ON A FREE-ENDPOINT ISOPERIMETRIC PROBLEM

On a Free-Endpoint Isoperimetric Problem

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A Thesis Submitted to the School of Graduate Studies in the Partial Fulfillment of the Requirements for the Degree Master of Science

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Department of Mathematics & Statistics
Master of Science, Mathematics (2023)
Hamilton, Ontario, Canada

Title: On a Free-Endpoint Isoperimetric Problem
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Number of pages: ix, 49

Abstract

Inspired by a planar partitioning problem involving multiple unbounded chambers, this thesis investigates using classical techniques what can be said of the existence, uniqueness, and regularity of minimizers in a certain free-endpoint isoperimetric problem. In two cases, a full existence-uniqueness-regularity result is proved using a convexity technique inspired by work of Talenti. The problem studied here can be interpreted physically as the identification of the equilibrium shape of a sessile liquid drop in half-space (in the absence of gravity). This is a well-studied variational problem whose full resolution requires the use of geometric measure theory, in particular the theory of sets of finite perimeter. A crash course on the theory required for the modern statement of the equilibrium shape theorem is presented in an appendix.

Keywords: Calculus of variations · Isoperimetric problem · Geometric measure theory · Sets of finite perimeter · Sessile drop · Equilibrium shape · Partitioning problem

Acknowledgements

I would like to sincerely thank my supervisors Dr. Stanley Alama and Dr. Lia Bronsard for their support and encouragement throughout the research process. Their mentorship was invaluable, and helped me to learn what it takes to do good research. Dr. Alama's laid-back charm and Dr. Bronsard's infectious enthusiasm are a wonderful complement to each other, and make for very enjoyable mathematical discussions. This thesis could not have been written without them.

Thank you to all of my instructors, Dr. Stanley Alama, Dr. Lia Bronsard, Dr. Greg Cousins, Dr. Ian Hambleton, Dr. Megumi Harada, Dr. Siyuan Lu, Dr. Dmitry Pelinovsky, Dr. Jenna Rajchgot, Dr. Adam Van Tuyl, Dr. McKenzie Wang, and Dr. Gail Wolkowicz, for showing me what clear and precise mathematical thinking looks like.

Thank you to my first supervisor Dr. Ian Hambleton for taking a chance on me and welcoming me to McMaster as an MSc project student during the online COVID year of 2020. I am indebted to you, and grateful for your support as I navigated my difficult first year at McMaster.

Thank you to the Math and Stats graduate team, especially Hanadi Attar-Elbard, Rabia Awan, Dr. Ben Bolker, Julie Fogarty, Dr. Bartosz Protas, and Emily Warnock, for all of their advice and administrative help.

Thank you to all of my mathematical friends here at McMaster, especially Dan Barake, Amy Bell, Kieran Bhaskara, Adrian Cook, Mike Cummings, Karim Eltanahy, Greg Forkutza, Fletcher Gates, Matthew How, Julie Jenkins, Michaela Kelly, Illya Kierkosz, Sullivan Francis MacDonald, Pip Matharu, Jessie Meanwell, Subhajit Mishra, Ana Mucalica, Jason Pekos, Tilly Pitt, Katarina Sacka, and Szymon Sobieszek, for being such a welcoming and supportive community. Our discussions throughout my time here, on matters both mathematical and personal, helped me to navigate the rollercoaster ride that is graduate school. I look forward to our continuing friendship.

Finally, I would like to thank my family: my parents Gerine de Jong and Phil Vriend; my stepmother Lianne Doucet; my brothers Jesse and Adam Vriend; my stepsisters Genevieve, Isabelle and Marie-Eve Smith; and most of all my partner Heather Lowe (and our funny little cat Bean), for their love, patience, and unwavering support. I am so grateful for you all.

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Declaration of Academic Achievement

I, Silas VRIEND, declare that this thesis titled "On a Free-Endpoint Isoperimetric Problem" and the work presented in it are my own.

To my parents
and
to Simon

Chapter 1

Introduction

1.1 Partitioning Problems

The isoperimetric problem has its roots in antiquity [3]. The traditional setup is as follows: given a compact domain S in the plane $\mathbb{R}^2$ with boundary ∂S of fixed length ℓ, what is the configuration of the boundary ∂S for which the area enclosed is maximal? The answer, perhaps unsurprisingly, is that ∂S should be a circle. This question, together with the brachistochrone problem of Bernoulli, gave impetus to the mathematical field known as the calculus of variations.

One can "dualize" the traditional isoperimetric problem as follows: we may ask, given a compact domain S of fixed area A in the plane $\mathbb{R}^2$, what is the configuration of the boundary ∂S for which the perimeter is minimal? The answer, once again, is that ∂S should be a circle. In general, the term "isoperimetric problem" is given to any variational problem wherein one geometrical quantity is maximized or minimized, while another is held fixed.

Similarly to the dual of the isoperimetric problem given above, the following question can be posed about two compact domains S_1, S_2 in the plane $\mathbb{R}^2$, of fixed areas A_1, A_2 respectively: which configuration of the boundaries of these domains yields minimal perimeter, if such a configuration even exists? Note that we allow the two domains to reduce their total perimeter by sharing a portion of their boundaries, so the minimal-perimeter configuration need not be two disjoint discs. We remark that in this setup, the plane $\mathbb{R}^2$ is partitioned (almost disjointly, i.e., with pairwise disjoint interiors) into three "chambers": two compact, and one non-compact. Such problems are known as *partitioning problems*, and their solutions are *minimizing clusters.*

The famed double bubble conjecture asserted that given two compact domains in $\mathbb{R}^n$, each with fixed n-dimensional volume, the so-called standard double bubble is the configuration which uniquely minimizes perimeter. The first partial resolution of this conjecture came in work of Foisy et al. [9] in 1993: they showed via ad hoc geometric methods that the standard double bubble in $\mathbb{R}^2$ uniquely minimizes perimeter. A resolution of the 3-dimensional case [14] and then the n-dimensional case [16, 23] followed not so long after. Beyond mere existence, further results have demonstrated the regularity and stability of such minimizing clusters [20, 21].

The 2-bubble problem can be generalized to the q-bubble problem, $q \geq 2$, but only the 3-bubble case in $\mathbb{R}^2$ has been resolved to date [31]. Sullivan [28] conjectured that the optimal configuration in all dimensions should be a certain "standard" bubble cluster. In 2022, Milman-Neeman [19] announced a proof of the q-bubble conjecture in $\mathbb{R}^n$ and S^n for all $q \leq \min(5, n+1)$.

Another natural partitioning problem is the following: given N compact domains $S_1, \dots, S_N$ in $\mathbb{R}^n$, each with fixed n-dimensional volume $V_1, \dots, V_N$, and M unbounded domains $U_1, \dots, U_M$, with all domains having pairwise disjoint interiors, what is the configuration of the interfaces which locally minimizes $(n-1)$-dimensional surface area? We say that a candidate configuration is *locally perimeter minimizing* if it minimizes $(n-1)$-dimensional surface area relative to any other candidate configuration when tested within arbitrary compact sets.

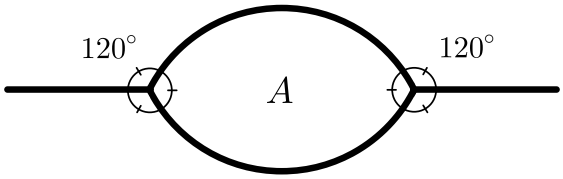

Figure 1.1: Conjectured optimal configuration for the planar partitioning problem with one compact and two unbounded chambers

The simplest case of this problem is in dimension $n = 2$, with $N = 1$ compact chamber with a fixed area A, and $M = 2$ unbounded chambers. We conjecture that the optimal configuration for this problem is given by a *vesica piscis* (the intersection of two discs with equal radii) of the desired area meeting a line at triple junctions with all angles equal to 120°; see Figure 1.1.

This conjecture is motivated by several factors: in other partitioning problems, one finds that the interfaces between the chambers are minimal surfaces (i.e. surfaces with zero mean curvature) which meet at certain standard junctions; see Maggi [18]. Furthermore, in the work of Bellettini-Novaga [1] and Schnürer et al. [25], this configuration appears as the limiting configuration of a planar network with two triple junctions under curve-shortening flow. One may also observe that it is conformally equivalent to the standard double-bubble via an inversion of the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ through an appropriately chosen circle.

We are therefore interested in solving the following variational problem in $\mathbb{R}^2$. Given a positive constant $A > 0$ (the area to enclose), we wish to partition the plane into almost disjoint measurable sets Ω_0, Ω_+, and Ω_- such that:

(i) Ω_0 is compact with Lebesgue measure $|\Omega_0| = A$,

(ii) Ω_+ and Ω_- are unbounded, and

(iii) for any other almost disjoint partition $(\tilde{\Omega}_0, \tilde{\Omega}_+, \tilde{\Omega}_-)$ satisfying properties (i) and (ii), the local perimeter of the interfaces, as measured within any compact test set, is at least that of $(\Omega_0, \Omega_+, \Omega_-)$.

We will refer to the individual sets Ω_0, Ω_+ and Ω_- as *chambers*, and the triple $(\Omega_0, \Omega_+, \Omega_-)$ as a *cluster*. In brief, we want to determine the configuration of the boundaries of $(\Omega_0, \Omega_+, \Omega_-)$ which locally minimizes the perimeter of the chambers. We may always rescale so that $A = 2$.

As posed, this variational problem raises several questions:

(i) What is meant by perimeter?

(ii) Does there exist a minimizer?

(iii) If a minimizer exists, is it unique? Are the chambers connected? Is there a line of symmetry?

(iv) Are the interfaces between Ω_+, Ω_- and Ω_0 smooth (i.e. 1-dimensional manifolds)?

(v) Is there a singular set where the interfaces fail to be 1-dimensional manifolds? If so, what does the singular set look like?

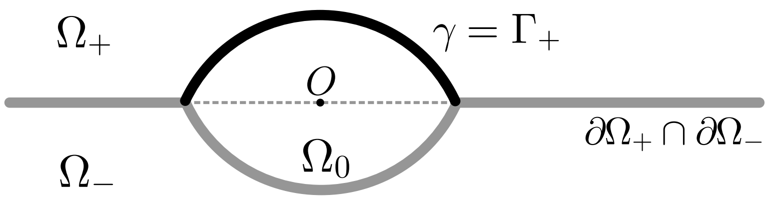

Figure 1.2: Symmetrized candidate configuration, with Ω_0 centered at the origin O, symmetric about the x-axis (dashed), and with $\gamma = \Gamma_+$ (in black)

In answer to question (i), a very general framework in which to work would be the theory of *sets of finite perimeter*; see, for instance, Maggi [18]. These are sets E for which the characteristic function χ_E has a distributional derivative representable by an $\mathbb{R}^2$-valued Radon measure with finite total variation; that is, for which the characteristic function is of *(locally) bounded variation*, $\chi_E \in BV_{\text{loc}}(\mathbb{R}^2)$. For the relevant definitions, we refer the reader to Appendix A.

Towards a proof within the framework of sets of finite perimeter, assume we have a candidate for such a locally perimeter minimizing cluster. We conjecture that the technique of *Steiner symmetrization* (a measure-preserving, perimeter-diminishing symmetrization with respect to a hyperplane; see Talenti [29]) would allow us to make the following simplifying assumptions:

- *Assumption:* away from Ω_0, the unbounded sets Ω_+, Ω_- meet in a straight line. (This seems intuitively plausible: deviations from straightness must necessarily increase arclength.)
- *Assumption:* Ω_0 shares boundary with both Ω_+ and Ω_-. (Also intuitively plausible: we can reduce the total arclength by having Ω_0 share boundary with the unbounded chambers.)
- *Assumption:* Ω_0 is convex, and is symmetrical about the axis formed by the interface between Ω_+ and Ω_- away from Ω_0.
- *Assumption:* By rotating and translating our coordinate system if necessary, we can assume that Ω_0 is centered at $O = (0, 0)$, and that

$$\Omega_+ = \{(x, y) : y \geq 0\} \setminus \text{int}(\Omega_0),$$
$$\Omega_- = \{(x, y) : y \leq 0\} \setminus \text{int}(\Omega_0),$$

  so that Ω_+ and Ω_- meet along the x-axis away from Ω_0.
- *Assumption:* Denoting the boundary of Ω_0 by $\partial\Omega_0 = \Gamma_+ \cup \Gamma_-$, where $\Gamma_+ = \partial\Omega_+ \cap \partial\Omega_0$ and $\Gamma_- = \partial\Omega_- \cap \partial\Omega_0$, we may assume that Γ_+ is a continuous parametrizable curve symmetric about the y-axis, lying in the upper half-space $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$. Denote by γ the upper curve Γ_+; see Figure 1.2 above. (Note that it is unclear *a priori* how regular γ would be after symmetrizing, so for the purposes of our discussion, we will assume it is at least continuous.)

We would therefore like to determine the shape of γ. *A priori*, we do not know the location of the endpoints of $\gamma = \Gamma_+$, which we may assume are of the form $(\pm p, 0)$ for some $p > 0$. We wish to minimize the length of γ and its reflection through the x-axis, while simultaneously locally minimizing the portion of the x-axis adjacent to γ, namely $\partial\Omega_+ \cap \partial\Omega_-$. To achieve the latter, it is equivalent to maximize the distance between the endpoints of γ.

Denote the length of γ by $\ell(\gamma)$, and the length of $\partial\Omega_0$ by $\ell(\partial\Omega_0)$; let $A(\gamma)$ denote the area enclosed by γ and the x-axis; and let $d(\gamma) = 2p$ denote the distance along the x-axis between the endpoints of γ. We therefore wish to minimize the functional

$$J[\gamma] = \ell(\gamma) - \frac{1}{2}d(\gamma) = \frac{1}{2}(\ell(\partial\Omega_0) - d(\gamma)), \tag{1.1}$$

for $\gamma = (x, y)$ a sufficiently regular parametrized curve satisfying

$$y \geq 0, \quad x(t_0) = -x(t_1) = p > 0, \quad A(\gamma) = 1. \tag{1.2}$$

This is a free-endpoint isoperimetric problem in the calculus of variations.

As it turns out, the solution to this variational problem is well-known: using the theory of sets of finite perimeter, the resolution to this problem appears in Maggi [18] in the following form. To interpret the statement, we note the following:

- The open upper half-space H is defined by $H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$.
- The notation $P(A; B)$ is to be read as "the perimeter of the set A as measured within the set B," while the notation $P(E)$ denotes the total perimeter $P(E; \mathbb{R}^n)$.
- The notation $B(x, r)$ indicates a ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$.
- The Euclidean space $\mathbb{R}^n$ is equipped with the standard basis $e_1, \ldots, e_n$, where e_i is the vector with a 1 in the i^{th} slot and 0s elsewhere.
- The vector ν_E should be thought of as the measure-theoretic outward unit normal to the set E of finite perimeter.

In our special case above, $n = 2$ and $\beta = \frac{1}{2}$.

Theorem 1.1 (Maggi Thm. 19.21, Liquid drops in the absence of gravity)**.** *For every $\beta \in (-1, 1)$, there exists a unique $\sigma(\beta) > 0$ with the following property: a set of finite perimeter $E \subset H$ with $|E| = 1$ is a minimizer in the variational problem*

$$\psi(\beta) = \inf\{\mathcal{F}_\beta(E; H) : E \subset H,\, P(E) < \infty,\, |E| = 1\}, \tag{1.3}$$

where

$$\mathcal{F}_\beta(E; H) = P(E; H) - \beta P(E; \partial H), \tag{1.4}$$

if and only if, up to horizontal translation, E is equivalent to the set

$$G_\beta = F_{\sigma(\beta)}, \tag{1.5}$$

where F_σ, $\sigma > 0$, is a set of the form

$$F_\sigma = B(se_n, r) \cap H, \tag{1.6}$$

with $s \in \mathbb{R}$ and $r > 0$ uniquely determined by the conditions

$$|F_\sigma| = 1, \quad P(F_\sigma; \partial H) = \sigma. \tag{1.7}$$

Moreover,

$$\nu_{G_\beta} \cdot e_n = \beta, \quad \textit{on } \operatorname{bdry}(H \cap \partial G_\beta). \tag{1.8}$$

To fully understand the statement and proof of this theorem, one must study a formidable amount of modern analytical machinery. (Indeed, this theorem appears in Chapter 19 of the text by Maggi!) For the interested reader, we present a crash course on this material in Appendix A, consisting of just those definitions and results which enter into the statement and proof of the theorem. This raises a question: can we find a setting where this machinery is not necessary while still obtaining a meaningful result?

1.2 Thesis Outline

The goal of this thesis is to examine what may be said using *classical* methods about existence, uniqueness, and regularity of minimizers in the variational problem addressed by Theorem 1.1, for the special case $n = 2$. In particular, can we determine existence-uniqueness-regularity for graphs of C^1 functions? What about graphs of $W^{1,1}$ functions? Or C^1 parametrized curves?

The outline of the thesis is as follows:

- In Chapter 2, as preparation, we derive general expressions for the first and second variation of a functional, for both graphs of functions and parametrized curves. The "naive" Taylor expansion approach presented here was inspired by that of Gelfand-Fomin [11]. This approach can be refined significantly by considering 1-parameter families of nearby admissible curves; see, for example, Morse [22].
- In Chapters 3 and 4, inspired by Talenti [29], we present existence-uniqueness-regularity proofs in the first and second cases mentioned above: for graphs of C^1 functions, the proof is given in Chapter 3; for graphs of $W^{1,1}$ functions, the proof is given in Chapter 4. These proofs are enabled by a strict convexity property of the integrand when the functional is restricted to the class of curves given by graphs of functions.
- In Chapter 5 we present a uniqueness-regularity result for C^1 parametrized curves. The question of existence poses a difficult problem here. One might argue that this very question motivated much of the development of the modern theory presented in Maggi; indeed, the direct method in the calculus of variations was developed with the express purpose of furnishing existence proofs.

