Hessenberg Patch Ideals of Codimension 1

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Abstract

A Hessenberg variety is a subvariety of the flag variety parametrized by two maps: a Hessenberg function on $[n]$ and a linear map on $\mathbb{C}^n$. We study regular nilpotent Hessenberg varieties in Lie type A by focusing on the Hessenberg function $h = (n-1, n, \ldots, n)$. We first state a formula for the $f_{n,1}^{w}$ which generates the local defining ideal $J_{w,h}$ for any $w \in \mathfrak{S}_n$. Second, we prove that there exists a convenient monomial order so that $\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h})$ is squarefree. As a consequence, we conclude that each codimension-1 regular nilpotent Hessenberg variety is locally Frobenius split (in positive characteristic).

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Contents

Dedicated to my dad.

Chapter 1

Introduction

The full flag variety $\mathrm{Flags}(\mathbb{C}^n)$ is a set consisting of nested sequences of subspaces of $\mathbb{C}^n$. Hessenberg varieties in type A are subvarieties of the full variety $\mathrm{Flags}(\mathbb{C}^n)$. They were introduced by De Mari, Procesi, and Shayman in [8]. Since the late 1980s, there has been considerable research into the geometry and equivariant topology of Hessenberg varieties [7, 8]. This field of study is situated at the point where several research domains, including geometric representation theory, combinatorics, and algebraic geometry and topology intersect, and it establishes links between them. In addition, Hessenberg varieties are intriguing because they appear in several contexts, such as the study of quantum cohomology of flag varieties, the generalization of Springer fibers in geometric representation theory, the integration of systems supported by the total spaces of certain families of Hessenberg varieties [1]. Their cohomology rings suggest that there is a significant Hessenberg analogue to the Schubert calculus theory on $\mathrm{Flags}(\mathbb{C}^n)$ [10]. The motivation for the current manuscript stems from the perspective of Schubert calculus, specifically the examination of the geometry of Schubert varieties. For instance, Miller and Knutson proved in another way by identifying the Stanley-Reisner complex as a special kind of subword complex in $\mathfrak{S}_n$ that Schubert varieties are Cohen-Macaulay [12].

The examination of local patches of Schubert varieties is a classical topic, and a significant amount of information is available concerning their corresponding local defining ideals. These ideals allow for the deduction of properties of Schubert varieties.

In this thesis, our focus is on the investigation of local patches of Hessenberg varieties which are the intersections of these varieties with particular choices of open affine Zariski subsets of $\mathrm{Flags}(\mathbb{C}^n)$. Due to the isomorphism $\mathrm{Flags}(\mathbb{C}^n) \cong GL_n(\mathbb{C})/B$ where B is the Borel subgroup of upper triangular matrices, $\mathrm{Flags}(\mathbb{C}^n)$ can be covered by coordinate charts centered at permutation flags $w \in \mathfrak{S}_n$. These coordinate charts are indeed Zariski-open subsets of $GL_n(\mathbb{C})/B$ each isomorphic to $\mathbb{A}^{\frac{n(n-2)}{2}}$. More precisely, we study a special case of the regular nilpotent Hessenberg variety that is parametrized by the Hessenberg function $h = (n-1, n, \ldots, n)$. Regular nilpotent Hessenberg varieties are defined in [2] as the intersections of

Hessenberg varieties with some choices of affine Zariski-open subsets of $\mathrm{Flags}(\mathbb{C}^n)$. There are several papers in which the authors focused on local defining ideals of local patches of Hessenberg varieties. In [6], the authors provided some algebraic properties of Hessenberg patch ideals of w_0-chart for any choice of indecomposable h and they proved $J_{w_0,h}$ of the regular nilpotent Hessenberg variety $\mathrm{Hess}(N, h)$ is Frobenius split.

In Chapter 3, we first give some helpful combinatorial results to examine the generators of Hessenberg patch ideals, and secondly, we focus on the Hessenberg patch ideal of codimension one. In Section 3.2, we define a monomial order so that the corresponding Hessenberg patch ideal $J_{w,h}$ with $h = (n-1, n, \dots, n)$ has a squarefree initial ideal for any permutation matrix $w \in \mathfrak{S}_n$. We use the result to show that each Hessenberg patch variety $\mathbb{V}(J_{w,h})$ for $h = (n-1, n \dots, n)$ is Frobenius split when working over a base field $\mathbb{K}$ which is perfect of characteristic $p > 0$.

Chapter 2

Background

In this chapter we state several results and definitions which will be helpful in the following chapters.

2.1 Ideals and Gröbner bases

First, we introduce some standard computational algebraic background. Our main reference is [5]. Let $\mathbb{K}$ be a field and $\mathbb{K}[x_{i,j}]$ be a polynomial ring in the indeterminates $x_{i,j}$ with $1 \leq i, j \leq n$.

Definition 2.1.1. *A monomial* $\mathbf{x}^{\mathbf{a}} \in \mathbb{K}[x_{i,j}]$ *is a product* $\mathbf{x}^{\mathbf{a}} = x_{11}^{a_{11}} x_{12}^{a_{12}} \dots x_{nn}^{a_{nn}}$*, where* $\mathbf{a} = (a_{11}, a_{12}, \dots, a_{nn}) \in \mathbb{N}^{n^2}$*.*

- *An ideal* $I \subset \mathbb{K}[x_{i,j}]$ *generated by monomials is said to be a* ***monomial ideal****.*
- *A monomial* $\mathbf{x}^{\mathbf{a}}$ *is called* ***squarefree*** *if every exponenent* a_{ij} *is either 0 or 1. An ideal generated by squarefree monomials is also called* ***squarefree****.*

Definition 2.1.2. *A* ***monomial order*** $<$ *on* $\mathbb{K}[x_{i,j}]$ *is a total order on the set of monomials* $\{x^a\}_{a \in \mathbb{N}^{n^2}}$ *which satisfies the following properties:*

1. *if* $\mathbf{x}^{\mathbf{a}} < \mathbf{x}^{\mathbf{b}}$*, then* $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}} < \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{a}}$ *for any* $\mathbf{x}^{\mathbf{c}} \in \mathbb{K}[x_{i,j}]$*, and*
2. $<$ *is a well-ordering, i.e., every non-empty set of monomials has a smallest element.*

Now, we will define several well-known examples of monomial orders on $\mathbb{K}[x_{i,j}]$. First, we specify an ordering of the variables as

$$x_{k,\ell} < x_{m,n} \text{ if } \ k > m, \text{ or, } k = m \text{ and } \ell > n.$$

Let $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathbb{K}[x_{i,j}]$ be two monomials.

- **The lexicographic order:** We say $\mathbf{x}^{\mathbf{a}} <_{lex} \mathbf{x}^{\mathbf{b}}$, if the leftmost nonzero coordinate of $\mathbf{b} - \mathbf{a}$ is positive.

- **The graded lexicographic order:** We say $\mathbf{x^a} <_{grlex} \mathbf{x^b}$, if either $\sum a_i < \sum b_i$, or $\sum a_i = \sum b_i$ and $\mathbf{x^a} <_{lex} \mathbf{x^b}$.

Example 2.1.3. Consider the polynomial ring $k[x, y, z]$ with $x > y > z$. Let $f = xyz + 2x$ and $g = x^2 - z^2$ in $k[x, y, z]$. Therefore, we observe that

- $g > f$ with respect to the lexicographic order, and
- $f > g$ with respect to the graded lexicographic order.

Definition 2.1.4. *Let $<$ be a monomial order on $\mathbb{K}[x_{i,j}]$. Given a polynomial $f \neq 0$ and an ideal $I \neq (0)$ in $\mathbb{K}[x_{i,j}]$, we define the following notions:*

1. *the largest term of f with respect to $<$ is called the **initial term** and denoted by $\mathrm{in}_<(f)$,*
2. *the coefficient $c \in \mathbb{K}$ of $\mathrm{in}_<(f)$ is called the **initial coefficient** of f, and $\frac{1}{c}.\mathrm{in}_<(f)$ is the **initial monomial** of f,*
3. *the **initial ideal** of I, $\mathrm{in}_<(I)$, is generated by $\mathrm{in}_<(g)$ for all nonzero $g \in I$.*

In Chapter 3.2, we will deal with initial terms with initial coefficient 1.
The obvious question to ask about $\mathrm{in}_<(I)$ is how to find a generating set. First, recall that we can find a finite generating set for any ideal of $\mathbb{K}[x_{i,j}]$ by the Hilbert basis theorem.

Theorem 2.1.5. *[5, §2.5, Theorem 4 (Hilbert Basis Theorem)] Let $I \subset \mathbb{K}[x_{i,j}]$ be a nonzero ideal. Then there are some polynomials $f_1, \ldots, f_s$ in I such that $I = \langle f_1, \ldots, f_s \rangle$.*

However, it is not always possible to compute $\mathrm{in}_<(I)$ by taking the initial terms of a given finite generator set for I. In particular, if $I = \langle f_1, \ldots, f_s \rangle \subset \mathbb{K}[x_{i,j}]$, $\mathrm{in}_<(I)$ might contain more elements than $\langle \mathrm{in}_<(f_1), \ldots, \mathrm{in}_<(f_s) \rangle$. Let's look at an example so that we can see this situation.

Example 2.1.6. Consider the lexicographic ordering on the polynomial ring $k[x, y, z]$ with $x > y > z$. Take an ideal $I = \langle f, g \rangle$ of $k[x, y, z]$ with $f = x^2y + 2yz$ and $g = y^2 - z$. Then $\mathrm{in}_<(f) = x^2y$ and $\mathrm{in}_<(g) = y^2$. Observe also that

$$yf - x^2g = y(x^2y + 2yz) - x^2(y^2 - z) = 2y^2z + x^2z \in I.$$

This implies that $\mathrm{in}_<(2y^2z+x^2z) = x^2z \in \mathrm{in}_<(I)$ by Definition 2.1.4. However, x^2z is divisible neither by $\mathrm{in}_<(f) = x^2y$ nor by $\mathrm{in}_<(g) = y^2$, so $x^2z \notin \langle \mathrm{in}_<(f), \mathrm{in}_<(g) \rangle$. Therefore, $\mathrm{in}_<(I)$ is not generated by $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$.

It turns out that one can find a finite generating set for any polynomial ideal I so that the corresponding initial terms generate its initial ideal $\mathrm{in}_<(I)$.

Definition 2.1.7. *Let $<$ be a monomial order on $\mathbb{K}[x_{i,j}]$. We say that $G = \{g_1, \dots, g_m\} \subset I$ is a **Gröbner basis** of I if*

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(g_1), \dots, \mathrm{in}_<(g_m) \rangle.$$

It can be shown that any Gröbner basis of I is also a generating set for I [5, §2.5 Corollary 6]. Gröbner bases are important tools in algebraic geometry as they resolve the problem we had encountered in Example 2.1.6. Furthermore, it resolves the Ideal Membership Problem which asks if a given polynomial f belongs to a given polynomial ideal I. There are some different ways to compute Gröbner basis such as Buchberger's Algorithm, see [5, §2.6, §2.7]. In Buchberger's Algorithm, we use S-polynomials for concrete calculations.

Definition 2.1.8. *[5, §2.6, Definition 4] Let $<$ be a monomial order on $\mathbb{K}[x_{i,j}]$. Let $f, g \in \mathbb{K}[x_{i,j}]$. Then the S-polynomial of f and g is defined as follows:*

$$S(f,g) := \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_1.\mathrm{in}_<(f)} f - \frac{\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))}{c_2.\mathrm{in}_<(g)} g,$$

where $\mathrm{lcm}(\mathrm{in}_<(f), \mathrm{in}_<(g))$ is the least common multiple of $\mathrm{in}_<(f)$ and $\mathrm{in}_<(g)$, and c_1, c_2 are the leading coefficients of f, g, respectively.

We say that a polynomial f in the Gröbner basis G is **irreducible** if no monomial of f lies in $\langle \mathrm{in}_<(G \setminus \{f\}) \rangle$. A Gröbner basis is **reduced** if every polynomial in it is irreducible by the other elements of the basis, and has 1 as initial coefficient. Since we will deal with the Hessenberg patch ideals of codimension one in this thesis, which are principal ideals, we only need to examine the Gröbner bases of polynomial ideals with one generator. By Buchberger's Algorithm, an ideal I generated by a single polynomial f with initial coefficient 1 has reduced Gröbner basis $\{f\}$.

Example 2.1.9. We define a monomial order $<$ on $k[x, y, z]$ by assuming $z > x > y$ and letting $<$ be a lexicographic order. Then take the polynomial $f = x^2y + 2yz$. With respect to the monomial order we fixed, we rewrite it as

$$f = 2yz + x^2y.$$

Therefore, for $I = \langle f \rangle$, we get a squarefree initial ideal

$$\mathrm{in}_{\prec_{\mathrm{w}}}(I) = \langle yz \rangle$$

by Definition 2.1.1. So the reduced Gröbner basis is $\frac{1}{2}f$.

