The Densities of Bounded Primes in Hypergeometric Series

The Densities of Bounded Primes in Hypergeometric Series

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Abstract

This thesis deals with the properties of the coefficients of Hypergeometric Series. Specifically, we are interested in which primes appear in the denominators to a bounded power. The first main result gives a method of categorizing the primes up to equivalence class which appear finitely many times in the denominators of generalized hypergeometric series ${}_nF_m\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}; z$ over the rational numbers $\mathbb{Q}$. Necessary and sufficient conditions for when the density is zero are provided as well as a categorization of the n and m for which the problem is interesting.

The second main result is a similar condition for the appearance of primes in the denominators of the hypergeometric series ${}_2F_1$ over number fields, specifically quadratic extensions $\mathbb{Q}(\sqrt{D})$. A novel conjecture to the study of p-adic numbers is also provided, which discusses the digits of irrational algebraic numbers' p-adic expansions. Both of these results build off previous research done by Franc et al. 2020, which was concerned with ${}_2F_1$ over the rational numbers, and expands it to general hypergeometric series, as well as fields other than the rational numbers.

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Declaration of Authorship

I, Nathan HEISZ, declare that this thesis titled, "The Densities of Bounded Primes in Hypergeometric Series" and the work presented in it are my own. I confirm that:

- Chapter 1: Introduction
  - This chapter is entirely my own work.
  - The purpose of this chapter is to introduce the problem and state the main results of the thesis.
- Chapter 2: Background
  - This chapter is my own work.
  - The purpose of this chapter is to provide existing definitions and results which are required for the thesis.
- Chapter 3: Generalised Hypergeometric Series
  - This chapter is entirely my own work.
  - This chapter provides some of the main results of the thesis, specifically Theorem 3.1.5
- Chapter 4: A Conjecture on Irrational p-adic Numbers
  - This chapter is entirely my own work.
  - This chapter provides the remaining main results and a novel conjecture to the field
- Chapter 5: Conclusion
  - This chapter is entirely my own work.
  - This chapter serves to summarize main results, as well as remark on some potential next steps

Chapter 1

Introduction

1.1 Basic Problem

The crux of this thesis lies in the examination of the arithmetic properties of the coefficients of hypergeometric series. We are interested in the density of bounded primes in the denominators of the series coefficients.

We provide two main results in this thesis. Chapter 3 contains the first, Theorem 3.1.5. This theorem gives a complete method to finding the density of bounded primes in the denominators of the series ${}_nF_{n-1}$. This is a generalization of the density result given as Theorem 4 of Franc et al. 2020 to higher values of n. Furthermore, necessary and sufficient conditions for the density of the bounded primes to be zero are shown, as well as conditions on the density depending on the values of n and m. Specifically, it is shown that the only case of interest for the density of bounded primes in the denominators of generalized hypergeometric series ${}_nF_m$ is when $n = m + 1$.

Chapter 4 contains the second set of main results. Theorem 4.2.4 is an analogous result to Theorem 4 of Franc et al. 2020 over quadratic number fields $\mathbb{Q}(\sqrt{D})$ rather than over the rational numbers $\mathbb{Q}$. As well, similar conditions to that of the previous chapter are given, providing conditions for when the density is zero. These results rely on a novel conjecture to the field, which presents an expected property of the p-adic expansions of irrational algebraic numbers.

1.2 Notation

In this section we will provide a quick reference for different notation used throughout the thesis. This will include definitions of functions as well as the shorthand that will be used. These will be explained in context in their individual sections.

Notation for writing p-adic integers: 2.2.1

p-adic Absolute Value: 2.2.5

p-adic Truncation: 2.2.2

p-adic Valuation: 2.2.3

p-adic Carry: 2.2.4

Fractional Part: 2.2.6

Greek letter notation for digits of expansion: 2.2.7

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Numerator Majorization: 3.1.2

Chapter 2

Background

2.1 Quadratic Number Fields

A field F is an extension field if it contains a subfield $R \subseteq F$ such that the operations on R are the same as those on F restricted to R. As an example, the complex numbers $\mathbb{C}$ are an extension field over the real numbers $\mathbb{R}$.

We define the degree of an extension field to be the dimension of that field over its subfield. So the field of complex numbers, which can be viewed as 2-dimensional vectors with real components, has degree 2 as an extension over the real numbers.

In this way, quadratic number fields are degree 2 field extensions of the rational numbers $\mathbb{Q}$. We denote one of these fields $\mathbb{Q}\left(\sqrt{D}\right)$, where $D \in \mathbb{Z}$ is square-free. By the quadratic formula, every quadratic field can be expressed in this way.

For a quadratic field $K = \mathbb{Q}\left(\sqrt{D}\right)$, we define the ring of integers $\mathcal{O}_K$ to be the ring composed of all elements of K that are roots of monic polynomials with integer coefficients Neukirch 2013. This ring can be expressed as $\mathcal{O}_K = \mathbb{Z}+\mathbb{Z}\left(\frac{1}{2}(d+\sqrt{d})\right)$, where

$$\begin{cases} d = D & \text{if } D \equiv 1 \bmod 4 \\ d = 4D & \text{if } D \equiv 2 \text{ or } 3 \bmod 4 \end{cases}$$

An important property of quadratic number fields is that an ideal in the ring of integers $\mathcal{O}_K$ of the field $K = \mathbb{Q}\left(\sqrt{D}\right)$ that is generated by a prime $p \in \mathbb{Z}$ is not necessarily prime in this field. For an ideal $\langle p \rangle$, either $\langle p \rangle$ is prime, in which case we call p inert, $\langle p \rangle$ is the square of a prime ideal, in which case p is ramified, or $\langle p \rangle$ is the product of two distinct prime ideals, and we say p splits in $\mathbb{Q}\left(\sqrt{D}\right)$.

We also note that, if D is not square-free, we can take the square-free part of it, as if $D = x^2 d$, then for any $a + b\sqrt{D}$, it is equal to taking $a + b'\sqrt{d}$, where $b' = bx$.

We provide the following method to determine when a rational prime splits in $\mathbb{Q}\left(\sqrt{D}\right)$, which will be important later in Chapter 4, along with the above listed property.

Theorem 2.1.1. *Take $\mathbb{Q}\left(\sqrt{D}\right)$ with D square-free. If p is an odd prime, then p splits in $\mathbb{Q}\left(\sqrt{D}\right)$ if and only if D is a quadratic residue mod p.*

Proof. This is adapted from Proposition 8.5 of Neukirch 2013.

2.2 p-adic numbers

We can first begin with the basic idea of a p-adic number. Consider a natural number x, and a prime p. The p-adic expansion of x is to write it as a sum of powers of p. For instance, if we take $x = 219, p = 7$, we get the p-adic expansion

$$2 + 3 \cdot 7 + 4 \cdot 7^2.$$

Writing natural numbers in base p this way is simple and intuitive. These numbers can be added and subtracted as expected. We call a number a p-adic integer when the lowest power of p in its expansion is greater than or equal to zero. While we can clearly see how natural numbers are p-adic integers, this becomes more complicated when attempting to extend the same to integers and rational numbers, and it helps to first introduce the p-adic absolute values, as well as the truncation, p-adic valuation, and the notation in which a p-adic expansion is written.

Definition 2.2.1. *For a p-adic integer a, we write the digits of its expansion as $a_0a_1a_2\cdots a_k$ if it has a finite expansion, and as $a_0a_1a_2\cdots a_k\overline{a_{k+1}\cdots a_n}$ where the over-lined part is the periodic part of an expansion.*

Definition 2.2.2. *If x is a p-adic integer, then $\tau_j(x)$ denotes the truncation of $x \bmod p^j$. This is the unique integer $0 \leq \tau_j(x) < p^j$ such that $x \equiv \tau_j(x) \bmod p^j$.*

Definition 2.2.3. *Let $\nu_p(n)$ denote the p-adic valuation of n, the exponent of the highest power of p that divides n. This can be negative if the highest power of p is in the denominator of n. By convention, we define $\nu_p(0) = \infty$.*

Definition 2.2.4. *We use $c_p(x, y)$ to denote the number of carries required to evaluate the sum of the p-adic numbers x and y.*

Definition 2.2.5. *Let x be a rational number. We define its p-adic absolute value for a prime p to be*

$$|x|_p = \frac{1}{p^n}$$

where $n = \nu_p(x)$ is the highest power of p that divides x.

These absolute values are non-Archimedean, meaning they fulfill the strong triangle inequality: for x, y,

$$|x + y|_p \leq \max(|x|_p, |y|_p)$$

This can be seen to be true by considering the fact that the p-adic absolute value is found from the lowest exponent term. When adding two numbers together, their sum cannot gain a term with a lower exponent than either summand.

Under this absolute value, we have the interesting property that a number that is very large under the Euclidean absolute value can be very small under the p-adic absolute value. As a result, this also lets two numbers that are very far away in the Euclidean sense be close using this norm. For example, p^{100} is quite large under the regular Archimedean norm, but under the p-adic absolute value, $|p^{100}|_p = \frac{1}{p^{100}}$.

Using this norm, we can now determine a p-adic number X that can act as -1. Since we need this number to be the additive inverse of 1, we need $X + 1$ to be zero. This is where we use the properties of the p-adic norm. When we take each term of the expansion to be $p - 1$, then when we add 1, we continually carry onto the next digit. Consider the following p-adic integer X. We will show how this acts like -1 does for the integers:

$$X = (p - 1) + (p - 1) \cdot p + (p - 1) \cdot p^2 + \cdots$$

If we consider the partial sum given by $1 + \tau_j(X)$, we can look at the limit as $j \to \infty$. The remaining term after the carries of this sum is $(p - 1)p^j$. Using the p-adic absolute value, $|1 + \tau_j(X)|_p = \frac{1}{p^j}$. As j increases, this quickly approaches zero, and so the limit is zero. As such $X + 1 = 0$ and $X = -1$. If one is to multiply X by itself, you will also find that by the same carry properties, $X^2 = 1$, and so X acts as desired. Thus any negative integer can be constructed by the multiplication of a positive integer by -1.