As a final note, we remark that there is much to be said about the classical theory of the calculus of variations which will not be presented here. In particular, the classical theory of sufficient conditions for a minimizer received much attention in the first half of the 20$^{\text{th}}$ century. For an introduction and a thorough bibliography, we refer the interested reader to the recent text by Kot [15]. For more advanced developments, the reader should consult texts such as Bliss [2], Bolza [4], Ewing [8], Fox [10], Gelfand-Fomin [11], Hestenes [13], Morse [22], and Sagan [24].

Chapter 2

First and Second Variation

In this chapter, we derive formulae for the first and second variation of a functional in two settings: for graphs of C^1 functions, and for regular C^1 parametrized curves. The formulae derived are general in nature, allowing for one or both endpoints of the curve to vary.

Definition 2.1 (Big-O and little-o)**.** Given functions $f, g : \mathbb{R} \to \mathbb{R}$ with $g > 0$, we say:

(i) f is *big-O* of g, and write $f(x) = O(g(x))$, if there exist $M > 0$ and $x_0 \in \mathbb{R}$ such that

$$|f(x)| \leq Mg(x) \quad \text{for all } x \text{ in a neighbourhood of } x_0. \tag{2.1}$$

(ii) f is *little-o* of g, and write $f(x) = o(g(x))$, if for every $\varepsilon > 0$ there exists $x_0 \in \mathbb{R}$ such that

$$|f(x)| \leq \varepsilon g(x) \quad \text{for all } x \text{ in a neighbourhood of } x_0. \tag{2.2}$$

Theorem 2.2 (Fundamental lemma of the calculus of variations)**.** *If $M(x) \in C[a, b]$ and if*

$$\int_a^b M(x)\eta(x)dx = 0 \tag{2.3}$$

for every $\eta(x) \in C^1[a, b]$ such that $\eta(a) = \eta(b) = 0$, then

$$M(x) = 0 \quad \text{for all } x \in [a, b]. \tag{2.4}$$

Proof. We refer the reader to Kot [15], p. 39, for the proof. □

2.1 Graphs of Functions

In this section we derive general formulae for the first and second variations of a functional of the form

$$J[u] = \int_{x_0}^{x_1} F(x, u, u')dx, \quad u \in C^2[x_0, x_1], \tag{2.5}$$

where $x_0 < x_1$ are real numbers, and $F : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 function.

Formulae for the first and second variation of such a functional can be obtained by using a 1-parameter family of nearby admissible curves; this approach is presented in Bliss [2], Fox [10], and Morse [22]. The formulae presented in these sources allow for nonlinearities in the endpoint variations, and include ours as a special case. Since we do not require this, we instead follow and expand upon the Taylor expansion approach of Gelfand-Fomin [11].

Let $I = [x_0, x_1]$ and $I^* = [x_0^*, x_1^*]$. Let $u : I \to \mathbb{R}$ and $u^* : I^* \to \mathbb{R}$ be C^2 functions, that is, for each $0 \leq k \leq 2$ the derivative $u^{(k)}$ exists and is continuous on (x_0, x_1), and admits a continuous extension to $[x_0, x_1]$ (and similarly for u^*).

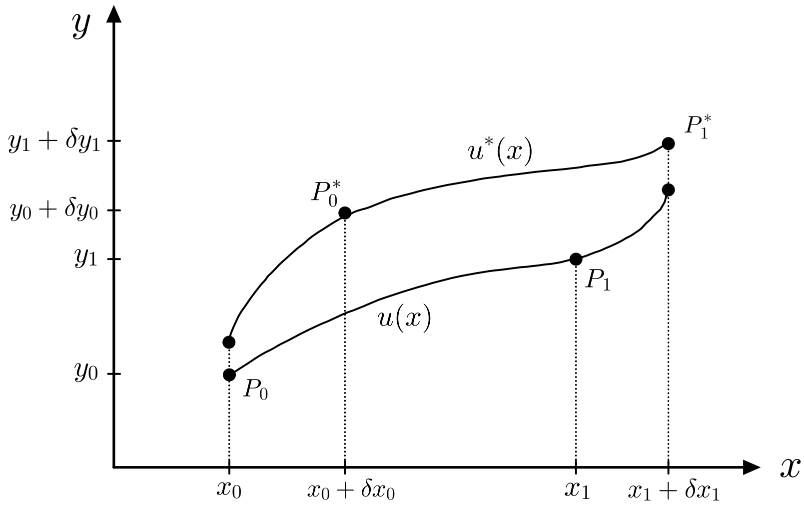

Figure 2.1: Two nearby curves in $\mathbb{R}^2$, after Gelfand-Fomin [11]

The *graph* of u is defined by

$$\Gamma(u) = \{(x, u(x)) \in \mathbb{R}^2 : x \in I\} \subset \mathbb{R}^2 \tag{2.6}$$

(and similarly for u^*).

Let P_0, P_1 (resp. P_0^*, P_1^*) denote the left and right endpoints of $\Gamma(u)$ (resp. $\Gamma(u^*)$). We set

$$P_0 = (x_0, y_0), \quad P_0^* = (x_0^*, y_0^*) = (x_0 + \delta x_0, y_0 + \delta y_0), \tag{2.7}$$
$$P_1 = (x_1, y_1), \quad P_1^* = (x_1^*, y_1^*) = (x_1 + \delta x_1, y_1 + \delta y_1), \tag{2.8}$$

where

$$\delta x_0 = x_0^* - x_0, \qquad \delta y_0 = y_0^* - y_0 = u^*(x_0 + \delta x_0) - u(x_0), \tag{2.9}$$
$$\delta x_1 = x_1^* - x_1, \qquad \delta y_1^* = y_1^* - y_1 = u^*(x_1 + \delta x_1) - u(x_1). \tag{2.10}$$

To facilitate a comparison of u and u^*, we extend each quadratically at the necessary endpoints as indicated in the following steps. See Figure 2.1 for a diagram.

(i) If $\delta x_0 > 0$, extend u^* on $[x_0, x_0 + \delta x_0]$ by the 2nd-order Taylor polynomial of u^* at $x_0 + \delta x_0$, so that for $x \in [x_0, x_0 + \delta x_0]$, we have

$$\begin{aligned} u^*(x) &= u^*(x_0 + \delta x_0) + u^{*\prime}(x_0 + \delta x_0)(x - x_0 - \delta x_0) \\ &\quad + \frac{1}{2} u^{*\prime\prime}(x_0 + \delta x_0)(x - x_0 - \delta x_0)^2. \end{aligned} \tag{2.11}$$

(ii) If $\delta x_0 < 0$, extend u on $[x_0 + \delta x_0, x_0]$ by the 2nd-order Taylor polynomial of u at x_0, so that for $x \in [x_0 + \delta x_0, x_0]$, we have

$$u(x) = u(x_0) + u'(x_0)(x - x_0) + \frac{1}{2} u''(x_0)(x - x_0)^2. \tag{2.12}$$

(iii) If $\delta x_1 > 0$, extend u on $[x_1, x_1 + \delta x_1]$ by the 2nd-order Taylor polynomial of u at x_1, so that for $x \in [x_1, x_1 + \delta x_1]$, we have

$$u(x) = u(x_1) + u'(x_1)(x - x_1) + \frac{1}{2}u''(x_1)(x - x_1)^2. \tag{2.13}$$

(iv) If $\delta x_1 < 0$, extend u^* on $[x_1 + \delta x_1, x_1]$ by the 2nd-order Taylor polynomial of u^* at $x_1 + \delta x_1$, so that for $x \in [x_1 + \delta x_1, x_1]$, we have

$$\begin{aligned} u^*(x) = u^*(x_1 + \delta x_1) &+ {u^*}'(x_1 + \delta x_1)(x - x_1 - \delta x_1) \\ &+ \frac{1}{2}{u^*}''(x_1 + \delta x_1)(x - x_1 - \delta x_1)^2. \end{aligned} \tag{2.14}$$

(v) For each i, if $\delta x_i = 0$, neither u nor u^* needs extension at the endpoint x_i.

Having extended u and u^* as needed, each is considered to be a function on the interval

$$K = [\min\{x_0, x_0 + \delta x_0\}, \max\{x_1, x_1 + \delta x_1\}] = \operatorname{conv}(I \cup I^*), \tag{2.15}$$

where conv denotes the convex hull. We will not distinguish between the original functions u, u^* and their quadratic extensions. Note that K is a closed and bounded interval, and the extensions of u, u^* are both $C^2(K)$ functions.

Let $\mathcal{F}$ be the family of functions given by $\mathcal{F} = \bigcup C^2(I)$, with the union taken over all closed bounded intervals $I \subset \mathbb{R}$. To quantify the *distance* between $u \in C^2(I)$ and $u^* \in C^2(I^*)$, we define a metric $\rho : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by the rule

$$\begin{aligned} \rho(u, u^*) = \|u - u^*\|_\infty &+ \left\|u' - {u^*}'\right\|_\infty + \left\|u'' - {u^*}''\right\|_\infty \\ &+ \|P_0 - P_0^*\|_2 + \|P_1 - P_1^*\|_2. \end{aligned} \tag{2.16}$$

For the first trio of terms, we extend u and u^* as above, and take the supremum norm $\|\cdot\|_\infty$ over the set $K = \operatorname{conv}(I \cup I^*)$. For the last pair of terms, $\|\cdot\|_2$ is the usual Euclidean distance in $\mathbb{R}^2$.

Let u and u^* be neighbouring curves in the sense of the distance defined by (2.16), that is, $\rho(u, u^*) << 1$. Define the *variation* $h : K \to \mathbb{R}$ by the rule

$$h(x) = u^*(x) - u(x), \tag{2.17}$$

where $K = \operatorname{conv}(I \cup I^*)$. We establish below some terminology about the variation h.

Definition 2.3. We call h a *weak variation* provided that its norm

$$\|h\| := \|h\|_\infty + \|h'\|_\infty + \|h''\|_\infty \tag{2.18}$$

is small, say $\|h\| << 1$. We will only consider weak variations in what follows.

Definition 2.4. We call h a *strong variation* provided that $\|h\|_\infty$ is small, say $\|h\|_\infty << 1$.

Lemma 2.5. *With the setup as above, we have*

$$h(x_i) = \delta y_i - u'(x_i)\delta x_i - h'(x_i)\delta x_i - \frac{1}{2}u''(x_i)\delta x_i^2 + O(\delta x_i^3, h''\delta x_i^2). \tag{2.19}$$

Proof. Fix $i \in \{1, 2\}$ and consider the compatibility condition at the endpoint x_i,

$$u^*(x_i + \delta x_i) = u(x_i + \delta x_i) + h(x_i + \delta x_i) = u(x_i) + \delta y_i. \tag{2.20}$$

Since u and h are C^2, we Taylor expand each to obtain

$$u(x_i + \delta x_i) = u(x_i) + u'(x_i)\delta x_i + \frac{1}{2}u''(x_i)\delta x_i^2 + O(\delta x_i^3)\ , \tag{2.21}$$

$$h(x_i + \delta x_i) = h(x_i) + h'(x_i)\delta x_i + O(h''\delta x_i^2)\ . \tag{2.22}$$

Substituting these into the compatibility condition and rearranging, we have

$$h(x_i) = \delta y_i - u'(x_i)\delta x_i - h'(x_i)\delta x_i^2 - \frac{1}{2}u''(x_i)\delta x_i^2 + O(\delta x_i^3, h''\delta x_i^2), \tag{2.23}$$

as was to be shown. □

The *total variation* from u to u^* of the functional $J[u]$ is the quantity

$$\Delta J := J[u^*|_{I^*}] - J[u|_I] = J[(u+h)|_{I^*}] - J[u|_I]. \tag{2.24}$$

In what follows, we consider ΔJ as being computed from the following data:

- a fixed function $u \in C^2(I)$ with domain $I = [x_0, x_1]$,
- the endpoint increments δx_0, δx_1, yielding an interval $I^* = [x_0 + \delta x_0, x_1 + \delta x_1]$,
- the endpoint value increments δy_0, δy_1, and
- a function $h \in C^2(K)$, with $K = \text{conv}(I \cup I^*)$, which satisfies the conditions

$$u(x_0 + \delta x_0) + h(x_0 + \delta x_0) = u(x_0) + \delta y_0, \tag{2.25}$$

$$u(x_1 + \delta x_1) + h(x_1 + \delta x_1) = u(x_1) + \delta y_1, \tag{2.26}$$

  where u is extended quadratically to K as needed. We call such a function h an *admissible variation.*

We wish to expand the total variation in the form

$$\Delta J = \delta J + \delta^2 J + o(\|h\|^2), \tag{2.27}$$

where:

- δJ consists of terms which are linear in the distance $\rho(u|_I, (u+h)|_{I^*})$, and
- $\delta^2 J$ consists of terms which are quadratic in the distance $\rho(u|_I, (u+h)|_{I^*})$.

Definition 2.6. We call δJ the *first variation*, and $\delta^2 J$ the *second variation.*

Theorem 2.7 (First and Second Variation, Graphs)**.** *Denote the endpoint increments by* $\delta x|_{x_i} = \delta x_i$ *and* $\delta y|_{x=x_i} = \delta y_i$*. Then, with the setup as above,*

(i) the first variation δJ *is given by*

$$\delta J = \int_{x_0}^{x_1} \left[F_u - \frac{d}{dx}F_{u'}\right] h dx + [(F - u'F_{u'})\delta x + F_{u'}\delta y]\Big|_{x=x_0}^{x=x_1}, \tag{2.28}$$

(ii) and the second variation $\delta^2 J$ *is given by*

$$\begin{aligned}\delta^2 J = \frac{1}{2}\int_{x_0}^{x_1} &\left[F_{uu}h^2 + 2F_{uu'}hh' + F_{u'u'}h'^2\right] dx \\ &+ \frac{1}{2}\left[(F_x - u'F_u)\delta x^2 + 2F_u \delta x \delta y\right]\Big|_{x=x_0}^{x=x_1},\end{aligned} \tag{2.29}$$

where F *and its derivatives are evaluated at* (x, u, u') *whenever the arguments are suppressed.*

Proof. By definition of the functional $J[u]$, we have

$$\begin{aligned}\Delta J &= J[u^*|_{I^*}] - J[u|_I] \\ &= \int_{x_0+\delta x_0}^{x_1+\delta x_1} F(x, u^*, u^{*\prime})dx - \int_{x_0}^{x_1} F(x, u, u')dx \\ &= \left(\int_{x_0+\delta x_0}^{x_0} + \int_{x_0}^{x_1} + \int_{x_1}^{x_1+\delta x_1}\right) F(x, u+h, u'+h')dx - \int_{x_0}^{x_1} F(x, u, u')dx \\ &= \int_{x_0}^{x_1} [F(x, u+h, u'+h') - F(x, u, u')]dx \qquad (2.30) \\ &\quad + \int_{x_1}^{x_1+\delta x_1} F(x, u+h, u'+h')dx - \int_{x_0}^{x_0+\delta x_0} F(x, u+h, u'+h')dx. \qquad (2.31)\end{aligned}$$

Applying Taylor's theorem to $F(x, u+h, u'+h')$, we have

$$\begin{aligned}F(x, u+h, u'+h') = F(x, u, u') &+ F_u(x, u, u')h + F_{u'}(x, u, u')h' \\ &+ \frac{1}{2}F_{uu}(x, u, u')h^2 + F_{uu'}(x, u, u')hh' \\ &+ \frac{1}{2}F_{u'u'}(x, u, u')h'^2 + O(h^3, h^2h', hh'^2, h'^3),\end{aligned} \tag{2.32}$$

Inserting (2.32) into (2.30), we obtain

$$\begin{aligned}&\int_{x_0}^{x_1} [F(x, u+h, u'+h') - F(x, u, u')]dx \\ &\quad = \frac{1}{2}\int_{x_0}^{x_1} \left[F_{uu}(x, u, u')h^2 + 2F_{uu'}(x, u, u')hh' + F_{u'u'}(x, u, u')h'^2\right] dx \\ &\qquad + \int_{x_0}^{x_1} [F_u(x, u, u')h + F_{u'}(x, u, u')h']dx + O(h^3, h^2h', hh'^2, h'^3).\end{aligned} \tag{2.33}$$

Integrating (2.33) by parts, we have

$$\int_{x_0}^{x_1} [F_u h + F_{u'} h']dx = \int_{x_0}^{x_1} \left[F_u - \frac{d}{dx}F_{u'}\right] hdx + F_{u'}h|_{x=x_1} - F_{u'}h|_{x=x_0}. \tag{2.34}$$

Thus (2.30) evaluates to

$$\begin{aligned}\int_{x_0}^{x_1} [F(x, u+h, u'+h') - F(x,u,u')]dx = &\int_{x_0}^{x_1} \left[F_u - \frac{d}{dx}F_{u'}\right] hdx + F_{u'}h\Big|_{x=x_0}^{x=x_1} \\ &+ \frac{1}{2}\int_{x_0}^{x_1} \left[F_{uu}h^2 + 2F_{uu'}hh' + F_{u'u'}h'^2\right] dx \\ &+ O(h^3, h^2h', hh'^2, h'^3).\end{aligned} \tag{2.35}$$

Applying Lemma 2.5 to the boundary term, we obtain

$$\begin{aligned}F_{u'}h\Big|_{x=x_0}^{x=x_1} &= F_{u'}[\delta y - u'\delta x - h'\delta x - \tfrac{1}{2}u''\delta x^2]\Big|_{x=x_0}^{x=x_1} \\ &= \left[F_{u'}\delta y - u'F_{u'}\delta x - F_{u'}h'\delta x - \tfrac{1}{2}u''F_{u'}\delta x^2\right]\Big|_{x=x_0}^{x=x_1}.\end{aligned}$$

Thus we have that

$$\begin{aligned}&\int_{x_0}^{x_1} [F(x, u+h, u'+h') - F(x,u,u')]dx \\ &= \int_{x_0}^{x_1} \left[F_u - \frac{d}{dx}F_{u'}\right] hdx + [F_{u'}\delta y - u'F_{u'}\delta x]\Big|_{x=x_0}^{x=x_1} \\ &+ \frac{1}{2}\int_{x_0}^{x_1} \left[F_{uu}h^2 + 2F_{uu'}hh' + F_{u'u'}h'^2\right] dx - \left[F_{u'}h'\delta x + \tfrac{1}{2}u''F_{u'}\delta x^2\right]\Big|_{x=x_0}^{x=x_1} \\ &+ O(h^3, h^2h', hh'^2, h'^3).\end{aligned} \tag{2.36}$$

Fixing i and inserting (2.32) into a generic term of (2.31), we obtain

$$\begin{aligned}\int_{x_i}^{x_i+\delta x_i} F(x, u+h, u'+h')dx = &\int_{x_i}^{x_i+\delta x_i} F(x,u,u')dx + \int_{x_i}^{x_i+\delta x_i} F_u(x,u,u')hdx \\ &+ \int_{x_i}^{x_i+\delta x_i} F_{u'}(x,u,u')h'dx + O(\delta x_i h^2, \delta x_i hh', \delta x_i h'^2).\end{aligned} \tag{2.37}$$

We proceed to Taylor expand the integrands of each term in (2.37).

