PORTFOLIO RISK MEASUREMENT

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Abstract

The main goal of this thesis is computing portfolio risks and finding the optimal portfolios. Four types of popular risk measures: variance (Var), semi variance (SVar), Value at Risk (VaR), and Average Value at Risk (AVaR), are reviewed and their computation process are preformed for portfolios consisting of single and multiple primary assets, as well as general portfolios consisting of primary and secondary assets. Finding the distribution for financial loss plays an important role here. In the general portfolio case, moment generating function (MGF) is computed first, then we apply Fourier inversion to obtain the cumulative distribution function (CDF) of the portfolio loss. Following research in [2], Cornish-Fisher approximation is applied to obtain VaR; Cornish-Fisher value at risk 2 ($CFVaR_2$) is an approximation of VaR with one term and Cornish-Fisher value at risk 3 ($CFVaR_3$) with two terms. Two numerical experiments are performed, with the aim to quantify risk in a portfolio of options, and to explore the effect of correlation on risk measurements.

We also find optimal portfolios in a fairly general setting. The Var optimal portfolio, and the $CFVaR_2$ optimal portfolios are obtained by means of quadratic programming. A numerical experiment shows that the optimal $CFVaR_2$ portfolio and the minimal Var portfolio are very similar due to the mean-variance type formula of $CFVaR_2$. However, different optimal portfolios are obtained by minimizing $CFVaR_3$.

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Chapter 1

Introduction

A *risk measurement* is a number that quantifies a risk. A *risk measure* is a map which assigns a number to a risk. Thanks to this, risk analysts can compare the risks in different situations under the same scale from a numerical point of view. It allows us to have an ordering relationship for different risks. A risk measurement cannot tell the whole story of a risk because a single number may not be enough to explain a risk. It may be useful to calculate multiple risk measures to better understand a risk.

In the field of financial mathematics finding the optimal portfolios, those which minimize a risk measure, is an important problem.

Calculating portfolio risks and minimizing them are the goals of this thesis.

We review four types of popular risk measures: Var, SVar, VaR and AVaR. Exact formulas of risk measures are found if the distribution of portfolio loss is normal or Student's t. In a first step, risk measurements are performed for portfolios consisting of one asset, a stock or a bond. Then in a follow up step risk measurements are done for portfolios consisting of multiple primary assets, stocks or bonds. Finally a general portfolio, consisting of primary and secondary assets are considered and their risk is computed resorting to the delta-gamma approximation. Based on this, the moment generating function (MGF) of the portfolio loss is computed analytically and Fourier inversion yields the distribution of the loss. This method is described in Chapter 9 of [4], and [10] by means of discrete Fourier inversion. The computation of SVar based on this approach is done in [8]. Inspired by these works we computed the risk measures, Var, VaR, and SVar applied to the portfolio loss.

Our first numerical experiment is aimed to quantify risk in a portfolio of options. It is performed by computing the risk measures (Var, VaR, and SVar) applied to the portfolio loss over one week. The risk measurements are done by simulating the losses, and analytically by means of delta-gamma approximation. In addition, for the case of VaR, the risk measurement is done using the Cornish-Fisher approximation. In a second experiment, we explore the effect of underlyings correlation on risk measurements (using Var, VaR, and SVar) in a portfolio of two options. Our finding is that all of these risk measures are increasing with the correlation. This is proved analytically in the case of Var.

Another goal of this thesis is to preform portfolio risk measurements for general portfolios which hold an arbitrary number of shares. This is necessary if we want to find the portfolios which minimize the risk. The risk is measured though the Var and VaR risk measures. Minimizing the Var of a portfolio of options was done in [6]; they use the same setting as we do in Chapter 3. Since a VaR portfolio formula is not available in closed form within this paradigm, $CFVaR_2$, $CFVaR_3$ are used. In the same vein [1] finds the optimal portfolio for $CFVaR_3$. Similar with these works ([6] and [1]), we solve the portfolio Var and $CFVaR_2$ optimization. We manage to obtain exact solutions for these optimization problems exploiting their quadratic structure. $CFVaR_3$ portfolio optimization is more complex and it is numerically solved for the special case of portfolios consisting of two options.

Our numerical experiment shows that optimal $CFVaR_2$ portfolio and the minimal Var portfolio are very close to each other. This is explained by the mean Var type formula of $CFVaR_2$, and by the normality assumption on the change of factors. However, we obtain different optimal portfolios by minimizing $CFVaR_3$.

This thesis is organized as follows. In Chapter 2 we perform risk measure for portfolios consisting primary assets. Chapter 3 shows the computation of risks for a general portfolio. Chapter 4 presents the risk minimization of a general portfolio. Numerical experiments are provided in this chapter as well. Our proof and Python code are delegated to an Appendix.

Chapter 2

Introduction to Risk Measurements

Risk measures should be easy to understand and communicate. Different risk measures have emerged over time. The Var (variance) is perhaps the most well known and used risk measure. One drawback of the Var is that it equally penalizes gains and losses. An alternative to Var is the SVar (semi variance) which fixes that drawback. In 1990s, due to its simplicity, VaR (value at risk) emerges as a popular risk measure. Besides being easy to communicate (it is a maximum loss given a confidence threshold) VaR has some disadvantages, it doesn't tell what may happened in the tail of the loss distribution. This drawback was addressed by introducing AVaR (average value at risk) risk measure. If the portfolio loss has a continuous distribution then AVaR coincides with Conditional value at risks (CVaR) or Expected shortfall (ES) risk measures. This chapter starts with reviewing these four risk measures.

2.1 Variance

The Var of a random variable X which has mean $E[X] = \mu$ is,

$$Var = E[(X - \mu)^2].$$

Sometimes the sample Var is measured; for example the sample Var of a stock price in one year if available data on monthly stock prices $S_i,\ i = 1, 2, \cdots, 12$,

$$Var = \frac{[\sum_{i=1}^{12}(\overline{S} - S_i)^2]}{12 - 1},$$

where

$$\overline{S} = \frac{\sum_1^{12} S_i}{12 - 1}.$$

2.2 Value at Risk

Let us move now to VaR. Given a confidence level, VaR is the maximum loss given that confidence threshold. Below we give the formal definition.

Definition 2.1 *Let $\alpha \in (0, 1)$ be a confidence threshold, and X be the gains of a financial position. Then VaR is defined by*

$$VaR_\alpha = inf\{m | P[X + m < 0] \leq \alpha\}.$$

In common practice, $\alpha = 1\%$ and $\alpha = 5\%$ are considered. Then for the later we are 95% confident that the loss, $-X$, will not exceed VaR.

2.3 Average Value at Risk

Let us define AVaR as an average of VaRs corresponding to different confidence thresholds. This will be made precise in the next formal definition.

Definition 2.2 *Let $\alpha \in (0,1)$ be a confidence threshold, and VaR_u be the VaR at confidence level u. Then the AVaR is defined by*

$$AVaR_\alpha = \frac{1}{\alpha}\int_0^\alpha VaR_u du.$$

2.4 Semi Variance

The definition of SVar is made precise below.

Definition 2.3 *Let X be the gains of a financial position. Then the SVar is defined by*

$$SVar = E\left[(X - E[X])^2 1_{\{X \leq EX\}}\right].$$

2.5 Normality Assumptions

In order to facilitate the computation of risk measurements, we assume the financial assets gains or returns are normally distributed. However, normality assumption on the gains implies unbounded losses. But the loss of a portfolio is typically bounded in real life. This problem can be fixed by assuming that the geometric asset returns are normally distributed. In this section, we denote the portfolio value at time t by P_t.

2.5.1 Normally Distributed Gains

Assume the financial gain X has a normal distribution with mean μ and Var σ^2. Then risk measurements of X are easy to obtain as shown in the following Proposition.

Proposition 2.1 *Assume $X \sim N(\mu, \sigma^2)$ and ϕ the standard normal pdf, then the following holds true*

$$\begin{aligned} Var &= \sigma^2 \\ SVar &= \frac{1}{2}\sigma^2 \\ VaR_\alpha &= -[\mu + N^{-1}(\alpha)\sigma] \\ AVaR_\alpha &= -\mu + \frac{\sigma\phi(N^{-1}(\alpha))}{\alpha}. \end{aligned}$$

Proof 2.1 *See Appendix A.1*

2.5.2 Normally Distributed Arithmetic Returns

The arithmetic return for the time interval $(t, t+\tau)$ is

$$r = \frac{P_{t+\tau} - P_t}{P_t}.$$

Then the gains X is given by

$$X = rP_t.$$

In the next proposition, we obtain the risk measurements of X in terms of r statistics.

Proposition 2.2 *Assume $r \sim N(\mu_r, \sigma_r^2)$ and ϕ be the standard normal pdf, then the* *following holds true*

$$Var = P_t^2 \sigma_r^2$$
$$SVar = \frac{1}{2} P_t^2 \sigma_r^2$$
$$VaR_\alpha = -P_t[\mu_r + N^{-1}(\alpha)\sigma_r]$$
$$AVaR_\alpha = P_t \left[-\mu_r + \frac{\sigma_r \phi\left(N^{-1}(\alpha)\right)}{\alpha} \right].$$

Proof 2.2 *See Appendix A.2*

2.5.3 Normally Distributed Geometric Returns

The geometric return for the time interval $[t, t+\tau]$ is

$$R = log\left(\frac{P_{t+\tau}}{P_t}\right).$$

Then the gain X is given by

$$X = P_t\left(e^R - 1\right).$$

In the next proposition, we obtain the risk measurements of X in terms of R statistics.

Proposition 2.3 *Assume $R \sim N(\mu_R, \sigma_R^2)$ and ϕ the standard normal pdf, then the* *following holds true*

$$\begin{aligned}
Var &= P_t^2[exp(\sigma_R^2) - 1]exp(2\mu_R + \sigma_R^2) \\
SVar &= P_t^2 e^{2\mu_R+\sigma_R^2}\left[e^{\sigma_R^2}N(E[X] - 2\sigma_R) + N(E[X]) - 2N(E[X] - \sigma_R)\right] \\
VaR_\alpha &= P_t(1 - exp[\mu_R + N^{-1}(\alpha)\sigma_R]) \\
AVaR_\alpha &= P_t\left(1 - \frac{1}{\alpha}exp\left(\mu_R + \frac{1}{2}\sigma_R^2\right)[N\left(N^{-1}(\alpha) - \sigma_R\right)]\right),
\end{aligned}$$

where

$$E[X] = P_t\left(exp\left(\mu_R + \frac{1}{2}\sigma_R^2\right) - 1\right).$$

Proof 2.3 *See Appendix A.3*

2.6 Cornish - Fisher Expansion

If we want to approximate the quantile of a random variable in terms of the normal quantile we can use the Cornish-Fisher expansion. Suppose the random variable Y has mean 0 and standard deviation 1. We have the following Cornish - Fisher approximation of the α quantile $\Phi_Y(\alpha)$

$$\begin{aligned}
\Phi_Y^{-1}(\alpha) \approx & N^{-1}(\alpha) + \frac{[N^{-1}(\alpha)]^2 - 1}{6}k_3 + \frac{[N^{-1}(\alpha)]^3 - 3N^{-1}(\alpha)}{24}k_4 \\
& - \frac{2[N^{-1}(\alpha)]^3 - 5N^{-1}(\alpha)}{36}k_3^2,
\end{aligned}$$

where

$$k_1 = \mu,\ k_2 = \mu_2,$$
$$k_3 = \mu_3,\ k_4 = \mu_4 - 3\mu_2^2,$$
$$\mu = E[Y],\ \text{and}\ \mu_r = E[(Y-\mu)^r].$$

We can derive a formula for VaR based on this quantile approximation. This is done in the proposition below.

Proposition 2.4 *Let α be a confidence level. Suppose gain X has mean μ_X and standard deviation σ_X. Define the normalized financial gain by*

$$Y = \frac{X - \mu_X}{\sigma_X}.$$

Then

$$VaR_\alpha = -[\mu_X + \Phi_Y^{-1}(\alpha)\sigma_X],$$

where $\Phi_Y(\alpha)$ is the Cornish - Fisher approximation of the α quantile of Y.

Proof 2.4 *See Appendix A.4*

2.6.1 Student t-Distribution

One drawback of the normality assumption for financial gains/returns is that underestimates the probability of extreme events due to the thickness of the normal tails. One resolution is to consider the t-distribution. If the financial gain X has a student-t distribution with mean μ and standard deviation σ, assume that

$$\frac{X-\mu}{\sigma}\sqrt{\frac{\nu}{\nu-2}} \sim student(\nu),$$

where ν is the number of degrees of freedom, then we have the following result.

Proposition 2.5 *Assume the normalized gain,*

$$\frac{X-\mu}{\sigma}\sqrt{\frac{\nu}{\nu-2}} \sim student(\nu).$$

Then

$$VaR_\alpha = -\Phi^{-1}(\alpha)\sqrt{\frac{\nu-2}{\nu}}\sigma - \mu.$$

Here $\Phi^{-1}(\alpha)$ *is the student-t quantile.*

Proof 2.5 *See Appendix A.5*

2.7 Examples of Portfolios Risk Measurements

In this section, we calculate risk measurements for portfolios consisting of one asset, and multiple assets.

2.7.1 One-Stock-Portfolio in Black Scholes Model

In the Black Scholes model, the price of a stock is given by

$$S_t = S_0 exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t).$$

Here μ is stock return, and σ is the volatility. The risk measures applied to the portfolio holding this stock are given in the proposition below.

Proposition 2.6 *Let α be a confidence level. We have following risk measurements for the portfolio consisting of one stock during t and $t+\tau$*

$$\begin{aligned} Var &= S_t^2[exp(\sigma^2\tau)-1]exp\,(2\mu\tau) \\ SVar &= S_t^2 e^{2\mu\tau}\left[e^{\sigma^2\tau}N(E[X]-2\sigma\sqrt{\tau})+N(E[X])-2N(E[X]-\sigma\sqrt{\tau})\right]. \\ VaR_\alpha &= S_t\left(1-exp\left[\left(\mu-\frac{1}{2}\sigma^2\right)\tau+N^{-1}(\alpha)\,\sigma\sqrt{\tau}\right]\right) \\ AVaR_\alpha &= S_t\left(1-\frac{1}{\alpha}exp\,(\mu\tau)\left[N(N^{-1}(\alpha)-\sigma\sqrt{\tau})\right]\right), \end{aligned}$$

where

$$E[X] = P_t\,(exp\,(\mu\tau)-1)\,.$$

Proof 2.6 *See Appendix A.6*

2.7.2 One-Bond-Portfolio in Vasicek model

We consider the portfolio which holds one zero-coupon bond priced in the Vasicek model. In this model, the short rate is given by

$$\begin{aligned} r(t) &= \epsilon_0+\epsilon_1 Y(t) \\ Y(t) &= e^{-\lambda t}Y(0)+e^{-\lambda t}\int_0^t e^{\lambda u}dW(u), \end{aligned}$$

where ϵ_0, ϵ_1, and λ are constants. Then we can find the distribution of geometric return of bond price on given time interval.

Proposition 2.7 *The geometric return of the bond on time interval $[t, t+\tau]$ is normally distributed with mean μ_R and standard deviation σ_R, where*

$$\mu_R = -C(T-t-\tau)Y(t)e^{\lambda\tau} + C(T-t)Y(t) + [A(T-t) - A(T-t-\tau)]$$
$$\sigma_R = \sqrt{C^2(T-t-\tau)\int_t^{t+\tau} e^{2\lambda(u-\tau)}du}.$$

The functions C and A are given in the appendix.

Proof 2.7 *See Appendix A.7*

Consider the portfolio consisting of one bond during t and $t+\tau$ the next proposition gives risk measurements of this portfolio.

Proposition 2.8 *Let α be a confidence level. We have following risk measurements for the portfolio consisting of one bond during t and $t+\tau$*

$$Var = [P(t,T)]^2 \left[exp\left(\sigma_R^2\right) - 1\right] exp\left(2\mu_R + \sigma_R^2\right)$$
$$SVar = [P(t,T)]^2 e^{2\mu_R + \sigma_R^2}\left[e^{\sigma_R^2} N(E[X] - 2\sigma_R) + N(E[X]) - 2N(E[X] - \sigma_R)\right]$$
$$VaR_\alpha = P(t,T)\left(1 - exp\left[\mu_R + N^{-1}(\alpha)\sigma_R\right]\right)$$
$$AVaR_\alpha = P(t,T)\left(1 - \frac{1}{\alpha} exp\left(\mu_R + \frac{1}{2}\sigma_R^2\right)\left[N\left(N^{-1}(\alpha) - \sigma\right)\right]\right),$$

where

$$E[X] = P(t,T)\left(exp\left(\mu_R + \frac{1}{2}\sigma_R^2\right) - 1\right).$$

Proof 2.8 *See Appendix A.8*

2.7.3 A Portfolio with Multivariate Normal Assets Returns

Let us consider now portfolios consisting of several assets in a one-period model. Let $S_1, S_2, \cdots, S_n$ denote the asset prices and

$$R_i = \frac{S_i(1) - S_i(0)}{S_i(0)},\ i = 1, 2, \cdots, n$$

their arithmetic returns. Assume that

$$(R_1, R_2, \cdots, R_n) \sim N(\mu, \Sigma),$$

where

$$\mu = (\mu_i)$$

is the mean vector and

$$\Sigma = (\Sigma_{ij})$$

is the Var covariance matrix. Consider the portfolio holding Δ_i shares for each asset.

Let P be the current value of this portfolio and the portfolio weight in each asset

$$w_i = \frac{\Delta_i S_i(0)}{P(0)}.$$

Define w the vector of weight,

$$w = (w_1, w_2, \cdots, w_n).$$

The joint normality of asset returns makes it possible to derive the distribution of portfolio return and compute portfolio risk measures. This is done in the next proposition.

Proposition 2.9 *Let α be a confidence level. We have following risk measurements for this portfolio*

$$\begin{aligned}
Var &= P^2 w\Sigma w^T \\
SVar &= \frac{1}{2}P^2 w\Sigma w^T \\
VaR_\alpha &= -P\left[w\mu + N^{-1}(\alpha)\sqrt{w\Sigma w^T}\right] \\
AVaR_\alpha &= P\left[-w\mu + \frac{\sqrt{w\Sigma w^T}\phi(N^{-1}(\alpha))}{\alpha}\right].
\end{aligned}$$

Proof 2.9 *See Appendix A.9*

We see in this model that the variance-covariance matrix Σ has a big effect on VaR. Let's consider a model with two stocks. We have the following result about correlation effect on VaR.

Proposition 2.10 *Assume $\alpha \leq 0.5$, and ρ the correlation of the two stocks. Then the VaR of the portfolio consisting of the two stocks is decreasing in ρ.*

Proof 2.10 *See Appendix A.10*

2.7.4 A Portfolio of Stocks in the Black Scholes Model

Now we consider portfolios consisting of n stocks driven by n independent Brownian motions in the Black Scholes model, and assume the interest rate r is constant. Let

$W(t) = (W_j(t))$, $j = 1, 2, \cdots, n$, be the column vector consisting of n Brownian motions. Then differential of stock prices are given by

$$dS_i(t) = S_i(t)\left[\alpha_i dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)\right], \ i = 1, 2, \cdots, n.$$

Here α_i is return of the stock i and σ_{ij} comprise the volatility. Then define the excess rates of return

$$\mu_i = \alpha_i - r,$$

and the column vector μ with components μ_i. The proportion of the portfolio value in stock i is denoted $\zeta_i(t)$, and the column vector of proportions

$$\zeta(t) = (\zeta_i(t)).$$

Assume the vector of proportions $\zeta(t)$ is a constant vector over the time interval $[t, t + \tau]$. We have the following risk measurements for the portfolio on the interval $[t, t + \tau]$.

Proposition 2.11 *Let the portfolio value at time t equal $X(t)$, and α be a confidence* *level. The following risk measurements hold for the portfolio on* $[t, t+\tau]$

$$Var = [X(t)]^2 \left[exp\left(\left|\zeta^T\sigma\right|^2\tau\right) - 1\right] exp\left[2G(\zeta)\tau + \left|\zeta^T\sigma\right|^2\tau\right]$$

$$SVar = [X(t)]^2 e^{2G(\zeta)\tau + \left|\zeta^T\sigma\right|^2\tau} \Big[e^{\left|\zeta^T\sigma\right|^2\tau} N(E[X] - 2\left|\zeta^T\sigma\right|\sqrt{\tau})$$
$$+ N(E[X]) - 2N(E[X] - \left|\zeta^T\sigma\right|\sqrt{\tau})\Big]$$

$$VaR_\alpha = X(t)\left[1 - exp\left\{G(\zeta)\tau + N^{-1}(\alpha)\left|\zeta^T\sigma\right|\sqrt{\tau}\right\}\right]$$

$$AVaR_\alpha = X(t)\left(1 - \frac{1}{\alpha}exp\left(G(\zeta)\tau + \frac{1}{2}\left|\zeta^T\sigma\right|\sqrt{\tau}^2\right)\left[N\left(N^{-1}(\alpha) - \left|\zeta^T\sigma\right|\sqrt{\tau}\right)\right]\right),$$

where

$$E[X] = X(t)(e^{(G(\zeta)\tau + \frac{1}{2}\left|\zeta^T\sigma\right|^2\tau)} - 1).$$

Here G is the quadratic

$$G(\zeta) = \zeta^T\mu + r - \frac{1}{2}\left|\zeta^T\sigma\right|^2,$$

and $\left|\zeta^T\sigma\right|$ *is the Euclidean norm of* $\zeta^T\sigma$.

Proof 2.11 *See Appendix A.11*

2.8 Conclusion

This chapter begins with examining four commonly used risk measures: Var, SVar, VaR, and AVaR. Then we show that when the financial loss distribution is normal or Student's t, exact formulas for the risk measures can be established. Firstly, risk measures for single-asset portfolios are computed, and afterwards risk measures for portfolios consisting of multiple primary assets are determined.

Chapter 3

Risk Measurements for General Portfolios

In this chapter, we consider portfolios consisting of primary and secondary assets (for example, stocks and options on stocks), and we refer to them as general portfolios. The risk measurements for these portfolios will be more difficult to obtain analytically because the distribution of the loss or gains is not explicitly available. For example consider a portfolio consisting of European-call option in the Black Scholes model. A European-call option with payoff

$$C(S,T) = max[S-K,0]$$

has the following the price given by the Black Scholes formula

$$C(S(t),t) = S(t)N(d_+(T-t)) - Ke^{-r(T-t)}N(d_-(T-t)),$$

where

$$d_{\pm}(x) = \frac{1}{\sigma\sqrt{x}}\left[log\frac{S\left(t\right)}{K} + \left(r \pm \frac{\sigma^2}{2}\right)x\right].$$

Then

$$C(S(t+1), t+1) = S(t+1)N(d_+(T-t+1)) - Ke^{-r(T-t+1)}N(d_-(T-t+1)).$$

We are unable to find the distribution of the gain from holding one share,

$$C(S(t+1), t+1) - C(S(t), t)$$

thus we can't compute VaR. The resolution is to use the delta-gamma approximation.

