ANALYSIS OF SECOND-ORDER RECURRENCES USING AUGMENTED PHASE PORTRAITS

Analysis of second-order recurrences using augmented phase portraits

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Abstract

The augmented phase portrait, introduced in [10], is used to analyze second order rational discrete maps of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \text{ for } n \in \mathbb{N}_0 = \{0, 1, 2, \ldots, \}$$

with parameters α, β, γ, A, B, $C \geq 0$, and initial conditions, x_0, $x_{-1} > 0$.

First we study the special case,

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n},$$

with α, γ, $B > 0$ and $A \geq 0$. Applying the change of variables, $y_n = x_{n-1}$, this equation can be rewritten as a planar system. We provide a new proof to show that oscillatory solutions have semicycles of length one, except possibly the first cycle, and that nonoscillatory solutions must converge monotonically to the equilibrium. This was originally done in [3] and [8].

We also show that when the unique positive equilibrium is a saddle point, there exist non-trivial positive solutions that increase and decrease monotonically to the equilibrium, proving Conjecture 5.4.6 from [8]. In particular, Theorem 1.2 from [8] defines the tangent vector to the stable manifold at the equilibrium. We show that specific regions defined by the augmented phase portrait have solutions that increase and decrease monotonically to the equilibrium along the stable manifold. While Conjecture 5.4.6 from [8] was previously proven in [5] and [12], our proof provides a more intuitive and elementary solution.

We then consider the case,

$$x_{n+1} = \frac{\alpha + \beta(x_n + x_{n-1})}{A + B(x_n + x_{n-1})},$$

with $\alpha, \beta, A, B > 0$. Again, using $y_n = x_{n-1}$, this system can be written as a planar system. Thus, applying the augmented phase plane from [10], we prove global asymptotic stability of the positive equilibrium for some cases. In other cases, we show this using other theorems from [8] as was previously done in [1].

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Contents

Chapter 1

Introduction

Over the past twenty years there has been a massive increase in interest in studying rational difference equations. Creating a general framework to understand solutions of difference equations has become a popular topic, with two methods created in [10] and [1]. Rational difference equations have a variety of applications and thus the development of methods to understand their solutions is important. The second order rational difference equation,

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}}, \text{ for } n \in \mathbb{N}_0 \tag{1.1}$$

with parameters α, β, γ, A, $B\,C \geq 0$ and initial conditions $x_0 > 0$, $x_{-1} > 0$, was presented in the monograph by Kulenovic and Ladas [8], where the authors provide a collection of known results and open problems. Various cases of (1.1) have been studied in numerous papers including [1], [3], [5], [6], [7], [8], and [12]. The results from [3] were published prior to [8] and used classical techniques such as linearization, to analyze the case, $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{\gamma+x_n}$, with α, β, $\gamma \geq 0$.

Gibbons, Kulenovic, and Ladas in [3] proved that solutions oscillating about the equilibrium, oscillate with semicycle of length one except for possibly the first semicycle. Additionally, for multiple sub-cases, they proved that if solutions do not oscillate about the equilibrium, then they converge monotonically to the equilibrium. However, they did not prove the existence of non-trivial solutions that converge monotonically to the equilibrium. Kulenovic and Ladas conjectured in their monograph [8] that $x_{n+1} = \frac{1+x_{n-1}}{x_n}$ has solutions which converge monotonically to the equilibrium; this conjecture was later proved in [5] and [12].

The proof from [12] showed the existence of solutions that converge monotonically to the equilibrium by using techniques that were topological in nature. Thus, the proof was creative and complex, using nontraditional methods for difference equations. In contrast, the proof from [5] used more conventional methods to obtain a similar conclusion. They invoked the Stable Manifold Theorem, showing that for some second order difference equations, including $x_{n+1} = \frac{1+x_{n-1}}{x_n}$, there exist solutions that converge monotonically to the equilibrium.

In this thesis, we analyze $x_{n+1} = \frac{\alpha+\beta x_{n-1}}{A+\gamma x_n}$, with α, β, $A \geq 0$ and $\gamma > 0$. We prove the existence of solutions that converge monotonically to the equilibrium, that all oscillatory solutions have semicycle of length one, and that all non-oscillatory solutions converge monotonically to the equilibrium. This generalizes the results from [3], which were also generalized in [8], and provides a new proof for the conjecture in [8] (proven in [5] and [12].) Due to the unconventional use of topological concepts in the proof by [12], our proof is vastly different. On the other hand, like the proof by [5], we relied on the Stable Manifold Theorem to show that such a solution converges to the equilibrium. Our methods to show monotonicity of such solutions, and to show that oscillatory solutions have semicycles of length one, are entirely different due to our use of the augmented phase portrait, which was introduced in [11] and refined in [10].

The goal of Streipert, Wolkowicz, and Bohner in [11] was to derive a discrete difference equation predator-prey model from first principles that satisfies biologically relevant conditions. The model they derived is

$$x_{n+1} = \frac{(1+r)x_n}{1 + \frac{r}{K}x_n + \alpha y_n} \quad \text{and} \quad y_{n+1} = \frac{(1+\gamma x_n)y_n}{1+d},$$

with initial conditions $x_0 \geq 0$ and $y_0 \geq 0$ and parameters r, K, α, $\gamma, d > 0$ and x_n denotes the number of prey and y_n the number of predator ant the nth time step. They defined root-curves associated with the prey-nullcline (x-nullcline) to help understand the dynamics of the solutions and the stability of the three possible equilibrium solutions that arise from this system.

An augmented phase plane approach for discrete planar maps: Introducing next-iterate operators, by Streipert and Wolkowicz [10] was the most influential paper for this thesis. The authors fully developed the augmented phase portrait for planar difference equations to overcome several issues faced when using standard phase portraits to analyze difference equations. Unlike for planar systems of ordinary differential equations (ODEs), orbits of planar difference equations can jump across nullclines and in and out of regions that would be positively invariant if the same standard phase portrait configuration was for a system of planar ODEs. These issues are solved by using the augmented phase portrait.

Standard phase portraits usually include only the nullclines and the direction field. However, for the augmented phase portrait, Streipert and Wolkowicz defined next-iterate operators and root-curves in [10]. By augmenting the standard phase portrait with root-curves that separate regions where the next-iterate operator has a constant sign, it is possible to determine on which side of the nullclines the next iterate lies.

In [1], Atawna, Abu-Saris, Ismail, and Hashim also aimed to develop a framework that is useful for showing global properties of solutions of (1.1). Much like for planar ODEs, the authors determined invariant regions in the x_n, x_{n-1} plane. After showing that eventually all solutions of (1.1) enter the invariant region containing the positive equilibrium, they proved global stability of the positive equilibrium of (1.1) by using previously developed theorems from [8]. However, unlike the invariant regions determined by nullclines for ODEs, the invariant regions used in [1] did not depend on nullclines. While the framework created in [1] is intuitive, augmented phase portraits are more like phase portraits for ODEs and thus more accessible.

In this thesis, we aim to continue showcasing the powerful method of the augmented phase portrait. We focus on the analysis of second order difference equations, contributing to the development of a framework for studying planar difference equations. This may prove to useful in future applications of second-order rational difference equations.

Chapter 2

Essentials of Difference Equations

In this thesis we explain and apply a newly-developed technique in order to analyze the behaviour of solutions to certain classes of second order difference equations. The difference equations that we consider are of the form,

$$x_{n+1} = F(x_n, x_{n-1}). \tag{2.1}$$

Equivalently, we can make the change of variables, $y_n = x_{n-1}$, which gives us that $y_{n+1} = x_n$. As such, we rewrite a second order difference equation (2.1) as a system of planar difference equations,

$$\begin{aligned} x_{n+1} &= F(x_n, y_n), \\ y_{n+1} &= x_n =: G(x_n, y_n). \end{aligned} \tag{2.2}$$

Using this change of variables, we turn the general second-order time-delayed difference equation (2.1) into a planar system of equations. Hence, we can apply the methods developed in [10] to analyze the behaviour of solutions. However, we must first discuss classical background knowledge about difference equations from [8] as well as the definitions from [10] before we can apply the novel methods developed in [10].

2.1 Basic Definitions

In this section we provide some important definitions. Henceforth, we write $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. To begin, we define a solution of a difference equation.

Definition 2.1.1 (Adapted from [8]). *[Solution] Given $(x_0, x_{-1}) \in \mathbb{R} \times \mathbb{R}$, a solution (or orbit) of (2.1) is the sequence $\{x_n\}$ that satisfies $x_{n+1} = F(x_n, x_{n-1})$ for every $n \in \mathbb{N}_0$.*

Since (2.1) and (2.2) are equivalent representations of the same difference equation, we now provide an equivalent definition of a solution in terms of (2.2).

Definition 2.1.2 (Solution in planar variables). *Given $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$, a solution (or orbit) of (2.2) is the sequence $\{(x_n, y_n)\}$ that satisfies $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = G(x_n, y_n)$ for every $n \in \mathbb{N}_0$.*

Definition 2.1.3 (Positively Invariant). *A region $\mathcal{R} \in \mathbb{R} \times \mathbb{R}$ is said to be positively invariant for a discrete system $(x_{n+1}, y_{n+1}) = (F(x_n, y_n), G(x_n, y_n))$, if $(x_N, y_N) \in \mathcal{R}$ implies that $(x_n, y_n) \in \mathcal{R}$ for all $n \in \mathbb{N}$, $n \geq N$.*

We now define a useful property that of solutions.

Definition 2.1.4 (Monotonically increasing/decreasing solution). *A solution of the difference equation $x_{n+1} = F(x_n, x_{n-1})$ is monotonically increasing if*

$$x_n < x_{n+1} \text{ for every } n \in \mathbb{N}_0,$$

and it is monotonically decreasing if

$$x_n > x_{n+1} \text{ for every } n \in \mathbb{N}_0.$$

Next, we define a few of the most important types of solutions.

Definition 2.1.5 (Adapted from [2])**.** *[Equilibrium point] An equilibrium point (fixed point)* x^* *of a difference equation,* $x_{n+1} = F(x_n, x_{n-1})$*, is a point that solves the equation,* $x^* = F(x^*, x^*)$*.*

Definition 2.1.6 (Adapted from [2])**.** *[Eventually fixed] A given point* $(x_0, x_{-1}) \in \mathbb{R} \times \mathbb{R}$*, of* (2.1) *is eventually fixed if it is not itself fixed, but* x_n *for some* $n > 0$ *is fixed.*

We now give an equivalent definition for an equilibrium point using the planar system of difference equations (2.2).

Definition 2.1.7 (Planar equilibrium point)**.** *An equilibrium point (fixed point)* (x^*, y^*) *of* (2.2) *is a solution that solves the equation,* $(x^*, y^*) = (F(x^*, y^*), G(x^*, y^*))$*.*

Remark 2.1.8. *Since* (2.2) *defines* $G(x, y) = x$*, we have that* $y^* = x^*$*. As such, the equilibrium point* (x^*, x^*) *of* (2.2) *is a solution that solves,* $(x^*, x^*) = (F(x^*, x^*), x^*)$*.*

Remark 2.1.9 (Monotonically increasing/decreasing to the equilibrium)**.** *A solution of* (2.2) *such that* $x_n > x^*$ *and* $y_n > x^*$ *for every* $n \in \mathbb{N}_0$ *is said to monotonically decrease towards the equilibrium* (x^*, x^*) *if*

$$x^* \leq x_{n+1} < x_n, \text{ and } x^* \leq y_{n+1} < y_n \text{ for every } n \in \mathbb{N}_0.$$

Note that it is sufficient to show that $x^* \leq x_{n+1} < x_n$ *for every* $n \in \mathbb{N}_0$ *since* $x_{n+1} = y_{n+2}$ *and* $x_n = y_{n+1}$ *and so* $x^* \leq y_{n+2} < y_{n+1}$ *for every* $n \in \mathbb{N}_0$*.*

A solution of (2.2) *such that* $x_n < x^*$ *and* $y_n < x^*$ *for every* $n \in \mathbb{N}_0$ *is said to monotonically increase towards the equilibrium* (x^*, x^*) *if*

$$x^* \geq x_{n+1} > x_n, \text{ and } x^* \geq y_{n+1} > y_n \text{ for every } n \in \mathbb{N}_0.$$

Note that it is sufficient to show that $x^* \geq x_{n+1} > x_n$ *for every* $n \in \mathbb{N}_0$ *since* $x_{n+1} = y_{n+2}$ *and* $x_n = y_{n+1}$ *and so* $x^* \geq y_{n+2} > y_{n+1}$ *for every* $n \in \mathbb{N}_0$*.*

Definition 2.1.10 (Adapted from [8])**.** *[Periodic/prime periodic solution] Given* $(x_0, x_{-1}) \in \mathbb{R} \times \mathbb{R}$*, a solution of* (2.1) *is periodic with period* p *if* $x_{n+p} = x_n$ *for every* $n \geq -1$*.*

If p *is the least positive integer such that the solution is periodic, then the solution is periodic with prime-period* p*.*

Definition 2.1.11 (Adapted from [2])**.** *[Eventually periodic] A given point* $(x_0, x_{-1}) \in \mathbb{R} \times \mathbb{R}$*, of* (2.1) *is eventually periodic if it is not itself periodic, but* $\{x_n\}$ *for some* $n > 0$ *is periodic.*

Definition 2.1.12 (Adapted from [8])**.** *[Oscillating about the equilibrium] A solution* $\{x_n\}$ *is said to oscillate about the equilibrium* x^* *if the sequence* $x_n - x^*$ *oscillates in sign. That is, the sign of* $x_n - x^*$ *is not the same for every* $n \in \mathbb{N}_0$*.*

Definition 2.1.13 (See [8])**.** *[Semicycle] Given a solution* $\{x_n\}$ *of* $x_{n+1} = F(x_n, x_{n-1})$ *that oscillates about the equilibrium, a positive semicycle consists of a "string" of terms,* $\{x_m, x_{m+1}, \ldots, x_k\}$ *such that* $x_j - x^* \geq 0$ *for all* $j \in m, m+1, \ldots, k$ *and*

- *either:* $m = -1$*, or* $m > -1$ *and* $x_{m-1} < 0$*;*
- *and, either:* $k = \infty$*, or* $k < \infty$ *and* $x_{k+1} < 0$*.*

Similarly, a negative semicycle consists of a "string" of terms, $\{x_m, x_{m+1}, \ldots, x_k\}$ *such that* $x_j - x^* < 0$ *for all* $j \in m, m+1, \ldots, k$ *and*

- *either:* $m = -1$*, or* $m > -1$ *and* $x_{m-1} \geq 0$*;*
- *and, either:* $k = \infty$*, or* $k < \infty$ *and* $x_{k+1} \geq 0$*.*

The following definition describes how a solution might oscillate between two regions; depending on the location of the two region relative to the equilibrium, it is sometimes equivalent to Definition 2.1.12.

Definition 2.1.14. *[Oscillating between regions] A sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$ is said to oscillate between two disjoint regions $\mathcal{R}_1, \mathcal{R}_2 \subset \mathbb{R}^2$, if $(x_{2n}, y_{2n}) \in \mathcal{R}_1$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_2$ for all $n \in \mathbb{N}_0$.*

Definition 2.1.15 (Eventually oscillating between regions)**.** *A sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$ is said to eventually oscillate between two disjoint regions $\mathcal{R}_1, \mathcal{R}_2 \subset \mathbb{R}^2$, if there exists a $K \in \mathbb{N}_0$ such that $(x_{2n}, y_{2n}) \in \mathcal{R}_1$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_2$ for all $n > K$.*

Later, we will see that there are cases where Definitions 2.1.12 and 2.1.14 are equivalent; if a solution is oscillating between two regions on opposite sides of an equilibrium, then we also have that the solution is oscillating about the equilibrium. In these cases, planar analysis is powerful in the sense that we can determine the behaviour of solutions in a visual and elementary way.

2.2 Stability

We now define what it means for an equilibrium to be stable, locally asymptotically stable, globally attractive, globally asymptotically stable, unstable, and a source. For the following definitions, we consider (2.1) with $F : I \times I \to I$ where I is any interval in the real numbers, $\mathbb{R}$.

Definition 2.2.1 (Adapted from [8])**.** *[Stable equilibrium] An equilibrium point x^* of (2.1) is called stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0, x_{-1} \in I$ with $|x_0 - x^*| + |x_{-1} - x^*| < \delta$, we have*

$$|x_n - x^*| < \epsilon \text{ for all } n \geq -1.$$

Definition 2.2.2 (Adapted from [8])**.** *[Locally asymptotically stable equilibrium] The equilibrium point x^* of (2.1) is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$ such that for all $x_0, x_{-1} \in I$ with $|x_0 - x^*| + |x_{-1} - x^*| < \gamma$, we have*

$$\lim_{n\to\infty} x_n = x^*.$$

Definition 2.2.3 (Adapted from [8])**.** *[Global attractor equilibrium] The equilibrium point x^* of (2.1) is called a global attractor if for all $x_0, x_{-1} \in I$ we have*

$$\lim_{n\to\infty} x_n = x^*.$$

Definition 2.2.4 (Adapted from [8])**.** *[Globally asymptotically stable] The equilibrium point x^* of (2.1) is called a globally asymptotically stable if it is locally stable and a global attractor.*

Definition 2.2.5 (Adapted from [8])**.** *[Unstable equilibrium] The equilibrium point x^* of (2.1) is called unstable if it is not stable.*

Definition 2.2.6 (Adapted from [8])**.** *[Source/repeller equilibrium] The equilibrium point x^* of (2.1) is called a source, or a repeller, if there exists $r > 0$ such that for all $x_0, x_{-1} \in I$ with $0 < |x_0 - x^*| + |x_{-1} - x^*| < r$, there exists $N \geq 1$ such that*

$$|x_N - x^*| \geq r.$$

A source is an unstable equilibrium.

Definition 2.2.7 (Adapted from [9])**.** *[Saddle point equilibrium] The equilibrium point x^* of (2.1) is called a saddle point if some trajectories are attracted to it and others are repelled by it.*

While these definitions are helpful in understanding the stability of equilibrium points of (2.2), it is best to simplify the problem by linearizing the difference equation about the equilibrium. This can be an extremely effective strategy to classify the equilibrium because there are existing stability theorems in [8] that address linear systems. Thus, we begin by explaining how to linearize the system (2.2) about an equilibrium point.

To linearize about the equilibrium point, would mean to linearly approximate the function, $T(x, y)$, near the equilibrium point where,

$$T(x,y) = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}.$$

As long as $T(x, y)$ is $C^1(\mathbb{R}^2)$, we have the following definition for linearization of a system of planar difference equations at the equilibrium, (x^*, y^*).

Definition 2.2.8 (Adapted from [13])**.** *[Linearization about an equilibrium] Assume that* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *is* $C^1(\mathbb{R}^2)$ *where,*

$$T(x,y) = \begin{pmatrix} F(x,y) \\ G(x,y) \end{pmatrix}.$$

Then the linearization of $T(x, y)$ *around an equilibrium* (x^*, y^*) *is given by*

$$T(x,y) \approx T(x^*,x^*) + J_T(x^*,x^*) \begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix},$$

where $J_T(x^*, y^*)$ *is the Jacobian of* $T(x, y)$ *evaluated at* (x^*, y^*)*. The Jacobian is given by*

$$J_T(x,y) = \begin{pmatrix} \frac{\partial F}{\partial x}(x,y) & \frac{\partial F}{\partial y}(x,y) \\ \frac{\partial G}{\partial x}(x,y) & \frac{\partial G}{\partial y}(x,y) \end{pmatrix}.$$

For (2.2), $G(x, y) = x$ and so $\frac{\partial G}{\partial x} = 1$ and $\frac{\partial G}{\partial y} = 0$. Thus, the linearization of (2.2) around the equilibrium (x^*, x^*) is given by

$$T(x,y) = \begin{pmatrix} F(x,y) \\ x \end{pmatrix} \approx \begin{pmatrix} x^* \\ x^* \end{pmatrix} + \begin{pmatrix} \frac{\partial F}{\partial x}(x^*,x^*) & \frac{\partial F}{\partial y}(x^*,x^*) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x - x^* \\ y - x^* \end{pmatrix}.$$

Recalling that $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = G(x_n, y_n) = x_n$, we make a change of variables,

$$z_n = x_n - x^*.$$

Thus we obtain the linearized equation for (2.2) as,

$$\begin{pmatrix} z_{n+1} \\ z_n \end{pmatrix} \approx \begin{pmatrix} \frac{\partial F}{\partial x}(x^*,x^*) & \frac{\partial F}{\partial y}(x^*,x^*) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_n \\ z_{n-1} \end{pmatrix}. \tag{2.3}$$

Letting $p := \frac{\partial F}{\partial x}(x^*, x^*)$ and $q := \frac{\partial F}{\partial y}(x^*, x^*)$, the associated characteristic equation of the linearized equation (2.3) is $0 = \det(\lambda I - J(x^*, x^*))$, or equivalently,

$$0 = \lambda^2 - p\lambda - q. \tag{2.4}$$

The roots of (2.4) are used to determine the local stability of the equilibrium in the following theorems. Note that the first theorem defines the criteria for the stability of equilibrium points of (2.2) and the subsequent theorem determines the necessary and sufficient conditions for classifying the stability of equilibrium points of (2.2).

Theorem 2.2.9. *[Adapted from [8, Theorem 1.1.1]] Let (x^*, x^*) be an equilibrium of (2.2).*
(a) If both roots of (2.4) lie in the open disk $|\lambda| < 1$, then (x^*, x^*) is locally asymptotically stable.

(b) If at least one of the roots of (2.4) lies outside the unit-circle, then (x^*, x^*) is unstable.

(c) If one of the roots of (2.4) lies outside the unit-circle and the other inside the unit circle, then (x^*, x^*) is a saddle point.

(d) If both roots of (2.4) lie outside of the unit circle, then (x^*, x^*) is a repeller.

Theorem 2.2.10. *[Adapted from [8, Theorem 1.1.1]] Let (x^*, x^*) be an equilibrium of (2.2).*
(a) A necessary and sufficient condition for both roots of (2.4) to lie in the open unit disk $|\lambda| < 1$ is

$$|p| < 1 - q < 2.$$

In this case, the locally asymptotically stable equilibrium, (x^*, x^*), is also called a sink.
(b) A necessary and sufficient condition for the roots of (2.4) to have modulus greater than one is

$$|q| > 1 \quad \text{and} \quad |p| < |1 - q|.$$

In this case, (x^*, x^*) is a repeller.
(c) A necessary and sufficient condition for one root of (2.4) to have modulus greater than one and the other to have absolute value less than one is

$$p^2 + 4q > 0 \quad \text{and} \quad |p| > |1 - q|.$$

In this case, (x^*, x^*) is a saddle point.
(d) A necessary and sufficient condition for a root of (2.4) to have modulus equal to one is

$$|p| = |1 - q| \qquad \text{or} \qquad q = -1 \text{ and } |p| \leq 2$$

In this case, (x^*, x^*) is called a nonhyperbolic point.

We now discuss and define the stable and unstable manifolds of a saddle point equilibrium.

Definition 2.2.11 (See [8])**.** *[Stable/Unstable Manifold] Let (x^*, x^*) be a saddle point equilibrium of (2.2). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by,*

$$T(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix}.$$

Then the stable manifold of (x^*, x^*) is the set of initial points whose forward orbit under the iteration of T,

$$(x, y),\, T(x, y),\, T^2(x, y), \ldots$$

converges to (x^*, x^*). The unstable manifold of (x^*, x^*) is the set of initial points whose backward orbit under the iteration of T,

$$(x, y),\, T^{-1}(x, y),\, T^{-2}(x, y), \ldots$$

converges to (x^*, x^*).

While local stability allows us to determine the behaviour of the solutions near the equilibrium, if we want to determine the global stability of an equilibrium, we require the use of other theorems. The following theorems are useful in proving global stability for a unique, positive equilibrium of difference equations.

Theorem 2.2.12. *[See [8, Theorem 1.4.8]] Let $I = [a, b]$ be an interval of real numbers and assume that*

$$f : I \times I \to I$$

is a continuous function satisfying the following properties:

a) $f(x, y)$ is nondecreasing in each of its arguments;
b) the equation, $f(x, x) = x$, has a unique positive solution.

Then, $x_{n+1} = f(x_n, x_{n-1})$ has a unique equilibrium $x^* \in I$ and every solution converges to x^*.

Theorem 2.2.13. *[See [8, Theorem 1.4.7]] Let $I = [a, b]$ be an interval of real numbers and assume that*

$$f : I \times I \to I$$

is a continuous function satisfying the following properties:

a) $f(x, y)$ is nonincreasing in each of its arguments;
b) if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system $f(m, m) = M$ and $f(M, M) = m$, then $m = M$.

Then, $x_{n+1} = f(x_n, x_{n-1})$ has a unique equilibrium $x^* \in I$ and every solution converges to x^*.

There are other theorems that can help show the global stability of an equilibrium in the textbook [8]. We chose to highlight these two because we will use them in later chapters.

While these classical theorems can be used to show global stability of an equilibrium point, the goal of this thesis is to showcase an alternative method for establishing properties of difference equations like the global stability of equilibrium points. Specifically, we will use nullclines and their associated root-curves introduced in [10] and we will identify cases where a classical theorem from [8] can also be used to show global stability.

2.3 Augmented Phase Portraits

We define nullclines and their associated root-curves for a system of planar difference equations since we will be using them to analyze special cases of planar systems of the form (2.2). For systems of planar differential equations, nullclines define where the rate of change with respect to one variable is zero. In discrete systems, nullclines define where the difference between two iterates of the same variable is zero.

For differential equations, if a solution exists on both sides of a nullcline, then by continuity, the solution intersects that nullcline. In contrast, the solution of a difference equation can have the next-iterate jump over nullclines. This difference has made standard phase portraits a less effective method for analyzing systems of difference equations; however, if we additionally use the pre-image of the nullclines, termed the root-curves associated with the nullclines in [10], then we can obtain a lot of information about the behaviour of solutions. The augmented phase portrait consists of the nullclines, their associated root-curves, the direction field, and the sign of the next-iterate operators in the regions bounded by the root-curves.

The nullclines together with an indication of the direction field in the regions bounded by the nullclines as well as the associated root-curves and the sign of the next-iterate operators in the regions bounded by the root-curves constitutes the augmented phase portrait.

Before defining the nullclines of difference equations, we first define the difference operator.

Definition 2.3.1 (Adapted from [7, Definition 2.1]). *[Difference Operator] The difference operator* Δ *is defined as* $\Delta x_n := x_{n+1} - x_n$.

Using this definition, we define nullclines for a planar system of difference equations (2.2).

Definition 2.3.2 (Adapted from [10]). *[Nullcline] An* x*-nullcline of* (2.2) *is a function* $y = h(x)$ *or* $x = \tilde{h}(y)$ *that satisfies* $F(x, h(x)) = x$ *or* $F(\tilde{h}(y), x) = x$.

A y*-nullcline of* (2.2) *is a function* $y = k(x)$ *or* $x = \tilde{k}(y)$ *that satisfies* $G(x, k(x)) = y$ *or* $G(\tilde{k}(y), x) = y$.

Remark 2.3.3. *For* (2.2)*, the* y*-nullcline can be defined explicitly as* $y = k(x) = x$*. However, sometimes it is only possible to define a nullcline implicitly.*

Remark 2.3.4. *Since* x *and* y*-nullclines satisfy* $x = F(x, y)$ *and* $y = G(x, y)$*, respectively, by Definition 2.1.7, any intersection of an* x*-nullcline and a* y*-nullcline occurs at an equilibrium solution.*

Definition 2.3.5 (Direction field). *The direction field is a grid of line segments indicating the slope and direction of the orbit to the next iterate.*

For the set $\{(x_n, y_n);\ \Delta x_n > 0,\ \Delta y_n > 0\}$*, the next-iterate will be up and to the right.*

For the set $\{(x_n, y_n);\ \Delta x_n > 0,\ \Delta y_n < 0\}$*, the next-iterate will be down and to the right.*

For the set $\{(x_n, y_n);\ \Delta x_n < 0,\ \Delta y_n > 0\}$*, the next-iterate will be up and to the left.*

For the set $\{(x_n, y_n);\ \Delta x_n < 0,\ \Delta y_n < 0\}$*, the next-iterate will be down and to the left.*

The nullclines separate the phase plane into regions where the signs of Δx_n at all points in the region are the same and the signs of Δy_n at all points in the region are the same. However, unlike for systems of ordinary differential equations, solutions of difference equations can jump over nullclines. Knowing where the next-iterate will be relative to the nullclines motivated the introduction of the next-iterate operators and the next-iterate root-curves associated with the nullclines in [10].