- *First term.* Since the domain of integration has length δx_i, we need only expand the integrand to first order. Centering the expansion at $x = x_i$, we have

$$F(x,u,u') = F(x,u,u')|_{x=x_i} + \frac{d}{dx}[F(x,u,u')]|_{x=x_i}(x - x_i) + O((x-x_i)^2).$$

  Then we have that

$$\int_{x_i}^{x_i+\delta x_i} F(x,u,u')dx = F|_{x=x_i}\delta x_i + \frac{1}{2}\frac{dF}{dx}\Big|_{x=x_i}\delta x_i^2 + O(\delta x_i^3).$$

Computing the derivative in the second term, we have

$$\frac{d}{dx}F(x,u(x),u'(x)) = F_x + F_u u' + F_{u'}u''.$$

Therefore we have

$$\int_{x_i}^{x_i+\delta x_i} F(x,u,u')dx = \left\{F\delta x + \frac{1}{2}[F_x + F_u u' + F_{u'}u'']\delta x^2\right\}\Big|_{x=x_i} + O(\delta x_i^3).$$

- *Second term.* Since the domain has length δx_i and the integrand contains the variation h, we expand to 0^{th} order. Centering the expansion at $x = x_i$, we have

$$F_u(x,u,u')h = F_u h|_{x=x_i} + O((x-x_i)).$$

Then we have that

$$\int_{x_i}^{x_i+\delta x_i} F_u(x,u,u')h dx = F_u h\delta x\Big|_{x=x_i} + O(h\delta x_i^2, h'\delta x_i^2, \delta x_i^3).$$

Applying Lemma 2.5, it follows that

$$\int_{x_i}^{x_i+\delta x_i} F_u h dx = F_u[\delta y - u'\delta x]\delta x\Big|_{x=x_i} + O(h\delta x_i^2, h'\delta x_i^2, \delta x_i^3). \tag{2.38}$$

Therefore we have

$$\int_{x_i}^{x_i+\delta x_i} F_u h dx = \left[F_u\delta x\delta y - u'F_u\delta x^2\right]\Big|_{x=x_i} + O(h\delta x_i^2, h'\delta x_i^2, \delta x_i^3). \tag{2.39}$$

- *Third term.* Since the domain has length δx_i and the integrand contains the variation h, we expand to 0^{th} order. Centering the expansion at $x = x_i$, we have

$$F_{u'}(x,u,u')h' = F_{u'}h'|_{x=x_i} + O((x-x_i)). \tag{2.40}$$

Thus to second order

$$\int_{x_i}^{x_i+\delta x_i} F_{u'}h' dx = F_{u'}h'\delta x\Big|_{x=x_i} + O(h'\delta x_i^2, h''\delta x_i^2, \delta x_i^3). \tag{2.41}$$

As such, we can write (2.37) as

$$\begin{aligned}\int_{x_i}^{x_i+\delta x_i} F(x,u+h,u'+h')dx = &\left\{F\delta x + \frac{1}{2}[F_x + F_u u' + F_{u'}u'']\delta x^2\right\}\Big|_{x=x_i} \\ &+ \left[F_u\delta x\delta y - u'F_u\delta x^2\right]\Big|_{x=x_i} + F_{u'}h'\delta x\Big|_{x=x_i} \\ &+ O(\delta x_i^3, h\delta x_i^2, h'\delta x_i^2, h''\delta x_i^2).\end{aligned}$$

Therefore our expression for the total variation becomes

$$\begin{aligned}
\Delta J = &\int_{x_0}^{x_1} \left[F_u - \frac{d}{dx}F_{u'}\right] h dx + [F_{u'}\delta y - u'F_{u'}\delta x]\Big|_{x=x_0}^{x=x_1} \\
&+ \frac{1}{2}\int_{x_0}^{x_1} \left[F_{uu}h^2 + 2F_{uu'}hh' + F_{u'u'}h'^2\right] dx - \left[F_{u'}h'\delta x + \tfrac{1}{2}u''F_{u'}\delta x^2\right]\Big|_{x=x_0}^{x=x_1} \\
&+ \left\{F\delta x + \frac{1}{2}[F_x + F_u u' + F_{u'}u'']\delta x^2\right\}\Big|_{x=x_0}^{x=x_1} \\
&+ \left[F_u\delta x\delta y - u'F_u\delta x^2\right]\Big|_{x=x_0}^{x=x_1} + F_{u'}h'\delta x\Big|_{x=x_0}^{x=x_1} \\
&+ O(\delta x_i^3, h\delta x_i^2, h'\delta x_i^2, h''\delta x_i^2).
\end{aligned} \tag{2.42}$$

Making note of the cancellations and regrouping the terms by order, we have

$$\begin{aligned}
\Delta J = &\int_{x_0}^{x_1} \left[F_u - \frac{d}{dx}F_{u'}\right] h dx + [(F - u'F_{u'})\delta x + F_{u'}\delta y]\Big|_{x=x_0}^{x=x_1} \\
&+ \frac{1}{2}\int_{x_0}^{x_1} \left[F_{uu}h^2 + 2F_{uu'}hh' + F_{u'u'}h'^2\right] dx \\
&+ \frac{1}{2}\left[(F_x - u'F_u)\delta x^2 + 2F_u\delta x\delta y\right]\Big|_{x=x_0}^{x=x_1} \\
&+ O(\delta x_i^3, h\delta x_i^2, h'\delta x_i^2, h''\delta x_i^2).
\end{aligned} \tag{2.43}$$

Thus the total variation can be written

$$\Delta J = \delta J + \delta^2 J + o(\|h\|^2), \tag{2.44}$$

where the *first variation* δJ is given by

$$\delta J[h] = \int_{x_0}^{x_1} \left[F_u - \frac{d}{dx}F_{u'}\right] h dx + [(F - u'F_{u'})\delta x + F_{u'}\delta y]\Big|_{x=x_0}^{x=x_1},$$

and the *second variation* $\delta^2 J$ is given by

$$\begin{aligned}
\delta^2 J[h] = &\frac{1}{2}\int_{x_0}^{x_1} \left[F_{uu}h^2 + 2F_{uu'}hh' + F_{u'u'}h'^2\right] dx \\
&+ \frac{1}{2}\left[(F_x - u'F_u)\delta x^2 + 2F_u\delta x\delta y\right]\Big|_{x=x_0}^{x=x_1},
\end{aligned}$$

as was to be shown. □

2.2 Parametrized Curves

In this section we derive general formulae for the first and second variations of a functional of the form

$$J[x,y] = \int_{t_0}^{t_1} F(x,y,x',y')dt, \quad x,y \in C^2[t_0,t_1], \tag{2.45}$$

where $t_0 < t_1$ are real numbers, x, y satisfy $x'(t)^2 + y'(t)^2 > 0$ for all $t_0 \leq t \leq t_1$, and $F : \mathbb{R}^4 \to \mathbb{R}$ is a C^2 function which is (positively) homogeneous of degree 1 in x' and y'. Here we follow and expand upon the text of Kot [15].

Let $I = [t_0, t_1]$. Let $\gamma = (x, y) : I \to \mathbb{R}^2$ and $\gamma^* = (x^*, y^*) : I \to \mathbb{R}^2$ be C^2 functions, that is, for each $0 \leq k \leq 2$ the derivatives $x^{(k)}, y^{(k)}$ exist and are continuous on (t_0, t_1), and admit continuous extensions to $[t_0, t_1]$ (and similarly for (x^*, y^*)).

The *trace* of γ is defined by

$$\Gamma(\gamma) = \{(x(t), y(t)) \in \mathbb{R}^2 : t \in I\} \subset \mathbb{R}^2, \tag{2.46}$$

(and similarly for γ^*).

Let P_0, P_1 (resp. P_0^*, P_1^*) denote the initial and final endpoints of $\Gamma(\gamma)$ (resp. $\Gamma(\gamma^*)$). Then

$$P_0 = (x(t_0), y(t_0)), \quad P_0^* = (x^*(t_0), y^*(t_0)) = (x_0 + \delta x_0, y_0 + \delta y_0), \tag{2.47}$$

$$P_1 = (x(t_1), y(t_1)), \quad P_1^* = (x^*(t_1), y^*(t_1)) = (x_1 + \delta x_1, y_1 + \delta y_1), \tag{2.48}$$

where

$$\delta x_0 = x^*(t_0) - x(t_0), \qquad \delta y_0 = y^*(t_0) - y(t_0), \tag{2.49}$$

$$\delta x_1 = x^*(t_1) - x(t_1), \qquad \delta y_1 = y^*(t_1) - y(t_1). \tag{2.50}$$

Compared to the graphs case, it is easier to compare γ and γ^* since they are both defined on the same interval I. This is without loss of generality, since we can always reparametrize γ^* to have the same domain as γ, per Kot [15]. Homogeneity of the integrand F ensures that the functional depends only on the trace, and not on the parametrization.

Let $\mathcal{F}$ be the family of curves γ given by $\mathcal{F} = C^2(I) \times C^2(I)$. To quantify the *distance* between $\gamma \in F$ and $\gamma^* \in \mathcal{F}$, we define a metric $\rho : \mathcal{F} \times \mathcal{F} \to [0, \infty)$ by the rule

$$\begin{aligned}\rho(\gamma, \gamma^*) = {} & \|x - x^*\|_\infty + \|y - y^*\|_\infty + \left\|x' - x^{*\prime}\right\|_\infty + \left\|y' - y^{*\prime}\right\|_\infty \\ & + \left\|x'' - x^{*\prime\prime}\right\|_\infty + \left\|y'' - y^{*\prime\prime}\right\|_\infty + \|P_0 - P_0^*\|_2 + \|P_1 - P_1^*\|_2,\end{aligned} \tag{2.51}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on the interval I, and $\|\cdot\|_2$ denotes the usual Euclidean norm in the plane $\mathbb{R}^2$.

Let γ and γ^* be neighbouring curves in the sense of the distance (2.51), that is, $\rho(\gamma, \gamma^*) << 1$. Define the *coordinate variations*

$$\begin{aligned}\xi : I \to \mathbb{R}, \quad & \xi(t) = x^*(t) - x(t), \\ \eta : I \to \mathbb{R}, \quad & \eta(t) = y^*(t) - y(t).\end{aligned}$$

Definition 2.8. We call $\delta = (\xi, \eta)$ a *weak variation* provided that its norm

$$\|\delta\| := \|\xi\|_\infty + \|\xi'\|_\infty + \|\xi''\|_\infty + \|\eta\|_\infty + \|\eta'\|_\infty + \|\eta''\|_\infty$$

is small, say $\|\delta\| << 1$. We shall only consider weak variations in what follows.

Definition 2.9. We call $\delta = (\xi, \eta)$ a *strong variation* provided that $\|\delta\|_\infty := \|\xi\|_\infty + \|\eta\|_\infty$ is small, say $\|\delta\|_\infty << 1$.

The *total variation* from γ to γ^* of the functional $J[u]$ is the quantity

$$\Delta J := J[\gamma^*] - J[\gamma]. \tag{2.52}$$

We wish to expand the total variation in the form

$$\Delta J = \delta J + \delta^2 J + o(\|\delta\|^2), \tag{2.53}$$

where:

- δJ consists of terms which are linear in $\rho(\gamma, \gamma^*)$, and
- $\delta^2 J$ consists of terms which are quadratic in $\rho(\gamma, \gamma^*)$.

Definition 2.10. We call δJ the *first variation*, and $\delta^2 J$ the *second variation.*

Theorem 2.11 (First and Second Variation, Parametrized Curves)**.** *Denote the endpoint increments by $\delta x|_{x_i} = \delta x_i$ and $\delta y|_{x=x_i} = \delta y_i$. Then, with the setup as above,*

(i) the first variation δJ is given by

$$\delta J = \int_{t_0}^{t_1} \left\{ \xi \left[F_x - \frac{d}{dt} F_{x'} \right] + \eta \left[F_y - \frac{d}{dt} F_{y'} \right] \right\} dt + F_{x'} \delta x \Big|_{t=t_0}^{t=t_1} + F_{y'} \delta y \Big|_{t=t_0}^{t=t_1}, \tag{2.54}$$

(ii) and the second variation $\delta^2 J$ is given by

$$\delta^2 J = \frac{1}{2} \int_{t_0}^{t_1} (\xi, \eta, \xi', \eta') \cdot \mathrm{Hess}(F)(\xi, \eta, \xi', \eta') dt,$$

where $\mathrm{Hess}(F)$ *denotes the Hessian matrix of F,*

and F and its derivatives are evaluated at (x, y, x', y') whenever the arguments are suppressed.

Proof. By definition of the functional $J[\gamma]$, we have

$$\Delta J = J[\gamma^*] - J[\gamma] = \int_{t_0}^{t_1} [F(x + \xi, y + \eta, x' + \xi', y' + \eta') - F(x, y, x', y')] dt.$$

Applying Taylor's theorem to $F(x + \xi, y + \eta, x' + \xi', y' + \eta')$, we have

$$\begin{aligned} F(x + \xi, y + \eta, x' + \xi', y' + \eta') &= F + \nabla F \cdot (\xi, \eta, \xi', \eta') \\ &\quad + \frac{1}{2} (\xi, \eta, \xi', \eta') \cdot \mathrm{Hess}(F)(\xi, \eta, \xi', \eta') + O(\delta^3 \cdot \partial^3 F), \end{aligned}$$

where $\delta^3 \cdot \partial^3 F$ denotes terms which are third order in both $\delta = (\xi, \eta)$ and in derivatives of F; ∇F denotes the gradient of F, with entries

$$\nabla F = (F_x, F_y, F_{x'}, F_{y'}); \tag{2.55}$$

and $\mathrm{Hess}(F)$ denotes the Hessian matrix of F, with entries

$$\mathrm{Hess}(F) = \left(\frac{\partial^2 F}{\partial z_i \partial z_j} \right), \tag{2.56}$$

where z_i, z_j each range over $\{x, y, x', y'\}$. Then we have that

$$\Delta J = \int_{t_0}^{t_1} \nabla F \cdot (\xi, \eta, \xi', \eta') dt + \frac{1}{2} \int_{t_0}^{t_1} (\xi, \eta, \xi', \eta') \cdot \mathrm{Hess}(F)(\xi, \eta, \xi', \eta') dt + O(\delta^3 \cdot \partial^3 F). \tag{2.57}$$

Integrating the first term by parts, we find that

$$\begin{aligned}
&\int_{t_0}^{t_1} \nabla F \cdot (\xi, \eta, \xi', \eta') dt \\
&\quad = \int_{t_0}^{t_0} \left[F_x \xi + F_y \eta + F_{x'} \xi' + F_{y'} \eta' \right] dt \\
&\quad = \int_{t_0}^{t_1} \left\{ \xi \left[F_x - \frac{d}{dt} F_{x'} \right] + \eta \left[F_y - \frac{d}{dt} F_{y'} \right] \right\} dt + \left[F_{x'} \xi + F_{y'} \eta \right] \Big|_{t=t_0}^{t=t_1} \\
&\quad = \int_{t_0}^{t_1} \left\{ \xi \left[F_x - \frac{d}{dt} F_{x'} \right] + \eta \left[F_y - \frac{d}{dt} F_{y'} \right] \right\} dt + F_{x'} \delta x \Big|_{t=t_0}^{t=t_1} + F_{y'} \delta y \Big|_{t=t_0}^{t=t_1}.
\end{aligned} \tag{2.58}$$

Thus the total variation can be written

$$\Delta J = \delta J + \delta^2 J + o(\|\delta\|^2), \tag{2.59}$$

where the *first variation* δJ is given by

$$\delta J = \int_{t_0}^{t_1} \left\{ \xi \left[F_x - \frac{d}{dt} F_{x'} \right] + \eta \left[F_y - \frac{d}{dt} F_{y'} \right] \right\} dt + F_{x'} \delta x \Big|_{t=t_0}^{t=t_1} + F_{y'} \delta y \Big|_{t=t_0}^{t=t_1}, \tag{2.60}$$

and the *second variation* $\delta^2 J$ is given by

$$\delta^2 J = \frac{1}{2} \int_{t_0}^{t_1} (\xi, \eta, \xi', \eta') \cdot \mathrm{Hess}(F)(\xi, \eta, \xi', \eta') dt, \tag{2.61}$$

as was to be shown. $\square$

Chapter 3

Proof for Graphs of C^1 Functions

In this section, we present a classical existence-uniqueness-regularity proof for the variational problem described in the introduction, with competition among graphs of C^1 functions. The variational problem is made precise in Table 3.1.

