ON THE COORDINATE TRANSFORMATION OF A VOA

On the Coordinate Transformation of a Vertex Operator Algebra

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A Thesis Submitted to the School of Graduate Studies in the Partial Fulfillment of the Requirements for the Degree Master of Science

Master of Science, Mathematics (2023)
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Title: On the Coordinate Transformation of a Vertex Operator Algebra
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Number of pages: vii, 55

Abstract

We provide first a purely VOA-theoretic guide to the theory of coordinate transformations for a VOA in direct accordance with its first appearance in a paper of Zhu. Among these results, we are able to obtain new closed-form expressions for the square-bracket Heisenberg modes. We then elaborate on the connection to p-adic modular forms which arise as characters of states in p-adic VOAs. In particular, we show that the image of the p-adic character map for the p-adic Heisenberg VOA contains infinitely-many p-adic modular forms of level one which are not quasi-modular. Finally, we introduce a new VOA structure obtained from the Artin-Hasse exponential, and serving as the p-adic analogue of the square-bracket formalism.

Acknowledgements

If I were to write the names of everyone who has had a hand in assisting me in any way with the process of writing this thesis, this document would be easily doubled in length. As such, I will remain quite concise. I would like to first thank Dr. Franc for always taking the time to listen, and to answer the many (sometimes absurd) questions I have had over the course of this program; I would not have gained the confidence I have today as a mathematician if it were not for his support. I owe my thanks also to Dr. Harada and to Dr. Kunduri for their kindness and encouragement during and after the defence of this thesis. I would also like to thank Dr. Geoffrey Mason for allowing me to proceed with the case $t \geq 2$ in a joint result with Dr. Franc, outlined in this work.

At last but certainly not least, I extend my deepest gratitude to my family and to each and every one of my friends for always offering their assistance to me in the form of good company, and for simply being present in my life at during this time. I genuinely could not have completed this thesis had it not been for their continuous encouragement.

Contents

Declaration of Academic Achievement

I, Daniel Barake, declare that this thesis titled "On the Coordinate Transformation of a Vertex Operator Algebra" and the work presented in it are my own, with support provided by my supervisor Dr. Cameron Franc.

Dedicated to my family, my friends, and to Damira

Chapter 1

Introduction

It is well known that vertex operator algebras (VOAs) occupy a central position in various developments in both pure mathematics and two-dimensional conformal field theory. Among these, we are presently interested in their somewhat-recent connection to modular forms established first in a celebrated paper of Zhu [Zhu96] and outlined in many works such as [DLM00], [Hua97], [MT10]. That is, characters of states in a VOA give rise to modular forms. Instrumental to proving this result was the idea of "rescaling" the elements of a VOA according to the holomorphic map $e^z - 1$ which in turn defines new vertex operators, the "square-bracket formalism":

$$Y(a, e^z - 1)e^z = Y[a, z] = \sum_{n\in\mathbb{Z}} [a]_n z^{-n-1}.$$

Geometrically, this formulates the vertex operators as existing on a torus instead of a sphere (cf. [FBZ04]), and operationally it greatly illuminates the link to modular forms. For example, in Section 5.2 we will see that the character of the conformal state ω yields the ambiguous formal power series given in eq. (5.10) whereas the square-bracket conformal state $[\omega]$ gives rise to the Eisenstein series quasi-modular form $G_2(q)$.

The aim of this discussion is twofold. We first wish to provide a rigorous "fill in the blanks" treatment of the theory of coordinate transformations for VOAs in direct accordance with [Zhu96]. Though this is a well-known idea which finds itself within many texts concerned with VOAs and modular forms, the only complete framework is provided in [FBZ04] wherein the authors emphasize a geometric viewpoint. Furthermore, a different expression (eq. (3.20)) for the coordinate transformation $\phi(z)$ is used than that (eq. (3.3)) given initially in [Zhu96]. Hence we make use of results given in [FBZ04] directly to the assumptions in [Zhu96] in order to obtain a purely VOA-theoretic guide to the theory of coordinate transformations. This in turn can be followed by those without expertise on algebraic geometry. We then proceed to outline the process of transforming the Heisenberg VOA via the exponential function yielding the aforementioned square-bracket formalism, and also give expressions for the transformed modes which have not appeared in print in such an explicit form prior to this work. These expressions (eqs. (4.5) and (4.9)) are combinatorial in flavour as they are constructed via the use of integer compositions (cf. [HM04]) as well as Stirling numbers of both kinds.

Our second goal is to shed light on the ties to modular forms from a p-adic perspective. The authors of [FM22] motivate and introduce the study of p-adic VOAs which arise via the p-adic completion of the axioms for "usual" VOAs. It is also shown that there exist p-adic variants of the Heisenberg, Virasoro and Monster VOAs. Notably, and what will be the focus of later discussion, the character map is extended such that p-adic states then give rise to p-adic modular forms. First introduced in [Ser73b], p-adic modular forms are p-adic limits of "classical" modular forms such as $G_k(q)$. In particular, they are power series with p-adic coefficients. One such example which emerges as the character of a state in the p-adic Heisenberg VOA is the p-adic Eisenstein series

$$G_2^\star(q) = \frac{p-1}{24} + \sum_{n\geq1} \sigma^\star(n) q^n \tag{1.1}$$

where $\sigma^\star(n)$ denotes the sum of all divisors of n which are coprime to p. Unlike the algebraic case in which it is known that every modular form of level one is realized as the character of some state in a VOA (cf. [MT10]), it is yet undetermined whether every p-adic modular form arises as the character of some state from a p-adic VOA. Until now we knew only that the image of the p-adic character map for the p-adic Heisenberg contains $G_2^\star(q)$, a result established in Section 10 of [FM22]. In Section 6.2 of this discussion, we expand on this fact. Using the theory established on the coordinate transformation, we show (Theorem 6.2.5) the following:

Theorem. *The image of the p-adic character map $\boldsymbol{f}\colon S \to \mathbb{Q}_p[E_2, E_4, E_6]$ for the p-adic Heisenberg VOA S contains infinitely many p-adic modular forms of level one which are not quasi-modular.*

Further inquiry into this subject via the square-bracket formalism rapidly becomes cumbersome and so we proceed to introduce a new formalism in the following way: The Artin-Hasse exponential series has product expansion given by

$$\mathrm{AH}_p(z) = \prod_{\gcd(p,i)=1} \left(1 - z^i\right)^{-\frac{\mu(i)}{i}} \tag{1.2}$$

where p is a fixed prime and μ is the familiar Mobius function. For $|z| < 1$, one has the identity

$$e^z = \prod_{i \geq 1} \left(1 - z^i\right)^{-\frac{\mu(i)}{i}}$$

and so the Artin-Hasse exponential can be seen as the p-adic analogue of the usual exponential function. Transforming a VOA via $\mathrm{AH}_p(z) - 1$ then yields a structure parallel to the square-bracket formalism which we call the *angle-bracket formalism*. For simpler computations, it is shown that these two formalisms often coincide, however in most cases one obtains additional structure. This makes the angle-bracket formalism the p-adic extension of the square-bracket formalism. These kinds of computations hint at more sophisticated patterns and thus a more intricate theory which could be used to explore further the ties to p-adic modular forms.

At last, this discussion serves the third and final goal of being an entirely self-contained overview for anyone interested in VOAs, the coordinate transformation, and their connection to modular forms. In fact, Chapter 2 gives the necessary operational background for many aspects of VOA theory, as well as the full construction of the simplest non-trivial example of a VOA. Chapter 3 serves to explain the coordinate transformation, with a concrete example provided in the subsequent Chapter 4. Chapter 5 then outlines the ties to modular forms, with Chapter 6 providing the p-adic perspective to the theory.

Chapter 2

Preliminaries

2.1 Formal Calculus

An overview of formal calculus and its usage is given first in [FLM88] and explored in great detail in [LL04]. As such, we provide the facts necessary to keep in mind when performing computations in VOA theory. Let V be a vector space and z a formal variable. Using the notation given in [LL04], the following spaces will be used throughout:

The space of *formal power series*:

$$V[[z]] = \left\{ \sum_{i\geq 0} v_n z^n \;\middle|\; v_n \in V \right\}, \tag{2.1}$$

the space of *V-valued polynomials in z*:

$$V[z] = \left\{ \sum_{i\geq 0} v_n z^n \;\middle|\; v_n \in V, \text{ all but finitely many } v_n = 0 \right\}, \tag{2.2}$$

the space of *formal Laurent series*:

$$V[[z,z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \;\middle|\; v_n \in V \right\}, \tag{2.3}$$

the space of *V-valued Laurent polynomials in z*:

$$V[z,z^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \;\middle|\; v_n \in V, \text{ all but finitely many } v_n = 0 \right\}, \tag{2.4}$$

the space of *truncated formal Laurent series*:

$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}} v_n z^n \;\middle|\; v_n \in V, \; v_n = 0 \text{ for } n \text{ sufficiently negative} \right\}. \tag{2.5}$$

As we will see, the most important identity (eq. (2.28)) for a vertex operator algebra involves the *formal delta series* given by

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \in V[[z,z^{-1}]]. \tag{2.6}$$

This is no more than the Laurent series expansion of the well-known Dirac-delta function distribution at $z = 1$, and in fact $\delta(z)$ carries the analogous property that $v(z)\delta(z) = v(1)\delta(z)$ for $v(z) \in V[[z,z^{-1}]]$, (observe that $z^n\delta(z) = \delta(z)$ for $n \in \mathbb{Z}$ and the result follows).

Operations involving formal series are commonplace in the theory and so should be examined in detail. Though one may take the product of a formal Laurent series with a Laurent polynomial for example, the product $\delta(z)\delta(z)$ is troublesome since there are infinitely many terms z^n for *all* $n \in \mathbb{Z}$. To remedy this, one must ensure that within operations involving formal series, only finitely-many terms are added together when computing any single one of the resulting coefficients of the formal variables. With this rule in place, we can make sense of the product $\delta(z)\delta(z)$ by instead considering $\delta(z_1)\delta(z_2)$ for commuting formal variables z_1, z_2. This trick of using multiple formal variables is quite common, and as such, eqs. (2.1) and (2.5) extend naturally to their multi-variable analogues. For example, one has the space

$$V[[z_1, z_1^{-1}\ z_2, z_2^{-1}]] = \left\{ \sum_{m,n\in\mathbb{Z}} v_{mn} z_1^m z_2^n \,\middle|\, v_{mn} \in V \right\}.$$

As the axioms of a vertex operator algebra are written in terms of formal generating series, the ability to extract specific coefficients will be of crucial importance. For $v(z) \in V[[z, z^{-1}]]$, there are the operators

$$\text{Coeff}_{z^n}\ v(z) = \text{the coefficient of the term } z^n \text{ in } v(z) \tag{2.7}$$

$$\text{Res}_n\ v(z) = \text{the coefficient of the term } z^{-1} \text{ in } v(z). \tag{2.8}$$

It is not difficult to see for example that for $v(z) \in V[[z]]$, we have the operator equality

$$\text{Res}_n\ z^{-n} v(z) = \text{Coeff}_{z^{n-1}}\ v(z),$$

an identity which will appear extensively in Chapters 4 and 6. The following "change of variable" lemma serves to further internalize these concepts, and will also be used when deriving expressions for the square-bracket Heisenberg modes in Chapter 4. We use the notation $zV[[z]]$ to denote the space of formal Laurent series with non-zero linear term.

Lemma 2.1.1. *Let* $f(z) = \sum_n v_n z^n \in V((z))$ *and* $g(z) \in zV[[z]]$. *Then for another formal variable* w, *we have* $\text{Res}_z\ (f(g(z))g'(z) = \text{Res}_w\ f(w)$.

Proof. First write $f(z) = v_{-1}z^{-1} + F'(z)$ for some $F(z) \in V[[z, z^{-1}]]$. With this, compute

$$\begin{aligned}\text{Res}_z\ (f(g(z))g'(z) &= \text{Res}_z\ \left(\left(v_{-1}(g(z))^{-1} + F'(g(z))\right) g'(z)\right)\\ &= \text{Res}_z\ \left(v_{-1}\frac{g'(z)}{g(z)} + (F(g(z))'\right),\end{aligned}$$

where the rightmost term in the last equality comes from the chain rule for derivatives. Since $F(g(z)) \in V[[z, z^{-1}]]$, its derivative consists of no powers of -1 and so

$$\text{Res}_z\ (f(g(z))g'(z) = v_{-1}\text{Res}_z\ \frac{g'(z)}{g(z)}.$$

Let $g(z) = zG(z)$ for some $G(z) \in V[[z]]$ with non-zero constant term. The expression becomes

$$\text{Res}_z\ (f(g(z))g'(z) = v_{-1}\text{Res}_z\ \left(\frac{1}{z} + \frac{G'(z)}{G(z)}\right)$$

Finally, since $G(z)$ has non-zero constant term, its multiplicative inverse also lies in $V[[z]]$ and so $\text{Res}_z\ G'(z)/G(z) = 0$. This establishes the result. □

We will be also considering the case where the coefficients of eqs. (2.1), (2.3) and (2.5) are in the space $\text{End}(V)$. Once again, care must be taken with operations involving such series. Here, we must ensure that coefficients of the formal variables act as finite sums of operators when applied to any fixed but arbitrary vector $v \in V$. The main objects of consideration here then, are *fields*:

$$\mathcal{F}(V) = \left\{ A(z) \in \text{End}(V)[[z, z^{-1}]] \,\middle|\, A(z)v \in V((z)) \text{ for all } v \in V \right\}. \tag{2.9}$$

Equivalently, $A(z) = \sum a_n z^n$ is a field if, for any $v \in V$, $a_n v = 0$ for n sufficiently large. Vertex operators which are central to us are fields themselves, which warrants the definition.

Finally, a convention which we will be using throughout is the expansion of a binomial series in non-negative powers of the *second variable*:

$$(z_1 + z_2)^n = \sum_{i \geq 0} \binom{n}{i} z_1^{n-i} z_2^i \tag{2.10}$$

where $n \in \mathbb{Z}$. Though usually it will not be explicitly written out, the case when $n < 0$ in eq. (2.10) should always be kept in mind during calculations. One such reason is the following: For $n > 0$, it is enough to remember the identity

$$\binom{-n}{i} = (-1)^i \binom{n+i-1}{i} \tag{2.11}$$

Applied to eq. (2.10) we see that unlike the positive case, $(z_1 + z_2)^{-n}$ is not necessarily equal to $(z_2 + z_1)^{-n}$ for $n > 0$.

2.2 The Witt and Virasoro Lie Algebras

We introduce two important Lie algebras which appear extensively in VOA theory. In particular, the coordinate invariance of vertex operators discussed in Sections 3.2 to 3.4 hinge on the action of the Virasoro algebra. We first describe the construction of the Witt algebra.

A *derivation* of an algebra A over a field k is defined as a k-linear map $D : A \to A$ satisfying the *Leibniz rule*:

$$D(ab) = aD(b) + D(a)b, \quad a, b \in A.$$

Proposition 2.2.1. *The space of derivations of formal Laurent series* $\mathbb{C}[[z, z^{-1}]]$ *forms the Witt algebra, defined as*

$$\mathfrak{w} = \left\{ v(z)\partial_z \,\middle|\, v(z) \in \mathbb{C}[[z, z^{-1}]] \right\}. \tag{2.12}$$

Proof. Let $D \in \mathrm{Der}(\mathbb{C}[[z, z^{-1}]])$. Notice first that for an arbitrary constant $c \in \mathbb{C}$ we have

$$D(c) = cD(1) = cD(1 \cdot 1) = c(1 \cdot D(1) + D(1) \cdot 1) = 2cD(1) = 2D(c)$$

and so $D(c) = 0$. Note also that

$$D(z^2) = D(z \cdot z) = zD(z) + D(z)z = 2zD(z).$$

It is then natural to speculate that for $n \geq 1$, we have $D(z^n) = nz^{n-1}D(z)$. Indeed inductively,

$$D(z^{n+1}) = D(z^n \cdot z) = z^n D(z) + z(nz^{n-1}D(z)) = (n+1)z^n D(z).$$

Likewise

$$0 = D(1) = D(z^n \cdot z^{-n}) = z^n D(z^{-n}) + D(z^n)z^{-n} = z^n D(z^{-n}) + nz^{-1}D(z).$$

It follows that $D(z^{-n}) = -nz^{-n-1}D(z)$ and so

$$D(z^n) = nz^{n-1}D(z) \quad n \in \mathbb{Z}.$$

Let $f(z) \in \mathbb{C}[[z, z^{-1}]]$. Then, by linearity

$$D(f(z)) = \sum_{n\in\mathbb{Z}} f_n D(z^n) = \sum_{n\in\mathbb{Z}} n f_n z^{-n-1} D(z) = D(z)\partial_z f(z)$$

and so by setting $D(z) = v(z) \in \mathbb{C}[[z, z^{-1}]]$, we have the operator equality $D = v(z)\partial_z$ which is what we wanted to show. □

Notice $\mathfrak{w}$ is in 1-1 correspondence with $\mathbb{C}[[z, z^{-1}]]$ as $v(z) \leftrightarrow v(z)\partial_z$ and so we may choose the basis $\{-z^{n+1}\partial_z \mid n \in \mathbb{Z}\}$ for $\mathfrak{w}$. It is easily seen that $\mathfrak{w}$ is non-commutative. Then with the basis chosen above, for fixed $m, n \in \mathbb{Z}$ and $v(z) \in \mathbb{C}[[z, z^{-1}]]$,

$$\begin{aligned}\left[-z^{i+1}\partial_z, -z^{j+1}\partial_z\right] v(z) &= \left(z^{i+1}\partial_z\right)\left(z^{j+1}\partial_z v(z)\right) - \left(z^{j+1}\partial_z\right)\left(z^{i+1}\partial_z v(z)\right) \\ &= z^{i+j+1}(j+1)\partial_z v(z) - z^{i+j+1}(i+1)\partial_z v(z) \\ &= (j-i) z^{i+j+1}\partial_z v(z)\end{aligned}$$

and so the Witt algebra has the structure of a Lie algebra with bracket

$$\left[-z^{i+1}\partial_z, -z^{j+1}\partial_z\right] = (j-i) z^{i+j+1}\partial_z. \tag{2.13}$$

The *Virasoro* Lie algebra $\mathfrak{v} = \mathfrak{w} \oplus \mathbb{C}\mathbf{k}$ is then defined as the unique central extension of $\mathfrak{w}$ (cf. Chapter 5 of [Sch08]), with a basis given by

$$\{\, \mathbf{k}, L_n \mid n \in \mathbb{Z}\,\}, \tag{2.14}$$

and bracket relations

$$\begin{aligned}&[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathbf{k}}{12}(m^3 - m)\delta_{m+n,0} \\ &[L_n, \mathbf{k}] = 0\end{aligned}$$

where $\mathbf{k}$ is a central element and $\delta_{m,n}$ is the Kronecker delta function. Comparing eq. (2.13) with the above relations, we have the representation $z^{n+1}\partial_z \mapsto -L_n$. Of course, specializing $\mathbf{k} = 0$ returns eq. (2.13). The Virasoro Lie algebra is significant in two-dimensional conformal field theory, and for our purposes, it is enough to know that it stems as the unique central extension of $\mathfrak{w}$. Finally, given a vector space V, define the set of *primary vectors of weight* n as

$$\mathcal{P}_n(V) = \{a \in V \mid L_i a = 0 \ \text{for} \ i \geq 1, \ L_0 a = na\}. \tag{2.15}$$

That is, $a \in \mathcal{P}_n(V)$ is a highest weight vector of weight n. Notice here how $L_0 \in \mathfrak{v}$ plays the special role of a gradation operator with integral eigenvalues n.

2.3 Vertex Operator Algebras

A *vertex operator algebra* is a quadruple $\{V, Y, \mathbf{1}, \omega\}$ consisting of a $\mathbb{Z}$-graded vector space V, the *Fock* space

$$V = \coprod_{n\in\mathbb{Z}} V_{(n)} \tag{2.16}$$

where we call vectors $v \in V$ *states* and say $v \in V_{(n)}$ if $\mathrm{wt}(v) = n$. These subspaces satisfy

$$\dim V_{(n)} < \infty \quad \text{for all } n \in \mathbb{Z} \tag{2.17}$$

$$V_{(n)} = 0 \qquad \text{for } n \text{ sufficiently negative,}. \tag{2.18}$$

There is also a linear map called the *vertex operator*

$$\begin{aligned} Y(\cdot, z) : & V \to \mathcal{F}(V) \\ & v \mapsto Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \end{aligned} \tag{2.19}$$

where the $v_n \in \mathrm{End}(V)$ are referred to as the *modes* of v. For $u, v \in V$, the *truncation condition* holds:

$$u_n v = 0 \qquad \text{for sufficiently large } n. \tag{2.20}$$

There are two distinguished states: The *vacuum state* $\mathbf{1} \in V_{(0)}$ satisfies the *vacuum property* and the *creation property*, respectively

$$Y(\mathbf{1}, z) = I_V \tag{2.21}$$

$$Y(v,z)\mathbf{1} \in V[[z]] \quad \text{and} \quad \lim_{z\to 0} Y(v,z)\mathbf{1} = v, \tag{2.22}$$

where I_V is the identity of V viewed as an element of $\mathrm{End}(V)$. The *conformal state* $\omega \in V_{(2)}$ spans a copy of the *Virasoro Lie algebra* $\mathfrak{v}$

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \quad \left(= \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1} \right) \tag{2.23}$$

i.e. $L_n = \omega_{n+1}$ where the $L_n \in \mathrm{End}(V)$ satisfy the Virasoro bracket relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c_V}{12}(m^3 - m)\delta_{m+n,0} \tag{2.24}$$

$$[L_n, c_V] = 0 \tag{2.25}$$

where $c_V \in \mathbb{C}$ is the *central charge* of V. The L_0*-eigenspace decomposition* of V coincides with the grading of V

$$L_0 v = nv = \mathrm{wt}(v)v \quad v \in V_{(n)} \tag{2.26}$$

and we also have the L_{-1}*-derivative property*

$$Y(L_{-1}v, z) = \partial_z Y(v,z) \quad \Longleftrightarrow \quad L_{-1}\mathbf{1} = 0. \tag{2.27}$$

Finally, the *Jacobi identity* holds: for $u, v \in V$ and formal variables z_0, z_1, z_2,

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1) \\ &= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \end{aligned} \tag{2.28}$$

The Jacobi identity is the most crucial expression in VOA theory, and as we will see in the next section, it can be expanded to illustrate an infinite list of intricate identities which the modes must satisfy. This completes the definition.

