Frobenius Splittings for the toric ideals of graphs and ladder determinantal ideals

by
Runyue Wang, B.Sc

A thesis submitted to the school of Graduate Studies
in the Partial Fulfillment of the Requirements for
the Degree Master of Science

McMaster University
Hamilton, Ontario, Canada

Abstract

Let $\mathbb{F}$ be a field and let $R = \mathbb{F}[x_1, \dots, x_n]$ be a polynomial ring. Given a polynomial $f \in R$ with a squarefree initial ideal (for some monomial order), one can build a class of ideals in R call the *Knutson ideals* associated to f. Each Knutson ideal is radical and the set of all Knutson ideals associated to $f \in R$ is closed under summation, intersection, and saturation. Each Knutson ideal Gröbner degenerates to a squarefree monomial ideal.

The goal of this thesis is to prove that certain classes of ideals are Knutson. The classes we focus on are toric ideals of graphs. We prove that toric ideals of certain classes of graphs are Knutson. We also show that if the toric ideal of a graph G is Knutson, and H is obtained from G by gluing an even cycle to an edge of G, then the toric ideal of H is Knutson. We also discuss the one-sided ladder determinantal ideals and prove that every one-sided ladder determinantal ideal is Knutson. In the last chapter, we discuss some future directions.

Acknowledgements

I would like to express my sincerest gratitude to my thesis supervisors, Jenna Rajchgot and Adam Van Tuyl, for their unwavering support, guidance, and patience throughout the entire research process.

I would also like to thank the Department of Mathematics and Statistics for providing the scholarship and learning environment during my Master's program.

Lastly, I would like to express my heartfelt appreciation to my friends Büşra Atar, Lan Wang, Haoshuai Wang and Xinyi Zhang for their love and support. I am also grateful for the financial support from my parents. Special thanks to my pets Bitcoin, Swiss and Yuan. Without them, I could not have enjoyed my past two years this much.

Contents

CHAPTER 1

Introduction

Radical ideals are studied in commutative algebra and algebraic geometry. An ideal I is radical if $I = \{r \in R \mid r^m \in I \text{ for some } m \in \mathbb{N}\}$. We study a subset of radical ideals, which are closed under summation, intersection, and saturation, in this thesis. Notice, such a subset is proper when R is a polynomial ring, for example, consider the ring $R = \mathbb{Z}[x]$, and let $I = \langle 3\rangle, J = \langle x^2+3\rangle$. Then $x^2 \in I+J$ while $x \notin I+J$. Thus, $I+J$ is not radical.

Let $\mathbb{F}$ be a field of characteristic $p > 0$ and let $R = \mathbb{F}[e_1, \ldots, e_d]$ be a polynomial ring. The Frobenius map is defined as the pth power map $R \to R$ where $r \mapsto r^p$. A Frobenius splitting is a map $\varphi : R \to R$ which satisfies $\varphi(f_1 + f_2) = \varphi(f_1) + \varphi(f_2)$, $\varphi(f_1^p f_2) = f_1\varphi(f_2)$, and $\varphi(1) = 1$. An ideal I is said to be compatibly split under φ if $\varphi(I) \subseteq I$. Every compatibly split ideal is radical.

The trace map $\mathrm{Tr}(f^{p-1}\bullet) : R \to R$ is defined as

$$\mathrm{Tr}(c_1^p\mathfrak{m}_1 + \cdots + c_s^p\mathfrak{m}_s) = c_1\mathrm{Tr}(\mathfrak{m}_1) + \cdots + c_s\mathrm{Tr}(\mathfrak{m}_s),$$

where $\mathfrak{m}_i$ are monomials, $c_i \in R$, and

$$\mathrm{Tr}(\mathfrak{m}) = \begin{cases} \frac{\sqrt[p]{\mathfrak{m}\prod_{i=1}^d e_i}}{\prod_{i=1}^d e_i} & \text{if } \mathfrak{m}\prod_{i=1}^d e_i \text{ is a } p\text{th power} \\ 0 & \text{otherwise} \end{cases}.$$

In [**10**], Knutson showed that $\mathrm{Tr}(f^{p-1}\bullet)$ defines a Frobenius splitting if there exists a monomial order $<$ such that $\mathrm{init}_<(f) = \prod_{i=1}^d e_i$. In addition, the set of compatibly split ideals is closed under summation, intersection, and prime decomposition. Relaxing the constraint of p being prime, an ideal $I \in R$ is said to be *Knutson* if it can be obtained from some $\langle f\rangle$ using summation, intersection, and prime decomposition, and $\mathrm{init}_< f$ is square-free (See Definition 2.18).

The goal of this thesis is to study the Knutson property of the toric ideals of graphs and ladder determinantal ideals. This is motivated by the following question:

QUESTION 1.1. *What families of ideals are Knutson?*

In [**10**, Section 7.2], Knutson showed that every Schubert determinantal ideal is Knutson by analyzing the corresponding Schubert varieties. Seccia used a purely commutative algebra approach to show that the determinantal ideal of every generic matrix is Knutson in [**14**].

Consider a graph $G = (V_G, E_G)$ where $V_G = \{x_1, \ldots, x_n\}$ is the vertex set and $E_G = \{e_1, \ldots, e_d\}$ is the edge set. The toric ideal I_G of G is defined as the kernel of the ring homomorphism $K[E_G] \to K[V_G]$ where e_i is mapped to the vertices $x_j x_k$ connected by e_i. In the third chapter, we will show two ways of constructing larger graphs while the toric ideals of the new graphs remain Knutson. The first is gluing an even cycle along one edge.

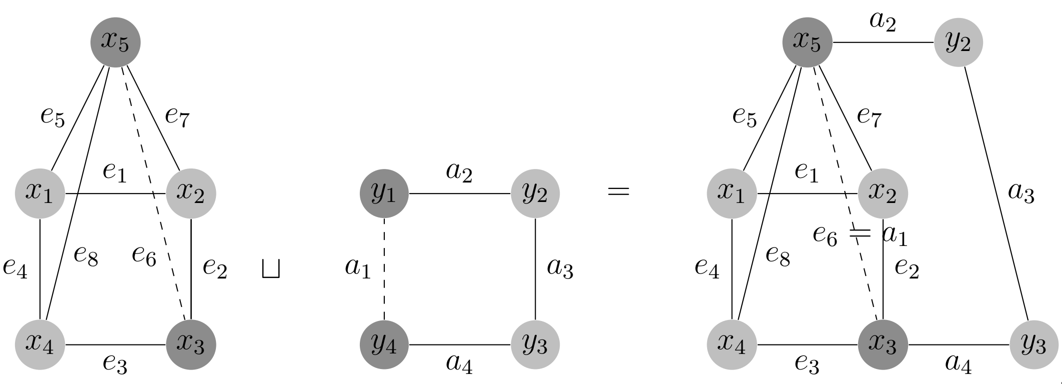


FIGURE 1. Gluing a 4-cycle along e_6 in the left graph.

THEOREM 1.2 (Theorem 3.2). *Let G be a finite simple graph and assume that its toric ideal I_G is Knutson. Let C_{2n} be an even cycle. Suppose H_1 (resp. H_2) is the subgraph in G (resp. C_{2n}) which only contains one edge and two vertices. Then we can construct a new graph H as the disjoint union $G \sqcup_\varphi C_{2n}$ under the identification $H_1 \sim \varphi(H_1)$, where $\varphi : H_1 \to H_2$ is a graph homomorphism. Then the toric ideal I_H of H is also Knutson.*

We also prove a related result about Frobenius splittings. In particular, we can induce an extension of the Frobenius splitting by gluing an even cycle along one edge.

THEOREM 1.3 (Theorem 3.5). *Define G, φ, and H as in Theorem 1.2. Then for any Frobenius splitting $\mathrm{Tr}_1(g\bullet)$ over $\mathbb{F}_p[E(G)]$, which compatibly splits I_G, we can extend it to a new splitting $\mathrm{Tr}_2((\mathfrak{a}C)^{p-1}g\bullet)$ such that I_H is compatibly split under $\mathrm{Tr}_2((\mathfrak{a}C)^{p-1}g\bullet)$, and $\mathfrak{a}, C$ only depends on φ and C_{2n}.*

We then study toric ideals of a special family of graphs which are obtained by attaching an even path to the vertices of degree m in the complete bipartite graph $K_{2,m}$. These graphs were first introduced in [**8**].

THEOREM 1.4 (Theorem 3.11). *Assume we have a complete bipartite graph $K_{2,m}$ with $V_{K_{2,m}} = \{x_1, x_2, y_1, \cdots, y_m\}$ where x_1 and x_2 are the only two vertices of degree m. Then $G_{r,m}$ is the graph obtained by attaching a path $P_{2r-2} = (\{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{2r-3}, x_2\})$*

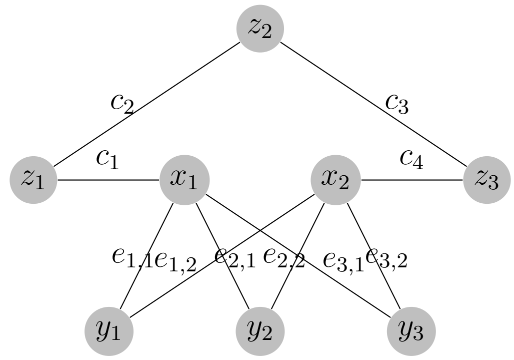


FIGURE 2. Attaching a 4-path to x_1 and x_2 in $K_{2,3}$

between x_1 and x_2 such that $c_i \notin E_{K_{2,m}}$ for all $2 \leq i \leq 2r-3$. Then the toric ideal $I_{G_{r,m}}$ is Knutson.

In the last part of this thesis, we look at ladder determinantal ideals. The ladder determinantal ideal is the ideal generated by all the $k-$minors of some matrices in the ladder shape. For example, a ladder matrix M can have the shape

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & \end{bmatrix},$$

and the ladder determinantal ideal of M generated by 2-minors is

$$I_2(M) = \langle ae - bd, ah - bg, dh - eg, af - cd, bf - ce \rangle.$$

The fourth chapter will focus on discussing the Plücker relation and proving the following theorem.

THEOREM 1.5 (Theorem 4.9). *Every (one-sided) ladder determinantal ideal is Knutson.*

The last chapter will give several related conjectures which may lead to further work on this topic.

CHAPTER 2

Background

This chapter will introduce the relevant background that is required for this thesis. We will first give some basic results about graph theory and toric ideals of graphs which are discussed in [**11**] and [**13**]. Then we will review the needed definitions and theorems about Frobenius splitting stated in [**10**] and [**1**].

1. Toric Ideals of Graphs

A **graph** G is defined by (V_G, E_G), where $V_G = \{x_1, \ldots, x_n\}$ is the set of all vertices in G, and

$$E_G = \{\{x_i, x_j\} \subset V_G \mid x_i \text{ and } x_j \text{ are connected by an edge in } G\}$$

is the complete collection of all edges in G. In this thesis, we will only consider finite simple connected graphs, i.e., graphs with finitely many edges, connected, and do not have more than one edge between any two vertices and no edge starts and ends at the same vertex.

Let $d = |E_G|$ and $n = |V_G|$, and label the elements in E_G as $e_1, e_2, \ldots, e_d$. The incidence matrix M_G of G is an $n \times d$ matrix which is defined as

$$M_G := (a_{ij})_{\substack{1 \le i \le n, \\ 1 \le j \le d}} \text{ where } a_{ij} = \begin{cases} 1 & \text{if } x_i \in e_j \\ 0 & \text{otherwise} \end{cases}.$$

We can treat M_G as a $\mathbb{Z}$-module homomorphism which takes $\mathbb{Z}^d$ to $\mathbb{Z}^n$, that is M_G defines a map $\mathbb{Z}^d \to \mathbb{Z}^n$ given by $\mathbf{v} \mapsto M_G\mathbf{v}$.

One way to define the toric ideal of a graph is by treating it as the lattice ideal associated with the kernel of M_G as discussed in [**11**].

Definition 2.1. The **toric ideal** of a graph G is a homogeneous binomial ideal of the form

$$I_G := \langle \mathbf{e^u} - \mathbf{e^v} \mid \mathbf{u}, \mathbf{v} \in \mathbb{Z}^d_{\ge 0}, \mathbf{u} - \mathbf{v} \in \ker(M_G) \rangle \subset K[e_1, \ldots, e_d],$$

where $\mathbf{e^u} = e_1^{u_1} \cdots e_d^{u_d}$ and K is any field.

The paper [**13**] gives an alternative definition of the toric ideal as the kernel of the ring homomorphism

$$\varphi : K[E_G] \to K[V_G] \text{ by } e_i \mapsto x_j x_k \text{ if } e_i = \{x_j, x_k\}.$$

Then $I_G = \ker(\varphi)$ is a prime ideal.

DEFINITION 2.2. Let $R = K[e_1, \dots, e_d]$ be any polynomial ring, and $I, J \subset R$ be ideals. Then the **saturation** of I with respect to J is defined as

$$I : J^\infty = \langle f \in R \mid fJ^m \subset I \text{ for some } m \in \mathbb{N} \rangle.$$

It can be computationally complicated to go through all the elements in $\ker(M_G)$ to guarantee that no generators of the toric ideal I_G are missing. However, I_G can be derived as follows.

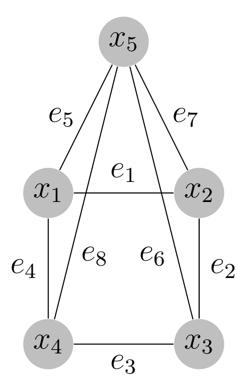
LEMMA 2.3 ([**11**, Lemma 7.6]). *Let G be a graph with d edges, let M_G be the incidence matrix of G, let I_G be the toric ideal of G, and let $\tilde{M}_G$ be the $\mathbb{Q}-$module homomorphism whose matrix representation is the same as M_G. Assume $\mathcal{B}_G = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is a basis of* $\ker(\tilde{M}_G)$. *Define*

$$I_L := \langle \mathbf{e}^{\mathbf{u}_i} - \mathbf{e}^{\mathbf{v}_i} \mid \mathbf{u}_i, \mathbf{v}_i \in \mathbb{Z}^d_{\geq 0}, \mathbf{u}_i - \mathbf{v}_i = \mathbf{b}_i, 1 \leq i \leq k \rangle.$$

Then

$$I_G = I_L : \langle e_1 \cdots e_d \rangle^\infty.$$

EXAMPLE 2.4. Let G be the graph



.

The incidence matrix of G is

$$M_G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Define a linear $\mathbb{Q}-$module homomorphism $\varphi : \mathbb{Q}^d \to \mathbb{Q}^n$ given by $\mathbf{v} \mapsto \tilde{M}_G\mathbf{v}$ where the matrix representation of $\tilde{M}_G$ is the same as M_G. Then

$$\ker(\tilde{M}_G) = \operatorname{span}(\mathcal{B}_G) \text{ with } \mathcal{B}_G = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Going through elements $\mathbf{b}_i$ in $\mathcal{B}_G$ to find pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^n_{\geq 0}$ such that $\mathbf{u} - \mathbf{v} = \mathbf{b}_i$, we get

$$\begin{aligned} \mathbf{u}_1 &= (0,1,0,0,1,0,0,0), \mathbf{v}_1 = (1,0,0,0,0,1,0,0); \\ \mathbf{u}_2 &= (0,0,0,1,0,1,0,0), \mathbf{v}_2 = (0,0,1,0,1,0,0,0); \\ \mathbf{u}_3 &= (1,0,0,0,0,0,0,1), \mathbf{v}_3 = (0,0,0,1,0,0,1,0). \end{aligned}$$

And,

$$\mathbf{e}^{\mathbf{u}_1} - \mathbf{e}^{\mathbf{v}_1} = e_2e_5 - e_1e_6, \mathbf{e}^{\mathbf{u}_2} - \mathbf{e}^{\mathbf{v}_2} = e_4e_6 - e_3e_5, \text{ and } \mathbf{e}^{\mathbf{u}_3} - \mathbf{e}^{\mathbf{v}_3} = e_1e_8 - e_4e_7.$$

Thus,

$$I_L = \langle e_2e_5 - e_1e_6, e_4e_6 - e_3e_5, e_1e_8 - e_4e_7 \rangle,$$

and

$$I_G = I_L : \langle e_1 \cdots e_8 \rangle^\infty = \langle e_2e_5 - e_1e_6, e_4e_6 - e_3e_5, e_1e_8 - e_4e_7, e_3e_7 - e_2e_8, e_1e_3 - e_2e_4 \rangle.$$

The toric ideal of a graph can be studied in a combinatorial way.