Theorem 3.1. *If $0 < \beta < 1$, then there exists a unique smooth minimizer $u(x)$ for the variational problem in Table 3.1. In particular, the unique minimizer $u(x)$ is an arc of a circle of the form*

$$u(x) = \sqrt{R^2 - x^2} - \beta R, \tag{3.1}$$

where the radius of curvature $R > 0$ is determined by

$$R^{-1} = \sqrt{\arccos\beta - \beta\sqrt{1-\beta^2}}, \tag{3.2}$$

and $u(x)$ is defined on $[-p, p]$ with $p = \sqrt{1-\beta^2}R$. Furthermore, the graph of u meets the x-axis at $\pm p$ with interior angle $\arccos\beta$.

Proof. (*Uniqueness*) Assume that a minimizer $u \in C^1[-p, p]$ exists for the problem in Table 3.1, for some $p > 0$. Let $v \in C^1[-q, q]$ be a competitor curve for some $q > 0$, nearby to u in the sense of the distance $\rho(u, v)$. Since v vanishes at the endpoints of its symmetric domain, the endpoint increments satisfy

$$\delta x(p) = -\delta x(-p), \quad \delta y = 0. \tag{3.3}$$

Since u is a constrained minimizer, there must exist a Lagrange multiplier $\lambda \in \mathbb{R}$ such that the augmented functional

$$\Lambda[u] := J[u] - \lambda K[u] \tag{3.4}$$

is stationary at u, i.e., the first variation $\delta\Lambda$ must vanish at u. Written out in full, we have

$$\Lambda[u] = \int_{-p}^{p} F(u, u')dx, \quad F(u, u') = \sqrt{1 + u'^2} - \beta - \lambda u. \tag{3.5}$$

Table 3.1: Our isoperimetric problem with competition among graphs of C^1 functions

`among`	nonnegative C^1 functions $u : [-p, p] \to \mathbb{R}$ with $p > 0$ free
`minimize`	$J[u] = \int_{-p}^{p} \left[\sqrt{1 + u'^2} - \beta\right] dx$
`subject to`	$K[u] = \int_{-p}^{p} u dx = 1$
`and`	$u(p) = u(-p) = 0.$

Since $\Lambda[u]$ is stationary at the minimizer u, the first variation $\delta\Lambda$ of Λ at u must vanish for all admissible variations. By Theorem 2.7, the first variation $\delta\Lambda$ is given by

$$\delta\Lambda = \int_{-p}^{p} \left[F_u - \frac{d}{dx}F_{u'}\right] hdx + [F - u'F_{u'}]\delta x\Big|_{x=-p}^{x=p} \tag{3.6}$$

where $h = v - u$ is defined on $K = [-p,p] \cup [-q,q]$, and we extend u and v linearly at the endpoints as needed so that h is well-defined. Then for all admissible variations h, we must have

$$\delta\Lambda = 0. \tag{3.7}$$

In particular, this holds for admissible variations h with $\delta x = 0$. Hence for all $h \in C^1[-p,p]$ with $h(p) = h(-p) = 0$, we have

$$\int_{-p}^{p} \left[F_u(u,u') - \frac{d}{dx}F_{u'}(u,u')\right] hdx = 0.$$

By the fundamental lemma of the calculus of variations, it follows that u satisfies

$$F_u(u,u') - \frac{d}{dx}F_{u'}(u,u') = 0, \tag{3.8}$$

the *Euler-Lagrange equation* of the functional Λ.

The remaining terms in the first variation yield the condition

$$\delta\Lambda = [F(u,u') - u'F_{u'}(u,u')]\delta x\Big|_{x=-p}^{x=p} = 0\ . \tag{3.9}$$

for all admissible variations $h \in C^1(K)$ with $\delta x(p) = -\delta x(-p) \neq 0$. This is the *transversality condition* for our problem. Since $\delta x(p) = -\delta x(-p) \neq 0$, u must satisfy the equation

$$[F(u,u') - u'F_{u'}(u,u')]\Big|_{x=p} + [F(u,u') - u'F_{u'}(u,u')]\Big|_{x=-p} = 0.$$

Euler-Lagrange Equation

The Euler-Lagrange equation reads

$$0 = F_u - \frac{d}{dx}F_{u'} = -\lambda - \frac{d}{dx}\left(\frac{u'}{\sqrt{1+u'^2}}\right). \tag{3.10}$$

We can immediately integrate this to find that

$$C_1 = -\lambda x - \frac{u'}{\sqrt{1+u'^2}} \tag{3.11}$$

for some constant of integration C_1. Solving for u', we find that

$$u'^2 = \frac{(\lambda x + C_1)^2}{1 - (\lambda x + C_1)^2}. \tag{3.12}$$

Integrating again, we see that

$$u(x) = \pm \frac{1}{\lambda}\sqrt{1-(\lambda x + C_1)^2} + C_2 \tag{3.13}$$

for some constant of integration C_2. Rearranging and absorbing the appropriate multiplicative constants into C_1 and C_2, we see that $u(x)$ must satisfy

$$(x + C_1)^2 + (u(x) + C_2)^2 = \left(\frac{1}{\lambda}\right)^2. \tag{3.14}$$

Hence $u(x)$ is an arc of a circle with radius $\frac{1}{|\lambda|}$ and centre (C_1, C_2).

We immediately see that $C_1 = 0$ by symmetry considerations. In more detail, we apply the endpoint conditions $u(p) = u(-p) = 0$ to obtain

$$(p + C_1)^2 + C_2^2 = 1 = (-p + C_1)^2 + C_2^2 \quad \Longrightarrow \quad |p + C_1| = |-p + C_1|, \tag{3.15}$$

from which we conclude that $C_1 = 0$. Hence $u(x)$ is an arc of a circle symmetric about the y-axis,

$$x^2 + (u(x) + C_2)^2 = \left(\frac{1}{\lambda}\right)^2. \tag{3.16}$$

Reapplying the endpoint condition $u(\pm p) = 0$, we see that

$$C_2^2 = \left(\frac{1}{\lambda}\right)^2 - p^2 \geq 0. \tag{3.17}$$

Note that this implies

$$\left(\frac{1}{\lambda}\right)^2 \geq p^2 \quad \Longrightarrow \quad \frac{1}{|\lambda|} \geq p. \tag{3.18}$$

Thus $u(x)$ is given by

$$u(x) = \pm\sqrt{\left(\frac{1}{\lambda}\right)^2 - x^2} \mp \sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2} \quad \text{for all } x \in [-p, p], \tag{3.19}$$

where the signs are such that $u(\pm p) = 0$. Since $u \geq 0$, we must have

$$u(x) = \sqrt{\left(\frac{1}{\lambda}\right)^2 - x^2} - \sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2}. \tag{3.20}$$

Returning to the Euler-Lagrange equation, we can determine the sign of λ: a computation shows that

$$\frac{u'}{\sqrt{1 + u'^2}} = -|\lambda| x, \tag{3.21}$$

from which we determine that

$$0 = -\lambda - \frac{d}{dx}\left(\frac{u'}{\sqrt{1 + u'^2}}\right) = -\lambda - \frac{d}{dx}(-|\lambda| x) = -\lambda + |\lambda|, \tag{3.22}$$

so that we have $|\lambda| = \lambda \geq 0$. Hence $u(x)$ is an arc of a circle with radius $\frac{1}{\lambda}$ and centre $(0, \sqrt{(1/\lambda)^2 - p^2})$, where $\lambda \geq 0$ and $\frac{1}{\lambda} \geq p > 0$. The remaining two parameters are determined by the area constraint and the transversality condition.

We now apply the area constraint. Computing the area functional using the change of variable $x = \frac{1}{\lambda}\sin\theta$, we have:

$$\begin{aligned}
K[u] &= \int_{-p}^{p} \sqrt{\left(\frac{1}{\lambda}\right)^2 - x^2} dx - \int_{-p}^{p} \sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2} dx \\
&= 2\int_{0}^{p} \sqrt{\left(\frac{1}{\lambda}\right)^2 - x^2} dx - 2p\sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2} \\
&= \frac{2}{\lambda^2}\int_{0}^{\arcsin(\lambda p)} \cos^2\theta d\theta - 2p\sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2} \\
&= \frac{1}{\lambda^2}[\theta + \sin\theta\cos\theta]\Big|_0^{\arcsin(\lambda p)} - 2p\sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2} \\
&= \frac{1}{\lambda^2}\arcsin(\lambda p) - p\sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2}.
\end{aligned}$$

We note that this is the area of the circular segment with arc $u(x)$ and chord the x-axis. Hence the area constraint reads

$$1 = \frac{1}{\lambda^2}\arcsin(\lambda p) - p\sqrt{\left(\frac{1}{\lambda}\right)^2 - p^2}. \tag{3.23}$$

This is a transcendental equation in the variables λ, p, and is well-defined since $\frac{1}{\lambda} \geq p$. To progress, we need the transversality condition.

Transversality condition

The transversality condition reads

$$[F(u,u') - u'F_{u'}(u,u')]\Big|_{x=p} + [F(u,u') - u'F_{u'}(u,u')]\Big|_{x=-p} = 0. \tag{3.24}$$

From the definition of the integrand F, we see that

$$\begin{aligned}
F - u'F_{u'} &= \sqrt{1+u'^2} - \beta - \lambda u - \frac{u'^2}{\sqrt{1+u'^2}} \\
&= \frac{1}{\sqrt{1+u'^2}} - \beta - \lambda u.
\end{aligned}$$

Furthermore, we see that from the expression

$$u'(x) = \frac{-x}{\sqrt{\left(\frac{1}{\lambda}\right)^2 - x^2}} \tag{3.25}$$

that $[u'(p)]^2 = [u'(-p)]^2$. Since $u(\pm p) = 0$, the transversality condition yields

$$0 = \left[\frac{1}{\sqrt{1+u'(p)^2}} - \beta - \lambda u(p)\right] + \left[\frac{1}{\sqrt{1+u'(-p)^2}} - \beta - \lambda u(-p)\right] = \frac{2}{\sqrt{1+u'(p)^2}} - 2\beta, \quad (3.26)$$

from which we derive the *angle condition*

$$\frac{1}{\sqrt{1+u'(p)^2}} = \beta. \quad (3.27)$$

Note that this shows the tangent vector $(1, u'(p))$ makes an angle of $\arccos\beta$ with the positive x-axis, as claimed. (Note also that this implies $0 < \beta < 1$, so our assumption was necessary.) From the definition of $u'(x)$, we find that

$$\frac{1}{\sqrt{1+u'(p)^2}} = \sqrt{1-(\lambda p)^2} = \beta, \quad (3.28)$$

from which we obtain

$$\lambda p = \sqrt{1-\beta^2}. \quad (3.29)$$

Inserting this into the area constraint (3.23), we find that

$$\lambda^2 = \arcsin(\lambda p) - \lambda p\sqrt{1-(\lambda p)^2} = \arccos\beta - \beta\sqrt{1-\beta^2}, \quad (3.30)$$

where we have used the identity $\arcsin\sqrt{1-\theta^2} = \arccos\theta$.

Define $R = \frac{1}{\lambda}$, the radius of the circular arc. We may then identify λ as the *curvature* of the arc, so that $R = \frac{1}{\lambda}$ is the *radius of curvature*. Written out explicitly, we have

$$R^{-1} = \sqrt{\arccos\beta - \beta\sqrt{1-\beta^2}}. \quad (3.31)$$

From (3.29), it follows that $p = \sqrt{1-\beta^2}R$ and $\sqrt{R^2-p^2} = \beta R$.

Therefore the minimizer $u(x)$ must have the form

$$u(x) = \sqrt{R^2-x^2} - \beta R, \quad (3.32)$$

where R is defined by (3.31), and $u(x)$ is defined on $[-p,p]$ where $p = \sqrt{1-\beta^2}R$. This proves uniqueness.

(*Existence*) To prove existence, we claim that the function $u : [-p,p] \to [0,\infty)$ given by

$$u(x) = \sqrt{R^2-x^2} - \beta R \quad (3.33)$$

with R and p as defined above, is a minimizer for the problem in Table 3.1.

To prove this, we first make some observations about the augmented functional $\Lambda[u]$ given by

$$\Lambda[u] = \int_{-p}^{p} [\sqrt{1+u'^2} - \beta - \lambda u]dx. \quad (3.34)$$

By the above, we see that for a minimizer to exist, λ is necessarily of the form

$$\lambda = \frac{\sqrt{1-\beta^2}}{p}. \tag{3.35}$$

This gives us a family of integrals

$$\Lambda[u,p] = \int_{-p}^{p} \left[\sqrt{1+u'^2} - \beta - \frac{\sqrt{1-\beta^2}}{p} u \right] dx, \quad p > 0. \tag{3.36}$$

By "Euler's rule" (see Kot [15], p. 119) our constrained optimization problem in Table 3.1 is equivalent to the problem of minimizing the augmented functional $\Lambda[u,p]$, then selecting the extremal which satisfies the area constraint $K[u] = 1$.

From the above, we know that the unique extremal of $\Lambda[u,p]$ is

$$u(x) = \sqrt{R^2 - x^2} - \beta R, \quad R = \frac{p}{\sqrt{1-\beta^2}}. \tag{3.37}$$

It satisfies the Euler-Lagrange equations and the transversality conditions. This is our candidate for a minimizer of $\Lambda[u,p]$.

Since we are only considering functions which vanish at the endpoints of the interval $[-p,p]$, an integration by parts shows that

$$\Lambda[u,p] = \int_{-p}^{p} \left[\sqrt{1+u'^2} - \beta + \frac{\sqrt{1-\beta^2}}{p} x u' \right] dx. \tag{3.38}$$

Observe now that the integrand of $\Lambda[u,p]$ is strictly convex with respect to u': denoting the integrand by $f(x,u')$, we have

$$\frac{\partial^2 f}{\partial u'^2}(x,u') = \frac{1}{(1+u'^2)^{\frac{3}{2}}} > 0. \tag{3.39}$$

As such, for any function $w \in C^1[-p,p]$ satisfying $w(-p) = w(p) = 0$, we have

$$f(x,w') \geq f(x,u') + (w'-u')\frac{\partial f}{\partial u'}(x,u') \tag{3.40}$$

with equality if and only if $w' = u'$. Since w and u both vanish at the endpoints, $w' = u'$ if and only if $w = u$.

Then observe that for $u(x) = \sqrt{R^2 - x^2} - \beta R$, we have

$$\frac{\partial f}{\partial u'} = \frac{u'}{\sqrt{1+u'^2}} + \frac{\sqrt{1-\beta^2}}{p} x = 0. \tag{3.41}$$

Therefore $f(x,w') \geq f(x,u')$ with equality if and only if $w = u$. Hence, for this choice of u,

$$\Lambda[w,p] \geq \Lambda[u,p], \tag{3.42}$$

with equality if and only if $w = u$, so u is the unique minimizer of $\Lambda[u, p]$.

This gives us a family of extremals $u(x, p)$, each of which minimizes the corresponding integral $\Lambda[u, p]$. It remains to select the extremal $u(x, p)$ which satisfies the area constraint $K[u] = 1$. Computing the integral of $u(x, p)$, we have

$$\begin{aligned}
K[u] &= \int_{-p}^{p} u(x,p)dx = \int_{-p}^{p} \left[\sqrt{R^2 - x^2} - \beta R\right] dx \\
&= 2\int_0^p \sqrt{R^2 - x^2}dx - 2\beta R p \\
&= R^2[\theta + \sin\theta\cos\theta]|_0^{\arcsin\sqrt{1-\beta^2}} - \frac{2\beta p^2}{\sqrt{1-\beta^2}} \\
&= \frac{p^2}{1-\beta^2}[\arcsin\sqrt{1-\beta^2} + \beta\sqrt{1-\beta^2}] - \frac{2\beta\sqrt{1-\beta^2}}{1-\beta^2}p^2 \\
&= \frac{\arccos\beta - \beta\sqrt{1-\beta^2}}{1-\beta^2}p^2,
\end{aligned}$$

where we have used the fact that $\arcsin\sqrt{1-\beta^2} = \arccos\beta$. Setting $K[u] = 1$ and solving for p, we obtain

$$p = \frac{\sqrt{1-\beta^2}}{\sqrt{\arccos\beta - \beta\sqrt{1-\beta^2}}} = \sqrt{1-\beta^2}R, \tag{3.43}$$

where R is determined by

$$R^{-1} = \sqrt{\arccos\beta - \beta\sqrt{1-\beta^2}}. \tag{3.44}$$

Thus the extremal $u(x) = \sqrt{R^2 - x^2} - \beta R$ with R and p as given above is a minimizer of the variational problem in Table 3.1. This proves existence.

