COMPARING INVARIANTS OF TORIC IDEALS OF BIPARTITE GRAPHS

Comparing Invariants of Toric Ideals of Bipartite Graphs

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A Thesis Submitted to the School of Graduate Studies in the Partial Fulfillment of the Requirements for the Degree Master of Science

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Abstract

Given a finite simple graph G, one can associate to G an ideal I_G, called the *toric ideal of* G. There are a number of algebraic invariants of ideals which are frequently studied in commutative algebra. In general, understanding these invariants is very difficult for arbitrary ideals. However, when the ideals are related to combinatorial objects, in this case, graphs, a deeper investigation can be conducted. If, in addition, the graph G is bipartite, even more can be said about these invariants. In this thesis, we explore a comparison of invariants of toric ideals of bipartite graphs. Our main result describes all possible values for the tuple

$$(\mathrm{reg}(\mathbb{K}[E]/I_G), \deg(h_{\mathbb{K}[E]/I_G}), \mathrm{pdim}(\mathbb{K}[E]/I_G), \mathrm{depth}(\mathbb{K}[E]/I_G), \dim(\mathbb{K}[E]/I_G))$$

when G is a bipartite graph on $n \geq 1$ vertices.

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Chapter 1

Introduction

This thesis takes place in the realm of combinatorial commutative algebra. This modern and active field of mathematics lies at the intersection of commutative algebra and combinatorics. A major goal of research in combinatorial commutative algebra is to study invariants of algebraic objects by gleaning information from their combinatorial structure. In our case, the algebraic objects are toric ideals, and the combinatorial structure comes from properties of an associated finite simple graph.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let G be a finite simple graph (hereafter referred to as a graph) with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{e_1, \dots, e_q\}$. The **toric ideal of** G, denoted I_G, is the kernel of the map

$$\begin{aligned} \varphi\colon \mathbb{K}[e_1, \dots, e_q] &\to \mathbb{K}[v_1, \dots, v_n] \\ e_i &\mapsto v_{i_1} v_{i_2} \end{aligned}$$

where $e_i = \{v_{i_1}, v_{i_2}\} \in E$. We write $\mathbb{K}[E]$ and $\mathbb{K}[V]$ for the polynomial rings $\mathbb{K}[e_1, \dots, e_q]$ and $\mathbb{K}[v_1, \dots, v_n]$ respectively. There are a number of homological invariants which can be associated to I_G. These include

- the regularity, $\operatorname{reg}(\mathbb{K}[E]/I_G)$;
- the degree of the h-polynomial, $\deg(h_{\mathbb{K}[E]/I_G})$;
- the projective dimension, $\operatorname{pdim}(\mathbb{K}[E]/I_G)$;
- the depth, $\operatorname{depth}(\mathbb{K}[E]/I_G)$; and
- the (Krull) dimension, $\dim(\mathbb{K}[E]/I_G)$.

Some properties of these homological invariants of I_G can be found in [3, 10, 14, 16, 18, 23, 28, 39].

In recent years, there has been a particular focus on comparing these invariants:

Question 1.1. *What can be said about the sets*

$$\{(\mathrm{reg}(\mathbb{K}[E]/I_G), \deg(h_{\mathbb{K}[E]/I_G})) \mid G \text{ is a graph}\},$$

$$\{(\mathrm{reg}(\mathbb{K}[E]/I_G), \deg(h_{\mathbb{K}[E]/I_G})) \mid G \text{ is a graph on } n \text{ vertices}\},$$

or similar sets for other pairs of invariants?

Questions of this type were first studied by Hibi and Matsuda in [26], in the case of monomial ideals in a polynomial ring instead of toric ideals. Comparisons of invariants of other graph-associated ideals (in particular, edge ideals and binomial edge ideals) have also been studied in the literature. For instance, Hibi, Matsuda, and Van Tuyl [27] showed that for any positive integers r and d, there is an edge ideal $I(G)$ such that $\mathrm{reg}(\mathbb{K}[V]/I(G)) = r$ and $\deg(h_{K[V]/I(G)}) = d$, while Hà and Hibi [17] determined, for any graph G on n vertices, all the pairs $(\mathrm{reg}(\mathbb{K}[V]/I(G)), \mathrm{pdim}([\mathbb{K}[V]/I(G)))$ for which each invariant attains its respective minimum value. On the other hand, for binomial edge ideals J_G, Ficarra and Sgroi [13] recently described (fairly comprehensively) the pairs $(\mathrm{reg}(J_G), \mathrm{pdim}(J_G))$, where G ranges over all graphs on n vertices with no isolated vertices.

However, things are not so simple for toric ideals of graphs. In [11] Favacchio, Keiper, and Van Tuyl proved that for any positive integers $d \geq r \geq 4$, there is a connected graph G such that $\mathrm{reg}(\mathbb{K}[E]/I_G) = r$ and $\deg(h_{K[E]/I_G}) = d$. However, they also showed that this result does not hold for all pairs of positive integers, since $\deg(h_{K[E]/I_G}) = 1$ if $\mathrm{reg}(\mathbb{K}[E]/I_G) = 1$. It is therefore of interest to understand which pairs (r, d) of positive integers can be realized in this way. One method to approach this problem is to restrict the class of graphs considered. Thus, we can refine Question 1.1 as follows:

Question 1.2. *Fix a class $\mathcal{G}$ of graphs. What can be said about the sets*

$$\{(\mathrm{reg}(\mathbb{K}[E]/I_G), \deg(h_{\mathbb{K}[E]/I_G})) \mid G \in \mathcal{G}\},$$

$$\{(\mathrm{reg}(\mathbb{K}[E]/I_G), \deg(h_{\mathbb{K}[E]/I_G})) \mid G \in \mathcal{G} \text{ has } n \text{ vertices}\},$$

or similar sets for other pairs of invariants?

This type of problem was first studied (in the context of edge ideals instead of toric ideals) for the class of Cameron-Walker graphs (see [24], [25]). Later, Erey and Hibi [9] determined all the pairs $(\mathrm{reg}(\mathbb{K}[V]/I(G)), \mathrm{pdim}(\mathbb{K}[V]/I(G)))$ as G ranges over all connected bipartite graphs on n vertices. Edge ideals and binomial edge ideals of bipartite graphs have been widely studied in the literature (e.g., see [2, 4, 12, 19, 30, 31, 32, 33, 34, 35, 38, 40, 41]).

The bounty of results concerning edge ideals of bipartite graphs suggest that toric ideals of bipartite graphs may be tractable objects of study. Investigations of invariants of such ideals are sparse in the literature (see [1, 3, 16, 21]). However, the structure provided by bipartite graphs allows for a comprehensive comparison of the aforementioned invariants. For each $n \geq 1$ and $1 \leq c \leq n$, we define $\mathrm{CBPT}(n)$ to be the set of all connected bipartite graphs on n vertices and $\mathrm{BPT}(n,c)$ to be the set of all bipartite graphs on n vertices with c connected components. Setting $\mathbb{K}[G] = \mathbb{K}[E]/I_G$, define the sets

$$\mathrm{CBPT}^{\deg}_{\mathrm{reg}}(n) = \{(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]})) : G \in \mathrm{CBPT}(n)\},$$

$$\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n) = \{(\mathrm{reg}(\mathbb{K}[G]), \mathrm{pdim}(\mathbb{K}[G])) : G \in \mathrm{CBPT}(n)\},$$

$$\mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n) = \{(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) : G \in \mathrm{CBPT}(n)\},$$

and

$$\mathrm{BPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n,c) = \{(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) : G \in \mathrm{BPT}(n,c)\}.$$

The following five theorems are the major results of this thesis.

Theorem 1.3 (Theorem 3.10)**.** *Let $n \geq 1$ be an integer. Then*

$$\mathrm{CBPT}^{\deg}_{\mathrm{reg}}(n) = \left\{(a,a) \in \mathbb{Z}^2 \mid 0 \leq a < \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Theorem 1.3 describes all the possible pairs $(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}))$ for connected bipartite graphs G on n vertices.

Theorem 1.4 (Theorem 4.14)**.** *Let $n \geq 1$ be an integer. Then*

$$\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n) = \left\{(r,p) \in \mathbb{Z}^2 \mid 0 < r < \left\lfloor \frac{n}{2} \right\rfloor,\ 1 \leq p \leq r(n-2-r)\right\} \cup \{(0,0)\}.$$

Theorem 1.4 gives a full characterization of all possible values for the tuple $(\mathrm{reg}(\mathbb{K}[E]/I_G), \mathrm{pdim}(\mathbb{K}[E]/I_G))$ among connected bipartite graphs with a fixed number of vertices, completely analogous to the edge ideal result of Erey and Hibi in [9].

Our proof of this result relies heavily on a classical graph theory result of Jackson [29].

Theorem 1.5 (Theorem 4.17)**.** *Let r and p be integers. Then there is a connected bipartite graph G on at least two vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$ if and only if $r = p = 0$ or $r, p \geq 1$. Equivalently, for $\mathbb{N} = \{1, 2, 3, \dots\}$, we have*

$$\bigcup_{n\geq 1}^{\infty} \mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{pdim}}(n) = \{(0,0)\} \cup \mathbb{N}^2.$$

Theorem 1.5 shows that if we allow for an arbitrary number of vertices, every possible tuple $(r, p) \in \{(0,0)\}\cup\mathbb{N}^2$ can be realized as $(\operatorname{reg}(\mathbb{K}[G]), \operatorname{pdim}(\mathbb{K}[G]))$ for some connected graph G. Recall that Hibi, Matsuda, and Van Tuyl showed in [27] that for any positive integers r and d, there is an edge ideal $I(G)$ such that $\operatorname{reg}(\mathbb{K}[V]/I(G)) = r$ and $\deg(h_{K[V]/I(G)}) = d$. Our theorem is an analogous result, for the case of toric ideals (instead of edge ideals) and projective dimension (instead of degree of the h-polynomial).

Theorem 1.6 (Theorem 4.18)**.** *Let $n \geq 1$ be an integer. Then the set $\mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n)$ is given by*

$$\left\{(r,r,p,n-1,n-1) \in \mathbb{Z}^5 \mid 0 < r < \left\lfloor \frac{n}{2} \right\rfloor,\ 1 \leq p \leq r(n-2-r)\right\}\cup\{(0,0,0,n-1,n-1)\}.$$

Theorem 1.6, combining Theorem 1.3 and Theorem 1.4, gives the values of the invariants $\operatorname{reg}(\mathbb{K}[E]/I_G)$, $\deg(h_{\mathbb{K}[E]/I_G})$, $\operatorname{pdim}(\mathbb{K}[E]/I_G)$, $\operatorname{depth}(\mathbb{K}[E]/I_G)$, and $\dim(\mathbb{K}[E]/I_G)$ as G ranges across connected bipartite graphs on n vertices.

Theorem 1.7 (Theorem 5.13)**.** *Let $n \geq 1$, $1 \leq c \leq n$ be integers. Then the set $\mathrm{BPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n, c)$ is given by*

$$\left\{(r,r,p,n-c,n-c) \in \mathbb{Z}^5 \mid 0 < r < \left\lfloor \tfrac{n-(c-1)}{2} \right\rfloor,\ 1 \leq p \leq r(n-(c+1)-r)\right\} \cup \{(0,0,0,n-c,n-c)\}.$$

Theorem 1.7 generalizes Theorem 1.6, now giving the values of the invariants $\operatorname{reg}(\mathbb{K}[E]/I_G)$, $\deg(h_{\mathbb{K}[E]/I_G})$, $\operatorname{pdim}(\mathbb{K}[E]/I_G)$, $\operatorname{depth}(\mathbb{K}[E]/I_G)$, and $\dim(\mathbb{K}[E]/I_G)$ as G ranges across all bipartite graphs on n vertices with c connected components.

The proofs of all four theorems rely on fundamental results concerning toric ideals of bipartite graphs. These results include the fact that $\mathbb{K}[E]/I_G$ is Cohen-Macaulay when G is bipartite, and recent work of Almousa, Dochtermann, and Smith [1] which derives novel properties of the regularity of toric ideals of graphs using tropical methods.

We now provide a brief overview of the structure of this thesis.

In Chapter 2, we introduce the majority of the definitions, results and concepts from both combinatorics and commutative algebra that we will require throughout the thesis. Some definitions will be saved for later chapters when they are required.

In Chapter 3, we initiate our first comparison of invariants of toric ideals of bipartite graphs, namely, the invariants $\text{reg}(\mathbb{K}[E]/I_G)$ and $\deg(h_{\mathbb{K}[E]/I_G})$. We also introduce the known results concerning regularity of bipartite graphs we will require for our arguments. Our main result in this chapter, Theorem 3.10, describes the pairs $(\text{reg}(\mathbb{K}[E]/I_G), \deg(h_{\mathbb{K}[E]/I_G}))$ as G ranges across all connected bipartite graphs on n vertices.

Chapter 4 then compares the invariants $\text{reg}(\mathbb{K}[E]/I_G)$ and $\text{pdim}(\mathbb{K}[E]/I_G)$ for connected bipartite graphs. The main result of this chapter is Theorem 4.14, which determines, in analogy to Theorem 3.10, the pairs $(\text{reg}(\mathbb{K}[E]/I_G), \text{pdim}(\mathbb{K}[E]/I_G))$ as G ranges across all connected bipartite graphs on n vertices. As a corollary, we determine the number of such pairs for each fixed n in Theorem 4.15. In addition, we show that each pair (r, p) of positive integers is realized as $r = \text{reg}(\mathbb{K}[E]/I_G)$ and $p = \text{pdim}(\mathbb{K}[E]/I_G)$ for some connected bipartite graph G. We conclude this chapter with Theorem 4.18, the main result of this thesis, combining Theorem 3.10 and Theorem 4.14.

In Chapter 5, we generalize our previous results by considering non-connected bipartite graphs G. The main result of this chapter is Theorem 5.13, which provides an analogue to Theorem 4.18 for bipartite graphs on n vertices with c connected components.

In Chapter 6, we summarize our results and suggest several directions for future study.

Finally, we include tables and *Macaulay2* [15] code that was used for this thesis in Appendix A.

Chapter 2

Background

In this chapter, we provide the basic definitions, notation, and concepts which will be used repeatedly throughout this thesis. Section 2.1 gives the relevant graph theory background we will draw from in our arguments. Section 2.2 outlines several key concepts and results in commutative algebra in the specific context of this thesis. The Hilbert-Serre theorem is one such fundamental result; we dedicate Subsection 2.2.1 to its proof. Finally, Section 2.3 introduces toric ideals of graphs, the main topic of this thesis. For the sake of readability, some definitions are saved until later chapters when they are required.

2.1 Graph Theory Background

In this section, we introduce the necessary graph theory terminology. We will restrict ourselves to discussing finite simple graphs, as most of the existing literature on toric ideals of graphs also makes this restriction. However, one should note that toric ideals of graphs can be defined for graphs with loops or multiple edges. The following information can be found in any standard graph theory book (e.g., [45]).