Definition 2.5. A **walk** W in G is a finite sequence of edges

$$W := (\{x_{i_1}, x_{i_2}\}, \{x_{i_2}, x_{i_3}\}, \ldots, \{x_{i_{r-1}}, x_{i_r}\}) \text{ where each } \{x_{i_j}, x_{i_{j+1}}\} \in E_G.$$

A walk W is said to be **even** (resp. **odd**) if $|W|$, the length of W, is even (resp. odd), and W is a **closed walk** if $x_{i_1} = x_{i_r}$. A **primitive even walk** is a minimal even closed walk, i.e., does not contain any other proper even closed walk.

Definition 2.6. An **n-path** P_n in G is a walk with $|P_n| = n$ and without repeated vertices nor repeated edges. An **n-cycle** C_n in G is a closed walk of length n such that

$$x_{i_a} = x_{i_b} \text{ if and only if } a = 1, b = n \text{ or } a = n, b = 1.$$

H. Ohsugi and T. Hibi [**13**] showed that the toric ideal of a graph is generated by walks of a special form.

Proposition 2.7 ([**13**, Lemma 3.2]). *Given a graph G, a closed even walk W is a primitive walk if it is in any of the following forms:*

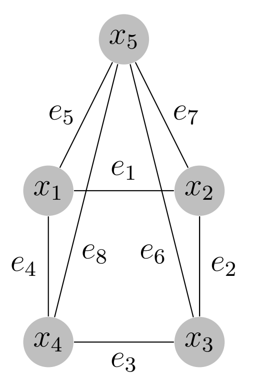
(1) An even cycle.
(2) (C_1, C_2) *where* C_1 *and* C_2 *are odd cycles and have exactly one vertex in common.*
(3) $(C_1, p, C_2, -p)$ *where* C_1 *and* C_2 *are odd cycles which are disjoint and* p *is a path running from a vertex of* C_1 *to a vertex of* C_2*.*

LEMMA 2.8 ([**13**, Lemma 3.2]). *The toric ideal* I_G *of a graph* G *is generated by*

$$\langle \prod_{k=1}^{p} e_{i_k} - \prod_{k=1}^{p} e_{j_k} \mid (e_{i_1}, e_{j_1}, \ldots, e_{i_p}, e_{j_p}) \text{ is a primitive closed even walk of } G \rangle.$$

REMARK. The generators derived from the primitive closed even walks may not be a minimal set of generators.

EXAMPLE 2.9. Let G be the graph defined in Example 2.4.



The primitive closed walks in G are

$$\{(e_1, e_2, e_3, e_4), (e_1, e_2, e_6, e_5), (e_1, e_4, e_8, e_7), (e_2, e_3, e_8, e_7), (e_3, e_4, e_5, e_6), (e_8, e_4, e_5, e_6, e_2, e_7)\}.$$

Since

$$e_8 e_5 e_2 - e_4 e_6 e_7 = e_8(e_2 e_5 - e_1 e_6) - e_6(e_4 e_7 - e_1 e_8),$$

the toric ideal of G is

$$\begin{aligned} I_G &= \langle e_1e_3 - e_2e_4, e_2e_5 - e_1e_6, e_4e_7 - e_1e_8, e_3e_7 - e_2e_8, e_3e_5 - e_4e_6, e_8e_5e_2 - e_4e_6e_7 \rangle \\ &= \langle e_1e_3 - e_2e_4, e_2e_5 - e_1e_6, e_4e_7 - e_1e_8, e_3e_7 - e_2e_8, e_3e_5 - e_4e_6 \rangle. \end{aligned}$$

Up to sign, these generators as exactly the same as those derived in Example 2.4.

2. Frobenius Splittings

Throughout this section, R will denote a commutative $\mathbb{F}_p$-algebra for some field $\mathbb{F}_p$ with prime characteristic p. The **Frobenius map** is defined as the pth power map

$$\psi : R \to R \text{ where } r \mapsto r^p,$$

whose image contains all elements of R that are of pth powers, and R acts on the domain with $a \circ b = ab$ while R acts on the codomain with $a * b = a^p b$. The map ψ is an $R-$module homomorphism and $\psi(a \circ b) = (ab)^p = a * \psi(b)$. Then a splitting of ψ can be defined.

Definition 2.10. A **Frobenius splitting** of R is a map $\varphi : R \to R$ such that for all $f_1, f_2 \in R$,

(1) $\varphi(f_1 + f_2) = \varphi(f_1) + \varphi(f_2)$,
(2) $\varphi(f_1^p f_2) = f_1 \varphi(f_2)$ and,
(3) $\varphi(1) = 1$.

We say $\varphi : R \to R$ is a **near-splitting** if it only satisfies the first two conditions.

Indeed, φ splits ψ because the composition of maps

$$R \xrightarrow{\psi} R \xrightarrow{\varphi} R$$

is the identity map on R.

Definition 2.11. Consider the ring $R = \mathbb{F}_p[e_1, \dots, e_n]$. The **trace map** $\mathrm{Tr}(\bullet) : R \to R$ is defined as

$$\mathrm{Tr}(\mathfrak{m}) = \begin{cases} \frac{\sqrt[p]{\mathfrak{m} \prod_i e_i}}{\prod_i e_i} & \text{if } m \prod_i e_i \text{ is a } p\text{th power} \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathfrak{m}$ is a monomial. This map extends R-linearly to all $g = c_1^p \mathfrak{m}_1 + \cdots + c_s^p \mathfrak{m}_s \in R$ with $c_i \in \mathbb{F}_p$, i.e.,

$$\mathrm{Tr}(g) = c_1 \mathrm{Tr}(\mathfrak{m}_1) + \cdots + c_s \mathrm{Tr}(\mathfrak{m}_s).$$

Since Fermat's Little Theorem tells us that for any $c \in \mathbb{F}_p$, $c^p \equiv c \mod p$. We have $g = c_1^p \mathfrak{m}_1 + \cdots + c_s^p \mathfrak{m}_s = c_1 \mathfrak{m}_1 + \cdots + c_s \mathfrak{m}_s$ in R. So,

$$\mathrm{Tr}(g) = \mathrm{Tr}(c_1 \mathfrak{m}_1 + \cdots + c_s \mathfrak{m}_s) = c_1 \mathrm{Tr}(\mathfrak{m}_1) + \cdots + c_s \mathrm{Tr}(\mathfrak{m}_s).$$

The trace map can induce another map $\varphi_f : R \to R$ given by $g \mapsto \mathrm{Tr}(f^{p-1} g)$, and Knutson gave an easy way to decide if certain φ_f defines a Frobenius splitting.

Theorem 2.12 ([**10**, Theorem 2]). *Let $f \in R = \mathbb{F}_p[e_1, \dots, e_n]$ with $\deg f \leq n$ and $<$ be a monomial order. If $\deg f < n$, then no polynomial multiple of $\mathrm{Tr}(f^{p-1} \bullet)$ is a Frobenius splitting. Denote the initial term of f under the monomial order $<$ with $\mathrm{init}_<(f)$. Then*

$$\mathrm{Tr}(f^{p-1}) = \mathrm{Tr}(\mathrm{init}_<(f)^{p-1}).$$

Thus, $\mathrm{Tr}(f^{p-1}\bullet)$ *defines a Frobenius splitting if and only if* $\mathrm{Tr}(\mathrm{init}_<(f)^{p-1}\bullet)$ *does. Moreover, if* $\mathrm{init}(f) = \prod_{i=1}^n e_i$*, then* $\mathrm{Tr}(f^{p-1}\bullet)$ *defines a Frobenius splitting on* $\mathbb{F}_p[e_1, \ldots, e_n]$ *with respect to which* $\langle f \rangle$ *is compatibly split.*

Example 2.13. Let $R = \mathbb{F}_3[e_1, e_2]$ and let $\mathfrak{m}$ be any monomial in R. Then the module homomorphism $\varphi : R \to R$ induced by $\mathrm{Tr}((e_1e_2)^2\bullet)$ has the property that

$$\varphi(\mathfrak{m}) = \begin{cases} \frac{\sqrt[3]{\mathfrak{m}e_1^3e_2^3}}{e_1e_2} = \sqrt[3]{\mathfrak{m}} & \text{if } \mathfrak{m} \text{ is a cube,} \\ 0 & \text{otherwise.} \end{cases}$$

The map φ is indeed a Frobenius splitting. Let $f_1, f_2 \in R$, we can check the conditions of Definition 2.10

(1) Since the map extends additively $\varphi(f_1 + f_2) = \varphi(f_1) + \varphi(f_2)$.
(2) For any $f_1 = c_1\mathfrak{m}_1 + \cdots + c_s\mathfrak{m}_s$, and $f_2 = a_1\mathfrak{m}_1 + \cdots + a_s\mathfrak{m}_s$, we have $f_1^3 = c_1^3\mathfrak{m}_1^3 + \cdots + c_s^3\mathfrak{m}_s^3$ by the Freshman's Dream. Also,

$$\begin{aligned} \varphi(f_1^3 f_2) &= \sum_{i=1}^s \varphi(c_i^3\mathfrak{m}_i^3 f_2) \\ &= \sum_{i=1}^s \sum_{j=1}^s \varphi(c_i^3\mathfrak{m}_i^3 a_j\mathfrak{m}_j) \\ &= \sum_{i=1}^s \sum_{\substack{j=1 \\ \mathfrak{m}_j \text{ a third power}}}^s \sqrt[3]{c_i^3\mathfrak{m}_i^3 a_j\mathfrak{m}_j} \\ &= \sum_{i=1}^s \sum_{\substack{j=1 \\ \mathfrak{m}_j \text{ a third power}}}^s c_i\mathfrak{m}_i \sqrt[3]{a_j\mathfrak{m}_j} \\ &= \sum_{i=1}^s c_i\mathfrak{m}_i\varphi(f_2) \\ &= f_1\varphi(f_2). \end{aligned}$$

(3) Since 1 is monomial and $1^3 = 1$ is a third power. Then $\varphi(1) = \varphi(1^3) = 1$.

Definition 2.14. Let I be an ideal of R with a Frobenius (near-)splitting φ. Then I is **compatibly (near-)split** with respect to φ if $\varphi(I) \subseteq I$.

A Frobenius splitting can define a collection of radical ideals which is closed under intersection, summation and taking prime components.

Definition 2.15. An ideal Q of R is **primary** if for any $a, b \in R$ with $ab \in Q$, either $a \in Q$, or $b^n \in Q$ for some $n \in \mathbb{N}$.

The radical of a primary ideal $P = \sqrt{Q}$ is prime, and Q is called P**-primary**. A **primary decomposition** of an ideal $I \subseteq R$ has the form:

$$I = \bigcap_{j \in J} Q_j$$

where Q_j are primary and J is a finite index set. The prime ideal $\sqrt{Q_j}$ is said to be a **prime component** of I if it is a minimal prime over I.

LEMMA 2.16 ([**10**, Section 1]). *Let R be an $\mathbb{F}_p-$algebra and let $\varphi : R \to R$ be a Frobenius splitting. For any two compatibly split ideals $I, J \subset R$ with respect to φ, the ideals $I + J$, $I \cap J$, and the prime components of I are all compatibly split. In addition, every compatibly split ideal is radical.*

EXAMPLE 2.17. Define R and φ as in Example 2.13. Let $I = \langle e_1 e_2 \rangle$. We want to show that $\varphi(I) \subseteq I$. Suppose $\mathfrak{m} = c e_1^n e_2^m \in I$ with $n, m \geq 1$, and $c \in R$. Then

$$\varphi(\mathfrak{m}) = \begin{cases} \sqrt[3]{c e_1^n e_2^m} = \sqrt[3]{c} e_1^{\frac{n}{3}} e_2^{\frac{m}{3}} & \text{if } 3 \mid n, 3 \mid m, \text{ and } c \text{ is a cube} \\ 0 & \text{otherwise} \end{cases} \in I.$$

Thus, I is compatibly split with respect to φ. Also, since

$$I = \langle e_1 \rangle \cap \langle e_2 \rangle,$$

Lemma 2.16 implies $\langle e_1 \rangle$ and $\langle e_2 \rangle$ are also compatibly split.

We now describe a closely related notion, which no longer requires that our base field has prime characteristic.

DEFINITION 2.18. Let $R = \mathbb{F}[e_1, \ldots, e_n]$ where $\mathbb{F}$ is a field of any characteristic. Let $f \in R$ be a polynomial of degree $d \leq n$ where $\text{init}_<(f)$ is square-free of degree d for some monomial order $<$. Then the poset $\mathcal{P}_f$ of f, which is partially ordered by inclusion, is the unique collection of ideals in R that satisfy

(1) $\langle f \rangle \in \mathcal{P}_f$.
(2) If $I, J \in \mathcal{P}_f$, then $I + J, I \cap J \in \mathcal{P}_f$.
(3) If $I \in \mathcal{P}_f$ and J is a prime component of I, then $J \in \mathcal{P}_f$.

An ideal $I \in \mathcal{P}_f$ is called a **Knutson ideal**.

We may compare two posets $\mathcal{P}_f$ and $\mathcal{P}_g$ if one of f, g divides the other.

LEMMA 2.19. *Let $R = \mathbb{F}[e_1, \ldots, e_n]$ where $\mathbb{F}$ is a field. Fix polynomials $f, g \in R$ of degree $\leq n$ such that* $\text{init}_<(f)$ *and* $\text{init}_<(g)$ *are square-free for some monomial order. If $f \mid g$, then $\mathcal{P}_f \subseteq \mathcal{P}_g$.*

PROOF. Assume

$$f = f_1 f_2 \cdots f_s, \text{ and } g = f_1 f_2 \cdots f_s r_1 r_2 \cdots r_t,$$

where $f_i, r_j \in R$ are irreducible and $i, j \in \mathbb{N}$. Then

$$\langle g \rangle = \langle f_1 \rangle \cap \cdots \cap \langle f_s \rangle \cap \langle r_1 \rangle \cap \cdots \cap \langle r_t \rangle.$$

So $\langle f_i \rangle \in \mathcal{P}_g$ for all $1 \le i \le s$. And thus,

$$\langle f \rangle = \langle f_1 \rangle \cap \cdots \cap \langle f_s \rangle \in \mathcal{P}_g.$$

By the construction of the poset, we have $\mathcal{P}_f \subseteq \mathcal{P}_g$. □

We next note that the principal ideal generated by any irreducible factor of f is an element of $\mathcal{P}_f$.

LEMMA 2.20. *Let $R = \mathbb{F}[e_1, \ldots, e_n]$ where $\mathbb{F}$ is a field. Fix a polynomial $f \in R$ such that $\operatorname{init}_<(f)$ is square-free for some monomial order $<$. If $g \mid f$ and g is irreducible, then $\langle g \rangle \in \mathcal{P}_f$.*

PROOF. Assume $f = gh$ where g is irreducible and h is some polynomial. Then $\operatorname{init}_< g$ is square-free due to $\operatorname{init}_< f = (\operatorname{init}_< g) \cdot (\operatorname{init}_< h)$ is square-free. Lemma 2.19 yields that $\langle g \rangle \in \mathcal{P}_g \subseteq \mathcal{P}_f$. □

Note that if the field $\mathbb{F}$ has positive character p, then the Knutson ideals are compatibly split under an appropriately chosen Frobenius splitting.

