GROUND STATES IN GROSS-PITAEVSKII THEORY

Ground states in Gross-Pitaevskii theory

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Abstract

We study ground states in the nonlinear Schrödinger equation (NLS) with an isotropic harmonic potential, in energy-critical and energy-supercritical cases. In both cases, we prove existence of a family of ground states parametrized by their amplitude, together with the corresponding values of the spectral parameter. Moreover, we derive asymptotic behavior of the spectral parameter when the amplitude of ground states tends to infinity. We show that in the energy-supercritical case the family of ground states converges to a limiting singular solution and the spectral parameter converges to a nonzero limit, where the convergence is oscillatory for smaller dimensions, and monotone for larger dimensions. In the energy-critical case, we show that the spectral parameter converges to zero, with a specific leading-order term behavior depending on the spatial dimension.

Furthermore, we study the Morse index of the ground states in the energy-supercritical case. We show that in the case of monotone behavior of the spectral parameter, that is for large values of the dimension, the Morse index of the ground state is finite and independent of its amplitude. Moreover, we show that it asymptotically equals to the Morse index of the limiting singular solution. This result suggests how to estimate the Morse index of the ground state numerically.

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Chapter 1

Introduction

1.1 Introduction

The main subject of this study is the focusing nonlinear Schrödinger equation (NLS) with an isotropic harmonic potential, given by

$$i\partial_t w = -\Delta w + |x|^2 w - |w|^{2p} w, \tag{1.1}$$

where $w(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and $p > 0$.

In physical dimensions $d = 1, 2, 3$, equation (1.1) is also known as the Gross-Pitaevskii equation. For cubic ($p = 1$) and quintic ($p = 2$) nonlinearities, it serves as a model for the behavior of the Bose-Einstein condensate with attractive atomic interactions under a magnetic trap [22, 3]. Originally, it was derived independently by Gross [24] and Pitaevskii [47] in 1961 by methods of the mean field theory. Starting with Lieb et al. [38, 39], the physical assumptions were mathematically justified with error bounds between the multi-particle wave function and the product of individual wave functions. More recently, equation (1.1) without the harmonic potential was justified in [51] for the cubic nonlinearity and $d = 1$, and in [12] for dimensions $d = 2, 3$. For the quintic nonlinearity, it was justified in [11] for $d = 1, 2$. From a mathematical viewpoint, (1.1) is a prototype model of dynamics of nonlinear dispersive waves subject to a confining potential, and from this perspective it is interesting to consider higher dimensions d and arbitrary nonlinearity powers $p > 0$.

Note that formally, by multiplying (1.1) by the complex conjugate $\bar{w}$, integrating over $\mathbb{R}^d$, and taking the imaginary part, we obtain the conservation of mass:

$$\frac{d}{dt} M(w) = \frac{d}{dt} \int_{\mathbb{R}} |w|^2 dx = 0. \tag{1.2}$$

Moreover, by multiplying equation (1.1) by $\partial_t \bar{w}$, integrating over $\mathbb{R}^d$, and taking the real part, we formally obtain the conservation of energy:

$$\frac{d}{dt} E(w) = \frac{d}{dt} \int_{\mathbb{R}^d} \left(|\nabla w|^2 + |x|^2 |w|^2 - \frac{1}{p+1} |w|^{2p+2} \right) dx = 0. \tag{1.3}$$

The above calculations suggest that the suitable space to look for solutions $w(t,\cdot)$ of (1.1) is the space $\mathcal{E}_{NLS}$, where both $M(w)$ and $E(w)$ are well-defined:

$$\mathcal{E}_{NLS} := \{u \in L^2(\mathbb{R}^d) : \quad \nabla u \in L^2(\mathbb{R}^d), \quad xu \in L^2(\mathbb{R}^d), \quad u \in L^{2p+2}(\mathbb{R}^d)\}. \tag{1.4}$$

This space is often referred to as the *energy space.*

By looking for standing wave solutions to (1.1) in the form $w(t,x) = e^{i\lambda t}u(x)$, we obtain the following stationary NLS

$$-\Delta u + |x|^2 u - |u|^{2p}u = \lambda u. \tag{1.5}$$

We are interested in studying *ground states* of (1.5), which are solutions corresponding to the lowest energy level. The study of excited states lies beyond the scope of this work, we refer to [52] and [17] for some of the recent developments on this subject. It is well-known [37] that ground states must be radially symmetric and decaying to zero at infinity. Thus, stationary equation (1.5) reduces to a radial equation

$$u''(r) + \frac{d-1}{r}u'(r) - r^2u(r) + \lambda u(r) + |u(r)|^{2p}u(r) = 0, \tag{1.6}$$

for $u(r) : \mathbb{R}^+ \mapsto \mathbb{R}$, where $r = |x|$. Hence, the principal problem of finding ground states, denoted as $\mathfrak{u}$, can be formulated as

$$\begin{cases} \mathfrak{u}''(r) + \frac{d-1}{r}\mathfrak{u}'(r) - r^2\mathfrak{u}(r) + \lambda\mathfrak{u}(r) + |\mathfrak{u}(r)|^{2p}\mathfrak{u}(r) = 0, & r > 0, \\ \mathfrak{u}(r) > 0, \qquad \mathfrak{u}'(r) < 0, \\ \lim_{r\to 0} \mathfrak{u}(r) < \infty, \quad \lim_{r\to\infty} \mathfrak{u}(r) = 0. \end{cases} \tag{1.7}$$

Weak solutions to the boundary-value problem (1.7) are defined in the energy space

$$\mathcal{E} := \left\{u \in L^2_r(\mathbb{R}^+) : \quad u' \in L^2_r(\mathbb{R}^+), \quad ru \in L^2_r(\mathbb{R}^+), \quad u \in L^{2p+2}_r(\mathbb{R}^+)\right\}, \tag{1.8}$$

where L^q_r denotes the space of radially symmetric $L^q(\mathbb{R}^d)$ functions with the norm defined as

$$\|u\|_{L^q_r} := \left(\int_0^\infty r^{d-1}|u(r)|^q dr\right)^{\frac{1}{q}}. \tag{1.9}$$

Note that (1.8) follows from (1.4) after transformation to the radial variable $r := |x|$.

1.2 Classification for the NLS without potential

For the NLS without the harmonic potential (free NLS)

$$i\partial_t w = -\Delta w - |w|^{2p}w, \tag{1.10}$$

it was proposed (see, e.g., [8]) to introduce different classification cases, based on the scaling properties of the conserved quantities of mass

$$M_f(w) := \int_{\mathbb{R}} |w|^2 dx, \tag{1.11}$$

and energy

$$E_f := \int_{\mathbb{R}^d} \left(|\nabla w|^2 - \frac{1}{p+1} |w|^{2p+2} \right) dx. \tag{1.12}$$

If $w(t,x)$ is a solution of the free NLS (1.10), then the scaled function $w_L(t,x)$ defined for $L > 0$ as

$$w_L(t,x) := L^{\frac{1}{p}} w(L^2 t, Lx) \tag{1.13}$$

is also a solution of (1.10), with the corresponding mass and energy rescaled according to

$$M_f(w_L) = L^{\frac{2}{p}-d} M(w), \quad E_f(w_L) = L^{\frac{2}{p}+2-d} E(w). \tag{1.14}$$

Scalings (1.14) suggest three distinct cases for each of the conserved quantities. When $dp < 2$, we have a *mass-subcritical* case, and the Cauchy problem for (1.10) is globally well-posed in $H^1(\mathbb{R}^d)$ from the conservation laws (1.2), (1.3), and the Gagliardo-Nirenberg inequality [21]. Case $dp = 2$ is called *mass-critical*. Here, local well-posedness is guaranteed in $H^1(\mathbb{R}^d)$, however global existence is available only in $L^2(\mathbb{R}^d)$ for sufficiently small initial data, and the solution might have a finite-time blow-up for large-norm initial conditions, or for initial data with negative energy $E_f(w)$ [57, 46, 60, 16]. Finally, the case $dp > 2$ is called *mass-supercritical*. Here, scaling (1.13) shows that the time evolution of (1.10) may break in a finite time for arbitrary small L^2-initial data [16].

When $dp > 2$, we need to distinguish between three additional cases, this time for the energy. Local well-posedness in $H^1(\mathbb{R}^d)$ is available, as long as $(d-2)p < 2$, or $(d-2)p = 2$ which correspond to the *energy-subcritical*, and *energy-critical* cases, respectively [10, 9]. The case $(d-2)p > 2$ for $d \geq 3$ is referred to as the *energy-supercritical* case. If $d = 1, 2$, or if $d \geq 3$ and $(d-2)p < 2$, solutions of (1.10) exist globally, as long as the initial condition is sufficiently small with respect to the H^1-norm [16]. In the energy-critical case and dimensions $d = 3, 4, 5$, it has been shown in [35] that solutions of the radial Cauchy problem for (1.10) are global and scatter, under the assumption that both the $\dot{H}^1$-norm and energy of the initial condition are sufficiently small. For $d \geq 3$, well-posedness of (1.10) in $H^1(\mathbb{R}^d)$ is lost for sufficiently large values of p. This is an implication of reaching the Sobolev conjugate exponent in the continuous Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, valid for any $2 \leq q \leq \frac{2d}{d-2} =: 2^*$. In our case, $q = 2p+2$, as seen in (1.8), so that $2p + 2 = \frac{2d}{d-2} \iff (d-2)p = 2$. To remedy this, local well-posedness in the energy-supercritical case $(d-2)p > 2$ is obtained in $H^s(\mathbb{R}^d)$, where $s > \frac{dp-2}{2p} > 1$, and the initial data for the Cauchy problem is taken in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ [60]. We also refer to [41] for a more detailed blow-up study for the energy-supercritical NLS.

1.3 Ground states with the harmonic potential

The terminology and classification regarding criticality of mass and energy in free NLS (1.10) is also used in the context of (1.1). Although the harmonic potential does not prevent blowup in finite time, it creates a possibility of existence of the standing waves (ground states). The linear part of the boundary-value problem (1.7) has the ground state given by the Gaussian function $\mathfrak{u}_0 = e^{-\frac{r^2}{2}}$, which exists when the eigenvalue parameter λ is given by $\lambda_0 := d$. By standard local bifurcation theory [42], there exists a nonlinear ground state $\mathfrak{u}_b(r)$ with small amplitude $b := \|\mathfrak{u}_b(0)\|_\infty$ that bifurcates from the linear ground state $\mathfrak{u}_0$ as $b \to 0$, with $\lambda < \lambda_0$ sufficiently close to the bifurcation point. The nonlinear ground state corresponds to $\lambda = \lambda(b)$ satisfying the limit $\lambda(b) \to \lambda_0$ as $b \to 0$, with the following asymptotics:

$$\mathfrak{u}_b(r) = b\mathfrak{u}_0(r) + \mathcal{O}(b^3), \quad \lambda(b) = d - 2^{-\frac{d}{2}} b^2 + \mathcal{O}(b^4), \quad \text{as } b \to 0. \tag{1.15}$$

One of the main objectives of our work is the study of the bifurcation curve in the (λ, b) plane as $b \to \infty$.

The behavior of the bifurcation curve depends on criticality of the energy functional. In the energy-subcritical case $(d-2)p < 2$, global behavior of the solution curve can be analyzed by using variational methods and global bifurcation theory [49] due to the fact that the Sobolev embedding $H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d) \subset L^{2p+2}(\mathbb{R}^d)$ is compact, where $L^{2,1}(\mathbb{R}^d)$ is defined as the space of functions $u : \mathbb{R}^d \to \mathbb{R}$, such that $|\cdot|u \in L^2(\mathbb{R}^d)$. Existence of solutions to (1.7) has been shown [34, 20, 52] for every $\lambda < d$. Uniqueness was shown in [29] for $d \geq 3$, and in [30] for $d = 1, 2$ (note that $d = 1, 2$ correspond to the energy-subcritical case for any $p > 0$).

In the limiting energy-critical case $(d-2)p = 2$, the Sobolev embedding is still compact, so that the global bifurcation theory applies. However, the admissible values of λ are further restricted by using Pohozaev [48] identity

$$4\|ru\|^2_{L^2_r} - 2\lambda\|u\|^2_{L^2_r} + \frac{(d-2)p-2}{p+1}\|u\|^{2p+2}_{L^{2p+2}_r} = 0, \tag{1.16}$$

which holds for every solution of (1.7) (for the proof of (1.16) see the proof of Proposition 2.2 in Section 2.2). Equation (1.16) shows that in the energy-critical case $(d-2)p = 2$, no nonzero solutions of (1.7) exist if $\lambda \leq 0$. In fact, it was proven in [52] that $\lambda \in (0, d)$ if $d \geq 4$ and $\lambda \in (1, 3)$ if $d = 3$, for every solution of (1.7). In this context, the pioneering work of Brezis and Nirenberg [5] in 1983 was the first breakthrough in characterizing ground states in the energy-critical case. The variational arguments applied there in the context of free NLS served as a basis for an extension to NLS with harmonic potential proposed in [52], where existence was proven for $d \geq 3$. Additionally, numerical evidence provided there served as a basis for more rigorous theory in [53], where it was shown that the solution branch in the (λ, b) plane becomes unbounded. Uniqueness of solutions to (1.7) was proven in more general setting in [55].

In the energy-supercritical case $(d-2)p > 2$, variational approach is no longer available, so that different methods need to be developed. Identity (1.16) shows that $\lambda > 0$ for every solution of (1.7). Furthermore, it was shown in [53] that solutions of (1.7) exist for the values of λ in a smaller subset of $(0, d)$. A much more striking property of the energy-supercritical case is that ground states $\mathfrak{u}_b$ converge to a *limiting singular solution* $\mathfrak{u}_\infty$ along the bifurcation curve in the (λ, b) plane, as $b \to \infty$. Existence of such $\mathfrak{u}_\infty \in \mathcal{E}$, $\mathfrak{u}_\infty \notin L^\infty$ was proven in [54]. The limiting solution satisfies the stationary NLS (1.6) for some $\lambda = \lambda_\infty \in (0, d)$, and diverges at the origin according to

$$\mathfrak{u}_\infty(r) = \frac{1}{p}\left(d - 2 - \frac{1}{p}\right)^{\frac{1}{2p}} r^{-\frac{1}{p}} \left(1 + \mathcal{O}(r^2)\right), \quad \text{as } r \to 0. \tag{1.17}$$

Convergence $\mathfrak{u}_b \to \mathfrak{u}_\infty$ as $b \to \infty$ in the energy space was proven in [54] by modifying arguments used in [40] in the context of stationary NLS in a ball and without a harmonic potential. Specifically, in [40] the following problem was considered:

$$\begin{cases} \Delta u + \nu u + |u|^{2p} u = 0, & x \in B, \\ u > 0, & x \in B \\ u = 0, & x \in \partial B, \end{cases} \tag{1.18}$$

where $\nu > 0$, $(d-2)p > 2$, and B denotes the unit ball in $\mathbb{R}^d$, $d \geq 3$. It was shown in [40] that there exists a unique value $\nu = \nu^*$, such that problem (1.18) has a unique radial singular solution with the same singular behaviour as in (1.17). Moreover, it was proven in [15, 25] that for ν sufficiently close to ν^*, problem (1.18) has a large number of classical solutions, as long as either $d \leq 10$, or $p < \frac{2}{d-2\sqrt{d-1}-4}$ if $d > 10$. These considerations, together with numerical evidence provided in [52], suggest that a similar phenomenon could occur in unbounded domains, and in the presence of harmonic potential, as in (1.7).

The nature of convergence of $\lambda(b)$ to λ_∞ is heavily related to the spatial dimension d. In fact, one can show that $\lambda(b)$ oscillates around λ_∞ as $b \to \infty$ for $2 + \frac{2}{p} < d < d_*(p)$, and converges to it monotonically for $d > d_*(p)$, where

$$d_*(p) := 6 + \frac{2}{p} + 2\sqrt{4 + \frac{2}{p}}. \tag{1.19}$$

This critical value of $d_*(p)$ is in agreement with the results of [40], as $d = d_*(p) \iff p = \frac{2}{d-2\sqrt{d-1}-4}$. The relation between d and p is visualized in Figure 1.1. In literature, this oscillating dependence of $\lambda(b)$ is also known as *snaking behavior*. Such dichotomy between snaking and monotone dependencies has been previously discovered for the classical Liouville-Bratu-Gelfand problem [32] in [33]. The methods used in [33] were based on Emden-Fowler transformation [18] and a rigorous shooting method combined with geometric approach, including phase plane analysis.

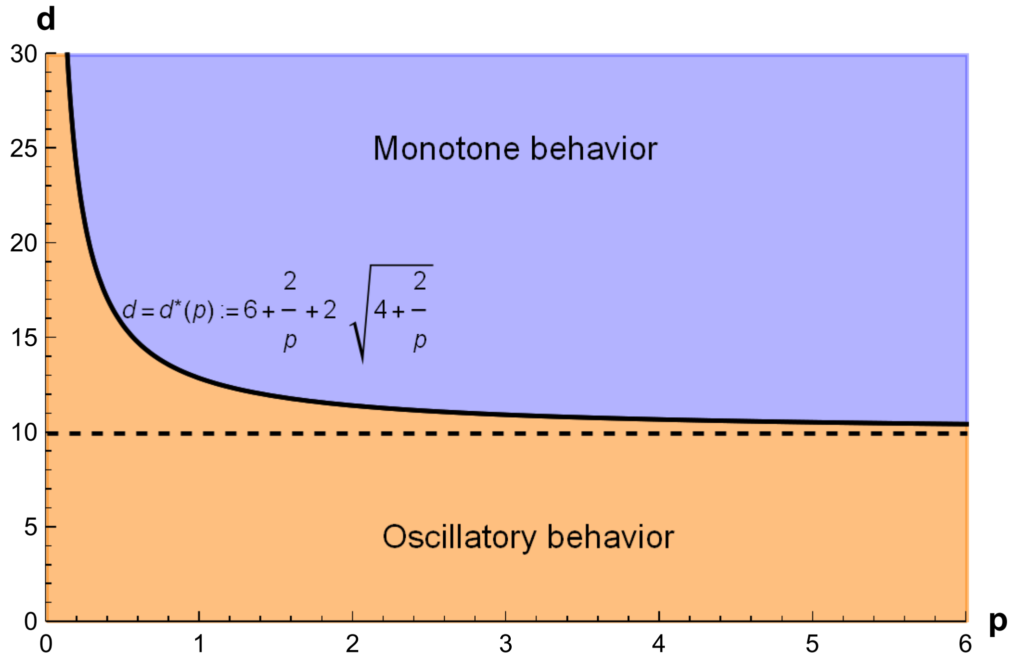

FIGURE 1.1: Boundary between regions with oscillatory versus monotone convergence of $\lambda(b)$ to λ_∞ as an interplay between dimension d and power of nonlinearity p.

1.4 Morse index

Knowing the monotonicity of $\lambda(b)$ is especially important when studying Morse index of the ground state. In this context, it is understood as the number of negative eigenvalues of the linearized operator $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$, defined as

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + r^2 - \lambda(b) - (2p+1)|\mathfrak{u}_b(r)|^{2p}, \tag{1.20}$$

where $\mathcal{E}^*$ denotes the dual space of $\mathcal{E}$. Note that if we assume C^1 dependence of $\mathfrak{u}_b$ and $\lambda(b)$ on b, then differentiating the ODE in (1.7) on $\mathfrak{u}_b$ with respect to b results in the equation $\mathcal{L}_b \partial_b \mathfrak{u}_b = \lambda'(b)\mathfrak{u}_b$, with $\partial_b \mathfrak{u}_b \in \mathcal{E}$. Thus, for every $b > 0$ such that $\lambda'(b) = 0$, the operator $\mathcal{L}_b$ has a zero eigenvalue present in its spectrum, with the corresponding eigenfunction $\partial_b \mathfrak{u}_b$. This shows that monotonicity of $\lambda(b)$ is closely related to spectral properties of $\mathcal{L}_b$. Specifically, it is reasonable to expect that the operator $\mathcal{L}_b$ gains an additional negative eigenvalue whenever $\lambda(b)$ passes a turning point for increasing values of b. These considerations become relevant in the energy-supercritical case, where the snaking behavior of $\lambda(b)$ is replaced by the monotone behavior, see Figure 1.1.

Knowing the Morse index of $\mathfrak{u}_b$ can provide us with information about stability of the ground state. For physical applications it is particularly important to know whether ground states survive in the time evolution under small perturbations. From the dynamical systems point of view, we can look at ground states as critical points of the augmented Hamiltonian $\Lambda_\lambda(w) := E(w) - \lambda M(w)$ associated with the NLS (1.1). Generally, we would expect that those critical points are not stable in time evolution in the usual sense, as they are saddle points. However, if the mass $M(w)$ in (1.2) is conserved independently of the energy $E(w)$ in (1.3), then the critical points could be stable if they are constrained minimizers of energy for fixed mass. Because the Gross-Pitaevskii equation (1.1) possesses a rotation symmetry: if w is a solution, so is $T(\theta)w$ with $T(\theta) = e^{i\theta}$ for $\theta \in \mathbb{R}$, stability of critical points has to be defined as the orbital stability of the family (orbit) $\{T(\theta)\mathfrak{u}_b\}_{\theta\in\mathbb{R}}$. We are interested whether solutions of (1.1) that are initially close to ground state $\mathfrak{u}_b$ in the energy space, stay close to the orbit generated by $\mathfrak{u}_b$. We provide the rigorous definition below.

Definition 1.1. *We say that the ground state $\mathfrak{u}_b$ is orbitally stable in the Banach space $(X, \|\cdot\|_X)$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|w(0) - \mathfrak{u}_b\|_X < \delta$, then*

$$\inf_{\theta\in\mathbb{R}} \|w(t,\cdot) - T(\theta)\mathfrak{u}_b\|_X < \epsilon, \tag{1.21}$$

for every $t > 0$. Otherwise, $\mathfrak{u}_b$ is orbitally unstable in X.

In our case, the Banach space X to work in is the energy space $\mathcal{E}_{NLS}$, which reduces to $\mathcal{E}$ under the radial symmetry assumption. Operator $\mathcal{L}_b$ defined by (1.20) represents the Hessian operator of the augmented Hamiltonian $\Lambda_\lambda(w)$ at $w = \mathfrak{u}_b$. We look at the spectral properties of $\mathcal{L}_b$ in the orthogonal complement of the ground state $\{\mathfrak{u}_b\}^\perp$, which from the variational point of view can be seen as imposing the constraint $M(w) = M_0$ for some $M_0 > 0$. The following theorem presents the orbital stability result under constrained mass.

Theorem 1.1 (Theorem 1 in [23])**.** *Suppose that the Cauchy problem for* (1.1) *is locally well-posed in $\mathcal{E}$, and that*

$$n(\mathcal{L}_b|_{\{\mathfrak{u}_b\}^\perp}) = 0, \quad z(\mathcal{L}_b|_{\{\mathfrak{u}_b\}^\perp}) = 1, \tag{1.22}$$

where $n(\mathcal{L}_b|_{\{\mathfrak{u}_b\}^\perp})$ denotes the number of negative eigenvalues, and $z(\mathcal{L}_b|_{\{\mathfrak{u}_b\}^\perp})$ denotes the multiplicity of the zero eigenvalue, of operator (1.20) *in the orthogonal complement of $\mathfrak{u}_b$. Then, $\mathfrak{u}_b$ is orbitally stable in time evolution of* (1.1)*.*

Remark 1.1. *Note that one of the requirements of Theorem 1.1 is local well-posedness of the Cauchy problem for* (1.1)*. As discussed in Section 1.2 in the context of free NLS, local well-posedness in the energy-supercritical case is not necessarily given a priori and needs to be considered as a problem of its own.*

One useful way of verifying spectral condition (1.22) in Theorem 1.1 is the so-called *Vakhitov-Kolokolov criterion*, which ensures that the condition (1.22) is satisfied if the Morse index is one and the mapping $\lambda \mapsto \|\mathfrak{u}_b\|^2_{L^2_r}$ is monotonically decreasing.

Theorem 1.2 ([59]). *Suppose that the mapping $\lambda \mapsto M(\mathfrak{u}_b)$ is C^1. Under the setting of Theorem 1.1, if $n(\mathcal{L}_b) = z(\mathcal{L}_b) = 1$, then*

$$\partial_\lambda M(\mathfrak{u}_b) < 0 \iff n(\mathcal{L}_b|_{\{\mathfrak{u}_b\}^\perp}) = 0, \quad z(\mathcal{L}_b|_{\{\mathfrak{u}_b\}^\perp}) = 1. \tag{1.23}$$

1.5 Outline of the thesis

Here, we give a brief overview of the content of all the chapters.

- In **Chapter 2**, we first define and describe basic properties of the linear quantum harmonic oscillator on the real line. We provide formulas for its eigenvalues and eigenfunctions, and show how to obtain them from the *ladder operator method*. Then, for the nonlinear NLS with harmonic potential (1.6), we derive preliminary bounds on the spectral parameter λ for the existence of the ground states.

- In **Chapter 3**, we consider the principal boundary-value problem (1.7) for ground states in the energy-supercritical case. We describe how a rigorous shooting method is used in order to prove existence of a family of ground states $\{\mathfrak{u}_b\}_{b>0}$, with the corresponding values $\lambda = \lambda(b)$ of the spectral parameter, and b defined as *the amplitude*: $b := \|\mathfrak{u}_b\|_\infty = \mathfrak{u}_b(0)$. Moreover, we analyze the behavior of this family as $b \to \infty$. We prove that there exists a *limiting singular solution* $\mathfrak{u}_\infty$, and a corresponding value $\lambda = \lambda_\infty$, such that $\mathfrak{u}_b \to \mathfrak{u}_\infty$ and $\lambda(b) \to \lambda_\infty$ as $b \to \infty$. Existence of $\mathfrak{u}_\infty$ and λ_∞ was previously shown in [54], however we provide an alternative proof of this result.

- In **Chapter 4**, we continue to work with the energy-supercritical case for (1.7). Having a family of ground states $\{\mathfrak{u}_b\}_{b>0}$, and the corresponding values of $\lambda = \lambda(b)$, we study the precise asymptotic behaviour of $\lambda(b)$ in the convergence $\lambda(b) \to \lambda_\infty$ as $b \to \infty$. We prove that for large values of b the convergence is oscillatory for dimensions $5 \leq d \leq 12$, and that it is monotone for $d \geq 13$ if the nonlinearity power is fixed at $p = 1$.

  The content of Chapters 3 and 4 is based on [4]: *P. Bizon, F. Ficek, D. E. Pelinovsky, and S. Sobieszek, "Ground state in the energy super-critical Gross-Pitaevskii equation with a harmonic potential", Nonlinear Analysis 210 (2021) 112358.*

- In **Chapter 5**, we study the Morse index of the ground state $\{\mathfrak{u}_b\}$ of (1.7) in the energy-supercritical case and the cubic nonlinearity ($p = 1$). We focus on the case of monotone behaviour of $\lambda(b)$ which, based on the results from Chapter 4, corresponds to $d \geq 13$. We prove that for sufficiently large values of b, the Morse index of $\mathfrak{u}_b$ is finite and independent of b. Moreover, we show that the Morse index of the ground state coincides with the Morse index of the limiting singular solution for sufficiently large values of b.

The content of Chapter 5 is based on [45]: *D. E. Pelinovsky and S. Sobieszek, "Morse index for the ground state in the energy supercritical Gross-Pitaevskii equation", Journal of Differential Equations 341 (2022) 380-401.*

- In **Chapter 6** we study the energy-critical case for (1.1), and general power $p > 0$. We consider the family of ground states $\{\mathfrak{u}_b\}_{b>0}$, as solutions of (1.7), and the corresponding values $\lambda = \lambda(b)$ of the spectral parameter. We explore the shooting method applied previously in the energy-supercritical case, to show that for sufficiently large $b > 0$ the ground state is pointwise close near the origin to the Aubin-Talenti solution of the energy-critical wave equation, and to the confluent hypergeometric function for large values of the argument. Moreover, we derive precise asymptotic behavior of $\lambda(b)$ as $b \to \infty$, for $p \in (0, 1)$. We explain why our method cannot be used for $p = 1$, and suggest its possible extensions.

  The content of Chapter 6 is based on [44]: *D. E. Pelinovsky and S. Sobieszek, "Ground state of the Gross-Pitaevskii equation with a harmonic potential in the energy-critical case", arXiv:2302.03865 (2023).*

1.6 Future study

We list several open problems which arise naturally from the results discussed in this thesis.

- Existence of ground states in the energy-critical and energy-supercritical cases is discussed in Chapter 3. In both cases, we have shown that for any fixed value $b := \|\mathfrak{u}_b\|_\infty$, there exists a corresponding λ which makes $\mathfrak{u}_b$ a solution to (1.7). Even though our numerical evidence suggests uniqueness of such λ, this assertion is still an open problem. Similarly, in the energy-supercritical case, it remains to prove uniqueness of λ_∞ corresponding to the limiting singular solution $\mathfrak{u}_\infty$.
- The precise asymptotic behavior of $\lambda(b)$ in the energy-supercritical case was obtained in Chapter 4, where we distinguish between snaking (for small dimensions) and monotone (for large dimensions) dependencies. The results were proven for $p = 1$ in (1.7). We believe that generalization of these results for any $p > 0$ can be obtained by a straightforward modification of the arguments from Chapter 4.
- Morse index of the family of ground states $\{\mathfrak{u}_b\}_{b>0}$ was studied in Chapter 5 in the energy-supercritical case and for the monotone behavior of the spectral parameter $\lambda(b)$. It remains to study the Morse index in the case of the oscillatory dependence of $\lambda(b)$. We believe that the linearized operator $\mathcal{L}_b$ gains an additional zero eigenvalue at each turning point of $\lambda(b)$, making the Morse index of $\mathfrak{u}_b$ infinite, see [43] for details. Moreover, the Morse index in Chapter 5 was studied for radially symmetric perturbations. It is interesting to study the number of negative eigenvalues of the linearized operator for general (non radially-symmetric) perturbations.
- The behavior of the spectral parameter $\lambda(b)$ as $b \to \infty$ was studied in Chapter 6 for the general power nonlinearity $p > 0$ in (1.7). The shooting method developed

by us can be used for any $p \in (0, 1)$, but fails for $p = 1$ (for details see Remark 6.2). We believe that the invariant manifold approach is still viable for $p \geq 1$, however it requires certain modifications. For a more detailed explanation, we refer to the discussion after the proof of Theorem 6.1 in Section 6.4.

- We believe that the tools developed in this dissertation in the context of the Gross-Pitaevskii equation could be used in order to study other models of theoretical physics, such as the Schrödinger-Newton-Hooke system, considered in [17].

Chapter 2

Preliminaries

2.1 Linear quantum harmonic oscillator

In this section, we review the basic properties of the one-dimensional linear harmonic oscillator operator H:

$$H := -\frac{d^2}{dx^2} + x^2, \tag{2.1}$$

where $x \in \mathbb{R}$, which plays a fundamental role in the theory of quantum mechanics.

Remark 2.1. *Note that it is sufficient to consider the one-dimensional case* (2.1)*, as the problem posed in* $\mathbb{R}^d$

$$(-\Delta + |x|^2)u = 0, \quad x \in \mathbb{R}^d,$$

decomposes into d *one-dimensional problems using separation of variables.*

Usually, we look for eigenfunctions of H in the space $H^2(\mathbb{R}) \cap L^{2,2}(\mathbb{R})$, where $L^{2,2}(\mathbb{R})$ is the space of functions $u : \mathbb{R} \to \mathbb{R}$ such that $|\cdot|^2 u \in L^2(\mathbb{R})$. This space is compactly embedded in $L^2(\mathbb{R})$, which makes densely defined operator (2.1) essentially self-adjoint, meaning that it is symmetric and its closure is self-adjoint. Such operators admit a unique extension that is self-adjoint, so that the spectral theorem applies. In particular, we know that the spectrum of (2.1) in $L^2(\mathbb{R})$ is real and purely discrete. Moreover, it is well-known, see [28], that the spectrum consists of eigenvalues $\{E_n\}_{n\in\mathbb{N}_0}$ given by $E_n = 2n + 1$. The corresponding eigenfunctions $\{\phi_n\}_{n\in\mathbb{N}_0}$ form an orthogonal basis for $L^2(\mathbb{R})$, and are given by

$$\phi_n(x) = H_n(x)e^{-\frac{x^2}{2}}, \tag{2.2}$$

where $H_n(x)$ are Hermite polynomials of degree $n \in \mathbb{N}_0$. Note that $\phi_n(x) \in S(\mathbb{R}, \mathbb{R})$ for all $n \in \mathbb{N}_0$, where $S(\mathbb{R}, \mathbb{R})$ is the Schwartz space of real-valued functions defined on the real line. The eigenvalues E_n in the physical context are called *energy levels*, and the eigenfunction $\phi_0(x)$ (corresponding to the lowest energy level) is called the *ground state*. For $n \geq 1$, $\phi_n(x)$ are referred to as *excited states*.

Knowing $\phi_0(x) = e^{-\frac{x^2}{2}}$, eigenfunctions (2.2) can be obtained by using the so called *ladder operator method*. Following [28] we define two operators, L^+ (called *creation*, or *raising* operator) and L^- (called *annihilation*, or *lowering* operator), as

$$L^{\pm} := \mp \frac{d}{dx} + x, \tag{2.3}$$

where we consider $S(\mathbb{R}, \mathbb{R})$ as the domain for both L^+ and L^-. Note that for any $f \in S(\mathbb{R}, \mathbb{R})$, we have

$$(L^+L^- + \mathbb{I})f = -f'' + xf' - f - xf' + x^2 f + f = Hf,$$

so that $H = L^+L^- + \mathbb{I}$. Thus, we also have

$$HL^+ = L^+(H + 2\mathbb{I}). \tag{2.4}$$

Using above identity, we see that $L^+\phi_0$ is the next eigenfunction of (2.1), with the eigenvalue $E_1 = 3$:

$$H(L^+\phi_0) = L^+(H + 2\mathbb{I})\phi_0 = 3L^+\phi_0,$$

and in general

$$\phi_n = (L^+)^n \phi_0, \quad E_n = 2n + 1, \quad n \geq 1. \tag{2.5}$$

Thus, Hermite polynomials in (2.2) appear naturally by recursively applying the creation operator L^+, and orthonormalizing the resulting set of eigenfunctions.

Remark 2.2. *By a symmetric identity to* (2.4)*,*

$$HL^- = L^-(H - 2\mathbb{I}),$$

we can see that for any $n \geq 1$

$$HL^-\phi_n = L^-(H - 2\mathbb{I})\phi_n = (E_n - 2)L^-\phi_n = E_{n-1}L^-\phi_n.$$

Thus, the terminology for operators $L^{\pm}$ becomes apparent: L^+ can be used in order to move to higher energy levels of operator (2.1)*, whereas L^- acts in the opposite manner.*

2.2 Bounds on the spectral parameter λ

In this section, we collect three well-known results regarding existence of nontrivial solutions to the boundary-value problem (1.7) in the energy space $\mathcal{E}$.

Proposition 2.1. *For every $d \geq 1$ and $\lambda \in [d, \infty)$, no solutions of the boundary-value problem (1.7) exist in $\mathcal{E}$.*

Proof. It is well known (see, e.g., [28]) that the operator $L_0 := -\Delta + |x|^2$ is self-adjoint in $L^2(\mathbb{R}^d)$. The ground state of L_0 is given up to a normalization by the Gaussian function $\mathfrak{u}_0(r) = e^{-\frac{1}{2}r^2}$ and corresponds to the smallest eigenvalue $\lambda_0 = d$. The linear ground state $\mathfrak{u}_0$ satisfies the following boundary value problem:

$$\begin{cases} \mathfrak{u}_0''(r) + \frac{d-1}{r}\mathfrak{u}_0'(r) - r^2\mathfrak{u}_0(r) = -d\mathfrak{u}_0(r), & r > 0, \\ \mathfrak{u}_0(r) > 0, \qquad \mathfrak{u}_0'(r) < 0, & r > 0, \\ \lim_{r\to 0} \mathfrak{u}_0(r) < \infty, \quad \lim_{r\to\infty} \mathfrak{u}_0(r) = 0. \end{cases} \tag{2.6}$$

By projecting (1.7) to $\mathfrak{u}_0$ and integrating by parts with the use of (2.6), we obtain:

$$-d\langle \mathfrak{u}_0, \mathfrak{u}\rangle_{L^2_r} + \lambda\langle \mathfrak{u}_0, \mathfrak{u}\rangle_{L^2_r} + \langle \mathfrak{u}_0, \mathfrak{u}^3\rangle_{L^2_r} = 0$$

which implies

$$d - \lambda = \frac{\langle \mathfrak{u}_0, \mathfrak{u}^3\rangle_{L^2_r}}{\langle \mathfrak{u}_0, \mathfrak{u}\rangle_{L^2_r}}.$$

Since $\langle \mathfrak{u}_0, \mathfrak{u}\rangle_{L^2_r} > 0$ and $\langle \mathfrak{u}_0, \mathfrak{u}^3\rangle_{L^2_r} > 0$, we must have $\lambda < d$ for every solution $\mathfrak{u} \in \mathcal{E}$ of the boundary-value problem (1.7). □

Proposition 2.2. *For every $d \geq 4$ and $\lambda \in (-\infty, d-4]$, no solutions of the boundary-value problem (1.7) exist in $\mathcal{E}$.*

Proof. It follows from multiplication of (1.7) by $r^{d-1}\mathfrak{u}$ that if $\mathfrak{u} \in \mathcal{E}$, then

$$\|\mathfrak{u}'\|^2_{L^2_r} + \|r\mathfrak{u}\|^2_{L^2_r} - \lambda\|\mathfrak{u}\|^2_{L^2_r} - \|\mathfrak{u}\|^4_{L^4_r} = 0. \tag{2.7}$$

Similarly, it follows from multiplication of (1.7) by $r^d\mathfrak{u}'(r)$ and integration by parts that

$$(d-2)\|\mathfrak{u}'\|^2_{L^2_r} + (d+2)\|r\mathfrak{u}\|^2_{L^2_r} - \lambda d\|\mathfrak{u}\|^2_{L^2_r} - \frac{1}{2}d\|\mathfrak{u}\|^4_{L^4_r} = 0. \tag{2.8}$$

Combining (2.7) and (2.8) yields the Pohozaev identity [48]:

$$4\|r\mathfrak{u}\|^2_{L^2_r} - 2\lambda\|\mathfrak{u}\|^2_{L^2_r} + \frac{1}{2}(d-4)\|\mathfrak{u}\|^4_{L^4_r} = 0. \tag{2.9}$$

Hence, no nonzero solution $\mathfrak{u} \in \mathcal{E}$ exists if $\lambda \leq 0$ and $d \geq 4$.

Furthermore, since d is the lowest eigenvalue of $L_0 = -\Delta + r^2$, we obtain similarly to [5]:

$$d\|\mathfrak{u}\|^2_{L^2} \leq \|\mathfrak{u}'\|^2_{L^2} + \|r\mathfrak{u}\|^2_{L_2} = \lambda\|\mathfrak{u}\|^2_{L^2} + \|\mathfrak{u}\|^4_{L^4}. \tag{2.10}$$

If $d \neq 4$, then $\|\mathfrak{u}\|^4_{L^4}$ can be expressed by using (2.9), after which inequality (2.10) yields

$$\lambda \geq d - 4 + \frac{8}{d}\frac{\|r\mathfrak{u}\|^2_{L_2}}{\|\mathfrak{u}\|^2_{L_2}}, \tag{2.11}$$

hence no nonzero solution $\mathfrak{u} \in \mathcal{E}$ exists if $\lambda \leq d-4$. □

Proposition 2.3. *For every $d \geq 1$, there exists a unique solution of the boundary-value problem (1.7) in $\mathcal{E} \cap L^\infty$ for $\lambda \in (d - \delta, d)$ with sufficiently small δ such that $\|\mathfrak{u}\|_{L^\infty_r} \to 0$ as $\lambda \to d$.*

Proof. For $1 \leq d \leq 4$, the proof follows by the standard Lyapunov–Schmidt theory (see Theorem 2.1 in [52] and references therein). For $d \geq 5$, the proof follows by the compactification of the nonlinear term for the standard Lyapunov–Schmidt theory and by the Moser's iteration argument to control the L^∞-norm of the bifurcating solution and thus the nonlinear term (see Theorem 5 in [53] and references therein). □

Chapter 3

Existence of the ground states

In this section, we prove existence of a family of ground states, i.e., solutions to (1.7), in the energy-critical and supercritical cases. For simplicity, we assume $p = 1$, however we believe that our methods could be easily modified and used for any $d \geq 3$ and $p > 0$ satisfying $(d-2)p \geq 2$. Note that for integer values of the dimension d, $p = 1$ implies $d \geq 4$.

Our main strategy consists of studying solutions to the following initial-value problem:

$$\begin{cases} f''(r) + \frac{d-1}{r} f'(r) - r^2 f(r) + \lambda f(r) + f(r)^3 = 0, & r > 0, \\ f(0) = b, \quad f'(0) = 0, \end{cases} \tag{3.1}$$

where $b \in \mathbb{R}$ is a free parameter (assumed to be positive without loss of generality) by means of the shooting method pioneered in [33].

We first prove that for each $b > 0$ and each $\lambda \in \mathbb{R}$, there exists a unique global classical solution to the initial-value problem (3.1); moreover, there exists $\lambda = \lambda(b) \in (d-4, d)$ such that the corresponding solution f decays to zero at infinity, so that it gives the ground state $\mathfrak{u} = \mathfrak{u}_b$ of the boundary-value problem (1.7). The following theorem presents this result.

Theorem 3.1. *Fix $d \geq 4$. For every $b > 0$, there exists $\lambda \in (d-4, d)$, labeled as $\lambda(b)$, such that the unique classical solution $f \in C^2(0, \infty)$ to the initial-value problem (3.1) with $\lambda = \lambda(b)$ is a solution $\mathfrak{u} = \mathfrak{u}_b \in \mathcal{E}$ to the boundary-value problem (1.7).*

Remark 3.1. *Uniqueness of λ in Theorem 3.1 for each given $b > 0$ is an open problem.*

Remark 3.2. *We believe that the shooting argument used to prove Theorem 3.1 can be generalized to prove the existence of the n-th excited state with n nodes on $\mathbb{R}^+$ for some $\lambda \in (\lambda_n - 4, \lambda_n)$, where $\lambda_n := d + 4n$ is the n^{th} eigenvalue of the linear problem, $n \in \mathbb{N}$. Such solutions were also considered in [52].*

Next, following [54], for $d \geq 5$ we introduce the limiting singular solution f_∞, which solves the ODE in (3.1), and is connected to the family $\{\mathfrak{u}_b\}_{b>0}$ by a convergence result from [54]. Specifically, it was shown in [54] that f_∞ exists for a unique value λ_∞ of λ, and that $\mathfrak{u}_b \to f_\infty$ in $\mathcal{E}$ and $\lambda(b) \to \lambda_\infty$ as $b \to \infty$. The limiting singular solution f_∞ is defined by the following divergent behavior:

$$f_\infty(r) = \frac{\sqrt{d-3}}{r}\left[1 + \mathcal{O}(r^2)\right] \qquad \text{as} \quad r \to 0. \tag{3.2}$$

Note that f_∞ corresponds to $\mathfrak{u}_\infty$ in (1.17) for $p = 1$. If $f_\infty \in C^2(0,\infty)$ and f_∞ decays to zero at infinity fast enough, then $f_\infty \in \mathcal{E}$ for $d \geq 5$.

