bmatrix}.$$

Using Lemma 2.1 and looking at the integral equations (2.31), (2.32), (2.33) and (2.34) we arrive to the following result

Lemma 2.8. *If $u \in L^1(\mathbb{R})$, then for every $k \in \mathbb{R}$ there exist unique bounded solutions $M^*, \widehat{M}^*, \widehat{N}^*, N^*$. Moreover M^*, N^* are analytic functions for $Im(k) < 0$ and continuous for $Im(K) \leq 0$, while $\widehat{M}^*, \widehat{N}^*$ are analytic functions for $Im(k) > 0$ and continuous for $Im(k) \geq 0$.*

Hence the large k asymptotics of $M^*, \widehat{M}^*, \widehat{N}^*, N^*$ are similar to those for $M, \widehat{M}, N, \widehat{N}$:

$$M^* \to \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \widehat{M}^* \to \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \widehat{N}^* \to \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad N^* \to \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{2.35}$$

Similarly to (2.20) with $d = \overline{a}$ and $c = -\overline{b}$, we can derive adjoint relations by applying complex conjugation. These are only true for $k \in \mathbb{R}$ since $b, \overline{b}$ do not extend to a complex plane.

$$\begin{cases} M^* = \overline{b}e^{-2ikx}N^* + \overline{a}\widehat{N}^*, \\ \widehat{M}^* = aN^* - be^{2ikx}\widehat{N}^*, \\ N^* = be^{2ikx}M^* + \overline{a}\widehat{M}^*, \\ \widehat{N}^* = aM^* - \overline{b}e^{-2ikx}\widehat{M}^*. \end{cases} \tag{2.36}$$

Remark 2.9. *From Corollary 2.7 we can deduce relations between* Φ, Ψ *and solutions of adjoint spectral problem:*

$$\Psi^{-1} = (J_+E)^{-1} = E^{-1}J_+^{-1} = E^{-1}\begin{bmatrix} \widehat{N}^* \\ N^* \end{bmatrix} = \begin{bmatrix} \widehat{N}^* e^{ikx} \\ N^* e^{-ikx} \end{bmatrix} = \begin{bmatrix} \widehat{\psi}^* \\ \psi^* \end{bmatrix},$$

and

$$\Phi^{-1} = (J_-E)^{-1} = E^{-1}J_-^{-1} = E^{-1}\begin{bmatrix} M^* \\ \widehat{M}^* \end{bmatrix} = \begin{bmatrix} M^* e^{ikx} \\ \widehat{M}^* e^{-ikx} \end{bmatrix} = \begin{bmatrix} \phi^* \\ \widehat{\phi}^* \end{bmatrix}, \tag{2.37}$$

where Ψ^{-1}, Φ^{-1} *satisfy the following adjoint problem*

$$(\Psi^{-1})_x = ik\Psi^{-1}\sigma_3 - \Psi^{-1}Q, \tag{2.38}$$

which is easily obtained from (2.30)*.*

Proposition 2.10. *The sets of solutions of* (2.8)-(2.11) *and* (2.31)-(2.34) *satisfy for* $k \in \mathbb{R}$*:*

$$\begin{cases} M^*M = \widehat{M}^*\widehat{M} = 1, \\ \widehat{M}^*M = M^*\widehat{M} = 0 \end{cases} \tag{2.39}$$

and

$$\begin{cases} \widehat{N}^*\widehat{N} = N^*N = 1, \\ \widehat{N}^*N = N^*\widehat{N} = 0. \end{cases} \tag{2.40}$$

Proof. It follows from $J_-^{-1}J_- = I$ that

$$\begin{bmatrix} M^* \\ \widehat{M}^* \end{bmatrix} \begin{bmatrix} M & \widehat{M} \end{bmatrix} = \begin{bmatrix} M^*M & M^*\widehat{M} \\ \widehat{M}^*M & \widehat{M}^*\widehat{M} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

From above we obtain (2.39). It follows from $J_+^{-1}J_+ = I$ that

$$\begin{bmatrix} \widehat{N}^* \\ N^* \end{bmatrix} \begin{bmatrix} \widehat{N} & N \end{bmatrix} = \begin{bmatrix} \widehat{N}^*\widehat{N} & \widehat{N}^*N \\ N^*\widehat{N} & N^*N \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

from which we obtain (2.40). □

2.5 Riemann-Hilbert problem

A classical Riemann-Hilbert problem aims to find a pair of functions that are analytic in two regions separated by a simple contour. Since our functions M, N and their adjoints M^*, N^* are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, we can formulate a Riemann-Hilbert problem in the following way by introducing new matrices P^-, P^+:

$$P^- := \begin{bmatrix} M^* \\ N^* \end{bmatrix}, P^+ := \begin{bmatrix} M & N \end{bmatrix}. \tag{2.41}$$

Proposition 2.11. *Matrices P^- and P^+ satisfy the following property:*

$$\det P^- = \overline{a}, \quad \det P^+ = a.$$

Proof. Using first equation from (2.36) we proceed as follows

$$\begin{aligned}
\det P^- &= \det \begin{bmatrix} \overline{b}e^{-2ikx}N^* + \overline{a}\widehat{N}^* \\ N^* \end{bmatrix} \\
&= \det \begin{bmatrix} \overline{b}e^{-2ikx}N_1^* + \overline{a}\widehat{N}_1^* & \overline{b}e^{-2ikx}N_2^* + \overline{a}\widehat{N}_2^* \\ N_1^* & N_2^* \end{bmatrix} \\
&= \overline{b}e^{2ikx}(N_1^*N_2^* - N_2^*N_1^*) + \overline{a}(\widehat{N}_1^*N_2^* - N_1^*\widehat{N}_2^*) \\
&= \overline{a}\det J_+^{-1} = \overline{a}.
\end{aligned}$$

Using first equation of (2.20) and following the same computations we get $\det P^+ = a$. ☐

Riemann Hilbert problem is defined by the conditions for $k \in \mathbb{R}$:

$$P^-P^+ = G. \tag{2.42}$$

From (2.41) we find that (2.42) can be written as

$$\begin{bmatrix} M^*M & M^*N \\ N^*M & N^*N \end{bmatrix} = G.$$

From (2.20), (2.39), (2.40) we obtain

$$\begin{bmatrix} 1 & \overline{b}e^{-2ikx} \\ be^{2ikx} & 1 \end{bmatrix} = G, \tag{2.43}$$

which can be rewritten as

$$G = I + \begin{bmatrix} 0 & \overline{b}e^{-2ikx} \\ be^{2ikx} & 0 \end{bmatrix} = I + \Delta.$$

Combining above with (2.42) we obtain

$$P^- P^+ = I + \Delta$$

If $a, \bar{a}$ do not have zeros on $\mathbb{R}$, P^+, P^- are invertible on $\mathbb{R}$. Multiplying both sides of the equation by $(P^-)^{-1}$ leads to Riemann-Hilbert problem for $k \in \mathbb{R}$:

$$P^+ - (P^-)^{-1} = (P^-)^{-1}\Delta. \tag{2.44}$$

The jump condition (2.44) defines analytic continuation of P^+ in the upper half-plane and $(P^-)^{-1}$ in the lower half-plane, provided that $a, \bar{a}$ do not have zeros in the upper, lower half-planes. We denote the two half-planes by $\mathbb{C}_+, \mathbb{C}_-$.

2.6 Solution to the regular Riemann-Hilbert problem

The Riemann-Hilbert problem accosiated with the jump condition (2.44) is considered regular if $\det P^\pm \neq 0$ in the regions of $\mathbb{C}$, where they are analytically extended.

Lemma 2.12. *The regular Riemann-Hilbert problem* (2.42) *has a unique solution subject to the boundary conditions $P^\pm \to I$ as $|k| \to \infty$ in their domains of analyticity.*

Proof. Suppose (2.42) has two solutions $P^\pm, \tilde{P}^\pm$. Then we have that

$$P^- P^+ = \tilde{P}^- \tilde{P}^+.$$

Multiplying both sides by $(P^-)^{-1}$ from the left and by $(\tilde{P}^+)^{-1}$ from the right we obtain

$$P^+(\tilde{P}^+)^{-1} = (P^-)^{-1}\tilde{P}^-. \tag{2.45}$$

Since Riemann Hilbert problem is regular, $\det P^\pm, \det \tilde{P}^\pm$ are nonzero in their domains of analyticity. Left hand side of (2.45) is analytic in $\mathbb{C}_+$, while right hand side of (2.45) is analytic in $\mathbb{C}_-$. Hence, they are equal to each other on the real line. This means that they are equal and analytic in $\mathbb{C}$. Due to the boundary conditions $P^{\pm} \to I$ as $|k| \to \infty$ and Liouville's theorem, both sides are equal to I in $\mathbb{C}$, which means that $P^+ = \tilde{P}^+$ and $P^- = \tilde{P}^-$. Thus, solution to the regular Riemann-Hilbert problem (2.42) is unique. □

The solution of the Riemann Hilbert problem (2.44) can be written by using Plemelj formula. Suppose that Γ is a line dividing the complex plane to D_+, D_- and f is continuous on Γ. Also suppose that ϕ is sectionally analytic in D_+, D_- and is vanishing at infinity. If $\phi_+ - \phi_- = f$ on Γ, then Plemelj formula yields

$$\phi(k) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - k} d\xi. \tag{2.46}$$

The regular Riemann-Hilbert problem (2.44) satisfies all the conditions stated above with $\phi_+ = P^+ - I, \phi_- = (P^-)^{-1} - I$, and $f = (P^-)^{-1}\Delta$. Since $P^{\pm}$ is analytic sectionally (P^+ on $\mathbb{C}_+$ and P^- on $\mathbb{C}_-$), Γ is $\mathbb{R}$ and $(P^-)^{-1}\Delta$ is continuous on Γ we have that

$$\begin{cases} (P^-)^{-1} - I = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(P^-)^{-1}\Delta}{\xi - (k - i0)} d\xi, \\ P^+ - I = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(P^-)^{-1}\Delta}{\xi - (k + i0)} d\xi. \end{cases}$$

These are solutions of the regular Riemann Hilbert problem (2.44) with $\det P^{\pm} \neq 0$ in $\mathbb{C}_{\pm}$ subject to the boundary conditions $P^{\pm} \to I$ as $|k| \to \infty$ in $\mathbb{C}_{\pm}$.

Chapter 3

Completeness of eigenfunctions

In this section we present a complete set of eigenfunctions for the Zakharov-Shabat spectral problem (1.2). The corresponding result is given by the following theorem.

Theorem 1. *The set of eigenfunctions* $\{\phi, \widehat{\phi}\}$ *defined by* (2.13) *is complete, i.e. every* $f(x) \in L^2(\mathbb{R})$ *can be written as follows:*

$$f(x) = \int_{\mathbb{R}} \Big[\tilde{c}(k)\phi(x,k) + \tilde{d}(k)\widehat{\phi}(x,k) \Big] dk, \tag{3.1}$$

where $\tilde{c}(k), \tilde{d}(k)$ *are given by*

$$\tilde{c}(k) = \frac{1}{2\pi a(k)} \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 f(y)dy, \qquad \tilde{d}(k) = -\frac{1}{2\pi \overline{a}(k)} \int_{\mathbb{R}} \psi^*(y,k)\sigma_3 f(y)dy, \tag{3.2}$$

where we use adjoint eigenfunctions $\{\psi^*, \widehat{\psi}^*\}$ *defined by* (2.37)

To prove the above Theorem 1 we proceed as follows:

- Introduce special functions $\mathcal{R}^+, \mathcal{R}^-$ that consists of eigenfunctions ϕ and $\widehat{\phi}$.
- Integrate $\mathcal{R}^+ + \mathcal{R}^-$ using contour integration to obtain completeness relation for the set of eigenfunctions $\{\phi, \widehat{\phi}\}$.
- Compute inner products between eigenfunctions and adjoint eigenfunctions.