LEMMA 2.21. *Let $R = \mathbb{F}_p[e_1, \ldots, e_n]$ where p is prime. Let $<$ be a monomial order on R. Let $f \in R$ be a polynomial where $\operatorname{init}_<(f)$ is square-free of degree $\deg f < n$. Then for any Knutson ideal $I \in \mathcal{P}_f$, there exists a $g \in R$ where $f \mid g$, such that I is compatibly split under the Frobenius splitting $\operatorname{Tr}(g^{p-1}\bullet)$.*

PROOF. Suppose for some index set $\alpha \subseteq \{1, 2, \ldots, n\}$ we have $\operatorname{init}_<(f) = \prod_{i \in \alpha} e_i$. Let $\beta = \{1, 2, \ldots, n\} \backslash \alpha$ be the complementary index set of α. Define the polynomial

$$g := (\prod_{i \in \beta} e_i) f.$$

Since $\operatorname{init}_<(g) = \prod_{i=1}^n e_i$, $\operatorname{Tr}(g^{p-1}\bullet)$ defines a Frobenius splitting on R by Theorem 2.12. Thus, $\langle g \rangle$ is compatibly split for this Frobenius splitting. Lemma 2.16 implies that every Knutson ideal in $\mathcal{P}_g$ is a compatibly split under $\operatorname{Tr}(g^{p-1}\bullet)$. Since $f \mid g$, we can deduce that $I \in \mathcal{P}_g$ using Lemma 2.19. Therefore, I is compatibly split under the Frobenius splitting $\operatorname{Tr}(g^{p-1}\bullet)$. □

Example 2.22. Let G be the graph defined in Example 2.9, and let I_G be its toric ideal. Recall, we have

$$I_G = \langle e_1e_3 - e_2e_4, e_2e_5 - e_1e_6, e_4e_7 - e_1e_8, e_3e_7 - e_2e_8, e_3e_5 - e_4e_6 \rangle.$$

Defining a lexicographic monomial order $<$ with

$$e_8 < e_7 < e_6 < e_5 < e_3 < e_4 < e_2 < e_1.$$

Consider $f \in \mathbb{F}_p[e_1, ..., e_8]$ with

$$f = (e_1e_3 - e_2e_4)(e_3e_5 - e_4e_6)(e_3e_7 - e_2e_8).$$

Then

$$\text{init}_<(f) = e_1e_2e_3e_4e_6e_8 \text{ is square-free.}$$

Notice, we need $e_3 < e_4$ to guarantee that $\text{init}_<(f)$ is square-free. And thus, $\langle f \rangle$ is Knutson.

The following is the list of prime components of $\langle f \rangle$. Notice, to get the full collection of $\mathcal{P}_f$, we only need to record all the intersections and summations of the prime components.

- $\langle e_1e_3 - e_2e_4 \rangle, \langle e_3e_5 - e_4e_6 \rangle,$ and $\langle e_3e_7 - e_2e_8 \rangle$: due to
$$\langle f \rangle = \langle e_1e_3 - e_2e_4 \rangle \cap \langle e_3e_5 - e_4e_6 \rangle \cap \langle e_3e_7 - e_2e_8 \rangle.$$
- $\langle e_3e_5 - e_4e_6, e_3e_7 - e_2e_8 \rangle$: due to it being prime, and
$$\langle e_3e_5 - e_4e_6, e_3e_7 - e_2e_8 \rangle = \langle e_3e_5 - e_4e_6 \rangle + \langle e_3e_7 - e_2e_8 \rangle.$$
- $\langle e_1e_3 - e_2e_4, e_3e_5 - e_4e_6, e_2e_5 - e_1e_6 \rangle$ and $\langle e_3, e_4 \rangle$: due to
$$\langle e_1e_3 - e_2e_4, e_3e_5 - e_4e_6 \rangle = \langle e_1e_3 - e_2e_4, e_3e_5 - e_4e_6, e_2e_5 - e_1e_6 \rangle \cap \langle e_3, e_4 \rangle.$$
- $\langle e_1e_3 - e_2e_4, e_3e_7 - e_2e_8, e_4e_7 - e_1e_8 \rangle$ and $\langle e_2, e_3 \rangle$: due to
$$\langle e_1e_3 - e_2e_4, e_3e_7 - e_2e_8 \rangle = \langle e_1e_3 - e_2e_4, e_3e_7 - e_2e_8, e_4e_7 - e_1e_8 \rangle \cap \langle e_2, e_3 \rangle.$$
- $\langle e_3, e_4, e_1e_6 - e_2e_5 \rangle, \langle e_2, e_3, e_1e_8 - e_4e_7 \rangle$: due to they are prime and
$$\langle e_3, e_4, e_1e_6 - e_2e_5 \rangle = \langle e_1e_3 - e_2e_4, e_3e_5 - e_4e_6, e_2e_5 - e_1e_6 \rangle + \langle e_3, e_4 \rangle,$$
$$\langle e_2, e_3, e_1e_8 - e_4e_7 \rangle = \langle e_1e_3 - e_2e_4, e_3e_7 - e_2e_8, e_4e_7 - e_1e_8 \rangle + \langle e_2, e_3 \rangle.$$
- $I_G, \langle e_3, e_4, e_8 \rangle, \langle e_2, e_3, e_4 \rangle,$ and $\langle e_2, e_3, e_6 \rangle$: due to
$$\langle (e_1e_3 - e_2e_4) \rangle + \langle (e_3e_5 - e_4e_6) \rangle + \langle (e_3e_7 - e_2e_8) \rangle = \langle (e_1e_3 - e_2e_4), (e_3e_5 - e_4e_6), (e_3e_7 - e_2e_8) \rangle,$$
$$\langle (e_1e_3 - e_2e_4), (e_3e_5 - e_4e_6), (e_3e_7 - e_2e_8) \rangle = I_G \cap \langle e_3, e_4, e_8 \rangle \cap \langle e_2, e_3, e_4 \rangle \cap \langle e_2, e_3, e_6 \rangle.$$
- $\langle e_1e_6 - e_2e_5, e_3, e_4, e_8 \rangle, \langle e_1, e_2, e_3, e_4 \rangle, \langle e_1e_8 - e_4e_7, e_2, e_3, e_6 \rangle, \langle e_2, e_3, e_4, e_6 \rangle$ and $\langle e_2, e_3, e_4, e_8 \rangle$: due to
$$I_G + \langle e_3, e_4 \rangle = \langle e_1e_6 - e_2e_5, e_3, e_4, e_8 \rangle \cap \langle e_1, e_2, e_3, e_4 \rangle,$$
$$I_G + \langle e_2, e_3 \rangle = \langle e_1e_8 - e_4e_7, e_2, e_3, e_6 \rangle \cap \langle e_1, e_2, e_3, e_4 \rangle,$$
$$\langle e_2, e_3, e_4 \rangle + \langle e_2, e_3, e_6 \rangle = \langle e_2, e_3, e_4, e_6 \rangle,$$
$$\langle e_3, e_4, e_8 \rangle + \langle e_2, e_3, e_4 \rangle = \langle e_2, e_3, e_4, e_8 \rangle.$$

- $\langle e_1, e_2, e_3, e_4, e_6\rangle, \langle e_1, e_2, e_3, e_4, e_8\rangle,$ and $\langle e_2, e_3, e_4, e_6, e_8\rangle$: due to

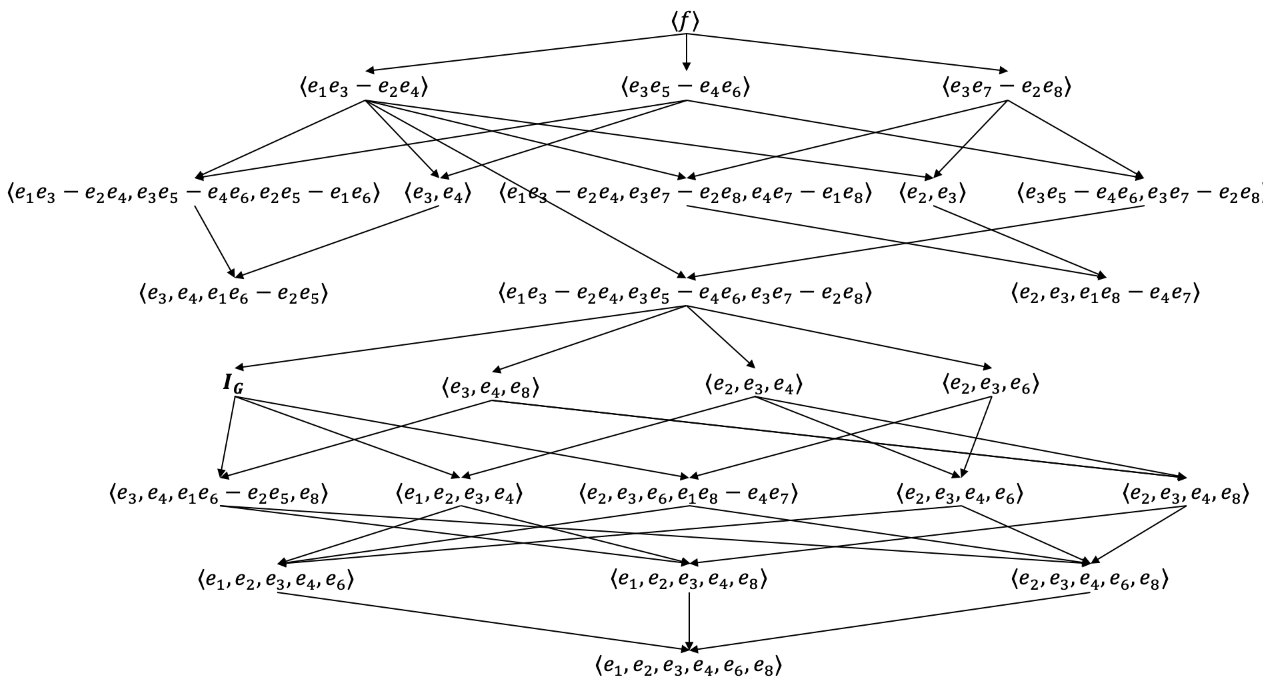
$$\langle e_1, e_2, e_3, e_4\rangle + \langle e_2, e_3, e_4, e_6\rangle = \langle e_1, e_2, e_3, e_4, e_6\rangle,$$
$$\langle e_1, e_2, e_3, e_4\rangle + \langle e_2, e_3, e_4, e_8\rangle = \langle e_1, e_2, e_3, e_4, e_8\rangle,$$
$$\langle e_2, e_3, e_4, e_6\rangle + \langle e_2, e_3, e_4, e_8\rangle = \langle e_2, e_3, e_4, e_6, e_8\rangle.$$

- $\langle e_1, e_2, e_3, e_4, e_6, e_8\rangle$: due to

$$\langle e_1, e_2, e_3, e_4, e_6\rangle + \langle e_1, e_2, e_3, e_4, e_8\rangle = \langle e_1, e_2, e_3, e_4, e_6, e_8\rangle$$

Then the toric ideal I_G is in the poset $\mathcal{P}_f$, thus it is Knutson.

The graph of the poset $\mathcal{P}_f$ is



.

CHAPTER 3

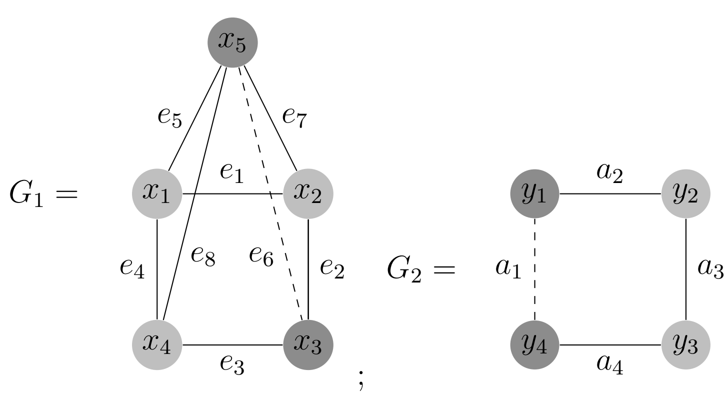
Graph constructions

In this chapter, we will give two ways of constructing larger graphs from those whose toric ideals are Knutson, such that the new toric ideals are also Knutson. The first approach is constructing a larger graph by gluing an even cycle along one edge. The second approach is attaching an even path to two vertices with degree m in the complete bipartite graph $K_{2,m}$.

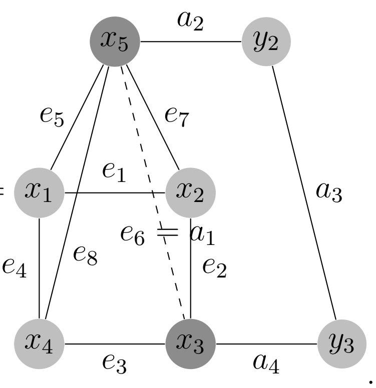
We will first explain the idea of "gluing" graphs. An example appears after the construction.

CONSTRUCTION 3.1. Let G_1, G_2 be two graphs with induced subgraphs $H_1 \subseteq G_1$, $H_2 \subseteq G_2$. Suppose $\varphi : H_1 \to H_2$ is a graph homomorphism. Define $G_1 \sqcup_\varphi G_2$ to be the disjoint union of G_1 and G_2 under the identification $H_1 \sim \varphi(H_1)$. We informally call this construction **gluing** G_1 and G_2 **along** H where $H_1 \cong H \cong H_2$.

EXAMPLE 3.2. Let G_1 be the graph defined in Example 2.4, and G_2 be a 4$-$cycle. Let $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ be the subgraphs whose edges are highlighted by the dashed line and the vertices are darkened.



Define a graph homomorphism $\varphi : H_1 \to H_2$ with $\varphi(x_5) = y_1, \varphi(x_3) = y_4$. Then the glued graph $G_1 \sqcup_\varphi G_2$ along H is given below

$G_1 \sqcup_\varphi G_2 =$ 

The next result shows that if we glue an even cycle onto a graph, the Knutson property is preserved.

THEOREM 3.3. *Let G be a finite simple graph and assume that its toric ideal I_G is Knutson. Let C_{2n} denote an even cycle, for some $n \in \mathbb{Z}_{\geq 2}$. Let φ be a graph isomorphism from a single edge in G to a single edge in C_{2n}. Then, the toric ideal of the glued graph $G \sqcup_\varphi C_{2n}$ is Knutson.*

PROOF. Let $E_G = \{e_1, \ldots, e_d\}$ be the edge set of G and let f be the polynomial with square-free initial term such that $I_G \in \mathcal{P}_f$. Without loss of generality, we can label the edges in the even cycle $E_{C_{2n}} = \{a_1, \ldots, a_{2n}\}$ where $a_1 = e_i$ for some $1 \leq i \leq d$. Let φ be a graph isomorphism from e_i in G to a_1 in C_{2n}, and denote $H = G \sqcup_\varphi C_{2n}$. We claim that $I_H \in \mathcal{P}_g$ for $g = f \cdot (a_1 a_3 \cdots a_{2n-1} - a_2 a_4 \cdots a_{2n}) \in \mathbb{F}[e_1, \ldots, e_d, a_2, \ldots, a_{2n}]$, and $\mathrm{init}_{<'}(g)$ is square-free for some monomial order $<'$.

We first check that there exists a monomial order $<'$ such that $\mathrm{init}_{<'}(g)$ is square-free. Suppose $e_{i_1} < \ldots < e_{i_d}$ is the monomial order that makes $\mathrm{init}_<(f)$ square-free. Then define the new monomial order $<'$ to be the product order in the new ring $R' = \mathbb{F}[e_1, \ldots, e_d, a_2, \ldots, a_{2n}]$ such that

$$e_{i_1} < \ldots < e_{i_d} < a_2 < a_3 < a_4 < \ldots < a_{2n}.$$

Because $\mathrm{init}_<(f)$ does not contain any variables in $\{a_2, \ldots, a_{2n}\}$, we have

$$\mathrm{init}_{<'}(g) = \mathrm{init}_<(f)(a_2 a_4 \cdots a_{2n})$$

which is also square-free.