2.2 Flag Varieties

A (full) **flag** $F_\bullet$ is a nested sequences of subspaces of $\mathbb{C}^n$, that is,

$$F_\bullet = (F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n = \mathbb{C}^n) \text{ such that } \dim_{\mathbb{C}}(F_i) = i \ \forall i.$$

The **flag variety** denoted $\mathrm{Flags}(\mathbb{C}^n)$, is the set consisting of such flags, i.e.

$$\mathrm{Flags}(\mathbb{C}^n) = \{F_\bullet = (F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(F_i) = i \quad \forall i\}.$$

Example 2.2.1. Assume $n = 4$. Then we have an example of a flag $F_\bullet$ as follows

$$F_\bullet = \left\{ \left\langle \begin{pmatrix} 2\\1\\0\\0 \end{pmatrix} \right\rangle \subset \left\langle \begin{pmatrix} 2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix} \right\rangle \subset \left\langle \begin{pmatrix} 2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 6\\0\\3\\0 \end{pmatrix} \right\rangle \subset \left\langle \begin{pmatrix} 2\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 6\\0\\3\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix} \right\rangle = \mathbb{C}^4$$

Example 2.2.2. The standard flag in $\mathrm{Flags}(\mathbb{C}^n)$ is

$$I_\bullet = (\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \langle e_1, e_2, \ldots, e_n \rangle = \mathbb{C}^n),$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{C}^n$.

We can simplify the notation of the flag by writing it as an $n \times n$ matrix. To illustrate, let's rewrite the flag in example 2.2.1 as follows

$$\begin{pmatrix} 2 & 0 & 6 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Note that we can represent a flag using different matrices. Let

$$\begin{pmatrix} 4 & 0 & 0 & 0 \\ 2 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{2.2}$$

The matrices 2.1 and 2.2 represent the same flag from Example 2.2.1 since the first i columns of either matrix generate the same subspace for each i, $1 \leq i \leq 4$. Note that we can find infinitely many bases for a subspace of $\mathbb{C}^k$ for all $1 \leq k \leq n$.

Therefore, one can find infinitely many distinct matrices to represent a given flag. To resolve this ambiguity in the notation, we apply a group action of $GL_n(\mathbb{C})$ on $\text{Flags}(\mathbb{C}^n)$, and it gives us a way to handle the structure of $\text{Flags}(\mathbb{C}^n)$. The action is defined for $g \in GL_n(\mathbb{C})$ and $F_\bullet \in \text{Flags}(\mathbb{C}^n)$ as follows:

$$\begin{aligned} g * F_\bullet &= g * (F_1 \subset F_2 \subset \ldots \subset F_n = \mathbb{C}^n) \\ &= (g.F_1 \subset g.F_2 \subset \ldots \subset g.F_n = g.\mathbb{C}^n) \end{aligned} \tag{2.3}$$

where $g.F_i := g.\langle v_1, v_2, \ldots, v_i\rangle = \langle g.v_1, g.v_2, \ldots, g.v_i\rangle$ for each $i = 1, \ldots, n$ with $F_i = \langle v_1, v_2, \ldots v_i\rangle$ for some vectors $v_i \in \mathbb{C}^n$ and $g.v_i$ is given by the linear action of $g \in GL_n(\mathbb{C})$ on $\mathbb{C}^n$.

Definition 2.2.3. *Let G be a group and A a set. Then the map $* : G \times A \to A$ is called a **group action** if it satisfies the following properties*

1. $(gh) * a = g * (h * a)$ *for all* $g, h \in G$ *and* $a \in A$*,*
2. $e * a = a$ *for all* $a \in A$*.*

We claim that 2.3 defines an action of $GL_n(\mathbb{C})$ on $\text{Flags}(\mathbb{C}^n)$. We first check that the action of $GL_n(\mathbb{C})$ on $\text{Flags}(\mathbb{C}^n)$ respects the definition of a flag, i.e., preserves the containment of the subspaces in a sequence and their dimensions.

Lemma 2.2.4. *The action of $GL_n(\mathbb{C})$ on $\text{Flags}(\mathbb{C}^n)$ respects the inclusions and dimensions of subspaces F_i in a flag $F_\bullet \in \text{Flags}(\mathbb{C}^n)$.*

Proof. Given $g \in GL_n(\mathbb{C})$ and $F_\bullet \in \text{Flags}(\mathbb{C}^n)$, we want to show that $\dim_{\mathbb{C}} gF_i = i$ for all $1 \leq i \leq n$ and $(gF_1 \subset gF_2 \subset \ldots \subset gF_n)$. Since any matrix g in $GL_n(\mathbb{C})$ can be seen as an invertible linear transformation G on $\mathbb{C}^n$, $\dim_{\mathbb{C}} F_i = \dim_{\mathbb{C}} G(F_i)$ for all i. Now, consider two consecutive subspaces $F_{j-1} = \langle v_1, \ldots, v_{j-1}\rangle$ and $F_j = \langle v'_1, \ldots, v'_j\rangle$ in $F_\bullet$. Since $F_{j-1} \subset F_j$, for $v_k \in F_{j-1}$ there are some scalars $\delta_1, \ldots, \delta_j$ such that

$$v_k = \delta_1 v'_1 + \ldots + \delta_j v'_j.$$

After applying the map G corresponding to g, we get

$$G(v_k) = G(\delta_1 v'_1 + \ldots + \delta_j v'_j) = \delta_1 G(v'_1) + \ldots + \delta_j G(v'_j)$$

by linearity of G. Hence, $G(F_{j-1}) \subset G(F_j)$ for all j. So we conclude that $(gF_1 \subset gF_2 \subset \ldots \subset gF_n)$. □

Next, we check the action is a valid group action.

Lemma 2.2.5. *The action defined in 2.3 is a group action.*

Proof. Let $g, h \in GL_n(\mathbb{C})$, and $F_\bullet \in \text{Flags}(\mathbb{C}^n)$. Therefore,

$$\begin{aligned}(gh) * F_\bullet &= (gh) * (F_1 \subset F_2 \subset \ldots \subset F_n) \\ &= ((gh).F_1 \subset (gh).F_2 \subset \ldots \subset (gh).F_n) \\ &= (g.(h.F_1) \subset g(h.F_2) \subset \ldots \subset g.(h.F_n)) \\ &= g * (h.F_1 \subset h.F_2 \subset \ldots \subset h.F_n) \\ &= g * (h * (F_1 \subset F_2 \subset \ldots \subset F_n)).\end{aligned}$$

So $(gh) * F_\bullet = g * (h * F_\bullet)$ as desired. Moreover, the identity element of $GL_n(\mathbb{C})$, denoted I_n, acts on $F_\bullet$ as follows

$$\begin{aligned}I_n * F_\bullet &= (I_n.F_1 \subset I_n.F_2 \subset \ldots \subset I_n.F_n) \\ &= (F_1 \subset F_2 \subset \ldots \subset F_n).\end{aligned}$$

Thus, $* : GL_n(\mathbb{C}) \times \text{Flags}(\mathbb{C}^n) \to \text{Flags}(\mathbb{C}^n)$ is a valid group action. □

Definition 2.2.6. *Let $* : G \times A \to A$ be a group action. If given $a_1, a_2 \in A$ there is a $g \in G$ such that $a_1 = g * a_2$, then the group action is said to be transitive.*

Lemma 2.2.7. *The action of $GL_n(\mathbb{C})$ on* $\text{Flags}(\mathbb{C}^n)$ *is transitive.*

Proof. Let $V_\bullet = (V_1 \subset \ldots \subset V_n = \mathbb{C}^n)$ and $W_\bullet = (W_1 \subset \ldots \subset W_n = \mathbb{C}^n)$ be two flags in $\text{Flags}(\mathbb{C}^n)$. We can find a basis $\langle v_1, \ldots, v_n \rangle$ for V_n such that $\langle v_1, \ldots, v_i \rangle = V_i$ by inclusions of subspaces. Similarly, $W_n = \langle w_1, \ldots, w_n \rangle$. We put $M := [v_1 \mid \ldots \mid v_n]$ and $N := [w_1 \mid \ldots \mid w_n]$. Since $\{v_1, \ldots, v_n\}$ is a basis $\mathbb{C}^n$, the columns of M are linearly independent, and so M is invertible. Then we can choose $g := NM^{-1}$ which implies that $g.M = N$. Thus, $g * V_\bullet = W_\bullet$. The conclusion is that the operation is transitive. □

Given a G-action on a set A, the **stabilizer** of $a \in A$ in G is

$$\text{Stab}(a) = \{g \in G : g * a = a\}.$$

Lemma 2.2.8. *The stabilizer of $I_\bullet$ is B, where $I_\bullet$ is $n \times n$ standard flag of Example 2.2.2.*

Proof. We want to show that $\mathrm{Stab}(I_\bullet) = B$. First, we let

$$g = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix} \in \mathrm{Stab}(I_\bullet).$$

First, $g * \langle e_1 \rangle = \langle g.e_1 \rangle = \langle e_1 \rangle$. Hence,

$$\langle g.e_1 \rangle = \left\langle \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{n1} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\rangle.$$

Therefore, $g_{21} = \ldots = g_{n1} = 0$. Secondly, $g * \langle e_1, e_2 \rangle = \langle g.e_1, g.e_2 \rangle = \langle e_1, e_2 \rangle$. Hence,

$$\langle g.e_1, g.e_2 \rangle = \left\langle \begin{pmatrix} g_{11} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} g_{12} \\ g_{22} \\ \vdots \\ g_{n2} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \right\rangle.$$

Similarly, we get that $g_{32} = g_{42} = \ldots = g_{n2} = 0$. If we continue this process, one can notice that we will have

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{13} & \cdots & g_{1n} \\ 0 & g_{22} & g_{23} & \cdots & g_{2n} \\ 0 & 0 & g_{33} & \cdots & g_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & g_{nn} \end{pmatrix}$$

where $g_{ii} \neq 0$ for all $1 \leq i \leq n$, and then $g \in B$. Conversely, let $M = [M_1 \mid M_2 \mid \ldots \mid M_n] \in B$. We note that M cannot have zero on the diagonal as it is invertible, therefore, $\langle M_1, \ldots, M_i \rangle = \langle e_1, \ldots, e_i \rangle$ for each i. Thus, $M \in \mathrm{Stab}(I_\bullet)$. $\square$

If G acts transitively on A, then $G/\mathrm{Stab}(a) \cong A$ for all $a \in A$ by the Orbit-Stabilizer Theorem.
Hence, we conclude by Lemmas 2.2.4, 2.2.5, 2.2.7 the following bijection coming from $GL_n(\mathbb{C})$-action on $\mathrm{Flags}(\mathbb{C}^n)$.

Fact 2.2.9. $\mathrm{Flags}(\mathbb{C}^n) \cong GL_n(\mathbb{C})/B$.

Denote by U^- the subgroup of $GL_n(\mathbb{C})$ containing lower triangular matrices with 1's along the diagonal,

$$U^- := \left\{ M = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \star & 1 & 0 & \dots & 0 \\ \star & \star & 1 & \dots & 0 \\ \vdots & \vdots & & & \\ \star & \star & \star & \dots & 1 \end{bmatrix} \right\} \cong \mathbb{A}^{\frac{n(n-1)}{2}}$$

where $\star$ are arbitrary complex scalars. Then $U^-B \subset GL_n(\mathbb{C})/B$ is the set of cosets uB for all $u \in U^-$. It turns out U^-B is a dense open set in $GL_n(\mathbb{C})/B$ and can be seen as a coordinate chart on $GL_n(\mathbb{C})/B$, see [6]. Let $n \geq 3$. We denote $[n]$ the set $\{1, 2, \dots, n\}$ and $\mathfrak{S}_n$ the symmetric group on $[n]$. Let $w \in \mathfrak{S}_n$. By abuse of notation, we denote by the same notation w the associated permutation matrix.

Example 2.2.10. Let $n = 4$, $w = (2314) \in \mathfrak{S}_n$. Then we represent w by the permutation matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{bmatrix} \in U^-.$$

Therefore, we can multiply w and M,

$$wM = \begin{bmatrix} a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ 1 & 0 & 0 & 0 \\ d & e & f & 1 \end{bmatrix}.$$

Note that multiplying M by w from the left permuted the rows of M.

Now, we define the coordinate chart containing the permutation flag $w \in \mathfrak{S}_n$ by

$$\mathcal{N}_w := wU^-B,$$

i.e. $\mathcal{N}_w$ is the left translate of U^-B by w. Then this allows us to consider the following open cover of $\mathrm{Flags}(\mathbb{C}^n)$:

$$\mathrm{Flags}(\mathbb{C}^n) = \bigcup_{w \in \mathfrak{S}_n} \mathcal{N}_w.$$

An element of $\mathcal{N}_w \subset \mathrm{Flags}(\mathbb{C}^n)$ can be identified with the w-translate of an element M in U^- by the map

$$\begin{aligned} U^- &\xrightarrow{\cong} \mathcal{N}_w \\ M &\mapsto wMB. \end{aligned} \tag{2.4}$$

It follows that each $\mathcal{N}_w$ is isomorphic to the affine space $\mathbb{A}^{\frac{n(n-1)}{2}}$. By the isomorphism 2.4, an element of $\mathcal{N}_w$ is uniquely determined by a matrix $wM = [x_{i,j}]_{1 \leq i,j \leq n}$ such that

$$\begin{aligned} x_{w(j),j} &= 1 \ \ \forall j \in [n] \\ x_{w(i),j} &= 0 \ \ \forall i,j \in [n]; \ j > i. \end{aligned} \tag{2.5}$$

Let I be an ideal of $\mathbb{C}[x_{i,j}]$ generated by those relations given in 2.5. We define $\mathbb{C}[x_w]$ as the coordinate ring of $\mathcal{N}_w$ with $\mathbb{C}[x_w] \cong \mathbb{C}[x_{i,j}]/I$.

2.3 Hessenberg Varieties

In this section, we define Hessenberg varieties which are subvarieties of the full flag varieties.