2.4 Consequences of the Definition

Here we provide and prove a selection of well-known identities, chosen for their relevance and use in future discussion. As a slight abuse of notation, we sometimes denote by the Fock space V the whole VOA $\{V, Y, \mathbf{1}, \omega\}$, though it will be made clear when V is simply a vector space.

Proposition 2.4.1. *For $a, b \in V$, we have*

$$[a_m, Y(b,z)] = \sum_{i\geq 0} \binom{m}{i} Y(a_i b, z) z^{m-i}.$$

Proof. We expand the three terms in the Jacobi identity as follows. Expanding the first term gives

$$\left(\sum_{i\geq 0}\sum_{l\in\mathbb{Z}} (-1)^i \binom{l}{i} z_0^{-l-1} z_1^{l-i} z_2^i\right)\left(\sum_{m\in\mathbb{Z}} a_m z_1^{-m-1}\right)\left(\sum_{n\in\mathbb{Z}} b_n z_2^{-n-1}\right)$$

and equating the coefficient of z_0^{-l-1} and z_1^{-m-1} yields

$$\sum_{i\geq 0} (-1)^i \binom{l}{i} a_{m+l-i} Y(b, z_2) z_2^{n+i}. \tag{2.29}$$

Likewise, expanding the second term of the Jacobi identity gives

$$-\left(\sum_{i\geq 0}\sum_{l\in\mathbb{Z}} (-1)^{l+i} \binom{l}{i} z_0^{-l-1} z_1^{i} z_2^{l-i}\right)\left(\sum_{n\in\mathbb{Z}} b_n z_2^{-n-1}\right)\left(\sum_{m\in\mathbb{Z}} a_m z_1^{-m-1}\right)$$

and equating the coefficient of z_0^{-l-1} and z_1^{-m-1} yields

$$-\sum_{i\geq 0} (-1)^{l+i} \binom{l}{i} Y(b, z_2) a_{m+i} z_2^{n+l-i}. \tag{2.30}$$

Finally, expanding the third term of the Jacobi identity and equating the coefficient of z_0^{-l-1} and z_1^{-m-1} yields

$$\sum_{i\geq 0} \binom{m}{i} Y(a_{l+i} b, z_2) z_2^{m+n-i}. \tag{2.31}$$

Putting together eqs. (2.29) to (2.31), the Jacobi identity is equivalent to

$$\begin{aligned}\sum_{i\geq 0} (-1)^i \binom{l}{i} a_{m+l-i} Y(b, z) z^{n+i} - \sum_{i\geq 0} (-1)^{l+i} \binom{l}{i} Y(b, z) a_{m+i} z^{n+l-i} \\ = \sum_{i\geq 0} \binom{m}{i} Y(a_{l+i} b, z) z^{m+n-i}.\end{aligned} \tag{2.32}$$

Note we have set $z_2 = z$, since we now deal with only one formal variable. Finally, set $l = 0$ and $n = 0$ in eq. (2.32) to obtain

$$[a_m, Y(b,z)] = \sum_{i\geq 0} \binom{m}{i} Y(a_i b, z) z^{m-i}.$$

□

Note that we may also take the coefficient of z^{-1} in eq. (2.32) to obtain the following expression purely in terms of the modes of a and b, one which is sometimes employed in the literature:

$$\begin{aligned}\sum_{i\geq 0} (-1)^i \binom{l}{i} a_{m+l-i} b_{n+i} - \sum_{i\geq 0} (-1)^{l+i} \binom{l}{i} b_{n+l-i} a_{m+i} \\ = \sum_{i\geq 0} \binom{m}{i} (a_{l+i} b)_{m+n-i}.\end{aligned} \tag{2.33}$$

The following corollary of Proposition 2.4.1 will be useful in Chapter 3:

Corollary 2.4.2. *For $a \in V_{(n)}$ and $L_i \in End(V)$ for $i \in \mathbb{Z}$, we have*

$$[L_i, Y(a,z)] = z^{i+1}\partial_z Y(a,z) + \sum_{j\geq 0} \frac{1}{(j+1)!}\left(\partial_z^{j+1} z^{i+1}\right) Y(L_j a, z).$$

Proof. We use Proposition 2.4.1, the L_{-1}-derivative property (eq. (2.27)), and $L_n = \omega_{n+1}$:

$$\begin{aligned}
[L_i, Y(a,z)] &= \sum_{j\geq 0} \binom{i+1}{j} Y(L_{j-1}a, z) z^{i+1-j} \\
&= \sum_{j\geq -1} \binom{i+1}{j+1} Y(L_j a, z) z^{i-j} \\
&= \sum_{j\geq -1} \frac{(i+1)!}{(i-j)!(j+1)!} z^{i-j} Y(L_j a, z) \\
&= \sum_{j\geq -1} \frac{1}{(j+1)!}\left(\partial_z^{j+1} z^{i+1}\right) Y(L_j a, z) \\
&= z^{i+1}\partial_z Y(a,z) + \sum_{j\geq 0} \frac{1}{(j+1)!}\left(\partial_z^{j+1} z^{i+1}\right) Y(L_j a, z).
\end{aligned}$$

□

By specializing Corollary 2.4.2 to $i = -1, 0$, we obtain the two important identities

$$[L_{-1}, Y(a,z)] = \partial_z Y(a,z) \tag{2.34}$$
$$[L_0, Y(a,z)] = z\partial_z Y(a,z) + nY(a,z). \tag{2.35}$$

Proposition 2.4.3. *Let $v \in V_{(k)}$ and $m \in \mathbb{Z}$. Then,*

$$v_n : V_{(m)} \to V_{(m+k-n-1)}.$$

Proof. Let $w \in V_{(m)}$ and consider the product

$$L_0 v_n w = ([L_0, v_n] + v_n L_0)\, w \tag{2.36}$$

Setting $l = 0$ in eq. (2.33) or taking the coefficient of z^{-n-1} in Proposition 2.4.1 gives the bracket

$$[a_m, b_n] = \sum_{i\geq 0} \binom{m}{i} (a_i b)_{m+n-i}. \tag{2.37}$$

Using this alongside the fact that $L_n = \omega_{n+1}$, eq. (2.36) becomes

$$\begin{aligned}
L_0 v_n w &= \left(\sum_{i\geq 0} \binom{1}{i} (\omega_i v)_{1+n-i} + v_n L_0\right) w \\
&= ((\omega_0 v)_{n+1} + (\omega_1 v)_n + v_n L_0)\, w \\
&= ((L_{-1} v)_{n+1} + (L_0 v)_n + v_n L_0)\, w. \qquad (2.38)
\end{aligned}$$

In order to simplify eq. (2.38), use the L_{-1}-derivative property (eq. (2.27)) to establish

$$\begin{aligned}
(L_{-1}v)_n &= \mathrm{Coeff}_{z^{-n-1}}\ Y(L_{-1}v, z) \\
&= \mathrm{Coeff}_{z^{-n-1}}\ \frac{d}{dz} Y(v,z) \\
&= \mathrm{Coeff}_{z^{-n-1}}\ \sum_{n\in\mathbb{Z}} (-n-1) v_n z^{-n-2} \\
&= \mathrm{Coeff}_{z^{-n-1}}\ \sum_{n\in\mathbb{Z}} -n v_{n-1} z^{-n-1} \\
&= -n v_{n-1}.
\end{aligned}$$

Using this derived relation on $n+1$,

$$(L_{-1}v)_{n+1} = -(n+1)v_n.$$

Now, since $v \in V_{(k)}$ and $w \in V_{(m)}$, eq. (2.38) becomes

$$\begin{aligned} L_0 v_n w &= \left(-(n+1)v_n + kv_n + v_n L_0\right) w \\ &= -nv_n w - v_n w + kv_n w + v_n L_0 w \\ &= -nv_n w - v_n w + kv_n w + mv_n w \\ &= (m+k-n-1)v_n w. \end{aligned}$$

Therefore by the L_0-eigenspace decomposition (eq. (2.26)), the above relation asserts that for homogeneous $v \in V_{(k)}$,

$$v_n : V_{(m)} \to V_{(m+k-n-1)}.$$

□

2.5 The Heisenberg VOA

We construct and outline perhaps the simplest non-trivial example of a VOA. It is useful to note that this process can be seen as a special case of the construction of a VOA from the representation theory of an affine Lie algebra. An overview of the Heisenberg can also be found in many texts such as [FLM88], [LL04], [MT10], [FBZ04], however for generalities on Lie algebras, including the construction of the universal enveloping algebra and its properties, we follow Chapter 9 of [Car05].

Let $\mathfrak{h}$ be a finite-dimensional vector space, viewed as an abelian Lie algebra i.e. $[\mathfrak{h}, \mathfrak{h}] = 0$. To $\mathfrak{h}$, we associate the (untwisted) affine Lie algebra

$$\widehat{\mathfrak{h}} = \left(\mathfrak{h} \otimes \mathbb{C}[t, t^{-1}]\right) \oplus \mathbb{C}\mathbf{k} \tag{2.39}$$

where $\mathbf{k}$ is a central vector, and with basis given by

$$\{\, h_1 \otimes t^n \mid \alpha \in \mathfrak{h},\ n \in \mathbb{Z} \,\} \cup \{\mathbf{k}\}$$

and bracket relations

$$[h_1 \otimes t^m, h_2 \otimes t^n] = m\delta_{m+n,0}\mathbf{k} \tag{2.40}$$

$$[\mathbf{k}, \widehat{\mathfrak{h}}] = 0, \tag{2.41}$$

for $h_1, h_2 \in \mathfrak{h}$ and $m, n \in \mathbb{Z}$. Once again, note the similarities with the *affinization* of a more general Lie algebra $\mathfrak{g}$ where the bracket relations are given in that case as

$$[a \otimes t^m, b \otimes t^n] = [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m+n,0}\mathbf{k} \tag{2.42}$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$ where $\langle \cdot, \cdot \rangle$ is the bilinear symmetric invariant (Killing) form. The vector space $\widehat{\mathfrak{h}}$, now viewed as an affine Lie algebra, is equipped with a clear $\mathbb{Z}$-grading structure:

$$\widehat{\mathfrak{h}} = \coprod_{n \in \mathbb{Z}} \widehat{\mathfrak{h}}_{(n)}$$

where we will denote the subspaces

$$\begin{aligned} \widehat{\mathfrak{h}}_{(n)} &= \mathfrak{h} \otimes t^{-n}, \quad n \neq 0 \\ \widehat{\mathfrak{h}}_{(0)} &= \mathfrak{h} \oplus \mathbf{k}. \end{aligned}$$

In view of eq. (2.40), there are two graded subalgebras which we write with respect to the notation given in [LL04] (by weights) as:

$$\widehat{\mathfrak{h}}^{+} = \mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}] \tag{2.43}$$

$$\widehat{\mathfrak{h}}^{-} = \mathfrak{h} \otimes t\mathbb{C}[t]. \tag{2.44}$$

In some texts, the notation above is inverted so that the sign in $\widehat{\mathfrak{h}}^{\pm}$ corresponds to the sign in $t^{\pm n}$, rather than the weight. Note the bracket (eq. (2.40)) vanishes when restricting to these two subspaces. We wish to embed $\widehat{\mathfrak{h}}$ into an associative polynomial algebra-like structure, while still respecting the Lie bracket relations (eqs. (2.40) and (2.41)). This is precisely the purpose of the *universal enveloping algebra* which is constructed as the quotient

$$\mathcal{U}(\widehat{\mathfrak{h}}) = T(\widehat{\mathfrak{h}})/I \tag{2.45}$$

where $T(\widehat{\mathfrak{h}})$ is the tensor algebra of $\widehat{\mathfrak{h}}$

$$T(\widehat{\mathfrak{h}}) = \bigoplus_{i\geq 1} \widehat{\mathfrak{h}}^{\otimes i} = \mathbb{C} \oplus \widehat{\mathfrak{h}} \oplus \left(\widehat{\mathfrak{h}} \otimes \widehat{\mathfrak{h}}\right) \oplus \left(\widehat{\mathfrak{h}} \otimes \widehat{\mathfrak{h}} \otimes \widehat{\mathfrak{h}}\right) \oplus \cdots$$

and I is the ideal

$$I = \langle h_1 \otimes h_2 - h_2 \otimes h_1 - [h_1, h_2]_{\widehat{\mathfrak{h}}} \rangle$$

for $h_1, h_2 \in \widehat{\mathfrak{h}}$. We will write $h_1 h_2$ for $h_1 \otimes h_2$ from now on. The *universal property* of $\mathcal{U}(\widehat{\mathfrak{h}})$ asserts that there is a 1-1 correspondence between representations of $\widehat{\mathfrak{h}}$ and $\mathcal{U}(\widehat{\mathfrak{h}})$-modules. As well, when passing from $\widehat{\mathfrak{h}}$ to $\mathcal{U}(\widehat{\mathfrak{h}})$, subalgebras are preserved via the canonical inclusion mapping. Now, we want to take $\mathcal{U}(\widehat{\mathfrak{h}})$ and turn it into a subalgebra of $\mathrm{End}(\mathcal{H})$ for some underlying (Fock) space $\mathcal{H}$. First we let

$$\widehat{\mathfrak{h}}_{\leq 0} = \widehat{\mathfrak{h}}^{-} \oplus \widehat{\mathfrak{h}}_{(0)},$$

and let $\mathbb{C}\mathbf{1}$ be the 1-dimensional $\widehat{\mathfrak{h}}_{\leq 0}$-module acting trivially as

$$\mathbf{k} \cdot \mathbf{1} = \mathbf{1} \tag{2.46}$$

$$\widehat{\mathfrak{h}}^{-} \cdot \mathbf{1} = 0. \tag{2.47}$$

Lift this to an action of $\mathcal{U}(\widehat{\mathfrak{h}}_{\leq 0})$ and construct the *induced module*

$$\mathcal{H} := \mathrm{Ind}_{\mathcal{U}(\widehat{\mathfrak{h}}_{\leq 0})}^{\mathcal{U}(\widehat{\mathfrak{h}})} \mathbb{C}\mathbf{1} = \mathcal{U}(\widehat{\mathfrak{h}}) \otimes_{\mathcal{U}(\widehat{\mathfrak{h}}_{\leq 0})} \mathbb{C}\mathbf{1}. \tag{2.48}$$

This gives an action of $\mathcal{U}(\widehat{\mathfrak{h}})$ on $\mathcal{H}$ given by left multiplication, and given a total ordering of $\widehat{\mathfrak{h}}$, elements of $\mathcal{U}(\widehat{\mathfrak{h}}_{\leq 0})$ "pass through" the tensor above and act on $\mathbb{C}\mathbf{1}$ in the previous manner given by eqs. (2.46) and (2.47). Denote henceforth by h_n the action of the element $h \otimes t^n \in \mathcal{U}(\widehat{\mathfrak{h}})$ on $\mathcal{H}$, for $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$. That is, let $h_n \in \mathrm{End}(\mathcal{H})$. Also identify the vacuum $\mathbf{1}$ with $1 \otimes \mathbf{1}$. Using the Poincare-Birkhoff-Witt theorem (cf. [Car05]) and eq. (2.47), elements $\mathcal{H}$ are of the form

$$h_{-n_1} \cdots h_{-n_r} \mathbf{1}$$

where $n_i > 0$, $r > 0$. This, along with the fact that $\widehat{\mathfrak{h}}^{+}$ is abelian, gives

$$\mathcal{H} = \mathcal{U}(\widehat{\mathfrak{h}}^{+}) = S(\widehat{\mathfrak{h}}^{+}), \tag{2.49}$$

and so we may think of $\mathcal{H}$ as the *commutative* algebra of polynomials in the variables $h_n = a \otimes t^n$ for $n < 0$. The Fock space $\mathcal{H}$ (which we will also refer to as the Heisenberg VOA itself) is equipped with a $\mathbb{Z}$-grading (by weights) given by

$$\mathrm{wt}(h_{-n_1} \cdots h_{-n_r} \mathbf{1}) = \sum_i n_i \tag{2.50}$$

From eq. (2.47), we see there are no elements of negative weight in $\mathcal{H}$, satisfying eq. (2.17). The h_n for $n > 0$ act on the elements of positive weight in the following way. Let $h_n, h_{-m_1}, h_{-m_2} \in \mathrm{End}(\mathcal{H})$ where $n, m_1, m_2 > 0$. We compute the following product, making use of the Lie bracket relations (eqs. (2.40) and (2.41)) and noting both eqs. (2.46) and (2.47):

$$\begin{aligned} h_n h_{-m_1} h_{-m_2}\mathbf{1} &= \left([h_n, h_{-m_1}] + h_{-m_1}h_n\right) h_{-m_2}\mathbf{1} \\ &= n\delta_{n-m_1,0}h_{-m_2}\mathbf{1} + h_{-m_1}\left([h_n, h_{-m_2}] + h_{-m_2}h_n\right)\mathbf{1} \\ &= n\delta_{n-m_1,0}h_{-m_2}\mathbf{1} + n\delta_{n-m_2,0}h_{-m_1}\mathbf{1}. \end{aligned}$$

From here we observe that

$$h_n h_{-m_1} h_{-m_2}\mathbf{1} = \begin{cases} m_1 h_{-m_2}\mathbf{1}, & n = m_1 \\ m_2 h_{-m_1}\mathbf{1}, & n = m_2 \end{cases}$$

and so the h_n for $n > 0$ act as operators $n\partial_{h_{-n}}$. Notice this implies that h_0 acts as 0 on $\mathcal{H}$. It is because of this that elements of negative weight are sometimes called *annihilation operators* whereas elements of positive weight which form a basis for $\mathcal{H}$ are called *creation operators.*

We construct the vertex operators on $\mathcal{H}$ following briefly the discussion given in Chapter 2.2 of [FBZ04]. First we assign $Y(\mathbf{1}, z) = I_V$, the identity element of $\mathcal{H}$. Define the *Heisenberg Field* as

$$h(z) = Y(h_{-1}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} h_n z^{-n-1} \in \mathrm{End}(\mathcal{H})[[z, z^{-1}]]. \tag{2.51}$$

This is no more than a generating function for the operators $h_n \in \mathrm{End}(\mathcal{H})$, and so the vertex operator maps for arbitrary states $h_{n_1}\cdots h_{n_r}\mathbf{1} \in \mathcal{H}$ will be given in terms of $h(z)$. Recall that $\mathcal{H}$ may be viewed as a polynomial ring in the variables h_{-n} for $n > 0$. With this analogy, we take $h_{-2}\mathbf{1} \sim t^{-2} = -\partial_t t^{-1}$. Thus

$$Y(h_{-2}\mathbf{1}, z) = \sum_{n\in\mathbb{Z}} (-n-1) h_n z^{-n-2}$$

and so inductively

$$Y(h_{-n}, z) = \frac{1}{(n-1)!}\partial_z^{n-1} h(z), \quad n > 0.$$

Vertex operators of more complex states in $\mathcal{H}$ require the *normally ordered product*

$$:A(z_1)B(z_2): = A(z_1)_+ B(z_2) + B(z_2)A(z_1)_- \tag{2.52}$$

for $A(z), B(z) \in \mathcal{F}(V)$ where, for $v(z) \in V[[z]]$, we have defined

$$v(z)_+ = \sum_{n\geq 0} v_n z^n, \quad v(z)_- = \sum_{n<0} v_n z^n.$$

This operation ensures that annihilation operators always appear to the right of creation operators. The normally ordered product of a field $A(z)$ with itself is given as

$$:A(z)^2: = \sum_{n\in\mathbb{Z}} \left(\sum_{k+l=n} :a_k a_l: \right) z^{-n-2}$$

where

$$:a_k a_l: = \begin{cases} a_l a_k, & l = -k,\ k \geq 0 \\ a_k a_l, & \text{otherwise.} \end{cases}$$

This only changes the coefficient of z^{-2} which becomes

$$\sum_{k\in\mathbb{Z}} {:}a_k a_{-k}{:} = 2\sum_{k\le -1} -k a_k \partial_{a_k}. \tag{2.53}$$

This allows us to construct vertex operators for arbitrary states in $\mathcal{H}$ as

$$Y(h_{n_1}h_{n_2}\cdots h_{n_k}, z) = \frac{1}{(-n_1-1)!\cdots(-n_k-1)!}\,{:}\partial_z^{-n_1-1}h(z)\cdots\partial_z^{-n_k-1}h(z){:} \tag{2.54}$$

for $n < 0$. In what remains of this section, we describe the Virasoro subalgebra $\mathfrak{v}$ in $\mathcal{H}$. Given $\lambda \in \mathbb{C}$, take

$$\omega_\lambda = \frac{1}{2}h_{-1}^2\mathbf{1} + \lambda h_{-2}\mathbf{1}$$

as the conformal state with central charge $c_V = 1 - 12\lambda^2$. This gives a family of representations of $\mathfrak{v}$ on $\mathcal{H}$, and though it is common to take $\lambda = 0$ to obtain a representation of central charge 1, we find it illustrative to keep λ as a variable. Then,

$$Y(\omega_\lambda, z) = \frac{1}{2}\,{:}h(z)^2{:}\, + \lambda\partial_z h(z) = \sum_{n\in\mathbb{Z}} L_{\lambda,n} z^{-n-2} \tag{2.55}$$

and so we must show that the modes $L_{\lambda,n}$ satisfy the Virasoro relations (eqs. (2.24) and (2.25)). This is equivalent to establishing the L_{-1}-derivative property (eq. (2.27)) and the L_0-eigenspace decomposition property (eq. (2.26)). First, using eq. (2.55),

$$\begin{aligned} L_{\lambda,n} &= \mathrm{Coeff}_{z^{-n-2}}\left(\frac{1}{2}\,{:}h(z)^2{:}\, + \lambda\partial_z h(z)\right) \\ &= \frac{1}{2}\sum_{m\in\mathbb{Z}} {:}h_m h_{n-m}{:}\, + (-n-1)\lambda h_n. \end{aligned} \tag{2.56}$$

Setting $n = -1$ in eq. (2.56), one obtains the operator equality

$$L_{\lambda,-1} = \frac{1}{2}\sum_{m\in\mathbb{Z}} {:}h_m h_{-m-1}{:} = \sum_{m\ge 0} h_{-m-1}h_m.$$

For $h_j \in \mathrm{End}(\mathcal{H})$ where $j \in \mathbb{Z}$,

$$[L_{\lambda,-1}, h_{-j}] = \left(\sum_{m\ge 0} h_{-m-1}h_m\right)h_j - h_j\left(\sum_{m\ge 0} h_{-m-1}h_m\right) = -j h_{j-1}.$$

Thus $[L_{\lambda,-1}, h(z)] = \partial_z h(z)$. In fact, $[L_{\lambda,-1}, \partial_z^n h(z)] = \partial_z^{n+1}h(z)$ and with the fact that the Leibniz rule holds for the normally ordered product, looking at eq. (2.54) we see that the L_{-1}-derivative property is satisfied. Also from eq. (2.56) at $n = 0$, using eq. (2.53) and the definition of the annihilation operators,

$$L_{\lambda,0} = \sum_{m\le -1} -m h_m \partial_{h_m} = \sum_{m\ge 1} h_{-m}h_m.$$

Applying this to $h_{-n_1}\cdots h_{-n_r}\mathbf{1} \in \mathcal{H}$ with $n_i > 0$ gives

$$L_{\lambda,0}\left(h_{-n_1}\cdots h_{-n_r}\mathbf{1}\right) = \sum_{m\ge 1} h_{-m}h_m\left(h_{-n_1}\cdots h_{-n_r}\mathbf{1}\right) = \left(\sum_{i=1}^{r} n_i\right)h_{-n_1}\cdots h_{-n_r}\mathbf{1},$$

thus the L_0-eigenspace decomposition property is satisfied.