Definition 2.1. A **finite simple graph** $G = (V(G), E(G))$ consists of a non-empty finite set $V(G)$, and a finite set $E(G) \subseteq \{\{u, v\} \mid u, v \in V(G), u \neq v\}$ of distinct unordered pairs of distinct elements of $V(G)$. The elements of $V(G)$ are called **vertices**, the elements of $E(G)$ are called **edges**, and the sets $V(G)$ and $E(G)$ are called the **vertex set** and **edge set** of G, respectively. An edge $e = \{u, v\} \in E(G)$ is said to **join** the vertices $u, v \in V(G)$. We say that two vertices $u, v \in V(G)$ are **adjacent** if $\{u, v\} \in E(G)$. We also say that two edges $e, f \in V(G)$ are **adjacent** if $e \cap f \neq \emptyset$. We say that a vertex $v \in V(G)$ and an edge $e \in E(G)$ are **incident** if $v \in e$. The

degree of a vertex $v \in V(G)$, denoted $\deg(v)$, is the number of edges of G incident with v.

Remark 2.2. The previous definition implies that finite simple graphs cannot have edges between identical vertices, or multiple edges between vertices. For the remainder of this thesis, we will refer to finite simple graphs simply as "graphs." Also, we refer to $V(G)$ and $E(G)$ as V and E, respectively, when there is no confusion of the associated graph.

The notion of a graph isomorphism allows us to identify graphs that are essentially the same after relabelling.

Definition 2.3. Two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are **isomorphic** if there exists a bijection $\varphi\colon V(G) \to V(H)$ such that $\{u, v\} \in V(G)$ if and only if $\{\varphi(u), \varphi(v)\} \in V(H)$.

Definition 2.4. A graph H is said to be a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In this case, we say that G **contains** H and write $G \subseteq H$.

Definition 2.5. A subgraph H of a graph G is said to be an **induced subgraph** of G if for every $u, v \in V(H)$, $\{u, v\} \in E(H)$ if $\{u, v\} \in E(G)$.

Definition 2.6. A subgraph H of a graph G is said to be a **spanning subgraph** of G if $V(H) = V(G)$.

Example 2.7. Let $G = (V, E)$ be the graph in Figure 2.1. Let H_1 be the subgraph of G consisting of the solid edges and all incident vertices, let H_2 be the subgraph of G consisting of the dashed edges and all incident vertices, and let H_3 be the subgraph of G consisting of all vertices and edges of G, except the edge e. The graphs H_1 and H_2 are induced subgraphs of G, but not spanning subgraphs of G as their vertex sets differ from the vertex set of G. The graph H_3 is a spanning subgraph of G, but not an induced subgraph of G, since vertices 1 and 2 are in the vertex set of H_3, but the edge $e \in E$ is not in the edge set of H_3.

We now define the concept of a walk on a graph, which is fundamental to elucidating the connection between a graph and its corresponding toric ideal.

Definition 2.8. Let $G = (V, E)$ be a graph. A **walk** of G is a sequence of edges $w = (e_1, e_2, \dots, e_m)$, where each $e_i = \{u_{i_1}, u_{i_2}\} \in E$ and $u_{i_2} = u_{(i+1)_1}$ for each $i = 1, \dots, m-1$. Equivalently, a walk is a sequence of vertices $(u_1, \dots, u_m, u_{m+1})$ such that $\{u_i, u_{i+1}\} \in E$ for all $i = 1, \dots, m$. Here, m is referred to as the **length** of the walk. A walk is said to be **even** if m is even. A walk is called **closed** if $u_{m+1} = u_1$.

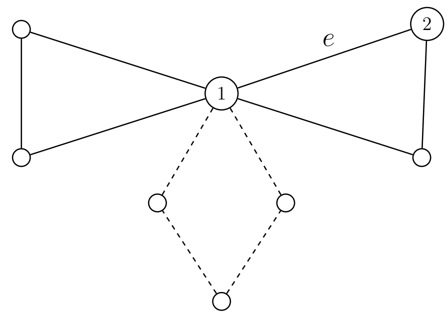

Figure 2.1: The graph G with subgraphs H_1, H_2, and H_3

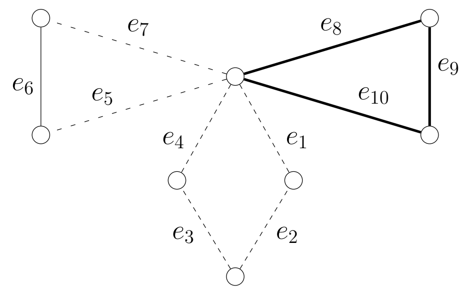

Figure 2.2: The graph G with walks w_1, w_2, and w_3

Example 2.9. Consider the graph G in Figure 2.2. The three sequences of edges $w_1 = (e_5, e_7)$, $w_2 = (e_8, e_9, e_{10})$, and $w_3 = (e_1, e_2, e_3, e_4)$ are all walks of G. The walk w_1 is an even walk, the walk w_2 is a closed walk, and the walk w_3 is both closed and even.

Definition 2.10. Two vertices u and v are said to be **connected** if there is a walk between them. A graph G is said to be **connected** if every two distinct vertices of G are connected. A **connected component** of G is a maximal connected subgraph of G.

Definition 2.11. A vertex of a graph G that is adjacent to only one other vertex is called a **pendant vertex**. An edge of G that is incident to a pendant vertex is called a **pendant edge**.

We conclude this section by giving definitions and examples of a few families of graphs that will appear throughout this thesis.

Definition 2.12. A **cycle** of a graph G is a closed walk $(u_1, \ldots, u_m, u_{m+1} = u_1)$ of vertices of G (with $m \geq 3$) such that the only vertices in the walk that are not pairwise distinct are u_1 and u_{m+1}. A cycle of length m is called an m-**cycle**.

Definition 2.13. A **cycle graph** is a graph that is itself an m-cycle for some $m \geq 3$. The **cycle graph on** $m \geq 3$ **vertices** is denoted C_m.

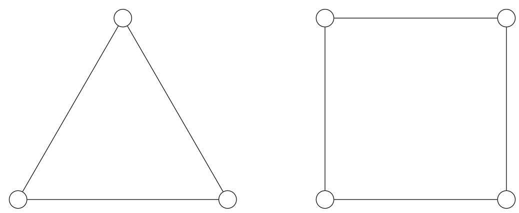

Figure 2.3: The cycle graphs C_3 and C_4

Two cycles graphs are shown in Figure 2.3.

Remark 2.14. For small m, m-cycles are commonly referred to by their shape (e.g., triangle, square, pentagon instead of 3-cycle, 4-cycle, 5-cycle).

Definition 2.15. A **tree** is a connected graph that contains no cycles. A pendant vertex of a tree is called a **leaf**.

Theorem 2.16 ([6, Theorem 1.5.1])**.** *Every connected graph contains a spanning subgraph which is a tree.*

Theorem 2.17 ([6, Corollary 1.5.3])**.** *A connected graph with n vertices is a tree if and only if it has $n-1$ edges.*

Definition 2.18. A **complete graph** is a graph where every two vertices are adjacent. The complete graph on n vertices is denoted K_n.

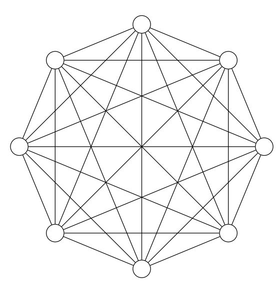

Figure 2.4: The complete graph K_8

Figure 2.4 shows K_8, the complete graph on eight vertices.

Definition 2.19. A graph $G = (V, E)$ is said to be **bipartite** if there exists a bipartition $V = V_1 \cup V_2$ of the vertex set of G such that every edge of E joins a vertex in V_1 and a vertex in V_2. In this case, we may write $G = (V_1, V_2, E)$ to indicate the bipartition. A graph H is said to be a **complete bipartite graph** if it has a bipartition $H = (V_1, V_2, E)$ such that $\{v_1, v_2\} \in E$ for every $v_1 \in V_1$ and $v_2 \in V_2$. The complete bipartite graph with vertex sets of size $|V_1| = m$ and $|V_2| = n$ is denoted $K_{m,n}$.

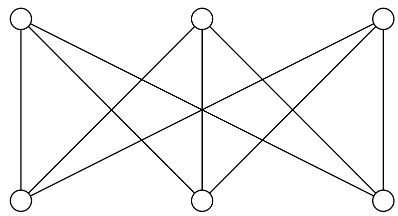

Figure 2.5: The complete bipartite graph $K_{3,3}$

Figure 2.5 shows the complete bipartite graph $K_{3,3}$.

Theorem 2.20 ([6, Proposition 1.6.1])**.** *A graph is bipartite if and only if it contains no odd cycles (i.e., cycles of odd length).*

We obtain the following as an immediate corollaries.

Corollary 2.21. *Any subgraph of a bipartite graph is itself bipartite.*

Corollary 2.22. *All trees are bipartite.*

Corollary 2.23. *Let G be a cycle graph. Then G is bipartite if and only if G is an even cycle.*

Definition 2.24. A **matching** of a graph G is a collection of pairwise non-adjacent edges of G. The **matching number** of G, denoted $\mathrm{mat}(G)$, is the largest size of any matching of G.

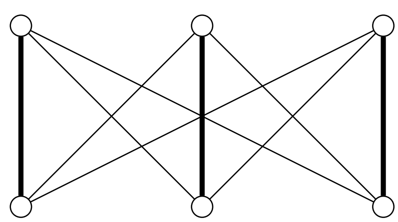

Figure 2.6: A matching of $K_{3,3}$

Figure 2.6 shows a matching of $K_{3,3}$ of size 3.

2.2 Commutative Algebra Background

In this section, we introduce the necessary algebra background needed to understand the work in the subsequent chapters. The following definitions can be found in [8] and [37].

Definition 2.25. Let R be a commutative ring with identity and A a monoid (a set with an associative binary operation "$\cdot$" and an identity element). The ring R is said to have an **A-grading** if there exists a direct sum decomposition (as groups) $R = \bigoplus_{a \in A} R_a$ such that $R_i R_j \subseteq R_{i \cdot j}$ for any $i, j \in A$. In this case, we say that R is an **A-graded ring**. An element of R_a for some $a \in A$ is called a **homogeneous element** of R and said to have **degree** a. An ideal I of R is a **homogeneous ideal** if it has a generating set consisting only of homogeneous elements. In this case, we can write $I = \bigoplus_{a \in A} I_a$, where we define $I_a \coloneqq I \cap R_a$ for each $a \in A$.

We can generalize this notion of grading for modules in a natural way.

Definition 2.26. Let R be an A-graded ring and M an R-module. M has an **A-grading** if there exists a direct sum decomposition (as groups) $M = \bigoplus_{a \in A} M_a$ such that $R_i M_j \subseteq M_{i \cdot j}$ for any $i, j \in A$. In this case, we say that M is an **A-graded R-module**. An element of M_a for some $a \in A$ is called a **homogeneous element** of M and said to have **degree** a.

Definition 2.27. Let M and N be graded R-modules. We say that an R-module homomorphism $\varphi \colon M \to N$ has **degree** i if $\deg(\varphi(m)) = i + \deg(m)$ for each homogeneous element $m \in M$.

Remark 2.28. For the remainder of the thesis, we concern ourselves only with $\mathbb{N}$-graded or $\mathbb{N}^n$-graded modules. Note that for a homogeneous ideal I of a graded polynomial ring R, we can view R/I as a graded R-module with decomposition $R/I = \bigoplus_{a \in A} R_a/I_a$ where A is $\mathbb{N}$ or $\mathbb{N}^n$ (depending on the grading). As we will see, this is important as it allows us to construct a graded free resolution of R/I.

We now introduce some definitions concerning polynomials, following the notation of [22].

Definition 2.29. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and $R = \mathbb{K}[x_1, \ldots, x_m]$ be a polynomial ring. We say that a **monomial** $u \in R$ is a polynomial of the form $u = x_1^{a_1} \cdots x_n^{a_n}$ for some $\alpha = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$. We call α the **multidegree** of u and $\sum_{i=1}^n a_i$ the **degree** of u. We say that a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ divides a monomial $v = x_1^{b_1} \cdots x_n^{b_n}$ if $a_i \leq b_i$ for all $1 \leq i \leq n$, and write $u \mid v$. A **binomial**

in R is a polynomial of the form $u - v$, where u and v are monomials in R. If, additionally, the monomials u and v have the same degree, the binomial is said to be a **homogeneous binomial**. A **binomial ideal** of R is an ideal of R generated by binomials.

Remark 2.30. Note that polynomial rings over a field are Noetherian and so all of their ideals are finitely generated (see [7], Chap. 9.6, Corollary 22).

The following background on resolutions and a more detailed exposition can be found in [37].

Definition 2.31. A **free resolution** of a finitely generated R-module M is an exact sequence

$$\cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_0} F_0 \xrightarrow{d_0} U \longrightarrow 0$$

of R-module homomorphisms such that each F_i is a free R-module and $U \cong F_0/\operatorname{Im}(d_1)$. If in addition, R is a graded ring, the F_i and U are graded modules, and each of the d_i and the isomorphism $F_0/\operatorname{Im}(d_1) \cong U$ are of degree 0, then the free resolution is said to be **graded**.

Definition 2.32. A **graded free resolution**

$$\cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_0} F_0 \xrightarrow{d_0} U \longrightarrow 0$$

is said to be **minimal** if $d_{i+1}(F_{i+1}) \subseteq \langle x_1, \ldots, x_n \rangle F_i$ for all $i \geq 0$.

Theorem 2.33 ([37, Theorem 7.5])**.** *Let U be a graded finitely generated R-module. Then there exists a unique (up to isomorphism) minimal graded free resolution of U.*

The previous theorem allows us to speak of *the* minimal graded free resolution of the finitely generated R-module R/I. Unless otherwise noted, I is a homogeneous ideal of the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ over an algebraically closed field $\mathbb{K}$ of characteristic 0.

Definition 2.34. One can associate to I a **minimal graded free resolution** of the form:

$$0 \to \bigoplus_j R(-j)^{\beta_{l,j}(I)} \to \bigoplus_j R(-j)^{\beta_{l-1,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0$$

where $R(-j)$ is the free R-module obtained by shifting the degrees of R by j (i.e., so that $R(-j)_a = R_{a-j}$). The number $\beta_{i,j}(I)$ is called the (i, j)th **graded Betti**

number of I and equals the number of minimal generators of degree j in the i-th syzygy module of I. The ith **total Betti number** is defined to be $\beta_i(I) = \sum_j \beta_{i.j}(I)$. The graded Betti numbers can be recorded in a **Betti diagram**, as below.

	β_0	β_1	β_2	$\dots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\dots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\dots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\dots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Definition 2.35. The **minimal graded free resolution of** R/I has the form:

$$0 \to \bigoplus_j R(-j)^{\beta_{l+1,j}(R/I)} \to \bigoplus_j R(-j)^{\beta_{l,j}(R/I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1,j}(R/I)} \to R \to R/I \to 0$$

where $\beta_{0,0}(R/I) = 1$, $\beta_{0,j}(R/I) = 0$ for all $j \neq 0$, and $\beta_{i,j}(R/I) = \beta_{i-1,j}(I)$ for all $i > 0$.