Let us introduce matrix functions

$$\mathcal{R}^{+}(x,y,k) := \chi^{+}(x,k)diag\Big[\theta(y-x), -\theta(x-y)\Big](\chi^{+})^{-1}(y,k), \tag{3.3}$$

$$\mathcal{R}^{-}(x,y,k) := \chi^{-}(x,k)diag\Big[\theta(x-y), -\theta(y-x)\Big](\chi^{-})^{-1}(y,k) \tag{3.4}$$

where $\chi^{\pm}$ are

$$\chi^{+} = \begin{bmatrix} M & N \end{bmatrix} E, \tag{3.5}$$

$$\chi^{-} = \begin{bmatrix} \widehat{N} & \widehat{M} \end{bmatrix} E, \tag{3.6}$$

and $\theta(x)$ is a Heavyside step function,

$$\theta(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} .$$

Proposition 3.1. *Let* $\chi^{\pm}$ *be defined as in* (3.5), (3.6), *then the following is true*

$$det\chi^{+} = a, \tag{3.7}$$

$$det\chi^{-} = \overline{a}. \tag{3.8}$$

Proof. We obtain from (3.5) that

$$\begin{aligned} \det \chi^{+} = \det \begin{bmatrix} M & N \end{bmatrix} \cdot \det E &= \det \begin{bmatrix} a\widehat{N} + be^{2ikx}N & N \end{bmatrix} \cdot 1 \\ &= \det \begin{bmatrix} a\widehat{N}_1 + be^{2ikx}N_1 & N_1 \\ a\widehat{N}_2 + be^{2ikx}N_2 & N_2 \end{bmatrix} \\ &= a\widehat{N}_1 N_2 + be^{2ikx}N_1N_2 - (a\widehat{N}_2N_1 + be^{2ikx}N_2N_1) \\ &= a(\widehat{N}_1N_2 - \widehat{N}_2N_1) = a \det \begin{bmatrix} \widehat{N} & N \end{bmatrix} = a \det J_{+} = a, \end{aligned}$$

where we used (2.7) and (2.20). Proof for (3.8) is analogous. □

Lemma 3.2. *Let* $\mathcal{R}^{\pm}$ *be defined as in* (3.3), (3.4) *and suppose that* $\det \chi^{\pm} \neq 0$. *Then the set of eigenfunctions* $\{\phi, \widehat{\phi}\}$ *satisfies the following completeness relation:*

$$\delta(x-y)\sigma_3 = \frac{1}{2\pi}\int_{\mathbb{R}} \left[\frac{1}{a}\phi(x,k)\widehat{\psi}^*(y,k) - \frac{1}{\bar{a}}\widehat{\phi}(x,k)\psi^*(y,k)\right]dk. \tag{3.9}$$

Proof. Due to large-k asymptotics of (2.12) of $M, N, \widehat{M}, \widehat{N}$ we have the following boundary conditions for $\chi^{\pm}$

$$\chi^{\pm}(x,k) \to E.$$

Thus, we have the following asymptotics for $R^{\pm}$ in $\mathbb{C}_{\pm}$ as $|k| \to \infty$:

$$\mathcal{R}^{+}(x,y,k) \to diag\Big[\theta(y-x)e^{ik(y-x)}, -\theta(x-y)e^{ik(x-y)}\Big] = \rho^{+}(x,y,k),$$

$$\mathcal{R}^{-}(x,y,k) \to diag\Big[\theta(x-y)e^{ik(y-x)}, -\theta(y-x)e^{ik(x-y)}\Big] = \rho^{-}(x,y,k),$$

which are bounded in $\mathbb{C}_{\pm}$ as $|k| \to \infty$. Let $C_R^{\pm}$ be semi-circles of radius R in $\mathbb{C}_{\pm}$. We define two contours: one is $[-R,R] \cup C_R^{+}$ in $\mathbb{C}_{+}$, the other is $[-R,R] \cup C_R^{-}$ in $\mathbb{C}_{-}$ as shown in Fig 3.1. Let us begin with integrating $\mathcal{R}^{+}(x,y,k)$ on the closed contour $[-R,R] \cup C_R^{+}$. By Cauchy theorem, since $\mathcal{R}^{+}(k)$ is analytic in this closed contour, we have the following

$$\int_{[-R,R]\cup(C_R^{+})} \mathcal{R}^{+}(x,y,k)dk = 0. \tag{3.10}$$

We can rewrite the integral from (3.10) in the limit $R \to \infty$ as follows:

$$\int_{\mathbb{R}} \mathcal{R}^{+}(x,y,k)dk + \lim_{R\to\infty}\int_{C_{R+}} \mathcal{R}^{+}(x,y,k)dk = 0.$$

Using the limiting asymptotics $\mathcal{R}^{\pm} \to \rho^{\pm}$ as $|k| \to \infty$ we obtain that

$$\int_{\mathbb{R}} \mathcal{R}^{+}(x,y,k)dk = -\lim_{R\to\infty}\int_{C_R^{+}} diag\Big[\theta(y-x)e^{ik(y-x)}, -\theta(x-y)e^{ik(x-y)}\Big]dk.$$

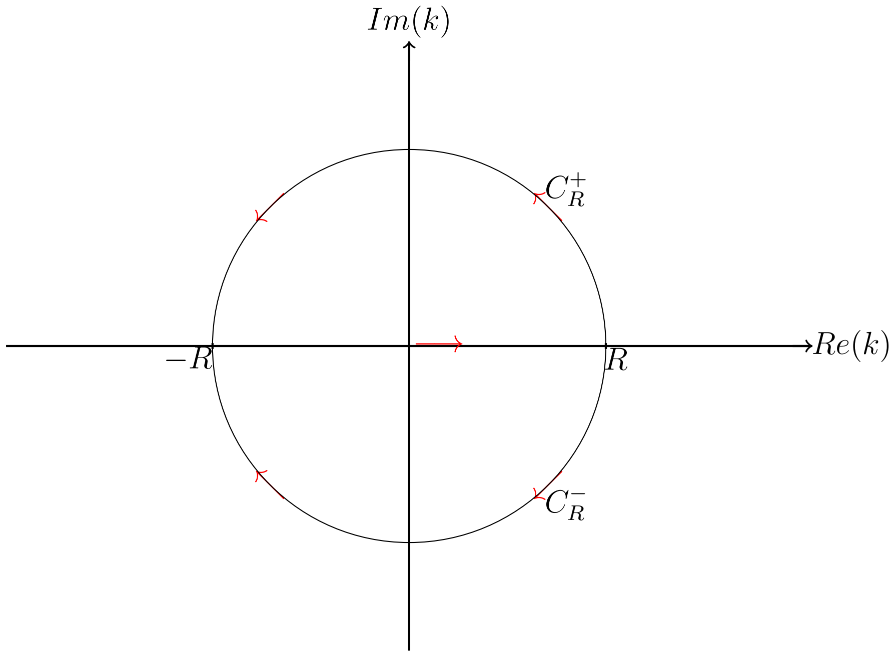

FIGURE 3.1: Contours of integration for $\mathcal{R}^{\pm}$.

Using the fact that $diag\Big[\theta(y-x)e^{ik(y-x)}, -\theta(x-y)e^{ik(x-y)}\Big]$ is analytic in $\mathbb{C}_+$, the relation above can be rewritten by integrating this quantity on the real line:

$$\int_{\mathbb{R}} \mathcal{R}^+(x,y,k)dk = \int_{\mathbb{R}} diag\Big[\theta(y-x)e^{ik(y-x)}, -\theta(x-y)e^{ik(x-y)}\Big]dk.$$

Analogously, for $\mathcal{R}^-(x,y,k)$ we have

$$\int_{\mathbb{R}} \mathcal{R}^-(x,y,k)dk = \int_{\mathbb{R}} diag\Big[\theta(x-y)e^{ik(y-x)}, -\theta(y-x)e^{ik(x-y)}\Big]dk.$$

Adding above integrals to each other we have the following

$$\int_{\mathbb{R}} \Big(\mathcal{R}^+(x,y,k) + \mathcal{R}^-(x,y,k)\Big)dk = \int_{\mathbb{R}} diag\Big[e^{ik(y-x)}, -e^{ik(x-y)}\Big]dk = 2\pi\delta(x-y)\sigma_3. \tag{3.11}$$

Now, to conclude a closure relation, we just need to rewrite $\mathcal{R}^+, \mathcal{R}^-$ using definitions (3.3), (3.4)

$$\mathcal{R}^+(x,y,k) = \chi^+(x,k)diag[1,0](\chi^+)^{-1}(y,k) - \theta(x-y)\chi^+(x,k)(\chi^+)^{-1}(y,k)$$

and

$$\mathcal{R}^-(x,y,k) = -\chi^-(x,k)diag[0,1](\chi^-)^{-1}(y,k) + \theta(x-y)\chi^-(x,k)(\chi^-)^{-1}(y,k).$$

Since on the real axis both $\chi^+(x,k)$ and $\chi^-(x,k)$ are fundamental solutions of system (2.1), χ^+ and χ^- are linear combinations of each other so that

$$\chi^+(x,k)(\chi^+)^{-1}(y,k) = \chi^-(x,k)(\chi^-)^{-1}(y,k).$$

Therefore, the left hand side of (3.11) is

$$\begin{aligned} &\int_{\mathbb{R}} \Big(\mathcal{R}^+(x,y,k) + \mathcal{R}^-(x,y,k)\Big)dk \qquad (3.12)\\ &= \int_{\mathbb{R}} \Big[\chi^+(x,k)diag[1,0](\chi^+)^{-1}(y,k) - \chi^-(x,k)diag[0,1](\chi^-)^{-1}(y,k)\Big]dk. \end{aligned}$$

To simplify the latter equation, we use solutions of adjoint equation in Lemma 2.8 and Corollary 2.7. Since χ^+ is a solution to equation (2.1), then $(\chi^+)^{-1}$ is a solution of the corresponding adjoint equation (2.38).

We claim that

$$(\chi^+)^{-1} = \begin{bmatrix} \frac{1}{a}\widehat{N}^* e^{ikx} \\ \frac{1}{a}\widehat{M}^* e^{-ikx} \end{bmatrix} = \begin{bmatrix} \frac{1}{a}\widehat{\psi}^* \\ \frac{1}{a}\widehat{\phi}^* \end{bmatrix}. \qquad (3.13)$$

To confirm (3.13), let us multiply it by its inverse from the right

$$(\chi^+)^{-1}(\chi^+) = \begin{bmatrix} \frac{\widehat{N}^*}{a} e^{ikx} \\ \frac{\widehat{M}^*}{a} e^{-ikx} \end{bmatrix} \begin{bmatrix} Me^{-ikx} & Ne^{ikx} \end{bmatrix} = \frac{1}{a} \begin{bmatrix} \widehat{N}^* M & \widehat{N}^* N e^{2ikx} \\ \widehat{M}^* M e^{-2ikx} & \widehat{M}^* N \end{bmatrix}. \qquad (3.14)$$

Using (2.39) and (2.40) we see that (3.14) is now

$$(\chi^+)^{-1}(\chi^+) = \frac{1}{a}\begin{bmatrix} \widehat{N}^* M & 0 \\ 0 & \widehat{M}^* N \end{bmatrix}.$$

Substituting second and fourth equations from (2.36) in the above, we can see the following

$$\frac{1}{a}\begin{bmatrix} aM^*M - \overline{b}e^{-2ikx}\widehat{M}^*M & 0 \\ 0 & aN^*N - be^{2ikx}\widehat{N}^*N \end{bmatrix} = \frac{1}{a}\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = I.$$

Thus, the claim (3.13) has been proved.

Analogously, we can prove that

$$(\chi^-)^{-1} = \begin{bmatrix} \frac{1}{a}M^*e^{ikx} \\ \frac{1}{a}N^*e^{-ikx} \end{bmatrix} = \begin{bmatrix} \frac{1}{a}\phi^* \\ \frac{1}{a}\psi^* \end{bmatrix}. \tag{3.15}$$

Substituting (3.13), (3.15) (3.5), (3.6) into (3.12) we obtain a closure relation (3.9). □

Lemma 3.3. *The sets of eigenfunctions* $\{\phi, \widehat{\phi}\}$ *and the adjoint eigenfunctions* $\{\psi^*, \widehat{\psi}^*\}$ *are orthogonal according to the following orthogonality conditions*

$$\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\sigma_3\phi(x,k')dx = 2\pi a(k)\delta(k-k'), \tag{3.16}$$

$$\int_{\mathbb{R}} \psi^*(x,k)\sigma_3\widehat{\phi}(x,k')dx = -2\pi\overline{a}(k)\delta(k-k'), \tag{3.17}$$

$$\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\sigma_3\widehat{\phi}(x,k')dx = 0, \tag{3.18}$$

$$\int_{\mathbb{R}} \psi^*(x,k)\sigma_3\phi(x,k')dx = 0. \tag{3.19}$$

Proof. Let us prove (3.16) as an example. Firstly, we write the equation (2.1) for $\phi(x,k')$

$$\phi_x(x,k') = -ik'\sigma_3\phi(x,k') + Q(u)\phi(x,k').$$

Multiply the above by $\widehat{\psi}^*(x,k)$ and integrate over $\mathbb{R}$

$$\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\phi_x(x,k')dx = -ik'\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\sigma_3\phi(x,k')dx + \int_{\mathbb{R}} \widehat{\psi}^*(x,k)Q(u)\phi(x,k')dx. \tag{3.20}$$

Integrating by parts the left hand side gives us the following

$$\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\phi_x(x,k')dx = \Big[\widehat{\psi}^*(x,k)\phi(x,k')\Big]_{x\to-\infty}^{x\to+\infty} - \int_{\mathbb{R}} \widehat{\psi}^*_x(x,k)\phi(x,k')dx. \tag{3.21}$$

