Tropical Mutation Schemes and Examples

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Lay Abstract

Informally, algebraic geometry is the study of solution sets to systems of polynomial equations, called algebraic varieties. Such systems are ubiquitous across the sciences, being found as biological models, optimization problems, revenue models, and much more. However, it is a difficult problem in general to ascertain salient properties of the solutions to these systems. One type of algebraic variety which is easier to work with is a toric variety. These varieties can be associated to simpler mathematical objects such as lattices, polytopes and fans, and important geometric properties of the variety can then be obtained via analyzing properties of these simpler objects. This thesis introduces the notion of a tropical mutation scheme, which is a generalization of a lattice. A broader class of algebraic varieties can be associated with tropical mutation schemes in a similar manner to how toric varieties are associated with lattices. We then compute this association explicitly in the case of the simplest non-trivial examples of a tropical mutation scheme, rank 2 tropical mutation schemes with 2 charts.

Abstract

This thesis provides an introduction to the theory of tropical mutation schemes, and computes explicit examples. Tropical mutation schemes generalize toric geometry. The study of toric varieties is a popular area of algebraic geometry, due to toric varieties' strong combinatorial interpretations. In particular, the characters and one-parameter subgroups of the rank r algebraic torus form a pair of dual lattices of rank r, isomorphic to $\mathbb{Z}^r$. We can then construct toric varieties from fans in these lattices, and compactifications of the algebraic torus are parametrized by full dimensional convex polytopes.

A tropical mutation scheme is a finite collections of lattices, equipped with bijective piecewise-linear functions between each pair of lattices, where these functions satisfy certain compatibility conditions. They generalize lattices in the sense that a lattice can be viewed as the trivial tropical mutation scheme. We also introduce the space of points of a tropical mutation scheme, which is the set of functions from a tropical mutation scheme to $\mathbb{Z}$ which satisfy a minimum condition. A priori, the structure of the space of points of a tropical mutation scheme is unknown, but in certain cases can be identified by the elements of another tropical mutation scheme, inducing a dual pairing between the two tropical mutation schemes. When we have a strict dual pairing of tropical mutation schemes, we can sometimes construct an algebra to be a detropicalization of the pairing. In the trivial case, the coordinate ring of the algebraic torus is a detropicalization of a single lattice and its dual. Thus, when we can construct a detropicalization for a non-trivial strict dual pairing, we recover much of the useful combinatorics from the toric case.

This thesis shows that all rank 2 tropical mutation schemes on two lattice charts are autodual, meaning there is a dual pairing between the tropical mutation scheme and its own space of points. Furthermore, we construct a detropicalization for these tropical mutation schemes. We end the thesis by reviewing open questions and future directions for the theory of tropical mutation schemes.

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There are so many to whom I owe so much for their support and guidance throughout my academic pursuits.

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On a personal note, my studies would not have been possible without the support of my parents, Ian and Carol. From an early age they emphasized the value of learning and the importance of curiosity. Though they do not understand my choice of mathematics as a scholarly discipline, their wholehearted support means the world to me. Thank you as well to my partner, Michelle. The value of your love and support cannot be expressed in words.

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Contents

Declaration of Authorship

I, Adrian Cook, declare that this thesis titled, "Tropical Mutation Schemes and Examples" and the work presented in it are my own. I wrote this thesis document, including an introduction to the theory of tropical mutation schemes and computations for rank-2 tropical mutation schemes on 2 charts. I explicitly computed the space of points for rank-2 tropical mutation schemes on 2 charts, and showed that all such tropical mutation schemes are autodual. I then proved that all these tropical mutation schemes are detropicalizatible, by explicitly constructing a detropicalization. I modeled this result from a proof of detropicalization of a different class of tropical mutation schemes by Dr. Laura Escobar, Dr. Megumi Harada and Dr. Chris Manon.

My supervisor, Dr. Harada, and Dr. Escobar and Dr. Manon first definied the notion of a tropical mutation scheme, and other preliminaries on which this thesis is built. Dr. Harada has provided detailed guidance and feedback throughout the entire process. Dr. Escobar and Dr. Manon have answered many clarifying questions over the course of this study. My commitee members, Dr. Adam Van Tuyl and Dr. Jenna Rajchgot have also provided edits and feedback on this manuscript.

To my teachers

"For whose benefit, then, did I learn it all? If it was for your own benefit that you learnt it you have no call to fear that your trouble may have been wasted."

Seneca

Chapter 1

Introduction

Toric Varieties are a well known and well studied subset of algebraic varieties due to their connections with combinatorial objects. In fact, many varieties with which we are most familiar, such as $\mathbb{P}^n$, $\mathbb{A}^n$, and $\mathbb{P}^k \times \mathbb{P}^{n-k}$, are examples of toric varieties. An algebraic torus is a space that is isomorphic to n copies of the multiplicative group of a field, $(k^*)^n$. In this thesis, we will work over $\mathbb{C}$ unless otherwise stated, so we will denote by $\mathbb{T}$ the algebraic torus over the complex numbers, i.e. $\mathbb{T} \cong \mathbb{C}^*$, and $\mathbb{T}^n \cong (\mathbb{C}^*)^n$. Because $(\mathbb{C}^*)^n$ forms an abelian group under component-wise multiplication, the affine variety $\mathbb{T}^n$ also inherits this group structure. The coordinate ring of the n-dimensional algebraic torus is the Laurent polynomial ring, $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$.

Tropical mutation schemes, the subject of this paper, seek to generalize the combinatorics of toric varieties, so it will be useful to summarize these now. Given an algebraic torus $\mathbb{T}^n$, a *character of* $\mathbb{T}$ is a group homomorphism $m : \mathbb{T}^n \to \mathbb{C}^*$ The characters of $\mathbb{T}^n$ are exactly the maps given by

$$(t_1, \dots, t_n) \mapsto t_1^{a_1} \dots t_n^{a_n}, \text{ for } (a_1, \dots, a_n) \in \mathbb{Z}^n$$

As a corollary, we find that the characters of the algebraic torus form a lattice, called the *character lattice* of $\mathbb{T}$, denoted by $M \cong \mathbb{Z}^n$. The above proposition also shows that the characters of the torus are exactly the Laurent monomials.

Given the lattice M, we can also construct the dual lattice $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^n$. The lattice N is called the *cocharacter lattice*, or the *lattice of one parameter subgroups* of the algebraic torus. It comprises all group homomorphisms

$$n : \mathbb{C}^* \to \mathbb{T}^n.$$

Similarly to the characters, the cocharacters of the algebraic torus are given by the morphisms

$$t \mapsto (t^{a_1}, \dots, t^{a_n}) \text{ for } (a_1, \dots, a_n) \in \mathbb{Z}^n.$$

Fixing coordinates for $\mathbb{T}^n$ gives an identification of M and N with $\mathbb{Z}^n$, and induces a natural bilinear product between elements of M and N. Explicitly,

$$\langle \cdot, \cdot \rangle : M \times N \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z}, \ \langle m, n \rangle \mapsto \lambda^m \circ \chi^n,$$

where λ^n is the cocharacter given by n, and χ^m is the character given by m. Conversely, given a lattice $N \cong \mathbb{Z}^n$, we can cannonically construct an algebraic torus whose cocharacter lattice is N, namely $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*$.

A *toric variety* is an irreducible variety which contains $(\mathbb{C}^*)^n$ as an open dense subset, such that the action of $(\mathbb{C}^*)^n$ on itself extends to an action on the whole variety. A simple example is $\mathbb{A}^1$, where the action of the torus $\mathbb{C}^*$ extends trivially to 0.

In general, a toric variety can be thought of as a union of $\mathbb{T}^n$-orbits. Information about these $\mathbb{T}^n$-orbits can be recorded using a polyhedral fan Σ in the character or cocharacter lattice of $\mathbb{T}^n$. Furthermore, projective toric varieties are parametrized by the full dimensional convex lattice polytopes. Therefore, by studying the combinatorial properties of these fans, cones and polytopes associated to toric varieties, we can recover geometric and algebraic properties of the underlying varieties and coordinate rings, respectively. Readers interested in learning more about this correspondence are referred to the book by Cox, Little, and Schenck [1].

Another place combinatorics and fans arise in algebraic geometry is in tropical geometry. Often called the "combinatorial shadow" of algebraic geometry, tropical geometry associates to each algebraic variety over a valued field (i.e. a field equipped with a valuation) a polyhedral complex called a *tropical variety* which can be studied to obtain information about the underlying variety. When the field over which we are working is trivially valued, which is usually the case when working over $\mathbb{C}$, the aforementioned polyhedral complex is actually a polyhedral fan. Furthermore, if the ideal I which defines a variety $\mathbb{V}(I)$ is homogeneous, then the associated tropical variety is a particular subfan of the Gröbner fan of $\mathbb{I}(V)$ [11].

To understand how we obtain a tropical variety from a classical one, first we will define how to obtain the tropicalization of a Laurent polynomial. Let K be a valued field and take $f = \sum C_i X_i^{\alpha_i}$ a Laurent polynomial in n variables over K. The *tropicalization of* f is

$$trop(f) : \mathbb{R}^n \to \mathbb{R}$$

where $trop(f)(w) = min\{val(C_i) + w \cdot \alpha_i\}$. The *tropical hypersurface* defined by a Laurent polynomial f is the non-differentiable locus of $trop(f)$, equivalently, the vectors $w \in \mathbb{R}^n$ such that the minimum is achieved at least twice in $min\{val(C_i) + w \cdot \alpha_i\}$.

1.0.1 Example. As a simple example, consider the polynomial $f(x, y) = x + y + 1$, $f \in \mathbb{C}[x^{\pm}, y^{\pm}]$, where we consider $\mathbb{C}$ with the trivial valuation. Then $trop(f)(w_1, w_2) = min\{w_1, w_2, 0\}$. The tropical hypersurface corresponding to f therefore is composed of three rays, $w_1 = w_2 < 0$, $w_1 = 0 < w_2$, and $w_2 = 0 < w_1$. For an arbitrary algebraic variety V, i.e. not a hypersurface, the corresponding tropical variety can be found by intersecting the tropical hypersurfaces for each $f \in \mathbb{I}(V)$. It is a fundamental result of tropical geometry that there is some finite set $\{f_1, \ldots, f_k\}$ which generate $\mathbb{I}(V)$ such that the intersection of the tropical hypersurfaces of each f_i is the tropical variety of V ([10] Theorem 2.6.6). Such a set is called a *tropical basis* of $\mathbb{I}(V)$. Readers interested in learning more about tropical geometry should refer to the book by Maclagan and Sturmfelds [10].

When looked at through the right lens, we see that cluster algebras and cluster varieties are a natural generalization of toric varieties. Cluster varieties are a type of algebraic variety which contain algebraic tori that are glued using specific bilinear maps called *cluster mutations* as an open dense subset. They were introduced by Fock and Goncharov in a series of papers approximately 15 years ago (see [5],[6], and [7]). They were introduced as the geometric counterpart of a class of commutative algebras called cluster algebras, which are similarly constructed by taking the union of (possibly infinitely many) generating sets related by cluster mutations. Cluster algebras were first defined in [8] by Fomin and Zelevinsky in the early 2000s, and have since become a popular object of study. A typical way of building cluster algebras is using quivers.

Given a quiver Q with mutable vertices $\{v_1, \dots, v_n\}$ and frozen vertices $\{n_1, \dots, n_k\}$, we obtain a new quiver by mutating at a vertex v_i, and we denote this quiver by $\mu_i(Q)$. The mutable vertices correspond to variables which form a free generating set called a cluster for the field of rational functions over $k(n_1, \dots, n_k)$. By mutating at a vertex v_i, we obtain a new cluster, $(v_1, \dots, v_{i-1}, \mu(v_i), \dots, v_n)$, and a relation on $\mu(v_i)$. The cluster algebra associated to the quiver Q is then the union of all possible clusters obtained from all possible mutations, and all relations obtained from this process. We will illustrate with a small example.

1.0.2 Example. Take Q to be the quiver with one mutable variable x, and one fixed variable y, with an arrow pointing from x to y, as shown below with the frozen vertex, which is immutable, filled in.

$$\circ_x \longrightarrow \bullet_y$$

The only possible mutation is at x, and doing so yields

$$\circ_{x'} \longleftarrow \bullet_y$$

and the relation $x' = \frac{y+1}{x}$. Since quiver mutation is an involution, the only two clusters are (x) and (x'). Thus, the cluster algebra associated to this quiver is the ring

$$A = \mathbb{C}[x, x', y^{\pm}]/\langle xx' - y - 1\rangle$$

To see that $Spec(A) = \{(x, x', y) \mid xx' - y - 1 = 0\}$ is contains an open dense subset which is covered by toric charts, consider the tori

$$\mathcal{A}_1 = \{(x, y) \mid xy \neq 0\}$$

and

$$\mathcal{A}_2 = \{(x', y) \mid x'y \neq 0\}.$$

The maps

$$\psi_i : A_1 \to Spec(A)$$

where $\psi_i(x, y) = (x, \frac{y+1}{x}, y)$ and

$$\psi_2 : A_2 \to Spec(A)$$

given by $\psi_2(x', y) = (\frac{y+1}{x'}, x', y)$ cover an open dense subset of $Spec(A)$ with toric charts. For a more detailed introduction to cluster algebras, see this expository paper by Williams [12].

We can also view cluster varieties through the lens of tropical geometry. By tropicalizing the mutation relations of a cluster algebra, we obtain piecewise linear maps between the lattices of the underlying tori. Therefore, the "combinatorial shadows" of cluster varieties are collections of lattices equiped with piecewise linear maps between them. Thus, a natural question becomes, "Can we set up a general theory from piecewise linear maps between lattices which allows us to recover an analogy to the toric dictionary for a broader class of varieties which includes cluster varieties?" Once we have obtained this combinatorial object, we can ask, "When did this combinatorial object come from a geometric one?"

Though in its infancy, the theory of Tropical Mutation Schemes attempts to do just that. Informally, a tropical mutation scheme is a finite collection of lattices, equipped with bijective piecewise linear maps between each pair, subject to some compatibility conditions. After providing a basic overview of the theory of tropical mutation schemes, we will focus our attention to a special case of rank-2 tropical mutation schemes with exactly 2 lattice charts. Though this type is the simplest type of not-trivial tropical mutation scheme, it still represents an infinite class of tropical mutation schemes, and is therefore an excellent place to begin. In this thesis, we will show that these rank-2 2-chart tropical mutation schemes satisfy many of the properties we desire in tropical mutation schemes, including fullness, dualizability, and detropicalizability. Tropical Mutation Schemes were first defined by Escobar, Harada, and Manon in [3],[4], and build off of work done by Kaveh and Manon [9], and Escobar and Harada [2] on algebraic and geometric wall crossing maps.

The structure of this thesis is as follows. First, we will summarize the theory of tropical mutation schemes, including the definition of a tropical mutations scheme, as well as their space of points, and other important concepts. We will also remind the reader of some classical concepts in order to set notation. In chapter 3, we will compute the space of points for a special class of a rank 2 tropical mutation scheme on 2 charts. As a consequence, we will show that these tropical mutation schemes are both full, and autodual. In the following chapter, we will show that any of these rank 2 tropical mutation schemes can be realized as a detropicalization of a particular polynomial ring. Finally, we will briefly discuss some future directions of study.

Chapter 2

Preliminaries and Definitions

In this chapter, we will remind the reader about some notions in order to set notation, as well as introduce definitions and concepts related to tropical mutation schemes which we will explore in the latter sections of this thesis. In particular, we will define a tropical mutation scheme, as well as the space of points, and the canonical semialgebra of a tropical mutation scheme. We also provide illustrative examples of these concepts.

2.1 Lattices

We begin by reviewing classical notions, mainly to set notation. We will define lattices, piecewise linear functions on lattices, and related concepts.

The starting point of this thesis is the notion of a lattice. Indeed, our main object of study, the tropical mutation scheme, is a generalization of a lattice. First, we will define a classical lattice, which will become the trivial tropical mutation scheme.

2.1.1 Definition. We say that M is a *lattice* if it is a free $\mathbb{Z}$-module of finite rank. Thus, $M \cong \mathbb{Z}^r$, for some $r \in \mathbb{Z}_{>0}$. Given a lattice M, we may define its dual lattice N, as $N := Hom_{\mathbb{Z}}(M, \mathbb{Z})$. There is then a canonical bilinear pairing $\langle -, - \rangle$, where $\langle m, n \rangle := n(m)$ for $n \in N$ and $m \in M$.