The limiting singular solution f_∞ can be introduced by the change of variables $f(r) = r^{-1}F(r)$, where $F(r)$ is defined as a bounded solution to the following initial value problem:

$$\begin{cases} F''(r) + \frac{d-3}{r}F'(r) - \frac{d-3}{r^2}F(r) - r^2F(r) + \lambda F(r) + \frac{1}{r^2}F(r)^3 = 0, \quad r > 0, \\ F(0) = \sqrt{d-3}, \quad F'(0) = 0. \end{cases} \tag{3.3}$$

For each $\lambda \in \mathbb{R}$, there exists a unique global classical solution to the initial value problem (3.3), moreover, there exists a value of λ denoted as λ_∞ such that the corresponding solution F decays to zero at infinity. This decaying solution F gives the limiting singular solution f_∞ after the transformation $f(r) = r^{-1}F(r)$. The following theorem was proven in [54].

Theorem 3.2. *Fix $d \geq 5$. There exists a value of $\lambda \in (0, d)$, labeled as λ_∞, such that the unique classical solution $F \in C^2(0,\infty)$ to the initial-value problem (3.3) with $\lambda = \lambda_\infty$ satisfies $F(r) > 0$ and $F'(r) < 0$ for every $r > 0$ and $F(r) \to 0$ as $r \to \infty$ such that $f_\infty(r) = r^{-1}F(r)$ belongs to $\mathcal{E}$.*

Remark 3.3. *Uniqueness of the value of λ_∞ in Theorem 3.2 was claimed in [54, Section 4] by analyzing the behavior of the quotient between two hypothetical solutions of (3.3) for two different values of λ. However, we believe the proof is incorrect, see Remark 3.8 below.*

Remark 3.4. *The proof of Theorem 3.1 is similar to the proof of Theorem 3.2 but we have to work with the different initial-value problem (3.1) compared to (3.3). We also prove the fast decay to zero at infinity and this allows us to simplify some arguments from [54].*

3.1 Existence of bounded solutions at the origin

Here we consider the differential equation

$$f''(r) + \frac{d-1}{r}f'(r) - r^2f(r) + \lambda f(r) + f(r)^3 = 0, \quad r > 0, \tag{3.4}$$

and prove several results regarding existence of classical solutions to this differential equation.

The first result shows that the additional condition $f'(0) = 0$ does not over-determine the initial-value problem (3.1) at the singularity point $r = 0$ as long as the classical solution $f(r)$ to the differential equation (3.4) is bounded as $r \to 0$.

Lemma 3.1. *For every $d \geq 1$ and every $\lambda \in \mathbb{R}$, assume that there exists a classical solution $f \in C^2(0, r_0)$, $r_0 > 0$ to the differential equation (3.4) such that $f(0) := \lim_{r\to 0} f(r) < \infty$. Then,*

$$f'(0) := \lim_{r\to 0} f'(r) = 0.$$

Proof. One can rewrite the differential equation (3.4) in the self-adjoint form:

$$\frac{d}{dr}\left[r^{d-1} f'(r)\right] = r^{d-1}\left[r^2 f(r) - \lambda f(r) - f(r)^3\right]. \tag{3.5}$$

The right-hand side of (3.5) is integrable as $r \to 0$ if f is bounded near $r = 0$. Then $\lim_{r\to 0} r^{d-1} f'(r) = 0$ and integration of (3.5) on $[0, r]$ yields

$$f'(r) = \frac{1}{r^{d-1}} \int_0^r s^{d-1}\left[s^2 f(s) - \lambda f(s) - f(s)^3\right] ds. \tag{3.6}$$

Since $\lim_{r\to 0} f'(r)$ is an indeterminate form $\left[\frac{0}{0}\right]$, we can apply L'Hospital's rule and obtain

$$\lim_{r\to 0} f'(r) = \lim_{r\to 0} \frac{r^{d-1}\left[r^2 f(r) - \lambda f(r) - f(r)^3\right]}{(d-1)\, r^{d-2}} = \frac{1}{d-1} \lim_{r\to 0} r\left[r^2 f(r) - \lambda f(r) - f(r)^3\right] = 0,$$

since f is bounded near $r = 0$. Hence, $f'(0) := \lim_{r\to 0} f'(r) = 0$. □

Remark 3.5. *A similar result but for $d \geq 5$ can be stated about the initial-value problem (3.3). If $F(0) = \sqrt{d-3}$ for a classical solution $F \in C^2(0, r_0)$ with $r_0 > 0$, then $F'(0) = 0$. Indeed, the differential equation in the initial-value problem (3.3) can be written in the self-adjoint form:*

$$\frac{d}{dr}\left[r^{d-3} F'(r)\right] = r^{d-5}\left[(d-3)F(r) - F(r)^3 - \lambda r^2 F(r) + r^4 F(r)\right]$$

The right-hand side is integrable for $d \geq 5$ so that integration gives

$$F'(r) = \frac{1}{r^{d-3}} \int_0^r s^{d-5}\left[(d-3)F(s) - F(s)^3 - \lambda s^2 F(s) + s^4 F(s)\right] ds$$

By using the L'Hospital's rule twice, we get if $F(0) = \sqrt{d-3}$:

$$\begin{aligned} \lim_{r\to 0} F'(r) &= \lim_{r\to 0} \frac{(d-3)F(r) - F(r)^3 - \lambda r^2 F(r) + r^4 F(r)}{(d-3)r} \\ &= \lim_{r\to 0} \frac{(d-3) - 3F(r)^2}{(d-3)} F'(r) \\ &= -2 \lim_{r\to 0} F'(r), \end{aligned}$$

so that $\lim_{r\to 0} F'(r) = 0$.

Singularity at $r = 0$ of the differential equation (3.4) is unfolded using the following Emden–Fowler transformation [18]:

$$r = e^t, \qquad f(r) = \psi(t), \qquad f'(r) = e^{-t}\psi'(t). \tag{3.7}$$

By chain rule, the second-order differential equation (3.4) for $f(r)$ becomes

$$\psi''(t) + (d-2)\psi'(t) = -e^{2t}\left(\lambda\psi(t) + \psi(t)^3\right) + e^{4t}\psi(t), \quad t \in \mathbb{R}. \tag{3.8}$$

The next result guarantees that there exists a unique local classical solution to the initial-value problem (3.1). The proof is developed from analysis of the existence of the bounded solutions of the differential equation (3.8) as $t \to -\infty$.

Lemma 3.2. *For every* $d \geq 3$, $\lambda \in \mathbb{R}$, *and* $b > 0$, *there exists* $r_0 > 0$ *and a unique classical solution* $f \in C^2(0, r_0)$ *to the initial-value problem (3.1) such that* $f(r) > 0$ *and* $f'(r) < 0$ *for* $r \in (0, r_0)$.

Proof. By using the Emden–Fowler transformation (3.7), the initial conditions $f(0) = b$ and $f'(0) = 0$ in the initial-value problem (3.1) become the following boundary conditions

$$\left\{ \begin{array}{l} \psi(t) \to b, \\ \psi'(t) \to 0, \end{array} \right. \quad \text{as} \quad t \to -\infty. \tag{3.9}$$

By the method of variation of parameters, we rewrite the differential equation (3.8) with the boundary conditions (3.9) as the following Volterra's integral equation:

$$\psi(t) = A(\psi)(t) := b + \frac{1}{d-2}\int_{-\infty}^{t} \left[1 - e^{-(d-2)(t-t')}\right] F(\psi(t'), t')dt', \tag{3.10}$$

where $F(\psi, t) := -e^{2t}\left(\lambda\psi + \psi^3\right) + e^{4t}\psi$. The integral operator A is considered on ψ in the Banach space $L^\infty(-\infty, t_0)$, where $-\infty < t_0 \ll -1$. It follows from (3.10) that

$$\|A(\psi)\|_{L^\infty} \leq b + \left[\frac{1}{2d}\left(|\lambda| + \|\psi\|^2_{L^\infty}\right) + \frac{1}{4(d+2)}e^{2t_0}\right] \|\psi\|_{L^\infty} e^{2t_0},$$

and

$$\|A(\psi) - A(\phi)\|_{L^\infty} \leq \left[\frac{1}{2d}\left(|\lambda| + (\|\psi\|_{L^\infty} + \|\phi\|_{L^\infty})^2\right) + \frac{1}{4(d+2)}e^{2t_0}\right] \|\psi - \phi\|_{L^\infty} e^{2t_0}.$$

If t_0 is a sufficiently large negative number, then $A : B_{2b} \to B_{2b}$ is a contraction operator in the ball $B_{2b} \subset L^\infty(-\infty, t_0)$ of a fixed radius $2b > 0$. By Banach's fixed-point theorem, there exists the unique solution $\psi \in B_{2b} \subset L^\infty(-\infty, t_0)$ to the integral equation (3.10).

Since $F(\psi(\cdot), \cdot) \in L^1(-\infty, t_0)$ if $\psi \in L^\infty(-\infty, t_0)$, the fixed point of the integral equation (3.10) is in $C^0(-\infty, t_0)$. Since $F(\psi(\cdot), \cdot) \in C^0(-\infty, t_0)$ if $\psi \in C^0(-\infty, t_0)$, the fixed point of the integral equation (3.10) is in $C^1(-\infty, t_0)$, so that differentiation of (3.10) yields

$$\psi'(t) = \int_{-\infty}^{t} e^{-(d-2)(t-t')} F(\psi(t'), t') dt'. \tag{3.11}$$

Finally, since $F(\psi(\cdot), \cdot) \in C^1(-\infty, t_0)$ if $\psi \in C^1(-\infty, t_0)$, the fixed point of the integral equation (3.10) is in $C^2(-\infty, t_0)$. By the chain rule, this implies that $f \in C^2(0, r_0)$ for small $r_0 > 0$.

By continuity of the solution, we have $\psi(t) > 0$ for $t \in (-\infty, t_0)$ if t_0 is a sufficiently large negative number. The transformation formula $f(r) = \psi(t)$ yields $f(r) > 0$ for $r \in (0, r_0)$ with small positive r_0. Furthermore, thanks to the bound

$$\|\psi - b\|_{L^\infty(-\infty, t_0)} \leq C_1 e^{2t_0}$$

with some $C_1 > 0$, it follows from (3.11) that

$$\|\psi' + (\lambda b + b^3) d^{-1} e^{2t}\|_{L^\infty(-\infty, t_0)} \leq C_2 e^{4t_0},$$

for some $C_2 > 0$. Hence $\psi'(t) < 0$ for $t \in (-\infty, t_0)$ if t_0 is a large negative number. By the transformation formula $f'(r) = e^{-t}\psi'(t)$, this yields $f'(r) < 0$ for $r \in (0, r_0)$ with small $r_0 > 0$. □

Remark 3.6. *The solution $\psi \in C^2(-\infty, t_0)$ in Lemma 3.2 satisfies the asymptotic expansion*

$$\psi(t) = b - \frac{\lambda b + b^3}{2d} e^{2t} + \mathcal{O}(e^{4t}) \quad \text{as} \quad t \to -\infty. \tag{3.12}$$

This expansion implies that

$$\lim_{r \to 0} f'(r) = \lim_{t \to -\infty} e^{-t} \psi'(t) = 0 \tag{3.13}$$

and

$$\lim_{r \to 0} f''(r) = \lim_{t \to -\infty} e^{-2t} \left[\psi''(t) - \psi'(t)\right] = -(\lambda b + b^3) d^{-1}. \tag{3.14}$$

where the first limit is in agreement with Lemma 3.1.

Remark 3.7. *The proof of Lemma 3.2 is based on classical fixed-point arguments, which is the main technical tool used in the rest of this work.*

3.2 Existence of the decaying solution at infinity

Another solution to the same differential equation (3.4) can be constructed from the condition that $f(r), f'(r) \to 0$ as $r \to \infty$. In order to construct such decaying solutions, we reformulate the second-order equation (3.8) as the following three-dimensional dynamical system:

$$\begin{cases} x' = 2x, \\ \psi' = \varphi, \\ \varphi' = (2-d)\varphi - x(\lambda\psi + \psi^3) + x^2\psi, \end{cases} \tag{3.15}$$

where $x(t) := e^{2t}$ and the prime stands for the derivative in t. The following lemma identifies the admissible behavior of classical solutions to the differential equation (3.4) such that $f(r), f'(r) \to 0$ as $r \to \infty$.

Lemma 3.3. *For every $d \geq 1$ and every $\lambda \in \mathbb{R}$, there exists $r_0 > 0$ and a one-parameter family of classical solutions $f \in C^2(r_0, \infty)$ to the differential equation (3.4) such that $f(r), f'(r) \to 0$ as $r \to \infty$. Moreover,*

$$f(r) \sim Cr^{\frac{\lambda-d}{2}} e^{-\frac{1}{2}r^2} \quad \text{as} \quad r \to \infty, \tag{3.16}$$

for some $C \in \mathbb{R}$, where $f(r) \sim g(r)$ is the asymptotic correspondence which can be differentiated.

Proof. The limit $r \to \infty$ corresponds to the limit $t \to +\infty$ due to the transformation (3.7). If $f(r), f'(r) \to 0$ as $r \to \infty$, then $x(t) \to \infty$, $\psi(t) \to 0$, and $\varphi(t)/\sqrt{x(t)} \to 0$ as $t \to +\infty$. We introduce the following transformation of variables:

$$x(t) = \frac{1}{y(\tau)}, \quad \psi(t) = \psi(\tau), \quad \varphi(t) = \frac{\phi(\tau)}{y(\tau)}, \tag{3.17}$$

where τ is the new time variable defined by the chain rule $dt = y(\tau)d\tau$. For convenience, we do not change the notation for ψ that now depends on τ. By integrating $dt = y(\tau)d\tau$ or equivalently, $d\tau = x(t)dt$ with the initial condition $\tau = 0$ at $t = 0$, we obtain

$$\tau = \frac{1}{2}\left(e^{2t} - 1\right). \tag{3.18}$$

Substitution of (3.17) into (3.15) yields the dynamical system

$$\begin{cases} \dot{y} = -2y^2, \\ \dot{\psi} = \phi, \\ \dot{\phi} = \psi - y\left(d\phi + \lambda\psi + \psi^3\right), \end{cases} \tag{3.19}$$

where the dot denotes the derivative in τ.

The only equilibrium point of system (3.19) is $(y, \psi, \phi) = (0, 0, 0)$. Linearization of system (3.19) at $(0, 0, 0)$ yields eigenvalues $\{-1, 0, 1\}$, which implies that the orbits approaching $(0, 0, 0)$ as $\tau \to \infty$ belongs to the two-dimensional stable-center manifold. Moreover, since $x(t) = e^{2t}$, the transformation (3.17) suggests that

$$y(\tau) = \frac{1}{1 + 2\tau} = \frac{1}{2\tau} + Y(\tau), \tag{3.20}$$

where $Y \in L^1(\tau_0, \infty)$ for any $\tau_0 > 0$. Since $y(0) = 1$ is uniquely determined, we are only looking for a unique orbit on the two-dimensional stable-center manifold that approaches $(0, 0, 0)$ as $\tau \to \infty$. By the theorem on invariant manifolds, this manifold is tangential to the stable-center manifold of the linearized system. Therefore, we study the analytical representation of solutions to the linearized system. With the help of (3.20), the linearized system is written in the form:

$$\frac{d}{d\tau} \begin{pmatrix} \psi \\ \phi \end{pmatrix} = [A + V(\tau) + R(\tau)] \begin{pmatrix} \psi \\ \phi \end{pmatrix}, \tag{3.21}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V(\tau) = -\frac{1}{2\tau} \begin{pmatrix} 0 & 0 \\ \lambda & d \end{pmatrix}, \quad R(\tau) = -Y(\tau) \begin{pmatrix} 0 & 0 \\ \lambda & d \end{pmatrix}.$$

The eigenvalues of A are $\mu_\pm = \pm 1$ with the eigenvectors $(1, \pm 1)$. Solving the characteristic equation for $A + V(\tau)$, we obtain the eigenvalues of $A + V(\tau)$ denoted by $\nu_\pm(\tau)$ in the form:

$$\nu_\pm(\tau) = -\frac{d}{4\tau} \pm \sqrt{1 - \frac{\lambda}{2\tau} + \frac{d^2}{16\tau^2}} = \mu_\pm - \frac{d \pm \lambda}{4\tau} + \nu_\pm^{(R)}(\tau), \tag{3.22}$$

where $\nu_\pm^{(R)} \in L^1(\tau_0, \infty)$ for any $\tau_0 > 0$. By Theorem 8.1 on p.92 in [13], for which the assumptions $V', R \in L^1(\tau_0, \infty)$ are satisfied, there exist two linearly independent classical solutions $(\psi_\pm, \phi_\pm)$ of the linearized system (3.21) satisfying the limit

$$\lim_{\tau \to \infty} \begin{pmatrix} \psi_\pm \\ \phi_\pm \end{pmatrix} e^{-\int_{\tau_0}^{\tau} \nu_\pm(\tau') d\tau'} = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}. \tag{3.23}$$

Thanks to the leading order of the eigenvalues in (3.22), the upper sign corresponds to the unstable solution and the lower sign corresponds to the stable solution. Since we are looking for the stable solution, we adopt the decomposition of (ψ, ϕ) over the eigenvectors of A together with the time-dependent factor which follows from the integration

$$e^{\int_{\tau_0}^{\tau} \nu_-(\tau') d\tau'} = C(\tau_0) \tau^{\frac{\lambda - d}{4}} e^{-\tau} \left[1 + \mathcal{O}(\tau^{-1})\right] \quad \text{as} \quad \tau \to \infty,$$

where the positive constant $C(\tau_0)$ depends on τ_0. Hence we write

$$\psi(\tau) = \tau^{\frac{\lambda-d}{4}} e^{-\tau} \left[\psi_+(\tau) + \psi_-(\tau)\right], \quad \phi(\tau) = \tau^{\frac{\lambda-d}{4}} e^{-\tau} \left[\psi_+(\tau) - \psi_-(\tau)\right], \tag{3.24}$$

where (ψ_+, ψ_-) are new variables satisfying the following system of equations:

$$\begin{cases} \dot{\psi}_+ = 2\psi_+ - \lambda(2\tau)^{-1}\psi_+ + (d-\lambda)(4\tau)^{-1}\psi_- - H(\psi_+, \psi_-, \tau), \\ \dot{\psi}_- = (d+\lambda)(4\tau)^{-1}\psi_+ + H, \end{cases} \tag{3.25}$$

where

$$H(\psi_+, \psi_-, \tau) := \frac{1}{2} Y(\tau) \left[(\lambda + d)\psi_+ + (\lambda - d)\psi_-\right] + \frac{1}{2} y(\tau) \tau^{\frac{\lambda-d}{2}} e^{-2\tau} \left(\psi_+ + \psi_-\right)^3.$$

If $\psi_+, \psi_- \in L^\infty(\tau_0, \infty)$ for $\tau_0 > 0$, then $H(\psi_+(\cdot), \psi_-(\cdot), \cdot) \in L^1(\tau_0, \infty)$ due to $Y \in L^1(\tau_0, \infty)$. This suggests that the remainder terms in the H-function remain small along the solution satisfying $(\psi, \phi) \to (0, 0)$ as $\tau \to \infty$. In order to make this analysis precise, we integrate the first equation of system (3.25) subject to the boundary condition $\lim_{\tau\to\infty} e^{-2\tau}\psi_+(\tau) = 0$ and obtain the integral equation:

$$\psi_+(\tau) = \int_\tau^\infty e^{-2(\tau'-\tau)} \left[\frac{\lambda}{2\tau'}\psi_+(\tau') + \frac{\lambda - d}{4\tau'}\psi_-(\tau') + H(\psi_+(\tau'), \psi_-(\tau'), \tau')\right] d\tau'. \tag{3.26}$$

On the other hand, integrating the second equation of system (3.25) subject to the boundary condition $\lim_{\tau\to\infty} \psi_-(\tau) = c$ for an arbitrary constant $c \in \mathbb{R}$ yields another integral equation:

$$\psi_-(\tau) = c - \int_\tau^\infty \left[\frac{\lambda + d}{4\tau'}\psi_+(\tau') + H(\psi_+(\tau'), \psi_-(\tau'), \tau')\right] d\tau'. \tag{3.27}$$

It is clear from the integral equation (3.27) that $\psi_+ \in L^\infty(\tau_0, \infty)$ is not sufficient for $\psi_- \in L^\infty(\tau_0, \infty)$. Therefore, we consider the Banach space $L^1(\tau_0, \infty) \cap L^\infty(\tau_0, \infty)$ for $\tau^{-1}\psi_+(\tau)$ and $L^\infty(\tau_0, \infty)$ for $\psi_-(\tau)$, where $1 \ll \tau_0 < \infty$. This suggest that one can obtain $\tilde{\psi}_+(\tau) := \tau^{-1}\psi_+(\tau)$ and $\psi_-(\tau)$ from solutions to the system of fixed-point equations:

$$\tilde{\psi}_+ = A_+(\tilde{\psi}_+, \psi_-), \qquad \psi_- = A_-(\tilde{\psi}_+, \psi_-), \tag{3.28}$$

where

$$A_+(\tilde{\psi}_+, \psi_-)(\tau) := \frac{1}{\tau} \int_\tau^\infty e^{-2(\tau'-\tau)} \left[\frac{\lambda}{2}\tilde{\psi}_+ + \frac{\lambda - d}{4\tau'}\psi_- + H(\tilde{\psi}_+, \psi_-, \tau')\right] d\tau'$$

and

$$A_-(\tilde{\psi}_+, \psi_-)(\tau) := c - \int_\tau^\infty \left[\frac{\lambda + d}{4}\tilde{\psi}_+ + H(\tilde{\psi}_+, \psi_-, \tau')\right] d\tau',$$

with $H(\tilde{\psi}_+, \psi_-, \tau)$ being redefined in new variables by

$$H = \frac{1}{2} Y(\tau) \left[(\lambda + d)\tau\tilde{\psi}_+ + (\lambda - d)\psi_- \right] + \frac{1}{2} y(\tau)\tau^{\frac{\lambda-d}{2}} e^{-2\tau} \left(\tau\tilde{\psi}_+ + \psi_- \right)^3.$$

We proceed with fixed-point estimates similarly to the proof of Lemma 3.2. By using the Young inequality for convolution integrals, we estimate the first and third term in A_+ as follows:

$$\begin{aligned} &\|\tfrac{1}{\tau}\textstyle\int_\tau^\infty e^{-2(\tau'-\tau)}[\frac{\lambda}{2}\tilde{\psi}_+ + H(\tilde{\psi}_+, \psi_-, \tau')]d\tau'\|_{L^1\cap L^\infty} \\ &\le \|\tau^{-1}\|_{L^\infty} \|e^{-2\tau}\|_{L^1(0,\infty)} \left[\tfrac{|\lambda|}{2} \|\tilde{\psi}_+\|_{L^1\cap L^\infty} + \|H(\tilde{\psi}_+, \psi_-, \cdot)\|_{L^1\cap L^\infty} \right], \end{aligned}$$

where all norms are defined on (τ_0, ∞) with $\tau_0 \gg 1$ except for $\|e^{-2\tau}\|_{L^1(0,\infty)} = \frac{1}{2}$. In addition, we estimate

$$\begin{aligned} \|H(\tilde{\psi}_+, \psi_-, \cdot)\|_{L^1\cap L^\infty} \quad \le \quad & \frac{|\lambda| + d}{2} \left(\|\tau Y(\tau)\|_{L^\infty} \|\tilde{\psi}_+\|_{L^1\cap L^\infty} + \|Y\|_{L^1\cap L^\infty} \|\psi_-\|_{L^\infty} \right) \\ & + \frac{1}{2} \|y(\tau)\tau^{\frac{\lambda-d}{2}} e^{-2\tau} (\tau\tilde{\psi}_+ + \psi_-)^3\|_{L^1\cap L^\infty}, \end{aligned}$$

with $Y \in L^1 \cap L^\infty$, $\tau Y \in L^\infty$ from (3.20) and $y(\tau)\tau^{\frac{\lambda-d}{2}} e^{-2\tau}$ being exponentially small on (τ_0, ∞) with $\tau_0 \gg 1$. For the second term in A_+, we use both the Young and Cauchy–Schwarz inequalities in order to obtain:

$$\|\frac{1}{\tau} \int_\tau^\infty e^{-2(\tau'-\tau)} \frac{(\lambda - d)}{4\tau'} \psi_- d\tau'\|_{L^1\cap L^\infty} \le \frac{|\lambda| + d}{4} \|\tau^{-1}\|_{L^2\cap L^\infty} \|e^{-2\tau}\|_{L^1(0,\infty)} \|\tau^{-1}\|_{L^2\cap L^\infty} \|\psi_-\|_{L^\infty}.$$

Finally, we estimate A_- as follows:

$$\|A_-(\tilde{\psi}_+, \psi_-)\|_{L^\infty} \quad \le \quad |c| + \frac{|\lambda| + d}{4} \|\tilde{\psi}_+\|_{L^1} + \|H(\tilde{\psi}_+, \psi_-, \cdot)\|_{L^1}.$$

If τ_0 is a sufficiently large positive number and if

$$\|\tilde{\psi}_+\|_{L^1\cap L^\infty} + \|\psi_-\|_{L^\infty} \le 2|c| \tag{3.29}$$

then the previous bounds imply that

$$\|A(\tilde{\psi}_+, \psi_-)\|_{L^1\cap L^\infty} + \|A_-(\tilde{\psi}_+, \psi_-)\|_{L^\infty} \le 2|c|,$$

due to smallness of $\|\tau Y\|_{L^\infty}$, $\|Y\|_{L^1\cap L^\infty}$, $\|\tau^{-1}\|_{L^2\cap L^\infty}$, and $\|y(\tau)\tau^{\frac{\lambda-d}{2}} e^{-2\tau}\|_{L^1\cap L^\infty}$ if $\tau_0 \gg 1$. In addition, by similar estimates, it is easy to prove that (A_+, A_-) is a contraction operator in the set (3.29) if $\tau_0 \gg 1$. By Banach's fixed-point theorem, there exists the unique solution for $\tilde{\psi}_+ \in L^1(\tau_0, \infty) \cap L^\infty(\tau_0, \infty)$ and $\psi_- \in L^\infty(\tau_0, \infty)$ to the system of integral equations (3.28) in the set (3.29). From $\tilde{\psi}_+$, we obtain ψ_+ by $\psi_+(\tau) = \tau\tilde{\psi}_+(\tau)$. Furthermore, bootstrapping arguments similar to those in the proof of Lemma 3.2 gives smoothness of ψ_+ and ψ_- on (τ_0, ∞). Thanks to the integrability of $\tau^{-1}\psi_+$

and continuity of ψ_+, we have $\psi_+(\tau) \to 0$ as $\tau \to \infty$.

By unfolding the transformations (3.7), (3.17), and (3.24), we obtain that if $f(r), f'(r) \to 0$, then $f(r)$ satisfies the asymptotic behavior (3.16), where

$$C := 2^{-\frac{\lambda-d}{4}} e^{\frac{1}{2}} c$$

and $c := \lim_{\tau\to\infty} \psi_-(\tau)$ is defined in the integral equation (3.27). □

The following lemma guarantees global continuation of classical solutions to the differential equation (3.4) from $r = 0$ to $r \to \infty$ and from $r \to \infty$ to $r = 0$.

Lemma 3.4. *For every $d \geq 1$ and $\lambda \in \mathbb{R}$, if $f \in C^2(0, r_0)$ is a solution of Lemma 3.2 for some $r_0 \in (0, \infty)$, then $f \in C^2(0, \infty)$ and if $f \in C^2(r_0, \infty)$ is a solution of Lemma 3.3 for some $r_0 \in (0, \infty)$, then $f \in C^2(0, \infty)$.*

Proof. Let us introduce the Lyapunov function in the form:

$$\Lambda(f, f', r) := \frac{1}{2}(f')^2 + \frac{1}{2}(\lambda - r^2)f^2 + \frac{1}{4}f^4. \tag{3.30}$$

It follows from (3.4) and (3.30) that

$$\frac{d}{dr}\Lambda(f, f', r) = -\frac{d-1}{r}(f')^2 - rf^2 < 0, \tag{3.31}$$

hence the map $r \mapsto \Lambda(f(r), f'(r), r)$ is strictly monotonically decreasing along the classical solution to the differential equation (3.4). It follows from (3.30) that

$$\frac{1}{2}(f')^2 + \frac{1}{4}(f^2 + \lambda - r^2)^2 \leq \Lambda(f, f', r) + \frac{1}{4}(\lambda - r^2)^2. \tag{3.32}$$

Let $f \in C^2(0, r_0)$ be a solution of Lemma 3.2 for some $r_0 > 0$ and assume that the solution blows up at a finite $R < \infty$. Since the map $r \mapsto \Lambda(f(r), f'(r), r)$ is decreasing, we obtain a contradiction from the bound (3.32):

$$\frac{1}{2}(f')^2 + \frac{1}{4}(f^2 + \lambda - r^2)^2 \leq \Lambda(f(r_0), f'(r_0), r_0) + \frac{1}{4}(\lambda - R^2)^2 < \infty, \quad r \in [r_0, R].$$

Hence, no finite R exists and the classical solution continues on $(0, \infty)$.

Let $f \in C^2(r_0, \infty)$ be a solution of Lemma 3.3 for some $r_0 > 0$. It follows from the fast decay of $f(r), f'(r) \to 0$ as $r \to \infty$ that $\Lambda(f(r), f'(r), r) \to 0$ as $r \to \infty$. It follows from (3.31) and (3.32) that there exist positive constants A_0 and B_0 such that

$$r\frac{d}{dr}\Lambda(f, f', r) \geq -A_0\Lambda(f, f', r) - B_0, \quad r \in (0, r_0],$$

or equivalently,

$$r\frac{d}{dr}r^{A_0}\Lambda(f,f',r) \geq -B_0 r^{A_0}, \quad r\in(0,r_0].$$

Integration on $[r, r_0]$ yields

$$\Lambda(f(r),f'(r),r) \leq \left(\frac{r_0}{r}\right)^{A_0}\left[\Lambda(f(r_0),f'(r_0),r_0)+\frac{B_0}{A_0}\right] < \infty, \quad r\in(0,r_0].$$

It follows from the bound (3.32) that the classical solution continues on $(0,\infty)$. □

3.3 Proof of Theorem 3.1

Here we develop the shooting method for the proof of Theorem 3.1.

The unique global solution $f \in C^2(0,\infty)$ of the initial value problem (3.1) is given by Lemmas 3.2 and 3.4. We define the following three sets:

$$I_+ := \left\{\lambda\in\mathbb{R}: \ \exists r_0\in(0,\infty): \ f(r_0)=0, \text{ while } f(r)>0, \ \ f'(r)<0, \ \ r\in(0,r_0)\right\}, \tag{3.33}$$

$$I_- := \left\{\lambda\in\mathbb{R}: \ \exists r_0\in(0,\infty): \ f'(r_0)=0, \text{ while } f(r)>0, \ \ f'(r)<0, \ \ r\in(0,r_0)\right\}, \tag{3.34}$$

and

$$I_0 := \left\{\lambda\in\mathbb{R}: \ f(r)>0, \ \ f'(r)<0, \ \ r\in(0,\infty)\right\}. \tag{3.35}$$

The sets I_+, I_-, and I_0 depend on parameters b and d, which are not written. We make the following partition of $\mathbb{R}$ for parameter λ:

$$\mathbb{R} = I_+ \cup I_0 \cup I_-. \tag{3.36}$$

By uniqueness of solutions to differential equations, if $f(r_0) = f'(r_0) = 0$ for some $r_0 \in (0,\infty)$, then $f(r) = 0$ for every $r\in(0,\infty)$, hence $I_+ \cap I_- = \emptyset$. By construction, it is also true that $I_+\cap I_0 = \emptyset$ and $I_-\cap I_0=\emptyset$, hence the three sets are disjoint.

In the following two lemmas, we prove that the sets I_+ and I_- are open and non-empty. These results imply that I_0 in the partition (3.36) is closed and non-empty.

Lemma 3.5. *For every $d\geq 1$, I_+ is open and, moreover, $[d,\infty)\subset I_+$.*

Proof. The unique solution $f\in C^2(0,\infty)$ depends smoothly on the parameter λ since the differential equation (3.4) is smooth in f and λ. Let f_λ denotes the unique λ-dependent solution and r_0 be a root of f_{λ_0} for a fixed $\lambda_0\in I_+$. By uniqueness of the zero solution, if $f_{\lambda_0}(r_0)=0$, then $f'_{\lambda_0}(r_0)\neq 0$. Since f_λ is smooth in λ, it follows from the implicit function theorem that for every λ in an open neighbourhood of λ_0 there exists r_λ near r_0 such that $f_\lambda(r_\lambda)=0$. Hence, the set I_+ is open. It remains to prove that such $r_\lambda\in(0,\infty)$ exists for every $\lambda\in[d,\infty)$.

Let $g(r) = e^{\frac{1}{2}r^2} f(r)$. Then, $g(r)$ satisfies the differential equation:

$$g''(r) + \left[\frac{d-1}{r} - 2r\right] g'(r) + e^{-r^2} g(r)^3 + (\lambda - d) g(r) = 0, \tag{3.37}$$

subject to the initial conditions $g(0) = b$ and $g'(0) = 0$. By using the transformation (3.7) and the asymptotic expansion (3.12), we obtain with the chain rule

$$\begin{aligned} e^{-\frac{1}{2}r^2} g'(r) &= f'(r) + r f(r) \\ &= e^{-t}\psi'(t) + e^{t}\psi(t) \\ &= \frac{b}{d}(d - \lambda - b^2)e^{t} + \mathcal{O}(e^{3t}) \quad \text{as} \quad t \to -\infty. \end{aligned}$$

Since $\lambda \geq d$, we have $g'(r) < 0$ for some small $r > 0$.

Let $r_0 := \inf\{r > 0 : \ g(r) = 0\}$. We need to show that $r_0 < \infty$. First, we show that $g'(r) < 0$ for all $r \in (0, r_0)$. Indeed, if there exists $r_1 \in (0, r_0)$ such that $g'(r_1) = 0$ and $g'(r) < 0$ for $r \in (0, r_1)$, then the differential equation (3.37) with $\lambda \geq d$ implies that $g''(r_1) < 0$, which is impossible. Hence, $g'(r) < 0$ for all $r \in (0, r_0)$.

It follows from (3.37) that

$$g''(r) \leq \left[2r - \frac{d-1}{r}\right] g'(r), \quad r \in (0, r_0).$$

If $r_0 \leq R := \frac{\sqrt{d-1}}{\sqrt{2}}$, we are done. Assume that $r_0 > R$ and define $G(r) := -g'(r)$. Then,

$$G'(r) \geq \left[2r - \frac{d-1}{r}\right] G(r), \quad r \in (R, r_0).$$

Since $G(r) > 0$ for $r \in [R, r_0)$, we have $G(r) \geq G(R)$ for $r \in [R, r_0)$, or alternatively, $g'(r) \leq g'(R) < 0$. The case $r_0 = \infty$ is impossible since $g(r)$ must hit zero for a finite r. Thus, $r_0 < \infty$ for every $\lambda \in [d, \infty)$. □

Lemma 3.6. *For every $d \geq 4$, I_- is open and, moreover, $(-\infty, 0] \subset I_-$.*

Proof. In order to prove that I_- is open, we extend the proof of Lemma 3.5 based on the implicit function theorem. Let f_λ denote the λ-dependent unique solution and r_0 be a root of f'_{λ_0} for a fixed $\lambda_0 \in I_-$. Then, the differential equation (3.4) implies that either $f''_{\lambda_0}(r_0) \neq 0$ or $f''_{\lambda_0}(r_0) = 0$ and $r_0^2 = \lambda_0 + f_{\lambda_0}(r_0)^2$. In the latter case, since f is smooth, the derivative of the differential equation (3.4) at the point r_0 for which $f'_{\lambda_0}(r_0) = 0$ and $f''_{\lambda_0}(r_0) = 0$ gives $f'''_{\lambda_0}(r_0) = 2r_0 f_{\lambda_0}(r_0) > 0$, which is impossible if $f'_{\lambda_0}(r) < 0$ for $r \in (0, r_0)$. This implies that if $f'_{\lambda_0}(r_0) = 0$, then $f''_{\lambda_0}(r_0) \neq 0$. Since f_λ is smooth in λ, it follows from the implicit function theorem that for every λ in an open neighbourhood of λ_0 there exists r_λ near r_0 such that $f'_\lambda(r_\lambda) = 0$. Hence, the set I_- is open. It remains to prove that such $r_\lambda \in (0, \infty)$ exists for every $\lambda \in (-\infty, 0]$.

First, we show that $f(r) > 0$ for $r > 0$ if $\lambda \leq 0$. It follows from (3.4) that

$$r^d f'(r)f''(r) + (d-1)r^{d-1}[f'(r)]^2 - r^{d+2}f(r)f'(r) + r^d f(r)^3 f'(r) + \lambda r^d f(r)f'(r) = 0.$$

Assuming $f(R) = 0$ for some $R \in (0,\infty)$ and integrating on $[0,R]$ yields

$$\frac{1}{2}R^d[f'(R)]^2 + \frac{d-2}{2}\int_0^R r^{d-1}[f'(r)]^2 dr + \frac{d+2}{2}\int_0^R r^{d+1}f(r)^2 dr \\ -\frac{d}{4}\int_0^R r^{d-1}f(r)^4 dr - \frac{d\lambda}{2}\int_0^R r^{d-1}f(r)^2 dr = 0.$$

Similarly, integrating equation

$$r^{d-1}f(r)f''(r) + (d-1)r^{d-2}f(r)f'(r) - r^{d+1}f(r)^2 + r^{d-1}f(r)^4 + \lambda r^{d-1}f(r)^2 = 0$$

on $[0,R]$ with $f(R) = 0$ yields

$$-\int_0^R r^{d-1}[f'(r)]^2 dr - \int_0^R r^{d+1}f(r)^2 dr + \int_0^R r^{d-1}f(r)^4 dr + \lambda\int_0^R r^{d-1}f(r)^2 dr = 0.$$

Eliminating $\int_0^R r^{d-1}[f'(r)]^2 dr$ from these two equations yields the constraint:

$$\frac{1}{2}R^d[f'(R)]^2 + 2\int_0^R r^{d+1}f(r)^2 dr + \frac{d-4}{4}\int_0^R r^{d-1}f(r)^4 dr - \lambda\int_0^R r^{d-1}f(r)^2 dr = 0.$$

If $d \geq 4$ and $\lambda \leq 0$, this constraint is never satisfied, hence no $R \in (0,\infty)$ exists and $f(r) > 0$ for every $r > 0$. Moreover, if $f \in C^2(0,\infty)$ and $f(r), f'(r) \to 0$ as $r \to \infty$, then the fast asymptotic decay (3.16) in Lemma 3.3 implies that $f \in L^2_r(\mathbb{R}^+)$, which is impossible if $\lambda \in (-\infty, 0]$ by Proposition 2.2. Hence, there exists a constant $c > 0$ such that $f(r) \geq c$ for $r > 0$.

Next, we show that there exists $r_0 \in (0,\infty)$ such that $f'(r_0) = 0$. To do so, we integrate the differential equation

$$\frac{d}{dr}\left[r^{d-1}f'(r)\right] = r^{d+1}f(r) - r^{d-1}f(r)^3 - \lambda r^{d-1}f(r) \tag{3.38}$$

on $[0,R]$ and obtain the estimate:

$$\begin{aligned} R^{d-1}f'(R) &= \int_0^R r^{d+1}f(r)dr - \int_0^R r^{d-1}f(r)^3 dr - \lambda\int_0^R r^{d-1}f(r)dr \\ &\geq \frac{c}{d+2}R^{d+2} - \frac{b^3}{d}R^d, \end{aligned}$$

where we have used that $\lambda \leq 0$ and $c \leq f(r) \leq b$ as long as $f'(r) < 0$. Hence for

$$R > \left(\frac{b^3(d+2)}{dc}\right)^{1/2},$$

we must have $f'(R) > 0$ so that there exists $r_0 \in (0, \infty)$ such that $f'(r_0) = 0$ if $\lambda \in (-\infty, 0]$. □

It follows from Lemmas 3.5 and 3.6 that I_0 is closed and non-empty. The following lemma states that the set I_0 in the partition (3.36) contains all values of λ for which the unique solution f to the initial-value problem (3.1) is a solution $\mathfrak{u} \in \mathcal{E}$ to the boundary-value problem (1.7).

Lemma 3.7. *If $\lambda \in I_0$, then $f(r), f'(r) \to 0$ as $r \to \infty$ and $f \in \mathcal{E} \subset L^2_r(\mathbb{R}^+)$.*

Proof. If $f \in C^2(0, \infty)$ satisfies $f(r) > 0$ and $f'(r) < 0$ for $r \in (0, \infty)$, then necessarily $f'(r) \to 0$ as $r \to \infty$ because $[0, b] \ni f$ is compact. Assume that $f(r) \to c$ as $r \to \infty$ with some $c \in (0, b)$. Then, integrating (3.38) on $[0, R]$ similarly to the proof of Lemma 3.6 yields

$$\begin{aligned} R^{d-1} f'(R) &= \int_0^R r^{d+1} f(r) dr - \int_0^R r^{d-1} f(r)^3 dr - \lambda \int_0^R r^{d-1} f(r) dr \\ &\geq \frac{c}{d+2} R^{d+2} - \frac{b(b^2 + \lambda)}{d} R^d, \end{aligned}$$

where $\lambda \in (0, d)$ if $\lambda \in I_0$. Hence for

$$R > \left(\frac{b(b^2 + \lambda)(d + 2)}{dc} \right)^{1/2},$$

we must have $f'(R) > 0$ which is a contradiction. This implies that $c = 0$, that is, $f(r) \to 0$ as $r \to \infty$. Since $f(r), f'(r) \to 0$ as $r \to \infty$, Lemma 3.3 implies that $f(r)$ satisfies the fast asymptotic decay (3.16) so that $f \in \mathcal{E} \subset L^2_r(\mathbb{R}^+)$ for this $\lambda \in I_0$. □

We collect all individual results together as the proof of Theorem 3.1.

Proof of Theorem 3.1. Fix $d \geq 4$ so that all previous results can be equally applied.

By Lemmas 3.2 and 3.4, there exists the unique global classical solution $f \in C^2(0, \infty)$ to the initial-value problem (3.1) for $\lambda \in \mathbb{R}$. The line $\mathbb{R}$ for the parameter λ in the differential equation (3.4) can be partitioned into the union of three disjoint sets I_+, I_-, and I_0 given by (3.33), (3.34), and (3.35) respectively. Suitable solutions to the boundary-value problem (1.7) in the function space $\mathcal{E} \subset L^2_r(\mathbb{R}^+)$ may only exist for $\lambda \in I_0$.

By Lemmas 3.5 and 3.6, the sets I_+ and I_- are open and non-empty, so that the set I_0 in the partition (3.36) is closed and non-empty. By Lemma 3.7, we proved that if $\lambda \in I_0$, then the corresponding function $f \in C^2(0, \infty)$ is a solution $\mathfrak{u} \in \mathcal{E}$ to the boundary-value problem (1.7). It follows by Propositions 2.1 and 2.2 that $I_0 \subset (d-4, d)$. □

Figure 3.1 illustrates the shooting method used in the proof of Theorem 3.1. For $d = 5$ and $b = 10$, we compute numerically the unique classical solution to the initial-value problem (3.1) for three different values of λ. For a special value of λ denoted as $\lambda(b)$, the solution gives the ground state of the boundary-value problem (1.7), which implies that $\lambda(b) \in I_0$. For another value of $\lambda < \lambda(b)$ the solution does not cross the zero level but grows with some oscillations as $r \to \infty$. Therefore, there is $r_0 \in (0, \infty)$ such that $f'(r_0) = 0$ and this $\lambda \in I_-$. For yet another value of $\lambda > \lambda(b)$, the solution crosses the zero level (and becomes large negative with some oscillations) so that there is $r_0 \in (0, \infty)$ such that $f(r_0) = 0$ and this $\lambda \in I_+$. We have confirmed numerically that the value of $\lambda(b) \in I_0$ is unique for every $b > 0$.