3.1 Delta-Gamma Approximation

Let $S(t)$ be a vector of factors (for example, stock prices at time t), and

$$c(t, S(t))$$

be the value of a portfolio at time t. Here

$$c = c(t, S)$$

is a function which maps the factors to the portfolio value. Let Δc be the change of this portfolio on time interval $[t, t + \Delta t]$,

$$\Delta c = c(t + \Delta t, S(t + \Delta t)) - c(t, S(t)).$$

Then the delta-gamma approximation of Δc is the second order Taylor approximation,

$$\Delta c \approx \Delta t \frac{\partial c}{\partial t} + \delta^T \Delta S + \frac{1}{2} \Delta S^T \Gamma \Delta S,$$

where

$$\begin{aligned} \Delta S &= S(t + \Delta t) - S(t) \\ \delta_i &= \frac{\partial c}{\partial S_i} \\ \Gamma_{ij} &= \frac{\partial^2 c}{\partial S_i \partial S_j}. \end{aligned}$$

3.2 Normal Returns

In order to facilitate the computation of portfolio $c(t, S(t))$ risk measures, let us assume that the change in factors $\Delta S(t)$ follows a multi-variate normal distribution. Let us make this formal.

Assumption: Following Chapter 9 of [4] we assume that the change of factors in time interval $[t, t + \Delta t]$ is normally distributed:

$$\Delta S \sim N(\vec{\mathbf{0}}, \Delta t \Sigma).$$

Here, $\vec{\mathbf{0}}$ is a $n \times 1$ zero column vector. $\Delta t \Sigma$ is the variance-covariance matrix of ΔS, where Σ is the variance-covariance matrix of annual change factors.

3.2.1 Diagonalization

Next we would like to compute the MGF of portfolio loss

$$L = -\Delta c.$$

This is done through a diagonalization procedure. Since

$$\Delta S \sim N(\vec{\mathbf{0}}, \Delta t \Sigma),$$

then

$$\Delta S = CZ$$

where

$$CC^T = \Delta t \Sigma,$$

and

$$Z \sim N(\vec{\mathbf{0}}, I_n).$$

The portfolio loss L is approximated by

$$L \approx a - \delta^T CZ - \frac{1}{2} Z^T (C^T \Gamma C) Z,$$

where

$$a = -\Delta t \frac{\partial c}{\partial t}.$$

At first, let $\widetilde{C}$ be a square matrix such that

$$\widetilde{C}\widetilde{C}^T = \Delta t \Sigma$$

(for example $\widetilde{C}$ can be the Cholesky decomposition of $\Delta t\Sigma$). The matrix

$$\frac{1}{2}\widetilde{C}^T \Gamma \widetilde{C}$$

is symmetric, because

$$(\widetilde{C}^T \Gamma \widetilde{C})^T = \widetilde{C}^T \Gamma^T \widetilde{C},$$

where

$$\Gamma_{ij} = \frac{\partial c^2}{\partial S_i \partial S_j} = \frac{\partial c^2}{\partial S_j \partial S_i} = \Gamma_{ji}.$$

Secondly, by eigenvalue decomposition,

$$-\frac{1}{2}\widetilde{C}^T \Gamma \widetilde{C} = U \Lambda U^T,$$

where

$$\Lambda = diag(\lambda_i),$$

and λ_i, $i = 1, 2, \cdots, n$, are eigenvalues of $-\frac{1}{2}\widetilde{C}^T \Gamma \widetilde{C}$. Next let

$$C = \widetilde{C} U,$$

then

$$CC^T = \widetilde{C} U U^T \widetilde{C}^T = \Delta t \Sigma.$$

Moreover,

$$-\frac{1}{2}C^T\Gamma C = -\frac{1}{2}U^T\widetilde{C}^T\Gamma\widetilde{C}U = U^TU\Lambda U^TU = \Lambda.$$

We obtain the following approximation of the loss L

$$L \approx a + b^TZ + Z^T\Lambda Z = Q,$$

where

$$b = -C^T\delta.$$

This makes it possible to compute the MGF of approximate loss Q, which we do next.

3.2.2 Moment Generating Function

Let us compute the MGF of Q (the approximation of the portfolio loss)

Proposition 3.1 *The MGF of Q is given by the formula*

$$\phi_Q(\theta) = E[exp(\theta Q)] = exp(\eta(\theta)),$$

where

$$\eta(\theta) = \theta a + \sum_{i=1}^{n}\frac{1}{2}\left(\frac{\theta^2 b_i^2}{1-2\theta\lambda_i} - log\left(1-2\theta\lambda_i\right)\right).$$

Proof 3.1 *See Appendix A.12*

3.2.3 Expectation, Variance and The Third Central Moment

Now let us derive expectation, Var, and the third central moment of the approximated portfolio loss.

Proposition 3.2 *The expectation of Q has the following formula*

$$E[Q] = \mu_Q = a + \sum_{i=1}^{n} \lambda_i.$$

The Var of Q has the following formula

$$Var[Q] = \sigma_Q^2 = \sum_{i=1}^{n} \left(b_i^2 + 2\lambda_i^2\right).$$

The third central moment of Q, $\mu_3(Q)$, has the following formula

$$\mu_3(Q) = \sum_{i=1}^{n} \left(6b_i^2\lambda_i + 8\lambda_i^3\right).$$

Proof 3.2 *See Appendix A.13*

3.2.4 Value at Risk

It is time to compute the VaR for the portfolio loss on time interval $[t, t+\Delta t]$, denoted by VaR.

Proposition 3.3 *Let α be a confidence level, and x the solution of the following equation*

$$\frac{1}{2} + \frac{1}{2\pi}\int_0^{\infty} \frac{e^{iux}\phi_Q(-iu) - e^{-iux}\phi_Q(iu)}{iu} du = 1 - \alpha,$$

where ϕ_Q is the MGF of Q. Then

$$VaR_\alpha = x.$$

Proof 3.3 *See Appendix A.14*

3.2.5 Cornish-Fisher Value at Risk

Next let us obtain a VaR approximation based on the Cornish-Fisher expansion. Define the normalized

$$q = \frac{Q - \mu_Q}{\sigma_Q},$$

and we take the Cornish-Fisher approximation

$$\Phi_q^{-1}(\alpha) \approx N^{-1}(\alpha) + \frac{[N^{-1}(\alpha)]^2 - 1}{6} k_3 + \cdots ,$$

where k_3 is the third central moment of q,

$$k_3 = \frac{\mu_3(Q)}{\sigma_Q^3}.$$

By **Proposition 2.4**

$$\begin{aligned} VaR_\alpha &= \mu_Q + \Phi_q^{-1}(\alpha)\sigma_Q \\ &= \mu_Q - \left[N^{-1}(\alpha) - \frac{1}{6}\left\{ \left[N^{-1}(\alpha)\right]^2 - 1 \right\} \frac{\mu_3(Q)}{\sigma_Q^3} + \cdots \right] \sigma_Q. \end{aligned}$$

Define $CFVaR_2(Q)$ and $CFVaR_3(Q)$ below

$$CFVaR_2(Q) = \mu_Q - N^{-1}(\alpha)\,\sigma_Q$$
$$CFVaR_3(Q) = \mu_Q - N^{-1}(\alpha)\,\sigma_Q + \frac{1}{6}\left\{\left[N^{-1}(\alpha)\right]^2 - 1\right\}\frac{\mu_3(Q)}{\sigma_Q^2}.$$

3.2.6 Semi Variance

Now we can find the SVar of portfolio on time interval $[t, t+\Delta t]$, and it is done in the following proposition.

Proposition 3.4 *The SVar has the following formula*

$$SVar[Q] = \int_{EQ}^{\infty} (x - E[Q])^2 f_Q(x)\,dx,$$

where

$$f_Q(x) = \frac{1}{\pi}\int_0^{\infty} e^{-iux}\phi_Q(u)du,$$

and ϕ_Q is the MGF of Q.

Proof 3.4 *See appendix A.15*

3.3 Student's t Returns

In this section, we want to compute our risk measurements if change in factors is assumed to be Student's t-distributed. This is made formal below.

Assumption: Let X be a multivariate Student's t distribution $(\nu, \Delta t\Sigma)$ represented as

$$X = \frac{AZ}{\sqrt{\frac{Y}{\nu}}} = A\widetilde{X},$$

where

$$AA^T = \Delta t\Sigma,\ Z \sim N(\vec{\mathbf{0}}, I_n),$$

and Y has a χ^2 distribution with ν degrees of freedom independent of X; the vector $\widetilde{X}$ has a multivariate t student distribution with (ν, I). Following Chapter 9 of [4] we assume that the change in factors in time interval $[t, t+\Delta t]$ is given by

$$\Delta S = CX,$$

where the matrix C is such that

$$CC^T = \Delta t\Sigma.$$

3.3.1 Approximated Loss

Now we would like to compute the MGF of portfolio approximated loss in this case. This will be done by a similar diagonalization operation as in the previous section. Recall the approximated portfolio loss in the normal case

$$L \approx Q = a + \sum_{i=1}^{n} \left(b_i X_i + \lambda_i X_i^2\right).$$

In the following step, let us rewrite the formula of the approximated loss Q for the Student's t case. Let Z be a vector of independent standard normals. Since

$$\Delta S = CX = \frac{CZ}{\sqrt{\frac{Y}{\nu}}},$$

then it follows that

$$Q = a + \sum_{i=1}^{n} \left(b_i \frac{Z_i}{\sqrt{\frac{Y}{\nu}}} + \lambda_i \frac{Z_i^2}{\frac{Y}{\nu}} \right).$$

In order to facilitate the calculation later, it is time to consider a fixed x, and define

$$Q_x = \frac{Y}{\nu}(Q - x) = \frac{Y}{\nu}(a - x) + \sum_{i=1}^{n} \left(b_i \sqrt{\frac{Y}{\nu}} Z_i + \lambda_i Z_i^2 \right).$$

Let us notice that

$$P(Q \leq x) \text{ if and only if } P(Q_x \leq 0).$$

3.3.2 Moment Generating Function

Let us compute the MGF of Q_x. Recall that the MGF of a χ^2 distributed with ν degrees of freedom random variable Y is

$$\phi_Y(\theta) = (1 - 2\theta)^{-\frac{\nu}{2}}.$$

Proposition 3.5 *The MGF of Q_x is given by the following formula*

$$\phi_{Q_x}(\theta) = E\left[exp\left(\theta Q_x\right)\right] = \phi_Y\left(\alpha\left(\theta\right)\right) \left[\prod_{i=1}^{n} \left(1 - 2\theta\lambda_i\right)^{-\frac{1}{2}} \right],$$

where

$$\alpha(\theta) = \frac{\theta}{\nu}(a - x) + \frac{1}{2\nu}\sum_{i=1}^{n}\frac{\theta^2 b_i^2}{1 - 2\theta\lambda_i}.$$

Proof 3.5 *See Appendix A.16*

3.3.3 Expectation and Variance

Now let us derive expectation, and Var of the approximated portfolio loss.

Proposition 3.6 *The expectation of Q_x, denoted $tE[Q_x]$, has the following formula*

$$tE[Q] = a + \frac{\nu}{\nu - 2}\sum_{i=1}^{n}\lambda_i.$$

The Var of Q, denoted $tVar[Q]$, has the following formula

$$tVar[Q] = \frac{\nu}{\nu - 2}\sum_{i=1}^{n} b_i^2 + \frac{2\nu^2}{(\nu - 2)(\nu - 4)}\sum_{i=1}^{n}\lambda_i^2 + \left(\frac{\nu^2}{(\nu - 2)(\nu - 4)} - \frac{\nu^2}{(\nu - 2)^2}\right)\left(\sum_{i=1}^{n}\lambda_i\right)^2.$$

Proof 3.6 *See Appendix A.18*

In addition, let us consider the comparison of the quantity between tVaR and VaR.

The following result states that given the same mean variance covariance matrix for factors the portfolio Var is higher in the student t-distribution setting. This is made precise below.

Proposition 3.7 *If the degree of freedom of a t-distribution is greater than 4, then*

$$tVar[Q] > Var[Q].$$

Proof 3.7 *See Appendix A.19*

3.3.4 Value at Risk

It is time to compute the VaR for the portfolio in the t-distribution case, which we denote $tVaR$.

Proposition 3.8 *Let α be a confidence level. Let x be the root of the equation*

$$\frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{\phi_{Q_x}(-u) - \phi_{Q_x}(u)}{iu} du = 1 - \alpha,$$

where ϕ_{Q_x} is the MGF of Q_x. Then

$$tVaR_\alpha = x.$$

Proof 3.8 *See Appendix A.20*

3.3.5 Semi Variance

Now we can find the portfolio SVar for the t-distribution.

Proposition 3.9 *The portfolio SVar denoted $tSVar$ is given by the following formula*

$$tSVar[Q] = \int_{tE[Q]}^\infty (x - tE[Q])^2 f_Q(x) dx,$$

where

$$f_Q(x) = \frac{1}{2\pi}\int_0^\infty \left[(1-2\alpha(-iu))^{(-\frac{\nu}{2}-1)}\left[\prod_{i=1}^{n}(1+2iu\lambda_i)^{-\frac{1}{2}}\right]\right.$$
$$\left. + (1-2\alpha(iu))^{(-\frac{\nu}{2}-1)}\left[\prod_{i=1}^{n}(1-2iu\lambda_i)^{-\frac{1}{2}}\right]\right]du.$$

Proof 3.9 *See Appendix A.21*

3.4 Portfolio Risk Measurement Example

In this numerical experiment, we consider a portfolio consisting of five European call options each written on one of the five stocks, Disney, Exxon, Pfizer, Altria, and Intel. We use the date set from Wenbo Hu and Alec N. Kercheval in [5]. The data set is given in the table below.

4/8/2005	Disney	Exxon	Pfizer	Altria	Intel
stock price	28.02	60.01	25.24	65.53	23.29
volatility	0.1699	0.2032	0.2064	0.1794	0.2476

Table 3.1: *Stock prices and volatility*

correlations	Disney	Exxon	Pfizer	Altria	Intel
Disney	1	0.367	0.337	0.189	0.42
Exxon	0.367	1	0.359	0.197	0.303
Pfizer	0.337	0.359	1	0.215	0.297
Altria	0.189	0.197	0.215	1	0.168
Intel	0.42	0.303	0.297	0.168	1

Table 3.2: *Correlation table of stock prices*

Options have maturity $T = 1$, and are at the money. They are priced by the Black Scholes formula. We perform the risk measurements for this portfolio over one week, which means $\Delta t = \frac{1}{52}$. We compute risk measures with confidence level $\alpha = 0.01$, and $\alpha = 0.05$. The interest rate is set at 0.05. C_0 is the initial value of this portfolio. The results are summarized in the tables below,

$\alpha = 0.01$	$\sqrt{Var}/C_0$	VaR/C_0	SVar/C_0
ExactSimulated	0.11089	0.24880	0.12235
DGSimulated	0.11145	0.25048	0.12254
Parametric	0.11126	0.25033	0.12348

Table 3.3: *Risk measurements at confidence level 0.01*

$\alpha = 0.05$	$\sqrt{Var}/C_0$	VaR/C_0	SVar/C_0
ExactSimulated	0.11054	0.18151	0.12191
DGSimulated	0.11111	0.18246	0.12320
Parametric	0.11126	0.18272	0.12348

Table 3.4: *Risk measurements at confidence level 0.05*

At first, we simulate the stock price changes given a Δt change in time 1,000,000 times. Then we compute portfolio values by the Black Scholes formula. We get 1000000 simulated losses and compute values of our risk measurements (Var, VaR and SVar) based on these simulated losses. We refer to these results as exact simulated.

Next, we compute Greeks of our options, then compute changes of portfolio values by delta-gamma approximation. We again get 1,000,000 losses simulated and compute the values of our risk measurements based on this simulations. We refer to these results as delta-gamma simulated.

Finally, we compute our risk measurements directly by the analytical formulas of this chapter. We call these results parametric.

The exact simulated, delta-gamma simulated, and parametric results turn out to be close to each other, which says that delta-gamma approximation is pretty good. Next let us talk about the effect of the confidence level α; as we expected, increasing the confidence level decreases VaR.

$\alpha = 0.01$	$VaR_{pa}/C0$	$CFVaR_2/C0$	$CFVaR_3/C0$
Parametric	0.25033	0.26383	0.25052

Table 3.5: *$CFVaR_2$ & $CFVaR_3$ at confidence level 0.01*

$\alpha = 0.05$	$VaR_{pa}/C0$	$CFVaR_2/C0$	$CFVaR_3/C0$
Parametric	0.18272	0.18800	0.18286

Table 3.6: *$CFVaR_2$ & $CFVaR_3$ at confidence level 0.05*

We then compute the values of $CFVaR_2$ and $CFVaR_3$ at different confidence levels and compare them with the corresponding parametric VaR in a same table. In

Table (3.5) and Table (3.6) we see that $CFVaR_2$ has significant error when compare with the true VaR, and $CFVaR_3$ is a more accurate approximation of VaR.

$\alpha = 0.01$	$\sqrt{Var}/C_0$	VaR/C_0	SVar/C_0
normal-distributed	0.11126	0.25033	0.12348
t-distributed	0.13696	0.31984	0.17533

Table 3.7: *T-distributed results at confidence level 0.01*

$\alpha = 0.05$	$\sqrt{Var}/C_0$	VaR/C_0	SVar/C_0
normal-distributed	0.11126	0.18272	0.12348
t-distributed	0.13696	0.20917	0.17533

Table 3.8: *T-distributed results at confidence level 0.05*

Then, we assume the change in factors is t-distributed. We refer to these results as t-distributed, and compare them with normal results.

In Table (3.7) we see the t-distributed risk measurements are higher than their corresponding normally distributed. For Var this finding was in fact established by **Proposition 3.7**. In Table (3.8) we see that if the confidence level increases, the difference between normally distributed VaR and t-distributed VaR decreases.

3.5 Two-Options Portfolio

In this section we find the relationship between correlation and risk measurements. For this we consider a portfolio consisting of two options each written on two different stocks, S_1, and S_2, with correlation ρ. Then we get the following result.

Proposition 3.10 *The Var of this portfolio is an increasing function of ρ.*

Proof 3.10 *See Appendix A.22*

Next we run a numerical experiment to explore the effect of correlation on other risk measures as well.

3.5.1 Portfolio Risk Measurement Example

Assume following parameters $T = 1, \Delta t = \frac{1}{52}$, stocks' initial price $S_1(0) = 50, S_2(0) = 30$, and options' stirke prices $K_1 = 40, K_2 = 20$, interest rate and volatility $r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.20$. The correlation ρ ranges from 0.1 to 0.5 in steps of 0.1. The confidence level $\alpha = 0.01$. C_0 denotes the initial value of this portfolio. The number of simulations $n = 1000000$.

ρ	$\sqrt{Var}/C_0$	VaR/C_0	SVar/C_0
0.5	0.06910	0.16266	0.05480
0.4	0.06668	0.15660	0.05109
0.3	0.06429	0.15143	0.04749
0.2	0.06187	0.14522	0.04388
0.1	0.05927	0.13929	0.04029

Table 3.9: *Exact simulated risk measurements*

ρ	$\sqrt{Var}/C_0$	VaR/C_0	SVar/C_0
0.5	0.06903	0.16182	0.05473
0.4	0.06675	0.15688	0.05118
0.3	0.06441	0.15148	0.04757
0.2	0.06191	0.14563	0.04397
0.1	0.05935	0.13939	0.04032

Table 3.10: *Delta-Gamma simulated risk measurements*

The results of Table (3.9) and Table (3.10), show that the simulated risk measurements decrease with correlation. For Var this finding was in fact established by **Proposition 3.10**.

ρ	$\sqrt{Var}/C_0$	$\sqrt{tVar}/C_0$	VaR/C_0	tVaR/C_0	SVar/C_0	tSVar/C_0
0.5	0.06910	0.08463	0.16211	0.21750	0.05473	0.08154
0.4	0.06679	0.08180	0.15679	0.21032	0.05114	0.07620
0.3	0.06440	0.07887	0.15127	0.20288	0.04754	0.07086
0.2	0.06191	0.07583	0.14554	0.19514	0.04395	0.06551
0.1	0.05932	0.07265	0.13957	0.18708	0.04035	0.06016

Table 3.11: *Parametric risk measurements*

In Table (3.11), we see the t-distributed risk measurements are higher than their corresponding normal results, and decrease with correlation as well.

ρ	$CFVaR_3/C_0$	Par VaR/C_0	Sim VaR/C_0	DGSim VaR/C_0
0.5	0.16215	0.16215	0.16188	0.16226
0.4	0.15682	0.15682	0.15643	0.15725
0.3	0.15131	0.15131	0.15121	0.15102
0.2	0.14546	0.14558	0.14578	0.14549
0.1	0.13960	0.13960	0.13973	0.13969

Table 3.12: *Cornish-Fisher VaR*

In Table (3.12), the $CFVaR_3$ approximation also has a negative relation with correlation.

3.6 Conclusion

In this chapter, we use the delta-gamma approximation to compute the portfolio loss. This procedure makes it possible to compute the MGF of loss. Then CDF and PDF of portfolio loss are found by applying Fourier inversion to MGF. Afterwards, we perform risk measurements in both when the change of factors is normally distributed or Student's t distributed.

In the first numerical experiment, Var, VaR, and SVar are computed for a portfolio consisting of five European options. This was done first by simulating the portfolio losses, and secondly by computing the parametric Var, VaR, and SVar though delta-gamma approximation and MGF inversion (VaR and SVar). VaR is also computed by means of Cornish-Fisher approximation. In the second numerical experiment, we consider a two-options portfolio. Our numerical experiment shows that risk increase with correlation; in the case of Var, an analytical formula was established for the risk dependence on correlation, and this shows the Var increase in correlation.

Chapter 4

Optimal Portfolios

Finding an optimal portfolio in a specific condition is a primary topic in financial mathematics, and it also another goal of this thesis. Since we have completed computing risk measurements of portfolios consisting of multiple options, now we are ready for finding portfolios which minimize these risks. They are parameterized by the vector of weights or shares invested in each asset. The portfolio risk is measured by Var and VaR. Let us consider portfolios which minimize Var.

4.1 Optimal Variance Portfolios

In this section, the goal is to minimize the Var of a portfolio of options. Let us consider a portfolio consisting of m instruments and let

$$x_k, \ k = 1, 2, \cdots, m$$

stand for the number of shares of instrument k held in the portfolio at time t. Denote by

$$V_k(S,t),\ k=1,2,\cdots,m$$

the value of instrument k held in the portfolio at time t. Then, the value of the portfolio is given by

$$V\left(S,t\right)=\sum_{k=1}^{m}x_kV_k(S,t).$$

Here

$$S=(S_1,S_2,\cdots,S_n)$$

is the vector of the underlyings.