The following lemma gives closed-form expressions for certain Virasoro modes of $\mathcal{H}$ of central charge 1. Though it does not appear in the literature, it may be known to experts. From eq. (2.56):

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} {:}h_m h_{n-m}{:}\ . \tag{2.57}$$

Lemma 2.5.1. *Fix $n \geq 0$ and let $a = h_{-k_1} \cdots h_{-k_r} \mathbf{1} \in \mathcal{H}$ where $c_V = 1$ and $k_i > 0$ for $1 \leq i \leq r$. If n is even, then*

$$L_n a = \sum_{i=1}^{r} \frac{k_i h_{n-k_i}}{h_{-k_i}} h_{-k_1} \cdots h_{-k_r} \mathbf{1} + \left(\sum_{m=1}^{n/2-1} h_m h_{n-m} + \frac{1}{2} h_{n/2}^2 \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1},$$

and if n is odd then

$$L_n a = \sum_{i=1}^{r} \frac{k_i h_{n-k_i}}{h_{-k_i}} h_{-k_1} \cdots h_{-k_r} \mathbf{1} + \left(\sum_{m=1}^{\lfloor n/2 \rfloor} h_m h_{n-m} \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1}.$$

Proof. Fix $n \geq 0$. We apply eq. (2.57) to a and break up the sum in cases. First when $m < 0$,

$$\frac{1}{2} \left(\sum_{m<0} h_m h_{n-m} \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1}.$$

Note $h_{n-m} h_{-k_i} = k_i$ if $n - m = k_i$ for each $1 \leq i \leq r$, and zero otherwise. In the former case, $h_m = h_{n-k_i}$, a creation operator, and so

$$\begin{aligned} \frac{1}{2} \left(\sum_{m \leq 0} h_m h_{n-m} \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1} &= \frac{1}{2} \left(k_1 h_{n-k_1} \cdots h_{-k_r} \mathbf{1} + \cdots + k_r h_{-k_1} \cdots h_{n-k_r} \mathbf{1} \right) \\ &= \frac{1}{2} \sum_{i=1}^{r} \frac{k_i h_{n-k_i}}{h_{-k_i}} h_{-k_1} \cdots h_{-k_r} \mathbf{1}. \end{aligned} \tag{2.58}$$

When $m = 0$, the operator h_0 annihilates a. When $m > n$, then

$$\frac{1}{2} \left(\sum_{m>n} h_{n-m} h_m \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1}.$$

Note again that $h_m h_{-k_i} = k_i$ if $m = k_i$ for $1 \leq i \leq r$, and zero otherwise. In the former case, $h_{n-m} = h_{n-k_i}$, a creation operator, and so

$$\begin{aligned} \frac{1}{2} \left(\sum_{m>n} h_{n-m} h_m \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1} &= \frac{1}{2} \left(k_1 h_{n-k_1} \cdots h_{-k_r} \mathbf{1} + \cdots + k_r h_{-k_1} \cdots h_{n-k_r} \mathbf{1} \right) \\ &= \frac{1}{2} \sum_{i=1}^{r} \frac{k_i h_{n-k_i}}{h_{-k_i}} h_{-k_1} \cdots h_{-k_r} \mathbf{1}, \end{aligned} \tag{2.59}$$

which is the same as the case $m < 0$. Finally, if $0 \leq m \leq n$ then both h_m and h_{n-m} are annihilation operators. Hence $k_i = n$ and $k_j = m - n$ in a for at least one i and j, or else a vanishes here. Notice that as m ranges from 0 to n, we get both $h_m h_{n-m}$ and $h_{n-m} h_m$, and these operators commute. If $m = n = 0$ then this particular sum vanishes entirely since we get h_0. Thus if n is even, we get

$$\frac{1}{2} \left(\sum_{m=0}^{n} h_m h_{n-m} \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1} = \left(\sum_{m=1}^{n/2-1} h_m h_{n-m} + \frac{1}{2} h_{n/2}^2 \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1}, \tag{2.60}$$

and if n is odd we get

$$\frac{1}{2} \left(\sum_{m=0}^{n} h_m h_{n-m} \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1} = \left(\sum_{m=1}^{\lfloor n/2 \rfloor} h_m h_{n-m} \right) h_{-k_1} \cdots h_{-k_r} \mathbf{1}. \tag{2.61}$$

By combining eqs. (2.58) to (2.61) we obtain the desired formulas. ☐

Chapter 3

The Coordinate Transformation

This chapter follows both Section 4.2 of [Zhu96] and Chapter 5 of [FBZ04]. Denote by $\mathcal{O}$ the set $\mathbb{C}[[z]]$ which we regard as the formal coordinate ring of the affine plane at the origin. Define a *coordinate transformation* $\phi \in \mathrm{Aut}(\mathcal{O})$ as a formal power series or holomorphic map about $0 \in \mathbb{C}$

$$\phi(z) = \sum_{n=1}^{\infty} k_n z^n \in \mathbb{C}[[z]], \quad k_1 \neq 0. \tag{3.1}$$

Indeed, every element in $\mathrm{Aut}(\mathcal{O})$ can be represented in the way above. The constant term being equal to zero ensures the origin is mapped to itself, however it gives the composition action for $\mathrm{Aut}(\mathcal{O})$, making it a infinite-dimensional Lie group. In order to illustrate this, we briefly recall the discussion on pages 42-44 of [Lan99].

Let $f, g \in V[[z]]$ (recall eq. (2.1)) where $f = \sum a_n z^n$ and $g = \sum a_n z^n$. The power series f and g are equal if $f \equiv g \mod z^r$ for all $r \geq 1$, i.e. if $a_n = b_n$ for all $0 \leq n \leq r-1$. Then, for both f and g, there are unique polynomials $P(z)$ and $Q(z)$ such that $f \equiv P \mod z^r$ and $g \equiv Q \mod z^r$. In particular,

$$\begin{aligned} P(z) &= a_1 z + a_2 z^2 + \cdots + a_{r-1} z^{r-1} \\ Q(z) &= b_1 z + b_2 z^2 + \cdots + b_{r-1} z^{r-1}. \end{aligned}$$

Proposition 3.0.1. *Let $f_1, f_2, g_1, g_2 \in V[[z]]$ where $f_1 \equiv f_2 \mod z^r$ and $g_1 \equiv g_2 \mod z^r$, and suppose that g_1, g_2 have zero constant term. Then, $f_1 \circ g_1 \equiv f_2 \circ g_2 \mod z^r$ and so the composition of such power series is a well-defined operation.*

Proof. Let P_1, P_2 be the polynomials of degree $r-1$ such that $f_1 \equiv P_1 \mod z^r$ and $f_2 \equiv P_2 \mod z^r$. Clearly then, $P_1 = P_2 = P$, the same polynomial. Furthermore, for arbitrary $g \in V[[z]]$ with zero constant term,

$$f_1(g) \equiv P_1(g) = P_2(g) \equiv f_2(g) \mod z^r. \tag{3.2}$$

The assertion that g has a zero constant term is used in the above line. If g had non-zero constant term, then as quotients of a higher power of z are taken, the constant terms from g would contribute to the constant term of f_1, yielding a different constant term for $f_1(g)$ at each subsequent quotient. The same is for $f_2(g)$. Let $Q(z)$ be the polynomial of degree $r-1$ such that $Q(z) \equiv g_1(z) \equiv g_2(z) \mod z^r$. Then

$$\begin{aligned} P(g_1) &= a_0 + a_1 g_1 + \cdots + a_{r-1} g_1^{r-1} \\ &\equiv a_0 + a_1 Q + \cdots + a_{r-1} Q^{r-1} \mod z^r \\ &\equiv a_0 + a_1 g_2 \cdots + a_{r-1} g_2^{r-1} \mod z^r \\ &\equiv P(g_2) \mod z^r. \end{aligned}$$

Combining this result with eq. (3.2) gives $f_1 \circ g_1 \equiv f_2 \circ g_2 \mod z^r$ and so the composition of such series is indeed a well-defined operation.

□

Let V be a vertex operator algebra. Given such a $\phi \in \mathrm{Aut}(\mathcal{O})$, it is natural to transform the variable $z \mapsto \phi(z)$ to obtain a new vertex operator $Y(a, \phi(z))$ and hence a new VOA structure entirely. In order to find an expression for $Y(a, \phi(z))$, we relate it to $Y(T_\phi(a), z)$ for some action

$$\begin{aligned} \mathrm{Aut}(\mathcal{O}) \times V &\to V \\ (\phi(z), v) &\mapsto T_\phi(a). \end{aligned}$$

That is, we wish to "rescale" the elements of V according to the action of ϕ, via the use of some operator T_ϕ associated to ϕ. Following [Zhu96], we do this by writing ϕ as the exponent of elements $z^{i+1}\partial_z \in z\mathbb{C}[[z]]\partial_z$; the Lie algebra coupled to $\mathrm{Aut}(\mathcal{O})$. In particular, we wish to write

$$\phi(z) = \exp\left(\sum_{i\geq 0} l_i z^{i+1}\partial_z\right) z. \tag{3.3}$$

Once we have done this, using both the representation $z^{i+1}\partial_z \mapsto -L_i$ (cf. Section 2.2) and the fact that the L_i are endomorphisms of V, we construct in Section 3.2 the $\mathrm{Aut}(\mathcal{O})$-action on V in the form of the operator $T_\phi : V \to V$ (cf. eq. (3.21)).

Of course, if this technique is to be employed, we must show that the writing of ϕ as in eq. (3.3) is a well-defined procedure, something which is not given in [Zhu96]. This is given in Proposition 2.1.1 of [Hua97], however we provide an alternative proof (Lemma 3.1.3) in Section 3.1 below. This new method yields expressions which will prove useful in establishing the coordinate invariance for the Virasoro field in Section 3.4. Then, Sections 3.2 to 3.4 are concerned with the operator T_ϕ and each conclude with expressions for the coordinate invariance of certain vertex operators.

3.1 Zhu's Equation

Here we establish that eq. (3.1) may be written as eq. (3.3). Looking at the latter, since $l_i z^{i+1}\partial_z$ are elements of the Witt algebra acting here on the space $z\mathbb{C}[[z]]$, we have the grading

$$z\mathbb{C}[[z]] = \coprod_{j\geq 1} \mathbb{C}z^j$$

and so for $i \geq 0$ and fixed $j \geq 1$, it is easy to see that

$$l_i z^{i+1}\partial_z : \mathbb{C}z^j \to \mathbb{C}z^{j+i}. \tag{3.4}$$

In order to come to eq. (3.3), we establish relations between the coefficients l_i and k_i by viewing both eq. (3.1) and eq. (3.3) over the quotient ring

$$Q_r = z\mathbb{C}[[z]]/\langle z^r \rangle \tag{3.5}$$

for $r \geq 2$. We then extract the coefficient of z^{r-1} and obtain our relations. These obtained relations (eqs. (3.6), (3.13) and (3.14)) will prove subsequently useful when establishing the coordinate invariance of the Virasoro field in Section 3.4. Because of eq. (3.4), working over Q_r ensures the operators $l_i z^{i+1}\partial_z$ are locally nilpotent for large enough i. The same is true for high enough powers of these operators.

The relation between l_0 and k_1 is established first. From eq. (3.1), k_1 is the coefficient of z in ϕ

and so we work over Q_2 and look at

$$\begin{aligned}\exp\left(\sum_{i\geq 0} l_i z^{i+1}\partial_z\right) z &\equiv \exp(l_0 z\partial_z)z \mod z^2 \\ &= \sum_{n\geq 0}\frac{1}{n!}(l_0 z\partial_z)^n(z) \\ &= \sum_{n\geq 0}\frac{1}{n!} l_0^n z \\ &= e^{l_0}z,\end{aligned}$$

and so in terms of k_1 and l_0 respectively, we have

$$k_1 = e^{l_0} \tag{3.6}$$

$$l_0 = \log k_1. \tag{3.7}$$

Notice l_0 is uniquely determined by k_1 by choosing the principal branch $0 \leq \mathrm{im}(l_0) \leq 2\pi$ of the complex logarithm. This assumption is stated explicitly in [Zhu96] and we assume it henceforth.

We will continue to derive relations such as eq. (3.6) in the same way, however the case where $k_1 = 1$ must be treated separately beforehand. In this case, eq. (3.7) indicates that $l_0 = 0$ and so eq. (3.3) reduces to

$$\begin{aligned}\phi(z) &= \exp\left(\sum_{i\geq 1} l_i z^{i+1}\partial_z\right) z \\ &= z + \left(\sum_{i\geq 1} l_i z^{i+1}\right) + \frac{1}{2}\left(\sum_{i\geq 1} l_i z^{i+1}\partial_z\right)\cdot\left(\sum_{i\geq 1} l_i z^{i+1}\right) + \cdots \end{aligned}\tag{3.8}$$

and so we may directly obtain relations between the l_i and k_i by equating coefficients:

$$k_1 = 1 \tag{3.9}$$

$$k_2 = l_1 \tag{3.10}$$

$$k_3 = l_2 + l_1^2 \tag{3.11}$$

$$\vdots$$

Recall briefly that an m-composition of n is an ordered m-tuple $(n_1, \ldots, n_m)$ with $n_i \geq 1$ for each i and $\sum_i n_i = n$. A detailed overview of integer compositions as well counting such compositions can be found in [HM04]. For example, the 2-compositions of 4 are

$$(1,3),\ (3,1),\ (2,2).$$

The following proposition gives a closed-form expression for these relations. Though it may be known to experts, since it is not given in either [Zhu96] or [FBZ04] it is likely new.

Proposition 3.1.1. *Set $n \geq 1$. Let $l_{(n_1,n_2,\ldots,n_m)} = l_{n_1}l_{n_2}\cdots l_{n_m}$ where $(n_1, n_2, \ldots, n_m)$ is an m-composition of n, and let $\mathcal{C}(n,m)$ be the set of all such m-compositions of n. Then with $n_0 = 1$,*

$$k_{n+1} = \sum_{m\geq 1}\frac{1}{m!}\sum_{(n_1,n_2,\ldots,n_m)\in\mathcal{C}(n,m)}\left(\prod_{i=1}^{m}\sum_{j=0}^{i-1} n_j\right) l_{(n_1,n_2,\ldots,n_m)}.$$

Proof. Fix n, and consider the m-th term in the series expansion eq. (3.8):

$$\frac{1}{m!}\left(\sum_{i\geq 1} l_i z^{i+1}\partial_z\right)^m z.$$

For each $m \geq 1$, we aim to find the coefficient of z^{n+1}. For integers $n_1, \dots, n_m$ all at least 1 and $n_0 = 1$, the functional composition of m of these operators applied to z is

$$l_{n_m} z^{n_m+1} \left(\cdots \left(l_{n_1} z^{n_1+1} \partial_z z \right) \right) = \left(\prod_{i=1}^{m} \sum_{j=0}^{i-1} n_j \right) l_{n_1} \cdots l_{n_m} z^{n_1+\cdots n_m+1}. \tag{3.12}$$

For k_{n+1}, we aim to find the coefficient of z^{n+1} in the m-th term for each $m \geq 1$. From eq. (3.12), the number of ways to construct a term of degree $n+1$ from m operators corresponds to the number of m-compositions $(n_1, n_2, \dots, n_m)$ of n. Thus we sum over the set $\mathcal{C}(n,m)$ of all such compositions for each m. Note $\mathcal{C}(n,m) = 0$ for $m > n$. Finally, the sum is taken over all $m \geq 1$ and the factorial term comes from the exponential in eq. (3.8). □

Note that $|\mathcal{C}(n,k)| = \binom{n-1}{k-1}$ and so the second sum in Proposition 3.1.1 consists of this many terms. To prove this, write

$$\underbrace{1 \square 1 \square \cdots \square 1}_{n}$$

and note that out of these $n-1$ boxes, by choosing $k-1$ boxes to write a comma and by adding a "+" in the remaining boxes, we obtain a k-composition of n.

For the rest of the section, assume that $k_1 \neq 1$ (or equivalently that $l_0 \neq 0$). We establish the relation between k_2 and l_1, l_0 by working over Q_3:

$$\begin{aligned} \exp\left(\sum_{i \geq 0} l_i z^{i+1} \partial_z \right) z &\equiv \exp\left(l_0 z \partial_z + l_1 z^2 \partial_z \right) z \mod z^3 \\ &= \sum_{n \geq 0} \frac{1}{n!} \left(l_0 z \partial_z + l_1 z^2 \partial_z \right)^n z. \end{aligned}$$

It can be shown inductively for $n \geq 0$ that

$$(l_0 z \partial_z + l_1 z^2 \partial_z)^n z \equiv l_0^n z + (2^n - 1) l_0^{n-1} l_1 z^2 \mod z^3,$$

Using this,

$$\begin{aligned} \mathrm{Coeff}_{z^2} \exp\left(\sum_{i \geq 0} l_i z^{i+1} \partial_z \right) z &\equiv \sum_{n \geq 0} \frac{1}{n!} (2^n - 1) l_0^{n-1} l_1 \mod z^3 \\ &= \sum_{n \geq 0} \frac{2^n}{n!} \frac{l_0^n}{l_0} l_1 - \sum_{n \geq 0} \frac{1}{n!} \frac{l_0^n}{l_0} l_1 \\ &= e^{2l_0} \frac{l_1}{l_0} - e^{l_0} \frac{l_1}{l_0}. \end{aligned}$$

Thus we have the relation

$$k_2 = e^{2l_0} \frac{l_1}{l_0} - e^{l_0} \frac{l_1}{l_0}. \tag{3.13}$$

It should now be clear as to why the case $l_0 = 0$ was treated separately. Of course setting $l_0 = 0$ in eq. (3.13) is an invalid operation, but notice by rearranging eq. (3.13) and taking the limit as $l_0 \to 0$ gives

$$\lim_{l_0 \to 0} \left(\frac{l_1}{l_0} \left(e^{2l_0} - e^{l_0} \right) \right) = l_1 \lim_{l_0 \to 0} \left(2e^{l_0} - e^{l_0} \right) = l_1$$

which coincides precisely with eq. (3.10) which is the case when we had let $l_0 = 0$ from the outset. We give one further relation, namely between k_3 and l_2, l_1, l_0. Working over Q_4 now:

$$\begin{aligned}\exp\left(\sum_{i\geq 0} l_i z^{i+1}\partial_z\right) z &\equiv \exp\left(l_0 z\partial_z + l_1 z^2\partial_z + l_2 z^3\partial_z\right) z \mod z^4 \\ &= \sum_{n\geq 0} \frac{1}{n!}\left(l_0 z\partial_z + l_1 z^2\partial_z + l_2 z^3\partial_z\right)^n z.\end{aligned}$$

Again, it can be shown inductively that $\left(l_0 z\partial_z + l_1 z^2\partial_z + l_2 z^3\partial_z\right)^n z$ for $n \geq 0$ is equivalent to

$$l_0^n z + (2^n - 1)l_0^{n-1} l_1 z^2 + \left((3^n - 2^{n+1} + 1)l_0^{n-2} l_1^2 + \left(\frac{3^n - 1}{2}\right) l_0^{n-1} l_2\right) z^3 \mod z^4.$$

Then, by extracting the appropriate coefficient,

$$\begin{aligned}\mathrm{Coeff}_{z^3} \exp\left(\sum_{i\geq 0} l_i z^{i+1}\partial_z\right) z &\equiv \sum_{n\geq 0} \frac{1}{n!}\left((3^n - 2^{n+1} + 1)l_0^{n-2} l_1^2 + \left(\frac{3^n - 1}{2}\right) l_0^{n-1} l_2\right) \mod z^4 \\ &= e^{3l_0}\frac{l_1^2}{l_0^2} - e^{2l_0}\frac{2l_1^2}{l_0^2} + e^{l_0}\frac{l_1^2}{l_0^2} + e^{3l_0}\frac{l_2}{2l_0} - e^{l_0}\frac{l_2}{2l_0} \\ &= e^{3l_0}\left(\frac{l_1^2}{l_0^2} + \frac{l_2}{2l_0}\right) - e^{2l_0}\frac{2l_1^2}{l_0^2} + e^{l_0}\left(\frac{l_1^2}{l_0^2} - \frac{l_2}{2l_0}\right).\end{aligned}$$

Thus we have the relation

$$k_3 = e^{3l_0}\left(\frac{l_1^2}{l_0^2} + \frac{l_2}{2l_0}\right) - e^{2l_0}\frac{2l_1^2}{l_0^2} + e^{l_0}\left(\frac{l_1^2}{l_0^2} - \frac{l_2}{2l_0}\right). \tag{3.14}$$

Notice as with the previous relation that by rearranging eq. (3.14) and taking the limit as $l_0 \to 0$, one obtains

$$\lim_{l_0\to 0} \frac{l_1^2\left(9e^{3l_0} - 8e^{2l_0} + e^{l_0}\right)}{2} + \lim_{l_0\to 0} \frac{l_2\left(3e^{3l_0} - e^{l_0}\right)}{2} = l_1^2 + l_2$$

which again coincides precisely with eq. (3.11) which was the case where $l_0 = 0$.