Definition 2.3.1. *A **Hessenberg function** is a function $h : [n] \to [n]$ such that $h(i) \geq i$ for all $i \in [n]$, and $h(i+1) \geq h(i)$ for all $i \in [n-1]$.*

For a Hessenberg function h, we use the one line notation $h = (h(1), h(2), \ldots, h(n))$ throughout.

Definition 2.3.2. *Let h be a Hessenberg function and $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map. Then A can be represented as an $n \times n$ complex matrix with respect to the standard basis of $\mathbb{C}^n$. The **Hessenberg variety** defined by the Hessenberg function h and the linear map A is*

$$\mathrm{Hess}(A, h) := \{F_\bullet \in \mathrm{Flags}(\mathbb{C}^n) \mid AF_i \subset F_{(h(i))} \quad \forall i\}.$$

Definition 2.3.3. *A Hessenberg function h is called* ***indecomposable*** *if $h(i) \geq i+1$ for all $1 \leq i \leq n-1$.*

Example 2.3.4. For $n = 4$, the indecomposable Hessenberg functions are

$$h_0 = (4,4,4,4), h_1 = (3,4,4,4),\ h_2 = (3,3,4,4),\ h_3 = (2,4,4,4),\ h_4 = (2,3,4,4).$$

2.4 Regular Nilpotent Hessenberg Patch Ideals

The geometric properties of the Hessenberg variety $\mathrm{Hess}(A, h)$ depend on the choice of A and h. Let A be a linear operator. We say that A is a **principal nilpotent operator** if the Jordan canonical form of A has a single Jordan block with eigenvalues zero, that is, up to change of basis A is of the form

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \tag{2.6}$$

Let N denote the matrix (2.6) throughout.

Definition 2.4.1. *Let A be a principal linear operator. Then the Hessenberg variety* $\mathrm{Hess}(A, h)$ *is said to be* ***regular nilpotent****.*

Example 2.4.2.

- Let $h = (n, n, \ldots, n)$. Then $\mathrm{Hess}(N, h) = \mathrm{Flags}(\mathbb{C}^n)$.
- Let $h = (2, 3, \ldots, n, n)$. Then $\mathrm{Hess}(N, h)$ is called a Peterson variety.

We examined the $GL_n(\mathbb{C})$-action on $\mathrm{Flags}(\mathbb{C}^n)$. The torus subgroup $S \subset GL_n(\mathbb{C})$ defined below acts naturally on $\mathrm{Flags}(\mathbb{C}^n)$.

$$S := \left\{ \begin{pmatrix} g & & & \\ & g^2 & & \\ & & \ddots & \\ & & & g^n \end{pmatrix} : g \in \mathbb{C}^* \right\}.$$

Indeed, since $Ns = sN$ for all $s \in S$, S-action on $\mathrm{Flags}(\mathbb{C}^n)$ preserves $\mathrm{Hess}(N, h)$, see [9].

Lemma 2.4.3. *[3, Lemma 2.3] The S-fixed point set* $\mathrm{Hess}(N, h)^S \subset \mathfrak{S}_n$ *of* $\mathrm{Hess}(N, h)$ *is given by*

$$\mathrm{Hess}(N, h)^S = \{w \in \mathfrak{S}_n \mid w^{-1}(w(j) - 1) \leq h(j) \quad \forall j \in [n]\}.$$

Restricting to the S-fixed points simplify the computations about the Hessenberg patch ideals $J_{w,h}$ as we narrow the set of permutation matrices to $\mathrm{Hess}(N, h)^S$.

Definition 2.4.4. *[2, Definition 3.3] Let $w \in \mathfrak{S}_n$ and let $i, j \in [n]$ with $i > j$. Define the polynomial in $\mathbb{C}[x_w]$ by*

$$f_{i,j}^{w} := \big((wM)^{-1}N(wM)\big)_{i,j}.$$

Lemma 2.4.5. *Let $w \in \mathfrak{S}_n$ and let $i, j \in [n]$ with $i > j$. Then the ideal of Hess(N,h) in $\mathcal{N}_w$ is defined by*

$$J_{w,h} := \big\langle f_{i,j}^{w} \mid i > h(j) \big\rangle \subset \mathbb{C}[x_w].$$

In [6], the authors defined explicitly the initial ideal of $J_{w,h}$ of $\mathrm{Hess}(N, h)$ with $w = (n, n-1, \ldots, 2, 1)$ for any indecomposable h with respect to the following monomial order:

(i) Assume that $x_{i,j} > x_{i',j'}$ if $i > i'$ or if $i = i'$ and $j < j'$.

(ii) Let $<$ be a lexicographic order.

Theorem 2.4.6. *[6, Theorem 5.15] Let $n \geq 3$, and $h : [n] \to [n]$ be an indecomposable Hessenberg function. Then the set of elements $f_{k,\ell}^{w_0}$ generating Hessenberg patch ideal $J_{w_0,h}$ of* $\mathrm{Hess}(N, h)$ *with $w = (n, n-1, \ldots, 2, 1)$, form a Gröbner basis for $J_{w_0,h}$ wiht respect to $<$. Moreover,*

$$\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w_0,h}) = \langle x_{n-i+1,j+1} : (i,j) \in H(h,0) \rangle,$$

where $H(h, 0) = \{(k, l) \in [n] \times [n] : h(l) < k < n\} \sqcup \{(k, l) \in [n] \times [n] : k = n, h(l) < n\}$.

2.4.1 Linear Algebra

Throughout this manuscript, let M denote the $n \times n$ matrix in U^- as follows:

$$M = \begin{pmatrix} 1 & & & & \\ x_{2,1} & 1 & & & \\ x_{3,1} & x_{3,2} & 1 & & \\ \vdots & \vdots & & & \\ x_{n,1} & x_{n,2} & x_{n,3} & \ldots & 1 \end{pmatrix}, \tag{2.7}$$

and M^{-1} denote its inverse is

$$M^{-1} = \begin{pmatrix} 1 & & & & \\ y_{2,1} & 1 & & & \\ y_{3,1} & y_{3,2} & 1 & & \\ \vdots & \vdots & & & \\ y_{n,1} & y_{n,2} & y_{n,3} & \dots & 1 \end{pmatrix} \tag{2.8}$$

where $x_{i,j}, y_{i,j} \in \mathbb{C}$, and $x_{i,i} = y_{i,i} = 1$, and $x_{i,j} = y_{i,j} = 0$ if $j > i$, for all $1 \leq i, j \leq n$. We denote by M^k the $k \times k$ submatrix of M intersecting at the leftmost k rows and the uppermost k columns of M, and $M^k_{\ell,k}$ the matrix obtained by deleting ℓ-th row and k-th column of M^k for any $3 \leq \ell < k \leq n$. To illustrate, for $n = 5$ we can take $k = 4, \ell = 2$. Then we get that

$$M = \begin{pmatrix} 1 & & & & \\ x_{2,1} & 1 & & & \\ x_{3,1} & x_{3,2} & 1 & & \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 & \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & 1 \end{pmatrix}$$

Therefore,

$$M^k = \begin{pmatrix} 1 & & & \\ x_{2,1} & 1 & & \\ x_{3,1} & x_{3,2} & 1 & \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 \end{pmatrix} \implies M^4_{2,4} = \begin{pmatrix} 1 & & \\ x_{3,1} & x_{3,2} & 1 \\ x_{4,1} & x_{4,2} & x_{4,3} \end{pmatrix}.$$

Note that for $w = (12 \dots n) \in \mathfrak{S}_n$, we get $(wM)^{-1}N(wM) = M^{-1}NM$. Now, we will give some linear algebraic results that will be helpful to conclude the combinatorial properties of the entries of M^{-1} and $(wM)^{-1}N(wM)$ later in Section 3.1.

Proposition 2.4.7. *Denote $M^n_{i,j}$ the $n-1$ minor obtained by deleting j-th row and i-th column of matrix M. Then the entries of M^{-1} are*

$$y_{i,j} = (-1)^{(i+j)} \det(M^n_{j,i}).$$

Moreover, $y_{i,j}$ is squarefree.

Proof. We have by [13, Theorem 8.9] that

$$M^{-1} = \frac{1}{\det(M)} \operatorname{adj}(M)$$

where adj(M) is the transpose of cofactor matrix of M. Since M is a lower triangular matrix with 1's along the diagonal, we have $\det(M) = 1$. Fix $1 \leq i, j \leq n$. Then the entries of M^{-1} can be computed as

$$[M^{-1}]_{i,j} = y_{i,j} = (-1)^{(i+j)} \det(M^n_{j,i})$$

for all $1 \leq i, j \leq n$. As the cofactors are multiple of several distinct variables, we conclude that $\det(M^n_{j,i})$ is squarefree, so is $y_{i,j}$. □

Example 2.4.8. For $n = 4$, consider

$$M = \begin{pmatrix} 1 & & & \\ x_{2,1} & 1 & & \\ x_{3,1} & x_{3,2} & 1 & \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 \end{pmatrix}.$$

If we take $w = (2314)$, then

$$wM = \begin{pmatrix} x_{2,1} & 1 & & \\ x_{3,1} & x_{3,2} & 1 & \\ 1 & & & \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 \end{pmatrix}.$$

Therefore, we compute its inverse by Proposition 2.4.7 as

$$\begin{aligned} (wM)^{-1} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -x_{2,1} & 0 \\ -x_{3,2} & 1 & x_{2,1}x_{3,2} - x_{3,1} & 0 \\ x_{3,2}x_{4,3} - x_{4,2} & -x_{4,3} & -x_{2,1}x_{3,2}x_{4,3} + x_{2,1}x_{4,2} + x_{3,1}x_{4,3} + x_{3,1}x_{4,3} - x_{4,1} & 1 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -x_{2,1} & 0 \\ -x_{3,2} & 1 & -x_{3,2}(-x_{2,1}) - x_{3,1} & 0 \\ x_{4,3}(x_{3,2}) - x_{4,2} & -x_{4,3} & -x_{4,3}(x_{2,1}x_{3,2} - x_{3,1}) - x_{4,2}(-x_{2,1}) - x_{4,1} & 1 \end{pmatrix}. \end{aligned}$$

Then we get that

$$y_{k,\ell} = \sum_{i=1}^{k-1} -x_{k,i}y_{i,\ell}$$

for each $k, \ell = 1, 2, 3, 4$ by assuming $y_{k,k} = 1$. Therefore, the Hessenberg patch ideal is $J_{w,h} = \left\langle \left[(wM)^{-1}N(wM)\right]_{4,1} \right\rangle$, and so

$$J_{w,h} = \langle -x_{2,1}^2 x_{3,2}x_{4,3} + x_{2,1}^2 x_{4,2} + x_{2,1}x_{3,1}x_{4,3} + x_{3,1}x_{3,2}x_{4,3} - x_{2,1}x_{4,1} - x_{3,1}x_{4,2} - x_{4,1}x_{4,3} \rangle.$$

Remark 2.4.9. By Proposition 2.4.7, we notice that $y_{i,j}$ is a multiple of distinct variables for all $1 < i \leq n$ and $1 \leq j < n$.

Proposition 2.4.10. *Let X be an $n \times n$ complex matrix in the form of*

$$\begin{pmatrix} Y & 0 \\ 0 & I \end{pmatrix}$$

where Y is a square matrix and I is the identity matrix. Then $\det(X) = \det(Y)$.

Proof. Consider a complex $n \times n$ matrix X. Let $2 \leq k \leq n$ such that X is of the form

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,k} & & & \\ \vdots & & \vdots & & & \\ x_{k,1} & \dots & x_{k,k} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where $x_{ii} = 1$ for all $i > k$ and $x_{i,j} = 0$ for all distinct $i, j > k$. If we compute the determinant of X along $k + 1$-th row, then we get that

$$\det(X) = \begin{vmatrix} x_{1,1} & \dots & x_{1,k} & & & \\ \vdots & & \vdots & & & \\ x_{k,1} & \dots & x_{k,k} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{vmatrix} = (-1)^{2(k+1)} \begin{vmatrix} x_{1,1} & \dots & x_{1,k} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \dots & x_{k,k} \end{vmatrix} = \begin{vmatrix} x_{1,1} & \dots & x_{1,k} \\ \vdots & \ddots & \vdots \\ x_{k,1} & \dots & x_{k,k} \end{vmatrix}.$$

□

Remark 2.4.11.

1. If X is of the form
$$\begin{pmatrix} I & 0 \\ 0 & Y \end{pmatrix}$$
then $\det(X) = \det(Y)$. The proof can proceed similarly to the previous proposition.