Rational numbers are more complicated. The following construction of rational numbers' p-adic expansions is adapted from Conrad 2019. First note that all rational numbers are an integer addition away from a rational between -1 and 0. So it suffices to look at rationals in the interval $(-1, 0)$. For a rational $-\frac{a}{b}$ with

$0 < a < b$, we can first take the order of $p \bmod b$. Call this order M. Then taking $(p^M - 1) = bb'$, we use this as the denominator of a geometric series, with ab' as the numerator.

$$-\frac{a}{b} = -\frac{ab'}{bb'}$$

Then, taking the p-adic expansion of the numerator, we get a string of digits with length M, which forms the periodic expansion of this number.

$$-\frac{a}{b} = \frac{a_0 a_1 a_2 \cdots a_{M-1}}{1 - p^M} = \overline{a_0 a_1 a_2 \cdots a_{M-1}}$$

Thus, any rational number can be expressed as the sum of an integer expansion and the expansion of a number between negative 1 and zero.

As an example, we will take the 3-adic expansion of $-\frac{a}{b} = -\frac{7}{11}$. We begin with finding the order of $3 \bmod 11$, which gives $M = 5$. We then get $3^5 - 1 = 22 \cdot 11$ which we can apply to the $-\frac{7}{11}$, and take the 3-adic expansion of the numerator:

$$-\frac{a}{b} = -\frac{7 \cdot 22}{11 \cdot 22} = \frac{10221}{1 - 3^5} = \overline{10221}$$

Since we have an absolute value, we can thus complete the rational numbers for some p, which gives us the p-adic number fields. There are infinitely many of these fields, as there are infinitely many primes. Interestingly, by a result of Ostrowski, the only distinct completions of the rationals are the real numbers, the completion with regard to the trivial norm, and the p-adic numbers Neukirch 2013. We will refer to the completion of the rational numbers with respect to a p-adic absolute value as $\mathbb{Q}_p$, and define $\mathbb{Z}_p$ to be the p-adic integers, which are any $x \in \mathbb{Q}_p$ with $\nu_p(x) \geq 0$.

It will be useful to have some functions on the p-adic numbers that we can use. We begin with the truncation of a p-adic integer.

Definition 2.2.6. *Let x be a rational number. We denote by $\{x\}$ the fractional part of x, such that*

$$\{x\} = x - \lfloor x \rfloor$$

where $\lfloor x \rfloor$ is the standard floor operator.

It will be valuable later to have a method for calculating the p-adic coefficients of a rational number. For this we introduce the following lemma, taken from Franc et al. 2018 with notation changed:

Lemma 2.2.7. *Let $a = \frac{n}{d}$ denote a rational number with $\gcd(n, d) = 1$ satisfying $0 < a < 1$, and let p denote a prime such that $a - 1 \in \mathbb{Z}_p^\times$. Let M denote the multiplicative order of $p \bmod d$, and let $a - 1 = \overline{\alpha_0\alpha_1 \cdots \alpha_{M-1}}$ denote the p-adic expansion of $a - 1$. Then for each index $0 \le j < M$,*

$$\alpha_j = \left\lfloor \left\{ -p^{M-1-j}\alpha \right\} p \right\rfloor.$$

In this way, we will refer to the digits of a rational number's p-adic expansion by the corresponding Greek letter, i.e. α_i refers to the digits of $a - 1$ and γ_i refers to the digits of $c - 1$, et cetera.

Notice that we are referring to the expansion of $a-1$ rather than the expansion of a. We can do this without worry, which is due to the fact that the expansions of a and $a - 1$ can only differ on finitely many leading terms. When we add 1 to $a - 1$, in almost every case it only changes the first term. If $\alpha_1 = p - 1$, we will carry to the next term, which will be the last time it carries unless that is also $p-1$. We saw earlier that $X = (p-1)+(p-1)p+(p-1)p^2+\cdots$ is the expansion of the integer -1, and this is also the only case where we could have infinitely many carries. However, this lemma restricts to $0 < a < 1$, and so $a = X + 1 = 0$ is outside the bounds of this lemma.

We will also occasionally refer to the digits of the expansion of $b\sqrt{D}$, which will be denoted by $\bar{\beta}_i$ for an index i.

We can also use these p-adic fields to determine the splitting properties of a prime. The following theorem is a consequence of Dedekind's theorem for factoring primes and Hensel's Lemma for p-adic number fields. This is also naturally related to Theorem 2.1.1.

Theorem 2.2.8. *Let $\mathbb{Q}\left(\sqrt{D}\right)$ be a quadratic number field and let p be an odd prime. Then p splits in $\mathbb{Q}\left(\sqrt{D}\right)$ if and only if $\mathbb{Q}_p$ contains $\mathbb{Q}\left(\sqrt{D}\right)$.*

2.3 Hypergeometric Series

The hypergeometric function is defined in terms of the hypergeometric series as follows:

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n$$

where $(x)_n$ is the rising Pochhammer symbol

$$(x)_n = \begin{cases} 1 & n = 0 \\ x(x+1)\cdots(x+n-1) & n > 0. \end{cases}$$

The hypergeometric function is commonly seen as a solution to second-order ODEs with three regular singular points. The function is algebraic on certain sets of parameters, discovered and categorized by Schwarz 1873. Table 2.1 presents this list of parameters for which the function is algebraic.

No.	a	b	c
1	$\frac{1}{2}$	$\frac{1}{2}$	ν
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
4	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$
5	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$
6	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$
7	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$
8	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$
9	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$
10	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$
11	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
12	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$
13	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
14	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$
15	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$

TABLE 2.1: Schwarz's list of parameters for which the hypergeometric function can be expressed algebraically. Schwarz 1873

There also exists a generalized hypergeometric function, which is as follows:

$$ {}_nF_{n-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_{n-1} \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{\prod_{i\le n}(a_i)_k}{\prod_{i\le n}(b_i)_k} \frac{1}{k!} z^k \tag{2.1}$$

Importantly, for all cases of the generalized hypergeometric function, we will be using a_i to refer to parameters in the numerator, and using b_i to refer to those in the denominators.

A specific property of these series that we are interested in is the primes that appear in the denominators. For any given prime p, we want to know if the prime is bounded.

Definition 2.3.1. *We call a prime unbounded in the denominators of the series if the p-adic valuation of all of the coefficients are not bounded below. We call a prime bounded if it is not unbounded.*

A simple example could be the geometric series $\sum_{r\geq 0} 2^{-r}x^r$. The prime $p = 2$ is clearly unbounded in the denominators of the coefficients, but any odd prime will never appear, and so is bounded.

For an example of a prime bounded in the denominators of a hypergeometric series, if we have a series ${}_2F_1(a, c; c; z)$, if the 7-adic valuation of every coefficient is always greater than some integer r, then it would be bounded below by r, and we would say that 7 is bounded in the denominators of the series.

In the following chapters, we will be referring to a concept of density of bounded primes in the denominators of the series, so it is important to properly define what that means.

Definition 2.3.2. *The* density of bounded primes *for a series F is the natural density given by*

$$\lim_{N\to\infty} \frac{\{p \leq N \mid p \text{ is bounded for } F\}}{\{p \leq N\}}$$

where p represents a prime number.

Considering ${}_2F_1$, when we have rational coefficients we find the density is accurately given by a computable finite set of primes, as proven in Franc et al. 2020. As this will be referenced often, we will restate it here:

Theorem 2.3.3. *Let $a, b,$ and c denote three rational numbers satisfying $0 < a, b, c < 1$ and $c \neq a, b$. Let M denote the least common multiple of the denominators of $a - 1, b - 1,$ and $c - 1$ when written in lowest terms, and define*

$$B(a, b; c) = \Big\{u \in (\mathbb{Z}/M\mathbb{Z})^\times \mid \text{ for all } j \in \mathbb{Z}, \Big\{-u^j c\Big\} \leq \max\Big(\Big\{-u^J a\Big\}, \Big\{-u^j b\Big\}\Big)\Big\}.$$

Then, for all primes $p > M$, the series ${}_2F_1(a, b; c)$ is p-adically bounded if and only if p is congruent to an element of $B(a, b; c) \bmod M$. Thus, the density of the set *of bounded primes for ${}_2F_1(a,b;c)$ is*

$$D(a,b;c) = \frac{|B(a,b;c)|}{|(\mathbb{Z}/M\mathbb{Z})^{\times}|}$$

Interestingly, we find that the only cases where this density is 1 are precisely those cases on Schwarz's list. That is, if the monodromy group is not finite, then there always exists a positive density of unbounded primes.

There are analogues for this density 1 case that can be given for some generalized hypergeometric functions. In the case of ${}_3F_2$, the parameters that give a density of 1 are given in Table 1 of Kato 2011. Other cases can likely be derived in a similar manner, making use of results from Beukers and Heckman 1989.

In the next chapter we will explore the density of bounded primes in the denominators of the generalized hypergeometric series over the rationals. Following that, we will discuss the densities for bounded primes in the denominators of the hypergeometric series for fields outside the rationals, specifically quadratic number fields.

Chapter 3

Generalized Hypergeometric Series

3.1 Results on ${}_nF_{n-1}$

It has been previously seen how the density of bounded primes in denominators of the standard hypergeometric series ${}_2F_1$ can be found over the rational numbers. In this chapter we present conditions for ${}_nF_{n-1}$ to have densities of zero, one, as well as provide a general construction of the set that gives the density for any set of rational parameters.

We begin with a valuable lemma for the proofs in this chapter.