We now show that $I_H \in \mathcal{P}_g$. Since $I_G \in \mathcal{P}_f$ and $f \mid g$, by Lemma 2.19, $I_G \in \mathcal{P}_g$. Also, $\langle a_1 a_3 \cdots a_{2n-1} - a_2 a_4 \cdots a_{2n} \rangle \in P_g$.

So using [**7**, Theorem 3.7] we have

$$I_H = I_G + \langle a_1 a_3 \cdots a_{2n-1} - a_2 a_4 \cdots a_{2n} \rangle \in \mathcal{P}_g.$$

Thus, I_H is Knutson. $\square$

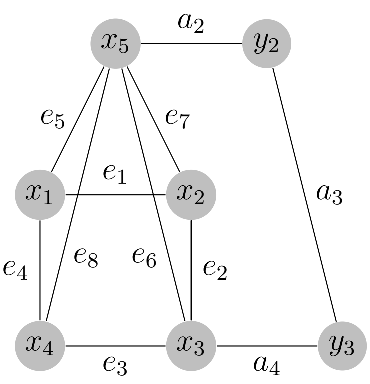
Example 3.4. Let G be the graph of Example 2.4. As shown in Example 2.22, the toric ideal I_G of G is Knutson with the lexicographical order given by

$$<: e_8 < e_7 < e_6 < e_5 < e_3 < e_4 < e_2 < e_1,$$

and

$$f = (e_1e_3 - e_2e_4)(e_3e_5 - e_4e_6)(e_3e_7 - e_2e_8).$$

Attaching C_4 to vertex 3 and 5 along the edge e_6, the new graph H is



Set the new monomial order to be the product order such that

$$<': e_8 < e_7 < e_6 < e_5 < e_3 < e_4 < e_2 < e_1 < a_2 < a_3 < a_4,$$

and set polynomial g to be

$$g = (e_1e_3 - e_2e_4)(e_3e_5 - e_4e_6)(e_3e_7 - e_2e_8)(e_6a_3 - a_2a_4).$$

Then

$$\mathrm{init}_{<'}(g) = a_2a_4e_1e_2e_3e_4e_6e_8$$

is square free, and

$$I_H = I_G + \langle e_6a_3 - a_2a_4\rangle \in \mathcal{P}_g.$$

Gluing an even cycle along one edge also gives a way to extend the Frobenius splitting.

Theorem 3.5. *Assume G is a graph such that its toric ideal I_G is compatibly split by a Frobenius splitting*

$$\mathrm{Tr}_1(g\bullet) : \mathbb{F}_p[E(G)] \to \mathbb{F}_p[E(G)],$$

where Tr_1 is the trace map of $\mathbb{F}_p[E(G)]$ to itself and g is homogeneous. If we glue an even cycle C_{2n}, for some $n \in \mathbb{Z}_{\geq 2}$, to G along an edge e to get a new graph H, the toric ideal of the new graph H is compatibly split by a Frobenius splitting

$$\varphi(\bullet) = \mathrm{Tr}_2((\mathfrak{a}C)^{p-1}g\bullet) : \mathbb{F}_p[E(H)] \to \mathbb{F}_p[E(H)],$$

where $C = \mathbf{a^u} - \mathbf{a^v}$ is the binomial representation of the cycle C_{2n}, $\mathfrak{a}$ is the product of some indeterminates in $E(C_{2n})$ such that

$$\mathfrak{a} = \begin{cases} \frac{\mathbf{a^u}}{e} & \text{if } e \mid \mathbf{a^u} \\ \frac{\mathbf{a^v}}{e} & \text{otherwise} \end{cases},$$

and Tr_2 *is the trace map of $\mathbb{F}_p[E(H)]$ to itself.*

Proof. Let $E_G = \{e_1, \ldots, e_d\}$ be the edge set of G. Then for some $c_i \in \mathbb{F}_p$ nonzero and distinct monomials $\mathfrak{m}_i$, we can write $g = c_1\mathfrak{m}_1 + \cdots + c_s\mathfrak{m}_s \in \mathbb{F}_p[e_1, \ldots, e_d]$ such that $\mathrm{Tr}_1(gI_G) \subseteq I_G$. Without loss of generality, we can label the edges in the even cycle $E_{C_{2n}} = \{a_1, \ldots, a_{2n}\}$ where $a_1 = e_i$ for some $1 \leq i \leq d$. Let $C = -a_1a_3\cdots a_{2n-1} + a_2a_4\cdots a_{2n}$ be the associated closed even walk C_{2n}. We then have $e_i \mid a_1a_3\cdots a_{2n-1}$, and $\mathfrak{a} = a_3a_5\cdots a_{2n-1}$. We claim that $\psi(\bullet) = \mathrm{Tr}_2((\mathfrak{a}C)^{p-1}g\bullet)$ is also a Frobenius splitting and the toric ideal of the new graph I_H is compatibly split by ψ, i.e., $\psi(I_H) \subseteq I_H$.

Let $h_1, h_2 \in \mathbb{F}_p[E(H)]$. Since ψ is a module homomorphism as defined at the beginning of Section 2, $\psi(h_1 + h_2) = \psi(h_1) + \psi(h_2)$, and $\psi(h_1^p h_2) = h_1\psi(h_2)$. Since $\mathrm{Tr}(g\bullet)$ is a Frobenius splitting, $\mathrm{Tr}(g) = 1$ implies that $\sum_{k=1}^{s} c_k\mathrm{Tr}(\mathfrak{m}_k) = 1$. Then there exists a unique term in g such that $c_t\mathrm{Tr}(\mathfrak{m}_t) = 1$, i.e., $c_t = 1$, $\mathfrak{m}_t = (\prod_{j=1}^{d} e_j)^{p-1}$, and $\mathfrak{m}_k \prod_{k=1}^{d} e_j$ is not a pth power if $k \neq t$. Notice, $\mathfrak{a}C = -(a_1a_3^2\cdots a_{2n-1}^2) + \prod_{i=2}^{2n} a_i$, and

$$(\mathfrak{a}C)^{p-1}\prod_{i=2}^{2n} a_i = (\prod_{i=2}^{2n} a_i)^p + \sum_{k=1}^{p-1}(-1)^{p-1-k}\binom{p-1}{k}(\prod_{i=2}^{2n} a_i)^{k+1}(a_1a_3^2\cdots a_{2n-1}^2)^{p-1-k}.$$

Since g does not involve any variable in $\{a_2, \ldots, a_{2n}\}$, the only term in $(\mathfrak{a}C)^{p-1}g$ which becomes a pth power upon multiplying by $\prod_{i=2}^{2n} a_i \prod_{j=1}^{d} e_j$ is $(\prod_{i=2}^{2n} a_i^{p-1})(\prod_{j=1}^{d} e_j)^{p-1}$. Therefore,

$$\psi(1) = \mathrm{Tr}_2((\mathfrak{a}C)^{p-1}g) = \frac{\sqrt[p]{(\prod_{i=1}^{2n} a_i)^{p-1}(\prod_{j=1}^{d} e_j)^{p-1}\prod_{i=2}^{2n} a_i \prod_{j=1}^{d} e_j}}{\prod_{i=2}^{2n} a_i \prod_{j=1}^{d} e_j} = 1,$$

and ψ is thus a Frobenius splitting.

To show $\psi(I_H) \subseteq I_H$, we first show that $\psi(I_G) \subseteq I_G \subseteq I_H$. Let $h \in I_G$. Since ψ extends linearly on $\mathbb{F}_p[E(H)]$, we can assume $h = \mathfrak{n}\mathfrak{g}$ where $\mathfrak{n}$ is a monomial and $\mathfrak{g}$ is a generator in I_G. Then $\mathrm{Tr}_1(gh) \in I_G$. Expand $(\mathfrak{a}C)^{p-1}$ we have

$$\begin{aligned}(\mathfrak{a}C)^{p-1} &= (-(a_1a_3^2\cdots a_{2n-1}^2) + \prod_{i=2}^{2n} a_i)^{p-1} \\ &= \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n} a_i)^{p-1-k},\end{aligned}$$

then

$$\begin{aligned}\psi(h) &= Tr_2(g(\mathfrak{a}C)^{p-1}h)\\ &= \sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}\mathrm{Tr}_2(g(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n}a_i)^{p-1-k}h).\end{aligned}$$

Suppose for some $0 \le k \le p-1$, $(\prod_{i=2}^{2n} a_i)^{p-1}$ divides $(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n} a_i)^{p-1-k}h$, then define a new polynomial

$$h' := \frac{(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n} a_i)^{p-1-k}h}{(\prod_{i=2}^{2n} a_i)^{p-1}}.$$

Since $\mathfrak{g}$ does not contain variables in $\{a_2,\dots,a_{2n}\}$, we can write h' as a multiple of $\mathfrak{g}$. Thus, $h' \in I_G$. Observe that for any monomial $\mathfrak{n} \in \mathbb{F}_p[E(H)]$,

$$\begin{aligned}\mathrm{Tr}_2(\mathfrak{n}(\prod_{i=2}^{2n}a_i)^{p-1}) &= \frac{\sqrt[p]{\mathfrak{n}(\prod_{i=2}^{2n}a_i)^{p-1}\prod_{i=2}^{2n}a_i\prod_{j=1}^{d}e_j}}{\prod_{i=2}^{2n}a_i\prod_{j=1}^{d}e_j}\\ &= \frac{\sqrt[p]{\mathfrak{n}\prod_{j=1}^{d}e_j}}{\prod_{j=1}^{d}e_j}\\ &= \mathrm{Tr}_1(\mathfrak{n}).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{Tr}_2(g(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n}a_i)^{p-1-k}h) &= \mathrm{Tr}_2(gh'(\prod_{i=2}^{2n}a_i)^{p-1}))\\ &= \mathrm{Tr}_1(gh') \in I_G.\end{aligned}$$

For $0 \le k \le p-1$ where $(\prod_{i=2}^{2n} a_i)^{p-1} \nmid (g(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n} a_i)^{p-1-k}h)$. Since g and $\mathfrak{g}$ does not contain any variable in $\{a_2,\dots,a_{2n}\}$, neither term in $g(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n} a_i)^{p-1-k}h$ can be divided by $(\prod_{i=2}^{2n} a_i)^p$. Therefore,

$$\mathrm{Tr}_2(g(a_1a_3^2\cdots a_{2n-1}^2)^k(\prod_{i=2}^{2n}a_i)^{p-1-k}h) = 0 \in I_G.$$

Therefore, $\psi(h) \in I_G$.

[**7**, Theorem 3.7] implies that $I_H = I_G + \langle C\rangle$, so it remains to check $\psi(hC) \in I_H$ for any $h \in \mathbb{F}_p[E(H)]$. Notice,

$$\psi(hC) = \mathrm{Tr}_2((\mathfrak{a}C)^{p-1}hC) = \mathrm{Tr}_2(\mathfrak{a}^{p-1}C^ph) = C\mathrm{Tr}_2(\mathfrak{a}^{p-1}h) \in I_H.$$

Therefore, I_H is compatibly split with respect to φ_{Cf}. □

The authors of [**8**] studied toric ideals of a specific family of graphs denoted as $G_{r,m}$. We now introduce this family. Assume we have a complete bipartite graph $K_{2,m}$

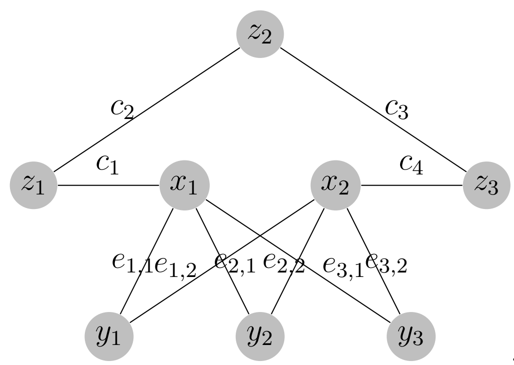
with $V_{K_{2,m}} = \{x_1, x_2, y_1, \ldots, y_m\}$ where x_1 and x_2 are the only two vertices of degree m. Then $G_{r,m}$ is the graph obtained by attaching a path $P_{2r-2} = \{c_1, \ldots, c_{2r-2}\} = (\{x_1, z_1\}, \{z_1, z_2\}, \ldots, \{z_{2r-3}, x_2\})$ between x_1 and x_2 such that $z_i \notin V_{K_{2,m}}$ for all $1 \leq i \leq 2r-3$. One of the reasons that we study this type of graphs is the existence of an explicit formula for a Gröbner basis for each toric ideal in this family.

Lemma 3.6 ([**8**, Corollary 3.3]). *For integers $r \geq 3$ and $m \geq 2$, a Gröbner basis for $I_{G_{r,m}}$ with respect to any monomial ordering is given by*

$$\{e_{i,1}e_{j,2} - e_{j,1}e_{i,2} \mid 1 \leq i < j \leq m\} \cup \{e_{i,2}c_1 \cdots c_{2r-3} - e_{i,1}c_2 \cdots c_{2r-2} \mid 1 \leq i \leq m\},$$

where $e_{i,k}$ is the edge between x_k and y_i, and $\{c_1, c_2, \ldots, c_{2r-2}\}$ is the walk of the attached path.

Example 3.7. The graph $G_{3,3}$ is the graph obtained from attaching an even path $\{c_1, c_2, c_3, c_4\}$ between vertices x_1 and x_2 in $K_{2,3}$. The graph is drawn as follows



The toric ideal of $G_{3,3}$ is

$$\begin{aligned} I_{G_{3,3}} =& \langle e_{1,1}e_{2,2} - e_{1,2}e_{2,1}, e_{1,1}e_{3,2} - e_{3,1}e_{1,2}, e_{2,1}e_{3,2} - e_{3,1}e_{2,2}\rangle + \\ &+ \langle e_{1,1}c_2c_4 - e_{1,2}e_1e_3, e_{2,1}c_2c_4 - e_{2,2}c_1c_3, e_{3,1}c_2c_4 - e_{3,2}c_1c_3\rangle. \end{aligned}$$

We will show that the ideal generated by the 2−minors of a generic $n \times 2$ matrix is Knutson.

Lemma 3.8. *Let $M = (e_{ij})$ be an $n \times 2$ matrix and let $<$ be the anti-diagonal order where*

$$e_{n,1} > e_{n,2} > e_{n-1,1} > \cdots > e_{1,2}.$$

Then for the polynomial $f := \prod_{i=2}^{n} d_i$ with $d_i = \det \begin{bmatrix} e_{i-1,1} & e_{i-1,2} \\ e_{i,1} & e_{i,2} \end{bmatrix}$, we have $\mathrm{init}_<(f)$ is square-free and the ideal generated by 2−minors of M is an element of $\mathcal{P}_f$.

We will first quote the following two theorems.

Lemma 3.9 (Krull's Height Theorem, [**5**, Corollary 10.5]). *Let R be a Noetherian ring, and let $J \subset R$ be an ideal of height n. For some prime ideal $I \subset R$, if $J \subset I$ and the height $ht(I) = n$, then I is a minimal prime component over J.*

LEMMA 3.10 ([**6**, Example 4.1]). *Suppose M is a $m \times k$ matrix with full rank and $m > k$. Let I_k be the ideal generated by the $k-$minors of M. Then*

$$ht(I_k) = m - k + 1.$$

Now we shall prove Lemma 3.8.