2. Let X be an $n \times n$ complex matrix such that

$$X = \left(\begin{array}{cccc|c} 1 & & & & \\ \star & 1 & & & \mathbf{0} \\ \vdots & & \ddots & & \\ \star & \star & \dots & 1 & \\ \hline & \bigstar & & & \bigstar \end{array}\right) = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right)$$

with given blocks X_1, X_2, X_3 and X_4 so that X_1 is a lower triangular square matrice and X_4 is a square matrix, X_2 is a zero matrix, and X_3 is a randomly chosen complex matrix. Note that X_1, X_4 are square matrices, however, X_2 and X_3 are not necessarily square. Then there exists $k \in \mathbb{N}$ such that X_1 is a $k \times k$ matrix. The column reduced echelon form of X_1 is the $k \times k$ identity matrix. Therefore,

$$\det(X) = \left|\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right| = \left|\begin{array}{c|c} I_k & 0 \\ \hline X_3 & X_4 \end{array}\right|.$$

Again by applying a finite number of row operations to X one can vanish the entries of X_3 by using the rows of X_4. During these operations, X_4 stays the same because X_2 is the zero matrix. Thus, by the previous remark, we get

$$\det(X) = \left|\begin{array}{c|c} I_k & 0 \\ \hline X_3 & X_4 \end{array}\right| = \left|\begin{array}{c|c} I_k & 0 \\ \hline 0 & X_4 \end{array}\right| = \det(X_4)$$

3. Let X be an $n \times n$ complex matrix such that

$$X = \left(\begin{array}{cccc|c|cccc} 1 & & & & & & & & \\ \star & 1 & & & \mathbf{0} & & \mathbf{0} & & \\ \vdots & & \ddots & & & & & & \\ \star & \star & \dots & 1 & & & & & \\ \hline & \bigstar & & & \bigstar & & \mathbf{0} & & \\ \hline & & & & & 1 & & & \\ & & & & & \star & 1 & & \\ & \bigstar & & & \bigstar & \vdots & & \ddots & \\ & & & & & \star & \star & \dots & 1 \end{array}\right) = \left(\begin{array}{c|c|c} X_1 & X_2 & X_3 \\ \hline X_4 & X_5 & X_6 \\ \hline X_7 & X_8 & X_9 \end{array}\right)$$

so that X_1, X_9 are a lower triangular square matrices and X_4, X_5, X_7, X_8 are randomly chosen complex matrices, X_2, X_3, X_6 are zero matrices. We want to show that $\det(X) = \det(X_5)$. First, we note that by applying some column operations we obtain identity matrices from the blocks X_1 and X_9 using the diagonal entries. Therefore,

$$\left(\begin{array}{c|c|c} X_1 & X_2 & X_3 \\ \hline X_4 & X_5 & X_6 \\ \hline X_7 & X_8 & X_9 \end{array}\right) \sim \left(\begin{array}{c|c|c} I_k & 0 & 0 \\ \hline X_4 & X_5 & 0 \\ \hline X_7 & X_8 & I_\ell \end{array}\right)$$

for some $2 \leq k, \ell \leq n-2$.

Again by applying a finite number of row operations to X one can vanish the entries of X_4 and X_7 by using the rows of I_k. During these operations, X_5, X_8 stays the same because X_2 is the zero matrix. Similarly, using the columns of I_ℓ, one can vanish X_8 by a finite number of column operations. Therefore,

$$\left(\begin{array}{c|c|c} I_k & 0 & 0 \\ \hline X_4 & X_5 & 0 \\ \hline X_7 & X_8 & I_\ell \end{array}\right) \sim \left(\begin{array}{c|c|c} I_k & 0 & 0 \\ \hline 0 & X_5 & 0 \\ \hline 0 & 0 & I_\ell \end{array}\right)$$

By replacing the columns I_ℓ, we can obtain

$$\det(X) = \left|\begin{array}{c|c|c} I_k & 0 & 0 \\ \hline 0 & X_5 & 0 \\ \hline 0 & 0 & I_\ell \end{array}\right| = (-1)^\ell \left|\begin{array}{c|c|c} I_k & 0 & 0 \\ \hline 0 & 0 & I_\ell \\ \hline 0 & X_5 & 0 \end{array}\right|.$$

Again, by replacing the rows of I_ℓ, we can get

$$\det(X) = (-1)^\ell(-1)^\ell \left|\begin{array}{c|c|c} I_k & 0 & 0 \\ \hline 0 & I_\ell & 0 \\ \hline 0 & 0 & X_5 \end{array}\right| = \left|\begin{array}{c|c} I_{k+\ell} & 0 \\ \hline 0 & X_5 \end{array}\right|.$$

Thus, by the previous remarks, we get

$$\det(X) = \det(X_5).$$

2.5 Frobenius Splitting

In this section, we recall the basics of Frobenius splitting. Our primary sources are [4, 6, 11].

Definition 2.5.1. *[11] Let R be a commutative $\mathbb{F}_p$-algebra. Then a map $\Phi : R \to R$ is* ***Frobenius splitting*** *if*

(i) $\Phi(a + b) = \Phi(a) + \Phi(b)$,

(ii) $\Phi(a^p b) = a\Phi(b)$,

(iii) $\Phi(1) = 1$.

In particular, we say that Φ is a **near-splitting** if it only satisfies $(ii), (iii)$ of Definition 2.5.1.
Furthermore, R is **split** if it is equipped with a Frobenius splitting Φ. Given an ideal $I \leq R$ with a Frobenius (near-)splitting Φ is **compatibly (near-)split** if $\Phi(I) \subset I$.

Theorem 2.5.2. *Let R be a Frobenius split with a splitting Φ, and $I, J \leq R$ ideals. Then:*

(i) R is reduced.

(ii) If I is compatibly split, then I is radical and $\Phi(I) = I$.

(iii) If I, J are compatibly split, then so are $I \cap J$ and $I + J$. (They are radical by (ii).)

(iv) If I is compatibly split, and J is arbitrary, then $I : J$ is compatibly split. In particular, the prime components of I are compatibly split.

Let $R = \mathbb{F}_p[x_{i,j}]$. Now, we define the **trace map** for any monomial $m \in \mathbb{F}_p[x_{i,j}]$ as follows.

$$\mathrm{Tr}(m) = \begin{cases} \frac{\sqrt[p]{m \prod_{i,j=1}^n x_{i,j}}}{\prod_{i,j=1}^n x_{i,j}} & \text{if } m \prod_{i,j=1}^n x_{i,j} \text{ is the } p\text{-th power} \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Theorem 2.5.3. *[11, Theorem 2] Let $f \in \mathbb{F}_p[x_{i,j}]$ such that $\deg f \leq n^2$. Then*

$$\mathrm{Tr}(f^{p-1}) = \mathrm{Tr}(\mathrm{in}_<(f^{p-1})),$$

i.e., $\mathrm{Tr}(f^{p-1})$ defines a Frobenius splitting if and only if $\mathrm{Tr}(\mathrm{in}_<(f^{p-1}))$ does. In particular, if $\deg f = n^2$, then no multiple of $\mathrm{Tr}(f^{p-1})$ is a Frobenius splitting.

For $f \in R$, it is known that the map $\varphi_f(g) := \mathrm{Tr}(fg)$ defines a Frobenius splitting of R if $\mathrm{Tr}(f) = 1$, see [4, section 1.3.1]. Recently, in [6], the authors examined the Frobenius splitting of $\mathcal{N}_{w_0}$.

Lemma 2.5.4. *[6, Lemma 5.6] Let $g \in R = \mathbb{K}[x_{i,j}]$ and $\mathbb{K}$ a field of positive characteristic p, and let $<$ be a lexicographic monomial order on R such that $\mathrm{in}_<(g) = \prod_{i,j=1}^n x_{i,j}$. Let $f := g^{p-1}$. Then φ_f defines a Frobenius splitting of R.*

In Section 3.2, we will show that there is a Frobenius splitting of the form $\mathrm{Tr}(f^{p-1}\bullet)$ for each height 1 Hessenberg patch ideal (see Theorem 3.2.10).

Chapter 3

Results on Regular Nilpotent Hessenberg Patch Ideals

3.1 Combinatorial Properties of the $y_{i,j}$ and $f^w_{k,\ell}$

In this section, we first will give a recursive formula for the entries $y_{i,j}$ of M^{-1} given in Definition 2.8. Then we will examine $(wM)^{-1}N(wM)$ whose entries $f^w_{k,\ell}$ given in 2.4.4 allow us to find the generators of Hessenberg patch ideals. Let us check an example before giving the formula.

Example 3.1.1. For $n = 4$, consider

$$M = \begin{pmatrix} 1 & & & \\ x_{2,1} & 1 & & \\ x_{3,1} & x_{3,2} & 1 & \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 \end{pmatrix}.$$

Then by Proposition 2.4.7 the inverse of M is

$$\begin{aligned} M^{-1} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_{2,1} & 1 & 0 & 0 \\ x_{2,1}x_{3,2} - x_{3,1} & -x_{3,2} & 1 & 0 \\ -x_{2,1}x_{3,2}x_{4,3} + x_{2,1}x_{4,2} + x_{3,1}x_{4,3} - x_{4,1} & x_{3,2}x_{4,3} - x_{4,2} & -x_{4,3} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x_{2,1} & 1 & 0 & 0 \\ -x_{3,2}(-x_{2,1}) - x_{3,1} & -x_{3,2} & 1 & 0 \\ -x_{4,3}(x_{2,1}x_{3,2} - x_{3,1}) - x_{4,2}(-x_{2,1}) - x_{4,1} & x_{4,3}(x_{3,2}) - x_{4,2} & -x_{4,3} & 1 \end{pmatrix}. \end{aligned}$$

Then we notice that

$$y_{k,\ell} = -\sum_{i=1}^{k-1} x_{k,i}y_{i,\ell}$$

for $k > 1$, and

$$y_{1,\ell} = \begin{cases} 1 \text{ if } \ell = 1, \\ 0 \text{ otherwise.} \end{cases}$$

If we take $w = (2314)$, then

$$wM = \begin{pmatrix} x_{2,1} & 1 & & \\ x_{3,1} & x_{3,2} & 1 & \\ 1 & & & \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 \end{pmatrix}.$$

Therefore, we compute its inverse by Proposition 2.4.7 as

$$\begin{aligned}(wM)^{-1} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -x_{2,1} & 0 \\ -x_{3,2} & 1 & x_{2,1}x_{3,2} - x_{3,1} & 0 \\ x_{3,2}x_{4,3} - x_{4,2} & -x_{4,3} & -x_{2,1}x_{3,2}x_{4,3} + x_{2,1}x_{4,2} + x_{3,1}x_{4,3} - x_{4,1} & 1 \end{pmatrix}, \\ &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -x_{2,1} & 0 \\ -x_{3,2} & 1 & -x_{3,2}(-x_{2,1}) - x_{3,1} & 0 \\ x_{4,3}(x_{3,2}) - x_{4,2} & -x_{4,3} & -x_{4,3}(x_{2,1}x_{3,2} - x_{3,1}) - x_{4,2}(-x_{2,1}) - x_{4,1} & 1 \end{pmatrix}.\end{aligned}$$

Then we get that

$$[(wM)^{-1}]_{k,\ell} = y_{k,w(\ell)} = \sum_{i=1}^{k-1} -x_{k,i}y_{i,w(\ell)}$$

for each $k, \ell = 1, 2, 3, 4$ by assuming $y_{k,k} = 1$. Let $h = (3, 4, 4, 4)$. Then the Hessenberg patch ideal at w is defined as $J_{w,h} = \left\langle \left[(wM)^{-1}N(wM)\right]_{4,1} \right\rangle$, which is computed in Macaulay2 to be

$$J_{w,h} = \langle -x_{2,1}^2x_{3,2}x_{4,3} + x_{2,1}^2x_{4,2} + x_{2,1}x_{3,1}x_{4,3} + x_{3,1}x_{3,2}x_{4,3} - x_{2,1}x_{4,1} - x_{3,1}x_{4,2} - x_{4,1}x_{4,3} \rangle.$$

We are now ready to treat the general case. Let

$$M = \begin{pmatrix} 1 & & & & \\ x_{2,1} & 1 & & & \\ x_{3,1} & x_{3,2} & 1 & & \\ \vdots & \vdots & & & \\ x_{n,1} & x_{n,2} & x_{n,3} & \dots & 1 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 1 & & & & \\ y_{2,1} & 1 & & & \\ y_{3,1} & y_{3,2} & 1 & & \\ \vdots & \vdots & & & \\ y_{n,1} & y_{n,2} & y_{n,3} & \dots & 1 \end{pmatrix}$$

as defined before in 2.7 and 2.8.

We note that $y_{k,\ell} = 0$ if $k < l$ as M^{-1} is lower triangular, and the diagonal entries $y_{k,k} = 1$ for all $1 \leq k \leq n$. Next, we give the general formula for the $y_{i,j}$ in the following proposition.