We want to minimize the Var of this portfolio. Denote by ΔV the return over time interval $[t,t+\Delta t]$,

$$\Delta V=V\left(S+\Delta S,t+\Delta t\right)-V\left(S,t\right).$$

Following Chapter 9 of [4] we assume ΔS, has a normal distribution

$$\Delta S\sim N(\vec{\mathbf{0}},\Delta t\Sigma).$$

4.1.1 Delta-Gamma Approximation

Now let us use delta-gamma approximation of **Chapter 3**

$$\Delta V\approx\Delta t\frac{\partial V}{\partial t}+\delta^T\Delta S+\frac{1}{2}\Delta S^T\Gamma\Delta S,$$

where

$$\delta = [\delta_1, \delta_2, \cdots, \delta_n]^T, \ \Gamma = [\Gamma_{ij}]_{i=1,2,\cdots,n;\, j=1,2,\cdots,n}$$
$$\delta_i = \frac{\partial V}{\partial S_i}, \text{ and } \Gamma_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j}.$$

Recall that the value of the portfolio at time t is

$$V(S,t) = \sum_{k=1}^{m} x_k V_k(S,t).$$

Next let us compute the Greeks of this portfolio

$$\delta_i = \frac{\partial V}{\partial S_i} = \sum_{k=1}^{m} x_k \delta_i^k$$
$$\delta_i^k := \frac{\partial V_k}{\partial S_i}, \quad i = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, m$$
$$\Gamma_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j} = \sum_{k=1}^{m} x_k \Gamma_{ij}^k$$
$$\Gamma_{ij}^k := \frac{\partial^2 V_k}{\partial S_i \partial S_j}, \quad i, \ j = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, m.$$

In addition, we would like to use the diagonalization procedure of **Chapter 3**, with the vector of independent normals

$$Z \sim N(\vec{\mathbf{0}}, I_n),$$

and matrix C that

$$CC^T = \Delta t \Sigma$$

to get the approximated loss

$$-\Delta V \approx a + b^T Z + Z^T \Lambda Z := L.$$

Here

$$\begin{aligned} a &= -\Delta t \frac{\partial V}{\partial t} \\ b &= -C^T \delta \\ \Lambda &= -\frac{1}{2} C^T \Gamma C. \end{aligned}$$

4.1.2 Formula of Variance

Now let us compute the Var of this portfolio. Apply **Proposition 3.3** to get

$$Var[L] = \sum_{i=1}^{n} \left(b_i^2 + 2\lambda_i^2 \right).$$

Next we rewrite this formula as a function of x, where x is the vector consisting of the number of shares in the portfolio. In order to ease the notation, assume the initial value of the portfolio $V(S,t) = 1$. Let us make this computation formal.

Proposition 4.1 *Let x represent the vector of the number of shares held. The Var of L is given by*

$$Var[L] = \frac{1}{2} x^T \left(\hat{\Sigma} + Q \right) x,$$

where

$$\hat{\Sigma} = 2M^T (\Delta t \Sigma) M \quad and$$
$$M = (M_{ik}) = \left(\delta_i^k\right), \quad i = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, m$$
$$Q = (Q_{ik}) = \left(tr\left(\Gamma^i (\Delta t \Sigma) \Gamma^k (\Delta t \Sigma)\right)\right), \quad i = 1, 2, \cdots, n, \quad k = 1, \cdots, m.$$

Proof 4.1 *See Appendix A.23*

Then to minimize the Var of L we need to find vector x which solves the quadratic programming problem

$$\min_x \left\{ \frac{1}{2} x^T \left(\hat{\Sigma} + Q\right) x \right\}$$
$$s.t.$$
$$\sum_{k=1}^{m} x_k V_k(S, t) = V(S, t) = 1.$$

Notice that since $Var(L) \geq 0$, then it follows that $\left(\hat{\Sigma} + Q\right)$ is positive definite. Now set

$$P = \left(\hat{\Sigma} + Q\right)$$
$$V = (V_i (S, t)), \quad i = 1, 2, \cdots, n.$$

The Lagrangian is

$$\mathcal{L}(x, \lambda) = \frac{1}{2} x^T P x + \lambda^T (V x - 1).$$

Then our optimization problem is reduced to the solution of the first order conditions (FOC),

$$Px + V^T\lambda = 0$$
$$Vx = 1.$$

Recall P is positive definite, and therefore it is invertible so

$$x = -P^{-1}V^T\lambda.$$

Plug this in

$$Vx = 1$$

to get

$$-VP^{-1}V^T\lambda = 1.$$

So

$$\lambda = -(VP^{-1}V^T)^{-1}.$$

Plug this in

$$x = -P^{-1}V^T\lambda$$

to get

$$x = P^{-1}V^T(VP^{-1}V^T)^{-1}.$$

4.2 Optimal Cornish-Fisher Value at Risk Portfolios

In this section, we continue the ideas from the previous section, and find the portfolios which minimizes VaR. However, we can foresee that is complicated to perform optimization of VaR found through Fourier Inversion in **Chapter 3**. In order to solve this problem, we consider $CFVaR_2$, and $CFVaR_3$ approximations of VaR.

4.2.1 Optimal CFVaR2 Portfolios

Let confidence level be α, then $CFVaR_2$ is given by next proposition.

Proposition 4.2 *Let x represent the vector of the number of shares. Then*

$$\begin{aligned} CFVaR_2 &= \mu_L - N^{-1}(\alpha)\,\sigma_L \\ &= -x^T\Theta - x^Tp - N^{-1}(\alpha)\sqrt{\frac{1}{2}x^T\left(\hat{\Sigma}+Q\right)x} \end{aligned}$$

where

$$\theta_i = \Delta t \frac{\partial V_i}{\partial t}, \quad i = 1, 2, \cdots, n$$

$$\Theta = (\theta_i), \quad i = 1, 2, \cdots, n$$

$$p = \frac{1}{2} \left(tr\left(\Gamma^1 \Sigma\right), \cdots, tr\left(\Gamma^m \Sigma\right) \right)^T$$

$$\mu_L = -x^T \Theta - x^T p$$

$$\sigma_L = \sqrt{\frac{1}{2} x^T \left(\hat{\Sigma} + Q \right) x}.$$

Proof 4.2 *See Appendix A.24*

We work under the assumption that the vector

$$\Theta + p$$

is not the zero vector, nor it is proportional to vector V, for otherwise $CFVaR_2$ will be just a scalar times the standard deviation plus a term free of x. The portfolio optimization problem is

$$\min_x \left\{ -x^T \Theta - x^T p - N^{-1}(\alpha) \sqrt{\frac{1}{2} x^T \left(\hat{\Sigma} + Q \right) x} \right\}$$

$$s.t.$$

$$\sum_{k=1}^{m} x_k V_k(S, t) = 1.$$

We solve this optimization problem in two steps. In a first step, set

$$x^T\Theta + x^T p = \epsilon,$$

and optimize over x for a fixed ϵ. Then our optimization problem becomes

$$\begin{aligned} &\underset{x}{min}\left\{-\epsilon - N^{-1}\left(\alpha\right)\sigma_L\right\} \\ &s.t. \\ &Ax = b, \end{aligned}$$

where

$$A = \begin{bmatrix} p_1 + \theta_1 & \cdots & p_n + \theta_n \\ V_1(S,t) & \cdots & V_n(S,t) \end{bmatrix}$$
$$b = \begin{bmatrix} \epsilon \\ 1 \end{bmatrix}.$$

Since

$$\alpha \leq \frac{1}{2},$$

then

$$N^{-1}\left(\alpha\right) \leq 0,$$

so we conclude that $CFVaR_2$ is an increasing function of σ_L. Therefore, let us find x that minimizes σ_L^2. Set

$$P = (p_{ij}) = \left(\hat{\Sigma} + Q\right), \quad i = 1, 2, \cdots, n; \quad j = 1, 2, \cdots, n.$$

Recall that in the previous subsection, we solve for x in the following quadratic programming problem to minimizing σ_L^2, where recall that

$$\sigma_L = \sqrt{\frac{1}{2} x^T P x}.$$

Now we need to solve

$$\begin{aligned} &\min_x \left\{ \frac{1}{2} x^T P x \right\} \\ &s.t. \\ &Ax = b. \end{aligned}$$

The Lagrangian is

$$\mathcal{L}(x, \lambda) = \frac{1}{2} x^T P x + \lambda^T (Ax - b),$$

then our optimization problem is reduced to the solution of the following system

$$\begin{aligned} Px + A^T \lambda = 0 \\ Ax = b. \end{aligned}$$

Because P is invertible we get

$$x = -P^{-1}A^T\lambda.$$

Plug this in

$$Ax = b$$

to get

$$-AP^{-1}A^T\lambda = b.$$

Because the matrix

$$AP^{-1}A^T$$

is invertible (since the vector $(\Theta + p)$ is not a zero vector nor proportional to the V vector), it follows that

$$\lambda = -(AP^{-1}A^T)^{-1}b.$$

Plug this in

$$x = -P^{-1}A^T\lambda$$

to get

$$x = P^{-1}A^T(AP^{-1}A^T)^{-1}b.$$

Let

$$P^{-1}A^T(AP^{-1}A^T)^{-1} = G = \begin{bmatrix} g_{11} & g_{12} \\ \cdots & \cdots \\ g_{n1} & g_{n2} \end{bmatrix},$$

then

$$x = (x_i) = (g_{i1}\epsilon + g_{i2}),$$

and

$$\begin{aligned} \sigma_L^2 &= \frac{1}{2}x^T P x = \frac{1}{2}\left[\sum_{i=1}^{n} x_i^2 p_{ii} + \sum_{i\neq j} x_i x_j p_{ij}\right] \\ &= \mathcal{A}\epsilon^2 + \mathcal{B}\epsilon + \mathcal{C}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A} &= \frac{1}{2}\sum_{i,j} g_{i1}g_{j1}p_{ij} \\ \mathcal{B} &= \sum_{i=1}^{n} g_{i1}g_{i2}p_{ii} + \frac{1}{2}\sum_{j\neq k}(g_{j1}g_{k_2}p_{jk} + g_{j2}g_{k1}p_{jk}) \\ \mathcal{C} &= \frac{1}{2}\sum_{i,j} g_{i2}g_{j2}p_{ij}. \end{aligned}$$

Now we can establish the exact formula of $CFVaR_2$ as a function of ϵ,

$$CFVaR_2 = -\epsilon - N^{-1}(\alpha)\sqrt{\mathcal{A}\epsilon^2 + \mathcal{B}\epsilon + \mathcal{C}}.$$

In the second step we preform optimization over ϵ. The first derivative of this function is given by

$$-1 - N^{-1}(\alpha)\frac{2\mathcal{A}\epsilon + \mathcal{B}}{2\sqrt{\mathcal{A}\epsilon^2 + \mathcal{B}\epsilon + \mathcal{C}}}.$$

The critical points are

$$\begin{aligned}
\epsilon &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
a &= \left[4\mathcal{A}^2\left(N^{-1}(\alpha)\right)^2 - 4\mathcal{A}\right] \\
b &= \left[4\mathcal{A}\mathcal{B}\left(N^{-1}(\alpha)\right)^2 - 4\mathcal{B}\right] \\
c &= \left[\mathcal{B}^2\left(N^{-1}(\alpha)\right)^2 - 4\mathcal{C}\right].
\end{aligned}$$

Since

$$-N^{-1}(\alpha) > 0,$$

then

$$\frac{2\mathcal{A}\epsilon + \mathcal{B}}{2\sqrt{\mathcal{A}\epsilon^2 + \mathcal{B}\epsilon + \mathcal{C}}} = \frac{1}{-N^{-1}(\alpha)} > 0.$$

The only ϵ that makes

$$2\mathcal{A}\epsilon + \mathcal{B} > 0$$

is the minimizer. Therefore,

$$\epsilon = \begin{cases} e_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}, & if \;\; 2\mathcal{A}e_1 + \mathcal{B} > 0 \\ e_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}, & if \;\; 2\mathcal{A}e_2 + \mathcal{B} > 0. \end{cases}$$

Finally, the optimal portfolio is x, where

$$x = P^{-1}A^T(AP^{-1}A^T)^{-1}b,$$

and

$$b = \begin{bmatrix} \epsilon \\ \\ 1 \end{bmatrix}.$$

4.3 Portfolio Optimization Example

In this example, we use the same data as in **Section 3.2.10**. Assume a portfolio investing in five European call options each written on one of the five stocks. The interest rate is $r = 0.05$. Options have maturity $T = 1$, and are at the money. They are priced by the Black-Scholes-Merton formula. We optimize the risk for this portfolio over one week, which means $\Delta t = \frac{1}{52}$. We set the initial value of the portfolio to 1 dollar. Next let us compute optimal weights vector w, $w = [w_1, \cdots, w_5]$,

$$w_i = x_i V_i, \;\; i = 1, 2, \cdots, 5$$

by using our result of the previous section. The optimal portfolio weights w for minimizing Var and $CFVaR_2$ are presented in the table below.

$\alpha = 0.01$	w_1	w_2	w_3	w_4	w_5
Variance	0.18587	0.17725	0.17662	0.29702	0.16323
$CFVaR_2$	0.18338	0.17766	0.17715	0.29614	0.16566

Table 4.1: *Optimal Portfolio under confidence level 0.01*

$\alpha = 0.05$	w_1	w_2	w_3	w_4	w_5
Variance	0.18587	0.17725	0.17662	0.29702	0.16323
$CFVaR_2$	0.18235	0.17784	0.17737	0.29578	0.16666

Table 4.2: *Optimal Portfolio under confidence level 0.05*

From Table 4.1 and Table 4.2 we see little variation in the two optimal portfolios (minimal Var, and $CFVaR_2$ portfolios). This is due to: 1) the normality assumption of the stock price changes and 2) the linear term,

$$-x^T\Theta - x^T p,$$

(in the formula of $CFVaR_2$), having little effect on the optimization.

Recall

$$\sigma_L^2 = \mathcal{A}\epsilon^2 + \mathcal{B}\epsilon + \mathcal{C},$$

then for Var, $\epsilon = \epsilon_\sigma$, where

$$\epsilon_\sigma = \frac{-\mathcal{B}}{2\mathcal{A}} = -0.005018.$$

For $CFVaR_2$, $\epsilon = \epsilon_{cfvar2}$, where ϵ_{cfvar2} is the root of the following equation

$$0 = -1 - N^{-1}(\alpha)\frac{2\mathcal{A}\epsilon + \mathcal{B}}{2\sqrt{\mathcal{A}\epsilon^2 + \mathcal{B}\epsilon + \mathcal{C}}}.$$

We get

$$\epsilon_{cfvar2} = -0.005013.$$

Since ϵ_σ and ϵ_{cfvar2} are close, then we obtain optimal portfolios with little variation because they are given by

$$x = P^{-1}A^T(AP^{-1}A^T)^{-1}\begin{bmatrix}\epsilon\\1\end{bmatrix}.$$

4.4 Two-Options Portfolio

Next we perform a more accurate approximation of the VaR which is the $CFVaR_3$. However, we can foresee a complication arising from the third moment of $CFVaR_3$. Thus, we consider a portfolio which only invests in two options. Let x_1 and x_2 be the number of shares held. In addition, assume the portfolio's initial value equals 1.

4.4.1 Delta-Gamma Approximation of Portfolio Loss

Denote S_1 the initial price of the first stock, and S_2 the initial price of the second stock. Following Chapter 9 of [4] let us assume change in stock prices have zero mean, and volatility σ_1, σ_2, and correlation ρ. Next let us consider the delta-gamma approximation of the portfolio loss

$$L \approx -\left(\Delta t \frac{\partial V}{\partial t} + \delta^T \Delta S + \frac{1}{2}\Delta S^T \Gamma \Delta S\right),$$

where

$$\delta_i = \frac{\partial V}{\partial S_i},$$
$$\Gamma_{ij} = \frac{\partial^2 V}{\partial S_i \partial S_j}.$$

Then the vector of δ, and the matrix of Γ is given by

$$\delta = \begin{bmatrix} \frac{\partial x_1 V_1}{\partial S_1} \\ \frac{\partial x_2 V_2}{\partial S_2} \end{bmatrix} = \begin{bmatrix} x_1\delta_1 \\ x_2\delta_2 \end{bmatrix}$$
$$\Gamma = \begin{bmatrix} \frac{\partial^2 x_1 V_1}{\partial S_1 \partial S_1} & 0 \\ 0 & \frac{\partial^2 x_2 V_2}{\partial S_2 \partial S_2} \end{bmatrix} = \begin{bmatrix} x_1\Gamma_1 & 0 \\ 0 & x_2\Gamma_2 \end{bmatrix}.$$

Set $z_1 \sim N(0,1)$, $z_2 \sim N(0,1)$, and $z_1 \perp z_2$. Let

$$C = \begin{bmatrix} \sigma_1 S_1 \sqrt{\Delta t} & 0 \\ \sigma_2 S_2 \sqrt{\Delta t}\rho & \sigma_2 S_2 \sqrt{\Delta t}\sqrt{1-\rho^2} \end{bmatrix}$$

then

$$CC^T = \begin{bmatrix} \sigma_1^2 S_1^2 \Delta t & \sigma_1 \sigma_2 S_1 S_2 \rho \Delta t \\ \sigma_1 \sigma_2 S_1 S_2 \rho \Delta t & \sigma_2^2 S_2^2 \Delta t \end{bmatrix} = \Sigma.$$

Since ΔS is normally distributed with variance-covariance matrix Σ, then

$$\Delta S = CZ,$$

where

$$Z = [z_1, z_2]^T.$$

Thus,

$$\begin{aligned} \delta^T \Delta S &= \sigma_1 S_1 \sqrt{\Delta t} \delta_1 x_1 z_1 + \sigma_2 S_2 \sqrt{\Delta t} \delta_2 x_2 (\rho z_1 + \sqrt{1-\rho^2} z_2) \\ \frac{1}{2} \Delta S^T \Gamma \Delta S &= \frac{1}{2} (\sigma_1 S_1 \sqrt{\Delta t})^2 \Gamma_1 x_1 z_1^2 + \frac{1}{2} (\sigma_2 S_2 \sqrt{\Delta t})^2 \Gamma_2 x_2 (\rho z_1 + \sqrt{1-\rho^2} z_2)^2. \end{aligned}$$

Next let us compute central moments of the portfolio loss

$$
\begin{aligned}
E[L] &= -\Delta t\frac{\partial V}{\partial t} + E\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right] \\
Var[L] &= Var\left[-\Delta t\frac{\partial V}{\partial t} - \delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right] \\
&= Var\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right] \\
\mu_3(L) &= E\left[\left(-\Delta t\frac{\partial V}{\partial t} - \delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S - E[L]\right)^3\right] \\
&= E\Bigg[\Bigg(- \Delta t\frac{\partial V}{\partial t} - \delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S \\
&\quad - \left(E\left[-\Delta t\frac{\partial V}{\partial t}\right] + E\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right]\right)\Bigg)^3\Bigg] \\
&= E\left[\left(-\Delta t\frac{\partial V}{\partial t} - \delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S - E[L]\right)^3\right] \\
&= E\left[\left(-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S - E\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right]\right)^3\right] \\
&= \mu_3\left(-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right).
\end{aligned}
$$

Therefore, the formula of $CFVaR_3$ is given by

$$
\begin{aligned}
CFVaR_3 = &-\Delta t\frac{\partial V}{\partial t} + E\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right] \\
&- N^{-1}(\alpha)\sqrt{Var\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right]} \\
&+ \frac{1}{6}\left[\left(N^{-1}(\alpha)\right)^2 - 1\right]\frac{\mu_3\left(-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right)}{Var\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right]}.
\end{aligned}
$$

We want to write the exact formula of $CFVaR_3$ as a function number of shares. Let

$$-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S = a + b + c + d.$$

Here

$$\begin{aligned}
a_1 &= -\sigma_1 \delta_1 S_1 x_1 \\
a_2 &= -\sigma_2 \delta_2 S_2 x_2 \\
b_1 &= -\frac{1}{2}\sigma_1^2 S_1^2 \Gamma_1 x_1 \\
b_2 &= -\frac{1}{2}\sigma_2^2 S_2^2 \Gamma_2 x_2 \\
a &= \sqrt{\Delta t} a_1 z_1 \\
b &= \sqrt{\Delta t} a_2 (\rho z_1 + \sqrt{1-\rho^2} z_2) \\
c &= \Delta t b_1 z_1^2 \\
d &= \Delta t b_2 (\rho z_1 + \sqrt{1-\rho^2} z_2)^2.
\end{aligned}$$

For tractability, let $\rho = 0$. We have the following formulas:

$$\begin{aligned}
E\left[-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right] &= \Delta t b_1 + \Delta t b_2 \\
E\left[\left(-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right)^2\right] &= \Delta t a_1^2 + \Delta t a_2^2 + 3[\Delta t]^2 b_1^2 + 3[\Delta t]^2 b_2^2 + 2[\Delta t]^2 b_1 b_2 \\
E\left[\left(-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right)^3\right] &= 15[\Delta t]^3 b_1^3 + 15[\Delta t]^3 b_2^3 + 9[\Delta t]^2 a_1^2 b_1 + 9[\Delta t]^2 a_2^2 b_2 \\
&\quad + 3[\Delta t]^2 a_1^2 b_2 + 3[\Delta t]^2 a_2^2 b_1 + 9[\Delta t]^3 b_1^2 b_2 + 9[\Delta t]^3 b_1 b_2^2.
\end{aligned}$$

These formulas are established in the Appendix A.25.

Next let us denote

$$E\left[-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right] = \mu.$$

Then the formula of the second central moment and the third central moment is given by

$$Var\left[-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right] = E\left[(-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S)^2\right] - \mu^2 = \sigma^2,$$

and

$$\mu_3\left(-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right) = E\left[\left(-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right)^3\right] - 3\mu\sigma^2 - \mu^3 = \mu_3$$

Now the formula of $CFVaR_3$ is given by

$$CFVaR_3 = -\Delta t \frac{\partial V}{\partial t} + \mu - N^{-1}(\alpha)\sigma + \frac{1}{6}\left[\left(N^{-1}(\alpha)\right)^2 - 1\right]\frac{\mu_3}{\sigma^2},$$

where

$$\frac{\partial V}{\partial t} = \frac{x_1 \partial V_1}{\partial t} + \frac{x_2 \partial V_2}{\partial t} = x_1\theta_1 + x_2\theta_2,$$

and

$$\theta_i = \frac{\partial V_i}{\partial t}\ , i = 1, 2.$$

Since

$$x_2 = \frac{1 - x_1 V_1}{V_2},$$

we can write Var, $CFVaR_2$, and $CFVaR_3$ as a function of x_1. Although this function is non-linear, we can use scipy.optimize to minimize it. This is done in the next subsection.

4.4.2 Two-Options Portfolio Example

In this example, we use the same data as in **Section 3.2.10**. However, we only invest in stock options written on Disney and Exxon. Options have maturity $T = 1$, $\Delta t = \frac{1}{52}$, and they are at the money. We set the initial portfolio value to 1 dollar. We set the interest rate at 0.05. We compute the optimal w, the vector of weights, $w = [w_1, w_2]$. Our results are displayed in the table below.