The next object we will need before we introduce the tropical mutation scheme is a piecewise linear map. Informally, this is a map of lattices which is linear when restricted to particular cones in the lattice. We will introduce some notions related to piecewise linear functions in order to set notation, the first being a hyperplane and a halfspace.

2.1.2 Definition. Let V be a finite-dimensional vector space over $\mathbb{R}$. Let $p \in V^*$, the dual vector space of V. A *hyperplane* of V defined by p, denoted by H_p is the set

$$H_p := \{x \in V \mid p(x) = 0\}.$$

Similarly, a *halfspace* of V is the set

$$H_p := \{x \in V \mid p(x) \geq 0\}$$

for some $p \in V^*$.

Essential to defining piecewise linear functions are cones and fans, which we will define below.

2.1.3 Definition. Let V be a finite-dimensional vector space over $\mathbb{R}$. A subset $\mathcal{C}$ of V is a *cone* if it is closed with respect to taking non-negative linear combinations, i.e., if $x, y \in \mathcal{C}$, then $\alpha x + \beta y \in \mathcal{C}$ for all $\alpha, \beta \in \mathbb{R}_{\geq 0}$. A cone $\mathcal{C}$ is said to be *polyhedral* if there exists a finite set of vectors $\{v_1, \ldots, v_n\} \in V$ such that

$$C = \{a_1 v_1 + \cdots + a_n v_n \mid a_i \in \mathbb{R}_{\geq 0}\}. \tag{2.1.1}$$

Conversely, given a finite set of vectors $\{v_1, \ldots, v_n\}$ in V, we define the cone generated by $\{v_1, \ldots, v_n\}$, denoted by $\mathcal{C}(v_1, \ldots, v_n)$ is the set given by the right hand side of equation (2.1.1).

Given a polyhedral cone, $\mathcal{C}$, a *face* σ of $\mathcal{C}$ is a subset of $\mathcal{C}$ which is supported on a hyperplane, i.e, $\sigma = \mathcal{C} \cap H_p$ for a hyperplane H_p, for some $p \in V^*$.

Collections of cones satisfying certain properties are called fans, which will be central to the theory of tropical mutation schemes.

2.1.4 Definition. Let V be a finite-dimensional vector space over $\mathbb{R}$. A collection of finitely many cones Σ in V is called a *fan* if

1. the intersection of any two cones σ and σ' is a face of each, and
2. if a cone σ is in Σ, then every face of σ is in Σ as well.

The *support*, $|\Sigma|$, of the fan Σ, is the set of all vectors $v \in V$ which are contained in some cone $\sigma \in \Sigma$, i.e. there exists a $\sigma \in \Sigma$ such that $v \in \sigma$. We say a fan is *complete* if $|\Sigma| = V$.

Given any two fans in a vector space, we can build a new fan called the common refinement.

2.1.5 Definition. Let V be a finite dimensional vector space over $\mathbb{R}$, and let Σ and Σ' be fans in V. We say that Σ' is a *refinement* of Σ if

1. each cone $C \in \Sigma'$ is contained in a cone of Σ and
2. $|\Sigma| = |\Sigma'|$.

If Σ' is a refinement of Σ, we say that Σ is *coarser* that Σ'. Given two fans Σ and Σ', the *common refinement* of both, $\Sigma + \Sigma'$, is the coarsest fan which is a refinement of both Σ and Σ', i.e. is some fan Σ^* is a refinement of both Σ and Σ', then Σ^* is a refinement of $\Sigma + \Sigma'$.

In general, when we say a fan Σ is the coarsest fan satisfying a certain property, we mean that any other fan Σ' which also satisfies said property is a refinement of Σ.

Now that we have set definitions for lattices and fans, we are ready to define piecewise linear functions over lattices, which are central to the theory of tropical mutation schemes.

2.1.6 Definition. Let M be a lattice. A *piecewise linear function* on M is a function

$$\Phi : M \to \mathbb{Z}$$

such that there exists a complete fan Σ over $M \otimes \mathbb{R}$ such that for each cone σ in Σ, and for any $m_1, m_2 \in \sigma$, we have

$$\Phi(m_1 + m_2) = \Phi(m_1) + \Phi(m_2).$$

In other words, for each cone $\sigma \in \Sigma$, the restriction $\Phi|_\sigma$ of Φ to σ is linear. We further require that the induced function obtained by tensoring with $\mathbb{R}$,

$$\tilde{\Phi} : M \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R},$$

is continuous with respect to the usual topology on $\mathbb{R}^n$. Given a piecewise linear function Φ, we use $\Sigma(\Phi)$ to denote the coarsest fan such that the restriction of Φ to each cone $\sigma \in \Sigma(\Phi)$ is linear, and we call this *the fan of* Φ.

The set of piecewise linear functions on a lattice M satisfies some additional algebraic structure, which we will describe below.

2.1.7 Definition. A *semialgebra* is a monoid A equipped with a binary operation $\odot$ which is associative and distributive over the addition operation of A.

We will now define two binary operations on the set of piecewise linear functions on a lattice M, min and $+$.

2.1.8 Definition. Given two piecewise linear functions f and g, over a lattice M, $f + g$ is the function $f + g : M \to \mathbb{Z}$ such that $f + g(m) = f(m) + g(m)$. This is called the *sum* of the piecewise linear functions. Given two piecewise linear functions, f and g, $min\{f, g\}$ is the function $min\{f, g\} : M \to \mathbb{Z}$ such that

$$min\{f, g\}(m) = \begin{cases} f(m) \text{ whenever } f(m) \leq g(m) \\ g(m) \text{ whenever } g(m) \leq f(m) \end{cases}$$

We denote the set of piecewise linear functions over a lattice M by $\mathcal{O}_M$.

2.1.9 Remark. Here, we are defining piecewise linear functions over $\mathbb{Z}$, but they are often defined over $\mathbb{R}$. The definitions and proofs to follow work identically when M is an $\mathbb{R}$ vector space, and the image of the piecewise linear function is $\mathbb{R}$.

2.1.10 Theorem. The set of piecewise linear functions on a lattice M forms a semialgebra under the operations min and $+$.

Proof. We will first show that $\mathcal{O}_M$ is a monoid under the min operation. Take ϕ and θ to be piecewise linear functions on M. For convention, we include ∞ as the additive identity, to make $\mathcal{O}_M$ a monoid, where for any $f \in \mathcal{O}_M$ we have that $min\{f, \infty\} = f$, and $f + \infty = \infty$. It is clear that $min\{f, g\}$ is continuous, therefore, we only need to show that there exists some complete fan on such that the restriction to each cone is linear. Take $\Sigma(g) + \Sigma(f)$ to be the common refinement of the fans of f and g. Now let Σ^* be the complete fan whose cones are given by the regions where $min\{f, g\} = g$ and where $min\{f, g\} = f$, and the boundaries of each cone are the regions where $f = g$. Then on the common refinement of $\Sigma(g) + \Sigma(f)$ and Σ^*, we will have that the restriction of $min\{f, g\} = f$ or g, and further that it is linear. Thus, $(\mathcal{O}_m, min)$ is a monoid.

Now we will show that sums of piecewise linear functions are piecewise linear. Let ϕ and θ be two piecewise linear functions on M, and let $\Sigma(\phi)$ and $\Sigma(\theta)$ be the fans of ϕ and θ, respectively. Take $\Sigma(\phi) + \Sigma(\theta)$, the common refinement of the two fans. Then certainly, for every cone $\sigma \in \Sigma(\phi) + \Sigma(\theta)$, both ϕ and θ are linear on σ. Therefore, $\phi + \theta$ is linear as well. Since both $\tilde{\phi}$ and $\tilde{\theta}$, are continuous, certainly $\tilde{\phi} + \tilde{\theta}$ will be continuous. Therefore, $\mathcal{O}_M$ is closed under addition.

It is easy to check that $+$ is associate and distributive over *min*. Therefore, the set of piecewise linear functions over a lattice M forms a semialgebra. □

We have now defined piecewise linear functions on lattices, and we will now analougously define piecewise linear maps between lattices.

2.1.11 Definition. Let r be a positive integer. Given two lattices of rank r, M and N we say a function

$$\Phi : M \to N$$

is a *piecewise linear map of lattices* if there exists some complete fan $\Sigma \in M \otimes_{\mathbb{Z}} \mathbb{R}$ such that for any cone $\sigma \in \Sigma$ we have the following property for the restriction map $\Phi|_\sigma$:

$$\Phi|_\sigma(m_1 + m_2) = \Phi|_\sigma(m_1) + \Phi|_\sigma(m_2) \text{ for any } m_1, m_2 \in M$$

In other words, the map is linear when restricted to each cone of Σ. As before, we further require that the induced map obtained by tensoring with $\mathbb{R}$

$$\tilde{\Phi} : M \otimes \mathbb{R} \to N \otimes \mathbb{R}$$

is continuous with respect to the usual topology of $\mathbb{R}^n$. As before, given a piecewise linear map of lattices Φ, we use $\Sigma(\Phi)$ to denote the coarsest fan such that the restriction of Φ to each cone $\sigma \in \Sigma(\Phi)$ is linear, all call this *the fan of* Φ.

We will use piecewise linear maps between lattices to define our main object of study, the tropical mutation scheme.

2.2 Tropical Mutation Schemes

We are now ready to define a tropical mutation scheme. A tropical mutation scheme is a generalization of a lattice in the sense that a classical lattice is a trivial tropical mutation scheme. Informally, it is a collection of lattices, equipped with mutation maps between any pair of lattices. A classical lattice is a tropical mutation scheme with only 1 chart and no mutation maps other than the identity. Tropical mutation schemes arose from the study of tropical geometry and toric geometry. Using this tool, we can recover combinatorial results from toric geometry in a more general setting, which we will see in later chapters in the context of rank-2 tropical mutation schemes with 2 charts.

2.2.1 Definition. Let r be an element of $\mathbb{Z}_{>0}$. A *tropical mutation scheme* over $\mathbb{Z}$ of rank r, denoted by $\mathcal{M}$, is a finite collection of lattices of rank r. We denote the set of lattices which comprise a tropical mutation scheme by $L(\mathcal{M})$ (so that $\mathcal{M} = (M_\sigma)_{\sigma \in L(\mathcal{M})}$). A tropical mutation scheme also requires a set of invertible piecewise linear maps:

$$\mu_{\sigma,\tau} : M_\sigma \to M_\tau \ \text{ for all } \sigma,\ \tau \in L(\mathcal{M}) \tag{2.2.1}$$

which fulfill the following conditions:

1) $\mu_{\sigma,\sigma} = id_{M_\sigma}$ for all $\sigma \in L(\mathcal{M})$
2) $\mu^{-1}_{\tau,\sigma} = \mu_{\sigma,\tau}$ for all $\sigma,\ \tau \in L(\mathcal{M})$
3) $\mu_{\sigma,\tau} \circ \mu_{\tau,\delta} = \mu_{\sigma,\delta}$ for all $\sigma,\ \tau,\ \delta \in L(\mathcal{M})$

We will call each $\mu_{\sigma,\tau}$ a *mutation*, or, *a mutation map of* $\mathcal{M}$. The set of all charts of a tropical mutation scheme will be denoted by $L(\mathcal{M})$. The lattices M_σ for each $\sigma \in L(\mathcal{M})$ are called the *charts* of $\mathcal{M}$, i.e. $L(\mathcal{M})$ is an indexing set for the underlying latties of a tropical mutation scheme.

We will now discuss how to think about the elements of each chart of a tropical mutation scheme as forming a cohesive structure.

2.2.2 Definition. Let $\mathcal{M}$ be a tropical mutation scheme and consider a chart M_i. Take $m_i \in M_i$, and consider the the images of m_i under all possible mutation functions. The set composed of m_i along with its images under all possible mutation functions is called an *element* of $\mathcal{M}$. The collection of all such elements is called the set of elements of $\mathcal{M}$. We abuse notation slightly to denote the set of elements of a tropical mutation scheme as $\mathcal{M}$.

Notice that the set of elements of a tropical mutation scheme is in bijection with each chart M_j, since for each j we have the projection π_j which reads of the part of an element corresponding to the j-th chart. Formally:

$$\pi_\tau : \mathcal{M} \to M_\tau$$

$$(m_\sigma)_{\sigma \in L(\mathcal{M})} \mapsto m_\tau \tag{2.2.2}$$

2.2.3 Remark. Unlike the situation of a classical lattice, there is not necessarily a well-defined operation of addition on (the set of elements of) $\mathcal{M}$. Given two elements $m_1, m_2 \in \mathcal{M}$, and σ, τ being different charts of $L(\mathcal{M})$, then addition in M_σ and M_τ may yield different results. On the other hand, if we fix some $\tau \in L(\mathcal{M})$, τ-addition is denoted by $m_1 +_\tau m_2$, as is defined by $m_1 +_\tau m_2 := \pi_\tau(m_1) + \pi_\tau(m_2)$, where $\pi_\tau(m_1), \pi_\tau(m_2) \in M_\tau$. For a fixed pair $m_1, m_2 \in \mathcal{M}$, the set of all elements which can be obtained by addition on some chart will be denoted by $\Delta(m_1, m_2)$. Concretely, $\Delta(m_1, m_2) = \{m_1 +_\tau m_2 \mid \tau \in L(\mathcal{M})\}$.

Many of the definitions we have introduced in the previous section can be easily extended to fit into the framework of tropical mutation schemes. We will now define the notions of cones, fans, and piecewise linear functions in the context of tropical mutation schemes.

2.2.4 Remark. While we have defined $\mathcal{M}$ using lattices as our models and mutations as piecewise linear functions over $\mathbb{Z}$, in what follows it will be natural to consider a similar object comprising vector spaces over $\mathbb{R}$. For a tropical mutation scheme given by lattices $(M_\sigma)_\sigma \in L(\mathcal{M})$, it is straightforward to extend the definitions to $M_\sigma \otimes \mathbb{R}$, for each $\sigma \in L(\mathcal{M})$

to obtain an object we denote by $\mathcal{M}\otimes\mathbb{R}$. Specifically, by this we mean the collection of vector spaces $M_\sigma\otimes\mathbb{R}$ for each $\sigma\in L(\mathcal{M})$, along with the mutation functions $\tilde{\mu}_{\sigma,\tau} : M_\sigma\otimes\mathbb{R}\to M_\tau\otimes\mathbb{R}$ for each $\sigma,\tau\in L(\mathcal{M})$ induced by the tensor product. Again by slight abuse of notation, we use $\mathcal{M}\otimes\mathbb{R}$ to also denote the set of elements of $\mathcal{M}\otimes\mathbb{R}$, which can be projected bijectively to each $M_\sigma\otimes\mathbb{R}$ via $\pi_\sigma\otimes\mathbb{R} : \mathcal{M}\otimes\mathbb{R}\to M_\sigma\otimes\mathbb{R}$.

2.2.5 Definition. A subset $\mathcal{C}\subseteq\mathcal{M}$ of the set of elements of $\mathcal{M}$ is a tropical mutation scheme *cone*, or $\mathcal{M}$-*cone* if for every $\tau\in S(M)$, the projection $\pi_\tau(\mathcal{C})\subset M_\tau\otimes\mathbb{R}$ is a cone in the usual sense of 2.1.3. A tropical mutation scheme *fan*, or $\mathcal{M}$-*fan* Σ, is a collection of $\mathcal{M}$-cones such that

1. For any cone $\sigma\in\Sigma$, any face of σ is also an element of Σ
2. For any $\sigma_1,\sigma_2\in\Sigma$, the intersection of both is a face of each.

2.2.6 Remark. Since each mutation map is piecewise linear and bijective, we may think of an $\mathcal{M}$-fan as a tuple $(\Sigma_\sigma)_{\sigma\in L(\mathcal{M})}$ where each Σ_σ is a fan in the classical sense on the lattice M_σ, along with the image of this fan on all possible mutation maps. Conversely, by taking some fan Σ on M_σ, we may obtain a corresponding $\mathcal{M}$-fan by considering all possible images of Σ as a tuple.

Similarly to piecewise linear functions, we can define a fan from each tropical mutation scheme.

2.2.7 Lemma. Given a tropical mutation scheme M with finitely many charts, there exists a fan Σ such that the restriction $\mu_{i,j}|_\sigma$ is linear for all i,j,σ. Furthermore, there exists a coarsest fan such that this is the case.