Proposition 3.1.2. *Let $1 \leq \ell < k \leq n$. Then*

$$y_{k,\ell} = -\sum_{i=\ell}^{k-1} x_{k,i} y_{i,\ell}.$$

Proof. By Proposition 2.4.7, we know

$$y_{k,\ell} = (-1)^{k+\ell} \det(M^n_{\ell,k}). \tag{3.1}$$

We can write $M^n_{\ell,k}$ with blocks, and compute $\det(M^n_{\ell,k})$ by Remark 2.4.11. The number of blocks of $M^n_{\ell,k}$ depends on the choice of ℓ and k, and we need to examine in different cases as follows:

(i) Assume $\ell = 1$, and $1 < k \leq n$. Then

$$M^n_{\ell,k} = \left(\begin{array}{ccccc|cccc} x_{2,1} & 1 & & & & & & & \\ x_{3,1} & x_{3,2} & 1 & & & & & & \\ \vdots & & & \ddots & & & \mathbf{0} & & \\ x_{k-1,1} & x_{k-1,2} & \dots & & 1 & & & & \\ x_{k,1} & x_{k,2} & \dots & & x_{k,k-1} & & & & \\ \hline & & & & & 1 & & & \\ & & & & & \star & 1 & & \\ & & \bigstar & & & \vdots & & \ddots & \\ & & & & & \star & \star & \dots & 1 \end{array}\right) = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right).$$

Hence, by Remark 2.4.11 $\det(M^n_{1,k}) = \det(X_1)$. We expand the determinant along the bottom row, and we obtain

$$
\begin{aligned}
\det(M^n_{1,k}) &= (-1)^k x_{k,1} \begin{vmatrix} 1 & & & \\ x_{3,2} & 1 & & \\ \vdots & & \ddots & \\ x_{k-1,2} & \dots & & 1 \end{vmatrix} + (-1)^{k+1} x_{k,2} \begin{vmatrix} x_{2,1} & & & \\ x_{3,1} & 1 & & \\ \vdots & & \ddots & \\ x_{k-1,1} & \dots & x_{k-1,k-2} & \end{vmatrix} + \cdots \\
&\quad + (-1)^{2k} x_{k,k-1} \begin{vmatrix} 1 & & & & \\ x_{2,1} & 1 & & & \\ x_{3,1} & x_{3,2} & 1 & & \\ \vdots & & & \ddots & \\ x_{k-1,1} & x_{k-1,2} & \dots & x_{k-1,k-2} & \end{vmatrix} \\
&= (-1)^k x_{k,1}.\det(\mathbf{M^{k-1}_{1,1}}) + (-1)^{k+1} x_{k,2} \det(\mathbf{M^{k-1}_{1,2}}) + \ldots + (-1)^{2k} x_{k,k-1} \det(\mathbf{M^{k-1}_{1,k-1}}) \\
&= (-1)^k x_{k,1}.(-1)^2 \mathbf{y_{1,1}} + (-1)^{k+1} x_{k,2} (-1)^3 \mathbf{y_{2,1}} + \ldots + (-1)^{2k} x_{k,k-1} (-1)^k \mathbf{y_{k-1,1}} \\
&= (-1)^{k+2} x_{k,1}.\mathbf{y_{1,1}} + (-1)^{k+4} x_{k,2} \mathbf{y_{2,1}} + \ldots + (-1)^{3k} x_{k,k-1} \mathbf{y_{k-1,1}} \\
&= (-1)^k x_{k,1}.\mathbf{y_{1,1}} + (-1)^k x_{k,2} \mathbf{y_{2,1}} + \ldots + (-1)^k x_{k,k-1} \mathbf{y_{k-1,1}} \\
&= (-1)^k \sum_{i=1}^{k-1} x_{k,i} y_{i,1}
\end{aligned}
$$

Thus, the expression 3.1 follows that

$$y_{k,1} = (-1)^{k+1} (-1)^k \sum_{i=1}^{k-1} x_{k,i} y_{i,1} = - \sum_{i=1}^{k-1} x_{k,i} y_{i,1}.$$

(ii) Assume $k = \ell + 1$. Then $M^n_{\ell,k}$ looks like

$$
\left(\begin{array}{cccccc|cccc}
1 & & & & & & & & & \\
x_{2,1} & 1 & & & & & & & & \\
x_{3,1} & x_{3,2} & 1 & & & & & \mathbf{0} & & \\
\vdots & & & \ddots & & & & & & \\
x_{\ell-1,1} & x_{\ell-1,2} & \dots & & 1 & & & & & \\
x_{\ell+1,1} & x_{\ell+1,2} & \dots & & x_{\ell+1,\ell-1} & x_{\ell+1,\ell} & & & & \\
\hline
 & & & & & & 1 & & & \\
 & & & \bigstar & & & \star & 1 & & \\
 & & & & & & \vdots & & \ddots & \\
 & & & & & & \star & \star & \dots & 1
\end{array}\right) = \left(\begin{array}{c|c} X_1 & X_2 \\ \hline X_3 & X_4 \end{array}\right).
$$

Therefore, by Remark 2.4.11, we obtain

$$\det(M^n_{\ell,k}) = \det(X_1) = x_{\ell+1,\ell} = \sum_{i=\ell}^{\ell} x_{\ell+1,i}\, y_{i,\ell}.$$

Hence, the expression 3.1 satisfies

$$y_{\ell+1,\ell} = (-1)^{2\ell+1} \det(M^n_{\ell,\ell+1}) = -x_{\ell+1,\ell}.$$

(iii) We examine other situations in this case, i.e., assume $\ell \neq 1$ or $k \neq \ell+1$.

$$\begin{aligned}
\det(M^n_{\ell,k}) &= \det\left(\begin{array}{c|c|c}
\begin{matrix} 1 & & & \\ \star & 1 & & \\ \vdots & & \ddots & \\ \star & \star & \dots & 1 \end{matrix} & \mathbf{0} & \mathbf{0} \\ \hline
\bigstar & \bigstar & \mathbf{0} \\ \hline
\bigstar & \bigstar & \begin{matrix} 1 & & & \\ \star & 1 & & \\ \vdots & & \ddots & \\ \star & \star & \dots & 1 \end{matrix}
\end{array}\right) \\
&= \det\left(\begin{array}{c|c|c} X_1 & X_2 & X_3 \\ \hline X_4 & X_5 & X_6 \\ \hline X_7 & X_8 & X_9 \end{array}\right) \\
&= \det(X_5)
\end{aligned}$$

where

$$X_5 = \begin{pmatrix}
x_{\ell+1,\ell} & 1 & & & \\
x_{\ell+2,\ell} & x_{\ell+2,\ell+1} & 1 & & \\
\vdots & \vdots & & & \\
x_{k-1,\ell} & x_{k-1,\ell+1} & \dots & x_{k-1,k-2} & 1 \\
x_{k,\ell} & x_{k,\ell+1} & \dots & x_{k,k-2} & x_{k,k-1}
\end{pmatrix}.$$

Then we expand the determinant along the bottom row, and we obtain

$$
\begin{aligned}
\det(M^n_{\ell,k}) =& (-1)^{k+\ell-1}x_{k,\ell}.1 + (-1)^{k+\ell}x_{k,\ell+1}\begin{vmatrix} x_{\ell+1,\ell} & & & \\ x_{\ell+2,\ell} & 1 & & \\ \vdots & \vdots & & \\ x_{k-1,\ell} & x_{k-1,\ell+1} & \dots & 1 \end{vmatrix} \\
&+ (-1)^{k+\ell+1}x_{k,\ell+2}\begin{vmatrix} x_{\ell+1,\ell} & 1 & & & \\ x_{\ell+2,\ell} & x_{\ell+2,\ell+1} & & & \\ x_{\ell+3,\ell} & x_{\ell+3,\ell+1} & 1 & & \\ \vdots & \vdots & & & \\ x_{k-1,\ell+1} & x_{k-1,\ell+3} & & \dots & 1 \end{vmatrix} + \cdots \\
&+ (-1)^{2k-2}x_{k,k-1}\begin{vmatrix} x_{\ell+1,\ell} & 1 & & & \\ x_{\ell+2,\ell} & x_{\ell+2,\ell+1} & 1 & & \\ \vdots & \vdots & & \ddots & \\ x_{k,1} & x_{k,2} & \dots & & 1 \\ x_{k-1,1} & x_{k-1,2} & \dots & & x_{k-1,k-2} \end{vmatrix} \\
=& (-1)^{k+\ell-1}x_{k,\ell}.\mathbf{y}_{\ell,\ell} + (-1)^{k+\ell}x_{k,\ell+1}\mathbf{M}^{\mathbf{k-1}}_{\ell,\ell+\mathbf{1}} + (-1)^{k+\ell-1}x_{k,\ell+2}\mathbf{M}^{\mathbf{k-1}}_{\ell,\ell+\mathbf{2}} + \cdots \\
&+ (-1)^2x_{k,k-1}\mathbf{M}^{\mathbf{k-1}}_{\ell,\mathbf{k-1}} \\
=& (-1)^{k+\ell-1}x_{k,\ell}.\mathbf{y}_{\ell,\ell} + (-1)^{k+\ell-1}x_{k,\ell+1}\mathbf{y}_{\ell+\mathbf{1},\ell} + (-1)^{k+\ell-1}x_{k,\ell+2}\mathbf{y}_{\ell+\mathbf{2},\ell} + \cdots \\
&+ (-1)^{k+\ell-1}x_{k,k-1}\mathbf{y}_{\mathbf{k-1},\ell} \\
=& (-1)^{k+\ell-1}\sum_{i=\ell}^{k-1}x_{k,i}y_{i,\ell}
\end{aligned}
$$

Thus, the expression 3.1 becomes

$$y_{k,\ell} = (-1)^{k+\ell}(-1)^{k+\ell-1}\sum_{i=\ell}^{k-1}x_{k,i}y_{i,\ell} = -\sum_{i=\ell}^{k-1}x_{k,i}y_{i,\ell}.$$

□

Remark 3.1.3.
Let $1 \leq \ell < k \leq n$. Then by the previous Proposition, we have

$$
\begin{aligned}
y_{k,\ell} &= -\sum_{m=\ell}^{k-1}x_{k,m}y_{m,\ell} \\
&= -(x_{k,\ell}y_{\ell,\ell} + x_{k,\ell+1}y_{\ell+1,\ell} + \ldots + x_{k,k-1}y_{k-1,\ell}).
\end{aligned}
$$

Then the terms of $y_{k,\ell}$ are in the form

$$-x_{k,m}y_{m,\ell}$$

for $\ell \leq m \leq k-1$. This implies that

(a) If $i = k$ and $\ell \leq j \leq k-1$, then $x_{i,j}y_{j,\ell}$ is a term of $x_{k,m}y_{m,\ell}$.

(b) By the previous proposition,

$$y_{m,\ell} = \sum_{m_0=\ell}^{m-1} x_{m,m_0}y_{m_0,m}.$$

Thus, $x_{i,j}$ divides a term of $y_{m,\ell}$ if $\ell \leq i \leq m \leq k-1$ and $\ell \leq j \leq m-1 \leq k-1$.

By (a) and (b), we conclude that $y_{k,\ell}$ has terms containing $x_{i,j}$ if $\ell \leq i \leq k$ and $\ell \leq j \leq k-1$.

Remark 3.1.4. Let $w \in \mathfrak{S}_n$. The matrix wM is acquired by permuting the rows of M with respect to the permutation matrix w. We also obtain $(wM)^{-1} = M^{-1}w^{-1}$ by permuting the columns of M^{-1} via w^{-1}. In short,

$$[wM]_{i,j} = x_{w(i),j}, \quad [(wM)^{-1}]_{i,j} = y_{i,w^{-1}(j)}.$$

Given $w \in \mathfrak{S}_n$, we found a formula for any entry of $(wM)^{-1}$ in terms of $x_{i,j}$ and $y_{k,\ell}$ in Proposition 3.1.2

$$y_{k,\ell} = -\sum_{i=\ell}^{k-1} x_{k,i}y_{i,\ell} \tag{3.2}$$

Since the formula 3.2 is recursive, we can write the entries of $(wM)^{-1}$ only in terms of $x_{i,j}$ when we expand all $y_{k,l}$. As all $y_{i,\ell}$ on the right-hand side of the expression 3.2 comes from the same column, we prove $y_{k,\ell} \in \mathbb{K}[x_{i,j}]$ with an induction on k.

Corollary 3.1.5. *Let $1 \leq \ell < k \leq n$. Then $y_{k,\ell}$ is a polynomial in $\mathbb{K}[x_{i,j}]$.*

Proof. For $k = 2$, there is only one nonzero entry in $(wM)^{-1}$ which is $y_{2,1}$.

$$y_{2,1} = -\sum_{i=1}^{1} x_{2,i}y_{i,1} = x_{2,1}y_{1,1} = x_{2,1}.$$

Let $1 \leq \ell < k$. Assume $y_{r,\ell}$ is a polynomial in $\mathbb{K}[x_{i,j}]$ for all $\ell \leq r \leq k$. Therefore,

$$\begin{aligned} y_{k+1,\ell} &= -\sum_{i=\ell}^{k} x_{k+1,i} y_{i,\ell} \\ &= x_{k+1,\ell} y_{\ell,\ell} + x_{k+1,\ell+1} y_{\ell+1,\ell} + \ldots + x_{k+1,k} y_{k,\ell} \end{aligned}$$

where $y_{\ell,\ell}, y_{\ell+1,\ell}, \ldots, y_{k,\ell}$ all are the poynomials in $\mathbb{K}[x_{i,j}]$ by induction. Hence, $y_{k+1,\ell}$ is also a polynomial in $\mathbb{K}[x_{i,j}]$. $\square$

Our next goal is to find a formula for the generators of $J_{w,h}$, i.e., the entries of $(wM)^{-1}N(wM)$.