Lemma 3.1.1. *Let a and b be rational numbers $0 < a, b < 1$. If $a < b$, then for sufficiently large prime p, there is at least one digit in the p-adic expansion of a that is greater than the digit in the same index of the p-adic expansion of b. By periodicity of the expansions, this then occurs infinitely often.*

We can note that if the prime is not sufficiently large, then the position where this would be true could instead have the digits be equal.

Proof. Recall the formula for the digits for a p-adic expansion given in Lemma 2.2.7. Denote the order of $p \bmod \operatorname{den}(a)$ as M_a, and the order of $p \bmod \operatorname{den}(b)$ as M_b. We can note that, if the expansion of a is 1-periodic, then, since $a < b$, the fractional part of $-a$ will be greater than the fractional part of $-b$. So in the b_{M_b-1} digit, we have

$$\lfloor \{-b\}p \rfloor < \lfloor \{-a\}p \rfloor.$$

Observe that in this step is where the statement relies on the prime being sufficiently large, as the rounding inherent to the floor function with a small prime can lead to equality of these digits.

In the cases where the expansion of a is not 1-periodic, there is still a biggest term, which will always be in the a_{M_a-1} digit, as, like above, we have

$$\lfloor \{-b\}p \rfloor < \lfloor \{-a\}p \rfloor$$

since M_a divides the exponent of p in the formula.

Then, we further divide this into cases. If the period of the expansion of b divides that of a, then we end with a digit where the prime power in the formula for the digits is p^0 in both a and b. The same occurs if the period of a divides that of b.

If the greatest common divisor of the two periods is 1, then we simply look at the digit at the lcm of the periods of a and b. Then, once again, M_a and M_b will divide the exponent of p in the formula, and we have that the digit for a is larger than the digit for b at that position. □

We can now introduce the following notation, which will form the basis of the upcoming theorem.

Definition 3.1.2. *We call a set of rational generalized hypergeometric parameters* numerator majorized *with respect to a prime p if a permutation $\sigma_j \in S_n$ can be defined for every position j up to the lowest common multiple M of the denominators of the parameters such that*

$$\beta_{i,j} < \alpha_{\sigma_j(i),j}$$

for all $0 \leq i \leq m, 0 \leq j \leq M$.

We provide the following table as an easy example of numerator majorized parameters with respect to $p = 11$. Each row in Table 3.1 represents the p-adic digits of a rational hypergeometric parameter such that all five parameters have 4-periodic 11-adic expansions. Hence it is a finite computation to check whether the parameters are numerator majorized with respect to the prime $p = 11$. The digits are coloured according to the pairing, where a red numerator digit is the distinct digit greater than the red denominator digit, and likewise for the blue digits.

	1	2	3	4
$a_1 = \frac{3}{10}$	7	7	7	7
$a_2 = \frac{1}{3}$	2	8	2	8
$a_3 = \frac{1}{3}$	8	2	8	2
$b_1 = \frac{8}{15}$	1	5	1	5
$b_2 = \frac{13}{15}$	5	1	5	1

Table 3.1: An example of hypergeometric parameters that are numerator majorized with respect to $p = 11$.

In this table we can see that at every position, each denominator digit has a distinct numerator parameter greater than it.

This is a property that can easily be checked for any given set of parameters and prime. Particularly, we can check this condition using an equivalent technique to that which appears in Theorem 2.3.3. The following lemma serves as an equivalent definition.

Lemma 3.1.3. *A set of parameters is numerator majorized with respect to a sufficiently large prime $p \equiv u \bmod M$ where M is the lowest common multiple of the denominators of the parameters if the following condition holds.*

$$\text{for all } j \in \mathbb{Z}_{\geq 0}, \text{ there exists a permutation } \sigma_j \in S_n \text{ such that}$$

$$\left\{-u^j b_i\right\} \leq \left\{-u^j a_{\sigma_j(i)}\right\}$$

for all b_i

Proof. As before, a prime that is not sufficiently large may be equal instead of less than. Note that we only need to check for j up to the multiplicative order of u in $(\mathbb{Z}/M\mathbb{Z})^\times$. This Lemma follows as a result of the earlier stated Lemma 2.2.7. □

This is a much easier definition to use when calculating the densities, as it allows for wholesale checking of an entire congruence class of prime numbers: notice that the lined formula in Lemma 3.1.3 no longer relies on p. As Lemma 2.2.7 is the simplest way of calculating rational p-adic expansions manually, the process of calculating the digits of the expansion includes the calculation of the fractional parts used in Lemma 3.1.3.

Let us now discuss in detail how to check numerator majorization using both the p-adic approach and the approach of Lemma 3.1.3. We begin with the p-adic approach. Take $p = 11$ and parameters $(a_i) = \left(\frac{2}{3}, \frac{1}{2}, \frac{1}{3}\right), (b_i) = \left(\frac{4}{5}, \frac{3}{5}\right)$. We can then see the expansions of these parameters in Table 3.2. We can see that at every position, each b parameter has at least one a parameter greater than it, and so with the manual method we can see that this set of parameters is numerator majorized.

Position	1	2	3	4	5
2/3	8	3	7	3	7
1/2	6	5	5	5	5
1/3	4	7	3	7	3
4/5	3	2	2	2	2
3/5	5	4	4	4	4

Table 3.2: The 11-adic expansions of $(a_i) = \left(\frac{2}{3}, \frac{1}{2}, \frac{1}{3}\right), (b_i) = \left(\frac{4}{5}, \frac{3}{5}\right)$

Now we will consider instead the approach given by Lemma 3.1.3. In fact, this method will determine numerator majorization for all primes $p \equiv 11 \bmod 30$, as 30 is the lowest common multiple of the denominators . This illustrates the usefulness of Lemma 3.1.3. For $u = 11$, we calculate $\{-u^j a_1\}, \{-u^j a_2\}, \{-u^j a_3\}, \{-u^j b_1\},$ and $\{-u^j b_2\}$. Note that the multiplicative order of 11 in $\mathbb{Z}/30\mathbb{Z}$ is 2, and so we only need to check $j = 0, 1$.

$$\begin{aligned}
\{-u^0 a_1\} &= \frac{1}{3} & \qquad \{-u^1 a_1\} &= \frac{2}{3} \\
\{-u^0 a_2\} &= \frac{1}{2} & \qquad \{-u^1 a_2\} &= \frac{1}{2} \\
\{-u^0 a_3\} &= \frac{2}{3} & \qquad \{-u^1 a_3\} &= \frac{1}{3} \\
\{-u^0 b_1\} &= \frac{1}{5} & \qquad \{-u^1 b_1\} &= \frac{1}{5} \\
\{-u^0 b_2\} &= \frac{2}{5} & \qquad \{-u^1 b_2\} &= \frac{2}{5}
\end{aligned}$$

Here we see that, just as in the table version, we have distinct $a_{\sigma_j(i)} > b_i$ for each b_i and each exponent j. So each method gives the same result, that this set of parameters is numerator majorized with respect to large enough primes $p \equiv u \bmod 30$. Since we are ultimately interested in densities of primes, it is sufficient for our purposes to work with sufficiently large primes, where the notion of sufficiently large depends only on the hypergeometric parameters.

Notice that in the case of $n = 2$, this is identical to the property given in Theorem 2.3.3. If there is only one b parameter, then you only need to check whether $\{-u^j b\} \leq \max(\{-u^j a_1\}, \{-u^j a_2\})$ for all j up to the multiplicative order of $u \in \mathbb{Z}/M\mathbb{Z}$. Thus, our theorem generalizes this to arbitrary n.

We also will state the following theorem from Franc et al. 2018 which shall be used in the proof of the main theorem for this chapter. Here, admissible refers to a rational parameter in the interval $(0, 1)$, and good primes are precisely those such that $a_i - 1$ and $b_i - 1$ have purely periodic expansions.

Theorem 3.1.4. *Let $(a_1, \ldots, a_n; b_1, \ldots, b_{n-1})$ denote rational hypergeometric parameters, and let p denote a prime such that $\nu_p(a_j - 1) \geq 0$ and $\nu_p(b_k - 1) \geq 0$ for all j and k. Then if A_m denotes the m^{th}coefficient of ${}_nF_{n-1}(a_j; b_k; z)$,*

$$\nu_p(A_m) = \sum_{j=1}^{n} c_p(a_j - 1, m) - \sum_{k=1}^{n-1} c_p(b_k - 1, m) \tag{3.1}$$

Further, assume that the parameters $(a_1, \ldots, a_n; b_1, \ldots, b_{n-1})$ are admissible, assume that p is a good prime for this data, and let M denote the corresponding common period of the p-adic expansions of the hypergeometric parameters. Then

$$\nu_p(A_{mp^M}) = \nu_p(A_m)$$

for all $m \in \mathbb{Z}_{\geq 0}$.

Proof. See Theorem 3.4 of Franc et al. 2018. □

We can now use the previously stated definition of numerator majorization to form the basis of the following theorem:

Theorem 3.1.5. *Consider a generalized hypergeometric series*

$$F = {}_nF_{n-1}\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_{n-1} \end{pmatrix}; z\Bigg).$$

Take M to be the lowest common multiple of the denominators of the parameters $\{a_1, \ldots, a_n, b_1, \ldots, b_{n-1}\}$. We can define a set

$$B = \left\{ u \in (\mathbb{Z}/M\mathbb{Z})^\times \,\middle|\, \begin{array}{l} \text{The parameters are numerator majorized} \\ \text{for sufficiently large primes } p \equiv u \bmod M \end{array} \right\}.$$

The density of bounded primes in the denominators of this series F is given by

$$\mathbf{D} = \frac{|B|}{|(\mathbb{Z}/M\mathbb{Z})^{\times}|}$$

Proof. This proof makes use of Theorem 3.1.4 to determine when an equivalence class is bounded. By applying Lemma 3.1.3, we are free to use the numerator majorization condition.