Proof. Since each $\mu_{\tau,\phi}$ for $\tau,\phi\in L(\mathcal{M})$ is a piecewise linear map between lattices, there exists a complete fan $\Sigma(\mu_{\tau,\phi})$ in $N_\tau\otimes\mathbb{R}$ such that the restriction of $\mu_{\phi,\tau}$ to each cone of $\Sigma(\mu_{\tau,\phi})$ is linear. Furthermore, $\Sigma(\mu_{\tau,\phi})$ lifts to an $\mathcal{M}$-fan by considering $(\mu_{\gamma,\beta}(\Sigma(\mu_{\tau,\phi}))_{\gamma,\beta\in L(\mathcal{M})}$, i.e., the image of this fan under all mutation maps. Then certainly if we consider the common refinement of these fans for all $\tau,\phi\in L(\mathcal{M})$, the resulting fan we will denote by $\Sigma(\mathcal{M})$ will have the property that for all $\sigma\in\Sigma(\mathcal{M})$, the restriction of $\mu_{\phi,\tau}$ to σ will be linear for all $\phi,\tau\in L(\mathcal{M})$. We also claim this is the coarsest possible fan such that this is the case. This follows from the fact that the fan of each $\mu_{\tau,\phi}$ is defined to be the coarsest possible fan for which $\mu_{\tau,\phi}$ is linear. □

We call this $\mathcal{M}$-fan, $\Sigma(\mathcal{M})$, obtained by taking the common refinement of all the fans of the mutation maps *the* $\mathcal{M}$*-fan of* $\mathcal{M}$. Furthermore, we note that since each mutation map is linear on each cone of $\Sigma(\mathcal{M})$, addition is well defined when we restrict to each cone. Specifically, take $m,m'\in\sigma$ for any $\sigma\in\Sigma(\mathcal{M})$. Then $m+_\tau m' = m+_\phi m'$ for all $\tau,\phi\in L(\mathcal{M})$.

For the remainder of this thesis, we will often refer back to or illustrate results using the following two examples of Tropical Mutation Schemes.

2.2.8 Example. The first example will be the rank n tropical mutation scheme with only one chart, and therefore no mutation functions. This corresponds to a single lattice, $M\cong\mathbb{Z}^n$. Here, the fan $\Sigma(\mathcal{M})$ is simply a fan with exactly one cone, namely all of $M\otimes\mathbb{R}$.

2.2.9 Example. For our second example $\mathcal{M}$ will be a rank-2 tropical mutation scheme with two charts, M_σ and $M_{\sigma'}$, for which we have fixed identifications with $\mathbb{Z}^2$ for each. There is one mutation map (and its inverse), given by:

$$\mu_{\sigma,\sigma'} : M_\sigma \to M'_\sigma$$
$$(x,y) \mapsto \begin{cases} (x,-y) & x \geq 0 \\ (x, x-y) & x < 0 \end{cases}$$

Here, the fan $\Sigma(\mathcal{M})$ has exactly two maximal-dimensional cones, namely the $x < 0$ half-space, and the $x \geq 0$ half-space in $\mathcal{M} \otimes \mathbb{R}$. In other words, $\pi_\sigma(\Sigma(\mathcal{M}))$ and $\pi'_\sigma(\Sigma(\mathcal{M}))$ are the $x \geq 0$ half-space and the $x < 0$ half-space in M_σ and M'_σ, respectively.

We can also define a piecewise linear function for tropical mutation schemes in a similar way.

2.2.10 Definition. Let $\mathcal{M}$ be a tropical mutation scheme. A *piecewise linear function on* $\mathcal{M}$ is a function $\Phi : \mathcal{M} \to \mathbb{Z}$ such that there exists a complete $\mathcal{M}$-fan Σ, such that for every chart $\tau \in L(\mathcal{M})$, and every cone $\sigma \in \pi_\tau(\Sigma)$, the restriction $\Phi|_\sigma$ is linear.

2.2.11 Remark. Like above with fans, we may think of a piecewise linear function on a tropical mutation scheme $\mathcal{M}$ as a tuple of piecewise linear maps $(\Phi_\sigma)_{\sigma \in L(\mathcal{M})}$. Conversely, given any piecewise linear map Φ on any chart M_σ, we obtain a piecewise linear map of tropical mutation schemes by considering the tuple of piecewise linear maps obtained first by mutating from M_τ to M_σ, then evaluating. We give a small example.

2.2.12 Example. Let $\mathcal{M}$ be the simple shear example we have seen in Example 2.2.9 . Consider the piecewise linear function

$$f(x,y) = min\{x,y\} \text{ on } M_\sigma.$$

Since the elements of $\mathcal{M}$ are in bijection with the elements of M_σ, this function defines a piecewise linear map of tropical mutation schemes, which we will call p, whose fan is the common refinement of $\Sigma(f)$ and $\Sigma(\mathcal{M})$. Furthermore, it defines a piecewise linear map g on M_τ by

$$g = f \circ \mu_{\tau,\sigma} = min\{x, -y, x-y\}.$$

Thus, we may think of p as $p = (f,g)$.

2.3 Spaces of points of tropical mutation schemes

A special kind of piecewise linear fuction on a tropical mutation scheme is called a point. The notion of a point is what will allow us to construct dual tropical mutation schemes, similarly to how we can construct the dual of a classical lattice.

2.3.1 Definition. Given a tropical mutation scheme $\mathcal{M}$, a function

$$p : \mathcal{M} \to \mathbb{Z}. \tag{2.3.1}$$

is a *point of* $\mathcal{M}$ if it satisfies the following property:

$$p(m) + p(m') = min\{p(m +_\tau m') \mid \tau \in L(\mathcal{M})\} \text{ for all } m, m' \in \mathcal{M} \tag{2.3.2}$$

The set of all points of $\mathcal{M}$ is called the space of points, and we denote it by $Sp(\mathcal{M})$.

2.3.2 Proposition. Let $p \in Sp(\mathcal{M})$. Then p is piecewise linear. In particular, the maps $p_\sigma : M_\sigma \to \mathbb{Z}$ given by $p_\sigma = p \circ \pi_\sigma^{-1}$ are piecewise linear for each $\sigma \in L(\mathcal{M})$.

Proof. Consider the $\mathcal{M}$-fan $\Sigma(\mathcal{M})$. Since for each cone $\sigma \in \Sigma(\mathcal{M})$, the restriction of $\mu_{\phi,\tau}$ is linear, we know that for $m, m' \in \sigma$, $m +_\tau m' = m +_\phi m'$ for all $\tau, \phi \in L(\mathcal{M})$. Therefore, if p is a point, and $m_1, m_2 \in \pi_\phi(\sigma)$ for any $\sigma \in \Sigma(\mathcal{M})$ and $\phi \in L(\mathcal{M})$ we have that $p(m_1) + p(m_2) = min\{p(m +_\tau m') \mid \tau \in L(\mathcal{M})\} = p(m_1 +_\phi m_2)$ for our specific ϕ, i.e. the value of $m +_\phi m'$ is independent of our choice of ϕ. Thus, for any cone $\sigma \in \Sigma(\mathcal{M})$, and any chart $\phi \in L(\mathcal{M})$, we have that if $m, m' \in \pi_\phi(\sigma)$, then $p \circ \pi_\phi^{-1}(m) + p \circ \pi_\phi^{-1}(m) = p \circ \pi_\phi^{-1}(m + m')$, and therefore $p \circ \pi_\phi^{-1}$ is piecewise linear for all $\phi \in L(\mathcal{M})$. $\square$

Given a tropical mutation scheme $\mathcal{M}$, there is a natural way to give some extra structure to the space of points, which we will explore here. Let $\mathcal{M}$ be a tropical mutation scheme, and $Sp(\mathcal{M})$ be its space of points. For $\sigma \in L(\mathcal{M})$, we define $Sp(\mathcal{M}, \sigma)$ for $\sigma \in L(\mathcal{M})$ to denote the set of points of $\mathcal{M}$ for which p is linear on M_σ. More precisely,

$$Sp(\mathcal{M}, \sigma) := \{p \in Sp(\mathcal{M}) \mid p_\sigma = p \circ \pi_\sigma^{-1} \; M_\sigma \to \mathbb{Z} \text{ is linear}\}. \tag{2.3.3}$$

In general, it is not clear whether the subsets $Sp(\mathcal{M}, \sigma)$ cover the space of points for a tropical mutation scheme. This prompts the following definition.

2.3.3 Definition. Let $\mathcal{M}$ be tropical mutation scheme. If $\bigcup_{\tau \in L(\mathcal{M})} Sp(\mathcal{M}, \tau) = Sp(\mathcal{M})$, then we say that $\mathcal{M}$ is *full*.

2.3.4 Remark. As we have seen with piecewise linear functions on tropical mutation schemes, it then makes sense to think about points as tuples of piecewise linear functions, i.e. $p = (p_\sigma)_{\sigma \in L(\mathcal{M})}$. Then we can restate the above as, a tropical mutation scheme is full if for every point $p = (p_\sigma)_{\sigma \in L(\mathcal{M})}$, p_τ is linear for some $\tau \in L(\mathcal{M})$.

Additionally, the space of points of a tropical mutation scheme then generates a subsemialgebra inside the semialgebra of piecewise linear function $\mathcal{O}_\mathcal{M}$. We will denote this subsemialgebra by $P_\mathcal{M} \subset \mathcal{O}_\mathcal{M}$. The subsemialgebra $P_\mathcal{M}$ is partially ordered by $\geq$, where $f \geq g$ if and only if for all $m \in \mathcal{M}$, $f(m) \geq g(m)$. For convention, $\infty \in P_\mathcal{M}$, and $\infty \geq f$ for all $f \in P_\mathcal{M}$.

If a tropical mutation scheme is full, we can sometime identify $Sp(\mathcal{M})$ with another tropical mutation scheme $\mathcal{N}$ via a dual pairing. Then each $Sp(\mathcal{M}, \sigma)$ will inherit a cone structure, and the union of these cones will form a complete fan. We will define and discuss dual pairings in the following section

2.3.5 Example. In Chapter 3, we show that an infinite class of tropical mutation schemes which are rank-2 and have 2 charts are full. We will also explicitly compute the space of points for these tropical mutation schemes. This class includes our example of the simple shear.

2.3.1 Dualizability

We will now introduce the notion of dualizability of tropical mutation schemes. It is this construction which generalizes the notion of duals from classical lattices.

2.3.6 Definition. Let $\mathcal{M}$ and $\mathcal{N}$ be full tropical mutation schemes of rank r. A *pairing* between $\mathcal{M}$ and $\mathcal{N}$ is a pair of functions $\mathfrak{v} : \mathcal{M} \to Sp(\mathcal{N})$ and $\mathbf{w} : \mathcal{N} \to Sp(\mathcal{M})$ such that for any $m \in \mathcal{M}$, $n \in \mathcal{N}$ we have that $\mathfrak{v}(m)[n] = \mathbf{w}(n)[m]$, i.e. the point $\mathfrak{v}(m)$ evaluated at n gives the same value as the point $\mathfrak{w}(n)$ evaluated at m. We call $\mathfrak{v}$, $\mathfrak{w}$ a *dual pairing* if both functions are bijections. Furthermore, if the preimages of each $Sp(\mathcal{M}, \sigma)$ and $Sp(\mathcal{N}, \sigma)$ are the maximal cones of $\Sigma(\mathcal{M})$ and $\Sigma(\mathcal{N})$, respectively, we say that the pairing is *strict*. This condition is enough to ensure that strict dual pairings are unique, up to isomorphism. If $\mathcal{M} = \mathcal{N}$, and there exists maps $\mathfrak{v}$ and $\mathbf{w}$, as above, we say $\mathcal{M}$ is *autodual*, i.e. it is strictly dual with itself.

2.3.7 Example. We can easily see that for the trivial tropical mutation scheme, the natural biliniear product $\langle -, - \rangle$ satisfies the condition of a strict dual pairing, thereby realizing dual lattices as a strict dual pairing of tropical mutation schemes. In Chapter 3, we will show that all rank-2 tropical mutation schemes with 2 charts are autodual, that is, there exists a strict dual pairing between $\mathcal{M}$ and $Sp(\mathcal{M})$.

When we have a dual pairing $\mathcal{M}$, and $\mathcal{N}$, $Sp(\mathcal{M})$ forms a semialgebraically additive basis for $P_{\mathcal{M}}$.

2.3.8 Theorem. Let $\mathcal{M}$ and $\mathcal{N}$ be full tropical mutation schemes such that there exist maps $\mathfrak{v}$, $\mathbf{w}$ such that $\mathcal{M}$ and $\mathcal{N}$ form a strict dual pairing. Then $Sp(\mathcal{M})$ forms a semialgebraically additive basis for $P_{\mathcal{M}}$. That is, for any $f \in P_{\mathcal{M}}$ we can write $f = min\{p_1, \ldots, p_n\}$ for some collection of $p_i \in Sp(\mathcal{M})$.

Proof. By the distributive property of $+$ over min, it suffices to show that for any two points $p_1, p_2 \in Sp(\mathcal{M})$, we can write $p_1 + p_2$ as a min combination of points. Because of the strict dual pairing, we can write $p_1 = \mathfrak{v}(n_1)$ and $p_2 = \mathfrak{v}(n_2)$ for some n_1 and n_2 in $\mathcal{N}$. Then we claim

$$\mathfrak{v}(n_1) + \mathfrak{v}(n_2) = min\{\mathfrak{v}(n_i) \mid n_i \in \Delta(n_1, n_2)\}.$$

For the left hand side of the equation, we simply note that $\mathfrak{v}(n_1)(m) + \mathfrak{v}(n_2)(m) = \mathbf{w}(m)(n_1) + \mathbf{w}(m)(n_2)$, by the definition of a dual pairing. For the right hand side, we similarly have $min\{\mathfrak{v}(n_i)(m) \mid n_i \in \Delta(n_1, n_2)\} = min\{\mathbf{w}(m)(n_i) \mid n_i \in \Delta(n_1, n_2)\}$, again by the properties of a dual pairing. However, $\mathbf{w}$ is a point, and therefore $min\{\mathfrak{v}(n_i)(m) \mid n_i \in \Delta(n_1, n_2)\} = min\{\mathbf{w}(m)(n_1 +_\tau n_2) \mid \tau \in L(\mathcal{M})\} = \mathbf{w}(m)(n_1) + \mathbf{w}(m)(n_2)$. Thus, the left side equals the right side, and we can write the sum of any two points as a min combination of points. $\square$

2.3.2 Convexity

The last ingredient we need in order to construct the canonical semialgebra of a tropical mutation scheme, which allows for detropicalizations, is a notion of convexity for subsets of $\mathcal{M} \otimes \mathbb{R}$. The notion of convexity we will define comes from the notion of $\mathcal{M}$-half-spaces defined by points.

2.3.9 Definition. Let $\mathcal{M}$ be a tropical mutation scheme, and let $Sp(\mathcal{M})$ be its space of points. Given a point $p \in Sp(\mathcal{M})$ and a value $a \in \mathbb{Z}$, *the $\mathcal{M}$-half-space defined by p and a*, denoted by $H_{p,a}$, is the subset of $\mathcal{M} \otimes \mathbb{R}$ defined as

$$H_{p,a} := \{m \in \mathcal{M} \otimes \mathbb{R} \mid p(m) \geq a\} \subset \mathcal{M} \otimes \mathbb{R}.$$

This notion of an $\mathcal{M}$-half-space allows us to define convexity for tropical mutation schemes analogous to how convexity can be defined via half-spaces in the classical setting.

2.3.10 Definition. A subset $U \subset \mathcal{M} \otimes \mathbb{R}$ is *pointwise convex* if it is the intersection of half-planes defined by points in $Sp(\mathcal{M})$, i.e

$$U = \bigcap H_{p,a}$$

for some collection of pairs $(p, a) \in Sp(\mathcal{M}) \times \mathbb{Z}$. The *pointwise convex hull* of a subset $S \subseteq \mathcal{M} \otimes \mathbb{R}$ is the intersection of all pointwise convex sets containing S.

2.3.11 Remark. It is easy to see that if a subset $S \subset \mathcal{M} \otimes \mathbb{R}$ it pointwise convex, then the projection $\pi_\tau(S)$ is convex in the classical sense of containing any line connecting points $m_1, m_2 \in S$ for any $\tau \in L(\mathcal{M})$. In fact, a pointwise convex set satisfies the stronger condition that if $m_1, m_2 \in S$, then S contains every broken line connecting m_1 and m_2. Precisely, if ℓ_σ is the line between $\pi_\sigma(m_1)$ and $\pi_\sigma(m_2)$, then $\pi_\sigma^{-1}(\ell_\sigma) \subset S$. If a set satisfies the broken line condition, we say it is $\mathcal{M}$*-convex*. It is currently unknown whether $\mathcal{M}$-convexity is equivalent to pointwise convexity. However, in general we see that convex hulls are required to be "larger" than we might expect, as the image of a convex set on one chart is not necessarily convex on every chart, and therefore not convex itself.