We first define the next-iterate operator associated with an x-nullcline, $y = h(x)$ and a y-nullcline, $y = k(x)$:

$$\mathcal{L}_h(x, y) := G(x, y) - h(F(x, y)) \quad \text{and} \quad \mathcal{L}_k(x, y) := G(x, y) - k(F(x, y)). \tag{2.5}$$

Evaluated at a point (x_n, y_n) in the phase plane, (2.5) is

$$\mathcal{L}_h(x_n, y_n) := y_{n+1} - h(x_{n+1}) \quad \text{and} \quad \mathcal{L}_k(x_n, y_n) := y_{n+1} - k(x_{n+1}). \tag{2.6}$$

By the definition of $\mathcal{L}_h(x, y)$ and $\mathcal{L}_k(x, y)$, it is evident that the sign of the function determines the placement of y_{n+1} relative to the associated nullcline. For example, if $\mathcal{L}_h(x_n, y_n) > 0$, then we know that $y_{n+1} > h(x_{n+1})$ and if $\mathcal{L}_k(x_n, y_n) > 0$, then we know that $y_{n+1} > k(x_{n+1})$. With this in mind the next-iterate root set (or root-set) and the next-iterate root-curve (or root-curve) associated with a nullcline is defined in [10] as follows.

Definition 2.3.6 (Adapted from [10]). *[Root-set] The root-set associated with the* $h(x)$ *nullcline is the set of points* $\{(x, y) \in \mathbb{R}^2\}$ *that satisfy*

$$G(x, y) = h(F(x, y)).$$

The root-set associated with the $k(x)$ *nullcline is the set of points* $\{(x, y) \in \mathbb{R}^2\}$ *that satisfy*

$$G(x, y) = k(F(x, y)).$$

Definition 2.3.7 (Adapted from [10])**.** *[Root-curves] A root-curve associated with the $h(x)$ nullcline is a function $y = r_h(x)$ that satisfies*

$$G(x, r_h(x)) = h(F(x, r_h(x))).$$

A root-curve associated with the $k(x)$ nullcline is a function $y = r_k(x)$ that satisfies

$$G(x, r_k(x)) = k(F(x, r_k(x))).$$

Remark 2.3.8. *Sometimes it is more convenient to express a root-curve associated with a nullcline a a function of x instead of a function f y and sometimes it is only possible to express a root-curve implicitly.*

Lemma 2.3.9 (Adapted from [10], Lemma 2.5)**.** *Assume that the root-curves associated with the nullclines of (2.2) can be defined explicitly as $y = r_h(x)$ and $y = r_k(x)$. If for some $\bar{x}$ $r_k(\bar{x}) = k(\bar{x})$, then $(\bar{x}, \bar{x})$ must be an equilibrium. Also, if $r_h(\bar{x}) = h(\bar{x})$, then $(\bar{x}, \bar{x})$ must be an equilibrium.*

We do not include the proof because it was proved in [10].

Lemma 2.3.10. *Assume that the x-nullcline can be defined explicitly as $y = h(x)$ and that $y = h(x)$ is one-to-one. Then, $r_k(x) = h(x)$.*

Proof. For (2.2), from Definition 2.3.2, $y = h(x)$ satisfies

$$x = F(x, h(x)).$$

From Remark 2.3.3, $k(x) = x$ and so $x = k(F(x, h(x)))$. Furthermore, $G(x, y) = x$, and so,

$$G(x, h(x)) = k(F(x, h(x))).$$

Therefore, $r_k(x) = h(x)$. □

Lemma 2.3.11. *Assume that the root-curves associated with the nullclines of (2.2) can be defined explicitly as $y = r_h(x)$ and $y = r_k(x)$. If for some $\bar{x}$, $r_h(\bar{x}) = r_k(\bar{x})$, then $(\bar{x}, \bar{x})$ must be an equilibrium.*

Proof. Let $r_h(\bar{x}) = r_k(\bar{x})$. By Lemma 2.3.10, $r_k(x) = h(x)$ and so

$$r_h(\bar{x}) = h(\bar{x}).$$

By Lemma 2.3.9, $r_h(x)$ and $h(x)$ can only intersect at equilibria and so $(\bar{x}, \bar{x})$ must be an equilibrium point. □

We define two additional functions that will provide more information about the behaviour of solutions to (2.2). We first define the operator,

$$\mathcal{J}(x_n, y_n) := F(x_n, y_n) - x^*,$$

or equivalently,

$$\mathcal{J}(x_n, y_n) := x_{n+1} - x^*.$$

The sign of $\mathcal{J}(x_n, y_n)$ will determine what side of the line $x = x^*$ the next-iterate is found on. Thus, the pre-image of the line $x = x^*$ is equivalent to where $\mathcal{J}(x_n, y_n) = 0$ and is defined as follows.

Definition 2.3.12. *The pre-image of the line $x = x^*$, is the set of points $\{(x, y) \in \mathbb{R}^2\}$ that satisfy*

$$F(x, y) = x^*.$$

When we can write the set of points as a function $y = S(x)$, the function satisfies $F(x, S(x)) = x^*$.

Now we define the pre-image of the root-curve associated with the $h(x)$ nullcline. Note that we could define a pre-image for each root-curve, however we only require the pre-image of $r_h(x)$ in this thesis. Thus, we define the operator,

$$\mathcal{L}_{r_h}(x, y) := G(x, y) - r_h(F(x, y)).$$

Evaluating the pre-image at (x_n, y_n) we obtain

$$\mathcal{L}_{r_h}(x_n, y_n) := y_{n+1} - r_h(x_{n+1}).$$

Note that we are interested in where $\mathcal{L}_{r_h}(x, y) = 0$.

Definition 2.3.13. *The pre-image of a root-curve $y = r_h(x)$ associated with a $y = h(x)$ nullcline is the set of points $\{(x, y) \in \mathbb{R}^2\}$ that satisfy*

$$G(x, y) = r_h(F(x, y)).$$

When we can write the set of points as a function $y = Q(x)$, this function satisfies $G(x, Q(x)) = r_h(F(x, Q(x)))$.

Now that we have defined the nullclines, the root-curves associated with the nullclines, the pre-image of the root-curves and the pre-image of the line $x = x^*$, we can construct the augmented phase portrait from [10]. Thus, we can obtain a good understanding of how solutions of (2.2) behave.

Chapter 3

Analysis of $x_{n+1} = \frac{\alpha+\gamma x_{n-1}}{A+Bx_n}$

We analyze,

$$x_{n+1} = \frac{\alpha + \gamma x_{n-1}}{A + Bx_n}, \qquad n \in \mathbb{N}_0 \tag{3.1}$$

with initial conditions $x_0 > 0$, $x_{-1} > 0$ and parameters $\alpha, \gamma, B > 0$, and $A \geq 0$. To simplify this equation, without loss of generality, we make the following change of variables. First, we divide both the numerator and denominator by γ. Then, we let $\tilde{x}_n = \frac{B}{\gamma} x_n$, and hence $\tilde{x}_{n+1} = \frac{B}{\gamma} x_{n+1}$. As such,

$$\tilde{x}_{n+1} = \frac{\tilde{\alpha} + \tilde{x}_{n-1}}{\tilde{A} + \tilde{x}_n},$$

where $\tilde{\alpha} = \frac{B\alpha}{\gamma^2}$ and $\tilde{A} = \frac{A}{\gamma}$. To simplify notation, henceforth we "drop the tilde" and refer to $\tilde{x}_n$ as x_n, $\tilde{A}$ as A, and $\tilde{\alpha}$ as α. We note that the change of variables we used is different from what was done in [8] so that we can include $A = 0$ in our analysis.

Now, letting $y_n = x_{n-1}$, we write (3.1) as the planar system,

$$x_{n+1} = \frac{\alpha + y_n}{A + x_n} =: F(x_n, y_n), \quad y_{n+1} = x_n =: G(x_n, y_n) \tag{3.2}$$

with parameters $A \geq 0$, $\alpha > 0$, and initial conditions $x_0 > 0$ and $y_0 > 0$. Since $x_{n+1} > 0$, the region $(0, \infty) \times (0, \infty)$ is positively invariant.

From Definition 2.1.5, the equilibria of (3.2) must satisfy $x^* = F(x^*, x^*)$. Solving this equation for x^* yields the only positive solution,

$$x^* = \frac{-(A-1) + \sqrt{(A-1)^2 + 4\alpha}}{2}. \tag{3.3}$$

Thus (3.2) has a unique positive equilibrium given by $E^* = (x^*, x^*)$, provided $A \neq 1$ and $\alpha \neq 0$.

The following theorems from [8] will be used in future sections and are adapted to suit the variables of (3.2).

Theorem 3.0.1. *[Adapted from [8, Theorem 6.5.1]] The equilibrium of (3.2) is locally asymptotically stable when $A > 1$ and is an unstable saddle point when $A < 1$.*

The proof is in Chapter 5, Section 5.1.

Theorem 3.0.2. *[Adapted from [8, Section 1.2]](Stable Manifold Theorem) Let T be a one-to-one, smooth mapping with a smooth inverse in $I \times I$. Assume that $(x^*, y^*) \in I \times I$ is a saddle point of T and that the Jacobian of T evaluated at (x^*, y^*) has eigenvalues s and u with $|s| < 1$ and $|u| > 1$. Let v_s and v_u be the eigenvectors corresponding to s and u, respectively. Let S be the stable manifold of (x^*, y^*) and U be the unstable manifold of (x^*, y^*) as defined in Definition 2.2.11.*

Then both S and U are one dimensional manifolds, or curves, that contain (x^*, y^*). Furthermore, the eigenvectors v_s and v_u are tangent to S and U, respectively at the point (x^*, y^*).

This theorem will allow us to find the tangent vector to the stable manifold at the equilibrium point $E^* = (x^*, x^*)$, whenever (x^*, x^*) is a saddle point. The application of this theorem will help us to locate the stable manifold relative to the root-curves and their associated nullclines.

In [4], $x_{n+1} = \frac{\alpha+\beta x_n+\gamma x_{n-1}}{A+x_n}$ is studied with A, α, β, and γ, as non-negative parameters and $n \in \mathbb{N}_0$. Setting $\beta = 0$ and $\gamma = 1$, we obtain (3.2). Thus, we present an adapted version of the main theorem from [4] for this special case.

Theorem 3.0.3. *[Adapted from [4, Theorem 1]] Consider* (3.2)*.*

> ***a)*** *Assume that $A = 1$. Then every solution converges to a period two solution.*
>
> ***b)*** *Assume that $A < 1$. Then there exist unbounded solutions.*
>
> ***c)*** *Assume that $A > 1$. Then every solution has a finite limit.*

From this theorem, we realize that it is important to consider $A < 1$, $A = 1$, and $A > 1$ separately. We first look at the case where $A \in [0, 1)$.

3.1 Case 1: $A \in [0, 1)$.

Our goal in this section is to prove the following theorem.

Theorem 3.1.1. *If $x_{n+1} = \frac{\alpha+x_{n-1}}{A+x_n}$ with $\alpha \geq 0$, $A \in [0, 1)$ and $n \in \mathbb{N}_0$, then*

a) there exists a nontrivial positive solution that decreases monotonically to (x^*, x^*),

b) there exists a nontrivial positive solution that increases monotonically to (x^*, x^*).

The motivation for this problem comes from Conjecture 5.4.6 in [8].

Conjecture 1. *[See [8, Conjecture 5.4.6]] Show that $x_{n+1} = \frac{1+x_{n-1}}{x_n}$ for $n \in \mathbb{N}_0$ has a nontrivial positive solution that decreases monotonically to the equilibrium.*

After the publication of Conjecture 1 in [8], it was proved by Hoag [5] and Sun and Xi [12]. However, since we prove this conjecture by using the augmented phase portrait, our proof is very different from that of Sun and Xi [12]. The proof done by Hoag [5] shares some similarities. We will highlight the similarities and differences after we provide our proof.

Despite the fact that Conjecture 1 has been previously proven, we find that the value in proving it using the augmented phase portrait is that our methods are intuitive compared to those in [5] and [12]. This proof shows the power of this elementary and accessible approach.

3.1.1 Analysis

By Definition 2.3.2, the x-nullcline of (3.2) is the function $y = h(x)$ that satisfies $F(x, y) = x$, and so

$$y = h(x) = x^2 + Ax - \alpha.$$

The y-nullcline is the function $y = k(x)$ that satisfies $G(x, y) = y$, and so

$$y = k(x) = x.$$

The calculations can be found in Chapter 5, Section 5.2.

We now observe how the nullclines divide the plane into component-wise monotone regions.

$$x_{n+1} - x_n \begin{cases} > 0, & y_n > h(x_n), \\ = 0, & y_n = h(x_n), \\ < 0, & y_n < h(x_n), \end{cases} \quad \text{and} \quad y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases} \tag{3.4}$$

The calculations justifying (3.4) are given in Chapter 5, Section 5.3. Since both nullclines can be expressed explicitly, we find the root-curves associated with the nullclines using Definition 2.3.7. The root-curve associated with the $h(x)$ nullcline is a function $y = r_h(x)$ that satisfies $G(x, y) = h(F(x, y))$. Solving, we obtain that the only positive function is,

$$y = r_h(x) = \frac{1}{2}\left(-2\alpha - A^2 - Ax + (A + x)\sqrt{4\alpha + A^2 + 4x}\right).$$

The details of this calculation are in Chapter 5, Section 5.4.

Since $h'(x) = 2x + A > 0$ for all $x > 0$, we have that it is strictly increasing for $x > 0$ and thus one-to-one for $x > 0$. Thus, from Lemma 2.3.10, the root-curve associated with the $k(x)$ nullcline is given by

$$y = r_k(x) = h(x) = x^2 + Ax - \alpha.$$

Lemma 3.1.2. *Consider* (3.2) *and assume that* $A \in [0, 1)$*.*

- ***a)*** *If* $y_n < r_h(x_n)$*, then* $y_{n+1} > h(x_{n+1})$ *and equivalently,* $\mathcal{L}_h(x_n, y_n) > 0$.
- ***b)*** *If* $y_n > r_h(x_n)$*, then* $y_{n+1} < h(x_{n+1})$ *and equivalently,* $\mathcal{L}_h(x_n, y_n) < 0$.
- ***c)*** *If* $y_n < r_k(x_n)$*, then* $y_{n+1} > k(x_{n+1})$ *and equivalently,* $\mathcal{L}_k(x_n, y_n) > 0$.
- ***d)*** *If* $y_n > r_k(x_n)$*, then* $y_{n+1} < k(x_{n+1})$ *and equivalently,* $\mathcal{L}_k(x_n, y_n) < 0$.

The proof is in Chapter 5, Section 5.5.

Property 3.1.3. *For* (3.2) *with* $A \in [0, 1)$*, any two of* $y = r_k(x)$*,* $y = r_h(x)$ *and* $y = k(x)$ *intersect only at the unique positive equilibrium,* $E^* = (x^*, x^*)$*, for* $x^* > 0$ *given in* (3.3)*.*

The proof of Property 3.1.3 can be found in Chapter 5, Section 5.6.

Lemma 3.1.4. *Consider* (3.2) *and assume* $A \in [0, 1)$*. Then,*

- ***a)*** $r_k(x) < r_h(x) < k(x)$ *for* $x \in (0, x^*)$*;*
- ***b)*** $k(x) < r_h(x) < r_k(x)$ *for* $x > x^*$*.*

Proof. By definition, $r_k(x^*) = r_h(x^*) = h(x^*) = x^*$ and by Property 3.1.3, $r_k(x)$, $r_h(x)$ and $k(x)$ can only intersect at $x = x^*$. Thus, we can show that $r_k(x) < r_h(x) < k(x)$ for $x \in (0, x^*)$ if for a particular $x \in [0, x^*)$, $r_k(x) < r_h(x) < k(x)$. At $x = 0$,

$$\begin{aligned} k(0) &= 0, \\ r_h(0) &= \frac{1}{2}\left(-2\alpha - A^2 + A\sqrt{4\alpha + A^2}\right), \text{ and} \\ r_k(0) &= -\alpha. \end{aligned}$$

Since $-A^2 + \sqrt{A^4} = 0$, we have that $\frac{1}{2}\left(-A^2 + \sqrt{4A^2\alpha + A^4}\right) > 0$, and so,

$$r_k(0) = -\alpha < -\alpha + \frac{1}{2}\left(-A^2 + \sqrt{4A^2\alpha + A^4}\right) = r_h(0).$$

Now, suppose for the sake of a contradiction that $r_h(0) > 0$. Then,

$$\sqrt{4A^2\alpha + A^4} > 2\alpha + A^2.$$

Since both sides of the inequality are positive, we can square both sides and simplify to obtain $4A^2\alpha + A^4 > 4\alpha^2 + 4\alpha A^2 + A^4$. Hence, this gives us the contradiction, $4\alpha^2 < 0$. Furthermore, we observe that $r_h(0) = 0$ if and only if $\alpha = 0$. As such, $r_h(0) < 0$ and so $r_k(0) < r_h(0) < k(0)$.

We now show $k'(x^*) < r'_h(x^*) < r'_k(x^*)$ to show that $k(x) < r_h(x) < r_k(x)$ for $x \in (x^*, \infty)$. First we notice that from (3.3),

$$2x^* + A = 1 + \sqrt{(A-1)^2 + 4\alpha} > 1. \tag{3.5}$$

We evaluate $k'(x)$, $r'_h(x)$, and $r'_k(x)$ at $x = x^*$ as,

$$\begin{aligned} k'(x^*) &= 1, \\ r'_h(x^*) &= \frac{1}{2}\left(-A + \sqrt{4\alpha + A^2 + 4x^*} + \frac{2(A+x^*)}{\sqrt{4\alpha + A^2 + 4x^*}} \right), \\ r'_k(x^*) &= 2x^* + A. \end{aligned} \tag{3.6}$$

Using $x^* = \frac{\alpha + x^*}{A + x^*} > 0$ or equivalently, $\alpha + x^* = x^*(A + x^*)$, we simplify $\sqrt{4\alpha + A^2 + 4x^*}$ so that,

$$\begin{aligned} \sqrt{A^2 + 4(x^* + \alpha)} &= \sqrt{A^2 + 4x^*(A + x^*)}, \\ &= \sqrt{A^2 + 4x^*A + (2x^*)^2}, \\ &= \sqrt{(A + 2x^*)^2}, \\ &= A + 2x^* > 0. \end{aligned}$$

Thus, $r'_h(x^*)$ simplifies to

$$r'_h(x^*) = x^* + \frac{A + x^*}{A + 2x^*}, \tag{3.7}$$

and so $r'_h(x^*) > 1 = k'(x^*)$.

Finally, to show that $r'_h(x^*) < r'_k(x^*)$, it suffices to show that $r'_h(x^*) = x^* + \frac{A+x^*}{A+2x^*} < 2x^* + A = r'_k(x^*)$. By (3.5) and (3.6),

$$\begin{aligned} r'_h(x^*) - r'_k(x^*) &= x^* + \frac{A + x^*}{A + 2x^*} - 2x^* - A, \\ &= \frac{A + x^*}{A + 2x^*} - (A + x^*), \\ &< 0. \end{aligned}$$

Thus, we also have that $k(x) < r_h(x) < r_k(x)$ for $x \in (x^*, \infty)$. □

We now take the information from the direction field in (3.4) and from Lemmas 3.1.2 and 3.1.4 to plot generic root-curves and their associated nullclines in Figure 3.1. Using this figure we define the regions bounded by the root-curves and their associated nullclines as,

$$\begin{aligned} \mathcal{R}_{1_1} &= \{r_k(x) < r_h(x) < y \le k(x),\, x \in (0, x^*)\}; \\ \mathcal{R}_{1_2} &= \{r_k(x) \le y \le r_h(x) < k(x),\, x \in (0, x^*)\}; \\ \mathcal{R}_2 &= \{y < \min\{k(x),\, r_k(x)\},\, y < r_h(x),\, x > 0\}; \\ \mathcal{R}_{3_1} &= \{k(x) \le y < r_h(x) < r_k(x),\, x > x^*\}; \\ \mathcal{R}_{3_2} &= \{k(x) < r_h(x) \le y \le r_k(x),\, x > x^*\}; \\ \mathcal{R}_4 &= \{y > \max\{k(x),\, r_k(x)\},\, y > r_h(x),\, x > 0\}. \end{aligned} \tag{3.8}$$

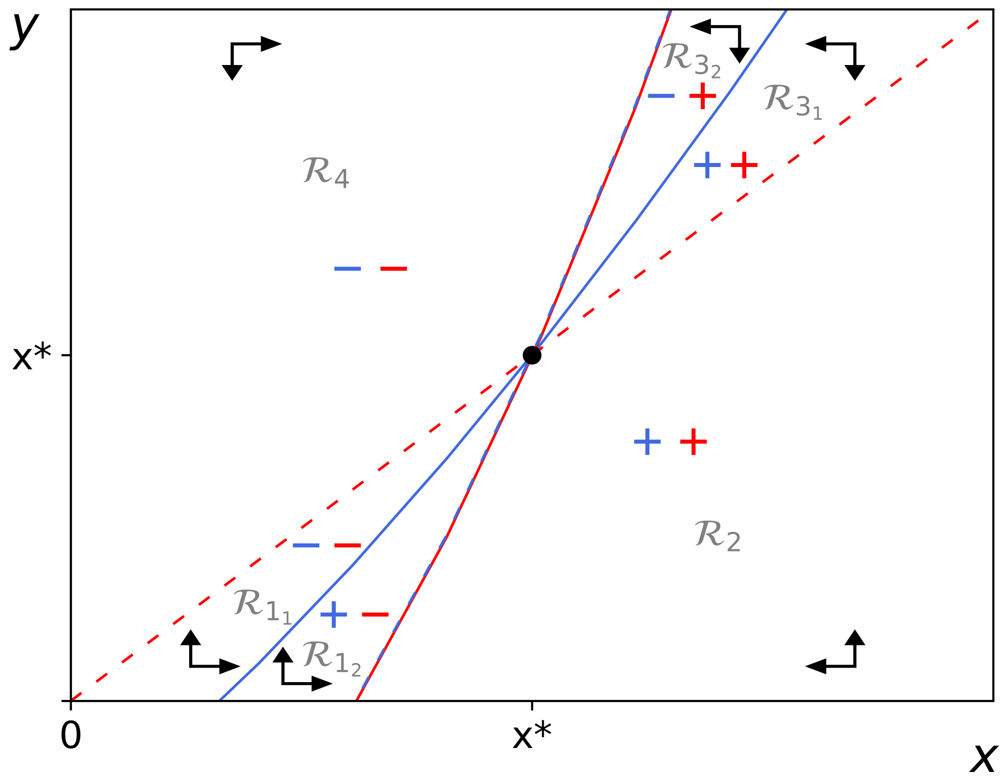

Figure 3.1: The augmented phase portrait of (3.2) for $A \in [0, 1)$. The x-nullcline, representing $y = h(x)$, is in dashed blue. The y-nullcline, $k(x) = x$, is in dashed red. The associated root-curves, $y = r_h(x)$ and $y = r_k(x)$ are the curves in solid blue and red, respectively. Recall that $h(x) = r_k(x)$. The vertical and horizontal arrows represent regions in which orbits have the same component-wise monotonicity. For example, for a point $(x_n, y_n) \in \mathcal{R}_2$, the left arrow indicates that $\Delta x_n < 0$ with equality only at $y_n = h(x_n)$ and the upwards arrow indicates that $\Delta y_n > 0$. Finally, regions containing a '+' ('−') indicate that the next-iterate lies above (below) the same colour nullcline. For example, if $(x_n, y_n) \in \mathcal{R}_{1_2}$, the blue plus sign indicates that the next-iterate, (x_{n+1}, y_{n+1}), lies above the blue nullcline and the red minus sign indicates that the next-iterate lies below the red nullcline.

3.1.2 Possible paths of solutions

To show there exists a solution of (3.2) that decreases monotonically to the equilibrium, $E^* = (x^*, x^*)$, we determine the possible paths a solution can take. We do this using the definition of the regions (3.8) and the previous lemmas. First we explain the following notation for a solution, $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$:

- if $(x_n, y_n) \in \mathcal{R}_i$ and $(x_{n+1}, y_{n+1}) \in \mathcal{R}_j$, we write, $\mathcal{R}_i \to \mathcal{R}_j$;
- if $(x_n, y_n) \in \mathcal{R}_i$ and there exists $\ell \geq 0$ such that $(x_{n+k}, y_{n+k}) \in \mathcal{R}_i$, for $0 \leq k < \ell$, before entering $\mathcal{R}_j$, we write, $\mathcal{R}_i \to^* \mathcal{R}_j$, (i.e., the orbit can remain in $\mathcal{R}_i$ for more than one iteration before entering $\mathcal{R}_j$);
- if a solution remains in region $\mathcal{R}_i$ indefinitely, we write, $\mathcal{R}_i \to \cdots \to \mathcal{R}_i$;
- if a solution oscillates between two regions indefinitely, we write, $\mathcal{R}_i \rightleftharpoons \mathcal{R}_j$.

Lemma 3.1.5. *i) Orbits that start in $\mathcal{R}_{1_2}$ either remain in $\mathcal{R}_{1_2}$ for all $n \in \mathbb{N}_0$, or they move from region to region eventually oscillating between regions $\mathcal{R}_2$ and $\mathcal{R}_4$ indefinitely as follows: $\mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_2 \rightleftharpoons \mathcal{R}_4$.*

ii) Orbits that start in $\mathcal{R}_{3_2}$ either remain in $\mathcal{R}_{3_2}$ for all $n \in \mathbb{N}_0$, or they move from region to region eventually oscillating between $\mathcal{R}_2$ and $\mathcal{R}_4$ indefinitely as follows: $\mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_4 \rightleftharpoons \mathcal{R}_2$.

Figure 3.2 provides visual intuition of the results in Lemma 3.1.5. We refer to it throughout the proof of Lemma 3.1.5 so that it is easier to follow.

a) $\mathcal{R}_{1_2} \to \cdots \to \mathcal{R}_{1_2}$

b) $\mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_2 \rightleftharpoons \mathcal{R}_4$

c) $\mathcal{R}_{3_2} \to \cdots \to \mathcal{R}_{3_2}$

d) $\mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_4 \rightleftharpoons \mathcal{R}_2$

Figure 3.2: A visual representation of the possible paths that a solution of (3.2) can take, as given in Lemma 3.1.5. Paths a) and b) correspond to i) in Lemma 3.1.5 and paths c) and d) correspond to ii) in Lemma 3.1.5.

Proof. If $(x_n, y_n) \in \mathcal{R}_{1_2}$, then by the definition of the region $\mathcal{R}_{1_2}$ (3.8), $r_k(x_n) \leq y_n \leq r_h(x_n)$ for $x_n \in (0, x^*)$. From Lemma 3.1.2 we have that $h(x_{n+1}) \leq y_{n+1} \leq k(x_{n+1})$. Thus, by the definition of the regions (3.8), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{1_1} \cup \mathcal{R}_{1_2}$. Thus, the solution can remain in $\mathcal{R}_{1_2}$ indefinitely or it can stay in $\mathcal{R}_{1_2}$ for a finite number of steps before entering $\mathcal{R}_{1_1}$. Thus, in Figure 3.2 we obtain path a) and the start of path b) as,

$$\begin{aligned} &\text{a) } \mathcal{R}_{1_2} \to \cdots \to \mathcal{R}_{1_2} \\ &\text{b) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \end{aligned} \tag{3.9}$$

If $(x_n, y_n) \in \mathcal{R}_{1_1}$, then by the definition of $\mathcal{R}_{1_1}$ (3.8), $r_k(x) < r_h(x_n) < y_n$ for $x \in (0, x^*)$. Hence, by Lemma 3.1.2, $y_{n+1} < h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} < k(x_{n+1})$ and so by Lemma

3.1.4, $y_{n+1} < r_h(x_{n+1})$. Thus, by the definition of the regions, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_2$ and using (3.9) we continue showing path b) in Figure 3.2,

$$\text{b) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_2. \tag{3.10}$$

If $(x_n, y_n) \in \mathcal{R}_2$, then by the definition of $\mathcal{R}_2$ (3.8), $y_n < r_k(x_n)$ and $y_n < r_h(x_n)$. By Lemma 3.1.2, $y_{n+1} > h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} > k(x_{n+1})$. Hence, by Lemma 3.1.4, $y_{n+1} > r_h(x_{n+1})$. Thus by definition of the regions (3.8), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_4$ and using (3.10) we continue to show path b) from Figure 3.2 as,

$$\text{b) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_2 \to \mathcal{R}_4. \tag{3.11}$$

If $(x_n, y_n) \in \mathcal{R}_4$, then by the definition of $\mathcal{R}_4$ (3.8), $y_n > r_k(x_n)$ and $y_n > r_h(x_n)$. By Lemma 3.1.2, $y_{n+1} < h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} < k(x_{n+1})$, and so by Lemma 3.1.4, $y_{n+1} < r_h(x_{n+1})$. Thus by definition of the regions (3.8), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_2$ and so we know that a solution will oscillate indefinitely between $\mathcal{R}_2$ and $\mathcal{R}_4$. Using (3.11), we obtain path b) from Figure 3.2,

$$\text{b) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_2 \rightleftharpoons \mathcal{R}_4.$$

Thus, we have shown i), that solutions starting in $\mathcal{R}_{1_2}$ either stay in $\mathcal{R}_{1_2}$ indefinitely or eventually oscillate indefinitely between regions $\mathcal{R}_2$ and $\mathcal{R}_4$.