In this proof, by an abuse of notation, when referring to the expansions of the parameters, we are referring to the expansions of the parameters fractional parts minus 1. This convention is used, as these expansions only differs from the 'actual' expansions by the addition of a rational integer. So we can simply consider the purely periodic expansion of $\{x\} - 1$ for a parameter x.

Take $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_{n-1}\}$ to be the numerator and denominator parameters for ${}_nF_{n-1}$. For each equivalence class $u \in (\mathbb{Z}/M\mathbb{Z})^{\times}$, we check whether these parameters are numerator majorized. We then have two cases. We will begin with the case where we are not numerator majorized.

If we are not numerator majorized, then there is some exponent j according to the inequality of Lemma 3.1.3 between 0 and the order of $u \bmod M$ where one of the denominator parameters has no distinct numerator parameter greater than it. This, by our definition of numerator majorization, corresponds to a position where that parameter's p-adic digit is greater than the remaining valid numerator parameter digits. We can renumber the parameters such that b_1 is the first parameter where numerator majorization fails and take j to be the lowest index position where this occurs. Then, taking β_j to be the value of b_1 at this position j, we can construct the following for some positive integer r:

$$m_r = \sum_{s=0}^{r} (p - \beta_j) p^{Ms+j-1}.$$

We now can apply the first statement of Theorem 3.1.4 taking A_{m_r}.

$$\nu_p(A_{m_r}) = \sum_{j=1}^{n} c_p(a_j - 1, m_r) - \sum_{k=1}^{n-1} c_p(b_k - 1, m_r)$$

Note that when we add m_r to the denominator parameters, since we are not numerator majorized, we will have more carries in the denominator than in the numerator. Since the definition of m_r gives a carry in the first r positions where majorization fails, we get the following bound on the value of $\nu_p(A_{m_r})$:

$$\nu_p(A_{m_r}) \leq -r - 1.$$

As r increases, the valuation of A_{m_r} with respect to p is bounded above, and unbounded below, and thus that prime is unbounded in the denominators of the series, as well as any prime in its equivalence class.

If we now look at the u where we do have numerator majorization, we see that regardless of the value of m used for A_m in (3.1.4), we will always carry at least an equal amount in the numerator parameters as in the denominators. Thus, the valuation with respect to any given prime $p \equiv u \bmod M$ is bounded below, and so these u are bounded in the denominators of the series.

We thus can compute the density of bounded primes in the denominators of the hypergeometric series ${}_nF_{n-1}$ as

$$\mathbf{D} = \frac{|B|}{\phi(M)}$$

since $\phi(M) = |(\mathbb{Z}/M\mathbb{Z})^\times|$. $\square$

We also can provide some examples of parameters and their corresponding densities. Table 3.3 provides several examples. Note that there are cases where we have density 1, which make use of some results shown in Kato 2011. We expect there to be similar cases in higher n that can be found with similar techniques. We also see cases with density zero, which occur precisely when the conditions for Theorem 3.1.6 (stated and proven below) are satisfied.

You can also notice that in general, these densities are very small. This is due to the fact that, the more parameters there are, the lower the likelihood that all of the denominator expansions are smaller than sufficiently many numerator parameters. Note that it only takes one digit at one position per $u \in (\mathbb{Z}/M\mathbb{Z})^\times$ for all primes $p \equiv u \bmod M$ to have valuation unbounded from below.

We also limit the denominators shown in this example. While this theorem works for any set of parameters, these examples were generated using no repeat parameters, and limiting the denominator to 15. As the LCM of the denominators

n	a_i	b_i	Density
3	$\left[\frac{1}{6}, \frac{2}{3}, \frac{5}{6}\right]$	$\left[\frac{5}{4}, \frac{3}{4}\right]$	1
3	$\left[\frac{1}{2}, \frac{3}{11}, \frac{2}{7}\right]$	$\left[\frac{4}{9}, \frac{7}{11}\right]$	$\frac{13}{360}$
3	$\left[\frac{1}{3}, \frac{1}{4}, \frac{13}{15}\right]$	$\left[\frac{5}{13}, \frac{11}{13}\right]$	$\frac{7}{96}$
3	$\left[\frac{1}{11}, \frac{4}{11}, \frac{3}{4}\right]$	$\left[\frac{1}{4}, \frac{1}{2}\right]$	$\frac{3}{20}$
4	$\left[\frac{3}{13}, \frac{10}{13}, \frac{5}{13}, \frac{2}{9}\right]$	$\left[\frac{3}{11}, \frac{1}{3}, \frac{3}{4}\right]$	$\frac{1}{80}$
4	$\left[\frac{1}{7}, \frac{4}{13}, \frac{2}{3}, \frac{7}{10}\right]$	$\left[\frac{5}{7}, \frac{1}{5}, \frac{3}{5}\right]$	$\frac{17}{576}$
4	$\left[\frac{3}{10}, \frac{1}{9}, \frac{1}{2}, \frac{7}{11}\right]$	$\left[\frac{11}{14}, \frac{2}{11}, \frac{7}{9}\right]$	$\frac{1}{72}$
5	$\left[\frac{6}{13}, \frac{2}{3}, \frac{2}{15}, \frac{3}{11}, \frac{5}{14}\right]$	$\left[\frac{9}{13}, \frac{3}{4}, \frac{1}{2}, \frac{1}{6}\right]$	$\frac{11}{1280}$
5	$\left[\frac{7}{10}, \frac{1}{14}, \frac{11}{14}, \frac{3}{11}, \frac{2}{15}\right]$	$\left[\frac{4}{5}, \frac{1}{2}, \frac{3}{10}, \frac{1}{4}\right]$	$\frac{7}{480}$
5	$\left[\frac{1}{14}, \frac{6}{11}, \frac{1}{2}, \frac{3}{5}, \frac{10}{13}\right]$	$\left[\frac{2}{3}, \frac{1}{8}, \frac{3}{4}, \frac{4}{7}\right]$	$\frac{1}{64}$
6	$\left[\frac{3}{5}, \frac{5}{7}, \frac{5}{13}, \frac{1}{2}, \frac{1}{6}, \frac{4}{11}\right]$	$\left[\frac{7}{9}, \frac{8}{15}, \frac{5}{11}, \frac{7}{8}, \frac{2}{3}\right]$	$\frac{371}{69120}$
6	$\left[\frac{4}{7}, \frac{2}{7}, \frac{9}{10}, \frac{1}{3}, \frac{6}{7}, \frac{2}{3}\right]$	$\left[\frac{1}{2}, \frac{9}{13}, \frac{3}{5}, \frac{2}{5}, \frac{7}{8}\right]$	$\frac{1}{32}$
6	$\left[\frac{2}{11}, \frac{2}{9}, \frac{7}{11}, \frac{8}{13}, \frac{1}{7}, \frac{5}{8}\right]$	$\left[\frac{9}{13}, \frac{4}{15}, \frac{5}{13}, \frac{11}{13}, \frac{1}{6}\right]$	$\frac{53}{69120}$
7	$\left[\frac{1}{13}, \frac{1}{2}, \frac{7}{15}, \frac{1}{14}, \frac{1}{6}, \frac{3}{4}, \frac{5}{8}\right]$	$\left[\frac{11}{15}, \frac{3}{5}, \frac{11}{13}, \frac{4}{9}, \frac{1}{9}, \frac{13}{15}\right]$	$\frac{77}{3456}$
7	$\left[\frac{8}{11}, \frac{3}{11}, \frac{8}{9}, \frac{4}{5}, \frac{5}{8}, \frac{2}{7}, \frac{1}{5}\right]$	$\left[\frac{5}{6}, \frac{5}{9}, \frac{1}{3}, \frac{1}{6}, \frac{4}{11}, \frac{1}{14}\right]$	0
7	$\left[\frac{2}{13}, \frac{3}{7}, \frac{1}{5}, \frac{6}{11}, \frac{9}{10}, \frac{1}{3}, \frac{13}{15}\right]$	$\left[\frac{5}{12}, \frac{9}{13}, \frac{1}{10}, \frac{1}{2}, \frac{3}{10}, \frac{4}{5}\right]$	0

Table 3.3: Example densities for various sets of parameters and values of n.

quickly increases with larger denominator values, the computing time becomes prohibitively long.

Furthermore, the sets of parameters given in this example are not chosen entirely at random. The parameters were generated such that there would be no duplicate parameters, and no parameters outside $(0,1)$. Values without this restriction can still work, as seen in the first row which contains $b_1 = \frac{5}{4}$, but for simplicities sake were not generated in the random parameter sets. As well, in most cases, randomly generating a set of parameters will give a density of zero in much higher proportion than is shown. In order to communicate more non-zero densities, sets of parameters were repeatedly generated until sufficiently few zeroes were present.

There are several corollaries to our main Theorem 3.1.5 above. We will begin with a sufficient condition for the density of bounded primes in the denominators of the series to be zero. This generalizes Theorem 4.2 of Franc et al. 2018.

Theorem 3.1.6. *Consider* ${}_nF_{n-1}\left(\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_{n-1} \end{matrix}; z\right)$*, and assume without loss of generality that both the sets of parameters are ordered from smallest to greatest fractional part. The density of bounded primes in the denominators of this series is zero if and only if*

$$\textit{there exists } i \in \{1, \ldots, n-1\} \textit{ such that } \{b_i\} < \{a_i\}.$$

Proof. Recall from the earlier stated Lemma 2.2.7 that if $b_i < a_i$, then the expansions of b_i and a_i have at least one position k where $\alpha_k < \beta_k$. As in the proof of Theorem 3.1.5, when referring to the digits of a parameter, we are truly referring to the digits of that parameter minus 1.