The following lemma with help us more easily compute the pointwise convex hull of a subset of a tropical mutation scheme.

2.3.12 Lemma. Let S be a subset of $\mathcal{M} \otimes \mathbb{R}$. We will define the following subset of $Sp(\mathcal{M}) \times \mathbb{Z}$:

$$\mathcal{B}_S := \{(p, a) \mid p \in Sp(\mathcal{M}) \text{ such that } a := \inf\{p(s)\}, s \in S \text{ exists}\} \subset Sp(\mathcal{M}) \times \mathbb{Z}.$$

Then the pointwise convex hull of S is equal to the set

$$U = \bigcap_{(p,a) \in \mathcal{B}_S} H_{p,a} \subset \mathcal{M} \otimes \mathbb{R}.$$

Proof. First we will show that any element m not satisfying the above is not in the point convex hull of S. Let $m \in \mathcal{M}$, and suppose there is some point p such that $p(m) < p(s)$ for all $s \in S$. There there is some $a \in \mathbb{R}$ such that m is not an element of the half-space $H_{p,a}$, but $S \subset H_{p,a}$. Thus, if m is not an element of U, it is not an element of S. Now we show the converse. Suppose for all $p \in Sp(\mathcal{M})$, $p(m) \geq min\{p(s) \mid s \in S\}$. Then for any half-space $H_{p,a}$ such that $S \subset H_{p,a}$, since $p(m) \geq min\{p(s) \mid s \in S\} \geq a$, by the definition of a half-space $m \in H_{p,a}$. Therefore, m must be in the point-convex hull of S. □

2.3.3 The Canonical Semiaglebra

Pointwise convexity also allows us to define *the canonical semialgebra* for a tropical mutation scheme.

2.3.13 Definition. Let $\mathcal{M}$ be a tropical mutation scheme. Let $\langle\mathcal{M}\rangle$ denote the free semi-group obtained by taking formal sums of the elements of $\mathcal{M}$, modulo the equivalence relation of the set of elements in the formal sum having the same pointwise convex hull. We include ∞ as the additive identity of this semi-group, to form a monoid. For $m_1, m_2 \in \mathcal{M}$, recall from Remark 2.2.3 that $\Delta(m_1, m_2)$ denotes the set $\Delta(m_1, m_2) = \{m_1 +_\sigma m_2 \mid \sigma \in L(\mathcal{M})\}$. Now, we will define the following binary product, denoted by $\star$ on the semigroup $\langle\mathcal{M}\rangle$ as follows:

$$m_1 \star m_2 := \bigoplus_{m \in \Delta(m_1, m_2)} m.$$

The *canonical semialgebra* of a tropical mutation scheme $\mathcal{M}$, denoted by $\mathcal{S}_\mathcal{M}$, is the free semigroup defined above together with the binary operation $\star$. The canonical semialgebra is partially ordered by $\succeq$, where $\bigoplus m_i \succeq \bigoplus m_j$ if and only if the pointwise convex hull of $\bigoplus m_i$ is contained in the pointwise convex hull of $\bigoplus m_j$. Furthermore, $\infty \succeq \bigoplus m_i$ for all $\bigoplus m_i \in \mathcal{S}_\mathcal{M}$.

2.3.14 Lemma. This product is both associative and distributive over the addition operation of $\langle\mathcal{M}\rangle$.

Proof. We have defined $\star$ such that $\bigoplus m_i \star \bigoplus m_j := \bigoplus m_i \star m_j$, so $\star$ is clearly distributive over the addition operation. Now we must show that $\star$ is associative. Here, it will suffice to show $\star$ is associative for elements, i.e. that $m_1 \star (m_2 \star m_3) = (m_1 \star m_2) \star m_3$, for $m_1, m_2, m_3 \in \mathcal{M}$. We will let Δ_1 denote the set of all elements obtain from $(m_1 +_\sigma m_2) +_\tau m_3$ for $\tau, \sigma \in L(\mathcal{M})$. Formally, $\Delta_1 = \bigcup_{m_i \in \Delta(m_1, m_2)} \Delta(m_i, m_3)$. Similarly, let $\Delta_2 = \bigcup_{m_i \in \Delta(m_2, m_3)} \Delta(m_i, m_1)$.

To complete the proof, we must show that Δ_1 and Δ_2 have the same point-convex hull. Let p be a point of $\mathcal{M}$. By the definition of a point, we know that

$$p(m_1) + p(m_2) + p(m_3) = min\{p(m_i) \mid m_i \in \Delta_1\}.$$

As a result, for any m' in the point-convex hull of Δ_1, we have that $p(m') \geq min\{p(m_i) \mid m_i \in \Delta_1\}$. However,

$$p(m_1) + p(m_2) + p(m_3) = min\{p(m_i) \mid m_i \in \Delta_2\},$$

as well. So for any m in the point-convex hull of Δ_1, we have that $p(m) \geq min\{p(m_i) \mid m_i \in \Delta_2\}$, and for m' in the point convex hull of Δ_2, we have that $p(m') \geq min\{p(m_i) \mid m_i \in \Delta_1\}$. The result then follows from Lemma 2.3.12. □

2.3.15 Example. In the simplest case of a single lattice interpreted as a tropical mutation scheme, the canonical semialgebra may be interpreted as the set of lattice polytopes, where the addition operation is given by the convex hull of the union of polytopes, and the multiplication operation is given by Minkowski sum. The canonical semialgebra is then the semialgebra of lattice polytopes under these operations.

The final key ingredient to constructing a detropicalization is the existence of a *valuation* to the cannonical semialgebra, which we will define below.

2.3.16 Definition. Let A be a k-algebra over some field k. We call a function $\mathfrak{v} : A \to \mathcal{S}_\mathcal{M}$ to the canonical semialgebra of a tropical mutation scheme a *valuation* if it satisfies the following properties:

1. $\mathfrak{v}(f) = \infty$ if an only if $f = 0$,
2. $\mathfrak{v}(fg) = \mathfrak{v}(f) \star \mathfrak{v}(g)$ for all $f, g \in A$, $f, g \neq 0$,
3. $\mathfrak{v}(cf) = \mathfrak{v}(f)$ for all $c \in k^*$ and $f \in A$,
4. $\mathfrak{v}(f + g) \geq \mathfrak{v}(f) \oplus \mathfrak{v}(g)$ for all $f, g \in A$.

2.3.17 Remark. In general, any function $\mathfrak{v}$ from a k-algebra to a partially ordered set $\Gamma \cup \{\infty\}$ satisfying (1-4) is a valuation, not just those whose image is the canonical semialgebra of a tropical mutation scheme.

Now we are ready to define a detropicalization of a tropical mutation scheme.

2.3.18 Definition. We say that a k-algebra $A_{\mathcal{M}}$ is a *detropicalization* of a tropical mutation scheme $\mathcal{M}$ if there exists a valuation $\mathfrak{v} : A_{\mathcal{M}} \to \mathcal{S}_{\mathcal{M}}$ such that every element of $\mathcal{M} \subset S_{\mathcal{M}}$ is in the image of $\mathfrak{v}$, and the Krull dimension of $A_{\mathcal{M}}$ is equal to the rank of $\mathcal{M}$. Thus a detropicalzation is a pair $(A_{\mathcal{M}}, \mathfrak{v})$ which satisfies the above conditions.

Identifying a particular type of basis for a k-algebra is useful for constructing valuations, prompting the following definition.

2.3.19 Definition. Let $(\mathcal{A}_{\mathcal{M}}, \mathfrak{v})$ be a detropicalization for a tropical mutation scheme $\mathcal{M}$. Then $\mathbb{B} \subset \mathcal{A}_{\mathcal{M}}$ is an *adapted basis* for the detropicalization if $\mathbb{B}$ is an additive basis for $\mathcal{A}_{\mathcal{M}}$ such that $\mathfrak{v}(\sum C_i b_i) = \bigoplus \mathfrak{v}(b_i)$ for $b_i \in \mathbb{B}$, and $\mathfrak{v}(b) \in \mathbb{M} \subset S_{\mathcal{M}}$ for all $b \in \mathbb{B}$, i.e. the image of any b is an element of $\mathcal{M}$.

2.3.20 Example. We can find a detropicalization $(A, \mathfrak{v})$ explicitly in the simplest example of the trivial tropical mutation scheme. Let M be a lattice of rank r, and think of it as a tropical mutation scheme as we have above. Let A denote the Laurent polynomial ring over r variables, $A = \mathbb{C}[x_1^{\pm}, \ldots, x_r^{\pm}]$. The valuation which realizes this detropicalization is the map which sends a Laurent polynomial to its Newton polytope. The reader may check that this map satisfied the axioms of a valuation presented in 2.3.16. Furthermore, the Laurent monomials form an adapted basis for this detropicalization, as the Newton polytope of any monomial is simply a point on the associated lattice, establishing a bijection between points of the character lattice and the Laurent monomials. This connection between the coordinate ring of the rank-r algebraic torus (the Laurent polynomial ring) and a lattice and its lattice polytopes is the basis for much of the rich combinatorics found in the study of toric geometry. Thus, we hope to recover some of these combinatorial properties for more general algebras through the study of tropical mutation schemes and their detropicalizations.

The following theorem illustrates why dualizablity is important when it comes to constructing detropicalizations of tropical mutation schemes.

Given a strict dual pairing of tropical mutation schemes $\mathcal{M}$ and $\mathcal{N}$, with maps $\mathfrak{v} : \mathcal{N} \to Sp(\mathcal{M})$ and $\mathbf{w} : \mathcal{M} \to Sp(\mathcal{N})$. We can define an extension of $\mathfrak{v}$, denoted by $\hat{\mathfrak{v}} : \mathcal{S}_{\mathcal{N}} \to P_{\mathcal{M}}$ and given by:

$$\hat{\mathfrak{v}}(\bigoplus n_i) := min\{\mathfrak{v}(n_i)\}, \ \hat{\mathfrak{v}}(\infty) = \infty.$$

2.3.21 Theorem. Let $\mathcal{M}$ and $\mathcal{N}$ be tropical mutation schemes such that there exists functions $\mathfrak{v} : \mathcal{N} \to Sp(\mathcal{M})$ and $\mathbf{w} : \mathcal{M} \to Sp(\mathcal{N})$ which realize $\mathcal{M}$ and $\mathcal{N}$ as a strict dual pairing of tropical mutation schemes. Then $\hat{\mathfrak{v}}$ is an order preserving semialgebra isomorphism between $\mathcal{S}_{\mathcal{N}}$ and $P_{\mathcal{M}}$.

Proof. The well-definedness of this function is equivalent to showing that $\bigoplus n_i$ has the same pointwise convex hull as $\bigoplus n_j$ if and only if for all $m \in \mathcal{M}$, we have that $min\{\mathfrak{v}(n_i)(m)\} = min\{\mathfrak{v}(n_j)(m)\}$. This is a special case of our argument that the function is order-preserving, so we will address this below.

Next, we will show that $\hat{\mathfrak{v}}$ is a semialgebra homomorphism. First, clearly the identity of the underlying monoids are mapped to each other by definition. Furthermore, the additive structure of $S_{\mathcal{M}}$ and $P_{\mathcal{N}}$ is respected by definition, since $\hat{\mathfrak{v}}(\bigoplus n_i) = min\{\mathfrak{v}(n_i)\}$. To show that

$$\hat{\mathfrak{v}}(\oplus n_i \star \oplus n_j) = min\{\mathfrak{v}(n_i)\} + min\{\mathfrak{v}(n_j)\},$$

it will suffice to consider the case of $\mathfrak{v}(n_1 \star n_2)$ for $n_1, n_2 \in \mathcal{N}$. We have $\mathfrak{v}(n_1 \star n_2) = \mathfrak{v}(\bigoplus n_i)$ for all $n_i \in \Delta(n_1, n_2)$. By Theorem 2.3.8, $\mathfrak{v}(\bigoplus n_i)$ for all $n_i \in \Delta(n_1, n_2)$ is equal to $\mathfrak{v}(n_1) + \mathfrak{v}(n_2)$, as required. Therefore, the function $\hat{\mathfrak{v}}$ defines a homomorphism of semialgebras between $\mathcal{S}_{\mathcal{M}}$ and $P_{\mathcal{N}}$.

It is immediately clear that the function $\hat{\mathfrak{v}}$ is surjective, since we have shown in theorem 2.3.8 that we can write any $f \in P_{\mathcal{N}}$ as $f = min\{\mathfrak{v}(n_i)\} = \hat{\mathfrak{v}}(\bigoplus n_i)$. To show that $\hat{\mathfrak{v}}$ in injective, suppose that $\hat{\mathfrak{v}}(\bigoplus n_i) = \hat{\mathfrak{v}}(\bigoplus n_j)$. We must show that $\bigoplus n_i$ and $\bigoplus n_j$ have the same point convex hull. Assume toward a contradiction that they do not, and assume without loss of generality that n_1 is in the point-convex hull of $\bigoplus n_i$, but not the point-convex hull of $\bigoplus n_j$. Then we must show there exists some $m \in \mathcal{M}$ such that $\mathfrak{v}(n_1)(m) < min\{\mathfrak{v}(n_i)(m)\}$ for all n_i. Using the definition of a dual pairing, we can rewrite $\mathfrak{v}(n_1)(m)$ as $\mathbf{w}(m)(n_1)$ and we can rewrite $min\{\mathfrak{v}(n_i)(m)\}$ for all n_i as $min\{\mathbf{w}(m)(n_i)\}$ for all n_i. The relation $\mathbf{w}(m)(n_1) < min\{\mathbf{w}(m)(n_i)\}$ is then implied by Lemma 2.3.12, contradicting our assumption that $\hat{\mathfrak{v}}(\bigoplus n_i) = \hat{\mathfrak{v}}(\bigoplus n_j)$. Therefore, $\hat{\mathfrak{v}}$ is injective, and so, a semialgebra isomorphism.

Finally, we must show that $\bigoplus n_i \succeq \bigoplus n_j$ if and only if $\hat{\mathfrak{v}}(\bigoplus n_i) \geq \hat{\mathfrak{v}}(\bigoplus n_j)$. First, suppose $\bigoplus n_i \succeq \bigoplus n_j$. If this is the case, then $\bigoplus n_i$ is contained in the convex hull of $\bigoplus n_j$, and thus every term in $min\{\mathfrak{v}(n_i)\}$ will appear in $min\{\mathfrak{v}(n_j)\}$. Consequently $\hat{\mathfrak{v}}(\bigoplus n_i) \geq \hat{\mathfrak{v}}(\bigoplus n_j)$. Now suppose $\hat{\mathfrak{v}}(\bigoplus n_i) \geq \hat{\mathfrak{v}}(\bigoplus n_j)$. Recall this means that $\min\{\mathfrak{v}(n_i)(m)\} \geq \min\{\mathfrak{v}(n_j)(m)\}$ for all $m \in \mathcal{M}$. Again, we can use the properties of a strict dual pairing to rewrite this as $\min\{\mathbf{w}(m)(n_i)\} \geq \min\{\mathbf{w}(m)(n_j)\}$, for every $m \in \mathcal{M}$. This means that for every point in $Sp(\mathcal{N})$, we have that $\min\{p(n_i)\} \geq \min\{p(n_j)\}$. Then, by Lemma 2.3.12, we know that the convex hull of $\bigoplus n_i$ is contained in the convex hull of $\bigoplus n_j$, and therefore $\bigoplus n_i \succeq \bigoplus n_j$.