(*Regularity*) We have shown that

$$u(x) = \sqrt{R^2 - x^2} - \beta R \tag{3.45}$$

is the unique minimizer of the problem in Table 3.1. The smoothness of $u(x)$ follows immediately, since $|x| \leq p < R$ and $u(x)$ is a composition of smooth functions. □

Remark 3.2. With respect to the parametrization $t \mapsto (t, u(t))$, the graph of $u(x)$ has (non-unit) tangent vector $(1, u'(x))$. Normalizing and rotating the tangent vector 90° counter-clockwise, we find that the outward unit normal ν of the graph of $u(x)$ is given by

$$\nu = \left(-\frac{u'(x)}{\sqrt{1 + u'(x)^2}}, \frac{1}{\sqrt{1 + u'(x)^2}}\right). \tag{3.46}$$

As such, we see that (3.27) tells us that

$$\nu \cdot e_2 = \beta \quad \text{at } x = \pm p, \tag{3.47}$$

in agreement with the modern statement of the theorem in Maggi [18]. For our special case of $\beta = \frac{1}{2}$, we obtain an interior angle of 60°, so the exterior angle is 120° as in Figure 1.1.

Chapter 4

Proof for Graphs of $W^{1,1}$ Functions

4.1 Definitions

We recall some of the basic definitions relating to the Sobolev space $W^{1,1}$. We assume the reader's familiarity with the Lebesgue integral; for more on this, we refer the reader to the text of Stein-Shakarchi [26]. A useful resource for the theory of Sobolev spaces is the book by Brezis [5], from which we draw the definitions and results which follow.

Let $I \subset \mathbb{R}$ be an open interval, possibly unbounded. We recall that $C_c^1(I)$ denotes the set of all $C^1(I)$ functions with compact support, i.e. functions for which the set

$$\operatorname{supp} \varphi = \overline{\{x \in I : \varphi(x) \neq 0\}} \subset I \tag{4.1}$$

is compact.

Definition 4.1. The *Sobolev space* $W^{1,1}(I)$ is defined by

$$W^{1,1}(I) = \left\{ u \in L^1(I) : \exists\, g \in L^1(I) \text{ such that } \int_I u\varphi' dx = -\int_I g\varphi \, dx \text{ for all } \varphi \in C_c^1(I) \right\}.$$

In the definition of $W^{1,1}$, we call the functions $\varphi \in C_c^1(I)$ *test functions*. We call the function g the *weak derivative* of u, and write $u' := g$.

If $u \in C^1(I) \cap L^1(I)$ and its classical derivative $u' \in L^1(I)$, integration by parts shows that

$$\int_I u\varphi' dx = -\int_I u'\varphi \, dx \quad \text{for all } \varphi \in C_c^1(I), \tag{4.2}$$

so that $u \in W^{1,1}(I)$. Hence our notation for u' is consistent! (This integration by parts procedure is what motivates the definition of a Sobolev space.)

In the language of distribution theory, we say that a function $u \in W^{1,1}(I)$ if $u \in L^1(I)$ and u has a distributional derivative which is representable by an L^1 function; for more on distribution theory, see Strichartz [27].

There is in fact a 1-parameter family of Sobolev spaces, denoted $W^{1,p}(I)$ for $1 \leq p \leq \infty$, which are defined as above, but with $u, g \in L^p(I)$. Sobolev spaces of higher (weak) regularity, denoted $W^{k,p}(I)$ for $k \in \mathbb{N}$, are defined inductively. For more details, see Brezis [5].

Theorem 4.2. *$W^{1,1}(I)$ is a separable Banach space when equipped with the norm*

$$\|u\|_{W^{1,1}(I)} = \|u\|_{L^1(I)} + \|u'\|_{L^1(I)}. \tag{4.3}$$

Proof. For the proof, see Brezis [5]. □

A significant advantage of working in 1 dimension is the following embedding theorem, which allows us to talk about pointwise values for Sobolev functions, where otherwise we must talk about equivalence classes of functions which agree almost everywhere.

Theorem 4.3. *Let $u \in W^{1,1}(I)$. Then there exists a function $\tilde{u} \in C(\bar{I})$ such that*

$$u = \tilde{u} \quad \text{a.e. on } I, \tag{4.4}$$

and

$$\tilde{u}(y) - \tilde{u}(x) = \int_x^y u'(t)dt \quad \text{for all } x, y \in \bar{I}. \tag{4.5}$$

As such, we may identify $W^{1,1}(I)$ with a subset of $C(\bar{I})$: symbolically,

$$W^{1,1}(I) \subset C(\bar{I}). \tag{4.6}$$

Proof. For the proof, see Brezis [5]. □

Note that this result is generally *not true* in higher dimensions. The question of whether a Sobolev space $W^{k,p}(\Omega)$ (with $\Omega \subset \mathbb{R}^n$ open) embeds into a Lebesgue space $L^q(\Omega)$ or a classical Hölder space $C^{k,\alpha}(\Omega)$ is answered by the class of theorems known as *Sobolev embedding theorems*; see Brezis [5]. (This tradeoff between regularity and integrability is lucidly described by Terence Tao in an article on function spaces appearing in the Princeton Companion to Mathematics [30].)

In higher dimensions, Sobolev embedding theorems are highly dependent on the geometry of the domain Ω, especially the regularity of the boundary $\partial\Omega$. Since we are working with an interval $I \subset \mathbb{R}$, regularity of the boundary is a non-issue.

The following lemma allows us to enter into the discussion of our variational problem in the context of the Sobolev space $W^{1,1}(I)$.

Lemma 4.4. *Fix $0 < \beta < 1$ and $p > 0$, and let $u \in W^{1,1}[-p, p]$ with $u(p) = u(-p) = 0$. Then the functional*

$$J[u] = \int_{-p}^{p} \left[\sqrt{1 + u'^2} - \beta\right] dx \tag{4.7}$$

is finite on u, i.e., $|J[u]| < \infty$.

Proof. Let β, p, and u be as given. Then

$$\begin{aligned} |J[u]| &\leq \int_{-p}^{p} \left|\sqrt{1 + u'^2} - \beta\right| dx \\ &\leq \int_{-p}^{p} \sqrt{1 + u'^2} dx + 2\beta p \\ &\leq \int_{-p}^{p} [1 + |u'|] dx + 2\beta p \\ &= 2(1 + \beta)p + \|u'\|_{L^1} < \infty \end{aligned}$$

since $u' \in L^1(-p, p)$.

□

Table 4.1: Our isoperimetric problem with competition among graphs of $W^{1,1}$ functions

`among`	nonnegative functions $u \in W^{1,1}[-p, p]$ with $p > 0$ free
`minimize`	$J[u] = \int_{-p}^{p} \left[\sqrt{1 + u'^2} - \beta\right] dx$
`subject to`	$K[u] = \int_{-p}^{p} u\, dx = 1$
`and`	$u(p) = u(-p) = 0.$

4.2 Proof of Existence-Uniqueness-Regularity

We wish to solve the variational problem in Table 4.1. Following the approach presented in Talenti [29], we prove an isoperimetric inequality, from which we establish the existence of a unique smooth minimizer.

Taken on its own, the proof below is perhaps a little philosophically unsatisfying, as it is essentially a mathematical sleight of hand. However, the hard work of identifying a candidate for a minimizer was carried out in Chapter 3, so we are free to pull the rabbit out of a hat!

Theorem 4.5 (Isoperimetric Inequality)**.** *Let $0 < \beta < 1$, and let $u \in W^{1,1}[-p, p]$ be a nonnegative real-valued function defined in an interval $[-p, p]$, with $p > 0$ a free parameter. Assume u vanishes at both endpoints, i.e.,*

$$u(-p) = u(p) = 0. \tag{4.8}$$

Define the length of the graph of u and the area under the graph of u by

$$L = \int_{-p}^{p} \sqrt{1 + u'^2} dx \quad \text{and} \quad A = \int_{-p}^{p} u dx, \tag{4.9}$$

respectively. Then

$$L \geq \left(\frac{\arccos \beta}{\sqrt{1 - \beta^2}} + \beta \right) p + \sqrt{1 - \beta^2} \frac{A}{p} \tag{4.10}$$

with equality if and only if the graph of u is an arc of a circle of the form

$$u(x) = \sqrt{R^2 - x^2} - \beta R, \tag{4.11}$$

where R is determined by

$$R = \frac{p}{\sqrt{1 - \beta^2}}. \tag{4.12}$$

Proof. Let $p > 0$ be fixed, and consider the functional

$$\Lambda[u, p] = \frac{\sqrt{1 - \beta^2}}{p} \int_{-p}^{p} \left[\sqrt{1 + u'^2} - \beta + \frac{\sqrt{1 - \beta^2}}{p} x u' \right] dx. \tag{4.13}$$

Note that the integrand $f(x, u') = \sqrt{1 + (u')^2} - \beta + \frac{\sqrt{1-\beta^2}}{p} x u'$ is strictly convex with respect to u': we have

$$\frac{\partial^2}{\partial (u')^2} f(x, u') = \frac{1}{(1 + u'^2)^{\frac{3}{2}}} > 0. \tag{4.14}$$

Thus for any nonnegative $u, w \in W^{1,1}[-p,p]$ with $u(p) = u(-p) = 0$ and $w(p) = w(-p) = 0$, we have

$$f(x,w') \geq f(x,u') + (w'-u')\frac{\partial}{\partial(u')}f(x,u') \tag{4.15}$$

with equality if and only if $w' = u'$.

Define

$$R = \frac{p}{\sqrt{1-\beta^2}} \quad \text{and} \quad u(x) = \sqrt{R^2-x^2} - \beta R. \tag{4.16}$$

Note since $0 < p < R$ that $u(x)$ is smooth up the boundary of $[-p,p]$, so $u \in W^{1,1}[-p,p]$. Furthermore, we have $u(p) = u(-p) = 0$ by construction, and

$$\frac{\partial}{\partial u'}f(x,u') = \frac{u'}{\sqrt{1+u'^2}} + \frac{\sqrt{1-\beta^2}}{p}x = \frac{-x}{R} + \frac{x}{R} = 0.$$

Therefore for any $w \in W^{1,1}[-p,p]$ with $w(p) = w(-p) = 0$, we have

$$f(x,w') \geq f(x,u') \tag{4.17}$$

with equality if and only if $w' = u'$. Since w and u both vanish at the endpoints, $w' = u'$ holds if and only if $w = u$. As such, we have

$$\Lambda[w,p] = \frac{\sqrt{1-\beta^2}}{p}\int_{-p}^{p} f(x,w')dx \geq \frac{\sqrt{1-\beta^2}}{p}\int_{-p}^{p} f(x,u')dx = \Lambda[u,p] \tag{4.18}$$

with equality if and only if $w = u$. Therefore u is the unique minimizer of the functional $\Lambda[u,p]$. *A posteriori*, we see that u is smooth on $[-p,p]$, as noted above.

We wish to compute the value of the functional attained by the unique minimizer $u(x)$ given by (4.16). Recalling that $p = \sqrt{1-\beta^2}R$, we have

$$\begin{aligned}
\Lambda[u,p] &= \frac{\sqrt{1-\beta^2}}{p}\int_{-p}^{p} f(x,u')dx \\
&= \frac{1}{R}\int_{-p}^{p}\left[\sqrt{1+u'^2} - \beta + \frac{1}{R}xu'\right]dx \\
&= \frac{1}{R}\int_{-p}^{p}\left[\frac{R}{\sqrt{R^2-x^2}} - \beta + \frac{1}{R}x\left(-\frac{x}{\sqrt{R^2-x^2}}\right)\right]dx \\
&= \frac{1}{R}\int_{-p}^{p}\left[\frac{R^2-x^2}{R\sqrt{R^2-x^2}} - \beta\right]dx = \frac{1}{R}\int_{-p}^{p}\left[\frac{\sqrt{R^2-x^2}}{R} - \beta\right]dx \\
&= \frac{2}{R^2}\int_{0}^{p}\sqrt{R^2-x^2}dx - 2\beta\sqrt{1-\beta^2}.
\end{aligned}$$

The remaining integral can be computed using the change of variables $x = R\sin\theta$. Note that $\arcsin\sqrt{1-\beta^2} = \arccos\beta$. We have

$$\int_0^p \sqrt{R^2-x^2}dx = \int_0^{\arcsin(p/R)} \sqrt{R^2-R^2\sin^2\theta}R\cos\theta d\theta$$

$$\begin{aligned}
&= R^2 \int_0^{\arcsin\sqrt{1-\beta^2}} \cos^2\theta d\theta = \frac{R^2}{2}\int_0^{\arccos\beta} [1+\cos(2\theta)]d\theta \\
&= \frac{R^2}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]\Big|_0^{\arccos\beta} = \frac{R^2}{2}\left[\theta + \sin(\theta)\cos(\theta)\right]\Big|_0^{\arccos\beta} \\
&= \frac{R^2}{2}\left[\arccos\beta + \sqrt{1-\beta^2}\beta\right].
\end{aligned}$$

Inserting this into our expression for $\Lambda[u,p]$, we obtain

$$\Lambda[u,p] = \arccos\beta - \beta\sqrt{1-\beta^2}. \tag{4.19}$$

Thus for all nonnegative $w \in W^{1,1}[-p,p]$ with $w(-p) = w(p) = 0$, we have

$$\Lambda[w,p] = \frac{\sqrt{1-\beta^2}}{p}\int_{-p}^{p}\left[\sqrt{1+w'^2} - \beta + \frac{\sqrt{1-\beta^2}}{p}xw'\right]dx \geq \arccos\beta - \beta\sqrt{1-\beta^2}, \tag{4.20}$$

with equality if and only if $w = u$ as given in (4.16).

Integrating by parts, the inequality (4.20) yields

$$\frac{\sqrt{1-\beta^2}}{p}\int_{-p}^{p}\left[\sqrt{1+w'^2} - \beta - \frac{\sqrt{1-\beta^2}}{p}w\right]dx \geq \arccos\beta - \beta\sqrt{1-\beta^2}, \tag{4.21}$$

with equality if and only if $w = u$. Written in terms of L and A, we have

$$\frac{\sqrt{1-\beta^2}}{p}\left[L - 2\beta p - \frac{\sqrt{1-\beta^2}}{p}A\right] \geq \arccos\beta - \beta\sqrt{1-\beta^2}, \tag{4.22}$$

with equality if and only if $w = u$. Rearranging, we obtain the isoperimetric inequality

$$L \geq \left(\frac{\arccos\beta}{\sqrt{1-\beta^2}} + \beta\right)p + \sqrt{1-\beta^2}\frac{A}{p}, \tag{4.23}$$

with equality if and only if $w = u$ as given in (4.16), as was to be shown. □

Theorem 4.6 (Existence-Uniqueness-Regularity in $W^{1,1}$)**.** *For the variational problem in Table 4.1, there exists a unique smooth minimizer, given by*

$$u(x) = \sqrt{R^2 - x^2} - \beta R, \tag{4.24}$$

where R is determined by

$$R^{-1} = \sqrt{\arccos\beta - \beta\sqrt{1-\beta^2}} \tag{4.25}$$

and $p = \sqrt{1-\beta^2}R$. Furthermore, the graph of u meets the x-axis at $\pm p$ with interior angle $\arccos\beta$*.*

Proof. Observe that the functional $J[u]$ in Table 4.1 has the form

$$J[u] = \int_{-p}^{p} [\sqrt{1+u'^2} - \beta]dx = L - 2\beta p. \tag{4.26}$$

By the isoperimetric inequality (4.10), we have that

$$J[u] = L - 2\beta p \geq \left(\frac{\arccos \beta}{\sqrt{1-\beta^2}} - \beta \right) p + \sqrt{1-\beta^2}\frac{A}{p}, \tag{4.27}$$

with equality if and only if u is given by (4.16).

We wish to determine when the right-hand side of (4.27) is minimized. For the function $g(p)$ which defines the right-hand side, we have

$$g'(p) = \left(\frac{\arccos \beta}{\sqrt{1-\beta^2}} - \beta \right) - \sqrt{1-\beta^2}\frac{A}{p^2},$$
$$g''(p) = 2\sqrt{1-\beta^2}\frac{A}{p^3} > 0 \quad \text{for all } p > 0,$$

so $g(p)$ is concave up. Furthermore, $g'(p)$ vanishes precisely when

$$p = p_0 := \sqrt{\frac{(1-\beta^2)A}{\arccos \beta - \beta\sqrt{1-\beta^2}}}. \tag{4.28}$$

Since $g(p)$ is concave up, it follows that $p = p_0$ yields a minimum of $g(p)$. The minimum value of $g(p)$ is then

$$g(p_0) = 2\sqrt{A(\arccos \beta - \beta\sqrt{1-\beta^2})}. \tag{4.29}$$

Therefore for $p > 0$ free and $w \in W^{1,1}[-p, p]$ with $w(p) = w(-p) = 0$, we have

$$J[u] = L - 2\beta p \geq 2\sqrt{A(\arccos \beta - \beta\sqrt{1-\beta^2})}, \tag{4.30}$$

with equality if and only if $p = p_0$ and $w = u$ with

$$u(x) = \sqrt{R^2 - x^2} - \beta R, \quad R = \frac{p_0}{\sqrt{1-\beta^2}} = \sqrt{\frac{A}{\arccos \beta - \beta\sqrt{1-\beta^2}}}. \tag{4.31}$$

As such, we see that among nonnegative functions $w \in W^{1,1}[-p, p]$ with $p > 0$ free, $u(p) = u(-p) = 0$, and $A = 1$, the functional $J[u] = L - 2\beta p$ has the unique minimizer

$$u(x) = \sqrt{R^2 - x^2} - \beta R, \quad \text{with } R^{-1} = \sqrt{\arccos \beta - \beta\sqrt{1-\beta^2}}, \tag{4.32}$$

which *a posteriori* is smooth up to the boundary on the interval $[-p, p]$ with

$$p = \sqrt{1-\beta^2}R. \tag{4.33}$$

Thus the variational problem in Table 4.1 has a unique smooth minimizer, as was to be shown. The angle condition is satisfied per Remark 3.2. $\square$

Chapter 5

Proof for C^1 Parametrized Curves

In this section, we present a uniqueness-regularity proof for the variational problem described in the introduction, with competition among C^1 parametrized curves. The variational problem is made precise in Table 5.1. The dot notation indicates a derivative with respect to t.