Since $\widehat{\psi}^*$ is a solution to the adjoint equation (2.38), we have the following

$$\widehat{\psi}^*_x(x,k) = ik\widehat{\psi}^*(x,k)\sigma_3 - \widehat{\psi}^*(x,k)Q(u).$$

Inserting this into the integral on the right hand side of (3.21) we obtain that

$$\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\phi_x(x,k')dx = \Big[\widehat{\psi}^*(x,k)\phi(x,k')\Big]_{x\to-\infty}^{x\to+\infty} - \int_{\mathbb{R}} \Big[ik\widehat{\psi}^*(x,k)\sigma_3 - \widehat{\psi}^*(x,k)Q(u)\Big]\phi(x,k')dx.$$

Substituting the above back into (3.20) we have that

$$\Big[\widehat{\psi}^*(x,k)\phi(x,k')\Big]_{x\to-\infty}^{x\to+\infty} - ik\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\sigma_3\phi(x,k')dx = -ik'\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\sigma_3\phi(x,k')dx,$$

which can be rewritten as

$$\Big[\widehat{\psi}^*(x,k)\phi(x,k')\Big]_{x\to-\infty}^{x\to+\infty} = i(k-k')\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\sigma_3\phi(x,k')dx. \tag{3.22}$$

The left-hand side of (3.22) can be simplified as follows

$$\Big[\widehat{\psi}^*(x,k)\phi(x,k')\Big]_{x\to-\infty}^{x\to+\infty} = \Big[\widehat{N}^* e^{ikx} M e^{-ik'x}\Big]_{x\to-\infty}^{x\to+\infty} = \Big[\widehat{N}^* M e^{i(k-k')x}\Big]_{x\to-\infty}^{x\to+\infty},$$

where we can use last equation of (2.36) to obtain

$$\begin{aligned}\Big[\widehat{\psi}^*(x,k)\phi(x,k')\Big]_{x\to-\infty}^{x\to+\infty} &= \Big[\Big(aM^* - \overline{b}e^{-2ikx}\widehat{M}^*\Big)Me^{i(k-k')x}\Big]_{x\to-\infty}^{x\to+\infty} \\ &= \Big[\Big(aM^*M - \overline{b}e^{-2ikx}\widehat{M}^*M\Big)e^{i(k-k')x}\Big]_{x\to-\infty}^{x\to+\infty}.\end{aligned}$$

Using equalities from (2.39) we finally find the following

$$\Big[\widehat{\psi}^*(x,k)\phi(x,k')\Big]_{x\to-\infty}^{x\to+\infty} = a(k)\Big[e^{i(k-k')x}\Big]_{x\to-\infty}^{x\to+\infty}.$$

Substituting it back to (3.22) to get

$$i(k-k')\int_{\mathbb{R}} \widehat{\psi}^*(x,k)\sigma_3\phi(x,k')dx = a(k)\Big[e^{i(k-k')x}\Big]_{x\to-\infty}^{x\to+\infty}. \tag{3.23}$$

Lastly, we can use generalized formula for $\delta(k-k')$:

$$2\pi\delta(k-k') = \int_{\mathbb{R}} e^{i(k-k')x}dx = \frac{1}{i(k-k')}\Big[e^{i(k-k')x}\Big]_{x\to-\infty}^{x\to+\infty}.$$

From which we see that (3.23) yields (3.16). The orthogonality relations (3.17), (3.18) and (3.19) can be proved using analogous strategy. □

Proof of Theorem 1 follows from Lemma 3.2 and 3.3 and is explained below.

Proof of Theorem 1. To justify (3.2) we multiply (3.1) by $\widehat{\psi}^*\sigma_3$ and integrate over $\mathbb{R}$

$$\begin{aligned}\int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 f(y)dy &= \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 \int_{\mathbb{R}} \Big[\tilde{c}(h)\phi(y,h) + \tilde{d}(h)\widehat{\phi}(y,h)\Big]dhdy \\ &= \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 \int_{\mathbb{R}} \tilde{c}(h)\phi(y,h)dhdy + \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 \int_{\mathbb{R}} \tilde{d}(h)\widehat{\phi}(y,h)dhdy\end{aligned}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 \tilde{c}(h)\phi(y,h)dhdy + \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 \tilde{d}(h)\widehat{\phi}(y,h)dhdy$$
$$= \int_{\mathbb{R}} \tilde{c}(h) \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 \phi(y,h)dydh + \int_{\mathbb{R}} \tilde{d}(h) \int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 \widehat{\phi}(y,h)dydh,$$

Using (3.16) and (3.18) we can rewrite the above as

$$\int_{\mathbb{R}} \widehat{\psi}^*(y,k)\sigma_3 f(y)dy = \int_{\mathbb{R}} \tilde{c}(h)2\pi a(k)\delta(k-h)dh = 2\pi a(k)\tilde{c}(k),$$

which is equivalent to the expression for $\tilde{c}$ in (3.2). To prove the expression for $\tilde{d}$ the same procedure suffices with the exception that (3.1) needs to be multiplied by $\psi^*\sigma_3$ and (3.17) and (3.19) are used. □

Let us illustrate that (3.9) is the completeness relation between eigenfunctions $\{\phi, \widehat{\phi}\}$ and the adjoint eigenfunctions $\{\psi^*, \widehat{\psi}^*\}$. Inserting both integrals from (3.2) into (3.1) we obtain the following

$$\begin{aligned}
f(x) &= \int_R \left(\left[\frac{1}{2\pi a(k)} \int_R \widehat{\psi}^*(y,k)\sigma_3 f(y)dy \right] \phi(x,k) \right. \\
&\qquad \left. - \left[\frac{1}{2\pi \overline{a}(k)} \int_{\mathbb{R}} \psi^*(y,k)\sigma_3 f(y)dy \right] \widehat{\phi}(x,k) \right) dk \\
&= \int_{\mathbb{R}} \left(\left[\int_{\mathbb{R}} \frac{1}{2\pi a(k)} \widehat{\psi}^*(y,k)\phi(x,k)dk \right] \sigma_3 f(y) \right. \\
&\qquad \left. - \left[\int_{\mathbb{R}} \frac{1}{2\pi \overline{a}(k)} \psi^*(y,k)\widehat{\phi}(x,k)dk \right] \sigma_3 f(y) \right) dy \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{1}{2\pi a(k)} \widehat{\psi}^*(y,k)\phi(x,k)dk - \int_{\mathbb{R}} \frac{1}{2\pi \overline{a}(k)} \psi^*(y,k)\widehat{\phi}(x,k)dk \right) \sigma_3 f(y)dy \\
&= \int_{\mathbb{R}} \delta(x-y)\sigma_3\sigma_3 f(y)dy = \int_{\mathbb{R}} \delta(x-y)f(y)dy = f(x),
\end{aligned}$$

where the completeness relation (3.9) has been used. Thus, (3.9) represents the decomposition of an identity operator.

Chapter 4

Completeness of squared eigenfunctions

In this section we present a complete set of squared eigenfunctions for the Zakharov-Shabat spectral problem (1.2). The corresponding result is given by the following theorem.

Theorem 2. *The set of squared eigenfunctions* $\{Z^+, Z^-\}$ *defined by* (4.43) *is complete, i.e. every* $f \in L^2(\mathbb{R})$ *can be written as follows:*

$$f(x) = \int_{\mathbb{R}} \Big[\tilde{C}(k) Z^-(x,k) + \tilde{D}(k) Z^+(x,k)\Big] dk, \tag{4.1}$$

where

$$\tilde{C}(k) = -\frac{1}{\pi \bar{a}^2(k)} \int_{\mathbb{R}} \Omega^-(y,k) f(y) dy, \qquad \tilde{D}(k) = -\frac{1}{\pi a^2(k)} \int_{\mathbb{R}} \Omega^+(y,k) f(y) dy, \tag{4.2}$$

where we use adjoint squared eigenfunctions $\{\Omega^+, \Omega^-\}$ *defined by* (4.19)

To prove Theorem 2, we proceed as follows:

- Construct adjoint squared eigenfunctions by expressing variation of scattering data in terms of variation of potential.

- Construct squared eigenfunctions by expressing variation of potential in terms of variation of scattering data.
- Obtain a completeness relation and orthogonality conditions for squared eigenfunctions.

4.1 Adjoint squared eigenfunctions

In this section perturbation of the potential δu in the spectral problem (2.1) is used to define the adjoint squared eigenfunctions. We consider a perturbed solution of the NLS equation (1.1) in the form:

$$u(x,t) + \delta u(x,t), \tag{4.3}$$

where $u(x,t)$ is a solution of NLS and δu is its perturbation.

Substituting $u + \delta u$ instead of u into

$$\Phi_x = -ik\sigma_3\Phi + Q(u)\Phi, \tag{4.4}$$

and writing $\Phi + \delta\Phi$, $Q(u+\delta u) = Q + \delta Q$ we obtain

$$\begin{aligned}
(\Phi + \delta\Phi)_x &= -ik\sigma_3(\Phi + \delta\Phi) + (Q + \delta Q)(\Phi + \delta\Phi) \\
&= -ik\sigma_3\Phi - ik\sigma_3\delta\Phi + Q\Phi + Q\delta\Phi + \delta Q\Phi + (\delta Q)(\delta\Phi) \\
&= (-ik\sigma_3\Phi + Q\Phi) - ik\sigma_3\delta\Phi + Q\delta\Phi + \delta Q\Phi + (\delta Q)(\delta\Phi) \\
&= \Phi_x - ik\sigma_3\delta\Phi + Q\delta\Phi + \delta Q\Phi + (\delta Q)(\delta\Phi).
\end{aligned}$$

Neglecting quadratic term $\delta Q\delta\Phi$ yields

$$(\delta\Phi)_x = -ik\sigma_3\delta\Phi + Q\delta\Phi + (\delta Q)\Phi, \tag{4.5}$$

where

$$\delta Q = \begin{bmatrix} 0 & \delta u \\ -\delta\overline{u} & 0 \end{bmatrix}$$

is a variation of potential.

Proposition 4.1. *Let* Φ *be a solution to spectral problem* (2.1) *subject to boundary conditions* (2.15)*, then solution to* (4.5) *is*

$$\delta\Phi(x) = \Phi(x)\int_{-\infty}^{x} \Phi^{-1}(y)\delta Q(y)\Phi(y)dy. \tag{4.6}$$

Proof. Recall that $\Phi(x) = J_{-}(x)E$. This implies that $\Phi(x) \to E$ and $\delta\Phi(x) \to 0$ as $x \to -\infty$. We solve (4.5) by method of variation of parameters. Thus, we write

$$\delta\Phi(x) = \Phi(x)C(x). \tag{4.7}$$

Inserting (4.7) into (4.5) we obtain the following

$$\Phi_x C + \Phi C_x = -ik\sigma_3\Phi C + Q\Phi C + \delta Q\Phi,$$

where we use (4.4) and obtain

$$\Phi C_x = \delta Q\Phi,$$

Since Φ^{-1} exists, we can apply it here to have

$$C_x = \Phi^{-1}\delta Q\Phi,$$

which if we integrate leads to

$$C(x) = \int_{-\infty}^{x} \Phi^{-1}(y,k)\delta Q(y)\Phi(y,k)dy,$$

returning to (4.7) we obtain (4.6). □

Lemma 4.2. *Let $\delta\Phi$ be defined as in* (4.6) *and let Φ and Ψ satisfy boundary conditions* (2.15)-(2.16)*, then we have the following*

$$\delta S = \int_{-\infty}^{+\infty} \Psi^{-1}(x)\delta Q(x)\Phi(x)dx, \tag{4.8}$$

$$\delta S^{-1} = -\int_{-\infty}^{+\infty} \Phi^{-1}(x)\delta Q(x)\Psi(x)dx. \tag{4.9}$$

Proof. Recall that $\Phi = \Psi S$ and $\Psi(x) \to E$ as $x \to +\infty$, therefore $\Phi(x) \to ES$ and $\delta\Phi(x) \to E(\delta S)$ as $x \to +\infty$. Hence, applying the limit of $x \to +\infty$ to the solution (4.6) we have the following

$$E(\delta S) = \lim_{x\to+\infty} \delta\Phi(x) = ES\int_{-\infty}^{+\infty} \Phi^{-1}(x)\delta Q(x)\Phi(x)dx$$

Cancelling exponential parts we continue as follows

$$\begin{aligned}
\delta S &= S\int_{-\infty}^{+\infty} \Phi^{-1}(x)\delta Q(x)\Phi(x)dx \\
&= S\int_{-\infty}^{+\infty} S^{-1}\Psi^{-1}(x)\delta Q(x)\Phi(x)dx \\
&= SS^{-1}\int_{-\infty}^{+\infty} \Psi^{-1}(x)\delta Q(x)\Phi(x)dx \\
&= \int_{-\infty}^{+\infty} \Psi^{-1}(x)\delta Q(x)\Phi(x)dx.
\end{aligned}$$

Proceeding similarly as $x \to -\infty$, we obtain (4.9) □

The formulae (4.8), (4.9) represent the variations of the scattering data $S(k)$ due to variation of the potential δQ. Let us write each entry of δS and δS^{-1} from (4.8)-(4.9) below.