The computations necessary to establish relations between k_r and l_i for $r \geq 4$ will henceforth always involve deriving formulae for expressions of the form

$$(l_0 z\partial_z + l_1 z^2\partial_z + \cdots + l_{r-1} z^r\partial_z)^n z \mod z^{r+1}$$

for $n \geq 0$. This is a tricky combinatorial problem, where it is difficult to determine emerging patterns without the aid of computer software. We conjecture that for $n \geq 0$, the full expression is of the form

$$\mathrm{Coeff}_{z^r} (l_0 z\partial_z + l_1 z^2\partial_z + \cdots + l_{r-1} z^r\partial_z)^n z \equiv \cdots + \frac{r^n - 1}{r - 1} l_0^{n-1} l_{r-1} \mod z^{r+1}$$

where the expression on the right hand side involves r terms.

We now implement more Lie-theoretic results in order to establish the main result of this section. First, if X and Y are *commuting* elements of the space $z(\mathrm{End}\, V)[[z]]$ for some vector space V, then one may write

$$e^{X+Y} = e^X e^Y. \tag{3.15}$$

The goal is to deconstruct eq. (3.3) using the above property. We have seen that the $l_i z^{i+1}\partial_z \in z\mathbb{C}[[z]]\partial_z$ form a representation of the Virasoro Lie algebra, and the bracket here is given by

$$[l_i z^{i+1}\partial_z, l_j z^{j+1}\partial_z] = (j-i)l_i l_j z^{i+j+1}\partial_z. \tag{3.16}$$

Clearly the operators $l_i z^{i+1}\partial_z$ do not commute. We remedy this by working over Q_r as before. In this case, two operators $l_i z^{i+1}\partial_z$ and $l_j z^{j+1}\partial_z$ commute if $i+j+1 \geq r$ by eq. (3.16). Not all operators will commute, however. We require an extension of eq. (3.15), which is given in the form of the *Zassenhaus formula*

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]} e^{\frac{1}{6}(2[Y,[X,Y]]+[X,[X,Y]])} e^{-\frac{1}{24}([X,[X,[X,Y]]]+3[[[X,Y],X],Y]+\cdots}$$

where X and Y are elements of a Lie algebra, and the exponent consists of Lie monomials of higher and higher degree. A broader discussion on the Zassenhaus formula can be found in [CMN12]. Directly applying this formula will not be very useful. With our current assumptions in place however, there are significant simplifications to be made to the Zassenhaus formula. First, taking eq. (3.3) over Q_r gives

$$\phi(z) = \exp\left(\sum_{i\geq 0} l_i z^{i+1}\partial_z\right) z \equiv \exp\left(\sum_{i=0}^{r-2} l_i z^{i+1}\partial_z\right) z \mod z^r,$$

and so define the terms

$$X = \sum_{i=0}^{r-3} l_i z^{i+1}\partial_z, \qquad Y = l_{r-2} z^{r-1}\partial_z.$$

Since the $l_1 z^2\partial_z, \ldots, l_{r-3} z^{r-2}\partial_z$ commute with $l_{r-2}z^{r-1}\partial_z$, by eq. (3.16),

$$[X,Y] = [l_0 z\partial_z, l_{r-2}z^{r-1}\partial_z] = (r-2)l_0 l_{r-2} z^{r-1}\frac{d}{dz} = (r-2)l_0 Y. \tag{3.17}$$

Here is a useful observation. If we assign ~ to mean "scalar multiple modulo z^r", then by eq. (3.17)

$$[Y,[X,[X,Y]]] \sim [Y,[X,Y]] \sim [Y,Y] = 0.$$

In fact, is is not difficult to see that any Lie monomial as above containing more than one Y vanishes when working over Q_r.

This observation is applied to the Zassenhaus formula. In [CMN12], the following recursive equation is given: Define

$$e^{X+Y} = e^X e^Y e^{C_2(X,Y)} e^{C_3(X,Y)} \cdots$$

where $C_n(X,Y)$ is a homogeneous Lie polynomial of degree n in arbitrary X and Y. Then for $n \geq 1$, we have

$$C_{n+1}(X,Y) = \frac{1}{n+1} \sum_{(i_0,i_1,\ldots,i_n)\in\mathcal{I}_n} \frac{(-1)^{i_0+i_1+\cdots+i_n}}{i_0! i_1! \cdots i_n!} \mathrm{ad}_{C_n}^{i_n} \cdots \mathrm{ad}_{C_2}^{i_2} \mathrm{ad}_Y^{i_1} \mathrm{ad}_X^{i_0} Y,$$

where $\mathrm{ad}_A^j B = [A, \mathrm{ad}_A^{j-1} B]$ denotes the adjoint representation ($\mathrm{ad}_A^0 B$ is to represent B) and

$$\mathcal{I}_n = \{(i_0, i_1, \ldots, i_n) \in \mathbb{N}^{n+1} \mid i_0 + i_1 + 2i_2 + \cdots + ni_n = n,\ i_0 \geq 1\}.$$

Notice that for any $n \geq 1$, there is a unique tuple of the form

$$(n, 0, \ldots, 0) \in \mathcal{I}_n$$

i.e. there exists a unique tuple with $i_1 = i_2 = \cdots = i_n = 0$. Thus for any $n \geq 1$, within the expression for C_{n+1}, there is always precisely one term of the form

$$\frac{1}{n+1}\left(\frac{(-1)^n}{n!}\mathrm{ad}_X^n Y\right) = \frac{(-1)^n}{(n+1)!}\underbrace{[X,[X,\ldots[X,Y]]]}_{n}$$

and all other terms involve a Lie monomials containing more than one Y. Therefore over Q_r and with our X and Y defined as before, the Zassenhaus formula reduces to

$$e^{X+Y} \equiv e^X e^Y e^{-\frac{1}{2}[X,Y]} e^{\frac{1}{6}[X,[X,Y]]} e^{-\frac{1}{24}[X,[X,[X,Y]]]} \cdots \mod z^r. \tag{3.18}$$

Proposition 3.1.2. *Let X and Y be defined as above. Then for $k \geq 1$,*

$$\underbrace{[X,[X,\ldots[X,Y]]]}_{k} \equiv ((r-2)l_0)^k Y \mod z^r$$

where the Lie monomial on the left hand side is of degree k.

Proof. We proceed by induction on k. The case $k = 1$ has been established in eq. (3.17). For $k = 2$,

$$\begin{aligned}
[X,[X,Y]] &= \left[\sum_{i=0}^{r-3} l_i z^{i+1}\partial_z,\ (r-2)l_0 l_{r-2} z^{r-1}\partial_z\right] \\
&= \left[l_0 z\partial_z,\ (r-2)l_0 l_{r-2} z^{r-1}\partial_z\right] + \cdots + \left[l_{r-3} z^{r-2}\partial_z,\ (r-2)l_0 l_{r-2} z^{r-1}\partial_z\right] \\
&= (r-2)l_0[l_0 z\partial_z, l_{r-2} z^{r-1}\partial_z] + \cdots + (r-2)l_0[l_{r-3} z^{r-2}\partial_z, l_{r-2} z^{r-1}\partial_z] \\
&\equiv ((r-2)l_0)^2 Y \mod z^r.
\end{aligned}$$

Suppose the result holds for $k = n$. Then

$$\begin{aligned}
\underbrace{[X,[X,\ldots[X,Y]]]}_{n+1} &\equiv [X, ((r-2)l_0)^n Y] \mod z^r \\
&= ((r-2)l_0)^n [X,Y] \\
&\equiv ((r-2)l_0)^{n+1} Y \mod z^r,
\end{aligned}$$

where we have used eq. (3.17) in the last line. □

Using Proposition 3.1.2 allows us to write eq. (3.18) as

$$\begin{aligned}
e^X e^Y &\equiv e^X e^Y e^{-\frac{1}{2}(r-2)l_0 Y} e^{\frac{1}{6}((r-2)l_0)^2 Y} e^{-\frac{1}{24}((r-2)l_0)^3 Y} \cdots \mod z^r \\
&= e^X \cdot \exp\left(\frac{1-e^{-(r-2)l_0}}{(r-2)l_0} Y\right).
\end{aligned}$$

Applying this to eq. (3.3) $\mod z^r$, we obtain

$$\begin{aligned}
\phi(z) &\equiv \exp\left(\sum_{i=0}^{r-3} l_i z^{i+1}\partial_z\right) \cdot \exp\left(\frac{1-e^{-(r-2)l_0}}{(r-2)l_0} l_{r-2} z^{r-1}\partial_z\right) z \mod z^r \\
&\equiv \exp\left(\sum_{i=0}^{r-3} l_i z^{i+1}\partial_z\right)\left(z + \frac{1-e^{-(r-2)l_0}}{(r-2)l_0} l_{r-2} z^{r-1}\right) \mod z^r \\
&\equiv \exp\left(\sum_{i=0}^{r-3} l_i z^{i+1}\partial_z\right) z + \exp(l_0 z\partial_z)\left(\frac{1-e^{-(r-2)l_0}}{(r-2)l_0} l_{r-2} z^{r-1}\right) \mod z^r \\
&\equiv \exp\left(\sum_{i=0}^{r-3} l_i z^{i+1}\partial_z\right) z + \left(e^{(r-1)l_0}\left(\frac{l_{r-2}}{(r-2)l_0}\right) - e^{l_0}\left(\frac{l_{r-2}}{(r-2)l_0}\right)\right) z^{r-1} \mod z^r.
\end{aligned} \tag{3.19}$$

With eq. (3.19), we are ready to state the main result of this section which has been implicitly used in [Zhu96].

Lemma 3.1.3. *For each $r \geq 1$, there is a unique expression for k_r, written in terms of $l_0, \ldots, l_{r-1}$ where $e^{l_0} = k_1$ with $0 \leq \operatorname{im}(l_0) \leq 2\pi$.*

Proof. Consider the case where $l_0 = 0$. From Proposition 3.1.1, the terms in the expression for k_r come from the m-compositions of $r-1$ for $m \geq 1$. These are unique for each r, and so the expressions for k_r are unique for each r.

Now consider the case where $l_0 \neq 0$. Notice that the rightmost term in eq. (3.19) consists of some, *but not all*, of the coefficients of z^{r-1}. This can be seen by comparison with the relations eqs. (3.6), (3.13) and (3.14). Nevertheless we have extracted the l_{r-2} terms from eq. (3.3), with the leftmost term in eq. (3.19) consisting of l_i for $0 \leq i \leq r-3$. Also from eq. (3.19) we see that for $r \geq 2$ and after having equated coefficients with eq. (3.1), each expression for k_{r-1} is written precisely in terms of $l_0, \ldots, l_{r-2}$. In other words, no further l_i occur within the expression. We can also see that within the expression for k_{r-1}, an l_{r-2} *must* appear. Thus each expression for k_r is unique. □

The first three relations eqs. (3.6), (3.13) and (3.14) are:

$$\begin{aligned}
k_1 &= e^{l_0} \\
k_2 &= e^{2l_0}\frac{l_1}{l_0} - e^{l_0}\frac{l_1}{l_0} \\
k_3 &= e^{3l_0}\left(\frac{l_1^2}{l_0^2} + \frac{l_2}{2l_0}\right) - e^{2l_0}\frac{2l_1^2}{l_0^2} + e^{l_0}\left(\frac{l_1^2}{l_0^2} - \frac{l_2}{2l_0}\right).
\end{aligned}$$

We prove the following fact which is a more precise statement about the rationality of the l_i than that given in Proposition 2.1.1 of [Hua97].

Corollary 3.1.4. *Let $\phi(z) = \sum_{i=1}^{\infty} k_i z^i \in \mathbb{Q}[[z]]$. Then, $l_r \in \mathbb{Q}[l_0, e^{l_0}]$ for all $r \geq 1$.*

Proof. Suppose $l_0 = 0$. It is clear that by re-arranging the expressions defined in Proposition 3.1.1, we obtain expressions for the l_r which are made up of at most rational coefficients.

Now let $l_0 \neq 0$. Since eqs. (3.13) and (3.14), for example, involve terms l_0^i and e^{jl_0} for $i, j \in \mathbb{Z}$ with $j \geq 1$, so do all expressions k_r for $r \geq 2$. We can re-arrange these to get expressions for the l_{r-1} which still contain these terms. If $k_1 = e^{l_0} \in \mathbb{Q}$, then we still have terms l_0^i in the expression, and these are not integral unless $l_0 = 0$. Thus in this case, the expression for l_{r-1} lies in $\mathbb{Q}[l_0]$. Likewise if $l_0 \in \mathbb{Q}$ or even $\mathbb{Q}[i]$, then we have terms e^{jl_0} which are not integral unless $l_0 = 0$. Thus in this case, the expression for l_{r-1} lies in $\mathbb{Q}[e^{l_0}]$. □

We may re-write the relations eqs. (3.6), (3.13) and (3.14) to get expressions for the l_{r-1} in order to illustrate the proof of Corollary 3.1.4:

$$\begin{aligned}
l_0 &= \ln k_1 \\
l_1 &= \frac{l_0 k_2}{e^{2l_0} - e^{l_0}} \\
l_2 &= \frac{-2l_1^2 e^{3l_0} + 4l_1^2 e^{2l_0} - 2l_1^2 e^{l_0} + 2l_0^2 k_3}{l_0 e^{3l_0} - l_0 e^{l_0}}.
\end{aligned}$$

The expression for l_2 above can be written in terms of l_0^i and e^{jl_0} for integers $i \in \mathbb{Z}$ and $j \geq 1$ by inserting the expression for l_1 accordingly. In fact, by Lemma 3.1.3, the expressions for the l_{r-1} can all be written in this way.

3.2 Coordinate Invariance for Primary Fields

Let V be a vertex operator algebra. As mentioned in the introduction, the results in this section and in the next are adapted from those [FBZ04], however the motivation and process is purely VOA-theoretic. We return to our discussion of T_ϕ. Though we will continue to use eq. (3.3), it is important to note that in [FBZ04] the authors instead work with the similar expression

$$\phi(z) = \exp\left(\sum_{i\geq 1} l_i z^{i+1}\partial_z\right) l_0^{z\partial_z} \cdot z. \tag{3.20}$$

It is shown there that any coordinate transformation may be written in this way, and the relations derived between the l_i and k_i in this case are similar to eqs. (3.6), (3.13) and (3.14), when $l_0 = 0$. In fact, they are

$$\begin{aligned} k_1 &= l_0 \\ k_2 &= l_1 l_0 \\ k_3 &= l_2 l_0 + l_1^2 l_0 \\ &\vdots \end{aligned}$$

The results of Lemma 3.1.3 and Corollary 3.1.4 hold if one adopts instead the conventions of [FBZ04], and the proofs are the same as when we dealt with the case $l_0 = 0$.

As explained in the beginning of this chapter, Lemma 3.1.3 and the representation $z^{i+1}\partial_z \mapsto -L_i$ allow us to define the linear operator $T_\phi : V \to V$ giving the $\mathrm{Aut}(\mathcal{O})$-action on V as

$$T_\phi = \exp\left(-\sum_{i\geq 0} l_i L_i\right) \tag{3.21}$$

Equivalently, and along with the following Proposition, $\phi \mapsto T_\phi$ gives a representation of $\mathrm{Aut}(\mathcal{O})$ on V i.e. for $\phi, \psi \in \mathrm{Aut}(\mathcal{O})$, we have $T_{\phi(\psi(z))} = T_\phi T_\psi$.

Proposition 3.2.1. *For any $a \in V$, $T_\phi(a)$ is a finite sum and so $T_\phi : V \to V$ is well-defined.*

Proof. Let $a \in V_{(n)}$. Proposition 2.4.3 with $k = 2$ gives

$$L_n = \omega_{n+1} : V_{(m)} \to V_{(m-n)}$$

for $m \in \mathbb{Z}$ and so the operators $-L_i$ for $i > 0$ are locally nilpotent (cf. eq. (2.18)) and so sufficently high powers of L_i for $i > 0$ vanish. The truncation condition (eq. (2.20)) ensures that $L_i a = 0$ for i sufficiently large.

Since $L_0 : V_{(m)} \to V_{(m)}$, it remains to show that L_0 doesn't cause $T_\phi(a)$ to diverge. We expand $T_\phi(a)$ and extract the terms involving the operator L_0:

$$\begin{aligned} T_\phi(a) &= \exp\left(-\sum_{i\geq 0} l_i L_i\right) a \\ &= a + (-l_0 L_0 - \cdots)a + \frac{1}{2}(-l_0 L_0 - \cdots)^2 + \frac{1}{6}(-l_0 L_0 - \cdots)^3 a + \cdots \\ &= a - nl_0 + \frac{(nl_0)^2}{2} - \frac{(nl_0)^3}{6} + \cdots \\ &= a + e^{-nl_0} - 1 + \cdots \end{aligned}$$

Therefore $T_\phi(a)$ is indeed a finite sum. □

Recall the space of primary vectors $\mathcal{P}_n(V)$ seen in eq. (2.15). The corresponding vertex operators $Y(a,z)$ for $a \in \mathcal{P}_n(V)$ are then called ***primary fields***. We derive an explicit equation for the invariant $\mathrm{Aut}(\mathcal{O})$-action of ϕ on these primary fields, giving clarity on this result which is stated in [Zhu96]. Another account of this result is stated in Proposition 5.3.4 of [FBZ04], and so we go over these details here. For $A(z) \in \mathcal{F}(V)$ and $\phi \in \mathrm{Aut}(\mathcal{O})$, define the linear operator Δ_ϕ on $\mathcal{F}(V)$:

$$\Delta_\phi \cdot A(z) = T_\phi A(\phi(z)) T_\phi^{-1} (\phi'(z))^n \tag{3.22}$$

Lemma 3.2.2. *The following hold for Δ_ϕ:*

(a). *$A(\phi(z))$ is a well-defined element of $\mathcal{F}(V)$,*

(b). *$\phi \mapsto \Delta_\phi$ gives a representation of $\mathrm{Aut}(\mathcal{O})$ on $\mathcal{F}(V)$.*

Proof. Let $A(z) \in \mathcal{F}(V)$ and $\phi, \psi \in \mathrm{Aut}(\mathcal{O})$.

(a). For $\phi(z) = \sum_j k_j z^j$,

$$A(\phi(z)) = \sum_{i\in\mathbb{Z}} A_i \left(k_1 z + k_2 z^2 + \cdots\right)^i.$$

For $i \geq 0$, it is clear that $A(\phi(z)) \in \mathcal{F}(V)$ is well-defined, as the sum splits linearly. The case $i < 0$ is made clear when we write

$$\phi(z)^{-1} = \left(k_1 z + k_2 z^2 + \cdots\right)^{-1} = k_1^{-1} z^{-1} \left(1 + k_2 k_1^{-1} z + \cdots\right)^{-1} \in z^{-1}\mathbb{C}[[z]]$$

and so $A(\phi(z)) \in \mathcal{F}(V)$ is a well-defined element.

(b). Using the fact that T_ϕ gives a representation of $\mathrm{Aut}(\mathcal{O})$ on V, let $A(z) \in \mathcal{F}(V)$ and compute

$$\begin{aligned}
\Delta_{\phi(\psi(z))} A(z) &= T_{\phi(\psi(z))} A(\phi(\psi(z))) T_{\phi(\psi(z))}^{-1} \left((\phi(\psi(z)))'\right)^n \\
&= T_\phi T_\psi A(\phi(\psi(z))) T_\psi^{-1} T_\phi^{-1} \left(\phi'(\psi(z))\right)^n \left(\psi'(z)\right)^n \\
&= \Delta_\phi \left(T_\psi A(\psi(z)) T_\psi^{-1} (\psi'(z))^n\right) \\
&= \Delta_\phi \left(\Delta_\psi A(z)\right).
\end{aligned}$$

Thus we have the operator equality $\Delta_{\phi(\psi(z))} = \Delta_\phi \Delta_\psi$ which establishes the result.

□

So far, $\mathcal{F}(V)$ has been shown to be an $\mathrm{Aut}(\mathcal{O})$-module with action Δ_ϕ. We now show that the subspace of primary fields of $\mathcal{F}(V)$ is $\mathrm{Aut}(\mathcal{O})$-invariant, that is, $\Delta_\phi \cdot Y(a,z) = Y(a,z)$.

Proposition 3.2.3. *Let $a \in \mathcal{P}_n(V)$ and $Y(a,z)$ its associated primary field. Then,*

$$Y(a,z) = T_\phi Y(a, \phi(z)) T_\phi^{-1} \left(\phi'(z)\right)^n$$

Proof. We establish a general expression for $[L_i, Y(a,z)]$ where $i \in \mathbb{Z}$. Corollary 2.4.2 gives the commutator

$$[L_i, Y(a,z)] = z^{i+1}\partial_z Y(a,z) + n\partial_z z^{i+1} Y(a,z).$$

Let $v(z) = \sum_{i\geq 0} l_i z^{i+1}$. Assign, then, the operator $\mathbf{v} = -\sum_{i\geq 0} l_i L_i$ for $v(z)\partial_z$. Summing over the l_i, we get

$$[\mathbf{v}, Y(a,z)] = -v(z)\partial_z Y(a,z) - n\partial_z v(z) Y(a,z). \tag{3.23}$$

Now, the equation for Δ_ϕ is given in terms of the operators T_ϕ and ϕ which each give an action of $\mathrm{Aut}(\mathcal{O})$ from elements $z\mathbb{C}[[z]]\partial_z$ by exponentiation. The exponential map is surjective, and so we need not evaluate higher than linear terms in the exponents of T_ϕ and ϕ in order to show the invariance of $Y(a,z)$ under Δ_ϕ. With this in mind, let $R(\epsilon)$ denote the *dual numbers*

$$R(\epsilon) = \mathbb{R}[\epsilon]/\langle \epsilon^2 \rangle,$$

that is, numbers of the form $a + b\epsilon$ where $a, b \in \mathbb{R}$ and $\epsilon^2 = 0$. Let $a \in \mathcal{P}_n(V)$. Then by expanding both ϕ and T_ϕ in powers of ϵ,

$$\begin{aligned} \phi(z) &= \exp\left(\epsilon v(z)\partial_z\right) z = z + \epsilon v(z) \\ T_\phi &= \exp\left(\epsilon \mathbf{v}\right) = \mathrm{Id} + \epsilon \mathbf{v}. \end{aligned}$$

With this, we have the expression

$$\Delta_\phi Y(a,z) = (\mathrm{Id} + \epsilon\mathbf{v}) Y\left(a, z + \epsilon v(z)\right)\left(\mathrm{Id} - \epsilon \mathbf{v}\right)\left(\partial_z (z + \epsilon v(z))\right)^n. \tag{3.24}$$

Notice for $i \in \mathbb{Z}$ that

$$(z + \epsilon v(z))^i = \sum_{j\geq 0} \binom{i}{j} z^{i-j} \left(\epsilon v(z)\right)^j = z^i + \partial_z z^i \epsilon v(z)$$

and so

$$Y(a, z + \epsilon v(z)) = Y(a,z) + \epsilon v(z) \partial_z Y(a,z)$$

which is a familiar property of $R(\epsilon)$. Thus by expanding eq. (3.24) and using the commutator eq. (3.23), we get

$$\begin{aligned} \Delta_\phi Y(a,z) &= Y(a,z) + \epsilon\left(\mathbf{v}Y(a,z) - Y(a,z)\mathbf{v} + v(z)\partial_z Y(a,z) + n\partial_z v(z) Y(a,z)\right) \\ &= Y(a,z) + \epsilon\left([\mathbf{v}, Y(a,z)] + v(z)\partial_z Y(a,z) + n\partial_z v(z) Y(a,z)\right) \\ &= Y(a,z) \end{aligned}$$

which proves the result. □

Proposition 3.2.3 outlines the conditions for the coordinate invariance of a primary field. Recall that in Lemma 3.1.3 we have established that $\phi \in \mathrm{Aut}(\mathcal{O})$ can be written uniquely as exponents of elements in $z\mathbb{C}[[z]]\partial_z$, giving a representation of $\mathrm{Aut}(\mathcal{O})$ on V. The associated operator $T_\phi : V \to V$ constructed via the representation $z^{i+1} \mapsto -L_i$ is thus an automorphism of V. We can then define the transformed vertex operator

$$Y_\phi(a,z) = T_\phi Y(a,z) T_\phi^{-1} \in \mathrm{End}(V)[[z, z^{-1}]]. \tag{3.25}$$

Then immediately from Proposition 3.2.3, we obtain equation (4.2.13) in [Zhu96]:

Theorem 3.2.4. *Let $\phi \in \mathrm{Aut}(\mathcal{O})$. If $a \in \mathcal{P}_n(V)$ and $Y(a,z)$ is its associated primary field, then*

$$Y_\phi(a,z) = Y(a, \phi(z))(\phi'(z))^n$$

defines a new field obtained from the action of ϕ.