$\alpha = 0.05$, $\Delta t = \frac{1}{52}$	Variance		$CFVaR_2$		$CFVaR_3$	
	w_1	w_2	w_1	w_2	w_1	w_2
	0.51789	0.48211	0.51551	0.48449	0.51515	0.48485

Table 4.3: *Optimal Portfolio of short time horizon under confidence level 0.05*

$\alpha = 0.01$, $\Delta t = \frac{1}{52}$	Variance		$CFVaR_2$		$CFVaR_3$	
	w_1	w_2	w_1	w_2	w_1	w_2
	0.51789	0.48211	0.51621	0.48379	0.51586	0.48414

Table 4.4: *Optimal Portfolio of short time horizon under confidence level 0.01*

We notice that the optimal portfolios (for Var, $CFVaR_2$, and $CFVaR_3$) are close to each other.

We would like to get an intuition for this result. Recall the formula of $CFVaR_3$

$$CFVaR_3 = -\theta_1 x_1 - \theta_2 x_2 + \mu - N^{-1}(\alpha)\sigma + \frac{1}{6}\left[\left(N^{-1}(\alpha)\right)^2 - 1\right]\frac{\mu_3}{\sigma^2}.$$

Since we take Δt to be a small number

$$\mu = \mathcal{O}(\Delta t)$$
$$\sigma = \mathcal{O}(\sqrt{\Delta t})$$
$$\mu_3 = \mathcal{O}(\Delta t^2).$$

Therefore,

$$-\frac{CFVaR_3}{\sqrt{\Delta t}N^{-1}(\alpha)} = \frac{x_1(\theta_1 - b_1) + x_2(\theta_2 - b_2)}{N^{-1}(\alpha)} + \frac{\sigma}{\sqrt{\Delta t}} - \frac{1}{6}\frac{\left[(N^{-1}(\alpha))^2 - 1\right]}{N^{-1}(\alpha)}\mathcal{O}(\Delta t).$$

It is obvious that

$$\left[\frac{x_1(\theta_1 - b_1) + x_2(\theta_2 - b_2)}{N^{-1}(\alpha)}\right]$$

is linear for x_1 and x_2. This part effects the minimisation little. If we want the term with μ_3 have more effect on minimizing $CFVaR_3$, we make

$$-\frac{1}{6}\frac{\left[(N^{-1}(\alpha))^2 - 1\right]}{N^{-1}(\alpha)}\sqrt{\Delta t}$$

large. Since

$$-\frac{1}{6}\frac{\left[\left(N^{-1}\left(\alpha\right)\right)^{2}-1\right]}{N^{-1}\left(\alpha\right)}$$

is a decreasing function of α this is achieved for very small values of α, and large enough values of Δt. The result of this parameters' choice is presented in the table below.

$\alpha = 10^{-4}$ $\Delta t = \frac{2}{12}$	Variance		$CFVaR_2$		$CFVaR_3$	
	w_1	w_2	w_1	w_2	w_1	w_2
	0.51809	0.48191	0.51662	0.48338	0.36059	0.63941

Table 4.5: *Optimal portfolio of long time horizon under confidence level* 10^{-4}

$\alpha = 10^{-7}$ $\Delta t = \frac{1}{12}$	Variance		$CFVaR_2$		$CFVaR_3$	
	w_1	w_2	w_1	w_2	w_1	w_2
	0.51789	0.48211	0.51714	0.48286	0.34034	0.65966

Table 4.6: *Optimal portfolio of long time horizon under confidence level* 10^{-7}

We notice that $CFVaR_3$ optimal portfolios are different. This result should be taken with a grain of salt because we set large time horizon (one month or two months), in this case, 1) the delta-gamma approximation may not be accurate anymore, and 2) the zero assumption of stock changes mean may not be realistic anymore. However, this result shows that different portfolios are optimal for $CFVaR_3$.

4.5 Conclusion

In this chapter, we consider portfolios which invest in options each written on an different stocks. We find the optimal Var and $CFVaR_2$ portfolios in closed form. Based on optimal portfolios' formulas we showcase the results on stock data.

Another goal of this chapter is to compare optimal Var, $CFVaR_2$, and $CFVaR_3$ portfolios. Our numerical results show that Var, and $CFVaR_2$ optimal portfolios are close to each other which means we are always minimizing the variance. However, we obtained different optimal portfolios using $CFVaR_3$ risk measure. This result implies optimization on $CFVaR_3$ is a process of minimizing VaR.

Appendix A

Appendix

A.1 Proof of Proposition 2.1.

Recall $X \sim N(\mu, \sigma)$.

The Var of X is

$$Var = \sigma^2.$$

Next let us compute the SVar of X directly from the definition

$$\begin{aligned} E\left[(X - E[X])^2 1_{\{X \leq E[X]\}}\right] &= \int_{-\infty}^{\mu} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} (x-\mu)^2 exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\} dx. \end{aligned}$$

Substitute $z = \frac{x-\mu}{\sigma}$, $dx = \sigma dz$

$$\begin{aligned} E\left[(X - E[X])^2 1_{\{X \leq E[X]\}}\right] &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} \sigma^2 z^2 exp\left\{-\frac{1}{2}(z)^2\right\} \sigma dz \\ &= \frac{1}{\sigma\sqrt{2\pi}} \sigma^3 \int_{-\infty}^{0} z^2 exp\left\{-\frac{1}{2}(z)^2\right\} dz. \end{aligned}$$

Rewrite

$$\int_{-\infty}^{0} z^2 exp\left\{-\frac{1}{2}(z)^2\right\} dz = \int_{-\infty}^{0} z(zexp\left\{-\frac{1}{2}(z)^2\right\})dz.$$

Integrate by parts with $u = z$, $dv = zexp\left\{-\frac{1}{2}(z)^2)\right\} dz$

$$\begin{aligned} v &= -exp\left\{-\frac{1}{2}(z)^2)\right\} \\ \int_{-\infty}^{0} udv &= uv|_{-\infty}^{0} - \int_{-\infty}^{0} vdu \\ &= -zexp\left\{-\frac{1}{2}(z)^2)\right\} |_{-\infty}^{0} - \int_{-\infty}^{0} -exp\left\{-\frac{1}{2}(z)^2)\right\} dz \\ &= \left[-zexp\left\{-\frac{1}{2}(z)^2)\right\} + \sqrt{\frac{\pi}{2}} erf\left(\frac{z}{\sqrt{2}}\right)\right] |_{-\infty}^{0} \\ &= \left[-0 \cdot exp(0) + \sqrt{\frac{\pi}{2}} erf(0)\right] - \left[0 + \sqrt{\frac{\pi}{2}} erf(-\infty)\right] \\ &= \sqrt{\frac{\pi}{2}}, \end{aligned}$$

since $erf(-\infty) = -1, erf(0) = 0$. Therefore, the SVar of X is

$$SVar = E\left[(X - E[X])^2 1_{\{X \leq E[X]\}}\right] = \frac{1}{\sigma\sqrt{2\pi}} \sigma^3 \sqrt{\frac{\pi}{2}} = \frac{1}{2}\sigma^2.$$

Next let us move to finding VaR of X. Since $X \sim N(\mu, \sigma)$ then

$$P\left[X+m<0\right] = P\left[X-\mu<-m-\mu\right] = P\left[\frac{X-\mu}{\sigma}<-\frac{m+\mu}{\sigma}\right] = N\left(-\frac{m+\mu}{\sigma}\right).$$

Next set $m = VaR_\alpha$, then by definition

$$P[X+m<0] = \alpha.$$

Therefore,

$$N\left(-\frac{m+\mu}{\sigma}\right) = \alpha.$$

Solve for m to get

$$m = -\left[\mu + N^{-1}(\alpha)\sigma\right]$$

which means

$$VaR_\alpha = -\left[\mu + N^{-1}(\alpha)\sigma\right].$$

Next we establish the formula of AVaR. By definition

$$\begin{aligned} AVaR_\alpha &= \frac{1}{\alpha}\int_0^\alpha VaR_u du \\ &= \frac{1}{\alpha}\int_0^\alpha -\left[\mu + N^{-1}(u)\sigma\right] du \\ &= \frac{1}{\alpha}\int_0^\alpha -\mu du - \frac{1}{\alpha}\int_0^\alpha N^{-1}(u)\sigma du \\ &= -\mu - \frac{1}{\alpha}\int_0^\alpha N^{-1}(u)\sigma du. \end{aligned}$$

To solve for $\int_0^\alpha N^{-1}(u)\sigma du$ we substitute $u = N(y)$, where N is the standard normal

CDF then $y = N^{-1}(u)$, and $du = \phi(y)dy$ (ϕ standard normal PDF), and

$$\begin{aligned}
\int_0^{\alpha} N^{-1}(u)\sigma du &= \int_{N^{-1}(0)}^{N^{-1}(\alpha)} N^{-1}(N(y))\sigma\phi(y)dy \\
&= \int_{-\infty}^{N^{-1}(\alpha)} y\sigma\phi(y)dy \\
&= \int_{-\infty}^{N^{-1}(\alpha)} \frac{y\sigma}{\sqrt{2\pi}} exp\left(-\frac{y^2}{2}\right) dy \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[-exp\left(\frac{-y^2}{2}\right)\right] |_{-\infty}^{N^{-1}(\alpha)} \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[-exp\left(\frac{-N^{-1}(\alpha)^2}{2}\right)\right] \\
&= -\sigma\phi\left(N^{-1}(\alpha)\right).
\end{aligned}$$

Therefore,

$$AVaR_{\alpha} = -\mu + \frac{\sigma\phi\left(N^{-1}(\alpha)\right)}{\alpha}.$$

A.2 Proof of Proposition 2.2.

First of all, we compute the Var of portfolio gains. Since

$$X = -Loss_{t+\tau} = rP_t,$$

then

$$Var = P_t^2\sigma_r^2.$$

Now let us compute the SVar. Recall P_t is normally distributed, then

$$X \sim N(\mu_x, \sigma_x),$$

where $\mu_x = P_t\mu_r$, $\sigma_x^2 = P_t^2\sigma_r^2$. Therefore,

$$Var = P_t^2\sigma_r^2.$$

By the same argument as in the **Proof of Proposition 2.2**, we get

$$SVar = E\left[(X - E[X])^2 1_{\{X \le EX\}}\right] = \frac{1}{\sigma_x\sqrt{2\pi}}\sigma_x^3\sqrt{\frac{\pi}{2}} = \frac{1}{2}\sigma_x^2 = \frac{1}{2}P_t^2\sigma_r^2.$$

Next, let us establish the formula of VaR. Since $X \sim N(\mu_x, \sigma_x)$ then

$$P\left[rP_t + m < 0\right] = P\left[r_t < -\frac{m}{P_t}\right] = N\left(-\frac{-\frac{m}{P_t} - \mu_r}{\sigma_r}\right).$$

Now, let $m = VaR_\alpha$ then

$$\alpha = P\left[rP_t + m < 0\right] = N\left(-\frac{-\frac{m}{P_t} - \mu_r}{\sigma_r}\right).$$

Solve for m to get

$$\frac{m}{P_t} = -\left[\mu_r + N^{-1}(\alpha)\sigma_r\right].$$

Therefore,

$$VaR_\alpha = -P_t\left[\mu_r + N^{-1}(\alpha)\sigma_r\right].$$

Now we can compute the AVaR directly from the definition

$$\begin{aligned}
AVaR_\alpha &= \frac{1}{\alpha}\int_0^\alpha VaR_u du \\
&= \frac{1}{\alpha}\int_0^\alpha -P_{t-1}\left[\mu_r + N^{-1}(u)\sigma_r\right] du \\
&= P_t\frac{1}{\alpha}\int_0^\alpha -\left[\mu_r + N^{-1}(u)\sigma_r\right] du.
\end{aligned}$$

We can use a the same argument as in the proof of **Proposition 2.2** to get

$$AVaR_\alpha = P_t\left[-\mu_r + \frac{\sigma_r\phi(N^{-1}(\alpha))}{\alpha}\right],$$

where ϕ is the standard normal pdf.

A.3 Proof of Proposition 2.3.

Let $Y = exp(R)$, where $R \sim N(\mu, \sigma)$. The MGF of Y has the formula:

$$E[Y^n] = exp\left(n\mu + \frac{1}{2}n^2\sigma^2\right).$$

Now we can establish the first and second moments of X, and obtain the formula of Var

$$E[Y] = exp\left(\mu + \frac{1}{2}\sigma^2\right)$$
$$E[Y^2] = exp(2\mu + 2\sigma^2)$$
$$Var[Y] = E[Y^2] - (E[Y])^2 = exp(2\mu + 2\sigma^2) - exp(2\mu + \sigma^2)$$
$$Var[Y] = [exp(\sigma^2) - 1]exp(2\mu + \sigma^2).$$

Substitute μ_R, and σ_R to get

$$Var = P_t^2[exp(\sigma_R^2) - 1]exp(2\mu_R + \sigma_R^2).$$

Next, let us compute the SVar. The expectation of gain has formula

$$E[X] = P_t e^{(\mu_R + \frac{1}{2}\sigma_R^2)} - P_t = P_t(e^{\left(\mu_R + \frac{1}{2}\sigma_R^2\right)} - 1).$$

Since R has log-normal distribution, rewrite

$$R = \mu_R + \sigma_R z.$$

Here z has standard normal distribution,

$$z \sim N(0, 1).$$

By definition of SVar,

$$
\begin{aligned}
SVar &= E\left[(X-E[X])^2 1_{\{X\leq E[X]\}}\right] \\
&= \int_{-\infty}^{E[X]} \left[P_t(e^{(\mu_R+\sigma_R z)}-1) - P_t(e^{\left(\mu_R+\frac{1}{2}\sigma_R^2\right)}-1)\right]^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \\
&= \int_{-\infty}^{E[X]} P_t^2 e^{2\mu_R} \left[e^{\sigma_R z} - e^{\frac{1}{2}\sigma_R^2}\right]^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz \\
&= \frac{P_t^2 e^{2\mu_R}}{\sqrt{2\pi}} \int_{-\infty}^{E[X]} \left[e^{\sigma_R z} - e^{\frac{1}{2}\sigma_R^2}\right]^2 e^{-\frac{z^2}{2}} dz.
\end{aligned}
$$

Now let us compute following integration,

$$
\begin{aligned}
\int_{-\infty}^{E[X]} \left[e^{\sigma_R z} - e^{\frac{1}{2}\sigma_R^2}\right]^2 e^{-\frac{z^2}{2}} dz &= \int_{-\infty}^{E[X]} \left[e^{2\sigma_R z} + e^{\sigma_R^2} - 2e^{(\frac{1}{2}\sigma_R^2+\sigma_R z)}\right] e^{-\frac{z^2}{2}} dz \\
&= \int_{-\infty}^{E[X]} e^{2\sigma_R z - \frac{z^2}{2}} + e^{\sigma_R^2 - \frac{z^2}{2}} - 2e^{\left(\frac{1}{2}\sigma_R^2+\sigma_R z - \frac{z^2}{2}\right)} dz.
\end{aligned}
$$

First of all, let us compute the first term in this integration.

$$
\begin{aligned}
\int_{-\infty}^{E[X]} e^{2\sigma_R z - \frac{z^2}{2}} dz &= \int_{-\infty}^{E[X]} e^{-\frac{1}{2}\left[(z-2\sigma_R)^2 - 4\sigma_R^2\right]} dz \\
&= \int_{-\infty}^{E[X]} e^{-\frac{1}{2}(z-2\sigma_R)^2 + 2\sigma_R^2} dz \\
&= e^{2\sigma_R^2} \int_{-\infty}^{E[X]} e^{-\frac{1}{2}(z-2\sigma_R)^2} dz.
\end{aligned}
$$

Substitute $u = z - 2\sigma_R$, and $du = dz$ to get

$$
\begin{aligned}
\int_{-\infty}^{E[X]} e^{2\sigma_R z - \frac{z^2}{2}} dz &= e^{2\sigma_R^2} \int_{-\infty}^{E[X]-2\sigma_R} e^{-\frac{1}{2}(u)^2} du \\
&= e^{2\sigma_R^2} \sqrt{2\pi} N(E[X] - 2\sigma_R).
\end{aligned}
$$

Here

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt$$

is the CDF of normal distribution. Then let us compute the second term in integration

$$\begin{aligned}\int_{-\infty}^{E[X]} e^{\sigma_R^2 - \frac{z^2}{2}} dz &= e^{\sigma_R^2} \int_{-\infty}^{E[X]} e^{-\frac{z^2}{2}} dz \\ &= e^{\sigma_R^2} \sqrt{2\pi} N(E[X]).\end{aligned}$$

We compute the third term in integration.

$$\begin{aligned}-\int_{-\infty}^{E[X]} 2e^{(\frac{1}{2}\sigma_R^2 + \sigma_R z - \frac{z^2}{2})} dz &= -2e^{\frac{1}{2}\sigma_R^2} \int_{-\infty}^{E[X]} e^{(\sigma_R z - \frac{z^2}{2})} dz \\ &= -2e^{\frac{1}{2}\sigma_R^2} \int_{-\infty}^{E[X]} e^{-\frac{1}{2}[(z-\sigma_R)^2 - \sigma_R^2]} dz \\ &= -2e^{\frac{1}{2}\sigma_R^2} \int_{-\infty}^{E[X]} e^{-\frac{1}{2}(z-\sigma_R)^2 + \frac{1}{2}\sigma_R^2} dz \\ &= -2e^{\sigma_R^2} \int_{-\infty}^{E[X]} e^{-\frac{1}{2}(z-\sigma_R)^2} dz.\end{aligned}$$

Substitute $u = z - \sigma_R$, $du = dz$ to get,

$$\begin{aligned}-\int_{-\infty}^{EX} 2e^{\left(\frac{1}{2}\sigma_R^2 + \sigma_R z - \frac{z^2}{2}\right)} dz &= -2e^{\sigma_R^2} \int_{-\infty}^{E[X]-\sigma_R} e^{-\frac{1}{2}u^2} du \\ &= -2e^{\sigma_R^2} \sqrt{2\pi} N(E[X] - \sigma_R).\end{aligned}$$

Therefore, the integration equals to

$$e^{2\sigma_R^2} \sqrt{2\pi} N(E[X] - 2\sigma_R) + e^{\sigma_R^2} \sqrt{2\pi} N(E[X]) - 2e^{\sigma_R^2} \sqrt{2\pi} N(E[X] - \sigma_R).$$

Next let us times it with the constant term outside to get

$$\frac{P_t^2 e^{2\mu_R}}{\sqrt{2\pi}} \left[e^{2\sigma_R^2}\sqrt{2\pi}N(E[X]-2\sigma_R)+e^{\sigma_R^2}\sqrt{2\pi}N(E[X])-2e^{\sigma_R^2}\sqrt{2\pi}N(E[X]-\sigma_R)\right]$$
$$= P_t^2 e^{2\mu_R+\sigma_R^2}\left[e^{\sigma_R^2}N(E[X]-2\sigma_R)+N(E[X])-2N(E[X]-\sigma_R)\right].$$

Therefore,

$$SVar = P_t^2 e^{2\mu_R+\sigma_R^2}\left[e^{\sigma_R^2}N(E[X]-2\sigma_R)+N(E[X])-2N(E[X]-\sigma_R)\right].$$

Now let us establish the VaR of of geometric loss. Consider

$$P\left[P_{t+\tau}-P_t+m<0\right] = P\left[P_t e^{R_t}-P_t<-m\right] = P\left[R<log\left(1-\frac{m}{P_t}\right)\right].$$

Let

$$x = log\left(1-\frac{m}{P_t}\right),$$

and $m = VaR_\alpha$ then

$$\alpha = P[P_{t+\tau}-P_t+m<0] = N\left(\frac{x-\mu_R}{\sigma_R}\right).$$

Solve for x to get

$$x = \mu_R + N^{-1}(\alpha)\sigma_R.$$

Now, let us solve for m

$$m = P_t(1 - exp[\mu_R + N^{-1}(\alpha)\sigma_R]).$$

Recall $m = VaR_\alpha$, the formula of VaR equals

$$VaR_\alpha = P_t(1 - exp[\mu_R + N^{-1}(\alpha)\sigma_R]).$$

Next let us compute AVaR directly. By definition,

$$\begin{aligned} AVaR_\alpha &= \frac{1}{\alpha}\int_0^\alpha VaR_u du \\ &= \frac{1}{\alpha}\int_0^\alpha P_t(1 - exp[\mu_R + N^{-1}(u)\sigma_R])du \\ &= P_t - P_t(exp(\mu_R)\frac{1}{\alpha}\int_0^\alpha exp(N^{-1}(u)\sigma_R)du. \end{aligned}$$

Now, let us solve the integration

$$\int_0^\alpha exp(N^{-1}(u)\sigma_R)du.$$

Let us substitute $u = N(y)$, where N is the standard normal CDF. Then $y = N^{-1}(u)$, and $du = \phi(y)dy$, where ϕ standard normal PDF. Consider following integration,

$$
\begin{aligned}
\int_0^{\alpha} exp(N^{-1}(u)\sigma_R)du &= \int_{N^{-1}(0)}^{N^{-1}(\alpha)} exp(N^{-1}(N(y)\sigma_R)\phi(y)dy \\
&= \int_{N^{-1}(0)}^{N^{-1}(\alpha)} exp(y\sigma_R)\phi(y)dy \\
&= \int_{N^{-1}(0)}^{N^{-1}(\alpha)} exp(y\sigma_R)\frac{1}{\sqrt{2\pi}}exp\left(-\frac{y^2}{2}\right)dy \\
&= \int_{N^{-1}(0)}^{N^{-1}(\alpha)} \frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}(y-\sigma_R)^2+\frac{1}{2}\sigma_R^2\right)dy \\
&= exp\left(\frac{1}{2}\sigma_R^2\right)\int_{N^{-1}(0)}^{N^{-1}(\alpha)} \frac{1}{\sqrt{2\pi}}exp\left(-\frac{1}{2}(y-\sigma_R)^2\right)dy \\
&= exp\left(\frac{1}{2}\sigma_R^2\right)\int_{N^{-1}(0)}^{N^{-1}(\alpha)} \phi(y-\sigma_R)dy \\
&= exp\left(\frac{1}{2}\sigma_R^2\right)[N(y-\sigma_R)]|_{-\infty}^{N^{-1}(\alpha)} \\
&= exp\left(\frac{1}{2}\sigma_R^2\right)[N(N^{-1}(\alpha)-\sigma_R)].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
AVaR_\alpha &= P_t - P_t(exp(\mu_R)\frac{1}{\alpha}\int_0^{\alpha} exp(N^{-1}(u)\sigma_R)du \\
&= P_t\left(1-\frac{1}{\alpha}exp\left(\mu_R+\frac{1}{2}\sigma_R^2\right)[N(N^{-1}(\alpha)-\sigma_R)]\right).
\end{aligned}
$$

A.4 Proof of Proposition 2.4.