PROOF. Define the matrix M, monomial order $<$ and f as in the statement. The initial term $\mathrm{init}_<(f) = \prod_{i=2}^{n} \mathrm{init}_<(d_i) = \prod_{i=2}^{n} e_{i-1,1}e_{i,2}$ is square-free. Let $I = I_2(M)$ be an ideal generated by the $2-$minors of M. [**11**, Section 16.4] yields that I is prime. Define the ideal J to be

$$J := \langle d_i \mid 2 \le i \le n \rangle.$$

Since the initial terms of $\{d_2, \ldots, d_n\}$ are relatively prime, the given generators form a Gröbner basis [**4**, Proposition 10.1, Corollary 10.7]. Then its initial ideal can be generated by relatively prime monomials, and thus form a regular sequence $\{d_2, \ldots, d_n\}$. Therefore, $\mathrm{ht}(J) = n - 1$. Also $\mathrm{ht}(I_2) = n - 1$ by Lemma 3.10. In addition, every generator of J is contained in I. Therefore, $J \subseteq I$ and I is a minimal prime component of J by Krull's Height Theorem. By Lemma 2.20, $\langle d_i \rangle \in \mathcal{P}_f$ for all $2 \le i \le n$. Then $J = \langle d_2 \rangle + \cdots + \langle d_n \rangle \in \mathcal{P}_f$, and so is I. □

THEOREM 3.11. *The toric ideal of $G_{r,m}$ has the Knutson property for all $r > 2$ and $m \ge 1$.*

PROOF. Let G be a complete bipartite graph $K_{2,m}$ with the vertex set

$$V_G = \{x_1, x_2, y_1, y_2, \ldots, y_m\}.$$

Label the edge in G as $e_{i,j}$ if it connects vertices x_j and y_i. Let $G_{r,m}$ be obtained by joining x_1 and x_2 along the path $P_{2r-2} = (c_1, c_2, \ldots, c_{2r-2})$.

The toric ideal of G is generated by the 2-minors whose column indices correspond to the edges of G. See the proof of [**3**, Proposition 5.1]. Then

$$I_G = \mathrm{minors}(2, T) \text{ where } T = \begin{bmatrix} e_{1,1} & e_{1,2} \\ e_{2,1} & e_{2,2} \\ \cdots & \cdots \\ e_{m,1} & e_{m,2} \end{bmatrix}.$$

Lemma 3.8 gives a polynomial f such that $I_G \in \mathcal{P}_f$ is Knutson:

$$f := \prod_{i=2}^{m} (e_{i,1}e_{i-1,2} - e_{i,2}e_{i-1,1}),$$

and $<$ is the lexicographic order

$$e_{1,2} < e_{1,1} < e_{2,2} < e_{2,1} < \cdots < e_{m,2} < e_{m,1}.$$

Now define a new polynomial

$$g := f \cdot (e_{m,1}c_2c_4 \cdots c_{2r-2} - e_{m,2}c_1c_3 \cdots c_{2r-3})$$

and a new monomial product order $<'$

$$e_{1,2} < e_{1,1} < e_{2,2} < e_{2,1} < \cdots < e_{m,2} < e_{m,1} < c_{2r-2} < c_{2r-3} < \cdots < c_2 < c_1.$$

Then $\mathrm{init}_{<'}(g)$ is a monomial. Indeed,

$$\mathrm{init}_{<'}(f) = e_{m,1}e_{m-1,2}e_{m-1,1}\cdots e_{2,2}e_{2,1}e_{1,2},$$

and

$$\mathrm{init}_{<'}(e_{m,1}c_2c_4\cdots c_{2k} - e_{m,2}c_1c_3\cdots c_{2r-3}) = c_1c_3\cdots c_{2r-1}e_{m,2}.$$

Thus,

$$\begin{aligned}\mathrm{init}_{<'}(g) &= \mathrm{init}_{<'}(f)\cdot \mathrm{init}_{<'}(e_{m,1}c_2c_4\cdots c_{2r-2} - e_{m,2}c_1c_3\cdots c_{2r-3})\\ &= e_{1,2}e_{2,1}e_{2,2}\cdots e_{m,1}e_{m,2}c_1c_3\cdots c_{2r-1},\end{aligned}$$

which is square-free.

Now it suffices to show that $I_{G_{r,m}} \in \mathcal{P}_g$. Since we attached the path from x_1 to x_2, Lemma 3.6 shows that the generators of $I_{G_{r,m}}$ is either a $2-$minor of T or $e_{i,2}c_1c_3\cdots c_{2r-3} - e_{i,1}c_2c_4\cdots c_{2r-2}$ for some $1 \le i \le m$. Thus, the toric ideal $I_{G_{r,m}}$ is generated by the 2-minors of the following matrix:

$$T' = \begin{bmatrix} e_{1,1} & e_{1,2}\\ e_{2,1} & e_{2,2}\\ \cdots & \cdots\\ e_{m,1} & e_{m,2}\\ c_1c_3\cdots c_{2r-3} & c_2c_4\cdots c_{2r-2}\end{bmatrix}.$$

Define a new ideal $J = \langle d_i \mid 1 \le i \le m\rangle$ where d_i is the determinant of the submatrix of T' involving rows i and $i+1$. Since $\langle d_i\rangle \in \mathcal{P}_g$ for all $1 \le i \le m$ by Lemma 2.20, we have $J \in \mathcal{P}_g$. Since the monomials $c_1c_3\cdots c_{2r-3}$ and $c_2c_4\cdots c_{2r-2}$ do not contain any variable $e_{i,j}$, the initial terms of d_i are relative coprime under the monomial order $<'$. Therefore, the given generators of J form a Gröbner basis [**4**, Proposition 10.1, Corollary 10.7] whose initial ideal can be generated by relatively prime monomials. Then we can build a regular sequence $S = \{d_1, d_2, \cdots, d_n\}$. Therefore, $\mathrm{ht}(J) = \mathrm{length}(S) = m$. Since $J \subseteq I_{G_{r,m}}$ and the height of an ideal generated by $2-$minors of a $m+1\times 2$ matrix is at most m, $\mathrm{ht}(I_{G_{r,m}}) = m$. By Krull's Height Theorem, $I_{G_{r,m}}$ is a minimal prime component of J. Therefore, $I_{G_{r,m}} \in \mathcal{P}_g$ is Knutson. □

EXAMPLE 3.12. Let $G = K_{2,3}$ where $V_G = \{x_1, x_2, y_1, y_2, y_3\}$ and $E_G = \{e_{i,j} \mid x_j$ and y_i is adjacent$\}$. Then the toric ideal of G is

$$\begin{aligned}I_G &= \langle e_{1,1}e_{2,2} - e_{1,2}e_{2,1}, e_{1,1}e_{3,2} - e_{3,1}e_{1,2}, e_{2,1}e_{3,2} - e_{3,1}e_{2,2}\rangle\\ &= \mathrm{minors}\left(2, \begin{bmatrix} e_{1,1} & e_{1,2}\\ e_{2,1} & e_{2,2}\\ e_{3,1} & e_{3,2}\end{bmatrix}\right).\end{aligned}$$

Define

$$f := (e_{1,1}e_{2,2} - e_{2,1}e_{1,2})(e_{2,1}e_{3,2} - e_{3,1}e_{2,2}),$$

and the lexicographic order

$$<: e_{1,2} < e_{1,1} < e_{2,2} < e_{2,1} < e_{3,2} < e_{3,1}.$$

Then, since by Macaulay2, we have the decomposition

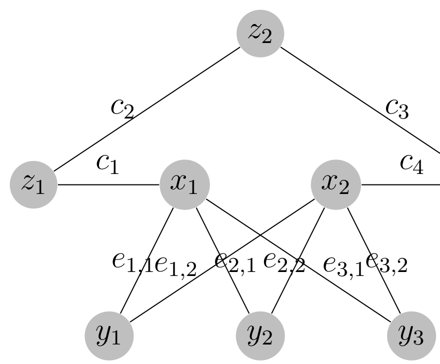
$$I_G \cap \langle e_{2,1}, e_{2,2}\rangle = \langle e_{1,1}e_{2,2} - e_{2,1}e_{1,2}\rangle + \langle e_{2,1}e_{3,2} - e_{3,1}e_{2,2}\rangle$$

and

$$\text{init}_< f = e_{3,1}e_{2,2}e_{2,1}e_{1,2} \text{ is square-free,}$$

we have $I_G \in \mathcal{P}_f$ and I_G is Knutson.

We join vertices x_1 and x_2 with a $4-$path P_4. Then the new graph $G_{3,3}$ is isomorphic to the following graph



.

The toric ideal of $G_{3,3}$ is

$$I_{G_{3,3}} = I_G + \langle e_{1,1}c_2c_4 - e_{1,2}e_1e_3, e_{2,1}c_2c_4 - e_{2,2}c_1c_3, e_{3,1}c_2c_4 - e_{3,2}c_1c_3\rangle.$$

Define the polynomial g be

$$g := f \cdot (e_{3,1}c_2c_4 - e_{3,2}c_1c_3) = (e_{1,1}e_{2,2} - e_{2,1}e_{1,2})(e_{2,1}e_{3,2} - e_{3,1}e_{2,2})(e_{3,1}c_2c_4 - e_{3,2}c_1c_3)$$

and product order $<'$

$$<': e_{1,2} < e_{1,1} < e_{2,2} < e_{2,1} < e_{3,2} < e_{3,1} < c_4 < c_3 < c_2 < c_1.$$

Then

$$\text{init}_{<'}(g) = \text{init}_{<'}(f)\text{init}_{<'}(e_{3,1}c_2c_4 - e_{3,2}c_1c_3) = e_{3,1}e_{2,2}e_{2,1}e_{1,2}e_{3,2}c_1c_3$$

is square-free. Using Macaulay2, we can see that the prime decomposition of $\langle e_1e_4 - e_2e_3, e_3e_6 - e_4e_5, e_5c_2c_4 - e_6c_1c_3\rangle$ is

$$\langle e_5, e_6, e_2e_3 - e_1e_4\rangle \cap \langle e_3, e_4, e_5c_2c_4 - e_6c_1c_3\rangle \cap I_{G_{3,3}}.$$

Thus, we have the following deductions:

$$\begin{aligned}
&\langle e_1e_4 - e_2e_3\rangle, \langle e_3e_6 - e_4e_5\rangle, \langle e_5c_2c_4 - e_6c_1c_3\rangle \in \mathcal{P}_g\\
\Longrightarrow\ &\langle e_1e_4 - e_2e_3, e_3e_6 - e_4e_5, e_5c_2c_4 - e_6c_1c_3\rangle \in \mathcal{P}_g\\
\Longrightarrow\ &\langle e_5, e_6, e_2e_3 - e_1e_4\rangle \cap \langle e_3, e_4, e_5c_2c_4 - e_6c_1c_3\rangle \cap I_{G_{3,3}} \in \mathcal{P}_g\\
\Longrightarrow\ &I_{G_{3,3}} \in \mathcal{P}_g.
\end{aligned}$$

Therefore, $I_{G_{3,3}} \in \mathcal{P}_g$ and is thus Knutson.

CHAPTER 4

Ladder Determinantal Ideal

Many classes of determinantal ideals are Knutson. Knutson in [**10**] showed that every Schubert determinantal ideal is Knutson using connections with Schubert varieties. A one-sided ladder determinantal ideal, see [**2**, Section 1], is an example of a Schubert determinantal ideal. Seccia used a commutative algebraic proof to show that the determinantal ideal of every generic matrix is Knutson in [**14**]. In this chapter, we refer to the proof in [**14**] and also give a commutative algebraic proof to show that a ladder determinantal ideal is Knutson.

Definition 4.1. A $\lambda_1 \times n$ matrix M is said to be a **ladder** if there is a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ for some $n \in \mathbb{N}$, such that:

(1) For any $1 \leq i < n$, $\lambda_i \geq \lambda_{i+1}$.
(2) The (i, j) entry of the matrix M satisfies:

$$\begin{cases} m_{ij} & 1 \leq j \leq n \text{ and } i \leq \lambda_j \\ \text{empty} & \text{otherwise} \end{cases},$$

where m_{ij} is an indeterminate.

Definition 4.2. An ideal I is said to be an **(unmixed) ladder determinantal ideal** if there exists some ladder M such that I is generated by the $k-$minors of M for some $k \in \mathbb{N}$, i.e., $I = \text{minors}(k, M)$.

Example 4.3. Let M be a ladder matrix which is defined by $\lambda = (5, 5, 4, 3, 2)$. Then

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & \\ m_{41} & m_{42} & m_{43} & & \\ m_{51} & m_{52} & & & \end{bmatrix},$$

and λ_i describes the number of nonempty entries of the $i-$th row in M. Let

$$I = \text{minors}(3, \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}) + \text{minors}(3, \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}).$$

Then I is a ladder determinantal ideal which is generated by the $3-$minors of M.

Before diving into the main theorem of this chapter, we introduce the Plücker relations.

THEOREM 4.4 (Plücker relations, [**11**, Theorem 14.6]). *Let T be a $k \times (k+n)$ matrix. Given two strictly ascending sequence $1 \le i_1 < \cdots < i_{k-1} \le k+n$ and $1 \le j_1 < \cdots < j_{k+1} \le k+n$, we have the following equation:*

$$\sum_{l=1}^{k+1} (-1)^k P^T_{[i_1,\dots,i_{k-1},j_l]} P^T_{[j_1,\dots,\hat{j}_l,\dots,j_{k+1}]} = 0, \tag{4.1}$$

where $\hat{j}_l$ represents the omitted term j_l, and $P^T_{[c_1,\dots,c_k]}$ is the determinant of the $k \times k$ submatrix of T which involves columns $c_1, c_2, \dots, c_k$.

Notice if $c_s = c_t$ for some s, t, then we have

$$P^T_{[c_1,\dots,c_k]} = 0,$$

since the submatrix does not have full rank.

LEMMA 4.5 ([**12**, Lemma 3.2.20]). *Let M be a $k \times n$ matrix, and let T be a $k \times (k+n)$ matrix formed by concatenating a $k \times k$ identity matrix to the last column of M. For $1 \le p \le \min\{k, n\}$, let $d^{[b_1,b_2,\dots,b_p],M}_{[a_1,a_2,\dots,a_p]}$ denote the determinant of the $p \times p$ submatrix of M which involves rows $a_1, \dots, a_p$ and columns $b_1, \dots, b_p$, and let*

$$S = \{1 \le s \le k : s \ne a_i \text{ for all } 1 \le i \le p\} = \{s_1, \dots, s_{k-p}\}.$$

Then we have

$$d^{[b_1,b_2,\dots,b_p],M}_{[a_1,a_2,\dots,a_p]} = (-1)^k P^T_{[b_1,\dots,b_p,s_1+n,\dots,s_{k-p}+n]},$$

where k is some integer, i.e., $d^{[b_1,b_2,\dots,b_p],M}_{[a_1,a_2,\dots,a_p]}$ is equal to $P^T_{[b_1,\dots,b_p,s_1+n,\dots,s_{k-p}+n]}$ up to a sign.

For simplicity, we will omit M in $d^{[-],M}_{[-]}$ and T in $P^T_{[-]}$.