Taking the special case that $\bigoplus n_i \succeq \bigoplus n_j$ and $\bigoplus n_j \succeq \bigoplus n_i$, respectively that $\min\{\mathfrak{v}(n_i)(m)\} \geq \min\{\mathfrak{v}(n_j)(m)\}$ and $\min\{\mathfrak{v}(n_j)(m)\} \geq \min\{\mathfrak{v}(n_i)(m)\}$, we show that $\bigoplus n_i$ has the same pointwise convex hull as $\bigoplus n_j$ if and only if for all $m \in \mathcal{M}$, we have that $min\{\mathfrak{v}(n_i)\}(m) = min\{\mathfrak{v}(n_j)\}(m)$, proving that $\tilde{\mathfrak{v}}$ is well defined. □

2.3.22 Corollary. Any valuation to $P_{\mathcal{N}}$ can be used to construct a valuation to $S_{\mathcal{N}}$. Thus, to find a detropicazation of $\mathcal{M}$, it suffices to find a valuation from an algebra A to $P_{\mathcal{N}}$, where $\mathcal{N}$ is the strict dual of $\mathcal{M}$

Due to the above corollary, it is usually easier to construct valuations to $P_{\mathcal{M}}$, as piecewise linear functions are often easier to work with than "tropical mutation scheme polytopes." For a single lattice, which is autodual, it is easy to check that the map which sends a Laurent polynomial f to $trop(f)$ satisfies the axioms of a valuation, and has image P_M. Now that we have introduced all background required, we will focus our attention to a particular class of rank-2 tropical mutation schemes on two charts, and show that they are all full, autodual, and detropicalizable.

Chapter 3

The Space of Points and Duality for Rank 2 Tropical Mutation Schemes on 2 Charts

3.1 Computing the Space of Points for Rank 2 Tropical Mutation Schemes with 2 charts

Recall our definition of a point of a tropical mutation scheme. A point is a map $p : \mathcal{M} \to \mathbb{R}$ such that the following equality holds:

$$p(m) + p(m') = min\{p(m'')|\pi_\phi(m) + \pi_\phi(m') = \pi_\phi(m'') \text{ for some } \phi \in L(\mathcal{M})\} \tag{3.1.1}$$

In this section, we will compute the space of points for an infinite class of rank 2 tropical mutation schemes with two cones of linearity, which we will denote by $\mathcal{M}_b$, where one cone is the positive x half-space, and the other is the negative x half-space, with the mutation map on the positive x half-space being the map $(x, y) \mapsto (x, -y)$.

For the remainder of this section, our tropical mutation schemes will have exactly two charts, i.e, $L(\mathcal{M}) = \{\phi, \tau\}$. We will fix identifications of $M_\phi \otimes \mathbb{R}$ and $M_\tau \otimes \mathbb{R}$ with $\mathbb{R}^2$ for our computations. We may assume without loss of generality that we have chosen these bases in such a way that the 2 domains of linearity in $M_\phi \otimes \mathbb{R}$ are separated by the y-axis, and moreover, the images of these 2 cones of linearity are the 2 half-spaces in $M_\tau \otimes \mathbb{R}$ separated by the y-axis in $M_\tau \otimes \mathbb{R}$. In particular, we can assume without loss of generality that the mutation map on the positive half-space is given by the matrix A $= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Since the mutation maps must agree on the vector $(0, 1)$, and since we have arranged that the mutation map is A on the positive half-plane, this means that on the negative half-space, the mutation map must be given by a matrix of the form $\begin{bmatrix} 1 & 0 \\ b & -1 \end{bmatrix}$. If the upper right entry was not zero, the matrix would no longer be a map of lattices and the diagonal entries must be 1 or -1 to ensure the determinant is ± 1. Furthermore, to ensure a bijective map of lattices, we may assume that $b \in \mathbb{Z} - \{0\}$. Thus, this family of tropical mutation schemes is parametrized by the integer parameter b. We will abuse notation slightly and let $\mathcal{M}_b$ denote the tropical mutation scheme obtained by setting our integer parameter equal to b. When writing elements of the tropical mutation scheme $\mathcal{M}_b$, as a tuple, we will always write the element of M_ϕ first, i.e. $m = ((x, y), (x, bx - y))$. Because of this, we will sometime refer to M_ϕ as the *first* chart, and M_τ as the *second* chart. The following proof computes the space of points for any rank 2 tropical mutation $\mathcal{M}_b$, for $b \in \mathbb{Z} \setminus \{0\}$.

3.1.2 Remark. As of now, there is not a consensus regarding the proper definition for the notion of isomorphism of tropical mutation schemes. However, we expect that the class $\mathcal{M}_b$ actually makes up all rank-2 tropical mutation schemes on exactly two charts, up to isomorphism. This is because, given any rank-2 tropical mutation scheme whose fan has exactly two domains of linearity, we can always make a linear change of coordinates on each chart in order to obtain a tropical mutation scheme of the form $\mathcal{M}_b$. Furthermore, it can be shown that no rank-2 tropical mutation scheme with two charts can have a fan with 3 full dimensional cones, and we conjecture that for $n \geq 3$, no rank-2 tropical mutation scheme can have a fan with n full dimensional cones.

We will make some preliminary observations.

3.1.1 Lemma. Let $\mathcal{M}$ be a tropical mutation scheme with fan $\Sigma(\mathcal{M})$, and let $p \in Sp(\mathcal{M})$. Then p is linear on every chart when restricted to each cone of Σ. Furthermore, the induced map $\hat{p} : \mathcal{M} \otimes \mathbb{R} \to \mathbb{R}$ is continuous.

Proof. Take any two elements m and m' both lying in some cone $\sigma \in \Sigma(\mathcal{M})$, and consider the projection of these elements to a chart $\tau \in L(\mathcal{M})$. Since every mutation map is linear on σ, we have that $m +_\tau m' = m +_\phi m'$ for all $\tau, \phi \in L(\mathcal{M})$. As a consequence,

$$p(m) + p(m') = min\{p(m +_\phi m')|\phi \in L(\mathcal{M})\} = p(m +_\tau m').$$

Since τ is an arbitrary chart, we have shown that a point is linear on any chart when restricted to each cone in $\Sigma(\mathcal{M})$, as required.

To see that $\hat{p}$ is continuous, we simply note that since the mutation maps $\mu_{\phi,\tau}$ are all continuous, the above argument shows that a point must actually be linear on the closure of each cone $\sigma \in \Sigma$, and as a result must agree on the shared boundary of any two adjacent cones. $\square$

As we have seen in Remark 2.3.4 a point is determined by its values on any single coordinate chart (since elements of the tropical mutation scheme are determined by its coordinate on a single chart). With respect to a different coordinate chart, the value of the point with respect to that chart is given by composition with an appropriate mutation, i.e. a point can be thought of as a tuple of piecewise linear maps for each chart in $L(\mathcal{M})$.

By the above lemma, for our example, a point is determined by two linear maps: namely, a map of the form $\begin{bmatrix} c & d \end{bmatrix}$ on the negative half-plane and $\begin{bmatrix} e & d \end{bmatrix}$ on the positive half-plane, with respect to the first coordinate chart M_1. (If we were to work with the second coordinate chart M_2, then equivalently, the point would be determined by the two linear maps

$\begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1}$ on the negative half-space, and $\begin{bmatrix} e & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ b & -1 \end{bmatrix}^{-1}$ on the positive half-space.)

As previously mentioned, the definition of a point implies that it is linear on each cone, but we will find further restrictions when we consider taking sums of a point evaluated at elements which fall on different cones of linearity. Consider two arbitrary elements, one with x-coordinate > 0, and one with x-coordinate < 0. These will be denoted by $m = ((x, y), (x, -y))$ and $m' = ((x', y'), (x', bx' - y'))$, so in particular $x > 0$ and $x' < 0$ respectively. By the nature of tropical mutation schemes, there is no well-defined notion of a "sum" of elements. What we can do instead is to take the projections of elements to different charts, and take the sum there, but there is no guarantee that if we were to take different charts, that the different sums are related by mutation. So we must consider different possible cases of what can happen in the two different coordinate charts.

Thus there are two cases to consider for the "sum" of these elements in the two different charts, according to as to whether the sum (in a particular chart) lands in the negative or the positive half-space. The following are the possibilities:

1. $x + x' < 0$
2. $x + x' > 0$

Recall that we assume $x < 0$ and $x' > 0$. Note that a priori, one might think this computation requires four cases to check, i.e. considering the cases where addition on the two coordinate charts yield two elements that lie on opposite cones of linearity. However, since the mutation map does not affect the first coordinate on either chart, we know that addition on each chart will always yield elements which lie in the same cone, regardless of our initial choice of elements.

Since our tropical mutation scheme has two charts ϕ and τ, for each point p, the value of $p(m) + p(m') = \min\{p((x + x', y + y'), (\mu_{\phi,\tau}(x + x', y + y'), p((\mu_{\tau,\phi}(x + x', bx' - y' - y), (x + x', bx' - y' - y))\}$ i.e. it is the minimum value between the point evaluated at the element of the sum of the first coordinates (denoted by s_1), and the element of the sum of the second coordinates (denoted by s_2).

3.1.3 Theorem. *Let the notation and setup be as above. In particular, b is fixed. Then p , specified by a choice of parameters c, d, e as above, is a point if and only if*

1. *$c = e$ and $d < 0$, or,*
2. *$d = \frac{e-c}{b}$ and $d > 0$.*

In particular,

(a) *this tropical mutation scheme is full, since a point satisfying condition (1) is linear on the first chart, and on satisfying condition (2) is linear on the second chart.*

(b) *and the space of points is homeomorphic to* $\mathbb{R}^2$*, and more specifically, it is the subset of* $\mathbb{R}^3$ *(with coordinates* c, d, e*) obtained by gluing the half-2-plane* $\{c = e, 0 > d\}$ *to the half-2-plane* $\{d = \frac{e+c}{b}, 0 < d\}$*.*

Proof. We first show that if p is a point, then the parameters c, d, e must satisfy either condition (1) or (2) in the statement of the theorem. Since addition agrees on every chart if two elements are taken from the same cone of $\Sigma(\mathcal{M})$, and thus the minimum condition for a point simply implies that a point is linear on each cone, we may consider only the case where the two elements come from each half-planes. Concretely, take two elements of $\mathcal{M}_b$, $m = ((x, y)(x, -y))$ with $x > 0$ and $m' = ((x', y'), (x', bx' - y'))$ such that $x' < 0$. We will show, over two cases, that the restrictions on a point imply that it is linear on either the first or second chart.

Case 1: In the first case, we assume that $x + x' > 0$. First we note that in this case,

$$s_1 = m +_\phi m' = ((x + x', y + y'), (x + x', -(y + y'))),$$

and

$$s_2 = m +_\tau m' = ((x + x', y - bx' + y'), (x + x', -y + bx' - y')).$$

(Note that since $x + x' > 0$ by assumption, the elements s_1 and s_2 have first coordinates lying in the positive half-space of M_1, and thus the mutation map identifying with M_2 simply negates the second coordinate). Thus, $p(s_1) = c(x + x') + d(y + y')$ and $p(s_2) = c(x+x')+d(y+y'+bx')$. For later, we note that $p(s_1) = p(s_2)$ when $0 = d$ and $p(s_1) < p(s_2)$ when d > 0.

Now we can compute the LHS and the RHS of the requirement (3.1.1) for being a point. For the LHS, we note that $p(m) + p(m') = cx + dy + ex' + dy'$. For the LHS, we compute $\min\{p(s_1), p(s_2)\} = \min\{c(x + x') + d(y + y'), c(x + x') + d(y + y' - bx')\}$. Hence we require

$$cx + dy + ex' + dy' = \min\{c(x + x') + d(y + y'), c(x + x') + d(y + y' - bx')\}.$$

We then consider the two subcases, $p(s_1) < p(s_2)$ and vice versa. Suppose $p(s_1) < p(s_2)$. Then $cx + dy + ex' + dy' = c(x + x') + d(y + y')$. This imposes the restriction that $c = e$.

Now suppose $p(s_1) > p(s_2)$. In this case, we have that $cx + dy + ex' + dy' = c(x + x') + d(y + y' - bx')$. Clearly, the $dy + dy'$ cancel, leaving us with $cx + ex' = cx + (c - db)x'$. Since this equality holds on a non-empty open set, the coefficients must be equal, and we get that $e = (c - db)$, which can be rearranged to get $d = \frac{c-e}{b}$. Note also that when we apply the inverse of the mutation map to $\begin{bmatrix} e & \frac{c-e}{b} \end{bmatrix}$ we get $\begin{bmatrix} c & \frac{e-c}{b} \end{bmatrix}$. Thus this restriction is equivalent to being linear with respect to the second chart.

In total, this case yields that for $0 > d$, a point is linear on the first chart, and for $0 < d$ it is linear on the second chart.

Case 2: Now we note that in this case, $s_1 = ((x + x', y + y'), (x + x', bx + bx' - y - y'))$, and

$s_2 = ((x+x', y+y'+bx), (x+x', -y+bx'-y'))$. Thus, $p(s_1) = e(x+x') + d(y+y')$ and $p(s_2) = e(x+x') + d(y+y'+bx)$. Similarly to the previous case, we note that $p(s_1) = p(s_2)$ when $0 = d$, and $p(s_1) < p(s_2)$ when $d > 0$.

Recall that $p(m) + p(m') = cx + dy + ex' + dy'$, and $p(m) + p(m') = \min\{p(s_1), p(s_2)\} = \min\{e(x+x') + d(y+y'), e(x+x') + d(y+y'+bx)\}$.

We then consider the two subcases, $p(s_1) < p(s_2)$ and vice versa. Suppose $p(s_1) < p(s_2)$. Then $cx + dy + ex' + dy' = c(x+x') + d(y+y')$. This imposes the restriction that $c = e$.

Now suppose $p(s_1) > p(s_2)$. In this case, we have that $cx + dy + ex' + dy' = e(x+x') + d(y+y'+bx)$. Clearly, the $dy + dy'$ cancel, leaving us with $cx + ex' = ex' + (e+db)x$. Since this equality holds on a non-empty open set, the coefficients must be equal, and we get that $c = e + db$, which can be rearranged to get $d = \frac{c-e}{b}$. Note also that when we apply the inverse of the mutation map to $\begin{bmatrix} e & -(\frac{c-e}{b}) \end{bmatrix}$ we get $\begin{bmatrix} c & -(\frac{c-e}{b}) \end{bmatrix}$. Thus this restriction is equivalent to being linear with respect to the second chart.

In total, this case yields that for $0 < d$, a point is linear on the first chart, and for $0 > d$ it is linear on the second chart, as before.

Thus, for any tropical mutation scheme of this form, we have a space of points isomorphic to $\mathbb{R}^2$, where the folded book living in $\mathbb{R}^3$ is glued together along the line $0 = d$. On one side of this line, the relation is $c = e$, and on the other it is $d = c - e$.

□

3.1.4 Remark. It's useful to notice that, although points can be linear in a given domain of linearity in a given coordinate chart, it is not true that an arbitrary linear function on a domain of linearity comes from a point. For instance, if $b = 1$, then condition (1) in the above theorem says that we must have $d < 0$, which means for instance that the linear function $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (i.e. $c = 1, d = 1$) on the first chart can never arise from a point. This is because the minimum condition fails. For example, take the elements $m = ((-1,1),(-1,-2))$ and $m' = ((3,2),(3,-2))$. Then $m +_\phi m' = ((2,3),(2,-3))$ and $m +_\tau m' = ((2,2),(2,-4))$. But then this linear function on the first chart does not satisfy the requirements of a point, as it does not satisfy the minimum condition. The image of $m +_\phi m'$ is 5, while the image of $m +_\tau m'$ is 4.

3.2 Autoduality of Rank-2 Tropical Mutation Schemes with 2 Charts

Now that we have computed the space of points, and have been able to identify it with a subset of $\mathbb{R}^3$, we are well on the way to establishing a strict dual pairing for tropical mutation schemes of rank 2. Recall that two tropical mutation schemes $\mathcal{M}$ and $\mathcal{N}$ form a strict dual pair if there exist bijective maps $\mathfrak{v} : \mathcal{M} \to Sp(\mathcal{N})$ and $\mathbf{w} : \mathcal{N} \to Sp(\mathcal{M})$ such that for any $n \in \mathcal{N}$ and any $m \in \mathcal{M}$ we have $\mathfrak{v}(m)[n] = \mathbf{w}(n)[m]$. A tropical mutation scheme is autodual if such pairings exists with its own space of points. The following theorem will show that all the examples $\mathcal{M}_b$ are autodual.