Having defined minimal graded free resolutions of R/I, we can now define several invariants of R/I. The following definitions can be found in [37] or [44].

Definition 2.36. We define the **(Castelnuovo-Mumford) regularity** of R/I to be $\text{reg}(R/I) \coloneqq \max\{j - i \mid \beta_{i,j}(R/I) \neq 0\}$.

Definition 2.37. We define the **projective dimension** of R/I to be the length of the minimal graded free resolution of R/I, that is,

$$\text{pdim}(R/I) \coloneqq \max\{i \mid \beta_{i,j}(R/I) \neq 0 \text{ for some } j\} = \max\{i \mid \beta_i(R/I) \neq 0\}.$$

Definition 2.38. The **Hilbert series** of R/I is the formal power series

$$\text{HS}_{R/I}(x) = \sum_{i \geq 0} [\dim_{\mathbb{K}}(R/I)_i] x^i$$

where $\dim_{\mathbb{K}}(R/I)_i$ is the dimension of the ith graded piece of R/I.

The Hilbert series of R/I is encoded into the minimal free resolution of R/I via the Betti numbers.

Theorem 2.39 ([20, Equation 6.3])**.** *The Hilbert series of R/I is given by*

$$\mathrm{HS}_{R/I}(x) = \frac{\sum_{i,j\geq 0}(-1)^i \beta_{i,j}(R/I)x^j}{(1-x)^n}$$

where n is the number of variables of the polynomial ring R.

Definition 2.40. Let M be an R-module. A sequence $X_1, \ldots, X_m$ of elements of R is called a **regular sequence** of M if $\langle X_1, \ldots, X_m \rangle M \neq M$, X_1 is not a zero divisor in M, and for all $i > 1$, X_i is not a zero divisor in $M/\langle X_1, \ldots, X_{i-1}\rangle M$. Here, m is referred to as the **length** of the sequence.

Example 2.41. Observe that $x_1, \ldots, x_n$ is a regular sequence of R.

Definition 2.42. Let M be a nonzero R-module. The **depth** of M, denoted depth(M), is the length of any maximal regular sequence of M that is contained in the maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle \subset R$.

Definition 2.43. A **chain of prime ideals** of the ring R (resp. the ring R/I) is a finite strictly increasing sequence of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_m$$

in R (resp. R/I). Here, m is called the **length** of the chain.

Definition 2.44. The **Krull dimension** (or simply **dimension)** of the ring R (resp. the ring R/I), denoted by $\dim(R)$ (resp. $\dim(R/I)$), is the supremum of the lengths of all chains of prime ideals in R (resp. R/I).

Theorem 2.45 ([8, Therorem 10.13])**.** $\dim(R) = \dim(\mathbb{K}[x_1, \ldots, x_n]) = n$.

Remark 2.46. More generally, one can define the **Krull dimension** $\dim(M)$ of an R-module M so that when M is itself a ring, it agrees with the above definition of $\dim(M)$. We always have the inequality depth$(M) \leq \dim(M)$ (see [5], Prop. 1.2.12).

Definition 2.47. An R-module M is said to be **Cohen-Macaulay** if depth$(M) = \dim(M)$ or $M = 0$.

Example 2.48. We have that $x_1, \ldots, x_n$ is a regular sequence of R by Example 2.41, so $n \leq$ depth(R). On the other hand depth$(R) \leq \dim(R) = n$ by Theorem 2.45 and Remark 2.46. Hence depth$(R) = n = \dim(R)$ so R is Cohen-Macaulay.

Our primary goal of this thesis is to compare several invariants of R/I. The following theorems allow us to make these comparisons. We begin with the celebrated Auslander-Buchsbaum formula.

Theorem 2.49 ([5, Theorem 1.3.3])**.** *If $R/I \neq 0$, we have* $\mathrm{pdim}(R/I) = \mathrm{depth}(R) - \mathrm{depth}(R/I)$.

We close out this section by stating the well-known Hilbert-Serre theorem, which we will prove in the next subsection.

Theorem 2.50 ([44, Theorem 5.1.4])**.** *There exists a unique polynomial $h_{R/I} \in \mathbb{Z}[x]$, called the **$h$-polynomial** of R/I, such that the Hilbert series $\mathrm{HS}_{R/I}$ can be written as*

$$\mathrm{HS}_{R/I}(x) = \frac{h_{R/I}(x)}{(1-x)^{\dim(R/I)}}$$

*with $h_{R/I}(1) \neq 0$. We denote the **degree of the h-polynomial** $h_{R/I}$ by $\deg(h_{R/I})$.*

We can study $\deg(h_{R/I})$ as an invariant of R/I. We always have the following useful inequality.

Theorem 2.51 ([42, Corollary B.4.1])**.** *We have* $\deg(h_{R/I}) - \mathrm{reg}(R/I) \leq \dim(R/I) - \mathrm{depth}(R/I)$*, and equality holds if R/I is Cohen-Macaulay.*

2.2.1 Proof of Hilbert-Serre Theorem

As the Hilbert-Serre theorem is fundamental in defining the invariant $\deg(h_{R/I})$, we dedicate this section to a proof of the theorem. Our proof will show that the Hilbert series is a rational function; see [44, pp. 172–174] for a proof that the numerator and denominator have the desired form.

Proof of Hilbert-Serre Theorem. Our goal is to show that we can write

$$\mathrm{HS}_{R/I}(x) = \frac{f(x)}{(1-x)^a}$$

for some polynomial $f(x) \in \mathbb{Z}[x]$ and nonnegative integer a. Let $M = R/I$. We proceed by induction on n, the number of generators of the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$.

If $n = 0$, then the graded pieces R_i of R are zero for all $i > 0$. Hence $R = R_0$ and M is a finitely generated R_0 module. It follows that that the graded pieces M_i of M are zero for all large i, so $\mathrm{HS}_{R/I}(x)$ is a polynomial, and hence rational.

Now let $n > 0$ and assume the result holds for $n-1$. One can easily check that multiplication by x_n gives rise to an R-module endomorphism, say f, of M. Let $K = \ker(f)$ and $L = \mathrm{coker}(f)$. Then K and L are graded R-modules and hence we have an exact sequence

$$0 \to K \to M \stackrel{x_n}{\to} M \to L \to 0,$$

and hence

$$0 \to K_i \to M_i \stackrel{x_n}{\to} M_{i+1} \to L_{i+1} \to 0,$$

is an exact sequence of R_0-modules for all $i \in \mathbb{Z}$. Properties of exact sequences then give

$$\lambda(K_i) - \lambda(M_i) + \lambda(M_{i+1}) - \lambda(L_{i+1}) = 0,$$

for each $i \in \mathbb{Z}$, where $\lambda(A_i) = \dim_{\mathbb{K}}(A_i)$ for any R_0-module A. Multiplying by x^{i+1} and summing with respect to i, we have

$$\sum_i x(x^i\lambda(K_i) - x^i\lambda(M_i)) + \sum_i (x^{i+1}\lambda(M_{i+1}) - x^{i+1}\lambda(L_{i+1})) = 0,$$

so

$$x\mathrm{HS}_K(x) - x\mathrm{HS}_M(x) + \mathrm{HS}_M(x) - \mathrm{HS}_L(x) = 0.$$

Hence,

$$\mathrm{HS}_M(x) = \frac{\mathrm{HS}_L(x) - x\mathrm{HS}_K(x)}{1-x}.$$

By construction, we see that x_n acts as multiplication by 0 on K and L, so we can view them as $R/(x_n)$-modules. Hence, the induction hypothesis applies to them, and we conclude from the previous equality that $\mathrm{HS}_M(x)$ is a rational function. □

2.3 Toric Ideals of Graphs

We now introduce the primary object of study of this thesis: the toric ideal of a graph. The following definitions can be found in [23] and [28].

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_q\}$. Let $\mathbb{K}[V] = \mathbb{K}[v_1, \ldots, v_n]$

and $\mathbb{K}[E] = \mathbb{K}[e_1, \ldots, e_q]$ be polynomial rings in the vertex and edge variables, respectively. Consider the ring homomorphism $\varphi \colon \mathbb{K}[E] \to \mathbb{K}[V]$ defined by $\varphi(e_i) \coloneqq v_{i_1}v_{i_2}$ for all $e_i = \{v_{i_1}, v_{i_2}\} \in E$, $1 \leq i \leq q$.

Definition 2.52. The **toric ideal of** G, denoted I_G, is defined to be the kernel of the homomorphism φ.

Remark 2.53. Observe that the image of φ is an integral domain, and since $\varphi(\mathbb{K}[E])$ is isomorphic to $\mathbb{K}[E]/I_G$ by the first isomorphism theorem, it follows that I_G is a prime ideal. As we will see, I_G is also a binomial ideal.

Remark 2.54. We write $\mathbb{K}[G]$ to denote the quotient ring $\mathbb{K}[E]/I_G$. Note that in the literature (see e.g., [1, 21, 23]), $\mathbb{K}[G]$ often denotes the *edge ring of* G (i.e. the image $\operatorname{im}(\varphi)$ of φ). As mentioned in the previous remark, $\operatorname{im}(\varphi)$ and $\mathbb{K}[E]/I_G$ are isomorphic as rings; however, we must take care when stating results about gradings on these rings, as they may differ. In all subsequent appearances of the notation $\mathbb{K}[G]$, we have ensured that results from the literature concerning $\mathbb{K}[G]$ remain true under our interpretation.

Remark 2.55. If $G = K_1$, then G has no edges. In this case, we make the convention that $\mathbb{K}[E] = \mathbb{K}$ so $\mathbb{K}[G] = \mathbb{K}$. In this case, we have

$$\operatorname{reg}(\mathbb{K}[G]) = \deg(h_{\mathbb{K}[G]}) = \operatorname{pdim}(\mathbb{K}[G]) = \operatorname{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = 0.$$

There is a well-known connection between the closed even walks of a graph G and the generators of the toric ideal I_G.

Theorem 2.56 ([43, Proposition 3.1])**.** *Let* $\Gamma = (e_{i_1}, \ldots, e_{i_{2m}})$ *be a closed even walk of a graph* G*. Define the binomial*

$$f_\Gamma = \prod_{2 \nmid j} e_{i_j} - \prod_{2 \mid j} e_{i_j}.$$

Then the toric ideal I_G *is generated by all the binomials* f_Γ*, where* Γ *is a closed even walk of* G*. If in addition* G *is bipartite, then* I_G *is generated by all the binomials* f_Γ*, where* Γ *is an even cycle of* G*.*

Remark 2.57. Note that Theorem 2.56 implies that I_G is a homogeneous ideal, so the definitions and theorems of Chapter 2 apply.

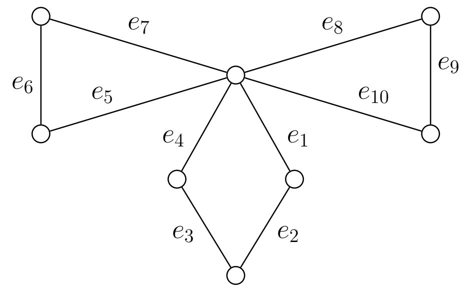

FIGURE 2.7: The graph G

We now give an example which demonstrates many of the definitions and theorems we have introduced so far.

Example 2.58. Consider the graph G in Figure 2.7. By Theorem 2.56, we have that $I_G = \langle e_1e_3 - e_2e_4, e_1e_3e_5e_7e_9 - e_2e_4e_6e_8e_{10}, e_5e_7e_9 - e_6e_8e_{10}\rangle$. We compute the minimal graded free resolution of I_G using *Macaulay2* [15]:

$$\underset{0}{R^1} \xleftarrow{\begin{pmatrix} e_1e_3-e_2e_4 & e_5e_7e_9-e_6e_8e_{10}\end{pmatrix}} \underset{1}{R^2} \xleftarrow{\begin{pmatrix} -e_5e_7e_9+e_6e_8e_{10} \\ e_1e_3-e_2e_4 \end{pmatrix}} \underset{2}{R^1} \xleftarrow{0} \underset{3}{0}$$

$$\left\{\begin{array}{l} 0 : R^1 \xleftarrow{\begin{pmatrix} e_1e_3-e_2e_4 & e_5e_7e_9-e_6e_8e_{10}\end{pmatrix}} R^2 : 1 \\ 1 : R^2 \xleftarrow{\begin{matrix}\{2\} \\ \{3\}\end{matrix}\begin{pmatrix} -e_5e_7e_9+e_6e_8e_{10} \\ e_1e_3-e_2e_4 \end{pmatrix}} R^1 : 2 \\ 2 : R^1 \xleftarrow{0} 0 : 3 \end{array}\right\}$$

We can also compute $\operatorname{reg}(\mathbb{K}[G])$, $\operatorname{HS}_{R/I}(x)$, $\deg(h_{\mathbb{K}[G]})$, $\operatorname{pdim}(\mathbb{K}[G])$, and the Betti diagram for $\mathbb{K}[G]$ using *Macaulay2*. (See Appendix A for the code.) We have

$$\operatorname{HS}_{R/I}(x) = \frac{1 + 2x + 2x^2 + x^3}{(1-x)^8}, \quad \deg(h_{\mathbb{K}[G]}) = 3.$$

We also have

$$\operatorname{reg}(\mathbb{K}[G]) = 3, \quad \operatorname{pdim}(\mathbb{K}[G]) = 2,$$

which agrees with the Betti table given below.

	0	1	2
total:	1	2	1
0 :	1	.	.
1 :	.	1	.
2 :	.	1	.
3 :	.	.	1

We end this chapter with a result concerning $\dim(\mathbb{K}[G])$ for bipartite graphs G, which we require in Chapter 4.

Theorem 2.59 ([44, Corollary 10.1.21])**.** *Let G be a connected bipartite graph on n vertices. Then* $\dim(\mathbb{K}[G]) = n - 1$.

Chapter 3

Comparing regularity and degree of bipartite graphs

As discussed in Chapter 1, the set of all possible values of $(\operatorname{reg}(R/I), \deg(h_{R/I}))$ is unknown when I is a toric ideal of a graph. In order to explore a comparison between these invariants, we will restrict the type of graphs considered to a smaller family. It turns out that connected bipartite graphs are a suitable family, as evidenced by the following theorem.

Theorem 3.1 ([22, Corollary 5.26])**.** *Let G be a connected bipartite graph. Then $\mathbb{K}[G]$ is Cohen-Macaulay.*

We therefore get the following important result.

Theorem 3.2. *If $G = (V, E)$ is a connected bipartite graph, then* $\operatorname{reg}(\mathbb{K}[G]) = \deg(h_{\mathbb{K}[G]})$.