Entries of δS:

$$\delta a = \int_{-\infty}^{+\infty} \widehat{\psi}^{*}\delta Q\phi dx, \qquad \delta\overline{a} = \int_{-\infty}^{+\infty} \psi^{*}\delta Q\widehat{\phi}dx,$$

$$-\delta\overline{b} = \int_{-\infty}^{+\infty} \widehat{\psi}^* \delta Q \widehat{\phi} dx, \qquad \delta b = \int_{-\infty}^{+\infty} \psi^* \delta Q \phi dx.. \tag{4.10}$$

Entries of δS^{-1}:

$$\begin{aligned} \delta\overline{a} &= -\int_{-\infty}^{+\infty} \phi^* \delta Q \widehat{\psi} dx, \qquad \delta a = -\int_{-\infty}^{+\infty} \widehat{\phi}^* \delta Q \psi dx, \\ -\delta b &= -\int_{-\infty}^{+\infty} \widehat{\phi}^* \delta Q \widehat{\psi} dx, \qquad \delta\overline{b} = -\int_{-\infty}^{+\infty} \phi^* \delta Q \psi dx. \end{aligned} \tag{4.11}$$

Remark 4.3. *The formulae* (4.10), (4.11) *suggest that $b, \overline{b}$ are given by products of eigenfunctions which are not analytic in the same region of $\mathbb{C}$, whereas $a, \overline{a}$ are given by products of eigenfunctions, which are analytic in the same regions of $\mathbb{C}$, namely $\mathbb{C}_+$ for a and $\mathbb{C}_-$ for $\overline{a}$. Since our aim is to obtain adjoint squared eigenfunctions that are analytic in either of the half-planes, we want variation of scattering data to be expressed in terms of product of eigenfunctions that are analytic in $\mathbb{C}_+$ or $\mathbb{C}_-$.*

Lemma 4.4. *Let ρ and $\tilde{\rho}$ be defined as*

$$\rho = \frac{b}{\overline{a}}, \qquad \tilde{\rho} = \frac{\overline{b}}{a}. \tag{4.12}$$

Also let $\delta\rho, \delta\tilde{\rho}$ be variations of $\rho, \tilde{\rho}$, then we can express variation of scattering data in terms of variation of potential in the following way

$$\delta\rho = \frac{1}{\overline{a}^2}\left\langle \Omega^-, \begin{bmatrix} \delta u \\ \delta\overline{u} \end{bmatrix} \right\rangle, \qquad \delta\tilde{\rho} = \frac{1}{a^2}\left\langle \Omega^+, \begin{bmatrix} \delta u \\ \delta\overline{u} \end{bmatrix} \right\rangle, \tag{4.13}$$

where

$$\Omega^- = \begin{bmatrix} \psi_1^* \widehat{\psi}_2 \\ -\psi_2^* \widehat{\psi}_1 \end{bmatrix}, \qquad \Omega^+ = \begin{bmatrix} -\widehat{\psi}_1^* \psi_2 \\ \widehat{\psi}_2^* \psi_1 \end{bmatrix}. \tag{4.14}$$

Proof. Taking variation of ρ we obtain:

$$\delta\rho = \delta\left(\frac{b}{\overline{a}}\right) = \frac{\delta b\overline{a} - \delta\overline{a} b}{\overline{a}^2} = \frac{1}{\overline{a}^2}\left(\overline{a}\delta b - b\delta\overline{a}\right)$$

$$
\begin{aligned}
&= \frac{1}{\overline{a}^2}\Big(\overline{a}\int_{-\infty}^{+\infty}\widehat{\phi}^*\delta Q\widehat{\psi}dx + b\int_{-\infty}^{+\infty}\phi^*\delta Q\widehat{\psi}dx\Big) \\
&= \frac{1}{\overline{a}^2}\Big(\int_{-\infty}^{+\infty}\overline{a}\widehat{\phi}^*\delta Q\widehat{\psi}dx + \int_{-\infty}^{+\infty}b\phi^*\delta Q\widehat{\psi}dx\Big) \\
&= \frac{1}{\overline{a}^2}\int_{-\infty}^{+\infty}(\overline{a}\widehat{\phi}^* + b\phi^*)\delta Q\widehat{\psi}dx \qquad (4.15)
\end{aligned}
$$

To proceed further we take the inverse of (2.17) to get $\Phi^{-1} = S^{-1}\Psi^{-1}$, which can be rewritten as

$$
\begin{cases}
\phi^* = \overline{a}\widehat{\psi}^* + \overline{b}\psi^* \\
\widehat{\phi}^* = -b\widehat{\psi}^* + a\psi^*
\end{cases}, \qquad (4.16)
$$

Using (4.16) in (4.15) we get

$$
\begin{aligned}
\delta\rho &= \frac{1}{\overline{a}^2}\int_{-\infty}^{+\infty}\Big(\overline{a}(-b\widehat{\psi}^* + a\psi^*) + b(\overline{a}\widehat{\psi}^* + \overline{b}\psi^*)\Big)\delta Q\widehat{\psi}dx \\
&= \frac{1}{\overline{a}^2}\int_{-\infty}^{+\infty}(\overline{a}a\psi^* + b\overline{b}\psi^*)\delta Q\widehat{\psi}dx,
\end{aligned}
$$

where we can use (2.18) and obtain

$$
\delta\rho = \frac{1}{\overline{a}^2}\int_{-\infty}^{+\infty}\psi^*\delta Q\widehat{\psi}dx, \qquad (4.17)
$$

where $\overline{a}, \psi^*, \widehat{\psi}$ are all analytic in $\mathbb{C}_-$. Analogously, by taking variation of $\tilde{\rho}$ and following the same procedure we have that

$$
\delta\tilde{\rho} = -\frac{1}{a^2}\int_{-\infty}^{+\infty}\widehat{\psi}^*\delta Q\psi dx, \qquad (4.18)
$$

where $a, \widehat{\psi}^*, \psi$ are all analytic in $\mathbb{C}_+$. Now rewriting (4.17) in a more convenient form we get

$$
\delta\rho = \frac{1}{\overline{a}^2}\int_{-\infty}^{+\infty}\begin{bmatrix}\psi_1^* & \psi_2^*\end{bmatrix}\begin{bmatrix}0 & \delta u \\ -\delta\overline{u} & 0\end{bmatrix}\begin{bmatrix}\widehat{\psi}_1 \\ \widehat{\psi}_2\end{bmatrix}dx
$$

$$= \frac{1}{\overline{a}^2}\int_{-\infty}^{+\infty}\begin{bmatrix}-\psi_2^*\delta\overline{u} & \psi_1^*\delta u\end{bmatrix}\begin{bmatrix}\widehat{\psi}_1 \\ \overline{\psi_2}\end{bmatrix}dx$$
$$= \frac{1}{\overline{a}^2}\int_{-\infty}^{+\infty}(\psi_1^*\widehat{\psi}_2\delta u - \psi_2^*\widehat{\psi}_1\delta\overline{u})dx$$
$$= \frac{1}{\overline{a}^2}\left\langle\begin{bmatrix}\psi_1^*\widehat{\psi}_2 \\ -\psi_2^*\widehat{\psi}_1\end{bmatrix}, \begin{bmatrix}\delta u \\ \delta\overline{u}\end{bmatrix}\right\rangle,$$

which is equivalent to (4.13) with (4.14) Performing the same with (4.18) we obtain

$$\delta\tilde{\rho} = \frac{1}{a^2}\left\langle\begin{bmatrix}-\widehat{\psi}_1^*\psi_2 \\ \widehat{\psi}_2^*\psi_1\end{bmatrix}, \begin{bmatrix}\delta u \\ \delta\overline{u}\end{bmatrix}\right\rangle,$$

which is equivalent to the second equation in (4.13) with (4.14). □

The entries of $\Omega^{\pm}$ are a products between Jost solutions to the Zakharov-Shabar system (2.1) and their adjoints. Note that superscript "±" denotes the plane of analyticity. From Remark 2.9

$$\Psi^{-1} = \begin{bmatrix}\widehat{\psi}_1 & \psi_1 \\ \widehat{\psi}_2 & \psi_2\end{bmatrix}^{-1} = \frac{1}{det\Psi}\begin{bmatrix}\psi_2 & -\psi_1 \\ -\widehat{\psi}_2 & \widehat{\psi}_1\end{bmatrix} = \begin{bmatrix}\psi_2 & -\psi_1 \\ -\widehat{\psi}_2 & \widehat{\psi}_1\end{bmatrix} = \begin{bmatrix}\widehat{\psi}_1^* & \widehat{\psi}_2^* \\ \psi_1^* & \psi_2^*\end{bmatrix},$$

since by Abel's identity $\det\Psi(x) = \lim_{x\to+\infty}\det\Psi(x_0) = \det J_+ \det E = 1$. Hence, $\widehat{\psi}_1^* = \psi_2, \widehat{\psi}_2^* = -\psi_1, \psi_1^* = -\widehat{\psi}_2, \psi_2^* = \widehat{\psi}_1$, so that we can rewrite $\Omega^{\pm}$ by using Jost solutions only

$$\Omega^- = -\begin{bmatrix}\widehat{\psi}_2^2 \\ \widehat{\psi}_1^2\end{bmatrix}, \qquad \Omega^+ = -\begin{bmatrix}\psi_2^2 \\ \psi_1^2\end{bmatrix}. \tag{4.19}$$

4.2 Squared eigenfunctions

In the previous section we considered the variation of the scattering data via variation of the potential of the Zakharov-Shabat spectral problem (2.1). Now we are following the opposite direction, and the main idea is to take variation of Riemann-Hilbert problem (2.42). We see from (2.43) that scattering data $b, \overline{b}$ are part of the Riemann-Hilbert problem on the real axis $k \in \mathbb{R}$. In order to extend variation of scattering data in $\mathbb{C}_{\pm}$, just as before, we will use some manipulations to transform the Riemann-Hilbert problem with $\rho, \widehat{\rho}$ instead of $b, \overline{b}$. To do so, we firstly introduce new matrices

$$F^{+} = P^{+} \begin{bmatrix} 1 & 0 \\ 0 & 1/a \end{bmatrix}, \qquad F^{-} = (P^{-})^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \overline{a} \end{bmatrix}. \tag{4.20}$$

Lemma 4.5. *Let $F^{\pm}$ be defined as above in (4.20) and assume that $a, \overline{a}$ do not have zeros in $\mathbb{C}_{\pm} \cup \mathbb{R}$, then $F^{\pm}$ are analytic in $\mathbb{C}_{\pm}$ and can be explicitly written in terms of Jost solutions and scattering data*

$$F^{+} = \begin{bmatrix} M & N/a \end{bmatrix}, \qquad F^{-} = \begin{bmatrix} \widehat{N^{*}}/\overline{a} & \widehat{M^{*}} \end{bmatrix}^{T}. \tag{4.21}$$

with the following large k asymptotics

$$F^{\pm} \to I, \qquad \text{as } |k| \to \infty \text{ in } \mathbb{C}_{\pm}. \tag{4.22}$$

Proof. We obtain from (4.20) that

$$P^{+} = F^{+} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \qquad (P^{-})^{-1} = F^{-} \begin{bmatrix} 1 & 0 \\ 0 & 1/\overline{a} \end{bmatrix}. \tag{4.23}$$

Thus, writing (2.42) as

$$P^{+} = (P^{-})^{-1} G.$$

Then, using (4.23) we have the following

$$F^{+} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} = F^{-} \begin{bmatrix} 1 & 0 \\ 0 & 1/\overline{a} \end{bmatrix} G,$$

which can be rewritten as

$$F^{+} = F^{-} \tilde{G}, \tag{4.24}$$

where

$$\tilde{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1/\overline{a} \end{bmatrix} G \begin{bmatrix} 1 & 0 \\ 0 & 1/a \end{bmatrix} = \begin{bmatrix} 1 & \tilde{\rho} e^{-2ikx} \\ \rho e^{2ikx} & 1 + \rho\tilde{\rho} \end{bmatrix}. \tag{4.25}$$

In the latter we used (2.18), (2.43) and (4.12). Concerning analyticity of $F^{\pm}$, both are analytic in $\mathbb{C}_{\pm}$ as long as $a, \overline{a}$ are nonzero. To find the large k asymptotics, we should rewrite $F^{\pm}$ in the form (4.21). From definition for F^{+} and $P^{\pm}$ it is easy to see that first equation in (4.21) is true. For the second one, we proceed as follows:

$$\begin{aligned} F^{-} &= (P^{-})^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \overline{a} \end{bmatrix} = \begin{bmatrix} M^{*} \\ N^{*} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \overline{a} \end{bmatrix} = \begin{bmatrix} M_1^{*} & M_2^{*} \\ N_1^{*} & N_2^{*} \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \overline{a} \end{bmatrix} \\ &= \frac{1}{\det P^{-}} \begin{bmatrix} N_2^{*} & -M_2^{*} \\ -N_1^{*} & M_1^{*} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{a} \end{bmatrix} = \frac{1}{\overline{a}} \begin{bmatrix} N_2^{*} & -M_2^{*} \\ -N_1^{*} & M_1^{*} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{a} \end{bmatrix} \\ &= \begin{bmatrix} N_2^{*}/\overline{a} & -M_2^{*} \\ -N_1^{*}/\overline{a} & M_1^{*} \end{bmatrix} = \begin{bmatrix} \overline{\widehat{N_1^{*}}}/\overline{a} & \overline{\widehat{M_1^{*}}} \\ \overline{\widehat{N_2^{*}}}/\overline{a} & \overline{\widehat{M_2^{*}}} \end{bmatrix} = \begin{bmatrix} \overline{\widehat{N^{*}}}/\overline{a} & \overline{\widehat{M^{*}}} \end{bmatrix}^{T}. \end{aligned}$$

Using (2.12), (2.35) we find (4.22).