If $(x_n, y_n) \in \mathcal{R}_{3_2}$, then by the definition of region $\mathcal{R}_{3_2}$ (3.8), $r_h(x_n) \leq y_n \leq r_k(x_n)$ for $x_n > x^*$. Thus by Lemma 3.1.2, $k(x_{n+1}) \leq y_{n+1} \leq h(x_{n+1}) = r_k(x_{n+1})$ and so by the definition of the regions (3.8), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{3_1} \cup \mathcal{R}_{3_2}$. Hence, the solution can remain in $\mathcal{R}_{3_2}$ for all $n \in \mathbb{N}_0$, or it can remain in $\mathcal{R}_{3_2}$ for a finite number of steps before entering $\mathcal{R}_{3_1}$. Thus, from Figure 3.2, we obtain path c) and the start of path d) as,

$$\begin{aligned} &\text{c) } \mathcal{R}_{3_2} \to \cdots \to \mathcal{R}_{3_2} \\ &\text{d) } \mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1}. \end{aligned} \tag{3.12}$$

If $(x_n, y_n) \in \mathcal{R}_{3_1}$, then by the definition of $\mathcal{R}_{3_1}$ (3.8), $y_n < r_h(x_n) < r_k(x_n)$ for $x > x^*$. Hence, by Lemma 3.1.2, $y_{n+1} > h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} > k(x_{n+1})$ and so by Lemma 3.1.4, $y_{n+1} > r_h(x_{n+1})$. Thus, by the definition of the regions (3.8), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_4$. We know solutions will oscillate between $\mathcal{R}_4$ and $\mathcal{R}_2$ indefinitely. Thus, using (3.12) we write path d) from Figure 3.2 as,

$$\text{d) } \mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_4 \rightleftharpoons \mathcal{R}_2.$$

Thus, we have shown that solutions starting in $\mathcal{R}_{3_2}$ will either stay in $\mathcal{R}_{3_2}$ for all $n \in \mathbb{N}_0$ or that they will eventually oscillate between $\mathcal{R}_4$ and $\mathcal{R}_2$ indefinitely. ☐

Lemma 3.1.6. *a) A solution that oscillates between $\mathcal{R}_2$ and $\mathcal{R}_4$ can neither increase nor decrease monotonically to the positive equilibrium $E^* = (x^*, x^*)$.*

b) A solution that remains in $\mathcal{R}_{3_2}$ for all $n \in \mathbb{N}_0$, decreases monotonically and converges to the positive equilibrium $E^* = (x^*, x^*)$.

c) A solution that remains in $\mathcal{R}_{1_2}$ for all $n \in \mathbb{N}_0$, increases monotonically and converges to the positive equilibrium $E^* = (x^*, x^*)$.

Proof. a) Suppose for the sake of a contradiction that there exists a solution that decreases monotonically to the equilibrium and oscillates between $\mathcal{R}_2$ and $\mathcal{R}_4$. Then by assumption, $x_n \geq x^*$ and $y_n \geq x^*$ for all $n \in \mathbb{N}_0$ and $x_n > x_{n+1}$ for all $n \in \mathbb{N}_0$.

Without loss of generality, assume that $(x_0, y_0) \in R_4$, so that $(x_1, y_1) \in R_2$. By the definition of $\mathcal{R}_2$ (3.8), $y_1 < k(x_1)$ and so $y_1 < x_1$. However, $y_1 = x_0$. This implies that $x_0 = y_1 < x_1$. This is a contradiction since by assumption, $x_n > x_{n+1}$ for all $n \in \mathbb{N}_0$.

Similarly, we can show that a solution oscillating between $\mathcal{R}_2$ and $\mathcal{R}_4$ cannot increase monotonically to the equilibrium.

b) Assume the solution $\{(x_n, y_n)\}_{n\in\mathbb{N}_0} \in R_{3_2}$. By the definition of R_{3_2} (3.8), $x_n > x^*$ and $y_n > k(x_n)$, and hence, $y_n > x_n > x^*$ for all $n \in \mathbb{N}_0$. Thus it suffices to show that the solution is decreasing monotonically to also show that it converges to the equilibrium.

By Remark 2.1.9, to show that the solution decreases monotonically to the equilibrium, we need to show that $x_n > x_{n+1} > x^*$ for all $n \in \mathbb{N}_0$. From (3.4), $\Delta x_n < 0$ in R_{3_2} and it immediately follows that,

$$x_n > x_{n+1} > x^* \text{ for all } n \in \mathbb{N}_0.$$

Thus, a solution that remains in R_{3_2} for all $n \in \mathbb{N}_0$, decreases monotonically and converges to the equilibrium $E^* = (x^*, x^*)$.

c) Similarly, we assume that $\{(x_n, y_n)\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{1_2}$ and by the definition of $\mathcal{R}_{1_2}$, $x_n < x^*$ and $y_n < k(x_n) = x_n$. Hence, $y_n < x_n < x^*$ for all $n \in \mathbb{N}_0$. Thus it suffices to show that the solution is monotonically increasing to show that it converges to the equilibrium.

By Remark 2.1.9, we need to show that $x_n < x_{n+1} < x^*$. From (3.4), $\Delta x_n > 0$ in $\mathcal{R}_{3_2}$ so,

$$x_n < x_{n+1} < x^*.$$

Thus a solution that remains in R_{1_2} for all $n \in \mathbb{N}_0$ increases monotonically towards the equilibrium, $E^* = (x^*, x^*)$ and converges. □

We now aim to show existence of a solution that monotonically decreases to the equilibrium in $\mathcal{R}_{3_2}$ and a solution that monotonically increases to the equilibrium in $\mathcal{R}_{1_2}$.

3.1.3 Proof of Theorem 3.1.1

Proof. By Theorem 3.0.1, (x^*, x^*) is a saddle point equilibrium for $A \in [0, 1)$. We show that the stable manifold lies entirely in $\mathcal{R}_{3_2}$ and $\mathcal{R}_{1_2}$. Since a solution on the stable manifold must stay on the stable manifold and from Lemma 3.1.5, solutions cannot jump between $\mathcal{R}_{3_2}$ and $\mathcal{R}_{1_2}$, this will show that there exist solutions that stay in either $\mathcal{R}_{3_2}$ or $\mathcal{R}_{1_2}$ for all time. Then, from Lemma 3.1.6, such solutions will monotonically decrease and increase and converge to (x^*, x^*).

To find the tangent to the stable manifold at the equilibrium, we use Theorem 3.0.2. First we verify the conditions of Theorem 3.0.2 and denote T with $\alpha \geq 0$ and $A \in [0, 1)$ as,

$$T(x, y) = \begin{pmatrix} F(x, y) \\ G(x, y) \end{pmatrix} = \begin{pmatrix} \frac{\alpha+y}{A+x} \\ x \end{pmatrix}.$$

We need $T(x, y)$ to be one-to-one. Suppose that $T(x_1, y_1) = T(x_2, y_2)$ or equivalently,

$$\begin{pmatrix} \frac{\alpha+y_1}{A+x_1} \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{\alpha+y_2}{A+x_2} \\ x_2 \end{pmatrix}.$$

It is obvious that $x_1 = x_2$ and this implies that $\alpha + y_1 = \alpha + y_2$ and hence $y_1 = y_2$.

The Jacobian of T evaluated at (x^*, x^*) is,

$$J_T(x^*, x^*) = \begin{pmatrix} \frac{\partial F}{\partial x}(x^*, x^*) & \frac{\partial F}{\partial y}(x^*, x^*) \\ \frac{\partial G}{\partial x}(x^*, x^*) & \frac{\partial G}{\partial y}(x^*, x^*) \end{pmatrix} = \begin{pmatrix} \frac{-x^*}{A+x^*} & \frac{1}{A+x^*} \\ 1 & 0 \end{pmatrix}.$$

Since $\det(J_T(x^*, x^*)) = \frac{-1}{A+x^*} < 0$, by the Inverse Function Theorem, the inverse of T exists and is smooth. Thus, we can use Theorem 3.0.2 to find the stable manifold.

Linearizing $T(x, y)$ around (x^*, x^*), we obtain the associated characteristic equation (2.4) of $J_T(x^*, x^*)$,

$$0 = \lambda^2 + \frac{x^*}{A + x^*}\lambda - \frac{1}{A + x^*}. \tag{3.13}$$

Solving for λ we obtain,

$$\lambda_+ := \lambda = \frac{-x^* + \sqrt{(x^*)^2 + 4(A + x^*)}}{2(A + x^*)} \quad \text{and} \quad \lambda_- := \lambda = \frac{-x^* - \sqrt{(x^*)^2 + 4(A + x^*)}}{2(A + x^*)}. \tag{3.14}$$

From the proof of Theorem 3.0.1 in Chapter 5, Section 5.1, for $A \in [0, 1)$, $|\lambda_+| < 1 < |\lambda_-|$. From Theorem 3.0.2, the eigenvector associated with the eigenvalue λ_+ is tangent to the stable manifold.

We solve $(\lambda_+ I - J_T(x^*, x^*))v = 0$ for the eigenvector v. This is equivalent to,

$$\begin{pmatrix} \lambda_+ + \frac{x^*}{A+x^*} & \frac{-1}{A+x^*} \\ -1 & \lambda_+ \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $\lambda_+ v_1 + \frac{x^* v_1}{A+x^*} - \frac{v_2}{A+x^*} = 0$ and $v_2 = \frac{v_1}{\lambda_+}$. Substituting the latter equation into the former,

$$\left(\lambda_+ + \frac{x^*}{A + x^*} - \frac{1}{\lambda_+(A + x^*)}\right)v_1 = 0.$$

Multiplying this by λ_+, we obtain the characteristic equation, (3.13), implying that v_1 is a free variable. Therefore, an eigenvector associated with λ_+ is

$$v_+ = \begin{pmatrix} \lambda_+ \\ 1 \end{pmatrix}$$

and it is tangent to the stable manifold at (x^*, x^*).

To show that the stable manifold is in region $\mathcal{R}_{3_2}$, we show that the slope of v_+ is greater than the slope of $r_h(x)$ at $x = x^*$ and less than the slope of $r_k(x)$ at $x = x^*$.

The slope of the eigenvector is $\frac{1}{\lambda_+}$. Additionally, from (3.6) and (3.7),

$$r_h(x^*) = x^* + \frac{A + x^*}{A + 2x^*} \text{ and } r_k(x^*) = 2x^* + A.$$

It therefore suffices to show that

$$x^* + \frac{A + x^*}{A + 2x^*} < \frac{1}{\lambda_+} < 2x^* + A.$$

We begin by showing $\frac{1}{\lambda_+} < 2x^* + A$. Since $|\lambda_+| < 1$,

$$\begin{aligned} -x^* + \sqrt{(x^*)^2 + 4(A + x^*)} &< 2(A + x^*) \Rightarrow \\ (x^*)^2 + 4(A + x^*) &< (2A + 3x^*)^2 \Rightarrow \\ 4(A + x^*) &< 4A^2 + 12Ax^* + 8(x^*)^2. \end{aligned}$$

Thus, $A + x^* < A^2 + 3Ax^* + 2(x^*)^2$. Adding $Ax^* + 2(x^*)^2$ to each side, we create a perfect square on the right hand side and the inequality becomes $A + x^* + x^*(A + 2x^*) < (A + 2x^*)^2$. Multiplying both sides by $\frac{4(A+x^*)}{(A+2x^*)^2}$ and then adding $(x^*)^2$ to each side, we get

$$\frac{4(A + x^*)^2}{(A + 2x^*)^2} + \frac{4(A + x^*)x^*}{A + 2x^*} + (x^*)^2 < 4(A + x^*) + (x^*)^2.$$

Notice that we can rewrite the left hand side as a perfect square. We can then take the square root of both sides to obtain,

$$\frac{2(A + x^*)}{A + 2x^*} + x^* < \sqrt{4(A + x^*) + (x^*)^2},$$

and so from (3.14),

$$\frac{1}{\lambda_+} = \frac{2(A + x^*)}{-x^* + \sqrt{4(A + x^*) + (x^*)^2}} < A + 2x^*.$$

Thus $\frac{1}{\lambda_+} < 2x^* + A$ and hence, the slope of the stable manifold is less than the slope of $r_k(x)$ at $x = x^*$.

We now have left to show that, $x^* + \frac{A+x^*}{A+2x^*} < \frac{1}{\lambda_+}$. Rearranging the characteristic equation (3.13), we have,

$$\frac{1}{\lambda_+} = x^* + \lambda_+(A + x^*).$$

Since $\frac{1}{\lambda_+} < 2x^* + A$, or equivalently, $\frac{1}{2x^*+A} < \lambda_+$,

$$\frac{1}{\lambda_+} = x^* + \lambda_+(A + x^*) > x^* + \frac{A + x^*}{A + 2x^*},$$

as required. Thus, the slope of the stable manifold is greater than the slope of $r_h(x)$ at $x = x^*$, and hence the stable manifold is in $\mathcal{R}_{3_2}$ for $x > x^*$, close to x^* and in $\mathcal{R}_{1_2}$ for $x < x^*$ for x close to x^*.

From Lemma 3.1.5 solutions cannot enter $\mathcal{R}_{3_2}$ from any other region. Additionally, solutions on a stable manifold cannot leave the stable manifold. Thus, the existence of the stable manifold locally in region $\mathcal{R}_{3_2}$ implies that the stable manifold must be located in region $\mathcal{R}_{3_2}$ for all $x > x^*$ and in $\mathcal{R}_{1_2}$ for all $x < x^*$.

A solution on the stable manifold in $\mathcal{R}_{3_2}$ stays in $\mathcal{R}_{3_2}$ indefinitely. Hence, by Lemma 3.1.6, the solution must monotonically decrease to the equilibrium and converge. Similarly, a solution on the stable manifold in $\mathcal{R}_{1_2}$ stays in $\mathcal{R}_{1_2}$ indefinitely. As such, by Lemma 3.1.6, the solution monotonically increases to the equilibrium and converges. □

3.1.4 Consequences

Conjecture 1 is an immediate consequence of Theorem 3.1.1 since it considers the specific case where $\alpha = 1$, $\beta = 1$, $A = 0$, and $\gamma = 1$. As previously mentioned, Conjecture 1 was also proved by Hoag [5] and by Sun and Xi [12]. While Sun and Xi's [12] proof is vastly different from our proof, the proof by Hoag [5] shares some similarities. We discuss the similarities and differences in the methods we used in our proof of Theorem 3.1.1. The following theorem is the main theorem Hoag used to prove Conjecture 1.

Theorem 3.1.7. *[See [5, Theorem 1]] Let I be an interval and let $F : I \times I \to I$ be a continuously differentiable function. Let $x^* \in I$ be a saddle point equilibrium of*

$$x_{n+1} = F(x_{n-1}, x_n) \tag{3.15}$$

with $x_{n-1}, x_n \in I$. Assume F satisfies the following conditions:

i) $\dfrac{\partial}{\partial x}F(x, y) \leq 0$ *for all* $(x, y) \in I \times I$;

ii) $\dfrac{\partial}{\partial y}F(x, y) > 0$ *for all* $(x, y) \in I \times I$;

iii) $(x - x^*)(x - F(x, x)) > 0$ *for all* $x \in I$ *and* $x \neq x^*$

Then every solution of (3.15) *that converges to x^* is monotone.*

Since (3.2) satisfies the criteria for Theorem 3.1.7, Hoag shows that the special case $\alpha = 1$, $\beta = 1$, $A = 0$, and $\gamma = 1$ is a consequence of Theorem 3.1.7. It is obvious that Theorem 3.1.7 uses a more general function than Theorem 3.1.1; however, Theorem 3.1.1 is shown using root-curves and their associated nullclines. This allows for a more intuitive and visual proof. This is the main difference between the two methods used to show Conjecture 1.

Much like in the proof of Theorem 3.1.1, Theorem 3.1.7 requires that the equilibrium is a saddle point. Since saddle points have stable manifolds and thus convergent solutions, the proof of Theorem 3.1.7 showed that solutions on the stable manifold will be monotonically increasing or decreasing. This is similar to the idea we used in our proof of Theorem 3.1.1. In our proof, to show that solutions on the stable manifold are monotonically increasing or decreasing, we show that it is located in a specific region that ensures this property. In contrast to this, Hoag [5] used four Lemmas not required in our proof. The theorem by Hoag is more general; however, our proof of Theorem 3.1.1 uses an analogue to the method of phase portraits for differential equations that allows for a more elementary and accessible proof.

3.1.5 Additional Results

Using Theorem 3.1.1 and the augmented phase portrait, we provide more information on the behaviour of solutions of (3.2) that do not converge monotonically to the equilibrium. By Lemma 3.1.5, such solutions eventually oscillate between $\mathcal{R}_2$ and $\mathcal{R}_4$ indefinitely. Thus, we subdivide regions $\mathcal{R}_2$ and $\mathcal{R}_4$ into,

$$\begin{aligned}
\mathcal{R}_{2_a} &= \mathcal{R}_2 \cap \{x < x^*,\, y < x^*\},\\
\mathcal{R}_{2_b} &= \mathcal{R}_2 \cap \{x \geq x^*,\, y < x^*\},\\
\mathcal{R}_{2_c} &= \mathcal{R}_2 \cap \{x \geq x^*,\, y \geq x^*\},\\
\mathcal{R}_{4_a} &= \mathcal{R}_4 \cap \{x \geq x^*,\, y \geq x^*\},\\
\mathcal{R}_{4_b} &= \mathcal{R}_4 \cap \{x < x^*,\, y \geq x^*\},\\
\mathcal{R}_{4_c} &= \mathcal{R}_4 \cap \{x < x^*,\, y < x^*\}.
\end{aligned} \tag{3.16}$$

The visual representation of these sub-regions can be found in Figure 3.3.

Now we explain the following notation for a solution $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$:

- if a solution can oscillate between two regions $\mathcal{R}_i$ and $\mathcal{R}_j$ either indefinitely or finitely before entering a region $\mathcal{R}_k$ from $\mathcal{R}_j$, we write $\mathcal{R}_i \rightleftharpoons^* \mathcal{R}_j \to \mathcal{R}_k$.

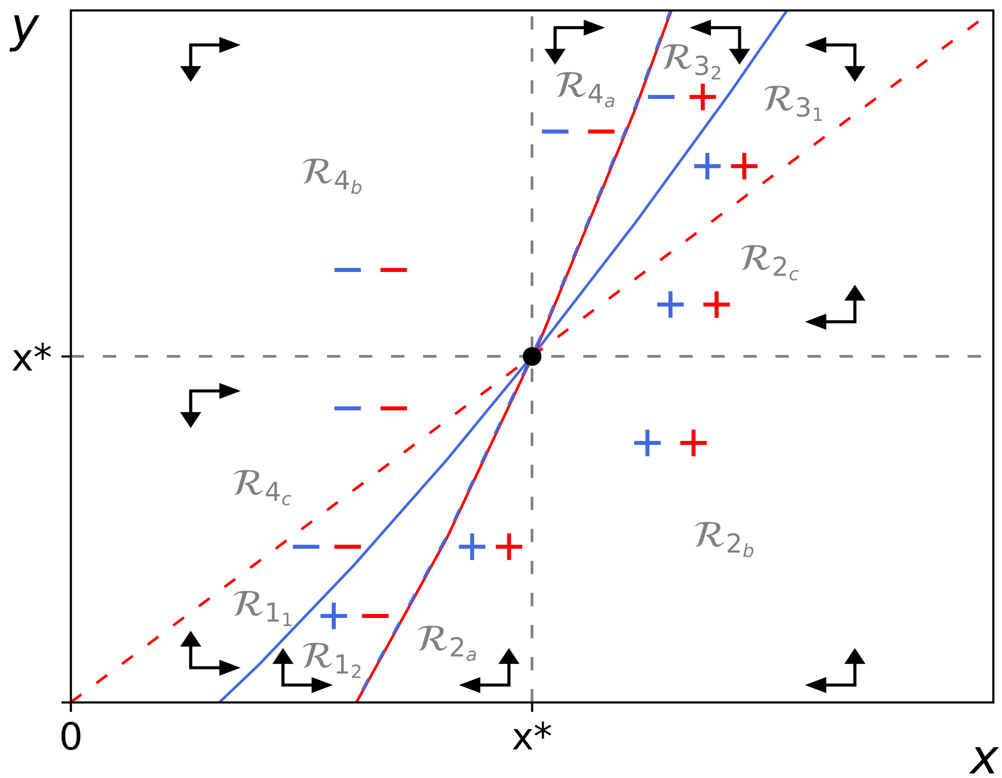

Figure 3.3: The augmented phase portrait of (3.2) with $A \in [0, 1)$ and the new regions, $\mathcal{R}_{2_a}, \mathcal{R}_{2_b}, \mathcal{R}_{2_c}, \mathcal{R}_{4_a}, \mathcal{R}_{4_b}$, and $\mathcal{R}_{4_c}$.

Lemma 3.1.8. *Consider a solution of* (3.2) *with* $A \in [0, 1)$. *Then the following holds.*

i) If $x_n < x^*$, *then* $y_{n+1} < x^*$.

ii) If $x_n \geq x^*$, *then* $y_{n+1} \geq x^*$.

iii) If $x_n < x^*$ *and* $y_n \geq x^*$, *then* $x_{n+1} > x^*$ *and* $y_{n+1} < x^*$

iv) If $x_n \geq x^*$ *and* $y_n < x^*$, *then* $x_{n+1} < x^*$ *and* $y_{n+1} \geq x^*$.

Proof. The proof of i) and ii) comes from the fact that $y_{n+1} = x_n$.

iii) If $x_n \geq x^*$ and $y_n < x^*$, then $x_{n+1} = \frac{\alpha+y_n}{A+x_n} < \frac{\alpha+x^*}{A+x^*} = x^*$. From ii), $y_{n+1} = x_n \geq x^*$.

iv) If $x_n < x^*$ and $y_n \geq x^*$, then $x_{n+1} = \frac{\alpha+y_n}{A+x_n} > \frac{\alpha+x^*}{A+x^*} = x^*$. From i), $y_{n+1} = x_n < x^*$. ☐

Lemma 3.1.9. *For solutions of* (3.2) *that do not converge monotonically to the equilibrium, one of the following is always true:*

i) the solution eventually oscillates between $\mathcal{R}_{2_b}$ *and* $\mathcal{R}_{4_b}$ *indefinitely;*

ii) the solution eventually oscillates between $\mathcal{R}_{2_a}$ *and* $\mathcal{R}_{4_c}$ *indefinitely;*

iii) the solution eventually oscillates between $\mathcal{R}_{2_c}$ *and* $\mathcal{R}_{4_a}$ *indefinitely.*

We begin by providing a visual representation of all possible paths a solution could take if it does not converge to the equilibrium in Figure 3.4. Additionally we provide a visual representation of regions that solutions cannot jump into based on Lemma 3.1.8 in Figure 3.5.

$$a)\ \mathcal{R}_{3_2} \rightarrow^* \mathcal{R}_{3_1} \rightarrow \mathcal{R}_{4_a} \stackrel{*}{\rightleftharpoons} \mathcal{R}_{2_c} \rightarrow \mathcal{R}_{4_b} \rightleftharpoons \mathcal{R}_{2_b}$$

$$b)\ \mathcal{R}_{3_2} \rightarrow^* \mathcal{R}_{3_1} \rightarrow \mathcal{R}_{4_b} \rightleftharpoons \mathcal{R}_{2_b}$$

$$c)\ \mathcal{R}_{1_2} \rightarrow^* \mathcal{R}_{1_1} \rightarrow \mathcal{R}_{2_a} \stackrel{*}{\rightleftharpoons} \mathcal{R}_{4_c} \rightarrow \mathcal{R}_{2_b} \rightleftharpoons \mathcal{R}_{4_b}$$

$$d)\ \mathcal{R}_{1_2} \rightarrow^* \mathcal{R}_{1_1} \rightarrow \mathcal{R}_{2_b} \rightleftharpoons \mathcal{R}_{4_b}$$

Figure 3.4: The possible paths a solution of (3.2) that does not converge monotonically to the equilibrium could take.

$$\begin{array}{ll}
\mathcal{R}_{1_1} \not\rightarrow \mathcal{R}_{2_c} & \quad \mathcal{R}_{2_a} \not\rightarrow \mathcal{R}_{4_a} \cup \mathcal{R}_{4_b} \\
\mathcal{R}_{3_1} \not\rightarrow \mathcal{R}_{4_c} & \quad \mathcal{R}_{2_b} \not\rightarrow \mathcal{R}_{4_a} \cup \mathcal{R}_{4_c} \\
\mathcal{R}_{2_c} \not\rightarrow \mathcal{R}_{4_c} & \quad \mathcal{R}_{4_a} \not\rightarrow \mathcal{R}_{2_a} \cup \mathcal{R}_{2_b} \\
\mathcal{R}_{4_c} \not\rightarrow \mathcal{R}_{2_c} & \quad \mathcal{R}_{4_b} \not\rightarrow \mathcal{R}_{2_a} \cup \mathcal{R}_{2_c}
\end{array}$$

Figure 3.5: The regions that solutions of (3.2) cannot jump into using Lemma 3.1.8.

Proof. If $(x_n, y_n) \in \mathcal{R}_{3_2}$ and the solution does not converge to $E^* = (x^*, x^*)$, from Lemma 3.1.5, there exists an $\ell > 0$ such that $(x_{n+k}, y_{n+k}) \in \mathcal{R}_{3_2}$ for all $0 \leq k < \ell$ and $(x_\ell, y_\ell) \in \mathcal{R}_{3_1}$. Thus, we write the start of paths a) and b) in Figure 3.4 as,

$$\mathcal{R}_{3_2} \rightarrow^* \mathcal{R}_{3_1}. \tag{3.17}$$

If $(x_n, y_n) \in \mathcal{R}_{3_1}$ then by the definition $\mathcal{R}_{3_1}$ (3.8), $x_n > x^*$ and $y_n \geq k(x_n) = x_n > x^*$, implying that, $y_{n+1} > x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_4$ and so by the definition of the sub-regions (3.16), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_a} \cup \mathcal{R}_{4_b}$. Using (3.17), we can continue to write paths a) and b), from Figure 3.4, respectively, as,

$$\begin{aligned} &a)\ \mathcal{R}_{3_2} \rightarrow^* \mathcal{R}_{3_1} \rightarrow \mathcal{R}_{4_a}, \\ &b)\ \mathcal{R}_{3_2} \rightarrow^* \mathcal{R}_{3_1} \rightarrow \mathcal{R}_{4_b}. \end{aligned} \tag{3.18}$$

If $(x_n, y_n) \in \mathcal{R}_{4_a}$ then by the definition of the sub-regions (3.16), $x_n \geq x^*$ and $y_n \geq x^*$, and so $y_{n+1} \geq x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_2$. Thus by definition of the sub-regions, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_c}$. Thus using (3.18) we continue to show path a) in Figure 3.4 as,

$$a)\ \mathcal{R}_{3_2} \rightarrow^* \mathcal{R}_{3_1} \rightarrow \mathcal{R}_{4_a} \rightarrow \mathcal{R}_{2_c}. \tag{3.19}$$

If $(x_n, y_n) \in \mathcal{R}_{2_c}$ then by the definition of the sub-regions (3.16), we know that $x_n \geq x^*$ and $y_n \geq x^*$ implying that, $y_{n+1} \geq x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_4$. Thus by definition of the sub-regions, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_a} \cup \mathcal{R}_{4_b}$. Using (3.19), we continue to write path a) from Figure 3.4 as,

$$a)\ \mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_{4_a} \overset{*}{\rightleftharpoons} \mathcal{R}_{2_c} \to \mathcal{R}_{4_b}. \tag{3.20}$$

If $(x_n, y_n) \in \mathcal{R}_{4_b}$ then by the definition of the sub-regions (3.16), we know that $x_n < x^*$ and $y_n \geq x^*$. By Lemma 3.1.8, $x_{n+1} > x^*$ and $y_{n+1} < x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_2$ and so by the definition of the sub-regions of $\mathcal{R}_2$, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_b}$. Thus using (3.20) and (3.18) we can continue paths a) and b) from Figure 3.4, respectively, as,

$$a)\ \mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_{4_a} \overset{*}{\rightleftharpoons} \mathcal{R}_{2_c} \to \mathcal{R}_{4_b} \to \mathcal{R}_{2_b},$$

$$b)\ \mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_{4_b} \to \mathcal{R}_{2_b}.$$

If $(x_n, y_n) \in \mathcal{R}_{2_b}$ then by the definition of the sub-regions (3.16), we know that $x_n \geq x^*$ and $y_n < x^*$. By Lemma 3.1.8 $x_{n+1} < x^*$ and $y_{n+1} \geq x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_4$. Thus by definition of the sub-regions of $\mathcal{R}_4$, we have $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_b}$. Finally, we complete paths a) and b) from Figure 3.4, respectively, as,

$$a)\ \mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_{4_a} \overset{*}{\rightleftharpoons} \mathcal{R}_{2_c} \to \mathcal{R}_{4_b} \rightleftharpoons \mathcal{R}_{2_b},$$

$$b)\ \mathcal{R}_{3_2} \to^* \mathcal{R}_{3_1} \to \mathcal{R}_{4_b} \rightleftharpoons \mathcal{R}_{2_b}.$$

Thus solutions on path a) can either oscillate between $\mathcal{R}_{4_a}$ and $\mathcal{R}_{2_c}$ indefinitely or oscillate between $\mathcal{R}_{4_b}$ and $\mathcal{R}_{2_b}$ indefinitely.