Proof. By Theorem 3.1 and Theorem 2.51, we have

$$\deg(h_{\mathbb{K}[E]/I_G}) - \operatorname{reg}(\mathbb{K}[E]/I_G) = \dim(\mathbb{K}[E]/I_G) - \operatorname{depth}(\mathbb{K}[E]/I_G) = 0. \qquad \square$$

For each $n > 0$, let $\operatorname{CBPT}(n)$ be the set of all connected bipartite graphs on n vertices. Define

$$\operatorname{CBPT}^{\deg}_{\operatorname{reg}}(n) = \{(\operatorname{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]})) : G \in \operatorname{CBPT}(n)\}.$$

The main goal of this section is to determine the elements of $\operatorname{CBPT}^{\deg}_{\operatorname{reg}}(n)$ for each $n \geq 1$. Theorem 3.2 shows that it suffices to determine the possible values of $\operatorname{reg}(\mathbb{K}[G])$

when G is a connected bipartite graph. We therefore state some preliminary results concerning regularity that we will require.

Theorem 3.3 ([21, Theorem 1])**.** *Let G be a connected bipartite graph on $n \geq 2$ vertices. Then* $\operatorname{reg}(\mathbb{K}[G]) \leq \operatorname{mat}(G) - 1$*.*

The following lemma establishes upper and lower bounds for the regularity of bipartite graphs.

Lemma 3.4. *Let $G = (V, E)$ be a connected bipartite graph on n vertices. If $n = 1$, then* $\operatorname{reg}(\mathbb{K}[G]) = 0$*. Otherwise, if $n \geq 2$, we have* $0 \leq \operatorname{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n}{2} \right\rfloor$*.*

Proof. Since $\beta_{0,0}(\mathbb{K}[E]/I_G) = 1 \neq 0$, we have $0 = 0 - 0 \leq \operatorname{reg}(\mathbb{K}[E]/I_G) = \operatorname{reg}(\mathbb{K}[G])$. If $n = 1$, then G contains no cycles, so $I_G = \langle 0 \rangle$ by Theorem 2.56. Hence $\mathbb{K}[G] = \mathbb{K}[E]/I_G \cong \mathbb{K}[E]$ so the minimal graded free resolution of $\mathbb{K}[G]$ is

$$0 \to \mathbb{K}[E] \to \mathbb{K}[E]/\langle 0 \rangle \to 0$$

and so $\operatorname{reg}(\mathbb{K}[G]) = 0$.

So let $n \geq 2$, and suppose that the sets $A, B \subseteq V$ form a bipartition of G. Since G is bipartite, $\operatorname{mat}(G) \leq \min\{|A|, |B|\} = \min\{|A|, n - |A|\}$. Hence, by Theorem 3.3,

$$\operatorname{reg}(\mathbb{K}[G]) \leq \operatorname{mat}(G) - 1 \leq \min\{|A|, n - |A|\} - 1.$$

Now, one of $|A|$, $n - |A|$ must not exceed $\left\lfloor \frac{n}{2} \right\rfloor$. Suppose towards a contradiction this is not the case. Then both $|A|$ and $n - |A|$ are larger than $\left\lfloor \frac{n}{2} \right\rfloor$, so

$$2 \left\lfloor \frac{n}{2} \right\rfloor < |A| + (n - |A|) = n.$$

So we see that n must be odd. But then $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2}$. Hence $|A| > \frac{n-1}{2}$ and $n - |A| > \frac{n-1}{2}$. But since n is odd, $|A|$ and $n - |A|$ must have different parities, so one must exceed $\frac{n-1}{2} + 1 = \frac{n+1}{2}$. Without loss of generality, say $n - |A| > \frac{n+1}{2}$. Then

$$n < |A| + (n - |A|) = n$$

which is impossible.

It follows that $\min\{|A|, n - |A|\} - 1 \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$ so $\operatorname{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n}{2} \right\rfloor$ as desired. □

To describe the set $\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{deg}}(n)$, we will make repeated use of the following three results from the literature.

Theorem 3.5 ([18, Lemma 3.10])**.** *Let $G = K_{a,b}$ $(a, b \geq 1)$ be a complete bipartite graph. Then* $\mathrm{reg}(\mathbb{K}[G]) = \min\{a, b\} - 1$*.*

Theorem 3.6 ([1, Theorem 6.11])**.** *Suppose $B \subseteq K_{a,b}$ $(a, b \geq 1)$ is a connected bipartite graph and let $B' \subseteq B$ be a connected subgraph. Then* $\mathrm{reg}(\mathbb{K}[B']) \leq \mathrm{reg}(\mathbb{K}[B])$*.*

Theorem 3.7 ([1, Theorem 6.13])**.** *Suppose B is a connected bipartite graph with bipartition $V = \{v_1, \ldots, v_a\} \cup \{w_1, \ldots, w_b\}$, where $a, b \geq 2$. Let $r = |\{v_i : deg(v_i) = 1\}|$ and $s = |\{w_j : deg(w_j) = 1\}|$. Then*

$$\mathrm{reg}(\mathbb{K}[B]) \leq \min\{a - r, b - s\} - 1.$$

Remark 3.8. The hypotheses of Theorem 3.7 were modified slightly: the condition that $a, b \geq 2$ was added to ensure that the star graph $K_{1,m}$ has regularity 0 instead of -1, as $K_{1,m}$ is a tree (see Lemma 3.13).

Note that the previous theorem improves a result of Biermann, O'Keefe, and Van Tuyl [3], who showed the statement is true for *chordal* bipartite graphs, that is, bipartite graphs such that every induced cycle has exactly four vertices.

Lemma 3.4 bounds the possible values of $\mathrm{reg}(\mathbb{K}[G])$ for bipartite graphs G. We now show that each value in this range is realized by some connected bipartite graph G.

Lemma 3.9. *Let n and r be integers with $n \geq 2$ and $0 \leq r < \left\lfloor \frac{n}{2} \right\rfloor$. Then there exists a connected bipartite graph G on n vertices with* $\mathrm{reg}(\mathbb{K}[G]) = r$*.*

Proof. Let n and r be as given. Then $r + 1 \leq \left\lfloor \frac{n}{2} \right\rfloor \leq \frac{n}{2}$, so $2(r + 1) \leq n$. Consider the complete bipartite graph $K_{r+1,r+1}$. Construct the graph $G_{n,r}$ from $K_{r+1,r+1}$ by adjoining a set of $(n - 2(r + 1)) \geq 0$ vertices and adding edges between some fixed vertex v of $K_{r+1,r+1}$ and each of the new vertices (see Figure 3.1 for an example). We show that the graph $G = G_{n,r}$ has the desired properties. First, since $K_{r+1,r+1}$ has $2(r + 1)$ vertices, it is clear that G is a connected graph with n vertices by construction. Additionally, since no odd cycles are created in the construction of G, G must be bipartite since $K_{r+1,r+1}$ is.

It remains to show that $\mathrm{reg}(\mathbb{K}[G]) = r$. If $r = 0$, then by our construction, G is the complete graph $K_{1,n-1}$ so $\mathrm{reg}(\mathbb{K}[G]) = \mathrm{reg}(\mathbb{K}[K_{1,n-1}]) = 0 = r$ by Theorem 3.5.

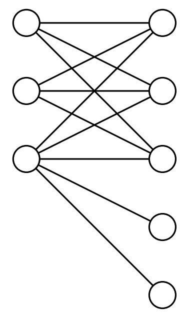

Figure 3.1: The graph $G_{8,2}$, constructed from the graph $K_{2+1,2+1}$

So we may assume that $r \geq 1$. Observe that G is a bipartite graph $G = (V_1, V_2, E)$ with sets of size $|V_1| = r + 1 \geq 2$ and $|V_2| = n - r - 1 \geq 2$, where this last inequality follows since

$$r + 3 = r + 2 + 1 \leq r + 2 + r = 2(r + 1) \leq n.$$

Since $r \geq 1$, $K_{r+1,r+1}$ does not contain any vertices of degree 1. It follows that all of the vertices of degree 1 in G are contained in V_2. (These vertices are precisely the $n - 2(r + 1)$ vertices added to $K_{r+1,r+1}$ to construct G.) Hence by Theorem 3.7,

$$\begin{aligned}\operatorname{reg}(\mathbb{K}[G]) &\leq \min\{(r + 1) - 0, n - r - 1 - (n - 2(r + 1))\} - 1 \\ &= \min\{r + 1, r + 1\} - 1 \\ &= r.\end{aligned}$$

Since $K_{r+1,r+1}$ is a connected subgraph of G, it follows by Theorem 3.5 and Theorem 3.6 that $r = \operatorname{reg}(\mathbb{K}[K_{r+1,r+1}]) \leq \operatorname{reg}(\mathbb{K}[G])$. Thus, $\operatorname{reg}(\mathbb{K}[G]) = r$ as desired. □

Finally, we reach our first major theorem.

Theorem 3.10. *Let $n \geq 1$ be an integer. Then*

$$\mathrm{CBPT}^{\deg}_{\mathrm{reg}}(n) = \left\{(a, a) \in \mathbb{Z}^2 \mid 0 < a < \left\lfloor \frac{n}{2} \right\rfloor \right\} \cup \{(0, 0)\}.$$

Proof. Let $n \geq 1$. By Lemma 3.4, any $G \in \mathrm{CBPT}(n)$ has $\operatorname{reg}(\mathbb{K}[G]) = 0$ if $n = 1$ and $0 \leq \operatorname{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n}{2} \right\rfloor$ if $n \geq 2$. Since $\operatorname{reg}(\mathbb{K}[G]) = \deg(h_{\mathbb{K}[G]})$ by Theorem 3.2, one inclusion follows.

When $n = 1$, the RHS set is $\{(0, 0)\}$ and since $\operatorname{reg}(\mathbb{K}[K_1]) = 0$ by Lemma 3.4, we have $(0, 0) \in \mathrm{CBPT}^{\deg}_{\mathrm{reg}}(1)$ by Theorem 3.2. So suppose $0 \leq a < \left\lfloor \frac{n}{2} \right\rfloor$ with $n \geq 2$. Then

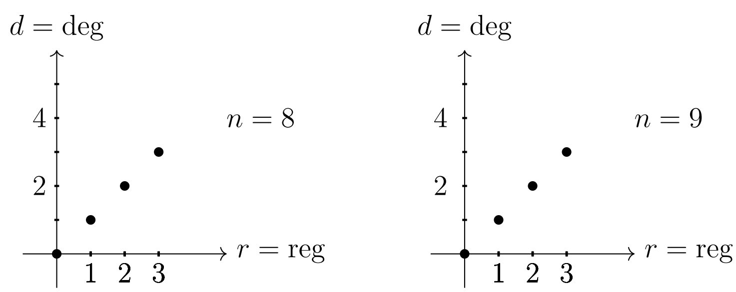

Figure 3.2: Possible $(r, d) = (\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}))$ for all connected bipartite graphs on 8 and 9 vertices

by Lemma 3.9, there is a connected bipartite graph G on n vertices with $\text{reg}(\mathbb{K}[G]) = a$. But again by Theorem 3.2, we also have $\deg(h_{\mathbb{K}[G]}) = a$, so $(a, a) \in \text{CBPT}_{\text{deg}}^{\text{reg}}(n)$. and the other inclusion holds. □

As an example of Theorem 3.10, Figure 3.2 shows the sets $\text{CBPT}_{\text{reg}}^{\text{deg}}(n)$ for $n = 8, 9$.

Corollary 3.11. *For each $n \geq 2$, $|\text{CBPT}_{\text{reg}}^{\text{deg}}(n)| = \left\lfloor \frac{n}{2} \right\rfloor$.*

For each $n > 0$, let $\text{TREE}(n)$ be the set of all trees on n vertices. Define

$$\text{TREE}_{\text{reg}}^{\text{deg}}(n) = \{(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]})) : G \in \text{TREE}(n)\}.$$

We conclude this chapter by describing the elements of $\text{TREE}_{\text{reg}}^{\text{deg}}(n)$ for $n \geq 1$. We require two lemmas.

Lemma 3.12. *Let $G = C_{2r}$ be a $2r$-cycle for some $r \geq 2$. Then $\text{reg}(\mathbb{K}[G]) = r - 1$.*

Proof. By Theorem 2.56, we have $I_G = \langle f \rangle$ for some homogeneous binomial $f \in \mathbb{K}[E(G)]$ of degree r. It follows that the minimal graded free resolution of $\mathbb{K}[G] = \mathbb{K}[E(G)]/\langle f \rangle$ is

$$0 \to \mathbb{K}[E(G)](-r) \to \mathbb{K}[E(G)] \to \mathbb{K}[E(G)]/\langle f \rangle \to 0.$$

Hence $\text{reg}(\mathbb{K}[G]) = r - 1$ as desired. □

Lemma 3.13. *Let G be a connected bipartite graph. Then $\text{reg}(\mathbb{K}[G]) = 0$ if and only if G is a tree.*

Proof. Let $\operatorname{reg}(\mathbb{K}[G]) = 0$. Suppose towards a contradiction that G is not a tree. Then G contains some cycle, and this cycle must be even of length ≥ 4 since G is bipartite. So G contains the cycle C_{2r+2} for some $r \geq 1$. Since $\operatorname{reg}(\mathbb{K}[C_{2r+2}]) = r$ by Lemma 3.12, it follows by Theorem 3.6 that $\operatorname{reg}(\mathbb{K}[G]) \geq r \geq 1$ which is impossible.

Conversely, suppose G is a tree. Since G contains no cycles, $I_G = \langle 0 \rangle$ by Theorem 2.56. Hence $\mathbb{K}[G] = \mathbb{K}[E(G)]/I_G \cong \mathbb{K}[E(G)]$ so the minimal graded free resolution of $\mathbb{K}[G]$ is

$$0 \to \mathbb{K}[E(G)] \to \mathbb{K}[E(G)]/\langle 0 \rangle \to 0$$

and so $\operatorname{reg}(\mathbb{K}[G]) = 0$. □

Remark 3.14. Note that in the previous lemma, the bipartite condition is required for the only if direction to hold: $\operatorname{reg}(\mathbb{K}[C_3]) = 0$ but C_3 is not a tree.

Theorem 3.15. *Let* $n \geq 1$. *Then* $\mathrm{TREE}^{\deg}_{\mathrm{reg}}(n) = \{(0,0)\}$.

Proof. Follows immediately from Lemma 3.13 and Theorem 3.2. □

Chapter 4

Comparing regularity and projective dimension of bipartite graphs

In this chapter, we continue our comparison of invariants of toric ideals of bipartite graphs. In particular, we characterize all possible values of $(\text{reg}(\mathbb{K}[G]), \text{pdim}(\mathbb{K}[G]))$ for connected bipartite graphs G. We begin with a fundamental result which we will use frequently.

Theorem 4.1. *Let $G = (V, E)$ be a connected bipartite graph with n vertices and q edges. Then* $\text{pdim}(\mathbb{K}[G]) = q - n + 1$.