3.2.1 Theorem. *Let $\mathcal{M} = \mathcal{M}_b$ be a rank 2 tropical mutation scheme with 2 charts, as above. Then $\mathcal{M}$ is autodual.*

Proof. We will construct the dual pairing between $\mathcal{M}$ and itself. Recall, since we can choose a basis for each coordinate chart, without loss of generality we can assume the mutation map from v_1 to v_2 simply negates the second coordinate on the $x \geq 0$ halfplane, and $\begin{bmatrix} 1 & 0 \\ b & -1 \end{bmatrix}$ on the other half-plane. Recall also that the space of points of $\mathcal{M}$, $Sp(\mathcal{M})$ is a folded book in $\mathbb{R}^3$, glued at $d = 0$, and with $c = e$ whenever $d > 0$, and $(c - d)/b = d$ for $d < 0$. Since we are claiming that $\mathcal{M}$ is autodual, we require only one map, $\mathbf{w} : \mathcal{M} \to \mathrm{Sp}(\mathcal{M})$, which satisfies the condition that for any $m, n \in \mathcal{M}$, $\mathbf{w}(m)[n] = \mathbf{w}(n)[m]$. We claim the function which realized $\mathcal{M}$ as autodual is given by:

$$\mathbf{w} : \mathcal{M} \to Sp(\mathcal{M})$$

$$(x, y)(x, y) \mapsto [y, x] \text{ as a linear map on } M_\phi$$

$$(x', y')(x', bx' - y') \mapsto [y, -x] \text{ as a linear map on } M_\sigma$$

Note that this map ensure that $\mathfrak{v}(m)$ will always be a point of $\mathcal{M}$. When $x > 0$ we map to a linear function on the first chart with $d > 0$, and when $x < 0$, when we mutate $\mathfrak{v}(m)$ to a piecewise linear map on the first chart, we get that $c - e = db$, and $d < 0$, ensuring that $\mathfrak{v}(m)$ is always a point. For this proof, $\langle m, m' \rangle$ will denote $\mathfrak{v}(m)[m']$, and $\langle (x, y), (x', y') \rangle_{std}$ will denote the standard inner product between two vectors in $\mathbb{R}^2$.

To show this pairing is strict, we simply need to show that $\mathbf{w}(m)(m') = \mathbf{w}(m')(m)$. We must test 3 cases.

Case (1): Suppose m, m' are both in $x \geq 0$. Suppose $m = ((x, y), (x, y))$ and $m' = ((x', y'), (x', y'))$. Compute:

$$\mathbf{w}(m)(m') = \langle ((x, y)(x, -y)), ((x', y'), (x', -y')) \rangle = \langle (y, x), (x', y') \rangle_{std} = yx' + xy'.$$

This is symmetric when we switch the primes, and thus has the property required.

Case (2): Suppose m, m' are both in $x \leq 0$. Suppose $m = ((x, y), (x, bx - y))$ and $m' = ((x', y'), (x', bx' - y'))$. Compute:

$$\mathbf{w}(m)(m') = \langle ((x, y), (x, bx - y)), ((x', y'), (x', bx - y')) \rangle = \langle (y, -x), (x', bx' - y') \rangle_{std} = x'y + y'x - bxx'$$

Likewise, this satisfies the dual pairing requirement.

Case (3): Finally, suppose m is in $x \geq 0$, and m' is in $x < 0$. Suppose $m = ((x, y), (x, -y))$ and $m' = ((x', y'), (x', bx' - y'))$. Compute:

$$\mathbf{w}(m)(m') = \langle ((x, y), (x, -y)), ((x', y'), (x', bx - y')) \rangle = \langle (y, x), (x', y') \rangle_{std} = yx' + xy'$$

Meanwhile, we have that

$$\mathbf{w}(m')(m) = \langle((x',y'),(x',bx-y')),((x,y),(x,-y))\rangle = \langle(y',-x'),(x,-y)\rangle_{std} = yx' + xy',$$

as required.

Finally, we must check that the preimages of the maximal cones of $Sp(\mathcal{M})$ are maximal cones of $\Sigma(\mathcal{M})$. This is clearly the case, since the half-plane $d > 0$ is identified with the maximal cone of $\Sigma(\mathcal{M})$, $x \leq 0$, and likewise the preimage of $d < 0$ is the maximal cone $x > 0$. Thus $\mathfrak{v}, \mathbf{w}$ as defined above are the requisite maps to define a strict dual pairing between $\mathcal{M}$ and itself, showing that all rank 2 tropical mutation schemes on 2 charts are (strictly) autodual. □

We have thus computed the space of points for all our examples $\mathcal{M}_b$ of rank two tropical mutation schemes on two charts, and further shown that all such tropical mutation schemes are autodual. In the following chapter, we will show that for each of these tropical mutation schemes, we can construct a detropicalization.

Chapter 4

The Rank 2 Tropical Mutation Schemes $\mathcal{M}_b$ are Detropicalizable

One of the ultimate goals in the study of tropical mutation schemes is to identify classes of tropical mutation schemes which admit detropicalizations. In such a situation, we may expect algebraic invariants of the detropicalization, or the geometic invariants of the underlying variety, to be encoded, wholly or in part, by the combinatorial and semialgebraic structure of the underlying tropical mutation scheme. Recall that the motivational example for this construction is the map which sends a Laurent polynomial to its associated Newton polytope is a valuation from $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ to the canonical semialgebra of the trivial tropical mutation scheme, see Example 2.3.20. Properties of this valuation form a part of the powerful "toric dictionary," where each lattice polytope gives rise an associated compactification of the algebraic torus.

In this section, we will show that all our examples $\mathcal{M}_b$ of rank-2 tropical mutations schemes on exactly two charts have a detropicalization. As we have seen in chapter 3, these tropical mutation schemes are classified up to isomorphism by an integer parameter $b \in \mathbb{Z} \setminus \{0\}$. Let $\mathcal{M}_b$ denote the rank-2 tropical mutation scheme on two charts corresponding to $b \in \mathbb{Z} \setminus \{0\}$. Specifically, In this chapter we will realize the quotient ring

$$\mathbb{C}[x_1, x_2, y_1, y_2, y_1^{-1}, y_2^{-1}]/\langle x_1 x_2 - y_1^b - y_2^b,\ y_2 - 1\rangle,$$

for $b \in \mathbb{Z} \setminus \{0\}$ as a detropicalization of a $\mathcal{M}_b$.

For this computation, we will find it useful to identify each of the two charts of $\mathcal{M}_b$ with two charts with two different subsets of $\mathbb{Z}^2 \times \mathbb{Z}^2$. the first chart will be identified with

$$M_b(1) := \{(w_1, w_2, v_1, v_2) \mid v_1 = 0,\ w_2 = 0\} \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$$

while the second chart will be identified by the set

$$M_b(2) := \{(w_1, w_2, v_1, v_2) \mid v_2 = 0,\ w_2 = 0\} \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2.$$

The mutation map $\mu_{1,2} : M_b(1) \to M_b(2)$ is given by the following:

$$\mu_{1,2}(w_1, 0, 0, v_2) = (w_1, 0, b \cdot \min\{w_1, 0\} - v_2, 0).$$

Although it may seem counter intuitive to identify this tropical mutation scheme with a subset of $\mathbb{Z}^2 \times \mathbb{Z}^2$ when $w_2 = 0$ for both charts, we will see that the extra variable is useful in our upcoming computations.

To see that these sets do indeed indentify the tropical mutation schemes from Chapter 2 consider the following. Let $\mathcal{M}$ be a tropical mutation scheme such that $L(\mathcal{M}) = \{\sigma, \tau\}$. Then we can assume without loss of generality that the mutation map $\mu_{\sigma,\tau}$ sends $(x, y) \mapsto (x, -y)$ on the $x > 0$ half-space, and sends $(x, y) \mapsto (x, bx - y)$ on the $x \leq 0$ half-space. Consider the maps:

$$\Theta_1 : M_\sigma \to M_b(1)$$
$$\Theta_1(x, y) = (x, 0, 0, y) \text{ and}$$
$$\Theta_2 : M_\tau \to M_b(2)$$
$$\Theta_2(x, y) = (x, 0, y, 0)$$

It is easy to see that the following diagram commutes:

$$\begin{tikzcd}
M_\sigma \arrow[r, "\mu_{\sigma,\tau}"] \arrow[d, "\Theta_1"] & M_\tau \arrow[d, "\Theta_2"] \\
M_b(1) \arrow[r, "\mu_{1,2}"] & M_b(2)
\end{tikzcd}$$

and therefore, and rank 2 tropical mutation scheme on two chart can be identified with $M_b(1)$ and $M_b(2)$, for some non-zero integer parameter b.

We notice that, independent of the value of b, the piecewise linear mutation map $\mu_{1,2}$ has exactly two regions of linearity. Namely, the regions of linearity are $\{w_1 > 0\}$, and $\{w_1 < 0\}$. The domains of linearity of $\mu_{1,2}^{-1}$, the inverse mutation map are identical to those of $\mu_{1,2}$. Therefore, $\Sigma(\mathcal{M}_b)$ can be identified by the fan $\Sigma(M_b(1) \otimes \mathbb{R})$ whose maximal cones are given by $\tau^+ = \{(u_1, u_2), (v_1, v_2) \in M_b(1) \otimes \mathbb{R} \mid u_1 > 0\}$, and $\tau^- = \{(u_1, u_2), (v_1, v_2) \in M_b(1) \otimes \mathbb{R} \mid u_1 < 0\}$.

We can now introduce a new subset of $\mathbb{Z}^2 \times \mathbb{Z}^2$. We will map this subset bijectively onto both $M_b(1)$ and $M_b(2)$. This bijection will help us to construct the detropicalization of $\mathcal{M}_b$ later on. Explicitly, consider the following subset of $\mathbb{Z}^2 \times \mathbb{Z}^2$:

$$\mathbb{M}_b := \{((w_1, w_2), (s_1, s_2)) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \mid \min\{w_1, w_2\} = 0, -bw_2 = s_1 + s_2\}. \tag{4.0.1}$$

Now, we can construct the following maps:

$$\psi_1 : \mathbb{M}_b \to M_b(1), \qquad \psi_1(((w_1, w_2), (s_1, s_2))) = (w_1 - w_2, 0, 0, s_2)$$
$$\psi_2 : \mathbb{M}_b \to M_b(2), \qquad \psi_1(((w_1, w_2), (s_1, s_2))) = (w_1 - w_2, 0, s_1, 0)$$

What these maps are actually doing is adding the sum of the s_i coordinates divided by b from each w_i coordinate. Note here that the condition that $v_1 + v_2 = -bu_2$ ensures that we are allowed to divide by b. It also ensures that $u_2 = 0$ in the image of ψ_i.

4.0.1 Lemma. The maps ψ_1 and ψ_2 are bijections to $M_b(1)$ and $M_b(2)$, respectively. Furthermore, we have that $\mu_{1,2} \circ \psi_1 = \psi_2$, and $\mu_{2,1} \circ \psi_2 = \psi_1$.

Proof. We can first show that ψ_1 is injective. Suppose $\psi_1((w_1, w_2)(s_1, s_2)) = \psi_1((w_1', w_2')(s_1', s_2'))$. Firstly, we can see immediately that $s_2 = s_2'$ since ψ_2 does not affect that coordinate. Additionally, we have the conditions $s_1 + s_2 = -bw_2$ and $\min(w_1, w_2) = \min(w_1', w_2') = 0$. If $w_1 = w_1' = 0$ or $w_2 = w_2' = 0$, the result follows immediately. Now suppose $w_1 = 0$ and $w_2' = 0$. Then $-w_2 = w_1'$. However, $w_1', w_2 \geq 0$ so we can conclude that $w_1' = u_2 = 0$. As a result, we have shown that if $\psi_1((w_1, w_2)(s_1, s_2)) = \psi_1((w_1', w_2')(s_1', s_2'))$ then $((w_1, w_2)(s_1, s_2)) = ((w_1', w_2')(s_1', s_2'))$ and therefore ψ_1 is a injection. An identical argument will show that ψ_2 is indeed also invective.

We will now show ψ_1, ψ_2 is surjective. We begin with ψ_1. Take some $(w_1, 0, 0, s_2) \in M_b(1)$. If $w_1 > 0$, then we can see that $\psi_1(w_1, 0, -s_2, s_2) = (w_1, 0, 0, s_2)$. If $w_1 \leq 0$, then $\psi_1(0, -w_1, bw_1 - s_2, s_2) = (w_1, 0, 0, s_2)$. Thus, ψ_1 is surjective.

Now we will show ψ_2 is surjective. Take some $(w_1, 0, s_1, 0) \in M_b(2)$. If $w_1 > 0$, then $\psi_2(w_1, 0, s_1, -s_1) = (w_1, 0, s_1, 0)$. If $w_1 \leq 0$, then $\psi_2(0, -w_1, s_1, -s_1 + bw_1) = (w_1, 0, s_1, 0)$. Thus, ψ_2 is surjective.

What remains to be shown is that $\mu_{1,2} \circ \psi_1 = \psi_2$. This can be seen by a straightforward computation.

$$\begin{aligned}
\mu_{1,2}(\psi_1((w_1, w_2)(-bw_2 - s_2, s_2))) &= \mu_{1,2}(w_1 - w_2, 0, 0, s_2) \\
&= (w_1 - w_2, 0, b \cdot \min\{w_1 - w_2, 0\} - s_2, 0) \\
&= \psi_2((w_1, w_2)(s_1, s_2))
\end{aligned}$$

Note that this gives us the following as well:

$$\begin{aligned}
\mu_{1,2} \circ \psi_1 &= \psi_2 \\
\mu_{1,2}^{-1} \circ \mu_{1,2} \circ \psi_1 &= \mu_{1,2}^{-1} \circ \psi_2 \\
\mu_{2,1} \circ \psi_2 &= \psi_1.
\end{aligned}$$

Therefore, the bijections ψ_1, ψ_2 identify the tropical mutation scheme charts $M_b(1)$ and $M_b(2)$ with $\mathbb{M}_b$, so that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{M}_b \arrow[r, "\psi_1"] \arrow[dr, "\psi_2"'] & M_b(1) \arrow[d, "\mu_{1,2}"] \\
& M_b(2)
\end{tikzcd}$$

$\square$

In fact, not only are the maps ψ_i bijective, we can explicitly write inverse functions.

4.0.2 Lemma. The functions ψ_1^{-1}, and ψ_2^{-1}, given below, are inverse functions to ψ_1 and ψ_2, respectively.

Proof. We claim that the following function is the inverse of ψ_1.

$$\psi_1^{-1} : M_b(1) \to \mathbb{M}_b$$

$$\psi_1^{-1}(u_1, 0, 0, v_2) = (u_1 - \min\{u_1, 0\}, -\min\{u_1, 0\}, b \cdot \min\{u_1, 0\} - v_2, v_2).$$

Certainly, the image of ψ_1^{-1} is contained in $\mathbb{M}_b$ since for any $(u_1, 0, v_1, v_2) \in M_1(1)$ we will have for $\psi_1^{-1}(u_1, 0, v_1, v_2)$ that $\min\{u_1, u_2\} = 0$, and that $v_1 + v_2 = -bu_2$.

Now we will consider $\psi_1^{-1} \circ \psi_1((w_1, w_2)(s_1, s_2))$ for some $((w_1, w_2)(s_1, s_2)) \in \mathbb{M}_b$. First suppose that $\min\{u_1, u_2\} = u_1 = 0$. Then we have $\psi_1((w_1, w_2)(s_1, s_2)) = \psi_1(0, u_2, -bu_2 - v_2, v_2) = (-u_2, 0, 0, v_2)$. Then $\psi_1^{-1}(-u_2, 0, 0, v_2) = (-u_2 + u_2, u_2, -bu_2 - v_2, v_2)$, as required. If $\min\{u_1, u_2\} = u_2 = 0$, then $\psi_1(u_1, 0, -v_2, v_2) = (u_1, 0, 0, v_2)$. Finally, we see that $\psi_1^{-1}(u_1, 0, 0, v_2) = (u_1, 0, -v_2, v_2)$, a required. Therefore, ψ_1^{-1} is indeed the inverse of ψ_1.