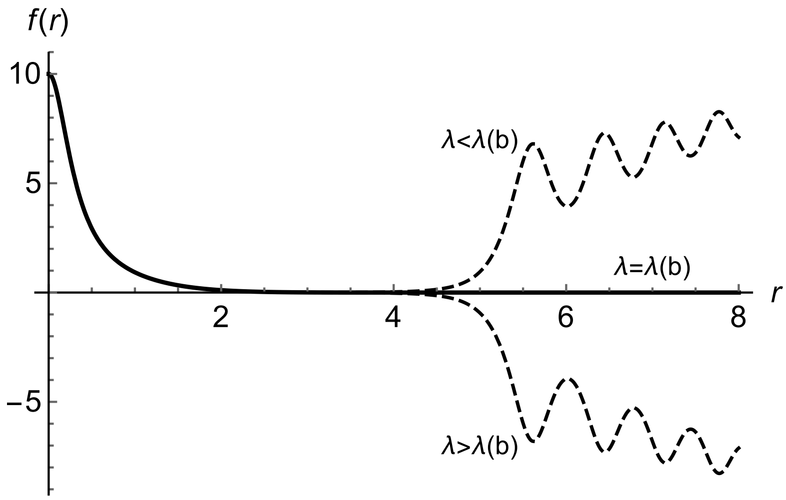

FIGURE 3.1: Plot of the unique solution f satisfying the initial-value problem (3.1) for $d = 5$, $b = 10$, and three values of λ. For $\lambda = \lambda(b)$, the solution f satisfies the boundary-value problem (1.7).

3.4 Proof of Theorem 3.2

Here we explain how the shooting method can be applied to the proof of Theorem 3.2. Our arguments basically reproduce the approach in [54] with some important modifications.

By using the transformation

$$r = e^t, \quad F(r) = \Psi(t) \quad F'(r) = e^{-t}\Psi'(t), \tag{3.39}$$

one can obtain solutions to the initial-value problem (3.3) from the second-order differential equation

$$\Psi''(t) + (d-4)\Psi'(t) + (3-d)\Psi(t) + \Psi(t)^3 = -\lambda e^{2t}\Psi(t) + e^{4t}\Psi(t), \quad t \in \mathbb{R} \tag{3.40}$$

completed with the boundary conditions

$$\begin{cases} \Psi(t) \to \sqrt{d-3}, \\ \Psi'(t) \to 0, \end{cases} \quad \text{as} \quad t \to -\infty. \tag{3.41}$$

We denote the unique solution of the second-order equation (3.40) satisfying the boundary conditions (3.41) by $\Psi_\lambda(t)$. One can prove by a simple extension of Lemma 3.2 that this solution satisfies the asymptotic behavior:

$$\Psi_\lambda(t) = \sqrt{d-3}\left[1 - \frac{\lambda}{4d-10}e^{2t} + \mathcal{O}(e^{4t})\right] \quad \text{as } t \to -\infty. \tag{3.42}$$

The solution of Theorem 3.2 arises for $\lambda = \lambda_\infty$, for which $\Psi_\infty := \Psi_{\lambda=\lambda_\infty}$ decays to zero as $t \to +\infty$. The following list contains relevant details how the shooting method is modified for the proof of Theorem 3.2.

- $(\sqrt{d-3}, 0, 0)$ is an equilibrium point of the three-dimensional dynamical system

$$\begin{cases} x' = 2x, \\ \Psi' = \Phi, \\ \Phi' = (4-d)\Phi + (d-3)\Psi - \Psi^3 - \lambda x \Psi + x^2 \Psi, \end{cases} \tag{3.43}$$

  where $x(t) := e^{2t}$ and the prime stands for the derivative in t. If $d \geq 5$, the equilibrium point $(\sqrt{d-3}, 0, 0)$ admits a one-dimensional unstable manifold and a two-dimensional stable manifold. The unique local classical solution Ψ_λ satisfying the differential equation (3.40) and the boundary conditions (3.41) corresponds to the one-dimensional unstable manifold of the dynamical system (3.43) with uniquely defined $x(t) = e^{2t}$. The existence and uniqueness of Ψ_λ follows by the unstable manifold theorem. In an analogue with Lemma 3.2, this gives the unique solution $F \in C^2(0, r_0)$ with $F(r) > 0$ and $F'(r) < 0$ for $r \in (0, r_0)$ to the initial-value problem (3.3) for $d \geq 5$.

- The proof of Lemma 3.3 does not depend on the behavior of $f(r)$ near $r = 0$ as long as $f(r), f'(r) \to 0$ as $r \to \infty$. By the transformation $F(r) = rf(r)$, if $F(r), F'(r) \to 0$ as $r \to \infty$, then $f(r), f'(r) \to 0$ as $r \to \infty$. By Lemma 3.3, there exists $C \in \mathbb{R}$ such that

$$F(r) \sim C r^{\frac{\lambda-d+2}{2}} e^{-\frac{1}{2}r^2} \quad \text{as} \quad r \to \infty. \tag{3.44}$$

- The proof of Lemma 3.4 is extended to $f(r) = r^{-1}F(r)$ verbatim.

- For the set I_+ in (3.36) defined by zeros of F, openness of I_+ follows from uniqueness of the zero solutions in (3.40) which implies that if $F(r_0) = 0$, then $F'(r_0) \neq 0$.

In order to show that $[d,\infty) \subset I_+$, we define

$$e^{-\frac{1}{2}r^2} g(r) = f(r) = r^{-1}F(r)$$

and

$$e^{-\frac{1}{2}r^2} g'(r) = r^{-1}F'(r) - \frac{1-r^2}{r^2}F(r),$$

hence $g'(r) < 0$ for small $r > 0$. The rest of the proof of Lemma 3.5 applies verbatim.

- For the set I_- in (3.36) defined by zeros of F', a special care should be taken to prove that the set is open. In the special case when $F'_{\lambda_0}(r_0) = F''_{\lambda_0}(r_0) = 0$, for which $d-3-F_{\lambda_0}(r_0)^2 = (\lambda_0 - r_0^2)r_0^2 > 0$, we obtain by differentiation in r:

$$F'''_{\lambda_0}(r_0) = 2(2r_0 - \lambda_0 r_0^{-1})F_{\lambda_0}(r_0),$$

hence the contradiction with $F'''_{\lambda_0}(r_0) > 0$ only holds if $\lambda_0 \in (r_0^2, 2r_0^2)$. If $\lambda_0 = 2r_0^2$ so that $F'''_{\lambda_0}(r_0) = 0$, then we obtain by another differentiation in r:

$$F''''_{\lambda_0}(r_0) = (6\lambda_0 r_0^{-2} - 4)F_{\lambda_0}(r_0) = 8F_{\lambda_0}(r_0) > 0,$$

so that the minimum of F_λ persists near r_0 when the solution is continued with respect to λ near λ_0. If $\lambda_0 > 2r_0^2$ and $F'''_{\lambda_0}(r_0) < 0$, then $F'_{\lambda_0}(r) \leq 0$ near $r = r_0$, so that if no other extremal points exist, then $F_{\lambda_0}(r) \in [0, \sqrt{d-3}]$ and $F'_{\lambda_0}(r) \leq 0$ for all $r > 0$. However, $F_{\lambda_0}(r) \to c$ as $r \to \infty$ is impossible for $c \neq 0$ (see the next item), hence $F_{\lambda_0}(r) \to 0$ as $r \to \infty$. However, if $F_{\lambda_0} \in C^2(0,\infty)$, $F_{\lambda_0}(r) > 0$ for $r > 0$, and $F_{\lambda_0}(r) \to 0$ as $r \to \infty$, then $F'_{\lambda_0}(r) < 0$ for $r > 0$ by the arguments from [37], which is a contradiction with $F'_{\lambda_0}(r_0) = 0$. Thus, either $F'_{\lambda_0}(r_0) = 0$ and $F''_{\lambda_0}(r_0) \neq 0$ or $F'_{\lambda_0}(r_0) = F''_{\lambda_0}(r_0) = F'''_{\lambda_0}(r_0) = 0$ and $F''''_{\lambda_0}(r_0) > 0$, in both cases the minimum of F_λ persists near $r = r_0$ in λ near λ_0.

  In order to show that $(-\infty, 0] \subset I_-$, we apply the proof of Lemma 3.6 to $f(r) = r^{-1}F(r)$, which holds due to the fast decay

$$r^d[f'(r)]^2 \to 0, \quad r^d[f(r)]^4 \to 0, \quad r^{d-1}f(r)f'(r) \to 0 \quad \text{as} \quad r \to 0$$

  if $d \geq 5$ (no decay holds if $d = 4$). Integrating (3.38) on $[0,R]$ with $f(r) = r^{-1}F(r)$ for $c \leq F(r) \leq \sqrt{d-3}$ and $\lambda \leq 0$ yields

$$R^{d-2}F'(R) \geq \frac{c}{d+1}R^{d+1} + (c - \sqrt{d-3})R^{d-3},$$

  due to the fast decay $r^{d-1}f'(r) \to 0$ as $r \to 0$. The lower bound implies that $F'(r) > 0$ for sufficiently large r. Hence, $(-\infty, 0] \in I_-$ if $d \geq 5$.

- The proof of Lemma 3.7 also applies to $f(r) = r^{-1}F(r)$. Integrating (3.38) on $[0, R]$ with $f(r) = r^{-1}F(r)$ for $c \le F(r) \le \sqrt{d-3}$ yields

$$R^{d-2}F'(R) \ge \frac{c}{d+1}R^{d+1} + (c - \sqrt{d-3})R^{d-3} - \frac{\lambda\sqrt{d-3}}{d-2}R^{d-2},$$

which is a contradiction with $F'(r) < 0$ for sufficiently large r. Hence $c = 0$ and $F(r), F'(r) \to 0$ as $r \to \infty$.

Remark 3.8. *Uniqueness of the solution in Theorem 3.2 was claimed in Section 4 of [54], however, we believe that the proof was incorrect. Indeed, assuming two solutions $\Psi_{\lambda_1}(r)$ and $\Psi_{\lambda_2}(r)$ for two values λ_1 and λ_2 in Theorem 3.2, we construct a quotient*

$$\rho(t) = \frac{\Psi_1(t)}{\Psi_2(t)},$$

which satisfies the differential equation

$$\rho''(t) + \left[d - 4 + \frac{2\Psi_2'(t)}{\Psi_2(t)}\right]\rho'(t) + \Psi_2(t)^2\rho(t)[\rho(t)^2 - 1] + (\lambda_1 - \lambda_2)e^{2t}\rho(t) = 0. \quad (3.45)$$

It follows from (3.42) that

$$\rho(t) = 1 - \frac{\lambda_1 - \lambda_2}{4d - 10}e^{2t} + \mathcal{O}(e^{4t}) \quad \text{as} \quad t \to -\infty. \quad (3.46)$$

A rescaling of time was applied in the arguments of [54] to make the last term in (3.45) small but was not applied to the second term of the expansion (3.46). As a result, the differential equation (3.45) was replaced by a differential inequality which led to a contradiction in [54]. With the proper scaling of time in both (3.45) and (3.46), transformation of the differential equation to a differential inequality cannot be justified.

Figure 3.2 illustrates the shooting method used in the proof of Theorem 3.2. The left panel shows $F(r)$ as the unique classical solution to the initial-value problem (3.3), whereas the right panel shows $\Psi(t)$ as a solution to the differential equation (3.40) with the boundary conditions (3.41). For $d = 5$, we compute numerically the solutions for three different values of λ. For a special value of $\lambda = \lambda_\infty$, the solution F gives the limiting singular solution $f_\infty \in \mathcal{E}$ after the transformation $f(r) = r^{-1}F(r)$. For values of λ above (below) λ_∞, the solution crosses the zero level and diverges to negative infinity (attains a minimum and diverges to positive infinity). We have found numerically that the value of λ_∞ is unique.

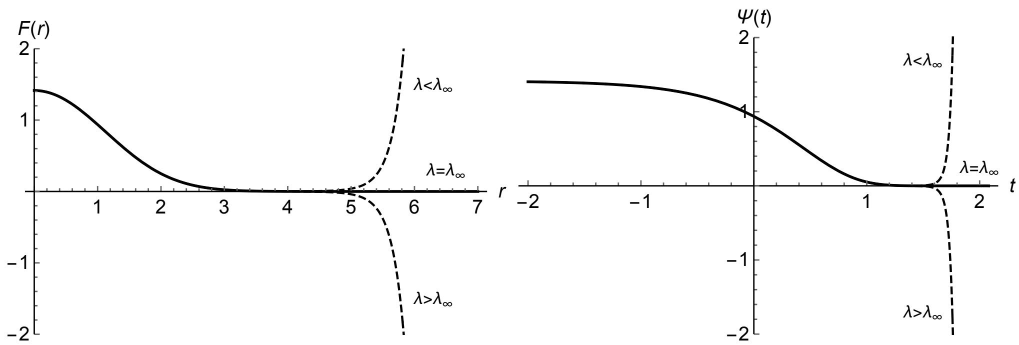

FIGURE 3.2: Plot of the unique solution $F(r)$ satisfying the initial-value problem (3.3), together with $\Psi(t)$ defined by the transformation (3.39), for $d = 5$ and three values of λ. For $\lambda = \lambda_\infty$, the solution gives the limiting singular solution $f_\infty \in \mathcal{E}$ after the transformation $f(r) = r^{-1}F(r)$. The dashed lines show solutions for values of λ slightly deviating from λ_∞.

Chapter 4

Snaking and monotone behaviors of the ground states

We consider the energy-supercritical case for (1.7) and assume for simplicity that $p = 1$, so that $(d-2)p > 2$ implies $d \geq 5$. In Chapter 3 we have shown that there exists a family of ground states $\{\mathfrak{u}_b\}_{b>0} \in \mathcal{E}$, and the main result of [54] proves existence of a limiting singular solution $f_\infty \in \mathcal{E}$, such that $\mathfrak{u}_b \to f_\infty$ as $b \to \infty$ in $\mathcal{E}$. By Theorems 3.1 and 3.2 we know that there exist $\lambda(b) \in (0, d)$ for every $b > 0$, and $\lambda_\infty \in (0, d)$, corresponding to $\mathfrak{u}_b$ and f_∞ respectively, such that $\lambda(b) \to \lambda_\infty$ as $b \to \infty$. Recall from Chapter 3 that $\mathfrak{u}_b$ is found from the solutions f of the initial-value problem

$$\begin{cases} f''(r) + \frac{d-1}{r} f'(r) - r^2 f(r) + \lambda f(r) + f(r)^3 = 0, \quad r > 0, \\ f(0) = b, \quad f'(0) = 0, \end{cases} \tag{4.1}$$

for $\lambda = \lambda(b)$, for which the solution f to the initial-value problem (4.1) decays to zero at infinity. Furthermore, f_∞ is found by the transformation $f_\infty(r) = r^{-1} F(r)$ from the solution F to the initial-value problem

$$\begin{cases} F''(r) + \frac{d-3}{r} F'(r) - \frac{d-3}{r^2} F(r) - r^2 F(r) + \lambda F(r) + \frac{1}{r^2} F(r)^3 = 0, \quad r > 0, \\ F(0) = \sqrt{d-3}, \quad F'(0) = 0, \end{cases} \tag{4.2}$$

for $\lambda = \lambda_\infty$, for which the solution F to the initial-value problem (4.2) decays to zero at infinity.

In this chapter, we study the convergence of $\lambda(b)$ to λ_∞ as $b \to \infty$, depending on the dimension $d \geq 5$. We show under a technical non-degeneracy assumption that the solution curve has an oscillatory (snaking) behavior for $5 \leq d \leq 12$ and a monotone behavior for $d \geq 13$. The following theorem presents the corresponding result.

Theorem 4.1. *Assume that λ_∞ is given by Theorem 3.2 and Assumptions 4.1 and 4.2 are satisfied. Then, there exists $b_0 \in [0, \infty)$ such that for every $b > b_0$ the value of λ in Theorem 3.1, denoted by $\lambda(b)$, is uniquely defined near λ_∞ such that $\lim_{b\to\infty} \lambda(b) = \lambda_\infty$.*

Moreover, for $5 \leq d \leq 12$*, there exist constants* $A_\infty > 0$ *and* $\delta_\infty \in \mathbb{R}$ *such that*

$$\lambda(b) - \lambda_\infty \sim A_\infty b^{-\beta} \sin(\alpha \ln b + \delta_\infty) \quad \text{as} \quad b \to \infty, \tag{4.3}$$

where

$$\alpha = \frac{\sqrt{-d^2 + 16d - 40}}{2}, \quad \beta = \frac{d-4}{2}, \tag{4.4}$$

whereas for $d \geq 13$*, there exists* $B_\infty > 0$ *such that*

$$\lambda(b) - \lambda_\infty \sim B_\infty b^{\kappa_+} \quad \text{as} \quad b \to \infty, \tag{4.5}$$

where

$$\kappa_+ = -\frac{d-4}{2} + \frac{\sqrt{d^2 - 16d + 40}}{2}. \tag{4.6}$$

Remark 4.1. *In (4.3) and (4.5),* $f(b) \sim g(b)$ *denotes the asymptotic correspondence in the sense* $g(b) \to 0$ *as* $b \to \infty$ *and* $\lim_{b\to\infty} \frac{|f(b)-g(b)|}{|g(b)|} = 0$*. Moreover, the asymptotic correspondence* $f(b) \sim g(b)$ *can be differentiated term by term.*

Remark 4.2. *If the value of* λ_∞ *in Theorem 3.2 is not unique, then for each* λ_∞*, which is isolated under Assumptions 4.1 and 4.2, there exists the solution curve of Theorem 4.1 with the oscillatory or monotone behavior. Our numerical results indicate that* λ_∞ *in Theorem 3.2 is unique; moreover,* $\lambda(b)$ *in Theorem 3.1 is unique for every* $b > 0$*.*

Remark 4.3. *The oscillatory behavior similar to the one in (4.3) was obtained in [6, 7, 15] for the stationary focusing nonlinear Schrödinger equation in a ball and without a harmonic potential. The similarity is explained by the same linearization of the stationary equation near the origin after the Emden–Fowler transformation [18]. While the previous works explore geometric methods, the main approach we undertake to prove Theorem 4.1 is based on the functional-analytical methods. In particular, we construct three families of solutions to the same differential equation: one family extends the solution of the initial value problem (4.1) in new variables, the other family extends the solution of the initial value problem (4.2), and the third family describes solution decaying to zero at infinity. By using our methods, we see necessity of adding technical non-degeneracy assumptions (Assumptions 4.1 and 4.2), which were not mentioned previously.*

Figure 4.1 illustrates the result of Theorem 4.1 and shows the numerically computed solution curve (the graph of λ as a function of b) for $d = 5$ (left) and $d = 13$ (right). In agreement with Theorem 4.1, we confirm the oscillatory behavior in the former case and the monotone behavior in the latter case. We also note that the unique value of $\lambda = \lambda(b)$ is found for every $b > 0$ in both cases (see Remarks 3.1, 3.3, and 4.2).

Notations. We denote $A = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$ if there exists an $\varepsilon_0 > 0$ and an ϵ-independent constant $C > 0$ such that $|A| \leq C\varepsilon$ if $\varepsilon \in (0, \varepsilon_0)$.

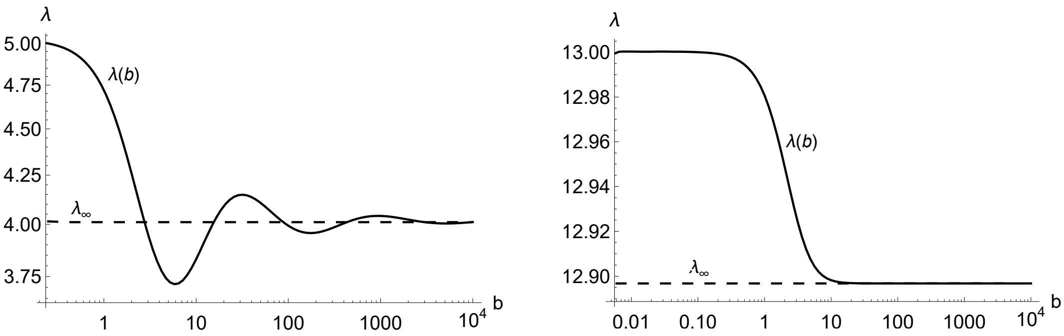

FIGURE 4.1: Graph of λ as a function of b for the ground state of the boundary-value problem (1.7) with $p = 1$ for $d = 5$ (left) and $d = 13$ (right).

4.1 Two solution families

For the reader's convenience, we begin by recalling several important functions defined in Chapter 3, which will be used in the proof of Theorem 4.1. By applying the Emden-Fowler transformation [18]

$$r = e^t, \quad F(r) = \Psi(t) \quad F'(r) = e^{-t}\Psi'(t), \tag{4.7}$$

to initial-value problem (4.2), we obtain the differential equation

$$\Psi''(t) + (d-4)\Psi'(t) + (3-d)\Psi(t) + \Psi(t)^3 = -\lambda e^{2t}\Psi(t) + e^{4t}\Psi(t), \quad t \in \mathbb{R}, \tag{4.8}$$

together with boundary conditions

$$\begin{cases} \Psi(t) \to \sqrt{d-3}, \\ \Psi'(t) \to 0, \end{cases} \quad \text{as} \quad t \to -\infty. \tag{4.9}$$

In order to prove Theorem 4.1, we study three particular solutions to differential equation (4.8). Similarly to Subsection 3.4, one solution to (4.8) is defined from the boundary conditions (4.9) and is denoted by Ψ_λ. It satisfies the asymptotic behavior

$$\Psi_\lambda(t) = \sqrt{d-3}\left[1 - \frac{\lambda}{4d-10}e^{2t} + \mathcal{O}(e^{4t})\right] \quad \text{as } t \to -\infty. \tag{4.10}$$

Another solution to (4.8) is obtained from the unique solution constructed in Lemma 3.2 after the scaling transformation $\Psi(t) = e^t\psi(t)$, where $\psi(t)$ satisfies the differential equation (3.8) and the boundary conditions (3.9). In order to distinguish this solution from Ψ_λ, we denote it by Ψ_b. It follows from (3.12) that Ψ_b satisfies the asymptotic behavior

$$\Psi_b(t) = be^t - (\lambda b + b^3)(2d)^{-1}e^{3t} + \mathcal{O}(e^{5t}) \quad \text{as} \quad t \to -\infty. \tag{4.11}$$

The truncated (autonomous) version to the second-order equation (4.8) is given by

$$\Theta''(t) + (d-4)\Theta'(t) + (3-d)\Theta(t) + \Theta(t)^3 = 0. \tag{4.12}$$

With an elementary exercise, we have the following lemma.

Lemma 4.1. *Fix $d \geq 5$. There exists a unique orbit of the truncated equation (4.12) on the phase plane (Θ, Θ') that connects the equilibrium points $(0,0)$ and $(\sqrt{d-3}, 0)$.*

Proof. The equilibrium point $(0,0)$ is a saddle point with two roots $\kappa_1 = 1$ and $\kappa_2 = 3-d < 0$ of the characteristic equation

$$\kappa^2 + (d-4)\kappa + (3-d) = 0. \tag{4.13}$$

By the unstable curve theorem, there exists a unique unstable curve on the plane (Θ, Θ') tangential to the direction $(1,1)$, along which two orbits exist satisfying $\Theta(t) \to 0$ as $t \to -\infty$. One orbit is connected to $(0,0)$ in the first quadrant of the (Θ, Θ')-plane and the other orbit is connected to $(0,0)$ in the third quadrant. Because the stable curve is connected to $(0,0)$ in the second and fourth quadrants and the orbits of the planar system do not intersect away from the equilibrium points, the unstable orbit connected to $(0,0)$ in the first quadrant stays in the right half-plane with positive Θ and the unstable orbit connected to $(0,0)$ in the third quadrant stays in the left half-plane with negative Θ. For the proof of the lemma, we only consider the former unstable orbit and introduce the energy function

$$V(\Theta, \Theta') := \frac{1}{2}(\Theta')^2 + \frac{1}{2}(3-d)\Theta^2 + \frac{1}{4}\Theta^4.$$

If $\Theta(t) \in C^2(\mathbb{R})$ is a solution to the second-order equation (4.12), then

$$\frac{d}{dt}V(\Theta, \Theta') = (4-d)(\Theta')^2 \leq 0. \tag{4.14}$$

Since $V(\Theta, \Theta')$ is bounded from below, and its value is monotonically decreasing, the unstable orbit stays in a compact region of the right-half of the phase plane (Θ, Θ'). No periodic orbits exist in this compact region, because if $\Theta(t+T) = \Theta(t)$ is periodic with the minimal period $T > 0$, then we get contradiction with (4.14):

$$0 = V(\Theta, \Theta')|_{t=T} - V(\Theta, \Theta')|_{t=0} = (4-d)\int_0^T (\Theta')^2\, dt < 0.$$

Hence, the unstable curve from $(0,0)$ has the limit set at the stable equilibrium point. The only stable equilibrium point in the right-half of the phase plane (Θ, Θ') is the point $(\sqrt{d-3}, 0)$, hence $(0,0)$ and $(\sqrt{d-3}, 0)$ are connected by the unique heteroclinic orbit. □

Since the truncated equation (4.12) is autonomous, the unique orbit of Lemma 4.1 can be parameterized by the time translation such that

$$\Theta(t+t_0) = e^{t+t_0} - (2d)^{-1} e^{3(t+t_0)} + \mathcal{O}(e^{5(t+t_0)}) \quad \text{as} \quad t \to -\infty, \tag{4.15}$$

where $t_0 \in \mathbb{R}$ is arbitrary and $\Theta(t)$ is uniquely defined. It follows by comparing the asymptotic behaviors (4.11) and (4.15) that the parameter b plays the same role as the translation parameter t_0 with the correspondence $t_0 = \log b$. The following lemma states that, when b is sufficiently large, the solution Ψ_b translated by $\log b$ converges to Θ on the negative half-line.

Lemma 4.2. *Fix $d \geq 5$ and $\lambda \in \mathbb{R}$. There exist $b_0 > 0$ (sufficiently large) and $C_0 > 0$ such that the unique solution Ψ_b to the second-order equation (4.8) with the asymptotic behavior (4.11) satisfies*

$$\sup_{t\in(-\infty,0]} |\Psi_b(t-\log b) - \Theta(t)| + \sup_{t\in(-\infty,0]} |\Psi_b'(t-\log b) - \Theta'(t)| \leq C_0 b^{-2}, \quad b \geq b_0, \tag{4.16}$$

where Θ is the uniquely defined solution to the truncated equation (4.12) with the asymptotic behavior (4.15) in Lemma 4.1.

Proof. By translating t, we rewrite (4.8) for $\Psi_b(t)$ in the form:

$$\Psi''(t)+(d-4)\Psi'(t)+(3-d)\Psi(t)+\Psi(t)^3 = -\lambda b^{-2}e^{2(t+\log b)}\Psi(t)+b^{-4}e^{4(t+\log b)}\Psi(t). \tag{4.17}$$

The solution $\Psi_b(t-\log b)$ is decomposed near the uniquely defined solution $\Theta(t)$ to the truncated equation (4.12) by using $\Psi_b(t-\log b) = \Theta(t) + \Upsilon(t)$, where $\Upsilon(t)$ satisfies the persistence problem

$$L\Upsilon = f_b(\Theta+\Upsilon) + N(\Theta,\Upsilon), \tag{4.18}$$

where

$$\begin{aligned} (L\Upsilon)(t) &= \Upsilon''(t) + (d-4)\Upsilon'(t) + (3-d)\Upsilon(t) + 3\Theta(t)^2\Upsilon(t), \\ f_b(t) &= -\lambda b^{-2}e^{2t} + b^{-4}e^{4t}, \\ N(\Theta,\Upsilon) &= -3\Theta\Upsilon^2 - \Upsilon^3. \end{aligned}$$

There exist two linearly independent solutions $\Theta'(t)$ and $\Xi(t)$ of the homogeneous equation $L\Upsilon = 0$, where $\Theta'(t)$ is due to translation of the truncated equation (4.12) and $\Xi(t)$ is the linearly independent solution satisfying the Wronskian relation from Liouville's theorem:

$$W(\Theta',\Xi)(t) := \Theta'(t)\Xi'(t) - \Theta''(t)\Xi(t) = W_\infty e^{(4-d)t}, \tag{4.19}$$

where W_∞ is an arbitrary nonzero constant. For unique normalization of $\Xi(t)$, we can just set $W_\infty = 1$. Since $\Theta'(t)$ decays to zero as $t \to -\infty$ according to

$$\Theta'(t) = e^t + \mathcal{O}(e^{3t}) \quad \text{as} \quad t \to -\infty,$$

it follows from the integration of (4.19) with $W_\infty = 1$ that $\Xi(t)$ grows as $t \to -\infty$ according to

$$\Xi(t) = (2-d)^{-1} e^{(3-d)t} + \mathcal{O}(e^{(5-d)t}) \quad \text{as} \quad t \to -\infty.$$

By solving the second-order differential equation (4.18) with the variation of parameters, we obtain the integral equation for $\Upsilon(t)$:

$$\Upsilon(t) = -\int_{-\infty}^{t} e^{(d-4)t'} \left[\Xi(t')\Theta'(t) - \Xi(t)\Theta'(t')\right] \left[f_b(t')(\Theta(t') + \Upsilon(t')) + N(\Theta(t'), \Upsilon(t'))\right] dt', \tag{4.20}$$

where the choice of integration from $-\infty$ to t ensures that $\Upsilon(t)$ does not grow as $t \to -\infty$ along the solution $\Xi(t)$ and does not introduce the additional translation in time along the solution $\Theta'(t)$. In order to prove existence of small solutions to the integral equation (4.20) on $(-\infty, 0]$ for large b, we introduce $\tilde{\Upsilon}(t) := e^{-t}\Upsilon(t)$, so that

$$\sup_{t\in(-\infty,0]} |\Upsilon(t)| \leq \sup_{t\in(-\infty,0]} |\tilde{\Upsilon}(t)|. \tag{4.21}$$

Then, $\tilde{\Upsilon}(t)$ is found from the integral equation

$$\tilde{\Upsilon}(t) = -\int_{-\infty}^{t} K(t,t') \left[f_b(t') \left(e^{-t'}\Theta(t') + \tilde{\Upsilon}(t')\right) + e^{2t'} N(e^{-t'}\Theta(t'), \tilde{\Upsilon}(t'))\right] dt', \tag{4.22}$$

where

$$K(t,t') = \left[e^{(d-3)t'}\Xi(t')\right]\left[e^{-t}\Theta'(t)\right] - e^{(d-2)(t'-t)}\left[e^{(d-3)t}\Xi(t)\right]\left[e^{-t'}\Theta'(t')\right].$$

Thanks to the exponential rates of $\Theta'(t)$ and $\Xi(t)$ as $t \to -\infty$, there exists a positive constant A_0 such that

$$\sup_{t\in(-\infty,0], t'\in(-\infty,0]} |K(t',t)| \leq A_0.$$

It is also clear that

$$\sup_{t\in(-\infty,0]} |f_b(t)| \leq (|\lambda| + 1)b^{-2}, \quad b \geq 1,$$

so that the inhomogeneous term of the integral equation (4.22) is small if b is large. By the same fixed-point iterations as in the proof of Lemma 3.2, it follows that there exists a sufficiently large b_0 such that for every $b \geq b_0$ there exists the unique solution $\tilde{\Upsilon}$ to the integral equation (4.22) in a closed subset of the Banach space $L^\infty(-\infty, 0)$ satisfying the bound

$$\sup_{t\in(-\infty,0]} |\tilde{\Upsilon}(t)| \leq C_0 b^{-2}, \tag{4.23}$$

where $C_0 > 0$ is a suitable chosen constant. Bounds (4.21) and (4.23) yield the first bound in (4.16). Since $\tilde{\Upsilon} \in C^1(-\infty, 0)$ by bootstrapping arguments similar to those in the proof of Lemma 3.2, the second bound in (4.16) follows by differentiating (4.22) in t and using bound (4.23). □

Remark 4.4. *It follows from the integral equation (4.22) with the account of exponential rates of $\Xi(t)$ and $\Theta'(t)$ that $\tilde{\Upsilon}(t) = \lambda b^{-2}(2d)^{-1}e^{2t} + \mathcal{O}(e^{4t})$ as $t \to -\infty$, in agreement with the asymptotic expansions (4.11) and (4.15) for $\tilde{\Upsilon}(t) = e^{-t}[\Psi_b(t - \log b) - \Theta(t)]$.*

In addition to the solutions Ψ_λ and Ψ_b to the differential equation (4.8), which are defined from the behavior as $t \to -\infty$, we define the third solution to (4.8) from the decaying behavior as $t \to +\infty$. This solution to (4.8) is denoted by $\Psi_c(t)$. Its existence follows from a modification of the result of Lemma 3.3.

Lemma 4.3. *Fix $d \geq 1$ and $\lambda \in \mathbb{R}$. There exists a one-parameter family of solutions to the second-order equation (4.8) denoted by $\Psi_c(t)$ such that $\Psi_c(t), \Psi_c'(t) \to 0$ as $t \to +\infty$ and*

$$\Psi_c(t) \sim c e^{\frac{\lambda - d + 2}{2}t} e^{-\frac{1}{2}e^{2t}} \quad \text{as} \quad t \to +\infty, \tag{4.24}$$

for some $c \in \mathbb{R}$, where the asymptotic correspondence can be differentiated. Moreover, the solution $\Psi_c(t)$ is extended globally for every $t \in \mathbb{R}$.

Proof. By the chain rule in (4.7), if $\Psi(t), \Psi'(t) \to 0$ as $t \to +\infty$, then $F(r), F'(r) \to 0$ as $r \to \infty$. Since $f(r) = r^{-1}F(r)$, this decay implies $f(r), f'(r) \to 0$ as $r \to \infty$. The precise asymptotic correspondence (4.24) is obtained from (3.16) in Lemma 3.3 and $\Psi_c(t) = e^t f(e^t)$. Global continuation of Ψ_c on $\mathbb{R}$ follows by Lemma 3.4 from the global continuation of the solution $f \in C^2(r_0, \infty)$ for some $r_0 \in (0, \infty)$ to $f \in C^2(0, \infty)$. □

Finally, we discuss linearization of the truncated equation (4.12) at the equilibrium point $(\sqrt{d-3}, 0)$. The characteristic equation

$$\kappa^2 + (d-4)\kappa + 2(d-3) = 0 \tag{4.25}$$

admits the following two roots

$$\kappa_\pm := -\frac{1}{2}(d-4) \pm \frac{1}{2}\sqrt{d^2 - 16d + 40} \tag{4.26}$$

If $5 \leq d \leq 12$, the roots are complex-conjugate and can be written as

$$\kappa_\pm = -\beta \pm i\alpha, \tag{4.27}$$

where real and positive α and β are given by (4.4). If $d \geq 13$, then the roots are real and negative with ordering

$$\kappa_- < \kappa_+ < 0. \tag{4.28}$$

The difference between the two cases can be observed when the solution ψ of Theorem 3.1 is transformed to the variable Ψ, in which case it becomes the intersection of the second solution Ψ_b defined as $t \to -\infty$ and the third solution Ψ_c for some $c = c(b) > 0$ defined as $t \to +\infty$. Both solutions satisfy the differential equation (4.8) for the particular value of $\lambda = \lambda(b)$.

Figure 4.2 shows both components $\psi(t)$ and $\Psi(t)$ for the solution of Theorem 3.1 for $d = 5$ (top) and $d = 13$ (bottom) that corresponds to $b = 14000$. Compared to the component $\psi(t)$ which is monotonically decreasing on $\mathbb{R}$, the component $\Psi(t)$ decays to zero at both infinities. In addition, $\Psi(t)$ develops oscillations at the intermediate range of t for $d = 5$ and no oscillations for $d = 13$.

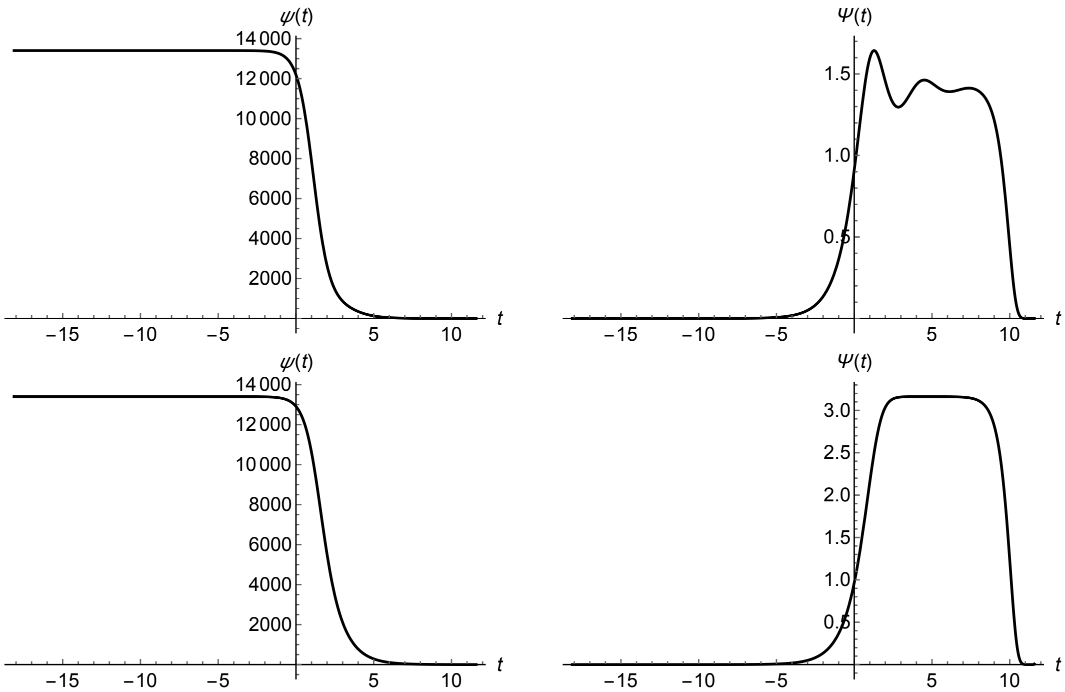

FIGURE 4.2: Components ψ (left) and Ψ (right) for the solution of Theorem 3.1 plotted versus t for $d = 5$ (top) and $d = 13$ (bottom) with $b = 14000$.

Because of the difference between the oscillatory and monotone behavior of the solutions of Theorem 3.1 in variable Ψ, the proof of Theorem 4.1 is developed separately for $5 \leq d \leq 12$ and $d \geq 13$.

4.2 Proof of Theorem 4.1 in the oscillatory case

By Lemma 4.1, there exists the unique solution Θ to the truncated equation (4.12) with the asymptotic behavior (4.15). The following lemma described the oscillatory behavior of $\Theta(t)$ as $t \to +\infty$.

Lemma 4.4. *Fix $5 \leq d \leq 12$. There exist $t_0 > 0$ (sufficiently large), $A_0 > 0$, $\delta_0 \in [0, 2\pi)$, and $C_0 > 0$ such that the unique solution Θ of Lemma 4.1 satisfies the following oscillatory behavior:*

$$\sup_{t\in[t_0,\infty)} |\Theta(t) - \sqrt{d-3} - A_0 e^{-\beta t} \sin(\alpha t + \delta_0)| \leq C_0 e^{-2\beta t_0}. \tag{4.29}$$

where α and β are given by (4.4).

Proof. The equilibrium point $(\sqrt{d-3}, 0)$ is a stable spiral point of the truncated equation (4.12) for $5 \leq d \leq 12$ due to the roots (4.27) of the characteristic equation (4.25). Quadratic terms beyond the linearization at $(\sqrt{d-3}, 0)$ can be removed by a near-identity transformation if $\kappa_\pm = -\beta \pm i\alpha$. By the Hartman–Grobman theorem, there exists a C^2-diffeomorphism, under which the dynamics of the truncated equation (4.12) near $(\sqrt{d-3}, 0)$ is conjugate to the dynamics of the linearized equation. The asymptotic behavior (4.29) follows from the solution of the linearized equation and the existence of the C^2-diffeomorphism. □

Remark 4.5. *It follows from the dynamical system theory that the bound (4.29) can be extended to $\Theta'(t)$ as follows:*

$$\sup_{t\in[t_0,\infty)} |\Theta'(t) - \alpha A_0 e^{-\beta t}\cos(\alpha t + \delta_0) + \beta A_0 e^{-\beta t}\sin(\alpha t + \delta_0)| \leq C_0 e^{-2\beta t_0}. \tag{4.30}$$

For simplicity of writing, we will not write henceforth the explicit bounds on the derivatives.

By extending Lemma 4.2 and using Lemma 4.4, we prove the oscillatory behavior of the solution $\Psi_b(t)$ at the intermediate values of t as $b \to \infty$.

Lemma 4.5. *Fix $5 \leq d \leq 12$ and $\lambda \in \mathbb{R}$. For fixed $T > 0$ and $a \in \left(0, \frac{4}{d}\right)$, there exist $b_{T,a} > 0$ and $C_{T,a} > 0$ such that the unique solution Ψ_b to the second-order equation (4.8) with the asymptotic behavior (4.11) satisfies*

$$\sup_{t\in[0,T+a\log b]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_{T,a} b^{-2(1-a)}, \quad b \geq b_{T,a}. \tag{4.31}$$

Consequently, it follows that

$$\begin{aligned} &|\Psi_b(T + (a-1)\log b) - \sqrt{d-3} - A_0 b^{-a\beta} e^{-\beta T}\sin(\alpha T + \delta_0 + a\alpha\log b)| \\ &\qquad \leq C_{T,a}\max\{b^{-2a\beta}, b^{-2(1-a)}\}, \quad b \geq b_{T,a}, \end{aligned} \tag{4.32}$$

where (α, β) are given by (4.4), (A_0, δ_0) are defined in (4.29), and $(b_{T,a}, C_{T,a})$ are adjusted appropriately.