Example 3.1.6. Let $w = (2314)$. Then

$$wM = \begin{pmatrix} x_{2,1} & 1 & 0 & 0 \\ x_{3,1} & x_{3,2} & 1 & 0 \\ 1 & 0 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 \end{pmatrix}, (wM)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & y_{2,1} & 0 \\ y_{3,2} & 1 & y_{3,1} & 0 \\ y_{4,2} & y_{4,3} & y_{4,1} & 1 \end{pmatrix}.$$

Here, we notice that $(wM)^{-1}_{i,j} = y_{i,w(j)}$ for any $1 \leq i, j \leq 4$. Let $h = (3, 4, 4, 4)$. Therefore,

$$J_{w,h} = \left\langle f^w_{i,j} = [(wM)^{-1}N(wM)]_{i,j} \mid i > h(j) \right\rangle = \left\langle f^w_{4,1} \right\rangle.$$

We compute the generator in Macaulay2,

$$\begin{aligned} f^w_{4,1} &= -x_{2,1}x_{3,2}x_{4,1}x_{4,2} + x_{2,1}x_{4,1}x_{4,2} + x_{3,1}x_{3,2}x_{4,2} + x_{3,1}x_{4,1} \\ &= x_{1,1}y_{4,3} + x_{3,1}y_{4,2} + x_{4,1}y_{4,1} \\ &= \sum_{k=1}^{3} x_{w(k+1),1}\, y_{4,w(k)}. \end{aligned}$$

Proposition 3.1.7. *Let $n \in \mathbb{Z}$ with $n \geq 3$, and $w \in \mathfrak{S}_n$. Then $\big((wM)^{-1}N(wM)\big)_{k,\ell}$ is*

$$f^w_{i,j} = \sum_{k=1}^{n-1} x_{w(k+1),j}\, y_{i,w(k)}.$$

Proof. We obtain wM by permuting the rows of M with respect to w as follows

$$wM = \begin{pmatrix} x_{w(1),1} & x_{w(1),2} & \cdots & x_{w(1),n} \\ x_{w(2),1} & x_{w(2),2} & \cdots & x_{w(2),n} \\ x_{w(3),1} & x_{w(3),2} & \cdots & x_{w(3),n} \\ \vdots & \vdots & & \\ x_{w(n),1} & x_{w(n),2} & \cdots & x_{w(n),n} \end{pmatrix}.$$

Also, we can compute $(wM)^{-1} = M^{-1}w^{-1}$ by permuting the columns of M^{-1} with respect to w as follows

$$(wM)^{-1} = \begin{pmatrix} y_{1,w(1)} & y_{1,w(2)} & \cdots & y_{1,w(n)} \\ y_{2,w(1)} & y_{2,w(2)} & \cdots & y_{2,w(n)} \\ y_{3,w(1)} & y_{3,w(2)} & \cdots & y_{3,w(n)} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n,w(1)} & y_{n,w(2)} & \cdots & y_{n,w(n)} \end{pmatrix}.$$

Recall the matrix N defined in (2.6). Then

$$N(wM) = \begin{pmatrix} x_{w(2),1} & x_{w(2),2} & \cdots & x_{w(2),n} \\ x_{w(3),1} & x_{w(3),2} & \cdots & x_{w(3),n} \\ \vdots & \vdots & & \\ x_{w(n),1} & x_{w(n),2} & \cdots & x_{w(n),n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, we conclude that

$$(wM)^{-1}N(wM) = \left[\sum_{k=1}^{n} x_{w(k+1),j}\, y_{i,w(k)} \right]_{i,j},$$

where $x_{w(n+1),j} = 0$ for all $1 \leq j \leq n$. □

3.2 Hessenberg Patch Ideals of Codimension 1

In this section, we define a monomial order such that the initial ideal of the Hessenberg patch ideal of codimension 1 is squarefree, i.e., the initial monomial is a product of distinct variables. Let $n \geq 3$, and $w \in \mathfrak{S}_n$. Throughout this section, we consider the indecomposable Hessenberg function $h = (n-1, n, \ldots, n)$. Then $J_{w,h} = \langle f^w_{n,1} \rangle$ by Lemma 2.4.5. Therefore, to find a squarefree $\text{in}_{\prec_w}(J_{w,h})$, we will focus on the terms of $f^w_{n,1}$, see 2.4.4, containing a square. Indeed, we will determine which variable of wM causes a square term in the generator.

To be able to check, the initial ideal we need a monomial order defined on $\mathbb{K}[x_{i,j}]$. We construct the monomial order for each $w \in \mathfrak{S}_n$ in the following steps.

Construction 1. 1. Let $n \geq 3$ and $w \in \mathfrak{S}_n$. Denote $s = w^{-1}(1)$. Put

$$a := \begin{cases} s-1 & \text{if } s > 1, \\ n & \text{if } s = 1. \end{cases}$$

2. In this step, we start ordering variables and order all the variables on and above the $w(a)$-th row of wM.
   i) Say R is the row which starts with the entry $x_{w(a),1}$ in wM. We first start ordering variables from the row R by sweeping the row from left to right.
   ii) Secondly, we order variables above the R-th row. In order to do that, we go one row up and keep ordering variables from left to right. After sweeping one row, we continue jumping one row up to visit all the rows of wM above the R-th row. When we finish ordering variables above the R-th row, we deal with the below rows in the 3rd step.
3. Note that 3rd step is not needed if s is 1 or n since in these cases we swept all the variables in the 2nd step. If $s \neq 1, n$, then we keep ordering variables by sweeping the entries of wM below the R-th row.
   i) Next, we keep adding the variables to our order that is started in the 3rd step. We add the variables from left to right on the $w(n)$-th row. Then we go one row up and keep ordering from left to right. We continue jumping one row up and ordering the variables from left to right till we reach the R-th row.

At the end of this process, we attain an order of variables in $\mathbb{K}[x_{i,j}]$.

4. Lastly, we let $\prec_w$ be the lexicographic order on $\mathbb{K}[x_{i,j}]$.

Example 3.2.1. Let $w = (2314)$. Then

$$wM = \begin{pmatrix} x_{2,1} & 1 & 0 & 0 \\ x_{3,1} & x_{3,2} & 1 & 0 \\ 1 & 0 & 0 & 0 \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 \end{pmatrix}, (wM)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & y_{2,1} & 0 \\ y_{3,2} & 1 & y_{3,1} & 0 \\ y_{4,2} & y_{4,3} & y_{4,1} & 1 \end{pmatrix}.$$

For Hessenberg function $h = (3, 4, 4, 4)$, by Definition 2.4.5 we compute in Macaulay2 the Hessenberg patch ideal.

$$J_{w,h} = \langle -x_{2,1}x_{3,2}x_{4,1}x_{4,3} + x_{2,1}x_{4,1}x_{4,2} + x_{3,1}x_{3,2}x_{4,3} + x_{3,1}x_{4,1}x_{4,3} - x_{4,1}^2 - x_{3,1}x_{4,2} - x_{4,3}\rangle.$$

We define the monomial order by the Construction 1.

1. Denote $s = w^{-1}(1) = 3$. Therefore, $a = s - 1 = 2$.
2. Since $w(a) = w(2) = 3$, $R = 2$ and we start to order variables from the 2nd row of wM. Then we obtain

$$x_{3,1} \succ_{\mathrm{w}} x_{3,2} \succ_{\mathrm{w}} x_{2,1}.$$

3. Next, we keep ordering the variables from the last row of wM starting with the entry $x_{w(4),1} = x_{4,1}$. Then we obtain

$$x_{3,1} \succ_{\mathrm{w}} x_{3,2} \succ_{\mathrm{w}} x_{2,1} \succ_{\mathrm{w}} x_{4,1} \succ_{\mathrm{w}} x_{4,2} \succ_{\mathrm{w}} x_{4,3}.$$

4. Let $\succ_{\mathrm{w}}$ be a lexicographic order.

Therefore, with respect to $\succ_{\mathrm{w}}$

$$\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h}) = \langle x_{3,1}x_{3,2}x_{4,2}\rangle.$$

Let

$$M = \begin{pmatrix} 1 & & & & \\ x_{2,1} & 1 & & & \\ x_{3,1} & x_{3,2} & 1 & & \\ \vdots & \vdots & & & \\ x_{n,1} & x_{n,2} & x_{n,3} & \dots & 1 \end{pmatrix} \in U^-$$

where $x_{i,j} \in \mathbb{C}$, $x_{i,i} = 1$ for all $i \in [n]$ and $x_{i,j} = 0$ if $j > i$ for all $i, j \in [n]$. Hence, by Proposition 3.1.7,

$$f^w_{n,1} = \sum_{k=1}^{n-1} x_{k+1,1}\, y_{n,k} = x_{2,1}y_{n,1} + x_{3,1}y_{n,2} + \ldots + x_{n,1}y_{n,n-1}. \tag{3.3}$$

Lemma 3.2.2. *Let $n \geq 3$ and $w = (12\ldots n)$. In the expression 3.3, there are only two terms $x_{n,1}y_{n,n-1}$ and $x_{2,1}y_{n,1}$ whose initial monomials with respect to $\succ_{\mathrm{w}}$ are divisible by $x_{n,1}$. Moreover, $\mathrm{in}_{\prec_{\mathrm{w}}}(x_{n,1}y_{n,n-1}) = x_{n,1}x_{n,n-1}$ and $\mathrm{in}_{\prec_{\mathrm{w}}}(x_{2,1}y_{n,1}) = -x_{2,1}x_{n,1}$.*

Proof. By Construction 1, we order the variables with respect to w,

$$x_{n,1} \succ_{\mathrm{w}} x_{n,2} \succ_{\mathrm{w}} \ldots \succ_{\mathrm{w}} x_{2,1}.$$

Note that by Proposition 3.1.2, the equation 3.3 becomes

$$\begin{aligned} f^w_{n,1} &= -x_{2,1}\left(\sum_{i=1}^{n-1} x_{n,i}y_{i,1}\right) - x_{3,1}\left(\sum_{i=2}^{n-1} x_{n,i}y_{i,2}\right) - \ldots - x_{n,1}\left(\sum_{i=n-1}^{n-1} x_{n,i}y_{i,n-1}\right) \\ &= -x_{2,1}\left(\sum_{i=1}^{n-1} x_{n,i}y_{i,1}\right) - x_{3,1}\left(\sum_{i=2}^{n-1} x_{n,i}y_{i,2}\right) - \ldots - x_{n,1}(x_{n,n-1}y_{n-1,n-1}). \end{aligned}$$

The last term of the above expression is $-x_{n,1}y_{n,n-1} = x_{n,1}(x_{n,n-1}y_{n-1,n-1}) = x_{n,1}x_{n,n-1}$, then we conclude that $x_{n,1}$ divides (the initial monomial of) $x_{n,1}y_{n,n-1}$. Also, by Proposition 3.1.2, we examine another term of $f^w_{n,1}$ as follows.

$$\begin{aligned} x_{2,1}y_{n,1} &= -x_{2,1}\left(\sum_{i=1}^{n-1} x_{n,i}y_{i,2}\right) \\ &= -x_{2,1}\left(x_{n,1}y_{1,1} + x_{n,2}y_{2,1} + x_{n,3}y_{3,1} + \ldots x_{n,n-1}y_{n-1,1}\right) \\ &= x_{2,1}\left(-x_{n,1} + x_{n,2}\left(\sum_{i=1}^{1} x_{2,i}y_{i,1}\right)y_{2,1} + x_{n,3}\left(\sum_{i=1}^{2} x_{3,i}y_{i,1}\right) + \ldots + x_{n,n-1}\left(\sum_{i=1}^{n-2} x_{n-1,i}y_{i,1}\right)\right) \end{aligned}$$

we get that $\mathrm{in}_{\prec_{\mathrm{w}}}(x_{2,1}y_{n,1}) = -x_{2,1}x_{n,1}$ which is divisible by $x_{n,1}$. Now, we need to check if the other terms of $f^w_{n,1}$ might be an initial monomial with respect to $\succ_{\mathrm{w}}$ By the equation 3.3, the other terms of $f^w_{n,1}$ are

$$x_{3,1}\left(\sum_{i=2}^{n-1} x_{n,i}y_{i,2}\right), x_{4,1}\left(\sum_{i=3}^{n-1} x_{n,i}y_{i,3}\right), \ldots, x_{n-1,1}\left(\sum_{i=n-2}^{n-1} x_{n,i}y_{i,n-2}\right).$$

As $i > 1$ in each sum, it is not possible to have $x_{n,1}$ from any of $x_{n,i}$. When we expand $y_{i,j}$ in each term for $2 \leq i \leq n-1$ and $2 \leq j < n-1$, we get by Proposition 3.1.2

$$y_{i,j} = -\sum_{t=j}^{i-1} x_{i,t} y_{t,j},$$

in which there is no term divisible by $x_{n,1}$ because $2 \leq i \leq n-1$ and $2 \leq j < n-1$. □

Proposition 3.2.3. *Let $n \geq 3$, $w = (12\dots n)$ and $h = (n-1, n, \dots, n)$. Then* $\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h}) = \langle x_{n,1} x_{n,n-1} \rangle$.

Proof. By Lemma 3.2.2, we have only two terms $x_{n,1} y_{n,n-1}$ and $x_{2,1} y_{n,1}$ such that either one of them gives us $\mathrm{in}_{\prec_{\mathrm{w}}}\left(f^w_{n,1}\right)$ with respect to $\prec_{\mathrm{w}}$. Recall that $\mathrm{in}_{\prec_{\mathrm{w}}}(x_{n,1} y_{n,n-1}) = x_{n,1} x_{n,n-1}$ and $\mathrm{in}_{\prec_{\mathrm{w}}}(x_{2,1} y_{n,1}) = -x_{2,1} x_{n,1}$. Therefore,

$$\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h}) = \langle \mathrm{in}_{\prec_{\mathrm{w}}}(x_{n,1} y_{n,n-1}) \rangle = \langle x_{n,1} x_{n,n-1} \rangle$$

as $x_{n,n-1} \succ_{\mathrm{w}} x_{2,1}$ by the Construction 1, and so $\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h})$ is squarefree. □

Lemma 3.2.4. *Let $n \geq 3$ and $h = (n-1, n, \dots, n)$. Consider $w \in \mathfrak{S}_n$ such that $w(1) = 1$ and $w(n) = n$. Then* $\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h}) = \langle x_{n,1} x_{n,w(n-1)} \rangle$.

Proof. By the Construction 1, we get that $a = n$, hence we order variables as $x_{n,1} \succ_{\mathrm{w}} x_{n,2} \succ_{\mathrm{w}} x_{n,3} \dots$.

By Definition 2.4.4 and Proposition 3.1.7, $J_{w,h}$ is generated by

$$f^w_{n,1} = \sum_{k=1}^{n-1} x_{w(k+1),1} y_{n,w(k)}.$$

Now, to find the initial monomial of $f^w_{n,1}$ it is enough to check the terms of $f^w_{n,1}$ that are divisible by $x_{n,1}$.