□

Proposition 4.6. *Let $\delta F^{\pm}$ be variations of $F^{\pm}$, then*

$$\left(\delta F F^{-1}\right)(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\Pi(x,\xi)}{\xi - k} d\xi, \tag{4.26}$$

where

$$\Pi(x,\xi) = F^{-}(x)\delta\tilde{G}(x,\xi)(F^{+})^{-1}(x), \qquad \xi \in \mathbb{R}. \tag{4.27}$$

Proof. Take variation of F^{+} in (4.24) to have the following

$$\delta F^{+} = \delta F^{-}\tilde{G} + F^{-}\delta\tilde{G} = \delta F^{-}(F^{-})^{-1}F^{+} + F^{-}\delta\tilde{G}.$$

Multiplying both sides by F^+ we have

$$\delta F^+(F^+)^{-1} = \delta F^-(F^-)^{-1} + F^-\delta\tilde{G}(F^+)^{-1},$$

rewriting which we obtain

$$\delta F^+(F^+)^{-1} - \delta F^-(F^-)^{-1} = F^-\delta\tilde{G}(F^+)^{-1}.$$

By applying Plemelj formula (2.46) for F^+, F^- we get (4.26) as desired. $\square$

To consider large-k asymptotics for $F^\pm$ we need to analyze asymptotics of $P^\pm$. Expanding solution to Riemann Hilbert problem as below

$$P^\pm = I + \frac{1}{k}P_1^\pm(x) + O(\frac{1}{k^2}) \tag{4.28}$$

allows us work with spectral problems, since entries of $P^\pm$ are Jost solutions.

Lemma 4.7. *Let $P^\pm$ be expanded as in* (4.28)*, then*

$$P_1^+ = \frac{1}{2i}\begin{bmatrix} \int_{-\infty}^{x} |u(y)|^2 dy & u \\ \overline{u} & \int_{x}^{+\infty} |u(y)|^2 dy \end{bmatrix} \tag{4.29}$$

$$P_1^- = -\frac{1}{2i}\begin{bmatrix} \int_{-\infty}^{x} |u(y)|^2 dy & u \\ \overline{u} & \int_{x}^{+\infty} |u(y)|^2 dy \end{bmatrix} \tag{4.30}$$

Proof. Substitute (4.28) into first equation of (2.4) to obtain the following

$$\frac{1}{k}(P_1^+)_x = -i\Big(\sigma_3 P_1^+ - P_1^+\sigma_3\Big) + Q + \frac{1}{k}QP_1^+.$$

From which we have the following equations

$$\begin{cases} (P_1^+)_x = QP_1^+ \\ Q = i[\sigma_3, P_1^+]. \end{cases} \tag{4.31}$$

Let's write P_1^+ in a matrix form

$$P_1^+ = \begin{bmatrix} (P_1^+)_{11} & (P_1^+)_{12} \\ (P_1^+)_{21} & (P_1^+)_{22}. \end{bmatrix} \tag{4.32}$$

This allows us to use second equation in (4.31) and obtain relations between u and entries of P_1^+, which are

$$\begin{cases} (P_1^+)_{12} = \frac{u}{2i} \\ (P_1^+)_{21} = \frac{\overline{u}}{2i}. \end{cases} \tag{4.33}$$

Then, solving first equation in (4.31) we have the following

$$\begin{cases} (P_1^+)_{11} = \frac{1}{2i} \int_{-\infty}^{x} |u(y)|^2 dy \\ (P_1^+)_{22} = \frac{1}{2i} \int_{x}^{+\infty} |u(y)|^2 dy \end{cases} \tag{4.34}$$

Now substituting (4.33), (4.34) into (4.32) we have (4.29) as needed. In a similar way one can prove (4.30). □

Corollary 4.8. *F^+ can be expanded as*

$$F^+ = \begin{bmatrix} 1 + \frac{1}{2ik} \int_{-\infty}^{x} |u|^2 dy + O(\frac{1}{k^2}) & \frac{u}{2ik} + O(\frac{1}{k^2}) \\ \frac{\overline{u}}{2ik} + O(\frac{1}{k^2}) & 1 + \frac{1}{2ik} \int_{x}^{+\infty} |u|^2 dy + O(\frac{1}{k^2}) \end{bmatrix}$$

Proof. We can write expansion of $P^\pm$

$$P^\pm = I \pm \frac{1}{2ik} \begin{bmatrix} \int_{-\infty}^{x} |u(y)|^2 dy & u \\ \overline{u} & \int_{x}^{+\infty} |u(y)|^2 dy \end{bmatrix} + O\left(\frac{1}{k^2}\right).$$

From (3.7) it is obvious that $\det P^+ = a$. Now, calculating determinant in 2×2 matrix P^+ we can find expansion for a

$$a = \left(1 + \frac{1}{2ik} \int_{x}^{+\infty} |u(y)|^2 dy\right)\left(1 + \frac{1}{2ik} \int_{-\infty}^{x} |u(y)|^2 dy\right) + \frac{|u|^2}{4k^2} + O\left(\frac{1}{k^2}\right)$$

$$= 1 + \frac{1}{2ik}\int_x^{+\infty} |u(y)|^2 dy + \frac{1}{2ik}\int_{-\infty}^{x} |u(y)|^2 dy + O\left(\frac{1}{k^2}\right)$$
$$= 1 + \frac{1}{2ik}\int_{\mathbb{R}} |u(y)|^2 dy + O\left(\frac{1}{k^2}\right).$$

Substituting expansions of P^+ and a in (4.20) we have expansion of F^+ as needed. □

Lemma 4.9. *Variation of potentials at $O(\frac{1}{k})$ are*

$$\begin{cases} \delta u = -\frac{1}{\pi}\int_{\mathbb{R}} \Pi_{12}(x,\xi) d\xi \\ \delta \overline{u} = -\frac{1}{\pi}\int_{\mathbb{R}} \Pi_{21}(x,\xi) d\xi. \end{cases} \tag{4.35}$$

Proof. For simplicity, let us write expansion of F^+ as follows

$$F^+ = \begin{bmatrix} 1 + O(\frac{1}{k}) & \frac{u}{2ik} + O(\frac{1}{k^2}) \\ \frac{\overline{u}}{2ik} + O(\frac{1}{k^2}) & 1 + O(\frac{1}{k}) \end{bmatrix}.$$

Taking variation of the above we have

$$\delta F^+ = \begin{bmatrix} O(\frac{1}{k}) & \frac{\delta u}{2ik} + O(\frac{1}{k^2}) \\ \frac{\delta \overline{u}}{2ik} + O(\frac{1}{k^2}) & O(\frac{1}{k}) \end{bmatrix}.$$

Now, substituting the above expansions into (4.26) we have the following

$$\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\Pi(x,\xi)}{\xi - k} d\xi = \begin{bmatrix} O(\frac{1}{k}) & \frac{\delta u}{2ik} + O(\frac{1}{k^2}) \\ \frac{\delta \overline{u}}{2ik} + O(\frac{1}{k^2}) & O(\frac{1}{k}) \end{bmatrix}. \tag{4.36}$$

We can also rewrite the left hand side of the above as follows

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{\xi - k}\Pi(x,\xi) d\xi &= \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{k}\frac{1}{(\xi/k - 1)}\Pi(x,\xi) d\xi \\ &= -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1}{k}\sum_{n=0}^{\infty}\left(\frac{\xi}{k}\right)^n \Pi(x,\xi) d\xi \\ &= -\frac{1}{2\pi i k}\int_{\mathbb{R}} \Pi(x,\xi) d\xi + O(\frac{1}{k^2}) \end{aligned}$$

$$= -\frac{1}{2\pi ik}\begin{bmatrix}\int_{\mathbb{R}}\Pi_{11}(x,\xi)d\xi & \int_{\mathbb{R}}\Pi_{12}(x,\xi)d\xi\\ \int_{\mathbb{R}}\Pi_{21}(x,\xi)d\xi & \int_{\mathbb{R}}\Pi_{22}(x,\xi)d\xi\end{bmatrix} + O\left(\frac{1}{k^2}\right)$$

combining it with (4.36) we obtain (4.35) as desired. □

Lemma 4.10. *Given* $\Pi(x,\xi)$ *is defined as in* (4.27) *we have the following*

$$\Pi = \Phi\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & 0\end{bmatrix}\Phi^{-1}. \tag{4.37}$$

Proof. Firstly, we take variation of (4.25) and rewrite it to get

$$\delta\tilde{G} = \begin{bmatrix}0 & \delta\tilde{\rho}e^{-2ikx}\\ \delta\rho e^{2ikx} & \delta\rho\tilde{\rho}+\rho\delta\tilde{\rho}\end{bmatrix} = E\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & \delta\rho\tilde{\rho}+\rho\delta\tilde{\rho}\end{bmatrix}E^{-1},$$

where

$$E = e^{-ikx\sigma_3} = \begin{bmatrix}e^{-ikx} & 0\\ 0 & e^{ikx}\end{bmatrix}$$

Now, substituting equation for $\delta\tilde{G}$ into (4.27) we have the following

$$\begin{aligned}\Pi &= F^{-}E\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & \delta\rho\tilde{\rho}+\rho\delta\tilde{\rho}\end{bmatrix}E^{-1}(F^{+})^{-1}\\ &= (P^{-})^{-1}\begin{bmatrix}1 & 0\\ 0 & \overline{a}\end{bmatrix}E\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & \delta\rho\tilde{\rho}+\rho\delta\tilde{\rho}\end{bmatrix}E^{-1}\begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix}(P^{+})^{-1},\end{aligned} \tag{4.38}$$

where we used definitions of $F^{\pm}$ in (4.20). Let us rewrite $P^{\pm}$ in a matrix form explicitly. Before that, for the sake of simplicity of the proof, let's define the following matrices

$$H_1 = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \qquad H_2 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Then, we proceed as

$$\begin{aligned} P^- = \begin{bmatrix} M^* \\ N^* \end{bmatrix} &= H_1 J_-^{-1} + H_2 J_+^{-1} = H_1 J_-^{-1} + H_2 J_+^{-1} J_- J_-^{-1} \\ &= (H_1 + H_2 J_+^{-1} J_-) J_-^{-1} = (H_1 + H_2 J_+^{-1} J_+ E S E^{-1}) J_-^{-1} \\ &= (H_1 + H_2 E S E^{-1}) J_-^{-1}. \end{aligned}$$

By taking inverse of the above we obtain

$$\begin{aligned} (P^-)^{-1} &= J_-(H_1 + H_2 E S E^{-1})^{-1} \\ &= J_- \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{bmatrix} \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} \begin{bmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{bmatrix} \right)^{-1} \\ &= J_- \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & -\overline{b} e^{-2ikx} \\ b e^{2ikx} & \overline{a} \end{bmatrix} \right)^{-1} \\ &= J_- \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ b e^{2ikx} & \overline{a} \end{bmatrix} \right)^{-1} = J_- \begin{bmatrix} 1 & 0 \\ b e^{2ikx} & \overline{a} \end{bmatrix}^{-1} \\ &= J_- \begin{bmatrix} 1 & 0 \\ -\rho e^{2ikx} & 1/\overline{a} \end{bmatrix} = \Phi E^{-1} \begin{bmatrix} 1 & 0 \\ -\rho e^{2ikx} & 1/\overline{a} \end{bmatrix}, \end{aligned} \tag{4.39}$$

where we used (2.13), (2.18) and (4.12). Performing the same manipulations with P^+ we have

$$(P^+)^{-1} = \begin{bmatrix} 1 & -\tilde{\rho} e^{-2ikx} \\ 0 & 1/a \end{bmatrix} E \Phi^{-1}. \tag{4.40}$$