Recall X has mean μ_X and standard deviation σ_X. Let us define normalized

$$Y = \frac{X-\mu_X}{\sigma_X}.$$

Then it follows that

$$\begin{aligned}P\left[X+m<0\right] &= P\left[X-\mu_X<-m-\mu_X\right] = P\left[\frac{X-\mu_X}{\sigma_X}<\frac{-m-\mu_X}{\sigma_X}\right] \\ &= P\left[Y<\frac{-m-\mu_X}{\sigma_X}\right].\end{aligned}$$

Then denote α by the confidence level. Then the Cornish - Fisher approximation of the α quantile $\Phi_Y(\alpha)$ is

$$\begin{aligned}\Phi_Y^{-1}(\alpha) \approx & N^{-1}(\alpha)+\frac{[N^{-1}(\alpha)]^2-1}{6}k_3+\frac{[N^{-1}(\alpha)]^3-3N^{-1}(\alpha)}{24}k_4 \\ & -\frac{2[N^{-1}(\alpha)]^3-5N^{-1}(\alpha)}{36}k_3^2.\end{aligned}$$

Here

$$k_1=\mu_1, k_2=\mu_2,$$

$$k_3=\mu_3, k_4=\mu_4-3\mu_2^2,$$

and

$$\mu_r=E[(Y-\mu)^r].$$

Now let us consider the formula of VaR. By directly computation,

$$P\left[Y<\frac{-m-\mu_X}{\sigma_X}\right]=\Phi_Y\left(\frac{-m-\mu_X}{\sigma_X}\right)=\alpha.$$

Recall $m = VaR_\alpha$, then solve for $m = VaR_\alpha$, to get

$$VaR_\alpha = -\left[\mu_X + \Phi_Y^{-1}(\alpha)\,\sigma_X\right].$$

A.5 Proof of Proposition 2.5.

Recall that

$$\frac{X-\mu}{\sigma}\sqrt{\frac{\nu}{\nu-2}} \sim student(\nu),$$

where ν is the number of degrees of freedom. Next let us compute directly,

$$P[X+m<0] = P\left[\frac{X-\mu}{\sigma}\sqrt{\frac{\nu}{\nu-2}} < -\frac{m+\mu}{\sigma}\sqrt{\frac{\nu}{\nu-2}}\right] = \Phi_Y\left(-\frac{m+\mu}{\sigma}\sqrt{\frac{\nu}{\nu-2}}\right) = \alpha.$$

Set $m = VaR_\alpha$, then solve for VaR_α to get

$$VaR_\alpha = -\Phi^{-1}(\alpha)\sqrt{\frac{\nu-2}{\nu}}\sigma - \mu.$$

Here Φ is the quantile of normalized X.

A.6 Proof of Proposition 2.6.

First, recall

$$S_t = S_0 exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right],$$
$$S_{t+\tau} = S_t exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma W_{t+\tau}\right].$$

Next let us establish the geometric return

$$R_t = log\left(\frac{S_{t+\tau}}{S_t}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma(W_{t+\tau} - W_t),$$

where

$$R_t \sim N\left(\left(\mu - \frac{1}{2}\sigma^2\right)\tau, \sigma\sqrt{\tau}\right).$$

Now let us show formulas for risk measurements, since the distribution of one-stock portfolio geometric return is obtained.

Substitute Var of geometric return with $\mu_R = \left(\mu - \frac{1}{2}\sigma^2\right)\tau$, $\sigma_R = \sigma\sqrt{\tau}$ into Var formula of **Proposition 2.3**

$$Var = S_t^2[exp(\sigma_R^2) - 1](exp(2\mu_R + \sigma_R^2)$$

to get

$$Var = S_t^2[exp(\sigma^2\tau) - 1]exp\left(2\mu\tau\right).$$

Substitute SVar of geometric return with $\mu_R = \left(\mu - \frac{1}{2}\sigma^2\right)\tau$, $\sigma_R = \sigma\sqrt{\tau}$ into SVar formula of **Proposition 2.3**

$$SVar = P_t^2 e^{2\mu_R + \sigma_R^2}\left[e^{\sigma_R^2}N(E[X] - 2\sigma_R) + N(E[X]) - 2N(E[X] - \sigma_R)\right]$$

to get

$$SVar = S_t^2 e^{2\mu\tau}\left[e^{\sigma^2\tau}N(E[X] - 2\sigma\sqrt{\tau}) + N(E[X]) - 2N(E[X] - \sigma\sqrt{\tau})\right].$$

Substitute VaR of geometric return with $\mu_R = \left(\mu - \frac{1}{2}\sigma^2\right)\tau$, $\sigma_R = \sigma\sqrt{\tau}$ into VaR formula of **Proposition 2.3**

$$VaR_\alpha = S_t \left(1 - exp\left[\mu_R + N^{-1}(\alpha)\sigma_R\right]\right)$$

to get

$$VaR_\alpha = S_t \left(1 - exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\tau + N^{-1}(\alpha)\sigma\sqrt{\tau}\right]\right).$$

Substitute AVaR of geometric return with $\mu_R = \left(\mu - \frac{1}{2}\sigma^2\right)\tau$, $\sigma_R = \sigma\sqrt{\tau}$ into AVaR formula of **Proposition 2.3**

$$AVaR_\alpha = S_t \left(1 - \frac{1}{\alpha} exp\left(\mu_R + \frac{1}{2}\sigma_R^2\right)\left[N(N^{-1}(\alpha) - \sigma_R)\right]\right)$$

to get

$$AVaR_\alpha = S_t \left(1 - \frac{1}{\alpha} exp\left(\mu\tau\right)\left[N(N^{-1}(\alpha) - \sigma\sqrt{\tau})\right]\right).$$

Here

$$E[X] = S_t\left[exp\left(\mu\tau\right) - 1\right].$$

A.7 Proof of Proposition 2.7.

In order to facilitate the computation, we let the market price of interest rate risk be zero. It is well known that the bond price under a risk-neutral measure in this model is given by

$$P(t,T) = exp[-Y(t)C(T-t) - A(T-t)],$$

where

$$\begin{aligned} Y(t) &= e^{-\lambda t}Y(0) + e^{-\lambda t}\int_0^t e^{\lambda u} dW(u) \\ C(x) &= \frac{\epsilon_1}{\lambda}\left(1 - e^{-\lambda x}\right) \\ A(x) &= \int_0^x \left(-\frac{1}{2}C'(u) + \epsilon_0\right) du. \end{aligned}$$

Let us consider the geometric return of the bond on the time interval $[t, t+\tau]$,

$$R_t = log\left(\frac{P(t+\tau, T)}{P(t,T)}\right) = log\left(P(t+\tau,T)\right) - log(P(t,T)),$$

then

$$R_t = [-Y(t+\tau)C(T-t-\tau) - A(T-t-\tau)] - [-Y(t)C(T-t) - A(T-t)].$$

Next let us establish the non stochastic term, and stochastic term of R_t, since

$$Y(t+\tau) = e^{-\lambda\tau}Y(t) + e^{-\lambda\tau}\int_t^{t+\tau} e^{\lambda u} dW(u),$$

the non stochastic term of R_t is

$$-C(T-t-\tau)Y(t)e^{\lambda\tau} + C(T-t)Y(t) + [A(T-t) - A(T-t-\tau)],$$

and the stochastic term of R_t is

$$-C(T-t-\tau)e^{-\lambda\tau}\int_t^{t+\tau} e^{\lambda u} dW(u).$$

Therefore, R_t has normal distribution, where

$$\mu_R = -C(T-t-\tau)Y(t)e^{\lambda\tau} + C(T-t)Y(t) + [A(T-t) - A(T-t-\tau)]$$
$$\sigma_R = \sqrt{C^2(T-t-\tau)\int_t^{t+\tau} e^{2\lambda(u-\tau)}du.}$$

A.8 Proof of Proposition 2.8.

Now since distribution of geometric turn is normal, next step is substituting μ_R, σ_R, and bond price $P(t,T)$ into formulas of **Proposition 2.3** to obtain

$$Var = [P(t,T)]^2 \left[exp\left(\sigma_R^2\right) - 1\right] exp\left(2\mu_R + \sigma_R^2\right)$$
$$SVar = [P(t,T)]^2 e^{2\mu_R+\sigma_R^2}\left[e^{\sigma_R^2}N(E[X]-2\sigma_R) + N(E[X]) - 2N(E[X]-\sigma_R)\right]$$
$$VaR_\alpha = P(t,T)\left(1 - exp\left[\mu_R + N^{-1}(\alpha)\sigma_R\right]\right)$$
$$AVaR_\alpha = P(t,T)\left(1 - \frac{1}{\alpha}exp\left(\mu_R + \frac{1}{2}\sigma_R^2\right)\left[N\left(N^{-1}(\alpha) - \sigma\right)\right]\right),$$

where

$$E[X] = P(t,T)\left(exp\left(\mu_R + \frac{1}{2}\sigma_R^2\right) - 1\right).$$

A.9 Proof of Proposition 2.9.

First, recall the weight of each stock

$$w_i = \frac{\Delta_i S_i(0)}{P(0)},$$

where $P(0)$ denotes the initial portfolio value. Recall that

$$w_i R_i = \frac{\Delta_i S_i(0)}{P(0)} \frac{(S_i(1) - S_i(0))}{S_i(0)} = \frac{\Delta_i (S_i(1) - S_i(0))}{P(0)}.$$

In light of this, the portfolio arithmetic return

$$\frac{P(1) - P(0)}{P(0)},$$

is given by

$$\frac{P(1) - P(0)}{P(0)} = \sum_{i=1}^{n} w_i R_i = R.$$

Since

$$\sum_{i=1}^{n} w_i R_i \sim N(w\mu, \sqrt{w\Sigma w^T}),$$

then

$$R \sim N(w\mu, \sqrt{w\Sigma w^T}).$$

Now we would like to establish formulas for risk measurements. Apply **Proposition 2.2** to get

$$\begin{aligned} Var &= P^2 w\Sigma w^T \\ SVar &= \frac{1}{2} P^2 w\Sigma w^T \\ VaR_\alpha &= -P \left[w\mu + N^{-1}(\alpha)\sqrt{w\Sigma w^T} \right] \\ AVaR_\alpha &= P \left[-w\mu + \frac{\sqrt{w\Sigma w^T}\phi(N^{-1}(\alpha))}{\alpha} \right], \end{aligned}$$

where ϕ is the standard normal pdf.

A.10 Proof of Proposition 2.10.

Assume equal weights,

$$w_1 = w_2 = \frac{1}{2},$$

in the two stocks, the Var of the stocks is $\sigma_1 = \sigma_2 = 1$, and their correlation is ρ. The portfolio Var is

$$\begin{aligned}\sigma_\rho^2 &= w\Sigma w^T \\ &= w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + w_2^2\sigma_{22} \\ &= \frac{1+\rho}{2}.\end{aligned}$$

By the previous proposition,

$$VaR_\alpha = -P\left[w\mu + N^{-1}(\alpha)\sqrt{\frac{1+\rho}{2}}\right]$$

which is increasing in ρ as long as $N^{-1}(\alpha)$ is negative or $\alpha \leq 0.5$.

A.11 Proof of Proposition 2.11.

The time t value of the portfolio is

$$X(t) = X(0)exp\left\{\int_0^t (\zeta^T\mu + r - \frac{1}{2}\left|\zeta^T\sigma\right|^2)ds + \int_0^t \zeta^T\sigma dW(s)\right\},$$

where $W(s)$ is the vector of Brownian motion. Recall that

$$\zeta^T\mu + r - \frac{1}{2}\left|\zeta^T\sigma\right|^2 = G(\zeta),$$

and consider portfolio value at time $t + \tau$,

$$\begin{aligned} X(t+\tau) &= X(t)exp\left\{ \int_t^{t+\tau} G(\zeta)ds + \int_t^{t+\tau} \zeta^T \sigma dW(s) \right\} \\ &= X(t)exp\left\{ G(\zeta)\tau + \zeta^T \sigma (W(t+\tau) - W(t)) \right\}. \end{aligned}$$

Now let us compute the geometric return for portfolio consisting of n stocks,

$$R_t = log\left(\frac{X(t+\tau)}{X(t)} \right) = G(\zeta)\tau + \zeta^T \sigma (W(t+\tau) - W(t)),$$

and

$$R_t \sim N(G(\zeta)\tau, \left|\zeta^T \sigma\right| \sqrt{\tau}).$$

Apply **Proposition 2.3** to get

$$\begin{aligned} Var &= [X(t)]^2 \left(exp((\left|\zeta^T\sigma\right|^2 \tau)) - 1 \right) exp\left(2G(\zeta)\tau + \left|\zeta^T\sigma\right|^2 \tau \right) \\ SVar &= [X(t)]^2 e^{2G(\zeta)\tau + \left|\zeta^T\sigma\right|^2 \tau} \Big[e^{\left|\zeta^T\sigma\right|^2 \tau} N(E[X] - 2\left|\zeta^T\sigma\right| \sqrt{\tau}) \\ &\quad + N(E[X]) - 2N(E[X] - \left|\zeta^T\sigma\right| \sqrt{\tau}) \Big] \\ VaR_\alpha &= X(t)[1 - exp\left\{ G(\zeta)\tau + N^{-1}(\alpha) \left|\zeta^T\sigma\right| \sqrt{\tau} \right\}] \\ AVaR_\alpha &= X(t)\left(1 - \frac{1}{\alpha} exp\left(G(\zeta)\tau + \frac{1}{2}\left|\zeta^T\sigma\right| \sqrt{\tau}^2 \right) [N(N^{-1}(\alpha) - \left|\zeta^T\sigma\right| \sqrt{\tau})] \right). \end{aligned}$$

Here

$$E[X] = X(t)(e^{(G(\zeta)\tau + \frac{1}{2}\left|\zeta^T\sigma\right|^2 \tau)} - 1),$$

and $\left|\zeta^T\sigma\right|$ is the Euclidean norm of $\zeta^T\sigma$.

A.12 Proof of Proposition 3.1.

Recall

$$Q = a + b^T Z + Z^T \Lambda Z,$$

then the MGF of Q is

$$\begin{aligned} E[exp(\theta Q)] &= E[exp(\theta(a + b^T Z + Z^T \Lambda Z)] \\ &= exp(\theta a) E[exp(\theta(b^T Z + Z^T \Lambda Z))]. \end{aligned}$$

Here

$$b^T Z = \sum_{i}^{n} b_i Z_i, \; Z^T \Lambda Z = \sum_{i=1}^{n} \lambda_j Z_j^2.$$

Let us focus on this expectation for now

$$E\left[exp(\theta(b^T Z + Z^T \Lambda Z))\right] = E\left[exp(\theta\left(\sum_{i=1}^{n} b_i Z_i + \lambda_j Z_j^2)\right)\right].$$

Since Z_i and Z_j are independent for $i \neq j$,

$$E\left[exp\left(\theta\left(\sum_{i=1}^{n} b_i Z_i + \lambda_j Z_j^2\right)\right)\right] = \prod_{i=1}^{n} E\left[exp(\theta(b_i Z_i + \lambda_j Z_j^2))\right].$$

Now we rewrite

$$E\left[exp(\theta(b_i Z_i + \lambda_j Z_j^2))\right] = E\left[exp(\theta\left(\lambda_i\left(Z_i + \frac{b_i}{2\lambda_i}\right)^2 - \frac{b_i^2}{4\lambda_i}\right)\right]$$

Substitute the MGF of the χ^2 distribution with $\theta = \theta\lambda_i$, $x = \frac{b_i}{2\lambda_i}$ into formula

$$E\left[exp(\theta(Z_i + x)^2)\right] = (1-2\theta)^{-\frac{1}{2}} exp\left(\frac{\theta x^2}{1-2\theta}\right)$$

to get

$$E\left[exp\left(\theta\lambda_i\left(Z_i + \frac{b_i}{2\lambda_i}\right)^2\right)\right] = (1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{\theta\lambda_i\frac{b_i^2}{4\lambda_i^2}}{1-2\theta\lambda_i}\right).$$

Then

$$\begin{aligned}
E\left[exp\left(\theta\left(\lambda_i\left(Z_i + \frac{b_i}{2\lambda_i}\right)^2 - \frac{b_i^2}{4\lambda_i}\right)\right)\right] &= (1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{\theta\lambda_i\frac{b_i^2}{4\lambda_i^2}}{1-2\theta\lambda_i}\right) exp\left(-\frac{\theta b_i^2}{4\lambda_i}\right) \\
&= (1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{\theta\lambda_i\frac{b_i^2}{4\lambda_i^2}}{1-2\theta\lambda_i} - \frac{\theta b_i^2}{4\lambda_i}\right) \\
&= (1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{\theta\frac{b_i^2}{4\lambda_i}}{1-2\theta\lambda_i} - \frac{\theta b_i^2}{4\lambda_i}\right) \\
&= (1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\left(\frac{1}{1-2\theta\lambda_i} - 1\right)\frac{\theta b_i^2}{4\lambda_i}\right) \\
&= (1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{2\theta\lambda_i}{1-2\theta\lambda_i} \cdot \frac{\theta b_i^2}{4\lambda_i}\right) \\
&= (1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{1}{2} \cdot \frac{\theta^2 b_i^2}{1-2\theta\lambda_i}\right)
\end{aligned}$$

Therefore, the MGF of Q is

$$E[exp(\theta Q)] = exp(\eta(\theta)),$$

where

$$\eta(\theta) = \theta a + \sum_{i=1}^{n} \frac{1}{2}\left(\frac{\theta^2 b_i^2}{1-2\theta\lambda_i} - log(1-2\theta\lambda_i)\right).$$

A.13 Proof of Proposition 3.2.

First of all, let us compute the expectation of Q by taking the first derivative of Q MGF,

$$E\left[Q\right] = \eta'\left(\theta\right) exp\left(\eta\left(\theta\right)\right)|_{\theta=0}.$$

Here

$$\begin{aligned}
\eta\left(0\right) &= 0a + \sum_{i=1}^{n} \frac{1}{2}\left(\frac{0^2 b_i^2}{1-2\cdot 0\lambda_i} - log\left(1-2\cdot 0\lambda_i\right)\right) = 0 \\
\eta'\left(0\right) &= a + \sum_{i=1}^{n} \frac{1}{2}\left(\frac{b_i^2\left(2\cdot 0 - 2\lambda_i\cdot 0^2\right)}{\left(1-2\lambda_i\cdot 0\right)^2} + \frac{2\lambda_i}{1-2\lambda_i\cdot 0}\right) = a + \sum_{i=1}^{n}\lambda_i.
\end{aligned}$$

Therefore, the obtained expectation is

$$E\left[Q\right] = \left(a + \sum_{i=1}^{n}\lambda_i\right) exp\left(0\right) = a + \sum_{i=1}^{n}\lambda_i.$$

Next let us calculate the second moment of Q by taking the second derivative of Q MGF,

$$E\left[Q^2\right] = \eta''\left(\theta\right) exp\left(\eta\left(\theta\right)\right) + \left(\eta'\left(\theta\right)\right)^2 exp\left(\eta\left(\theta\right)\right)|_{\theta=0}.$$

Here the second derivative of η at $\theta = 0$ is given by

$$\begin{aligned}
\eta''\left(0\right) &= \sum_{i=1}^{n} \frac{1}{2}\left(\frac{2b_i^2}{\left(1-2\lambda_i\cdot 0\right)^3} + \frac{4\lambda_i^2}{\left(1-2\lambda_i\cdot 0\right)^2}\right) \\
&= \sum_{i=1}^{n}\left(b_i^2 + 2\lambda_i^2\right).
\end{aligned}$$

Therefore, the second moment equals

$$E\left[Q^2\right] = M_Q^{(2)}(0) = \sum_{i=1}^{n}\left(b_i^2 + 2\lambda_i^2\right) + \left(a + \sum_{i=1}^{n}\lambda_i\right)^2.$$

Then let us compute the Var of portfolio loss

$$\begin{aligned} Var\left[Q\right] = E\left[Q^2\right] - E\left[Q\right]^2 &= \sum_{i=1}^{n}\left(b_i^2 + 2\lambda_i^2\right) + \left(a + \sum_{i=1}^{n}\lambda_i\right)^2 - \left(a + \sum_{i=1}^{n}\lambda_i\right)^2 \\ &= \sum_{i=1}^{n}\left(b_i^2 + 2\lambda_i^2\right). \end{aligned}$$

Last but not least, consider the third central moment

$$\mu_3\left(Q\right) = E\left[\left(Q - \mu_Q\right)^3\right] = E\left[Q^3\right] - 3\mu_Q\sigma_Q^2 - \mu_Q^3.$$

Now let us compute value of the third moment by taking the third derivative of Q MGF,

$$\begin{aligned} E\left[Q^3\right] &= \left[\frac{d}{d\theta}\eta''\left(\theta\right)exp\left(\eta\left(\theta\right)\right) + \frac{d}{d\theta}\left(\eta'\left(\theta\right)\right)^2 exp\left(\eta\left(\theta\right)\right)\right]|_{\theta=0} \\ &= \left[\eta'''\left(\theta\right)exp\left(\eta\left(\theta\right)\right) + \eta''\left(\theta\right)\eta'\left(\theta\right)exp\left(\eta\left(\theta\right)\right)\right]|_{\theta=0} \\ &+ \left[2\eta'\left(\theta\right)\eta''\left(\theta\right)exp\left(\eta\left(\theta\right)\right) + \left(\eta'\left(\theta\right)\right)^3 exp\left(\eta\left(\theta\right)\right)\right]|_{\theta=0} \\ &= \left[\eta'''\left(\theta\right)exp\left(\eta\left(\theta\right)\right) + 3\eta''\left(\theta\right)\eta'\left(\theta\right)exp\left(\eta\left(\theta\right)\right) + \eta''\left(\theta\right)\eta'\left(\theta\right)exp\left(\eta\left(\theta\right)\right)\right]|_{\theta=0} \end{aligned}$$

Here the third derivative of η at $\theta = 0$ is given by

$$\eta'''(0) = \sum_{i=1}^{n} \frac{6b_i^2\lambda_i}{(1-2\lambda_i \cdot 0)^4} + \frac{8\lambda^3}{(1-2\lambda_i \cdot 0)^3}.$$

Recall

$$\eta'(0) = E[Q] = a + \sum_{i=1}^{n} \lambda_i = \mu_Q$$
$$\eta''(0) = Var(Q) = \sum_{i=1}^{n} \left(b_i^2 + 2\lambda_i^2\right) = \sigma_Q^2.$$

Therefore,

$$\begin{aligned} E\left[Q^3\right] &= \eta'''(0) + 3\mu_Q\sigma_Q^3 + \mu_Q^3 \\ \mu_3(Q) &= E\left[(Q-\mu_Q)^3\right] = E\left[Q^3\right] - 3\mu_Q\sigma_Q^3 - \mu_Q^3 \\ &= \eta'''(0) + 3\mu_Q\sigma_Q^3 + \mu_Q^3 - 3\mu_Q\sigma_Q^3 - \mu_Q^3 \\ &= \eta'''(0) \\ &= \sum_{i=1}^{n} 6b_i^2\lambda_i + 8\lambda_i^3 \end{aligned}$$

A.14 Proof of Proposition 3.3.