1. If $x_{w(k+1),1} y_{n,w(k)} = x_{n,1} y_{n,w(k)}$, then $k = n-1$. Therefore, there is term $x_{n,1} y_{n,w(n-1)}$ divisible by $x_{n,1}$. By Proposition 3.1.2

$$y_{n,w(n-1)} = \sum_{i=w(n-1)}^{n-1} x_{n,i} y_{i,w(n-1)}.$$

Assume that there is a term of $y_{n,w(n-1)}$ which is divisible by $x_{n,1}$. Then we must have $i = 1$ or $w(n-1) = 1$. In either case, we obtain $w(n-1) = 1 = w(1)$ which implies that $n-1$-th and 1st columns are the same column. Hence, $n-1 = 1$ implies that $n = 2$ which contradicts with the statement $n \geq 3$. Hence, $x_{n,1}y_{n,w(n-1)}$ is squarefree.

2. By Proposition 3.1.2,

$$y_{n,w(k)} = \sum_{i=w(k)}^{n-1} x_{n,i}y_{i,w(k)}.$$

If there is a k such that $x_{n,1}$ divides a term of $y_{n,w(k)}$, then we have either $i = 1$ or $w(k) = 1$. In either case, we get that $i = 1$, and then there is term $x_{n,1}y_{1,w(1)} = x_{n,1}$ in $x_{w(k+1),1}y_{n,w(k)}$.

By the monomial order $x_{n,1}y_{n,w(n-1)} \succ_{\mathrm{w}} x_{n,1}$, thus, $\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h}) = \langle x_{n,1}y_{n,w(n-1)} \rangle$. $\square$

Example 3.2.5. Let $n = 5$. Choose $w = (21534)$, and $h = (4, 5, 5, 5, 5)$. Then

$$wM = \begin{pmatrix} x_{2,1} & 1 & & & \\ 1 & & & & \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & 1 \\ x_{3,1} & x_{3,2} & 1 & & \\ x_{4,1} & x_{4,2} & x_{4,3} & 1 & \end{pmatrix}.$$

Then by the Construction 1, we have $x_{2,1} \succ_{\mathrm{w}} x_{4,1} \succ_{\mathrm{w}} x_{4,2} \succ_{\mathrm{w}} x_{4,3} \succ_{\mathrm{w}} \ldots$.
The Hessenberg patch ideal is

$$\begin{aligned} J_{w,h} = \langle & x_{2,1}x_{3,2}x_{4,2}x_{5,1}x_{5,4} - x_{2,1}x_{3,2}x_{5,1}x_{5,3} - x_{2,1}x_{42}x_{5,1}x_{5,4} - x_{3,1}x_{4,2}x_{5,1}x_{5,4} + x_{2,1}x_{5,1}x_{5,2} + x_{3,1}x_{5,1} \\ & + x_{4,1}x_{4,2}x_{5,4} + x_{4,1}x_{5,1}x_{5,4} - x_{5,1}^2 + x_{3,2}x_{5,3} - x_{4,1}x_{5,3} + x_{4,2}x_{5,4} + x_{3,1} - x_{5,2} \rangle. \end{aligned}$$

Then $\mathrm{in}_{\prec_{\mathrm{w}}}(J_{w,h}) = \langle x_{2,1}x_{42}x_{5,1}x_{5,4} \rangle$.

Example 3.2.6. Let $w = (41352)$, and $h = (4, 5, 5, 5, 5)$. Then

$$wM = \begin{pmatrix} x_{4,1} & x_{4,2} & x_{4,3} & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ x_{3,1} & x_{3,2} & 1 & 0 & 0 \\ x_{5,1} & x_{5,2} & x_{5,3} & x_{5,4} & 1 \\ x_{2,1} & 1 & 0 & 0 & 0 \end{pmatrix} \implies (wM)^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & y_{2,1} & 0 & 0 & 1 \\ 0 & y_{3,1} & 1 & 0 & y_{3,2} \\ 1 & y_{4,1} & y_{4,3} & 0 & y_{4,2} \\ y_{5,4} & y_{5,1} & y_{5,3} & 1 & x_{5,2} \end{pmatrix}.$$

Thus, the Hessenberg patch ideal is

$$J_{w,h} = \left\langle f^w_{5,1} := [(wM)^{-1}N(wM)]_{5,1} \right\rangle .$$

Then $\mathrm{in}_{\prec_\mathrm{w}}(J_{w,h}) = \langle x_{2,1}x_{4,2}x_{5,1}x_{5,4}\rangle$. By Proposition 3.1.7, we get that

$$f^w_{5,1} = y_{5,4}x_{1,1} + x_{3,1}y_{5,1} + x_{5,1}y_{5,3} + x_{2,1}y_{5,5}$$

where $x_{1,1} = x_{5,5} = 1$. Hence,

$$\begin{aligned} J_{w,h} =& \langle x_{2,1}x_{3,1}x_{3,2}x_{4,3}x_{5,4} - x_{2,1}x_{3,1}x_{3,2}x_{5,3} - x_{2,1}x_{3,1}x_{4,2}x_{5,4} - x^2_{3,1}x_{4,3}x_{5,4} + x_{2,1}x_{3,1}x_{5,2} + x^2_{3,1}x_{5,3} \\ &+ x_{3,1}x_{4,1}x_{5,4} + x_{4,3}x_{5,1}x_{5,4} - x_{3,1}x_{5,1} - x_{5,1}x_{5,3} + x_{2,1} - x_{5,4}\rangle. \end{aligned}$$

Now, we examine which terms of $f^w_{5,1}$ are not square-free. We note that $y_{5,j}$ is square-free for $j = 1, 3, 4$. Then we can only have some square terms if $x_{3,1}$ appears in $y_{5,1}$ or $x_{5,1}$ appears in $y_{5,3}$. By Proposition 3.1.2,

$$\begin{aligned} y_{5,1} &= \sum_{i=1}^{4} x_{5,i}y_{i,1} \\ &= x_{5,1} + x_{5,2}y_{2,1} + x_{5,3}y_{3,1} + x_{5,4}y_{4,1} \\ &= x_{5,1} + x_{5,2}(-x_{2,1}) + x_{5,3}(x_{2,1}x_{3,2} - \mathbf{x_{3,1}}) + x_{5,4}(-x_{2,1}x_{3,2}x_{4,3} + x_{2,1}x_{4,2} + \mathbf{x_{3,1}}x_{4,3} - x_{4,1}). \end{aligned}$$

Similarly, we get

$$y_{5,3} = x_{5,3} + x_{5,4}y_{4,1}.$$

By Remark 3.1.3, $x_{5,1}$ cannot appear in $y_{4,1}$. So we have got two square terms in $f^w_{5,1}$:

$$x_{5,3}x^2_{3,1},\ x_{5,4}x^2_{3,1}x_{4,3}$$

which come from $x_{3,1}y_{5,1}$. If we order variables as built in the Construction 1

$$x_{4,1} \succ_\mathrm{w} x_{4,2} \succ_\mathrm{w} x_{4,3} \succ_\mathrm{w} x_{2,1} \succ_\mathrm{w} x_{5,1} \succ_\mathrm{w} \dots \succ_\mathrm{w} x_{3,2}$$

then

$$\mathrm{in}_{\prec_\mathrm{w}}(J_{w,h}) = \langle x_{3,1}x_{4,1}x_{5,4}\rangle$$

with respect to $\succ_\mathrm{w}$.

Remark 3.2.7. We note that there are other monomial orders $<$ such that the corresponding $\mathrm{in}_<(J_{w,h})$ is squarefree. We can take $x_{2,1}$ or $x_{5,1}$ as the most expensive variable. For example, we take $x_{5,1}$ as the most expensive variable and order the rest of the variables randomly after that, then we let $<$ be the lexicographic order. Therefore, $\mathrm{in}_<(J_{w,h})$ can be generated one of the following monomials

$$x_{4,3}x_{5,1}x_{5,4},\ x_{3,1}x_{5,1},\ x_{5,1}x_{5,3}$$

which are all squarefree. The choice of the initial monomial depends on the order of the rest of the variables, but we did not specify their ordering in this situation.

Let $n \geq 3$ and $w \in \mathfrak{S}_n$ be an S-fixed point. Then the generator of $J_{w,h}$ is given by Proposition 3.1.7

$$f^w_{n,1} = \sum_{k=1}^{n-1} x_{w(k+1),1}\, y_{n,w(k)}.$$

Lemma 3.2.8. *Let $n \geq 3$ and $w \in \mathfrak{S}_n$. Then $x_{w(k+1),1}$ divides $\mathrm{in}_{\prec_w}(y_{n,w(k)})$ only if*

$$k \neq \begin{cases} a+1 & \text{if } s > 1, \\ 1 & \text{if } s = 1. \end{cases}$$

where a is as defined in Construction 1.

Proof. Let k be such that $x_{w(k+1),1}$ divides $\mathrm{in}_{\prec_w}(y_{n,w(k)})$. Suppose in order to obtain a contradiction that

$$k \neq \begin{cases} a+1 & \text{if } s > 1, \\ 1 & \text{if } s = 1. \end{cases}$$

where a is as defined in Construction 1. Since $s = w^{-1}(1)$, we get that

$$k \neq \begin{cases} s & \text{if } s > 1, \\ 1 & \text{if } s = 1. \end{cases}$$

Then in either case we obtain $k \neq w^{-1}(1)$. Thus,

$$w(k) \neq 1. \tag{3.4}$$

By Proposition 3.1.2,

$$y_{n,w(k)} = -\sum_{i=w(k)}^{n-1} x_{n,i}y_{i,w(k)}.$$

By the expression 3.4, $y_{i,w(k)} \neq y_{i,1}$ for any $w(k) \leq i \leq n-1$. Since $w(k) > 1$, by Remark 3.1.3, $x_{w(k+1),1}$ cannot divide any term of $y_{i,w(k)}$ for any $1 \leq i \leq n-1$.

Therefore, $x_{w(k+1),1}$ can divide a term of $y_{n,w(k)}$ if there exists $w(k) \leq i \leq n-1$ such that $x_{w(k+1),1} = x_{n,i}$. In this case, $w(k+1) = n$, $i = 1$ and $x_{w(k+1),1}$ divides $x_{n,1}y_{1,w(k)}$. However, this contradicts with the expression 3.4.

□

Now, we can state our main result.

Theorem 3.2.9. *Let $n \geq 3$ and $w \in \mathfrak{S}_n$. Consider the Hessenberg function $h = (n-1, n, \ldots, n)$. Then $\mathrm{in}_{\prec_w}(J_{w,h})$ is squarefree.*

Proof. We have by Proposition 3.1.7 that

$$J_{w,h} = \left\langle f^w_{n,1} = \sum_{k=1}^{n-1} x_{w(k+1),1}\, y_{n,w(k)} \right\rangle.$$

Note that $y_{n,w(k)}$ is square-free for all k by Proposition 3.1.2. Then we can get square terms from $f^w_{n,1}$ only when $w(k) = 1$ by Lemma 3.2.8. Fix k so that $w(k) = 1$. We take

$$a := \begin{cases} s-1 & \text{if } s > 1, \\ n & \text{if } s = 1. \end{cases}$$

where $s = w^{-1}(1)$ as defined in Construction 1. Therefore, we get that

$$w(k) = 1 \implies w^{-1}(1) = k \implies s = k.$$

Similarly, we define

$$b := \begin{cases} a+2 & \text{if } a \leq n-2, \\ 1 & \text{if } a = n-1, \\ 2 & \text{if } a = n. \end{cases}$$

So the square terms come from

$$x_{w(k+1),1}y_{n,1} = x_{w(b),1}y_{n,1}. \tag{3.5}$$

Moreover, by Proposition 3.1.2, there are $2^{n-w(b)-1}$ many terms in $y_{n,1}$ containing $x_{w(b),1}$. By Proposition 2.4.7, since $y_{i,j}$ is computed by the determinant, it is clear that we cannot see the multiple of the entries from the same column as a term of $y_{i,j}$, therefore, any term of $y_{n,1}$ might contain $x_{k,1}$ for only one $1 \leq k \leq n$. Hence, neither of terms of $y_{n,1}$ contains $x_{k,1}$ for all $1 \leq k \leq n$ so that k-th and $k+2$-th rows are the same. If k-th and $k+2$-th rows are the same, then $k \equiv k+2 \mod n$ which contradicts with the assumption that $n \geq 3$. □

Using the above theorem, we can construct a Frobenius splitting of the polynomial ring $\mathbb{K}[x_{i,j}]$, when characteristic of $\mathbb{K}$ is prime which compatibly splits the Hessenberg patch ideal $J_{w,h}$ for $h = (n-1, n, \ldots, n)$. By Theorem 3.2.9, $\mathrm{in}_{\prec_{\mathrm{w}}}\left(J_{w,h}\right)$ is generated by a squarefree single generator monomial $\mathrm{in}_{\prec_{\mathrm{w}}}\left(f_{n,1}^{w}\right)$, i.e., $\mathrm{in}_{\prec_{\mathrm{w}}}\left(f_{n,1}^{w}\right)$ is product of several distinct variables in $\mathbb{K}[x_{i,j}]$,

$$\mathrm{in}_{\prec_{\mathrm{w}}}\left(f_{n,1}^{w}\right) = \prod_{i,j} x_{i,j}.$$

We put

$$\begin{aligned} f &:= \mathrm{in}_{\prec_{\mathrm{w}}}\left(f_{n,1}^{w}\right) \prod_{x_{i,j} \nmid \mathrm{in}_{\prec_{\mathrm{w}}}\left(f_{n,1}^{w}\right)} x_{i,j}, \\ g &:= (f)^{p-1}. \end{aligned}$$

Theorem 3.2.10. *Let $n \geq 3$, and $w \in \mathfrak{S}_n$. Then $J_{w,h}$ for $h = (n-1, n, \ldots, n)$ is compatibly split for the F-splitting $\mathrm{Tr}(fg)$ in characteristic $p > 0$ where f, g are defined as above.*

Proof. Note that

$$\mathrm{in}_{\prec_{\mathrm{w}}}(g) = \mathrm{in}_{\prec_{\mathrm{w}}}\left((f)^{p-1}\right) = \left(\prod_{i,j} x_{i,j}\right)^{p-1}$$

then by [6, Lemma 5.6], $\mathrm{Tr}\,(fg)$ defines a Frobenius splitting of $\mathbb{F}_p[x_{i,j}]$. $\square$

Chapter 4

Future Directions

Let F be a polynomial in $\mathbb{F}_p[x_{i,j}]$ such that

(i) there is exactly one term of F of the form

$$\left(\prod_{i,j=1}^{n} x_{i,j}\right)^{p-1}, \tag{4.1}$$

(ii) no other terms of F have the form of

$$m^p \left(\prod_{i,j=1}^{n} x_{i,j}\right)^{p-1} \tag{4.2}$$

where m is a monomial in $\mathbb{F}_p[x_{i,j}]$.