We remark that in this case, existence is harder to come by. To the best of the author's present knowledge, sufficient conditions for a problem of this type (parametric problem with one isoperimetric constraint and free endpoints) are not given in the standard classical sources (see, for example, Bliss [2] or Bolza [4]).

A recent article of Licea [17] provides sufficient conditions for a wide class of free endpoint problems, including ones with isoperimetric and mixed inequality constraints, but our problem fails to meet their criteria: the integrand of the objective functional does not have continuous partial derivatives due to a singularity at $(\dot{x}, \dot{y}) = (0, 0)$.

Theorem 5.1. *If $-1 < \beta < 1$ and a minimizer exists for the variational problem in Table 5.1, then that minimizer must be the smooth curve $\gamma(t) = (x(t), y(t))$ given by*

$$x(t) = R \sin\left(\frac{\ell(\frac{1}{2} - t)}{R}\right), \tag{5.1}$$

$$y(t) = R \cos\left(\frac{\ell(\frac{1}{2} - t)}{R}\right) - \beta R, \tag{5.2}$$

where R is the radius of curvature of the arc, determined by

$$R^{-1} = \sqrt{\arccos \beta - \beta\sqrt{1 - \beta^2}}, \tag{5.3}$$

ℓ is the arclength of the arc, given by

$$\ell = 2R \arccos \beta, \tag{5.4}$$

and $t \in [0, 1]$.

Table 5.1: Our isoperimetric problem with competition among C^1 parametrized curves

`among`	regular C^1 curves $\gamma(t) = (x(t), y(t))$, $t \in [0, 1]$, $y \geq 0$,
`minimize`	$J[\gamma] = \int_0^1 \left[\sqrt{\dot{x}^2 + \dot{y}^2} + \beta\dot{x}\right] dt$
`subject to`	$K[\gamma] = \int_0^1 \frac{1}{2}[x\dot{y} - y\dot{x}]dt = 1$
`and`	$\begin{cases} x(0) = -x(1) > 0, \\ y(0) = y(1) = 0. \end{cases}$

Proof. (*Uniqueness*) Assume that $\beta \in (-1, 1)$ and a minimizer $\gamma = (x, y)$ exists for the problem in Table 5.1. Let $\tilde{\gamma} = (\tilde{x}, \tilde{y})$ be a competitor curve, close to γ in the sense of the distance $\rho(\gamma, \tilde{\gamma})$. We define the coordinate variations $\xi, \eta \in C^1[0, 1]$ by setting

$$\xi(t) = \tilde{x}(t) - x(t), \quad \eta(t) = \tilde{y}(t) - y(t). \tag{5.5}$$

Since $\tilde{\gamma}$ must also have its endpoints on the x-axis, symmetric about 0, the coordinate variations must satisfy

$$\xi(0) = -\xi(1), \quad \eta(0) = \eta(1) = 0. \tag{5.6}$$

Since γ is a constrained minimizer, there must exist a Lagrange multiplier $\lambda \in \mathbb{R}$ such that the augmented functional

$$\Lambda[\gamma] := J[\gamma] - \lambda K[\gamma] \tag{5.7}$$

is stationary at γ, i.e., the first variation $\delta\Lambda$ must vanish at u. Written out in full, we have

$$\Lambda[u] = \int_0^1 F(x, y, \dot{x}, \dot{y})dt, \quad F(x, y, \dot{x}, \dot{y}) = \sqrt{\dot{x}^2 + \dot{y}^2} + \beta\dot{x} - \frac{\lambda}{2}[x\dot{y} - y\dot{x}]. \tag{5.8}$$

Since $\Lambda[\gamma]$ is stationary at the minimizer γ, the first variation $\delta\Lambda$ must vanish for all admissible variations. By Theorem 2.11, the first variation $\delta\Lambda$ is given by

$$\delta\Lambda = \int_0^1 \left\{ \xi \left[F_x - \frac{d}{dt} F_{\dot{x}} \right] + \eta \left[F_y - \frac{d}{dt} F_{\dot{y}} \right] \right\} dt + \left(F_{\dot{x}}\xi + F_{\dot{y}}\eta \right) \Big|_{t=0}^{t=1}. \tag{5.9}$$

For all admissible variations $\delta = (\xi, \eta)$, we must have $\delta\Lambda = 0$.

In particular, this holds for admissible variations (ξ, η) with $\delta x = 0$. Hence for all $\xi, \eta \in C^1[0, 1]$ with $\xi(0) = \xi(1) = 0$, we have

$$\delta\Lambda = \int_0^1 \left\{ \xi \left[F_x - \frac{d}{dt} F_{\dot{x}} \right] + \eta \left[F_y - \frac{d}{dt} F_{\dot{y}} \right] \right\} dt. \tag{5.10}$$

By separately considering variations where $\xi = 0$ or $\eta = 0$, the fundamental lemma of the calculus of variations allows us to conclude that $\gamma = (x, y)$ satisfies

$$F_x - \frac{d}{dt} F_{\dot{x}} = 0, \quad F_y - \frac{d}{dt} F_{\dot{y}} = 0,$$

the *Euler-Lagrange equations* of the functional Λ.

The remaining terms in the first variation yield the condition

$$\delta\Lambda = \left(F_{\dot{x}}\xi + F_{\dot{y}}\eta \right) \big| \Big|_{t=0}^{t=1} = 0, \tag{5.11}$$

for all admissible variations (ξ, η) with $\xi(0) = -\xi(1)$ and $\eta(0) = \eta(1) = 0$. This is the *transversality condition* for our problem. Since $\xi(0) = -\xi(1)$, it follows that $\gamma = (x, y)$ satisfies

$$F_{\dot{x}}|_{t=1} + F_{\dot{x}}|_{t=0} = 0. \tag{5.12}$$

Euler-Lagrange equations

The Euler-Lagrange equations read

$$\begin{aligned} 0 = F_x - \frac{d}{dt}F_{\dot{x}} &= -\frac{\lambda \dot{y}}{2} - \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} + \beta + \frac{\lambda y}{2}\right), \\ 0 = F_y - \frac{d}{dt}F_{\dot{y}} &= \frac{\lambda \dot{x}}{2} - \frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} - \frac{\lambda x}{2}\right). \end{aligned}$$

Expanding and differentiating, we have

$$0 = -\lambda \dot{y} - \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = \dot{y}\left[\lambda - \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}\right], \tag{5.13}$$

$$0 = \lambda \dot{x} - \frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = \dot{x}\left[\lambda - \frac{\dot{x}\ddot{y}-\ddot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}\right]. \tag{5.14}$$

Since γ is a regular curve, $\dot{x}$ and $\dot{y}$ are never simultaneously zero, so we conclude that

$$\lambda = \frac{\dot{x}\ddot{y}-\dot{x}\ddot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \tag{5.15}$$

for all $t \in [0,1]$. The quantity on the right is the *signed curvature* of the curve γ, which we see is constant. Hence we expect that γ is an arc of a circle with signed curvature λ and radius $\frac{1}{|\lambda|}$. To sweep out positive area, we can expect that γ must be oriented counter-clockwise, so we would have $\lambda > 0$ (this will be proven below).

Since γ is a regular C^1 curve, we are free to reparametrize it by arclength, so that it has unit speed. Let s be the arclength parameter,

$$s(t) = \int_0^t \sqrt{\dot{x}(u)^2+\dot{y}(u)^2}du \quad \Longrightarrow \quad \frac{ds}{dt} = \sqrt{\dot{x}^2+\dot{y}^2}. \tag{5.16}$$

Multiplying the first expressions in (5.13) and (5.14) by dt/ds, we obtain

$$-\lambda y' - x'' = 0 \quad \Longrightarrow \quad \left(y + \frac{1}{\lambda}x'\right)' = 0, \tag{5.17}$$

$$\lambda x' - y'' = 0 \quad \Longrightarrow \quad \left(x - \frac{1}{\lambda}y'\right)' = 0, \tag{5.18}$$

where the prime notation indicates a derivative with respect to s. From these equations we obtain the first integrals

$$x - \frac{1}{\lambda}y' = a, \quad y + \frac{1}{\lambda}x' = b, \tag{5.19}$$

where a, b are constants of integration. Substituting these back into the Euler-Lagrange equations, we obtain the uncoupled pair of equations

$$x'' + \lambda^2(x-a) = 0, \quad y'' + \lambda^2(y-b) = 0. \tag{5.20}$$

The general solution to each of these ODEs is a sum of sinusoids of the form

$$c_0 + c_1 \sin(\lambda s) + c_2 \cos(\lambda s), \quad \text{where } c_0, c_1, c_2 \text{ are some constants.} \tag{5.21}$$

As x and y are related by (5.19), we are free to take the solution to have the form

$$\begin{cases} x = a - R\sin(\lambda s + \varphi), \\ y = b + R\cos(\lambda s + \varphi), \end{cases} \tag{5.22}$$

where $R > 0$ and $\varphi \in (-\pi, \pi]$ are constants to be determined. Hence γ is an arc of a circle with radius R. Let ℓ denote the arclength of γ,

$$\ell = \int_0^1 \sqrt{\dot{x}^2 + \dot{y}^2}dt. \tag{5.23}$$

We proceed to determine the constants a, b, R, ℓ and φ from the constraints of the problem.

First we establish the relationship between R and λ. Since the curve γ parametrized by arclength has unit speed, we have

$$1 = x'^2 + y'^2 = (-R\lambda\cos(\lambda s + \varphi))^2 + (-R\lambda\sin(\lambda s + \varphi))^2 = R^2\lambda^2, \tag{5.24}$$

so we have $R = \frac{1}{|\lambda|}$.

Next we apply the endpoint conditions. At the right endpoint $s = 0$, we have

$$y(0) = b + \frac{1}{|\lambda|}\cos(\varphi) = 0, \tag{5.25}$$

from which we determine that $b = -\frac{1}{|\lambda|}\cos(\varphi)$. At the left endpoint $s = \ell$, we have

$$0 = y(\ell) = \frac{1}{|\lambda|}[\cos(\lambda\ell + \varphi) - \cos(\varphi)], \tag{5.26}$$

from which we obtain the condition

$$\cos(\lambda\ell + \varphi) = \cos(\varphi). \tag{5.27}$$

As such, we have either

$$\lambda\ell + \varphi \equiv \varphi \pmod{2\pi} \quad \text{or} \quad \lambda\ell + \varphi \equiv -\varphi \pmod{2\pi}. \tag{5.28}$$

From the condition $x(0) = -x(\ell) > 0$, we see that

$$a - \frac{1}{|\lambda|}\sin(\varphi) > 0 \quad \text{and} \quad a - \frac{1}{|\lambda|}\sin(\lambda\ell + \varphi) < 0, \tag{5.29}$$

from which we determine that

$$\frac{1}{|\lambda|}\sin(\varphi) < a < \frac{1}{|\lambda|}\sin(\lambda\ell + \varphi). \tag{5.30}$$

Thus $\sin(\varphi) < \sin(\lambda\ell + \varphi)$. Hence we cannot have $\lambda\ell \equiv 0 \pmod{2\pi}$, so we must have $\lambda\ell \equiv -2\varphi \pmod{2\pi}$. Then there exists $k \in \mathbb{Z}$ such that

$$\lambda\ell = -2\varphi + 2\pi k \quad \Longleftrightarrow \quad \varphi = -\frac{\lambda\ell}{2} + \pi k. \tag{5.31}$$

Applying the fact that $x(0) + x(\ell) = 0$, we have

$$\begin{aligned} 0 &= a - \frac{1}{|\lambda|}\sin\left(-\frac{\lambda\ell}{2} + \pi k\right) + a - \frac{1}{|\lambda|}\sin\left(\frac{\lambda\ell}{2} - \pi k\right) \\ &= 2a + \frac{1}{|\lambda|}\left[\sin\left(\frac{\lambda\ell}{2} - \pi k\right) - \sin\left(\frac{\lambda\ell}{2} + \pi k\right)\right] = 2a. \end{aligned}$$

Therefore $a = 0$, so the minimizer $\gamma = (x, y)$ has the form

$$\begin{aligned} x(s) &= -\frac{1}{|\lambda|}\sin\left(\lambda s - \frac{\lambda\ell}{2} + \pi k\right), \\ y(s) &= \frac{1}{|\lambda|}\left[\cos\left(\lambda s - \frac{\lambda\ell}{2} + \pi k\right) - \cos\left(-\frac{\lambda\ell}{2} + \pi k\right)\right]. \end{aligned}$$

It remains to determine $\operatorname{sgn}(\lambda)$, λ, ℓ, and k.

Returning to the endpoint conditions, the condition $x(0) > 0$ tells us that

$$-\frac{1}{|\lambda|}\sin(\varphi) > 0 \quad \Longrightarrow \quad \sin(\varphi) < 0. \tag{5.32}$$

Therefore $\varphi \in (-\pi, 0)$. Since $y(s) \geq 0$ for all s, we must have

$$\cos(\lambda s + \varphi) \geq \cos(\varphi) \tag{5.33}$$

for all $0 \leq s \leq \ell$. As $\varphi \in (-\pi, 0)$, we must have $\lambda > 0$, for otherwise we have a contradiction. As a further consequence, we must have

$$\varphi \leq \lambda s + \varphi \leq \lambda\ell + \varphi \leq -\varphi, \tag{5.34}$$

from which we determine that

$$0 < \lambda\ell < 2\pi \quad \Longleftrightarrow \quad 0 < \frac{\lambda\ell}{2} < \pi. \tag{5.35}$$

Since $\varphi = -\frac{\lambda\ell}{2} + \pi k \in (-\pi, 0)$ and $-\frac{\lambda\ell}{2} \in (-\pi, 0)$, we must have $k = 0$. Putting it all together and using the parity of cosine and sine, the minimizer $\gamma = (x, y)$ must have the form

$$\begin{aligned} x(s) &= \frac{1}{\lambda}\sin\left(\lambda\left(\frac{\ell}{2} - s\right)\right), \\ y(s) &= \frac{1}{\lambda}\left[\cos\left(\lambda\left(\frac{\ell}{2} - s\right)\right) - \cos\left(\frac{\lambda\ell}{2}\right)\right]. \end{aligned}$$

It remains to determine λ and ℓ.

We apply the area constraint. Computing the area functional, we have

$$\begin{aligned}
K[\gamma] &= \frac{1}{2}\int_0^\ell [xy' - yx']ds \\
&= \frac{1}{2}\int_0^\ell \Big\{ [\tfrac{1}{\lambda}\sin\big(\lambda(\tfrac{\ell}{2}-s)\big)][\sin\big(\lambda(\tfrac{\ell}{2}-s)\big)] \\
&\qquad\qquad - \tfrac{1}{\lambda}[\cos\big(\lambda(\tfrac{\ell}{2}-s)\big) - \cos\big(\tfrac{\lambda\ell}{2}\big)][-\cos\big(\lambda(\tfrac{\ell}{2}-s)\big)] \Big\} ds \\
&= \frac{1}{2\lambda}\int_0^\ell \big[1 - \cos\big(\tfrac{\lambda\ell}{2}\big)\cos\big(\lambda(\tfrac{\ell}{2}-s)\big)\big]\, ds \\
&= \frac{\ell}{2\lambda} - \frac{1}{2\lambda}\cos\bigg(\frac{\lambda\ell}{2}\bigg)\int_0^\ell \cos\big(\lambda(\tfrac{\ell}{2}-s)\big)ds \\
&= \frac{\ell}{2\lambda} - \frac{1}{2\lambda}\cos\bigg(\frac{\lambda\ell}{2}\bigg)\bigg[-\frac{1}{\lambda}\sin\big(\lambda(\tfrac{\ell}{2}-s)\big)\bigg]\bigg|_0^\ell \\
&= \frac{\ell}{2\lambda} + \frac{1}{2\lambda^2}\cos\bigg(\frac{\lambda\ell}{2}\bigg)\bigg[\sin\bigg(-\frac{\lambda\ell}{2}\bigg) - \sin\bigg(\frac{\lambda\ell}{2}\bigg)\bigg] \\
&= \frac{\ell}{2\lambda} - \frac{1}{\lambda^2}\cos\bigg(\frac{\lambda\ell}{2}\bigg)\sin\bigg(\frac{\lambda\ell}{2}\bigg) \\
&= \frac{\ell}{2\lambda} - \frac{1}{2\lambda^2}\sin(\lambda\ell).
\end{aligned}$$

Since we require that $K[\gamma] = 1$, it follows that the area constraint reads

$$1 = \frac{\ell}{2\lambda} - \frac{1}{2\lambda^2}\sin(\lambda\ell). \tag{5.36}$$

To proceed, we need the transversality condition.