Proof. We start with the proof of the bound (4.31). This can be done by rewriting the integral equation (4.20) in an equivalent form which is useful for $t \in [0, T + a\log b]$. To do so, we solve the second-order equation (4.18) with the variation of parameters from $t = 0$ towards $t = T + a\log b > 0$. This gives us the integral equation in the form:

$$\begin{aligned} \Upsilon(t) &= \Upsilon(0)\left[\Xi'(0)\Theta'(t) - \Xi(t)\Theta''(0)\right] + \Upsilon'(0)\left[\Xi(t)\Theta'(0) - \Xi(0)\Theta'(t)\right] \\ &- \int_0^t e^{(d-4)t'}\left[\Xi(t')\Theta'(t) - \Xi(t)\Theta'(t')\right]\left[f_b(t')(\Theta(t') + \Upsilon(t')) + N(\Theta(t'), \Upsilon(t'))\right]dt'. \end{aligned} \tag{4.33}$$

By the bound (4.16), there exist $b_0 > 0$ and $C_0 > 0$ such that

$$|\Upsilon(0)| + |\Upsilon'(0)| \leq C_0 b^{-2}, \quad b \geq b_0.$$

It follows from the definition of f_b that

$$\sup_{t\in[0,T+a\log b]} |f_b(t)| \leq (|\lambda| + 1)b^{-2(1-a)}e^{4T}, \quad b \geq 1,$$

where $T > 0$ is fixed independently of b. Since $(\sqrt{d-3}, 0)$ is a stable spiral point of the truncated equation (4.12) for $5 \leq d \leq 12$ with the roots (4.27), both $\Theta'(t)$ and $\Xi(t)$ decays to 0 exponentially fast as $t \to +\infty$ such that

$$|\Theta'(t)| + |\Xi(t)| \leq C_0 e^{-\beta t}, \quad t \geq 0,$$

for some b-independent $C_0 > 0$. The kernel of the integral equation (4.33) behaves like $e^{-\beta(t-t')}$ and decays exponentially as $t \to +\infty$. By the same fixed-point iterations as in the proof of Lemma 3.2, it follows that there exists a sufficiently large $b_{T,a}$ such that for every $b \geq b_{T,a}$ there exists the unique solution Υ to the integral equation (4.33) in a closed subset of Banach space $L^\infty(0, T + a\log b)$ satisfying the bound

$$\sup_{t\in[0,T+a\log b]} |\Upsilon(t)| \leq C_{T,b} b^{-2(1-a)}, \tag{4.34}$$

where $C_{T,b} > 0$ is a suitable chosen constant and $a \in (0, 1)$. Bound (4.34) yields (4.31).

Bound (4.32) follows from (4.29) and (4.31) since $a \log b \to +\infty$ as $b \to \infty$ if $a > 0$ and $b^{-a\beta} \gg b^{-2(1-a)}$ if $a < \frac{4}{d} < 1$. □

Remark 4.6. *The bound (4.16) was used in [7] without improvement given by the bound (4.31). The bound (4.16) is not sufficient for our purpose because if $a = 0$ in Lemma 4.5 then we are not allowed to use the asymptotic behavior (4.29) in order to derive the bound (4.32).*

Remark 4.7. *The constraint $a \in \left(0, \frac{4}{d}\right) \subset (0, 1)$ needed to control the small approximation error in the bound (4.31) implies that the oscillatory behavior (4.32) is observed in $\Psi_b(t)$ for sufficiently large negative t, yet not in the limit $t \to -\infty$. Indeed, $\Psi_b(t)$ satisfies the asymptotic behavior (4.11) and decays to zero as $t \to -\infty$.*

Let us now turn to the one-parameter solution $\Psi_c(t)$ defined by the asymptotic behavior (4.24) as $t \to +\infty$. This solution to the differential equation (4.8) is extended globally for every $t \in \mathbb{R}$ by Lemma 4.3. For $\lambda = \lambda_\infty$, there exists a uniquely defined $c = c_\infty$ such that $\Psi_{c=c_\infty}$ coincides with the unique solution $\Psi_\infty := \Psi_{\lambda=\lambda_\infty}$ which satisfies the asymptotic behavior (4.10) as $t \to -\infty$. Thus, $\Psi_\infty = \Psi_{c=c_\infty} = \Psi_{\lambda=\lambda_\infty}$ is a bounded function on $\mathbb{R}$. However, the functions $\Psi_{c\neq c_\infty}$ and $\Psi_{\lambda\neq\lambda_\infty}$ are not globally bounded on $\mathbb{R}$ due to divergence as $t \to -\infty$ and $t \to +\infty$ respectively.

The unique solution Ψ_c is differentiable in (λ, c) due to the smooth asymptotic behavior (4.24) and the smoothness of the differential equation (4.8). Therefore, we can define

$$\Psi_1 := \partial_\lambda \Psi_c|_{(\lambda,c)=(\lambda_\infty,c_\infty)}, \quad \Psi_2 := \partial_c \Psi_c|_{(\lambda,c)=(\lambda_\infty,c_\infty)}. \tag{4.35}$$

Functions $\Psi_{1,2}$ satisfy the linear second-order equations written in the form

$$\mathcal{L}_0\Psi_1 = f + g\Psi_1, \qquad \mathcal{L}_0\Psi_2 = g\Psi_2, \tag{4.36}$$

where

$$\begin{aligned}(\mathcal{L}_0\Psi)(t) &:= \Psi''(t) + (d-4)\Psi'(t) + 2(d-3)\Psi(t), \\ f(t) &:= -e^{2t}\Psi_\infty(t), \\ g(t) &:= 3(d-3-\Psi_\infty(t)^2) - \lambda_\infty e^{2t} + e^{4t}.\end{aligned}$$

We add the following technical assumption.

Assumption 4.1. *Uniquely defined functions Ψ_∞ and Ψ_2 are assumed to satisfy the following non-degeneracy assumption:*

$$\int_{-\infty}^{\infty} e^{(d-2)t}\Psi_\infty(t)\Psi_2(t)dt \neq 0. \tag{4.37}$$

Remark 4.8. *The non-degeneracy assumption (4.37) can be equivalently written as*

$$\frac{\partial}{\partial c}\int_{-\infty}^{\infty} e^{(d-2)t}\Psi_c(t)^2 dt\Big|_{\lambda=\lambda_\infty, c=c_\infty} \neq 0,$$

or

$$\frac{\partial}{\partial c}\int_0^{\infty} r^{d-3}F_c(r)^2 dr\Big|_{\lambda=\lambda_\infty, c=c_\infty} \neq 0,$$

or

$$\frac{\partial}{\partial c}\int_0^{\infty} r^{d-1}f_c(r)^2 dr\Big|_{\lambda=\lambda_\infty, c=c_\infty} \neq 0,$$

where $f_c(r) = r^{-1}F_c(r) = r^{-1}\Psi_c(\log r)$.

Remark 4.9. *One can reformulate the constraint (4.37) from a different point of view. Recall the solution Ψ_λ to the second-order equation (4.8) satisfying the asymptotic behavior (4.10) as $t \to -\infty$ and extended globally. Derivative $\partial_\lambda\Psi_\lambda$ satisfies the same differential equation (4.36) as Ψ_1 but compared to Ψ_1, $\partial_\lambda\Psi_\lambda(t)$ generally diverges as $t \to +\infty$. The condition (4.37) ensures that $\partial_\lambda\Psi_\lambda$ is not spanned by the derivatives of the solution $\Psi_c(t)$ in λ and c, which decays to zero as $t \to \infty$. Hence, the constraint*

(4.37) is a transversality condition between the two C^1 families of solutions to the differential equation (4.8) given by Ψ_λ and Ψ_c. Note that this transversality condition was not mentioned in the previous works in [6, 7, 15] on a related subject.

The following lemma determines the behavior of the solutions $\Psi_{1,2}$ for large negative t and the solution Ψ_c for parameters (λ, c) near the point $(\lambda_\infty, c_\infty)$.

Lemma 4.6. *Fix $5 \le d \le 12$. For fixed $T > 0$ and $a \in (0,1)$, there exist $b_{T,a} > 0$, $C_{T,a} > 0$, $A_{1,2}$, $B_{1,2}$ such that $\Psi_{1,2}$ in (4.35) satisfy for every $t \in (-\infty, (a-1)\log b + T]$:*

$$|\Psi_{1,2}(t) - A_{1,2}e^{-\beta t}\sin(\alpha t) - B_{1,2}e^{-\beta t}\cos(\alpha t)| \le C_{T,a} b^{-2(1-a)} e^{-\beta t}, \quad b \ge b_{T,a}, \tag{4.38}$$

where (α, β) are given by (4.4). Consequently, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ and for every $(\lambda, c) \in \mathbb{R}^2$ satisfying

$$(\lambda - \lambda_\infty)^2 + (c - c_\infty)^2 \le \epsilon^2 b^{-2\beta(1-a)}, \tag{4.39}$$

it is true for every $b \ge b_{T,a}$ and every $t \in [(a-1)\log b, (a-1)\log b + T]$ that

$$\begin{aligned} &|\Psi_c(t) - \sqrt{d-3} - [A_1(\lambda-\lambda_\infty) + A_2(c - c_\infty)]e^{-\beta t}\sin(\alpha t) \\ &\qquad\qquad - [B_1(\lambda - \lambda_\infty) + B_2(c-c_\infty)]e^{-\beta t}\cos(\alpha t)| \\ &\le C_{T,a}\Big(b^{-2(1-a)} + (\lambda-\lambda_\infty)b^{-(2-\beta)(1-a)} + (c-c_\infty)b^{-(2-\beta)(1-a)} \\ &\qquad\qquad + (\lambda-\lambda_\infty)^2 b^{2\beta(1-a)} + (c-c_\infty)^2 b^{2\beta(1-a)}\Big), \end{aligned} \tag{4.40}$$

where $b_{T,a}$ and $C_{T,a}$ are adjusted appropriately. If Assumption 4.1 is satisfied, then

$$A_1 B_2 \ne A_2 B_1. \tag{4.41}$$

Proof. Since $\Psi_\infty(t) = \sqrt{d-3} + \mathcal{O}(e^{2t})$ as $t \to -\infty$, there exist $b_{T,a} > 0$ and $C_{T,a} > 0$ such that

$$\sup_{t \in (-\infty, (a-1)\log b + T]} (|f(t)| + |g(t)|) \le C_{T,a} b^{-2(1-a)}, \quad b \ge b_{T,a}, \tag{4.42}$$

where $T > 0$ and $a \in (0,1)$ are fixed independently of b. The left-hand side of linear equations (4.36) coincides with the linearized equation near the stable spiral point $(\sqrt{d-3}, 0)$ with two roots (4.27). By variation of parameters, we can rewrite the linear equations for $\Psi_{1,2}$ in the integral form:

$$\begin{aligned} \Psi_{1,2}(t) &= A_{1,2}e^{-\beta t}\sin(\alpha t) + B_{1,2}e^{-\beta t}\cos(\alpha t) \\ &\quad + \alpha^{-1}\int_{-\infty}^{t} e^{-\beta(t-t')}\sin(\alpha(t-t'))\left[f(t')e_{1,2} + g(t')\Psi_{1,2}(t')\right] dt', \end{aligned} \tag{4.43}$$

where $A_{1,2}$, $B_{1,2}$ are some constant coefficients and $e_1 = 1$, $e_2 = 0$. The kernel of the integral equations (4.43) is bounded in the variable $\tilde{\Psi}_{1,2}(t) = e^{\beta t}\Psi_{1,2}(t)$, for which we can write

$$\tilde{\Psi}_{1,2}(t) = A_{1,2}\sin(\alpha t) + B_{1,2}\cos(\alpha t) \\ +\alpha^{-1}\int_{-\infty}^{t}\sin(\alpha(t-t'))\left[f(t')e^{\beta t'}e_{1,2} + g(t')\tilde{\Psi}_{1,2}(t')\right]dt'. \quad (4.44)$$

By the same fixed-point iterations as in the proof of Lemma 3.2, there exists the unique solutions $\tilde{\Psi}_{1,2}$ to the integral equations (4.44) in a closed subset of Banach space $L^\infty(-\infty, T+(a-1)\log b)$ satisfying the bounds

$$\sup_{t\in(-\infty,T+(a-1)\log b]} |\tilde{\Psi}_{1,2}(t) - A_{1,2}\sin(\alpha t) - B_{1,2}\cos(\alpha t)| \leq C_{T,a}b^{-2(1-a)}, \qquad b \geq b_{T,a},$$

due to bounds (4.42). By using the transformation $\tilde{\Psi}_{1,2}(t) = e^{\beta t}\Psi_{1,2}(t)$, we obtain (4.38).

In order to justify (4.40), we substitute the decomposition $\Psi_c = \Psi_\infty + \Sigma$ into (4.8) and obtain the following persistence problem:

$$\mathcal{L}_\infty\Sigma = \mathcal{F}, \quad (4.45)$$

where

$$\begin{aligned} (\mathcal{L}_\infty\Sigma)(t) &:= \Sigma''(t) + (d-4)\Sigma'(t) + (3-d)\Sigma(t) + 3\Psi_\infty(t)^2\Sigma(t) + \lambda_\infty e^{2t}\Sigma(t) - e^{4t}\Sigma(t), \\ \mathcal{F}(t) &:= -(\lambda-\lambda_\infty)e^{2t}(\Psi_\infty(t)+\Sigma(t)) - 3\Psi_\infty(t)\Sigma(t)^2 - \Sigma(t)^3. \end{aligned}$$

Let $\{\Sigma_1, \Sigma_2\}$ be the fundamental system of the homogeneous equation $\mathcal{L}_\infty\Sigma = 0$ subject to the normalization

$$\begin{cases} \Sigma_1(0) = 1, \\ \Sigma_1'(0) = 0, \end{cases} \qquad \begin{cases} \Sigma_2(0) = 0, \\ \Sigma_2'(0) = 1. \end{cases}$$

Since $\mathcal{L}_\infty = \mathcal{L}_0 - g$ and $g(t) = \mathcal{O}(e^{2t})$ as $t \to -\infty$, the functions $\Sigma_1(t)$ and $\Sigma_2(t)$ diverge like $\mathcal{O}(e^{-\beta t})$ as $t \to -\infty$, so that there exists a positive constant C such that

$$\sup_{t\in(-\infty,0]} e^{\beta t}\left(|\Sigma_1(t)| + |\Sigma_2(t)|\right) \leq C. \quad (4.46)$$

The Wronskian relation from Liouville's theorem yields

$$W(\Sigma_1,\Sigma_2)(t) = \Sigma_1(t)\Sigma_2'(t) - \Sigma_1'(t)\Sigma_2(t) = e^{-2\beta t}, \quad (4.47)$$

where $2\beta = d-4$. By variation of parameters, we can rewrite the differential equation (4.45) in the integral form:

$$\Sigma(t) = \Sigma(0)\Sigma_1(t) + \Sigma'(0)\Sigma_2(t) \\ + \int_t^0 e^{2\beta t'}\left[\Sigma_1(t)\Sigma_2(t') - \Sigma_1(t')\Sigma_2(t)\right]\mathcal{F}(t')dt', \quad (4.48)$$

where $t < 0$.

For simplicity, let us set $c = c_\infty$ and consider λ satisfying $|\lambda - \lambda_\infty| \leq \epsilon e^{-\beta(1-a)}$. The proof of the general case under the bound (4.39) is similar. We set $\tilde{\Sigma}(t) := e^{\beta t}\Sigma(t)$ as before and rewrite the integral equation (4.48) in the form:

$$\begin{aligned}\tilde{\Sigma}(t) = \Sigma(0)e^{\beta t}\Sigma_1(t) + \Sigma'(0)e^{\beta t}\Sigma_2(t)& \\ -(\lambda - \lambda_\infty)\int_t^0 K(t,t')e^{2t'}\left[e^{\beta t'}\Psi_\infty(t') + \tilde{\Sigma}(t')\right]dt'& \\ -\int_t^0 K(t,t')\left[3\Psi_\infty(t')e^{-\beta t'}\tilde{\Sigma}(t')^2 - e^{-2\beta t'}\tilde{\Sigma}(t')^3\right]dt',&\end{aligned} \tag{4.49}$$

where the kernel $K(t,t') := e^{\beta(t+t')}\left[\Sigma_1(t)\Sigma_2(t') - \Sigma_1(t')\Sigma_2(t)\right]$ satisfies the bound

$$\sup_{t\in(-\infty,0],t'\in(-\infty,0]} |K(t,t')| \leq C. \tag{4.50}$$

which follows from (4.46). By the smoothness of Ψ_c in λ, we have $|\Sigma(0)|+|\Sigma'(0)| \leq C|\lambda-\lambda_\infty|$. The nonlinear terms grow as $t \to -\infty$, therefore, the fixed-point arguments cannot be closed in $L^\infty(-\infty,0)$. However, they can be closed in the ball $B_\delta \subset L^\infty((a-1)\log b, 0)$ provided that $\delta = C\epsilon e^{-\beta(1-a)}$ with sufficiently small $\epsilon > 0$ and some $C > 0$. In particular, the nonlinear terms are contractive if ϵ is sufficiently small. By using the first-point iterations, there exists the unique solution to the integral equation (4.49) satisfying

$$\sup_{t\in[(a-1)\log b,0]} |\tilde{\Sigma}(t)| \leq C|\lambda - \lambda_\infty| \leq C\epsilon b^{-\beta(1-a)}. \tag{4.51}$$

Since $\tilde{\Sigma}$ is smooth in λ and $\partial_\lambda\tilde{\Sigma}|_{\lambda-\lambda_\infty} = \tilde{\Psi}_1$ constructed above, we then conclude that

$$\sup_{t\in[(a-1)\log b,0]} |\tilde{\Sigma}(t) - (\lambda - \lambda_\infty)\tilde{\Psi}_1| \leq C(\lambda - \lambda_\infty)^2 b^{\beta(1-a)} \leq C\epsilon^2 b^{-\beta(1-a)}. \tag{4.52}$$

Bound (4.40) follows from the decomposition $\Psi_c = \Psi_\infty + \Sigma$, the expansion $\Psi_\infty(t) = \sqrt{d-3} + \mathcal{O}(e^{2t})$ as $t \to -\infty$, the bound (4.38) on the first derivatives, and the bound (4.52) on the higher-order terms.

It remains to prove that $A_1B_2 \neq A_2B_1$ under Assumption 4.1. Since the differential equation (4.36) is homogeneous and Ψ_2 in (4.35) is nonzero due to the boundary conditions (4.24), it follows that $(A_2, B_2) \neq (0,0)$ by uniqueness of the zero solution in the integral equation (4.44) for $\tilde{\Psi}_2$. If $A_1B_2 = A_2B_1$, then there exists $\mu \in \mathbb{R}$ such that $(A_1, B_1) = \mu(A_2, B_2)$ and $\Delta(t) := \Psi_1(t) - \mu\Psi_2(t)$ satisfies the integral equation that follows from (4.43):

$$\Delta(t) = \alpha^{-1}\int_{-\infty}^t e^{-\beta(t-t')}\sin(\alpha(t-t'))\left[f(t') + g(t')\Delta(t')\right]dt'. \tag{4.53}$$

By the previous arguments, there exists the unique solution to the integral equation (4.53) for $\tilde{\Delta}(t) = e^{\beta t}\Delta(t)$ in a closed subset of Banach space $L^\infty(-\infty, T + (a-1)\log b)$. Moreover, $\tilde{\Delta}(t) \to 0$ as $t \to -\infty$. The Wronskian between Δ and Ψ_2 satisfies the inhomogeneous equation

$$\frac{d}{dt}e^{(d-4)t}W(\Delta,\Psi_2) = -e^{(d-4)t}f(t)\Psi_2(t). \tag{4.54}$$

Due to the fast decay of $\Psi_c(t)$ as $t \to +\infty$ in (4.24), we integrate the inhomogeneous equation (4.54) on $\mathbb{R}$ and obtain the contradiction with the constraint (4.37) in Assumption 4.1:

$$0 = \lim_{t\to-\infty} e^{(d-4)t}W(\Delta,\Psi_2) = \int_{-\infty}^{\infty} e^{(d-4)t}f(t)\Psi_2(t)dt = -\int_{-\infty}^{\infty} e^{(d-2)t}\Psi_\infty(t)\Psi_2(t)dt \neq 0,$$

where for the first equality we have used that $\tilde{\Psi}_2(t)$ is bounded and $\tilde{\Delta}(t)$ is decaying to zero as $t \to -\infty$. The contradiction implies that $A_1B_2 \neq A_2B_1$ under Assumption 4.1. □

The proof of Theorem 4.1 for $5 \leq d \leq 12$ is developed based on Lemmas 4.5 and 4.6.

Proof of Theorem 4.1 for $5 \leq d \leq 12$.

By Theorem 3.1, the solution $\Psi_b(t)$ exists for a certain value of λ denoted by $\lambda(b)$ for every $b > 0$. By Lemma 3.3, it satisfies the asymptotic behavior (4.24) for uniquely selected $c = c(b)$. Therefore, for this value of $\lambda = \lambda(b)$, we have

$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R}. \tag{4.55}$$

By comparing the bound (4.32) of Lemma 4.5 for any fixed $T \in \mathbb{R}$ and $a \in \left(0, \frac{4}{d}\right)$ with the bound (4.40) of Lemma 4.6 at the time instance $t = T + (a-1)\log b$, we obtain the system of nonlinear equations:

$$\left\{ \begin{array}{l} A_1(\lambda(b)-\lambda_\infty) + A_2(c(b)-c_\infty) = A_0 b^{-\beta}\cos(\delta_0 + \alpha\log b) + E_1 \\ B_1(\lambda(b)-\lambda_\infty) + B_2(c(b)-c_\infty) = A_0 b^{-\beta}\sin(\delta_0 + \alpha\log b) + E_2, \end{array} \right. \tag{4.56}$$

where coefficients (A_1, A_2) and (B_1, B_2) are the same as in (4.38) and (E_1, E_2) are error terms satisfying

$$\begin{aligned} E_{1,2} = \mathcal{O}(&b^{-(1+a)\beta}, b^{-(2+\beta)(1-a)}, (\lambda(b)-\lambda_\infty)b^{-2(1-a)}, (c(b)-c_\infty)b^{-2(1-a)}, \\ &(\lambda(b)-\lambda_\infty)^2 b^{\beta(1-a)}, (c(b)-c_\infty)^2 b^{\beta(1-a)}) \end{aligned}$$

as $b \to \infty$, provided that $(\lambda(b), c(b))$ satisfy the bound (4.39) for some $\epsilon > 0$. By Lemma 4.6, it follows that $A_1B_2 \neq A_2B_1$ so that the matrix in (4.56) is invertible. By the implicit function theorem, there exist constants A_∞, B_∞, δ_∞, and ν_∞ such that the unique solution to the system (4.56) is given by

$$\left\{ \begin{array}{l} \lambda(b)-\lambda_\infty = A_\infty b^{-\beta}\sin(\delta_\infty + \alpha\log b) + \mathcal{O}(b^{-(1+a)\beta}, b^{-(2+\beta)(1-a)}, b^{-\beta-2(1-a)}), \\ c(b)-c_\infty = B_\infty b^{-\beta}\sin(\nu_\infty + \alpha\log b) + \mathcal{O}(b^{-(1+a)\beta}, b^{-(2+\beta)(1-a)}, b^{-\beta-2(1-a)}). \end{array} \right. \tag{4.57}$$

The solution (4.57) satisfies the bound (4.39) since $b^{-\beta} \ll b^{-\beta(1-a)}$ for $a > 0$. On the other hand, if $a < \frac{4}{d}$, then $b^{-\beta} \gg e^{-(2+\beta)(1-a)}$, and the error terms in (4.57) are smaller compared to the leading-order terms. Expansion (4.57) justifies the expansion (4.3). □

Remark 4.10. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence of roots of $\lambda(b) = \lambda_\infty$. It follows from* (4.57) *that*

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = e^{\frac{\pi}{\alpha}}. \tag{4.58}$$

We verified the asymptotic limit (4.58) *numerically. The results for* $d = 5$ *are given in Table 4.1, where* $e^{\frac{\pi}{\alpha}} \approx 5.06478$.

n	b_n	b_{n+1}/b_n
1	3.7733455	5.37388
2	20.277514	5.07167
3	102.84079	5.08211
4	522.64782	5.06062
5	2644.9194	5.06744
6	13402.960	5.06352
7	67866.139	5.06588
8	343801.49	5.06317
9	1740725.8	

TABLE 4.1: Approximate values of b_n such that $\lambda(b_n) = \lambda_\infty$ for $d = 5$.

Figure 4.3 illustrates the solutions Ψ_b of the second-order equation (4.8) for $d = 5$ and $b = b_1, b_3, b_6$, where $\{b_n\}_{n\in\mathbb{N}}$ are defined in Table 4.1. The left panel shows that the solutions Ψ_b translated in t by $\log b$ in comparison with the solution Θ of the truncated equation (4.12). The right panel shows the solutions Ψ_b without translation in comparison with the limiting singular solution Ψ_∞ satisfying (4.8) and (4.9). The left panel confirms convergence of $\{\Psi_{b_n}(\cdot - \log b_n)\}_{n\in\mathbb{N}}$ to Θ on $(-\infty, t_0]$ for a fixed $t_0 > 0$. The right panel confirms convergence of $\{\Psi_{b_n}\}_{n\in\mathbb{N}}$ to Ψ_∞ on $[t_0, \infty)$ for a fixed $t_0 < 0$.

Figure 4.4 shows solutions Ψ_b for $b = b_1$ and $b = b_6$ on the phase plane (Ψ, Ψ') together with the solution Θ of the truncated equation (4.12) and the limiting singular solution Ψ_∞ satisfying (4.8) and (4.9). The difference of $\Psi_{b=b_6}$ (red dotted line) from Θ and Ψ_∞ is almost invisible, whereas the difference is large in the case of $\Psi_{b=b_1}$ (blue dotted line).

4.3 Proof of Theorem 4.1 in the monotone case

Here we state and prove the corresponding modifications of results of Lemmas 4.4, 4.5, and 4.6 in the case $d \geq 13$. The following lemma described the exponential behavior of $\Theta(t)$ as $t \to +\infty$.

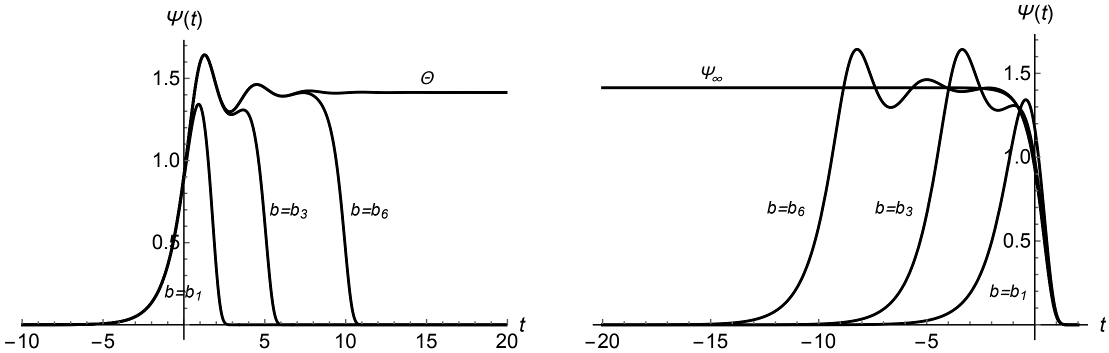

FIGURE 4.3: Plots of the solutions Ψ_b for $d = 5$ and $b = b_1, b_3, b_6$ in comparison with Θ after translation of t by $\log b$ (left) and with Ψ_∞ (right).

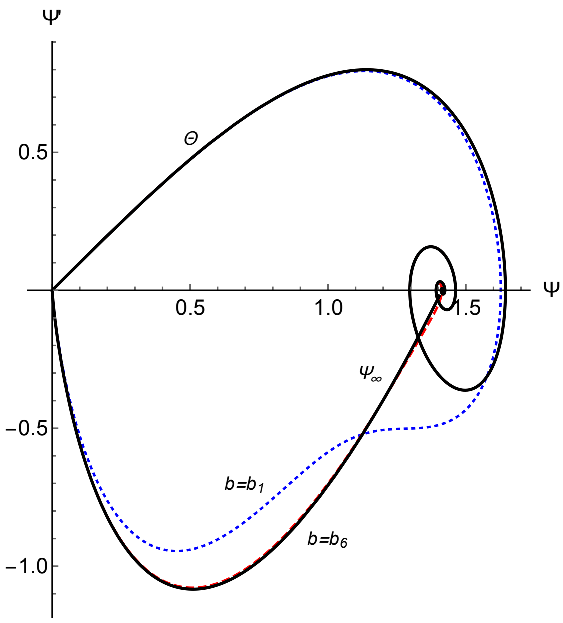

FIGURE 4.4: Solutions Ψ_{b_1}, Ψ_{b_6}, Θ, and Ψ_∞ on the phase plane (Ψ, Ψ') for $d = 5$.

Lemma 4.7. *Fix $d \geq 13$. There exist $t_0 > 0$ (sufficiently large), $A_0 > 0$, $B_0 > 0$, and $C_0 > 0$ such that the unique solution Θ of Lemma 4.1 satisfies the following behavior:*

$$\sup_{t\in[t_0,\infty)} |\Theta(t) - \sqrt{d-3} - A_0 e^{\kappa_+ t} - B_0 e^{\kappa_- t}| \leq C_0 e^{2\kappa_+ t_0}, \tag{4.59}$$

where (κ_+, κ_-) are given by (4.26).

Proof. The equilibrium point $(\sqrt{d-3}, 0)$ is a stable sink of the truncated equation (4.12) for $d \geq 13$ due to the roots (4.26) of the characteristic equation (4.25) satisfying (4.28). Quadratic terms beyond the linearization at $(\sqrt{d-3}, 0)$ can be removed by a near-identity transformation under the non-resonance condition $\kappa_- \neq 2\kappa_+$ which is satisfied since there are no integer solutions of the quadratic equation $d^2 - 17d + 43 = 0$. By the Hartman–Grobman theorem, there exists a C^2-diffeomorphism, under which the dynamics of the truncated equation (4.12) near $(\sqrt{d-3}, 0)$ is conjugate to the dynamics of the linearized equation. The asymptotic behavior (4.59) follows from the solution of the linearized equation and the existence of the C^2-diffeomorphism. □

Remark 4.11. *Because $\kappa_- < \kappa_+ < 0$, the function $\Theta(t)$ approaches $\sqrt{d-3}$ monotonically and the bound (4.59) can be rewritten in a simpler way:*

$$\sup_{t\in[t_0,\infty)} |\Theta(t) - \sqrt{d-3} - A_0 e^{\kappa_+ t}| \leq C_0 \max\{e^{\kappa_- t_0}, e^{2\kappa_+ t_0}\}, \tag{4.60}$$

from which the monotone behavior of $\Theta(t)$ as $t \to +\infty$ is obvious.

Using Lemma 4.7, the statement of Lemma 4.5 is modified to yield the exponential behavior of the solution $\Psi_b(t)$ at the intermediate values of t as $b \to \infty$.

Lemma 4.8. *Fix $d \geq 13$ and $\lambda \in \mathbb{R}$. For fixed $T > 0$ and $a \in (0, a_0)$, where $a_0 \in (0,1)$ is defined by (4.62), there exists $b_{T,a} > 0$ and $C_{T,a} > 0$ such that the unique solution Ψ_b to the second-order equation (4.8) with the asymptotic behavior (4.11) satisfies for $b \geq b_{T,a}$:*

$$|\Psi_b(T + (a-1)\log b) - \sqrt{d-3} - A_0 b^{a\kappa_+} e^{\kappa_+ T}| \leq C_{T,a} \max\{b^{a\kappa_-}, b^{2a\kappa_+}, b^{-2(1-a)}\}, \tag{4.61}$$

where (κ_+, κ_-) are given by (4.26) and A_0 is defined in (4.60).

Proof. The proof of the bound (4.31) remains the same for every $d \geq 5$. Bound (4.61) follows from (4.31) and (4.60) since $a \log b \to +\infty$ as $b \to \infty$ if $a > 0$ and $b^{a\kappa_+} \gg b^{-2(1-a)}$ if $a < a_0$, where

$$a_0 := \frac{2}{2 + |\kappa_+|} = \frac{4}{d - \sqrt{d^2 - 16d + 40}} = \frac{d + \sqrt{d^2 - 16d + 40}}{2(2d-5)}. \tag{4.62}$$

Note that $a_0 < \frac{1}{2}$ for every $d \geq 13$. □

Finally, we recall again that Ψ_c coincides with Ψ_∞ for $(\lambda, c) = (\lambda_\infty, c_\infty)$ and define $\Psi_{1,2}$ as in (4.35). We add the following technical assumption.

Assumption 4.2. *Uniquely defined functions Ψ_∞ and Ψ_2 are assumed to satisfy the following non-degeneracy assumptions:*

$$\int_{-\infty}^{\infty} e^{(d-2)t}\Psi_\infty(t)\Psi_2(t)dt \neq 0 \tag{4.63}$$

and

$$\lim_{t\to-\infty} e^{-\kappa_- t}\Psi_2(t) \neq 0. \tag{4.64}$$

Remark 4.12. *Compared to Assumption 4.1, we have an additional assumption (4.64) in Assumption 4.2. This additional condition excludes solutions of the homogeneous equation $\mathcal{L}_\infty \Psi_2 = 0$ decaying to zero as $t \to +\infty$ to grow slowly as $\mathcal{O}(e^{\kappa_+ t})$ as $t \to -\infty$.*

The following lemma determines the exponential behavior of the solution Ψ_c for (λ, c) near the point $(\lambda_\infty, c_\infty)$.

Lemma 4.9. *Fix $d \geq 13$. There exist $L_1, L_2 \in \mathbb{R}$ such that*

$$L_{1,2} = \lim_{t\to-\infty} e^{-\kappa_- t}\Psi_{1,2}(t). \tag{4.65}$$

If Assumption 4.2 is satisfied, then $L_2 \neq 0$ and for fixed $T > 0$ and $a \in (0,1)$, there exist $b_{T,a} > 0$, $C_{T,a} > 0$, and $\Delta_0 \neq 0$ such that $\Delta(t) := \Psi_1(t) - L_2^{-1}L_1\Psi_2(t)$ satisfy for every $t \in (-\infty, (a-1)\log b + T]$:

$$|\Delta(t) - \Delta_0 e^{\kappa_+ t}| \leq C_{T,a} b^{-2(1-a)} e^{\kappa_+ t}, \qquad b \geq b_{T,a}, \tag{4.66}$$

where (κ_+, κ_-) are given by (4.26). Consequently, there exists $\epsilon_0 > 0$ such that for every $\epsilon \in (0, \epsilon_0)$ and for every $\lambda \in \mathbb{R}$ satisfying

$$|\lambda - \lambda_\infty| \leq \epsilon b^{\kappa_+(1-a)}, \tag{4.67}$$

it is true for every $b \geq b_{T,a}$ and every $t \in [(a-1)\log b, (a-1)\log b + T]$ that

$$\begin{aligned} &|\Psi_{c_\infty - L_2^{-1}L_1(\lambda-\lambda_\infty)}(t) - \sqrt{d-3} - \Delta_0(\lambda-\lambda_\infty)e^{\kappa_+ t}| \\ &\quad \leq C_{T,a}\left(b^{-2(1-a)} + (\lambda(b)-\lambda_\infty)b^{-(2+\kappa_+)(1-a)} + (\lambda(b)-\lambda_\infty)^2 b^{-2\kappa_+(1-a)}\right) \end{aligned} \tag{4.68}$$

where $b_{T,a}$ and $C_{T,a}$ are adjusted appropriately.

Proof. The proof of Lemma 4.9 follows the same steps as the proof of Lemma 4.6 but incorporates the different exponential behavior of the solutions $\Psi_{1,2}(t)$ in (4.35) as $t \to -\infty$. By variation of parameters, the linear equations for $\Psi_{1,2}$ can be rewritten in the integral form:

$$\Psi_{1,2}(t) = A_{1,2}e^{\kappa_+ t} + B_{1,2}e^{\kappa_- t} + \frac{1}{\kappa_+ - \kappa_-}\int_t^{+\infty} \left[e^{\kappa_-(t-t')} - e^{\kappa_+(t-t')}\right] \left[f(t')e_{1,2} + g(t')\Psi_{1,2}(t')\right] dt' \quad (4.69)$$

where $A_{1,2}$, $B_{1,2}$ are some constant coefficients and $e_1 = 1$, $e_2 = 0$. Since $\kappa_- < \kappa_+ < 0$, whereas $f(t)$ and $g(t)\Psi_{1,2}(t)$ decays to zero fast as $t \to +\infty$, the integral kernel in (4.69) becomes bounded in the variable $\tilde{\Psi}_{1,2}(t) := e^{-\kappa_- t}\Psi_{1,2}(t)$ on $[t_0, +\infty)$ for every $t_0 \in \mathbb{R}$. The existence of $\tilde{\Psi}_{1,2}$ in $L^\infty(t_0, \infty)$ is guaranteed by the Banach fixed-point theorem and the solutions $\tilde{\Psi}_{1,2}$ are extended globally on $\mathbb{R}$. In the limit $t \to -\infty$, we obtain

$$\begin{aligned} L_{1,2} &:= \lim_{t\to-\infty} e^{-\kappa_- t}\Psi_{1,2}(t) \\ &= B_{1,2} + \frac{1}{\kappa_+ - \kappa_-}\int_{-\infty}^{+\infty} e^{-\kappa_- t'} \left[f(t')e_{1,2} + g(t')\Psi_{1,2}(t')\right] dt'. \end{aligned}$$

Hence, $L_{1,2}$ are bounded. Since $L_2 \neq 0$ due to the constraint (4.64) in Assumption 4.2, we can define

$$\Delta(t) := \Psi_1(t) - L_2^{-1}L_1\Psi_2(t),$$

so that $\lim_{t\to-\infty} e^{-\kappa_- t}\Delta(t) = 0$. By variation of parameters, the linear equation for Δ can be rewritten in the integral form:

$$\Delta(t) = \Delta_0 e^{\kappa_+ t} + \frac{1}{\kappa_+ - \kappa_-}\int_{-\infty}^{t} \left[e^{\kappa_+(t-t')} - e^{\kappa_-(t-t')}\right] \left[f(t') + g(t')\Delta(t')\right] dt', \quad (4.70)$$

where Δ_0 is some constant coefficient.

Remark 4.13. *The integral equation (4.70) is different from the one which would follow from the integral equation (4.69) in the variable $\Delta(t)$ so that $\Delta_0 \neq A_1 - L_2^{-1}L_1A_2$ generally. While (4.69) is useful in the limit $t \to +\infty$, (4.70) is useful in the limit $t \to -\infty$.*

The integral kernel in (4.70) becomes bounded in the variable $\tilde{\Delta}(t) := e^{-\kappa_+ t}\Delta(t)$, for which it can written in the form

$$\tilde{\Delta}(t) = \Delta_0 + \frac{1}{\kappa_+ - \kappa_-}\int_{-\infty}^{t} \left[1 - e^{-(\kappa_+ - \kappa_-)(t-t')}\right] \left[f(t')e^{-\kappa_+ t'} + g(t')\tilde{\Delta}(t')\right] dt', \quad (4.71)$$

By the same fixed-point iterations as in the proof of Lemma 3.2, there exist the unique solutions $\tilde{\Delta}$ to the integral equation (4.71) in a closed subset of Banach space $L^\infty(-\infty, T + (a-1)\log b)$ satisfying the bounds

$$\sup_{t\in(-\infty,T+(a-1)\log b)} |\tilde{\Delta}(t) - \Delta_0| \leq C_{T,a}b^{-2(1-a)}, \qquad b \geq b_{T,a},$$

due to bounds (4.42). Since $\tilde{\Delta}(t) = e^{-\kappa_+ t}\Delta(t)$, we obtain the bounds (4.66).

The linear combination in $\Delta = \Psi_1 - L_2^{-1} L_1 \Psi_2$ corresponds to the choice of

$$c - c_\infty = -L_2^{-1} L_1 (\lambda - \lambda_\infty).$$

The second derivatives of Ψ_c in (λ, c) grow like $\mathcal{O}(e^{2\kappa_+ t})$ as $t \to -\infty$. Similarly to the bound (4.52) in the proof of Lemma 4.6, one can justify the bound

$$\sup_{t \in [(a-1)\log b, 0]} |\Psi_{c_\infty - L_2^{-1} L_1 (\lambda - \lambda_\infty)}(t) - \Psi_\infty(t) - (\lambda - \lambda_\infty)\Delta(t)| \leq C(\lambda - \lambda_\infty)^2 e^{-2\kappa_+(1-a)} \tag{4.72}$$

if $|\lambda - \lambda_\infty| \leq \epsilon e^{\kappa_+(1-a)}$ with sufficiently small $\epsilon > 0$. Bound (4.68) follows from the expansion $\Psi_\infty(t) = \sqrt{d-3} + \mathcal{O}(e^{2t})$ as $t \to -\infty$ and the bounds (4.66) and (4.72).

Finally, it is proven similarly to the proof of Lemma 4.6 that $\Delta_0 = 0$ is in contradiction with the condition (4.63) of Assumption 4.2. Hence, $\Delta_0 \neq 0$. □

We end this section with the formal proof of Theorem 4.1 for $d \geq 13$.

Proof of Theorem 4.1 for $d \geq 13$.

We match again the solutions $\Psi_b(t)$ and $\Psi_c(t)$ as in (4.55). By comparing the bound (4.61) of Lemma 4.8 for any fixed $T \in \mathbb{R}$ and $a \in (0, a_0)$ with the bound (4.68) of Lemma 4.9 at the time instance $t = T + (a-1)\log b$, we obtain the nonlinear equation:

$$\Delta_0(\lambda(b) - \lambda_\infty) = A_0 b^{\kappa_+} + E, \tag{4.73}$$

where coefficients A_0 and Δ_0 are the same as in (4.61) and (4.66) respectively and the error term E satisfies

$$E = \mathcal{O}(b^{(1-a)\kappa_+ + a\kappa_-}, b^{(1+a)\kappa_+}, b^{-(2-\kappa_+)(1-a)}, (\lambda(b) - \lambda_\infty) b^{-2(1-a)}, (\lambda(b) - \lambda_\infty)^2 b^{-\kappa_+(1-a)})$$

as $b \to \infty$, provided that $\lambda(b)$ satisfies the bound (4.67) for some $\epsilon > 0$. Since $(1-a)\kappa_+ + a\kappa_- < \kappa_+ < 0$, it follows that $b^{(1-a)\kappa_+ + a\kappa_-} \ll b^{\kappa_+}$. Similarly, we have already checked that $b^{-(2-\kappa_+)(1-a)} \ll b^{\kappa_+}$ if $a < a_0$, where a_0 is given by (4.62).

Since $\Delta_0 \neq 0$ by Lemma 4.9, there exists the unique solution to the nonlinear equation (4.73) by the implicit function theorem and the unique solution for $\lambda(b)$ satisfies

$$\lambda(b) - \lambda_\infty = \Delta_0^{-1} A_0 b^{\kappa_+} + \mathcal{O}(b^{(1-a)\kappa_+ + a\kappa_-}, b^{(1+a)\kappa_+}, b^{-(2-\kappa_+)(1-a)}, b^{-2(1-a)+\kappa_+}). \tag{4.74}$$

Since $a > 0$ and $\kappa_+ < 0$, it follows that $b^{\kappa_+} \ll b^{\kappa_+(1-a)}$ so that $\lambda(b)$ in (4.74) belongs to the bound (4.67). The expansion (4.74) justifies the expansion (4.5).

Figure 4.5 illustrates the solutions Ψ_b of the second-order equation (4.8) with $\lambda = \lambda(b)$ for $d = 13$ and $b = 1, 10^2, 10^4$. The left panel shows that the solutions Ψ_b translated in t by $\log b$ in comparison with the solution Θ of the truncated equation (4.12). The right panel shows the solutions Ψ_b without translation in comparison with the limiting singular solution Ψ_∞ satisfying (4.8) and (4.9). Convergence $\Psi_b(\cdot - \log b) \to \Theta$ as $b \to \infty$ on $(-\infty, t_0]$ for a fixed $t_0 > 0$ is obvious from the left panel, whereas convergence $\Psi_b \to \Psi_\infty$ as $b \to \infty$ on $[t_0, \infty)$ for a fixed $t_0 < 0$ is obvious from the right panel.