Proof. Since $\mathbb{K}[G] = \mathbb{K}[E]/I_G \neq 0$, we have $\text{pdim}(\mathbb{K}[G]) = \text{depth}(\mathbb{K}[E]) - \text{depth}(\mathbb{K}[G])$ by Theorem 2.49. As seen in Example 2.48, the polynomial ring $\mathbb{K}[E]$ is Cohen-Macaulay, so $\text{depth}(\mathbb{K}[E]) = \dim(\mathbb{K}[E]) = q$ by Theorem 2.45. Additionally, $\mathbb{K}[G]$ is Cohen-Macaulay by Theorem 3.1, so $\text{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = n - 1$ by Theorem 2.59. It follows that $\text{pdim}(\mathbb{K}[G]) = q - n + 1$ as desired. □

Note that the previous theorem recovers the following result in the literature.

Theorem 4.2 ([18, Lemma 3.10])**.** *Let $G = K_{a,b}$ $(a, b \geq 1)$ be a complete bipartite graph. Then* $\text{pdim}(\mathbb{K}[G]) = (a - 1)(b - 1) = ab - (a + b) + 1$.

Corollary 4.3. *Let G be a connected bipartite graph. Then* $\text{pdim}(\mathbb{K}[G]) \geq 0$.

Note that the previous corollary also follows by definition of $\text{pdim}(\mathbb{K}[G])$ as the length of a minimal graded free resolution of $\mathbb{K}[G]$.

Proof. Since G is connected, G contains a spanning subgraph which is a tree by Theorem 2.16. This subgraph must have n vertices, where n is the number of vertices of G, and hence must contain $n-1$ edges by Theorem 2.17. But then G must contain at least $n-1$ edges. Hence $\mathrm{pdim}(\mathbb{K}[G]) \geq 0$ by Theorem 4.1. □

Define

$$\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{pdim}}(n) = \{(\mathrm{reg}(\mathbb{K}[G]), \mathrm{pdim}(\mathbb{K}[G]) : G \in \mathrm{CBPT}(n)\}.$$

We prove several preliminary lemmas on our way to describe this set.

Lemma 4.4. *Let G be a connected bipartite graph. Then $\mathrm{pdim}(\mathbb{K}[G]) = 0$ if and only if G is a tree.*

Proof. Let G be a connected bipartite graph with n vertices and q edges. Suppose $\mathrm{pdim}(\mathbb{K}[G]) = 0$. Then $0 = \mathrm{pdim}(\mathbb{K}[G]) = q - n + 1$, by Theorem 4.1. Consequently, G has $n-1$ edges. Since G is connected, it follows by Theorem 2.17 that G must be a tree.

Conversely, suppose G is a tree. Then $q = n-1$ by Theorem 2.17, so $\mathrm{pdim}(\mathbb{K}[G]) = q - n + 1 = 0$ by Theorem 4.1. □

Remark 4.5. Note that just as in Lemma 3.13, the bipartite condition is required for the only if direction of the previous lemma to hold: $\mathrm{pdim}(\mathbb{K}[C_3]) = 0$ but C_3 is not a tree.

Lemmas 3.13 and 4.4 together yield the following corollary.

Corollary 4.6. *Let G be a connected bipartite graph. Then $\mathrm{reg}(\mathbb{K}[G]) = 0$ if and only if $\mathrm{pdim}(\mathbb{K}[G]) = 0$.*

Define $\mathrm{TREE}_{\mathrm{reg}}^{\mathrm{pdim}}(n) = \{(\mathrm{reg}(\mathbb{K}[G]), \mathrm{pdim}(\mathbb{K}[G])) : G \in \mathrm{TREE}(n)\}$.

Theorem 4.7. *Let $n \geq 1$. Then $\mathrm{TREE}_{\mathrm{reg}}^{\mathrm{pdim}}(n) = \{(0,0)\}$.*

Proof. Follows immediately from Lemma 4.4 and Corollary 4.6. □

We now give a simple inequality that we will use in subsequent lemmas.

Lemma 4.8. *Let n and r be integers and suppose that $n \geq 2$ and $r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Then $0 \leq n - 2 - 2r$.*

Proof. Observe that
$$r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq \frac{n}{2} - 1 = \frac{n-2}{2}.$$
Hence $2r \leq n-2$, so $0 \leq n-2-2r$. ☐

The following two lemmas give ranges of positive integers r and p that can be realized as $r = \operatorname{reg}(\mathbb{K}[G])$ and $p = \operatorname{pdim}(\mathbb{K}[G])$ for some connected bipartite graph G. More precisely, Lemma 4.9 (resp. Lemma 4.10) shows that any $n \geq 4$, $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$ and $1 \leq p \leq r^2$ (resp. $r^2 \leq p \leq r(n-2-r)$) can be realized as $r = \operatorname{reg}(\mathbb{K}[G])$ and $p = \operatorname{pdim}(\mathbb{K}[G])$ for some connected bipartite graph G on n vertices. (Note that the $p = r^2$ case is covered twice, in fact, by two different graphs, as we will see.) Later, we show that these ranges describe the *only* such positive integers with this property.

Lemma 4.9. *Let n, r, p be integers with $n \geq 4$, $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$, and $1 \leq p \leq r^2$. Then there exists a connected bipartite graph G on n vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$.*

Proof. We describe the construction of the desired graph. Let n, r, p be as given. Consider the cycle graph C_{2r+2}, viewed as a bipartite graph with vertex sets of size $r+1$, as a subgraph of the bipartite graph $K_{r+1,r+1}$. Construct the graph $G_{n,r,p}$ from C_{2r+2} as follows:

(1) add any choice of $p-1 \geq 0$ edges from the possible $r^2-1 = (r+1)(r-1)$ edges between the vertices on each side of the partition so that the resulting graph remains a subgraph of $K_{r+1,r+1}$,

(2) add $n-2-2r \geq 0$ pendant edges to some fixed vertex v of the graph obtained after step (1).

(Note that $n-2-2r \geq 0$ by Lemma 4.8.) See Figure 4.1 for an example of the graph $G_{10,3,2}$. We claim that $G = G_{n,r,p}$ is a connected bipartite graph on n vertices that satisfies $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$.

Since no edges are removed in the construction of G and any new vertices added are connected to the cycle by pendant edges, G is connected. Note that the connected subgraph H of G formed after step (1) is a subgraph of $K_{r+1,r+1}$, and is hence bipartite. Since adding pendant edges in step (2) does not create any odd cycles, it follows that G is bipartite also. Since C_{2r+2} has $2r+2$ vertices, it is clear that G has $n = (2r+2)+(n-2-2r)$ vertices.

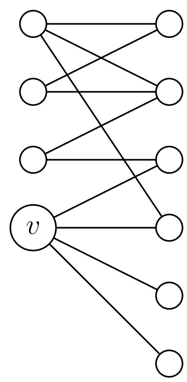

Figure 4.1: The graph $G_{10,3,2}$, constructed from the graph $C_{2\cdot 3+2}$

Now, since G is a connected bipartite graph with bipartite sets of size $n-r-1 = (r+1)+(n-2r-2)$ and $r+1$, it follows by Theorem 3.7 that

$$\begin{aligned} \operatorname{reg}(\mathbb{K}[G]) &\leq \min\{n-r-1-(n-2r-2), r+1\}-1 \\ &= \min\{r+1, r+1\}-1 \\ &= r \end{aligned}$$

as the leaves of G are precisely the $n-2r-2$ vertices added to one of the bipartite sets in step (2). Since G contains the cycle C_{2r+2}, $r = \operatorname{reg}(\mathbb{K}[C_{2r+2}]) \leq \operatorname{reg}(\mathbb{K}[G])$ by Theorem 3.6 and Lemma 3.12, and so $\operatorname{reg}(\mathbb{K}[G]) = r$.

Since C_{2r+2} has $2r+2$ edges, G has $(2r+2)+(p-1)+(n-2-2r) = p+n-1$ edges. Hence by Theorem 4.1, we have $\operatorname{pdim}(\mathbb{K}[G]) = (p+n-1)-n+1 = p$ as desired. □

Lemma 4.10. *Let n, r, p be integers with $n \geq 4$, $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$, and $r^2 \leq p \leq r(n-2-r)$. Then there exists a connected bipartite graph G on n vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$.*

Proof. First, note that since $r \leq n-2-r$ by Lemma 4.8, we have $r^2 \leq r(n-2-r)$. We describe the construction of the desired graph. Let n, r, p be as given. Consider the complete bipartite graph $K_{r+1,n-r-1}$. Construct the graph $H_{n,r,p}$ as follows:

(1) noting that $n-2-2r < n-r-1$ as $r > 0$, choose $n-2-2r = (n-r-1)-(r+1) \geq 0$ vertices $v_1, \ldots v_{n-2-2r}$ of $K_{r+1,n-r-1}$ from the $n-r-1$ vertices on one side of the bipartition,

(2) choose some fixed vertex v from the $r+1>1$ vertices on the other side of the bipartition,

(3) remove $r(n-2-r)-p \geq 0$ of the total $r(n-2-2r)$ edges connecting the vertices $v_1,\ldots,v_{n-2-2r}$ to the other side of the bipartition that are *not* incident to v.

(Note that $r(n-2-r)-p \leq r(n-2-2r)$ follows since $r^2 \leq p$. See Figure 4.2 for an example of the graph $H_{10,3,12}$. We claim that $G = H_{n,r,p}$ is a connected bipartite graph on n vertices that satisfies $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$.

Since no vertices are removed in the construction of G from $K_{r+1,n-r-1}$, G also has $n = (r+1)+(n-r-1)$ vertices. Observe that each edge of $K_{r+1,n-r-1}$ removed in the construction of G is incident to both a vertex in $\{v_1,\ldots,v_{n-2-2r}\}$ and a vertex other than v in the part of $K_{r+1,n-r-1}$ with $r+1$ vertices. By construction, each of $v_1,\ldots,v_{n-2-2r}$ remain connected to the vertex v. And v remains connected to each of the other $n-r-1-(n-2-2r) = r+1 > 1$ vertices from the part of $K_{r+1,n-r-1}$ with $n-r-1$ vertices. Additionally, each vertex $x \neq v$ in the part of $K_{r+1,n-r-1}$ with $r+1$ vertices has $\deg(x) = n-r-1$. Since we only remove at most $n-2-2r$ edges incident to x in the construction of G and $n-2-2r < n-r-1$, it follows that x cannot become disconnected through this process. Together, these observations imply G is connected.

Since G is a connected subgraph of the connected bipartite graph $K_{r+1,n-r-1}$, G is bipartite and $\operatorname{reg}(\mathbb{K}[G]) \leq \operatorname{reg}(\mathbb{K}[K_{r+1,n-r-1}]) = r$ by Theorem 3.5 and Theorem 3.6. Also, G contains the connected subgraph $K_{r+1,r+1}$ since $r+1 = (n-r-1)-(n-2-2r)$ vertices in the part of $K_{r+1,n-r-1}$ with $n-r-1$ vertices are not incident to any of the edges removed in the construction of G, and hence each of these vertices remains connected to the $r+1$ vertices on the other side of the partition. Hence $r = \operatorname{reg}(\mathbb{K}[K_{r+1,r+1}]) \leq \operatorname{reg}(\mathbb{K}[G])$ by Theorem 3.5 and Theorem 3.6, and so $\operatorname{reg}(\mathbb{K}[G]) = r$.

Since $K_{r+1,n-r-1}$ has $(r+1)(n-r-1)$ edges, G has $(r+1)(n-r-1)-r(n-2-r)+p = p+n-1$ edges. Hence $\operatorname{pdim}(\mathbb{K}[G]) = p+n-1-n+1 = p$ as desired. □

Before proving the main result, we prove one last lemma which makes use of the following result from the graph theory literature.

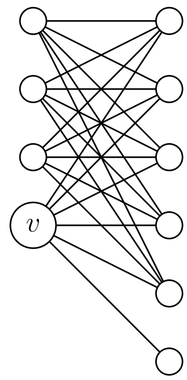

Figure 4.2: The graph $H_{10,3,12}$, constructed from the graph $K_{3+1,10-3-1}$

Theorem 4.11 ([29, Theorem 3])**.** *Let m be an integer and G be a bipartite graph with bipartition (A, B), where $|A| = a$, $|B| = b$, and $2 \leq m \leq a \leq b$. If*

$$|E(G)| > \begin{cases} a + (b-1)(m-1), & a \leq 2m-2, \\ (a+b-2m+3)(m-1), & a \geq 2m-2, \end{cases}$$

then G contains a cycle of length at least $2m$.

Remark 4.12. Note that the previous theorem was recently improved by Li and Ning [36].

Lemma 4.13. *Let G be a connected bipartite graph on $n \geq 1$ vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$. Then G has at most $(r+1)(n-r-1)$ edges.*

Proof. If $n = 1$, then $\operatorname{reg}(\mathbb{K}[G]) = 0$ by Lemma 3.4, and we see that G has at most $(0+1)(1-0-1)) = 0$ edges. So let $n \geq 2$. Suppose towards a contradiction that $|E(G)| > (r+1)(n-r-1)$. Since G is bipartite, we can view G as a subgraph of $K_{a,b}$ for some positive integers a, b with $a \leq b$ and $a + b = n$. Observe that if $a = 1$, then G is a tree since G is connected. Hence $r = 0$ by Theorem 3.13, so $|E(G)| = n - 1 = (r+1)(n-r-1)$ and we obtain a contradiction as desired.

So we may assume $2 \leq a \leq b$. Since $G \subseteq K_{a,b}$ and $\operatorname{reg}(\mathbb{K}[K_{a,b}]) = a-1$ by Theorem 3.5, it follows by Theorem 3.6 that $a \geq r+1$. If $a = r+1$, then $b = n-r-1$ and so G is contained in the graph $K_{r+1,n-r-1}$ with $(r+1)(n-r-1)$ edges, which contradicts the assumption that $|E(G)| > (r+1)(n-r-1)$. So we can assume $a \geq r+2$. Hence $2 \leq r+2 \leq a \leq b$, and so we can apply Theorem 4.11. We have two cases:

Case I: $a \leq 2(r+2)-2$. Since $0 \leq a-r-1$, we have $b \leq a+b-r-1 = n-r-1$. Hence $br \leq rn - r^2 - r$, so

$$br + (n-r-1) \leq rn - r^2 - r + (n-r-1) = (r+1)(n-r-1).$$

It follows that $a + (b-1)(r+1) = br + (n-r-1) \leq (r+1)(n-r-1) < |E(G)|$, so we conclude by Theorem 4.11 that G contains a cycle of length at least $2(r+2)$.

Case II: $a \geq 2(r+2)-2$. Then $(a+b-2(r+2)+3)(r+1) = (n-2r-1)(r+1) \leq (n-r-1)(r+1) < |E(G)|$, so again we conclude by Theorem 4.11 that G contains a cycle of length at least $2(r+2)$.