Inserting (4.39), (4.40) back into (4.38) we have

$$\Pi = \Phi E^{-1} \begin{bmatrix} 1 & 0 \\ \rho e^{2ikx} & 1/\overline{a} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \overline{a} \end{bmatrix} E \begin{bmatrix} 0 & \delta\tilde{\rho} \\ \delta\rho & \delta\rho\tilde{\rho} + \rho\delta\tilde{\rho} \end{bmatrix} E^{-1} \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} 1 & -\tilde{\rho} e^{-2ikx} \\ 0 & 1/a \end{bmatrix} E \Phi^{-1}$$

$$= \Phi E^{-1} \begin{bmatrix} 1 & 0 \\ -\rho e^{2ikx} & 1 \end{bmatrix} E \begin{bmatrix} 0 & \delta\tilde{\rho} \\ \delta\rho & \delta\rho\tilde{\rho} + \rho\delta\tilde{\rho} \end{bmatrix} E^{-1} \begin{bmatrix} 1 & -\tilde{\rho} e^{-2ikx} \\ 0 & 1 \end{bmatrix} E \Phi^{-1}$$

$$= \Phi E^{-1}\left(E\begin{bmatrix}1 & 0\\ -\rho & 1\end{bmatrix}E^{-1}\right)E\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & \delta\rho\tilde{\rho}+\rho\delta\tilde{\rho}\end{bmatrix}E^{-1}\left(E\begin{bmatrix}1 & -\tilde{\rho}\\ 0 & 1\end{bmatrix}E^{-1}\right)E\Phi^{-1}$$
$$= \Phi\begin{bmatrix}1 & 0\\ -\rho & 1\end{bmatrix}\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & \delta\rho\tilde{\rho}+\rho\delta\tilde{\rho}\end{bmatrix}\begin{bmatrix}1 & -\tilde{\rho}\\ 0 & 1\end{bmatrix}\Phi^{-1} = \Phi\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & 0\end{bmatrix}\Phi^{-1},$$

which concludes the proof. □

Lemma 4.11. *Perturbation δu of spectral problem* (2.1) *is expressed in terms of $\delta\rho$ as follows*

$$\begin{bmatrix}\delta u\\ \delta\overline{u}\end{bmatrix} = \frac{1}{\pi}\int_{\mathbb{R}}\Big(Z^-(x,\xi)\delta\rho(\xi) + Z^+(x,\xi)\delta\tilde{\rho}(\xi)\Big)d\xi, \tag{4.41}$$

where

$$Z^- = \begin{bmatrix}-\widehat{\phi}_1\phi_2^*\\ -\widehat{\phi}_2\phi_1^*\end{bmatrix}, \qquad Z^+ = \begin{bmatrix}-\phi_1\widehat{\phi}_2^*\\ -\phi_2\widehat{\phi}_1^*\end{bmatrix}. \tag{4.42}$$

Proof. Writing (4.37) explicitly results in

$$\Pi = \begin{bmatrix}\phi_1 & \widehat{\phi}_1\\ \phi_2 & \widehat{\phi}_2\end{bmatrix}\begin{bmatrix}0 & \delta\tilde{\rho}\\ \delta\rho & 0\end{bmatrix}\begin{bmatrix}\phi_1^* & \phi_2^*\\ \widehat{\phi}_1^* & \widehat{\phi}_2^*\end{bmatrix} = \begin{bmatrix}\widehat{\phi}_1\delta\rho & \phi_1\delta\tilde{\rho}\\ \widehat{\phi}_2\delta\rho & \phi_2\delta\tilde{\rho}\end{bmatrix}\begin{bmatrix}\phi_1^* & \phi_2^*\\ \widehat{\phi}_1^* & \widehat{\phi}_2^*\end{bmatrix}$$
$$= \begin{bmatrix}\widehat{\phi}_1\phi_1^*\delta\rho + \phi_1\widehat{\phi}_2^*\delta\tilde{\rho} & \widehat{\phi}_1\phi_2^*\delta\rho + \phi_1\widehat{\phi}_2^*\delta\tilde{\rho}\\ \widehat{\phi}_2\phi_1^*\delta\rho + \phi_2\widehat{\phi}_2^*\delta\tilde{\rho} & \widehat{\phi}_2\phi_2^*\delta\rho + \phi_2\widehat{\phi}_2^*\delta\tilde{\rho}\end{bmatrix} = \begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{21} & \Pi_{22}\end{bmatrix}$$

Combining this with results of Lemma 4.9 we have the following

$$\delta u = -\frac{1}{\pi}\int_{\mathbb{R}}\Big(\widehat{\phi}_1\phi_2^*\delta\rho + \phi_1\widehat{\phi}_2^*\delta\tilde{\rho}\Big)d\xi,$$
$$\delta\overline{u} = -\frac{1}{\pi}\int_{\mathbb{R}}\Big(\widehat{\phi}_2\phi_1^*\delta\rho + \phi_2\widehat{\phi}_1^*\delta\tilde{\rho}\Big)d\xi,$$

from which (4.41) follows directly. □

Note that from linear relation between Φ and Φ^{-1} we have

$$\begin{bmatrix} \phi_1^* & \phi_2^* \\ \widehat{\phi}_1^* & \widehat{\phi}_2^* \end{bmatrix} = \Phi^{-1} = \begin{bmatrix} \phi_1 & \widehat{\phi}_1 \\ \phi_2 & \widehat{\phi}_2 \end{bmatrix}^{-1} = \begin{bmatrix} \widehat{\phi}_2 & -\widehat{\phi}_1 \\ -\phi_2 & \phi_1 \end{bmatrix}.$$

Using the above we can rewrite (4.42) as follows

$$Z^- = \begin{bmatrix} \widehat{\phi}_1^2 \\ -\widehat{\phi}_2^2 \end{bmatrix}, \qquad Z^+ = \begin{bmatrix} -\phi_1^2 \\ \phi_2^2 \end{bmatrix}. \tag{4.43}$$

Lemma 4.12. *The sets* $\{\Omega^+, \Omega^-\}$ *and* $\{Z^+, Z^-\}$ *introduced in* (4.19) *and* (4.43) *satisfy the following completeness relation if* $a, \overline{a} \neq 0$*:*

$$\delta(x-y)I = \frac{1}{\pi}\int_{\mathbb{R}} \left[\frac{1}{\overline{a}^2(\xi)} Z^-(x,\xi)\Omega^-(y,\xi) + \frac{1}{a^2(\xi)} Z^+(x,\xi)\Omega^+(y,\xi) \right] d\xi. \tag{4.44}$$

Proof. For completeness relation, we adopt both formulas connecting variation of potential and variation of scattering data: (4.17)-(4.18) and (4.41). Then, we obtain

$$\begin{aligned} \begin{bmatrix} \delta u(x) \\ \delta \overline{u}(x) \end{bmatrix} &= \frac{1}{\pi}\int_{\mathbb{R}} \left[Z^-(x,\xi)\frac{1}{\overline{a}^2(\xi)} \left\langle \Omega^-, \begin{bmatrix} \delta u \\ \delta \overline{u} \end{bmatrix} \right\rangle + Z^+(x,\xi)\frac{1}{a^2(\xi)} \left\langle \Omega^+, \begin{bmatrix} \delta u \\ \delta \overline{u} \end{bmatrix} \right\rangle \right] d\xi \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \left[Z^-(x,\xi)\frac{1}{\overline{a}^2(\xi)} \int_{\mathbb{R}} \Omega^-(y,\xi) \begin{bmatrix} \delta u(y) \\ \delta \overline{u}(y) \end{bmatrix} dy \right. \\ &\qquad \left. + Z^+(x,\xi)\frac{1}{a^2(\xi)} \int_{\mathbb{R}} \Omega^+(y,\xi) \begin{bmatrix} \delta u(y) \\ \delta \overline{u}(y) \end{bmatrix} dy \right] d\xi \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{1}{\overline{a}^2(\xi)} Z^-(x,\xi)\Omega^-(y,\xi) \begin{bmatrix} \delta u(y) \\ \delta \overline{u}(y) \end{bmatrix} dy \right. \\ &\qquad \left. + \int_{\mathbb{R}} \frac{1}{a^2(\xi)} Z^+(x,\xi)\Omega^+(y,\xi) \begin{bmatrix} \delta u(y) \\ \delta \overline{u}(y) \end{bmatrix} dy \right] d\xi \end{aligned}$$

$$= \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{1}{\overline{a}^2(\xi)}Z^-(x,\xi)\Omega^-(y,\xi)\begin{bmatrix}\delta u(y)\\ \delta\overline{u}(y)\end{bmatrix} + \frac{1}{a^2(\xi)}Z^+(x,\xi)\Omega^+(y,\xi)\begin{bmatrix}\delta u(y)\\ \delta\overline{u}(y)\end{bmatrix}\right)dyd\xi$$

$$= \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\frac{1}{\overline{a}^2(\xi)}Z^-(x,\xi)\Omega^-(y,\xi) + \frac{1}{a^2(\xi)}Z^+(x,\xi)\Omega^+(y,\xi)\right)d\xi\begin{bmatrix}\delta u(y)\\ \delta\overline{u}(y)\end{bmatrix}dy,$$

which implies (4.44) □

Lemma 4.13. *The squared eigenfunctions* $\{Z^+, Z^-\}$ *and the adjoint squared eigenfunctions* $\{\Omega^+, \Omega^-\}$ *satisfy the following orthogonality relations:*

$$\begin{aligned}
\left\langle \Omega^-(x,\xi), Z^-(x,\xi')\right\rangle &= \pi\overline{a}^2(\xi)\delta(\xi-\xi'),\\
\left\langle \Omega^+(x,\xi), Z^+(x,\xi')\right\rangle &= \pi a^2(\xi)\delta(\xi-\xi'),\\
\left\langle \Omega^-(x,\xi), Z^+(x,\xi')\right\rangle &= 0,\\
\left\langle \Omega^+(x,\xi), Z^-(x,\xi')\right\rangle &= 0.
\end{aligned}$$

Proof. For inner products, we substitute (4.41) into (4.17) and (4.18). Then

$$\begin{aligned}
\delta\rho &= \frac{1}{\overline{a}^2(\xi)}\int_{\mathbb{R}}\Omega^-(x,\xi)\left[\frac{1}{\pi}\int_{\mathbb{R}}\Big(Z^-(x,\xi')\delta\rho(\xi') + Z^+(x,\xi')\delta\tilde{\rho}(\xi')\Big)d\xi'\right]dx\\
&= \frac{1}{\pi\overline{a}^2(\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\Omega^-(x,\xi)Z^-(x,\xi')\delta\rho(\xi') + \Omega^-(x,\xi)Z^+(x,\xi')\delta\tilde{\rho}(\xi')\Big)d\xi'dx\\
&= \frac{1}{\pi\overline{a}^2(\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\Omega^-(x,\xi)Z^-(x,\xi') \quad \Omega^-(x,\xi)Z^+(x,\xi')\Big]\begin{bmatrix}\delta\rho(\xi')\\ \delta\tilde{\rho}(\xi')\end{bmatrix}d\xi'dx\\
&= \frac{1}{\pi\overline{a}^2(\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\Omega^-(x,\xi)Z^-(x,\xi') \quad \Omega^-(x,\xi)Z^+(x,\xi')\Big]dx\begin{bmatrix}\delta\rho(\xi')\\ \delta\tilde{\rho}(\xi')\end{bmatrix}d\xi'.
\end{aligned}$$

Performing the same for $\delta\tilde{\rho}$ we have

$$\delta\tilde{\rho} = \frac{1}{\pi a^2(\xi)}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big[\Omega^+(x,\xi)Z^-(x,\xi') \quad \Omega^+(x,\xi)Z^+(x,\xi')\Big]dx\begin{bmatrix}\delta\rho(\xi')\\ \delta\tilde{\rho}(\xi')\end{bmatrix}d\xi'.$$

Hence, we have

$$\begin{bmatrix}\delta\rho(\xi)\\ \delta\tilde{\rho}(\xi)\end{bmatrix} = \int_{\mathbb{R}} B(\xi,\xi') \begin{bmatrix}\delta\rho(\xi')\\ \delta\tilde{\rho}(\xi')\end{bmatrix} d\xi'$$

with

$$B(\xi,\xi') = \begin{bmatrix} \frac{1}{\pi\overline{a}^2(\xi)}\Big\langle \Omega^-(x,\xi), Z^-(x,\xi')\Big\rangle & \frac{1}{\pi\overline{a}^2(\xi)}\Big\langle \Omega^-(x,\xi), Z^+(x,\xi')\Big\rangle \\ \frac{1}{\pi a^2(\xi)}\Big\langle \Omega^+(x,\xi), Z^-(x,\xi')\Big\rangle & \frac{1}{\pi a^2(\xi)}\Big\langle \Omega^+(x,\xi), Z^+(x,\xi')\Big\rangle \end{bmatrix}.$$

These relations imply $B(\xi,\xi') = \delta(\xi-\xi')I$, which are equivalent to the four orthogonality relations. □

Proof of Theorem 2 follows from Lemmas 4.12-4.13 and is provided below.