Figure 4.6 shows two solutions Ψ_b with $b = 1$ and $b = 10^2$ on the phase plane (Ψ, Ψ') together with the solution Θ of the truncated equation (4.12) and the limiting singular solution Ψ_∞ satisfying (4.8) and (4.9). The difference of $\Psi_{b=10^2}$ (red dotted line) from Θ and Ψ_∞ is almost invisible, whereas the difference is large in the case of $\Psi_{b=1}$ (blue dotted line).

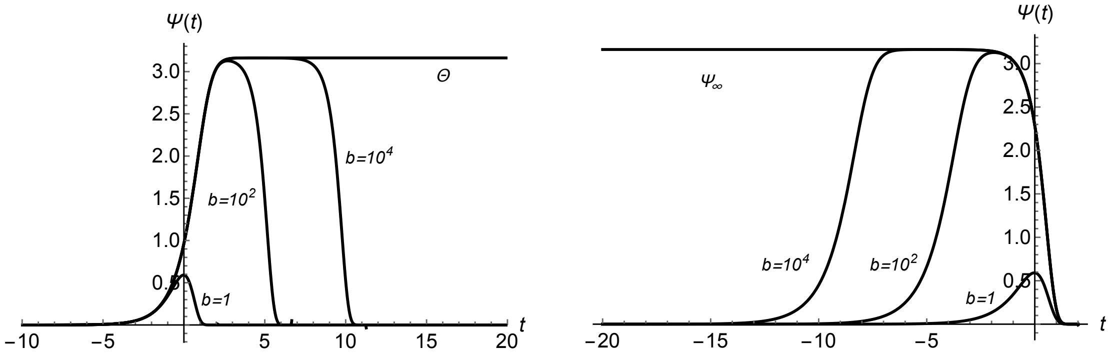

FIGURE 4.5: Plots of the solutions Ψ_b for $\lambda = \lambda(b)$ and specific values of b for $d = 13$ in comparison with Θ after translation of t by $\log b$ (left) and with Ψ_∞ (right).

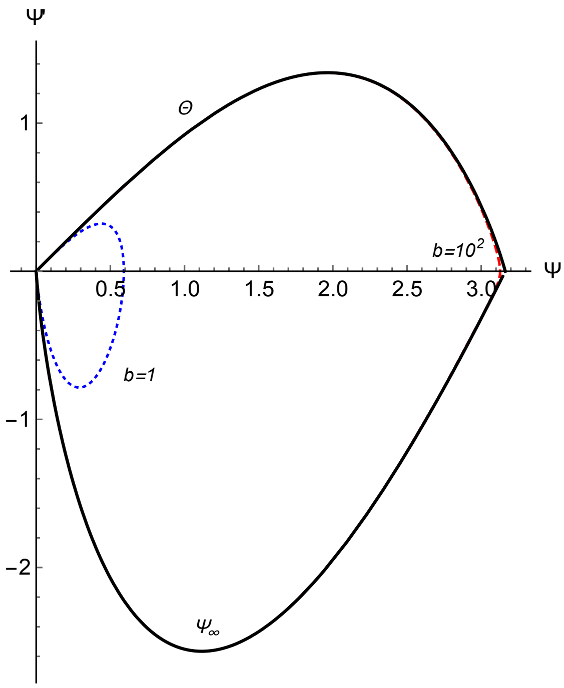

FIGURE 4.6: The solutions $\Psi_{b=1}$, $\Psi_{b=10^2}$, Θ, and Ψ_∞ on the phase plane (Ψ, Ψ') for $d = 13$.

Chapter 5

Morse index of the ground states in the monotone case

In this chapter, we study the Morse index of the ground states, i.e., solutions of (1.7), in the energy-supercritical case $(d-2)p > 2$, $d \geq 3$. For simplicity, we assume that $p = 1$ (so that we work with the cubic nonlinearity), which allows us to utilize the results from Chapters 3 and 4. Note that $p = 1$ implies that the energy-supercritical case occurs for $d \geq 5$, which will be assumed throughout this chapter.

For reader's convenience, we rewrite explicitly the most important differential equations of Chapters 3 and 4 that will be utilized in this work as well. Ground states were obtained in Chapter 3 by applying a rigorous shooting method to the initial-value problem

$$\begin{cases} f''(r) + \frac{d-1}{r} f'(r) - r^2 f(r) + \lambda f(r) + f(r)^3 = 0, \quad r > 0, \\ f(0) = b, \quad f'(0) = 0. \end{cases} \tag{5.1}$$

It has been shown in Theorem 3.1 that for every $b > 0$ there exists $\lambda(b) \in (d-4, d)$, such that the unique C^2 classical solution f to (5.1) is a solution $\mathfrak{u}_b \in \mathcal{E}$ to the boundary-value problem (1.7), where the energy space $\mathcal{E}$ was defined in (1.8). The parameter $b := \mathfrak{u}_b(0)$ is often referred to as *the amplitude*. Furthermore, it has been shown in [54] that there exists a limiting singular solution $\mathfrak{u}_\infty \in \mathcal{E}$, $\mathfrak{u}_\infty \notin L^\infty$, such that $\mathfrak{u}_b \to \mathfrak{u}_\infty$ as $b \to \infty$ in $\mathcal{E}$, and that it satisfies the following divergent behaviour

$$\mathfrak{u}_\infty(r) = \frac{\sqrt{d-3}}{r} \left[1 + \mathcal{O}\left(r^2\right)\right], \quad \text{as } r \to 0. \tag{5.2}$$

Existence of such solution for certain $\lambda = \lambda_\infty \in (d-4, d)$ was proven in Theorem 3.2 by applying shooting method to the following initial-value problem

$$\begin{cases} F''(r) + \frac{d-3}{r} F'(r) - \frac{d-3}{r^2} F(r) - r^2 F(r) + \lambda F(r) + \frac{1}{r^2} F(r)^3 = 0, \quad r > 0, \\ F(0) = \sqrt{d-3}, \quad F'(0) = 0, \end{cases} \tag{5.3}$$

where $F(r)$ was introduced by $F(r) = r f(r)$. Figure 5.1 shows the ground state $\mathfrak{u}_b(r)$ for two values of b and the limiting singular solution $\mathfrak{u}_\infty(r)$ for $d = 13$. The discrepancy between the two solutions moves to smaller values of r if the value of b is increased.

When $b = 10$, the difference between $\mathfrak{u}_b$ and $\mathfrak{u}_\infty$ becomes invisible on the scale used in Figure 5.1.

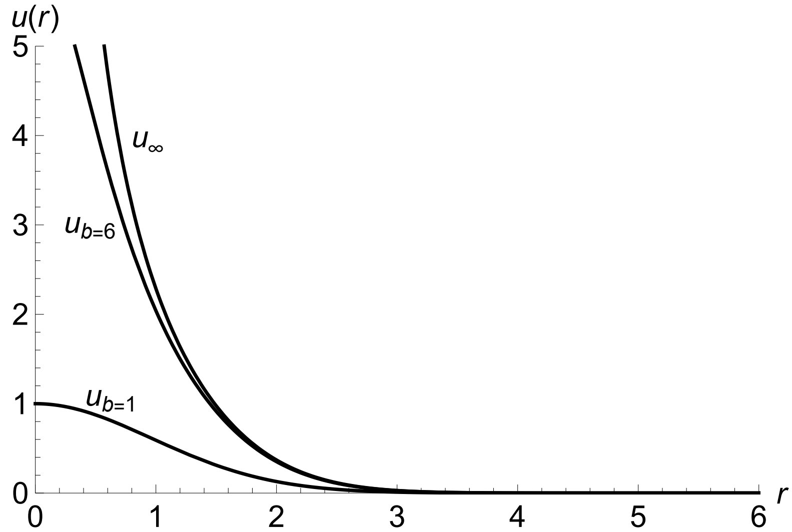

FIGURE 5.1: Graph of the ground state $\mathfrak{u}_b$ for $b = 1$ and $b = 6$ in comparison with the limiting singular solution $\mathfrak{u}_\infty$ for $d = 13$.

It was shown in Theorem 4.1 that $\lambda(b) \to \lambda_\infty$ as $b \to \infty$, however a more striking feature explored in Chapter 4, is the nature of this convergence. It was shown in Theorem 4.1 that for large values of b the monotonicity of $\lambda(b)$ depends on the dimension d. Specifically, if $5 \leq d \leq 12$ then $\lambda(b)$ oscillates around λ_∞ infinitely many times, whereas for $d \geq 13$ the convergence is monotone. These dependencies are visualized in Figure 4.1. Monotonicity of $\lambda(b)$ is closely related to the Morse index $\mathfrak{m}(\mathfrak{u}_b)$ of the ground state $\mathfrak{u}_b$. It is defined as the number of negative eigenvalues of the linearized operator $\mathcal{L}_b$ given by

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + r^2 - \lambda(b) - 3\mathfrak{u}_b^2(r). \tag{5.4}$$

Since $\mathcal{E}$ is the form domain of $\mathcal{L}_b$, we can write $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$, where $\mathcal{E}^*$ is the dual of $\mathcal{E}$ with respect to the scalar product in L^2_r.

Assuming C^1 property of $\mathfrak{u}_b$ in b and differentiating the initial-value problem (5.1) with $\lambda = \lambda(b)$ in b, we can see that $\mathcal{L}_b \partial_b \mathfrak{u}_b = \lambda'(b)\mathfrak{u}_b$, where $\partial_b \mathfrak{u}_b \in \mathcal{E}$. Hence, any value of b for which $\lambda'(b) = 0$ corresponds to zero eigenvalue being in the spectrum of $\mathcal{L}_b$ in L^2_r. Although the converse is not known, this property implies that the oscillatory case is very different from the monotone case, where the former has infinitely many crossing of zero eigenvalue of $\mathcal{L}_b$ in the parameter continuation in b as $b \to \infty$ whereas the latter does not have any eigenvalue crossing as $b \to \infty$, see also Figure ??. This suggests that the Morse index should be well defined in the monotone case, independently of b for large values of b. This is in fact the main result of this chapter, which we formulate as the following theorem.

Theorem 5.1. *Under non-degeneracy Assumptions 5.1 and 5.2, for every $d \geq 13$, there exists $b_0 > 0$ such that the Morse index $\mathfrak{m}(\mathfrak{u}_b)$ is finite and is independent of b for every $b \in (b_0, \infty)$.*

Remark 5.1. *Regarding the Morse index for the ground state in the energy supercrticial case, we are only aware of the works [25, 36], where the Morse index was estimated in the monotone case for the limiting singular solutions of the Dirichlet problem in a ball. We believe that the conclusion of Theorem 5.1 and the technique behind its proof remain valid for other problems in the monotone case, e.g. for the nonlinear Schrödinger equation in a ball.*

Remark 5.2. *By the Lyapunov–Schmidt reduction technique (see, e.g., [53]), the solution curve satisfies $\lambda(b) \to d$ and $\mathfrak{u}_b \to 0$ as $b \to 0$, where the Morse index $\mathfrak{m}(\mathfrak{u}_b)$ is equal to one for small $b > 0$. If $\mathfrak{m}(\mathfrak{u}_b) = 1$ for $b > b_0$ in Theorem 5.1, then it is quite possible that $\mathfrak{m}(\mathfrak{u}_b) = 1$ for every $b \in (0, \infty)$. Since the ground state is energetically stable with respect to the radial perturbation in $\mathcal{E} \cap L^4_r$ if $\mathfrak{m}(\mathfrak{u}_b) = 1$ and the mapping of $\lambda \mapsto \|\mathfrak{u}_b\|^2_{L^2_r}$ is monotonically decreasing (see Theorem 1.2 in Chapter 1), it is rather interesting that the transition from the oscillatory case for $5 \leq d \leq 12$ to the monotone case $d \geq 13$ may enforce stability of the ground state.*

In order to characterize the Morse index of $\mathcal{L}_b$, we use the Emden–Fowler transformation [18] for the nonlinear equation in (5.3) and study two families of solutions. One family is obtained from $F_b(r) := r\mathfrak{u}_b(r)$ and is parametrized by its parameter b from the behavior as $r \to 0$. The other family is parametrized by another parameter c from the decaying behavior as $r \to \infty$. The second family is considered in a local neighborhood of the limiting singular solution $F_\infty(r) = r\mathfrak{u}_\infty(r)$. Both families have C^1 property with respect to their parameters and their derivatives with respect to these parameters are solutions of the homogeneous equation $\mathcal{L}_b v = 0$ after the inverse Emden–Fowler transformation, e.g., $v(r) = r^{-1}\partial_b F_b(r)$. The proof of Theorem 5.1 is achieved from the Sturm's Oscillation Theorem (see, e.g., Theorem 3.5 in [56]) by showing that the two derivatives have finitely many oscillations and there exists $b_0 > 0$ such that the two derivatives are linearly independent for every $b \in (b_0, \infty)$.

As a by-product of our approach, we establish the equivalence of the Morse indices $\mathfrak{m}(\mathfrak{u}_b)$ and $\mathfrak{m}(\mathfrak{u}_\infty)$, where $\mathfrak{m}(\mathfrak{u}_\infty)$ is defined by the number of negative eigenvalues of the limiting operator $\mathcal{L}_\infty := \lim_{b\to\infty} \mathcal{L}_b$ computed at the limiting singular solution for $d \geq 5$:

$$\mathcal{L}_\infty := -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + r^2 - \lambda_\infty - 3\mathfrak{u}^2_\infty(r). \tag{5.5}$$

Compared to $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$, where the potential $-3\mathfrak{u}_b^2(r)$ is bounded from below, the potential $-3\mathfrak{u}^2_\infty(r)$ is unbounded from below. Because of that, we would normally take $\mathcal{E}_\infty = \{u \in \mathcal{E} : \ r^{-1}u \in L^2_r\}$ as the domain of $\mathcal{L}_\infty$. However, by using the following Hardy's inequality valid for $d \geq 3$:

$$\||\cdot|^{-1}u\|_{L^2(\mathbb{R}^d)} \leq \frac{2}{d-2}\|\nabla u\|_{L^2(\mathbb{R}^d)}, \quad u \in H^1(\mathbb{R}^d), \tag{5.6}$$

we can see that $\mathcal{E}_\infty = \mathcal{E}$. Thus, the operator $\mathcal{L}_\infty : \mathcal{E} \mapsto \mathcal{E}^*$ has the same domain and codomain as $\mathcal{L}_b$.

The following theorem gives the precise result on the Morse index of the two linear operators.

Theorem 5.2. *Under non-degeneracy Assumptions 5.1 and 5.2, for every $d \geq 13$, there exists $b_0 > 0$ such that $\mathfrak{m}(\mathfrak{u}_b) = \mathfrak{m}(\mathfrak{u}_\infty)$ for $b \in (b_0, \infty)$.*

Remark 5.3. *If the norm convergence of the resolvent for $\mathcal{L}_b$ to the resolvent for $\mathcal{L}_\infty$ can be established as $b \to \infty$, this would imply the result of Theorem 5.2. We do not study the norm convergence of resolvents here as our methods are based on analysis of differential equations.*

Remark 5.4. *The result of Theorem 5.2 suggests a simple way to obtain $\mathfrak{m}(\mathfrak{u}_b)$ in the monotone case for large b from $\mathfrak{m}(\mathfrak{u}_\infty)$, which can be approximated numerically with good accuracy.*

Figure 5.2 shows uniquely normalized solutions $v(r)$ of $\mathcal{L}_b v = 0$ with $b = 1$ and $\mathcal{L}_\infty v = 0$ such that $v(r) \to 0$ as $r \to \infty$. Both solutions diverge as $r \to 0$ with different divergence rates. Since there exists only one zero for each solution on $(0, \infty)$, Sturm's Oscillation Theorem (Theorem 3.5 in [56]) asserts that $\mathfrak{m}(\mathfrak{u}_b) = \mathfrak{m}(\mathfrak{u}_\infty) = 1$.

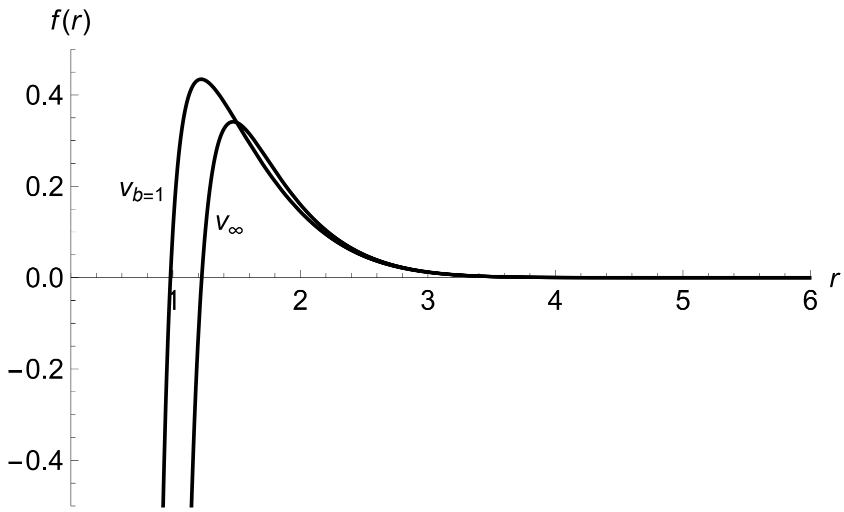

FIGURE 5.2: Graph of the uniquely normalized solutions $v(r)$ of $\mathcal{L}_b v = 0$ with $b = 1$ and $\mathcal{L}_\infty v = 0$ satisfying $v(r) \to 0$ as $r \to \infty$ for $d = 13$.

By the Vakhitov–Kolokolov stability criterion (Theorem 1.2 in Chapter 1), if $\mathfrak{m}(\mathfrak{u}_b) = 1$ and the mapping of $\lambda \mapsto \|\mathfrak{u}_b\|^2_{L^2_r}$ is monotonically decreasing, then the ground state $\mathfrak{u}_b$ is energetically stable with respect to radial perturbations in $\mathcal{E}$. Figure 5.3 shows the dependence of the mass $M(\mathfrak{u}_b) = \|\mathfrak{u}_b\|^2_{L^2_r}$ versus λ for $\lambda = \lambda(b)$. The red dot depicts the finite value of the limiting mass $M(\mathfrak{u}_\infty) = \|\mathfrak{u}_\infty\|^2_{L^2_r}$. Since the mapping is monotonically decreasing, the Vakhitov–Kolokolov stability criterion asserts that the ground state $\mathfrak{u}_b$

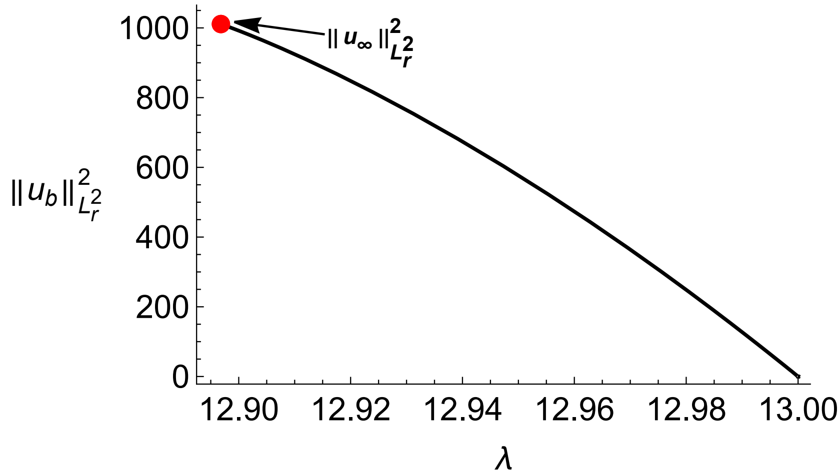

FIGURE 5.3: Mass $\|\mathfrak{u}_b\|^2_{L^2_r}$ of the ground state $\mathfrak{u}_b$ for $d = 13$ as a function of λ together with the mass $\|\mathfrak{u}_\infty\|^2_{L^2_r}$ of the limiting singular solution $\mathfrak{u}_\infty$.

is energetically stable for $d = 13$. The energetic stability is equivalent to the orbital stability if the time evolution of the Gross–Pitaevskii equation is locally well-posed in $\mathcal{E}$.

The same conclusion holds for other values of d in the monotone case $d \geq 13$. We have also checked other values of b and found no points of bifurcations along the solution family $\lambda(b)$ where $\mathcal{L}_b$ admits zero eigenvalue in L^2_r. This suggests that the monotone dependence of $\lambda(b)$ with no critical points, where $\lambda'(b)$ vanishes, implies no bifurcation points. This useful property has not been proven in the literature.

5.1 Two families of solutions

In this section, we recall and extend several preliminary estimates from Chapter 3. We begin by introducing the Emden-Fowler transformation

$$r = e^t, \quad F(r) = \Psi(t), \quad F'(r) = e^{-t}\Psi'(t), \tag{5.7}$$

which transforms the differential equation

$$F''(r) + \frac{d-3}{r}F'(r) - \frac{d-3}{r^2}F(r) - r^2F(r) + \lambda F(r) + \frac{1}{r^2}F(r)^3 = 0, \quad r > 0 \tag{5.8}$$

to the equivalent form

$$\Psi''(t) + (d-4)\Psi'(t) + (3-d)\Psi(t) + \Psi(t)^3 = -\lambda e^{2t}\Psi(t) + e^{4t}\Psi(t), \quad t \in \mathbb{R}. \tag{5.9}$$

For fixed $d \geq 5$ and $\lambda \in (0, d)$, two one-parameter families of solutions to the second-order differential equation (5.9) have been constructed in Chapter 3, according to their asymptotic behaviors as $t \to -\infty$ and $t \to +\infty$, respectively.

The first family of solutions to the differential equation (5.9), denoted as $\{\Psi_b\}_{b\in\mathbb{R}}$, corresponds to solutions of the initial-value problem (5.1) after applying the transformation $\Psi_b(t) = e^t f_b(e^t)$. By Lemmas 3.2 and 3.4 in Subsection 3.1, $\Psi_b \in C^2(\mathbb{R})$ satisfies the asymptotic behavior

$$\Psi_b(t) = be^t - (\lambda b + b^3)(2d)^{-1}e^{3t} + \mathcal{O}(e^{5t}), \quad \text{as } t \to -\infty, \tag{5.10}$$

where the expansion can be differentiated in t. These solutions depend on λ as well, and for $\lambda = \lambda(b)$ and $b > 0$, $\Psi_b(t)$ gives a solution to the boundary-value problem (1.7), after the transformation $\mathfrak{u}_b(r) = r^{-1}\Psi_b(\log r)$. For other values of λ, $\Psi_b(t)$ generally diverges as $t \to +\infty$.

The second family of solutions to the differential equation (5.9), denoted as $\{\Psi_c\}_{c\in\mathbb{R}}$, decays to zero as $t \to +\infty$. By Lemmas 3.3 and 3.4 in Subsection 3.1, $\Psi_c \in C^2(\mathbb{R})$ satisfies the asymptotic behavior

$$\Psi_c(t) \sim ce^{\frac{\lambda-d+2}{2}t}e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \to +\infty, \tag{5.11}$$

where the sign $\sim$ denotes the asymptotic correspondence which can be differentiated in t. Each $\Psi_c(t)$ generally diverges as $t \to -\infty$, except when $\lambda = \lambda(b)$ and $c = c(b)$ for some value of $c(b)$ for which it coincides with $\Psi_b(t) = e^t\mathfrak{u}_b(e^t)$:

$$\lambda = \lambda(b): \quad \Psi_b(t) = \Psi_{c(b)}(t), \quad \text{for all } \ t \in \mathbb{R}. \tag{5.12}$$

Each family of solutions is differentiable with respect to parameters λ and either b or c due to smoothness of the differential equation (5.9). Their derivatives decay to zero as $t \to -\infty$ and $t \to +\infty$ respectively, but generally diverge at the other infinities.

Let us define linearizations of the second-order equation (5.9) at the two families of solutions:

$$\mathcal{M}_b := \frac{d^2}{dt^2} + (d-4)\frac{d}{dt} + (3-d) + 3\Psi_b^2 + \lambda e^{2t} - e^{4t}, \tag{5.13}$$

$$\mathcal{M}_c := \frac{d^2}{dt^2} + (d-4)\frac{d}{dt} + (3-d) + 3\Psi_c^2 + \lambda e^{2t} - e^{4t}. \tag{5.14}$$

Then, differentiating the second-order equation (5.9) with respect to b and c at fixed λ yields

$$\mathcal{M}_b\partial_b\Psi_b = 0, \quad \mathcal{M}_c\partial_c\Psi_c = 0, \tag{5.15}$$

where $\partial_b\Psi_b(t) \to 0$ as $t \to -\infty$ and $\partial_c\Psi_c(t) \to 0$ as $t \to +\infty$.

The first family $\{\Psi_b\}_{b\in\mathbb{R}}$ is defined in a neighborhood of a heteroclinic orbit Θ connecting the saddle point $(0,0)$ and the stable point $(\sqrt{d-3}, 0)$ of the truncated autonomous version of equation (5.9) given by

$$\Theta''(t) + (d-4)\Theta'(t) + (3-d)\Theta(t) + \Theta(t)^3 = 0. \tag{5.16}$$

By Lemma 4.1 in Chapter 4, there exists a heteroclinic orbit between $(0,0)$ and $(\sqrt{d-3}, 0)$ which is uniquely defined (module to the translation in t) by the asymptotic behavior

$$\Theta(t) = e^t - (2d)^{-1}e^{3t} + \mathcal{O}(e^{5t}), \quad \text{as } t \to -\infty. \tag{5.17}$$

The following proposition presents the combined results of Lemmas 4.2, 4.5, and 4.8 from Chapter 4.

Proposition 5.1. *Fix $d \geq 5$ and $\lambda \in \mathbb{R}$. For every $T > 0$ and $a \in (0,1)$, there exist (T,a)-independent constants $b_0 > 0$, $C_0 > 0$ and (T,a)-dependent constants $b_{T,a} > 0$, $C_{T,a} > 0$ such that the unique solution Ψ_b to the differential equation (5.9) with the asymptotic behavior (5.10) satisfies for every $b \in (b_0, \infty)$*

$$\sup_{t \in (-\infty, 0]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_0 b^{-2} e^{3t}, \tag{5.18}$$

and for every $b \in (b_{T,a}, \infty)$

$$\sup_{t \in [0, T + a\log b]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_{T,a} b^{-2(1-a)}. \tag{5.19}$$

The heteroclinic orbit of the truncated equation (5.16) connects the saddle point $(0,0)$ associated with the characteristic exponents $\kappa_1 = 1$ and $\kappa_2 = 3-d$ and the stable point $(\sqrt{d-3}, 0)$ associated with the characteristic exponentis κ_+ and κ_- given by

$$\kappa_\pm = -\frac{1}{2}(d-4) \pm \frac{1}{2}\sqrt{d^2 - 16d + 40}. \tag{5.20}$$

For $d \geq 13$, the characteristic exponents are real and satisfy $\kappa_- < \kappa_+ < 0$. We make the following assumption on how the heteroclinic orbit converges to the stable point $(\sqrt{d-3}, 0)$.

Assumption 5.1. *Assume that there exists $A_0 \neq 0$ such that*

$$\Theta(t) = \sqrt{d-3} + A_0 e^{\kappa_+ t} + \mathcal{O}(e^{\kappa_- t}, e^{2\kappa_+ t}) \quad \text{as} \quad t \to +\infty. \tag{5.21}$$

Remark 5.5. *Assumption 5.1 implies that $\Theta(t)$ converges to $\sqrt{d-3}$ as $t \to +\infty$ according to the slowest decay rate given by κ_+. It is not a priori clear why the constant A_0 could not be zero in exceptional cases, for which $\Theta(t)$ converges to $\sqrt{d-3}$ as $t \to +\infty$ according to the fastest decay rate given by κ_-.*

The second family $\{\Psi_c\}_{c \in \mathbb{R}}$ is defined in a neighborhood of the special solution $\Psi_\infty(t) := e^t \mathfrak{u}_\infty(e^t)$ obtained from the limiting singular solution $\mathfrak{u}_\infty \in \mathcal{E}$. This special solution corresponds to the values of $\lambda = \lambda_\infty$ and $c = c_\infty$ so that

$$\lambda = \lambda_\infty : \quad \Psi_\infty(t) = \Psi_{c_\infty}(t), \quad \text{for all} \ \ t \in \mathbb{R}. \tag{5.22}$$

The solution Ψ_∞ satisfies the asymptotic behaviors

$$\Psi_\infty(t) = \sqrt{d-3}\left[1 - \frac{\lambda_\infty}{4d-10}e^{2t} + \mathcal{O}(e^{4t})\right], \quad \text{as } t \to -\infty \tag{5.23}$$

and

$$\Psi_\infty(t) \sim c_\infty e^{\frac{\lambda_\infty - d+2}{2}t} e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \to +\infty. \tag{5.24}$$

The following proposition presents a modification of Lemmas 4.6 and 4.9 from Chapter 4.

Proposition 5.2. *Fix $d \geq 13$ and $a \in (0,1)$. There exist constants $b_0 > 0$, $C_0 > 0$, and $\epsilon_0 > 0$, such that for every $\epsilon \in (0, \epsilon_0)$, $b \in (b_0, \infty)$, and $(\lambda, c) \in \mathbb{R}^2$ satisfying*

$$|\lambda - \lambda_\infty| + |c - c_\infty| \leq \epsilon b^{\kappa_- (1-a)}, \tag{5.25}$$

it is true for every $t \in [(a-1)\log b, 0]$ that

$$|\Psi_c(t) - \Psi_\infty(t)| \leq C_0 \epsilon b^{\kappa_-(1-a)} e^{\kappa_- t}. \tag{5.26}$$

Remark 5.6. *Note that the divergent behavior of $e^{\kappa_- t}$ for large negative t in (5.26) is cancelled by the decay of $b^{\kappa_-(1-a)}$ on any fixed interval $[(a-1)\log b, 0]$. Thus, bound (5.26) implies*

$$\sup_{t \in [(a-1)\log b, 0]} |\Psi_c(t) - \Psi_\infty(t)| \leq C_0 \epsilon, \tag{5.27}$$

for every $(\lambda, c) \in \mathbb{R}^2$ satisfying (5.25).

Remark 5.7. *Since Ψ_c is smooth in λ and c and has the same decay (5.11) as $t \to +\infty$ in comparison with (5.24) for Ψ_∞, it is true for every (λ, c) in a local neighborhood of $(\lambda_\infty, c_\infty)$ that*

$$\sup_{t \in [0,\infty)} |\Psi_c(t) - \Psi_\infty(t)| \leq C_0(|\lambda - \lambda_\infty| + |c - c_\infty|). \tag{5.28}$$

Proof of Proposition 5.2. Let $\Sigma(t) := \Psi_c(t) - \Psi_\infty(t)$. It follows from (5.9) that Σ satisfies the following equation:

$$\mathcal{M}_\infty \Sigma = \mathcal{F}(\Sigma)(t), \tag{5.29}$$

where $\mathcal{M}_\infty$ is defined by (5.38) and

$$\mathcal{F}(\Sigma)(t) := -(\lambda - \lambda_\infty)e^{2t}(\Psi_\infty(t) + \Sigma(t)) - 3\Psi_\infty(t)\Sigma(t)^2 - \Sigma(t)^3.$$

Since $\Psi_\infty(t) \to \sqrt{d-3}$ as $t \to -\infty$, as it follows from (5.23), we can pick two linearly independent solutions r_1, r_2 to $\mathcal{M}_\infty r = 0$ such that

$$r_1(t) = \mathcal{O}(e^{\kappa_- t}), \quad r_2(t) = \mathcal{O}(e^{\kappa_+ t}), \qquad \text{as } t \to -\infty, \tag{5.30}$$

where $\kappa_- < \kappa_+ < 0$ are given by (5.20). Using the Liouville's formula, we normalize the Wronskian according to the relation:

$$W(r_1, r_2)(t) = r_1(t) r_2'(t) - r_1'(t) r_2(t) = e^{-(d-4)t}. \tag{5.31}$$

By the variation of parameters method, we rewrite the differential equation (5.29) as an integral equation for every $t \in [(a-1)\log b, 0]$:

$$\begin{aligned}\Sigma(t) = \Sigma(0) \left[r_1(t) r_2'(0) - r_1'(0) r_2(t)\right] + \Sigma'(0) \left[r_1(0) r_2(t) - r_1(t) r_2(0)\right] \\ + \int_t^0 e^{(d-4)t'} \left[r_1(t) r_2(t') - r_1(t') r_2(t)\right] \mathcal{F}(\Sigma)(t') dt'.\end{aligned} \tag{5.32}$$

In order to use Banach fixed-point iterations, we introduce $\tilde{\Sigma}(t) := e^{-\kappa_- t}\Sigma(t)$, which satisfies $\tilde{\Sigma}(t) = \mathcal{A}(\tilde{\Sigma})(t)$, where

$$\begin{aligned}\mathcal{A}(\tilde{\Sigma})(t) = \Sigma(0) e^{-\kappa_- t} \left[r_1(t) r_2'(0) - r_1'(0) r_2(t)\right] + \Sigma'(0) e^{-\kappa_- t} \left[r_1(0) r_2(t) - r_1(t) r_2(0)\right] \\ - (\lambda - \lambda_\infty) \int_t^0 K_2(t,t') e^{2t'} \left[e^{-\kappa_- t'} \Psi_\infty(t') + \tilde{\Sigma}(t')\right] dt' \\ - \int_t^0 K_2(t,t') \left[3 e^{\kappa_- t'} \Psi_\infty(t') \tilde{\Sigma}^2(t') + e^{2\kappa_- t'} \tilde{\Sigma}^3(t')\right] dt',\end{aligned} \tag{5.33}$$

where the kernel $K_2(t,t')$ is defined as

$$K_2(t,t') := e^{\kappa_-(t'-t) - (\kappa_+ + \kappa_-)t'} \left[r_1(t) r_2(t') - r_1(t') r_2(t)\right]. \tag{5.34}$$

It follows from (5.30) that $K_2(t,t') \sim 1 + e^{(\kappa_+ - \kappa_-)(t-t')}$ as $t - t' \to -\infty$, which means that there exists some constant $K_0 > 0$, such that

$$\sup_{-\infty < t \le t' \le 0} |K_2(t,t')| \le K_0. \tag{5.35}$$

It follows from (5.28) that there exists some constant $C_0 > 0$ such that

$$|\Sigma(0)| + |\Sigma'(0)| \le C_0 \left(|\lambda - \lambda_\infty| + |c - c_\infty|\right).$$

The integral operator $\mathcal{A}(\tilde{\Sigma})$ in (5.33) is estimated for every $\tilde{\Sigma} \in L^\infty((a-1)\log b, 0)$ as

$$\begin{aligned}\|\mathcal{A}(\tilde{\Sigma})\|_\infty \le C_0 \Big[|\lambda - \lambda_\infty| + |c - c_\infty| + |\lambda - \lambda_\infty| \|\tilde{\Sigma}\|_\infty \\ + b^{-\kappa_-(1-a)} \|\tilde{\Sigma}\|_\infty^2 + b^{-2\kappa_-(1-a)} \|\tilde{\Sigma}\|_\infty^3\Big].\end{aligned} \tag{5.36}$$

Similar estimate applies to $\|\mathcal{A}(\tilde{\Sigma}_1) - \mathcal{A}(\tilde{\Sigma}_2)\|_\infty$. The estimates show that the integral operator $\mathcal{A}(\tilde{\Sigma})$ is closed and is a contraction in the ball $B_\delta \subset L^\infty((a-1)\log b, 0)$ of the small radius $\delta := 2C_0\epsilon b^{\kappa_- -(1-a)}$, provided that (λ, c) satisfy the bound (5.25) and $\epsilon > 0$ is sufficiently small. By the Banach fixed-point theorem, there exists a unique fixed point of $\mathcal{A}(\tilde{\Sigma})$ satisfying

$$\|\tilde{\Sigma}\|_\infty \leq 2C_0\epsilon b^{\kappa_- -(1-a)}, \tag{5.37}$$

which proves the bound (5.26) after going back to the original variable Σ and redefining C_0. □

Linearization of the second-order equation (5.9) at Ψ_∞ is given by

$$\mathcal{M}_\infty := \frac{d^2}{dt^2} + (d-4)\frac{d}{dt} + (3-d) + 3\Psi_\infty^2 + \lambda_\infty e^{2t} - e^{4t}. \tag{5.38}$$

Differentiating the second-order equation (5.9) with respect to c at fixed λ and then substituting $c = c_\infty$ and $\lambda = \lambda_\infty$ gives

$$\mathcal{M}_\infty \partial_c \Psi_\infty = 0, \tag{5.39}$$

where $\partial_c\Psi_\infty$ is a short notation for $\partial_c\Psi_c|_{(\lambda,c)=(\lambda_\infty,c_\infty)}$. The function $\partial_c\Psi_\infty(t)$ decays fast as $t \to +\infty$ according to (5.11), but generally diverges as $t \to -\infty$. Since $\Psi_\infty(t) \to \sqrt{d-3}$ as $t \to -\infty$, the divergence of $\partial_c\Psi_\infty(t)$ as $t \to -\infty$ is defined by the same two characteristic exponents κ_+ and κ_- given by (5.20). We make the following assumption on the divergence of this solution.

Assumption 5.2. *Assume that there exists $L_\infty \neq 0$ such that*

$$\partial_c\Psi_\infty(t) = L_\infty e^{\kappa_- t} + \mathcal{O}(e^{\kappa_+ t}, e^{(\kappa_- +2)t}) \quad \text{as} \quad t \to -\infty. \tag{5.40}$$

Remark 5.8. *Assumption 5.2 implies that $\partial_c\Psi_\infty(t)$ diverges as $t \to -\infty$ with the fastest growth rate given by κ_-. Again, it is not a priori clear why the constant L_∞ could not be zero in exceptional cases, for which $\partial_c\Psi_\infty(t)$ diverges as $t \to -\infty$ with the slowest growth rate given by κ_+.*

Figure 5.4 shows $\Psi_b(t)$ for two values of b and $\Psi_\infty(t)$ for $d = 13$. After the inverse Emden-Fowler transformation (5.7) and the transformation $\mathfrak{u}(r) = r^{-1}F(r)$, these functions correspond to $\mathfrak{u}_b(r)$ and $\mathfrak{u}_\infty(r)$ shown on Figure 5.1.

Figure 5.5 shows $\partial_c\Psi_{c(b)}(t)$ with $b = 1$ and $\partial_c\Psi_\infty(t)$ for $d = 13$. These functions are solutions of the homogeneous equaitons $\mathcal{M}_b\partial_c\Psi_{c(b)} = 0$ and $\mathcal{M}_\infty\partial_c\Psi_\infty = 0$. After the transformation $v(r) = r^{-1}\partial_c\Psi_{c(b)}(\log r)$, these functions correspond to solutions of $\mathcal{L}_b v = 0$ and $\mathcal{L}_\infty v = 0$ that decay to zero as $r \to \infty$ shown in Figure 5.2. Since $\partial_c\Psi_{c(b)}(t)$ and $\partial_c\Psi_\infty(t)$ have only one zero on $\mathbb{R}$, the corresponding functions $v(r)$ have only one zero on $(0,\infty)$, so that $\mathfrak{m}(\mathfrak{u}_b) = \mathfrak{m}(\mathfrak{u}_\infty) = 1$.

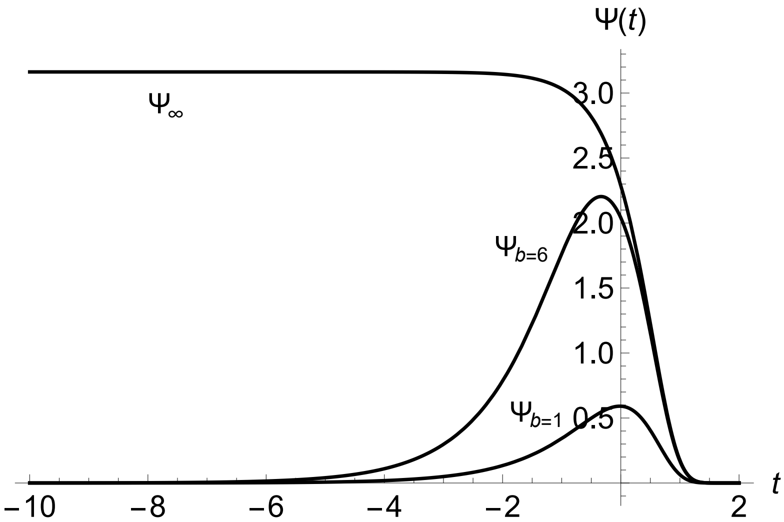

FIGURE 5.4: Graph of the solution Ψ_b for $b = 1$ and $b = 6$ in comparison with the solution Ψ_∞ for $d = 13$.

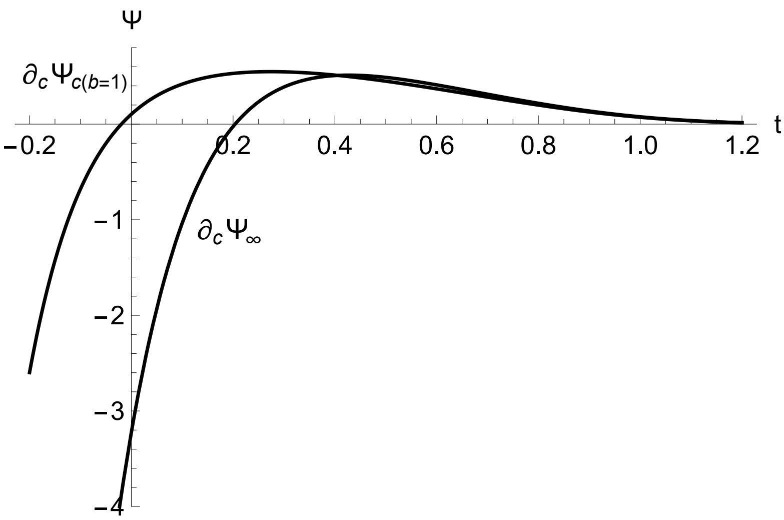

FIGURE 5.5: Graph of $\partial_c \Psi_{c(b)}$ with $b = 1$ and $\partial_c \Psi_\infty$ for $d = 13$.

5.2 Derivative of the b-family of solutions

Here we describe the asymptotic behavior of $\partial_b \Psi_b$. The following lemma shows that after translation by $-\log b$, $\partial_b \Psi_b$ converges to $b^{-1}\Theta'$ on the negative t-axis. Moreover, the estimate can be extended from $(-\infty, 0]$ to $[0, T + a\log b]$ for fixed $T \in \mathbb{R}$ and $a \in (0,1)$ and for sufficiently large b at the expense of slower convergence rate.