In either case, G contains a cycle of length at least $2(r+2)$. Hence G contains a connected subgraph H such that $\operatorname{reg}(\mathbb{K}[H]) \geq r+1$ by Theorem 3.12. But then $\operatorname{reg}(\mathbb{K}[G]) \geq \operatorname{reg}(\mathbb{K}[H]) \geq r+1$ by Theorem 3.6, contradicting the fact that $\operatorname{reg}(\mathbb{K}[G]) = r$. This final contradiction concludes the proof. ☐

Finally, we reach the main comparison of $\operatorname{reg}(\mathbb{K}[G])$ and $\operatorname{pdim}(\mathbb{K}[G])$.

Theorem 4.14. *Let $n \geq 1$ be an integer. Then*

$$\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{pdim}}(n) = \left\{(r,p) \in \mathbb{Z}^2 \mid 0 < r < \left\lfloor \frac{n}{2} \right\rfloor,\ 1 \leq p \leq r(n-2-r)\right\} \cup \{(0,0)\}.$$

Proof. We show both inclusions, starting with $\supseteq$. If $n \in \{1,2,3\}$, then the RHS set is $\{(0,0)\}$, and by Lemma 3.13 and Lemma 4.4, we know that there is a tree G on n vertices with $\operatorname{reg}(\mathbb{K}[G]) = \operatorname{pdim}(\mathbb{K}[G]) = 0$. So suppose $n \geq 4$ and take an element (r,p) of the RHS set. If $(r,p) = (0,0)$, then again by Lemma 3.13 and Lemma 4.4, there is a tree G on n vertices with $\operatorname{reg}(\mathbb{K}[G]) = \operatorname{pdim}(\mathbb{K}[G]) = 0$. Otherwise, if $(r,p) \neq (0,0)$, we must have $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$, and $1 \leq p \leq r(n-2-r)$. Now, we see that Lemma 4.9 and Lemma 4.10 together imply that there is a connected bipartite graph G on n vertices with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\operatorname{pdim}(\mathbb{K}[G]) = p$, which concludes the proof of the first inclusion.

Now, let $n \geq 1$ and $G \in \mathrm{CBPT}(n)$. If $n = 1$, then G is a tree, so $\operatorname{reg}(\mathbb{K}[G]) = \operatorname{pdim}(\mathbb{K}[G]) = 0$ by Lemma 3.13 and Lemma 4.4. So suppose $n \geq 2$. By Lemma 3.4, we have $0 \leq \operatorname{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n}{2} \right\rfloor$. If $\operatorname{reg}(\mathbb{K}[G]) = 0$, then $\operatorname{pdim}(\mathbb{K}[G]) = 0$ by Corollary 4.6. Hence letting $r = \operatorname{reg}(\mathbb{K}[G])$ and $p = \operatorname{pdim}(\mathbb{K}[G])$, we just need to show that if

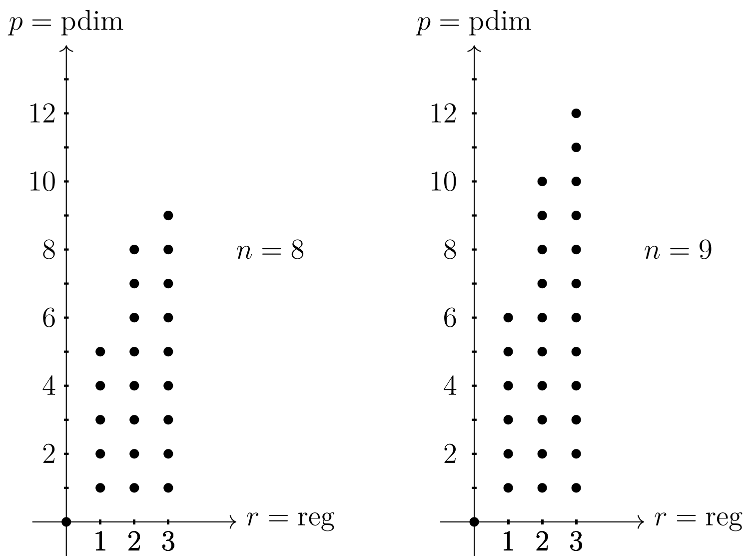

FIGURE 4.3: Possible $(r, p) = (\text{reg}(\mathbb{K}[G]), \text{pdim}(\mathbb{K}[G]))$ for all connected bipartite graphs on 8 and 9 vertices

$0 < r < \left\lfloor \frac{n}{2} \right\rfloor$, then $1 \leq p \leq r(n-2-r)$. So suppose $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$. Since $r \neq 0$, we must have $p \neq 0$ by Corollary 4.6. So $1 \leq p$ by Corollary 4.3.

It remains to show that $p \leq r(n-2-r)$. By Lemma 4.13, we have that $q \leq (r+1)(n-r-1)$, where q is the number of edges of G. Hence Theorem 4.1 gives

$$\begin{aligned} p &= q - n + 1 \\ &\leq (r+1)(n-r-1) - n + 1 \\ &= rn - r^2 - r + n - r - 1 - n + 1 \\ &= rn - r^2 - 2r \\ &= r(n-r-2) \end{aligned}$$

as desired, which concludes the proof. □

As an example of Theorem 4.14, Figure 4.3 shows the sets $\text{CBPT}_{\text{reg}}^{\text{pdim}}(n)$ for $n = 8, 9$. Now that we have a description of $\text{CBPT}_{\text{reg}}^{\text{pdim}}(n)$ for each n, we can ask how many elements this set contains.

Theorem 4.15. *For each $n \geq 1$,*

$$|\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)| = 1 - \frac{1}{6}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(2\left\lfloor \frac{n}{2} \right\rfloor - 3n + 5\right)$$
$$= \begin{cases} 1 + \frac{n(n-2)(2n-5)}{24}, & n \text{ is even,} \\ 1 + \frac{(n-1)(n-3)(2n-4)}{24}, & n \text{ is odd.} \end{cases}$$

Proof. It is clear that the formulas hold for $n = 1, 2, 3$ by Theorem 4.14 as $|\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)| = |\{(0,0)\}| = 1$. So let $n \geq 4$. By Theorem 4.14, we have

$$\begin{aligned} |\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)| - 1 &= \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} r(n-2-r) \\ &= (n-2)\sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} r - \sum_{r=1}^{\lfloor \frac{n}{2} \rfloor - 1} r^2 \\ &= \frac{n-2}{2}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right) - \frac{1}{6}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(2\left\lfloor \frac{n}{2} \right\rfloor - 1\right) \\ &= -\frac{1}{6}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(2\left\lfloor \frac{n}{2} \right\rfloor - 3n + 5\right). \end{aligned}$$

If n is even, we have

$$\begin{aligned} -\frac{1}{6}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(2\left\lfloor \frac{n}{2} \right\rfloor - 3n + 5\right) &= -\frac{1}{6}\left(\frac{n}{2}\right)\left(\frac{n}{2} - 1\right)\left(2\left(\frac{n}{2}\right) - 3n + 5\right) \\ &= -\frac{1}{6}\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)(-2n+5) \\ &= \frac{n(n-2)(2n-5)}{24}. \end{aligned}$$

If n is odd, we have

$$\begin{aligned} -\frac{1}{6}\left\lfloor \frac{n}{2} \right\rfloor \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)\left(2\left\lfloor \frac{n}{2} \right\rfloor - 3n + 5\right) &= -\frac{1}{6}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2} - 1\right)\left(2\left(\frac{n-1}{2}\right) - 3n + 5\right) \\ &= -\frac{1}{6}\left(\frac{n-1}{2}\right)\left(\frac{n-3}{2}\right)(-2n+4) \\ &= \frac{(n-1)(n-3)(2n-4)}{24}. \end{aligned}$$

Putting this together, we have

$$|\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)| = 1 - \frac{1}{6}\left\lfloor\frac{n}{2}\right\rfloor\left(\left\lfloor\frac{n}{2}\right\rfloor - 1\right)\left(2\left\lfloor\frac{n}{2}\right\rfloor - 3n + 5\right)$$
$$= \begin{cases} 1 + \frac{n(n-2)(2n-5)}{24}, & n \text{ is even} \\ 1 + \frac{(n-1)(n-3)(2n-4)}{24}, & n \text{ is odd} \end{cases}$$

as desired. □

The next result is an immediate corollary of Theorem 4.15.

Corollary 4.16.

$$\lim_{n\to\infty} \frac{|\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)|}{n^3} = \frac{1}{12}.$$

The following theorem shows that if we allow for an arbitrary number of vertices, every possible tuple $(r, p) \in \{(0,0)\} \cup \mathbb{N}^2$ can be realized as $(\mathrm{reg}(\mathbb{K}[G]), \mathrm{pdim}(\mathbb{K}[G]))$ for some connected bipartite graph G.

Theorem 4.17. *Let r and p be integers. Then there is a connected bipartite graph G with $\mathrm{reg}(\mathbb{K}[G]) = r$ and $\mathrm{pdim}(\mathbb{K}[G]) = p$ if and only if $r = p = 0$ or $r, p \geq 1$. Equivalently, with $\mathbb{N} = \{1, 2, 3, \dots\}$, we have*

$$\bigcup_{n\geq 1}^{\infty} \mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n) = \{(0,0)\} \cup \mathbb{N}^2.$$

Proof. Let G be a connected bipartite graph with at least two vertices with $\mathrm{reg}(\mathbb{K}[G]) = r$ and $\mathrm{pdim}(\mathbb{K}[G]) = p$. Then $r \geq 0$ by Lemma 3.4, and $p \geq 0$ by Corollary 4.3. By Corollary 4.6, it follows that $r = p = 0$ or $r, p \geq 1$.

Conversely, suppose $r = p = 0$ or $r, p \geq 1$. If $r, p = 0$, we have $(r, p) = (0, 0) \in \mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(1) = \{(0,0)\}$ by Theorem 4.14. So assume $r, p \geq 1$. Let $N = 2 + r + \max\{r, p\}$. Then $N \geq 2 + r + r = 2 + 2r$. Hence

$$0 < r < r + 1 = \left\lfloor\frac{2r+2}{2}\right\rfloor \leq \left\lfloor\frac{N}{2}\right\rfloor$$

and thus $0 < r < \left\lfloor\frac{N}{2}\right\rfloor$. Also, observe that since $r \geq 1$,

$$r(N - 2 - r) = r\max\{r, p\} \geq rp \geq p,$$

so $1 \leq p \leq r(N-2-r)$. It follows by Theorem 4.14 that $(r,p) \in \mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(N)$ so we are done. $\square$

Let G be a connected bipartite graph on $n \geq 1$ vertices. We know that the Krull dimension $\dim(\mathbb{K}[G])$ of $\mathbb{K}[G]$ is $n-1$ by Theorem 2.59, and since $\mathbb{K}[G]$ is Cohen-Macaulay by Theorem 3.1, we have $\mathrm{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = n-1$. We also recall that $\mathrm{reg}(\mathbb{K}[G]) = \deg(h_{\mathbb{K}[G]})$ by Theorem 3.2. So defining

$$\mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n) = \{(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) : G \in \mathrm{CBPT}(n)\}$$

in the analogous way, we completely determine the possible values for the major invariants as we range over connected bipartite graphs on $n \geq 1$ vertices.

Theorem 4.18. *Let $n \geq 1$ be an integer. Then $\mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n)$ is given by*

$$\left\{(r,r,p,n-1,n-1) \in \mathbb{Z}^5 \mid 0 < r < \left\lfloor \frac{n}{2} \right\rfloor,\ 1 \leq p \leq r(n-2-r)\right\} \cup \{(0,0,0,n-1,n-1)\}.$$

Proof. Fix an $n \geq 1$ and let T denote the set in the statement. We will first show that all the elements of $\mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n)$ belong to T.

Let G be any connected bipartite graph on n vertices, and set

$$(r, d_1, p, d_2, d_3) = (\mathrm{reg}(\mathbb{K}[G], \deg(h_{\mathbb{K}[G]}(t)), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])).$$

By Theorem 3.2 and the above remark, we have $r = d_1$ and $d_2 = d_3 = n-1$, i.e., $(r, d_1, p, d_2, d_3) = (r, r, p, n-1, n-1)$. Since $G \in \mathrm{CBPT}(n)$, by Theorem 4.14 we have $r = p = 0$, or $0 < r < \left\lfloor \frac{n}{2} \right\rfloor$ and $1 \leq p \leq r(n-2-r)$. Consequently, $(r, d_1, p, d_2, d_3) \in T$.

For the reverse containment, note that $(0,0,0,n-1,n-1) \in \mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n)$ since any connected tree G on n vertices satisfies

$$(\mathrm{reg}(\mathbb{K}[G], \deg(h_{\mathbb{K}[G]}(t)), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) = (0,0,0,n-1,n-1)$$

by Theorem 3.2, Lemma 3.13, and Corollary 4.4. So, consider any $(r,r,p,n-1,n-1) \in T$ with $0 < r$. Because the tuple (r,p) belongs to $\mathrm{CPBT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)$ by Theorem 4.14, there exists a connected bipartite graph G on n vertices with $\mathrm{reg}(\mathbb{K}[G]) = r$ and $\mathrm{pdim}(\mathbb{K}[G]) = p$. But by Theorem 3.2 and the above remark, this graph G also has $\deg(h_{\mathbb{K}[G]}) = r$ and $\dim(\mathbb{K}[G]) = \mathrm{depth}(\mathbb{K}[G]) = n-1$. Thus $(r,r,p,n-1,n-1) \in \mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n)$, as desired. $\square$

Chapter 5

A generalization to non-connected bipartite graphs

In the previous two chapters, we compared the invariants $\operatorname{reg}(\mathbb{K}[G])$, $\deg(h_{\mathbb{K}[G]})$, $\operatorname{pdim}(\mathbb{K}[G])$, $\operatorname{depth}(\mathbb{K}[G])$, and $\dim(\mathbb{K}[G])$ for connected bipartite graphs G. Our main result was Theorem 4.18, which describes all possible tuples which can be realized as $(\operatorname{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \operatorname{pdim}(\mathbb{K}[G]), \operatorname{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G]))$ for some connected bipartite graph G. The goal of this chapter is to generalize this result to bipartite graphs with more than one connected component. We begin by showing how the values of the above invariants for a bipartite graph G relate to the connected components of G.

Theorem 5.1. *Let G be a bipartite graph with connected components $G_1, \ldots, G_c$. Then*

(a) $\operatorname{reg}(\mathbb{K}[G]) = \sum_{i=1}^{c} \operatorname{reg}(\mathbb{K}[G_i])$;

(b) $\operatorname{pdim}(\mathbb{K}[G]) = \sum_{i=1}^{c} \operatorname{pdim}(\mathbb{K}[G_i])$;

(c) $\deg(h_{\mathbb{K}[G]}) = \sum_{i=1}^{c} \deg(h_{\mathbb{K}[G_i]})$;

(d) $\operatorname{depth}(\mathbb{K}[G]) = \sum_{i=1}^{c} \operatorname{depth}(\mathbb{K}[G_i])$;

(e) $\dim(\mathbb{K}[G]) = \sum_{i=1}^{c} \dim(\mathbb{K}[G_i])$.