An identical argument shows that the map:

$$\psi_2^{-1} : M_b(2) \to \mathbb{M}_b$$

$$\psi_2^{-1}(u_1, 0, v_1, 0) = (u_1 - \min\{u_1, 0\}, -\min\{u_1, 0\}, v_1, b \cdot \min(u_1, 0) - v_1)$$

is the inverse of the function ψ_2. □

We have seen in the previous chapters that any rank-2 tropical mutation scheme with 2 charts $\mathcal{M}_b$ is both full and autodual, and we have described its space of points. In chapter 4 it will be useful to have an identification of the space of points of $\mathcal{M}_b$ in terms of the set $\mathbb{M}_b$ which we have defined above. To do so, we will define the following set:

$$Sp(\mathbb{M}_b) := \Big\{ f : \mathbb{M}_b \to \mathbb{Z} \mid f((w_1, w_2)(s_1, s_2)) + f((w_1', w_2')(s_1', s_2')) \tag{4.0.2}$$

$$= \min_{i \in \{1,2\}}\{ f \circ \psi_i^{-1}(\psi_i((w_1, w_2)(s_1, s_2)) + \psi_i((w_1', w_2')(s_1', s_2')))\} \Big\}$$

We will also define for each $i \in \{1, 2\}$:

$$Sp(\mathbb{M}_b, i) := \Big\{ f : \mathbb{M}_b \to \mathbb{Z} \mid f((w_1, w_2)(s_1, s_2)) + f((w_1', w_2')(s_1', s_2')) \tag{4.0.3}$$

$$= f \circ \psi_i^{-1}(\psi_1((w_1, w_2)(s_1, s_2)) + \psi_i((w_1', w_2')(s_1', s_2'))) \Big\}$$

Now the following lemma identifies the space of points of $\mathcal{M}_b$ with the set $Sp(\mathbb{M}_b)$.

4.0.3 Lemma. The space of points of $\mathcal{M}_b$, $Sp(\mathcal{M}_b)$, is in bijection with the set $Sp(\mathbb{M}_b)$. Furthermore, the sets $Sp(\mathcal{M}_b, i)$ are in bijection with $\mathrm{Sp}(\mathbb{M}_b, i)$ for each $i \in \{1, 2\}$.

Proof. We define the function:

$$\phi : Sp(\mathcal{M}_b) \to Sp(\mathbb{M}_b),\ p = (p_1, \mu_{1,2}(p_1)) \mapsto p_1 \circ \psi_1$$

Because each ψ_1 is a bijection, this map will be well defined and injective. Since a fuction p is a point in $Sp(\mathcal{M}_b)$ if and only if for all $m, m' \in M_b(1)$

$$p_1(m) + p_1(m') = \min\{p_1(m + m'),\ p_1(\mu_{1,2}(m) + \mu_{1,2}(m')\},$$

the map must have image exactly $Sp(\mathbb{M}_b)$. Furthermore, the preimage under this map of $Sp(\mathbb{M}_b, 1)$ (respectively $Sp(\mathbb{M}_b, 2)$, is exactly $Sp(\mathcal{M}_b, 1)$ (respectively $Sp(\mathcal{M}, 2)$. □

Since a point $p \in Sp(\mathcal{M})$ restricts to a piecewise linear map on each chart of a tropical mutation scheme, we may think of a point as a tuple of piecewise linear maps $p = (p_i)_{i \in L(\mathcal{M})}$, where p_i is a piecewise linear map on the i-th chart of $\mathcal{M}$. Thus, the map ϕ simply sends a point to its corresponding piecewise linear map on the sublattice $M_b(1)$, composed with ψ_1.

We will now introduce a new space which we can use to identify the space of points of $\mathcal{M}_b$, similarly to how we used $\mathbb{M}_b$ to identify the elements of $\mathcal{M}_b$.

Let

$$T_b := \{(a_1, a_2)(c_1, c_2) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \mid a_1 + a_2 = b \cdot \min\{c_1, c_2\},\ c_2 = 0\}.$$

Furthermore, let

$$T_b(i) := \{(a_1, a_2)(c_1, c_2) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \mid a_1 + a_2 = b \cdot c_i\}.$$

We will first identify $Sp(\mathcal{M}_b)$ with T_b. Given an element $(\mathbf{a}, \mathbf{c}) = ((a_1, a_2), (c_1, 0)) \in T_b$ we will let $f_{(\mathbf{a},\mathbf{c})}$ denote the linear map

$$f_{(\mathbf{a},\mathbf{c})} : \mathbb{M}_b \to \mathbb{Z}, \qquad f_{(\mathbf{a},\mathbf{c})}((w_1, w_2)(s_1, s_2)) := a_1 w_1 + a_2 w_2 + c_1 \cdot s_1 + 0 \cdot s_2.$$

Thus, for each element $(\mathbf{a}, \mathbf{c})$, we obtain an element of $\mathcal{O}_{\mathcal{M}_b}$. Formally, we have the map

$$\Psi(\mathbf{a}, \mathbf{c}) : T_b \to \mathcal{O}_{\mathcal{M}_b} \quad \Psi(\mathbf{a}, \mathbf{c}), := (f_{(\mathbf{a},\mathbf{c})} \circ \psi_i^{-1})_{i \in [1,2]}.$$

Clearly, each element of T_b defines a piecewise linear map on $\mathcal{M}_b$ through the map Ψ, but the following lemma will show that they in fact identify points of $\mathcal{M}_b$.

4.0.4 Lemma. The map $\Psi(\mathbf{a}, \mathbf{c}) : T_b \to \mathcal{O}_{\mathcal{M}_b}$ is injective, and its image is exactly $Sp(\mathcal{M}_b)$. Furthermore, $\Psi(T_b(1)) = Sp(\mathcal{M}_b, 1)$ and $\Psi(T_b(2)) = Sp(\mathcal{M}_b, 2)$.

Proof. It follows immediately that Ψ is injective, since each $(\mathbf{a}, \mathbf{c}) \in T_b$ will define a different piecewise linear map on $\mathcal{M}_b$. To see this, note that the only way the map $f_{(\mathbf{a},\mathbf{c})}$ is identically zero is if $(\mathbf{a}, \mathbf{c}) = (0, 0, 0, c_1)$, but this tuple will only satisfy the relation of T_b if $c_1 = 0$. Therefore, the kernel of Φ is trivial, and we can conclude it is injective. Next we will show that $\mathrm{Im}(\Psi) \subseteq Sp(\mathcal{M})$. To do this, we will verify that for each $((w_1, w_2)(s_1, s_2)), ((w_1', w_2')(s_1', s_2')) \in \mathbb{M}_b$, we have that

$$f_{(\mathbf{a},\mathbf{c})}(\mathbf{u},\mathbf{v}) + f_{(\mathbf{a},\mathbf{c})}(\mathbf{u}',\mathbf{v}') = \\ \min\{f_{\mathbf{a},\mathbf{c}}(\psi_1^{-1}(\psi_1(\mathbf{u},\mathbf{v}) + \psi_1(\mathbf{u}',\mathbf{v}')), f_{\mathbf{a},\mathbf{c}}(\psi_2^{-1}(\psi_2(\mathbf{u},\mathbf{v}) + \psi_2(\mathbf{u}',\mathbf{v}'))\}.$$

First, we note that

$$\psi_1((w_1,w_2)(-bw_2-s_2,s_2))+\psi_1(u_1',u_2',-bu_2-v_2',v_2') = \pi_1(u_1+u_1'-(u_2+u_2'),0,-bu_2-v_2-bu_2'-v_2',v_2+v_2')$$

then

$$\begin{aligned}&\psi_1^{-1}(\psi_1(u_1,u_2,-bu_2-v_2,v_2)+\psi_1(u_1',u_2',-bu_2'-v_2',v_2'))= \\ &(u_1+u_1'-\min\{u_1+u_1',u_2+u_2'\}, u_2+u_2'-\min\{u_1+u_1',u_2+u_2'\}, \\ &v_1+v_1'+b\cdot\min\{u_1+u_1',u_2+u_2'\}, v_2+v_2'),\end{aligned} \tag{4.0.4}$$

and similarly,

$$\begin{aligned}&\psi_2^{-1}(\psi_2(u_1,u_2,-bu_2-v_2,v_2)+\psi_2(u_1',u_2',-bu_2'-v_2',v_2'))= \\ &(u_1+u_1'-\min\{u_1+u_1',u_2+u_2'\}, u_2+u_2'-\min\{u_1+u_1',u_2+u_2'\}, \\ &v_1+v_1', v_2+v_2'+b\cdot\min\{u_1+u_1',u_2+u_2'\}),\end{aligned} \tag{4.0.5}$$

Thus, by factoring out $b\cdot c_i-(a_1+a_2)$ we have the following for $f_{(\mathbf{a},\mathbf{c})}$.

$$\begin{aligned}&f_{(\mathbf{a},\mathbf{c})}(\psi_i^{-1}(\psi_i((w_1,w_2)(s_1,s_2))+\psi_i((w_1',w_2')(s_1',s_2'))) = \\ &a_1(u_1+u_1')+a_2(u_2+u_2')+c_1(v_1+v_1')+c_2(v_2+v_2')+(b\cdot c_i-(a_1+a_2))\min\{u_1+u_1',u_2+u_2'\})\end{aligned} \tag{4.0.6}$$

However, since $(\mathbf{a},\mathbf{c})\in T_b$, we have that $\min_{i\in[1,2]}\{(b\cdot c_i-(a_1+a_2))\}=0$. Then, we see that $\min\{f_{(\mathbf{a},\mathbf{c})}(\psi_i^{-1}(\psi_i((w_1,w_2)(s_1,s_2))+\psi_i((w_1',w_2')(s_1',s_2')))\} = f_{(\mathbf{a},\mathbf{c})}((w_1,w_2)(s_1,s_2)) + f_{(\mathbf{a},\mathbf{c})}((w_1',w_2')(s_1',s_2'))$. Therefore, $Im(\Psi)\subseteq Sp(\mathcal{M}_b)$ as required.

Now, we will show that $Sp(\mathcal{M}_b)\subseteq Im(\Psi)$. We will show that for each $p\in Sp(\mathcal{M}_b)$, there exists some $(\mathbf{a},\mathbf{c})\in T_b$ such that $\Psi(\mathbf{a},\mathbf{c})=p$. Because we have identified $Sp(\mathcal{M}_b)$ with $Sp(\mathbb{M}_b)$, it will suffice to show that $Sp(\mathbb{M}_b)\subseteq Im(\Psi)$. First, we will partition $\mathbb{M}_b$. Let

$$\mathbb{M}_b(1)=\{((w_1,w_2)(s_1,s_2))\in\mathbb{M}_b|u_1=0\} \text{ and } \mathbb{M}_b(2)=\{((w_1,w_2)(s_1,s_2))\in\mathbb{M}_b|u_2=0\}.$$

Clearly, $\mathbb{M}_b=\mathbb{M}_b(1)\cup\mathbb{M}_b(2)$. Furthermore, since $\min\{u_1+u_1',u_2+u_2'\}=u_i+u_i'=0$ for elements of $\mathbb{M}_b(i)$, we have that for all $f\in Sp(\mathbb{M}_b)$, $f|_{\mathbb{M}_b(k)}$ is linear. Therefore, f will be determined by its values on $e_1=(1,0,0,0), e_2=(0,1,-b,0), e_3=(0,0,-1,1)\in\mathbb{M}_b$.

We will let

$$a_1=f(1,0,0,0),\ a_2=f(0,1,-b,0),\ c_1=f(0,0,-1,1),\ c_2=0.$$

Since each each $(\mathbf{u},\mathbf{v})\in\mathbb{M}_b(1)$ can be written as $m_1e_1+n_1e_3,\ m_1\in\mathbb{Z}_{\geq 0},\ n_1\in\mathbb{Z}$, and similarly each $(\mathbf{u},\mathbf{v})\in\mathbb{M}_b(2)$ can we written as $m_1e_2+n_1e_3,\ m_1\in\mathbb{Z}_{\geq 0},\ n_1\in\mathbb{Z}$, we have that

$$a_1 + a_1 = f(e_1) + f(e_2) = \min\{\psi_1^{-1}(\psi_1(e_1) + \psi_1(e_2)), \psi_2^{-1}(\psi_2(e_1) + \psi_2(e_2))\} = \min\{f(0), f((0,0,-b,b))\} = \min\{0, b \cdot c_1\} \tag{4.0.7}$$

Therefore, $(a_1, a_2, c_1, 0) \in T_b$, as required, and Ψ identifies T_b with $Sp(\mathcal{M}_b)$. □

Although we have already shown that $\mathcal{M}_b$ is autodual, it will be useful to construct a dual pairing which realizes $\mathcal{M}_b$ as autodual in the context of $\mathbb{M}_b$ and T_b. The following propostion will do this.

4.0.5 Proposition. Let $\mathbf{v} : \mathbb{M}_b \to T_b$ be the map

$$\mathbf{v}(u_1, u_2, -bu_2 - v_2, v_2) := (-bu_2 - v_2, v_2, u_1 - u_2, 0).$$

Then $\mathbf{v}$ is bijective, and induces a strict autodual pairing between $\mathcal{M}_b$ and $Sp(\mathcal{M}_b)$.

Proof. We will show $\mathbf{v}$ is bijective by producing a well defined inverse. Let $\mathbf{v}^{-1} : T_b \to \mathbb{M}_b$ such that

$$\mathbf{v}^{-1}(a_1, a_2, 0, c_2) = \begin{cases} (c_1, 0, -a_2, a_2) & \text{if } a_1 + a_2 = 0 \\ (0, -c_1, bc_1 - a_2, a_2) & \text{if } a_1 + a_2 = bc_2 \end{cases} \tag{4.0.8}$$

It is easily verified that $\mathbf{v}^{-1} \circ \mathbf{v} = Id_{\mathbb{M}_b}$, and therefore $\mathbf{v}$ is a bijection. Now, we must show that

$$f_{\mathbf{v}(m)}(m') = f_{\mathbf{v}(m')}(m), \text{ for } m, m' \in \mathbb{M}_b$$

This is easily verified. let $m = (u_1, u_2, -bu_2 - v_2, v_2)$, and let $m' = (u_1', u_2', -bu_2' - v_2', v_2)$. Then

$$f_{\mathbf{v}(m)}(m') = (-bu_2 - v_2)u_1' + v_2u_2' + (u_1 - u_2)(-bu_2' - v_2') =$$
$$-bu_2u_1' - v_2u_1' + v_2u_2' - bu_2'u_1 + bu_2'u_2 - v_2'u_1 + v_2'u_2$$

Conversely,

$$f_{\mathbf{v}(m')}(m) = (-bu_2' - v_2')u_1 + v_2'u_2 + (u_1' - u_2')(-bu_2 - v_2) =$$
$$-bu_2u_1' - v_2u_1' + v_2u_2' - bu_2'u_1 + bu_2'u_2 - v_2'u_1 + v_2'u_2.$$

Therefore, we have that $f_{\mathbf{v}(m)}(m') = f_{\mathbf{v}(m')}(m)$, for $m, m' \in \mathbb{M}_b$, and thus $\mathbf{v}$ is a strict autodual pairing for $\mathbb{M}_b$. □

For the rank 2 tropical mutation scheme $\mathcal{M}_b$ we will consider the following algebra:

$$\mathcal{A}_b = \mathbb{C}[x_1, x_2, y_1, y_2, y_1^{-1}, y_2^{-1}]/\langle x_1x_2 - y_1^b - y_2^b,\ y_2 - 1\rangle.$$

We are now ready to realize the algebra $\mathcal{A}_b$ as a detropicalization of the tropical mutation scheme $\mathcal{M}_b$.

The first step in this process is to identify an adapted basis for $\mathcal{A}_b$. Let us consider the following set:

$$\mathbb{B}_b = \{x_1^{u_1}x_2^{u_2}y_1^{v_1}y_2^{v_2} \mid (u_1, u_2, v_2, v_2) \in \mathbb{M}_b\}.$$

We will first show that $\mathbb{B}$ does indeed form an additive basis for $\mathcal{A}_b$. Then, we will realize $\mathbb{B}$ as an adapted basis by constructing a valuation $\mathfrak{v} : \mathcal{A}_b \to P_{\mathcal{M}_b}$ such that for all $f = \sum C_i b_i \in \mathcal{A}_b$, we have that $\mathfrak{v}(f) = \bigoplus \mathfrak{v}(b_i)$.

4.0.6 Lemma. The set $\mathbb{B}_b$ forms an additive basis for the algebra $\mathcal{A}_b$.

Proof. Consider an element $f \in \mathcal{A}_b$. We must show we can write f as a linear combination of elements of $\mathbb{B}_b$. Consider an arbitrary polynomial $f = \sum C_i x_1^{w_1} x_2^{w_2} y_1^{s_1} y_2^{s_2}$ for some arbitrary exponent vectors in $\mathbb{Z}^4$. By the relation on $\mathcal{A}_b$, we can factor out $min\{w_1, w_2\}$ from each monomial in f, obtaining

$$f = \sum C_i (y_1^b + y_2^b)^{min\{w_1,w_2\}} x_1^{w_1'} x_2^{w_2'} y_1^{s_1} y_2^{s_2} \text{ where } \min\{w_1', w_2'\} = 0.$$

Since $y_2 = 1$, we can always set $s_2 = -bw_2 - s_1$ without changing the polynomial. Thus, so long as all monomials in the expansion of $(y_1^b + y_2^b)^{min\{w_1,w_2\}} x_1^{w_1'} x_2^{w_2'} y_1^{s_1} y_2^{s_2}$ are elements of $\mathbb{B}_b$, we can write f in terms of linear combinations of elements of $\mathbb{B}_b$.