Firstly, let us obtain the CDF of approximated portfolio loss Q. The characteristic function of Q is

$$\phi_Q(iu) = exp(\eta(iu)).$$

By Fourier inversion in **Chapter 4** of [7], Q has the following CDF

$$F_Q(x) = P(Q \leq x) = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{iux}\phi_Q(-iu) - e^{-iux}\phi_Q(iu)}{iu}du.$$

Set

$$VaR_\alpha = x,$$

by definition

$$\begin{aligned} P(-Q + x < 0) &= \alpha \\ P(x < Q) &= \alpha \\ P(Q \leq x) &= 1 - \alpha = \frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{e^{iux}\phi_Q(-iu) - e^{-iux}\phi_Q(iu)}{iu}du \end{aligned}$$

Then we solve for x in the above equation.

A.15 Proof of Proposition 3.4.

First let us compute the formula of $f_Q(x)$, the PDF of Q, by

$$\begin{aligned} f_Q(x) &= \frac{\partial}{\partial x}F_Q(x) \\ &= \frac{1}{2\pi}\frac{d}{dx}\int_0^\infty \frac{e^{iux}\phi_Q(-iu) - e^{-iux}\phi_Q(iu)}{iu}du \\ &= \frac{1}{\pi}\int_0^\infty e^{-iux}\phi_Q(iu)du. \end{aligned}$$

Then by **Definition 2.3.**

$$SVar[Q] = \int_{EQ}^{\infty} (x - EQ)^2 f_Q(x)\, dx$$

A.16 Proof of Proposition 3.5.

First let us consider conditional MGF of Q_x. If Y is known, then

$$E[exp(\theta Q_x)|Y] = exp\left(\frac{\theta Y}{\nu}(a-x)\right) E\left[exp\left(\sum_{i=1}^{n}\left(b_i\sqrt{\frac{Y}{\nu}}Z_i + \lambda_i Z_i^2\right)\right)\right].$$

In **Appendix 5.12** we have already obtained

$$\begin{aligned} E\left[exp\left(\theta\left(\sum_{i=1}^{n} b_i Z_i + \lambda_j Z_j^2\right)\right)\right] &= \prod_{i=1}^{n} E\left[exp\left(\theta\left(b_i Z_i + \lambda_j Z_j^2\right)\right)\right] \\ &= \prod_{i=1}^{n}(1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{1}{2}\cdot\frac{\theta^2 b_i^2}{1-2\theta\lambda_i}\right). \end{aligned}$$

Now let us replace b_i with $b_i\sqrt{\frac{Y}{\nu}}$ in previous formula. Then the following conditional MGF holds true

$$\begin{aligned} E[exp(\theta Q_x)|Y] &= exp\left(\frac{\theta Y}{\nu}(a-x)\right)\prod_{i=1}^{n}(1-2\theta\lambda_i)^{-\frac{1}{2}} exp\left(\frac{1}{2\nu}\cdot\frac{\theta^2 Y b_i^2}{1-2\theta\lambda_i}\right) \\ &= exp\left(\frac{\theta Y}{\nu}(a-x) + \frac{1}{2\nu}\sum_{i=1}^{n}\frac{\theta^2 Y b_i^2}{1-2\theta\lambda_i}\right)\prod_{i=1}^{n}(1-2\theta\lambda_i)^{-\frac{1}{2}}, \end{aligned}$$

In addition let us factor out Y to get

$$\frac{\theta Y}{\nu}(a-x) + \frac{1}{2\nu}\sum_{i=1}^{n}\frac{\theta^2 Y b_i^2}{1-2\theta\lambda_i} = \left(\frac{\theta}{\nu}(a-x) + \frac{1}{2\nu}\sum_{i=1}^{n}\frac{\theta^2 b_i^2}{1-2\theta\lambda_i}\right) Y,$$

and set

$$\frac{\theta}{\nu}(a-x)+\frac{1}{2\nu}\sum_{i=1}^{n}\frac{\theta^2 b_i^2}{1-2\theta\lambda_i}=\alpha(\theta).$$

Now the conditional MGF is

$$E[exp(\theta Q_x)|Y]=exp(\alpha(\theta)Y)\left[\prod_{i=1}^{n}(1-2\theta\lambda_i)^{-\frac{1}{2}}\right].$$

Recall that $\phi_Y(\theta)$ is the MGF of Y, and it is given by

$$\phi_Y(\theta)=(1-2\theta)^{-\frac{\nu}{2}}.$$

By the law of total expectation

$$\begin{aligned}E[exp(\theta Q_x)]&=E[E[exp(\theta Q_x)|Y]]\\&=E[exp(\alpha(\theta)Y)]\left[\prod_{i=1}^{n}(1-2\theta\lambda_i)^{-\frac{1}{2}}\right]\\&=\phi_Y(\alpha(\theta))\left[\prod_{i=1}^{n}(1-2\theta\lambda_i)^{-\frac{1}{2}}\right].\end{aligned}$$

A.17 MGF of Inverse Chi-Squared Distribution

The result of this section are needed in the proof of **Proposition 3.6**. Assume X has a chi-squared distribution, with ν degrees of freedom. Let

$$Y=\frac{1}{X}.$$

Recall an inverse-chi-squared distributed random variable have following PDF:

$$f_Y(y) = \frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} y^{\left(-\frac{\nu}{2}-1\right)} exp\left(-\frac{1}{2y}\right).$$

The expectation and Var of Y are given by the following formulas

$$E[Y] = \frac{1}{\nu - 2}$$
$$E[Y^2] = \frac{1}{(\nu-2)(\nu-4)}.$$

Now let us establish these formulas. The expectation of Y is

$$\begin{aligned}
E[Y] &= \int_0^\infty y \frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} y^{\left(-\frac{\nu}{2}-1\right)} exp\left(-\frac{1}{2y}\right) dy \\
&= \int_0^\infty \frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{y}\right)^{\frac{\nu}{2}} exp\left(-\frac{1}{2y}\right) dy \\
\text{substitute } u &= \frac{1}{y}, dy = -y^2 du, -y^2 = -u^{-2} \\
&= -\int_\infty^0 \frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} u^{\frac{\nu}{2}} exp\left(-\frac{1}{2}u\right) y^2 du \\
&= \int_0^\infty \frac{2^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} u^{\frac{\nu}{2}} exp\left(-\frac{1}{2}u\right) u^{-2} du \\
&= \int_0^\infty \frac{\frac{1}{2}^{-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} u^{\left(\frac{\nu}{2}-2\right)} exp\left(-\frac{1}{2}u\right) du.
\end{aligned}$$

By property of the gamma function,

$$\Gamma\left(\frac{\nu}{2}\right) = \Gamma\left(\frac{\nu}{2}-1\right)\left(\frac{\nu}{2}-1\right), \tag{A.17.1}$$

we get

$$E[Y] = \int_0^\infty \frac{\frac{1}{2}^{(\frac{\nu}{2}-1)}\frac{1}{2}}{\Gamma(\frac{\nu}{2}-1)(\frac{\nu}{2}-1)} u^{((\frac{\nu}{2}-1)-1)} exp\left(-\frac{1}{2}u\right) du$$
$$= \frac{\frac{1}{2}}{\frac{\nu}{2}-1} \int_0^\infty \frac{\frac{1}{2}^{(\frac{\nu}{2}-1)}}{\Gamma(\frac{\nu}{2}-1)} u^{((\frac{\nu}{2}-1)-1)} exp\left(-\frac{1}{2}u\right) du.$$

Note that

$$\frac{\frac{1}{2}^{(\frac{\nu}{2}-1)}}{\Gamma(\frac{\nu}{2}-1)} u^{((\frac{\nu}{2}-1)-1)} exp\left(-\frac{1}{2}u\right)$$

is the the PDF of

$$Gamma\left(u, \frac{\nu}{2}-1, \frac{1}{2}\right).$$

Then it is obvious,

$$\int_0^\infty \frac{\frac{1}{2}^{(\frac{\nu}{2}-1)}}{\Gamma(\frac{\nu}{2}-1)} u^{((\frac{\nu}{2}-1)-1)} exp\left(-\frac{1}{2}u\right) du = 1$$

Therefore,

$$E[Y] = \frac{\frac{1}{2}}{\frac{\nu}{2}-1} = \frac{1}{\nu-2}$$

Next let us compute $E[Y^2]$

$$\begin{aligned}
E[Y^2] &= \int_0^\infty \frac{2^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} y^{(-\frac{\nu}{2}+1)} exp\left(-\frac{1}{2y}\right) dy \\
&= \int_0^\infty \frac{2^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \left(\frac{1}{y}\right)^{(\frac{\nu}{2}-1)} exp\left(-\frac{1}{2y}\right) dy \\
\text{substitute } u &= \frac{1}{y}, dy = -y^2 du, -y^2 = -u^{-2} \\
&= -\int_\infty^0 \frac{2^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} u^{(\frac{\nu}{2}-1)} exp\left(-\frac{1}{2}u\right) y^2 du \\
&= \int_0^\infty \frac{2^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} u^{(\frac{\nu}{2}-1)} exp\left(-\frac{1}{2}u\right) u^{-2} du \\
&= \int_0^\infty \frac{\frac{1}{2}^{-\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} u^{(\frac{\nu}{2}-3)} exp\left(-\frac{1}{2}u\right) du
\end{aligned}$$

By applying (A.17.1) twice we get

$$\begin{aligned}
E[Y^2] &= \int_0^\infty \frac{\frac{1}{2}^{(\frac{\nu}{2}-2)}\frac{1}{4}}{\Gamma(\frac{\nu}{2}-2)(\frac{\nu}{2}-2)(\frac{\nu}{2}-1)} u^{((\frac{\nu}{2}-2)-1)} exp\left(-\frac{1}{2}u\right) du \\
&= \frac{\frac{1}{4}}{(\frac{\nu}{2}-1)(\frac{\nu}{2}-2)} \int_0^\infty \frac{\frac{1}{2}^{(\frac{\nu}{2}-2)}}{\Gamma(\frac{\nu}{2}-2)} u^{((\frac{\nu}{2}-2)-1)} exp\left(-\frac{1}{2}u\right) du.
\end{aligned}$$

Again

$$\frac{\frac{1}{2}^{(\frac{\nu}{2}-2)}}{\Gamma(\frac{\nu}{2}-2)} u^{((\frac{\nu}{2}-2)-1)} exp\left(-\frac{1}{2}u\right)$$

is the the PDF of

$$Gamma\left(u, \frac{\nu}{2}-1, \frac{1}{2}\right),$$

so

$$\int_0^\infty \frac{\frac{1}{2}^{(\frac{\nu}{2}-2)}}{\Gamma(\frac{\nu}{2}-2)} u^{((\frac{\nu}{2}-2)-1)} exp\left(-\frac{1}{2}u\right) du = 1$$

Therefore,

$$E[Y^2] = \frac{\frac{1}{4}}{(\frac{\nu}{2}-1)(\frac{\nu}{2}-2)} = \frac{1}{(\nu-2)(\nu-4)}$$

A.18 Proof of Proposition 3.6.

Recall

$$Q = a + \sum_{i=1}^{n} \left(b_i \frac{Z_i}{\sqrt{\frac{Y}{\nu}}} + \lambda_i \frac{Z_i^2}{\frac{Y}{\nu}} \right).$$

First we would like to apply the law of total expectation, to conditional expectation of Q if Y is known. Let us represent this process formally.

$$\begin{aligned} E[Q|Y] &= a + \frac{\nu}{Y}\sum_{i=1}^{n} \lambda_i \\ E[Q] &= E\left[a + \frac{\nu}{Y}\sum_{i=1}^{n} \lambda_i\right] \\ &= a + \nu \sum_{i=1}^{n} \lambda_i E\left[\frac{1}{Y}\right]. \end{aligned}$$

By **Appendix 5.17**,

$$E\left[\frac{1}{Y}\right] = \frac{1}{\nu-2}.$$

Therefore,

$$E[Q] = a + \frac{\nu}{\nu-2}\sum_{i=1}^{n} \lambda_i.$$

Next let us compute Var of Q. First, consider

$$
\begin{aligned}
Q^2 = a^2 + a\sum_{i=1}^{n} b_i W_i + a\sum_{i=1}^{n}\lambda_i W_i^2 \\
+ a\sum_{i=1}^{n} b_i W_i + \left(\sum_{j=1}^{n} b_j W_j\right)^2 + \left(\sum_{j=1}^{n} b_j W_j\right)\left(\sum_{i=1}^{n}\lambda_i W_i^2\right) \\
+ a\sum_{i=1}^{n}\lambda_i W_i^2 + \left(\sum_{j=1}^{n}\lambda_j W_j^2\right)\left(\sum_{i=1}^{n} b_i W_i\right) + \left(\sum_{j=1}^{n}\lambda_j W_j^2\right)^2.
\end{aligned}
$$

Here

$$
W_i = \sqrt{\frac{\nu}{Y}} Z_i \sim N\left(0, \sqrt{\frac{\nu}{Y}}\right).
$$

Expectations of the first three terms are given by

$$
\begin{aligned}
E[a^2] &= a^2 \\
E\left[a\sum_{i=1}^{n} b_i W_i\right] &= 0 \\
E\left[a\sum_{i=1}^{n}\lambda_i W_i^2\right] &= a\frac{\nu}{Y}\sum_{i=1}^{n}\lambda_i.
\end{aligned}
$$

Next let us compute the expectation for the fifth term

$$
\begin{aligned}
E\left[\left(\sum_{i=1}^{n} b_i W_i\right)^2\right] &= E\left[b_1 W_1 \sum_{i=1}^{n} b_i W_i + \cdots + b_n W_n \sum_{i=1}^{n} b_i W_i\right] \\
&= \sum_{i=j}^{n} E\left[b_j W_j \sum_{i=1}^{n} b_i W_i\right] \\
&= \sum_{i=j}^{n}\left[\sum_{i=1}^{n} b_j W_j E[W_j W_i]\right].
\end{aligned}
$$

Since W_j,W_i are independent for $i \neq j$,

$$E[W_jW_i] = (E[W_i])(E[W_j]) = 0,\ i \neq j,$$
$$E[W_jW_i] = (E[W_j^2]) = \frac{\nu}{Y},\ i = j,$$

then

$$E\left[b_jW_j\sum_{i=1}^{n} b_iW_i\right] = \frac{\nu}{Y}b_j^2$$
$$E\left[\left(\sum_{j=1}^{n} b_jW_j\right)\left(\sum_{i=1}^{n} b_iW_i\right)\right] = \frac{\nu}{Y}\sum_{i=1}^{n} b_i^2.$$

Now let us consider the remaining terms.

$$\begin{aligned}E\left[\left(\sum_{j=1}^{n} b_jW_j\right)\left(\sum_{i=1}^{n} \lambda_iW_i^2\right)\right] &= E\left[b_1W_1\sum_{i=1}^{n} \lambda_iW_i^2\right] + \cdots + E\left[b_nW_n\sum_{i=1}^{n} \lambda_iW_i^2\right]\\ &= \sum_{j=1}^{n} E\left[b_jW_j\sum_{i=1}^{n} \lambda_iW_i^2\right]\\ &= \sum_{j=1}^{n}\left[\sum_{i=1}^{n} b_jb_iE[W_jW_i^2]\right].\end{aligned}$$

Since W_j,W_i are independent for $i \neq j$,

$$E[W_jW_i^2] = E[W_j]E[W_i^2] = 0 \cdot 1 = 0,\ i \neq j,$$
$$E[W_jW_i^2] = E[W_j^3] = 0,\ i = j,$$

then

$$E\left[b_jW_j\sum_{i=1}^{n}\lambda_iW_i^2\right]=0$$
$$E\left[\left(\sum_{j=1}^{n}b_jW_j\right)\left(\sum_{i=1}^{n}\lambda_iW_i^2\right)\right]=\sum_{i=1}^{n}0=0.$$

Finally let us compute the expectation for the last term.

$$\begin{aligned}E\left[\left(\sum_{j=1}^{n}\lambda_jW_j^2\right)^2\right]&=E\left[\lambda_1W_1^2\sum_{i=1}^{n}\lambda_iW_i^2\right]+\cdots+E\left[\lambda_nW_n^2\sum_{i=1}^{n}\lambda_iW_i^2\right]\\&=\sum_{j=1}^{n}E\left[\lambda_jW_j^2\sum_{i=1}^{n}\lambda_iW_i^2\right]\\&=\sum_{j=1}^{n}\left[\lambda_J^2[EW_j^4]+\sum_{i=1,i\neq j}^{n}\lambda_j\lambda_iE[W_i^2W_j^2]\right].\end{aligned}$$

Since

$$E[W_j^4]=3\sigma^4=3\frac{\nu^2}{Y^2},$$
$$E[W_i^2W_j^2]=\frac{\nu}{Y}\cdot\frac{\nu}{Y}=\frac{\nu^2}{Y^2},\ i\neq j,$$

then

$$E\left[\left(\sum_{j=1}^{n}\lambda_jW_j^2\right)^2\right]=\frac{\nu^2}{Y^2}\left(2\sum_{i=1}^{n}\lambda_i^2+\left(\sum_{i=1}^{n}\lambda_i\right)^2\right).$$

Next if Y is known, the conditional expectation of Q is given by

$$E[Q^2|Y]=a^2+2a\frac{\nu}{Y}\sum_{i=1}^{n}\lambda_i+\frac{\nu}{Y}\sum_{i=1}^{n}b_i^2+\frac{\nu^2}{Y^2}\left(2\sum_{i=1}^{n}\lambda_i^2+\left(\sum_{i=1}^{n}\lambda_i\right)^2\right).$$

Then the expectation of the value above is given by

$$\begin{aligned} E[E[Q^2|Y]] &= E\left[a^2 + 2a\frac{\nu}{Y}\sum_{i=1}^{n}\lambda_i + \frac{\nu}{Y}\sum_{i=1}^{n}b_i^2 + \frac{\nu^2}{Y^2}\left(2\sum_{i=1}^{n}\lambda_i^2 + \left(\sum_{i=1}^{n}\lambda_i\right)^2\right)\right] \\ &= a^2 + 2a\nu E\left[\frac{1}{Y}\right]\sum_{i=1}^{n}\lambda_i + \nu E\left[\frac{1}{Y}\right]\sum_{i=1}^{n}b_i^2 \\ &+ \nu^2 E\left[\frac{1}{Y^2}\right]\left(2\sum_{i=1}^{n}\lambda_i^2 + \left(\sum_{i=1}^{n}\lambda_i\right)^2\right). \end{aligned}$$

Substitute

$$E\left[\frac{1}{Y}\right] = \frac{1}{\nu-2},$$

$$E\left[\frac{1}{Y^2}\right] = \frac{1}{(\nu-2)(\nu-4)},$$

(see Appendix A.17) to get

$$\begin{aligned} E[Q^2] =& a^2 + 2a\frac{\nu}{\nu-2}\sum_{i=1}^{n}\lambda_i + \frac{\nu}{\nu-2}\sum_{i=1}^{n}b_i^2 \\ &+ \frac{\nu^2}{(\nu-2)(\nu-4)}\left(2\sum_{i=1}^{n}\lambda_i^2 + \left(\sum_{i=1}^{n}\lambda_i\right)^2\right). \end{aligned}$$

Before our last step, let us compute $(E[Q])^2$.

$$(E[Q])^2 = a^2 + 2\frac{a\nu}{\nu-2}\sum_{i=1}^{n}\lambda_i + \frac{\nu^2}{(\nu-2)^2}\left(\sum_{i=1}^{n}\lambda_i\right)^2.$$

Finally, the Var of Q is given by

$$
\begin{aligned}
Var[Q] = & \frac{\nu}{\nu-2}\sum_{i=1}^{n} b_i^2 + \frac{2\nu^2}{(\nu-2)(\nu-4)}\sum_{i=1}^{n}\lambda_i^2 \\
& + \left(\frac{\nu^2}{(\nu-2)(\nu-4)} - \frac{\nu^2}{(\nu-2)^2}\right)\left(\sum_{i=1}^{n}\lambda_i\right)^2 .
\end{aligned}
$$

A.19 Proof of Proposition 3.7

Recall formulas of Var, and tVar.

$$
\begin{aligned}
Var[Q] = & \sum_{i=1}^{n}\left(b_i^2 + 2\lambda_i^2\right) \\
tVar[Q] = & \frac{\nu}{\nu-2}\sum_{i=1}^{n} b_i^2 + \frac{2\nu^2}{(\nu-2)(\nu-4)}\sum_{i=1}^{n}\lambda_i^2 \\
& + \left(\frac{\nu^2}{(\nu-2)(\nu-4)} - \frac{\nu^2}{(\nu-2)^2}\right)\left(\sum_{i=1}^{n}\lambda_i\right)^2 .
\end{aligned}
$$

Since $\nu > 4$, then

$$
\begin{aligned}
\frac{\nu}{\nu-2} &> 1 \\
\frac{2\nu^2}{(\nu-2)(\nu-4)} &> 2 \\
\left(\frac{\nu^2}{(\nu-2)(\nu-4)} - \frac{\nu^2}{(\nu-2)^2}\right) &> 1,
\end{aligned}
$$

proves the claim.

A.20 Proof of Proposition 3.8.

Let us first consider

$$\phi_{Q_x}(iu) = \phi_Y(\alpha(iu)) \prod_{i=1}^{n} (1 - 2iu\lambda_i))^{-\frac{1}{2}}$$

is the characteristic function of Q_x. By Fourier inversion, Q_x has following CDF

$$F_{Q_x}(a) = P(Q_x \leq a) = \frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{e^{iua}\phi_{Q_x}(-iu) - e^{-iua}\phi_{Q_x}(iu)}{iu} du.$$

Recall

$$P(Q_x \leq 0) \text{ if and only if } P(Q \leq x).$$

Set

$$VaR = x,$$

then by definition:

$$\begin{aligned}
&P(-Q + x < 0) = \alpha \\
&\quad P(x < Q) = \alpha \\
&\quad P(Q \leq x) = 1 - \alpha = P(Q_x \leq 0) = F_{Q_x}(0) = \frac{1}{2} + \frac{1}{2\pi} \int_0^\infty \frac{\phi_{Q_x}(-u) - \phi_{Q_x}(u)}{iu} du.
\end{aligned}$$

Then we solve for x for the above equation

A.21 Proof of Proposition 3.9.