Proof. To justify the choice of coefficients (4.2) we multiply (4.1) by $\Omega^-(y,k)$ and integrate over $y \in \mathbb{R}$. We obtain the following:

$$\begin{aligned}
\int_{\mathbb{R}}\Omega^-(y,k)f(y)dy &= \int_{\mathbb{R}}\Omega^-(y,k)\int_{\mathbb{R}}\Big[\tilde{C}(k')Z^-(y,k') + \tilde{D}(k')Z^+(y,k')\Big]dk'dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\tilde{C}(k')\Omega^-(y,k)Z^-(y,k')dk'dy + \int_{\mathbb{R}}\int_{\mathbb{R}}\tilde{D}(k')\Omega^-(y,k)Z^+(y,k')dk'dy \\
&= \int_{\mathbb{R}}\tilde{C}(k')\int_{\mathbb{R}}\Omega^-(y,k)Z^-(y,k')dydk' + \int_{\mathbb{R}}\tilde{D}(k')\int_{\mathbb{R}}\Omega^-(y,k)Z^+(y,k')dydk' \\
&= \int_{\mathbb{R}}\tilde{C}(k')\left[\pi\overline{a}^2(k)\delta(k-k')\right]dk' = \pi\overline{a}^2(k)\int_{\mathbb{R}}\tilde{C}(k')\delta(k-k')dk' \\
&= \pi\overline{a}^2(k)\tilde{C}(k),
\end{aligned}$$

where orthogonality relations from Lemma 4.13 were used. This yields (4.2) for $\tilde{C}(k)$. In the similar way we justify (4.2) for $\tilde{D}(k)$, by multiplying (4.1) by $\Omega^+(y,k)$ and integrating over $y \in \mathbb{R}$. □

Now inserting both coefficients (4.2) into (4.1) we have the following illustration that (4.44) is the completeness relation between squared eigenfunctions $\{Z^+, Z^-\}$ and adjoint squared eigenfunctions $\{\Omega^+, \Omega^-\}$:

$$
\begin{aligned}
f(x) &= \int_{\mathbb{R}} \Bigg[\left(\frac{1}{\pi \overline{a}^2(k)} \int_{\mathbb{R}} \Omega^-(y,k) f(y) dy \right) Z^-(x,k) \\
&\qquad\qquad + \left(\frac{1}{\pi a^2(k)} \int_{\mathbb{R}} \Omega^+(y,k) f(y) dy \right) Z^+(x,k) \Bigg] dk \\
&= \int_{\mathbb{R}} \Bigg[\frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\overline{a}^2(k)} \Omega^-(y,k) Z^-(x,k) f(y) dy \\
&\qquad\qquad + \frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{a^2(k)} \Omega^+(y,k) Z^+(x,k) f(y) dy \Bigg] dk \\
&= \int_{\mathbb{R}} \left[\frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{\overline{a}^2(k)} \Omega^-(y,k) Z^-(x,k) dk \right] f(y) dy \\
&\qquad\qquad + \int_{\mathbb{R}} \left[\frac{1}{\pi} \int_{\mathbb{R}} \frac{1}{a^2(k)} \Omega^+(y,k) Z^+(x,k) dk \right] f(y) dy \\
&= \int_{\mathbb{R}} \left(\frac{1}{\pi} \int_{\mathbb{R}} \left[\frac{1}{\overline{a}^2(k)} \Omega^-(y,k) Z^-(x,k) + \frac{1}{a^2(k)} \Omega^+(y,k) Z^+(x,k) \right] dk \right) f(y) dy \\
&= \int_{\mathbb{R}} \delta(x-y) I f(y) dy = f(x),
\end{aligned}
$$

where the completeness relation (4.44) has been used.

Chapter 5

Relation of squared eigenfunctions to the linearized NLS equation

In this section we explicitly show that the squared eigenfunctions of the linear system (1.2)-(1.3) solve the linearized NLS equation (1.5) and the adjoint squared eigenfunctions solve the corresponding adjoint linearized NLS equation. To do so, we follow the procedure below:

- Show dependence of squared eigenfunctions and adjoint squared eigenfunctions on t.
- Show that the time-dependent squared eigenfunctions are solution to the linearized NLS equation.
- Find an adjoint linearized NLS equation and prove that time-dependent adjoint squared eigenfunctions solve it.

Since in all previous computations, we have set $t = 0$ and omitted t from arguments of the fundamental solutions Φ, Φ^{-1}. Let us now augment all expressions by explicitly writing their dependence on t. In particular, the time-dependent squared eigenfunctions and the adjoint squared eigenfunctions similar to (4.19) and (4.43) are denoted as follows:

$$Z^{-(t)} = \begin{bmatrix} (\widehat{\phi}_1^{(t)})^2 \\ -(\widehat{\phi}_2^{(t)})^2 \end{bmatrix}, \qquad Z^{+(t)} = \begin{bmatrix} -(\phi_1^{(t)})^2 \\ (\phi_2^{(t)})^2 \end{bmatrix},$$

$$\Omega^{-(t)} = -\begin{bmatrix} (\widehat{\psi}_2^{(t)})^2 \\ (\widehat{\psi}_1^{(t)})^2 \end{bmatrix}, \qquad \Omega^{+(t)} = -\begin{bmatrix} (\psi_2^{(t)})^2 \\ (\psi_1^{(t)})^2 \end{bmatrix}.$$

Proposition 5.1. *Let u be a solution of the NLS equation* (1.1)*. Then, variation* $(\delta u, \delta\overline{u})$ *is a solution of the following linearized NLS equation:*

$$\mathcal{L} \begin{bmatrix} \delta u(x,t) \\ \delta\overline{u}(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where

$$\mathcal{L} = \begin{bmatrix} i\partial_t + \partial_{xx} + 4|u|^2 & 2u^2 \\ -2\overline{u}^2 & i\partial_t - \partial_{xx} - 4|u|^2 \end{bmatrix} \tag{5.1}$$

is the linearization operator.

Proof. Substitute (4.3) into (1.1) and proceeding as in (1.5) we obtain the linearized NLS for $(\delta u, \delta\overline{u})$:

$$i\delta u_t + \delta u_{xx} + 2u^2\delta\overline{u} + 4|u|^2\delta u = 0, \tag{5.2}$$

The complex conjugate equation is given by

$$\begin{aligned} 0 &= \overline{i\delta u_t} + \overline{\delta u_{xx}} + 2\overline{u^2\delta\overline{u}} + 4\overline{|u|^2\delta u} \\ &= -i\delta\overline{u}_t + \delta u_{xx} + 2\overline{u}^2\delta u + 4|u|^2\delta\overline{u} = 0. \end{aligned}$$

Multiplying the above by -1 we obtain

$$i\delta\overline{u}_t - \delta\overline{u}_{xx} - 4|u|^2\delta\overline{u} - 2\overline{u}^2\delta u = 0, \tag{5.3}$$

Combining (5.2) and (5.3) yields (5.1). □

Theorem 3. *The time-dependent squared eigenfunctions $Z^{-(t)}, Z^{+(t)}$ satisfy the linearized NLS equation:*

$$\mathcal{L}Z^{-(t)}(x,k,t) = \mathcal{L}Z^{+(t)}(x,k,t) = 0. \tag{5.4}$$

Proof. We will prove the result for $Z^{-(t)}$, since the proof for $Z^{+(t)}$ is identical. For simplicity, we denote $v_1 = \widehat{\phi}_1^{(t)}$, $v_2 = \widehat{\phi}_2^{(t)}$. Substituting these into (5.4) and using (5.2), (5.3) we want to show

$$\mathcal{L}Z^{-(t)} = \begin{bmatrix} \Big[i\partial_t + \partial_{xx} + 4|u|^2, \quad 2u^2\Big] \begin{pmatrix} v_1^2 \\ -v_2^2 \end{pmatrix} \\ \Big[-2|u|^2, \quad i\partial_t - \partial_{xx} - 4|u|^2\Big] \begin{pmatrix} v_1^2 \\ -v_2^2 \end{pmatrix} \end{bmatrix} = 0.$$

Consider the following

$$\begin{aligned} \Big[i\partial_t + \partial_{xx} + 4|u|^2, \quad 2u^2\Big] \begin{pmatrix} v_1^2 \\ -v_2^2 \end{pmatrix} &= i\partial_t(v_1^2) + \partial_{xx}(v_1^2) + 4|u|^2(v_1^2) - 2u^2v_2^2 \\ &= 2iv_1(v_1)_t + 2v_1(v_1)_{xx} + 2(v_1)_x^2 + 4|u|^2v_1^2 - 2u^2v_2^2. \end{aligned} \tag{5.5}$$

Since $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a solution to (1.2) and (1.3) we have the following

$$\begin{aligned} &(v_1)_{xx} = (v_{1x})_x = (-ikv_1 + uv_2)_x = -ik(v_1)_x + u_xv_2 + u(v_2)_x, \\ &(v_2)_x = -\overline{u}v_1 + ikv_2 \\ &(v_1)_t = -2ik^2v_1 + i|u|^2v_1 + iu_xv_2 + 2kuv_2. \end{aligned}$$

Substituting the above into (5.5) we have the following

$$\begin{aligned} &2iv_1\Big\{-2ik^2v_1 + i|u|^2v_1 + iu_xv_2 + 2kuv_2\Big\} + 2v_1\Big\{-ik(v_1)_x + u_xv_2 + u(v_2)_x\Big\} \\ &\quad + 2(-ikv_1 + uv_2)^2 + 4|u|^2v_1^2 - 2u_2v_2^2 \end{aligned}$$

$$= 4k^2v_1^2 - 2|u|^2v_1^2 - 2u_xv_1v_2 + 4ikuv_1v_2 + 2v_1\Big\{ - k^2v_1 + u_xv_2 - |u|^2v_1\Big\}$$
$$- 2k^2v_1^2 + 2u^2v_2^2 - 4ikuv_1v_2 + 4|u|^2v_1^2 - 2u^2v_2^2$$
$$= \Big\{4k^2 - 2|u|^2 - 2k^2 - 2|u|^2 - 2k^2 + 4|u|^2\Big\}v_1^2 + \Big\{2u^2 - 2u^2\Big\}v_2^2$$
$$+ \Big\{ - 2u_x + 4iku + 2u_x - 4iku\Big\}v_1v_2 = 0$$

Following the same procedure, we have $\begin{bmatrix}-2|u|^2, & i\partial_t - \partial_{xx} - 4|u|^2\end{bmatrix}\begin{pmatrix} v_1^2 \\ -v_2^2\end{pmatrix} = 0.$ □

Proposition 5.2. *The adjoint linearized NLS equation is written as*

$$\mathcal{L}^* \begin{bmatrix}\delta v(x,t) \\ \delta\overline{v}(x,t)\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix},$$

where

$$\mathcal{L}^* = \begin{bmatrix}-i\partial_t + \partial_{xx} + 4|u|^2 & -2\overline{u}^2 \\ 2u^2 & -i\partial_t - \partial_{xx} - 4|u|^2\end{bmatrix}.$$

Proof. Adjoint operator satisfies the following

$$\langle \mathcal{L}u, v\rangle = \langle u, \mathcal{L}^* v\rangle.$$

Using (2.24) we write above as

$$\langle \mathcal{L}u, v\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} \begin{bmatrix} i\partial_t + \partial_x x + 4|u|^2 & 2u^2 \\ -2\overline{u}^2 & i\partial_t - \partial_{xx} - 4|u|^2\end{bmatrix}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}\cdot\begin{bmatrix}v_1 \\ v_2\end{bmatrix} dxdt$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(i(u_1)_tv_1 + (u_1)_{xx}v_1 + 4|u|^2u_1v_1 + 2u^2u_2v_1$$
$$- 2\overline{u}^2u_1v_2 + i(u_2)_tv_2 - (u_2)_{xx}v_2 - 4|u|^2u_2v_2\Big)dxdt$$

$$= \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(i(u_1)_t v_1 + i(u_2)_t v_2 \Big) dtdx}_{=1} + \underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} \Big((u_1)_{xx} v_1 - (u_2)_{xx} v_2 \Big) dxdt}_{=2}$$

$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(4|u|^2 u_1 v_1 + 2u^2 u_2 v_1 - 2\overline{u}^2 u_1 v_2 - 4|u|^2 u_2 v_2 \Big) dxdt$$

Integrating first integral by parts once, and second integral by parts twice we obtain the following