3.3 Coordinate Invariance for General Vertex Operators

It is here that we generalize the findings of the previous section by deriving an expression for Y_ϕ for any vertex operator. The main result here is a modification of Lemma 5.4.6 in [FBZ04], a result which is attributed to [Hua97].

With $v(z)$ and $\mathbf{v}$ defined as before and for any $Y(a,z) \in \mathcal{F}(V)$ with $a \in V_{(n)}$, we can compute as in Proposition 3.2.3 the natural transformation

$$T_\phi Y(a,\phi(z))T_\phi^{-1} = Y(a,z) - \epsilon \sum_{j\geq 0} \frac{\left(\partial_z^{j+1} v(z)\right)}{(j+1)!} Y(L_j a, z).$$

Without the assumption that $a \in \mathcal{P}_n(V)$ i.e. that $L_i a = 0$ for $i \geq 1$ and without the term $(\phi'(z))^n$, the rightmost sum above does not vanish. Thus, we must work with a different operator than Δ_ϕ if we are to establish coordinate invariance in a general case. We wish to find an operator similar to Δ_ϕ such that expansion by powers of ϵ gives the negative of rightmost sum above. Notice that we may view the rightmost sum as the ϵ-linear term of the operator

$$T_{\phi_z} = \exp\left(-\epsilon \sum_{j\geq 0} \frac{\left(\partial_z^{j+1} v(z)\right)}{(j+1)!} L_j\right) = \mathrm{Id} - \epsilon \sum_{j\geq 0} \frac{\left(\partial_z^{j+1} v(z)\right)}{(j+1)!} L_j + \cdots \tag{3.26}$$

acting on $Y(a,z)$ for some yet undetermined coordinate transformation ϕ_z, when expanding in powers of ϵ. One would naively take the representation $z^{i+1}\partial_z \mapsto -L_i$ as in eq. (3.21) to obtain ϕ_z, however this would not yield a valid coordinate transformation since the resulting ϕ_z would not be in the form eq. (3.3). Take instead the representation $t^{i+1}\partial_t \mapsto -L_i$ for some new formal variable t to obtain the coordinate transformation

$$\begin{aligned}\phi_z(t) &= \exp\left(\epsilon \sum_{j\geq 0} \frac{\left(\partial_z^{j+1} v(z)\right)}{(j+1)!} t^{j+1}\partial_t\right) t \\ &= t + \epsilon \sum_{j\geq 0} \frac{\left(\partial_z^{j+1} v(z)\right)}{(j+1)!} t^{j+1} + \cdots\end{aligned}$$

The next lemma relates $\phi_z(t)$ to $\phi(z)$.

Lemma 3.3.1. *Let $\phi_z(t)$ be defined as above, and let $\phi = \exp\left(\sum_i l_i z^{i+1}\partial_z\right) z$ as before. Then, the sum in ϕ_z is the ϵ-linear term of $\phi(z+t) - \phi(z)$.*

Proof. By expanding in powers of ϵ, and with $v(z)$ defined as in Proposition 3.2.3,

$$\begin{aligned}\phi(z+t) - \phi(z) &= (z+t) + \epsilon v(z+t) + \cdots - z - \epsilon v(z) - \cdots \\ &= t + \epsilon\left(\sum_{i\geq 0} l_i \left(\sum_{j\geq 0} \binom{i+1}{j} z^{i-j+1} t^j - z^{i+1}\right)\right) + \cdots \\ &= t + \epsilon\left(\sum_{i\geq 0} l_i \left(\sum_{j\geq 0} \binom{i+1}{j+1} z^{i-j} t^{j+1}\right)\right) + \cdots \\ &= t + \epsilon\left(\sum_{i\geq 0} l_i \left(\sum_{j\geq 0} \frac{(i+1)!}{(i-j)!(j+1)!} z^{i-j} t^{j+1}\right)\right) + \cdots \\ &= t + \epsilon \frac{\left(\partial_z^{j+1} v(z)\right)}{(j+1)!} t^{j+1} + \cdots\end{aligned}$$

which establishes the result. ☐

Proposition 3.3.2. *Let $a \in V_{(n)}$ and $\phi \in \mathrm{Aut}(\mathcal{O})$. Then with $\phi_z(t) = \phi(z+t) - \phi(z)$,*

$$Y(a,z) = T_\phi Y(T_{\phi_z}^{-1}(a), \phi(z)) T_\phi^{-1}.$$

Proof. We proceed as in Proposition 3.2.3. Define the operator Θ_ϕ on $\mathrm{Hom}(V, \mathcal{F}(V))$ by

$$\Theta_\phi \cdot Y(a,z) = T_\phi Y(T_{\phi_z}^{-1}(a), \phi(z)) T_\phi^{-1}.$$

Let $\psi \in \mathrm{Aut}(\mathcal{O})$ and thus $\psi_z(t) = \psi(z+t) - \psi(z)$. Note that

$$\phi_{\psi(z)}(\psi_z) = \phi(\psi(z) + (\psi(z+t) - \psi(z))) - \phi(\psi(z)) = \phi(\psi(z))_z.$$

Using this,

$$\begin{aligned}
\Theta_{\phi(\psi(z))} Y(a,z) &= T_{\phi(\psi(z))} Y(T^{-1}_{\phi(\psi(z))_z}(a), \phi(\psi(z))) T^{-1}_{\phi(\psi(z))} \\
&= T_\phi T_\psi Y(T^{-1}_{\phi_{\psi(z)}}(T^{-1}_{\psi_z}(a)), \phi(\psi(z))) T_\psi^{-1} T_\phi^{-1} \\
&= \Theta_\phi \left(T_\psi Y(T^{-1}_{\psi_z}(a), \psi(z)) T_\psi^{-1} \right) \\
&= \Theta_\phi \left(\Theta_\psi Y(a,z) \right)
\end{aligned}$$

and so Θ_ϕ is a representation of $\mathrm{Aut}(\mathcal{O})$ on $\mathrm{Hom}(V, \mathcal{F}(V))$. Finally, by expanding in powers of ϵ and with the aid of Lemma 3.3.1 and with $v(z)$ and $\mathbf{v}$ defined as before,

$$\begin{aligned}
\Theta_\phi Y(a,z) &= (\mathrm{Id} + \epsilon \mathbf{v}) Y(T_{\phi_z}^{-1}(a), (z + \epsilon v(z)))(\mathrm{Id} - \epsilon \mathbf{v}) \\
&= Y(a,z) + \epsilon \left([\mathbf{v}, Y(a,z)] + \sum_{j \geq 0} \frac{1}{(j+1)!} \partial_z^{j+1} v(z) Y(L_j a, z) + v(z) \partial_z Y(a,z) \right) \\
&= Y(a,z)
\end{aligned}$$

where the last equality comes directly from Corollary 2.4.2. Therefore $Y(a,z)$ is invariant under Θ_ϕ, which establishes the result. □

As in the primary field case, we are able to reformulate the statement of Proposition 3.3.2 in terms of Y_ϕ:

Theorem 3.3.3. *Let $\phi \in \mathrm{Aut}(\mathcal{O})$. Then, with Y_ϕ defined as in Theorem 3.2.4,*

$$Y_\phi(a,z) = Y(T_{\phi_z}^{-1}(a), \phi(z))$$

defines a new vertex operator algebra isomorphic to V obtained from the action of ϕ.

3.4 Coordinate Invariance for the Virasoro Field

The main result of this section provides the necessary computations required for deriving the expression of the transformed Virasoro field $Y_\phi(\omega, z)$ under the action of ϕ. Use of the relations eqs. (3.6), (3.13) and (3.14) established in Chapter 3 is central to the proof.

Lemma 3.4.1. *For $\phi(z) = \exp(v(z)\partial_z)z \in \mathrm{Aut}(\mathcal{O})$ and $v(z) = \sum_{i\geq 0} l_i z^{i+1}$, we have*

$$\begin{aligned}
\phi'(z) &= \exp(\partial_z v(z)) \\
\phi''(z) &= (\phi'(z))^2 \frac{\partial_z^2 v(z)}{\partial_z v(z)} - \phi'(z)\frac{\partial_z^2 v(z)}{\partial_z v(z)} \\
\phi'''(z) &= (\phi'(z))^3 \left(\frac{3(\partial_z^2 v(z))^2}{2(\partial_z v(z))^2} + \frac{\partial_z^3 v(z)}{2\partial_z v(z)} \right) \\
&\quad - (\phi'(z))^2 \left(\frac{3(\partial_z v(z))^2}{(\partial_z v(z))^2} \right) \\
&\quad + \phi'(z) \left(\frac{3(\partial_z^2 v(z))^2}{2(\partial_z v(z))^2} - \frac{\partial_z^3 v(z)}{2\partial_z v(z)} \right).
\end{aligned}$$

Proof. Recall the coordinate transformation $\phi_z(t) = \phi(z+t) - \phi(z)$. Let $\phi_z(t) = \sum_{i\geq 1} \tilde{k}_i t^i$ and $\phi_z(t) = \exp(\sum_{i\geq 0} \tilde{l}_i t^{i+1}\partial_t)t$ for some coefficients $\tilde{k}_i$ and $\tilde{l}_i$ analogous to those in eqs. (3.1) and (3.3). Notice for $n \geq 1$ that

$$\partial_t^n \phi_z(0) = \partial_t^n \left(\sum_{i\geq 1} k_i (z+t)^i - \sum_{i\geq 1} k_i z^i \right)\Bigg|_{t=0} = \phi^{(n)}(z).$$

That is, $\phi^{(n)}(z) = n!\tilde{k}_n$. Thus we may use the aforementioned relations eqs. (3.6), (3.13) and (3.14) along with the expressions

$$\begin{aligned}
\tilde{l}_0 &= \partial_z v(z) \\
\tilde{l}_1 &= \frac{1}{2}\partial_z^2 v(z) \\
\tilde{l}_2 &= \frac{1}{6}\partial_z^3 v(z)
\end{aligned}$$

which are seen directly from eq. (3.26) to obtain the required derivatives. □

As a short remark, we are now able to give a corollary showing that the two previous theorems on coordinate invariance are consistent.

Corollary 3.4.2. *Theorem 3.2.4 is a special case of Theorem 3.3.3.*

Proof. Let $a \in \mathcal{P}_n(V)$. Then,

$$T_{\phi_z}^{-1}(a) = \exp\left(\partial_z v(z) L_0\right) a = \exp\left(n\partial_z v(z)\right) a$$

and so using Theorem 3.3.3 and $\phi'(z) = \exp(\partial_z v(z))$ from Lemma 3.4.1, we obtain

$$Y_\phi(a,z) = Y\left(\exp\left(n\partial_z v(z)\right) a, \phi(z)\right) = Y(a,\phi(z))(\phi'(z))^n.$$

□

Define the *Schwarzian derivative* of ϕ with respect to z as

$$\{\phi(z), z\} = \frac{\phi'''(z)}{\phi'(z)} - \frac{3}{2}\left(\frac{\phi''(z)}{\phi'(z)}\right)^2.$$

Lemma 3.4.3. *Let $\phi(z) = \exp(v(z)\partial_z)z \in \mathrm{Aut}(\mathcal{O})$ where $v(z) = \sum_{i\geq 0} l_i z^{i+1}$. Then,*

$$\{\phi(z), z\} = \frac{\partial_z^3 v(z)}{2\partial_z v(z)}\left((\phi'(z))^2 - 1\right).$$

Proof. Computing directly using the results of Lemma 3.4.1 gives

$$-\frac{3}{2}\left(\frac{\phi''(z)}{\phi'(z)}\right)^2 = -(\phi'(z))^2\frac{3(\partial_z^2 v(z))^2}{2(\partial_z v(z))^2} + \phi'(z)\frac{3(\partial_z^2 v(z))^2}{(\partial_z v(z))^2} - \frac{3(\partial_z^2 v(z))^2}{2(\partial_z v(z))^2},$$

and so from the definition of $\{\phi(z), z\}$ and the expression for $\phi'''(z)$, it follows that

$$\{\phi(z), z\} = \frac{\partial_z^3 v(z)}{2\partial_z v(z)}\left((\phi'(z))^2 - 1\right).$$

□

We may now use the results of this section along with Theorem 3.3.3 to derive the desired expression for the transformed Virasoro field $Y(\omega, z)$ under the action of ϕ. This is stated without proof, in equation (4.2.14) of [Zhu96].

Theorem 3.4.4. *Let $\phi \in \mathrm{Aut}(\mathcal{O})$. Then, with Y_ϕ defined as in Theorem 3.2.4,*

$$Y_\phi(\omega, z) = Y(\omega, \phi(z))(\phi'(z))^2 + \frac{c_V}{12}\{\phi(z), z\}.$$

Proof. The creation property (eq. (2.22)) applied in our case

$$\lim_{z\to 0} Y(\omega, z)\mathbf{1} = \omega$$

gives $L_{-1}\mathbf{1} = L_0\mathbf{1} = L_n\mathbf{1} = 0$ for $n \geq 1$. Also from the creation property, we see that $\omega = \omega_{-1}\mathbf{1} = L_{-2}\mathbf{1}$. These facts along with the Virasoro bracket relations (eqs. (2.24) and (2.25)) allow us to compute the following:

$$\begin{aligned} L_0\omega &= L_0 L_{-2}\mathbf{1} = 2L_{-2}\mathbf{1} = 2\omega \\ L_1\omega &= L_1 L_{-2}\mathbf{1} = 3L_{-1}\mathbf{1} = 0 \\ L_2\omega &= L_2 L_{-2}\mathbf{1} = \left(4L_0 + \frac{c_V}{2}\right)\mathbf{1} = \frac{c_V}{2}\mathbf{1} \\ L_i\omega &= 0 \quad \text{for } i \geq 3. \end{aligned}$$

Using these relations alongside Theorem 3.3.3, we have

$$\begin{aligned} T_{\phi_z}^{-1}(\omega) &= \exp\left(\partial_z v(z) L_0 + \frac{1}{6}\partial_z^3 v(z) L_2\right)\omega \\ &= \sum_{j\geq 0}\frac{1}{j!}\left(\partial_z v(z) L_0 + \frac{1}{6}\partial_z^3 v(z) L_2\right)^j \omega \\ &= \exp(2\partial_z v(z))\omega + \frac{c_V}{12}\partial_z^3 v(z)\sum_{j\geq 0}\frac{(2\partial_z v(z))^j}{(j+1)!}\mathbf{1} \\ &= \exp(2\partial_z v(z))\omega + \frac{c_V}{12}\left(\frac{\partial_z^3 v(z)}{2\partial_z v(z)}\left((\phi'(z))^2 - 1\right)\right)\mathbf{1}. \end{aligned}$$

Thus using Lemma 3.4.3 and the vacuum property (eq. (2.21)) we obtain

$$Y_\phi(\omega, z) = Y(\omega, \phi(z))(\phi'(z))^2 + \frac{c_V}{12}\{\phi(z), z\}.$$

□

Chapter 4

The Square-Bracket Formalism

We apply the results of the previous chapter to construct a vertex operator algebra structure from (and isomorphic to) the Heisenberg VOA of central charge $c_V = 1$. This resulting algebra is well-known and first introduced in [Zhu96] with details of the construction provided in [MT10]. We will also be using this algebra extensively in order to prove the results of Chapter 6.

4.1 Heisenberg Modes

Recall the Heisenberg VOA $\mathcal{H}$ defined in Section 2.5, and set $c_V = 1$. The creation operators $h_{-n} \in \mathrm{End}(\mathcal{H})$ which form a basis for $\mathcal{H}$ exist as coefficients of the Heisenberg field $h(z)$, or equivalently, the vertex operator of the state $h_{-1}\mathbf{1}$. Applying Lemma 2.5.1 to the state $h_{-1}\mathbf{1}$ gives the relation $L_n(h_{-1}\mathbf{1}) = h_{n-1}\mathbf{1}$ and so $h_{-1}\mathbf{1}$ is a primary state of weight 1. We can thus use the result of Theorem 3.2.4 to define

$$Y_\phi(h_{-1}\mathbf{1}, z) = Y(h_{-1}\mathbf{1}, \phi(z))\phi'(z). \tag{4.1}$$

Let $\phi(z) = e^z - 1 \in \mathrm{Aut}(\mathcal{O})$. Denote the square-bracket vertex operator transformed under $\phi(z)$ as the series

$$Y[a, z] = \sum_{n\in\mathbb{Z}} [a]_n z^{-n-1}. \tag{4.2}$$

Using eq. (4.1), we may proceed to derive explicit expressions for the square-bracket modes $[h]_n = [h_{-1}\mathbf{1}]_n$ of the square-bracket Heisenberg field $h[z]$ first by writing

$$Y_\phi(h_{-1}\mathbf{1}, z) = Y(h_{-1}\mathbf{1}, e^z - 1)e^z,$$

then by multiplying eq. (4.2) by z^n to obtain the formula

$$[h]_n = \mathrm{Res}_z\, Y(h_{-1}\mathbf{1}, e^z - 1)(e^z)(z^n). \tag{4.3}$$

By Lemma 3.2.2 (a), $Y(h_{-1}\mathbf{1}, e^z-1) \in \mathcal{F}(V)$, and so eq. (4.3) truncates when applied to any state in $\mathcal{H}$. We can then extend Lemma 2.1.1 to this case with $w = e^z - 1$, noting that $e^z - 1 \in z\mathbb{C}[[z]]$:

$$[h]_n = \mathrm{Res}_w\, h(w)\,(\log(w+1))^n$$

Define for $k \in \mathbb{Z}$, $k \neq 0$ the coefficient

$$c(n,k) = \mathrm{Coeff}_{w^k}\,(\log(w+1))^n\,.$$

With this, we can set $c(0,k) = 1$ for all k and so $[h]_0 = h_0$, where h_0 acts as the zero operator on $\mathcal{H}$ (we shall omit writing h_0 and its coefficient from now on). If we can find expressions for $c(n,k)$, then we can write

$$[h]_n = \mathrm{Res}_w \left(\sum_{m\in\mathbb{Z}} h_m w^{-m-1}\right)(\log(w+1))^n = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} c(n,k)h_k.$$

Consider first the case where $n \geq 1$. The geometric series is used to show that

$$\frac{1}{1+w} = \sum_{k\geq 0}(-1)^k w^k$$

and so by integrating, shifting indices, and taking the n-th power we obtain

$$(\log(w+1))^n = \left(\sum_{k\geq 1} \frac{(-1)^{k-1} w^k}{k}\right)^n \tag{4.4}$$

From eq. (4.4), $c(n,k) = 0$ for $k \leq -1$, since the expansion above never consists of any negative powers of w. To find $c(n,k)$ given a fixed value n, eq. (4.4) tells us that we must consider all possible ways to multiply n summands to produce a term of degree k. Then, we take the sum of these terms and extract the necessary coefficient. All such possible ways are in correspondence with the set $\mathcal{C}(k,n)$ of all n-compositions $(k_1, k_2, \ldots, k_n)$ of k (cf. Proposition 3.1.1), and so for $n \geq 1$ we have

$$c(n,k) = \sum_{(k_1,k_2,\ldots,k_n)\in\mathcal{C}(k,n)} \left(\prod_{i=1}^{n} \frac{(-1)^{k_i-1}}{k_i}\right).$$

Hence for $n \geq 1$, we have the following possibly new expression for the square-bracket modes:

$$[h]_n = \sum_{k\geq 1}\left(\sum_{(k_1,k_2,\ldots,k_n)\in\mathcal{C}(k,n)} \left(\prod_{i=1}^{n} \frac{(-1)^{k_i-1}}{k_i}\right)\right) h_k \tag{4.5}$$

Finally, note that $c(n,k) = 0$ for $n > k$, as one cannot form a composition of k of n parts here. The modes can then computed simply using eq. (4.5):

$$\begin{aligned}
[h]_1 &= h_1 - \frac{1}{2}h_2 + \frac{1}{3}h_3 - \frac{1}{4}h_4 + \cdots \\
[h]_2 &= h_2 - h_3 + \frac{11}{12}h_4 - \frac{1}{6}h_5 + \cdots \\
[h]_3 &= h_3 - \frac{3}{2}h_4 + \frac{7}{4}h_5 - \frac{15}{8}h_6 + \cdots \\
&\vdots
\end{aligned}$$

Next we have the following Laurent series expansion, raised to the n-th power for $n \geq 1$:

$$\frac{1}{(\log(w+1))^n} = \left(\frac{1}{w} - \sum_{k\geq 0}\left(\sum_{j=1}^{k+1} \frac{B_j s_k^{(j-1)}}{k!j}\right) w^k\right)^n \tag{4.6}$$

where B_j are the Bernoulli numbers, defined by the following Laurent series expansion about $z = 0$

$$\frac{1}{e^z - 1} = \sum_{j\geq 0} \frac{B_j}{j!} z^{j-1}, \tag{4.7}$$

and $s_k^{(j)}$ are the signed Stirling numbers of the first kind given by

$$s_k^{(j)} = (-1)^{k-j}(\#\text{ of permutations of length } k \text{ with } j \text{ disjoint cycles}). \tag{4.8}$$

Finding $c(n,k)$ for $n < 0$ requires some combinatorial techniques, as we have terms of degree -1 and 0 appearing above.