First let us compute the formula of $f_Q(x)$, the PDF of Q, by

$$\begin{aligned} f_Q(x) &= \frac{\partial}{\partial x} F_Q(x) \\ &= \frac{\partial}{\partial x} F_{Q_x} \\ &= \frac{\partial}{\partial x}\left(\frac{1}{2} + \frac{1}{2\pi}\int_0^\infty \frac{\phi_{Q_x}(-iu) - \phi_{Q_x}(iu)}{iu} du\right) \\ &= \frac{1}{2\pi}\int_0^\infty \left[(1 - 2\alpha(-iu))^{(-\frac{\nu}{2}-1)}\left[\prod_{i=1}^{n}(1+2iu\lambda_i)^{-\frac{1}{2}}\right]\right. \\ &\left. + (1-2\alpha(iu))^{(-\frac{\nu}{2}-1)}\left[\prod_{i=1}^{n}(1-2iu\lambda_i)^{-\frac{1}{2}}\right]\right] du \end{aligned}$$

Then by **Definition 2.3.**

$$tSVar[Q] = \int_{EQ}^{\infty} (x - EQ)^2 f_Q(x) dx.$$

A.22 Proof of Proposition 3.10.

Let us prove that the portfolio Var is an increasing function of the correlation. Let

$$z_1 \sim N(0,1),\ z_2 \sim N(0,1),$$

z_1 and z_2 are independent. Here

$$\delta_1 = \frac{\partial c_1}{\partial S_1}$$
$$\delta_2 = \frac{\partial c_2}{\partial S_2}$$
$$\Gamma_1 = \frac{\partial^2 c_1}{\partial S_1^2}$$
$$\Gamma_2 = \frac{\partial^2 c_2}{\partial S_2^2},$$

where c_1, and c_2 are the option prices. Define

$$\Delta v = \delta_1 z_1 + \delta_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right) + \Gamma_1 z_1^2 + \Gamma_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right)^2,$$

then

$$Gain = \Delta v + deterministic\ term$$
$$Var\left[Loss\right] = Var\left[-Gain\right] = Var\left[Gain\right]$$
$$Var\left[Gain\right] = Var\left[\Delta v + deterministic\ term\right] = Var\left[\Delta v\right].$$

Now we want to show $Var(\Delta v)$ is an increasing function of ρ.

$$\begin{aligned} E\left[\Delta v\right] &= \delta_1 E\left[z_1\right] + \delta_2 \rho E\left[z_1\right] + \delta_2 \sqrt{1-\rho^2} E\left[z_2\right] \\ &\quad + \Gamma_1 E\left[z_1^2\right] + \Gamma_2 \rho^2 E\left[z_1^2\right] + 2\Gamma_2 \rho \sqrt{1-\rho^2} E\left[z_1 z_2\right] \\ &\quad + \Gamma_2 \left(1-\rho\right)^2 E\left[z^2\right] \\ &= \Gamma_1 + \Gamma_2. \end{aligned}$$

Let $\Delta v = a + b + c + d$

$$
\begin{aligned}
a &= \delta_1 z_1 \\
b &= \delta_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right) \\
c &= \Gamma_1 z_1^2 \\
d &= \Gamma_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right)^2 \\
&= \Gamma_2 \left(\rho^2 z_1^2 + \left(1-\rho^2\right) z_2^2 + 2\rho\sqrt{1-\rho^2} z_1 z_2\right)
\end{aligned}
$$

$(a+b+c+d)^2 =$

$$
\begin{aligned}
&(\delta_1 z_1)^2 + \left(\delta_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right)\right)^2 \\
&+ \left(\Gamma_1 z_1^2\right)^2 + \left(\Gamma_2 \left(\rho^2 z_1^2 + \left(1-\rho^2\right) z_2^2 + 2\rho\sqrt{1-\rho^2} z_1 z_2\right)\right)^2 \\
&+ 2\left(\delta_1 z_1\right)\left(\delta_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right)\right) \\
&+ 2\left(\Gamma_1 z_1^2\right)\left(\Gamma_2 \left(\rho^2 z_1^2 + \left(1-\rho^2\right) z_2^2 + 2\rho\sqrt{1-\rho^2} z_1 z_2\right)\right) + 2\left(\delta_1 z_1\right)\left(\Gamma_1 z_1^2\right) \\
&+ 2\left(\delta_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right)\right)\left(\Gamma_1 z_1^2\right) \\
&+ 2\left(\delta_1 z_1\right)\left(\Gamma_2 \left(\rho^2 z_1^2 + \left(1-\rho^2\right) z_2^2 + 2\rho\sqrt{1-\rho^2} z_1 z_2\right)\right) \\
&+ 2\left(\delta_2 \left(\rho z_1 + \sqrt{1-\rho^2} z_2\right)\right)\left(\Gamma_2 \left(\rho^2 z_1^2 + \left(1-\rho^2\right) z_2^2 + 2\rho\sqrt{1-\rho^2} z_1 z_2\right)\right)
\end{aligned}
$$

By independence of z_1 and z_2, their zero skewness and kurtosis of three,

$$\begin{aligned} E\left[\Delta v^2\right] &= \delta_1^2 + \delta_2^2 + 3\Gamma_1^2 + 3\Gamma_2^2 + 2\delta_1\delta_2\rho + 4\Gamma_1\Gamma_2\rho^2 + 2\Gamma_1\Gamma_2 \\ Var\left[\Delta v\right] &= E\left[\Delta v^2\right] - E\left[\Delta v\right]^2 \\ &= \delta_1^2 + \delta_2^2 + 3\Gamma_1^2 + 3\Gamma_2^2 + 2\delta_1\delta_2\rho + 4\Gamma_1\Gamma_2\rho^2 + 2\Gamma_1\Gamma_2 - \left(\Gamma_1 + \Gamma_2\right)^2 \\ &= f\left(\rho\right) \end{aligned}$$

Calculate the derivative of $f(\rho)$ to get

$$f'\left(\rho\right) = 2\delta_1\delta_2 + 8\Gamma_1\Gamma_2\rho \geq 0.$$

Therefore Var of loss is an increasing function of ρ.

A.23 Proof of Proposition 4.1

Firstly, recall **Proposition 3.2**, Var of the portfolio loss is

$$Var[L] = \sum_{i=1}^{n}\left(b_i^2 + 2\lambda_i^2\right).$$

Now let us compute the value of

$$\sum_{i=1}^{n} b_i^2.$$

Since

$$-b = C^T\delta,$$

and

$$\delta_i = \sum_{k=1}^{m} x_k \delta_i^k,$$

then

$$\begin{aligned}
\sum_{i=1}^{n} b_i^2 &= b^T b \\
&= (C^T\delta)^T (C^T\delta) \\
&= \delta^T C C^T \delta \\
&= \delta^T \Delta t \Sigma \delta \\
&= x^T M^T \Delta t \Sigma M x
\end{aligned}$$

Set

$$\hat{\Sigma} = 2M^T \Delta t \Sigma M,$$

then

$$\sum_{i=1}^{n} b_i^2 = \frac{1}{2} x^T \hat{\Sigma} x.$$

Next let us compute

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \frac{1}{4} tr\left(\left(C^T \Gamma C\right)^T \left(C^T \Gamma C\right)\right) \\
&= \frac{1}{4} tr\left(C^T \Gamma C C^T \Gamma C\right) \\
&= \frac{1}{4} tr\left(\Gamma C C^T \Gamma C C^T\right) \\
&= \frac{1}{4} tr\left(\Gamma \Delta t \Sigma \Gamma \Delta t \Sigma\right)
\end{aligned}$$

Since

$$\Gamma_{ij} = \sum_{k=1}^{m} x_k \Gamma_{ij}^k,$$

then rewrite

$$\Gamma = \sum_{k=1}^{m} x_k \Gamma^k.$$

Now

$$\begin{aligned} \frac{1}{4} tr\left(\Gamma \Delta t \Sigma \Gamma \Delta t \Sigma\right) &= \frac{1}{4} tr\left(\left(\sum_{k=1}^{m} x_k \Gamma^k \Delta t \Sigma\right)^2\right) \\ &= \frac{1}{4}\left(\sum_{k=1}^{m} x_k^2 tr\left(\left(\Gamma^k \Delta t \Sigma\right)^2\right) + 2\sum_{i \neq k} x_i x_k tr\left(\Gamma_i \Delta t \Sigma \Gamma_k \Delta t \Sigma\right)\right) \\ &= \frac{1}{4} x^T Q x, \end{aligned}$$

where

$$Q = Q_{ik} = tr\left(\Gamma_i \Delta t \Sigma \Gamma_k \Delta t \Sigma\right), \quad i = 1, 2, \cdots, n, \quad k = 1, 2, \cdots, m.$$

Thus,

$$\begin{aligned} Var[L] &= \sum_{i=1}^{n} \left(b_i^2 + 2\lambda_i^2\right) \\ &= \frac{1}{2} x^T \left(\hat{\Sigma} + Q\right) x \end{aligned}$$

A.24 Proof of Proposition 4.2

Recall in **Section 3.2.5** we obtain

$$CFVaR_2 = \mu_L - N^{-1}(\alpha)\,\sigma_L,$$

where μ_L, and σ_L are the mean and standard deviation of loss L. Now combing **Proposition 3.2** and trace properties, we have

$$\begin{aligned}
\mu_L &= a + \sum_{i=1}^{n} \lambda_i \\
&= -x^T\Theta - tr\left(\frac{1}{2}C^T\Gamma C\right) \\
&= -x^T\Theta - \frac{1}{2}tr\left(\Gamma C C^T\right) \\
&= -x^T\Theta - \frac{1}{2}tr\left(\Gamma \Delta t \Sigma\right) \\
&= -x^T\Theta - \frac{1}{2}tr\left(\sum_{k=1}^{m} x_k \Gamma^k \Delta t \Sigma\right) \\
&= -x^T\Theta - x^T p,
\end{aligned}$$

where

$$p = \frac{1}{2}\left(tr\left(\Gamma^1 \Delta t \Sigma\right), \cdots, tr\left(\Gamma^m \Delta t \Sigma\right)\right)^T.$$

Recall by **Proposition 4.1** that

$$\sigma_L^2 = Var[L] = \frac{1}{2}x^T\left(\hat{\Sigma} + Q\right)x.$$

Thus, $CFVaR_2$ is given by

$$CFVR_2 = -x^T\Theta - x^T p - N^{-1}(\alpha)\sqrt{\frac{1}{2}x^T\left(\hat{\Sigma} + Q\right)x}.$$

A.25 Proof of Moments in 4.4.1

Firstly, let us compute the first moment. It's formula is given by

$$E\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right] = \sqrt{\Delta t}a_1 E\left[z_1\right] + \sqrt{\Delta t}a_2 E\left[z_2\right] + \Delta t b_1 E\left[z_1^2\right] + \Delta t b_2 E\left[z_2^2\right].$$

Since z_1, z_2 are independent, have zero means, and have Vars equal one, then

$$E\left[-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right] = \Delta t b_1 + \Delta t b_2.$$

Next, let us compute the second moment. Since

$$E\left[(a+b+c+d)^2\right] = E\left[\left(a^2+b^2+c^2+d^2+2ab+2ac+2bd+2bc+2cd\right)\right],$$

then it's formula is given by

$$\begin{aligned} E\left[\left(-\delta^T\Delta S - \frac{1}{2}\Delta S^T\Gamma\Delta S\right)^2\right] &= \Delta t a_1^2 E\left[z_1^2\right] + \Delta t a_2^2 E\left[z_1^2\right] + [\Delta t]^2 b_1^2 E\left[z_1^4\right] + [\Delta t]^2 b_2^2 E\left[z_1^4\right] \\ &+ 2\Delta t a_1 a_2 E\left[z_1 z_2\right] + 2[\Delta t]^{\frac{3}{2}} a_1 b_1 E\left[z_1^3\right] + 2[\Delta t]^{\frac{3}{2}} a_1 b_2 E\left[z_1 z_2^2\right] \\ &+ 2[\Delta t]^{\frac{3}{2}} a_2 b_1 E\left[z_1^2 z_2\right] + 2[\Delta t]^{\frac{3}{2}} a_2 b_2 E\left[z_2^3\right] + 2[\Delta t]^2 b_1 b_2 E\left[z_1^2 z_2^2\right]. \end{aligned}$$

Since z_1, z_2 are independent, have zero means, have Var equals one, have skewness equals zero, and kurtosis equals three then, the formula of the second order moment is given by

$$E\left[\left(-\delta^T\Delta S-\frac{1}{2}\Delta S^T\Gamma\Delta S\right)^2\right]=\Delta t a_1^2+\Delta t a_2^2+3[\Delta t]^2b_1^2+3[\Delta t]^2b_2^2+2[\Delta t]^2b_1b_2.$$

Finally, we compute the third moment. Since

$$\begin{aligned}E\left[(a+b+c+d)^3\right]=E\big[&a^3+b^3+c^3+d^3+3ab^2+3ac^2+3ad^2+3ba^2+3bc^2+3bd^2\\&+3ca^2+3cb^2+3cd^2+3da^2+3db^2+3dc^2+6abc+6abd+6acd+6bcd\big]\end{aligned}$$

It's formula is given by

$$\begin{aligned}E\left[\left(-\delta^T\Delta S-\frac{1}{2}\Delta S^T\Gamma\Delta S\right)^3\right]&=[\Delta t]^{\frac{3}{2}}a_1^3E\left[z_1^3\right]+[\Delta t]^{\frac{3}{2}}a_2^3E\left[z_2^3\right]\\&+[\Delta t]^3b_1^3E\left[z_1^6\right]+[\Delta t]^3b_2^3E\left[z_2^6\right]\\&+3[\Delta t]^{\frac{3}{2}}a_1a_2^2E\left[z_1z_2^2\right]+3[\Delta t]^{\frac{5}{2}}a_1b_1^2E\left[z_1^5\right]\\&+3[\Delta t]^{\frac{5}{2}}a_1b_2^2E\left[z_1z_2^4\right]+3[\Delta t]^{\frac{3}{2}}a_1^2a_2E\left[z_1^2z_2\right]\\&+3[\Delta t]^{\frac{5}{2}}a_2b_1^2E\left[z_1^4z_2\right]+3[\Delta t]^{\frac{5}{2}}a_2b_2^2E\left[z_2^5\right]\\&+3[\Delta t]^2a_1^2b_1E\left[z_1^4\right]+3[\Delta t]^3a_2^2b_1E\left[z_1^2z_2^2\right]\\&+3[\Delta t]^3b_1b_2^2E\left[z_1^2z_2^4\right]+3[\Delta t]^2a_1^2b_2E\left[z_1^2z_2^2\right]\\&+3[\Delta t]^2a_2^2b_2E\left[z_2^4\right]+3[\Delta t]^3b_1^2b_2E\left[z_1^4z_2^2\right]\\&+6[\Delta t]^2a_1a_2b_1E\left[z_1^3z_2\right]+6[\Delta t]^2a_1a_2b_2E\left[z_1z_2^3\right]\\&+6[\Delta t]^{\frac{5}{2}}a_1b_1b_2E\left[z_1^3z_2^2\right]+6[\Delta t]^{\frac{5}{2}}a_2b_1b_2E\left[z_1^2z_2^3\right].\end{aligned}$$

Since z_1, z_2 are independent, have zero means, have Var equals one, have skewness equals zero, kurtosis equals three, 5th moment equals zero, and 6th moment equals 15 then, the formula of the third order moment is given by

$$E\left[\left(-\delta^T \Delta S - \frac{1}{2}\Delta S^T \Gamma \Delta S\right)^3\right] = 15[\Delta t]^3 b_1^3 + 15[\Delta t]^3 b_2^3 + 9[\Delta t]^2 a_1^2 b_1 + 9[\Delta t]^2 a_2^2 b_2$$
$$+ 3[\Delta t]^2 a_1^2 b_2 + 3[\Delta t]^2 a_2^2 b_1 + 9[\Delta t]^3 b_1^2 b_2 + 9[\Delta t]^3 b_1 b_2^2.$$