If $(x_n, y_n) \in \mathcal{R}_{1_2}$ and the solution doesn't converge to the equilibrium, from Lemma 3.1.5, there exists an $\ell > 0$ such that $(x_{n+k}, y_{n+k}) \in \mathcal{R}_{1_2}$ for all $0 \leq k < \ell$ and $(x_\ell, y_\ell) \in \mathcal{R}_{1_1}$. Then, we can start writing paths c) and d) from Figure 3.4 as,

$$\mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1}. \tag{3.21}$$

If $(x_n, y_n) \in \mathcal{R}_{1_1}$ then by the definition of $\mathcal{R}_{1_1}$ (3.8), $x_n < x^*$ and $y_n \leq k(x_n) = x_n < x^*$ and so, $y_{n+1} < x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_2$, and by the definition of the sub-regions (3.16), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_a} \cup \mathcal{R}_{2_b}$. Using (3.21), we continue to write solutions c) and d) from Figure 3.4, respectively, as,

$$\begin{aligned} &\text{c) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_{2_a}, \\ &\text{d) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_{2_b}. \end{aligned} \tag{3.22}$$

If $(x_n, y_n) \in \mathcal{R}_{2_a}$ then by the definition of the sub-regions (3.16), $x_n < x^*$ and $y_n < x^*$. Thus, $y_{n+1} < x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_4$. By the definition of the sub-regions (3.16), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_c}$. Using (3.22), we continue writing path c) from Figure 3.4 as,

$$\text{c) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_{2_a} \to \mathcal{R}_{4_c}. \tag{3.23}$$

If $(x_n, y_n) \in \mathcal{R}_{4_c}$ then by the definition of the sub-regions (3.16), $x_n < x^*$ and $y_n < x^*$. Thus, $y_{n+1} < x^*$. From Lemma 3.1.5, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_2$. Thus by the definition of the sub-regions (3.16), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_a} \cup \mathcal{R}_{2_b}$. We continue to write path c) from Figure 3.4 using (3.23) as,

$$\text{c) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_{2_a} \overset{*}{\rightleftharpoons} \mathcal{R}_{4_c} \to \mathcal{R}_{2_b}.$$

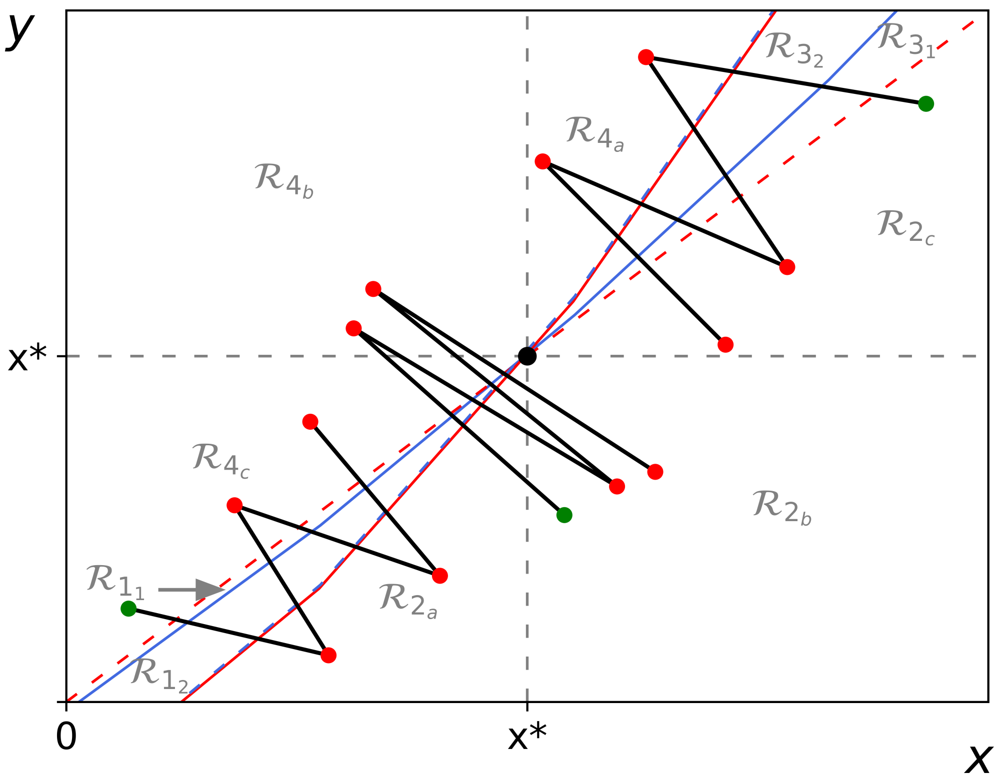

Figure 3.6: The augmented phase portrait of (3.2) with $A = 0.9$ and $\alpha = 0.1$. Depicted are five iterations of three solutions, each starting at one of the green dots. One solution is oscillating between $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$, another is oscillating between $\mathcal{R}_{2_a}$ and $\mathcal{R}_{4_c}$, and the third is oscillating between $\mathcal{R}_{2_c}$ and $\mathcal{R}_{4_a}$.

Since solutions that enter $\mathcal{R}_{2_b}$ oscillate between $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_6}$ indefinitely, we can finish writing paths c) and d) from Figure 3.4 as

$$\text{c) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_{2_a} \overset{*}{\rightleftharpoons} \mathcal{R}_{4_c} \to \mathcal{R}_{2_b} \rightleftharpoons \mathcal{R}_{4_b},$$

$$\text{d) } \mathcal{R}_{1_2} \to^* \mathcal{R}_{1_1} \to \mathcal{R}_{2_b} \rightleftharpoons \mathcal{R}_{4_b}.$$

Thus, for solutions that do not converge monotonically to the equilibrium, they either oscillate between $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$ indefinitely, oscillate between $\mathcal{R}_{2_a}$ and $\mathcal{R}_{4_c}$ indefinitely, or oscillate between $\mathcal{R}_{2_c}$ and $\mathcal{R}_{4_a}$ indefinitely. See Figure 3.6 for an example of each case. □

To say more about solutions of (3.2) that do not converge monotonically to the equilibrium, using Definition 2.3.12, we find the pre-image of the line $x = x^*$ to be a function $y = S(x)$ that satisfies $F(x, S(x)) = x$. Equivalently, $\frac{\alpha+y}{A+x} = x^*$, and so

$$S(x) = xx^* + Ax^* - \alpha.$$

Lemma 3.1.10. *Consider (3.2) with $A \in [0, 1)$. If $S(\bar{x}) = k(\bar{x})$, then $\bar{x} = x^*$ and the two functions intersect at $E^* = (x^*, x^*)$.*

Proof. Assume that $S(\bar{x}) = k(\bar{x})$. Then, $\bar{x}x^* + Ax^* - \alpha = \bar{x}$. Thus,

$$\bar{x} = \frac{Ax^* - \alpha}{1 - x^*}.$$

Since $x^* = \frac{\alpha + x^*}{A + x^*}$, we have that $1 - x^* = \frac{A-\alpha}{A+x^*}$ and so,

$$\bar{x} = \left(A\frac{\alpha + x^*}{A + x^*} - \alpha\right)\left(\frac{A + x^*}{A - \alpha}\right) = \left(\frac{A\alpha + Ax^* - \alpha A - \alpha x^*}{A + x^*}\right)\left(\frac{A + x^*}{A - \alpha}\right) = \frac{x^*(A - \alpha)}{A - \alpha} = x^*.$$

Therefore $S(x^*) = k(x^*) = x^*$. Hence, the functions intersect at $E^* = (x^*, x^*)$. □

Lemma 3.1.11. *Consider* (3.2) *with* $A \in [0, 1)$. *The the following holds.*

i) If $y_n < S(x_n)$, *then* $x_{n+1} < x^*$. *If* $y_n > S(x_n)$ *then,* $x_{n+1} > x^*$*;*

ii) If $\alpha > A$, *then* $S(x) < k(x)$ *for* $x \in (0, x^*)$ *and* $k(x) < S(x)$ *for* $x > x^*$.

The proof of Lemma 3.1.11 is in Chapter 5, Section 5.7. Now we prove Theorem 3.1.12 from [3] and [8] for $A \in [0, 1)$. We note that this theorem was first proved in [3] by Gibbons, Kulenovic, and Ladas and later included in [8].

Theorem 3.1.12. *[Adapted from [3, Theorem 3.1] and [8, Theorem 6.5.3]] Let* $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$ *be a nontrivial solution of* (3.2) *and let* $E^* = (x^*, x^*)$ *denote the unique positive equilibrium. Then the following statements hold.*

a) After the first semicycle, a solution that oscillates about the equilibrium will have semicycles of length one.

b) Assume $\alpha > A$. *Then every solution that does not eventually oscillate about the equilibrium converges monotonically to the equilibrium.*

Proof. a) By Lemma 3.1.9 all solutions that do not monotonically converge to $E^* = (x^*, x^*)$, eventually oscillate either between $\mathcal{R}_{2_a}$ and $\mathcal{R}_{4_c}$, or $\mathcal{R}_{2_c}$ and $\mathcal{R}_{4_a}$, or $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$.

By the definition of the sub-regions (3.16), since $x < x^*$ for $\mathcal{R}_{2_a}$ and $\mathcal{R}_{4_c}$, if a solution oscillates between $\mathcal{R}_{2_a}$ and $\mathcal{R}_{4_c}$, then it is not oscillating about the equilibrium. Similarly, if a solution oscillates between $\mathcal{R}_{2_c}$ and $\mathcal{R}_{4_a}$, then it is not oscillating about the equilibrium.

Thus, we consider a solution that oscillates between $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$. Without loss of generality, assume $\{(x_{2n}, y_{2n})\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{2_b}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{4_b}$. Then by the definition of the sub-regions (3.16),

$$x_{2n} \geq x^* \text{ and } x_{2n+1} < x^* \text{ for all } n \in \mathbb{N}_0.$$

Thus, the solution oscillates about the equilibrium with semicycle of length one.

Now we must show that any solution that eventually oscillates between $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$ does not oscillate about the equilibrium before it enters regions $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$. By the definition of the regions (3.8) and sub-regions (3.16), if $(x_n, y_n) \in \mathcal{R}_{3_2} \cup \mathcal{R}_{3_1} \cup \mathcal{R}_{4_a} \cup \mathcal{R}_{2_c}$, then

$$x_n \geq x^*.$$

Thus, a solution on path a) or b) from Figure 3.4 that eventually oscillates between $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$ does not start to oscillate about the equilibrium until it enters $\mathcal{R}_{4_b}$. Thus, it has semicycles of length one except for possibly the first semicycle.

Additionally, we have that from the definition of the regions (3.8) and sub-regions (3.16), if $(x_n, y_n) \in \mathcal{R}_{1_2} \cup \mathcal{R}_{1_1} \cup \mathcal{R}_{4_c} \cup \mathcal{R}_{2_a}$ then,

$$x_n < x^*.$$

Thus, a solution on path c) or d) from Figure 3.4 does not start to oscillate about the equilibrium until it enters $\mathcal{R}_{2_b}$. Thus, the solution has semicycles of length one except for possibly the first one.

b) If $\alpha > A$, then we eliminate the possibility that a solution that does not converge monotonically to the equilibrium, can oscillate between $\mathcal{R}_{2_c}$ and $\mathcal{R}_{4_a}$ indefinitely, or between $\mathcal{R}_{2_a}$ and $\mathcal{R}_{4_c}$ indefinitely.

From the proof of Lemma 3.1.9, if $(x_n, y_n) \in \mathcal{R}_{2_c}$, then $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_a} \cup \mathcal{R}_{4_b}$. By definition of $\mathcal{R}_2$ (3.8), $y_n < k(x_n)$ and by the definition of $\mathcal{R}_{2_c}$ (3.16), $x_n \geq x^*$. Since $\alpha > A$, by Lemma 3.1.11 ii), $y_n < k(x_n) < S(x_n)$. Thus by Lemma 3.1.11 i), $x_{n+1} < x^*$. By the definition of the sub-regions of $\mathcal{R}_{4_a}$ and $\mathcal{R}_{4_b}$, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_b}$. Thus, the solution cannot oscillate between $\mathcal{R}_{2_c}$ and $\mathcal{R}_{4_a}$.

Similarly, by Lemma 3.1.9 if $(x_n, y_n) \in \mathcal{R}_{4_c}$ then $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_a} \cup \mathcal{R}_{2_b}$. By the definition of $\mathcal{R}_4$ (3.8), $y_n > k(x_n)$ and by the definition of $\mathcal{R}_{4_c}$ (3.16), $x_n < x^*$. Since $\alpha > A$, by Lemma 3.1.11 ii), $y_n > k(x_n) > S(x_n)$ and so by Lemma 3.1.11 i), $x_{n+1} > x^*$. By the definition of the sub-regions $\mathcal{R}_{2_a}$ and $\mathcal{R}_{2_b}$, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_b}$. Thus a solution cannot oscillate between $\mathcal{R}_{4_c}$ and $\mathcal{R}_{2_a}$.

Therefore, a solution of (3.2) with $\alpha > A$ must either eventually oscillate between $\mathcal{R}_{4_b}$ and $\mathcal{R}_{2_b}$ and thus also oscillate about the equilibrium or must converge monotonically to the equilibrium. See Figure 3.7 for examples of solutions oscillating between $\mathcal{R}_{4_b}$ and $\mathcal{R}_{2_b}$. $\square$

In [3] and [8], the proof for Theorem 3.1.12 used more classical techniques for difference equations. In comparison, our proof did not use any classical theorems and provided a proof that is intuitive and accessible.

3.2 Case 2: $A > 1$

Since there exists a theorem that analyzes the behaviour of solutions of (3.2) with parameters $A > 1$ and $\alpha > 0$, we do not explore this system of equations any further. However, the following theorem, presented in the variables used in this chapter, provides the global dynamics.

Theorem 3.2.1. *[Adapted from [3, Theorem 6.1]] Assume that $A > 1$. Then every positive solution of $x_{n+1} = \frac{\alpha + y_n}{A + x_n}$ converges to the positive equilibrium, (x^*, x^*).*

Note that this theorem is similar to Theorem 3.0.1 from [8]. The difference is that in Theorem 3.0.1, we only get local asymptotic stability when $A > 1$ and Theorem 3.2.1 gives us global asymptotic stability of the unique positive equilibrium.

3.3 Case 3: $A = 1$

As in the case for $A > 1$, we simply provide information that is already known to present a thorough analysis of (3.2).

By Theorem 3.0.3 *a*) from [4], every solution of (3.2) converges to a period two solution. This result is also found in [3] as Theorem 5.1. We do not provide a proof for these theorems as the proof from [3] is short and comprehensive.

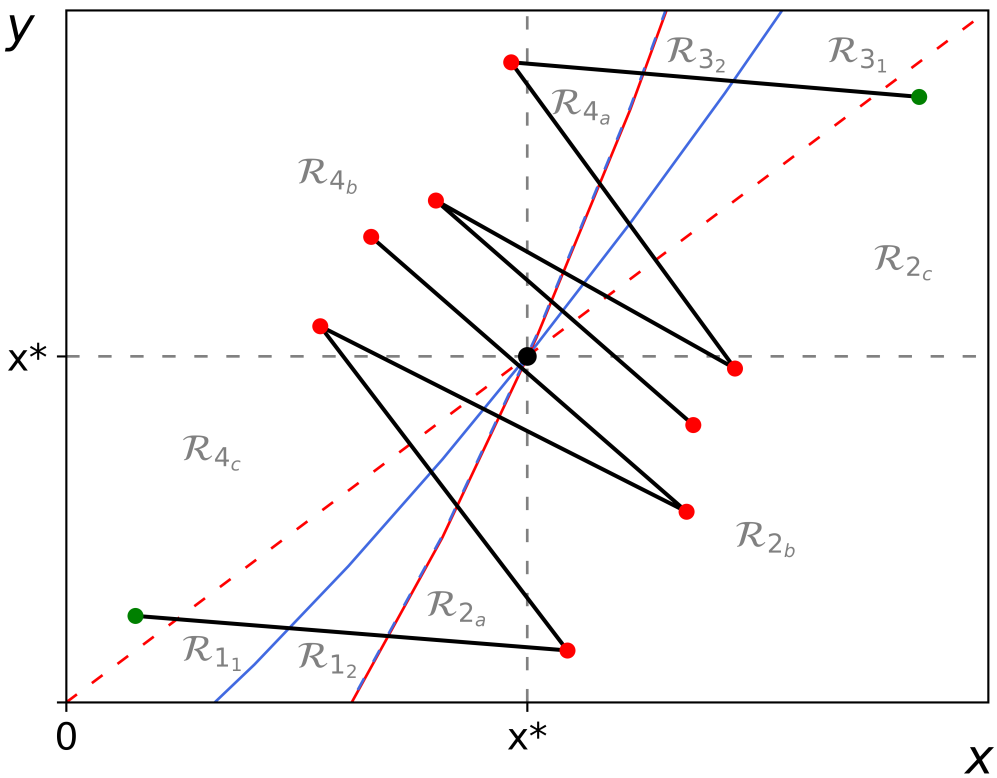

Figure 3.7: The augmented phase portrait of (3.2) with $A = 0.1$ and $\alpha = 0.9$. Depicted are five iterations of two solutions that are eventually oscillating between $\mathcal{R}_{2_b}$ and $\mathcal{R}_{4_b}$. One solution starts at the green dot in $\mathcal{R}_{4_c}$ and the others starts at the green dot in $\mathcal{R}_{2_c}$.

Chapter 4

Analysis of $x_{n+1} = \frac{\alpha+\beta(x_n+x_{n-1})}{A+B(x_n+x_{n-1})}$

We analyze

$$x_{n+1} = \frac{\alpha + \beta(x_n + x_{n-1})}{A + B(x_n + x_{n-1})}, \qquad n \in \mathbb{N}_0 \tag{4.1}$$

for initial conditions $x_0 > 0$, and $x_{-1} > 0$ and parameters $\alpha, \beta, A, B > 0$. Without loss of generality, we simplify (4.1) by dividing the numerator and denominator by β, and applying the change of variables, $\tilde{x}_n := \frac{B}{\beta} x_n$. Then,

$$\tilde{x}_{n+1} = \frac{B}{\beta} x_{n+1} = \frac{\tilde{\alpha} + \tilde{x}_n + \tilde{x}_{n-1}}{\tilde{A} + \tilde{x}_n + \tilde{x}_{n-1}}, \tag{4.2}$$

where $\tilde{\alpha} = \frac{B\alpha}{\beta^2} > 0$ and $\tilde{A} = \frac{A}{\beta} > 0$, with initial conditions $\tilde{x}_0 > 0$ and $\tilde{x}_{-1} > 0$.

To simplify the notation we henceforth "omit the tilde" by referring to $\tilde{\alpha}$ as α, $\tilde{A}$ as A, and $\tilde{x}_n$ as x_n. We can express (4.2) as a planar system by letting $y_n := x_{n-1}$. so that

$$x_{n+1} = \frac{\alpha + x_n + y_n}{A + x_n + y_n} =: F(x_n, y_n), \qquad y_{n+1} = x_n =: G(x_n, y_n), \qquad n \in \mathbb{N}_0 \tag{4.3}$$

with initial conditions $x_0 > 0$ and $y_0 > 0$ and parameters $A > 0$ and $\alpha > 0$.

By Definition 2.1.5, equilibria of (4.3) must satisfy $x^* = F(x^*, x^*)$, or equivalently,

$$x^* = \frac{\alpha + 2x^*}{A + 2x^*}. \tag{4.4}$$

Solving (4.4) for x^* yields the only positive solution,

$$x^* = \frac{-(A-2) + \sqrt{(A-2)^2 + 8\alpha}}{4}. \tag{4.5}$$

Since $\alpha > 0$, (4.3) has a unique positive equilibrium given by (4.3) $E^* = (x^*, x^*)$.

4.1 Case 1: $A > \alpha$

4.1.1 Analysis

In this section, we analyze (4.3) for $A > \alpha > 0$. For this case, our initial analysis using the augmented phase portrait provides the framework that allows us to prove that the unique positive equilibrium is globally asymptotically stable, independent of existing stability theorems.

We first remark that for $A > \alpha$,

$$0 < x_{n+1} = \frac{\alpha + x_n + y_n}{A + x_n + y_n} < 1.$$

This implies that solutions with non-negative and non-trivial initial conditions (that is, $x_0, y_0 > 0$), $(x_n, y_n) \in (0,1) \times (0,1)$ for $n \geq 2$. Hence, it suffices to only consider solutions with initial conditions $(x_0, y_0) \in (0,1) \times (0,1)$.

Furthermore, since $A > \alpha$, the unique positive equilibrium,

$$x^* = \frac{\alpha + 2x^*}{A + 2x^*} < 1.$$

We now discuss the dynamics of (4.3) using the augmented phase portrait. We do not apply classical methods to discuss the local dynamics, nor do we apply global stability theorems, such as Theorems 2.2.12 and 2.2.13, as was done in [1]. We prove global stability of $E^* = (x^*, x^*)$ by using root-curves and their associated nullclines.

By Definition 2.3.2, the x-nullcline is the function $y = h(x)$ that solves $F(x, y) = x$. Solving $F(x, y) = x$ in (4.3) for y yields,

$$\begin{aligned}
\frac{\alpha + x + y}{A + x + y} &= x, \\
\alpha + x + y &= x(A + x) + xy, \\
h(x) := y &= \frac{x^2 + (A-1)x - \alpha}{1 - x}.
\end{aligned}$$

From Remark 2.3.3, the y-nullcline is $k(x) = x$. Nullclines divide the space into regions of component-wise monotonicity (see Chapter 6, Section 6.1.1.)

Proposition 4.1.1. *Consider* (4.3) *and assume $A > \alpha$. Then,*

$$\Delta x_n = x_{n+1} - x_n \begin{cases} > 0, & y_n > h(x_n), \\ = 0, & y_n = h(x_n), \\ < 0, & y_n < h(x_n), \end{cases} \quad \textit{and} \quad \Delta y_n = y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases}$$

By Definition 2.6, any root-curve associated with the x-nullcline is a function $y = r_h(x)$ that satisfies $G(x, y) = h(F(x, y))$. Thus, we solve

$$x = h\left(\frac{\alpha + x + y}{A + x + y}\right),$$

for y. We rearrange using algebraic manipulations to obtain the only positive function,

$$y = r_h(x) := -\frac{(A - 1 + x)}{2} + \sqrt{\frac{4\alpha + 4x + (A - 1 + x)^2}{4}}. \tag{4.6}$$

The details of this calculation are found in Chapter 6, Section 6.1.2.

By Definition 2.3.7, a root-curve associated with the y-nullcline is a function $y = r_k(x)$ that satisfies $G(x, r_k(x)) = k(F(x, r_k(x)))$. This is equivalent to,

$$\begin{aligned}
x = F(x, r_k(x)) &= \frac{\alpha + x + r_k(x)}{A + x + r_k(x)}, \\
x(A + x) + x r_k(x) &= \alpha + x + r_k(x), \\
r_k(x) &= \frac{x^2 + (A-1)x - \alpha}{1 - x} = h(x).
\end{aligned}$$

Proposition 4.1.2. *Consider* (4.3) *and assume $A > \alpha$.*

a) *If $y_n < r_h(x_n)$ then $y_{n+1} > h(x_{n+1})$, or equivalently $\mathcal{L}_h(x_n, y_n) > 0$.*

b) *If $y_n > r_h(x_n)$ then $y_{n+1} < h(x_{n+1})$, or equivalently $\mathcal{L}_h(x_n, y_n) < 0$.*

c) *If $y_n < r_k(x_n)$ then $y_{n+1} > k(x_{n+1})$, or equivalently $\mathcal{L}_k(x_n, y_n) > 0$.*

d) *If $y_n > r_k(x_n)$ then $y_{n+1} < k(x_{n+1})$, or equivalently $\mathcal{L}_k(x_n, y_n) < 0$.*

The proof of Proposition 4.1.2 is in Chapter 6, Section 6.1.3.

Property 4.1.3. *For* (4.3) *with any* $A > 0$ *and* $\alpha > 0$*, any two of* $y = r_k(x)$*,* $y = r_h(x)$*, and* $y = k(x)$ *only intersect at* $E^* = (x^*, x^*)$*.*

The proof can be found in Chapter 6, Section 6.1.4.

Proposition 4.1.4. *Consider* (4.3) *and assume* $A > \alpha$*. Then,*

a) $r_k(x) < k(x) < r_h(x)$ *for* $x \in (0, x^*)$ *and* $r_h(x) < k(x) < r_k(x)$ *for* $x \in (x^*, 1)$*.*

b) $y = k(x)$ *and* $y = r_h(x)$ *are increasing functions for all* $x \in (0, 1)$*.*

c) $y = r_k(x)$ *is increasing for* $\max\{0, 1 - \sqrt{A-\alpha}\} < x < 1$*. In fact, if* $y = r_k(x) > 0$*, then* $y = r_k(x)$ *is increasing.*

d) *The only positive solution of* $r_k(x) = 1$ *is* $x = \sqrt{1 + \alpha + \frac{A^2}{4}} + \frac{A}{2} < 1$*. The only positive solution of* $r_k(x) = 0$ *is* $x = \frac{1}{2}\big(\sqrt{(A-1)^2 - 4\alpha} + 1 - A\big) > 0$*.*

Proof. **(a)** By Property 4.1.3, any pair of the three functions, $h(x)$, $r_h(x)$, and $r_k(x)$, only intersect at the equilibrium, (x^*, x^*). If we show $r_k(x) < k(x) < r_h(x)$ for a specific point $x \in [0, x^*)$, then $r_k(x) < k(x) < r_h(x)$ for all $x \in (0, x^*)$. Similarly, if we show that $r_h(x) < k(x) < r_k(x)$ for a specific point $x \in (x^*, 1]$, then $r_h(x) < k(x) < r_k(x)$ for all $x \in (x^*, 1)$.

Thus we evaluate $h(x), r_h(x)$, and $r_k(x)$ at $x = 0$. Since,

$$\begin{aligned}
r_k(0) &= -\alpha, \\
r_h(0) &= \frac{1 - A + \sqrt{(1-A)^2 + 4\alpha}}{2} > 0, \\
k(0) &= 0,
\end{aligned}$$

$r_k(x) < k(x) < r_h(x)$ for all $x \in (0, x^*)$.

We now evaluate $h(x)$, $r_h(x)$, and $r_k(x)$ at $x = 1$. Since,

$$\begin{aligned}
\lim_{x \to 1^-} r_k(x) &= \lim_{x \to 1^-} \frac{x^2 + (A-1)x - \alpha}{1 - x} \longrightarrow +\infty, \\
r_h(1) &= \frac{-A + \sqrt{(-A)^2 + 4\alpha + 4}}{2} < \frac{-A + \sqrt{A^2 + 4A + 4}}{2} = 1, \\
k(1) &= 1,
\end{aligned}$$

$r_h(x) < k(x) < r_k(x)$ for all $x \in (x^*, 1)$.