Proof. Let $R = \mathbb{K}[E]$ and $R_i = \mathbb{K}[E(G_i)]$ for each $i = 1, \ldots, c$. Since the polynomial rings R_i are in pairwise disjoint sets of variables, we have

$$R/I_G \cong \bigotimes_{i=1}^{c} R_i/I_{G_i}.$$

The result follows by tensoring the resolutions of each R_i/I_{G_i} to construct a resolution of R/I_G. (For (c), note that $h_{R/I_G} = \prod_{i=1}^{c} h_{R_i/I_{G_i}}$.) □

The following two theorems show that relations between several of the invariants of $\mathbb{K}[G]$ still hold in the case that G is not connected.

Theorem 5.2. *Let G be a bipartite graph with connected components $G_1, \ldots, G_c$. Then* $\operatorname{reg}(\mathbb{K}[G]) = \deg(\mathbb{K}[G])$.

Proof. Since G is bipartite, each connected component of G is bipartite so Theorem 3.2 gives $\operatorname{reg}(\mathbb{K}[G_i]) = \deg(h_{\mathbb{K}[G_i]})$ for each $i = 1, \ldots, c$. Hence,

$$\operatorname{reg}(\mathbb{K}[G]) = \sum_{i=1}^{c} \operatorname{reg}(\mathbb{K}[G_i]) = \sum_{i=1}^{c} \deg(h_{\mathbb{K}[G_i]}) = \deg(h_{\mathbb{K}[G]})$$

by Theorem 5.1. □

Theorem 5.3. *Let G be a bipartite graph on n vertices with connected components $G_1, \ldots, G_c$. Then $\mathbb{K}[G]$ is Cohen-Macaulay, and* $\operatorname{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = n - c$.

Proof. Since each connected component of G is bipartite, $\mathbb{K}[G_i]$ is Cohen-Macaulay for each $i = 1, \ldots, c$ by Theorem 3.1. Hence $\operatorname{depth}(\mathbb{K}[G_i]) = \dim(h_{\mathbb{K}[G_i]})$ for each $i = 1, \ldots, c$. Hence,

$$\operatorname{depth}(\mathbb{K}[G]) = \sum_{i=1}^{c} \operatorname{depth}(\mathbb{K}[G_i]) = \sum_{i=1}^{c} \dim(h_{\mathbb{K}[G_i]}) = \dim(h_{\mathbb{K}[G]})$$

by Theorem 5.1, so G is Cohen-Macaulay. Since $\dim(\mathbb{K}[G_i]) = |V(G_i)| - 1$ for each $i = 1, \ldots, c$ by Theorem 2.59, we have $\operatorname{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = n - c$ as desired. □

For each $n \geq 1$ and $1 \leq c \leq n$, let $\mathrm{BPT}(n, c)$ be the set of all bipartite graphs on n vertices with c connected components. Define

$$\mathrm{BPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n, c) = \{(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) : G \in \mathrm{BPT}(n, c)\}$$

The main goal of this section is to determine the elements of $\mathrm{BPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n, c)$ for each $n \geq 1$, $1 \leq c \leq n$. We begin by determining the possible values of $\mathrm{reg}(\mathbb{K}[G])$ when G is a bipartite graph. For this, we will make use of the following inequality.

Lemma 5.4. *Let G be a bipartite graph on $n \geq 1$ vertices, with $1 \leq c \leq n$ connected components. If $c = n$, then* $\mathrm{reg}(\mathbb{K}[G]) = 0$. *Otherwise,* $0 \leq \mathrm{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n-(c-1)}{2} \right\rfloor$.

Proof. Fix $n \geq 1$ and let G be a bipartite graph on n vertices, with c connected components $G_1, \ldots, G_c$. Let $|V(G_i)| = n_i$ for each $i = 1, \ldots, c$. If $c = n$, then $G_i = K_1$ for each $i = 1, \ldots, c$ so $\mathrm{reg}(\mathbb{K}[G_i]) = 0$ by Lemma 3.4. Hence $\mathrm{reg}(\mathbb{K}[G]) = 0$ by Theorem 5.1. So suppose $1 \leq c < n$. Then at least one connected component of G has two or more vertices. Without loss of generality, suppose $G_1 = \cdots = G_d = K_1$ and $G_{d+1}, \ldots, G_c$ each have at least two vertices. Note that since $c < n$, we must have $d \leq c - 1$. Observe that

$$\begin{aligned}
\mathrm{reg}(\mathbb{K}[G]) &= \sum_{i=1}^{c} \mathrm{reg}(\mathbb{K}[G_i]) && \text{(by Theorem 5.1)} \\
&= \sum_{i=d+1}^{c} \mathrm{reg}(\mathbb{K}[G_i]) && \text{(since } \mathrm{reg}(\mathbb{K}[K_1]) = 0) \\
&\leq \sum_{i=d+1}^{c} \left(\left\lfloor \frac{n_i}{2} \right\rfloor - 1 \right) && \text{(by Lemma 3.4)} \\
&= \sum_{i=d+1}^{c} \left\lfloor \frac{n_i}{2} - 1 \right\rfloor \\
&\leq \left\lfloor \frac{n_{d+1} + \cdots + n_c}{2} - (c - d) \right\rfloor \\
&= \left\lfloor \frac{n - d}{2} - (c - d) \right\rfloor && \text{(since } n_1 = \cdots = n_d = 1) \\
&= \left\lfloor \frac{n - (c-1)}{2} + \frac{d - c + 1}{2} - 1 \right\rfloor \\
&\leq \left\lfloor \frac{n - (c-1)}{2} - 1 \right\rfloor && \text{(since } \tfrac{d-c+1}{2} \leq 0)
\end{aligned}$$

$$= \left\lfloor \frac{n-(c-1)}{2} \right\rfloor - 1$$

so $\operatorname{reg}(\mathbb{K}[G]) \leq \left\lfloor \frac{n-(c-1)}{2} \right\rfloor - 1$. On the other hand, since $\operatorname{reg}(\mathbb{K}[G]) = \sum_{i=1}^{c} \operatorname{reg}(\mathbb{K}[G_i])$ by Theorem 5.1 and $\operatorname{reg}(\mathbb{K}[G_i]) \geq 0$ for each $i = 1, \ldots, c$ by Lemma 3.4, we have $\operatorname{reg}(\mathbb{K}[G]) \geq 0$, which concludes the proof. □

Although we will not require this result for our proofs, we observe that Theorem 3.6 remains true for non-connected bipartite graphs.

Lemma 5.5. *Suppose G is a connected bipartite graph and let $H \subseteq G$ be a subgraph of G. Then* $\operatorname{reg}(\mathbb{K}[H]) \leq \operatorname{reg}(\mathbb{K}[G])$.

Proof. Let G be a connected bipartite graph and H be any subgraph. Suppose G has n_G vertices, H has n_H vertices, and H has connected components $H_1, \ldots, H_c$. Since H is also bipartite, then if $c = n_H$, we have $\operatorname{reg}(\mathbb{K}[H]) = 0$ by Lemma 5.4, so $\operatorname{reg}(\mathbb{K}[H]) = 0 \leq \operatorname{reg}(\mathbb{K}[G])$ by Lemma 3.4. If $c \neq n_H$, Lemma 3.4 and Lemma 5.4 give

$$\operatorname{reg}(\mathbb{K}[H]) \leq \left\lfloor \frac{n_H-(c-1)}{2} \right\rfloor - 1 \leq \left\lfloor \frac{n_H}{2} \right\rfloor - 1 \leq \left\lfloor \frac{n_G}{2} \right\rfloor - 1 = \operatorname{reg}(\mathbb{K}[G])$$

as desired. □

Theorem 5.6. *Suppose G is a bipartite graph and let $H \subseteq G$ be a subgraph of G. Then* $\operatorname{reg}(\mathbb{K}[H]) \leq \operatorname{reg}(\mathbb{K}[G])$.

Proof. Let G be a bipartite graph with connected components $G_1, \ldots, G_c$. Let H be a subgraph of G. We can write the connected components of H as

$$H_{1,1}, \ldots, H_{1,k_1}, \ldots, H_{c,1}, \ldots, H_{c,k_c}$$

where $H_{i,j}$ is the jth connected component of H contained in G_i. For each $i = 1, \ldots, c$ define H_i to be the graph with connected components $H_{i,1}, \ldots, H_{i,k_i}$, and view H_i as a subgraph of G_i. We then have

$$\begin{aligned}
\operatorname{reg}(\mathbb{K}[H]) &= \sum_{i=1}^{c}(\operatorname{reg}(\mathbb{K}[H_{i,1}]) + \cdots + \operatorname{reg}(\mathbb{K}[H_{i,k_i}])) \\
&= \sum_{i=1}^{c}\operatorname{reg}(\mathbb{K}[H_i]) && \text{(by Theorem 5.1)} \\
&\leq \sum_{i=1}^{c}\operatorname{reg}(\mathbb{K}[G_i]) && \text{(by Lemma 5.5)} \\
&= \operatorname{reg}(\mathbb{K}[G]) && \text{(by Theorem 5.1)}
\end{aligned}$$

as desired. □

We also see that Lemma 3.13 also has an analogue for non-connected bipartite graphs.

Lemma 5.7. *Let G be a bipartite graph. Then* $\operatorname{reg}(K[G]) = 0$ *if and only if G is a forest.*

Proof. Let $G_1, \ldots, G_c$ be the connected components of G. Suppose $\operatorname{reg}(\mathbb{K}[G]) = 0$. Then $0 = \operatorname{reg}(\mathbb{K}[G]) = \sum_{i=1}^{c}\operatorname{reg}(\mathbb{K}[G_i])$ by Theorem 5.1. But since each G_i is connected, $\operatorname{reg}(\mathbb{K}[G_i]) \geq 0$ by Lemma 3.4. It follows that $\operatorname{reg}(\mathbb{K}[G_i]) = 0$ for each $i = 1, \ldots, c$, so each G_i is a tree by Lemma 3.13. Hence G is a forest.

Conversely, suppose G is a forest. Then each G_i is a tree for each $i = 1, \ldots, c$, and hence has $\operatorname{reg}(\mathbb{K}[G_i]) = 0$ by Lemma 3.13. Thus, $\operatorname{reg}(\mathbb{K}[G]) = 0$ by Theorem 5.1. □

We now give a few non-connected analogues of results from previous chapters concerning projective dimension.

Theorem 5.8. *Let G be a bipartite graph with n vertices, q edges, and c connected components $G_1, \ldots, G_c$. Then* $\operatorname{pdim}(\mathbb{K}[G]) = q - n + c$.

Proof. Suppose for each $i = 1, \ldots, c$, G_i has n_i vertices and q_i edges. Since each G_i is bipartite and connected, we have $\operatorname{pdim}(\mathbb{K}[G_i]) = n_i - q_i + 1$ by Theorem 4.1. It follows by Theorem 5.1 that $\operatorname{pdim}(\mathbb{K}[G]) = n - q + c$. □

Lemma 5.9. *Let G be a bipartite graph with connected components $G_1, \ldots, G_c$. Then* $\operatorname{pdim}(\mathbb{K}[G]) \geq 0$.

Proof. By Corollary 4.3, $\operatorname{pdim}(\mathbb{K}[G_i]) \geq 0$ for each $i = 1, \ldots, c$. Hence, $\operatorname{pdim}(\mathbb{K}[G]) \geq 0$ by Theorem 5.1. □

Lemma 5.10. *Let G be a bipartite graph. Then $\operatorname{pdim}(K[G]) = 0$ if and only if G is a forest.*

Proof. Let $G_1, \ldots, G_c$ be the connected components of G. Suppose $\operatorname{pdim}(\mathbb{K}[G]) = 0$. Then $0 = \operatorname{pdim}(\mathbb{K}[G]) = \sum_{i=1}^{c} \operatorname{pdim}(\mathbb{K}[G_i])$ by Theorem 5.1. But since each G_i is connected, $\operatorname{pdim}(\mathbb{K}[G_i]) \geq 0$ by Corollary 4.3. It follows that $\operatorname{pdim}(\mathbb{K}[G_i]) = 0$ for each $i = 1, \ldots, c$, so each G_i is a tree by Lemma 4.4. Hence G is a forest.

Conversely, suppose G is a forest. Then each G_i is a tree for each $i = 1, \ldots, c$, and hence has $\operatorname{reg}(\mathbb{K}[G_i]) = 0$ by Lemma 4.4. Thus, $\operatorname{reg}(\mathbb{K}[G]) = 0$ by Theorem 5.1. □

The previous two theorems give the following corollary.

Corollary 5.11. *Let G be a bipartite graph. Then $\operatorname{reg}(\mathbb{K}[G]) = 0$ if and only if $\operatorname{pdim}(\mathbb{K}[G]) = 0$.*

We can bound the number of edges of a bipartite graph with fixed regularity just as in Lemma 4.13.

Lemma 5.12. *Let G be a bipartite graph on $n \geq 1$ vertices with c connected components $G_1, \ldots, G_c$. Suppose $\operatorname{reg}(\mathbb{K}[G]) = r$. Then G has at most $(r+1)(n-c-r)$ edges.*

Proof. For each $i = 1, \ldots, c$, let $n_i = |V(G_i)|$ and $r_i = \operatorname{reg}(\mathbb{K}[G_i])$. Observe that

$$\begin{aligned}
|E(G)| &= \sum_{i=1}^{c} |E(G_i)| \\
&\leq \sum_{i=1}^{c} (r_i+1)(n_i - r_i - 1) && \text{(by Lemma 4.13)} \\
&\leq \sum_{i=1}^{c} (r_1 + \cdots + r_c + 1)(n_i - r_i - 1) && \text{(since } r_i \geq 0 \text{ for all } i \text{ by Lemma 5.4)} \\
&= (r+1) \sum_{i=1}^{c} (n_i - r_i - 1) && \text{(by Theorem 5.1)} \\
&= (r+1)(n-r-c) && \text{(by Theorem 5.1)}
\end{aligned}$$

so G has at most $(r+1)(n-r-c)$ edges. □

Finally, we have the main result of this chapter.