It suffices to show that for any $m, m' \in \mathbb{B}_b$, the monomials in mm' are all elements of $\mathbb{B}_b$. Let $((w_1, w_2)(s_1, s_2))$ be the exponent vector of m, and $((w_1', w_2')(s_1', s_2'))$ be the exponent vector of m'. If $u_1 = u_1' = 0$, or $u_2 = u_2' = 0$, then $mm' \in \mathbb{B}_b$ follows immediately.

Now, let us assume that $u_2 = 0$, and $u_1' = 0$. We will divide this into two cases. First, with $u_1 > u_2'$ and then with $u_2' \geq u_1$.

Case 1: If $u_1 > u_2'$, then

$$mm' = x_1^{u_1} x_2^{u_2'} y_1^{-v_2-bu_2-v_2'} y_2^{v_2+v_2'} =$$
$$(y_1^b + y_2^b)^{u_2'} x_1^{u_1-u_2'} y_1^{-v_2-bu_2'-v_2'} y_2^{v_2+v_2'} =$$
$$(y_1^{bu_2'} + \binom{u_2'}{1} y_1^{b(u_2-1)} y_2^b + \cdots + y_2^{bu_2'}) x_1^{u_1-u_2'} y_1^{-v_2-bu_2'-v_2'} y_2^{v_2+v_2'}.$$

Note that we will always be adding some exponent vector $(0, 0, b(u_2' - k), b(k))$ for $0 \leq k \leq u_2'$ to the exponent vector $(u_1 - u_2', 0, -v_2 - bu_2' - v_2', v_2 + v_2')$. We can see that $b(u_2' - k) - v_2 - bu_2' - v_2' = -(v_2 + v_2' + b(k))$, and thus every monomial is an element of $\mathbb{B}_b$, as required.

Now let us suppose that $u_2' \geq u_1$. In this case,

$$mm' = x_1^{u_1} x_2^{u_2'} y_1^{-v_2-bu_2'-v_2'} y_2^{v_2+v_2'} =$$
$$(y_1^b + y_2^b)^{u_1} x_2^{u_2'-u_1} y_1^{-v_2-bu_2'-v_2'} y_2^{v_2+v_2'} =$$
$$(y_1^{bu_1} + \binom{u_1}{1} y_1^{b(u_1-1)} y_2^b + \cdots + y_2^{bu_1}) x_2^{u_2'-u_1} y_1^{-v_2-bu_2-v_2'} y_2^{v_2+v_2'}.$$

Once again, we will be adding some exponent vector $(0, 0, b(u_1 - k), b(k))$ for $0 \leq k \leq u_1$ to the exponent vector $(0.u_2' - u_1, -v_2 - bu_2' - v_2', v_2 + v_2')$, we get that $v_2 + v_2' + bk - v_2 - bu_2' - v_2' + b(u_1 - k) = -b(u_2' - u_1)$, as required. Thus, the set $\mathbb{B}_b$ forms an additive basis. $\square$

We are now ready to construct a valuation $\mathfrak{v}$ which realizes $\mathcal{A}_b$ as a detropicalization of $\mathcal{M}_b$. Let $\mathfrak{v} : \mathbb{B}_b \to Sp(\mathcal{M}_b)$ be the function

$$\mathfrak{v}(x_1^{u_1}x_2^{u_2}y_1^{-bu_2-v_2}y_2^{v_2}) := \Psi \circ \mathbf{v}(u_1, u_2, -bu_2 - v_2, v_2).$$

Furthermore, we will extend $\mathfrak{v}$ to be a function on $\mathcal{A}_b$ by taking $\mathfrak{v}(\sum c_i b_i) := \bigoplus \mathfrak{v}(b_i)$. Recall that for $p, p' \in Sp(\mathcal{M}_b)$, $p \oplus p'(m) = \min\{p(m), p'(m)\}$, and $p \otimes p'(m) = p(m) + p(m)$.

4.0.7 Lemma. The function $\mathfrak{v} : \mathcal{A}_b \to P_{\mathcal{M}_b}$ is a valuation.

Proof. We will first verify that $\mathfrak{v}(fg) = \mathfrak{v}(f) \otimes \mathfrak{v}(g)$. To do so, start by taking $m, m' \in \mathbb{B}_b$. The result follows immediately from the definitions of Ψ and $\mathbf{v}$ if $mm' \in \mathbb{B}_b$, so assume that $mm' \notin \mathbb{B}_b$. We have two cases. First, let $mm' = (y_1^b + y_2^b)^{u_2'}(x_1^{u_1-u_2'}y_1^{-v_2-bu_2'-v_2'}y_2^{v_2+v_2'})$. Thus, mm' is the sum of the monomials $x_1^{u_1-u_2'}y_1^{-v_2-bu_2'-v_2'+b(u_2'-k)}y_2^{v_2+v_2'+bk}$ for $0 \leq k \leq u_2'$. So we must show that $\mathfrak{v}(m) + \mathfrak{v}(m') = f_{\mathbf{v}(m)} + f_{\mathbf{v}(m')} = \min\{\mathfrak{v}(mm'(i))\}$, where $mm'(i)$ is the monomial obtained above by setting $k = i$.

We will denote by $g_{(k)}$ the map $g_{(k)} : \mathbb{M}_b \to \mathbb{Z}$ such that $g_{(k)}(a_1, a_2, b_1, b_2) = b(u_2' - k)a_1 + b(k)a_2$. We know then that

$$f_{\mathbf{v}(m)} + f_{\mathbf{v}(m')} + g_{\mathbf{i}} = \mathfrak{v}(mm'(i)).$$

Furthermore, since $\min(a_1, a_2) = 0$, we have that $\min_{1 \leq k \leq u_2'}\{g_{\mathbf{i}}\} = 0$ and the desired result holds. An identical argument also shows the above is true if we assume that $u_2' \geq u_1'$. We can then conclude that for $m, m' \in \mathbb{M}_b$, we have that $\mathfrak{v}(mm') = \mathfrak{v}(m) \otimes \mathfrak{v}(m')$.

Now take arbitrary $f, g \in \mathcal{A}_b$. Suppose $f = \sum C_i m_i$ and $g = \sum D_j m_j$, for monomials $m_i, m_j \in \mathbb{B}_b$, and non-zero scalars $C_i, D_j \in \mathbb{C}$. Certainly, we know that

$$\mathfrak{v}(fg) = \min\{\mathfrak{v}(m_i m_j) \mid m_i m_j \text{ appears with a non-zero coefficient in } fg\}$$

Thus, using our previous result, we can see that

$$\mathfrak{v}(fg) \geq \bigoplus_{1 \leq i,j \leq n} \mathfrak{v}(m_i m_j) = \bigoplus_{1 \leq i,j \leq n} (\mathfrak{v}(m_i) \otimes \mathfrak{v}(m_j)) = \bigoplus_{1 \leq i,j \leq n} \mathfrak{v}(m_i) \otimes \bigoplus_{1 \leq i,j \leq n} \mathfrak{v}(m_j) = \mathfrak{v}(f) \otimes \mathfrak{v}(g)$$

Assume towards a contradiction that $\mathfrak{v}(fg) > \bigoplus \mathfrak{v}(m_i m_j)$. Therefore, by the definition of $>$ and piecewise linear functions, there must exist a full dimensional cone σ such that for any $n \in \sigma$ we have that $\mathfrak{v}(fg)(n) > \bigoplus \mathfrak{v}(m_i m_j)(n)$. As a result, we can find some monomial $\hat{m}_i \in supp(f)$ and $\hat{m}_j \in supp(g)$ such that on this cone, $\mathfrak{v}(f) = \mathfrak{v}(\hat{m}_i)$, and $\mathfrak{v}(g) = \mathfrak{v}(\hat{m}_j)$. Since

$$\mathfrak{v}(fg) > \bigoplus_{1 \leq i,j \leq n} \mathfrak{v}(m_i m_j) = \mathfrak{v}(f) \otimes \mathfrak{v}(g)$$

which on the cone σ is equal to

$$\mathfrak{v}(\hat{m}_1) \otimes \mathfrak{v}(\hat{m}_j) = \mathfrak{v}(\hat{m}_i \hat{m}_j),$$

the monomial $\hat{m}_i\hat{m}_j$ cannot appear in fg, and so must have a zero coefficient in fg. But this implies that there is also some $b_i \neq \hat{b_i}$ and $b_j \neq \hat{b_j}$ in the support of f and g, respectively, such that $m_im_j = \hat{m}_i\hat{m}_j$, since there must be some cancellation. It must also be the case that $\mathfrak{v}(b_i) > \mathfrak{v}(\hat{b_i})$ and $\mathfrak{v}(b_j) > \mathfrak{v}(\hat{b_j})$ on the full dimensional cone σ. However, recall that since m_i and m_j are monomial coming from $\mathbb{B}_b$, their image under the $\mathfrak{v}$ must be a point. Thus the values of $\mathfrak{v}(m_i), \mathfrak{v}(m_j), \mathfrak{v}(\hat{m}_i)$ and $\mathfrak{v}(\hat{m}_j)$ are completely determined by their values on σ. Therefore, this is a contradiction, since $\mathfrak{v}(b_ib_j) = \mathfrak{v}(\hat{b_i}\hat{b_j})$. We can therefore conclude that $\mathfrak{v}(fg) = \mathfrak{v}(f) \otimes \mathfrak{v}(g)$, as required.

The final axioms of a valuation, namely that $\mathfrak{v}(cf) = \mathfrak{v}(f)$ and that $\mathfrak{v}(f+g) \geq \mathfrak{v}(f) \oplus \mathfrak{v}(g)$ follow immediately from the definition of the function $\mathfrak{v}$. We have shown that $\mathfrak{v}$ is a valuation from the algebra $\mathcal{A}_b$ to $P_{\mathcal{M}_b}$. □

From this point, it is easy to show that $\mathcal{A}_b$ is indeed a detropicalization of the tropical mutation scheme $\mathcal{M}_b$.

4.0.8 Theorem. The polynomial ring $\mathcal{A}_b$ along with the valuation $\mathfrak{v}$ is a detropicalization of $\mathcal{M}_b$

Proof. The polynomial ring $\mathcal{A}_b$ is clearly isomorphic to $\mathbb{C}[x_1, x_2, y^{\pm}]/\langle x_1x_2 - y^b - 1\rangle$. Since the ideal $\langle x_1x_2 - y^b - 1\rangle$ is principal, and prime for all $b \in \mathbb{Z}$, it follows that $\mathcal{A}_b$ has Krull dimension 2, which is the rank of $\mathcal{M}_b$. Combine with the above lemma, we have that $(\mathcal{A}_b, \mathfrak{v})$ is a detropicalization. Furthermore, $\mathbb{B}_b$ is an adapted basis for this detropicalization, by construction. □

Now that we have completed the story for rank 2 tropical mutation schemes $\mathcal{M}_b$, in the final chapter we will discuss open questions and future directions for research on tropical mutation schemes.

Chapter 5

Future Directions

We will now discuss future directions for the theory of tropical mutation schemes. Since this theory is in its infancy, there are innumerable direction in which to go, so we will only discuss a few.

We mentioned already in Remark 3.1.2 that there is not yet consensus on the notion of isomorphism for tropical mutation schemes. An immediate follow-up to this thesis, once this has been resolved, would be to prove that the class of $\mathcal{M}_b$ does indeed include all rank-2 tropical mutation schemes on exactly two charts, up to isomorphism. This would complete the story for rank-2 tropical mutation schemes on 2 charts. There are two obvious directions we could go from here. First would be to investigate rank-2 tropical mutation schemes on n charts, for $n > 2$. The second would be to investigate rank r tropical mutation schemes on 2 charts with exactly two domains of linearity. Even in rank 3, a mutation map between two charts can have many full dimesional cones, so in terms of complexity it would be best to first restrict to the case of two full dimensional cones.

Another infinite class of dualizable and detropicalizable tropical mutation schemes are computed by Manon, Escobar, and Harada in [3],[4]. They define a tropical mutation scheme $\mathcal{M}_{d,r}$ for each pair of integers $d, r \geq 2$, where $\mathcal{M}_{d,r}$ is strictly dual to $\mathcal{M}_{r,d}$. The tropical mutation scheme $\mathcal{M}_{r,d}$ has rank $r + d - 1$, and has exactly d coordinate charts. Each chart of $\mathcal{M}_{d,r}$ can be identified with the following subset of $\mathbb{Z}^d \times \mathbb{Z}^r$,

$$M_i = \{(\mathbf{u}, \mathbf{v}) \mid v_i = 0\}$$

and each mutation map of $\mathcal{M}_{d,r}$ is given by

$$\mu_{i,j} : M_i \to M_{i+1}$$

$$\mu_{i,j}(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, v_1, \ldots, v_{i-1}, min(u_1, \ldots, u_d) - \sum v_k, 0, v_{i+2}, \ldots, v_r) \text{ for } 1 \leq i \leq r - 1,$$

and the induced compositions for general $\mu_{i,j}$. When $b = 1$, the rank 2 tropical mutation scheme $\mathcal{M}_1$ can be realized as a sub-tropical mutation scheme of the rank 3, autodual $M_{2,2}$. Two natural questions arise from this.

5.0.1 Question. When are sub-tropical mutation schemes of dualizable and detropicalizable tropical mutation schemes are themselves dualizable and detropicalizable?

In the case of the tropical mutation scheme $\mathcal{M}_b$ where $b = 1$, the dualization and detropicalization maps hold reasonably well. We just need to keep track of a certain linear relation on the variables to make sure nothing about the valuation or dual pairing map breaks. It seems likely that in most cases, a sub-tropical mutation scheme, i.e. a collection of sub-lattices which are images of one another under the mutation maps, would also satisfy the nice properties we look for in tropical mutation schemes, so long as the larger one did. It would be interesting to work out when this is possible, as a way to build new examples of detropicalizable tropical mutation schemes from old ones. The second question is this.

5.0.2 Question. Can we introduce an integer parameter to $\mathcal{M}_{2,2}$ such that all $\mathcal{M}_b$ are sub tropical mutation schemes for some $\mathcal{M}^b_{2,2}$? Can the parameter be extended to create a new class of detropicalizable tropical mutation schemes $\mathcal{M}^b_{d,r}$?

The proof of the detropicalizability of $\mathcal{M}_b$ is quite similar to that for $\mathcal{M}_{2,2}$, only we must keep track of the linear relation mentioned above, as well as the integer parameter b. In the case of the rank-2 sub-tropical mutation scheme, the integer parameter can be accounted for, and it gives rise to an infinite class of detropicalizable tropical mutation schemes. It would be interesting to see if an integer parameter can be added to each $\mathcal{M}_{d,r}$, to get a new infinite class of detropicalizable examples for each d, r.

As of now, specific examples of detropicalizable tropical mutation schemes have been computed, but an important step in the development of the theory would be to obtain some more general results, such as the following questions.

5.0.3 Question. Can we find necessary and sufficent conditions for a given tropical mutation scheme to be full? Dualizable? Detropicalizable?

Fullness, dualizability, and detropicalizabiliy are fundament properties of tropical mutation schemes. When they do not have these properties, they lose almost all of their useful analogies with lattices. As of yet, we do not have a general way to tell whether a given tropical mutation scheme satisfies these properties, other that to explicitly compute the space of points and a valuation. Having a combinatorial way to determine whether a tropical mutation scheme satisfies the above would be invaluable. On the other hand, we have similar questions about the extent detropicalizations generalize cluster varieties.

5.0.4 Question. Which k-algebras are tropical mutation scheme detropicalizations?

A good starting point here would be a way of building a tropical mutation scheme from a cluster algebra, such that the cluster algebra is its detropicalization. This would verify that in the archetypal example of cluster varieties, tropical mutation schemes do generalize the connection between tori and lattices. From there, it would be interesting to see what kinds of algebras which are not cluster algebras can fit into this framework.

In this thesis, we have introduced the reader to the basics of the theory of tropical mutation schemes, as well as computed fullness, dualizability, and detropicalizability for the infinite class $\mathcal{M}_b$ as a proof of concept for the above. We have left the reader with a brief list of future research questions on which to build. It will be interesting to see which directions the nascent theory goes in the years to come.

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