EXAMPLE 4.6. Let M' be the 5×5 matrix with indeterminates as entries. Then

$$M' = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} \end{bmatrix},$$

and we can construct T as

$$T = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & 1 & 0 & 0 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & 0 & 1 & 0 & 0 & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & 0 & 0 & 1 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & 0 & 0 & 0 & 1 & 0 \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then we notice

$$d_{[1,2]}^{[1,5]} = \det\left(\begin{bmatrix} m_{11} & m_{15} \\ m_{21} & m_{25} \end{bmatrix}\right) = \det\left(\begin{bmatrix} m_{11} & m_{15} & 0 & 0 & 0 \\ m_{21} & m_{25} & 0 & 0 & 0 \\ m_{31} & m_{35} & 1 & 0 & 0 \\ m_{41} & m_{45} & 0 & 1 & 0 \\ m_{51} & m_{55} & 0 & 0 & 1 \end{bmatrix}\right) = P_{[1,5,8,9,10]}.$$

Now consider the ascending sequence

$$\{i_1, i_2, i_3, i_4\} = \{5, 8, 9, 10\}, \text{ and } \{j_1, j_2, j_3, j_4, j_5, j_6\} = \{1, 2, 6, 7, 8, 9\}.$$

By the Plücker relations 4.1 in Lemma 4.4, we have

$$\begin{aligned} & - P_{[5,8,9,10,1]}P_{[2,6,7,8,9]} + P_{[5,8,9,10,2]}P_{[1,6,7,8,9]} - P_{[5,8,9,10,6]}P_{[1,2,7,8,9]} + P_{[5,8,9,10,7]}P_{[1,2,6,8,9]} \\ & - P_{[5,8,9,10,8]}P_{[1,2,6,7,9]} + P_{[5,8,9,10,9]}P_{[1,2,6,7,8]} \\ = & - P_{[1,5,8,9,10]}P_{[2,6,7,8,9]} + P_{[2,5,8,9,10]}P_{[1,6,7,8,9]} + P_{[5,6,8,9,10]}P_{[1,2,7,8,9]} - P_{[5,7,8,9,10]}P_{[1,2,6,8,9]} \\ = & \; d_{[1,2]}^{[1,5]}d_{[5]}^{[2]} - d_{[1,2]}^{[2,5]}d_{[5]}^{[1]} - d_{[2]}^{[5]}d_{[1,5]}^{[1,2]} + d_{[1]}^{[5]}d_{[2,5]}^{[1,2]} = 0. \end{aligned}$$

And we can observe

$$d_{[1,2]}^{[1,5]}d_{[5]}^{[2]} = d_{[1,2]}^{[2,5]}d_{[5]}^{[1]} + d_{[2]}^{[5]}d_{[1,5]}^{[1,2]} - d_{[1]}^{[5]}d_{[2,5]}^{[1,2]}. \tag{4.2}$$

Considering the ladder matrix in Example 4.3:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & \\ m_{41} & m_{42} & m_{43} & & \\ m_{51} & m_{52} & & & \end{bmatrix}.$$

Since the equation 4.2 does not involve any empty entry in M, we call such relations **valid** on the ladder matrix M. The determinant $d_{[a_1,a_2,\dots,a_p]}^{[b_1,b_2,\dots,b_p]}$ is **valid** on M if $a_i \leq \lambda_{\max\{b_j | 1 \leq j \leq p\}}$ for all $1 \leq i \leq p$. In Example 4.3, $d_{[1,2,3]}^{[2,3,4]} = P_{[2,3,4,9,10]}$ is valid on M, but $d_{[1,2,3]}^{[3,4,5]} = P_{[3,4,5,9,10]}$ is not.

A Plücker relation is valid on the ladder matrix if every one of its terms is valid. There is a criterion to test if P_α is valid given the sequence $\alpha = [i_1, i_2, \dots, i_m]$.

Lemma 4.7. *Let M be a $\lambda_1 \times n$ ladder matrix defined by $\lambda = (\lambda_1, \dots, \lambda_n)$, and let N be a $\lambda_1 \times (n + \lambda_1)$ matrix formed by concatenating a $\lambda_1 \times \lambda_1$ identity matrix to the last column of M. Then for $1 \leq i_1, \dots, i_p \leq n$ and $1 \leq s_1, \dots, s_{\lambda_1 - p} \leq \lambda_1$, the determinant $P_{[i_1,\dots,i_p,s_1+n,\dots,s_{\lambda_1-p}+n]}$ is valid on M if and only if*

$$\{s_1, \dots, s_{\lambda_1 - p}\} \supseteq \begin{cases} \{\lambda_{i_p} + 1, \lambda_{i_p} + 2, \dots, \lambda_1\} & \text{if } \lambda_1 > \lambda_{i_p} \\ \varnothing & \text{if } \lambda_1 = \lambda_{i_p} \end{cases}.$$

Proof. If $i_\alpha = i_\beta$ or $s_\alpha = s_\beta$ for some α, β in the range, then $P_{[i_1,\ldots,i_p,s_1+n,\ldots,s_{\lambda_1-p}+n]} = 0$ as the matrix is no longer full rank. Since the sign of $P_{[i_1,\ldots,i_p,s_1+n,\ldots,s_{\lambda_1-p}+n]}$ does not matter, we may assume $1 \le i_1 < \ldots < i_p \le n$, and $1 \le s_1 < \ldots < s_{\lambda_1-p} \le \lambda_1$. Define a sequence $b_1 \le b_2 \le \ldots \le b_p$ such that $\{b_1, \ldots, b_p\}$ is the complement of $\{s_1, \ldots, s_{\lambda_1-p}\}$ in $\{1, 2, \ldots, \lambda\}$, i.e.,

$$\{b_1, \ldots, b_p\} = \{1 \le b \le \lambda_1 : b \ne s_l \text{ for all } l\}.$$

Then, by Lemma 4.5, we have

$$P_{[i_1,\ldots,i_p,s_1+n,\ldots,s_{\lambda_1-p}+n]} = (-1)^k d^{[i_1,\ldots,i_p]}_{[b_1,\ldots,b_p]} \text{ for some } k \in \mathbb{N}.$$

If we assume that $\{s_1, \ldots, s_{\lambda_1-p}\} \supseteq \{\lambda_{i_p} + 1, \ldots, \lambda_1\}$, we have

$$\{b_1, \ldots, b_p\} \subseteq \{1, 2, \ldots, \lambda_{i_p}\}.$$

By construction of the ladder matrix M, we observe that there is no empty entry in the submatrix of M which takes columns from i_1 to i_p and rows from 1 to λ_{i_p}. Thus, $P_{[i_1,\ldots,i_p,s_1+n,\ldots,s_{\lambda_1-p}+n]}$ is valid.

Now assume $P_{[i_1,\ldots,i_p,s_1+n,\ldots,s_{\lambda_1-p}+n]}$ is valid. Then the submatrix of M which involves columns $\{i_1, \ldots, i_p\}$ and rows $\{b_1, \ldots, b_p\}$ has no empty entry. By construction of the ladder matrix, we must have

$$\{b_1, \ldots, b_p\} \subseteq \{1, 2, \ldots, \lambda_{i_p}\}.$$

Hence

$$\{s_1, \ldots, s_{\lambda_1-p}\} \supseteq \{\lambda_{i_p} + 1, \ldots, \lambda_1\}.$$

$\square$

Lemma 4.8. *Let M be a $\lambda_1 \times n$ ladder matrix defined by $\lambda = (\lambda_1, \ldots, \lambda_n)$, and let N be a $\lambda_1 \times (n + \lambda_1)$ matrix formed by concatenating a $\lambda_1 \times \lambda_1$ identity matrix to the last column of M. Given two valid determinants $D_1 = d^{[1,b_2,\ldots,b_{p-1},n]}_{[a_1,\ldots,a_p]}$ and $D_2 = d^{[d_1,\ldots,d_{p-1}]}_{[c_1,\ldots,c_{p-1}]}$ on M with $p > 1$ and $1 < d_1 < d_{p-1} < n$, then D_1D_2 appears as a term in a Plücker relation on N.*

Proof. Define two strictly ascending sequences

$$S = \{s_1, \ldots, s_{\lambda_1-p}\} = \{1, \ldots, \lambda_1\} - \{a_1, \ldots, a_p\},$$
$$S' = \{s'_1, \ldots, s'_{\lambda_1-p+1}\} = \{1, \ldots, \lambda_1\} - \{c_1, \ldots, c_{p-1}\}.$$

Then for some $k_1, k_2 \in \mathbb{N}$, we have

$$D_1 = (-1)^{k_1} P_{[1,b_2,\ldots,b_{p-1},n,n+s_1,\ldots,n+s_{\lambda_1-p}]}, \text{ and } D_2 = (-1)^{k_2} P_{[d_1,\ldots,d_{p-1},n+s'_1,\ldots,n+s'_{\lambda_1-p+1}]}.$$

Since both of the expressions are valid, by Lemma 4.7, we have

$$\{s_1, s_2, \ldots, s_{\lambda_1-p}\} \supseteq \{\lambda_n + 1, \ldots, \lambda_1\},$$
$$\{s'_1, s'_2, \ldots, s'_{\lambda_1-p+1}\} \supseteq \{\lambda_{d_{p-1}} + 1, \ldots, \lambda_1\}.$$

Now define two strictly ascending sequences with

$$(i_n) = \{i_1, \ldots, i_{\lambda_1 - 1}\} = \{b_2, \ldots, b_{p-1}, n, n + s_1, \ldots, n + s_{\lambda_1 - p}\},$$

$$(j_n) = \{j_1, \ldots, j_{\lambda_1 + 1}\} = \{1, d_1, \ldots, d_{p-1}, n + s'_1, \ldots, n + s'_{\lambda_1 - p+1}\},$$

where we move the index 1 from D_1 to D_2. Then we claim that for all $1 \leq l \leq \lambda_1 + 1$,

$$P_{[i_1,\ldots,i_{\lambda_1-1},j_l]} P_{[j_1,\ldots,\hat{j_l},\ldots,j_{\lambda_1+1}]} \text{ is valid on } N.$$

We may assume that $j_l \notin (i_n)$, otherwise $P_{[i_1,\ldots,i_{\lambda_1-1},j_l]} = 0$. If $l \leq p$, then

$$\{s_1, \ldots, s_{\lambda_1 - p}\} \supseteq \{\lambda_n + 1, \ldots, \lambda_1\}, \text{ and}$$

$$\{s'_1, \ldots, s'_{\lambda_1 - p+1}\} \supseteq \{\lambda_{d_{p-1}} + 1, \ldots, \lambda_1\}.$$

While for $l > p$, i.e., $j_l > n$, let $s = j_l - n$. Then since $j_l \notin (i_n)$, $s \leq \lambda_n \leq \lambda_{d_{p-1}}$. Therefore,

$$\{s_1, \ldots, s_{\lambda_1 - p}, j_l - n\} \supseteq \{s_1, \ldots, s_{\lambda_1 - p}\} \supseteq \{\lambda_n + 1, \ldots, \lambda_1\}, \text{ and}$$

$$\{s'_1, \ldots, \hat{s}, \ldots, s'_{\lambda_1 - p+1}\} \supseteq \{\lambda_{d_{p-1}} + 1, \ldots, \lambda_1\}.$$

By Lemma 4.7, we can conclude the claim, since $D_1 D_2$ equals to

$$P_{[i_1,\ldots,i_{\lambda_1-1},n]} P_{[j_1,\ldots,\hat{j_p},\ldots,j_{\lambda_1+1}]}$$

up to a sign, it is involved in a valid Plücker relation on N. $\square$

We now come to our main theorem.

Theorem 4.9. *Every ladder determinantal ideal is Knutson. In particular, consider the ladder matrix M determined by $\lambda = (\lambda_1, \ldots, \lambda_n)$. If I is a ladder determinantal ideal with respect to M, then $I \in \mathcal{P}_f$, where*

$$f := \prod_{j=1}^{n} \prod_{i=\lambda_j}^{\lambda_{j+1}} d_{(i,j)}^{\min\{i,j\}},$$

and $d_{(i,j)}^k$ denotes the determinant of the $k \times k$ submatrix with the south-east corner located at (i, j) and $\lambda_{n+1} = 1$, together with the diagonal term order $<$, i.e., $m_{i,j} < m_{i,j+1}$ and $m_{i,n} < m_{i+1,1}$.

We will first illustrate the proof with the following example.

Example 4.10. Let M be as defined in Example 4.3 with $\lambda = (5, 5, 4, 3, 2)$:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & \\ m_{41} & m_{42} & m_{43} & & \\ m_{51} & m_{52} & & & \end{bmatrix}$$

Then the desired polynomial f will be

$$
\begin{aligned}
f = &\det\left(\begin{bmatrix} m_{51} \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} m_{41} & m_{42} \\ m_{51} & m_{52} \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} m_{31} & m_{32} \\ m_{41} & m_{42} \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}\right) \cdot \\
&\det\left(\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} m_{12} & m_{13} & m_{14} \\ m_{22} & m_{23} & m_{24} \\ m_{32} & m_{33} & m_{34} \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \end{bmatrix}\right) \cdot \\
&\det\left(\begin{bmatrix} m_{14} & m_{15} \\ m_{24} & m_{25} \end{bmatrix}\right) \cdot \det\left(\begin{bmatrix} m_{15} \end{bmatrix}\right) \\
= &d^1_{(5,1)} \cdot d^2_{(5,2)} \cdot d^2_{(4,2)} \cdot d^3_{(4,3)} \cdot d^3_{(3,3)} \cdot d^3_{(3,4)} \cdot d^2_{(2,4)} \cdot d^2_{(2,5)} \cdot d^1_{(1,5)}
\end{aligned}
$$

Set $<$ be the lexicographic diagonal term order, i.e.,

$$m_{11} < m_{12} < \ldots < m_{15} < m_{21} < \cdots < m_{51} < m_{52}.$$

Then

$$\text{init}_<(f) = m_{51}m_{52}m_{41}m_{42}m_{31}m_{43}m_{32}m_{21}m_{33}m_{22}m_{11}m_{34}m_{23}m_{12}m_{24}m_{13}m_{25}m_{14}m_{15}$$

is square-free.

Let $\mathcal{D}_f = \{d^1_{(5,1)}, d^2_{(5,2)}, d^2_{(4,2)}, d^3_{(4,3)}, d^3_{(3,3)}, d^3_{(3,4)}, d^3_{(2,4)}, d^2_{(2,5)}, d^1_{(1,5)}\}$ and $M^{[t_1,t_2]}$ denote the submatrix of M which involves the t_1-th to t_2-th columns. We have the following two observations:

(1) Suppose

$$
\begin{aligned}
I = \text{minors}(2, M^{[3,4]}) &= \text{minors}\left(2, \begin{bmatrix} m_{13} & m_{14} \\ m_{23} & m_{24} \\ m_{33} & m_{34} \\ m_{43} & \end{bmatrix}\right) \\
&= \langle m_{13}m_{24} - m_{14}m_{23}, m_{13}m_{34} - m_{14}m_{33}, m_{23}m_{34} - m_{24}m_3 \rangle.
\end{aligned}
$$

Let J be the ideal whose generators are of the form $d^k_{(i,j)} \in \mathcal{D}_f$ where $k \geq 2$ and the matrix of $d^k_{(i,j)}$ contains the 3-rd and the 4-th columns of M, i.e.,

$$J = \langle d^3_{(3,4)}, d^2_{(2,4)} \rangle.$$

Then $J \subseteq I$ since $d^2_{(2,4)} = m_{13}m_{24} - m_{14}m_{23} \in I$ and

$$d^3_{(3,4)} = m_{12}(m_{23}m_{34} - m_{24}m_{33}) - m_{22}(m_{13}m_{34} - m_{14}m_{33}) + m_{32}(m_{13}m_{24} - m_{14}m_{23}) \in I.$$

Lemma 3.10 yields that $\text{ht}(I) = 2$. In addition, since $\text{init}(d^3_{(3,4)}) = m_{12}m_{23}m_{34}$ and $\text{init}(d^2_{(2,4)}) = m_{13}m_{24}$ are distinct, $\text{ht}(J) = 2$. By Krull's height theorem, we have I is a minimal prime ideal over J. Due to $d^3_{(3,4)}$ and $d^2_{(2,4)}$ are two irreducible factors of f, Lemma 2.20 implies that $d^3_{(3,4)} \in \mathcal{P}_f$ and $d^2_{(2,4)} \in \mathcal{P}_f$. Then $J = \langle d^3_{(3,4)} \rangle + \langle d^2_{(2,4)} \rangle \in \mathcal{P}_f$.

Using similar technique, we have $\text{minors}(2, M^{[t,t+1]}) \in \mathcal{P}_f$ for $t = 1, 2, 3, 4$.

(2) One can claim that minors$(2, M^{[1,5]}) \in \mathcal{P}_f$ by checking the followings using computational program Macaulay2.

- minors$(2, M^{[t,t+2]}) \in \mathcal{P}_f$ for all $t = 1, 2, 3$ due to

$$\text{minors}(2, M^{[t,t+1]}) + \text{minors}(2, M^{[t+1,t+2]}) = \text{minors}(2, M^{[t+1,t+3]}) \cap \text{minors}(1, M^{[t+2,t+2]}).$$

- minors$(2, M^{[t,t+3]}) \in \mathcal{P}_f$ for all $t = 1, 2$ due to

$$\text{minors}(2, M^{[t,t+2]}) + \text{minors}(2, M^{[t+1,t+3]}) = \text{minors}(2, M^{[t,t+3]}) \cap \text{minors}(1, M^{[t+1,t+2]}).$$

- minors$(2, M^{[1,5]}) \in \mathcal{P}_f$ due to

$$\text{minors}(2, M^{[1,4]}) + \text{minors}(2, M^{[2,5]}) = \text{minors}(2, M^{[1,5]}) \cap \text{minors}(1, M^{[2,4]}).$$

Therefore the determinantal ideal minors$(2, M) =$ minors$(2, M^{[1,5]})$ of M is Knutson.