Transversality condition

We recall from (5.12) that the transversality condition reads

$$F_{\dot{x}}|_{t=1} + F_{\dot{x}}|_{t=0} = 0. \tag{5.37}$$

Written out in full, we have

$$0 = \frac{\dot{x}(1)}{\sqrt{\dot{x}^2(1) + \dot{y}^2(1)}} + \beta + \frac{\dot{x}(0)}{\sqrt{\dot{x}^2(0) + \dot{y}^2(0)}} + \beta, \tag{5.38}$$

or in terms of the arclength parametrization,

$$0 = x'(\ell) + x'(0) + 2\beta. \tag{5.39}$$

Since we have $x'(s) = -\cos\big(\lambda(\frac{\ell}{2}-s)\big)$, the transversality condition yields the *angle condition*

$$\cos\bigg(\frac{\lambda\ell}{2}\bigg) = \beta. \tag{5.40}$$

Using the standard branch of arccos with values in $[0, \pi]$, we have

$$\lambda\ell = 2\arccos\beta. \tag{5.41}$$

Plugging (5.41) into the area constraint, we have

$$\begin{aligned}
1 &= \frac{\lambda\ell}{2\lambda^2} - \frac{1}{2\lambda^2}\sin(\lambda\ell) \\
&= \frac{2\arccos\beta}{\lambda^2} - \frac{1}{\lambda^2}\sin\left(\frac{\lambda\ell}{2}\right)\cos\left(\frac{\lambda\ell}{2}\right) \\
&= \frac{\arccos\beta}{\lambda^2} - \frac{1}{\lambda^2}\sin(\arccos\beta)\cos(\arccos\beta) \\
&= \frac{\arccos\beta}{\lambda^2} - \frac{1}{\lambda^2}\sqrt{1-\beta^2}\beta.
\end{aligned}$$

Solving for λ, we find that

$$\lambda = \sqrt{\arccos\beta - \beta\sqrt{1-\beta^2}}, \tag{5.42}$$

so the arclength ℓ of the curve γ is given by

$$\ell = \frac{2\arccos\beta}{\lambda} = \frac{2\arccos\beta}{\sqrt{\arccos\beta - \beta\sqrt{1-\beta^2}}}. \tag{5.43}$$

Setting $R = \lambda^{-1}$, we conclude that a minimizer $\gamma = (x, y)$ must have the form

$$\begin{aligned}
x(s) &= R\sin\left(\frac{\frac{\ell}{2} - s}{R}\right), \\
y(s) &= R\cos\left(\frac{\frac{\ell}{2} - s}{R}\right) - \beta R,
\end{aligned}$$

where $0 \le s \le \ell$ with $\ell = 2R\arccos\beta$. Reparametrizing by setting $s = \ell t$ yields the expression given in the statement. This proves uniqueness.

(*Regularity*) We have shown that the curve $\gamma = (x, y)$ given by

$$\begin{aligned}
x(t) &= R\sin\left(\frac{\ell(\frac{1}{2} - t)}{R}\right), \\
y(t) &= R\cos\left(\frac{\ell(\frac{1}{2} - t)}{R}\right) - \beta R,
\end{aligned}$$

is the only candidate for a minimizer of the problem in Table 5.1. The smoothness of γ follows immediately, since cos and sin are smooth. □

Remark 5.2. One would like at this point to argue that the curve $\gamma = (x, y)$ determined above is a minimizer for the variational problem in Table 5.1. We will not do so here, and instead we gather some observations about the problem. While not sufficient, they are worth noting.

We recall that the augmented functional $\Lambda[\gamma]$ is given by

$$\Lambda[\gamma] = \int_0^1 \left[\sqrt{\dot{x}^2 + \dot{y}^2} + \beta\dot{x} - \frac{\lambda}{2}[x\dot{y} - y\dot{x}]\right] dt. \tag{5.44}$$

By the above, we see that for a minimizer to exist, λ is necessarily of the form

$$\lambda = \frac{2\arccos\beta}{\ell}, \tag{5.45}$$

for $\ell > 0$ a fixed constant. This gives us a family of integrals

$$\Lambda[\gamma,\ell] = \int_0^1 \left[\sqrt{\dot{x}^2 + \dot{y}^2} + \beta\dot{x} - \frac{\arccos\beta}{\ell}[x\dot{y} - y\dot{x}]\right] dt. \tag{5.46}$$

By "Euler's rule" (see Kot [15], p. 119) our constrained optimization problem in Table 5.1 is equivalent to the problem of minimizing the augmented functional $\Lambda[\gamma,\ell]$, and then selecting the value of ℓ which yields an extremal satisfying the area constraint $K[\gamma] = 1$.

From the above, we know that the unique extremal of $\Lambda[\gamma,\ell]$ is

$$\begin{aligned} x(t) &= R\sin\left(\frac{\ell(\frac{1}{2} - t)}{R}\right), \\ y(t) &= R\cos\left(\frac{\ell(\frac{1}{2} - t)}{R}\right) - \beta R, \end{aligned}$$

where $R = \frac{\ell}{2\arccos\beta}$. This is our candidate for a minimizer of $\Lambda[\gamma,\ell]$.

Since we are only considering curves for which $y(0) = y(1) = 0$, integration by parts shows that

$$\begin{aligned} \Lambda[\gamma,\ell] &= \int_0^1 \left[\sqrt{\dot{x}^2 + \dot{y}^2} + \beta\dot{x} + \frac{2\arccos\beta}{\ell}y\dot{x}\right] dt \\ &= \int_0^1 \left[\sqrt{\dot{x}^2 + \dot{y}^2} + \left(\beta + \frac{2\arccos\beta}{\ell}y\right)\dot{x}\right] dt. \end{aligned}$$

We observe that the integrand is convex with respect to $\dot{x}$: denoting the integrand by $f(x,y,\dot{x},\dot{y})$, we have

$$\frac{\partial^2 f}{\partial\dot{x}^2} = \frac{\dot{y}^2}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} \geq 0, \tag{5.47}$$

with equality if and only $\dot{y} \equiv 0$. But since $y(0) = y(1) = 0$, we have $\dot{y} \equiv 0$ if and only if $y \equiv 0$, and this does not occur for curves which satisfy the area constraint. Hence f is strictly convex with respect to $\dot{x}$ for the class of curves we are considering.

Furthermore, the integrand is convex with respect to $\dot{y}$: we have

$$\frac{\partial^2 f}{\partial\dot{y}^2} = \frac{\dot{x}^2}{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}} \geq 0, \tag{5.48}$$

with equality if and only if $\dot{x} \equiv 0$. But $\dot{x} \equiv 0$ if and only if $x \equiv C$ for some constant C, and since we require $x(0) = -x(1)$, the only possibility is that $C = 0$, and this does not occur for curves which satisfy the area constraint. Hence f is strictly convex with respect to $\dot{y}$ for the class of curves we are considering. However, a computation of the Hessian with respect to $\dot{x}, \dot{y}$ shows that the determinant of the Hessian is 0, so we do not have strict convexity with respect to $\dot{x}, \dot{y}$.

We can compute the second variation of the given extremal. First we make the observation that for a nearby competitor curve $\tilde{\gamma} = (\tilde{x}, \tilde{y})$ with $\tilde{x} = x + \varepsilon\xi$, $\tilde{y} = y + \varepsilon\eta$, and $\varepsilon \in \mathbb{R}$ small, we have

$$\begin{aligned} 1 &= \int_0^\ell [(x+\varepsilon\xi)(y'+\varepsilon\eta') - (y+\varepsilon\eta)(x'+\varepsilon\xi')]ds \\ &= \int_0^\ell [xy' - yx']ds + \varepsilon \int_0^\ell [x\eta' - y\xi' + \xi y' - \eta x']ds + \varepsilon^2 \int_0^\ell [\xi\eta' - \eta\xi']ds \end{aligned}$$

for all ε. Since $\gamma = (x, y)$ satisfies the area constraint and this holds for all ε, it follows that

$$\int_0^\ell [x\eta' - y\xi']ds = -\int_0^\ell [\xi y' - \eta x']ds \quad \text{and} \quad \int_0^\ell [\xi\eta' - \eta\xi']ds = 0. \tag{5.49}$$

On the other hand, integration by parts shows that

$$\int_0^\ell [x\eta' - y\xi']ds = \int_0^\ell [\xi y' - \eta x']ds, \tag{5.50}$$

so we conclude that

$$\int_0^\ell [x\eta' - y\xi']ds = \int_0^\ell [\xi y' - \eta x']ds = 0. \tag{5.51}$$

To compute the second variation, we compute the Hessian matrix of $F(x, y, \dot{x}, \dot{y})$. We have

$$Hess(F) = \left(\frac{\partial^2 F}{\partial z_i z_j}\right)_{z_i, z_j \in \{x, y, \dot{x}, \dot{y}\}} = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{\dot{y}^2}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} & \frac{-\dot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \\ -\frac{1}{2} & 0 & \frac{-\dot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} & \frac{\dot{x}^2}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}} \end{bmatrix}. \tag{5.52}$$

Then the second variation is given by

$$\begin{aligned} \delta^2 J &= \frac{1}{2}\int_0^1 (\xi, \eta, \dot{\xi}, \dot{\eta}) \cdot Hess(F)(\xi, \eta, \dot{\xi}, \dot{\eta})dt \\ &= \frac{1}{2}\int_0^1 \Bigg\{ -\frac{1}{2}\xi\dot{\eta} + \frac{1}{2}\eta\dot{\xi} + \dot{\xi}\left[\frac{1}{2}\eta + \frac{\dot{y}^2}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}\dot{\xi} - \frac{\dot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}\dot{\eta}\right] \\ &\qquad\qquad + \dot{\eta}\left[-\frac{1}{2}\xi - \frac{\dot{x}\dot{y}}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}\dot{\xi} + \frac{\dot{x}^2}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}\dot{\eta}\right]\Bigg\}dt \\ &= \frac{1}{2}\int_0^1 \left[(\eta\dot{\xi} - \xi\dot{\eta}) + \frac{(\dot{y}\dot{\xi} - \dot{x}\dot{\eta})^2}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}\right]dt \end{aligned}$$

$$= \underbrace{\frac{1}{2}\int_0^\ell [\eta\xi' - \xi\eta']ds}_{=0} + \frac{1}{2}\int_0^\ell (y'\xi' - x'\eta')^2 ds$$
$$= \frac{1}{2}\int_0^\ell (x'\eta' - y'\xi')^2 ds \geq 0.$$

Hence $\delta^2 J \geq 0$ for all admissible variations $\delta = (\xi, \eta)$, so γ is consistent with that necessary condition of being a minimizer. However, from this computation, it is not apparent that the second variation will be strictly positive for all nonnull variations (ξ, η).

As remarked upon above, we see in this setting that the classical theory can only take us so far. An existence proof here would essentially require us to solve the variational problem of Theorem 1.1 with competition among sets with C^1 boundary, which form a proper subset of the family of sets of finite perimeter.

The problem of proving the existence of minimizers spurred much development in the calculus of variations, as can be seen in texts such as Dacorogna [6] and Maggi [18]. In particular, the direct method and various symmetrization techniques play a key role in many modern existence proofs, including that of the theorem on equilibrium shape under discussion in this thesis.

Chapter 6

Conclusion

In this thesis, we set out to answer the following question: can we restrict the setting of Theorem 1.1 to one where geometric measure theory is not necessary and still say something meaningful about existence, uniqueness, and regularity of minimizers?

It was not clear *a priori* whether the answer was yes, but in Chapters 3 and 4 we saw that restricting the setting to the class of curves which are expressible as graphs of C^1 or $W^{1,1}$ functions gives the augmented functional a useful strict convexity property. In these restricted settings, a full proof of existence-uniqueness-regularity is possible using only classical (i.e. non-direct) methods, adapting work of Talenti [29].

However, in the setting of C^1 parametrized curves, we saw that the corresponding convexity property does *not* hold, so we were not able to prove the existence of a minimizer using the same method. Furthermore, the classical literature surveyed by the author did not contain sufficient conditions for this particular class of parametric free-endpoint isoperimetric problems. This begs the question: are such sufficient conditions available?

We conclude with further open questions related to this research:

(i) Can Theorem 1.1 be proved using "classical" techniques in dimension $n > 2$?

(ii) For the partitioning problem with $N = 1$ compact chamber and $M = 2$ unbounded chambers, we conjectured that Steiner symmetrization would allow us to reduce to the variational problem solved by Theorem 1.1. Is this correct? Are our simplifying assumptions valid? Is the same true in dimension $n > 2$?

(iii) What can be said about the planar partitioning problem with $N = 1$ compact chamber and $M = 3$ unbounded chambers? We conjecture that the optimal configuration is the one depicted in Figure 6.1; this configuration is given by stereographic projection of the standard 4-bubble on the 2-sphere, with the point at infinity placed at one of the junctions. What can be said in dimension $n > 2$?

(iv) What can be said about the planar partitioning problem with $N = 2$ compact chambers of equal area and $M = 2$ unbounded chambers? We conjecture that the optimal configuration is the one depicted in Figure 6.2; this configuration is given by stereographic projection of the standard 4-bubble on the 2-sphere, with the point at infinity placed at the midpoint of one of the interfaces. What can be said in dimension $n > 2$?

(v) What can be said in general about N compact chambers of equal area and M unbounded chambers in n dimensions, with $M, N \geq 1$ and $n \geq 2$? Do minimizing clusters exist?

(vi) In problems with $N \geq 2$ compact chambers, what is true in the unequal area case?

(vii) In all of the above problems, can the setting be generalized by adding a parameter like β in Theorem 1.1 which controls the energetic preferences of the interfaces between regions?

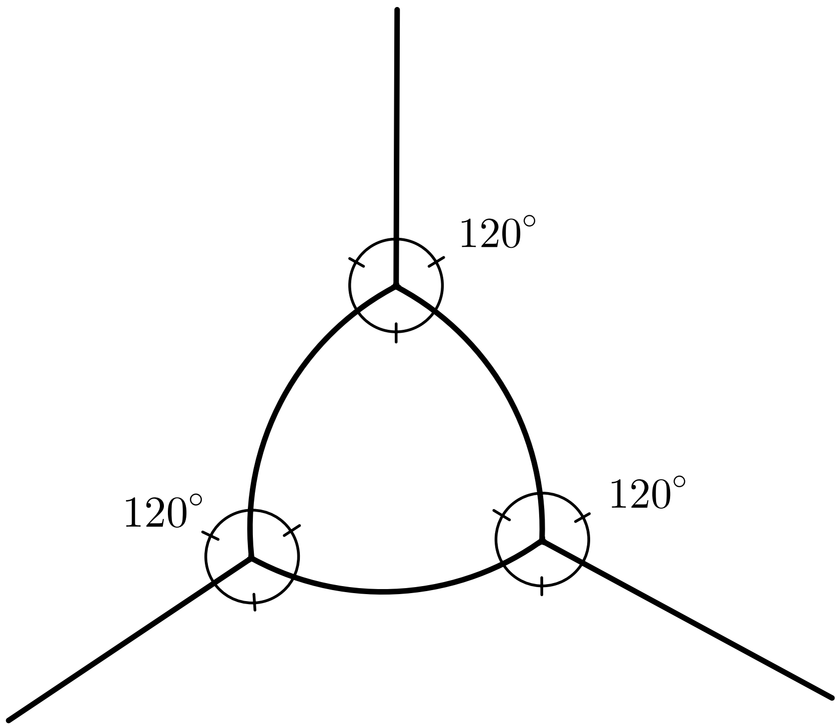

Figure 6.1: Conjectured optimal configuration for the planar partitioning problem with one compact and three unbounded chambers

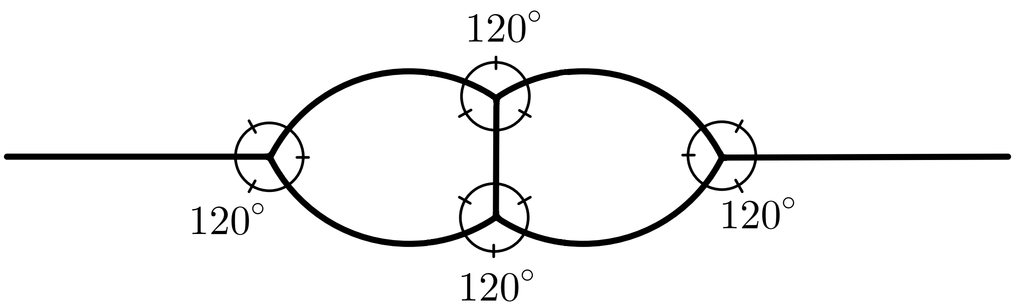

Figure 6.2: Conjectured optimal configuration for the planar partitioning problem with two compact chambers of equal area and two unbounded chambers

Appendix A

Sets of Finite Perimeter

Here we present a crash course consisting of the definitions and theorems which are needed to understand the statement and proof of Theorem 1.1, as presented in Maggi [18]. We assume the reader's familiarity with the construction of the Lebesgue integral; for more detail see Maggi [18] or Stein-Shakarchi [26]. For more on the theory of functions of bounded variation, the reader should consult texts such as Evans-Gariepy [7], Giusti [12], or Ziemer [32].