(b) Since $k(x) = x$, it is an increasing function. To show that $r_h(x)$ is an increasing function, we want to show that

$$r_h'(x) = \frac{1}{2}\left(\frac{1 + A + x - \sqrt{4(\alpha + x) + (1 - A - x)^2}}{\sqrt{4(\alpha + x) + (1 - A - x)^2}}\right) > 0.$$

Simplifying the numerator and noting that $A > \alpha$

$$\begin{aligned}
(1 + A + x) - \sqrt{4(\alpha + x) + (1 - A - x)^2} &= (1 + A + x) - \sqrt{4(\alpha + x) + 1 - 2(A + x) + (A + x)^2}, \\
&> (1 + A + x) - \sqrt{4(A + x) + 1 - 2(A + x) + (A + x)^2}, \\
&= (1 + A + x) - \sqrt{(1 + A + x)^2} \\
&= 0.
\end{aligned}$$

Thus, $r'_h(x) > 0$ and so $y = r_h(x)$ is increasing.

(c) The critical points of $y = h(x) = r_k(x)$ satisfy

$$r'_k(x) = \frac{A-\alpha}{(1-x)^2} - 1 = 0.$$

That is, $x = 1 \pm \sqrt{A-\alpha}$. Since $x = 1 + \sqrt{A-\alpha} > 1$, we only consider the critical point $x = 1 - \sqrt{A-\alpha}$. Now, for $x \in (0,1)$,

$$r''_k(x) = \frac{2(A-\alpha)}{(1-x)^3} > 0.$$

As such, $y = r_k(x)$ decreases for all $x \in (0, 1-\sqrt{A-\alpha})$ if $1 > \sqrt{A-\alpha}$. Otherwise, $y = r_k(x)$ increases for all $x \in (0,1)$.

If $1 - \sqrt{A-\alpha} > 0$, then $r_k(x)$ is increasing for all $x \in (1-\sqrt{A-\alpha}, 1)$. However, $r_k(0) = -\alpha < 0$. Since $r_k(x)$ is decreasing for $x < 1-\sqrt{A-\alpha}$, $r_k(x) < 0$ for $x \in (0, 1-\sqrt{A-\alpha})$. Thus, $r_k(x)$ is increasing when $r_k(x) > 0$ and $x > 1-\sqrt{A-\alpha}$.

d) If $r_k(x) = 1$, then $x^2 + Ax - (1+\alpha) = 0$ and the only positive solution is,

$$x = \frac{-A + \sqrt{A^2+4+4\alpha}}{2} > 0.$$

Since $A > \alpha$, $x = \frac{-A+\sqrt{A^2+4+4\alpha}}{2} < \frac{-A+\sqrt{A^2+4+4A}}{2} = \frac{-A+\sqrt{(A+2)^2}}{2} = 1$.

If $r_k(x) = 0$, then $x^2 + (A-1)x - \alpha = 0$. Thus the only positive solution is,

$$x = \frac{1-A+\sqrt{(A-1)^2+4\alpha}}{2} > 0.$$

□

Now applying the information from Proposition 4.1.4, we plot generic root-curves in Figure 4.1. We define the following regions.

$$\begin{aligned}
\mathcal{R}_1 &= \{(x,y) \in \mathbb{R}^2 |\, r_k(x) \le y \le k(x) < r_h(x),\ 0 < x < x^*\},\\
\mathcal{R}_{2_1} &= \left\{(x,y) \in \mathbb{R}^2 |\, y < \min\{r_k(x),\ r_h(x)\},\ y < k(x),\ \frac{1-A+\sqrt{(A-1)^2+4\alpha}}{2} < x < 1\right\},\\
\mathcal{R}_{2_2} &= \{(x,y) \in \mathbb{R}^2 |\, r_h(x) \le y < k(x) < r_k(x),\ x^* < x < 1\},\\
\mathcal{R}_3 &= \{(x,y) \in \mathbb{R}^2 |\, r_h(x) < k(x) \le y \le r_k(x),\ x^* < x < 1\},\\
\mathcal{R}_{4_1} &= \left\{(x,y) \in \mathbb{R}^2 |\, y > \max\{r_k(x),\ r_h(x)\},\ y > k(x),\ 0 < x < \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right\},\\
\mathcal{R}_{4_2} &= \{(x,y) \in \mathbb{R}^2 |\, r_k(x) < k(x) < y \le r_h(x),\ 0 < x < x^*\}.
\end{aligned} \tag{4.7}$$

4.1.2 Global Stability

Lemma 4.1.5. *Consider* (4.3) *with* $A > \alpha$.

a) $\mathcal{R}_1$ *is positively invariant. Every solution that enters* $\mathcal{R}_1$ *converges to* $E^* = (x^*, x^*)$.

b) $\mathcal{R}_3$ *is positively invariant. Every solution that enters* $\mathcal{R}_3$ *converges to* $E^* = (x^*, x^*)$.

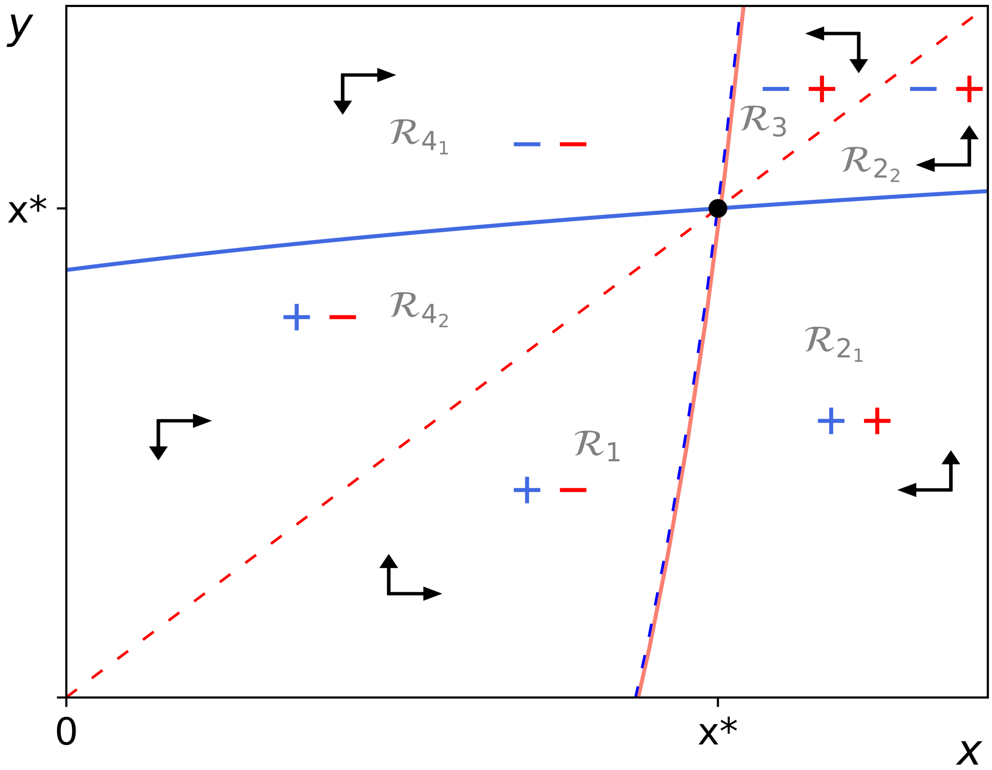

Figure 4.1: A generic augmented phase portrait of (4.3) with $A > \alpha$. The x-nullcline, representing $y = h(x)$ is in dashed blue. The y-nullcline, $k(x) = x$ is in dashed red. The associated root-curves, $y = r_h(x)$ and $y = r_k(x)$ are the curves in solid blue and red, respectively. Recall that $h(x) = r_k(x)$. The vertical and horizontal arrows represent the regions of component-wise monotonicity. For example, for a point $(x_n, y_n) \in \mathcal{R}_{4_1}$, the right arrow in $\mathcal{R}_{4_1}$ indicates that $\Delta x_n > 0$ and the downwards arrow indicates that $\Delta y_n < 0$. Finally, regions containing a '+' ('−') indicate that the next-iterate lies found above (below) the same coloured nullcline. For example, a blue plus sign in $\mathcal{R}_{2_1}$ indicates that for a point $(x_n, y_n) \in \mathcal{R}_{2_1}$, the next-iterate, (x_{n+1}, y_{n+1}), will lie above the blue nullcline. A red minus sign in $\mathcal{R}_{4_1}$ indicates that for $(x_n, y_n) \in \mathcal{R}_{4_1}$, the next iterate, (x_{n+1}, y_{n+1}) will lie below the red nullcline.

Proof. a) If $(x_n, y_n) \in \mathcal{R}_1$, then by the definition of $\mathcal{R}_1$ (4.7), $r_k(x_n) \leq y_n \leq k(x) < r_h(x_n)$ and $x_n \in (0, x^*)$. Hence, by Proposition 4.1.2, $h(x_{n+1}) = r_k(x_{n+1}) < y_{n+1} \leq k(x_{n+1})$. Thus, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_1$ and $\mathcal{R}_1$ is a positively invariant region.

Since $(x_n, y_n) \in \mathcal{R}_1$, by Proposition 4.1.1, $\Delta x_n > 0$ and $\Delta y_n > 0$, so the solution is increasing within $\mathcal{R}_1$. From the definition of the regions (4.7), $y_n \leq k(x_n) = x_n < x^*$, so the solution increases monotonically and converges to $E^* = (x^*, x^*)$.

b) If $(x_n, y_n) \in \mathcal{R}_3$, then by the definition of $\mathcal{R}_3$ 4.7, $r_h(x_n) < k(x_n) \leq y_n \leq r_k(x_n)$ and $x_n \in (x^*, 1)$. By Proposition 4.1.2, $k(x_{n+1}) \leq y_{n+1} < h(x_{n+1}) = r_k(x_{n+1})$. Thus $(x_{n+1}, y_{n+1}) \in \mathcal{R}_3$ and $\mathcal{R}_3$ is a positively invariant region.

Since $(x_n, y_n) \in \mathcal{R}_3$, by Proposition 4.1.1, $\Delta x_n < 0$ and $\Delta y_n < 0$. Additionally, from the definition of the regions (4.7), $y_n \geq k(x_n) = x_n > x^*$. Thus the solution decreases monotonically and therefore converges to $E^* = (x^*, x^*)$. □

Lemma 4.1.6. *Consider a solution $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$ of (4.3) with $A > \alpha$. Then one of following is always true:*

a) the solution eventually enters $\mathcal{R}_1$ and converges to the positive equilibrium, $E^* = (x^*, x^*)$.

b) the solution eventually enters $\mathcal{R}_3$ and converges to the positive equilibrium, $E^* = (x^*, x^*)$.

c) the solution oscillates between $\mathcal{R}_{2_1}$ and $\mathcal{R}_{4_1}$, indefinitely.

We first include a figure of all the possible paths a solution can take to provide visual intuition to the reader.

a) $\mathcal{R}_{2_1} \to \mathcal{R}_{4_2} \to \mathcal{R}_1 \to \cdots \to \mathcal{R}_1$

b) $\mathcal{R}_{4_1} \to \mathcal{R}_{2_2} \to \mathcal{R}_3 \to \cdots \to \mathcal{R}_3$

c) $\mathcal{R}_{2_1} \rightleftharpoons \mathcal{R}_{4_1}$ (oscillates).

Figure 4.2: The possible paths a solution of (4.3) can take.

Proof. Consider $(x_n, y_n) \in \mathcal{R}_{2_1}$. Then by the definition of $\mathcal{R}_{2_1}$ (4.7), $y_n < \min\{r_h(x_n), r_k(x_n)\}$ for all $x_n \in \left(\frac{1-A+\sqrt{(A-1)^2+4\alpha}}{2}, 1\right)$. By Proposition 4.1.2, $y_{n+1} > k(x_{n+1})$ and $y_{n+1} > h(x_{n+1}) = r_k(x_{n+1})$, and so $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_2} \cup \mathcal{R}_{4_1}$. We begin writing paths a) and c) from Figure 4.2, respectively, as

$$\begin{aligned} &\text{a) } \mathcal{R}_{2_1} \to \mathcal{R}_{4_2}, \\ &\text{c) } \mathcal{R}_{2_1} \to \mathcal{R}_{4_1}. \end{aligned} \tag{4.8}$$

Letting $(x_n, y_n) \in \mathcal{R}_{4_2}$, the by the definition of the regions (4.7), $r_k(x_n) < k(x_n) < y_n \leq r_h(x_n)$ and $x_n \in (0, x^*)$. Thus, by Proposition 4.1.2, $h(x_{n+1}) = r_k(x_{n+1}) \leq y_{n+1} < k(x_{n+1})$. As such, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_1$. Recall from Lemma 4.1.5 that solutions stay in $\mathcal{R}_1$ indefinitely and converge to the equilibrium. Thus, using (4.8) we complete path a) from Figure 4.2 as

$$\text{a) } \mathcal{R}_{2_1} \to \mathcal{R}_{4_2} \to \mathcal{R}_1 \to \cdots \to \mathcal{R}_1.$$

If $(x_n, y_n) \in \mathcal{R}_{4_1}$, then by definition (4.7), $y_n > \max\{r_h(x_n), r_k(x_n)\}$ for $x_n \in \left(0, \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right)$. Thus by Proposition 4.1.2, $y_{n+1} < h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} < k(x_{n+1})$, and so $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_1} \cup \mathcal{R}_{2_2}$. Using (4.8), solutions can oscillate between $\mathcal{R}_{2_1}$ and $\mathcal{R}_{4_1}$ indefinitely. The start of path b) and path c) from Figure 4.2, can be written, respectively as,

$$\begin{aligned} &\text{b) } \mathcal{R}_{4_1} \to \mathcal{R}_{2_2}, \\ &\text{c) } \mathcal{R}_{2_1} \rightleftharpoons \mathcal{R}_{4_1}. \end{aligned} \tag{4.9}$$

Let $(x_n, y_n) \in \mathcal{R}_{2_2}$. By definition of $\mathcal{R}_{2_2}$ (4.7), $r_h(x_n) \leq y_n < k(x_n) < r_k(x_n)$ and $x_n \in (x^*, 1)$. From Proposition 4.1.2, $k(x_{n+1}) < y_{n+1} \leq h(x_{n+1}) = r_k(x_{n+1})$ and so $(x_{n+1}, y_{n+1}) \in$

$\mathcal{R}_3$. From Lemma 4.1.5, solutions that enter $\mathcal{R}_3$, stay in $\mathcal{R}_3$ for all time and converge to the equilibrium. Using (4.9), we complete path d) from Figure 4.2 as,

$$\text{b) } \mathcal{R}_{4_1} \to \mathcal{R}_{2_2} \to \mathcal{R}_3 \cdots \mathcal{R}_3.$$

Thus, solutions either oscillate between $\mathcal{R}_{2_1}$ and $\mathcal{R}_{4_1}$ indefinitely or eventually enter $\mathcal{R}_3$ or $\mathcal{R}_1$ and converge to the equilibrium $E^* = (x^*, x^*)$. □

By Lemma 4.1.6, a solution in $\mathcal{R}_{4_1}$ could enter $\mathcal{R}_{2_1}$ or $\mathcal{R}_{2_2}$. We want to define the regions where the solution will enter $\mathcal{R}_{2_1}$ versus $\mathcal{R}_{2_2}$. In the generic augmented phase portrait given in Figure 4.1 and from the definition of the regions (4.7), we see that $\mathcal{R}_{2_1}$ and $\mathcal{R}_{2_2}$ are divided by $y = r_h(x)$. From Definition 2.3.13, to define the pre-image of the root-curve, $y = r_h(x)$, we find a function $y = Q(x)$ that satisfies $G(x, y) = r_h(F(x, y))$. Thus we solve,

$$x = \frac{1 - A - y + \sqrt{(1 - A - y)^2 + 4(\alpha + y)}}{2}$$

for y. Rearranging (see Chapter 6 Section 6.1.5), we obtain,

$$Q(x) := y = \frac{h(x)(A + x) - (\alpha + x)}{1 - h(x)}. \tag{4.10}$$

Lemma 4.1.7. *Consider* (4.3) *with $A > \alpha$. If $x_n < \frac{-A+\sqrt{A^2+4+4\alpha}}{2}$, then*

$$y_n - Q(x_n) \begin{cases} > 0, & y_{n+1} < r_h(x_{n+1}), \\ = 0, & y_{n+1} = r_h(x_{n+1}), \\ < 0, & y_{n+1} > r_h(x_{n+1}), \end{cases}$$

where $Q(x)$ is defined in (4.10) *and $r_h(x)$ in* (4.6).

The proof of Lemma 4.1.7 can be found in Chapter 6, Section 6.1.6.

Property 4.1.8. *For* (4.3) *with any $A > 0$ and $\alpha > 0$, $y = Q(x)$ intersects $y = r_k(x)$ only at $E^* = (x^*, x^*)$.*

Proof. If $Q(\bar{x}) = r_k(\bar{x})$ then $\frac{h(\bar{x})(A+\bar{x})-(\alpha+\bar{x})}{1-h(\bar{x})} = h(\bar{x})$, and equivalently,

$$h(\bar{x})^2 + h(\bar{x})(A - 1 + \bar{x}) - (\alpha + \bar{x}) = 0.$$

It follows that,

$$h(\bar{x}) = \frac{-(A - 1 + \bar{x}) + \sqrt{(A - 1 + \bar{x})^2 + 4(\alpha + \bar{x})}}{2} = r_h(\bar{x}).$$

From Lemma 2.3.9, if $h(\bar{x}) = r_h(\bar{x})$ then $\bar{x} = x^*$. Since $x^* = h(x^*) = r_k(x^*)$, $y = Q(x)$ only intersects $y = r_k(x)$ at $E^* = (x^*, x^*)$. □

Lemma 4.1.9. *Consider* (4.3) *with $A > \alpha$. Then $Q(x) < r_k(x)$ for $x \in \left(\frac{1-A+\sqrt{(A-1)^2+4\alpha}}{2}, x^*\right)$ and $r_k(x) < Q(x)$ for $x \in \left(x^*, \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right)$.*

Proof. **a)** From Property 4.1.8, $Q(x)$ and $r_k(x)$ intersect only at $E^* = (x^*, x^*)$. Thus if we can show that $r_k'(x^*) < Q'(x^*)$, then we know that $r_k(x) < Q(x)$ for all $x \in \left(x^*, \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right)$. Since by Lemma 2.3.10, $r_k(x) = h(x)$, we have,

$$\begin{aligned}
Q'(x) &= \frac{(h'(x)(A+x)+h(x)-1)(1-h(x)) + (h(x)(A+x)-(\alpha+x))h'(x)}{(1-h(x))^2}, \\
&= \frac{(h'(x)(A+x)+h(x)-1)(1-h(x))}{(1-h(x))^2} + \frac{(h(x)(A+x)-(\alpha+x))h'(x)}{(1-h(x))^2}, \\
&= \frac{h'(x)(A+x)+h(x)-1}{(1-h(x))} + \frac{Q(x)h'(x)}{(1-h(x))}, \text{ and,} \\
h'(x) &= \frac{(2x+A-1)(1-x)+(x^2+(A-1)x-\alpha)}{(1-x)^2}.
\end{aligned}$$

Now we evaluate $Q'(x)$ and $h'(x)$ at $x = x^*$. Since $h(x^*) = Q(x^*) = x^*$,

$$\begin{aligned}
Q'(x^*) &= \frac{h'(x^*)(A+x^*)+h(x^*)-1}{(1-h(x^*))} + \frac{Q(x^*)h'(x^*)}{(1-h(x^*))}, \\
&= \frac{h'(x^*)(A+x^*)+x^*-1}{(1-x^*)} + \frac{x^*h'(x^*)}{(1-x^*)}, \\
&= \frac{h'(x^*)(A+2x^*)+x^*-1}{(1-x^*)}, \text{ and,} \\
h'(x^*) &= \frac{A+3x^*-1}{1-x^*}.
\end{aligned}$$

From Remark 2.3.4 $y = h(x)$ and $y = k(x)$ intersect at $E^* = (x^*, x^*)$ and from Proposition 4.1.4, $k(x) < h(x)$ for $x \in \left(x^*, \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right) \subset (x^*, 1)$. Thus, $k'(x^*) < h'(x^*)$. Furthermore, $k(x) = x$ and so $k'(x) = 1$. Hence, $1 < h'(x^*)$. Using this, we show,

$$h'(x^*) = \frac{A+2x^*+x^*-1}{1-x^*} < \frac{h'(x^*)(A+2x^*)+x^*-1}{1-x^*} = Q'(x^*).$$

Thus, $h(x) < Q(x)$ for all $x \in \left(x^*, \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right)$. Similarly, we can show that $Q(x) < h(x)$ for all $x \in \left(\frac{1-A+\sqrt{(A-1)^2+4\alpha}}{2}, x^*\right)$. □

By Definition 2.3.12, the pre-image of the line $x = x^*$, with y arbitrary, is a function $y = S(x)$ that solves $F(x,y) = \frac{\alpha+x+y}{A+x+y} = x^*$. Rearranging, we get $\alpha + x + y = x^*(A+x) + yx^*$. Solving for y, we obtain,

$$\begin{aligned}
y(1-x^*) &= x(x^*-1) + Ax^* - \alpha, \\
S(x) := y &= -x + \frac{Ax^*-\alpha}{1-x^*}.
\end{aligned}$$

Lemma 4.1.10. *Consider* (4.3) *with* $A > \alpha$. *Then,*

$$x_{n+1} - x^* \begin{cases} > 0, & y_n > S(x_n), \\ = 0, & y_n = S(x_n), \\ < 0, & y_n < S(x_n). \end{cases}$$

Proof. Assume that $y_n > S(x_n)$. Then $y_n > -x_n + \frac{x^*A-\alpha}{1-x^*}$ and so,

$$\begin{aligned}(y_n + x_n)(1 - x^*) &> x^*A - \alpha,\\ y_n + x_n - (y_n + x_n)x^* &> x^*A - \alpha,\\ \alpha + x_n + y_n &> (A + y_n + x_n)x^*,\\ x_{n+1} &> x^*.\end{aligned}$$

The proof to show that $y_n < S(x_n)$ implies, $x_{n+1} - x^* < 0$ is similar. By Definition 2.3.12, $y_n = S(x_n)$ implies that $x_{n+1} - x^* = 0$. $\square$

Lemma 4.1.11. *Consider* (4.3) *with* $A > \alpha$.

a) If $S(x) = x^*$, *then* $x = x^*$, *where* x^* *is given by* (4.5).

b) If $S(\bar{x}) = r_h(\bar{x})$ *then* $\bar{x} = x^*$, *where* x^* *is given by* (4.5). *The two functions pass through* $E^* = (x^*, x^*)$.

Proof. a) First note that since $x^* = \frac{\alpha+2x^*}{A+2x^*}$, $1 - x^* = \frac{A-\alpha}{A+2x^*}$ and so we can rewrite $S(x)$ as,

$$\begin{aligned}S(x) = -x + \frac{Ax^* - \alpha}{1 - x^*} &= -x + \left(\frac{A(\alpha + 2x^*) - \alpha(A + 2x^*)}{A + 2x^*}\right)\left(\frac{A + 2x^*}{A - \alpha}\right)\\ &= -x + \frac{A\alpha + 2Ax^* - \alpha A - 2\alpha x^*}{A - \alpha} = -x + \frac{2x^*(A - \alpha)}{A - \alpha} = -x + 2x^*.\end{aligned}$$

Thus, If $S(x) = x^*$, then $-x + 2x^* = x^*$, and so $x = x^*$.

b) Assume that $S(\bar{x}) = r_h(\bar{x})$. By definition $F(\bar{x}, S(\bar{x})) = x^*$ and $G(\bar{x}, r_h(\bar{x})) = h(F(\bar{x}, r_h(\bar{x})))$ are satisfied. Since $G(\bar{x}, y) = \bar{x}$ and $S(\bar{x}) = r_h(\bar{x})$,

$$\bar{x} = G(\bar{x}, r_h(\bar{x})) = h(F(\bar{x}, r_h(\bar{x}))) = h(F(\bar{x}, S(\bar{x}))) = h(x^*) = x^*.$$

Thus, $\bar{x} = x^*$. Since $S(x^*) = x^*$, $S(x) = r_h(x)$ only at $E^* = (x^*, x^*)$. $\square$

Lemma 4.1.12. *Consider* (4.3) *with* $A > \alpha$. *Then:*

a) $S(x) > r_h(x) > k(x) > r_k(x)$ *for* $x \in (0, x^*)$ *and* $S(x) < r_h(x) < k(x) < r_k(x)$ *for* $x \in (x^*, 1)$.

b) $S(x) > x^*$ *for* $x \in (0, x^*)$ *and* $S(x) < x^*$ *for* $x \in (x^*, 1)$.

Proof. a) Since $y = S(x)$ is decreasing and $y = r_h(x)$ is increasing and by Lemma 4.1.11 they only intersect at (x^*, x^*), $S(x) > r_h(x)$ for $x \in (0, x^*)$ and $S(x) < r_h(x)$ for $x \in (x^*, 1)$. By Proposition 4.1.4, $S(x) > r_h(x) > k(x) > r_k(x)$ for $x \in (0, x^*)$ and $S(x) < r_h(x) < k(x) < r_k(x)$ for $x \in (x^*, 1)$.

b) Since $S(x)$ is strictly decreasing and $S(x^*) = x^*$, $S(x) > x^*$ for $x \in (0, x^*)$ and $S(x) < x^*$ for $x \in (x^*, 1)$. $\square$

Using the properties discussed in Lemmas 4.1.9 and 4.1.12 we are able to include generic curves for $y = Q(x)$ and $y = S(x)$ in the augmented phase portrait. We portray this in Figure 4.3. We need to redefine regions $\mathcal{R}_{4_1}$ and $\mathcal{R}_{2_1}$. We separate them into three regions, bounded

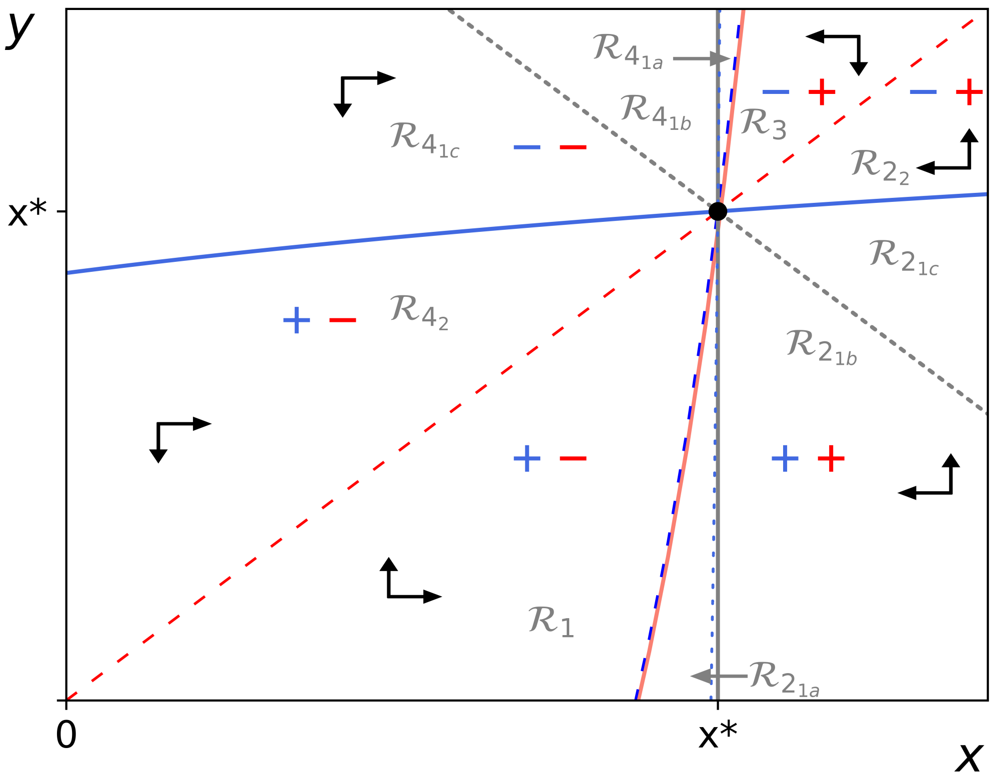

Figure 4.3: The augmented phase portrait of (4.3) with $A > \alpha$, updated to include $y = Q(x)$, as the dotted blue curve, $y = S(x)$, as the dashed grey line and $x = x^*$ as the solid grey line. The regions $\mathcal{R}_{2_1}$ and $\mathcal{R}_{4_1}$ are separated into three regions each, $\mathcal{R}_{2_{1a}}$, $\mathcal{R}_{2_{1b}}$, and $\mathcal{R}_{2_{1c}}$, and $\mathcal{R}_{4_{1a}}$, $\mathcal{R}_{4_{1b}}$, and $\mathcal{R}_{4_{1c}}$, respectively.

by adding $x = x^*$ and $y = S(x)$. We define $\mathcal{R}_{4_{1a}}$, $\mathcal{R}_{4_{1b}}$, $\mathcal{R}_{4_{1c}}$, $\mathcal{R}_{2_{1a}}$, $\mathcal{R}_{2_{1b}}$, and $\mathcal{R}_{2_{1c}}$ as follows:

$$\begin{aligned}
\mathcal{R}_{4_{1a}} &= \mathcal{R}_{4_1} \cap \{(x,y) : y > S(x),\, x \geq x^*\},\\
\mathcal{R}_{4_{1b}} &= \mathcal{R}_{4_1} \cap \{(x,y) : y > S(x),\, x < x^*\},\\
\mathcal{R}_{4_{1c}} &= \mathcal{R}_{4_1} \cap \{(x,y) : y \leq S(x),\, x < x^*\},\\
\mathcal{R}_{2_{1a}} &= \mathcal{R}_{2_1} \cap \{(x,y) : y < S(x),\, x \leq x^*\},\\
\mathcal{R}_{2_{1b}} &= \mathcal{R}_{2_1} \cap \{(x,y) : y < S(x),\, x > x^*\},\\
\mathcal{R}_{2_{1c}} &= \mathcal{R}_{2_1} \cap \{(x,y) : y \geq S(x),\, x > x^*\}.
\end{aligned} \tag{4.11}$$

Consider a solution $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$ that oscillates between $\mathcal{R}_{2_1}$ and $\mathcal{R}_{4_1}$ indefinitely. Without loss of generality, we let $(x_{2n}, y_{2n}) \in \mathcal{R}_{4_1}$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_{2_1}$ for all $n \in \mathbb{N}_0$. We can show that both subsequences, $\{(x_{2n}, y_{2n})\}_{n\in\mathbb{N}_0}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\in\mathbb{N}_0}$, converge to the equilibrium.