Lemma 5.1. *Let $d \geq 13$ and $\lambda \in \mathbb{R}$. For every $T > 0$ and $a \in (0,1)$, there exist (T,a)-independent constants $b_0 > 0$, $C_0 > 0$ and (T,a)-dependent constants $b_{T,a} > 0$, $C_{T,a} > 0$ such that $\partial_b \Psi_b$ satisfies for every $b \in (b_0, \infty)$*

$$\sup_{t\in(-\infty,0]} |\partial_b \Psi_b(t-\log b) - b^{-1}\Theta'(t)| + \sup_{t\in(-\infty,0]} |\partial_b \Psi_b'(t-\log b) - b^{-1}\Theta''(t)| \leq C_0 b^{-3} \quad (5.41)$$

and for every $b \in (b_{T,a}, \infty)$

$$\sup_{t\in[0,T+a\log b]} |\partial_b \Psi_b(t - \log b) - b^{-1}\Theta'(t)| + \sup_{t\in[0,T+a\log b]} |\partial_b \Psi_b'(t - \log b) - b^{-1}\Theta''(t)| \leq C_{T,a} b^{-2(1-a)-1}. \quad (5.42)$$

Proof. We begin by introducing $\gamma(t) := \partial_b \Psi_b(t - \log b) - b^{-1}\Theta'(t)$, where Ψ_b is the unique solution to (5.9) with the asymptotic behavior (5.10) and Θ is the unique solution to (5.16) with the asymptotic behavior (5.17). Since $\partial_b \Psi_b$ satisfies $\mathcal{M}_b \partial_b \Psi_b = 0$ and Θ' satisfies $\mathcal{M}_0 \Theta' = 0$, where $\mathcal{M}_b$ is given by (5.13) and

$$\mathcal{M}_0 := \frac{d^2}{dt^2} + (d-4)\frac{d}{dt} + (3-d) + 3\Theta^2, \quad (5.43)$$

the difference term $\gamma(t)$ satisfies the following equation:

$$\mathcal{M}_0 \gamma = f_b(b^{-1}\Theta' + \gamma), \quad (5.44)$$

where

$$f_b(t) := 3(\Theta(t)^2 - \Psi_b(t - \log b)^2) - \lambda b^{-2} e^{2t} + b^{-4} e^{4t}. \quad (5.45)$$

Note that $\mathcal{M}_0 - f_b$ gives $\mathcal{M}_b$ after translation $t \mapsto t + \log b$.

Proof of the bound (5.41). Two linearly independent solutions of $\mathcal{M}_0 \gamma = 0$ are given by $\Theta'(t)$ and another function $\Xi(t)$, which can be found from the Wronskian relation

$$W(\Theta', \Xi)(t) = \Theta'(t)\Xi'(t) - \Theta''(t)\Xi(t) = W_0 e^{(4-d)t}, \quad (5.46)$$

for some constant $W_0 \neq 0$. We take $W_0 = 1$ in order to normalize $\Xi(t)$ uniquely. Since $\Theta'(t) \to 0$ as $t \to -\infty$ according to the asymptotic expansion

$$\Theta'(t) = e^t + \mathcal{O}(e^{3t}), \quad \text{as } t \to -\infty, \tag{5.47}$$

we have $\Xi(t) \to \infty$ as $t \to -\infty$ according to the asymptotic expansion

$$\Xi(t) = (2-d)^{-1} e^{(3-d)t} + \mathcal{O}(e^{(5-d)t}), \quad \text{as } t \to -\infty. \tag{5.48}$$

In order to estimate the supremum-norm of $\gamma(t)$ for $t \in (-\infty, 0]$, we first rewrite the differential equation (5.44) as an integral equation

$$\gamma(t) = \int_{-\infty}^{t} e^{(d-4)t'} [\Theta'(t')\Xi(t) - \Theta'(t)\Xi(t')] f_b(t') [b^{-1}\Theta'(t') + \gamma(t')] dt', \tag{5.49}$$

where the free solution $c_1\Theta'(t) + c_2\Xi(t)$ is set to zero from the requirement that $\gamma(t) = \mathcal{O}(e^{3t})$ as $t \to -\infty$. The integral kernel in (5.49) becomes bounded if we introduce the transformation $\tilde{\gamma}(t) = e^{-t}\gamma(t)$. The integral equation corresponding to $\tilde{\gamma}$ is

$$\tilde{\gamma}(t) = \int_{-\infty}^{t} K_1(t,t') f_b(t') [b^{-1} e^{-t'} \Theta'(t') + \tilde{\gamma}(t')] dt', \tag{5.50}$$

where

$$K_1(t,t') := [e^{-t'}\Theta'(t')][e^{(d-3)t}\Xi(t)] e^{(d-2)(t'-t)} - [e^{-t}\Theta'(t)][e^{(d-3)t'}\Xi(t')]. \tag{5.51}$$

Since $\Theta'(t) = \mathcal{O}(e^t)$ and $\Xi(t) = \mathcal{O}(e^{(3-d)t})$ as $t \to -\infty$, the integral kernel $K_1(t,t')$ is a bounded function for all $t, t' \in (-\infty, 0]$.

By using bound (5.18) of Proposition 5.1, we obtain from (5.45) that

$$|f_b(t)| \leq C_0 b^{-2} e^{2t}, \quad t \in (-\infty, 0], \tag{5.52}$$

where the constant C_0 is independent of b for sufficiently large b and may change from one line to another line. Boundedness of K_1 in the integral equation (5.50) on $(-\infty, 0] \times (-\infty, 0]$ and the estimate (5.52) allow us to estimate the supremum norm of $\tilde{\gamma}(t)$ on $(-\infty, 0]$ as follows

$$\|\tilde{\gamma}\|_\infty \leq C_0 b^{-2} (b^{-1} \|e^{-t}\Theta'\|_\infty + \|\tilde{\gamma}\|_\infty). \tag{5.53}$$

Due to smallness of b^{-2} and boundness of b-independent $\|e^{-t}\Theta'\|_\infty$, this estimate implies that

$$\sup_{t \in (-\infty, 0]} |\tilde{\gamma}(t)| \leq C_0 b^{-3}. \tag{5.54}$$

Since $|\gamma(t)| \leq |\tilde{\gamma}(t)|$ for all $t \in (-\infty, 0]$, we obtain the first part of bound (5.41). Since $\tilde{\gamma} \in C^1(-\infty, 0)$, we obtain the second part of bound (5.41) by differentiting equation (5.50) and using (5.54).

Proof of the bound (5.42). In order to estimate $|\gamma(t)|$ for sufficiently large positive t, we need to define solutions to $\mathcal{M}_0\gamma = 0$ from their behavior as $t \to +\infty$. Since $(\sqrt{d-3}, 0)$ is a stable node of the nonlinear equation (5.16), we can pick two linearly independent solutions $\gamma_1(t)$ and $\gamma_2(t)$ from their decaying behavior

$$\gamma_1(t) = \mathcal{O}(e^{\kappa_- t}), \quad \gamma_2(t) = \mathcal{O}(e^{\kappa_+ t}) \quad \text{as } t \to +\infty, \tag{5.55}$$

where $\kappa_- < \kappa_+ < 0$ are given by (5.20). The Liouville's formula yields the Wronskian relation

$$W(\gamma_1, \gamma_2)(t) = \gamma_1(t)\gamma_2'(t) - \gamma_1'(t)\gamma_2(t) = W_0 e^{(4-d)t}, \tag{5.56}$$

for some constant $W_0 \neq 0$, and by normalizing $\gamma_1(t)$ and $\gamma_2(t)$ we can assume that $W_0 = 1$. In order to derive supremum-norm estimates for $\gamma(t)$, we once again rewrite differential equation (5.44) as an integral equation

$$\begin{aligned}\gamma(t) = \gamma(0)[\gamma_1(t)\gamma_2'(0) - \gamma_1'(0)\gamma_2(t)] + \gamma'(0)[\gamma_1(0)\gamma_2(t) - \gamma_1(t)\gamma_2(0)] \\ + \int_0^t e^{(d-4)t'}[\gamma_1(t')\gamma_2(t) - \gamma_1(t)\gamma_2(t')] f_b(t')[b^{-1}\Theta'(t') + \gamma(t')]dt', \end{aligned} \tag{5.57}$$

this time for $t \in [0, T + a\log b]$. From bound (5.41) we obtain existence of a constant $C_0 > 0$ and $b_0 > 0$, such that

$$|\gamma(0)| + |\gamma'(0)| \leq C_0 b^{-3}, \quad \text{for all } b \geq b_0. \tag{5.58}$$

Due to the decay of $\gamma_1(t)$ and $\gamma_2(t)$ as $t \to +\infty$, the kernel of the integral equation (5.57) behaves like $e^{\kappa_+(t-t')}$ and $e^{\kappa_-(t-t')}$, and is thus bounded as $t \to +\infty$ since $t' \leq t$. By using bound (5.19) of Proposition 5.1, we obtain from (5.45) that

$$\sup_{t\in[0,T+a\log b]} |f_b(t)| \leq C_{T,a} b^{-2(1-a)}, \tag{5.59}$$

where the b-independent constant $C_{T,a}$ may change from one line to another line. Using estimates (5.55), (5.58), and (5.59), we obtain from the integral equation (5.57) the following bound on the supremum-norm of $\gamma(t)$ on $[0, T + a\log b]$:

$$\|\gamma\|_\infty \leq C_{T,a}\left(b^{-3} + b^{-2(1-a)-1}\|\Theta'\|_{L^1} + (T + a\log b)b^{-2(1-a)}\|\gamma\|_\infty\right). \tag{5.60}$$

Due to smallness of $(T + a\log b)b^{-2(1-a)}$ and boundedness of b-independent $\|\Theta'\|_{L^1}$, this estimate implies that

$$\sup_{t\in[0,T+a\log b]} |\gamma(t)| \leq C_{T,a} b^{-2(1-a)-1}. \tag{5.61}$$

By differentiting equation (5.57) and using (5.61), we obtain a similar bound on $\gamma'(t)$, which together with (5.61) gives us bound (5.42). □

Bound (5.42) of Lemma 5.1 and the expansion (5.21) imply the following important representation of $\partial_b \Psi_b(t)$ at $t = T + (a-1)\log b$.

Corollary 5.1. *Under Assumption 5.1, there exist some constant* $a_0 \in (0, 0.5)$ *such that for every* $a \in (0, a_0)$ *and* $T > 0$*, there exist some* (T, a)*-dependent constants* $b_{T,a} > 0$ *and* $C_{T,a} > 0$ *such that for every* $b \in (b_{T,a}, \infty)$ *we have*

$$|\partial_b \Psi_b(T + (a-1)\log b) - A_0 \kappa_+ b^{a\kappa_+ - 1} e^{\kappa_+ T}| \leq C_{T,a} \max\left\{ b^{-2(1-a)-1}, b^{a\kappa_- - 1}, b^{2a\kappa_+ - 1} \right\}. \tag{5.62}$$

Proof. Bound (5.62) follows from the expansion (5.21) at large positive $t = T + a\log b$, the bound (5.42) at $t = T + (a-1)\log b$, and the triangle inequality if

$$b^{a\kappa_+ - 1} \gg b^{-2(1-a)-1}.$$

This constraint is satisfied if $a \in (0, a_0)$, where

$$a_0 := \frac{2}{2 + |\kappa_+|} = \frac{4}{d - \sqrt{d^2 - 16d + 40}} = \frac{d + \sqrt{d^2 - 16d + 40}}{2(2d-5)}. \tag{5.63}$$

Note that $a_0 \in (0, 0.5)$ for every $d \geq 13$. □

5.3 Derivative of the c-family of solutions

Here, we describe the asymptotic behavior of $\partial_c \Psi_c$. Since Ψ_c is smooth in λ and c and has the same decay as $t \to +\infty$ as the limiting solution Ψ_∞ according to (5.11) and (5.24), $\partial_c \Psi_c$ converges to $\partial_c \Psi_\infty$ on $[0, \infty)$. To be precise, there exists a constant $C_0 > 0$ such that for every (λ, c) in a local neighborhood of $(\lambda_\infty, c_\infty)$, we have

$$\sup_{t\in[0,\infty)} |\partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)| + \sup_{t\in[0,\infty)} |\partial_c \Psi_c'(t) - \partial_c \Psi_\infty'(t)| \leq C_0(|\lambda - \lambda_\infty| + |c - c_\infty|). \tag{5.64}$$

The following lemma extends the estimate on the difference $|\partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)|$ from $[0, \infty)$ to $[(a-1)\log b, 0]$ for fixed $a \in (0, 1)$ and for sufficiently large b provided that (λ, c) are sufficiently close to $(\lambda_\infty, c_\infty)$.

Lemma 5.2. *Fix* $d \geq 13$*. For fixed* $a \in (0, 1)$*, there exist* $b_0 > 0$*,* $C_0 > 0$*, and* $\epsilon_0 > 0$*, such that for every* $\epsilon \in (0, \epsilon_0)$ *and for every* $(\lambda, c) \in \mathbb{R}^2$ *satisfying*

$$|\lambda - \lambda_\infty| + |c - c_\infty| \leq \epsilon b^{\kappa_-(1-a)}, \tag{5.65}$$

it is true for every $b \geq b_0$ *and every* $t \in [(a-1)\log b, 0]$ *that*

$$|\partial_c \Psi_c(t) - \partial_c \Psi_\infty(t)| + |\partial_c \Psi_c'(t) - \partial_c \Psi_\infty'(t)| \leq C_0 \epsilon e^{\kappa_- t}. \tag{5.66}$$

Proof. Let $r(t) := \partial_c\Psi_c(t) - \partial_c\Psi_\infty(t)$. Since $\partial_c\Psi_c$ satisfies $\mathcal{M}_c\partial_c\Psi_c = 0$ and $\partial_c\Psi_\infty$ satisfies $\mathcal{M}_\infty\partial_c\Psi_\infty = 0$, the difference term r satisfies the following equation:

$$\mathcal{M}_\infty r = f_c + g_c r, \tag{5.67}$$

where

$$f_c(t) := 3(\Psi_\infty(t)^2 - \Psi_c(t)^2)\partial_c\Psi_\infty(t) - (\lambda - \lambda_\infty)e^{2t}\partial_c\Psi_\infty(t), \tag{5.68}$$

$$g_c(t) := 3(\Psi_\infty(t)^2 - \Psi_c(t)^2) + (\lambda_\infty - \lambda)e^{2t}. \tag{5.69}$$

Note that $\mathcal{M}_c = \mathcal{M}_\infty - g_c$.

As in the proof of Proposition 5.2, we pick two linearly independent solutions r_1, r_2 to $\mathcal{M}_\infty r = 0$, such that

$$r_1(t) = \mathcal{O}(e^{\kappa_- t}), \quad r_2(t) = \mathcal{O}(e^{\kappa_+ t}), \quad \text{as } t \to -\infty, \tag{5.70}$$

where $\kappa_- < \kappa_+ < 0$ are given by (5.20). Using the method of variation of parameters, we rewrite the differential equation (5.67) as an integral equation for every $t \in [(a-1)\log b, 0]$:

$$\begin{aligned} r(t) = r(0)[r_1(t)r_2'(0) - r_1'(0)r_2(t)] + r'(0)[r_1(0)r_2(t) - r_1(t)r_2(0)] \\ + \int_t^0 e^{(d-4)t'}[r_1(t)r_2(t') - r_1(t')r_2(t)][f_c(t') + g_c(t')r(t')]dt', \end{aligned} \tag{5.71}$$

where we have used the normalization of the Wronskian $W(r_1, r_2)(t) = e^{-(d-4)t}$ between the two solutions r_1 and r_2 as in (5.31).

In order to elliminate the divergent behavior of the kernel in (5.71) as $t \to -\infty$, we introduce the transformation $\tilde{r}(t) = e^{-\kappa_- t}r(t)$, which results in the following integral equation for $\tilde{r}$:

$$\begin{aligned} \tilde{r}(t) = r(0)e^{-\kappa_- t}[r_1(t)r_2'(0) - r_1'(0)r_2(t)] + r'(0)e^{-\kappa_- t}[r_1(0)r_2(t) - r_1(t)r_2(0)] \\ + \int_t^0 K_2(t,t')[e^{-\kappa_- t'}f_c(t') + g_c(t')\tilde{r}(t')]dt', \end{aligned} \tag{5.72}$$

where the kernel $K_2(t,t')$ is the same as in (5.34):

$$K_2(t,t') := e^{\kappa_-(t'-t)-(\kappa_+ + \kappa_-)t'}\left[r_1(t)r_2(t') - r_1(t')r_2(t)\right]. \tag{5.73}$$

The kernel is bounded for every $-\infty < t \le t' \le 0$ as in (5.35). It follows from (5.64) that

$$|r(0)| + |r'(0)| \le C_0(|\lambda - \lambda_\infty| + |c - c_\infty|) \le C_0\epsilon b^{\kappa_-(1-a)}, \tag{5.74}$$

where the (ϵ, b)-independent constant C_0 can change from one line to another line. It follows from the expansion (5.40) in Assumption 5.2 that $\partial_c\Psi_\infty(t) = \mathcal{O}(e^{\kappa_- t})$ as $t \to -\infty$.

Therefore, we get by using bounds (5.26) and (5.27):

$$\int_t^0 e^{-\kappa_- t'}|f_c(t')|dt' \leq C_0\left(\epsilon b^{\kappa_-(1-a)}\int_{(a-1)\log b}^0 e^{\kappa_- t'}dt' + |\lambda-\lambda_\infty|\right) \leq C_0\epsilon. \tag{5.75}$$

On the other hand, for every $\tilde{r} \in L^\infty((a-1)\log b, 0)$, we get by using bounds (5.26) and (5.27):

$$\int_{(a-1)\log b}^0 |g_c(t')\tilde{r}(t')|dt' \leq C_0\epsilon||\tilde{r}||_\infty. \tag{5.76}$$

Putting estimates (5.70), (5.74), (5.75), and (5.76) together in the integral equation (5.72) yields

$$\sup_{t\in[(a-1)\log b,0]} |\tilde{r}(t)| \leq C_0\epsilon, \tag{5.77}$$

which is the first part of bound (5.66) after going back to the original variable $r(t)$. The second part of bound (5.66) is obtained by differentiating (5.72) in t and using bound (5.77). □

Bound (5.66) of Lemma 5.2 and the expansion (5.40) imply the following important representation of $\partial_c\Psi_c(t)$ at $t = T + (a-1)\log b$.

Corollary 5.2. *Under Assumption 5.2, for every $a \in (0,1)$ and $T > 0$, there exist some (T,a)-dependent constants $b_{T,a} > 0$ and $C_{T,a} > 0$ such that for every $b \in (b_{T,a},\infty)$ we have*

$$|\partial_c\Psi_c(T+(a-1)\log b) - L_\infty e^{\kappa_- T}b^{-\kappa_-(1-a)}| \leq C_{T,a}\max\{\epsilon b^{-\kappa_-(1-a)}, b^{-\kappa_+(1-a)}, b^{-(2+\kappa_-)(1-a)}\}. \tag{5.78}$$

Proof. Bound (5.78) follows from the bound (5.66) at $t = T+(a-1)\log b$ for fixed $T > 0$, $a \in (0,1)$, and sufficiently large $b > 0$ after $\partial_c\Psi_\infty(t)$ for large negative t is expressed from the expansion (5.40). □

Remark 5.9. *Lemma 5.2 can be obtained from the C^1 property of Ψ_c in (λ, c) after some transformations. It follows from the proof of Proposition 5.2 that*

$$|\Psi_c(t) - \Psi_\infty(t)| = \mathcal{O}(\epsilon) \quad t \in [(a-1)\log b, 0],$$

and the asymptotic expansion can be differentiated in ϵ. Parameter ϵ determines the size of the distance $|\lambda-\lambda_\infty|$ and $|c - c_\infty|$ so that we can write $c - c_\infty = \mathcal{O}(\epsilon b^{-\kappa_-(1-a)})$ and differentiate it in ϵ. By taking derivative in c and using the chain rule, this yields

$$|\partial_c\Psi_c(t) - \partial_c\Psi_\infty(t)| = \mathcal{O}(\epsilon b^{-\kappa_-(1-a)}) \quad t \in [(a-1)\log b, 0],$$

which is equivalent to the bound (5.66). Since taking derivatives in c and using the chain rule are not obvious from the proof of Proposition 5.2, we provided the precise proof of Lemma 5.2.

5.4 Proofs of Theorems 5.1 and 5.2

We recall that $\mathcal{M}_b = \mathcal{M}_{c(b)}$ for $\lambda = \lambda(b)$ since $\Psi_b(t) = \Psi_{c(b)}(t)$ for every $t \in \mathbb{R}$. Hence, both $\partial_b \Psi$ and $\partial_c \Psi_{c(b)}$ are solutions of the same homogeneous equation $\mathcal{M}_b \gamma = 0$ for $\lambda = \lambda(b)$. The following lemma shows that these two solutions are linearly independent for sufficiently large values of b.

Lemma 5.3. *Fix $d \geq 13$. Under Assumptions 5.1 and 5.2, for every $a \in (0, a_0)$ with a_0 given by (5.63), there exists $b_0 > 0$ such that for every $b \in (b_0, \infty)$, there exists no $C \in \mathbb{R}$ such that*

$$C \partial_c \Psi_{c(b)}(t) = \partial_b \Psi_b(t), \quad \textit{for all } t \in \mathbb{R}. \tag{5.79}$$

Proof. In order to get a contradiction, suppose that relation (5.79) holds for some constant $C \in \mathbb{R}$. The results of Corollaries 5.1 and 5.2 apply for $t = T + (a-1)\log b$ for fixed $a \in (0, a_0)$, $T > 0$, and sufficiently large b. Substituting bounds (5.62) and (5.78) into (5.79) yields

$$\begin{aligned} C L_\infty e^{\kappa_- T} b^{-\kappa_-(1-a)} &\left[1 + \mathcal{O}(\epsilon, b^{-(\kappa_+ - \kappa_-)(1-a)}, b^{-2(1-a)})\right] \\ &= A_0 \kappa_+ b^{a\kappa_+ - 1} e^{\kappa_+ T} \left[1 + \mathcal{O}(b^{-2(1-a) - a\kappa_+}, b^{-a(\kappa_+ - \kappa_-)}, b^{a\kappa_+})\right], \end{aligned} \tag{5.80}$$

where we recall that $2(1-a) + a\kappa_+ > 0$ if $a \in (0, a_0)$, where a_0 is given by (5.63) in Corollary 5.1. Since $A_0 \neq 0$ and $L_\infty \neq 0$ by Assumptions 5.1 and 5.2, we obtain from (5.80) that

$$\begin{aligned} &C\left[1 + \mathcal{O}(\epsilon, b^{-(\kappa_+ - \kappa_-)(1-a)}, b^{-2(1-a)})\right] \\ &= L_\infty^{-1} A_0 \kappa_+ b^{a(\kappa_+ - \kappa_-) + \kappa_- - 1} e^{(\kappa_+ - \kappa_-)T} \left[1 + \mathcal{O}(b^{-2(1-a) - a\kappa_+}, b^{-a(\kappa_+ - \kappa_-)}, b^{a\kappa_+})\right]. \end{aligned} \tag{5.81}$$

Since the remainder terms on both sides of (5.81) are smaller than the leading-order terms and $\kappa_+ \neq \kappa_-$, this gives a T-dependent coefficient C, which is a contradiction with the relation (5.79) for all $t \in \mathbb{R}$ and hence for all $T > 0$. □

From Lemma 5.3, we can now prove Theorem 5.1 which states that the Morse index $\mathfrak{m}(\mathfrak{u}_b)$ is finite and is independent of b for every $b \in (b_0, \infty)$.

Proof of Theorem 5.1. For every $b \in (0, \infty)$, the potential $-3\mathfrak{u}_b^2(r)$ in $\mathcal{L}_b$ is bounded from below on $[0, \infty)$. The Schrödinger operator $-\Delta_r + r^2 : \mathcal{E} \mapsto \mathcal{E}^*$ is strictly positive with a purely discrete spectrum. Since $\mathcal{L}_b = -\Delta_r + r^2 - \lambda(b) - 3\mathfrak{u}_b^2(r)$ is bounded from below, the number of negative eigenvalues (the Morse index) of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ is finite by Theorem 10.7 in [31]. Thus, $\mathfrak{m}(\mathfrak{u}_b) < \infty$. It remains to show that $\mathfrak{m}(\mathfrak{u}_b)$ is independent of b for every $b \in (b_0, \infty)$.

Let us recall the Emden–Fowler transformation (5.7), which relates solutions of $\mathcal{L}_b v = 0$ with solutions of $\mathcal{M}_b \gamma = 0$ by $v(r) = r^{-1}\gamma(\log r)$. The spectrum of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ includes the zero eigenvalue if and only if there exists $v \in \mathcal{E}$ satisfying $\mathcal{L}_b v = 0$. This is impossible due to Lemma 5.3 according to the following arguments.

As $t \to -\infty$, there exist two linearly independent solutions to $\mathcal{M}_b\gamma = 0$ and the decaying solution is

$$\partial_b \Psi_b(t) = e^t + \mathcal{O}(e^{3t}), \quad \text{as} \quad t \to -\infty.$$

The other solution is growing as $e^{(3-d)t}$ which corresponds to $v(r) \sim r^{2-d}$ so that

$$r^{d-1}|v(r)|^2 \sim r^{3-d}$$

is not integrable near $r = 0$ for $d \geq 4$. Hence, the corresponding $v(r)$ is not in L^2_r and if there exists nonzero $v \in \mathcal{E}$ satisfying $\mathcal{L}_b v = 0$, then there exists a constant $C_- \neq 0$ such that

$$v(r) = C_- r^{-1} \partial_b \Psi_b(\log r).$$

As $t \to +\infty$, there exist two linearly independent solutions to $\mathcal{M}_b\gamma = 0$ and the decaying solution is

$$\partial_c \Psi_{c(b)}(t) \sim e^{\frac{\lambda(b)-d+2}{2}} e^{-\frac{1}{2}e^{2t}}, \quad \text{as} \quad t \to \infty.$$

The other solution is growing as $e^{\frac{1}{2}e^{2t}}$, which corresponds to $v(r) \sim e^{\frac{1}{2}r^2}$, clearly not in L^2_r. If there exists nonzero $v \in \mathcal{E}$ satisfying $\mathcal{L}_b v = 0$, then there exists a constant $C_+ \neq 0$ such that

$$v(r) = C_+ r^{-1} \partial_c \Psi_{c(b)}(\log r).$$

Since $C_-, C_+ \neq 0$, if there exists nonzero $v \in \mathcal{E}$, then $\partial_b \Psi_b$ and $\partial_c \Psi_{c(b)}$ are linearly dependent, which results in a contradiction with Lemma 5.3 for every $b \in (b_0, \infty)$. Hence $0 \notin \sigma(\mathcal{L}_b)$ for every $b \in (b_0, \infty)$ so that $\mathfrak{m}(\mathfrak{u}_b)$ is independent of b for every $b \in (b_0, \infty)$. □

By Lemma 5.2, $\partial_c \Psi_c(t)$ converges to $\partial_c \Psi_\infty$ on $[(a-1)\log b, \infty)$ as $(\lambda, c) \to (\lambda_\infty, c_\infty)$. Each zero of either $\partial_c \Psi_c$ or $\partial_c \Psi_\infty$ is simple since they are solutions of the second-order linear homogeneous equations $\mathcal{M}_c \partial_c \Psi_c = 0$ and $\mathcal{M}_\infty \partial_c \Psi_\infty = 0$. Consequently, the number of nodal domains of $\partial_c \Psi_c$ in $[(a-1)\log b, \infty)$ coincides with that of $\partial_c \Psi_\infty$ in $[(a-1)\log b, \infty)$.

The following lemma shows that $\partial_c \Psi_{c(b)}$ does not have additional nodal domains in the interval $(-\infty, (a-1)\log b)$ for sufficiently large b.

Lemma 5.4. *Fix $d \geq 13$. Under Assumption 5.2, for every $a \in (0,1)$, there exists $b_0 > 0$ such that for every $b \in (b_0, \infty)$, there exists $C_0 > 0$ such that*

$$e^{(d-3)t}|\partial_c \Psi_{c(b)}(t)| \geq C_0, \quad t \in (-\infty, (a-1)\log b). \tag{5.82}$$

Proof. Recall that $\mathcal{M}_b = \mathcal{M}_{c(b)}$ for $\lambda = \lambda(b)$ and $\mathcal{M}_b \partial_b \Psi_b = 0$ with

$$\partial_b \Psi_b(t) = e^t + \mathcal{O}(e^{3t}), \quad \text{as } t \to -\infty. \tag{5.83}$$

Similar to the proof of Lemma 5.1, we denote the second linearly independent solution of $\mathcal{M}_b \gamma = 0$ by Φ and normalize it such that

$$\Phi(t) = (2-d)^{-1} e^{(3-d)t} + \mathcal{O}(e^{(5-d)t}), \quad \text{as } t \to -\infty. \tag{5.84}$$

We are interested in the behavior of $\partial_c \Psi_{c(b)}$ for $t \in (-\infty, t_0)$, where $t_0 := (a-1)\log b$. It follows from (5.40) and (5.66) that for sufficiently large b, we have

$$\partial_c \Psi_{c(b)}(t_0) = L_\infty e^{\kappa_- t_0} \left[1 + \mathcal{O}\left(\epsilon, e^{(\kappa_+ - \kappa_-)t_0}, e^{2t_0}\right)\right], \tag{5.85}$$

$$\partial_c \Psi'_{c(b)}(t_0) = L_\infty \kappa_- e^{\kappa_- t_0} \left[1 + \mathcal{O}\left(\epsilon, e^{(\kappa_+ - \kappa_-)t_0}, e^{2t_0}\right)\right]. \tag{5.86}$$

Since $\partial_c \Psi_{c(b)}$ is a linear combination of $\partial_b \Psi_b$ and Φ by the linear superposition principle, we can express $\partial_c \Psi_{c(b)}$ as

$$\begin{aligned} \partial_c \Psi_{c(b)}(t) = e^{(d-4)t_0} [&(\partial_c \Psi_{c(b)}(t_0) \Phi'(t_0) - \partial_c \Psi'_{c(b)}(t_0) \Phi(t_0)) \partial_b \Psi_b(t) \\ &+ \left(\partial_c \Psi'_{c(b)}(t_0) \partial_b \Psi_b(t_0) - \partial_c \Psi_{c(b)}(t_0) \partial_b \Psi'_b(t_0)\right) \Phi(t)], \end{aligned} \tag{5.87}$$

where we have used the normalization $W(\partial_b \Psi_b, \Phi) = e^{(4-d)t}$ of the Wronskian between the two solutions $\partial_b \Psi_b$ and Φ. Since $t_0 \to -\infty$ as $b \to \infty$, we can use asymptotics (5.83) and (5.84) as well as the boundary condtions (5.85) and (5.86) to obtain for every $t \in (-\infty, t_0)$:

$$\begin{aligned} \partial_c \Psi_{c(b)}(t) = (d-2)^{-1} L_\infty e^{\kappa_- t_0} &\left[(\kappa_- + d - 3) e^{t - t_0} + (1 - \kappa_-) e^{(3-d)(t-t_0)}\right] \\ &\times \left[1 + \mathcal{O}\left(\epsilon, e^{(\kappa_+ - \kappa_-)t_0}, e^{2t_0}\right)\right] \left[1 + \mathcal{O}(e^{2t})\right], \end{aligned} \tag{5.88}$$

where $1 - \kappa_- > 0$ and

$$\kappa_- + d - 3 = \frac{1}{2}\left(d - 2 - \sqrt{d^2 - 16d + 40}\right) > 0$$

for every $d \geq 13$. Thus, the sign of $\partial_c \Psi_{c(b)}(t)$ for every $t \in (-\infty, t_0)$ coincides with the sign of L_∞. Multiplying (5.88) by $e^{(d-3)t}$ yields the bound (5.82). $\square$

Remark 5.10. *It is interesting to know that $\partial_c \Psi_{c(b)}$ and $\partial_c \Psi_\infty$ diverge as $t \to -\infty$ with different growth rates: $d-3$ for the former and $|\kappa_-|$ for the latter, as is seen from (5.40) and (5.88). This difference is explained by the different behavior of the t-dependent coefficients of $\mathcal{M}_{c(b)}$ and $\mathcal{M}_\infty$ as $t \to -\infty$ in (5.14) and (5.38).*

From Lemmas 5.2 and 5.4, we can now prove Theorem 5.2 which states that $\mathfrak{m}(\mathfrak{u}_b) = \mathfrak{m}(\mathfrak{u}_\infty)$ for every $b \in (b_0, \infty)$.

Proof of Theorem 5.2. By Sturm's Oscillation Theorem (see, e.g., Theorem 3.5 in [56]), the Morse index $\mathfrak{m}(\mathfrak{u}_b)$ coincides with the number of zeros of the function $v(r)$ on $(0, \infty)$ satisfying $\mathcal{L}_b v = 0$ and $v(r) \to 0$ as $r \to \infty$. Due to the Emden–Fowler transformation (5.7), the number of zeros of $v(r)$ on $(0, \infty)$ coincides with the number of zeros of $\partial_c \Psi_{c(b)}(t)$ on $\mathbb{R}$ since $\mathcal{M}_b \partial_c \Psi_{c(b)} = 0$ and $\partial_c \Psi_{c(b)}(t) \to 0$ as $t \to \infty$.

By Lemma 5.4, all zeros of $\partial_c \Psi_{c(b)}$ are located in the interval $[(a-1)\log b, \infty)$ for fixed $a \in (0,1)$ and sufficiently large $b > 0$. By Lemma 5.2 and simplicity of the zeros of $\partial_c \Psi_{c(b)}$ and $\partial_c \Psi_\infty$, the number of zeros of $\partial_c \Psi_{c(b)}$ and $\partial_c \Psi_\infty$ in $[(a-1)\log b, \infty)$ coincides since $\lambda(b) \to \lambda_\infty$ and $c(b) \to c_\infty$ as $b \to \infty$. All zeros of $\partial_c \Psi_\infty$ are located in the interval $[(a-1)\log b, \infty)$ by Assumption 5.2 with the expansion (5.40) and give $\mathfrak{m}(\mathfrak{u}_\infty) = \mathfrak{m}(\mathfrak{u}_b)$ for every $b \in (b_0, \infty)$. □

Chapter 6

Asymptotic behavior of the ground states in the energy-critical case

In this chapter, we consider the ground states of (1.7) in the energy-critical case $(d-2)p = 2$, $d \geq 3$. Based on the preliminary results of Chapter 3, there exists a family $\{\mathfrak{u}_b\}_{b>0}$ of ground states (i.e., solutions of (1.7)), together with corresponding values $\lambda = \lambda(b)$. The purpose of this chapter is to extend the shooting method utilized in Chapter 3 and Chapter 4 to the energy-critical case and to obtain the asymptotic representation of $\lambda(b)$ as $b \to \infty$. As far as we are aware, the shooting method has not been previously developed in the context of the energy-critical case, for which the variational approximations are more common.

For the shooting method in the energy-critical case with $d = 2 + \frac{2}{p}$, we introduce the same two analytic families of solutions as in Chapters 3–5. The b-family is defined by parameter $b := u(0)$ and the c-family is defined by parameter c in the asymptotic behavior

$$c := \lim_{r\to\infty} u(r) e^{\frac{1}{2}r^2} r^{\frac{d-\lambda}{2}}. \tag{6.1}$$

Contrary to the energy-supercritical case, the c-family exists in a local neighborhood of a spatially decaying solution to the stationary Schrödinger equation

$$V''(r) + \frac{d-1}{r} V'(r) - r^2 V(r) + \lambda V(r) = 0,$$

which is satisfied by $V(r) = ce^{-\frac{1}{2}r^2}\mathfrak{U}(r^2;\alpha,\beta)$, where $c \in \mathbb{R}$ is arbitrary and $\mathfrak{U}(z;\alpha,\beta)$ is the Tricomi function (see [1]) with

$$\alpha := \frac{p+1}{2p} - \frac{\lambda}{4}, \qquad \beta := 1 + \frac{1}{p}. \tag{6.2}$$

Furthermore, contrary to the energy-supercritical case, the b-family exists in a local neighborhood of the algebraic soliton

$$U_b(r) = \frac{b}{(1+\alpha_p b^{2p} r^2)^{\frac{1}{p}}}, \qquad \alpha_p := \frac{p^2}{4(1+p)}, \tag{6.3}$$

where the parameter b has been introduced from the condition $b = U_b(0)$. The algebraic soliton (also called the Aubin-Talenti solution [2, 58]) satisfies the nonlinear wave equation $-\Delta U_b = U_b^{2p+1}$ for every $b > 0$. It has been used in many studies of the energy-critical wave equations in bounded domains as in the pioneering work [5] and in follow-up works [19, 14, 26, 27, 50]. In the context of the stationary Gross–Pitaevskii equation (1.5), it was used in [52] in order to obtain the lower bound on the dependence $\lambda(b)$ from a variational method. Recently in [43], the variational methods and the elliptic estimates were extended in order to get the upper bound on the dependence $\lambda(b)$. We will use the shooting method to justify the relevance of the algebraic soliton (6.3) for the asymptotic behavior of $\lambda(b)$ as $b \to \infty$.

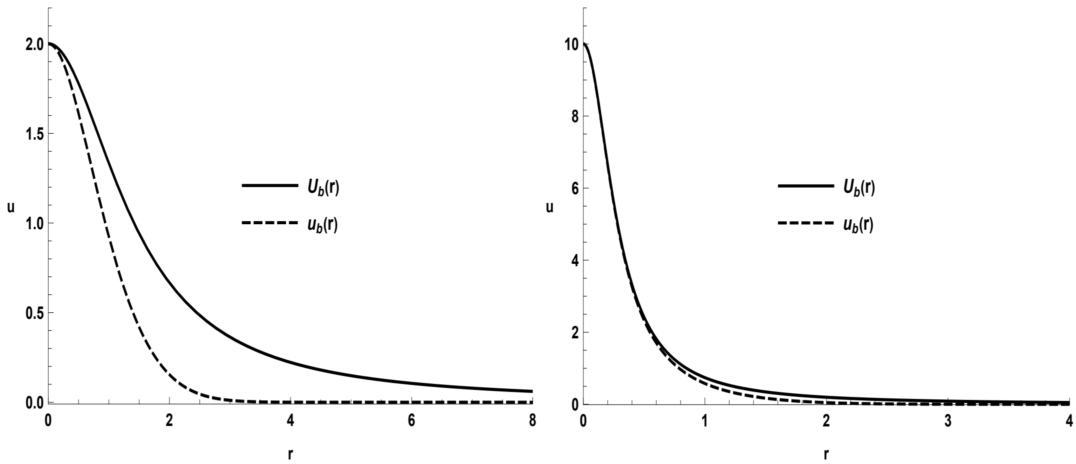

FIGURE 6.1: Ground states of the stationary equation (1.7) with $p = 1$ and $d = 4$ compared with the algebraic soliton (6.3) for $b = 2$ (left) and $b = 10$ (right).

Figure 6.1 shows the numerically obtained profile u_b versus r in comparison with the profile U_b for $b = 2$ (left) and $b = 10$ (right). Visualization is given for $p = 1$ (that corresponds to $d = 4$). Results for other values of $p \in (0, 1)$ are similar. The two profiles are different for $b = 2$ but the discrepancy gets smaller for $b = 10$ and becomes invisible for larger values of b. The values of λ are uniquely defined in terms of b along the curve $\lambda = \lambda(b)$ which is shown in Figure 6.2 for $p = 1$.

The following theorem presents outcomes of the shooting method, which is the main result of this chapter. We use the following notations:

- $\lambda(b) \sim \lambda_0(b)$ denotes the asymptotic equivalence in the sense $\lim_{b\to\infty} \lambda_0(b)^{-1}\lambda(b) = 1$,

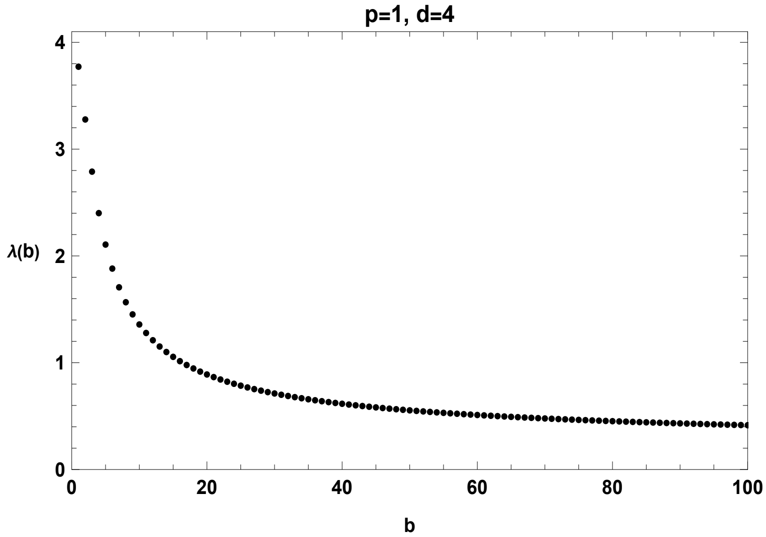

FIGURE 6.2: The dependence $\lambda = \lambda(b)$ for $p = 1$ and $d = 4$.

- $\lambda(b) = \mathcal{O}(b^q)$ denotes the order of magnitude in the sense that $|\lambda(b)| \leq Cb^q$ for some $C > 0$ and all sufficiently large b.

Theorem 6.1. *Fix $p = \frac{2}{d-2} \in (0,1)$ for $d > 4$ and let $\lambda = \lambda(b)$ be the solution curve for the ground state $u = u_b$ of the stationary Gross–Pitaevskii equation* (1.7) *satisfying $u_b(0) = b$, $u_b'(r) < 0$ for $r > 0$, and $u_b(r) \to 0$ as $r \to \infty$. Then,*

$$\lambda(b) \sim C_p \begin{cases} b^{-2(1-p)}, & \frac{1}{2} < p < 1, \\ b^{-1}\log b, & p = \frac{1}{2}, \\ b^{-2p}, & 0 < p < \frac{1}{2}, \end{cases} \quad \text{as} \quad b \to \infty, \tag{6.4}$$

with

$$C_p = \begin{cases} -\dfrac{\Gamma\left(\frac{p+1}{2p}\right)\Gamma\left(-\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)}{(1+p)\Gamma\left(\frac{p-1}{2p}\right)\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{p}-1\right)\Gamma\left(\frac{1}{p}+1\right)}\left[\dfrac{4(1+p)}{p^2}\right]^{\frac{1}{p}}, & \frac{1}{2} < p < 1, \\ 144, & p = \frac{1}{2}, \\ \dfrac{8(1+p)^2}{p^2(1-2p)}, & 0 < p < \frac{1}{2}. \end{cases}$$

Moreover, for every $a \in (0, \frac{p}{1+p})$, there exist $B_a, C_a > 0$ such that for every $b \geq B_a$, we have

$$\sup_{r\in[0,b^{-p(1-a)}]} b^{-1}\left|u_b(r) - U_b(r)\right| \leq C_a b^{-2p(1-a)}, \tag{6.5}$$

$$\sup_{r\in[b^{-p(1-a)},1]} r^{\frac{2}{p}}\left|u_b(r) - c(b)e^{-\frac{1}{2}r^2}\mathfrak{U}(r^2;\alpha,\beta)\right| \leq C_a|c(b)|^{2p+1}b^{2p(1-a)} \tag{6.6}$$

and

$$\sup_{r\in[1,\infty)} \left| e^{\frac{1}{2}r^2} r^{\frac{d-\lambda(b)}{2}} u_b(r) - c(b) r^{\frac{d-\lambda(b)}{2}} \mathfrak{U}(r^2;\alpha,\beta) \right| \leq C_a |c(b)|^{2p+1}, \tag{6.7}$$

where $c = c(b) \sim A_p b^{-1}$ *as* $b \to \infty$ *for some* $A_p > 0$.