We introduce here a *very weak k-composition* of $n \in \mathbb{Z}$ as a k-composition of n where we allow for the parts -1 and 0. These are related to *weak k-compositions*, (in which the parts are non-negative (cf. [HM04])), and so we have extended the definition here. For example, the very weak 2-compositions of 3 are

$$(2,1),\ (1,2),\ (3,0),\ (0,3),\ (4,-1),\ (-1,4).$$

Note case where $n < 0$. Let $\mathcal{C}'(n,k)$ denote the set of all very weak k-compositions of n.

Lemma 4.1.1. *Let $n \in \mathbb{Z}$ and $k \geq 0$.*

(a). *If $n \geq 0$ then $|\mathcal{C}'(n,k)| = \binom{n+2k-1}{k-1}$.*

(b). *If $n < 0$ and $k \geq |n|$ then $|\mathcal{C}'(n,k)| = \binom{n+2k-1}{k-1}$.*

(c). *If $n < 0$ and $k < |n|$ then $|\mathcal{C}'(n,k)| = 0$.*

Proof. Suppose $n \geq 0$. A very weak k-composition of n can be written as

$$(n_1 - 2) + (n_2 - 2) + \cdots + (n_k - 2) = n$$

for integers $n_1, \ldots, n_k \geq 1$. Rearranging gives

$$n_1 + n_2 + \cdots + n_k = n + 2k$$

which is a k-composition of $n+2k$. Recall from the note following Proposition 3.1.1 that $|\mathcal{C}(n,k)| = \binom{n-1}{k-1}$ and so $|\mathcal{C}'(n,k)| = \binom{n+2k-1}{k-1}$ which proves (a). The same argument works for $n < 0$ provided that we do not have $k < |n|$. In this case,

$$\underbrace{-1 - 1 - \cdots - 1}_{k} \neq n$$

and so $|\mathcal{C}'(n,k)| = 0$ here. This proves (b) and (c). □

From Lemma 4.1.1 above, we may also see that for integers $n_1, \ldots, n_k \geq 1$ we have

$$\begin{aligned}(n_1 - 2) + (n_2 - 2) + \cdots + (n_k - 2) &= n \\ (n_1 - 1) + (n_2 - 1) + \cdots + (n_k - 1) &= n + k\end{aligned}$$

and so the number of very weak k-compositions of n is the same as the number of weak k-compositions of $n + k$. In subsequent arguments, it will be useful to retain the -1 parts and so we continue to use very weak compositions despite the above relation between the two concepts.

Define the function $g : \mathbb{Z} \to \mathbb{Q}$ by the rules

$$g(x) = \begin{cases} -\sum_{j=1}^{x+1} \frac{B_j s_x^{(j-1)}}{x!j}, & x \geq 0 \\ 1, & x < 0. \end{cases}$$

We are then able to define the following expression for $n \geq 1$:

$$[h]_{-n} = \sum_{k \geq -n} \left(\sum_{(k_1,k_2,\ldots,k_n) \in \mathcal{C}'(k,n)} \left(\prod_{i=1}^{n} g(k_i) \right) \right) h_k \tag{4.9}$$

where the lower bound on the first sum above comes from Lemma 4.1.1. We can then compute the modes algorithmically:

$$\begin{aligned}[h]_{-1} &= h_{-1} - \frac{1}{12}h_1 + \frac{1}{24}h_2 - \frac{19}{720}h_3 + \cdots \\ [h]_{-2} &= h_{-2} + h_{-1} - \frac{1}{240}h_2 + \frac{1}{240}h_3 - \cdots \\ [h]_{-3} &= h_{-3} + \frac{3}{2}h_{-2} + \frac{1}{2}h_{-1} + \frac{1}{240}h_1 + \cdots \\ &\vdots\end{aligned}$$

Computations involving the square-bracket modes will usually not be this cumbersome; we have found explicit expressions for the modes in their entirety, however in most cases these expressions will be heavily truncated. One such computation (which will be of importance in Chapter 6) is the proof of Lemma 10.2 of [FM22] which gives an expression for $(r-1)![h]_{-r}[h]_{-1}\mathbf{1}$ for an odd integer $r \geq 1$.

4.2 Virasoro Modes

It should be noted that the process of obtaining square-bracket modes for certain primary states in a VOA is also possible (cf. Section 2.7 of [MT10]) and the idea is similar to previous computations and to what will follow. Since further discussion will only involve the Heisenberg VOA, we do not concern ourselves extensively with these generalized computations. In this short section however, we do mention briefly some facts about the square-bracket Virasoro modes.

Proposition 4.2.1. *Let $[\omega]$ denote the conformal state transformed under the action of $\phi = e^z - 1$ for any VOA $\{V, Y, \mathbf{1}, \omega\}$. We have $[\omega] = \omega - (c_V/24)\mathbf{1}$.*

Proof. We apply the statement of Theorem 3.4.4 to $\mathbf{1}$ and extract the constant term. It is easy to see that $\{e^z - 1, z\} = -1/2$. Moreover, the constant term of $Y(\omega, e^z - 1)(e^{2z})$ is just $\omega_{-1}\mathbf{1}$. It follows then, that $[\omega] = \omega - (c_V/24)\mathbf{1}$. □

We obtain expressions for the square-bracket Virasoro modes using Theorem 3.4.4. Doing so, and then applying Lemma 2.1.1 we get

$$\begin{aligned} [L]_n &= \mathrm{Res}_z \left(Y(\omega, e^z - 1)(e^{2z})(z^{n+1}) - (c_V/24)\left(z^{n+1}\right)\right) \\ &= \mathrm{Res}_w \left(Y(\omega, w)\left(\log(w+1)\right)^{n+1}(w+1) - (c_V/24)\left(\log(w+1)\right)^{n+1}\right). \end{aligned} \tag{4.10}$$

It is not hard to see that $[L]_{-1} = L_{-1} + L_0$. More generally, using eq. (4.4) we can write $\log(w+1)^{n+1} = w^{n+1} + o(w^{n+2})$ and so if $n > -2$,

$$[L]_n = L_n + \sum_{i \geq 1} c(n, i) L_{n+i} \tag{4.11}$$

for some coefficients $c(n, i)$. In fact, this expression appears in [Zhu96]. The square-bracket mode $[L]_0$ is of importance. Using eq. (4.10), we get that

$$\begin{aligned} [L]_0 &= \mathrm{Res}_w \left(\left(\sum_{k \geq 1} \frac{(-1)^{k-1} w^k}{k}\right)(Y(\omega, w) + Y(\omega, w)w)\right) \\ &= L_0 + \sum_{k \geq 1} \frac{(-1)^{k-1}}{k(k+1)} L_k. \end{aligned} \tag{4.12}$$

Deriving expressions for other square-bracket Virasoro modes involves extracting coefficients from integer powers of the logarithmic series above, and is similar to what was done in the Heisenberg case. Noteworthy to us is the fact that a square-bracket VOA, which we will denote henceforth by $[V]$, is equipped with a different grading than that of the original VOA structure: If v is homogeneous in $[V]$, then define

$$[V]_{(n)} = \{\, v \in [V] \mid [L]_0 v = nv \,\}. \tag{4.13}$$

That is, we write $v \in [V]_{(n)}$ to mean v has square-bracket weight n. It will be important to distinguish between which grading is being used in later discussions, and so care must be taken when discussing weights. The following short fact about the square-bracket grading is stated without proof in [DLM00]:

Lemma 4.2.2. *For any $N \in \mathbb{Z}$, we have*

$$\bigoplus_{n \leq N} V_{(n)} = \bigoplus_{n \leq N} [V]_{(n)}.$$

Proof. Using 2.18 in the definition of a VOA, let $M \in \mathbb{Z}$ be the smallest integer such that $V_{(M)} \neq 0$. Recall from Proposition 2.4.3 that $L_n : V_{(m)} \to V_{(m-n)}$. If $v \in V_{(M)}$, then using eq. (4.12) we get

$[L]_0 v = L_0 v = Mv$ and so $V_{(M)} = [V]_{(M)}$. If $v \in V_{(M+S)}$ for some $S > 0$, then also from eq. (4.12) we get

$$[L]_0 v = L_0 v + \left(\sum_{k=1}^{S} \frac{(-1)^{k+1}}{k(k+1)} L_k \right) v \in \bigoplus_{i=M}^{M+S} V_{(i)}.$$

□

It is also worth noting that since the square-bracket Virasoro field is constructed via Theorem 3.4.4, and thus is isomorphic to the "regular" Virasoro field, the square-bracket Virasoro modes satisfy their respective properties, i.e. we have the analogous relations

$$[[L]_m, [L]_n] = (m-n)[L]_{m+n} + \frac{c_V}{12}(m^3 - m)\delta_{m+n,0}$$
$$[[L]_n, c_V] = 0.$$

Chapter 5

The Relation To Number Theory

Up until now, our discussion has been almost purely VOA-theoretic. Many papers cited here such as [Zhu96], [MT10] and [FM22] however, are primarily interested in the connection between VOAs and modular forms. As such, we follow [Ser73a] and [MT10] to introduce elementary notions on modular forms of level one and characters of VOAs, and then proceed to outline how the coordinate transformation plays an important role in the theory. As in many related texts, we adopt henceforth the following *q-convention*:

$$q_z = e^z, \quad q = q_{2\pi i\tau} = e^{2\pi i\tau}. \tag{5.1}$$

5.1 Modular Forms

Define the *modular group* as the space of all 2×2 matrices consisting of integer entries and determinant one

$$\Gamma = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc = 1 \right\} \tag{5.2}$$

generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let $\mathfrak{H}$ denote the complex upper-half plane i.e. $\{ z \in \mathbb{C} \mid \mathrm{im}(z) > 0 \}$ and define the action of Γ on $\mathfrak{H}$ as

$$(\gamma, z) \mapsto \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

With $z = x + iy$ for $x, y \in \mathbb{R}$, it is easy to see that

$$\mathrm{im}\left(\frac{az+b}{cz+c}\right) = \frac{y}{(cx+d)^2 + (cy)^2} = \frac{\mathrm{im}(z)}{|cz+d|^2},$$

and so $\mathfrak{H}$ is invariant under the action of Γ. A *modular form of weight $k \in \mathbb{Z}$ of level one* is a function $f : \mathfrak{H} \to \mathbb{C}$ satisfying the following conditions:

(a). f is holomorphic

(b). For $\tau \in \mathfrak{H}$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

(c). As $\mathrm{im}(z) \to \infty$, $f(z)$ is bounded.

For $\tau \in \mathfrak{H}$, inserting the generators S and T into condition (b) above (often called the *modularity* condition) we obtain, respectively, the relations

$$f(\tau+1) = f(\tau) \tag{5.3}$$

$$f(-1/\tau) = z^k f(\tau) \tag{5.4}$$

and so this condition can be verified if the above two relations hold for all $\tau \in \mathfrak{H}$. As a short remark, notice that by inserting the negative of the identity matrix, we obtain the relation $f(z) = (-1)^k f(z)$ and so the only modular forms of odd weight k are those identically equal to zero. Equation (5.3) asserts that modular forms are periodic and so we may consider their Fourier series expansion, or *q-expansion*

$$f(\tau) = \sum_{n\in\mathbb{Z}} \alpha_n q^n,$$

for some coefficients α_n (note the use of 5.1). The simplest and perhaps most uninteresting examples of modular forms are the identically zero and constant functions on $\mathbb{C}$, (it is not difficult to see that they are modular forms of weight 0 and satisfy the necessary conditions (a)-(c)). For even $k \geq 4$, the simplest non-trivial examples of modular forms are the *Eisenstein Series*

$$G_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k} \tag{5.5}$$

These are indeed modular forms, and as an example, it is not difficult to establish the modularity condition, which we do via eqs. (5.3) and (5.4). Using the absolute convergence of the sum in eq. (5.5) and summing over the pair $(m, m+n)$, we get

$$G_k(\tau+1) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+m+n)^2} = G_k(\tau)$$

$$G_k(-1/\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{((-m/\tau)+n)^k} = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{\tau^k}{(n\tau-m)^k} = \tau^k G_k(\tau).$$

After normalizing, it can be shown (Proposition 8, pg. 92 of [Ser73a]) that $G_k(\tau)$ has the following Fourier series expansion for $k \geq 2$ even:

$$G_k(q) = -\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n \tag{5.6}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the classical *divisor function*. Notice how we write the series now in the variable q rather than τ. Since B_k never vanishes for k even, we may define the *normalized weight k Eisenstein series* as

$$E_k(q) = -\frac{2kG_k(q)}{B_k} = 1 - \frac{2k}{B_k}\sum_{n\geq 1} \sigma_{k-1}(n) q^n. \tag{5.7}$$

Note also that in eq. (5.5), $k \geq 4$ and so $G_2(q)$ and $E_2(q)$ are indeed *not* modular forms but rather *quasi-modular* forms. These satisfy slightly different modularity conditions which we will not discuss here, since $E_2(q)$ is the only quasi-modular form of relevance to this discussion. The first few normalized modular Eisenstein series are

$$\begin{aligned}
E_4(q) &= 1 + 240q + 2160q^2 + 6720q^3 + \cdots \\
E_6(q) &= 1 - 504q - 16632q^2 - 122976q^3 - \cdots \\
E_8(q) &= 1 + 480q + 61920q^2 + 1050240q^3 + \cdots \\
E_{10}(q) &= 1 - 264q - 135432q^2 - 5196576q^3 - \cdots
\end{aligned}$$

It is not hard to see that the expressions $E_4(q)E_2(q) = E_{10}(q)$ and $E_4(q)^2 = E_8(q)$ should hold, for example. The following theorem confirms our suspicions:

Theorem 5.1.1 ([MT10], Theorem 3.9)**.** *The weighted polynomial algebra* $\mathfrak{Q} = \mathbb{C}[E_2, E_4, E_6]$ *is the algebra of quasi-modular forms of level one. Its graded subalgebra* $\mathfrak{M} = \mathbb{C}[E_4, E_6]$ *is the algebra of all holomorphic modular forms of level one.*

5.2 The Character of a VOA

Recall the statement of Proposition 2.4.3: if $v \in V_{(k)}$ and $w \in V_{(m)}$ are homogeneous states, then

$$v_n : V_{(m)} \to V_{(m+k-n-1)}.$$

For any $v \in V_{(k)}$, define the *zero mode* $o(v)$ as v_{k-1}. By extending the definition of $o(v)$ to all of V additively, for any states v and integers m, we get

$$o(v) : V_m \to V_m. \tag{5.8}$$

The zero mode is thus an endomorphism on each graded piece $V_{(m)}$ and as such, we may consider its *trace* which we write in the form of a generating function in q:

$$Z_V(v,q) = \mathrm{Tr}_V o(v) q^{L_0 - c_V/24}.$$

For each subspace $V_{(n)}$ we have the operator equality $q^{L_0} = q^n$ and so by standard properties of the trace, we may re-write the above expression as

$$Z_V(v,q) = q^{-c_V/24} \sum_{n\in\mathbb{Z}} \mathrm{Tr}_{V_{(n)}} o(v) q^n. \tag{5.9}$$

The property (eq. (2.18)) in the definition of a VOA ensures the sum above is truncated i.e. $Z_V(v,q) \in q^{-c_V/24}\mathbb{C}[[q]][q^{-1}]$. The *character* of V is then the linear map

$$\begin{aligned} Z_V : & V \to q^{-c_V/24}\mathbb{C}[[q]][q^{-1}] \\ & v \mapsto Z_V(v,q). \end{aligned}$$

Take $v = \mathbf{1}$. Then due to the vacuum property (eq. (2.21)) it is easy to see that $o(\mathbf{1}) = I_V$. Since $\mathrm{Tr}_{V_{(n)}} \mathrm{Id}_V = \dim V_{(n)}$, we obtain

$$Z(\mathbf{1},q) = q^{-c_V/24} \sum_{n\in\mathbb{Z}} \dim_{V_{(n)}} q^n.$$

This is often called the *graded dimension* of V or the *partition function* of V.

Proposition 5.2.1. *For the Heisenberg VOA of central charge $c_V = 1$,*

$$Z_{\mathcal{H}}(\mathbf{1},q) = \eta(q)^{-1}$$

where $\eta(q)^{-1}$ is the inverse eta-function

$$\eta(q)^{-1} = q^{-1/24} \sum_{n\geq 0} p(n) q^n$$

and where $p(n)$ is the unrestricted partition function.

Proof. Recall that $\mathcal{H} \cong \mathbb{C}[h_{-1}, h_{-2}, h_{-3}, \ldots]$. As such, $\mathcal{H}_{(n)} = 0$ for $n < 0$. The weight of a state in $\mathcal{H}$ written in this basis is then

$$\mathrm{wt}(h_{-n_1} h_{-n_2} \cdots h_{-n_r} \mathbf{1}) = \sum_i n_i$$

for integers $n_i > 0$. The number of elements having weight $k \geq 0$ corresponds to the number of ways in which $\sum_i n_i = k$ for integers $n_i > 0$. This is precisely the definition of the unrestricted partition function $p(n)$ and so it follows that $\dim \mathcal{H}_{(n)} = p(n)$. This gives

$$Z_{\mathcal{H}}(\mathbf{1},q) = q^{-1/24} \sum_{n\geq 0} p(n) q^n = \eta(q)^{-1}.$$

□

Despite having discussed much of the theory of coordinate transformations, we are just now able to explore its intended purpose outlined first in [Zhu96]. Suppose we wish to take characters of more complicated states in order to obtain further results akin to that of Proposition 5.2.1. Following [MT10], consider the conformal state ω of an arbitrary VOA. Clearly, $o(\omega) = L_0$ and $\mathrm{Tr}_{V_{(n)}} L_0 = n \dim V_{(n)}$ thus

$$Z_V(\omega, q) = q^{-c_V/24} \sum_{n \in \mathbb{Z}} \mathrm{Tr}_{V_{(n)}} L_0 q^n = q^{-c_V/24} \sum_{n \in \mathbb{Z}} n \dim V_{(n)} q^n. \tag{5.10}$$

Unfortunately, it is difficult to see the desired phenomenon by using the standard conformal state ω. Consider instead the square-bracket conformal state $[\omega] = \omega - (c_V/24)\mathbf{1}$ (cf. Proposition 4.2.1). We require the zero mode of $[\omega]$, that is, the mode which preserves the grading of V (not the grading of $[V]$). As such, compute the zero mode of $[\omega]$ using the usual vertex operator to obtain $o([\omega]) = L_0 - c_V/24$. Then, $\mathrm{Tr}_{V_{(n)}} c_V/24 = c_V/24 \dim V_{(n)}$ which gives

$$Z_V([\omega], q) = q^{-c_V/24} \sum_{n \in \mathbb{Z}} (n - c_V/24) \dim V_{(n)} q^n.$$

This result along with a small observation can be summarized in the following lemma:

Proposition 5.2.2. *Let $[\omega]$ be the square-bracket conformal state for an arbitrary VOA. Then,*

$$Z_V([\omega], q) = q\partial_q Z_V(\mathbf{1})$$

It can be shown (cf. Exercise 3.18 of [MT10]) that $q\partial_q \eta(q) = (-1/2)\eta(q)G_2(q)$. We can apply this to the Heisenberg case using the statement of Proposition 5.2.1 and obtain

$$Z_{\mathcal{H}}([\omega], q) = q\partial_q \eta(q)^{-1} = \left(-\eta(q)^{-2}\right)\left((-1/2)\eta(q)G_2(q)\right) = \frac{G_2(q)}{2\eta(q)}.$$

Thus characters of square-bracket states outline the connection to modular forms, this being the central result of [Zhu96]. In fact, in the Heisenberg case, we have:

Theorem 5.2.3 ([MT10], Theorem 4.5)**.** *Denote by $[\mathcal{H}]$ the Fock space of the Heisenberg VOA equipped with the square-bracket grading. Let $\mathfrak{Q}$ be the graded algebra of quasi-modular forms of level one. There is a surjection of graded linear spaces*

$$\begin{aligned} [\mathcal{H}] &\to \mathfrak{Q} \\ v &\mapsto Q_v(q) \end{aligned}$$

such that $Z_{\mathcal{H}}(v, q) = Q_v(q)/\eta(q)$ and $Q_v(q)$ is the quasi-modular form of weight k attached to $v \in [V]_{(k)}$.

Thus after normalization by $\eta(q)^{-1}$, characters of states in the Heisenberg VOA give rise to quasimodular forms. In Proposition 5.2.1 we obtained the constant modular form 1 of weight zero, and in Proposition 5.2.2 we obtained the quasi-modular form $G_2(q)/2$ of weight 2.

An explicit description of the quasi-modular form $Q_v(q)$ above is given by equation (44) of [MT10] in the following way. For the state $v = [h]_{-k_1}[h]_{-k_2} \cdots [h]_{-k_r}\mathbf{1}$ with $k_i \geq 1$, we have

$$Q_v(q) = \sum_{\varphi = \ldots (s,t) \ldots} \prod_{(s,t)} \frac{2(-1)^{s+1}}{(s-1)!(t-1)!} G_{s+t}(q). \tag{5.11}$$

The equation is read as follows: Take the set $\Phi = \{k_1, k_2, \ldots, k_r\}$ and consider all fixed-point-free involutions φ in the symmetric group $\Sigma(\Phi)$. Represent each φ as a product of transpositions $\ldots (s,t) \ldots$ where $s, t \in \Phi$, and take the product over all such transpositions.