Based on the observations in Example 4.10, we have the following properties.

PROPOSITION 4.11. *Assume M, f, and $<$ as in Theorem 4.9. Let $I_{[t_1,t_2],p}$ denote the ideal generated by the p-minors of the submatrix which takes columns from t_1 to t_2 of M. Then*

(1) For any $1 \le t < n$, $I_{[t,t+p-1],p} \in P_f$.
(2) For any $s, t \in \mathbb{N}$ such that $1 \le t < t + s \le n$, exactly one of the following properties is satisfied:
- *If $p = 1$, then*

$$I_{[1,n],1} = \sum_{t=1}^{n} I_{[t,t],1} \in \mathcal{P}_f.$$

- *If $p > 1$, then*

$$I_{[t,t+s-1],p} + I_{[t+1,t+s],p} = \begin{cases} I_{[t,t+s],p} \cap I_{[t+1,t+s-1],p-1} & \text{otherwise}, \\ I_{[t,t+s],p} & \text{if } I_{[t,t+s-1],p} = \langle 0 \rangle \text{ or } I_{[t+1,t+s],p} = \langle 0 \rangle. \end{cases}$$

Thus, by induction on s, $I_{[t,t+s],p} \in \mathcal{P}_f$ for all $s, t, p \in \mathbb{N}$ where $1 \le t < t + s \le n$.

PROOF. (1) Let $\mathcal{D}_f$ denote the set of $d^k_{(i,j)}$ which divides f. We first define

$$J_{t,p} := \langle d^k_{(i,j)} \mid d^k_{(i,j)} \in \mathcal{D}_f, t + p - 1 \le j \le t + k - 1 \rangle$$

and we can interpret the generators in $J_{t,p}$ as the elements in $\mathcal{D}_f$ whose matrices contain columns from t to $t+p-1$. For $\lambda_{t+p-1} \ge p$, the number of generators of $J_{t,p}$ is $\lambda_{t+p-1} - p + 1$. Notice, each pair of elements in $\mathcal{D}_f$ have distinct indeterminates in their leading terms. Therefore,

$$\text{ht}(J_{t,p}) = \max\{0, \lambda_{t+p-1} - p + 1\}.$$

Then we claim that

$$I_{[t,t+p-1],p} \supseteq J_{t,p}.$$

Suppose $d^k_{(i,j)}$ is a generator in $J_{t,p}$, then due to $k \geq p$, the cofactor expansion of the matrix of $d^k_{(i,j)}$ is a combination of p-minors of the submatrix containing columns from t to $t+p-1$ of M, i.e., the generators of $I_{[t,t+p-1],p}$.

Now, we want to show that

$$\mathrm{ht}(I_{[t,t+p-1],p}) = \max\{0, \lambda_{t+p-1} - p + 1\} = \mathrm{ht}(J_{t,p}).$$

Let $N_{[t,t+p-1]}$ be the submatrix of M which takes rows from 1 to λ_{t+p-1}, and columns from t to $t+p-1$, i.e., the largest rectangle matrix without empty entries contained in the submatrix $M^{[t,t+p-1]}$. Then it is easy to see that

$$I_{[t,t+p-1],p} = \mathrm{minors}(p, M^{[t,t+p-1]}) = \mathrm{minors}(p, N_{[t,t+p-1]}).$$

Then we have the following two cases:

- $\lambda_{t+p-1} < p$: Then the p-minors of $N_{[t,t+p-1]}$ are $\langle 0 \rangle$, and there does not exist such (i,j,k) that $d^k_{(i,j)} \in \mathcal{D}_f$ and $t+p-1 \leq j \leq t+k-1$. Thus,
$$\mathrm{ht}(I_{[t,t+p-1],p}) = 0 = \mathrm{ht}(J_{t,p}).$$
- $\lambda_{t+p-1} \geq p$: Lemma 3.10 yields that
$$\mathrm{ht}(\mathrm{minors}(p, N_{[t,t+p-1]})) = \lambda_{t+p-1} - p + 1.$$
And $J_{t,p}$ is then non-zero. Thus,
$$\mathrm{ht}(I_{[t,t+p-1],p}) = \lambda_{t+p-1} + 1 - p = \mathrm{ht}(J_{t,p}).$$

Notice, $I_{[t,t+p-1],p}$ is indeed a prime ideal by [**11**, Corollary 16.29]. Using Krull's Height Theorem 3.9, we can conclude that $I_{[t,t+p-1],p}$ is a minimal prime over $J_{t,p}$. Moreover, since each $d^k_{(i,j)} \in \mathcal{D}_f$ is an irreducible factor of f, Lemma 2.20 indicates that $d^k_{(i,j)} \in \mathcal{P}_f$. Then $J_{t,p} \in \mathcal{P}_f$ as it is a sum of some $d^k_{(i,j)}$. Therefore, $I_{[t,t+p-1],p}$ is also in $\mathcal{P}_f$, since it is a minimal prime of $J_{t,p}$.

(2) First, we want to show that the $1-$minor of M is in $\mathcal{P}_f$. Since the generators of $I_{[t,k],1}$ are all the indeterminates in M, we have

$$I_{[1,n],1} = \langle m_{ij} \mid 1 \leq j \leq n, 1 \leq i \leq \lambda_j \rangle = \sum_{t=1}^{n} \langle m_{i,t} \mid 1 \leq i \leq \lambda_t \rangle = \sum_{t=1}^{n} I_{[t,t],1}.$$

This shows $I_{[1,n],1} \in \mathcal{P}_f$ since each $I_{[t,t],1} \in \mathcal{P}_f$ by part (1).

For p-minors with $p > 1$, we first assume that both of $I_{[t,t+s-1],p}$ and $I_{[t+1,t+s],p}$ are non-zero for some $p > 1$ and live inside $\mathcal{P}_f$. We want to show

$$I_{[t,t+s-1],p} + I_{[t+1,t+s],p} = I_{[t,t+s],p} \cap I_{[t+1,t+s-1],p-1}.$$

The direction $\subseteq$ is easy. Without loss of generality, suppose d is a generator of $I_{[t,t+s-1],p}$, then $d \in I_{[t,t+s],p}$. And the cofactor expansion of d shows that d is a combination of the generators of $I_{[t+1,t+s-1],p-1}$, i.e., $d \in I_{[t+1,t+s-1],p-1}$. Therefore,

$$I_{[t,t+s-1],p} \subseteq I_{[t,t+s],p} \cap I_{[t+1,t+s-1],p-1}.$$

Similarly, one has

$$I_{[t+1,t+s],p} \subseteq I_{[t,t+s],p} \cap I_{[t+1,t+s-1],p-1}.$$

Therefore,

$$I_{[t,t+s-1],p} + I_{[t+1,t+s],p} \subseteq I_{[t,t+s],p} \cap I_{[t+1,t+s-1],p-1}.$$

To show the other direction $I_{[t,t+s-1],p} + I_{[t+1,t+s],p} \supseteq I_{[t,t+s],p} \cap I_{[t+1,t+s-1],p-1}$, we first show that $I_{[t,t+s-1],p} + I_{[t+1,t+s],p} \supseteq I_{[t,t+s],p}I_{[t+1,t+s-1],p-1}$. Let $d^{[b_1,\ldots,b_p]}_{[a_1,\ldots,a_p]} \in I_{[t,t+s],p}$ and $d^{[d_1,\ldots,d_{p-1}]}_{[c_1,\ldots,c_{p-1}]} \in I_{[t+1,t+s-1],p-1l}$ be two generators. If $b_i \neq t$ for all i, then

$$d^{[b_1,\ldots,b_p]}_{[a_1,\ldots,a_p]} \in I_{[t+1,t+s],p} \implies d^{[b_1,\ldots,b_p]}_{[a_1,\ldots,a_p]} d^{[d_1,\ldots,d_{p-1}]}_{[c_1,\ldots,c_{p-1}]} \in I_{[t+1,t+s],p}.$$

Then $I_{[t,t+s],p}I_{[t+1,t+s],p-1} \subseteq I_{[t,t+s-1],p} + I_{[t+1,t+s],p}$. Similarly for the cases when $b_i \neq t+s$ for all i. Now, without loss of generality, assume $b_1 = t$ and $b_p = t+s$. Then applying Lemma 4.8 to the submatrix $M^{[t,t+s],p}$ there exists a valid Plücker relation which involves $d^{[t,b_2,\ldots,b_{p-1},t+s-1]}_{[a_1,\ldots,a_p]} d^{[d_1,\ldots,d_{p-1}]}_{[c_1,\ldots,c_{p-1}]}$:

$$\sum_{l=1}^{\lambda_t+1} (-1)^l P_{[i_1,\ldots,i_{\lambda_t-1},j_l]} P_{[j_1,\ldots,\hat{j}_l,\ldots,j_{\lambda_t+1}]} = 0.$$

where

$$\begin{gathered}
\{s_1,\ldots,s_{\lambda_t-p}\} = \{1,\ldots,\lambda_t\} - \{a_1,\ldots,a_p\}, \\
\{s'_1,\ldots,s'_{\lambda_t-p+1}\} = \{1,\ldots,\lambda_t\} - \{c_1,\ldots,c_{p-1}\}, \\
(i_n) = \{i_1,\ldots,i_{\lambda_t-1}\} = \{b_2,\ldots,b_{p-1},t+s,t+s+s_1,\ldots,t+s+s_{\lambda_1-p}\}, \text{ and} \\
(j_n) = \{j_1,\ldots,j_{\lambda_t+1}\} = \{t,d_1,\ldots,d_{p-1},t+s+s'_1,\ldots,t+s+s'_{\lambda_1-p+1}\}.
\end{gathered}$$

For $1 < l \leq p$, $P_{[i_1,\ldots,i_{\lambda_t-1},j_l]}$ equals to (up to a sign) the determinant of the $p \times p$ submatrix which involves row $b_2,\ldots,b_{p-1},t+s,d_{p-1}$, and is thus inside $I_{[t+1,t+s],p}$. For $p < l \leq \lambda_t+1$, $P_{[j_1,\ldots,\hat{j}_l,\ldots,j_{\lambda_t+1}]}$ equals to (up to a sign) the determinant of the $p \times p$ submatrix which involves row $t,d_1,\ldots,d_{p-1}$, and is thus inside $I_{[t,t+s-1],p}$. Therefore,

$$\sum_{l=2}^{\lambda_t+1} (-1)^l P_{[i_1,\ldots,i_{\lambda_t-1},j_l]} P_{[j_1,\ldots,\hat{j}_l,\ldots,j_{\lambda_t+1}]} \in I_{[t,t+s-1],p} + I_{[t+1,t+s],p}$$

Since for $l = 1$, $P_{[i_1,\ldots,i_{\lambda_t-1},t]} P_{[j_2,\ldots,j_{\lambda_t+1}]}$ equals to $d^{[t,b_2,\ldots,b_{p-1},t+s]}_{[a_1,\ldots,a_p]} d^{[d_1,\ldots,d_{p-1}]}_{[c_1,\ldots,c_{p-1}]}$ up to a sign, for some $m \in \mathbb{N}$, we have

$$d^{[t,b_2,\ldots,b_{p-1},t+s]}_{[a_1,\ldots,a_p]} d^{[d_1,\ldots,d_{p-1}]}_{[c_1,\ldots,c_{p-1}]} = (-1)^m \sum_{l=2}^{\lambda_t+1} (-1)^l P_{[i_1,\ldots,i_{\lambda_t-1},j_l]} P_{[j_1,\ldots,\hat{j}_l,\ldots,j_{\lambda_t+1}]},$$

and is thus contained in $I_{[t,t+s-1],p} + I_{[t+1,t+s],p} \in I_{[t,t+s-1],p} + I_{[t+1,t+s],p}$ Therefore,

$$I_{[t,t+s],p}I_{[t+1,t+s],p-1} \subseteq I_{[t,t+s-1],p} + I_{[t+1,t+s],p}.$$

Since $I_{[t,t+s-1],p}$ and $I_{[t+1,t+s],p}$ are Knutson, they are radical by Lemma 2.16. Then $\sqrt{I_{[t,t+s-1],p}+I_{[t+1,t+s],p}} = I_{[t,t+s-1],p}+I_{[t+1,t+s],p}$. In addition, [**11**, Section 16.4] yields that the determinantal ideals $I_{[t,t+s],p}$ and $I_{[t+1,t+s],p-1}$ are radical. So $\sqrt{I_{[t,t+s],p}\cap I_{[t+1,t+s],p-1}} = I_{[t,t+s],p}\cap I_{[t+1,t+s],p-1}$. One can thus derive that

$$\begin{aligned}
I_{[t,t+s],p}\cap I_{[t+1,t+s],p-1} &= \sqrt{I_{[t,t+s],p}\cap I_{[t+1,t+s],p-1}}\\
&= \sqrt{I_{[t,t+s],p}I_{[t+1,t+s],p-1}}\\
&\subseteq \sqrt{I_{[t,t+s-1],p}+I_{[t+1,t+s],p}}\\
&= I_{[t,t+s-1],p}+I_{[t+1,t+s],p}.
\end{aligned}$$

Now assume either $I_{[t,t+s-1],p}=\langle 0\rangle$ or $I_{[t+1,t+s],p}=\langle 0\rangle$. It is easy to see that $I_{[t,t+s],p} = I_{[t,t+s-1],p}+I_{[t+1,t+s],p}$. □

Now we shall prove Theorem 4.9.

Proof. Since $<$ is the diagonal term order, we have

$$\begin{aligned}
\mathrm{init}_{<}f &= \mathrm{init}\Big(\prod_{j=1}^{n}\prod_{i=\lambda_j}^{\lambda_{j+1}} d_{(i,j)}^{\min\{i,j\}}\Big)\\
&= \prod_{j=1}^{n}\prod_{i=\lambda_j}^{\lambda_{j+1}} \mathrm{init}_{<}\big(d_{(i,j)}^{\min\{i,j\}}\big)\\
&= \prod_{j=1}^{n}\prod_{i=\lambda_j}^{\lambda_{j+1}}\prod_{p=0}^{\min\{i,j\}} m_{i-p,j-p}\\
&= \prod_{1\le j\le n, 1\le i\le \lambda_j} m_{ij}
\end{aligned}$$

which is square-free. Suppose I is some determinantal ideal of M which is generated by p-minors. Then by Proposition 4.11, for $t=1, t+s=n$, $I = I_{[t,t+s],p}\in\mathcal{P}_f$. Thus, I is Knutson.

□

CHAPTER 5

Future Directions

In this chapter, we will discuss two observations that we were not able to prove. These observations may lead to future work on this topic.

Given a graph G, there are various ways to find a polynomial f such that the toric ideal $I_G \in \mathcal{P}_f$. However, in general, such f cannot be a product of a subset of the binomials represented by the primitive walks in G.

CONJECTURE 5.1. *Let $K_{n,m}$ be the complete bipartite graph, and let $I_{K_{n,m}}$ be the toric ideal. When $n \geq 3$ and $m \geq 4$, there does not exist an f, which is a product of some binomials represented by the primitive walks in $K_{n,m}$, together with a monomial order $<$, such that* $\operatorname{init}_< f$ *is square-free and $I_{K_{n,m}} \in \mathcal{P}_f$.*

This conjecture gives rise to another question.

QUESTION 5.2. *What properties of G need to hold so that we can write f as a product of some of the binomials, which are represented by the primitive walks in G, such that* $\operatorname{init}_< f$ *is square-free for some monomial order $<$ and $I_G \in \mathcal{P}_f$?*

In Chapter 4, we showed that every unmixed one-sided ladder determinantal ideal is Knutson. Such determinantal ideals are examples of the mixed two-sided ladder determinantal ideal, which is first introduced in [**9**].