Remark 6.1. *The asymptotic result (6.4) coincides with Theorem 1.1 in [43] obtained by the variational theory and elliptic estimates. It follows from Remark 1.2 in [43] that there exists* C_d *such that*

$$\lambda(\varepsilon) \sim C_d \begin{cases} \varepsilon & d = 5, \\ \varepsilon^2 |\log \varepsilon| & d = 6, \\ \varepsilon^2 & d \geq 7, \end{cases} \tag{6.8}$$

where $\varepsilon = b^{-p}$ *is defined from the algebraic soliton (6.3). The case* $d \geq 7$ *corresponds to* $0 < p < \frac{1}{2}$ *as in (6.4). For* $d = 6$*, we have* $p = \frac{1}{2}$ *so that* $\varepsilon^2 |\log \varepsilon| \sim b^{-1} \log b$ *as in (6.4). For* $d = 5$*, we have* $p = \frac{2}{3}$ *so that* $\varepsilon \sim b^{-2/3} = b^{-2(1-p)}$ *as in (6.4).*

Remark 6.2. *It also follows from Remark 1.2 in [43] that* $\lambda(\varepsilon) \sim C|\log \varepsilon|^{-1}$ *for* $d = 4$ *and* $\lambda(\varepsilon) - 1 \sim C\varepsilon$ *for* $d = 3$*. In our notations with* $\varepsilon = b^{-p}$*, this would correspond to* $\lambda(b) \sim C(\log b)^{-1}$ *for* $p = 1$ *and* $\lambda(b) - 1 \sim Cb^{-2}$ *for* $p = 2$*. However, we have found that the shooting method based on the b-family and the c-family can be applied for* $p \in (0,1)$ *but needs some further modifications for* $p \geq 1$*.*

Remark 6.3. *Since* $\mathfrak{U}(r^2;\alpha,\beta) = \mathcal{O}(r^{-\frac{2}{p}})$ *as* $r \to b^{-p(1-a)}$*, bound (6.6) shows that* $u_b(r) = \mathcal{O}(b^{1-2a})$ *as* $r \to b^{-p(1-a)}$*. This is smaller than* $u_b(r) = \mathcal{O}(b)$ *as* $r \to 0$ *in the bound (6.5). Since* $\mathfrak{U}(r^2;\alpha,\beta) = \mathcal{O}(r^{-\frac{d-\lambda(b)}{p}})$ *as* $r \to \infty$*, bound (6.7) shows that* $u_b(r)$ *satisfies the asymptotic behavior (6.1) with* $c = c(b) = \mathcal{O}(b^{-1})$ *as* $b \to \infty$*.*

Figure 6.3 illustrates relevance of the asymptotic result (6.4) for the solution curve $\lambda = \lambda(b)$. For a given dimension d and the critical exponent $p = \frac{2}{d-2}$, we numerically find $\lambda(b)$ and plot it versus b in comparison with the asymptotic dependence (6.4). The left and right panels show the plots for $d = 7$ when $p = \frac{2}{5}$ and $\lambda(b) \sim C_p b^{-4/5}$ and for $d = 5$ when $p = \frac{2}{3}$ and $\lambda(b) \sim C_p b^{-2/3}$, where C_p is obtained from the best least square fit. The proximity between the numerical and analytical curves becomes obvious in the log-log plot for larger values of b.

Our strategy to prove Theorem 6.1 is as follows. Section 6.1 contains preliminary results where the existence problem is reformulated after the Emden–Fowler transformation and the two solution families and their truncated limits are clearly identified. Section 6.2 gives analysis of the b-family in a local neighborhood of the algebraic soliton (6.3) which becomes the exponentially decaying soliton after the Emden–Fowler transformation. Section 6.3 describes analysis of the c-family in a local neighborhood of the confluent hypergeometric functions. Theorem 6.1 is proven in Section 6.4 where the two families are considered in the common asymptotic region with parameters c and λ obtained uniquely in the asymptotic limit $b \to \infty$. Besides the asymptotic dependence

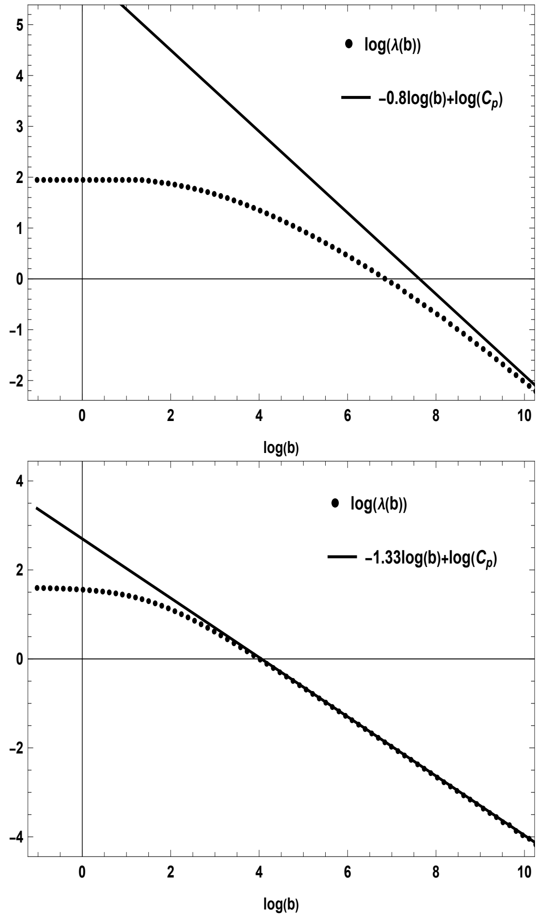

FIGURE 6.3: Log-log graphs of $\lambda(b)$ versus b for the ground state of the stationary equation (1.7) for $d = 7$, $p = \frac{2}{5}$ (left) and $d = 5$, $p = \frac{2}{3}$ (right) compared with the analytical dependence given in (6.4).

(6.4) which recovers independently the result (6.8) obtained in [43] with different methods, the main outcome of this work is the precise asymptotic construction of the ground state with pointwise estimates (6.5), (6.6), and (6.7) near the Aubin-Talenti solution and the confluent hypergeometric function.

6.1 Two families of solutions

As in Chapter 3, we reformulate the existence problem for the ground state of the stationary Gross–Pitaevskii equation (1.7) as the following initial-value problem:

$$\begin{cases} f''(r) + \frac{d-1}{r} f'(r) - r^2 f(r) + \lambda f(r) + |f(r)|^{2p} f(r) = 0, & r > 0, \\ f(0) = b, \quad f'(0) = 0, \end{cases} \tag{6.9}$$

where $b > 0$ is the free parameter and $d = 2 + \frac{2}{p}$ is defined in terms of $p > 0$ in the energy-critical case. We say that the solution of the initial-value problem (6.9) is a ground state if $f'(r) < 0$ for $r > 0$ and $f(r) \to 0$ as $r \to \infty$. Similarly to Lemmas 3.2 and 3.4 in Subsection 3.1 obtained in the particular case $p = 1$, the existence of a unique classical solution to the initial-value problem (6.9) can be concluded by using the integral equation formulation and the Lyapunov function method. We skip the proof since it is standard and state that for every $p > 0$, $\lambda \in \mathbb{R}$, and $b > 0$, there exists a unique classical solution $f \in C^2(0, \infty)$ to the initial-value problem (6.9) satisfying the asymptotic behavior:

$$f(r) = b - \frac{p(\lambda + b^{2p})}{4(p+1)} b r^2 + \mathcal{O}(r^4), \quad \text{as} \quad r \to 0. \tag{6.10}$$

The singularity of the stationary equation (6.9) at $r = 0$ is unfolded by introducing the Emden-Fowler transformation:

$$r = e^t, \quad \Psi(t) = e^{\frac{t}{p}} f(e^t). \tag{6.11}$$

After the transformation of variables, Ψ satisfies the second-order nonautonomous equation

$$\Psi''(t) - \frac{1}{p^2}\Psi(t) + |\Psi(t)|^{2p}\Psi(t) = -\lambda e^{2t}\Psi(t) + e^{4t}\Psi(t). \tag{6.12}$$

We say that the b-family of solutions to equation (6.12) is defined by applying the transformation (6.11) to the unique solution of the initial-value problem (6.9). The corresponding b-solution, denoted as $\Psi_b(t)$, satisfies the asymptotic behaviour

$$\Psi_b(t) = b e^{\frac{t}{p}} \left[1 - \frac{p(\lambda + b^{2p})}{4(p+1)} e^{2t} + \mathcal{O}(e^{4t}) \right], \quad \text{as} \quad t \to -\infty, \tag{6.13}$$

which follows from (6.10). Thus, the b-family of solutions decays to zero as $t \to -\infty$. We will show in Section 6.2 that the b-family stays close to the positive homoclinic orbit of the truncated version of equation (6.12) given by the second-order autonomous ODE

$$\Theta''(t) - \frac{1}{p^2}\Theta(t) + |\Theta(t)|^{2p}\Theta(t) = 0. \tag{6.14}$$

The second-order equation (6.14) is integrable with the first-order invariant

$$\frac{1}{2}(\Theta')^2 - \frac{1}{2p^2}\Theta^2 + \frac{1}{2(p+1)}\Theta^{2(p+1)} = E, \tag{6.15}$$

where E is constant along the classical solutions of equation (6.14). The origin in the (Θ, Θ')-plane is a saddle point. The unique (up to translation) positive homoclinic orbit exists at the energy level $E = 0$ for every $p > 0$. The homoclinic orbit can be found explicitly in the form

$$\Theta_h(t+t_0) = \frac{e^{\frac{t+t_0}{p}}}{(1+\alpha_p e^{2(t+t_0)})^{\frac{1}{p}}}, \qquad \alpha_p := \frac{p^2}{4\,(1+p)}, \tag{6.16}$$

where $t_0 \in \mathbb{R}$ is an arbitrary parameter of translation. Since

$$\Theta_h(t) = \begin{cases} e^{\frac{t}{p}}[1+\mathcal{O}(e^{2t})] & \text{as} \quad t \to -\infty, \\ \alpha_p^{-\frac{1}{p}} e^{-\frac{t}{p}}[1+\mathcal{O}(e^{-2t})] & \text{as} \quad t \to +\infty, \end{cases} \tag{6.17}$$

it follows by comparison with (6.13) that $\Psi_b(t) \sim b\Theta_h(t)$ as $t \to -\infty$ for which the translation parameter t_0 in (6.16) is uniquely selected as $t_0 = p \log b$. With this choice for t_0, we observe that $\Theta_h(t + p\log b)$ after transformation (6.11) coincides with the algebraic soliton $U_b(r)$ given by (6.3).

Next, we introduce another analytical family of solutions to equation (6.12) that decay to zero as $t \to +\infty$, which we call the c-family and denote as $\Psi_c(t)$. We show in Section 6.3 that the c-family stays close to the decaying solutions of the linearized version of equation (6.12) given by the linear second-order nonautonomous ODE

$$\Upsilon''(t) - \frac{1}{p^2}\Upsilon(t) + \lambda e^{2t}\Upsilon(t) - e^{4t}\Upsilon(t) = 0. \tag{6.18}$$

By using the change of variables

$$z = e^{2t}, \quad \Upsilon(t) = z^{\frac{1}{2p}} e^{-\frac{1}{2}z} u(z), \tag{6.19}$$

the second-order equation (6.18) becomes the confluent hypergeometric equation (also known as the Kummer equation):

$$zu''(z) + (\beta - z)\,u'(z) - \alpha u(z) = 0, \tag{6.20}$$

with parameters α and β given by (6.2). Two special solutions of the Kummer equation (6.20) are given by the Kummer function $\mathfrak{M}(z;\alpha,\beta)$ and the Tricomi function $\mathfrak{U}(z;\alpha,\beta)$, which are defined as follows [1]. The Kummer function is defined by the power series

$$\begin{aligned} \mathfrak{M}(z;\alpha,\beta) &= \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k}\frac{z^k}{k!} \\ &= 1 + \frac{\alpha}{\beta}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\frac{z^3}{3!} + \dots, \end{aligned} \tag{6.21}$$

hence it is bounded as $z \to 0$. The Tricomi function satisfies the asymptotic behavior

$$\mathfrak{U}(z;\alpha,\beta) \sim z^{-\alpha}\left[1 + \mathcal{O}(z^{-1})\right] \quad \text{as} \quad z \to +\infty, \tag{6.22}$$

hence it is decaying as $z \to +\infty$ if $\alpha > 0$. In fact, for every $\lambda \in (-\infty, d)$, we have

$$\alpha = \frac{p+1}{2p} - \frac{\lambda}{4} > \frac{p+1}{2p} - \frac{d}{4} = \frac{p+1}{2p} - \frac{1}{2} - \frac{1}{2p} = 0, \tag{6.23}$$

so that $\alpha > 0$ is satisfied in the energy-critical case. By [1, 13.1.3], the Tricomi function can be represented in the superposition form

$$\mathfrak{U}(z;\alpha,\beta) = \frac{\pi}{\sin \pi\beta}\left[\frac{\mathfrak{M}(z;\alpha,\beta)}{\Gamma(1+\alpha-\beta)\Gamma(\beta)} - z^{1-\beta}\frac{\mathfrak{M}(z;1+\alpha-\beta,2-\beta)}{\Gamma(\alpha)\Gamma(2-\beta)}\right], \tag{6.24}$$

which is true for $\beta \notin \mathbb{Z}$ but can also be used in the limit $\beta \to \mathbb{Z}$. By using the identity

$$\frac{\pi}{\sin \pi z} = \Gamma(1-z)\Gamma(z), \quad z \notin \mathbb{Z}. \tag{6.25}$$

we can rewrite (6.24) for $\beta \notin \mathbb{N}$ as

$$\mathfrak{U}(z;\alpha,\beta) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)}\mathfrak{M}(z;\alpha,\beta) + z^{1-\beta}\frac{\Gamma(\beta-1)}{\Gamma(\alpha)}\mathfrak{M}(z;1+\alpha-\beta,2-\beta). \tag{6.26}$$

By [1, 13.1.6], if $\beta = n+1 \in \mathbb{N}$, then

$$\begin{aligned}\mathfrak{U}(z;\alpha,n+1) = \frac{(-1)^{n+1}}{n!\Gamma(\alpha-n)}&\Big(\mathfrak{M}(z;\alpha,n+1)\log z \\ &+ \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(n+1)_k}\frac{z^k}{k!}[\psi(\alpha+k)-\psi(1+k)-\psi(1+n+k)]\Big) \\ &+ \frac{1}{\Gamma(\alpha)}\sum_{k=1}^{n}\frac{(k-1)!(1-\alpha+k)_{n-k}}{(n-k)!}z^{-k},\end{aligned} \tag{6.27}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

By means of the transformation (6.19), Tricomi function determines a suitable solution of the linear equation (6.18):

$$\Upsilon_h(t) = e^{\frac{t}{p}}e^{-\frac{1}{2}e^{2t}}\mathfrak{U}(e^{2t};\alpha,\beta). \tag{6.28}$$

This solution is considered to be the leading-order approximation of the c-family such that $\Psi_c(t) \sim c\Upsilon_h(t)$ as $t \to +\infty$ satisfies the asymptotic behavior

$$\Psi_c(t) \sim c e^{-\frac{(2-\lambda)t}{2}}e^{-\frac{1}{2}e^{2t}} \quad \text{as} \ \ t \to +\infty. \tag{6.29}$$

The ground state of Theorem 6.1 is the connection of the unique solution of the initial-value problem (6.9) satisfying (6.10) with the unique solution satisfying the decay behavior

$$f(r) \sim c r^{-(1+\frac{1}{p}-\frac{\lambda}{2})}e^{-\frac{1}{2}r^2} \quad \text{as} \ \ r \to \infty, \tag{6.30}$$

which follows from (6.11) and (6.29). The connection between (6.10) and (6.30) only exists for some specific values of $c = c(b)$ and $\lambda = \lambda(b)$. Thus, the main question is to find and to justify the analytical expressions for $c(b)$ and $\lambda(b)$ in the asymptotic limit $b \to \infty$.

6.2 Persistence of the b-family of solutions

The b-family of solutions Ψ_b of the differential equation (6.12) satisfying (6.13) is considered in a neighborhood of the homoclinic orbit Θ_h of the differential equation (6.14) satisfying (6.17). Since the comparison gives $\Psi_b(t) \sim b\Theta_h(t) \sim \Theta_h(t+p\log b)$ as $t \to -\infty$, we translate $\Psi_b(t)$ by $-p\log b$ and introduce the perturbation term

$$\gamma(t) := \Psi_b(t - p\log b) - \Theta_h(t),$$

which satisfies

$$\mathcal{L}\gamma = f_b(\Theta_h + \gamma) - N(\Theta_h, \gamma), \tag{6.31}$$

where $f_b(t) := -\lambda b^{-2p} e^{2t} + b^{-4p} e^{4t}$,

$$(\mathcal{L}\gamma)(t) := \gamma''(t) - \frac{1}{p^2}\gamma(t) + (2p+1)|\Theta_h(t)|^{2p}\gamma(t),$$

and

$$N(\Theta_h, \gamma) := |\Theta_h + \gamma|^{2p}(\Theta_h + \gamma) - |\Theta_h|^{2p}\Theta_h - (2p+1)|\Theta_h|^{2p}\gamma.$$

Remark 6.4. *Since $\Theta_h(t)$ is positive for all $t \in \mathbb{R}$ and $\Theta_h(t) + \gamma(t)$ is shown to be positive in the region of t where we analyze persistence of the b-family of solutions, we can neglect writing modulus signs in $\mathcal{L}\gamma$ and $N(\Theta_h, \gamma)$.*

The nonlinear term $N(\Theta_h, \gamma)$ is superlinear in γ if $p \in (0, \frac{1}{2})$ and quadratic if $p \geq \frac{1}{2}$, according to the following proposition.

Proposition 6.1. *Fix $p > 0$ and $a > 0$. If $F : [-a, a] \to \mathbb{R}$ is defined as*

$$F(x) := (a + x)^{2p+1} - a^{2p+1} - (2p+1)a^{2p}x,$$

then there exists a positive constant $C > 0$, such that for all $x \in [-a, a]$

$$|F(x)| \leq \begin{cases} C|x|^{2p+1}, & \text{if } p \in (0, \frac{1}{2}), \\ C|x|^2, & \text{if } p \in [\frac{1}{2}, \infty). \end{cases} \tag{6.32}$$

Proof. Without the loss of generality, we assume that $a = 1$ due to the scaling transformation:

$$F(x) = a^{2p+1}\left[\left(1 + \frac{x}{a}\right)^{2p+1} - 1 - (2p+1)\frac{x}{a}\right].$$

Note that $F''(x) = 2p(2p+1)(1+x)^{2p-1}$, so that if $2p-1 \geq 0$, then F'' is bounded on $[-1,1]$, and the second line of (6.32) follows from Taylor's theorem. If $2p-1 < 0$, then F'' is bounded on $[-\frac{1}{2}, \frac{1}{2}]$ so that

$$|F(x)| \leq C|x|^2 \leq C|x|^{2p+1}, \quad x \in \left[-\frac{1}{2}, \frac{1}{2}\right],$$

for some positive constant C. For $|x| \in [\frac{1}{2}, 1]$, we have $\left(\frac{1}{2}\right)^{2p+1} \leq |x|^{2p+1}$ so that

$$|F(x)| = \left(\frac{1}{2}\right)^{2p+1} 2^{2p+1}|F(x)| \leq C|x|^{2p+1}, \quad |x| \in \left[\frac{1}{2}, 1\right],$$

for another positive constant C. The two estimates above give the second line of (6.32). □

The homogeneous equation $\mathcal{L}\gamma = 0$ admits two linearly independent solutions. The first one is given by $\Theta_h'(t)$ due to the translation symmetry of the autonomous equation (6.14). The other solution denoted by $\Sigma(t)$ can be obtained from the Wronskian relation

$$\Theta_h'(t)\Sigma'(t) - \Theta_h''(t)\Sigma(t) = \Sigma_0, \qquad t \in \mathbb{R}, \tag{6.33}$$

where $\Sigma_0 \neq 0$ is constant. We take $\Sigma_0 = 1$ for normalizing $\Sigma(t)$. Using (6.17) in (6.33), we obtain that

$$\Sigma(t) = -\frac{p^2}{2}\begin{cases} e^{-\frac{t}{p}}[1 + \mathcal{O}(e^{2t})] & \text{as} \quad t \to -\infty, \\ \alpha_p^{\frac{1}{p}} e^{\frac{t}{p}}[1 + \mathcal{O}(e^{-2t})] & \text{as} \quad t \to +\infty. \end{cases} \tag{6.34}$$

Using the two linearly independent solutions Θ_h' and Σ of the homogeneous equation $\mathcal{L}\gamma = 0$, we rewrite (6.31) as an integral equation for γ:

$$\gamma(t) = \int_{-\infty}^{t} \left(\Theta_h'(t')\Sigma(t) - \Theta_h'(t)\Sigma(t')\right) \left[f_b(t')\left(\Theta_h(t') + \gamma(t')\right) - N\left(\Theta_h(t'), \gamma(t')\right)\right] dt', \tag{6.35}$$

where the free solution $c_1\Theta_h'(t) + c_2\Sigma(t)$ has been set to zero from the requirement that $\gamma(t)$ decays to zero as $t \to -\infty$ faster than $\Theta_h'(t)$.

The perturbation term γ can be estimated to be small in the L^∞ norm on the semi-infinite interval $(-\infty, T + ap\log b]$ with fixed $T > 0$ and $a > 0$, where the right end point diverges asymptotically to $+\infty$ as $b \to \infty$. The following lemma gives the persistence result for the solution $\Psi_b(t - p\log b)$ to stay close to the leading-order term $\Theta_h(t)$ for $t \in (-\infty, T + ap\log b]$.

Lemma 6.1. *Fix $p \in (0,1]$ and $\lambda \in \mathbb{R}$. For any fixed $T > 0$ and $a \in (0, \frac{p}{1+p})$ there exist $b_{T,a} > 0$ and $C_{T,a} > 0$ such that the unique solution $\Psi_b(t)$ to the second-order equation* (6.12) *with asymptotic behaviour* (6.13) *satisfies for $t \in (-\infty, T + ap\log b]$ and* *all* $b \geq b_{T,a}$*:*

$$|\Psi_b(t - p\log b) - \Theta_h(t)| \leq C_{T,a} b^{-2p(1-a)} e^{\frac{t}{p}}, \tag{6.36}$$

where the bound can be differentiated in t*.*

Remark 6.5. *The remainder term in the bound (6.36) is small on* $[ap\log b, T + ap\log b]$ *for every* $a \in (0, \frac{2p}{1+2p})$ *for which* $b^{-2p+a(2p+1)} \to 0$ *as* $b \to \infty$*. However,* $\Theta_h(t) = \mathcal{O}(b^{-a})$ *on the same interval so that the remainder term* $\gamma(t)$ *is smaller than the leading-order term* $\Theta_h(t)$ *for* $t \in [ap\log b, T + ap\log b]$ *if* $a \in (0, \frac{p}{1+p})$ *for which* $b^{-2p+a(2p+1)} \ll b^{-a}$ *for sufficiently large* b*.*

Proof. In order to eliminate the divergence of the integral kernel in the integral equation (6.35) as $t \to -\infty$, we introduce a change of variables: $\tilde{\Theta}_h(t) := e^{-\frac{t}{p}}\Theta_h(t)$ and $\tilde{\gamma}(t) := e^{-\frac{t}{p}}\gamma(t)$. The new integral equation for $\tilde{\gamma}$ can be considered as the fixed-point equation $\tilde{\gamma} = A\tilde{\gamma}$, where

$$(A\tilde{\gamma})(t) := \int_{-\infty}^{t} K(t,t') \left[f_b(t')(\tilde{\Theta}_h(t') + \tilde{\gamma}(t')) - e^{2t'} N(\tilde{\Theta}_h(t'), \tilde{\gamma}(t')) \right] dt', \tag{6.37}$$

where the integral kernel is defined as

$$K(t,t') := e^{\frac{t'}{p}} \Theta_h'(t') e^{-\frac{t}{p}} \Sigma(t) - e^{-\frac{t}{p}} \Theta_h'(t) e^{\frac{t'}{p}} \Sigma(t'), \quad t' \leq t.$$

It follows from the asymptotic behaviours (6.17) and (6.34) for Θ_h and Σ that the integral kernel is bounded for every $t \in \mathbb{R}$ by

$$|K(t,t')| \leq C\left(1 + e^{-\frac{2}{p}(t-t')}\right), \quad t' \leq t,$$

for some positive constant C. Thus, as the integration in (6.37) is done in t' from $-\infty$ to t, the kernel $K(t,t')$ is bounded. In addition, the lower bound on $\tilde{\Theta}_h$ follows from (6.17):

$$\tilde{\Theta}_h(t) \geq C_{T,a} b^{-2a}, \quad t \in (-\infty, T + ap\log b),$$

for some positive constant $C_{T,a}$ that depends on T and a for all large b. We shall prove that the integral operator A is a contraction in a small closed ball in the Banach space $L^\infty(-\infty, T + ap\log b)$ equipped with the norm $\|\cdot\|_\infty$.

Case $p \in (0, \frac{1}{2})$**.** Since $\tilde{\Theta}_h(t)$ is bounded from below for $t \in (-\infty, T + ap\log b)$, it follows by Proposition 6.1 if $\|\tilde{\gamma}\|_\infty \ll b^{-2a}$ for all large b, then

$$\|N(\tilde{\Theta}_h, \tilde{\gamma})\|_\infty \leq C\|\tilde{\gamma}\|_\infty^{2p+1}, \tag{6.38}$$

for some positive constant C. We use $f_b(t) := b^{-2p}e^{2t}[-\lambda + b^{-2p}e^{2t}]$ and estimate

$$\begin{aligned}\|A\tilde{\gamma}\|_\infty &\leq C\left[(1+\|\tilde{\gamma}\|_\infty)\int_{-\infty}^{T+ap\log b}|f_b(t')|dt' + \|\tilde{\gamma}\|_\infty^{2p+1}\int_{-\infty}^{T+ap\log b}e^{2t'}dt'\right]\\ &\leq C\left[(1+\|\tilde{\gamma}\|_\infty)\,b^{-2p(1-a)} + b^{2ap}\|\tilde{\gamma}\|_\infty^{2p+1}\right],\end{aligned}$$

where the positive constant C can change from one line to the other line. If $\|\tilde{\gamma}\|_\infty \leq 2Cb^{-2p(1-a)}$, then

$$\begin{aligned}\|A\tilde{\gamma}\|_\infty &\leq C\left[b^{-2p(1-a)} + 2Cb^{-4p(1-a)} + (2C)^{2p+1}b^{-2p(1-2a(p+1)+2p)}\right]\\ &\leq 2Cb^{-2p(1-a)},\end{aligned}$$

where we have used $2p(1-a) < 2p(1-2a(p+1)+2p)$ if $a \in (0, \frac{2p}{1+2p})$. Since $2a < 2p(1-a)$ if $a \in (0, \frac{p}{1+p})$ with $\frac{p}{1+p} < \frac{2p}{1+2p}$, the bound $\|\tilde{\gamma}\|_\infty \leq 2Cb^{-2p(1-a)}$ ensures validity of the bound $\|\tilde{\gamma}\|_\infty \ll b^{-2a}$ for which the bound (6.38) can be used. Similar calculations show that for two functions $\tilde{\gamma}$ and $\tilde{\gamma}'$ in the same small closed ball in $L^\infty(-\infty, T+ap\log b)$, we have

$$\|A\tilde{\gamma} - A\tilde{\gamma}'\|_\infty \leq Cb^{-2p(1-a)}\|\tilde{\gamma}-\tilde{\gamma}'\|_\infty,$$

so that A is a contraction for sufficiently large b. By the Banach fixed-point theorem, there exists a unique fixed point $\tilde{\gamma}$ of A such that

$$\sup_{t\in(-\infty,T+ap\log b)}|\tilde{\gamma}(t)| \leq 2Cb^{-2p(1-a)}.$$

Since $\gamma(t) = e^{\frac{t}{p}}\tilde{\gamma}(t)$, we obtain the bound (6.36) for the unique solution $\gamma(t)$ to the integral equation (6.35).

Case $p \in [\frac{1}{2}, 1]$**.** The only difference in the proof is that, by Proposition 6.1, the bound (6.38) is replaced by the bound

$$\|N(\tilde{\Theta}_h, \tilde{\gamma})\|_\infty \leq C\|\tilde{\gamma}\|_\infty^2, \tag{6.39}$$

if $\|\tilde{\gamma}\|_\infty \ll b^{-2a}$ for all large b. In this case, we get the estimate

$$\begin{aligned}\|A\tilde{\gamma}\|_\infty &\leq C\left[(1+\|\tilde{\gamma}\|_\infty)\,b^{-2p(1-a)} + b^{2ap}\|\tilde{\gamma}\|_\infty^{2}\right],\\ &\leq C\left[b^{-2p(1-a)} + 2Cb^{-4p(1-a)} + (2C)^2b^{-2p(2-3a)}\right]\\ &\leq 2Cb^{-2p(1-a)},\end{aligned}$$

where we have used $2p(1-a) < 2p(2-3a)$ if $a \in (0, \frac{1}{2})$. Since $2a < 2p(1-a)$ if $a \in (0, \frac{p}{1+p})$ with $\frac{p}{1+p} \leq \frac{1}{2}$, the bound $\|\tilde{\gamma}\|_\infty \leq 2Cb^{-2p(1-a)}$ ensures validity of the bound $\|\tilde{\gamma}\|_\infty \ll b^{-2a}$ for which the bound (6.39) can be used. The rest of the proof is verbatim to the case of $p \in (0, \frac{1}{2})$. □

Remark 6.6. *The bound (6.36) can be extended for every $p \geq 1$ if the values of a are restricted to $a \in (0, \frac{1}{2})$ as follows from the proof of Lemma 6.1 in the case of $p \in [\frac{1}{2}, 1]$.*

The result of Lemma 6.1 allows us to justify the validity of

$$\Psi_b(t - p\log b) \sim \Theta_h(t) \sim \alpha_p^{-\frac{1}{p}} e^{-\frac{t}{p}}, \quad t \in [ap\log b, T + ap\log b]$$

due to (6.17) and (6.36). In order to obtain the correction term which behaves like $e^{\frac{t}{p}}$ in the same asymptotic region, we need to analyze γ in more details and obtain the leading-order part of γ. To do so, we write $\gamma = \gamma_h + \delta$, where the leading-order term γ_h satisfies $\mathcal{L}\gamma_h = f_b\Theta_h$ and is given explicitly by

$$\gamma_h(t) = \Sigma(t)\int_{-\infty}^{t} f_b(t')\Theta_h'(t')\Theta_h(t')dt' - \Theta_h'(t)\int_{-\infty}^{t} f_b(t')\Sigma(t')\Theta_h(t')dt', \tag{6.40}$$

whereas the correction term δ satisfies

$$\mathcal{L}\delta = f_b(\gamma_h + \delta) - N(\Theta_h, \gamma_h + \delta). \tag{6.41}$$

The following lemma gives a sharper bound on γ_h compared to the bound (6.36). The sharper bound holds on $[ap\log b, T + ap\log b]$, where the asymptotic behavior of $\Theta_h(t)$ and $\Sigma(t)$ as $t \to +\infty$ is relevant.

Lemma 6.2. *Fix $p \in (0,1)$ and $\lambda \in \mathbb{R}$. For any fixed $T > 0$ and $a \in (0, \frac{p}{1+p})$ there exist $b_{T,a} > 0$ and $C_{T,a} > 0$ such that γ_h in (6.40) satisfies for $t \in [ap\log b, T + ap\log b]$ and all $b \geq b_{T,a}$:*

- *if $p \in (0, \frac{1}{2})$, then*

$$|\gamma_h(t)| \leq C_{T,a}\left[(|\lambda| b^{-2p(1-a)} + b^{-4p(1-a)})e^{-\frac{t}{p}} + (|\lambda| b^{-2p} + b^{-4p})e^{\frac{t}{p}}\right], \tag{6.42}$$

- *if $p \in [\frac{1}{2}, 1)$, then*

$$|\gamma_h(t)| \leq C_{T,a}\left[(|\lambda| b^{-2p(1-a)} + b^{-4p(1-a)})e^{-\frac{t}{p}} + (|\lambda| b^{-2p} + b^{-4p(1-a)})e^{\frac{t}{p}}\right], \tag{6.43}$$

where the bounds can be differentiated in t.

Proof. Since $\Sigma(t)\Theta_h(t)$ is bounded for every $t \in \mathbb{R}$ independently of b, the second integral term in (6.40) is controlled by

$$\begin{aligned}\left|\int_{-\infty}^{T+ap\log b} f_b(t)\Sigma(t)\Theta_h(t)dt\right| &\leq C\int_{-\infty}^{T+ap\log b}\left(|\lambda| b^{-2p}e^{2t} + b^{-4p}e^{4t}\right)dt \\ &\leq C_{T,a}(|\lambda| b^{-2p(1-a)} + b^{-4p(1-a)}).\end{aligned}$$

This estimate yields the first term in the bounds (6.42) and (6.43) due to the asymptotic behavior (6.17). On the other hand, since $\Theta_h(t)^2 = \mathcal{O}(e^{-\frac{2t}{p}})$ as $t \to +\infty$, the first integral term in (6.40) is controlled by

$$\left| \int_{-\infty}^{T+ap\log b} f_b(t)\Theta_h'(t)\Theta_h(t)dt \right| \leq C \int_{-\infty}^{T+ap\log b} \left(|\lambda| b^{-2p} e^{-\frac{2(1-p)}{p}t} + b^{-4p} e^{-\frac{2(1-2p)}{p}t} \right) dt$$
$$\leq C_{T,a}(|\lambda| b^{-2p} + b^{-4p+2a\nu_p}),$$

where $\nu_p = 0$ for $p \in (0, \frac{1}{2})$ and $\nu_p = 2p - 1$ for $p \in [\frac{1}{2}, 1)$. This yields the second term in the bounds (6.42) and (6.43) due to the asymptotic behavior (6.34), where we have also used that $4p(1-a) < 4p - 2a(2p-1)$. □

The sharper bounds (6.42) and (6.43) are compatible with the bound (6.36) on the semi-infinite interval $(-\infty, T + ap\log b]$, which can be rewritten in the form:

$$|\gamma_h(t)| \leq C_{T,a} b^{-2p(1-a)} e^{\frac{t}{p}}, \qquad t \in (-\infty, T + ap\log b]. \tag{6.44}$$

The correction term δ is estimated to be smaller than γ_h according to the following lemma.

Lemma 6.3. *Fix $p \in (0,1]$ and $\lambda \in \mathbb{R}$. For any fixed $T > 0$, $a \in (0, \frac{p}{1+p})$, there exist $b_{T,a} > 0$ and $C_{T,a} > 0$ such that for $t \in (-\infty, T + ap\log b]$ and all $b \geq b_{T,a}$:*

- *if $p \in (0, \frac{1}{2})$, then*

$$|\Psi_b(t - p\log b) - \Theta_h(t) - \gamma_h(t)| \leq C_{T,a} b^{-2p[(2p+1)(1-a)-a]} e^{\frac{t}{p}}, \tag{6.45}$$

- *if $p \in [\frac{1}{2}, 1]$, then*

$$|\Psi_b(t - p\log b) - \Theta_h(t) - \gamma_h(t)| \leq C_{T,a} b^{-2p(2-3a)} e^{\frac{t}{p}}, \tag{6.46}$$

where the bounds can be differentiated in t.

Remark 6.7. *If $a \in (0, \frac{2p}{1+2p})$ for $p \in (0, \frac{1}{2})$, we have*

$$2p(1-a) < 2p\left[(2p+1)(1-a) - a\right],$$

so that comparison (6.44) and (6.45) *shows that δ is smaller than γ_h for sufficiently large b. Similarly, if $a \in (0, \frac{1}{2})$ for $p \geq \frac{1}{2}$, we have*

$$2p(1-a) < 2p(2-3a),$$

so that the comparison of (6.44) and (6.46) *shows that δ is smaller than γ_h for sufficiently large b. In both cases, by Lemma 6.1, we also have γ_h being smaller than Θ_h if $a \in (0, \frac{p}{1+p})$, where $\frac{p}{1+p} \leq \min\{\frac{2p}{1+2p}, \frac{1}{2}\}$ if $p \in (0,1]$.*

Proof. Equation (6.41) for δ can be written similarly to (6.35) as the integral equation

$$\delta(t) = \int_{-\infty}^{t} \left(\Theta_h'(t')\Sigma(t) - \Theta_h'(t)\Sigma(t')\right) \left[f_b(t')\left(\gamma_h(t') + \delta(t')\right) - N\left(\Theta_h(t'), \gamma_h(t') + \delta(t')\right)\right] dt'. \tag{6.47}$$

We proceed in a similar way to the proof of Lemma 6.1. Using the change of variables

$$\tilde{\Theta}_h(t) := e^{-\frac{t}{p}}\Theta_h(t), \quad \tilde{\gamma}_h(t) := e^{-\frac{t}{p}}\gamma_h(t), \quad \tilde{\delta}(t) := e^{-\frac{t}{p}}\delta(t),$$

we rewrite the integral equation for δ as the fixed-point equation $\tilde{\delta} = B\tilde{\delta}$, where

$$(B\tilde{\delta})(t) := \int_{-\infty}^{t} K(t,t') \left[f_b(t')(\tilde{\gamma}_h(t') + \tilde{\delta}(t')) - e^{2t'} N(\tilde{\Theta}_h(t'), \tilde{\gamma}_h(t') + \tilde{\delta}(t'))\right] dt'. \tag{6.48}$$

The only essential difference between A in (6.37) and B in (6.48) is the source term which dictates the size of the closed ball in $L^\infty(-\infty, T + ap\log b]$, where the fixed-point iterations are closed. In (6.48), it consists of the linear term $f_b\tilde{\gamma}_h$ and the contribution from nonlinearity term $N(\tilde{\Theta}_h, \tilde{\gamma}_h)$. The linear term is estimated from (6.44) as

$$\left|\int_{-\infty}^{T+ap\log b} f_b(t)\tilde{\gamma}_h(t)dt\right| \le C\|\tilde{\gamma}_h\|_\infty \int_{-\infty}^{T+ap\log b} |f_b(t)|dt \le Cb^{-4p(1-a)}. \tag{6.49}$$

Estimates for the nonlinear term depend on the value of p. To proceed with the estimates, we decompose

$$N(\tilde{\Theta}_h, \tilde{\gamma}_h + \tilde{\delta}) = N(\tilde{\Theta}_h, \tilde{\gamma}_h) + \tilde{N}(\tilde{\Theta}_h, \tilde{\gamma}_h, \tilde{\delta}),$$

where

$$\begin{aligned} \tilde{N}(\tilde{\Theta}_h, \tilde{\gamma}_h, \tilde{\delta}) &:= N(\tilde{\Theta}_h, \tilde{\gamma}_h + \tilde{\delta}) - N(\tilde{\Theta}_h, \tilde{\gamma}_h) \\ &= (\tilde{\Theta}_h + \tilde{\gamma}_h + \tilde{\delta})^{2p+1} - (\tilde{\Theta}_h + \tilde{\gamma}_h)^{2p+1} - (2p+1)(\tilde{\Theta}_h)^{2p}\tilde{\delta}. \end{aligned}$$

Case $p \in (0, \frac{1}{2})$. By Proposition 6.1, we have

$$\|N(\tilde{\Theta}_h, \tilde{\gamma}_h)\|_\infty \le C'\|\tilde{\gamma}_h\|_\infty^{2p+1}. \tag{6.50}$$

Since $\tilde{N}|_{\delta=0} = 0$ and

$$\left.\frac{\partial \tilde{N}}{\partial \tilde{\delta}}\right|_{\tilde{\delta}=0} = (2p+1)(\tilde{\Theta}_h + \tilde{\gamma}_h)^{2p} - (2p+1)(\tilde{\Theta}_h)^{2p},$$

we obtain by a minor modification of the proof of Proposition 6.1 that

$$\|\tilde{N}(\tilde{\Theta}_h, \tilde{\gamma}_h, \tilde{\delta})\|_\infty \le C\|\tilde{\gamma}_h\|_\infty^{2p}\|\tilde{\delta}\|_\infty. \tag{6.51}$$

Putting together estimates (6.49), (6.50), and (6.51), we obtain that

$$\|B\tilde{\delta}\|_\infty \leq C\left(b^{-4p(1-a)} + b^{-2p(1-a)}\|\tilde{\delta}\|_\infty + b^{-2p[(2p+1)(1-a)-a]} + b^{-2p[2p(1-a)-a]}\|\tilde{\delta}\|_\infty\right).$$

Since $4p(1-a) > 2p[(2p+1)(1-a)-a]$ for every $p \in (0, \frac{1}{2})$, we have $b^{-4p(1-a)} \ll b^{-2p[(2p+1)(1-a)-a]}$ for sufficiently large b, hence the source term coming from the nonlinearity $N(\tilde{\Theta}_h, \tilde{\gamma}_h)$ is much larger than the source term coming from $f_b\tilde{\gamma}_h$ as $b \to \infty$. As a result, if $\|\tilde{\delta}\|_\infty \leq 2Cb^{-2p[2p(1-a)-a]}$, then $\|B\tilde{\delta}\|_\infty \leq 2Cb^{-2p[2p(1-a)-a]}$. Moreover, B is a contraction in the same small closed ball in $L^\infty(-\infty, T + ap\log b)$ for sufficiently large b. Hence, there exists a unique fixed point $\tilde{\delta}$ of B satisfying $\|\tilde{\delta}\|_\infty \leq 2Cb^{-2p[(2p+1)(1-a)-a]}$, which yields (6.45) for $\delta(t) = e^{\frac{t}{p}}\tilde{\delta}(t)$.

Case $p \in [\frac{1}{2}, 1]$**.** By Proposition 6.1, we have

$$\|N(\tilde{\Theta}_h, \tilde{\gamma}_h)\|_\infty \leq C\|\tilde{\gamma}_h\|_\infty^2,$$

and similarly,

$$\|\tilde{N}(\tilde{\Theta}_h, \tilde{\gamma}_h, \tilde{\delta})\|_\infty \leq C\|\tilde{\gamma}_h\|_\infty\|\tilde{\delta}\|_\infty.$$

Proceeding similarly to the previous computations, we obtain

$$\|B\tilde{\delta}\|_\infty \leq C'\left(b^{-4p(1-a)} + b^{-2p(1-a)}\|\tilde{\delta}\|_\infty + b^{-2p(2-3a)} + b^{-2p(1-2a)}\|\tilde{\delta}\|_\infty\right).$$

Since $4p(1-a) > 2p(2-3a)$, we have $b^{-4p(1-a)} \ll b^{-2p(2-3a)}$ for sufficiently large b, hence again the source term coming from the nonlinearity $N(\tilde{\Theta}_h, \tilde{\gamma}_h)$ is much larger than the source term coming from $f_b\tilde{\gamma}_h$ as $b \to \infty$. Proceeding similarly, for sufficiently large b, there exists a unique fixed point $\tilde{\delta}$ of B satifying $\|\tilde{\delta}\|_\infty \leq 2Cb^{-2p(2-3a)}$, which yields (6.46) for $\delta(t) = e^{\frac{t}{p}}\tilde{\delta}(t)$. □

Similarly to Lemma 6.2, we can find a sharper bound on δ compared to the bounds (6.45) and (6.46). This is given by the following lemma, the proof of which follows from the estimates obtained in Lemma 6.3.