We begin with the assumption that there exists $i \in \{1, \ldots, n-1\}$ such that $\{b_i\} < \{a_i\}$ holds. Then, we show that as long as this condition holds, we cannot be numerator majorized for any $u \in (\mathbb{Z}/M\mathbb{Z})^\times$. Take i to be the smallest index such that this condition holds. Assume first that $i = 1$, and so the smallest numerator parameter is greater than the smallest denominator parameter. Then there is necessarily a position within their expansions where $\alpha_k < \beta_k$ regardless of the prime we are expanding with. Thus, the sets of parameters is not numerator majorized for any $u \in (\mathbb{Z}/M\mathbb{Z})^\times$, and so the density must be zero.

Now assume that the smallest index is greater than 1. Then, for all lower indexed parameters, we have a further two cases, either they themselves would cause numerator majorization to fail for all primes $p \equiv u$ for some specific $u \in (\mathbb{Z}/M\mathbb{Z})^\times$, or we have $\alpha_k < \beta_k$ for all k and all $p \equiv u$. Note that both of those cases can be ignored. In the first case, we can just consider the u for which the lower indexed parameters do not fail numerator majorization, as if those primes fail, they will still fail when considering the expansions of a_i and b_i. In the second case, we note that each of the digits of the a_j expansion must be used to 'counteract' those of the b_j expansion. Each α_k carries with a smaller m than the corresponding β_k, and since one carry can only cancel out one carry, the values of these α_k will not affect whether we will cancel a carry for the expansions of b_i.

Then, just as above, since $b_i < a_i$, there exists a position k where $\alpha_k < \beta_k$ regardless of the prime $p \equiv u \in (\mathbb{Z}/M\mathbb{Z})^\times$. Thus, these also fail numerator majorization (keeping in mind that we do not need to consider any u for which a pair of parameters has already caused a failure) so if our condition holds for a set of parameters, the density of bounded primes in the denominators of the series is always zero.

We now assume that the density is zero. Since the density is zero, we know that for all equivalence classes $u \in (\mathbb{Z}/M\mathbb{Z})^\times$, all but finitely many primes $p \equiv u$

appear with valuation $\nu_p(A_m)$ unbounded below for coefficients A_m of the series. Recall that

$$\nu_p(A_m) = \sum_{j=1}^{n} c_p(a_j - 1, m) - \sum_{k=1}^{n-1} c_p(b_k - 1, m).$$

Since this is unbounded below, and the expansions are periodic, we know that we can construct m_r such that A_{m_r} has unbounded negative valuation for all but finitely many primes as $r \to \infty$.

If we take $p \equiv 1 \bmod M$, then all expansions are 1-periodic, since the digits of its expansion are then given by $x_0 = \lfloor \{-p^{1-1-0}x\}p \rfloor = \lfloor \{-x\}p \rfloor$. Notice as well that the larger the digit, the smaller the parameter, as the digit is a result of the fractional part of the negative of the parameter. So for numerator majorization to fail, there must be some denominator parameter that, when the lists of parameters are ordered from smallest to largest, is itself smaller than its numerator parameter with the same index. So if $u = 1$ fails, the condition stated in the theorem must hold, and so if the density is zero, there exists $i \in \{1, \ldots, n-1\}$ such that $\{b_i\} < \{a_i\}$. □

We also have the following on the potential densities:

Theorem 3.1.7. *The set of p-adically bounded primes for a given set of parameters is a union of cyclic subgroups of $(\mathbb{Z}/M\mathbb{Z})^\times$.*

Proof. This follows naturally from the definition of numerator majorization given in Lemma 3.1.3 and Theorem 3.1.5. Notice that if the condition in Lemma 3.1.3 holds for u, it holds for all powers of u, and in particular for the cyclic subgroup of $(\mathbb{Z}/M\mathbb{Z})^\times$ generated by u. □

While these theorems work well for generalized hypergeometric series of the form ${}_nF_{n-1}$, they need some adjustment to be used for series of the form ${}_nF_m$ for $n, m \in \mathbb{N}$. We will first make the argument that Theorem 3.1.4 can be adjusted in a somewhat natural way and used for the valuation of the coefficients of ${}_nF_m\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_m \end{pmatrix}; z\Big)$.

The following theorem shows that the only interesting cases of ${}_nF_m$ are the cases where $n = m + 1$.

Theorem 3.1.8. *Consider ${}_nF_m\left(\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_m \end{matrix}; z\right)$. If $n > m+1$, the density of bounded primes in the denominators is 1, and if $n < m+1$, the density of bounded primes in the denominators is 0.*

Proof. This proof relies on the use of the Gamma function, with properties taken from Abramowitz and Stegun 1965. Specifically 6.1.21 and 6.1.22 from Abramowitz and Stegun, which give the definitions for the binomial coefficient and Pochhammer symbol respectively. We begin by recalling that the Pochhammer symbol can be described as

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$$

and the binomial coefficient can be described as

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

We can then manipulate the coefficients A_k of ${}_nF_m\left(\begin{matrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_m \end{matrix}; z\right)$ as follows:

$$\begin{aligned}
A_k &= \frac{\prod_{i=1}^{n}(a_i)_k}{\prod_{j=1}^{m}(b_i)_k}\frac{1}{k!} \\
&= \frac{\prod_{i=1}^{n}\frac{\Gamma(a_i+k)}{\Gamma(a_i)}}{\prod_{j=1}^{m}\frac{\Gamma(b_j+k)}{\Gamma(b_j)}}\frac{1}{\Gamma(k+1)} \\
&= \frac{\prod_{i=1}^{n}\frac{\Gamma((a_i-1+k)+1)\Gamma(k+1)}{\Gamma((a_i-1+k)-k+1)\Gamma(k+1)}}{\prod_{j=1}^{m}\frac{\Gamma((b_j-1+k)+1)\Gamma(k+1)}{\Gamma((b_j-1+k)-k+1)\Gamma(k+1)}}\frac{1}{\Gamma(k+1)} \\
&= \frac{\prod_{i=1}^{n}\binom{a_i-1+k}{k}\Gamma(k+1)}{\prod_{j=1}^{m}\binom{b_j-1+k}{k}\Gamma(k+1)}\frac{1}{\Gamma(k+1)} \\
&= \frac{\prod_{i=1}^{n}\binom{a_i-1+k}{k}}{\prod_{j=1}^{m}\binom{b_j-1+k}{k}}(k!)^{n-m-1}
\end{aligned}$$

Now, by the p-adic analogue of Kummer's theorem given in Franc et al. 2018, we can see that the valuation of this term is given by

$$\nu_p(A_k) = \sum_{i=1}^{n} c_p(a_i-1,k) - \sum_{j=1}^{m} c_p(b_j-1,k) + (n-m-1)\nu_p(k!) \tag{3.2}$$

We now need to show that, if k is large enough, then $\nu_p(k!)$ dominates either of the sums in this formula. Note that, if we look at the A_kth coefficient, we can have at most as many carries as the highest exponent in the expansion of k.

This shows that

$$c_p(a_i - 1, k) = O(\log_p(k))$$
$$c_p(b_i - 1, k) = O(\log_p(k))$$

On the other hand, it is well known by Legendre's formula that $\nu_p(k!) = O(k)$ Neukirch 2013.

Call this exponent $\ell \approx \log_p(k)$. We can then see that the values of the first two terms of Formula 3.2 above are bounded by $n\ell$ and $m\ell$ respectively. Thus, we can see that the asymptotic behaviour of Equation 3.2 depends only on the sign of $n - m - 1$, except in the interesting case where $n = m + 1$). If this sign is negative we get unboundedness and if positive we get boundedness. This concludes the proof. $\square$

Chapter 4

A Conjecture on Irrational p-adic Numbers

4.1 Conjecture

We are now considering the subject of ${}_2F_1$ over quadratic extensions. We begin by stating the following conjecture:

Conjecture 4.1.1. *Let $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$, and let S be the set of unramified primes in $\mathbb{Q}(\alpha)$. Let r and s be integers, and let $0 \leq u < v \leq 1$. Let $\alpha_n(p)$ denote the nth p-adic digit of α for $p \in S$. Then for all but finitely many primes $p \in S$ (where the finite exceptional set of primes depends on α, r, s, u and v), $\alpha_n(p)$ is contained in the interval $(u(p-1), v(p-1))$ for infinitely many integers $n \geq 0$ in the arithmetic progression $n = rm + s$.*

There are several possible rationales in support of this conjecture. We can use Hensel's lemma to approximate the digits of a p-adic irrational number. With this, we can look at whether or not we find the digits in any interval we can define. As can be seen in Figure 4.1, we have digits frequently appearing in the interval.

When we look in detail at the digits that appear, we find that every digit appears with a natural density of roughly $\frac{1}{p}$. The further into the approximation we search, the closer we get to this density. We see in Figure 4.2 that along an arithmetic progression (here every fourth digit of the first 40000) we get a roughly equal density for each digit. That is to say, that there are no biases in the coefficients of these square roots.