Lemma 4.1.13. *i) If $\{(x_{2n}, y_{2n})\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{4_1}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\in\mathbb{N}} \in \mathcal{R}_{2_1}$, then for all $n \in \mathbb{N}_0$,*

a) $x_{2n} < x_{2n+2}$,
b) $y_{2n} > y_{2n+2}$,
c) $x_{2n+1} > x_{2n+3}$,
d) $y_{2n+1} < y_{2n+3}$.

ii) If $\{(x_{2n+1}, y_{2n+1})\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{4_1}$ and $\{(x_{2n}, y_{2n})\}_{n\in\mathbb{N}} \in \mathcal{R}_{2_1}$, then for all $n \in \mathbb{N}_0$,

a) $x_{2n+1} < x_{2n+3}$,
b) $y_{2n+1} > y_{2n+3}$,
c) $x_{2n} > x_{2n+2}$,
d) $y_{2n} < y_{2n+2}$.

Proof. For this proof, we assume $\{(x_{2n}, y_{2n})\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{4_1}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\in\mathbb{N}} \in \mathcal{R}_{2_1}$ since the equivalent statement in ii) is a trivial change in notation.

a) Using a contrapositive argument, we assume $x_{2n} \geq x_{2n+2}$. This is equivalent to

$$\begin{aligned}
y_{2n+1} &\geq \frac{\alpha + y_{2n+1} + x_{2n+1}}{A + y_{2n+1} + x_{2n+1}},\\
y_{2n+1}A + y_{2n+1}^2 + y_{2n+1}x_{2n+1} &\geq \alpha + y_{2n+1} + x_{2n+1},\\
y_{2n+1}^2 + y_{2n+1}(A + x_{2n+1} - 1) &\geq (\alpha + x_{2n+1}),\\
\left(y_{2n+1} + \frac{(A + x_{2n+1} - 1)}{2}\right)^2 &\geq \frac{(A + x_{2n+1} - 1)^2}{4} + (\alpha + x_{2n+1}),\\
y_{2n+1} &\geq -\frac{(A + x_{2n+1} - 1)}{2} + \sqrt{\frac{(A + x_{2n+1} - 1)^2}{4} + (\alpha + x_{2n+1})},\\
y_{2n+1} &\geq r_h(x_{2n+1}).
\end{aligned}$$

By the definition of $\mathcal{R}_{2_1}$ (4.7), this implies that $(x_{2n+1}, y_{2n+1}) \notin \mathcal{R}_{2_1}$.

b) Using a contrapositive argument, we let $y_{2n} \leq y_{2n+2}$. This is equivalent to

$$\begin{aligned}
y_{2n} &\leq x_{2n+1} = \frac{\alpha + x_{2n} + y_{2n}}{A + x_{2n} + y_{2n}}, \\
y_{2n}(A + x_{2n} + y_{2n}) &\leq \alpha + x_{2n} + y_{2n}, \\
y_{2n}A + y_{2n}x_{2n} + y_{2n}^2 &\leq \alpha + x_{2n} + y_{2n}, \\
y_{2n}^2 + y_{2n}(A + x_{2n} - 1) &\leq \alpha + x_{2n}, \\
(y_{2n} + \frac{(A + x_{2n} - 1)}{2})^2 &\leq \frac{(A + x_{2n} - 1)^4}{4} + \alpha + x_{2n}, \\
y_{2n} &\leq -\frac{(A + x_{2n} - 1)}{2} + \sqrt{\frac{(A + x_{2n} - 1)^4}{4} + \alpha + x_{2n}}\,, \\
y_{2n} &\leq r_h(x_{2n}).
\end{aligned}$$

By the definition of $\mathcal{R}_{4_1}$, $(x_{2n}, y_{2n}) \notin \mathcal{R}_{4_1}$.

The proofs for c) and d) are similar to a) and b), respectively. □

We now explain the following notation for a solution $\{(x_n, y_n)\}_{n\in\mathbb{N}_0}$:

- If a solution oscillates between two regions for a finite number of time steps, $(x_{2n}, y_{2n}) \in \mathcal{R}_i$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_j$ for all $n = 0, 1, \ldots, N-1 < \infty$, where $(x_{2N}, y_{2N}) \in \mathcal{R}_j$, and $(x_{2N+1}, y_{2N+1}) \in \mathcal{R}_k$, we write, $\mathcal{R}_i \overset{**}{\rightleftharpoons} \mathcal{R}_j \to \mathcal{R}_k$.

Lemma 4.1.14. *Consider a solution of* (4.3) *with $A > \alpha$. Then the following holds.*

i) If $(x_n, y_n) \in \mathcal{R}_{2_{1a}}$, then either $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_2}$ or $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_{1c}}$ and the solution oscillates between $\mathcal{R}_{2_{1a}}$ and $\mathcal{R}_{4_{1c}}$ for a finite number of steps and then eventually enters $\mathcal{R}_{4_2}$ from $\mathcal{R}_{2_{1a}}$, i.e., $(x_{2n}, y_{2n}) \in \mathcal{R}_{2_{1a}}$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_{4_{1c}}$ for all $n = 0, 1 \ldots, N-1 < \infty$ with $(x_{2N}, y_{2N}) \in \mathcal{R}_{2_{1a}}$ and $(x_{2N+1}, y_{2N+1}) \in \mathcal{R}_{4_2}$. Solutions in $\mathcal{R}_{2_{1a}}$ and $\mathcal{R}_{4_{1c}}$ converge to $E^* = (x^*, x^*)$.

ii) If $(x_n, y_n) \in \mathcal{R}_{4_{1a}}$, then either $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_2}$ or $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_{1c}}$ and the solution oscillates between $\mathcal{R}_{4_{1a}}$ and $\mathcal{R}_{2_{1c}}$ for a finite number of steps and eventually enters $\mathcal{R}_{2_2}$ from $\mathcal{R}_{4_{1a}}$, i.e., $(x_{2n}, y_{2n}) \in \mathcal{R}_{4_{1a}}$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_{2_{1c}}$ for all $n = 0, 1 \ldots, N-1 < \infty$ with $(x_{2N}, y_{2N}) \in \mathcal{R}_{4_{1a}}$ and $(x_{2N+1}, y_{2N+1}) \in \mathcal{R}_{2_2}$. Thus, solutions in $\mathcal{R}_{4_{1a}}$ and $\mathcal{R}_{2_{1c}}$ converge to $E^* = (x^*, x^*)$.

We first include a figure that will help the reader follow the proof.

a) $\mathcal{R}_{4_{1c}} \overset{**}{\rightleftharpoons} \mathcal{R}_{2_{1a}} \to \mathcal{R}_{4_2}$.

b) $\mathcal{R}_{2_{1c}} \overset{**}{\rightleftharpoons} \mathcal{R}_{4_{1a}} \to \mathcal{R}_{2_2}$.

Figure 4.4: The behaviour of solutions in $\mathcal{R}_{2_{1a}}$ and $\mathcal{R}_{4_{1a}}$. Path a) corresponds to i) from Lemma 4.1.14 and path b) corresponds to ii) Lemma 4.1.14.

Proof. i) If $(x_n, y_n) \in \mathcal{R}_{2_{1a}}$, then since $\mathcal{R}_{2_{1a}} \subset \mathcal{R}_{2_1}$, by Lemma 4.1.6, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_1} \cup \mathcal{R}_{4_2}$. By the definition of $\mathcal{R}_{2_{1a}}$, $x_n \in \big(\frac{1-A+\sqrt{(A-1)^2+4\alpha}}{2}, x^*\big]$ and by the definition of $\mathcal{R}_{2_1}$ and Proposition 4.1.4, $y_n < r_k(x_n) < r_h(x_n)$. Since by Lemma 4.1.9, $Q(x_n) < r_k(x_n)$, we consider two cases: $Q(x_n) \leq y_n < r_k(x_n)$ and $y_n < Q(x_n) < r_k(x_n)$.

First assume that $Q(x_n) \leq y_n < r_k(x_n)$. By Lemma 4.1.7, $y_{n+1} \leq r_h(x_{n+1})$ and so $(x_{n+1}, y_{n+1}) \notin \mathcal{R}_{4_1}$. Thus $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_2}$.

Now assume that $y_n < Q(x_n) < r_k(x_n)$. By Lemma 4.1.7, $y_{n+1} > r_h(x_{n+1})$ and so $(x_{n+1}, y_{n+1}) \notin \mathcal{R}_{4_2}$. Thus, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_1}$. Recall that by the definition of $\mathcal{R}_{2_{1a}}$, $x_n \leq x^*$. This is equivalent to $y_{n+1} \leq x^*$. Thus, by Lemma 4.1.12 b), $y_{n+1} \leq x^* < S(x_{n+1})$ and by definition of the new regions (4.11) $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_{1c}}$.

If $(x_n, y_n) \in \mathcal{R}_{4_{1c}} \subset \mathcal{R}_{4_1}$, then since $\mathcal{R}_{4_{1c}} \subset \mathcal{R}_{4_1}$, from Lemma 4.1.6, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_2} \cup \mathcal{R}_{2_1}$. By definition of $\mathcal{R}_{4_{1c}}$ (4.11), $x_n < x^*$ and $y_n \leq S(x_n)$. Thus, by Lemma 4.1.10, $x_{n+1} \leq x^*$ and so $(x_{n+1}, y_{n+1}) \notin \mathcal{R}_{2_2}$. Additionally, from the definition of the sub-regions of $\mathcal{R}_{2_1}$ (4.11), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_{1a}}$.

Thus, without loss of generality, suppose towards a contradiction that $(x_{2n}, y_{2n}) \in \mathcal{R}_{2_{1a}}$ such that $y_{2n} < Q(x_{2n}) < r_k(x_{2n})$ for all $n \in \mathbb{N}$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_{4_{1c}}$. By Lemma 4.1.13, $y_{2n} < y_{2n+2}$ and $x_{2n} > x_{2n+2}$ and so there exists a point $(\bar{x}, \bar{y}) \in \mathcal{R}_{2_{1a}}$ with $\bar{y} < Q(\bar{x})$ where $\lim_{n\to\infty} y_{2n} = \bar{y}$ and $\lim_{n\to\infty} x_{2n} = \bar{x}$. This implies that there exists a point $(\tilde{x}, \tilde{y}) \in \mathcal{R}_{4_{1c}}$ such that $\lim_{n\to\infty} y_{2n+1} = \tilde{y}$ and $\lim_{n\to\infty} x_{2n+1} = \tilde{x}$. Thus the solution will converge to a prime-period 2 orbit. However, this is not possible since by Lemma 4.1.13, $y_{2n} < y_{2n+2}$ and $x_{2n} > x_{2n+2}$ for any point in $\mathcal{R}_{2_{1a}}$, including $(\bar{x}, \bar{y}) \in \mathcal{R}_{2_{1a}}$. Thus, there must exist N such that $Q(x_{2N}) \leq y_{2N} < r_k(x_{2N})$. Thus, $(x_{2N+1}, y_{2N+1}) \in \mathcal{R}_{4_2}$. From Lemma 4.1.6, the solution converges to $E^* = (x^*, x^*)$.

Thus the solution can only oscillate between $\mathcal{R}_{2_{1a}}$ and $\mathcal{R}_{4_{1c}}$ for a finite number of steps before entering $\mathcal{R}_{4_2}$ from $\mathcal{R}_{2_{1a}}$. From Figure 4.4 we write path a) as

$$\text{a) } \mathcal{R}_{4_{1c}} \overset{**}{\rightleftharpoons} \mathcal{R}_{2_{1a}} \to \mathcal{R}_{4_2}.$$

ii) If $(x_n, y_n) \in \mathcal{R}_{4_{1a}}$, then since $\mathcal{R}_{4_{1a}} \subset \mathcal{R}_{4_1}$, from Lemma 4.1.6, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_1} \cup \mathcal{R}_{2_2}$. By the definition of $\mathcal{R}_{4_{1a}}$ (4.11), $x \in \left(x^*, \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right)$. By definition of $\mathcal{R}_{4_1}$, $y_n > r_k(x_n) > r_h(x_n)$. By Lemma 4.1.9, $Q(x) > r_k(x)$ for $x \in \left(x^*, \frac{-A+\sqrt{A^2+4+4\alpha}}{2}\right)$. Thus for $(x_n, y_n) \in \mathcal{R}_{4_{1a}}$, there are two cases: $r_k(x_n) < y_n \leq Q(x_n)$ and $r_k(x_n) < Q(x_n) < y_n$.

First consider $r_k(x_n) < y_n \leq Q(x_n)$. By Lemma 4.1.7, $y_{n+1} \geq r_h(x_{n+1})$. Thus by definition of the regions (4.7), $(x_{n+1}, y_{n+1}) \notin \mathcal{R}_{2_1}$ and so $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_2}$.

If $(x_n, y_n) \in \mathcal{R}_{4_{1a}}$ such that $r_k(x_n) < Q(x_n) < y_n$, then by Lemma 4.1.7, $y_{n+1} < r_h(x_{n+1})$. Thus, $(x_{n+1}, y_{n+1}) \notin \mathcal{R}_{2_2}$ and so $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_1}$. By the definition of $\mathcal{R}_{4_{1a}}$, $x_n \geq x^*$ and hence, $y_{n+1} \geq x^*$. Thus by Lemma 4.1.12 b), $y_{n+1} \geq x^* > S(x_{n+1})$. Thus, by the definition of the sub-regions of $\mathcal{R}_{2_1}$ (4.11), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_{1c}}$.

If $(x_n, y_n) \in \mathcal{R}_{2_{1c}}$, then since $\mathcal{R}_{2_{1c}} \subset \mathcal{R}_{2_1}$, from Lemma 4.1.6, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_2} \cup \mathcal{R}_{4_1}$. By the definition of $\mathcal{R}_{2_{1c}}$ (4.11), $y_n \geq S(x_n)$ and so by Lemma 4.1.10, $x_{n+1} \geq x^*$. Thus by the definition of the regions (4.11), $(x_{n+1}, y_{n+1}) \notin \mathcal{R}_{4_2}$ and so by the definition of the sub-regions of $\mathcal{R}_{4_1}$ (4.11), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_{1a}}$.

Thus, without loss of generality, suppose or the sake of contradiction that $(x_{2n}, y_{2n}) \in \mathcal{R}_{4_{1a}}$ with $y_{2n} > Q(x_{2n}) > r_k(x_{2n})$ and $(x_{2n+1}, y_{2n+1}) \in \mathcal{R}_{2_{1c}}$. By Lemma 4.1.13, $y_{2n} > y_{2n+2}$ and $x_{2n} < x_{2n+2}$ for all $n \in \mathbb{N}_0$. Thus, similar to the proof in i), there exists N such that $(x_{2N}, y_{2N}) \in \mathcal{R}_{4_{1a}}$ with $r_k(x_{2N}) < y_{2N} \leq Q(x_{2N})$ and so $(x_{2N+1}, y_{2N+1}) \in \mathcal{R}_{2_2}$. From Lemma 4.1.6, the solution converges to the equilibrium.

Thus the solution can oscillate between $\mathcal{R}_{4_{1a}}$ and $\mathcal{R}_{2_{1c}}$ for a finite number of steps before entering $\mathcal{R}_{2_2}$ from $\mathcal{R}_{4_{1a}}$. From Figure 4.4 we write path b) as,

$$\text{b) } \mathcal{R}_{2_{1c}} \overset{**}{\rightleftharpoons} \mathcal{R}_{4_{1a}} \to \mathcal{R}_{2_2}.$$

□

Theorem 4.1.15. *Consider* (4.3) *with* $A > \alpha$. *The nontrivial unique positive equilibrium,* $E^* = (x^*, x^*)$ *is globally asymptotically stable.*

Proof. Using Lemma 4.1.6, all solutions converge to the equilibrium except those oscillating in $\mathcal{R}_{2_1} \cup \mathcal{R}_{4_1}$ indefinitely. From Lemma 4.1.14, solutions that enter $\mathcal{R}_{2_{1a}}$, $\mathcal{R}_{2_{1c}}$, $\mathcal{R}_{4_{1a}}$, and $\mathcal{R}_{4_{1c}}$, converge to the equilibrium, $E^* = (x^*, x^*)$. Thus, we need to show that if a solution oscillates between $\mathcal{R}_{2_{1b}}$ and $\mathcal{R}_{4_{1b}}$ indefinitely, it converges to the equilibrium.

If $(x_n, y_n) \in \mathcal{R}_{2_{1b}} \subset \mathcal{R}_{2_1}$, then by Lemma 4.1.6, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_1} \cup \mathcal{R}_{4_2}$. By definition of the region $\mathcal{R}_{2_{1b}}$, $y_n < S(x_n)$. By Lemma 4.1.10, $x_{n+1} < x^*$. Thus, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_{1b}} \cup \mathcal{R}_{4_{1c}} \cup \mathcal{R}_{4_2}$. By Lemma 4.1.14, solutions in $\mathcal{R}_{4_{1c}}$ and $\mathcal{R}_{4_2}$ converge to the equilibrium, $E^* = (x^*, x^*)$. We only need to see what happens when $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_{1b}}$.

If $(x_n, y_n) \in \mathcal{R}_{4_{1b}} \subset \mathcal{R}_{4_1}$, then by Lemma 4.1.6, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_1} \cup \mathcal{R}_{2_2}$. By definition of the region $\mathcal{R}_{4_{1b}}$, $y_n > S(x_n)$, and so by Lemma 4.1.10, $x_{n+1} > x^*$. Thus, by definition of the sub-regions (4.11) $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_{1b}} \cup \mathcal{R}_{2_{1c}} \cup \mathcal{R}_{2_2}$. From Lemma 4.1.14, solutions in $\mathcal{R}_{2_{1c}}$ and $\mathcal{R}_{2_2}$ converge to the equilibrium, $E^* = (x^*, x^*)$. Thus, we only need to consider a solution that oscillates between $\mathcal{R}_{4_{1b}}$ and $\mathcal{R}_{2_{1b}}$ for all $n \in \mathbb{N}_0$.

Without loss of generality, let $\{(x_{2n}, y_{2n})\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{4_{1b}}$ and $\{(x_{2n+1}, y_{2n+1})\}_{n\in\mathbb{N}_0} \in \mathcal{R}_{2_{1b}}$. From the definition of $\mathcal{R}_{2_{1b}}$ (4.11), $y_{2n+1} < S(x_{2n+1}) < x^*$ and $x_{2n+1} > x^*$. From the definition of $\mathcal{R}_{4_{1b}}$ (4.11), $y_{2n} > S(x_{2n}) > x^*$ and $x_{2n} < x^*$.

From Lemma 4.1.13, the subsequence in $\mathcal{R}_{2_{1b}}$ increases in the y-coordinate and decreases in the x-coordinate. Thus, $x_{2n+1} > x_{2n+3} > x^*$ and $y_{2n+1} < y_{2n+3} < x^*$ for all $n \in \mathbb{N}_0$. Thus, the subsequence in $\mathcal{R}_{2_{1b}}$ converges to $E^* = (x^*, x^*)$.

By Lemma 4.1.13 the subsequence in $\mathcal{R}_{4_{1b}}$ decreases in the y-coordinate and increases in the x-coordinate. Thus, $x_{2n} < x_{2n+2} < x^*$ and $y_{2n} > y_{2n+2} > x^*$ for all $n \in \mathbb{N}_0$. Thus the subsequence in $\mathcal{R}_{4_{1b}}$ also converges to the equilibrium $E^* = (x^*, x^*)$.

Thus, if a solution oscillates between in $\mathcal{R}_{2_{1b}}$ and $\mathcal{R}_{4_{1b}}$ indefinitely, then the solution must converge to the equilibrium. Thus, every solution of (4.3) converges to the equilibrium (x^*, x^*), given by (4.5) and the equilibrium is globally asymptotically stable. □

In Figure 4.5, we show an example of two solutions that converge to the equilibrium.

4.2 Case 2: $A < \alpha$

In this section we analyze (4.3) for $0 < A < \alpha$ using the augmented phase portrait. Due to a lack of invariant regions, proving global asymptotic stability of the positive equilibrium, $E^* = (x^*, x^*)$ using only the augmented phase portrait was not possible. However, we provide the augmented phase portrait so that we can compare it with the case of $A > \alpha$.

We first remark that for $A < \alpha$,

$$1 < \frac{\alpha + x_n + y_n}{A + x_n + y_n} = x_{n+1} = \frac{\alpha - A}{A + x_n + y_n} + \frac{A + x_n + y_n}{A + x_n + y_n} \leq \frac{\alpha - A}{A} + 1 = \frac{\alpha}{A},$$

implying that solutions with non-negative and non-trivial initial conditions, enter the interval $(1, \frac{\alpha}{A})$ within one iteration, and so $(x_n, y_n) \in (1, \frac{\alpha}{A}) \times (1, \frac{\alpha}{A})$ for all $n \geq 2$. Thus, it suffices to only consider solutions with initial conditions $(x_0, y_0) \in (1, \frac{\alpha}{A}) \times (1, \frac{\alpha}{A})$.

Since $A < \alpha$, the unique positive equilibrium (4.4),

$$1 < \frac{\alpha + 2x^*}{A + 2x^*} = x^* = \frac{\alpha - A}{A + 2x^*} + \frac{A + 2x^*}{A + 2x^*} \leq \frac{\alpha - A}{A} + 1 = \frac{\alpha}{A}.$$

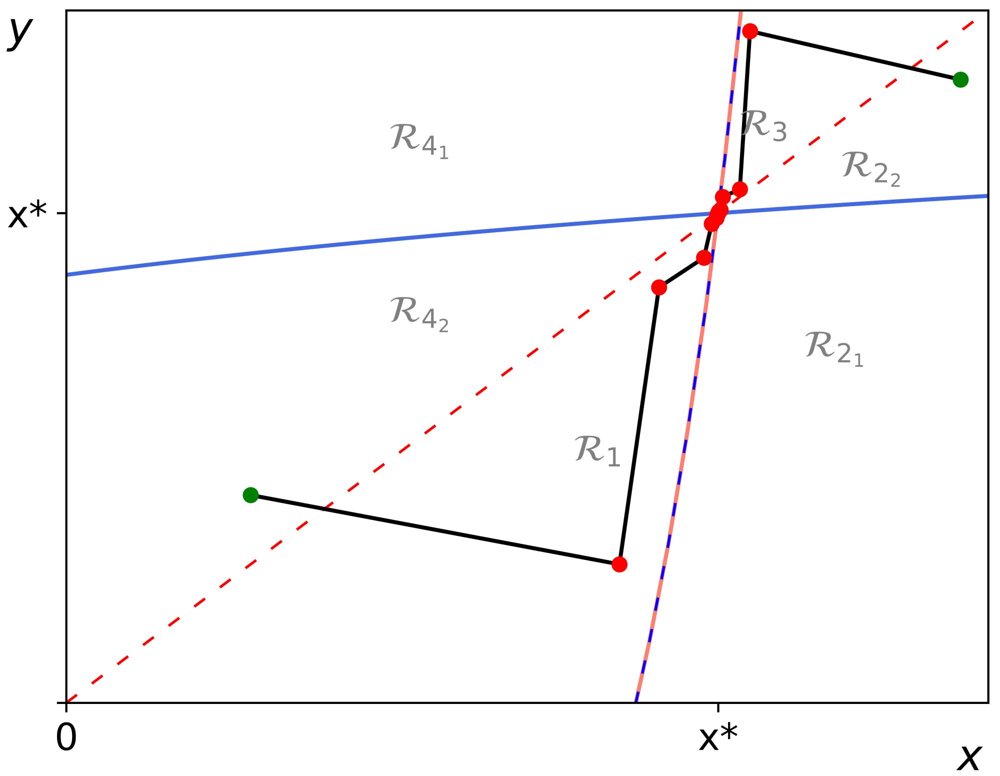

Figure 4.5: The augmented phase portrait of (4.3) with $A = 2$ and $\alpha = 1$. Depicted are five iterations of two solutions converging to (x^*, x^*). One solution begins at the green dot in $\mathcal{R}_{4_2}$ and the other at the green dot in $\mathcal{R}_{2_2}$.

The root-curves and their associated nullclines do not change from the $A > \alpha$ case, however the direction field does. We find that,

$$x_{n+1} - x_n \begin{cases} > 0, & y_n < h(x_n), \\ = 0, & y_n = h(x_n), \\ < 0, & y_n > h(x_n), \end{cases} \quad \text{and,} \quad y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases}$$

and the details of the calculations for this can be found in Chapter 6, Section 6.2.1.

Proposition 4.2.1. *Assume that $A < \alpha$. Then the following hold.*

a) *If $y_n < r_h(x_n)$, then $y_{n+1} > h(x_{n+1})$ and equivalently $\mathcal{L}_h(x_n, y_n) > 0$.*
b) *If $y_n > r_h(x_n)$, then $y_{n+1} < h(x_{n+1})$ and equivalently $\mathcal{L}_h(x_n, y_n) < 0$.*
c) *If $y_n < r_k(x_n)$, then $y_{n+1} < k(x_{n+1})$ and equivalently $\mathcal{L}_k(x_n, y_n) < 0$.*
d) *If $y_n > r_k(x_n)$, then $y_{n+1} > k(x_{n+1})$ and equivalently $\mathcal{L}_k(x_n, y_n) > 0$.*

The details of the proof can be found in Chapter 6, Section 6.2.2.

We now prove the following lemma in order to create a figure with generic root-curves and their associated nullclines.

Proposition 4.2.2. *If $A < \alpha$ then:*

a) *$r_k(x)$ and $r_h(x)$ are decreasing for all $x \in (1, \frac{\alpha}{A})$ and $k(x)$ is increasing for all $x \in (1, \frac{\alpha}{A})$.*

b) *$k(x) < r_h(x) < r_k(x)$ for $x \in (1, x^*)$ and $r_k(x) < r_h(x) < k(x)$ for $x \in (x^*, \frac{\alpha}{A})$.*

Proof. **a)** Since $A - \alpha < 0$ and $-(x-1)^2 < 0$,

$$r_k'(x) = \frac{A - \alpha - (x-1)^2}{(1-x)^2} < 0.$$

If $r_h(x)$ is decreasing, then,

$$r_h'(x) = \frac{1}{2}\left(\frac{1 + A + x - \sqrt{4(\alpha + x) + 1 - 2(A + x) + (A + x)^2}}{\sqrt{4(\alpha + x) + (1 - A - x)^2}}\right) < 0.$$

Since $A < \alpha$,

$$\begin{aligned} r_h'(x) &< \frac{1}{2}\left(\frac{1 + A + x - \sqrt{4(A + x) + 1 - 2(A + x) + (A + x)^2}}{\sqrt{4(A + x) + (1 - A - x)^2}}\right), \\ &= \frac{1}{2}\left(\frac{1 + A + x - \sqrt{(1 + A + x)^2}}{\sqrt{4(A + x) + (1 - A - x)^2}}\right), \\ &= 0. \end{aligned}$$

Since $k(x) = x$, it is increasing.

b) By Property 4.1.3, any two of $k(x)$, $r_h(x)$, and $r_k(x)$ can only intersect at $x = x^*$. Since $k(x)$ is increasing and $r_h(x)$ and $r_k(x)$ are decreasing, $k(x) < \min\{r_h(x), r_k(x)\}$ for every $x \in (1, x^*)$ and $k(x) > \max\{r_h(x), r_k(x)\}$ for every $x \in (x^*, \frac{\alpha}{A})$.