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} \Big(-iu_1(v_1)_t - iu_2(v_2)_t \Big) dtdx + \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(u_1(v_1)_{xx} - u_2(v_2)_{xx} \Big) dxdt \\
&\qquad + \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(4|u|^2 u_1 v_1 + 2u^2 u_2 v_1 - 2\overline{u}^2 u_1 v_2 - 4|u|^2 u_2 v_2 \Big) dxdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(-iu_1(v_1)_t + u_1(v_1)_{xx} + 4|u|^2 u_1 v_1 + 2u^2 u_2 v_1 \\
&\qquad - 2\overline{u}^2 u_1 v_2 - iu_2(v_2)_t - u_2(v_2)_{xx} - 4|u|^2 u_2 v_2 \Big) dxdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} -i\partial_t v_1 + \partial_{xx} v_1 + 4|u|^2 v_1 & -2\overline{u}^2 v_2 \\ 2u^2 v_1 & -i\partial_t v_2 - \partial_{xx} v_2 - 4|u|^2 v_2 \end{bmatrix} dxdt \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} -i\partial_t + \partial_{xx} + 4|u|^2 & -2\overline{u}^2 \\ 2u^2 & -i\partial_t - \partial_{xx} - 4|u|^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} dxdt = \langle u, \mathcal{L}^* v \rangle,
\end{aligned}$$

which yields the result. □

Theorem 4. *The time-dependent adjoint squared eigenfunctions* $\Omega^{-(t)}, \Omega^{+(t)}$ *satisfy the adjoint linearized NLS equation:*

$$\mathcal{L}^* \Omega^{-(t)}(x,k,t) = \mathcal{L}^* \Omega^{+(t)}(x,k,t) = 0.$$

Proof. We will prove for $\Omega^{-(t)}$, since $\Omega^{+(t)}$ case is identical. For simplicity, let's use the following notation within our proof $w_1 = \widehat{\psi}_1^{(t)}$, $w_2 = \widehat{\psi}_2^{(t)}$. Now, consider the following

$$\begin{bmatrix} -i\partial_t + \partial_{xx} + 4|u|^2 & -2\overline{u}^2 \end{bmatrix} \begin{pmatrix} -w_2^2 \\ -w_1^2 \end{pmatrix} = -i(-w_2^2)_t + (-w_2^2)_{xx} - 4|u|^2 w_2^2 + 2\overline{u}^2 w_1^2$$

$$= 2iw_2(w_2)_t - 2w_2(w_2)_{xx} - 2(w_2)_x^2 - 4|u|^2w_2^2 + 2\overline{u}^2w_1^2 \tag{5.6}$$

Since $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is a solution to (1.2) and (1.3), then w_1, w_2 satisfy the following

$$\begin{aligned}
&(w_2)_{xx} = (w_{2x})_x = (-\overline{u}w_1 + ikw_2)_x = -\overline{u}_x w_1 - \overline{u}(w_1)_x + ik(w_2)_x \\
&(w_1)_x = -ikw_1 + uw_2 \\
&(w_2)_t = i\overline{u}_x w_1 - 2k\overline{u}w_1 + 2ik^2w_2 - i|u|^2w_2
\end{aligned}$$

Using this in (5.6) we obtain

$$\begin{aligned}
&2iw_2\Big\{i\overline{u}_x w_1 - 2k\overline{u}w_1 + 2ik^2w_2 - i|u|^2w_2\Big\} - 2w_2\Big\{-\overline{u}_x - \overline{u}(w_1)_x + ik(w_2)_x\Big\} \\
&\quad - 2(-\overline{u}w_1 + ikw_2)^2 - 4|u|^2w_2^2 + 2\overline{u}^2w_1^2 \\
&= -2\overline{u}_x w_1 w_2 - 4ik\overline{u}w_1w_2 - 4k^2w_2^2 + 2|u|^2w_2^2 - 2w_2\Big\{-\overline{u}_x w_1 - |u|^2w_2 - k^2w_2\Big\} \\
&\quad - 2\overline{u}^2w_1^2 + 2k_2^2w_2^2 + 4ik\overline{u}w_1w_2 - 4|u|^2w_2^2 + 2\overline{u}^2w_1^2 \\
&= \Big\{-2\overline{u}^2 + 2\overline{u}^2\Big\}w_1^2 + \Big\{-4k^2 + 2|u|^2 + 2|u|^2 + 2k^2 + 2k^2 - 4|u|^2\Big\}w_2^2 \\
&\quad + \Big\{-2\overline{u}_x - 4ik\overline{u} + 2\overline{u}_x + 4ik\overline{u}\Big\}w_1w_2 = 0.
\end{aligned}$$

Following the same procedure, we have $\Big[2u^2, \quad -i\partial_t - \partial_{xx} - 4|u|^2\Big]\begin{pmatrix} -w_2^2 \\ -w_1^2 \end{pmatrix} = 0.$ □

Chapter 6

Conclusion and future directions

This thesis is centered around completeness of eigenfunctions of the spectral problem (1.2). In this study, a complete set of eigenfunctions is a basis in the L^2 space. We proved completeness of eigenfunctions of (1.2) in Chapter 3. To do so, we integrated special functions $\mathcal{R}^{\pm}$ and obtained a completeness relation in Lemma 3.2, then we found orthogonality relations between eigenfunctions and adjoint eigenfunctions in Lemma 3.3. These results lead to Theorem 1, where we provided a proof of completeness of a set $\{\phi, \widehat{\phi}\}$.

We extend our study to completeness of squared eigenfunctions in Chapter 4. First, we perturb our potential by adding a vartiation to it in (4.3). To derive adjoint squared eigenfunctions $\Omega^{\pm}$ in (4.19), we express variation of potential in terms of variation of scattering data. Then, by expressing variation of scattering data in terms of variation of potential we derive squared eigenfunctions $Z^{\pm}$ in (4.43). We obtain orthogonality relations between squared eigenfunctions and adjoint squared eigenfunctions in Lemma 4.13 that helps us to prove completeness of a set $\{Z^{-}, Z^{+}\}$ in Theorem 2.

In Chapter 5 we explain a connection between the squared eigenfunctions and the linearized NLS equation (1.5). Precisely, we manually show that squared eigenfunctions $Z^{\pm}$ are solutions to the linearized NLS equation in Theorem 3 and that adjoint squared eigenfunctions $\Omega^{\pm}$ are solutions to the adjoint linearized NLS equation in Theorem 4.

It is crucial to note that throughout the whole thesis we assumed that scattering data a and $\overline{a}$ are nonzero in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. Allowing these to have simple or multiple zeros complicate the problem to the next level. In particular, when integrating special functions $\mathcal{R}^{\pm}$ we see that this creates poles in $\mathbb{C}_{\pm}$, simple or multiple depending on the assumption. This means that the completeness relation would have contribution from residues. In the case of squared eigenfunctions, by looking at the completeness relation (4.1) we see that simple zeros of $a, \overline{a}$ are automatically double zeros because of the exponent of $a, \overline{a}$. In this case, we can expect that contribution from residues would involve derivatives of squared eigenfunctions with respect to k.

The case with the simple zeros of a, $\bar{a}$ was considered and discussed in [1]. In addition to continuous eigenfunctions $\{\phi(x,k), \widehat{\phi}(x,k)\}$ for $k \in \mathbb{R}$, a new complete set includes discrete eigenfunctions $\{\phi(x,k_j), \widehat{\phi}(x,\widehat{k}_j)\}$ at the zeros $k_j, \widehat{k}_j$ of $a, \overline{a}$ respectively. For squared eigenfunctions, the complete set $\{Z^-, Z^+\}$ is modified by addition of four additional eigenfunctions at the zeros $k_j, \widehat{k}_j$ of $a, \overline{a}$, respectively.

In this thesis all our computations were done for the zero background, i.e. the decay of the potential u to zero. Even though the results of this study are not novel, we present detailed proofs and thorough explanations that could act as a literature review and a good foundation for further research. In particular, this thesis may be used to tackle an open problem on completeness of the squared eigenfunctions for the potentials at the nonzero background (constant and/or periodic boundary conditions at $x \to \pm\infty$).

Bibliography

[1] J. Yang. Nonlinear waves in integrable and nonintegrable systems. SIAM, 2010.

[2] V. E. Zakharov and A. B. Shabat. Exact Theory of Two-dimensional Self-focusing and One-dimensional Self-modulation of Waves in Nonlinear Media. *Soviet Journal of Experimental and Theoretical Physics* 34 (Jan. 1972), 62.

[3] D. H. Peregrine. Water waves, nonlinear Schrödinger equations and their solutions. *The ANZIAM Journal* 25(1) (1983), 16–43.

[4] N. N. Akhmediev, V. M. Eleonskii, and N. Kulagin. Exact first-order solutions of the nonlinear Schrödinger equation. *Theoretical and mathematical physics* 72(2) (1987), 809–818.

[5] Y.-C. Ma. The perturbed plane-wave solutions of the cubic Schrödinger equation. *Studies in Applied Mathematics* 60(1) (1979), 43–58.

[6] G. P. Agrawal. Nonlinear fiber optics. In: *Nonlinear Science at the Dawn of the 21st Century*. Springer, 2000, 195–211.

[7] G. Fibich. The nonlinear Schrödinger equation: singular solutions and optical collapse. Vol. 192. Springer, 2015.

[8] C. Kharif, E. Pelinovsky, and A. Slunyaev. Rogue waves in the ocean. Springer Science & Business Media, 2008.

[9] L. Pitaevskii and S. Stringari. Bose-Einstein condensation and superfluidity. Vol. 164. Oxford University Press, 2016.

[10] M. Gedalin, T. C. Scott, and Y. B. Band. Optical Solitary Waves in the Higher Order Nonlinear Schrödinger Equation. *Phys. Rev. Lett.* 78 (3 Jan. 1997), 448–451.

[11] S. L. Palacios, A. Guinea, J. M. Fernández-Dıaz, and R. D. Crespo. Dark solitary waves in the nonlinear Schrödinger equation with third order dispersion, self-steepening, and self-frequency shift. *Phys. Rev. E* 60 (1 July 1999), R45–R47.

[12] Z. Li, L. Li, H. Tian, and G. Zhou. New Types of Solitary Wave Solutions for the Higher Order Nonlinear Schrödinger Equation. *Phys. Rev. Lett.* 84 (18 May 2000), 4096–4099.

[13] H. Segur, D. Henderson, J. Carter, J. Hammack, C.-M. Li, D. Pheiff, and K. Socha. Stabilizing the benjamin–feir instability. *Journal of Fluid Mechanics* 539 (2005), 229–271.

[14] A. G. Athanassoulis, G. A. Athanassoulis, and T. P. Sapsis. Localized instabilities of the Wigner equation as a model for the emergence of Rogue Waves. *Journal of Ocean Engineering and Marine Energy* 3(4) (Nov. 2017), 353–372. ISSN: 2198-6452.

[15] B. Hall, M. Lisak, D. Anderson, R. Fedele, and V. Semenov. Statistical theory for incoherent light propagation in nonlinear media. *Physical Review E* 65(3) (2002), 035602.

[16] A.-T. Assaubay, A. J. Castro, and A. A. Valido. Wigner instability analysis of the damped Hirota equation. *Physica D: Nonlinear Phenomena* 411 (2020), 132587.

[17] D. E. Pelinovsky. Instability of double-periodic waves in the nonlinear Schrödinger equation. *Frontiers in Physics* 9 (2021), 599146.

[18] M. Haragus and D. E. Pelinovsky. Linear instability of breathers for the focusing nonlinear Schrödinger equation. *Journal of Nonlinear Science* 32(5) (2022), 66.

[19] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Studies in applied mathematics* 53(4) (1974), 249–315.

[20] D. Kaup. Closure of the squared Zakharov-Shabat eigenstates. *Journal of Mathematical Analysis and Applications* 54(3) (1976), 849–864.

[21] D. Kaup. Perturbation theory for solitons in optical fibers. *Physical Review A* 42(9) (1990), 5689.

[22] D. Kaup and T. Lakoba. The squared eigenfunctions of the massive Thirring model in laboratory coordinates. *Journal of Mathematical Physics* 37(1) (1996), 308–323.

[23] D. Kaup, T. Lakoba, and Y. Matsuno. Complete integrability of the Benjamin-Ono equation by means of action-angle variables. *Physics Letters A* 238(2-3) (1998), 123–133.

[24] J. Yang and D. Kaup. Squared eigenfunctions for the Sasa–Satsuma equation. *Journal of mathematical physics* 50(2) (2009), 023504.

[25] D. Bilman and P. D. Miller. A robust inverse scattering transform for the focusing nonlinear Schrödinger equation. *Communications on Pure and Applied Mathematics* 72(8) (2019), 1722–1805.

[26] P. Grinevich and P. Santini. The linear and nonlinear instability of the Akhmediev breather. *Nonlinearity* 34(12) (2021), 8331.

[27] M. J. Ablowitz, B. Prinari, and A. D. Trubatch. Discrete and continuous nonlinear Schrödinger systems. Vol. 302. Cambridge University Press, 2004.