It is valuable to note the relationship between this conjecture and what would be commonly referred to as *normality* of an irrational number. We define the equivalent concept for an irrational p-adic number. Though the second clause

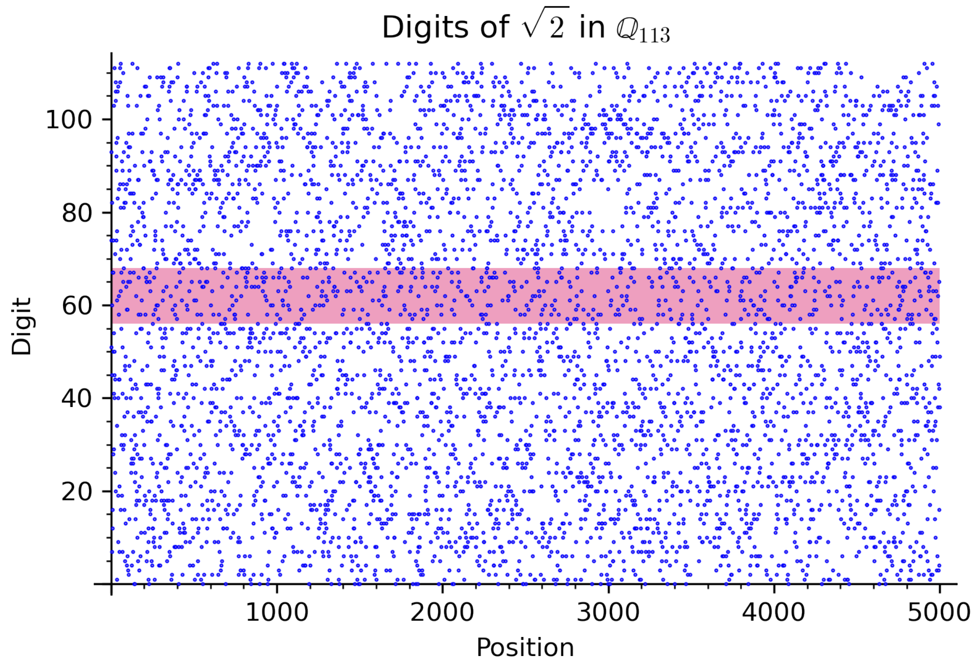

FIGURE 4.1: The first 5000 digits of $\sqrt{2}$ in $\mathbb{Q}_{113}$, with a coloured band representing the interval $(0.5(113-1), 0.65(113-1))$

below is a strengthening of the first clause in this definition, we phrase it in such a way that we can compare this with the analogue in the real numbers.

Definition 4.1.2. *An irrational p-adic number $x \in \mathbb{Q}_p$ is called* normal *if every digit in its expansion appears with equal natural density $\frac{1}{p}$ and, more generally if every string of n digits appears asymptotically with frequency p^{-n}.*

If we were instead working in the real numbers, the second part of this definition would come automatically as a result of being able to change bases, as can be seen in Theorem 1.3 of Harman 1998. In the p-adic case, however, results to this end do not exist, and so the stronger definition of normality must be taken.

With this definition of normality in mind, we can now see that our Conjecture holds for normal p-adic numbers.

Proposition 4.1.3. *If a p-adic number x is normal, then Conjecture 4.1.1 also holds for that number.*

Proof. Consider an arithmetic progression of positions $\{a_n\} = \{nk + \ell\}$, which looks at every k^{th} digit of the expansion after ℓ. We want to show that any digit appears infinitely many times on this progression. Since x is normal, we know that each digit has an equal probability of appearing on this progression of digits. However, we need to show that it is not possible for a digit to only appear a finitely many times. We can see that in certain cases this must be true. For example, when $\{a_n\} = \{1, 2, \ldots\}$, there can be no number that appears finitely many times, as this would contradict normality.

Assume towards a contradiction that there exists a digit b that appears finitely many times along this progression. Then, if we break this expansion into strings of length k, there are no occurrences of this digit in the k^{th}position of the string. However, there are finitely many strings of this nature. By the definition of normality each n-length string appears with equal density. Then we must have digits in the p^k expansion that correspond to strings containing b in its final position appearing. Thus there can be no final occurrence of b along $\{a_n\}$, and thus the conjecture holds. $\square$

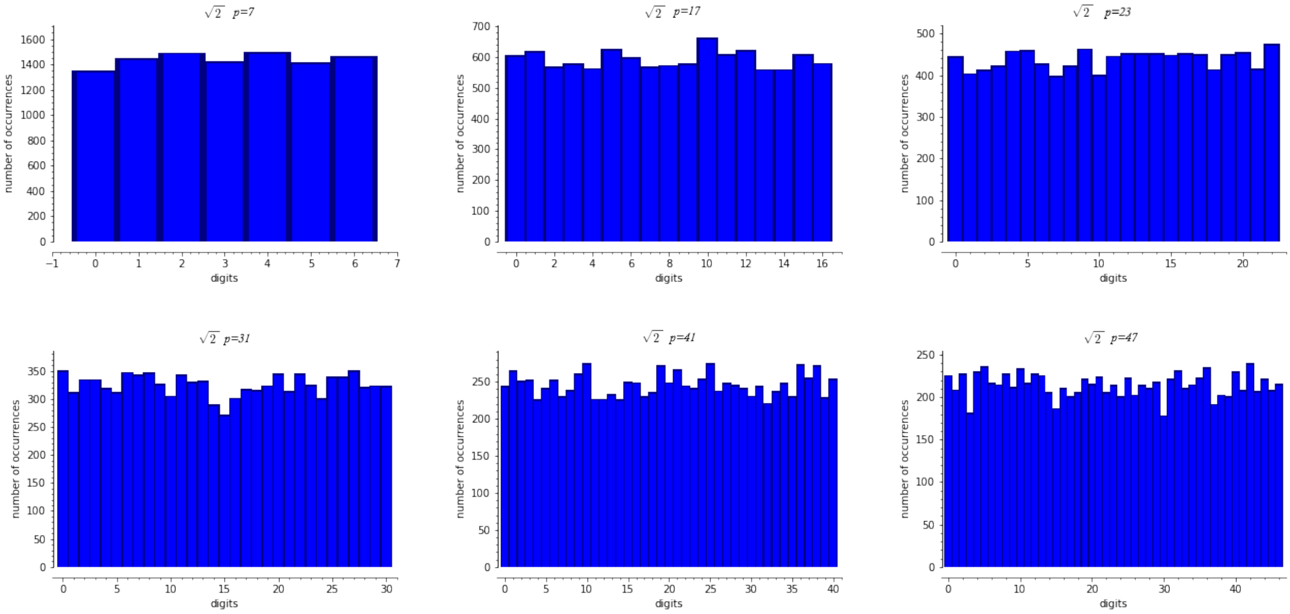

FIGURE 4.2: The number of occurrences of digits along the arithmetic progression $\{4n\}_{n\in\mathbb{Z}_+}$ in the expansions of $\sqrt{2}$ with respect to various primes.

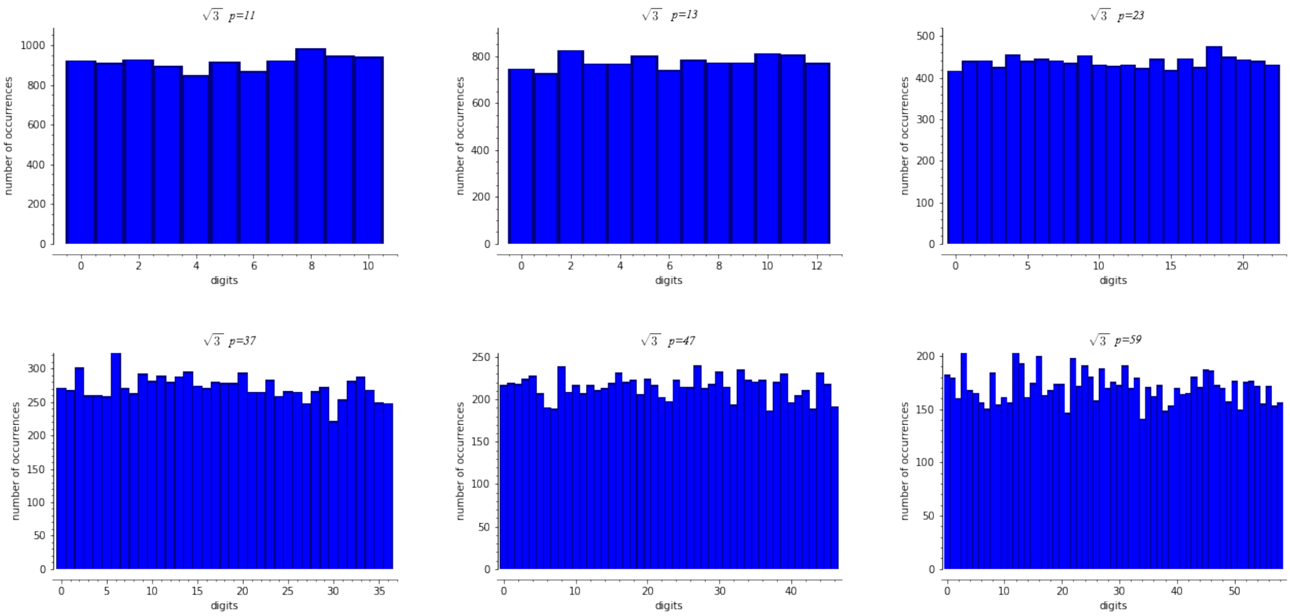

FIGURE 4.3: The number of occurrences of digits along the arithmetic progression $\{4n\}_{n\in\mathbb{Z}_+}$ in the expansions of $\sqrt{3}$ with respect to various primes.

4.2 Applications of the Conjecture to Hypergeometric Series

We define the hypergeometric series used in this section as below:

$$F = {}_2F_1\left(a + b\sqrt{D}, a - b\sqrt{D}; c; z\right) \tag{4.1}$$

for rational parameters a, b, c. Though this series has irrational parameters, it has rational coefficients. In this section, we will stufy the densities of this F. We begin with a necessary condition for a prime to be bounded in the denominators of F

Theorem 4.2.1. *A prime p can only be bounded in the denominators of F defined over a number field $\mathbb{Q}\left(\sqrt{D}\right)$ if p splits in $\mathbb{Q}\left(\sqrt{D}\right)$.*

Proof. Consider the formula for a coefficient of the hypergeometric series.

$$\begin{aligned} A_n &= \frac{(a + b\sqrt{D})_n (a - b\sqrt{D})_n}{(c)_n m!} \\ &= \frac{\prod_{j=0}^{n-1}(a + b\sqrt{D} + j)(a - b\sqrt{D} + j)}{\prod_{j=0}^{n-1}(c + j)(j + 1)} \\ &= \frac{\prod_{j=0}^{n-1}(a^2 - b^2 D + 2aj + j^2)}{\prod_{j=0}^{n-1}(c + j)(j + 1)} \end{aligned}$$

Defining $P(x) = a^2 - b^2D + 2ax + x^2$. We see that:

$$A_n = \frac{\prod_{j=0}^{n-1} P(j)}{\prod_{j=0}^{n-1}(c + j)(j + 1)}$$

The discriminant of $P(x)$ can be seen to be $4b^2D$, which is a perfect square multiple of D, and so defines the same number field. Then, this polynomial splits completely over the p-adic field $\mathbb{Q}_p$ if and only if its discriminant is a square mod p, as stated in Theorem 2.1.1.