If we show that for some $x \in [1, x^*)$, $r_h(x) < r_k(x)$ then $r_h(x) < r_k(x)$ for all $x \in (1, x^*)$. Similarly, if we show that $r_k(x) < r_h(x)$ for some $x \in (x^*, \frac{\alpha}{A}]$, then $r_k(x) < r_h(x)$ for every $x \in (x^*, \frac{\alpha}{A}]$.

Thus we evaluate $r_h(x)$ and $r_k(x)$ at $x = 1$. Since,

$$\lim_{x\to 1^+} r_k(x) = \lim_{x\to 1^+} \frac{x^2 + (A-1)x - \alpha}{1-x} \longrightarrow +\infty,$$
$$r_h(1) = \frac{-A + \sqrt{(-A)^2 + 4\alpha + 4}}{2} < \frac{-A + \sqrt{A^2 + 4A + 4}}{2} = 1,$$

$k(x) < r_h(x) < r_k(x)$ for all $x \in (1, x^*)$.

We evaluate $r_h(x)$ and $r_k(x)$ at $x = \frac{\alpha}{A}$. Since,

$$r_k\left(\frac{\alpha}{A}\right) = \frac{(\frac{\alpha}{A})^2 + (A-1)(\frac{\alpha}{A}) - \alpha}{1 - \frac{\alpha}{A}} = \frac{(\frac{\alpha}{A})^2 + \alpha - \frac{\alpha}{A} - \alpha}{1 - \frac{\alpha}{A}} = \frac{(\frac{\alpha}{A})(\frac{\alpha}{A} - 1)}{1 - \frac{\alpha}{A}} = -\frac{\alpha}{A},$$
$$r_h\left(\frac{\alpha}{A}\right) = \frac{1 - A - \frac{\alpha}{A} + \sqrt{(1 - A - \frac{\alpha}{A})^2 + 4(\alpha + \frac{\alpha}{A})}}{2} > \frac{1 - A - \frac{\alpha}{A} + \sqrt{(1 - A - \frac{\alpha}{A})^2}}{2} > 0,$$

$r_k(x) < r_h(x) < k(x)$ for all $x \in (x^*, \frac{\alpha}{A})$. □

Using Proposition 4.2.2, we make Figure 4.6 with generic root-curves and their associated nullclines. Furthermore, we define the following regions.

$$\begin{aligned}
\mathcal{R}_1 &= \{(x,y) \in \mathbb{R}^2 |\, y \le k(x),\, y < r_k(x),\, y < r_h(x_n),\, 1 < x < \frac{\alpha}{A}\}, \\
\mathcal{R}_{2_1} &= \{(x,y) \in \mathbb{R}^2 |\, r_k(x) \le y \le r_h(x) < k(x),\, x^* < x < \frac{\alpha}{A}\}, \\
\mathcal{R}_{2_2} &= \{(x,y) \in \mathbb{R}^2 |\, r_k(x) < r_h(x) < y < k(x),\, x^* < x < \frac{\alpha}{A}\}, \\
\mathcal{R}_3 &= \{(x,y) \in \mathbb{R}^2 |\, y \ge k(x),\, y > r_k(x),\, y > r_h(x),\, 1 < x < \frac{\alpha}{A}\}, \\
\mathcal{R}_{4_1} &= \{(x,y) \in \mathbb{R}^2 |\, k(x) < r_h(x) < y \le r_k(x),\, 1 < x < x^*\} \\
\mathcal{R}_{4_2} &= \{(x,y) \in \mathbb{R}^2 |\, k(x) < y \le r_h(x) < r_k(x),\, 1 < x < x^*\}.
\end{aligned} \tag{4.12}$$

Unfortunately, we cannot obtain any conclusive results about the behaviour of solutions using the augmented phase portrait in this case. This is because no region is positively invariant like in the case where $A > \alpha$. However, if the reader is interested, we have included an additional lemma and proof in Chapter 6, Section 6.2.3 that shows no region is positively invariant.

4.2.1 Global Stability

To complete the analysis of (4.3), we prove global stability of the positive equilibrium using Theorem 2.2.13 from [8]. We note that proving global asymptotic stability of $E^* = (x^*, x^*)$ using this Theorem 2.2.13 and method was done in [1]. We provide this proof for the sake of completeness, but acknowledge that we adapted it from [1].

Theorem 4.2.3. *All solutions of* (4.3) *with $A < \alpha$ converge to the equilibrium (x^*, x^*), given by* (4.5)*. Thus, the equilibrium is globally stable.*

Proof. Theorem 2.2.13 requires that $F(x,y) = \frac{\alpha+x+y}{A+x+y}$ with $F : [1, \frac{\alpha}{A}] \times [1, \frac{\alpha}{A}] \to [1, \frac{\alpha}{A}]$, be non-increasing in each of its arguments. Thus,

$$\frac{\partial F}{\partial x} = \frac{(A+x+y) - (\alpha+x+y)}{A+x+y} = \frac{A-\alpha}{A+x+y} < 0 \text{ and},$$
$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x} < 0,$$

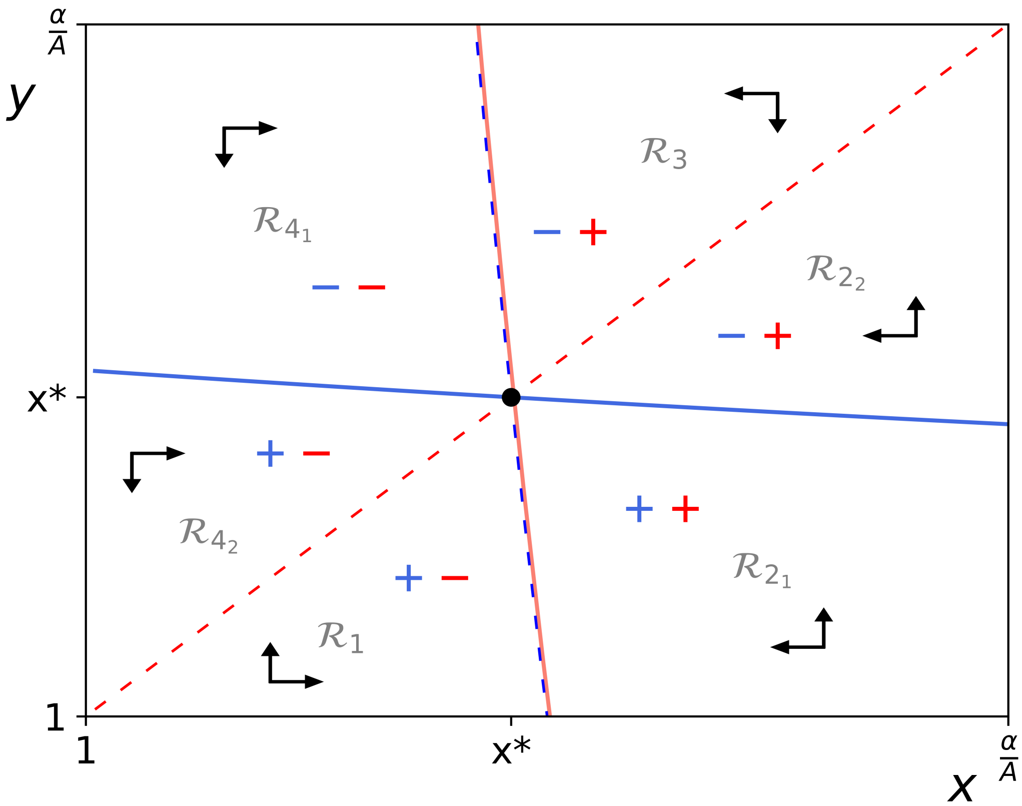

Figure 4.6: The augmented phase portrait of (4.3) with $A < \alpha$. The x-nullcline, representing $y = h(x)$ is in dashed blue. The y-nullcline, $k(x) = x$ is in dashed red. The associated root-curves, $y = r_h(x)$ and $y = r_k(x)$ are the curves in solid blue and red, respectively. Recall that $h(x) = r_k(x)$. The vertical and horizontal arrows represent the component-wise monotone regions. Finally, regions containing a '+' ('−') have that the next-iterate can be found above (below) the same colour nullcline.

as required. Furthermore, Theorem 2.2.13 requires that if $(m, M) \in [1, \frac{\alpha}{A}] \times [1, \frac{\alpha}{A}]$ such that $M = F(m, m)$ and $m = F(M, M)$, then $m = M$. Thus, assume that $(m, M) \in [1, \frac{\alpha}{A}] \times [1, \frac{\alpha}{A}]$ with,

$$\frac{\alpha + 2m}{A + 2m} = M \text{ and } \frac{\alpha + 2M}{A + 2M} = m.$$

The first equation is equivalent to $\alpha + 2m = AM + 2mM$ and so $m = \frac{AM - \alpha}{2(1-M)}$. Using the second equation, this is equivalent to,

$$\frac{\alpha + 2M}{A + 2M} = \frac{AM - \alpha}{2(1 - M)}.$$

Thus, $(AM - \alpha)(A + 2M) = 2(1 - M)(\alpha + 2M)$ and we obtain,

$$\begin{aligned} A^2 M - \alpha A + 2AM^2 - 2\alpha M &= 2\alpha + 4M - 2M\alpha - 4M^2, \\ M^2(2A + 4) + M(A^2 - 4) + (-\alpha A - 2\alpha) &= 0, \\ 2M^2 + M(A - 2) - \alpha &= 0. \end{aligned}$$

This yields the only positive solution, $M = \frac{(2-A)+\sqrt{(A-2)^2+8\alpha}}{4} = x^*$. Thus,

$$x^* = \frac{\alpha + 2m}{A + 2m},$$

can be rearranged so that $m = \frac{1}{2}\left(\frac{x^* A - \alpha}{1 - x^*}\right)$. From the proof of Lemma 4.1.14, $\frac{x^* A - \alpha}{1 - x^*} = 2x^*$. Thus $m = x^*$ and so $M = m$. By Theorem 2.2.13 the equilibrium, $E^* = (x^*, x^*)$ is globally stable. $\square$

4.3 Case 3: $A = \alpha$

Since $A = \alpha$, the system of difference equations simplifies to,

$$F(x_n, y_n) = 1, \quad G(x_n, y_n) = x_n.$$

where the equilibrium is $x^* = 1$.

Theorem 4.3.1. *All solutions of* (4.3) *with $A = \alpha$ converge to the equilibrium (x^*, x^*) given by* (4.4). *Thus, the equilibrium is globally stable.*

Proof. Let $(x_0, y_0) \in \mathbb{R}^2_+$. Thus, $(x_1, y_1) = (1, x_0)$ and $(x_2, y_2) = (1, 1)$. □

To complete the analysis of (4.3) we plot the root-curves and their associated nullclines. When $A = \alpha$, the y-nullcline is $k(x) = x$. From Definition 2.3.2, the x-nullcline is a function that satisfies $F(x, y) = x$. Thus we obtain the x-nullcline

$$h(y) = 1.$$

Thus the component-wise monotone regions are,

$$x_{n+1} - x_n \begin{cases} > 0, & x_n < 1, \\ = 0, & x_n = 1, \\ < 0, & x_n > 1, \end{cases} \quad \text{and,} \quad y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases}$$

The calculations are in Chapter 6, Section 6.3.1.

Since the nullcline $x = 1$ is a vertical line, we cannot find its root-curve as defined in Definition 2.3.7. However, using Remark 2.2 from [10], we find a root-curve $x = r_h(y)$ that satisfies $F(r_h(y), y) = h(G(r_h(y), y))$. Since $F(x, y) = h(y) = 1$ for every (x, y), the root-set is $\{(x, y) \in \mathbb{R}^2\}$.

Now, we find a function that satisfies, $G(x, y) = k(F(x, y))$. This is equivalent to $x = k(1)$ and so the root-curve of $y = k(x)$ is,

$$r_k(y) := 1.$$

Proposition 4.3.2. *Consider* (4.3) *with $A = \alpha$. Then,*

a) *If $x_n < 1$, then $y_{n+1} < k(x_{n+1})$ and equivalently, $\mathcal{L}_k(x_n, y_n) < 0$.*
b) *If $x_n > 1$, then $y_{n+1} > k(x_{n+1})$ and equivalently, $\mathcal{L}_k(x_n, y_n) > 0$.*

The proof is in Chapter 6, Section 6.3.2. We now plot the root-curves and their associated nullclines in Figure 4.7.

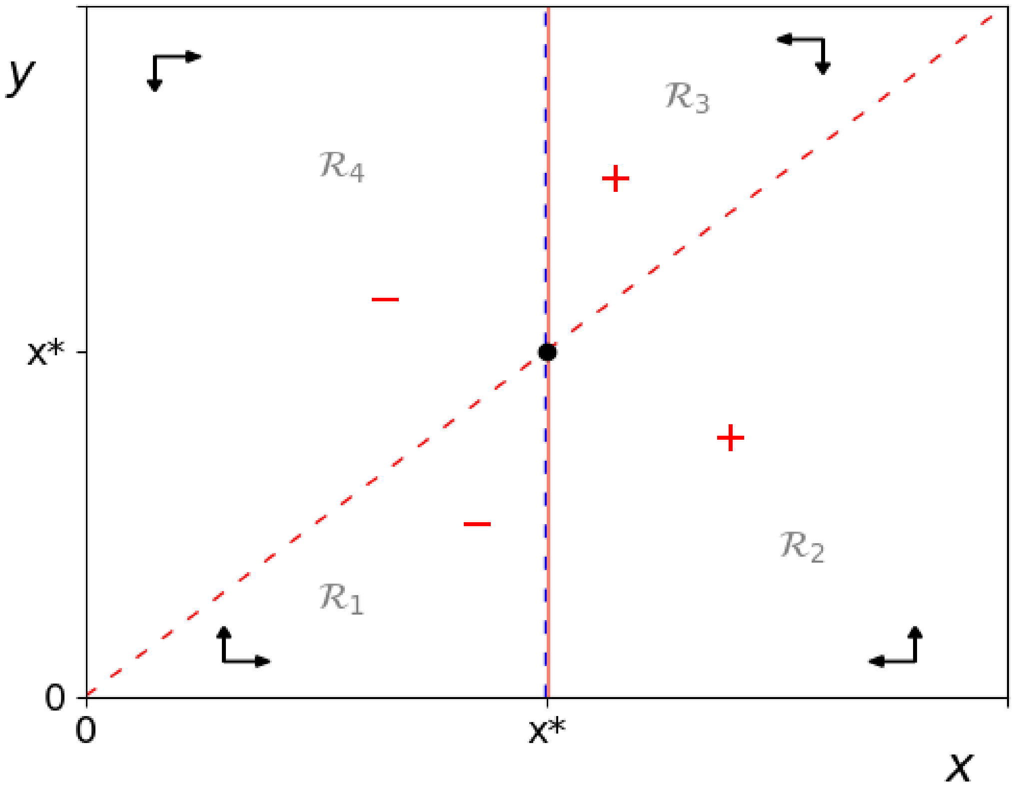

Figure 4.7: The augmented phase portrait of (4.3) with $A > \alpha$. The x-nullcline, $x = 1$ is in dashed blue. The y-nullcline, $k(x) = x$ is in dashed red and its associated root-curves $y = r_k(x)$ is the solid blue line, $x = 1$. The vertical and horizontal arrows represent the component-wise monotone regions. Finally, regions containing a '+' ('−') have that the next-iterate can be found above (below) the same colour nullcline.

4.4 Comparison of all three cases

Having analyzed all three cases, $A > \alpha$, $A = \alpha$, and $A < \alpha$, we summarize our results by comparing the figures from each case.

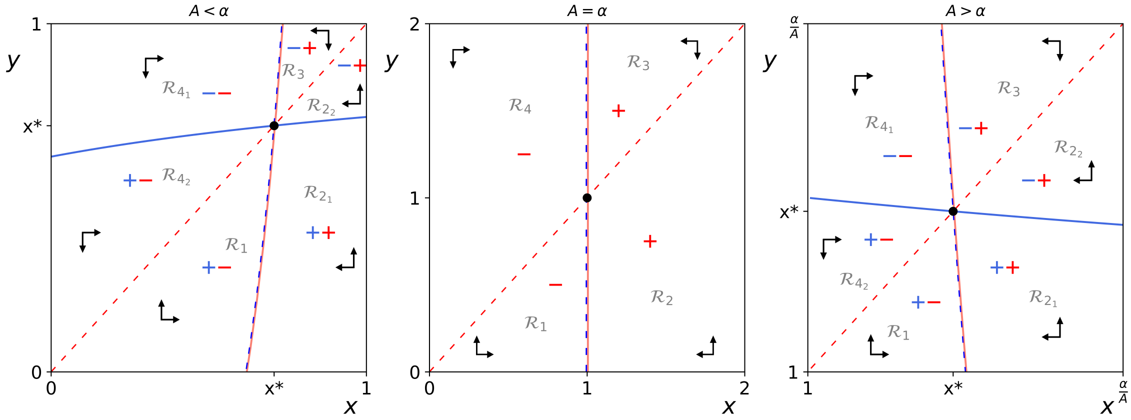

Figure 4.8: The augmented phase portrait for the three cases of (4.3). The solid red line is $r_k(x)$ and the dashed red line is the associated nullcline, $k(x)$. The solid blue line is $r_h(x)$ and the dashed blue line is the associated nullcline, $h(x)$.

In all three cases, it was possible to show global stability of the equilibrium. Holding α constant, as A increases, $r_k(x)$ rotates counter-clockwise and changes from increasing, to a vertical line, to decreasing and $r_h(x)$ changes from increasing to decreasing. However, at the transition it is no loner a single function. Instead, the root-set associated with $h(x)$ includes all of the points in the $x - y$ plane. We were able to use the augmented phase portrait from [10] to prove the global stability for $A > \alpha$. In the case of $A = \alpha$, since the root-set associated with the nullcline $h(x)$ contains all of the points in the $x - y$ plane and $r_k(x) = h(x)$, it follows immediately from the augmented phase portrait that every initial point is eventually fixed. However, the global stability in this case also follows immediately from the equations, noting that $F(x_n, y_n) = 1$. For the case with $A < \alpha$, we used existing theorems as was done in [1].

Chapter 5

Additional Proofs for Chapter 3

In this section we provide supplementary proofs to show various calculations, propositions, and lemmas from Chapter 3. This chapter is not meant to be read through as the previous ones; it is meant to be used and accessed at the interest of the reader and to clarify earlier results. The proofs and calculations in this chapter add the rigour necessary to show the main results from Chapter 3, but would have taken away from the overall story.

5.1 Proof of Theorem 3.0.1

Proof. We use Theorem 2.2.9 and Theorem 2.2.10 to show that the equilibrium of (3.2) is locally asymptotically stable when $A > 1$ and is an unstable saddle point when $A < 1$. To linearize (3.2) about the equilibrium we find the partial derivatives of $F(x, y) = \frac{\alpha+y}{A+x}$ as,

$$\frac{\partial F}{\partial x} = \frac{-(\alpha + y)}{(A + x)^2}, \text{ and } \frac{\partial F}{\partial y} = \frac{1}{A + x}.$$

The associated characteristic equation is given by (2.4) as $\lambda^2 - p\lambda - q = 0$ where $p = \frac{\partial F}{\partial x}(x^*, x^*)$ and $q = \frac{\partial F}{\partial y}(x^*, x^*)$. Thus, $\lambda^2 + \frac{x^*}{A+x^*}\lambda - \frac{1}{A+x^*} = 0$ and equivalently,

$$(A + x^*)\lambda^2 + x^*\lambda - 1 = 0.$$

As such, we get two roots, $\lambda_+ = \frac{-x^*+\sqrt{(x^*)^2+4(A+x^*)}}{2(A+x^*)}$ and $\lambda_- = \frac{-x^*-\sqrt{(x^*)^2+4(A+x^*)}}{2(A+x^*)}$.

Firstly, assume that $A > 1$ and by Theorem 2.2.9, to show (x^*, x^*) is locally asymptotically stable, we need to show that $|\lambda_+| < 1$ and $|\lambda_-| < 1$. Note that since $|\lambda_+| < |\lambda_-|$, we only need to show that $|\lambda_-| < 1$ or equivalently,

$$x^* + \sqrt{(x^*)^2 + 4(A + x^*)} < 2(A + x^*).$$

Beginning with $A > 1$, multiply both sides by $(4A + 4X^*)$ and add $(x^*)^2$ to obtain,

$$(x^*)^2 + 4A + 4x^* < 4A^2 + 4Ax^* + (x^*)^2.$$

The right hand side is a perfect square and so by taking the square root of both sides we get that $\sqrt{(x^*)^2 + 4(A + x^*)} < 2A + x^*$. Adding x^* to both sides, we obtain,

$$x^* + \sqrt{(x^*)^2 + 4(A + x^*)} < 2(A + x^*), \text{ as required.}$$

Thus when $A > 1$, by Theorem 2.2.9 (x^*, x^*) is locally asymptotically stable.

Now assume $A < 1$ and by Theorem 2.2.10, to show $E^* = (x^*, x^*)$ is a saddle point, we show that $|p| > |1 - q|$ and $p^2 + 4q > 0$. Since $A - 1 < 0$,

$$|p| = \frac{x^*}{A + x^*} > \frac{x^* + A - 1}{A + x^*}.$$

From the definition, $x^* = \frac{\alpha + x^*}{A+x^*}$,

$$A + x^* = \frac{\alpha + x^*}{x^*} = \frac{\alpha}{x^*} + 1.$$

As such, $x^* + A - 1 > 0$ and,

$$|p| = \frac{x^*}{A+x^*} \geq \frac{x^* + A - 1}{A+x^*} = 1 - \frac{1}{A+x^*} = \left|1 - \frac{1}{A+x^*}\right| = |1-q|.$$

Since $p^2 > 0$ and $q > 0$, we have $p^2 + 4q = \frac{(x^*)^2}{(A+x^*)^2} + \frac{4}{A+x^*} > 0$. Thus, by Theorem 2.2.10, the positive equilibrium $E^* = (x^*, x^*)$ is a saddle point when $A < 1$. □

5.2 Nullcline calculation

To find the x-nullcline, we find a function $y = h(x)$ that satisfies $F(x, y) = x$ as,

$$\begin{aligned} \frac{\alpha + y}{A + x} &= x, \\ h(x) := y &= x^2 + Ax - \alpha. \end{aligned}$$

Thus $h(x) = x^2 + Ax - \alpha$ is the x-nullcline. By Remark 2.3.3, $y = x$, satisfies $G(x, y) = x$ so the y-nullcline is $k(x) = x$.

5.3 Component-wise monotone regions

We want to show that:

$$x_{n+1} - x_n \begin{cases} > 0, & y_n > h(x_n), \\ = 0, & y_n = h(x_n), \\ < 0, & y_n < h(x_n), \end{cases} \quad \text{and} \quad y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases}$$

If $y_n < h(x_n)$, then this is equivalent to $y_n < x_n^2 + Ax_n - \alpha$ and,

$$\begin{aligned} \frac{\alpha + y_n}{A + x_n} &< x_n, \\ F(x_n, y_n) &< x_n, \\ \Delta x_n &< 0. \end{aligned}$$

By the definition of the x-nullcline, the equality case is obvious and showing that $y_n > h(x_n)$ implies $x_{n+1} - x_n < 0$ is similar to the above proof. Now, if $y_n < k(x_n)$, then $y_n < x_n$ and so,

$$\begin{aligned} y_n &< G(x_n, y_n), \\ \Delta y_n &> 0. \end{aligned}$$

Similarly, $y_n > k(x_n)$ implies $y_{n+1} - y_n > 0$. The equality case comes out of the definition of the y-nullcline.

5.4 Root-curve associated with x-nullcline

We show the calculation for the root-curve associated with the $h(x)$ nullcline. We want to find a function $y = r_h(x)$ that satisfies, $G(x, y) = h(F(x, y))$. Equivalently,

$$\begin{aligned} x &= F(x,y)^2 + AF(x,y) - \alpha, \\ x + \alpha &= \left(\frac{\alpha + y}{A + x}\right)^2 + A\left(\frac{\alpha + y}{A + x}\right), \\ x + \alpha + \frac{A^2}{4} &= \left(\frac{\alpha + y}{A + x} + \frac{A}{2}\right)^2. \end{aligned}$$

Solving for y, we find that,

$$y = \frac{1}{2}\left(-2\alpha - A^2 - Ax \pm (A + x)\sqrt{4\alpha + A^2 + 4x}\right).$$

We only take the positive square root since $(x, y) \in (0, \infty) \times (0, \infty)$ and so,

$$r_h(x) := y = \frac{1}{2}\left(-2\alpha - A^2 - Ax + (A + x)\sqrt{4\alpha + A^2 + 4x}\right).$$

5.5 Proof of Lemma 3.1.2

Proof. a) If $y_n < r_h(x_n)$, then equivalently,

$$\begin{aligned} 2y_n &< -2\alpha - A^2 - Ax_n + (A + x_n)\sqrt{4\alpha + A^2 + 4x}, \\ 2y_n + 2\alpha + A^2 + Ax_n &< (A + x_n)\sqrt{4\alpha + A^2 + 4x}. \end{aligned}$$

Simplifying the left hand side and squaring both sides, we obtain,

$$\begin{aligned} (2(y_n + \alpha) + A(A + x_n))^2 &< (A + x_n)^2(4\alpha + A^2 + 4x_n), \\ \left(\frac{2(y_n + \alpha) + A(A + x_n)}{(A + x_n)}\right)^2 &< (4\alpha + A^2 + 4x_n). \end{aligned}$$

Substituting $x_{n+1} = \frac{\alpha + y_n}{A + x_n}$, we obtain $(2x_{n+1} + A)^2 < (4\alpha + A^2 + 4x_n)$ and,

$$\begin{aligned} 4x_{n+1}^2 + 4Ax_{n+1} + A^2 &< 4\alpha + A^2 + 4x_n, \\ x_{n+1}^2 + Ax_{n+1} &< \alpha + x_n, \\ x_{n+1}^2 + Ax_{n+1} - \alpha &< y_{n+1}. \end{aligned}$$

Thus, $y_{n+1} > h(x_{n+1})$. The proof for b) is similar so we leave it out.

c) If $y_n < r_k(x_n)$, then equivalently $y_n < x_n^2 + Ax_n - \alpha$ and,

$$\begin{aligned} \frac{\alpha + y_n}{A + x_n} &< x_n, \\ x_{n+1} &< y_{n+1}. \end{aligned}$$

As such, $y_{n+1} > k(x_{n+1})$. The proof for d) is similar. □

5.6 Proof of Property 3.1.3

Proof. From Lemma 2.3.9, if $r_k(\bar{x}) = k(\bar{x})$, then $\bar{x}$ is an equilibrium point, and so $\bar{x} = x^*$. Since $k(\bar{x}) = \bar{x}$, then the two functions intersect at $E^* = (x^*, x^*)$.

From Lemma 2.3.11, if $r_k(\bar{x}) = r_h(\bar{x})$, then $\bar{x}$ is an equilibrium point, and so $\bar{x} = x^*$. Since $r_k(x^*) = x^*$, the two functions intersect at $E^* = (x^*, x^*)$.