Theorem 5.13. *Let $n \geq 1$, $1 \leq c \leq n$ be integers. Then* $\mathrm{BPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n, c)$ *is given by*

$$\left\{(r, r, p, n-c, n-c) \in \mathbb{Z}^5 \mid 0 < r < \left\lfloor \tfrac{n-(c-1)}{2} \right\rfloor,\ 1 \leq p \leq r(n-(c+1)-r)\right\} \cup \{(0, 0, 0, n-c, n-c)\}.$$

Proof. Let $(x_1, x_2, x_3, x_4, x_5) \in \left\{(r, r, p, n-c, n-c) \in \mathbb{Z}^5 \mid 0 < r < \left\lfloor \tfrac{n-(c-1)}{2} \right\rfloor,\ 1 \leq p \leq r(n-(c+1)-r)\right\} \cup \{(0, 0, 0, n-c, n-c)\}$. Observe that by Theorem 4.18, we have $(x_1, x_2, x_3, x_4, x_5) \in \mathrm{CBPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n-c+1)$. So there exists a connected bipartite graph H on $n-c+1 \geq 1$ vertices such that

$$(x_1, x_2, x_3, x_4, x_5) = (\mathrm{reg}(\mathbb{K}[H]), \deg(h_{\mathbb{K}[H]}), \mathrm{pdim}(\mathbb{K}[H]), \mathrm{depth}(\mathbb{K}[H]), \dim(\mathbb{K}[H])).$$

Let G be the graph with connected components $G_1, \ldots, G_c$, where $G_1 = H$ and $G_i = K_1$ for all $i = 2, \ldots, c$. Then G is a bipartite graph on n vertices with c connected components. Moreover,

$$(x_1, x_2, x_3, x_4, x_5) = (\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G]))$$

by Theorem 5.1 and the fact that for K_1, all the above invariants take value zero by Theorem 3.2, Theorem 5.3, Theorem 5.4, and Theorem 5.11. Thus, $(x_1, x_2, x_3, x_4, x_5) \in \mathrm{BPT}_{\mathrm{reg,deg,pdim,depth,dim}}(n, c)$.

Now, let $n \geq 1$, $1 \leq c \leq n$, and $G \in \mathrm{BPT}(n, c)$. If $c = n$, then $0 = \mathrm{reg}(\mathbb{K}[G]) = \deg(h_{\mathbb{K}[G]})$ by Theorem 5.2 and Lemma 5.4. So, $\mathrm{pdim}(\mathbb{K}[G]) = 0$ by Theorem 5.11. Since $\mathrm{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = n - c$ by Theorem 5.3, we have

$$(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) = (0, 0, 0, n-c, n-c).$$

So suppose $c \neq n$. By Lemma 5.4, we have $0 \leq \mathrm{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n-(c-1)}{2} \right\rfloor$. If $\mathrm{reg}(\mathbb{K}[G]) = 0$, we have

$$(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]), \mathrm{depth}(\mathbb{K}[G]), \dim(\mathbb{K}[G])) = (0, 0, 0, n-c, n-c)$$

as above. So, since $\mathrm{reg}(\mathbb{K}[G]) = \deg(h_{\mathbb{K}[G]})$ by Theorem 5.2 and $\mathrm{depth}(\mathbb{K}[G]) = \dim(\mathbb{K}[G]) = n - c$ by Theorem 5.3, it suffices to show that when $0 < \mathrm{reg}(\mathbb{K}[G]) <$

$\left\lfloor \frac{n-(c-1)}{2} \right\rfloor$, we have $1 \leq \text{pdim}(\mathbb{K}[G]) \leq \text{reg}(\mathbb{K}[G])(n-(c+1)-\text{reg}(\mathbb{K}[G]))$. So let $r = \text{reg}(\mathbb{K}[G])$, $p = \text{pdim}(\mathbb{K}[G])$ and suppose $0 < r < \left\lfloor \frac{n-(c-1)}{2} \right\rfloor$. Since $r \neq 0$, we must have $p \neq 0$ by Corollary 5.11. So $1 \leq p$ by Lemma 5.9.

It remains to show that $p \leq r(n-(c+1)-r)$. By Lemma 5.12, we have that $q \leq (r+1)(n-c-r)$, where q is the number of edges of G. Hence Theorem 5.8 gives

$$\begin{aligned} p &= q - n + c \\ &\leq (r+1)(n-c-r) - n + c \\ &= rn - rc - r^2 + n - c - r - n + c \\ &= rn - rc - r - r^2 \\ &= r(n-(c+1)-r) \end{aligned}$$

as desired, which concludes the proof. □

Remark 5.14. Note that Theorem 5.13 agrees with Theorem 4.18 when G is assumed to be connected (i.e. $c = 1$).

One should note that Theorem 5.13 shows that there are no values of the tuple $(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \text{pdim}(\mathbb{K}[G]))$ that can only be realized by *connected* bipartite graphs G. More precisely, if we define

$$\begin{aligned} \text{BPT}_{\text{reg,deg,pdim}}(n) &= \{(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \text{pdim}(\mathbb{K}[G])) : G \in \text{BPT}(n)\}, \\ \text{CBPT}_{\text{reg,deg,pdim}}(n) &= \{(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \text{pdim}(\mathbb{K}[G])) : G \in \text{CBPT}(n)\}, \end{aligned}$$

we have the following result.

Corollary 5.15. *For each* $n \geq 1$, $\text{CBPT}_{\text{reg,deg,pdim}}(n) = \text{BPT}_{\text{reg,deg,pdim}}(n)$.

Proof. Let $n \geq 1$. Since $\text{CBPT}(n) \subseteq \text{BPT}(n)$, we have $\text{CBPT}_{\text{reg,deg,pdim}}(n) \subseteq \text{BPT}_{\text{reg,deg,pdim}}(n)$. So let $G \in \text{BPT}(n)$ and $(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \text{pdim}(\mathbb{K}[G])) \in \text{BPT}_{\text{reg,deg,pdim}}(n)$. By Theorem 5.13, $(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \text{pdim}(\mathbb{K}[G])) = (0,0,0)$ or $(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \text{pdim}(\mathbb{K}[G])) = (r,r,p)$ for some $0 < r < \left\lfloor \frac{n-(c-1)}{2} \right\rfloor$ and $1 \leq p \leq r(n-(c+1)-r)$, where c is the number of connected components of G. Since $c \geq 1$, we see that $\left\lfloor \frac{n-(c-1)}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor$ and $r(n-(c+1)-r) \leq r(n-2-r)$. Hence $(\text{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \text{pdim}(\mathbb{K}[G])) \in \text{CBPT}_{\text{reg,deg,pdim}}(n)$ by Theorem 4.18. □

Chapter 6

Conclusion and future directions

The main result of this thesis is Theorem 5.13, which completely describes the values of $\operatorname{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \operatorname{pdim}(\mathbb{K}[G]), \operatorname{depth}(\mathbb{K}[G])$, and $\dim \mathbb{K}[G])$ for bipartite graphs G on $n \geq 1$ vertices. One possible direction for future study is to determine the possible values of these invariants for other classes of graphs. One such class of graphs is those that satisfy the odd cycle condition (e.g. see [22, pp. 131]).

Definition 6.1. Let G be a connected graph. We say that G satisfies the **odd cycle condition** if, for any two odd cycles C_1 and C_2 of G with $V(C_1) \cap V(C_2) = \emptyset$, there is an edge $e = \{i.j\}$ of G such that $i \in V(C_1)$ and $j \in V(C_2)$.

Theorem 6.2 ([22, Corollary 5.26])**.** *Let G be a connected graph satisfying the odd cycle condition. Then $\mathbb{K}[G]$ is Cohen-Macaulay.*

One could use the previous theorem to compare the aforementioned invariants for graphs satisfying the odd cycle condition, in much the same fashion as in this thesis. Other classes of graphs, such as chordal graphs or chordal bipartite graphs, could also be similarly considered.

One should note that Lemma 4.13 is fundamental to our proof of Theorem 4.14, which describes the set $\mathrm{CBPT}^{\mathrm{pdim}}_{\mathrm{reg}}(n)$ for each $n \geq 1$. Lemma 4.13 describes an upper bound on the number of edges of a connected bipartite graph G on n vertices in terms of $\operatorname{reg}(\mathbb{K}[G])$. This result is essentially a Turán-type result, rephrased in terms of the regularity of the corresponding graph. It would be interesting to see if similar formulae exist for other families of graphs.

As mentioned in the introduction, toric ideals of graphs are one of many classes of graph-associated ideals. Edge ideals and binomial ideals, for instance, have been widely studied in the literature. However, a full comparison of the invariants mentioned above has not been determined for these ideals. Restricting the graphs to one of the previously mentioned classes may be a suitable starting point to study these invariants.

As we have seen, the property of $\mathbb{K}[G]$ being Cohen-Macaulay allows much to be said about the associated invariants. It is therefore of great interest to understand what properties of G or $\mathbb{K}[G]$ force Cohen-Macaulayness. As an example, computations performed while working on this thesis have provided evidence for the following conjecture, which may serve as a starting point for future research.

Conjecture 6.3. *Let G be a connected graph. If $\operatorname{reg}(\mathbb{K}[G]) \leq 3$, then $\mathbb{K}[G]$ is Cohen-Macaulay.*

One can also examine the convexity of the sets $\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{deg}}(n)$ and $\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{pdim}}(n)$.

Definition 6.4. A subset $\mathcal{M} \subseteq \mathbb{Z}_{\geq 0}{}^2$ is said to be **convex** if the following conditions hold:

(1) if $(a, b_1), (a, b_2) \in \mathcal{M}$ with $b_1 < b_2$, then $(a, b) \in \mathcal{M}$ for all $b_1 < b < b_2$;

(2) if $(a_1, b), (a_2, b) \in \mathcal{M}$ with $a_1 < a_2$, then $(a, b) \in \mathcal{M}$ for all $a_1 < a < a_2$.

Theorem 6.5. *For each $n \geq 2$, the sets $\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{deg}}(n)$ and $\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{pdim}}(n)$ are convex.*

Proof. Theorem 3.10 shows that both conditions (1) and (2) are vacuously true for $\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{deg}}(n)$. Likewise, Theorem 4.14 shows that for $\mathrm{CBPT}_{\mathrm{reg}}^{\mathrm{pdim}}(n)$, condition (1) holds, and condition (2) holds if $b = 0$. To show that condition (2) holds generally, it suffices to show that for fixed $n \geq 2$, we have $r(n-2-r) \leq (r+1)(n-2-(r+1))$ for each $0 < r \leq \left\lfloor \frac{n}{2} \right\rfloor - 1$. Hence, it suffices to show that the function $f_n(x) = x(n-2-x)$ is increasing on $(0, \left\lfloor \frac{n}{2} \right\rfloor - 1]$, for each fixed $n \geq 2$. We have $0 \leq n-2-2x$ by Lemma 4.8 (note that the result indeed holds for all real numbers x). On the other hand, we see that $f_n'(x) = n-2-2x$. Hence $f_n'(x) \geq 0$, so we conclude that f_n is increasing on $(0, \left\lfloor \frac{n}{2} \right\rfloor - 1]$ as desired. □

Question 6.6. *Are there other invariants, classes of graphs, or types of ideals for which the sets comparing associated invariants describe convex lattice polytopes?*

Remark 6.7. One should note that these sets are not always convex. In [24], the authors show that the set consisting of the pairs $(\operatorname{depth}(\mathbb{K}[E]/I(G)), \dim(\mathbb{K}[E]/I(G)))$, where $I(G)$ is the edge ideal of a Cameron-Walker graph on n vertices, is not convex when $n \geq 9$ is odd.

Appendix A

Code and Tables

The following *Macaulay2* code and similar variants were used to conjecture the main theorems of this thesis. Below, the options `OnlyBipartite => true` and `OnlyConnected => true` are used, but these can be removed to search for graphs more generally.

```
loadPackage "Nauty"
toricEdgeIdeal = G -> (
--number of vertices
n := max unique flatten G;
--number of edges
m := length G;
--ring used for computing the toric edge ideal via elimination
--x variables represent vertices and are ordered via GRexLex
--e variables represent edges and are ordered via Lex
R := ZZ/(1493)[x_1..x_n,e_1..e_m,MonomialOrder=>{n,Lex=>m}];
--form monomials from edges and sort via GRevLex
g := rsort apply(G,i->x_(i_0)*x_(i_1));
--form the ideal generated by x_i*x_j-e_k,
--where {i,j} is the k-th edge in the GRevLex sorting
I := ideal(g-apply(m,i->(e_(i+1))));
--eliminate x_1,..,x_n to get the toric edge ideal
J := eliminate(apply(n,i->x_(i+1)),I);
--now form a new ring with only the e_1,..,e_m variables
--and obtain the toric edge ideal in it via a ring map S<-R
--that sends the x variables to zero
S := QQ[e_1..e_m,MonomialOrder=>Lex];
phi := map(S,R,{n:0_S}|apply(m,i->e_(i+1)));
```

```
return phi J;
)

f= G -> (
   ig = toricEdgeIdeal G;
   t=betti res ig;
             s=hilbertSeries ig;
             y=reduceHilbert s;
             G, r=regularity t,d=degree(numerator y),p=[pdim t])

P = QQ[x_1..x_7];  --- number of vertices
connected_bipartite = (n) -> (
     A = "7_connected_bipartite" << "";
          g = generateGraphs(P,0,n, OnlyBipartite => true,
          OnlyConnected => true);  --- n is number of edges
           for i from 0 to #g-1 do
        (
       l = g_i;
        y=edges l;
       z=apply(y, j -> apply(j, k -> 1+ index k));
        a=f(z);
    A << " " << a << endl;
 );
         A << close;
            );
   print(connected_bipartite(21));

quit;
```

Experimentation with this code shows that the set of all values of the invariants which are realized by bipartite graphs is a proper subset of those invariants which can be realized by graphs more generally. Table A.1 shows which tuples can be realized as $(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]))$ for connected bipartite graphs G and connected non-bipartite graphs G on $n = 7$ vertices. Notice that when we drop the bipartite assumption on G, we no longer have $\mathrm{reg}(\mathbb{K}[G]) < \left\lfloor \frac{n}{2} \right\rfloor$, and $\mathrm{pdim}(\mathbb{K}[G])$ can exceed $\mathrm{reg}(\mathbb{K}[G])(n - 2 - \mathrm{reg}(\mathbb{K}[G]))$.

$n = 7$	G connected and bipartite	G connected and not bipartite
$r = 0$	$(0, 0, 0)$	$(0, 0, 0)$
$r = 1$	$(1, 1, 1)$, $(1, 1, 2)$, $(1, 1, 3)$, $(1, 1, 4)$	$(1, 1, 1)$, $(1, 1, 2)$, $(1, 1, 3)$, $(1, 1, 4)$
$r = 2$	$(2, 2, 1)$, $(2, 2, 2)$, $(2, 2, 3)$, $(2, 2, 4)$, $(2, 2, 5)$, $(2, 2, 6)$	$(2, 2, 1)$, $(2, 2, 2)$, $(2, 2, 3)$, $(2, 2, 4)$, $(2, 2, 5)$, $(2, 2, 6)$, $(2, 2, 7)$, $(2, 2, 8)$
$r = 3$		$(3, 3, 1)$, $(3, 3, 2)$, $(3, 3, 3)$, $(3, 3, 4)$, $(3, 3, 5)$, $(3, 3, 6)$, $(3, 3, 7)$, $(3, 3, 8)$, $(3, 3, 9)$, $(3, 3, 10)$, $(3, 3, 11)$, $(3, 3, 12)$, $(3, 3, 13)$, $(3, 3, 14)$
$r = 4$		$(4, 4, 1)$

Table A.1: Values of $(\mathrm{reg}(\mathbb{K}[G]), \deg(h_{\mathbb{K}[G]}), \mathrm{pdim}(\mathbb{K}[G]))$ for connected bipartite graphs G (resp. connected non-bipartite graphs G) on 7 vertices

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