Then, we know that a prime splits completely in $\mathbb{Q}\left(\sqrt{D}\right)$ if and only if D is a quadratic residue mod p. Since $P(x)$ over $\mathbb{Q}_p$ splits only when D is a square mod p, by the definition of the Legendre symbol, we can only potentially cancel out a prime in the denominator of A_n when p splits in $\mathbb{Q}(\sqrt{D})$. Neukirch 2013.

□

We now introduce the following theorem, which needs some adaptation from how it appears in Franc et al. 2018. We are setting parameters in a specific way such that we have rational coefficients, which must be done since we are working over a number field instead of the rational numbers. The following theorem suppresses the need to consider the $\sqrt{D}$ term. Here "admissible hypergeometric parameters" are positive rational numbers $0 < a, b, c, < 1$ in least form.

Theorem 4.2.2. *Assuming 4.1.1 to be true, the following holds. Let $(a, b; c)$ denote admissible hypergeometric parameters, and p denote a good prime. Then the following are equivalent:*

(i) for some index j, we have $\tau_j(c-1) > \tau_j(a-1)$

(ii) ${}_2F_1(a + b\sqrt{D}, a - b\sqrt{D}, c; z)$ has p-adically unbounded coefficients.

Proof. We will explain how to adapt the proof from Franc et al. 2018 to the number field setting. The primary difference between the settings is that, in the rationals, the expansions all of the parameters are purely periodic, and in the case of number fields, the presence of the $\sqrt{D}$ factor means that we do not have periodicity of the expansions, and we must substitute Conjecture 4.1.1 for this lack of periodicity.

In the rational case, since the terms are periodic, it suffices to show that there is one index where condition (i) holds. Each index corresponds to one occurrence of the prime p in the denominator at some term of the series, and since they are periodic, a single index necessarily corresponds to an arithmetic sequence of indices that all fit the condition.

In our case however, since the $b\sqrt{D}$ term is irrational, we must consider that there is not necessarily more than one index along the arithmetic progression that has the condition holds. Take j to be the smallest index where condition (i) holds. Then we can look at an arithmetic progression $\{j + Mn\}$ where M is the lowest common multiple of the denominators of a and c. Along this progression, we know that if $b\sqrt{D}$ is sufficiently small at that index, we still have that digit of $c-1$ greater than the digits of $a+b\sqrt{D}$ and $a-b\sqrt{D}$. Take Δ_j to be the difference $\Delta_j = \gamma_j - \alpha_j$.

We can then fix an interval $(0, \Delta_j)$, and by the conjecture, $b\sqrt{D}$ must contain infinitely many occurrences of digits within this interval on our arithmetic progression. Let the indices where $\bar{\beta}_i \in (0, \Delta_j)$ along the arithmetic progression $\{j + Mn\}$ be denoted by $\{k_i\}$. We then define $m_r = \sum_{k\in\{k_i\}_{i\leq r}}(p-\gamma_j)p^k$. Then, as in the proofs in the previous chapter using Theorem 3.1.4, if we look at the $A_{m_r}{}^{\text{th}}$coefficient, we will have a carry in the denominator and not in the numerator for each $i \leq r$. Since r is not bounded, the p-adic valuation of this prime is not bounded below, and so ${}_2F_1(a + b\sqrt{D}, a - b\sqrt{D}, c; z)$ has p-adically bounded coefficients.

The rest of the proof is not affected by the change to number fields. □

We now, in pursuit of understanding how parameters affect the density, prove the analogue, in our setting, of Theorem 4.14 of Franc et al. 2018.

Theorem 4.2.3. *Assume Conjecture 4.1.1 to be true and let a, c denote the relevant hypergeometric parameters. The following are equivalent:*

(i) $c < \max\left(a, \left\{a + \frac{1}{2}\right\}\right)$.

(ii) All but finitely many primes are unbounded in the denominators of F.

Proof. We can assume without loss of generality that $a \geq \frac{1}{2}$, as if a is less than $\frac{1}{2}$, we look at $a + \frac{1}{2}$, and if a is greater, then $a + \frac{1}{2} \bmod 1$ is less than a.

So assume $c < a$, and $a \geq \frac{1}{2}$. Recall the formula for the terms of the p-adic expansions given in Lemma 2.2.7. Then, defining M_a and M_c to be the order of $p \bmod d_a, p \bmod d_c$ where d_a and d_c are the denominators of a and c in least form, respectively.

Then, notice that $\{c\} < \{a\}$, and by extension $-\{a\} < -\{c\}$. We then look at the p-adic expansions of $a - 1$ and $c - 1$, taking n such that $n \equiv 1 \bmod M_a M_c$:

$$\gamma_n = \left\lfloor \left\{ -p^{M_c - 1 - n} c \right\} p \right\rfloor = \lfloor \{-c\} p \rfloor = p + \lfloor -cp \rfloor$$
$$\alpha_n = \left\lfloor \left\{ -p^{M_a - 1 - n} a \right\} p \right\rfloor = \lfloor \{-a\} p \rfloor = p + \lfloor -ap \rfloor$$

If we take the primes p such that $p > \frac{1}{a-c}$, then we have $-ap + 1 < -cp$. Therefore,

$$\gamma_n = p + \lfloor -cp \rfloor > p + \lfloor -ap + 1 \rfloor > p + \lfloor -ap \rfloor = \alpha_n.$$

We can then apply the same $\Delta_n = \gamma_n - \alpha_n$ as in the proof of Theorem 4.2.2, which gives that this prime p is unbounded in the given series. Since this works for all sufficiently large primes, all but finitely many primes are unbounded.

To prove the other direction, we assume that all but finitely many primes are unbounded. Then, clearly infinitely many primes such that $p = 1 \bmod d$ are unbounded. Thus there exists a prime p such that $p \equiv 1 \bmod d$ and $\{-c\} > \{-a\}$. Therefore $c < a$, and so c is smaller than the maximum of $a, a + \frac{1}{2}$. □

We can now prove the following, making use of the framework provided in Franc et al. 2020.

Theorem 4.2.4. *Assume Conjecture 4.1.1. Let $D \in \mathbb{Z}$ be square-free. Take $a, b, c \in \mathbb{Q}, b \neq 0$. We define*

$$f = {}_2F_1\left(a + b\sqrt{D}, a - b\sqrt{D}; c\right).$$

Then f has a density of bounded primes independent of b. There exists a constant N such that if $p > N$ then f is p-adically bounded if and only if p is congruent to an element of

$$B(a,c) := \left\{u \in S \mid \left\{-u^j c\right\} \leq \max\left(\left\{-u^j a\right\}, \left\{-u^j\left(a + \frac{1}{2}\right)\right\}\right) for\ j = 0, \ldots, \phi(M)\right\}$$

where S is the set of classes of primes mod M *that split in $\mathbb{Q}\left(\sqrt{D}\right)$ and M is given by*

$$M = \operatorname{lcm}(4|D|, \operatorname{denom}(a), \operatorname{denom}(c)).$$

Therefore the density of bounded primes for f is given by $\mathbf{D} = \frac{|B(a,c)|}{|(\mathbb{Z}/M\mathbb{Z})^\times|}$.

Proof. We first demonstrate where the formula arises from. Consider the formula for the p-adic expansion of $a - 1$:

$$\alpha_n = \left\lfloor\left\{-p^{M_a-1-n}c\right\}p\right\rfloor.$$

If we 'normalize' this by dividing by p, and take the limit of all $p \equiv u \bmod d_a$ for a fixed u, we get the following:

$$\begin{aligned}
\lim_{p \equiv u (\bmod d_a)} \frac{\alpha_n}{p} &= \lim_{p \equiv u (\bmod d_a)} \frac{\left\lfloor\left\{-p^{M_a-1-n}a\right\}p\right\rfloor}{p} \\
&= \lim_{p \equiv u (\bmod d_a)} \frac{\left\{-p^{M_a-1-n}a\right\}p - \left\{\left\{-p^{M_a-1-n}a\right\}p\right\}}{p} \\
&= \lim_{p \equiv u (\bmod d_a)} \left\{-p^{M_a-1-n}a\right\} - \frac{\left\{\left\{-p^{M_a-1-n}a\right\}p\right\}}{p}.
\end{aligned}$$

Since the fractional part is less than 1, as p increases, the second term vanishes

$$\lim_{p \equiv u (\bmod d_a)} \frac{\alpha_n}{p} = \lim_{p \equiv u (\bmod d_a)} \left\{-p^{M_a-1-n}a\right\}.$$

Let $a = a'/d$. Since $p = u(\bmod d)$, we can write $p = u + td_a$ for some t. Continuing after making this substitution we find:

$$\begin{aligned}\lim_{p\equiv u(\bmod d_a)} \frac{\alpha_n}{p} &= \lim_{p\equiv u(\bmod d_a)} \left\{-(u+td_a)^{M_a-1-n} a'/d_a\right\} \\ &= \lim_{p\equiv u(\bmod d_a)} \left\{-(u)^{M_a-1-n} a'/d_a - \sum_{k=0}^{M_a-1-n} u^{M_a-1-n} t^{M_a-1-n-k} d_a^{M_a-2-n-k} a'\right\}.\end{aligned}$$

Notice that u, t, d_a, a' are all integers. So the second term of this can be safely dropped from the fractional part, and the above equation reduces to

$$\lim_{p\equiv u(\bmod d_a)} \frac{\alpha_n}{p} = \lim_{p\equiv u(\bmod d_a)} \left\{-(u)^{M_a-1-n} a\right\}.$$

Since there is no longer a dependency on p, we can drop the limit, which gives us

$$\lim_{p\equiv u(\bmod d_a)} \frac{\alpha_n}{p} = \left\{-u^{M_a-1-n} a\right\}.$$

When we use this technique, we can make the substitution of $j = M - 1 - n$, giving us

$$\lim_{p\equiv u(\bmod d)} \frac{\alpha_n}{p} = \left\{-u^j a\right\}$$

for large enough primes p. We can apply the same to c and $a+\frac{1}{2}$. Now we compare the terms to show that a class of primes is unbounded. Fix $u \in S$. Note that if $\gamma_j/p \le \alpha_j/p$, then $\gamma_j \le \alpha_j$, and so if $\{-u^j c\} > \max\left(\{-u^j a\}, \left\{-u^j\left(a+\frac{1}{2}\right)\right\}\right)$, then $\gamma_j > \alpha_j$.