Appendix B

Python Codes

B.1 Options-Portfolio Risk Measurement

```
from scipy.stats import norm
from scipy import integrate # Compute a definite integral.
from scipy.optimize import fsolve
from numpy.linalg import inv
import matplotlib.pyplot as plt
from scipy.optimize import minimize_scalar
import numpy as np
import math
import cmath
import scipy
import scipy.linalg
pi = math.pi
```

```
def BLSCPRICE(s,T,t,K,sig,r):
    d1=(np.log(s/K)+(r+sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    d2=(np.log(s/K)+(r-sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    call = s*norm.cdf(d1)-K*np.exp(-r*(T-t))*norm.cdf(d2)
    return call

def BLSCALLGREEK(s,T,t,K,sig,r):
    d1=(np.log(s/K)+(r+sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    d2=(np.log(s/K)+(r-sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    calldelta = norm.cdf(d1)
    callgamma = norm.pdf(d1)/(s*sig*np.sqrt(T-t))
    calltheta=-r*K*np.exp(-r*(T-t))*norm.cdf(d2)-((sig*s*norm.pdf(d1)
    ↪ )/(2*np.sqrt(T-t)))
    return[calldelta,callgamma,calltheta]

def Diag(cov):
    L=np.linalg.cholesky(cov)
    A=(-1/2)*np.matmul(np.matmul(L.transpose(),GAMMA),L)
    eigenvalues, U = np.linalg.eig(A)
    LAMBDA=np.zeros((5,5))
    for i in range(0,5):
        LAMBDA[i,i]=eigenvalues[i]
    C = np.matmul(L,U)
    A = -t*THETA
```

```
    B = -np.matmul(C.transpose(),DELTA)
    return [LAMBDA,A,B]

def intg1(x):
    F_1= lambda u: (np.exp(1j*u*x-A*1j*u+1/2*(-u**2*B[0]**2/(1+2*1j*u
    ↪   *LAMBDA[0,0])-np.log(1+2*1j*u*LAMBDA[0,0]))+1/2*(-u**2*B[1]**
    ↪   2/(1+2*1j*u*LAMBDA[1,1])-np.log(1+2*1j*u*LAMBDA[1,1]))+1/2*(-
    ↪   u**2*B[2]**2/(1+2*1j*u*LAMBDA[2,2])-np.log(1+2*1j*u*LAMBDA[2,
    ↪   2]))+1/2*(-u**2*B[3]**2/(1+2*1j*u*LAMBDA[3,3])-np.log(1+2*1j*
    ↪   u*LAMBDA[3,3]))+1/2*(-u**2*B[4]**2/(1+2*1j*u*LAMBDA[4,4])-np.
    ↪   log(1+2*1j*u*LAMBDA[4,4]))))/(1j*u)
    s_1=integrate.quad(F_1, 0, 20)
    F_2= lambda u: (np.exp(-1j*u*x+A*1j*u+1/2*(-u**2*B[0]**2/(1-2*1j*
    ↪   u*LAMBDA[0,0])-np.log(1-2*1j*u*LAMBDA[0,0]))+1/2*(-u**2*B[1]*
    ↪   *2/(1-2*1j*u*LAMBDA[1,1])-np.log(1-2*1j*u*LAMBDA[1,1]))+1/2*(
    ↪   -u**2*B[2]**2/(1-2*1j*u*LAMBDA[2,2])-np.log(1-2*1j*u*LAMBDA[2
    ↪   ,2]))+1/2*(-u**2*B[3]**2/(1-2*1j*u*LAMBDA[3,3])-np.log(1-2*1j
    ↪   *u*LAMBDA[3,3]))+1/2*(-u**2*B[4]**2/(1-2*1j*u*LAMBDA[4,4])-np
    ↪   .log(1-2*1j*u*LAMBDA[4,4]))))/(1j*u)
    s_2=integrate.quad(F_2, 0, 20)
    return (s_1[0]-s_2[0])

def pdf(x):
```

```
    g = lambda u: (1/pi)*(np.exp(-1j*u*x+A*1j*u+1/2*(-u**2*B[0]**2/(1
     ↪  -2*1j*u*LAMBDA[0,0])-np.log(1-2*1j*u*LAMBDA[0,0]))+1/2*(-u**2
     ↪  *B[1]**2/(1-2*1j*u*LAMBDA[1,1])-np.log(1-2*1j*u*LAMBDA[1,1]))
     ↪  +1/2*(-u**2*B[2]**2/(1-2*1j*u*LAMBDA[2,2])-np.log(1-2*1j*u*LA
     ↪  MBDA[2,2]))+1/2*(-u**2*B[3]**2/(1-2*1j*u*LAMBDA[3,3])-np.log(
     ↪  1-2*1j*u*LAMBDA[3,3]))+1/2*(-u**2*B[4]**2/(1-2*1j*u*LAMBDA[4,
     ↪  4])-np.log(1-2*1j*u*LAMBDA[4,4]))))
    pdf = integrate.quad(g,0,20)
    return pdf[0]

#initial stock price
S=[28.02,60.01,25.24,65.53,23.29]
#stirke price
K=S
#interest rate
r=0.05
sig=[0.1699,0.2032,0.2064,0.1794,0.2476]
#maturity
T=1
t=1/52
#number of similuations
n=1000000
#correlation
```

```
Rho=[[1,0.367,0.337,0.189,0.420],[0.367,1,0.359,0.197,0.303],[0.337,0
 ↪  .359,1,0.215,0.297],[0.189,0.197,0.215,1,0.168],[0.42,0.303,0.297
 ↪  ,0.168,1]]
#confidence level
alpha=0.01
index = n-n*alpha - 1

mean=[0,0,0,0,0]
COV=np.zeros((5,5))
for i in range(0,5):
    for j in range(0,5):
        COV[i,j]=t*S[i]*S[j]*sig[i]*sig[j]*Rho[i][j]
dS = np.random.multivariate_normal(mean, COV, n).T

C0=0
for i in range (0,5):
    C0=C0+BLSCPRICE(S[i],T,0,K[i],sig[i],r)
C1=0
for i in range (0,5):
    C1=C1+BLSCPRICE(S[i]+dS[i],T,t,K[i],sig[i],r)

# exact simulated
L_ex=C0-C1
```

```
Var_ex=np.var(L_ex)
LS_ex=np.sort(L_ex)
VaR_ex=LS_ex[int(index)]
Q_ex=-L_ex
E_ex=np.mean(Q_ex)
S_ex=Q_ex[Q_ex<E_ex]
S1_ex=Q_ex[Q_ex>E_ex]
SVar_ex=(1/Q_ex.shape[0])*np.sum((S_ex-E_ex)**2)

# delta gamma
DELTA = np.zeros(5)
for i in range (0,5):
    DELTA[i]=BLSCALLGREEK(S[i],T,0,K[i],sig[i],r)[0]
GAMMA = np.zeros((5,5))
for i in range (0,5):
    GAMMA[i,i]=BLSCALLGREEK(S[i],T,0,K[i],sig[i],r)[1]
THETA = 0
for i in range (0,5):
    THETA=THETA+BLSCALLGREEK(S[i],T,0,K[i],sig[i],r)[2]

dC=np.zeros(n)
for i in range (0,n):
    dC[i]=t*THETA+np.matmul(DELTA.T,dS[:,i])+0.5*np.matmul(dS[:,i].T, ⌋
     ↪  np.matmul(GAMMA,dS[:,i]))
```

```
L_dg=-dC

Var_dg=np.var(L_dg)
LS_dg=np.sort(L_dg)
VaR_dg=LS_dg[int(index)]
Q_dg=-L_dg
E_dg=np.mean(Q_dg)
S_dg=Q_dg[Q_dg<E_dg]
S1_dg=Q_dg[Q_dg>E_dg]
SVar_dg=(1/Q_dg.shape[0])*np.sum((S_dg-E_dg)**2) #SVar in note

[LAMBDA,A,B]=Diag(COV)

# parametric
E_pa=A
for i in range (0,5):
    E_pa=E_pa+LAMBDA[i,i]
Var_pa=0
for i in range (0,5):
    Var_pa=Var_pa+(B[i]**2+2*LAMBDA[i,i]**2)

s=2*pi*(0.5-alpha) # the integral in formula of CDF equals to this
↪  value
```

```
tol = 1e-6
a=5.00
b=6.00
while (b - a) > tol:
    m = (a + b) / 2
    if (intg1(a)-s>=0>= intg1(m)-s) or (intg1(a)-s <= 0 <=
     ↪  intg1(m)-s):
        # f changes sign in going from a to m, so there is a root
         ↪  in [a,m]. set b = m
        b = m

    else:
        # f must change sign in going from m to b, so there is a
         ↪  root in [m,b]. set a = m
        a = m
VaR_pa = a

S= lambda x: (x-E_pa)**2*pdf(x)
SVar_pa=integrate.quad(S,E_pa,20)
SVar_pa=SVar_pa[0]

# cfvar 2 and cfvar3
cfE = -E_pa
```

```
mu3 = 0
for i in range (0,5):
    mu3=mu3+(6*B[i]**2*LAMBDA[i,i] + 8*math.pow(LAMBDA[i,i],3))
mu3 = -mu3

cfvar_2 = -cfE - norm.ppf(alpha)*np.sqrt(Var_pa)

cfvar_3 = -cfE - norm.ppf(alpha)*np.sqrt(Var_pa) -
↪  (1/6)*((norm.ppf(alpha))**2 - 1)*(mu3/Var_pa)

print('Parametric risk measurements over initial
↪  wealth:',[np.sqrt(Var_pa)/C0, VaR_pa/C0, SVar_pa/C0])
print('Simulated risk measurements over initial
↪  wealth:',[np.sqrt(Var_ex)/C0, VaR_ex/C0, SVar_ex/C0])
print('D-G Simulated risk measurements over initial
↪  wealth:',[np.sqrt(Var_dg)/C0, VaR_dg/C0, SVar_dg/C0])
print('Cornish-Fisher VaR
↪  estimation:',[VaR_pa/C0,cfvar_2/C0,cfvar_3/C0])

# t student
nu = 6
E_t=A
for i in range (0,5):
    E_t=E_t+(nu/(nu-2))*LAMBDA[i,i]
```

```
bsquare = 0
l = 0
lsquare = 0

for i in range (0,5):
    bsquare = bsquare+B[i]**2
    l = l+LAMBDA[i,i]
    lsquare = lsquare+LAMBDA[i,i]**2

var_t = (nu/(nu-2))*bsquare + ((2*nu**2)/((nu-2)*(nu-4)))*lsquare +
↪  (((nu**2)/((nu-2)*(nu-4)))-((nu**2)/(nu-2)**2)) * l**2

def inteloft(x):
    F_1 = lambda u :
     ↪  np.exp(-3*np.log(1+2*1j*u*(A-x)/nu+(u**2*B[0]**2)/(nu+nu*2*1j ⌋
     ↪  *u*LAMBDA[0,0])+(u**2*B[1]**2)/(nu+nu*2*1j*u*LAMBDA[1,1])+(u* ⌋
     ↪  *2*B[2]**2)/(nu+nu*2*1j*u*LAMBDA[2,2])+(u**2*B[3]**2)/(nu+nu* ⌋
     ↪  2*1j*u*LAMBDA[3,3])+(u**2*B[4]**2)/(nu+nu*2*1j*u*LAMBDA[4,4]) ⌋
     ↪  ))*np.sqrt(1/(1+2*1j*u*LAMBDA[0,0]))*np.sqrt(1/(1+2*1j*u*LAMB ⌋
     ↪  DA[1,1]))*np.sqrt(1/(1+2*1j*u*LAMBDA[2,2]))*np.sqrt(1/(1+2*1j ⌋
     ↪  *u*LAMBDA[3,3]))*np.sqrt(1/(1+2*1j*u*LAMBDA[4,4]))/(1j*u)
    s_1 = integrate.quad(F_1, 0, 20)
```

```
    F_2 = lambda u :
     ↪  np.exp(-3*np.log(1-2*1j*u*(A-x)/nu+(u**2*B[0]**2)/(nu-nu*2*1j
     ↪  *u*LAMBDA[0,0])+(u**2*B[1]**2)/(nu-nu*2*1j*u*LAMBDA[1,1])+(u*
     ↪  *2*B[2]**2)/(nu-nu*2*1j*u*LAMBDA[2,2])+(u**2*B[3]**2)/(nu-nu*
     ↪  2*1j*u*LAMBDA[3,3])+(u**2*B[4]**2)/(nu-nu*2*1j*u*LAMBDA[4,4])
     ↪  ))*np.sqrt(1/(1-2*1j*u*LAMBDA[0,0]))*np.sqrt(1/(1-2*1j*u*LAMB
     ↪  DA[1,1]))*np.sqrt(1/(1-2*1j*u*LAMBDA[2,2]))*np.sqrt(1/(1-2*1j
     ↪  *u*LAMBDA[3,3]))*np.sqrt(1/(1-2*1j*u*LAMBDA[4,4]))/(1j*u)
    s_2 = integrate.quad(F_2, 0, 20)
    return (s_1[0]-s_2[0])


tol = 1e-6
a=6
b=7
while (b - a) > tol:
    m = (a + b) / 2
    if (inteloft(a)-s>=0>= inteloft(m)-s) or (inteloft(a)-s <= 0 <=
     ↪  inteloft(m)-s):
        # f changes sign in going from a to m, so there is a root
         ↪  in [a,m]. set b = m
        b = m


    else:
```

```
        # f must change sign in going from m to b, so there is a
        ↪  root in [m,b]. set a = m
        a = m
VaR_t = a

def pdfoft(x):
    F_1 = lambda u :
    ↪  np.exp(-4*np.log(1+2*1j*u*(A-x)/nu+(u**2*B[0]**2)/(nu+nu*2*1j
    ↪  *u*LAMBDA[0,0])+(u**2*B[1]**2)/(nu+nu*2*1j*u*LAMBDA[1,1])+(u*
    ↪  *2*B[2]**2)/(nu+nu*2*1j*u*LAMBDA[2,2])+(u**2*B[3]**2)/(nu+nu*
    ↪  2*1j*u*LAMBDA[3,3])+(u**2*B[4]**2)/(nu+nu*2*1j*u*LAMBDA[4,4])
    ↪  ))*np.sqrt(1/(1+2*1j*u*LAMBDA[0,0]))*np.sqrt(1/(1+2*1j*u*LAMB
    ↪  DA[1,1]))*np.sqrt(1/(1+2*1j*u*LAMBDA[2,2]))*np.sqrt(1/(1+2*1j
    ↪  *u*LAMBDA[3,3]))*np.sqrt(1/(1+2*1j*u*LAMBDA[4,4]))
    s_1 = integrate.quad(F_1, 0, 20)
    F_2 = lambda u :
    ↪  np.exp(-4*np.log(1-2*1j*u*(A-x)/nu+(u**2*B[0]**2)/(nu-nu*2*1j
    ↪  *u*LAMBDA[0,0])+(u**2*B[1]**2)/(nu-nu*2*1j*u*LAMBDA[1,1])+(u*
    ↪  *2*B[2]**2)/(nu-nu*2*1j*u*LAMBDA[2,2])+(u**2*B[3]**2)/(nu-nu*
    ↪  2*1j*u*LAMBDA[3,3])+(u**2*B[4]**2)/(nu-nu*2*1j*u*LAMBDA[4,4])
    ↪  ))*np.sqrt(1/(1-2*1j*u*LAMBDA[0,0]))*np.sqrt(1/(1-2*1j*u*LAMB
    ↪  DA[1,1]))*np.sqrt(1/(1-2*1j*u*LAMBDA[2,2]))*np.sqrt(1/(1-2*1j
    ↪  *u*LAMBDA[3,3]))*np.sqrt(1/(1-2*1j*u*LAMBDA[4,4]))
    s_2 = integrate.quad(F_2, 0, 20)
```

```
    return (s_1[0]+s_2[0])/(2*pi)


St= lambda x: (x-E_t)**2*pdfoft(x)
SVar_t=integrate.quad(St,E_t,20)
SVar_t=SVar_t[0]
print("t student risk
↪   measurements",[np.sqrt(var_t)/C0,VaR_t/C0,SVar_t/C0])
```

B.2 Options-Portfolio Optimization

B.2.1 Five-Options-Portfolio case

```
from scipy.stats import norm
from scipy import integrate
from scipy.optimize import fsolve
from numpy.linalg import inv
import matplotlib.pyplot as plt
from scipy.optimize import minimize_scalar
import numpy as np
import math
import scipy
import scipy.linalg
pi = math.pi


#initial stock price
S=[28.02,60.01,25.24,65.53,23.29]
```

```
#stirke price
#K=[40,40,40,40,40]
K=S

#interest rate
r=0.05
sig=[0.1699,0.2032,0.2064,0.1794,0.2476]

#maturity
T=1
t=1/52

#number of similuations
n=100000

#correlation
Rho=[[1,0.367,0.337,0.189,0.420],[0.367,1,0.359,0.197,0.303],[0.337,0
 ↪ .359,1,0.215,0.297],[0.189,0.197,0.215,1,0.168],[0.42,0.303,0.297
 ↪ ,0.168,1]]

#confidence level
alpha=0.01
index = n-n*alpha
```

```
mean=[0,0,0,0,0]
COV=np.zeros((5,5))
for i in range(0,5):
    for j in range(0,5):
        COV[i,j]=t*S[i]*S[j]*sig[i]*sig[j]*Rho[i][j]
dS = np.random.multivariate_normal(mean, COV, n).T

def BLSCPRICE(s,T,t,K,sig,r):
    d1=(np.log(s/K)+(r+sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    d2=(np.log(s/K)+(r-sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    call = s*norm.cdf(d1)-K*np.exp(-r*(T-t))*norm.cdf(d2)
    return call

C0=0
V = np.zeros(5)
for i in range (0,5):
    C0=C0+BLSCPRICE(S[i],T,0,K[i],sig[i],r)
    V[i] = BLSCPRICE(S[i],T,0,K[i],sig[i],r)

def BLSCALLGREEK(s,T,t,K,sig,r):
    d1=(np.log(s/K)+(r+sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    d2=(np.log(s/K)+(r-sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    calldelta = norm.cdf(d1)
```

```
    callgamma = norm.pdf(d1)/(s*sig*np.sqrt(T-t))
    calltheta=-r*K*np.exp(-r*(T-t))*norm.cdf(d2)-((sig*s*norm.pdf(d1)
     ↪  )/(2*np.sqrt(T-t)))
    return[calldelta,callgamma,calltheta]

DELTA = np.zeros(5)
for i in range (0,5):
    DELTA[i]=BLSCALLGREEK(S[i],T,0,K[i],sig[i],r)[0]
GAMMA = np.zeros((5,5))
for i in range (0,5):
    GAMMA[i,i]=BLSCALLGREEK(S[i],T,0,K[i],sig[i],r)[1]
THETA = np.zeros(5)
for i in range (0,5):
    THETA[i]=BLSCALLGREEK(S[i],T,0,K[i],sig[i],r)[2]

# Matrix Of M
M=np.zeros((5,5))
for i in range (0,5):
    M[i,i]=DELTA[i]

# Matrix of Gamma^i s
GAMMAS=np.zeros((25,5))
GAMMAS[0,0]=GAMMA[0,0]
GAMMAS[6,1]=GAMMA[1,1]
```

```
GAMMAS[12,2]=GAMMA[2,2]
GAMMAS[18,3]=GAMMA[3,3]
GAMMAS[24,4]=GAMMA[4,4]
GAMMAS=GAMMAS.reshape(5,5,5)

# Matrix of SIGMAHAT
SIGMAHAT=2*np.matmul(np.matmul(M.transpose(),COV),M)
#M atrix of Q
Q=np.zeros((5,5))
for i in range (0,5):
    for k in range(0,5):
        Q[i,k]=np.trace(np.matmul(GAMMAS[i],np.matmul(COV,np.matmul(G
         ↪  AMMAS[k],COV))))
P = SIGMAHAT+Q

# variance optimal portfolio
P_1 = np.matmul(np.matmul(V,inv(P)),V.transpose())
x_variance = (np.matmul(inv(P),V.transpose()))*(1/P_1)

# cfvar2 optimal portfolio
# P matrix
P = SIGMAHAT+Q
A = np.zeros((2,5))
for i in range (0,5):
```

```
    A[0,i] = (1/2)*np.trace(np.matmul(GAMMAS[i],COV))+THETA[i]*t
    A[1,i] = V[i]

# Matrix of G(ALPHA Here)
P_1 = np.matmul(np.matmul(A,inv(P)),A.transpose())
ALPHA = np.matmul(np.matmul(inv(P),A.transpose()),inv(P_1))

# coefficients in sigma function
funcA = 0
for j in range (0,5):
    for k in range(0,5):
        funcA = funcA + 0.5*(ALPHA[j,0]*ALPHA[k,0]*P[j,k])

funcB = 0
for i in range (0,5):
    funcB = funcB + ALPHA[i,0]*ALPHA[i,1]*P[i,i]
for j in range (0,5):
    for k in range (0,5):
        funcB = funcB + 0.5*(ALPHA[j,0]*ALPHA[k,1]*P[j,k]+ALPHA[j,1]*
        ↪  ALPHA[k,0]*P[j,k])
for l in range (0,5):
    funcB = funcB - 0.5*(ALPHA[l,0]*ALPHA[l,1]*P[l,l]+ALPHA[l,1]*ALPH
    ↪  A[l,0]*P[l,l])
```

```
funcC = 0
for j in range (0,5):
    for k in range (0,5):
        funcC = funcC + 0.5*ALPHA[j,1]*ALPHA[k,1]*P[j,k]

# another way to optimize variance
elp_variance = -funcB/(2*funcA)

b = np.array([elp_variance,1])
x_variance_2 = np.matmul(ALPHA,b)

# find the root of the first derivative
a_1 = 4*funcA*funcA*norm.ppf(alpha)*norm.ppf(alpha)-4*funcA
b_1 = 4*funcA*funcB*norm.ppf(alpha)*norm.ppf(alpha)-4*funcB
c_1 = funcB*funcB*norm.ppf(alpha)*norm.ppf(alpha)-4*funcC
elp=[(-b_1+np.sqrt(b_1**2-4*a_1*c_1))/(2*a_1),(-b_1-np.sqrt(b_1**2-4*
↪  a_1*c_1))/(2*a_1)]
if 2*funcA*elp[0]+funcB > 0:
    elp_cfvar = elp[0]
else:
    elp_cfvar = elp[1]
b = np.array([elp_cfvar,1])
x_cfvar = np.matmul(ALPHA,b)
```

```
# results
print(x_variance)
print(x_variance_2)
print(x_cfvar)
# compute the weights
print(x_variance*V,"variance optimal portfolio")
print(x_variance_2*V,"variance optimal portfolio")
print(x_cfvar*V,"cfvar2 optimal portfolio")

# comparision of elp
print(elp_variance,elp_cfvar)
```

B.2.2 Two-Options-Portfolio case

```
from scipy.stats import norm
from scipy import integrate # Compute a definite integral.
from scipy.optimize import fsolve
from scipy.optimize import minimize_scalar
import matplotlib.pyplot as plt
import numpy as np
import math
pi = math.pi

#parameters of stocks
```

```
#interest rate
r=0.05

#stock1 initial price
S1=28.02
#stock1 strike price
K1=S1

#stock1 volatility
sig1=0.1699

#sig1 = 1

#stock2 initial price
S2=60.01
#stock2 strike price
K2=S2

#stock2 volatility
sig2=0.2032

#sig2 = 1

#maturity
```

```
T=1
t=2/12

#number of similuations
n=100000

#correlation
rho=0

#alpha=0.01
alpha=math.pow(10,-4)

mean = [0,0]
cov12 = rho*sig1*sig2
cov =
↪ [[t*S1**2*sig1**2,t*S1*S2*cov12],[t*S1*S2*cov12,t*S2**2*sig2**2]]

def BLSCPRICE(s,T,t,K,sig,r):
    d1=(np.log(s/K)+(r+sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    d2=(np.log(s/K)+(r-sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    call = s*norm.cdf(d1)-K*np.exp(-r*(T-t))*norm.cdf(d2)
    return call

C0=BLSCPRICE(S1,T,0,K1,sig1,r) + BLSCPRICE(S2,T,0,K2,sig2,r)
```

```
C0
V1 = BLSCPRICE(S1,T,0,K1,sig1,r)
V2 = BLSCPRICE(S2,T,0,K2,sig2,r)

def BLSCALLGREEK(s,T,t,K,sig,r):
    d1=(np.log(s/K)+(r+sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    d2=(np.log(s/K)+(r-sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    calldelta = norm.cdf(d1)
    callgamma = norm.pdf(d1)/(s*sig*np.sqrt(T-t))
    calltheta=-r*K*np.exp(-r*(T-t))*norm.cdf(d2)-((sig*s*norm.pdf(d1)
    ↪ )/(2*np.sqrt(T-t)))
    return[calldelta,callgamma,calltheta]

def BLSPUTGREEK(s,T,t,K,sig,r):
    d1=(np.log(s/K)+(r+sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    d2=(np.log(s/K)+(r-sig**2/2)*(T-t))/(sig*np.sqrt(T-t))
    putdelta = norm.cdf(d1)-1 #-N(-d1)=N(d1)-1
    putgamma=norm.pdf(d1)/(s*sig*np.sqrt(T-t)) #d(N(d1)-1)=dN(d1)
    puttheta=r*K*np.exp(-r*(T-t))*norm.cdf(-d2)-((sig*s*norm.pdf(d1))
    ↪ /(2*np.sqrt(T-t)))
    return[putdelta,putgamma,puttheta]

greek_call1=BLSCALLGREEK(S1,T,0,K1,sig1,r)
greek_call2=BLSCALLGREEK(S2,T,0,K2,sig2,r)
```

```
DELTA=np.array([greek_call1[0],greek_call2[0]])
GAMMA=np.array([[greek_call1[1],0],[0,greek_call2[1]]])
THETA = greek_call1[2] + greek_call2[2]

def Diag(cov):
    L=np.linalg.cholesky(cov)
    A=(-1/2)*np.matmul(np.matmul(L.transpose(),GAMMA),L)
    eigenvalues, U = np.linalg.eig(A)
    LAMBDA = np.array([[eigenvalues[0],0],[0,eigenvalues[1]]])
    C = np.matmul(L,U)
    A = -t*(greek_call1[2]+greek_call2[2])
    #A=t*(greek_call1[2]+greek_call2[2])
    B = -np.matmul(C.transpose(),DELTA)
    return [LAMBDA,A,B]

[LAMBDA,A,B]=Diag(cov)

D_1 = greek_call1[0]
D_2 = greek_call2[0]
G_1 = greek_call1[1]
G_2 = greek_call2[1]
T_1 = greek_call1[2]
T_2 = greek_call2[2]
```

```
def var(x):
    x1 = x
    x2=(1-V1*x1)/V2

    s1 = sig1*S1*np.sqrt(t)
    s2 = sig2*S2*np.sqrt(t)

    a1 = s1*D_1*x1
    a2 = s2*D_2*x2
    b1 = 0.5*s1**2*G_1*x1
    b2 = 0.5*s2**2*G_2*x2

    E1 = b1+b2
    E2 = a1**2 + a2**2 + 3*b1**2 + 3*b2**2 + 2*a1*a2*rho +
    ↪  4*b1*b2*rho**2 + 2*b1*b2

    mu = E1
    var = E2 - mu**2

    return var

def CFVaR(x):
    x1 = x
```

```
x2=(1-V1*x1)/V2

s1 = sig1*S1*np.sqrt(t)
s2 = sig2*S2*np.sqrt(t)

a1 = s1*D_1*x1
a2 = s2*D_2*x2
b1 = 0.5*s1**2*G_1*x1
b2 = 0.5*s2**2*G_2*x2

E1 = b1+b2
E2 = a1**2 + a2**2 + 3*b1**2 + 3*b2**2 + 2*a1*a2*rho +
↪ 4*b1*b2*rho**2 + 2*b1*b2
E3 = 15*math.pow(b1,3) + 15*math.pow(b2,3) + 9*a1**2*b1 +
↪ 9*a2**2*b2 + 3*a1**2*b2 + 3*a2**2*b1 + 9*b1**2*b2 + 9*b1*b2**2

A = x1*T_1*t + x2*T_2*t
mu = E1
var = E2 - mu**2
mu3 = E3 - 3*mu*var - math.pow(mu,3)


#theta term
```

```
    cfvar = -mu - A - norm.ppf(alpha)*np.sqrt(var) -
     ↪  (1/6)*((norm.ppf(alpha))**2-1)*(mu3/var)
    #1/6

    return cfvar

def CFVaR2(x):
    x1 = x
    x2=(1-V1*x1)/V2

    s1 = sig1*S1*np.sqrt(t)
    s2 = sig2*S2*np.sqrt(t)

    a1 = s1*D_1*x1
    a2 = s2*D_2*x2
    b1 = 0.5*s1**2*G_1*x1
    b2 = 0.5*s2**2*G_2*x2

    E1 = b1+b2
    E2 = a1**2 + a2**2 + 3*b1**2 + 3*b2**2 + 2*a1*a2*rho +
     ↪  4*b1*b2*rho**2 + 2*b1*b2
    E3 = 15*math.pow(b1,3) + 15*math.pow(b2,3) + 9*a1**2*b1 +
     ↪  9*a2**2*b2 + 3*a1**2*b2 + 3*a2**2*b1 + 9*b1**2*b2 + 9*b1*b2**2
```

```
    mu = E1
    var = E2 - mu**2
    mu3 = E3 - 3*mu*var - math.pow(mu,3)

    A = x1*T_1*t + x2*T_2*t
    #theta term
    cfvar = -mu - A - norm.ppf(alpha)*np.sqrt(var)

    return cfvar

result = minimize_scalar(CFVaR, method="brent")
resultvar = minimize_scalar(var, method="brent")
reslutcfvar2=minimize_scalar(CFVaR2, method="brent")

print(result['x']*V1,(1-V1*result['x']),"cfvar3")
print(resultvar['x']*V1,(1-V1*resultvar['x']),"var")
print(reslutcfvar2['x']*V1,(1-V1*reslutcfvar2['x']),"cfvar2")
```

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