For a polynomial $F \in \mathbb{F}_p[x_{i,j}]$ satisfying (i) and (ii), $\mathrm{Tr}(F\bullet)$ defines a Frobenius splitting. One future direction is to build such an F for Hessenberg patch ideals.

Example 4.0.1. Let $n = 4$ and $\mathbb{F}_5[x_{i,j} : 1 \leq i, j \leq 4]$ be a polynomial ring. Therefore, by Lemma 2.4.3, the S-fixed points in $\mathfrak{S}_4$ are

$$\{(1234), (1243), (1324), (1432), (2134), (2143), (3214), (4321)\}.$$

Consider $h = (2, 3, 4, 4)$. Then by Lemma 2.4.5 the Hessenberg patch ideal is

$$J_{w,h} = \langle f^w_{4,1}, f^w_{3,1}, f^w_{4,2}\rangle,$$

and we put

$$g := f^w_{4,1} \cdot f^w_{3,1} \cdot f^w_{4,2}.$$

As we computed in Macaulay2, there exists a Frobenius splitting of g for each w. Let $F = g^4$.

1. Let $w = (1234)$. Then F has the term

$$(x_{2,1}x_{3,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4.$$

   Also, F does not have a term in the form of $m^5(x_{2,1}x_{3,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4$ for some monomial $m \in \mathbb{F}_5[x_{i,j}]$.

2. Let $w = (1243)$. Then F has the term
$$(x_{2,1}x_{3,1}x_{3,2}x_{4,1}x_{4,2})^4.$$
3. Let $w = (1324)$. Then F has the term
$$(x_{2,1}x_{3,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4.$$
Also, F does not have a term in the form of $m^5(x_{2,1}x_{3,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4$ for some monomial $m \in \mathbb{F}_5[x_{i,j}]$.
4. Let $w = (1432)$. Then F has the term
$$(x_{2,1}x_{3,1}x_{3,2}x_{4,1}x_{4,3})^4.$$
Also, F does not have a term in the form of $m^5(x_{2,1}x_{3,1}x_{3,2}x_{4,1}x_{4,3})^4$ for some monomial $m \in \mathbb{F}_5[x_{i,j}]$.
5. Let $w = (2134)$. Then F has the term
$$(x_{3,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4.$$
Also, F does not have a term in the form of $m^5(x_{3,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4$ for some monomial $m \in \mathbb{F}_5[x_{i,j}]$.
6. Let $w = (2143)$. Then F has the term
$$(x_{3,1}x_{3,2}x_{4,1}x_{4,2})^4.$$
Also, F does not have a term in the form of $m^5(x_{3,1}x_{3,2}x_{4,1}x_{4,2})^4$ for some monomial $m \in \mathbb{F}_5[x_{i,j}]$.
7. Let $w = (3214)$. Then F has the term
$$(x_{2,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4.$$
Also, F does not have a term in the form of $m^5(x_{2,1}x_{3,2}x_{4,1}x_{4,2}x_{4,3})^4$ for some monomial $m \in \mathbb{F}_5[x_{i,j}]$.
8. Let $w = (4321)$. Then F has the term
$$(x_{2,1}x_{3,1}x_{3,2})^4.$$
Also, F does not have a term in the form of $m^5(x_{2,1}x_{3,1}x_{3,2})^4$ for some monomial $m \in \mathbb{F}_5[x_{i,j}]$.

Chapter 5

Appendix

There is a function "isFPure(I)" which is a command in the package of "TestIdeals" in Macaulay2 checking that if an ideal I in a given ring R is Frobenius split. Therefore, we can check if Hessenberg patch ideals $J_{w,h}$ corresponding to patches $w \in \mathfrak{S}_n$ and Hessenberg functions h on $[n]$ are F-split in some finite fields $\mathbb{F}_p[x_{i,j}]$. The following tableaux consist of the results obtained for the case of $n = 4$ in Macaulay2. We checked if the Hessenberg patch ideals $J_{w,h}$ are F-split for any S-fixed point $w \in \mathfrak{S}_4$ and each indecomposable h for $n = 4$. We left some boxes empty because it was not possible to compute the F-splitting as p increased for some patches in Macaulay2. For $n = 4$ consider M, N two real matrices as follows

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{bmatrix}, \qquad N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $a, b, c, d, e, f \in \mathbb{F}$.

F-Split, $h = (3,4,4,4)$		
w	$J_{w,h} = \langle f_{41} \rangle$	In $\mathbb{Z}/p$ for prime $p \leq 23$
$(1,2,3,4)$	$\langle -a^2cf + a^2e + abf + bcf - ad - be - df \rangle$	Yes
$(1,3,2,4)$	$\langle -abcf + abe + b^2f + cdf - bd - de - af \rangle$	Yes
$(1,4,3,2)$	$\langle -acdf + ade + bdf - d^2 - af + b \rangle$	Yes
$(1,4,2,3)$	$\langle -acdf + ade + bcf + bdf - d^2 - be + a \rangle$	Yes
$(2,1,4,3)$	$\langle -acdf + ade + bdf - d^2 + cf + b - e \rangle$	Yes
$(2,1,3,4)$	$\langle -abcf + abe + b^2f - bd + cf - df - e \rangle$	Yes
$(2,3,1,4)$	$\langle -acdf + ade + bcf + bdf - d^2 - be - f \rangle$	Yes
$(2,4,1,3)$	$\langle -abcf + abe + b^2f + cdf - bd - de + 1 \rangle$	Yes
$(3,1,2,4)$	$\langle -a^2cf + a^2e + abf + cdf - ad - de - f \rangle$	Yes
$(3,2,4,1)$	$\langle cdf - de - af + 1 \rangle$	Yes
$(3,4,2,1)$	$\langle cf - df + a - e \rangle$	Yes
$(4,1,3,2)$	$\langle -abcf + abe + b^2f - bd - af + 1 \rangle$	Yes
$(4,2,3,1)$	$\langle bcf - be + a - f \rangle$	Yes
$(4,3,1,2)$	$\langle -a^2cf + a^2e + abf - ad + b - f \rangle$	Yes
$(4,3,2,1)$	$\langle -af + cf + b - e \rangle$	Yes

F-Split, $h=(2,4,4,4)$				
w	$J_{w,h}=\langle f_{41},f_{31}\rangle$	$\mathbb{Z}/3$	$\mathbb{Z}/5$	$\mathbb{Z}/7$
$(1,2,3,4)$	$\langle -a^2cf+a^2e+abf+bcf-ad-be-d, a^2c-ab-bc+d\rangle$	Yes	Yes	
$(1,2,4,3)$	$\langle -a^2cf+a^2e+abf+cdf-ad-de+bz, a^2c-ab-cd\rangle$	Yes	Yes	Yes
$(1,3,2,4)$	$\langle -abcf+abe+b^2f+cdf-bd-de-af, abc-b^2-cd+a\rangle$	Yes	Yes	Yes
$(1,3,4,2)$	$\langle -abcf+abe+b^2f-bd-df+a, abc-b^2+d\rangle$	Yes	Yes	Yes
$(1,4,2,3)$	$\langle -acdf+ade+bcf+bdf-d^2-be+a, acd-bc-bd\rangle$	Yes	Yes	Yes
$(1,4,3,2)$	$\langle -acdf+ade+bdf-d^2-af+b, acd-bd+a\rangle$	Yes	Yes	Yes
$(2,1,3,4)$	$\langle -abcf+abe+b^2f-bd+cf-df-e, abc-b^2-c+d\rangle$	Yes	Yes	Yes
$(2,1,4,3)$	$\langle -acdf+ade+bdf-d^2+cf+b-e, acd-bd-c\rangle$	Yes	Yes	Yes
$(3,2,1,4)$	$\langle -acdf+ade+bdf-d^2-af+cf-e, acd-bd+a-c\rangle$	Yes	Yes	Yes
$(3,2,4,1)$	$\langle cf-df+a-e, -c+d\rangle$	Yes	Yes	Yes
$(4,3,1,2)$	$\langle -a^2cf+a^2e+abf-ad+b-f, a^2c-ab+1\rangle$	Yes	Yes	Yes
$(4,3,2,1)$	$\langle -af+cf+b-e, a-c\rangle$	Yes	Yes	Yes

F-Split, $h=(3,3,4,4)$				
w	$J_{w,h}=\langle f_{41},f_{42}\rangle$	$\mathbb{Z}/3$	$\mathbb{Z}/5$	$\mathbb{Z}/7$
$(1,2,3,4)$	$\langle -a^2cf+a^2e+abf+bcf-ad-be-df, -acf+c^2f+ae-ce+bf-ef-d\rangle$	Yes		
$(1,2,4,3)$	$\langle -a^2cf+a^2e+abf+cdf-ad-de+b, -acf+cef+ae-e^2+bf+c-d\rangle$	Yes		
$(1,3,2,4)$	$\langle -abcf+abe+b^2f+cdf-bd-de-af, -ac^2f+ace+bcf+cef-cd-e^2-f\rangle$	Yes		
$(1,4,3,2)$	$\langle -acdf+ade+bdf-d^2-af+b, -acef+ae^2+bef-de+c-f\rangle$	Yes		
$(1,4,3,2)$	$\langle -acdf+ade+bdf-d^2-af+b, -acef+ae^2+bef-de+c-f\rangle$	Yes		
$(2,1,3,4)$	$\langle -abcf+abe+b^2f-bd+cf-df-e, -ac^2f+ace+bcf-cd-ef\rangle$	Yes		
$(2,1,4,3)$	$\langle -acdf+ade+bdf-d^2+cf+b-e, -acef+ae^2+bef-de+c\rangle$	Yes		
$(2,3,1,4)$	$\langle -acdf+ade+bcf+bdf-d^2-be-f, -acef+ae^2+c^2f+bef-ce-de\rangle$	Yes		
$(3,1,2,4)$	$\langle -a^2cf+a^2e+abf+cdf-ad-de-f, -acf+cef+ae-e^2+bf-d\rangle$	Yes		
$(3,2,1,4)$	$\langle -acdf+ade+bdf-d^2-af+cf-e, -acef+ae^2+bef-de-f\rangle$	Yes		
$(3,4,2,1)$	$\langle cf-df+a-e, -ef+1\rangle$	Yes	Yes	Yes
$(4,1,3,2)$	$\langle -abcf+abe+b^2f-bd-af+1, -ac^2f+ace+bcf-cd-f\rangle$	Yes		
$(4,3,2,1)$	$\langle -af+cf+b-e, c-f\rangle$	Yes	Yes	Yes

F-Split, $h = (2,3,4,4)$				
w	$J_{w,h} = \langle f_{41}, f_{31}, f_{42}\rangle$	$\mathbb{Z}/3$	$\mathbb{Z}/5$	$\mathbb{Z}/7$
$(1,2,3,4)$	$\langle -a^2cf + a^2e + abf + bcf - ad - be - d, a^2c - ab - bc + d, -acf + c^2f + ae - ce + bf - ef - d\rangle$	Yes		
$(1,2,4,3)$	$\langle -a^2cf + a^2e + abf + cdf - ad - de + bz, a^2c - ab - cd, -acf + cef + ae - e^2 + bf + c - d\rangle$	Yes		
$(1,3,2,4)$	$\langle -abcf + abe + b^2f + cdf - bd - de - af, abc - b^2 - cd + a, -ac^2f + ace + bcf + cef - cd - e^2 - f\rangle$	Yes		
$(1,4,3,2)$	$\langle -acdf + ade + bdf - d^2 - af + b, acd - bd + a, -acef + ae^2 + bef - de + c - f\rangle$	Yes		
$(2,1,3,4)$	$\langle -abcf + abe + b^2f - bd + cf - df - e, abc - b^2 - c + d, -ac^2f + ace + bcf - cd - ef\rangle$	Yes		
$(2,1,4,3)$	$\langle -acdf + ade + bdf - d^2 + cf + b - e, acd - bd - c, -acef + ae^2 + bef - de + c\rangle$	Yes		
$(3,2,1,4)$	$\langle -acdf + ade + bdf - d^2 - af + cf - e, acd - bd + a - c, -acef + ae^2 + bef - de - f\rangle$	Yes		
$(4,3,2,1)$	$\langle -af + cf + b - e, a - c, c - f\rangle$	Yes	Yes	Yes

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