Assume that $r_h(\bar{x}) = k(\bar{x})$, then equivalently $2\bar{x} = -2\alpha - A^2 - A\bar{x} + (A+\bar{x})\sqrt{4(\alpha+\bar{x}) + A^2}$ and,

$$\begin{aligned}
2(\bar{x}+\alpha) + A(A+\bar{x}) &= (A+\bar{x})\sqrt{4(\alpha+\bar{x}) + A^2},\\
2F(\bar{x},\bar{x}) + A &= \sqrt{4(\alpha+\bar{x}) + A^2},\\
4F(\bar{x},\bar{x})^2 + 4AF(\bar{x},\bar{x}) + A^2 &= 4(\alpha+\bar{x}) + A^2,\\
F(\bar{x},\bar{x})^2 + AF(\bar{x},\bar{x}) &= \alpha + \bar{x},\\
\left(\frac{\alpha+\bar{x}}{A+\bar{x}}\right)^2 + A\frac{\alpha+\bar{x}}{A+\bar{x}} &= \alpha + \bar{x},\\
\frac{\alpha+\bar{x}}{(A+\bar{x})^2} + \frac{A}{A+\bar{x}} &= 1,\\
\alpha + \bar{x} + A(A+\bar{x}) &= (A+\bar{x})^2,\\
\alpha + \bar{x} + A^2 + A\bar{x} &= A^2 + 2A\bar{x} + \bar{x}^2,\\
\alpha + \bar{x} &= \bar{x}^2 + A\bar{x},\\
0 &= \bar{x}^2 + \bar{x}(A-1) - \alpha,\\
\bar{x} &= \frac{-(A-1) \pm \sqrt{(A-1)^2 + 4\alpha}}{2}.
\end{aligned}$$

Thus the only positive value of $\bar{x} = \frac{-(A-1)+\sqrt{(A-1)^2+4\alpha}}{2} = x^*$ Since $k(x) = x$, the two functions intersect at $E^* = (x^*, x^*)$. ☐

5.7 Proof of Lemma 3.1.11

Proof. i) If $y_n < S(x_n)$, then $y_n < x_n x^* + Ax^* - \alpha$ which is equivalent to,

$$\begin{aligned}
y_n + \alpha &< x^*(x_n + A),\\
x_{n+1} &< x^*.
\end{aligned}$$

Similarly, if $y_n < S(x_n)$, then $x_{n+1} > x^*$.

ii) Since $S(x)$ and $k(x)$ are both linear functions that by Lemma 3.1.10 intersect only at $x = x^*$, we compare the values of $S(x)$ and $k(x)$ at one point where $x \neq x^*$. At $x = -A < x^*$,

$$\begin{aligned}
S(-A) &= -\alpha\\
k(-A) &= -A.
\end{aligned}$$

Since $\alpha > A$, $S(-A) = -\alpha < -A = k(-A)$. Since $-A < x^*$ and the functions are linear and intersect at (x^*, x^*), $S(x) < k(x)$ for $x < x^*$. Additionally, $S(x) > k(x)$ for $x > x^*$. ☐

Chapter 6

Additional Proofs for Chapter 4

This section provides supplementary proofs and calculations of various propositions and lemmas from Chapter 4. Like Chapter 5, this chapter is not meant to be read in order, rather, it is made to complement and provide the necessary rigour for the claims in Chapter 4. These proofs were omitted earlier because they hindered the presentation of the arguments; many of these proofs are long and tedious algebraic manipulations. However, they are necessary to ensure the mathematical rigour, hence we included them in this section.

6.1 Equation (4.3) with $A > \alpha$

6.1.1 Component-wise monotone regions

For (4.3) with $A > \alpha$, we want to show that,

$$\Delta x_n = x_{n+1} - x_n \begin{cases} > 0, & y_n > h(x_n), \\ = 0, & y_n = h(x_n), \\ < 0, & y_n < h(x_n), \end{cases} \quad \text{and} \quad \Delta y_n = y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases}$$

Proof. Assuming that $y_n > h(x_n)$, this is equivalent to $y_n > \frac{x_n^2 + (A-1)x_n - \alpha}{1 - x_n}$ and so,

$$\begin{aligned} y_n - y_n x_n &> x_n(A + x_n) - (x_n + \alpha), \\ \alpha + x_n + y_n &> x_n(A + x_n + y_n), \\ x_{n+1} - x_n &> 0. \end{aligned}$$

Similarly, $y_n < h(x_n)$ implies $x_{n+1} - x_n < 0$. Finally the equality case holds directly from the definition of an x-nullcline.

Next, assume $y_n > k(x_n)$, then equivalently $y_n > x_n$ and,

$$\begin{aligned} y_n &> y_{n+1}, \\ y_{n+1} - y_n &< 0. \end{aligned}$$

Similarly, $y_n < k(x_n)$ implies $y_{n+1} - y_n > 0$. The equality case come directly from the definition of a y-nullcline. $\square$

6.1.2 Calculating root-curves

To find a root-curve associated with the $y = h(x)$ nullclines, by Definition 2.3.7, we find a function $y = r_h(x)$ that satisfies, $G(x, y) = h(F(x, y))$. Solving for y,

$$\begin{aligned}
x &= h\left(\frac{\alpha + x + y}{A + x + y}\right), \\
&= \frac{\left(\frac{\alpha+x+y}{A+x+y}\right)^2 + (A-1)\left(\frac{\alpha+x+y}{A+x+y}\right) - \alpha}{\left(\frac{A-\alpha}{A+x+y}\right)}, \\
&= \left(\frac{\frac{-\alpha(A+x+y)^2+(\alpha+x+y)^2+(A-1)(\alpha+x+y)(A+x+y)}{(A+x+y)^2}}{\frac{A-\alpha}{A+x+y}}\right), \\
&= \left(\frac{-\alpha(A+x+y)^2 + (\alpha+x+y)^2 + (A-1)(\alpha+x+y)(A+x+y)}{(A-\alpha)(A+x+y)}\right), \\
&= \left(\frac{(A-1)(A-\alpha)(x+y) + (A-\alpha)(x+y)^2 - \alpha(A-\alpha)}{(A-\alpha)(A+x+y)}\right), \\
&= \left(\frac{(A-1)(x+y) + (x+y)^2 - \alpha}{(A+x+y)}\right),
\end{aligned}$$

Multiplying both sides by $(A + x + y)$, this yields,

$$\begin{aligned}
Ax + x(x+y) &= (A-1)(x+y) + (x+y)^2 - \alpha, \\
0 &= (x+y)^2 + (A-1-x)(x+y) - (\alpha + Ax), \\
0 &= y^2 + y(A-1+x) - (\alpha + x), \\
0 &= \left(y + \frac{(A-1+x)}{2}\right)^2 - \frac{(A-1+x)^2}{4} - (\alpha + x), \\
y + \frac{(A-1+x)}{2} &= \pm\sqrt{\frac{(A-1+x)^2}{4} + (\alpha + x)}, \\
y &= \frac{1 - A - x \pm \sqrt{(1-A-x)^2 + 4(\alpha + x)}}{2}.
\end{aligned}$$

Thus, the only positive root-curve is,

$$r_h(x) := y = \frac{1 - A - x + \sqrt{(1-A-x)^2 + 4(\alpha + x)}}{2}.$$

6.1.3 Proof of Proposition 4.1.2

Proof. **a)** By a contrapositive argument, we show that $y_{n+1} \le h(x_{n+1})$ implies $y_n \ge r_h(x_n)$. Thus $y_{n+1} \le h(x_{n+1})$ is equivalent to,

$$\begin{aligned}
\left(\frac{\frac{-\alpha(A+x_n+y_n)^2+(\alpha+x_n+y_n)^2+(A-1)(\alpha+x_n+y_n)(A+x_n+y_n)}{(A+x_n+y_n)^2}}{\frac{A-\alpha}{A+x_n+y_n}}\right) &\ge x_n, \\
\left(\frac{-\alpha(A+x_n+y_n)^2 + (\alpha+x_n+y_n)^2 + (A-1)(\alpha+x_n+y_n)(A+x_n+y_n)}{(A-\alpha)(A+x_n+y_n)}\right) &\ge x_n, \\
\frac{(A-1)(A-\alpha)(x_n+y_n) + (A-\alpha)(x_n+y_n)^2 - \alpha(A-\alpha)}{(A-\alpha)(A+x_n+y_n)} &\ge x_n.
\end{aligned}$$

Multiplying both sides by $(A + x_n + y_n)$, we obtain,

$$\begin{aligned}(A-1)(x_n+y_n)+(x_n+y_n)^2-\alpha &\geq x_n(A+x_n+y_n),\\ (x_n+y_n)^2+(A-1-x_n)(x_n+y_n)-(\alpha+Ax_n) &\geq 0,\\ y_n^2+y_n(A-1+x_n)-(\alpha+x_n) &\geq 0.\end{aligned}$$

Solving for y_n,

$$\begin{aligned}\left(y_n+\frac{(A-1+x_n)}{2}\right)^2 &\geq \frac{(A-1+x_n)^2}{4}+(\alpha+x_n),\\ y_n+\frac{(A-1+x_n)}{2} &\geq \sqrt{\frac{(A-1+x_n)^2}{4}+(\alpha+x_n)},\\ y_n &\geq \frac{1-A-x_n+\sqrt{(1-A-x_n)^2+4(\alpha+x_n)}}{2},\\ y_n &\geq r_h(x_n).\end{aligned}$$

Equivalently, if $y_n < r_h(x_n)$, then $y_{n+1} > h(x_{n+1})$. The proof for b) is similar so we leave it out.

c) If $y_n < r_k(x_n)$, then equivalently, $y_n < \frac{x_n^2+(A-1)x_n-\alpha}{1-x_n}$ and since $x_n \in (0,1)$,

$$\begin{aligned}y_n - y_n x_n &< x_n^2 + Ax_n - x_n - \alpha,\\ \alpha + x_n + y_n &< x_n(A + x_n + y_n),\\ \frac{\alpha + x_n + y_n}{A + x_n + y_n} &< x_n,\\ k(x_{n+1}) &< y_{n+1}.\end{aligned}$$

The proof of d) is similar. □

6.1.4 Proof of Property 4.1.3

Proof. From Lemma 2.3.9, if $r_k(\bar{x}) = k(\bar{x})$, then $\bar{x}$ is an equilibrium point, and so $\bar{x} = x^*$. Since $k(\bar{x}) = \bar{x}$, then the two functions intersect at $E^* = (x^*, x^*)$.

From Lemma 2.3.11, if $r_k(\bar{x}) = r_h(\bar{x})$, then $\bar{x}$ is an equilibrium point, and so $\bar{x} = x^*$. Since $r_k(x^*) = x^*$, we know that the two functions intersect at $E^* = (x^*, x^*)$.

Suppose that $r_h(\bar{x}) = k(\bar{x})$. Then, $G(\bar{x}, k(\bar{x})) = G(\bar{x}, r_h(\bar{x}))$. Since both $G(\bar{x}, r_h(\bar{x})) =$

$h(F(\bar{x}, r_h(\bar{x})))$ and $G(\bar{x}, k(\bar{x})) = k(\bar{x})$ are satisfied,

$$\begin{aligned}
k(\bar{x}) &= h(F(\bar{x}, r_h(\bar{x}))),\\
\bar{x} &= h(F(\bar{x}, k(\bar{x}))),\\
&= h(F(\bar{x}, \bar{x})),\\
&= h\left(\frac{\alpha + 2\bar{x}}{A + 2\bar{x}}\right),\\
&= \frac{\left(\frac{\alpha+2\bar{x}}{A+2\bar{x}}\right)^2 + (A-1)\left(\frac{\alpha+2\bar{x}}{A+2\bar{x}}\right) - \alpha}{\left(\frac{A-\alpha}{A+2\bar{x}}\right)},\\
&= \left(\frac{\frac{-\alpha(A+2\bar{x})^2+(\alpha+2\bar{x})^2+(A-1)(\alpha+2\bar{x})(A+2\bar{x})}{(A+2\bar{x})^2}}{\frac{A-\alpha}{A+2\bar{x}}}\right),\\
&= \left(\frac{-\alpha(A+2\bar{x})^2 + (\alpha+2\bar{x})^2 + (A-1)(\alpha+2\bar{x})(A+2\bar{x})}{(A-\alpha)(A+2\bar{x})}\right),\\
&= \left(\frac{(A-1)(A-\alpha)(2\bar{x}) + (A-\alpha)(2\bar{x})^2 - \alpha(A-\alpha)}{(A-\alpha)(A+2\bar{x})}\right),\\
&= \left(\frac{(A-1)(2\bar{x}) + (2\bar{x})^2 - \alpha}{(A+2\bar{x})}\right).
\end{aligned}$$

Multiplying both sides by $(A + 2\bar{x})$, this yields,

$$\begin{aligned}
A\bar{x} + \bar{x}(2\bar{x}) &= (A-1)(2\bar{x}) + (2\bar{x})^2 - \alpha,\\
0 &= (2\bar{x})^2 + (A - 1 - \bar{x})(2\bar{x}) - \alpha - A\bar{x},\\
0 &= 4\bar{x}^2 + 2A\bar{x} - 2\bar{x} - 2\bar{x}^2 - \alpha - A\bar{x},\\
0 &= 2\bar{x}^2 + \bar{x}(A-2) - \alpha,\\
\bar{x} &= \frac{-(A-2) \pm \sqrt{(A-2)^2 + 8\alpha}}{2}.
\end{aligned}$$

Thus the only positive solution is $\bar{x} = \frac{-(A-2)+\sqrt{(A-2)^2+8\alpha}}{2} = x^*$. Since $k(x) = x$, we have that the two functions intersect at $E^* = (x^*, x^*)$. □

6.1.5 Pre-image of the root-curve

By Definition 2.3.12, we want to find a function $y = Q(x)$ that satisfies $G(x, y) = r_h(F(x, y))$. Equivalently, $x = \frac{1-A-F(x,y)+\sqrt{(1-A-F(x,y))^2+4(\alpha+F(x,y))}}{2}$, and so,

$$\begin{aligned}
(2x - (1 - A - F(x,y)))^2 &= (1 - A - F(x,y))^2 + 4(\alpha + F(x,y)),\\
4x^2 - 4x(1 - A - F(x,y)) &= 4(\alpha + F(x,y)),\\
x^2 - x(1 - A - F(x,y)) &= (\alpha + F(x,y)),\\
x^2 - x(1-A) - \alpha &= F(x,y)(1-x),\\
h(x) &= F(x,y),\\
h(x) &= \frac{\alpha + x + y}{A + x + y},\\
y(h(x) - 1) &= \alpha + x - h(x)(A + x),\\
Q(x) := y &= \frac{h(x)(A+x) - (\alpha + x)}{1 - h(x)}.
\end{aligned}$$

6.1.6 Proof of Lemma 4.1.7

Proof. We first show that if $x < \frac{\sqrt{4+4\alpha+A^2}-A}{2}$, then equivalently,

$$\begin{aligned}
2x + A &< \sqrt{4+4\alpha+A^2}, \\
4x^2 + 4Ax + A^2 &< 4 + 4\alpha + A^2, \\
x^2 + Ax &< 1 + \alpha, \\
x^2 + Ax - x - \alpha &< 1 - x.
\end{aligned}$$

Since $x \in (0,1)$, we divide both sides by $1-x$ and $h(x) = \frac{x^2+(A-1)x-\alpha}{1-x} < 1$.

Now, let $y_n > Q(x_n)$, then $y_n > \frac{h(x_n)(A+x_n)-(\alpha+x_n)}{1-h(x_n)}$. Since $x_n < \frac{\sqrt{4+4\alpha+A^2}-A}{2}$, we have that $h(x_n) < 1$. Thus,

$$\begin{aligned}
y_n(1-h(x_n)) &> h(x_n)(A+x_n) - (\alpha + x_n), \\
\alpha + x_n + y_n &> h(x_n)(A + x_n + y_n), \\
\frac{\alpha + x_n + y_n}{A + x_n + y_n} &> h(x_n), \\
x_{n+1} &> \frac{x_n^2 + x_n(A-1) - \alpha}{1 - x_n}.
\end{aligned}$$

Since $x_n \in (0,1)$, we multiply both sides by $(1-x_n)$ and obtain,

$$\begin{aligned}
x_{n+1}(1-x_n) &> x_n^2 + x_n(A-1) - \alpha, \\
x_{n+1} + \alpha &> x_n^2 - x_n(1 - A - x_{n+1}), \\
(1 - A - x_{n+1})^2 + 4(\alpha + x_{n+1}) &> (2x_n - (1 - A - x_{n+1}))^2, \\
\frac{1 - A - x_{n+1} + \sqrt{(1 - A - x_{n+1})^2 + 4(\alpha + x_{n+1})}}{2} &> x_n, \\
r_h(x_{n+1}) &> y_{n+1}.
\end{aligned}$$

Showing $y_n < Q(x_n)$ implies $y_{n+1} > r_h(x_{n+1})$ is similar and the proof for the equality case come from the definition of $y = Q(x)$. □

6.1.7 Calculation of $y = S(x)$

We want to find a function $y = S(x)$ that satisfies $F(x,y) = x^*$. Thus we rearrange,

$$\begin{aligned}
F(x,y) &= x^*, \\
\frac{\alpha + x + y}{A + x + y} &= x^*, \\
\alpha + x + y &= x^*(A + x) + x^* y, \\
y(1 - x^*) &= x^*(A + x) - x - \alpha, \\
y(1 - x^*) &= x^*(A + x) - x - \alpha, \\
y &= \frac{x^* A + x(x^* - 1) - \alpha}{1 - x^*}, \\
S(x) := y &= -x + \frac{x^* A - \alpha}{1 - x^*}.
\end{aligned}$$

6.2 Additional Proofs for Chapter 4: $A < \alpha$

6.2.1 Component-wise monotone regions

We want to show that,

$$x_{n+1} - x_n \begin{cases} > 0, & y_n < h(x_n), \\ = 0, & y_n = h(x_n), \\ < 0, & y_n > h(x_n), \end{cases} \quad \text{and,} \quad y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases}$$

Proof. Assuming that $y_n > h(x_n)$, this is equivalent to,

$$y_n > \frac{Ax_n + x_n^2 - (\alpha + x_n)}{(1 - x_n)},$$

Note that $1 - x_n < 0$ and thus,

$$\begin{aligned} y_n(1 - x_n) &< Ax_n + x_n^2 - (\alpha + x_n), \\ \alpha + x_n + y_n &< Ax_n + x_n^2 + x_n y_n, \\ \alpha + x_n + y_n &< x_n(A + x_n + y_n), \\ x_{n+1} &< x_n. \end{aligned}$$

Similarly, $y_n < h(x_n)$ implies that $\Delta x_n > 0$. The equality case comes from the definition of an x-nullcline. For the y-component monotone regions, refer to previous calculations for the direction field in the $A > \alpha$ case in Chapter 6, Section 6.1.1. □

6.2.2 Proof for Proposition 4.2.1

Proof. **a)** We argue by the contrapositive; we show that $y_{n+1} \leq h(x_{n+1})$ implies $y_n \geq r_h(x_n)$. Starting with $y_{n+1} \leq h(x_{n+1})$, we have that,

$$\begin{aligned} \frac{\frac{-\alpha(A+x_n+y_n)^2 + (\alpha+x_n+y_n)^2 + (A-1)(\alpha+x_n+y_n)(A+x_n+y_n)}{(A+x_n+y_n)^2}}{\frac{A-\alpha}{A+x_n+y_n}} &\geq x_n, \\ \frac{-\alpha(A + x_n + y_n)^2 + (\alpha + x_n + y_n)^2 + (A - 1)(\alpha + x_n + y_n)(A + x_n + y_n)}{(A - \alpha)(A + x_n + y_n)} &\geq x_n, \\ \frac{(A - 1)(A - \alpha)(x_n + y_n) + (A - \alpha)(x_n + y_n)^2 - \alpha(A - \alpha)}{(A - \alpha)(A + x_n + y_n)} &\geq x_n, \\ \frac{(A - 1)(x_n + y_n) + (x_n + y_n)^2 - \alpha}{(A + x_n + y_n)} &\geq x_n. \end{aligned}$$

Thus, we have that,

$$\begin{aligned} (A - 1)(x_n + y_n) + (x_n + y_n)^2 - \alpha &\geq x_n(A + x_n + y_n), \\ (x_n + y_n)^2 + (A - 1 - x_n)(x_n + y_n) - (\alpha + Ax_n) &\geq 0, \\ y_n^2 + y_n(A - 1 + x_n) - (\alpha + x_n) &\geq 0. \end{aligned}$$

Solving for y_n,

$$\begin{aligned} \left(y_n + \frac{(A - 1 + x_n)}{2}\right)^2 &\geq \frac{(A - 1 + x_n)^2}{4} + (\alpha + x_n), \\ y_n &\geq \frac{1 - A - x_n + \sqrt{(1 - A - x_n)^2 + 4(\alpha + x_n)}}{2}, \\ y_n &\geq r_h(x_n). \end{aligned}$$

Similarly, we show can b) by the contrapositive.

c) If $y_n < r_k(x_n)$, then equivalently, $y_n < \frac{x_n^2+(A-1)x_n-\alpha}{1-x_n}$. Since $1 - x_n < 0$,

$$\begin{aligned} y_n - y_n x_n &> x_n^2 + Ax_n - x_n - \alpha, \\ \alpha + x_n + y_n &> x_n(A + x_n + y_n), \\ \frac{\alpha + x_n + y_n}{A + x_n + y_n} &> x_n, \\ k(x_{n+1}) &> y_{n+1}. \end{aligned}$$

The proof of d) is similar. □

6.2.3 Additional Lemma

Lemma 6.2.1. *Consider* (4.3) *with $A < \alpha$ and the regions defined in* (4.12)*. Then, no region defined in* (4.12) *by the augmented phase portrait is positively invariant.*

Proof. If $(x_n, y_n) \in \mathcal{R}_1$, by definition (4.12), $y_n \le k(x_n)$, $y_n < r_k(x_n)$, and $y_n < r_h(x_n)$ for $x \in (1, \frac{\alpha}{A})$. By Lemma 4.2.1, $h(x_{n+1}) = r_k(x_{n+1}) < y_{n+1} < k(x_{n+1})$. Hence, by definition of the regions (4.12), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_1} \cup \mathcal{R}_{2_2}$ and $\mathcal{R}_1$ is not positively invariant.

If $(x_n, y_n) \in \mathcal{R}_{2_1}$, then by definition (4.12), $r_k(x_n) < y_n \le r_h(x_n)$. From Lemma 4.2.1, $y_{n+1} \ge h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} > k(x_{n+1})$. Thus by definition of the regions (4.12) $(x_{n+1}, y_{n+1}) \in \mathcal{R}_3$ and $\mathcal{R}_{2_1}$ is not positively invariant.

If $(x_n, y_n) \in \mathcal{R}_{2_2}$, then by definition (4.12), $r_k(x_n) < r_h(x_n) < y_n < k(x_n)$ for $x_n \in (x^*, \frac{\alpha}{A})$. By Lemma 4.2.1, $k(x_{n+1}) < y_{n+1} < h(x_{n+1}) = r_k(x_{n+1})$ and by definition of the regions (4.12), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_1} \cup \mathcal{R}_{4_2}$. Thus, $\mathcal{R}_{2_2}$ is not positively invariant.

If $(x_n, y_n) \in \mathcal{R}_3$, then by definition (4.12) $y_n \ge k(x_n)$, $y_n > r_h(x_n)$, and $y_n > r_k(x_n)$ for $x \in (1, \frac{\alpha}{A})$. By Lemma 4.2.1, $k(x_{n+1}) < y_{n+1} < h(x_{n+1}) = r_k(x_{n+1})$ and by definition of the regions (4.12), $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{4_2} \cup \mathcal{R}_{4_1}$. Thus, $\mathcal{R}_3$ is not positively invariant.

If $(x_n, y_n) \in \mathcal{R}_{4_1}$, by definition of the region (4.12) we have $r_h(x_n) < y_n \le r_k(x_n)$ for $x \in (1, x^*)$. By Lemma 4.2.1, $y_{n+1} < h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} \le k(x_{n+1})$. Thus, by definition of the regions (4.12) $(x_{n+1}, y_{n+1}) \in \mathcal{R}_1$ and $\mathcal{R}_{4_1}$ is not positively invariant.

Letting $(x_n, y_n) \in \mathcal{R}_{4_2}$, by the definition of the region (4.12) that $y_n \le r_h(x_n) < r_k(x_n)$ for $x \in (1, \frac{\alpha}{A})$. By Lemma 4.2.1 $y_{n+1} \ge h(x_{n+1}) = r_k(x_{n+1})$ and $y_{n+1} < k(x_{n+1})$ and, by the definition of the regions, $(x_{n+1}, y_{n+1}) \in \mathcal{R}_{2_1} \cup \mathcal{R}_{2_2}$. Thus, $\mathcal{R}_{4_2}$ is not positively invariant. □

6.3 Additional Proofs for Chapter 4: $A = \alpha$

6.3.1 Component-wise monotone regions

We want to show,

$$x_{n+1} - x_n \begin{cases} > 0, & x_n < 1, \\ = 0, & x_n = 1, \\ < 0, & x_n > 1, \end{cases} \quad \text{and,} \quad y_{n+1} - y_n \begin{cases} > 0, & y_n < k(x_n), \\ = 0, & y_n = k(x_n), \\ < 0, & y_n > k(x_n). \end{cases}$$

Proof. If $x_n < 1$, then $x_n < x_{n+1}$ since $x_{n+1} = 1$. Similarly, $x_n > 1$ implies $x_{n+1} - x_n < 0$. The equality case comes from the definition of x-nullclines.

If $y_n < k(x_n)$, then $y_n < x_n$ and $y_n < y_{n+1}$. Similarly, $y_n > k(x_n)$ implies $y_{n+1} - y_n < 0$. The equality case comes from the definition of y-nullclines. □

6.3.2 Proof for Proposition 4.3.2

Proof. **a)** Let $x_n < 1$. Then $y_{n+1} < x_{n+1}$ and so

$$y_{n+1} < k(x_{n+1}).$$

b) The proof of b) is similar to a). □

Chapter 7

Conclusion

Applying the methods introduced by Streipert and Wolkowicz in [10], we were able to analyze various cases of

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma y_n}{A + Bx_n + Cy_n}, \text{ and } y_{n+1} = x_n,$$

with $n \in \mathbb{N}_0$, parameters $A, B, C, \alpha, \beta, \gamma \geq 0$, and initial conditions, $x_0, y_0 > 0$. Using nullclines and their associated root-curves as defined in [10] allowed us to prove that for the case,

$$x_{n+1} = \frac{\alpha + y_n}{A + x_n}, \text{ and } y_{n+1} = x_n,$$

with parameters $\alpha > 0$, $A, \geq 0$, and initial condition x_0, $y_0 > 0$, there exist solutions that increase and decrease monotonically to the equilibrium. By using the augmented phase portrait, we were able to determine regions in the plane where such solutions could exist. Furthermore, since the unique positive equilibrium is a saddle point, we determined the regions in which the stable manifold lies. Since the stable manifold is found in the regions where monotonically increasing and decreasing solutions could exist, we proved that such solutions exist along the stable manifold and converge to the equilibrium.

As a consequence, we were able to show Conjecture 1 since this would be a special case. While this conjecture has been previously shown by [5] and [12], the methods we used to show this conjecture were unlike any previous methods. Our method was visually intuitive and provided a holistic understanding of the behaviour of solutions of this equation. Additionally, the methods themselves are elementary. This thesis can be read as a guide to applying these methods.

While showing Conjecture 1 motivated the theorem stating the existence of solutions that increase and decrease monotonically towards the equilibrium, we also provide a new proof for Theorem 3.1.12 from [3] and [8] for a particular case. This theorem discusses the behaviour of solutions that do not converge monotonically to the equilibrium. Specifically, it says that after the first semi-cycle, every oscillatory solution oscillates about the equilibrium with semi-cycle of length one. Furthermore, we find that non-oscillatory solutions must converge monotonically to the equilibrium. The dynamics of this system become clear with the augmented phase portrait. While Theorem 3.1.12 is an existing theorem from [3] and [8], our proof shows the power in using the augmented phase portrait, from [10], to analyze recursions.

The second case we analyzed,

$$x_{n+1} = \frac{\alpha + x_n + y_n}{A + x_n + y_n}, \text{ and } y_{n+1} = x_n,$$

with parameters $\alpha > 0$, $A, > 0$, and initial conditions x_0, $y_0 > 0$, offered a different problem when showing the behaviour of the solutions. We examined various cases and were able to prove global asymptotic stability of the unique positive equilibrium by understanding the behaviour of solutions from the augmented phase portrait. For some cases, we also introduced two new curves, the pre-image of the line (x^*, y), with y arbitrary, and the pre-image of the root-curve associated with the nullcline. Using these two new curves in conjunction with the phase portrait, we proved the global asymptotic stability of the equilibrium.

In some cases, the planar analysis was less effective and left inconclusive results. However, we were still able to give a full picture of the behaviour of solutions since we could invoke theorems from [8] and proofs from [1] to show that the unique positive equilibrium is globally stable. Despite the inability to fully determine the behaviour of solutions using the augmented phase portrait in all cases, we still provided insight into how the system was changing as the parameters changed.

From this inquiry into applying nullclines and their associated root-curves, we found that augmented phase portraits are a powerful tool for understanding the general behaviour of solutions of difference equations. Similar to the existing theory on phase portraits for differential equations, the theory being develop by Streipert and Wolkowicz in [10] offers a new and promising method to better understand difference equations. This fresh perspective provided accurate visual intuition for the problems laid out in this thesis and allowed us to determine information about the behaviour of solutions of planar systems.

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