DEFINITION 5.3. A $\lambda_1 \times n$ matrix M is said to be **two-sided ladder** if there are two partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ for some $n \in \mathbb{N}$, such that:

(1) For any $1 \leq i < n$, $\lambda_i \geq \lambda_{i+1}$, $\mu_i \geq \mu_{i+1}$, and $\mu_i \leq \lambda_i$.
(2) The (i, j) entry of the matrix M satisfies:

$$\begin{cases} m_{ij} & 1 \leq j \leq n \text{ and } \mu_j \leq i \leq \lambda_j \\ \text{empty} & \text{otherwise} \end{cases},$$

where m_{ij} is an indeterminate.

DEFINITION 5.4. Let M be a two-sided ladder defined by λ and μ. Let

$$\Lambda = ((\lambda_{i_1}, i_1), \ldots, (\lambda_{i_s}, i_s))$$

be a sequence with $\lambda_{i_1} > \lambda_{i_2} > \cdots > \lambda_{i_s}$ a strictly descending subsequence of λ and for all $1 \leq k \leq n$, if $i_j < k$ then $\lambda_{i_j} < \lambda_k$. I.e., Λ records the positions of south-east corners in M.

Define L_m to be a submatrix of M that contains columns from μ_m to λ_m and rows from 1 to m, i.e.,

$$L_m := M^{[1,m]}_{[\mu_m,\lambda_m]}.$$

Then the **mixed ladder determinantal ideal** defined on $t = (t_1, \ldots, t_s)$ is

$$I_t(M) = \sum_{(\lambda_{i_k}, i_k) \in \Lambda} I_{t_k}(L_{i_k}),$$

where $I_{t_k}(L_{i_k})$ is the ideal generated by the t_k-minors of L_{i_k}.

Example 5.5. Let M be a two-sided ladder matrix which is defined by $\lambda = (5, 5, 4, 3, 3)$ and $\mu = (3, 2, 1, 1, 1)$. Then

$$M = \begin{bmatrix} & & m_{13} & m_{14} & m_{15} \\ & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \\ m_{41} & m_{42} & m_{43} & & \\ m_{51} & m_{52} & & & \end{bmatrix}.$$

We then have $\Lambda = ((\lambda_2, 2), (\lambda_3, 3), (\lambda_5, 5)) = ((5, 2), (4, 3), (5, 3))$,

$$L_2 = \begin{bmatrix} & m_{22} \\ m_{31} & m_{32} \\ m_{41} & m_{42} \\ m_{51} & m_{52} \end{bmatrix}, L_3 = \begin{bmatrix} & & m_{13} \\ & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \\ m_{41} & m_{42} & m_{43} \end{bmatrix}, \text{ and } L_5 = \begin{bmatrix} & & m_{13} & m_{14} & m_{15} \\ & m_{22} & m_{23} & m_{24} & m_{25} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} \end{bmatrix}.$$

Let $t = (2, 2, 3)$. Then the mixed ladder determinantal ideal defined on t is

$$I_t(M) = \text{minors}(2, L_2) + \text{minors}(2, L_3) + \text{minors}(3, L_5).$$

Conjecture 5.6. *Every mixed ladder determinantal ideal is Knutson. In particular, consider the two-sided ladder M determined by $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. If I is a mixed ladder determinantal ideal with respect to M, then $I \in \mathcal{P}_f$, where*

$$f := \prod_{j=1}^{n} \prod_{i=\lambda_j}^{\lambda_{j+1}} d_{(i,j)}^{\max\{k: i-k \geq \mu_{j-k+1}\}},$$

and $d^k_{(i,j)}$ denotes the determinant of the $k \times k$ submatrix with the south-east corner located at (i, j) and $\lambda_{n+1} = \mu_n$, together with the diagonal term order $<$.

Example 5.7. Define M and $I_t(M)$ as in Example 5.5. Then the desired polynomial f will be

$$f := d^1_{(5,1)} \cdot d^2_{(5,2)} \cdot d^2_{(4,2)} \cdot d^2_{(4,3)} \cdot d^2_{(3,3)} \cdot d^2_{(3,4)} \cdot d^3_{(3,5)} \cdot d^2_{(2,5)} \cdot d^1_{(1,5)}.$$

The initial term of f with respect to the diagonal term order is square-free since

$$\operatorname{init}(f) = \prod_{j=1}^{n} \prod_{i=\lambda_j}^{\lambda_{j+1}} \operatorname{init}(d_{(i,j)}^{\max\{k: i-k \geq \mu_{j-k+1}\}}) = \prod_{j=1}^{n} \prod_{i=\mu_j}^{\lambda_j} m_{ij}.$$

Then we can show that $\operatorname{minors}(2, L_2), \operatorname{minors}(2, L_3) \in \mathcal{P}_f$ using a method similar to the proof of part (1) in Proposition 4.11. The ideal $\operatorname{minors}(3, L_5) = \langle d_{3,5}^3 \rangle \in \mathcal{P}_f$ due to the fact that $d_{3,5}^3$ is irreducible. Therefore,

$$I_t(M) = \operatorname{minors}(2, L_2) + \operatorname{minors}(2, L_3) + \langle d_{3,5}^3 \rangle \in \mathcal{P}_f,$$

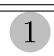
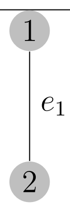
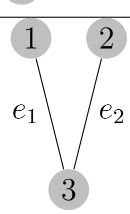
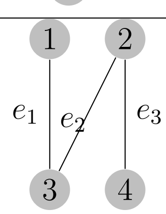
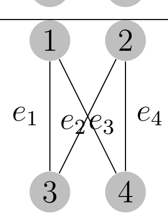
and is thus Knutson.

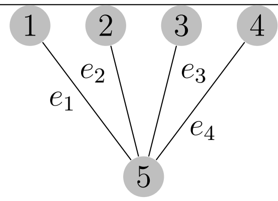
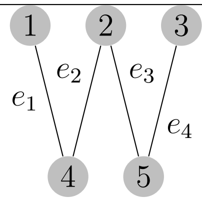
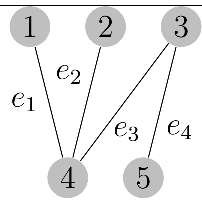
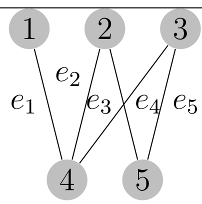
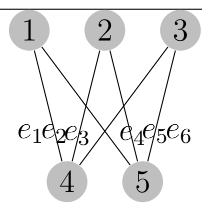
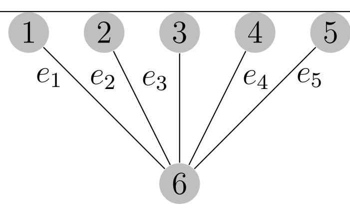
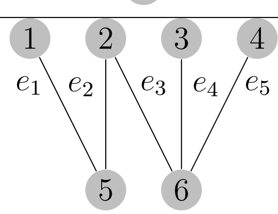
It has been tested in over 10 different cases that the mixed ladder determinantal ideal is Knutson.

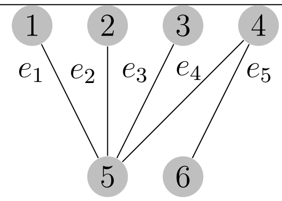
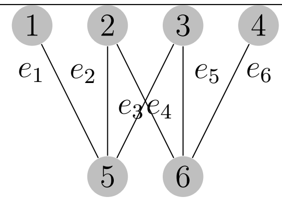
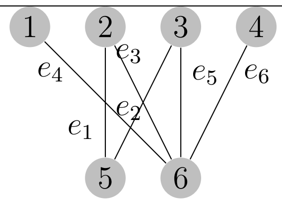
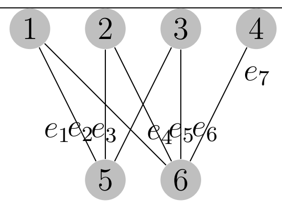
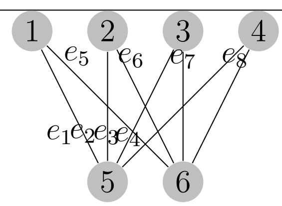
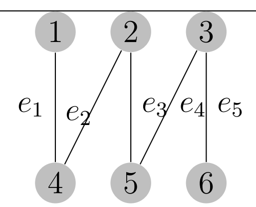
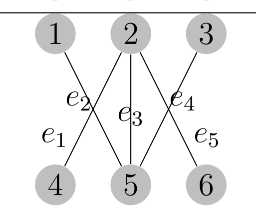
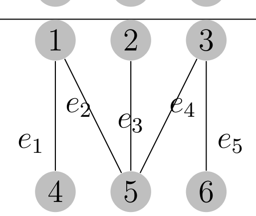
CHAPTER 6

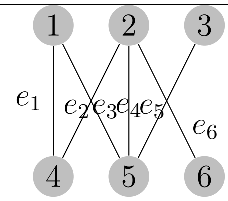
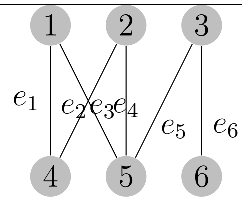
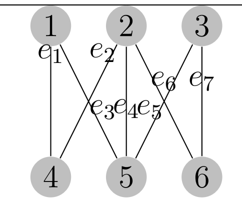
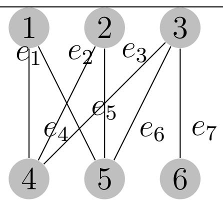
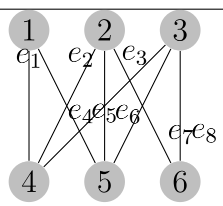
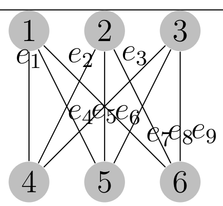
Appendix

A bipartite graph is said to be chordal if it has no induced cycles of length six or more. In other words, every closed primitive walk in the graph is of length four. This appendix illustrates that the toric ideal of every chordal bipartite graph with vertices no more than six is Knutson by providing an example for the desired f. The default lexicographic order is $e_1 > e_2 > e_3 > \cdots$ unless it is otherwise stated. With the help of Macaulay2, we can check that $I_G \in \mathcal{P}_f$ and $\text{init}_<(f)$ is square-free.

Chordal Bipartite Graph			
$\|V\|$	Graph	I_G	f
1		(0)	e_1
2		(0)	e_1
3		(0)	e_1e_2
4		(0)	$e_1e_2e_3$
4		$(e_1e_3 - e_2e_4)$	$(e_1e_3 - e_2e_4)e_2e_4$

Chordal Bipartite Graph			
$\|V\|$	Graph	I_G	f
5		(0)	$e_1e_2e_3e_4$
5		(0)	$e_1e_2e_3e_4$
5		(0)	$e_1e_2e_3e_4$
5		$(e_2e_5 - e_3e_4)$	$(e_2e_5 - e_3e_4)e_1e_2e_5$
5		$(e_1e_4 - e_2e_5, e_2e_6 - e_3e_5,$ $e_1e_6 - e_3e_4)$	$(e_1e_4 - e_2e_5)(e_2e_6 - e_3e_5)e_3e_5$
6		0	$e_1e_2e_3e_4e_5$
6		0	$e_1e_2e_3e_4e_5$

Chordal Bipartite Graph			
$\|V\|$	Graph	I_G	f
6		0	$e_1e_2e_3e_4e_5$
6		$(e_2e_5 - e_3e_4)$	$(e_2e_5 - e_3e_4)e_1e_3e_4$
6		$(e_1e_5 - e_2e_3)$	$(e_1e_5 - e_2e_3)e_2e_3e_4$
6		$(e_1e_4 - e_2e_5, e_2e_6 - e_3e_5,$ $e_1e_6 - e_3e_4)$	$(e_1e_4 - e_2e_5)(e_2e_6 - e_3e_5)e_3e_5e_7$
6		$(e_1e_6 - e_2e_5, e_1e_7 - e_3e_5,$ $e_1e_8 - e_4e_5, e_2e_7 - e_3e_6,$ $e_2e_8 - e_4e_6, e_3e_8 - e_4e_7)$	$(e_1e_6 - e_2e_5)(e_2e_7 - e_3e_6)(e_3e_8 - e_4e_7)e_4e_5$
6		(0)	$e_1e_2e_3e_4e_5$
6		(0)	$e_1e_2e_3e_4e_5$
6		(0)	$e_1e_2e_3e_4e_5$

Chordal Bipartite Graph			
$\|V\|$	Graph	I_G	f
6		$(e_1e_4 - e_2e_3)$	$(e_1e_4 - e_2e_3)e_2e_3e_5$
6		$(e_1e_4 - e_2e_3)$	$(e_1e_4 - e_2e_3)e_2e_3e_5$
6		$(e_1e_4 - e_2e_3, e_4e_7 - e_5e_6)$	$(e_1e_4 - e_2e_3)(e_4e_7 - e_5e_6)e_2e_3e_7$
6		$(e_1e_5 - e_2e_4, e_1e_6 - e_3e_4, e_2e_6 - e_3e_5)$	$(e_1e_5 - e_2e_4)(e_2e_6 - e_3e_5)e_3e_4e_7$
6		$(e_1e_5 - e_2e_4, e_1e_6 - e_3e_4, e_2e_6 - e_3e_5, e_4e_8 - e_5e_7, e_1e_2 - e_7e_8)$	$(e_1e_5 - e_2e_4)(e_2e_6 - e_3e_5)(e_4e_8 - e_5e_7)e_3e_7$
6		$(e_1e_5 - e_2e_4, e_1e_6 - e_3e_4, e_2e_6 - e_3e_5, e_4e_8 - e_5e_7, e_1e_2 - e_7e_8, e_2e_9 - e_3e_8, e_5e_9 - e_6e_8, e_4e_9 - e_6e_7, e_1e_9 - e_3e_7)$	$(e_1e_5 - e_2e_4)(e_1e_8 - e_2e_7)$ $(e_2e_6 - e_3e_5)(e_4e_9 - e_6e_7)e_9$ with $e_8 > e_3 > e_6 > e_2 > e_1 > e_4 > e_5 > e_7 > e_9$

Bibliography

[1] Michel Brion and Shrawan Kumar. *Frobenius splitting methods in geometry and representation theory*, volume 231 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2005.

[2] Aldo Conca. Ladder determinantal rings. *Journal of Pure and Applied Algebra*, 98(2):119–134, 1995.

[3] Alberto Corso and Uwe Nagel. Monomial and toric ideals associated to Ferrers graphs. *Trans. Amer. Math. Soc.*, 361(3):1371–1395, 2009.

[4] David A. Cox, John Little, and Donal O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Undergraduate Texts in Mathematics. Springer International Publishing Switzerland, 2015.

[5] David Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[6] David Eisenbud, Craig Huneke, and Bernd Ulrich. Heights of ideals of minors. *Amer. J. Math.*, 126(2):417–438, 2004.

[7] Giuseppe Favacchio, Johannes Hofscheier, Graham Keiper, and Adam Van Tuyl. Splittings of toric ideals. *J. Algebra*, 574:409–433, 2021.

[8] Federico Galetto, Johannes Hofscheier, Graham Keiper, Craig Kohne, Adam Van Tuyl, and Miguel Eduardo Uribe Paczka. Betti numbers of toric ideals of graphs: a case study. *J. Algebra Appl.*, 18(12):1950226, 14, 2019.

[9] Elisa Gorla. Mixed ladder determinantal varieties from two-sided ladders. *J. Pure Appl. Algebra*, 211(2):433–444, 2007.

[10] Allen Knutson. Frobenius splitting, point-counting, and degeneration. arXiv:0911.4941, 2009.

[11] Ezra Miller and Bernd Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[12] Emmanuel Neye. *Grobner Bases Via Linkage for Classes of Generalized Determinantal Ideals*. PhD thesis, University of Saskatchewan, 2022.

[13] Hidefumi Ohsugi and Takayuki Hibi. Toric ideals generated by quadratic binomials. *J. Algebra*, 218(2):509–527, 1999.

[14] Lisa Seccia. Knutson ideals of generic matrices. *Proc. Amer. Math. Soc.*, 150(5):1967–1973, 2022.