For example, let $r \geq 1$ be odd and consider the square-bracket state $[h]_{-r}[h]_{-1}\mathbf{1}$. We have $\Phi = \{r, 1\}$ and clearly out of the two elements of $\Sigma(\Phi)$, only $(r, 1)$ is a fixed-point-free involution. Since it is already a single transposition, we get that

$$Z_{\mathcal{H}}([h]_{-r}[h]_{-1}\mathbf{1}, q) = \frac{2(-1)^{r+1}}{(r-1)!(1-1)!} G_{r+1}(q) = \frac{2}{(r-1)!} G_{r+1}(q).$$

Chapter 6

The p-adic Perspective

The following aforementioned lemma gives an expression for a family of square-bracket states in terms of the usual basis of the Heisenberg VOA:

Lemma 6.0.1 ([FM22], Lemma 10.2)**.** *For an odd integer $r \geq 1$, we have*

$$(r-1)![h]_{-r}[h]_{-1}\mathbf{1} = \sum_{n\geq 0} n! S_r^{(n+1)} h_{-n-1}h_{-1}\mathbf{1} - \frac{B_{r+1}}{r+1}\mathbf{1},$$

where

$$S_r^{(n+1)} = \frac{1}{n!}\sum_{j\geq 0}\binom{n}{j}(-1)^{n+j}(j+1)^{r-1} \tag{6.1}$$

denote Stirling numbers of the second kind.

With $r = \{p^a(p-1)+1\}_{a\geq 0}$, the authors take the p-adic limit of this sequence of states and obtain a state u_1 in the p-adic Heisenberg VOA S. When considering p-adic VOAs, the character map is extended to a map giving rise to p-adic modular forms. In the Heisenberg case, the process is as follows. Theorem 9.6 of [FM22] first states that there is a surjective $\mathbb{Q}_p$-linear map

$$\begin{aligned} \boldsymbol{f} : \mathcal{H} &\to \mathbb{Q}_p[E_2, E_4, E_6] \\ v &\mapsto \eta(q)Z_{\mathcal{H}}(v,q) \end{aligned} \tag{6.2}$$

By taking p-adic limits, it is then shown that $\boldsymbol{f}(u_1) = 2G_2^\star(q)$ where $G_2^\star$ is the *p-adic Eisenstein series* seen in eq. (1.1). Such series are constructed in [Ser73b] as follows: Let $\{k_i\}_{i\geq 0}$ be a sequence of natural numbers with $k_i \geq 4$ for each i and with p-adic limit $k \geq 2$. Then, in the p-adic topology we have

$$\lim_{i\to\infty} G_{k_i}(q) = G_k^\star(q) = \frac{(1-p^{k-1})\zeta(1-k)}{2} + \sum_{n\geq 1}\sigma_{k-1}^\star(n)q^n \tag{6.3}$$

where $\zeta(1-k)$ denotes the *Riemann Zeta function* and $\sigma_k^\star(n)$ is the sum of all divisors of n raised to the power k, which are coprime to p.

6.1 Some Square-Bracket States

We first generalize Lemma 10.2 of [FM22] in order to obtain a broader family of states. To do this, we require two technical lemmas.

Lemma 6.1.1. *For an odd integer $t \geq 1$, we have*

$$[h]_{-1}^t\mathbf{1} = \sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k}\mathbf{1}.$$

Proof. We proceed by induction. Recall the derived expression for the square-bracket state encountered in Section 4.1:

$$[h]_{-1} = h_{-1} - \frac{1}{12}h_1 + \frac{1}{24}h_2 - \frac{19}{720}h_3 + \cdots$$

For $t = 1$ it is clear that $[h]_{-1}\mathbf{1} = h_{-1}\mathbf{1}$ which agrees with

$$[h]_{-1}\mathbf{1} = \sum_{k\geq 0}\binom{1}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{1-2k}\mathbf{1} = h_{-1}\mathbf{1}.$$

Suppose the claim holds up to some $t > 1$. Then, we have

$$\begin{aligned}
&[h]_{-1}\sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k}\mathbf{1} \\
&= \sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k+1}\mathbf{1} + \sum_{k\geq 0}\binom{t}{2k}\frac{2(2k)!(t-2k)}{k!(-24)^{k+1}}h_{-1}^{t-2k-1}\mathbf{1} \\
&= \sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k+1}\mathbf{1} + \sum_{k\geq 0}\frac{2(t!)(t-2k)}{k!(t-2k)!(-24)^{k+1}}h_{-1}^{t-2k-1}\mathbf{1} \\
&= \sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k+1}\mathbf{1} + \sum_{k\geq 0}\frac{2(t!)(k+1)}{(k+1)!(t-2k-1)!(-24)^{k+1}}h_{-1}^{t-2k-1}\mathbf{1} \\
&= \sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k+1}\mathbf{1} + \sum_{k\geq 0}\binom{t}{2k+1}\frac{(2k+2)!}{(k+1)!(-24)^{k+1}}h_{-1}^{t-2k-1}\mathbf{1}.
\end{aligned}$$

Re-indexing the second sum, and making use of Pascal's rule, we get

$$\begin{aligned}
&= h_{-1}^{t+1} + \sum_{k\geq 1}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k+1}\mathbf{1} + \sum_{k\geq 1}\binom{t}{2k-1}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k+1}\mathbf{1} \\
&= h_{-1}^{t+1} + \sum_{k\geq 1}\binom{t+1}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k+1}\mathbf{1} \\
&= [h]_{-1}^{t+1}\mathbf{1}
\end{aligned}$$

which is what we wanted to show. □

It is interesting to note that the expression given in Lemma 6.1.1 is related to *Hermite polynomials* $\mathrm{He}_n(x)$. These can be written in the form

$$\mathrm{He}_n(x) = (x - \partial_x)^n \cdot 1$$

for $n \geq 0$. By adding the factor $1/12$ to the partial derivative operator, and expanding via the binomial formula one obtains the desired expression.

Lemma 6.1.2. *For odd integer $t \geq 1$, we have*

$$h_1[h]_{-1}^t\mathbf{1} = \sum_{k\geq 0}\binom{t}{2k+1}\frac{(2k+1)!}{k!(-24)^k}h_{-1}^{t-2k-1}\mathbf{1}.$$

Proof. The proof is similar to the computation of the second sum in Lemma 6.1.1. □

We may now give the desired generalization which makes use of this new parameter t. Aside from some extra coefficients which we have already calculated above, the process is similar to that of the original lemma.

Proposition 6.1.3. *For odd integers $r, t \geq 1$, we have*

$$\begin{aligned}
(r-1)![h]_{-r}[h]_{-1}^t\mathbf{1} = &\sum_{n\geq 0}\sum_{k\geq 0}\binom{t}{2k}\frac{n!S_r^{(n+1)}(2k)!}{k!(-24)^k}h_{-n-1}h_{-1}^{t-2k}\mathbf{1} \\
&- \sum_{k\geq 0}\binom{t}{2k+1}\frac{B_{r+1}(2k+1)!}{k!(r+1)(-24)^k}h_{-1}^{t-2k-1}\mathbf{1}
\end{aligned}$$

Proof. First, using Lemma 6.1.1 we compute

$$\begin{aligned}
&(r-1)![h]_{-r}[h]_{-1}^{t}\mathbf{1}\\
&=(r-1)![h]_{-r}\sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k}\mathbf{1}\\
&=(r-1)!\mathrm{Res}_z\ z^{-r}e^{z}\sum_{n\in\mathbb{Z}}h_n\left(\sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-1}^{t-2k}\mathbf{1}\right)(e^z-1)^{-n-1}.
\end{aligned}$$

Applying Lemma 6.1.2, we obtain the expression

$$\begin{aligned}
&=(r-1)!\mathrm{Res}_z\ z^{-r}e^{z}\left(\sum_{n\geq 0}\sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-n-1}h_{-1}^{t-2k}\mathbf{1}\,(e^z-1)^{n}\right)\\
&\quad+(r-1)!\mathrm{Res}_z\ z^{-r}e^{z}\sum_{k\geq 0}\binom{t}{2k+1}\frac{(2k+1)!}{k!(-24)^k}h_{-1}^{t-2k-1}\mathbf{1}\,(e^z-1)^{-2},
\end{aligned}$$

and so we must compute the coefficients of the terms

$$\sum_{n\geq 0}\sum_{k\geq 0}\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}h_{-n-1}h_{-1}^{t-2k}\mathbf{1} \tag{6.4}$$

$$\sum_{k\geq 0}\binom{t}{2k+1}\frac{(2k+1)!}{k!(-24)^k}h_{-1}^{t-2k-1}\mathbf{1}. \tag{6.5}$$

For the coefficient of eq. (6.4), we have

$$\begin{aligned}
(r-1)!\mathrm{Res}_z\ z^{-r}e^{z}(e^z-1)^n &=(r-1)!\mathrm{Res}_z\ z^{-r}\sum_{j=0}^{n}\binom{n}{j}(-1)^{n+j}e^{(j+1)z}\\
&=(r-1)!\mathrm{Coeff}_{z^{r-1}}\ \sum_{j=0}^{n}\binom{n}{j}(-1)^{n+j}\left(\sum_{i\geq 0}\frac{(j+1)^i}{i!}z^i\right)\\
&=(r-1)!\sum_{j=0}^{n}\binom{n}{j}(-1)^{n+j}\left(\frac{(j+1)^{r-1}}{(r-1)!}\right)\\
&=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n+j}(j+1)^{r-1}\\
&=n!S_r^{(n+1)}
\end{aligned}$$

which agrees with the statement. For the coefficient of eq. (6.5), we make use of eq. (4.7) to obtain

$$\begin{aligned}
(r-1)!\mathrm{Res}_z\ z^{-r}e^{z}(e^z-1)^{-2} &=-(r-1)!\mathrm{Res}_z\ z^{-r}\partial_z\,(e^z-1)^{-1}\\
&=-(r)!\mathrm{Coeff}_{z^r}\ \left(\sum_{j\geq 0}\frac{B_j}{j!}z^{j-1}\right)\\
&=-\frac{B_{r+1}}{r+1}
\end{aligned}$$

which agrees with the statement and we are done. □

6.2 Characters and p-adic Convergence

We will now take the character of the family of states considered in Proposition 6.1.3 by using eq. (5.11). Then under a suitable choice of sequence in r, we establish the p-adic convergence of these of states, giving rise to the main result of this chapter. First, a combinatorial lemma:

Lemma 6.2.1. *Let $n \geq 0$ be an integer. The number of ways of partitioning a set of $2n$ elements into exactly n pairs is $(2n-1)!!$. This is the product of all odd integers no greater than $2n-1$, and if $n = 0$ then $(-1)!! = 1$.*

Proof. Clearly if $n = 0$ then there is only one way of partitioning a set of no elements into no pairs, and so $(2(0)-1)!! = (-1)!! = 1$. Now let $n \geq 1$. We begin with $2n$ elements and choose 2. From the remaining $2n-2$, we choose another 2 and so on, until there are no elements left. Since there are $n!$ ways of doing this and we are not concerned with order, we obtain

$$\begin{aligned}
\frac{1}{n!}\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2} &= \frac{1}{n!}\left(\frac{(2n)!(2n-2)!(2n-4)!\cdots 2!}{2^n(2n-2)!(2n-4)!\cdots 2!}\right) \\
&= \frac{(2n)!}{2^n n!} \\
&= \frac{(2n)(2n-1)(2n-2)(2n-4)\cdots(2)}{2^n n!}.
\end{aligned}$$

Since the numerator consists of $2n$ terms starting with $2n$ which is even, each even term is divided preciely by one of the 2s in the denominator. After rearranging, we get

$$\begin{aligned}
&= \frac{(n)(2n-1)(n-1)(2n-3)\cdots(1)(1)}{n!} \\
&= (2n-1)(2n-3)\cdots(1) \\
&= (2n-1)!!
\end{aligned}$$

which is what we wanted to show. □

Proposition 6.2.2. *For odd integers $r, t \geq 1$, we have*

$$\eta(q)Z_{\mathcal{H}}([h]_{-r}[h]_{-1}^{t}\boldsymbol{1}, q) = \frac{2^{(t+1)/2}t(t-2)!!}{(r-1)!}G_2^{(t-1)/2}(q)G_{r+1}(q).$$

Proof. We make use of eq. (5.11). In this case, we have the set

$$\Phi = \{r, \underbrace{1_1, 1_2, \ldots, 1_t}_{t \text{ times}}\}$$

where we have labelled the 1s for clarity, and we wish to consider all fixed-point-free involutions of $\Sigma(\Phi)$. That is, we want all ways of partitioning Φ into parts of size 2, something which can be done since t is odd here.

Suppose first that r is paired with 1_1, and the remaining $1_2, \ldots, 1_t$ are paired amongst themselves. Each way of partitioning these remaining 1s into parts of size 2 yields a fixed-point-free involution. For one such partition (and thus involution), since there are $t-1$ remaining 1s which get put in $(t-1)/2$ parts of size 2, we get

$$\left(\frac{2}{(r-1)!}G_{r+1}(q)\right)\left(\frac{2}{(1-1)!}G_2(q)\right)^{(t-1)/2} = \frac{2^{(t+1)/2}}{(r-1)!}G_2^{(t-1)/2}(q)G_{r+1}(q)$$

where we recall that r is odd here. Using Lemma 6.2.1, there are then $(t-2)!!$ distinct ways of partitioning the remaining $1_2, \ldots, 1_t$ into $(t-1)/2$ parts of size 2. Each way yields the same expression as above and so we multiply by the factor $(t-2)!!$. Finally, we may repeat this process

t times, since r can be paired with 1_2 then with 1_3 and so on, until r is paired with 1_t. This yields the expression

$$\frac{2^{(t+1)/2}t(t-2)!!}{(r-1)!}G_2^{(t-1)/2}(q)G_{r+1}(q)$$

which establishes the result. □

With this, we denote (in the notation of [FM22]) the state

$$v_{r,t} = \frac{(r-1)!}{2^{(t+1)/2}t(t-2)!!}[h]_{-r}[h]_{-1}^{t}\mathbf{1}$$

and so it is now immediate for Proposition 6.1.3 that

$$v_{r,t} = \frac{1}{2^{(t+1)/2}t(t-2)!!}\left(\sum_{n\geq 0}\sum_{k\geq 0}\binom{t}{2k}\frac{n!S_r^{(n+1)}(2k)!}{k!(-24)^k}h_{-n-1}h_{-1}^{t-2k}\mathbf{1}\right.$$
$$\left. - \sum_{k\geq 0}\binom{t}{2k+1}\frac{B_{r+1}(2k+1)!}{k!(r+1)(-24)^k}h_{-1}^{t-2k-1}\mathbf{1}\right)$$

and

$$\boldsymbol{f}(v_{r,t}) = G_2^{(t-1)/2}(q)G_{r+1}(q)$$

where $\boldsymbol{f}$ is the map defined in eq. (6.2). The goal is now to consider *sequences* in r of these states for some fixed odd $t \geq 1$, and assess convergence in the p-adic topology. We rescale the states $v_{r,t}$ as in [FM22] by defining $u_{r,t} = (1-p^r)(2^{(t+1)/2}t(t-2)!!)v_{r,t}$ for some odd prime p. That is, define

$$u_{r,t} = (1-p^r)\left(\sum_{n\geq 0}\sum_{k\geq 0}\binom{t}{2k}\frac{n!S_r^{(n+1)}(2k)!}{k!(-24)^k}h_{-n-1}h_{-1}^{t-2k}\mathbf{1}\right.$$
$$\left. - \sum_{k\geq 0}\binom{t}{2k+1}\frac{B_{r+1}(2k+1)!}{k!(r+1)(-24)^k}h_{-1}^{t-2k-1}\mathbf{1}\right)$$

We will be considering the sequence of states $(u_{p^a(p-1)+1,t})_{a\geq 0}$, again for $r,t \geq 1$ odd. The following is an extension of Lemma 10.4 of [FM22]:

Lemma 6.2.3. *Fix $t \geq 1$ odd, and let p be an odd prime with $r = p^a(p-1)+1$, $s = p^b(p-1)+1$ and $a \leq b$. Then for any fixed k within the range $0 \leq k \leq \lfloor t/2 \rfloor$ and any $n \geq 0$ we have*

$$(1-p^r)\binom{t}{2k}\frac{n!S_r^{(n+1)}(2k)!}{k!(-24)^k} \equiv (1-p^s)\binom{t}{2k}\frac{n!S_s^{(n+1)}(2k)!}{k!(-24)^k} \mod p^{a+x+1}.$$

for some fixed integer x.

Proof. Let $|\cdot|_p$ denote the p-adic absolute value. We have

$$\left|\binom{t}{2k}\frac{n!S_r^{(n+1)}(2k)!}{k!(-24)^k} - \binom{t}{2k}\frac{n!S_s^{(n+1)}(2k)!}{k!(-24)^k}\right|_p = \left|n!S_r^{(n+1)} - n!S_s^{(n+1)}\right|_p \left|\binom{t}{2k}\frac{(2k)!}{k!(-24)^k}\right|_p.$$

The rightmost term is dependent only on t and k which are fixed, and so denote by x the p-adic valuation of this term, which is also fixed. We show now that $n!S_r^{(n+1)} \equiv n!S_s^{(n+1)} \mod p^{a+1}$. Recall the formula

$$n!S_r^{(n+1)} = \sum_{j=0}^{n}\binom{n}{j}(-1)^{n+j}(j+1)^{r-1}.$$

There are two cases to consider. If $p \mid (j+1)$ then of course $(j+1)^{r-1} \equiv 0 \mod p^{a+1}$. Suppose $p \nmid (j+1)$. This means that p and $(j+1)$ are coprime, and subsequently that p^{a+1} and $(j+1)$ are coprime. Recall that the classical *Euler totient function* $\varphi(n)$ counts the number of positive integers up to n which are coprime to n, and that $\varphi(p^{a+1}) = p^a(p-1)$. So by Euler's theorem,

$$(j+1)^{p^a(p-1)} = (j+1)^{r-1} \equiv 1 \mod p^{a+1}.$$

which means that in this case we have

$$n!S_r^{(n+1)} \equiv \sum_{\substack{j=0 \\ p\nmid(j+1)}}^{n} \binom{n}{j}(-1)^{n+j} \mod p^{a+1}$$

and since the right hand side above does not depend on r, this establishes that $n!S_r^{(n+1)} \equiv n!S_s^{(n+1)} \mod p^{a+1}$. Putting everything together, we get

$$\binom{t}{2k}\frac{n!S_r^{(n+1)}(2k)!}{k!(-24)^k} \equiv \binom{t}{2k}\frac{n!S_s^{(n+1)}(2k)!}{k!(-24)^k} \mod p^{a+x+1}.$$

Finally since x is fixed, for sufficently large a we have

$$1 - p^{p^a(p-1)+1} \equiv 1 - p^{p^b(p-1)+1} \mod p^{a+x+1},$$

and so by combining the above two congruences together, we obtain the desired result.

□

The convergence of the terms of $u_{r,t}$ involving the Bernoulli numbers requires *Kummer's congruence* (cf. [Kum51]) which states that if $r+1$ and $s+1$ are positive even integers not divisible by $p-1$, (or equivalently, $r, s \not\equiv -1 \mod p-1$) then

$$(1-p^r)\frac{B_{r+1}}{r+1} \equiv (1-p^s)\frac{B_{s+1}}{s+1} \mod p^{m+1}$$

whenever $r+1 \equiv s+1 \mod \varphi(p^{m+1})$.

Lemma 6.2.4. *Fix $t \geq 1$ odd and let $p > 3$ be a prime with $r = p^a(p-1)+1$, $s = p^b(p-1)+1$ and $a \leq b$. Then for k in the range $0 \leq k \leq \lfloor (t-1)/2 \rfloor$, we have*

$$(1-p^r)\binom{t}{2k+1}\frac{B_{r+1}(2k+1)!}{k!(r+1)(-24)^k} \equiv (1-p^s)\binom{t}{2k+1}\frac{B_{s+1}(2k+1)!}{k!(s+1)(-24)^k} \mod p^{a+y+1}$$

for some fixed integer y.

Proof. We proceed similarly to Lemma 6.2.3. First write

$$\left|(1-p^r)\frac{B_{r+1}}{r+1} - (1-p^s)\frac{B_{s+1}}{s+1}\right|_p \left|\binom{t}{2k+1}\frac{(2k+1)!}{k(-24)^k}\right|_p.$$

Once again the rightmost term is dependent only on t and k which are fixed, and so denote by y the p-adic valuation of this term, which is also fixed. Notice that

$$p^a(p-1)+2 \equiv p^b(p-1)+2 \mod p^a(p-1)$$

and $p^a(p-1) = \varphi(p^{a+1})$. Since $r+1$ and $s+1$ are even and clearly not divisible by $p-1$ (since $p > 3$), by Kummer's congruence we obtain

$$\left|(1-p^r)\frac{B_{r+1}}{r+1} - (1-p^s)\frac{B_{s+1}}{s+1}\right|_p \left|\binom{t}{2k+1}\frac{(2k+1)!}{k(-24)^k}\right|_p = \frac{1}{p^{a+y+1}}$$

which gives us the right congruence. □

Putting everything together thus far, we are able to establish the following theorem:

Theorem 6.2.5. *Denote by $\mathfrak{Q}$ the ring of quasi-modular forms and im($\boldsymbol{f}$) the image of the p-adic character map for the Heisenberg VOA in the ring of p-adic modular forms. Then, the quotient im($\boldsymbol{f}$)/$\mathfrak{Q}$ is an infinite dimensional vector space.*

Proof. The congruences established in Lemmas 6.2.3 and 6.2.4 imply that the sequence of states $(u_{p^a(p-1)+1,t})_{a\geq 0}$ for any odd $t \geq 1$ and prime $p > 3$ converges to some state which we will denote u_t, following [FM22]. Then, using Proposition 6.2.2 and in the p-adic topology, we get

$$\begin{aligned} \boldsymbol{f}(u_t) &= \lim_{a\to\infty}(1 - p^{p^a(p-1)+1})2^{(t+1)/2}t(t-2)!!G_2^{(t-1)/2}(q)G_{p^a(p-1)+2}(q) \\ &= 2^{(t+1)/2}t(t-2)!!G_2^{(t-1)/2}(q)\lim_{a\to\infty}(1 - p^{p^a(p-1)+1})G_{p^a(p-1)+2}(q). \end{aligned}$$

Notice that $\lim_{a\to\infty} 1 - p^{p^a(p-1)+1} = 1$ and $\lim_{a\to\infty} p^a(p-1) + 2 = 2$ in the p-adic sense. Making use of 6.3 we obtain

$$\boldsymbol{f}(u_t) = 2^{(t+1)/2}t(t-2)!!G_2^{(t-1)/2}(q)G_2^{\star}(q)$$

where $G_2^{\star}(q)$ is the p-adic modular form encountered in eq. (1.1). After taking the quotient by $\mathfrak{Q}$, any linear combination $\sum_j \alpha_j \boldsymbol{f}(u_{t_j})$ for odd $t_j \geq 1$ and scalars α_j is never quasi-modular. Thus we have shown that in the Heisenberg case, im($\boldsymbol{f}$) contains infinitely many p-adic modular forms of level one which are not quasi-modular. □

6.3 The Angle-Bracket Formalism for the Heisenberg VOA

In this final section, we introduce a new kind of VOA structure obtained from a slight modification of the *Artin-Hasse exponential*:

$$\mathbf{AH}_p(z) = \mathrm{AH}_p(z) - 1 = \exp\left(\sum_{i\geq 0}\frac{z^{p^i}}{p^i}\right) - 1 \in \mathrm{Aut}(\mathcal{O}) \tag{6.6}$$

for a fixed prime p. Denote by $Y\langle a, z\rangle$ the vertex operator transformed under the coordinate transformation $\mathbf{AH}_p(z)$. Then

$$Y\langle a, z\rangle = \sum_{n\in\mathbb{Z}}\langle a\rangle_n z^{-n-1}. \tag{6.7}$$

We wish to proceed in a similar fashion as in the square-bracket formalism, that is, we would like to obtain expressions for certain states in the angle-bracket Heisenberg VOA $\langle\mathcal{H}\rangle$. We use eq. (4.1) to write

$$h\langle z\rangle = Y\langle h_{-1}\mathbf{1}, z\rangle = Y(h_{-1}\mathbf{1}, \mathbf{AH}_p(z))\mathbf{AH}_p'(z). \tag{6.8}$$

We immediately conclude the following facts, the second holding true for any VOA V.