Lemma 6.4. *Fix $p \in (0, 1)$ and $\lambda \in \mathbb{R}$. For any fixed $T > 0$ and $a \in (0, \frac{p}{1+p})$ there exist $b_{T,a} > 0$ and $C_{T,a} > 0$ such that δ in (6.47) satisfies for $t \in [ap\log b, T + ap\log b]$ and all $b \geq b_{T,a}$:*

- *if $p \in (0, \frac{1}{2})$, then*

$$|\delta(t)| \leq C_{T,a}(|\lambda|b^{-2p} + b^{-4p})^{2p+1}b^{2ap}e^{\frac{t}{p}}, \tag{6.52}$$

- *if $p \in [\frac{1}{2}, 1)$, then*

$$|\gamma_h(t)| \leq C_{T,a}(|\lambda|b^{-2p} + b^{-4p(1-a)})^2 b^{2ap}e^{\frac{t}{p}}, \tag{6.53}$$

where the bounds can be differentiated in t*.*

Proof. For $p \in (0, \frac{1}{2})$, the first term in the bound (6.42) with $e^{-\frac{t}{p}}$ is much smaller than the second term with $e^{\frac{t}{p}}$ on $[ap\log b, T + ap\log b]$. As a result, it can be neglected. On the other hand, the second term in the bound (6.42) can be extended for the semi-infinite interval $(-\infty, T + ap\log b]$ such that the sharper bound compared to (6.36) can be written in the form

$$|\gamma_h(t)| \le C_{T,a}(|\lambda| b^{-2p} + b^{-4p}) e^{\frac{t}{p}}, \quad t \in (-\infty, T + ap\log b].$$

The bound (6.52) follows from analysis of the integral equation (6.47) by using the transformation to the tilde variables in the proof of Lemma 6.3 and the estimate (6.50) on the nonlinear term N which is much larger than the source term from f_b.

For $p \in [\frac{1}{2}, 1)$, the proof is analogous but we use

$$|\gamma_h(t)| \le C_{T,a}(|\lambda| b^{-2p} + b^{-4p(1-a)}) e^{\frac{t}{p}}, \quad t \in (-\infty, T + ap\log b].$$

and the estimate (6.51) on the nonlinear term N which is still much larger than the source term from f_b. □

6.3 Persistence of the c-family of solutions

The c-family of solutions Ψ_c of the differential equation (6.12) satisfying (6.29) is considered near the solution Υ_h of the linear equation (6.18) given by (6.28). The comparison gives $\Psi_c(t) \sim c\Upsilon_h(t)$ as $t \to +\infty$. The correction term $\eta(t) := \Psi_c(t) - c\Upsilon_h(t)$ satisfies

$$\mathcal{M}\eta = -\left|c\Upsilon_h + \eta\right|^{2p} (c\Upsilon_h + \eta), \tag{6.54}$$

where

$$(\mathcal{M}\eta)(t) := \eta''(t) - \frac{1}{p^2}\eta(t) + \lambda e^{2t}\eta(t) - e^{4t}\eta(t).$$

The homogeneous equation $\mathcal{M}\eta = 0$ has two linearly independent solutions. One solution is Υ_h given by (6.28). The other solution, denoted as Υ_g, can be obtained from the normalized Wronskian relation

$$\Upsilon_h(t)\Upsilon_g'(t) - \Upsilon_h'(t)\Upsilon_g(t) = 1. \tag{6.55}$$

Since it follows from (6.29) that

$$\Upsilon_h(t) \sim e^{-\left(1-\frac{\lambda}{2}\right)t} e^{-\frac{1}{2}e^{2t}}, \quad \text{as } t \to +\infty, \tag{6.56}$$

integrating the Wronskian relation (6.55) yields

$$\Upsilon_g(t) \sim \frac{1}{2} e^{-\left(1+\frac{\lambda}{2}\right)t} e^{\frac{1}{2}e^{2t}}, \quad \text{as } t \to +\infty. \tag{6.57}$$

With two linearly independent solutions Υ_h and Υ_g, we rewrite (6.54) as an integral equation for η:

$$\eta(t) = \int_t^\infty \left(\Upsilon_h(t')\Upsilon_g(t) - \Upsilon_h(t)\Upsilon_g(t')\right) |c\Upsilon_h(t') + \eta(t')|^{2p}(c\Upsilon_h(t') + \eta(t'))dt', \quad (6.58)$$

where the free solution $c_1\Upsilon_h + c_2\Upsilon_g$ has been set to zero in order to guarantee that $\eta(t)$ decays to zero as $t \to +\infty$ faster than $\Upsilon_h(t)$.

The following lemma describes the size of $\eta(t)$ for $t \in [0, \infty)$.

Lemma 6.5. *Fix $\lambda \in (-\infty, 2]$ and $p > 0$. Then, there exist some constants $C > 0$ and $c_0 > 0$, such that for $t \in [0, \infty)$ and $c \in (-c_0, c_0)$, we have*

$$|\Psi_c(t) - c\Upsilon_h(t)| \le C|c|^{2p+1}e^{-(1-\frac{\lambda}{2})t}e^{-\frac{1}{2}e^{2t}}, \quad (6.59)$$

where the bound can be differentiated term by term.

Proof. In order to obtain a bounded kernel in the integral equation (6.58), we first introduce the change of variables

$$\tilde{\Upsilon}_h(t) := e^{(1-\frac{\lambda}{2})t}e^{\frac{1}{2}e^{2t}}\Upsilon_h(t), \qquad \tilde{\eta}(t) := e^{(1-\frac{\lambda}{2})t}e^{\frac{1}{2}e^{2t}}\eta(t),$$

which applied to (6.58) results in the integral equation $\tilde{\eta} = E\tilde{\eta}$, where

$$(E\tilde{\eta})(t) := \int_t^\infty \hat{K}(t,t')e^{-p(2-\lambda)t' - pe^{2t'} - 2t'}\hat{N}(c\tilde{\Upsilon}_h(t'), \tilde{\eta}(t'))dt', \quad (6.60)$$

and where the kernel $\hat{K}$ and the nonlinearity $\hat{N}$ are given by

$$\hat{K}(t,t') := e^{-(1-\frac{\lambda}{2})(t'-t) + 2t' + \frac{1}{2}(e^{2t} - e^{2t'})}(\Upsilon_h(t')\Upsilon_g(t) - \Upsilon_h(t)\Upsilon_g(t')),$$

and

$$\hat{N}(c\tilde{\Upsilon}_h, \tilde{\eta}) := |c\tilde{\Upsilon}_h + \tilde{\eta}|^{2p}(c\tilde{\Upsilon}_h + \tilde{\eta}).$$

Using asymptotic behaviours (6.56) and (6.57), we get that there exists $C > 0$ such that

$$|\hat{K}(t,t')| \le C(1 + e^{\phi(t,t')}), \qquad 0 \le t \le t'. \quad (6.61)$$

where $\phi(t,t') := \lambda(t'-t) - e^{2t'}(1 - e^{-2(t'-t)})$. Since $\phi(t,t') \to -\infty$ as $t' \to +\infty$ and $\phi(t,t')$ has an extremum in t' at $\lambda - 2e^{2t'} = 0$, which does not belong to $\mathbb{R}$ if $\lambda \in (-\infty, 0]$ and is located on $\mathbb{R}_-$ if $\lambda \in (0,2)$, we conclude that $\max\limits_{t' \in [t,\infty)} e^{\phi(t,t')} = e^{\phi(t,t)} = 1$ for $t \ge 0$. Hence, the kernel $\hat{K}(t,t')$ is bounded for every $0 \le t \le t' < \infty$. On the other hand, since $\hat{N}(c\Upsilon_h, \tilde{\eta})$ is a C^1 function for every $p > 0$, the nonlinear term satisfies the following bound:

$$|\hat{N}(c\tilde{\Upsilon}_h, \tilde{\eta})| \le C|c|^{2p}(|c| + |\tilde{\eta}|), \quad \text{as long as } |\tilde{\eta}| \le C, \quad (6.62)$$

where the constant $C > 0$ is independent of c.

In order to use the Banach fixed-point theorem, we first estimate the size of $E\tilde{\eta}$ for $\tilde{\eta}$ in a small closed ball in $L^\infty(0,\infty)$. Since $e^{-p(2-\lambda)t-pe^{2t}-2t}$ in (6.60) is absolutely integrable on $[0,\infty)$, we obtain by using (6.61) and (6.62) that

$$\|E\tilde{\eta}\|_\infty \leq C|c|^{2p}(|c| + \|\tilde{\eta}\|_\infty),$$

where the constant $C > 0$ is independent of c. Thus, the operator E maps a closed ball of radius $2C|c|^{2p+1}$ into itself as long as $|c|$ is chosen sufficiently small.

Similarly, by Proposition 6.1, we get for every $\tilde{\eta}_1$ and $\tilde{\eta}_2$ in the same small closed ball in $L^\infty(0,\infty)$ that $\hat{N}$ is a Lipschitz function satisfying

$$\|\hat{N}(c\tilde{\Upsilon}_h, \tilde{\eta}_1) - \hat{N}(c\tilde{\Upsilon}_h, \tilde{\eta}_2)\|_\infty \leq C|c|^{2p}\|\tilde{\eta}_1 - \tilde{\eta}_2\|_\infty, \tag{6.63}$$

which yields

$$\|E\tilde{\eta}_1 - E\tilde{\eta}_2\|_\infty \leq C|c|^{2p}\|\tilde{\eta}_1 - \tilde{\eta}_2\|_\infty,$$

so that the operator E is a contraction for sufficiently small values of $|c|$. By the Banach fixed-point theorem, there exists a unique solution $\tilde{\eta}(t) \in L^\infty(0,\infty)$ of the integral equation $\tilde{\eta} = E\tilde{\eta}$ satisfying $\|\tilde{\eta}\|_\infty \leq 2C|c|^{2p+1}$. This estimate yields (6.59) after unfolding the transformation for $\eta(t)$. □

Using the result of Lemma 6.5, we can now extend the estimates for $\eta(t)$ for large negative values of t.

Lemma 6.6. *Fix $\lambda \in (-\infty, 2]$, $p > 0$, and $a \in (0,1)$. Then, there exist $b_0 > 0$ and $C > 0$, such that for every $b \geq b_0$ there exists $c_0 > 0$, such that for $t \in [-(1-a)p\log b, 0]$ and $c \in (-c_0 b^{-(1-a)}, c_0 b^{-(1-a)})$, we have*

$$|\Psi_c(t) - c\Upsilon_h(t)| \leq C|c|^{2p+1}b^{2p(1-a)}e^{-\frac{t}{p}}, \tag{6.64}$$

where the bound can be differentiated term by term.

Proof. We rewrite equation (6.54) as an integral equation for $t \in [-(1-a)p\log b, 0]$:

$$\begin{aligned}\eta(t) &= \left(\Upsilon_g'(0)\Upsilon_h(t) - \Upsilon_h'(0)\Upsilon_g(t)\right)\eta(0) + \left(\Upsilon_h(0)\Upsilon_g(t) - \Upsilon_g(0)\Upsilon_h(t)\right)\eta'(0)\\ &\quad + \int_t^0 \left(\Upsilon_h(t')\Upsilon_g(t) - \Upsilon_h(t)\Upsilon_g(t')\right)\left(c\Upsilon_h(t') + \eta(t')\right)\left|c\Upsilon_h(t') + \eta(t')\right|^{2p} dt', \end{aligned} \tag{6.65}$$

where $|\eta(0)| + |\eta'(0)| \leq C|c|^{2p+1}$ by Lemma 6.5. By using the scattering relation (6.26) and the transformation (6.28), we obtain the following asymptotic behavior for $\Upsilon_h(t)$:

$$\Upsilon_h(t) \sim \frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma(\alpha)}e^{-\frac{t}{p}}, \quad \text{as } t \to -\infty, \tag{6.66}$$

where $\alpha > 0$ by (6.23). Wronskian relation (6.55) implies the following asymptotic behavior for $\Upsilon_g(t)$:

$$\Upsilon_g(t) \sim \frac{p\Gamma(\alpha)}{2\Gamma\left(\frac{1}{p}\right)} e^{\frac{t}{p}}, \quad \text{as } t \to -\infty. \tag{6.67}$$

The divergent behaviour of $\Upsilon_h(t)$ as $t \to -\infty$ dictates the correct form of the transformation to use, which in this case is given by:

$$\tilde{\Upsilon}_h(t) := e^{\frac{t}{p}}\Upsilon_h(t), \quad \tilde{\Upsilon}_g(t) := e^{\frac{t}{p}}\Upsilon_g(t), \quad \tilde{\eta}(t) := e^{\frac{t}{p}}\eta(t).$$

Applying it to the integral equation (6.65) results in the fixed point equation $\tilde{\eta} = \mathcal{F}\tilde{\eta}$, where

$$(\mathcal{F}\tilde{\eta})(t) := \left(\Upsilon_g'(0)\tilde{\Upsilon}_h(t) - \Upsilon_h'(0)\tilde{\Upsilon}_g(t)\right)\eta(0) + \left(\tilde{\Upsilon}_h(0)\Upsilon_g(t) - \Upsilon_g(0)\tilde{\Upsilon}_h(t)\right)\eta'(0) + \int_t^0 e^{-2t'}\tilde{K}(t,t')\hat{N}(c\tilde{\Upsilon}_h(t'), \tilde{\eta}(t'))dt', \tag{6.68}$$

where

$$\tilde{K}(t,t') := e^{\frac{t-t'}{p}}\left(\Upsilon_h(t')\Upsilon_g(t) - \Upsilon_h(t)\Upsilon_g(t')\right)$$

and $\hat{N}$ is the same as in the proof of Lemma 6.5.

We proceed by estimating each term of $\mathcal{F}\tilde{\eta}$ in the space $L^\infty(-(1-a)p\log b, 0)$. Since $\tilde{\Upsilon}_h$ and $\tilde{\Upsilon}_g$ are bounded for $t \in (-\infty, 0]$ and $|\eta(0)| + |\eta'(0)| \leq C|c|^{2p+1}$, we obtain

$$\left|\left(\Upsilon_g'(0)\tilde{\Upsilon}_h(t) - \Upsilon_h'(0)\tilde{\Upsilon}_g(t)\right)\eta(0) + \left(\tilde{\Upsilon}_h(0)\Upsilon_g(t) - \Upsilon_g(0)\tilde{\Upsilon}_h(t)\right)\eta'(0)\right| \leq C|c|^{2p+1}.$$

Furthermore, if $t \ll -1$, asymptotics (6.66) and (6.67) allow us to estimate size of the last term in (6.68) as

$$\begin{aligned}\left|\int_t^0 e^{-2t'}\tilde{K}(t,t')\hat{N}(c\tilde{\Upsilon}_h(t'), \tilde{\eta}(t'))dt'\right| &\leq C|c|^{2p}\int_t^0 e^{-2t'}(1 + e^{-\frac{2}{p}(t'-t)})(|c| + |\tilde{\eta}(t')|)dt' \\ &\leq C|c|^{2p}b^{2p(1-a)}(|c| + \|\tilde{\eta}\|_\infty),\end{aligned}$$

where we have used the C^1 property of $\hat{N}(c\tilde{\Upsilon}_h, \tilde{\eta})$ satisfying (6.62). These two estimates yield

$$\|\mathcal{F}\tilde{\eta}\|_\infty \leq C|c|^{2p}b^{2p(1-a)}\left(|c| + \|\tilde{\eta}\|_\infty\right),$$

for sufficiently large values of b. The divergent behaviour of $b^{2p(1-a)}$ for large b is controlled by appropriately reducing the value of $|c|$ satisfying $|c| < c_0 b^{-(1-a)}$ for sufficiently small $c_0 > 0$. Thus, we see that the operator $\mathcal{F}$ maps the closed ball of the radius $2C|c|^{2p+1}b^{2p(1-a)}$ in $L^\infty(-(1-a)p\log b, 0)$ into itself. Moreover, since $\hat{N}$ is a Lipschitz function satisfying (6.63), we get that if $\tilde{\eta}_1, \tilde{\eta}_2$ belong to the same ball, then

$$\|\mathcal{F}\eta_1 - \mathcal{F}\eta_2\|_\infty \leq C|c|^{2p}b^{2p(1-a)}\|\tilde{\eta}_1 - \tilde{\eta}_2\|_\infty,$$

so that the operator $\mathcal{F}$ is a contraction as long as $|c| < c_0 b^{-(1-a)}$ for sufficiently small $c_0 > 0$. By the Banach fixed-point theorem, there exists a unique $\tilde{\eta} \in L^\infty(-(1-a)p\log b, 0)$ satisfying

$$\sup_{t\in[-(1-a)p\log b,0]} |\tilde{\eta}(t)| \leq C|c|^{2p+1} b^{2p(1-a)},$$

which yields the bound (6.64) due to the transformation $\eta(t) = e^{-\frac{t}{p}}\tilde{\eta}(t)$. □

6.4 Matching the two solutions: the proof of Theorem 6.1

We are now equipped with all the necessary estimates to prove Theorem 6.1. The ground state $u = u_b$ of the stationary Gross–Pitaevskii equation (1.7) in the Emden-Fowler variables (6.11) exhibits decaying behaviour both as $t \to -\infty$ and as $t \to +\infty$ for every $b > 0$ if $\lambda = \lambda(b)$. In other words, it appears at the intersection of the two solution families with

$$\Psi_b(t) = \Psi_{c(b)}(t), \qquad t \in \mathbb{R} \tag{6.69}$$

for some $\lambda = \lambda(b)$ and $c = c(b)$. This allows us to use the asymptotic behaviours (6.45) and (6.46) for $\Psi_b(t)$, and the asymptotic behavior (6.64) for $\Psi_c(t)$ at the times $t = T - (1-a)p\log b$ with varying $T > 0$ and sufficiently large values of b. Equaling the asymptotic behaviors due to (6.69) yields two implicit equations for parameters λ and c.

Bound (6.5) follows from the bound (6.36) with $T = 0$ after the transformation (6.11). Bounds (6.6) and (6.7) follow from the bounds (6.59) and (6.64) in the reversed order after the transformation (6.11). The proof of Theorem 6.1 is completed after obtaining the asymptotic representation for $\lambda(b)$ and $c(b)$ for large b.

We fix $p \in (0,1)$, $T > 0$, and $a \in (0, \frac{p}{1+p})$. For sufficiently large $b \geq b_{T,a}$, we consider (λ, c) in the rectangle $[0,2] \times [0, c_0 b^{-(1-a)}]$ for which both Lemmas 6.4 and 6.6 can be applied.

By Lemma 6.4, we have for $t \in [ap\log b, T + ap\log b]$,

$$\Psi_b(t - p\log b) = \Theta_h(t) + \gamma_h(t) + \mathcal{O}(F(\lambda,b) b^{2ap} e^{\frac{t}{p}}), \tag{6.70}$$

where

$$F(\lambda,b) := \begin{cases} (|\lambda| b^{-2p} + b^{-4p})^{2p+1}, & p \in (0, \frac{1}{2}), \\ (|\lambda| b^{-2p} + b^{-4p(1-a)})^2, & p \in [\frac{1}{2}, 1), \end{cases}$$

and the asymptotic expansion can be differentiated in t. Using (6.40), we can write (6.70) as

$$\begin{aligned}\Psi_b(t - p\log b) =& \Theta_h(t) + \Sigma(t)\int_{-\infty}^t f_b(t')\Theta_h'(t')\Theta(t')dt' \\ &- \Theta_h'(t)\int_{-\infty}^t f_b(t')\Sigma(t')\Theta_h(t')dt' + \mathcal{O}(F(\lambda,b) b^{2ap} e^{\frac{t}{p}}).\end{aligned}$$

Evaluating these expressions at $t = T + ap\log b$ and using the asymptotic relations (6.17) and (6.34) for $\Theta_h(t)$ and $\Sigma(t)$ as $t \to +\infty$, we obtain

$$\begin{aligned}\Psi_b(T-(1-a)p\log b) =& \alpha_p^{-1/p} e^{-\frac{T}{p}} b^{-a}[1+\mathcal{O}(b^{-2ap})] \\ &- \frac{1}{2}p^2\alpha_p^{1/p}e^{\frac{T}{p}}b^a[1+\mathcal{O}(b^{-2ap})]\int_{-\infty}^{T+ap\log b} f_b(t)\Theta_h'(t)\Theta_h(t)dt \\ &+ \frac{1}{p}\alpha_p^{-1/p}e^{-\frac{T}{p}}b^{-a}[1+\mathcal{O}(b^{-2ap})]\int_{-\infty}^{T+ap\log b} f_b(t)\Sigma(t)\Theta_h(t)dt \\ &+ \mathcal{O}(F(\lambda,b)b^{2ap+a}e^{\frac{T}{p}}),\end{aligned}$$

where $f_b(t) = -\lambda b^{-2p}e^{2t} + b^{-4p}e^{4t}$. Since

$$\left|\int_{-\infty}^{T+ap\log b} f_b(t)\Sigma(t)\Theta_h(t)dt\right| \leq C_{T,a}b^{-2p(1-a)}$$

is obtained in the proof of Lemma 6.2, we finally obtain the asymptotic formula:

$$\begin{aligned}\Psi_b(T-(1-a)p\log b) &= \alpha_p^{-1/p}e^{-\frac{T}{p}}b^{-a}[1+\mathcal{O}(b^{-2ap}, b^{-2p(1-a)})] \\ -\frac{1}{2}p^2\alpha_p^{1/p}e^{\frac{T}{p}}b^a&\left[\int_{-\infty}^{T+ap\log b} f_b(t)\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2ap})] + \mathcal{O}(F(\lambda,b)b^{2ap})\right].\end{aligned} \tag{6.71}$$

By Lemma 6.6, we have for $t \in (-(1-a)p\log b, 0]$,

$$\Psi_c(t) = c\Upsilon_h(t) + \mathcal{O}(|c|^{2p+1}b^{2p(1-a)}e^{-\frac{t}{p}}), \tag{6.72}$$

where Υ_h is given by (6.28) and the asymptotic expansion can be differentiated in t. Since the expansion (6.72) is used for $t \to -\infty$, we can use either (6.26) or (6.27) for asymptotic expansions of the Tricomi function $\mathfrak{U}(e^{2t};\alpha,\beta)$ in (6.28), where $\alpha = \frac{p+1}{2p} - \frac{\lambda}{4} > 0$ due to (6.23) and $\beta = 1 + \frac{1}{p}$. If $p \neq \frac{1}{n}$ with $n \in \mathbb{N}$, then the asymptotic formula for the solution Ψ_c evaluated at $t = T - (1-a)p\log b$ is obtained with the help of (6.26), (6.28), and (6.72) in the form:

$$\begin{aligned}\Psi_c(T-(1-a)p\log b) =& ce^{\frac{T}{p}}\frac{\Gamma\left(-\frac{1}{p}\right)}{\Gamma\left(\frac{p-1}{2p}-\frac{\lambda}{4}\right)}b^{-(1-a)}[1+\mathcal{O}(b^{-2p(1-a)})] \\ &+ ce^{-\frac{T}{p}}\frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma\left(\frac{p+1}{2p}-\frac{\lambda}{4}\right)}b^{1-a}[1+\mathcal{O}(b^{-2p(1-a)}, |c|^{2p}b^{2p(1-a)})].\end{aligned} \tag{6.73}$$

If $p = \frac{1}{n}$ with $n \in \mathbb{N}$, then the asymptotic formula for the solution Ψ_c evaluated at $t = T - (1-a)p\log b$ is obtained with the help of (6.27), (6.28), and (6.72) in the form:

$$\begin{aligned}\Psi_c(T-(1-a)p\log b) =& ce^{nT}\frac{2(-1)^{n+1}}{n!\Gamma\left(\frac{1-n}{2}-\frac{\lambda}{4}\right)}b^{-(1-a)}[(T-(1-a)p\log b)[1+\mathcal{O}(b^{-2p(1-a)})]+\mathcal{O}(1)]\\ &+ce^{-nT}\frac{(n-1)!}{\Gamma(\alpha)}b^{1-a}[1+\mathcal{O}(b^{-2p(1-a)},|c|^{2p}b^{2p(1-a)})],\end{aligned} \tag{6.74}$$

where $\alpha = \frac{1+n}{2} - \frac{\lambda}{4} > 0$.

When we use the connection equation (6.69), it sets up a system of two equations for two unknowns λ and c. These two equations can be obtained by equaling $\Psi_b(t)$ and $\Psi_c(t)$ as well as their first derivatives at the time $t = T - (1-a)p\log b$. Alternatively, since the asymptotic expansions are differentiable in t term by term, we can set up the system by equaling coefficients in front of the exponential functions $e^{\frac{T}{p}}$ and $e^{-\frac{T}{p}}$. Equaling the coefficients for the $e^{-\frac{T}{p}}$ terms in (6.71) with either (6.73) or (6.74) yields the following equation:

$$\alpha_p^{-1/p}b^{-a}[1+\mathcal{O}(b^{-2ap},b^{-2p(1-a)})] = c\frac{\Gamma\left(\frac{1}{p}\right)}{\Gamma(\alpha)}b^{1-a}[1+\mathcal{O}(b^{-2p(1-a)},|c|^{2p}b^{2p(1-a)})]. \tag{6.75}$$

The nonlinear equation (6.75) is defined for $(\lambda, c) \in [0,2] \times [0, c_0 b^{-(1-a)}]$ and the remainder terms are C^1 functions with respect to (λ, c). Since the leading-order part of the nonlinear equation (6.75) is linear in c and suggests the solution $c = \mathcal{O}(b^{-1})$, which clearly exists inside $|c| \leq c_0 b^{-(1-a)}$, we have by an application of the implicit function theorem the existence of a C^1 function $c = c(\lambda, b)$ for $\lambda \in [0,2]$ and sufficiently large $b \geq b_{T,a}$, which is given asymptotically as

$$c(\lambda, b) = \alpha_p^{-1/p}\frac{\Gamma(\alpha)}{\Gamma\left(\frac{1}{p}\right)}b^{-1}[1+\mathcal{O}(b^{-2ap},b^{-2p(1-a)})]. \tag{6.76}$$

Equaling the coefficients for the $e^{\frac{T}{p}}$ terms in (6.71) with either (6.73) or (6.74) and substituting the expression (6.76) for c yields a nonlinear equation for λ, which we can also solve with an application of the implicit function theorem. However, details of computations depend on the value of $p \in (0,1)$ and hence are reported separately for different values of p.

Case $p \in (0, \frac{1}{2})$. If $p \neq \frac{1}{n}$ for every $n \in \mathbb{N}$, we use (6.71) and (6.73) in (6.69), equal the coefficients for the $e^{\frac{T}{p}}$ terms, and substitute the expression (6.76) for $c = c(\lambda, b)$.

This yields the nonlinear equation for λ:

$$\begin{aligned}&\frac{1}{2}p^2\alpha_p^{1/p}b^a[\lambda b^{-2p}\int_{-\infty}^{T+ap\log b}e^{2t}\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2ap})]\\&-b^{-4p}\int_{-\infty}^{T+ap\log b}e^{4t}\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2ap})]+\mathcal{O}((|\lambda|b^{-2p}+b^{-4p})^{2p+1}b^{2ap})]\\&=\alpha_p^{-1/p}\frac{\Gamma\left(\frac{p+1}{2p}-\frac{\lambda}{4}\right)\Gamma\left(-\frac{1}{p}\right)}{\Gamma\left(\frac{p-1}{2p}-\frac{\lambda}{4}\right)\Gamma\left(\frac{1}{p}\right)}b^{-2+a}[1+\mathcal{O}(b^{-2p(1-a)})].\end{aligned}\tag{6.77}$$

If $p\in(0,\frac{1}{2})$, both integrals in the left-hand-side of (6.77) converge due to the asymptotic expansion (6.17) so that they can be expanded as

$$\begin{aligned}\int_{-\infty}^{T+ap\log b}e^{2t}\Theta_h'(t)\Theta_h(t)dt&=-\int_{-\infty}^{+\infty}e^{2t}\Theta_h(t)^2dt+\mathcal{O}(b^{-2a(1-p)}),\\\int_{-\infty}^{T+ap\log b}e^{4t}\Theta_h'(t)\Theta_h(t)dt&=-2\int_{-\infty}^{+\infty}e^{4t}\Theta_h(t)^2dt+\mathcal{O}(b^{-2a(1-2p)}),\end{aligned}$$

which implies that the nonlinear equation (6.77) for λ can be rewritten in the equivalent form:

$$\begin{aligned}&\lambda\int_{-\infty}^{+\infty}e^{2t}\Theta_h(t)^2dt[1+\mathcal{O}(b^{-2ap},b^{-2a(1-p)})]\\&-2b^{-2p}\int_{-\infty}^{+\infty}e^{4t}\Theta_h(t)^2dt[1+\mathcal{O}(b^{-2ap},b^{-2a(1-2p)})]+\mathcal{O}((|\lambda|b^{-2p}+b^{-4p})^{2p+1}b^{2p(1+a)})\\&=-2p^{-2}\alpha_p^{-2/p}\frac{\Gamma\left(\frac{p+1}{2p}-\frac{\lambda}{4}\right)\Gamma\left(-\frac{1}{p}\right)}{\Gamma\left(\frac{p-1}{2p}-\frac{\lambda}{4}\right)\Gamma\left(\frac{1}{p}\right)}b^{-2(1-p)}[1+\mathcal{O}(b^{-2p(1-a)})].\end{aligned}\tag{6.78}$$

If $p\neq\frac{1}{n}$ for every $n\in\mathbb{N}$, then $\frac{p-1}{2p},-\frac{1}{p}\neq -m$ for every $m\in\mathbb{N}$ so that the arguments of the Gamma functions at $\lambda=0$ are away from their pole singularities. Hence, all terms of the nonlinear equation (6.78) are C^1 functions of λ at $\lambda=0$. For $p\in(0,\frac{1}{2})$, $b^{-2(1-p)}\ll b^{-2p}$ for sufficiently large b. Since the leading-order part of the nonlinear equation (6.78) is linear in λ and suggests the solution $\lambda=\mathcal{O}(b^{-2p})$, we have by an application of the implicit function theorem the existence of a C^1 function $\lambda=\lambda(b)$ for sufficiently large b which is given asymptotically by

$$\lambda(b)=2\frac{\int_{-\infty}^{+\infty}e^{4t}\Theta_h(t)^2dt}{\int_{-\infty}^{+\infty}e^{2t}\Theta_h(t)^2dt}b^{-2p}+\mathcal{O}(b^{-2(1-p)},b^{-2p(1+a)},b^{-2(p+a(1-2p))},b^{-2p(4p+1-a)}).\tag{6.79}$$

The integrals in (6.79) can be computed by using the explicit expression for Θ_h given by (6.16) with $t_0=0$. Using the substitution $s=(1+\alpha_pe^{2t})^{-1}$ we express the integrals in terms of the Beta function

$$B(z_1,z_2) := \int_0^1 s^{z_1-1}(1-s)^{z_2-1}ds = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}, \quad z_1,z_2>0$$

and obtain

$$\int_{-\infty}^{+\infty} e^{2t}\Theta_h(t)^2dt = \frac{1}{2\alpha_p^{1+1/p}}\int_0^1 s^{\frac{1}{p}-2}(1-s)^{\frac{1}{p}}ds = \frac{\Gamma\left(\frac{1}{p}-1\right)\Gamma\left(\frac{1}{p}+1\right)}{2\alpha_p^{1+1/p}\Gamma\left(\frac{2}{p}\right)},$$
$$\int_{-\infty}^{+\infty} e^{4t}\Theta_h(t)^2dt = \frac{1}{2\alpha_p^{2+1/p}}\int_0^1 s^{\frac{1}{p}-3}(1-s)^{\frac{1}{p}+1}ds = \frac{\Gamma\left(\frac{1}{p}-2\right)\Gamma\left(\frac{1}{p}+2\right)}{2\alpha_p^{2+1/p}\Gamma\left(\frac{2}{p}\right)}.$$

Substituting these expressions into (6.79) yields the final formula for $p \in (0, \frac{1}{2})$ and $p \neq \frac{1}{n}$ for every $n \in \mathbb{N}$:

$$\lambda(b) = \frac{2(1+p)}{\alpha_p(1-2p)}b^{-2p} + \mathcal{O}(b^{-2(1-p)}, b^{-2p(1+a)}, b^{-2(p+a(1-2p))}, b^{-2p(4p+1-a)}), \tag{6.80}$$

where we have used the property $\Gamma(z+1) = z\Gamma(z)$.

If $p = \frac{1}{n}$ for some $n \in \mathbb{N}\backslash\{1,2\}$, we use (6.71) and (6.74) in (6.69), equal the coefficients for the $e^{\frac{T}{p}}$ terms, and substitute the expression (6.76) for $c = c(\lambda, b)$. This yields the nonlinear equation for λ:

$$\begin{aligned}&\frac{1}{2}p^2\alpha_p^{1/p}b^a[\lambda b^{-2p}\int_{-\infty}^{T+ap\log b} e^{2t}\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2ap})]\\ &- b^{-4p}\int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2ap})] + \mathcal{O}((|\lambda|b^{-2p}+b^{-4p})^{2p+1}b^{2ap})]\\ &= \frac{2(-1)^{n+1}\Gamma\left(\frac{1+n}{2}-\frac{\lambda}{4}\right)}{\alpha_p^n n!(n-1)!\Gamma\left(\frac{1-n}{2}-\frac{\lambda}{4}\right)}b^{-2+a}[(T-(1-a)p\log b)[1+\mathcal{O}(b^{-2p(1-a)})]+\mathcal{O}(1)].\end{aligned} \tag{6.81}$$

After dividing it by b^{a-2p}, this equation can be rewritten in the form (6.78), where the right-hand side has the order of

$$\frac{\log b\, b^{-2(1-p)}}{\Gamma(\frac{1-n}{2}-\frac{\lambda}{4})},$$

which is much smaller than the leading-order term of the order of $\mathcal{O}(b^{-2p})$ for $p \in (0, \frac{1}{2})$. For even n, the final formula (6.79) for $\lambda(b)$ is modified as follows:

$$\lambda(b) = \frac{2(1+p)}{\alpha_p(1-2p)}b^{-2p} + \mathcal{O}(\log b b^{-2(1-p)}, b^{-2p(1+a)}, b^{-2(p+a(1-2p))}, b^{-2p(4p+1-a)}).$$

For odd n, we also have $\Gamma(\frac{1-n}{2}) = \infty$. Since

$$\Gamma(z) = \frac{(-1)^n}{n!(z+n)} + \mathcal{O}(1) \quad \text{as} \;\; z \to -n$$

and $\lambda(b) = \mathcal{O}(b^{-2p})$, we have

$$\frac{1}{\Gamma(\frac{1-n}{2} - \frac{\lambda}{4})} = \mathcal{O}(\lambda) = \mathcal{O}(b^{-2p}),$$

which modifies the final formula (6.79) for $\lambda(b)$ according to

$$\lambda(b) = \frac{2(1+p)}{\alpha_p(1-2p)} b^{-2p} + \mathcal{O}(\log b b^{-2}, b^{-2p(1+a)}, b^{-2(p+a(1-2p))}, b^{-2p(4p+1-a)}).$$

In both formulas for $\lambda = \lambda(b)$, we have $p = \frac{1}{n}$ with either even or odd $n \in \mathbb{N}\backslash\{1,2\}$. In all cases, $\lambda(b) > 0$ for suffiently large values of b.

Case $p \in (\frac{1}{2}, 1)$. Since $p \neq \frac{1}{n}$ for every $n \in \mathbb{N}$ if $p \in (\frac{1}{2}, 1)$, the nonlinear equation (6.77) can be used. However, the integral $\int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h(t)^2 dt$ diverges exponentially in the upper limit since $\Theta_h(t)^2 = \mathcal{O}(e^{-\frac{2t}{p}})$ as $t \to +\infty$. Consequently, there is a positive constant $C_{T,a}$ such that for all $b \geq b_{T,a}$, we have

$$\left| \int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h'(t)\Theta_h(t)dt \right| \leq C_{T,a} b^{2a(2p-1)}.$$

The nonlinear equation (6.77) can be rewritten in the equivalent form:

$$\begin{aligned} &\lambda \int_{-\infty}^{+\infty} e^{2t}\Theta_h(t)^2 dt [1 + \mathcal{O}(b^{-2ap}, b^{-2a(1-p)})] \\ &+ b^{-2p} \int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2ap})] + \mathcal{O}((|\lambda| b^{-2p} + b^{-4p(1-a)})^2 b^{2p(1+a)}) \\ &= -2p^{-2}\alpha_p^{-2/p} \frac{\Gamma\left(\frac{p+1}{2p} - \frac{\lambda}{4}\right)\Gamma\left(-\frac{1}{p}\right)}{\Gamma\left(\frac{p-1}{2p} - \frac{\lambda}{4}\right)\Gamma\left(\frac{1}{p}\right)} b^{-2(1-p)}[1 + \mathcal{O}(b^{-2p(1-a)})], \end{aligned} \tag{6.82}$$

where the second term on the left-hand side is of the order of $b^{-2p+2a(2p-1)} \to 0$ as $b \to \infty$ since $-2p + 2a(2p-1) < 0$ for $a < \frac{p}{2p-1}$, which is satisfied automatically, since $\frac{p}{2p-1} > 1$ for $p \in (\frac{1}{2}, 1)$. Moreover, since $b^{-2p+2a(2p-1)} \ll b^{-2(1-p)}$ for $p \in \left(\frac{1}{2}, 1\right)$ and $a \in (0,1)$, the right-hand side dominates in the nonlinear equation (6.82). Solving the nonlinear equation (6.82) by an application of the implicit function theorem, we have the existence of a C^1 function $\lambda = \lambda(b)$ for sufficiently large b which is given asymptotically by

$$\lambda(b) = -\frac{4\alpha_p^{1-1/p}\Gamma\left(\frac{p+1}{2p}\right)\Gamma\left(-\frac{1}{p}\right)\Gamma\left(\frac{2}{p}\right)}{p^2\Gamma\left(\frac{p-1}{2p}\right)\Gamma\left(\frac{1}{p}\right)\Gamma\left(\frac{1}{p}-1\right)\Gamma\left(\frac{1}{p}+1\right)} b^{-2(1-p)} + \mathcal{O}(b^{-2p+2a(2p-1)}, b^{-2(1-p)-2ap}, b^{-2(1+a)(1-p)}, b^{-2(1-pa)}, b^{-2p(3-5a)}). \quad (6.83)$$

Since $-\frac{1}{p} \in (-2,-1)$ and $\frac{p-1}{2p} \in (-\frac{1}{2}, 0)$ if $p \in (\frac{1}{2}, 1)$, we have $\Gamma\left(-\frac{1}{p}\right) > 0$ and $\Gamma\left(\frac{p-1}{2p}\right) < 0$. Hence, $\lambda(b) > 0$ for sufficiently large values of b.

Case $p = \frac{1}{2}$. This case corresponds to $n = 2$ in the nonlinear equation (6.81), which we can rewrite in the equivalent form:

$$\begin{aligned} &\lambda \int_{-\infty}^{+\infty} e^{2t}\Theta_h(t)^2 dt[1 + \mathcal{O}(b^{-a})] \\ &+ b^{-1} \int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h'(t)\Theta_h(t)dt[1 + \mathcal{O}(b^{-a})] + \mathcal{O}((|\lambda| b^{-1} + b^{-2}\log b)^2 b^a) \\ &= \frac{8\Gamma\left(\frac{3}{2} - \frac{\lambda}{4}\right)}{\alpha_p^4 \Gamma\left(-\frac{1}{2} - \frac{\lambda}{4}\right)} b^{-1}[(T - (1-a)p\log b)[1 + \mathcal{O}(b^{-(1-a)})] + \mathcal{O}(1)], \end{aligned} \quad (6.84)$$

where $\alpha_{p=\frac{1}{2}} = \frac{1}{4!}$. The second integral $\int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h'(t)\Theta_h(t)dt$ diverges linearly in the upper limit since $\Theta_h(t)^2 = \mathcal{O}(e^{-4t})$ as $t \to +\infty$. The exact computations with the help of the explicit formula (6.16) yield the following asymptotic expression:

$$\begin{aligned} \int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h'(t)\Theta_h(t)dt &= \frac{1}{2}e^{4t}\Theta_h(t)^2\Big|_{t\to-\infty}^{t=T+ap\log b} - 2\int_{-\infty}^{T+ap\log b} e^{4t}\Theta_h(t)^2 dt \\ &= \frac{1}{2\alpha_p^4} - \frac{1}{\alpha_p^4}\left[2(T + ap\log b) + \log(\alpha_p) - \frac{11}{6} + \mathcal{O}(b^{-2ap})\right] \\ &= -\frac{2}{\alpha_p^4}(T + ap\log b) + \frac{7}{3\alpha_p^4} - \frac{\log(\alpha_p)}{\alpha_p^4} + \mathcal{O}(b^{-2ap}). \end{aligned}$$

On the other hand, we use (6.25) and obtain

$$\frac{8\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)} = -\frac{8}{\pi}\Gamma^2\left(\frac{3}{2}\right) = -2,$$

so that the leading-order terms of the nonlinear equation (6.84) can be collected together as

$$\lambda \int_{-\infty}^{+\infty} e^{2t}\Theta_h(t)^2 dt[1+\mathcal{O}(b^{-a})] + \mathcal{O}(b^{-1}, \log b\, b^{-1-a}) + \mathcal{O}((|\lambda| b^{-1} + b^{-2}\log b)^2 b^a)$$
$$= \frac{1}{\alpha_p^4}\log b\, b^{-1}[1+\mathcal{O}(\lambda)][1+\mathcal{O}(b^{-(1-a)})] + \mathcal{O}(b^{-1}).$$

By using the implicit function theorem, we have the existence of a C^1 function $\lambda = \lambda(b)$ for sufficiently large b which is given asymptotically by

$$\lambda(b) = 144 \log b\, b^{-1} + \mathcal{O}(b^{-1}, \log b\, b^{-1-a}, (\log b)^2\, b^{-2}), \tag{6.85}$$

where we have used $p = \frac{1}{2}$, $\alpha_{p=\frac{1}{2}} = \frac{1}{4!}$, and

$$\int_{-\infty}^{+\infty} e^{2t}\Theta_h(t)^2 dt = \frac{1}{6\alpha_p^3}.$$

Hence, $\lambda(b) > 0$ for sufficiently large values of b.

Theorem 6.1 is proven. For details in Remark 6.2, we give the following computations.

Case $p = 1$. This case corresponds to $n = 1$ in the nonlinear equation (6.81), which we can rewrite in the equivalent form:

$$\lambda \int_{-\infty}^{T+a\log b} e^{2t}\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2a})]$$
$$- b^{-2}\int_{-\infty}^{T+a\log b} e^{4t}\Theta_h'(t)\Theta_h(t)dt[1+\mathcal{O}(b^{-2a})] + \mathcal{O}((|\lambda| b^{-2}\log b + b^{-4(1-a)})^2 b^{2a})$$
$$= \frac{4\Gamma\left(1-\frac{\lambda}{4}\right)}{\alpha_p^2\Gamma\left(-\frac{\lambda}{4}\right)}[(T-(1-a)\log b)[1+\mathcal{O}(b^{-2(1-a)})] + \mathcal{O}(1)],$$

where $\alpha_{p=1} = \frac{1}{8}$. Since

$$\Gamma(z) = \frac{1}{z} + \mathcal{O}(1) \quad \text{as} \quad z \to 0$$

and

$$\int_{-\infty}^{T+a\log b} e^{2t}\Theta_h'(t)\Theta_h(t)dt = -\frac{1}{\alpha_p^2}(T + a\log b - 1) - \frac{1}{2\alpha_p^2}\log(\alpha_p) + \mathcal{O}(b^{-2a}),$$

the leading-order terms contain only $\lambda \log b$, which are not balanced by the terms of the order of $\mathcal{O}(1)$ to get the asymptotic balance $\lambda = \mathcal{O}((\log b)^{-1})$ according to Remark 6.2. This failure of the shooting method is due to only one exponential term that appears in (6.74) for $n = 1$ and $\lambda = 0$. The way to handle the asymptotic balance is to obtain the second exponential terms from the higher-order (nonlinear) terms of the expansion for $\Psi_c(t)$ beyond the leading order. However, this adds complexity to the shooting method beyond the scopes of this work.

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