As in the earlier proof of Theorem 4.2.3, if $\gamma_j > \alpha_j$, then we can define Δ_j to be the difference, and by Theorem 4.2.2, which itself relies on the assumption of Conjecture 4.1.1, f has p-adically unbounded coefficients. Thus this value for u should not be in the set B. If we check for all $u \in S$ and remove any for which the condition of B fails, then we have a set only composed of the classes of primes mod M for which f does not have unbounded coefficients. We can thus get a density of bounded primes independent of the value of b by taking the fraction

$$\mathbf{D} = \frac{|B(a,c)|}{\phi(M)}.$$

□

We can note some interesting properties about this theorem. For one, we have a maximum possible density of $\frac{1}{2}$. In particular, there are always unbounded primes for these series. This is consistent with the fact that the corresponding

a	c	a	c	a	c
$\frac{l}{k}$	$\frac{l}{k}$	$\frac{2}{3}$	$\frac{13}{2}$	$\frac{12}{5}$	$\frac{3}{2}$
$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{14}{3}$	$\frac{7}{3}$
$\frac{7}{13}$	$\frac{5}{2}$	$\frac{1}{10}$	$\frac{6}{5}$	$\frac{7}{10}$	$\frac{1}{2}$
$\frac{1}{5}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{7}{3}$
$\frac{2}{9}$	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{12}{13}$	$\frac{7}{2}$
$\frac{7}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{1}{8}$	$\frac{11}{8}$
$\frac{15}{4}$	$\frac{13}{3}$	$\frac{8}{3}$	$\frac{1}{2}$	$\frac{4}{5}$	$\frac{1}{2}$
$\frac{7}{5}$	$\frac{1}{2}$	$\frac{1}{14}$	$\frac{1}{2}$	$\frac{11}{15}$	$\frac{1}{2}$
$\frac{1}{3}$	$\frac{4}{3}$	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{6}{5}$	$\frac{13}{2}$
$\frac{4}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{7}{3}$	$\frac{1}{3}$	$\frac{11}{3}$
$\frac{5}{3}$	$\frac{1}{3}$	$\frac{9}{2}$	$\frac{1}{2}$	$\frac{11}{2}$	$\frac{3}{2}$
$\frac{13}{9}$	$\frac{3}{2}$	$\frac{5}{14}$	$\frac{12}{7}$	$\frac{7}{12}$	$\frac{3}{2}$

Table 4.1: Sets of parameters a, c over the field $Q\left(\sqrt{2}\right)$ where the density is $\frac{1}{2}$.

monodromy groups are infinite, since the hypergeometric parameters are irrational (recalling that Schwarz's list of parameters such that the monodromy groups are finite is composed only of rational hypergeometric parameters). This is a result of Chebotarev's density theorem, which for our application states that the density of primes that can split in a quadratic number field $\mathbb{Q}\left(\sqrt{D}\right)$ is $\frac{1}{2}$ Neukirch 2013.

We also know some examples where we get this density. Note that since we are only dependent on a and c for our density, any set (a, c) defines an infinite class of parameters (a, b, c) which give an equal density. We provide the Table 4.1 that gives some, but not all, of the cases where we get density of $\frac{1}{2}$.

We can notice that whenever we have $a = c$, then we get a density of $\frac{1}{2}$. This makes sense, as when the parameters are equal, then they automatically satisfy the condition of Theorem 4.2.4. We also see that many of these fit with those that appear on Schwarz's list Schwarz 1873, as seen in Figure 2.1. This is not universal for the entries on that list, however. For example, $(a, b, c) = (1/2, 1/3, 1/5)$ is on Schwarz's list, but neither $(a, c) = (1/2, 1/5)$ or $(a, c) = (1/3, 1/5)$ gives a density of $\frac{1}{2}$ for our quadratic analogue. In fact, both of these sets of parameters gives a density of zero in the quadratic analogue of Schwarz's list.

Chapter 5

Conclusions and Next Steps

5.1 Summary

We have generalized results of Franc et al. 2018 and Franc et al. 2020 to different hypergeometric series, being ${}_nF_m$ over the rationals, and ${}_2F_1$ over quadratic number fields. Existing results were limited to the case of ${}_2F_1$ over the rational numbers, but with the results of Chapter 3, we can now precisely calculate a density for the bounded primes in the denominators of the generalized hypergeometric series ${}_nF_m$ for any set of rational parameters and any values of n and m. We also have presented a new conjecture which presents a probable property on the expansions of all algebraic p-adic algebraic numbers. We presented evidence and justifications towards the truth of the conjecture, and will discuss in the next section some possibly relevant results from the literature.

5.2 Next Steps

We will first discuss potential avenues towards a proof for Conjecture 4.1.1. The easiest case to prove it is likely the case of $p = 2$. In this case, the conjecture is simply that there are an infinite number of zeroes or ones along the arithmetic progression. There are several results that are related to this that could be useful. Kaneko 2016 gives a lower bound for the number of non-zero digits of an algebraic number's expansion over a real base b, which could have potential applications to an analogous result over the p-adic numbers. Similarly, Bugeaud 2007 gives bounds and results on the Hensel expansion of an algebraic irrational p-adic number. These results do give a spark of hope towards a proof of the conjecture, at least in the $p = 2$ case. They unfortunately only give bounds of the number of non-zero digits, which is not in general applicable to the conjecture where we need information on the positioning of these digits as well.

These results could potentially be applied to the simplest case, $p = 2$. In this case, since the only non-zero digit is 1, the bound directly gives the number of non-zero digits. The issue comes up when attempting to ensure that these non-zero digits appear on a given arithmetic progression. One can easily imagine an expansion that has a finite number of non-zero digits on any given arithmetic progression, though evidence suggests that an algebraic irrational number can never have this occur.

Of course, as proven earlier, the conjecture would be proven were normality of algebraic irrational p-adic numbers to be proven, but this seems to be a very difficult problem at present.

Looking elsewhere, we can consider possible results on number fields that are not quadratic. For instance, we could look at a cyclotomic field, with n roots of unity. These become very complicated, however. Recalling Chebotarev's Density Theorem, we know that the density of splitting primes in an Abelian extension is $1/n$ if n is the degree of the extension over $\mathbb{Q}$ (this was why we had a maximum density of $\frac{1}{2}$ in the results from chapter 4).

We believe that the only cases where primes can be bounded are when the hypergeometric parameters of the series multiply out to leave no irrational terms in the coefficients of the series. As the degree of the extension increases, however, it becomes increasingly difficult to consider all cases. For instance, a degree 5 extension could have ${}_5F_4$ with numerator parameters that are all rational, and denominator parameters that contain the powers of roots of unity, or the denominator parameters can be all rational with the roots of unity in the numerator, or both. There are simply too many cases to consider, so any study of this area will require a strict culling of the possibilities.

Bibliography

Abramowitz, M. and Stegun, I. (1965). *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*. Applied mathematics series. Dover Publications. ISBN: 9780486612720.

Beukers, F. and Heckman, G. (1989). Monodromy for the hypergeometric function ${}_nF_{n-1}$. *Inventiones mathematicae* 95, 325–354.

Bugeaud, Y. (2007). On the b-ary expansion of an algebraic number. *Rendiconti del Seminario Matematico della Universitā di Padova* 118, 217–233.

Conrad, K. (2019). *The p-adic expansion of rational numbers*. URL: https://kconrad.math.uconn.edu/blurbs/gradnumthy/rationalsinQp.pdf (visited on 04/14/2023).

Franc, C., Gannon, T., and Mason, G. (2018). On unbounded denominators and hypergeometric series. *Journal of Number Theory* 192, 197–220.

Franc, C., Gill, B., Goertzen, J., Pas, J., and Tu, F. (2020). Densities of bounded primes for hypergeometric series with rational parameters. *Research in Number Theory* 6(2), 1–14.

Harman, G. (1998). *Metric Number Theory, London Mathematical Society Monographs*. Oxford University Press Oxford.

Kaneko, H. (2016). On the number of nonzero digits in the beta-expansions of algebraic numbers. *Rendiconti del Seminario Matematico della Università di Padova* 136, 205–223.

Kato, M. (2011). Schwarz Images of Algebraic 3F2. *Kyushu Journal of Mathematics* 65(2), 299–333. ISSN: 1340-6116.

Neukirch, J. (2013). *Algebraic number theory*. Vol. 322. Springer Science & Business Media.

Schwarz, H. (1873). Ueber diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt. *Journal für die reine und angewandte Mathematik* 75, 292–335.