Note: A reference to "Maggi *X.Y*" indicates that the corresponding definition or theorem is found in Chapter X, Section Y of Maggi [18].

A.1 Borel and Radon Measures

Definition A.1 (Maggi 1, outer measure)**.** An *outer measure* μ on $\mathbb{R}^n$ is a set function on $\mathbb{R}^n$ with values in $[0,\infty]$, $\mu : \mathcal{P}(\mathbb{R}^n) \to [0,\infty]$, with $\mu(\varnothing) = 0$, and

$$E \subset \bigcup_{h\in\mathbb{N}} E_h \qquad \Longrightarrow \qquad \mu(E) \le \sum_{h\in\mathbb{N}} \mu(E_h). \tag{A.1}$$

This property is called *σ-subadditivity*, and it implies the *monotonicity* of μ,

$$E \subset F \qquad \Longrightarrow \qquad \mu(E) \le \mu(F). \tag{A.2}$$

Definition A.2 (Maggi 1.1, Dirac measure)**.** The *Dirac measure* δ_x at $x \in \mathbb{R}^n$ is defined on $E \subset \mathbb{R}^n$ as

$$\delta_x(E) = \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases} \tag{A.3}$$

Definition A.3 (Maggi 1.1, Lebesgue measure)**.** The *Lebesgue measure* of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{L}^n(E) = \inf_{\mathcal{F}} \sum_{Q\in\mathcal{F}} r(Q)^n, \tag{A.4}$$

where $\mathcal{F}$ is a countable covering of E by cubes with sides parallel to the coordinate axes, and $r(Q)$ denotes the side length of Q (the cubes Q are not assumed to be open, nor closed).

The Lebesgue measure $\mathcal{L}^n(E)$ is to be interpreted as the n-dimensional volume of E. Usually, we write

$$\mathcal{L}^n(E) = |E|, \tag{A.5}$$

and refer to $|E|$ as the *volume* of E.

Definition A.4 (Maggi 1.1, k-dimensional Hausdorff measure)**.** Given $n, k \in \mathbb{N}$, $\delta > 0$, the *k-dimensional Hausdorff measure of step δ* of a set $E \subset \mathbb{R}^n$ is defined as

$$\mathcal{H}^k_\delta(E) = \inf_{\mathcal{F}} \sum_{F \in \mathcal{F}} \omega_k \left(\frac{\operatorname{diam}(F)}{2} \right)^k, \tag{A.6}$$

where $\mathcal{F}$ is a countable covering of E by sets $F \subset \mathbb{R}^n$ such that $\operatorname{diam}(F) < \delta$. The *$k$-dimensional Hausdorff measure* of E is then

$$\mathcal{H}^k(E) = \sup_{\delta \in (0,\infty]} \mathcal{H}^k_\delta(E) = \lim_{\delta \to 0^+} \mathcal{H}^k_\delta(E). \tag{A.7}$$

Definition A.5 (Maggi 1.2, σ-additivity)**.** Given a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$, we say that the outer measure μ on $\mathbb{R}^n$ is *σ-additive* on $\mathcal{F}$, provided

$$\mu \left(\bigcup_{h \in \mathbb{N}} E_h \right) = \sum_{h \in \mathbb{N}} \mu(E_h), \tag{A.8}$$

for every disjoint sequence $\{E_h\}_{h \in \mathbb{N}} \subset \mathcal{F}$.

Definition A.6 (Maggi Rmk. 1.5, σ-algebra)**.** We call $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^n)$ a *σ-algebra* on $\mathbb{R}^n$ if $E \in \mathcal{M}$ implies $\mathbb{R}^n \setminus E \in \mathcal{M}$, $\{E_h\}_{h \in \mathbb{N}} \subset \mathcal{M}$ implies $\cup_{h \in \mathbb{N}} E_h \in \mathcal{M}$, and $\mathbb{R}^n \in \mathcal{M}$.

Definition A.7 (Maggi Rmk. 1.5, measure)**.** If $\mathcal{M}$ is a σ-algebra, then a set function $\mu : \mathcal{M} \to [0, \infty]$ is a *measure* on $\mathcal{M}$ if $\mu(\varnothing) = 0$ and μ is σ-additive on $\mathcal{M}$.

Theorem A.8 (Maggi 1.4, Carathéodory's theorem)**.** *If μ is an outer measure on $\mathbb{R}^n$, and $\mathcal{M}(\mu)$ is the family of those $E \subset \mathbb{R}^n$ such that*

$$\mu(F) = \mu(E \cap F) + \mu(F \setminus E), \quad \forall F \subset \mathbb{R}^n, \tag{A.9}$$

then $\mathcal{M}(\mu)$ is a σ-algebra, and μ is a measure on $\mathcal{M}(\mu)$.

Definition A.9 (Maggi Rmk. 1.6, μ-measurable sets)**.** Given an outer measure μ and the family of sets $\mathcal{M}(\mu)$ defined above, we call an element $E \in \mathcal{M}(\mu)$ a *μ-measurable set.*

Definition A.10 (Maggi 1.3, μ-measurable functions)**.** Let μ be a measure on the σ-algebra $\mathcal{M}$. A function $u : E \to [-\infty, \infty]$ is a *μ-measurable function* on $\mathbb{R}^n$ if its domain E covers μ-almost all of $\mathbb{R}^n$, that is $\mu(\mathbb{R}^n \setminus E) = 0$, and if, for every $t \in \mathbb{R}$, the super-level sets

$$\{u > t\} = \{x \in E : u(x) > t\} \tag{A.10}$$

belong to $\mathcal{M}$.

Definition A.11 (Maggi 2.1, Borel sets)**.** The *Borel sets* $\mathcal{B}(\mathbb{R}^n)$ are defined as the σ-algebra generated by the open sets of $\mathbb{R}^n$.

Definition A.12 (Maggi 2.1, Borel measure)**.** A *Borel measure* is an outer measure μ on $\mathbb{R}^n$ such that $\mathcal{B}(\mathbb{R}^n) \subset \mathcal{M}(\mu)$.

Theorem A.13 (Maggi 2.1, Carathéodory's criterion)**.** *If μ is an outer measure on $\mathbb{R}^n$, then μ is a Borel measure on $\mathbb{R}^n$ if and only if*

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2), \tag{A.11}$$

for every $E_1, E_2 \subset \mathbb{R}^n$ such that $\operatorname{dist}(E_1, E_2) > 0$.

Definition A.14 (Maggi 2.2, regular Borel measure)**.** We say that a Borel measure μ is *regular* if for every $F \subset \mathbb{R}^n$ there exists a Borel set E such that

$$F \subset E, \quad \mu(E) = \mu(F). \tag{A.12}$$

Definition A.15 (Maggi 2.3, locally finite measure)**.** An outer measure μ on $\mathbb{R}^n$ is *locally finite* if $\mu(K) < \infty$ for every compact set $K \subset \mathbb{R}^n$.

Definition A.16 (Maggi 2.4, Radon measure)**.** An outer measure μ is a *Radon measure* on $\mathbb{R}^n$ if it is a locally finite, Borel regular measure on $\mathbb{R}^n$.

Example A.17. The Dirac measure δ_x, the Lebesgue measure $\mathcal{L}^n$, and the k-dimensional Hausdorff measure $\mathcal{H}^k$ are all important examples of Radon measures.

Definition A.18 (Maggi 2.4, restriction of an outer measure)**.** Given an outer measure μ on $\mathbb{R}^n$, and $E \subset \mathbb{R}^n$, the *restriction of μ to E* is the outer measure $\mu \llcorner E$ defined as

$$(\mu \llcorner E)(F) = \mu(E \cap F), \quad F \subset \mathbb{R}^n. \tag{A.13}$$

Definition A.19 (Maggi 2.4, concentration and support of a measure)**.** An outer measure μ on $\mathbb{R}^n$ is *concentrated on $E \subset \mathbb{R}^n$* if $\mu(\mathbb{R}^n \setminus E) = 0$. The intersection of the closed sets E such that μ is concentrated on E is denoted by $\operatorname{spt}\mu$, and is called the *support of μ*.

Definition A.20 (Maggi 4.2, total variation of a functional)**.** We define the *total variation* $|L|$ of a linear functional L on $C_c^0(\mathbb{R}^n; \mathbb{R}^m)$ as the set function $|L| : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ such that, for $A \subset \mathbb{R}^n$ open,

$$|L|(A) = \sup\left\{\langle L, \varphi\rangle : \varphi \in C_c^0(A; \mathbb{R}^m), |\varphi| \leq 1\right\}, \tag{A.14}$$

and for $E \subset \mathbb{R}^n$ arbitrary,

$$|L|(E) = \inf\{|L|(A) : E \subset A \text{ and } A \text{ is open}\}. \tag{A.15}$$

Theorem A.21 (Maggi Thm. 4.7, Riesz's theorem for the dual of $C_c^0(\mathbb{R}^n; \mathbb{R}^m)$)**.** *If*

$$L : C_c^0(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R} \tag{A.16}$$

is a bounded linear functional, then its total variation $|L|$ is a Radon measure on $\mathbb{R}^n$ and there exists a $|L|$-measurable function $g : \mathbb{R}^n \to \mathbb{R}^m$ with $|g| = 1$ $|L|$-a.e. on $\mathbb{R}^n$ and

$$\langle L, \varphi\rangle = \int_{\mathbb{R}^n} (\varphi \cdot g) d|L|, \quad \forall \varphi \in C_c^0(\mathbb{R}^n; \mathbb{R}^m), \tag{A.17}$$

that is, $L = g|L|$. Moreover, for every open set $A \subset \mathbb{R}^n$,

$$|L|(A) = \sup\left\{\int_{\mathbb{R}^n} (\varphi \cdot g) d|L| : \varphi \in C_c^0(A; \mathbb{R}^m), |\varphi| \leq 1\right\}. \tag{A.18}$$

Definition A.22 (Maggi Rmk. 4.11, $\mathbb{R}^m$-valued Radon measure)**.** An $\mathbb{R}^m$*-valued Radon measure on* $\mathbb{R}^n$ is a bounded linear functional on $C_c^0(\mathbb{R}^n;\mathbb{R}^m)$. When $m = 1$ we speak of *signed Radon measures* on $\mathbb{R}^n$. We shall always adopt Greek symbols μ, ν, etc. in place of L to denote vector-valued Radon measures, and also set

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^n} \varphi \cdot d\mu \tag{A.19}$$

to denote the value of the $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ at $\varphi \in C_c^0(\mathbb{R}^n;\mathbb{R}^m)$.

Definition A.23 (Maggi Rmk. 4.12, polar decomposition of a vector-valued Radon measure)**.** By Riesz's theorem, every $\mathbb{R}^m$-valued Radon measure μ on $\mathbb{R}^n$ admits a *polar decomposition* $\mu = g|\mu|$, so that we can write

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^n} (\varphi \cdot g) d|\mu|. \tag{A.20}$$

A.2 Sets of Finite Perimeter

Definition A.24 (Maggi 12, set of locally finite perimeter)**.** Let E be a Lebesgue measurable set in $\mathbb{R}^n$. We say that E is a *set of locally finite perimeter* in $\mathbb{R}^n$ if for every compact set $K \subset \mathbb{R}^n$ we have

$$\sup \left\{ \int_E \operatorname{div} T(x) dx : T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n), \operatorname{spt} T \subset K, \sup_{\mathbb{R}^n} |T| \leq 1 \right\} < \infty. \tag{A.21}$$

Definition A.25 (Maggi 12, set of finite perimeter)**.** If the quantity above is bounded independently of K, then we say that E is a *set of finite perimeter* in $\mathbb{R}^n$.

Theorem A.26 (Maggi Prop. 12.1, Structure Theorem for Sets of Finite Perimeter)**.** *If E is a Lebesgue measurable set in $\mathbb{R}^n$, then E is a set of locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure μ_E on $\mathbb{R}^n$ such that*

$$\int_E \operatorname{div} T = \int_{\mathbb{R}^n} T \cdot d\mu_E, \quad \forall T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n). \tag{A.22}$$

Moreover, E is a set of finite perimeter if and only if $|\mu_E|(\mathbb{R}^n) < \infty$.

Definition A.27 (Maggi Rmk. 12.2, Gauss-Green measure μ_E of a set of finite perimeter, relative perimeter)**.** We call μ_E the *Gauss-Green measure* of E, and define the *relative perimeter* of E in $F \subset \mathbb{R}^n$, and the *perimeter* of E, as

$$P(E;F) = |\mu_E|(F), \quad P(E) = |\mu_E|(\mathbb{R}^n), \tag{A.23}$$

respectively.

Definition A.28 (Maggi 15, reduced boundary $\partial^* E$ of a set of finite perimeter)**.** The *reduced boundary* $\partial^* E$ of a set of locally finite perimeter E in $\mathbb{R}^n$ is the set of those $x \in \operatorname{spt} \mu_E$ such that the limit

$$\lim_{r \to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E(B(x,r))|} \tag{A.24}$$

exists and belongs to S^{n-1}.

Definition A.29 (Maggi 15, outer unit normal ν_E to a set of finite perimeter)**.** The *(measure-theoretic) outer unit normal to* E is the Borel function $\nu_E : \partial^* E \to S^{n-1}$ defined by

$$\nu_E(x) = \lim_{r\to 0^+} \frac{\mu_E(B(x,r))}{|\mu_E(B(x,r))|}, \quad x \in \partial^* E. \tag{A.25}$$

Theorem A.30 (Maggi Thm. 15.9, De Giorgi's structure theorem)**.** *If* E *is a set of locally finite perimeter in* $\mathbb{R}^n$*, then the Gauss-Green measure* μ_E *of* E *satisfies*

$$\mu_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E, \quad |\mu_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \tag{A.26}$$

and the generalized Gauss-Green formula holds true:

$$\int_E \nabla \varphi = \int_{\partial^* E} \varphi \nu_E d\mathcal{H}^{n-1}, \quad \forall \varphi \in C_c^1(\mathbb{R}^n). \tag{A.27}$$

Moreover, there exist countably many C^1*-hypersurfaces* M_h *in* $\mathbb{R}^n$*, compact sets* $K_h \subset M_h$*, and a Borel set* F *with* $\mathcal{H}^{n-1}(F) = 0$*, such that*

$$\partial^* E = F \cup \bigcup_{h\in\mathbb{N}} K_h, \tag{A.28}$$

and, for every $x \in K_h$*,* $\nu_E(x)^\perp = T_x M_h$*, the tangent space to* M_h *at* x*.*

A.3 Equilibrium Shape of a Liquid Drop

Given $\beta \in \mathbb{R}$, an open set $A \subset \mathbb{R}^n$, and a set of finite perimeter $E \subset A$, we shall set

$$\mathcal{F}_\beta(E; A) = P(E; A) - \beta P(E; \partial A) \tag{A.29}$$

for the total surface energy, and denote by

$$\mathcal{G}(E) = \int_E g(x)\, dx \tag{A.30}$$

the potential energy associated with a given Borel function $g : \mathbb{R}^n \to \mathbb{R}$.

Theorem A.31 (Maggi Thm. 19.8, Young's law)**.** *If* $\beta \in \mathbb{R}$*,* $g \in L^1(\mathbb{R}^n)$*),* A *is an open set with* C^1*-boundary in* $\mathbb{R}^n$*,* $E \subset A$ *is an open set with finite perimeter and measure,* $A \cap \partial E$ *is a* C^2*-hypersurface with boundary, and*

$$\mathcal{F}_\beta(E; A) + \mathcal{G}(E) \le \mathcal{F}_\beta(F; A) + \mathcal{G}(F), \tag{A.31}$$

for every $F \subset A$ *with* $|F| = |E|$*, then*

$$\nu_E \cdot \nu_A = -\beta \quad \textit{on} \quad \mathrm{bdry}(A \cap \partial E). \tag{A.32}$$

In particular, necessarily $|\beta| \le 1$*.*

Let $H = \{x_n > 0\}$ denote the open upper half-space in $\mathbb{R}^n$.

Theorem A.32 (Maggi Prop. 19.15, constrained perimeter minimizers in half-space)**.** *If $\sigma > 0$, then E is a minimizer in the variational problem*

$$\inf\{P(E;H) : E \subset H, |E| = 1, P(E;\partial H) = \sigma\} \tag{A.33}$$

if and only if, up to horizontal translations, it is equivalent to the set

$$F_\sigma = B(se_n, r) \cap H, \tag{A.34}$$

where $s \in \mathbb{R}$ and $r > 0$ are uniquely determined by the constraints

$$|F_\sigma| = 1, \quad P(F_\sigma; \partial H) = \sigma. \tag{A.35}$$

Theorem A.33 (Maggi Thm. 19.21, Liquid drops in the absence of gravity)**.** *For every $\beta \in (-1, 1)$, there exists a unique $\sigma(\beta) > 0$ with the following property: a set of finite perimeter $E \subset H$ with $|E| = 1$ is a minimizer in the variational problem*

$$\psi(\beta) = \inf\{\mathcal{F}_\beta(E;H) : E \subset H, P(E) < \infty, |E| = 1\}, \tag{A.36}$$

if and only if, up to horizontal translation, E is equivalent to the set

$$G_\beta = F_\sigma(\beta), \tag{A.37}$$

where F_σ, $\sigma > 0$, is defined as in Theorem A.32. Moreover,

$$\nu_{G_\beta} \cdot e_n = \beta, \quad \textit{on } \mathrm{bdry}(H \cap \partial G_\beta). \tag{A.38}$$

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