Lemma 6.3.1. *Let $h\langle z\rangle$ be the Heisenberg field transformed under the action of $\mathbf{AH}_p(z)$. Then, $\langle h\rangle_{-1}\boldsymbol{1} = h_{-1}\boldsymbol{1}$.*

Proof. We compute

$$\begin{aligned} \langle h\rangle_{-1}\mathbf{1} &= \mathrm{Res}_z\, z^{-1}\mathbf{AH}_p'(z)\sum_{n\in\mathbb{Z}} h_n\mathbf{1}\,(\mathbf{AH}_p(z))^{-n-1} \\ &= \mathrm{Coeff}_{z^0}\ \mathbf{AH}_p'(z)\sum_{n\geq 0} h_{-n-1}\mathbf{1}\,(\mathbf{AH}_p(z))^{n} \\ &= h_{-1}\mathbf{1} \end{aligned}$$

since only $n = 0$ gives a constant term and $\mathbf{AH}_p'(z) = 1 + o(z)$ in the last equality. □

Lemma 6.3.2. *Let $\langle\omega\rangle$ denote the conformal state transformed under the action of $\mathbf{AH}_p(z)$. We have $\langle\omega\rangle = \omega - (c_V/24)\mathbf{1}$.*

Proof. The n-th derivative of $\mathbf{AH}_p(z)$ for $n \geq 1$ is given by the expression

$$\mathbf{AH}_p^{(n)}(z) = \left(\sum_{i\geq 0} \frac{z^{p^i-1}}{p^i}\right)^n \mathrm{AH}_p(z).$$

Using this, we compute

$$\{\mathbf{AH}_p(z), z\} = \left(\sum_{i\geq 0} \frac{z^{p^i-1}}{p^i}\right)^2 - \frac{3}{2}\left(\sum_{i\geq 0} \frac{z^{p^i-1}}{p^i}\right)^2 = -\frac{1}{2}\left(\sum_{i\geq 0} \frac{z^{p^i-1}}{p^i}\right)^2$$

Making use of the statement of Theorem 3.4.4 results in

$$Y(\omega, \mathbf{AH}_p(z))\left(\sum_{i\geq 0} \frac{z^{p^i-1}}{p^i}\mathrm{AH}_p(z)\right)^2 - \frac{c_V}{24}\left(\sum_{i\geq 0} \frac{z^{p^i-1}}{p^i}\right)^2,$$

and applying this to $\mathbf{1}$ and extracting the constant term as in Proposition 4.2.1 gives the desired expression $\langle\omega\rangle = \omega - (c_V/24)\mathbf{1}$. □

Unlike the square-bracket case, deriving explicit expressions for the angle-bracket modes themselves is a much more difficult task, even when working with the Heisenberg VOA. We illustrate the difficulties here. To obtain expressions for $\langle h\rangle_n = \langle h_{-1}\mathbf{1}\rangle_n$, we multiply eq. (6.7) by z^n to get

$$\langle h\rangle_n = \mathrm{Res}_z\ Y(h_{-1}\mathbf{1}, \mathbf{AH}_p(z))\mathbf{AH}_p'(z)z^n. \tag{6.9}$$

Once again, by Lemma 3.2.2 (a), $Y(h_{-1}\mathbf{1}, \mathbf{AH}_p(z)) \in \mathcal{F}(V)$, and so eq. (6.8) truncates when applied to any element of $\mathcal{H}$. We may thus extend Lemma 2.1.1 to this case with $w = \mathbf{AH}_p(z)$, noting that $\mathbf{AH}_p(z) \in z\mathbb{C}[[z]]$, which gives

$$\langle h\rangle_n = \mathrm{Res}_w\ h(w)\left(\mathbf{AH}_p^{-1}(w)\right)^n \tag{6.10}$$

where $\mathbf{AH}_p^{-1}(w)$ denotes the compositional inverse. Theorem 5.4.2 of [Sta99] gives the following variant of the Lagrange inversion theorem for formal power series: If $v(z) \in zV[[z]]$ then for $n, k \in \mathbb{Z}$ with $k \neq 0$,

$$\mathrm{Coeff}_{z^k}\ \left(v^{-1}(z)\right)^n = \frac{n}{k}\mathrm{Coeff}_{z^{-n}}\ v(z)^{-k}$$

Applied to our case, one would have to look at

$$c(n,k) = \mathrm{Coeff}_{w^k}\left(\mathbf{AH}_p^{-1}(w)\right)^n = \frac{n}{k}\mathrm{Coeff}_{w^{-n}}\left(\mathbf{AH}_p(w)\right)^{-k}.$$

Notice from above and from eq. (6.10) that we can set $c(n,0) = 1$ for all $n \in \mathbb{Z}$, as a term w of degree zero is multiplied by $h_0 w^{-1}$ to yield a term z^{-1} of degree -1 and h_0 acts as the zero operator on $\mathcal{H}$. If we can obtain $c(n,k)$ given any k, then we can write

$$\langle h\rangle_n = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} c(n,k)h_k.$$

If $n = 0$, then $\langle h\rangle_0 = h_0$. Due to the complicated behaviour of the Artin-Hasse exponential, these $c(n,k)$ are challenging to compute for general n, and so we go no further in attempting to derive such explicit expressions.

If we are interested in computations done in the angle-bracket formalism rather than explicit expressions for the modes, we are capable of obtaining the following results. These are done within the context of Section 6.2.

Proposition 6.3.3. *Let $h\langle z\rangle$ be the Heisenberg field transformed under the action of $\mathbf{AH}_p(z)$ for a fixed odd prime p. Then,*

$$(p-1)!\langle h\rangle_{-p}\mathbf{1} = \sum_{n\geq 0} n!S_p^{(n+1)}h_{-n-1}\mathbf{1}.$$

Proof. Using Lemma 6.3.1, we compute

$$\begin{aligned}&(p-1)!\langle h\rangle_{-p}\mathbf{1}\\ &= \mathrm{Res}_z\ z^{-p}\mathbf{AH}_p'(z)\sum_{n\in\mathbb{Z}} h_n\mathbf{1}\left(\mathbf{AH}_p(z)\right)^{-n-1}\\ &= \mathrm{Res}_z\ z^{-p}\mathbf{AH}_p'(z)\sum_{n\geq 0} h_{-n-1}\mathbf{1}\left(\mathbf{AH}_p(z)\right)^{n}.\end{aligned}$$

We compute the coefficient of $\sum_{n\geq 0} h_{-n-1}\mathbf{1}$ as follows:

$$\begin{aligned}&(p-1)!\mathrm{Res}_z\ z^{-p}\mathbf{AH}_p'(z)\left(\mathbf{AH}_p(z)\right)^n\\ &= (p-1)!\mathrm{Res}_z\ z^{-p}\left(\sum_{i\geq 0}\frac{z^{p^i-1}}{p^i}\right)\mathrm{AH}_p(z)\left(\mathrm{AH}_p(z)-1\right)^n\\ &= (p-1)!\mathrm{Res}_z\ z^{-p}\left(\sum_{i\geq 0}\frac{z^{p^i-1}}{p^i}\right)\left(\sum_{m\geq 0}\binom{n}{j}(-1)^{n+m}\left(\mathrm{AH}_p(z)\right)^{m+1}\right)\\ &= (p-1)!\mathrm{Coeff}_{z^{p-1}}\left(\sum_{i\geq 0}\frac{z^{p^i-1}}{p^i}\right)\left(\sum_{m\geq 0}\binom{n}{m}(-1)^{n+m}\left(\exp\left((m+1)\sum_{l\geq 0}\frac{z^{p^l}}{p^l}\right)\right)\right).\end{aligned}$$

There are two instances above where one obtains a term z^{p-1}: First, when the summand corresponding to $i = 1$ is multiplied with the constant term 1 from the exponential series, and second when the summand corresponding to $i = 0$ (which is 1) is multiplied with the $p-1$-th term in the exponential series with $l = 0$. Thus the coefficient of $\sum_{n\geq 0} h_{-n-1}h_{-1}\mathbf{1}$ becomes

$$\begin{aligned}&(p-1)!\left(\frac{1}{p}\sum_{m=0}^{n}\binom{n}{m}(-1)^{n+m}+\sum_{m=0}^{n}\binom{n}{m}(-1)^{n+m}\frac{(m+1)^{p-1}}{(p-1)!}\right)\\ &= \sum_{m=0}^{n}\binom{n}{m}(-1)^{n+m}(m+1)^{p-1}\\ &= n!S_p^{(n+1)}\end{aligned}$$

which agrees with the statement and we are done. □

Once again, along with Lemmas 6.3.1 and 6.3.2 we see that the two formalisms seem to agree:

$$(p-1)!\langle h\rangle_{-p}\mathbf{1} = (p-1)![h]_{-p}\mathbf{1}$$

where we have made use of a simplified version of the result from Proposition 6.1.3. This is not always the case, as the following result will establish. In order to show this, we require a simple preliminary lemma:

Lemma 6.3.4. *For any integer $n\geq 3$ and $l = (n+1)/(n-1)$, $l\in\mathbb{Z}$ if and only if $n = 3$.*

Proof. Clearly if $n = 3$ then $l = 2$. If $l\in\mathbb{Z}$ and $n > 3$, notice that $(n+1)/(n-1)$ is a decreasing function and that $\lim_{n\to\infty}(n+1)/(n-1) = 1$. Thus $l\notin\mathbb{Z}$ if $n > 3$ and so $l\in\mathbb{Z}$ implies $n = 3$. □

Now compare the following result with that of Proposition 6.1.3 where $t = 0$:

Proposition 6.3.5. *Let $h\langle z\rangle$ be the Heisenberg field transformed under the action of $\mathbf{AH}_p(z)$ for a fixed odd prime $p \neq 3$. Then,*

$$(p-1)!\langle h\rangle_{-p}\langle h\rangle_{-1}\mathbf{1} = \sum_{n\geq 0} n!S_p^{(n+1)}h_{-n-1}\mathbf{1} - \left(\frac{B_{p+1}}{p+1} + \frac{(p-1)!}{12}\right)\mathbf{1}.$$

Proof. Using Lemma 6.3.1 once again,

$$\begin{aligned}
&(p-1)!\langle h\rangle_{-p}\langle h\rangle_{-1}\mathbf{1}\\
&= (p-1)!\langle h\rangle_{-p}h_{-1}\mathbf{1}\\
&= (p-1)!\mathrm{Res}_z\, z^{-p}\mathbf{AH}_p'(z)\sum_{n\in\mathbb{Z}} h_n h_{-1}\mathbf{1}\left(\mathbf{AH}_p(z)\right)^{-n-1}\\
&= (p-1)!\mathrm{Res}_z\, z^{-p}\mathbf{AH}_p'(z)\left(\sum_{n\geq 0} h_{-n-1}h_{-1}\mathbf{1}\left(\mathbf{AH}_p(z)\right)^{n} + \left(\mathbf{AH}_p(z)\right)^{-2}\mathbf{1}\right).
\end{aligned}$$

Computing the coefficient of the term $\sum_{n\geq 0} h_{-n-1}h_{-1}$ is the same as in Proposition 6.3.5, and the resulting term is identical. For the coefficient of $\mathbf{1}$, we proceed as in Proposition 6.1.3:

$$\begin{aligned}
&(p-1)!\mathrm{Res}_z\, z^{-p}\mathbf{AH}_p(z)\left(\mathbf{AH}_p(z)\right)^{-2}\\
&= -(p-1)!\mathrm{Res}_z\, z^{-p}\partial_z\left(\mathbf{AH}_p(z)\right)^{-1}\\
&= -(p)!\mathrm{Coeff}_{z^p}\left(\sum_{j\geq 0}\frac{B_j}{j!}\left(\sum_{i\geq 0}\frac{z^{p^i}}{p^i}\right)^{j-1}\right).
\end{aligned}$$

Consider the case $j = 0$. Using the geometric series we get

$$\left(\sum_{i\geq 0}\frac{z^{p^i}}{p^i}\right)^{-1} = \frac{1}{z}\sum_{l\geq 0}\left(-\sum_{i\geq 1}\frac{z^{p^i-1}}{p^i}\right)^{l}.$$

Looking above, for $i \geq 2$, we know that $l(p^i-1)-1 = lp^i - l - 1 > p$ and so no terms z^p appear here. If $i = 0$, then we need to see if $l(p-1)-1 = p$, which is equivalent to the condition that $l = (p+1)/(p-1)$. By Lemma 6.3.4, we must have that $p = 3$, since $l \in \mathbb{Z}$, however we have omitted this case. Thus there are no terms z^p when $j = 0$.

Now, if $j = 1$, then certainly there are no terms z^p. For $j \geq 2$, we have

$$\left(\sum_{i\geq 0}\frac{z^{p^i}}{p^i}\right)^{j-1} = \left(z + \frac{z^p}{p} + \frac{z^{p^2}}{p^2} + \cdots\right)^{j-1}.$$

Thus terms z^p only appear when $j = 2$ and when $j = p+1$. The coefficient of $\mathbf{1}$ is therefore

$$-(p)!\left(\frac{1}{12p} + \frac{B_{p+1}}{(p+1)!}\right) = -\left(\frac{B_{p+1}}{p+1} + \frac{(p-1)!}{12}\right)$$

which agrees with the statement and we are done. □

The above computation establishes that the angle-bracket formalism might bring about new results when considering sequences of states which give rise to p-adic modular forms. Such computations are tedious, however, and so it is natural to speculate whether there exists an ameliorated theory which is better suited to this task.

One such possibility is to develop a theory of *Hecke operators* arising from $\mathcal{H}$. These play an important role in the theory of modular forms as well as p-adic modular forms, and so a natural realization of Hecke operators on $\mathcal{H}$ (or equivalently $[\mathcal{H}]$ or $\langle\mathcal{H}\rangle$) would greatly assist in these kinds of computations. Though this is only hypothetical, the authors of [KSU89a], [KSU89b], and [KSU91] have shown that such operators do arise algebraically in the context of conformal field theory, namely out of the Fock space of the Fermionic algebra. It is then an open question whether similar structures exist in the Heisenberg algebra, in any of its various incarnations.

List of Symbols

Notation	Description
$V[[z]]$	space of formal power series. (2.1)
$V[z]$	space of V-valued polynomials in z. (2.2)
$V[[z,z^{-1}]]$	space of formal Laurent series. (2.3)
$V[z,z^{-1}]$	space of V-valued Laurent polynomials in z. (2.4)
$V((z))$	space of truncated formal Laurent series. (2.5)
$\delta(z)$	formal delta series. (2.6)
Coeff_{z^n}	coefficient of the term z^n. (2.7)
Res_n	coefficient of the term z^{-1}. (2.8)
$\mathcal{F}(V)$	space of fields on V. (2.9)
D	derivation on a ring. (see Proposition 2.2.1)
$\mathfrak{w}$	Witt Lie algebra. (2.12)
$\mathfrak{v}$	Virasoro Lie algebra. (2.14)
L_n	element of Virasoro Lie algebra. (2.14)
$\mathbf{k}$	central element of Virasoro Lie algebra. (2.14)
$\delta_{m,n}$	Kronecker delta function. (2.14)
$\mathcal{P}_n(V)$	space of primary states of weight n. (2.15)
$Y(\cdot,\cdot)$	vertex operator map. (2.19)
$\mathbf{1}$	vacuum state. (2.21)
ω	conformal state. (2.23)
I_V	identity of V. (2.21)
c_V	central charge of VOA. (2.24)
$\widehat{\mathfrak{h}}$	affine Lie algebra associated to $\mathfrak{h}$. (2.39)
$\widehat{\mathfrak{h}}^+$	subalgebra of $\widehat{h}$ of elements of positive weight. (2.43)
$\widehat{\mathfrak{h}}^-$	subalgebra of $\widehat{h}$ of elements of negative weight. (2.44)
$\mathcal{U}(V)$	tensor algebra of V. (2.45)
$S(V)$	symmetric algebra of V. (2.49)
$\mathcal{H}$	Fock space of Heisenberg VOA. (2.48)
$h(z)$	Heisenberg field. (2.51)
h_n	element of $\mathrm{End}(\mathcal{H})$. (2.48)
Ind	induced module. (2.48)
wt	weight. (2.5)
$:A(z)B(z):$	normally ordered product of fields $A(z), B(z)$. (2.52)
$\mathcal{O}$	the set $\mathbb{C}[[z]]$. (3.1)
$\phi(z)$	coordinate transformation, element of $\mathrm{Aut}(\mathcal{O})$. (3.1)
T_ϕ	linear isomorphism associated to ϕ. (3.21)
Y_ϕ	vertex operator transformed under the action of ϕ. (3.25)
$\mathcal{C}(n,m)$	set of all m-compositions of n. (see Proposition 3.1.1)
$\mathcal{C}'(n,m)$	set of all very weak m-compositions of n. (see Lemma 4.1.1)
Q_r	the quotient ring $z\mathbb{C}[[z]]/\langle z^r\rangle$. (3.5)
$v(z)$	the power series $\sum_{i\geq 0} l_i z^{i+1}$ in the exponent of ϕ. (see Proposition 3.2.3)
$\mathbf{v}$	the operator $-\sum_{i\geq 0} l_i L_i$ associated to $v(z)$. (see Proposition 3.2.3)
$R(\epsilon)$	dual numbers. (see Proposition 3.2.3)
Δ_ϕ	representation of $\mathrm{Aut}(\mathcal{O})$ on $\mathcal{F}(V)$ associated to ϕ. (3.22)
Θ_ϕ	representation of $\mathrm{Aut}(\mathcal{O})$ on $\mathrm{Hom}(V,\mathcal{F}(V))$ associated to ϕ. (see Proposition 3.3.2)

Notation	Description
$\phi_z(t)$	the coordinate transformation $\phi(z+t)-\phi(z)$. (see Lemma 3.3.1)
$\{\phi(z), z\}$	Schwarzian derivative of $\phi(z)$ with respect to z. (see Lemma 3.4.3)
$[V]$	Fock space of square-bracket VOA. (4.13)
$[\mathcal{H}]$	Fock space of square-bracket Heisenberg VOA. (see Theorem 5.2.3)
$Y[\cdot,\cdot]$	vertex operator transformed under the action of e^z-1. (4.2)
$h[z]$	Heisenberg field transformed under the action of e^z-1. (4.2)
$[h]_n$	element of $\mathrm{End}([\mathcal{H}])$. (4.3)
$[\omega]$	square-bracket conformal state. (see Proposition 4.2.1)
B_n	n-th Bernoulli number. (4.7)
$s_n^{(k)}$	signed Stirling number of the first kind on n permutations with k disjoint cycles. (4.8)
$S_n^{(m)}$	Stirling number of the second kind on n objects in m subsets. (6.1)
$\mathfrak{H}$	complex upper-half plane. (see Section 5.1)
Γ	the modular group $SL_2(\mathbb{Z})$. (5.2)
$G_k(q)$	weight k Eisenstein series. (5.5)
$E_k(q)$	normalized weight k Eisenstein series. (5.7)
$\mathfrak{Q}$	algebra of quasi-modular forms of level one. (see Theorem 5.1.1)
$\mathfrak{M}$	algebra of holomorphic modular forms of level one. (see Theorem 5.1.1)
$Q_v(q)$	quasi-modular form of weight k attached to the state v. (see Theorem 5.1.1)
$Z_V(\cdot, q)$	character map of VOA. (5.9)
Tr	trace. (5.9)
$o(v)$	zero mode of v. (5.8)
$\sigma_k(n)$	divisor function. (5.6)
$\eta(q)$	Dedekind eta-function. (see Proposition 5.2.1)
$\varphi(\cdot)$	Euler totient function. (see Lemma 6.2.3)
$\mu(\cdot)$	mobius function. (1.2)
$p(n)$	unrestricted partition function on n elements. (see Proposition 5.2.1)
$\sigma^\star(n)$	sum of divisors of n coprime to a prime p. (1.1)
$\Sigma(A)$	symmetric group on the set A. (5.11)
$\mathbb{Q}_p$	ring of p-adic numbers. (6.2)
$\boldsymbol{f}$	p-adic character map $\eta(q)Z_{\mathcal{H}}(\cdot, q)$. (6.2)
S	p-adic Heisenberg VOA. (6.2)
$G_k^\star(q)$	weight k p-adic Eisenstein series. (1.1)
$n!!$	product of all odd integers no greater than n. (see Lemma 6.2.1)
$\lvert\cdot\rvert_p$	p-adic absolute value. (see Lemma 6.2.3)
$\mathrm{AH}_p(z)$	Artin-Hasse exponential. (1.2)
$\mathbf{AH}_p(z)$	coordinate transformation $\mathrm{AH}_p(z)-1$. (6.6)

Notation	Description
$\langle\mathcal{H}\rangle$	Fock space of angle-bracket Heisenberg VOA. (6.7)
$Y\langle\cdot,\cdot\rangle$	vertex operator transformed under the action of $\mathbf{AH}_p(z)$. (6.7)
$h\langle z\rangle$	Heisenberg field transformed under the action of $\mathbf{AH}_p(z)$. (6.8)
$\langle h\rangle_n$	element of $\mathrm{End}(\langle\mathcal{H}\rangle)$. (6.9)
$\langle\omega\rangle$	angle-bracket conformal state. (see Lemma 6.3.2)

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