Estimating the Population Standard Deviation based on the Sample Range for Non-normal Data

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TITLE:	Estimating the Population Standard Deviation based on the Sample Range for Non-normal Data
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Abstract

Recently, an increasing number of researchers have attempted to overcome the constraints of size and scope in individual medical studies by estimating the overall treatment effects based on a combination of studies. A commonly used method is meta-analysis which combines results from multiple studies. The population standard deviation in primary studies is an essential quantitative value which is absent sometimes, especially when the outcome has a skewed distribution. Instead, the sample size and the sample range of the whole dataset is reported. There are several methods to estimate the standard deviation of the data based on the sample range if we assume the data are normally distributed. For example: *Tippett Method*[2], *Ramirez and Cox Method*[3], *Hozo et al Method*[4], *Rychtar and Taylor Method*[5], *Mantel Method*[6], *Sokal and Rohlf Method*[7] as well as *Chen and Tyler Method*[8]. Only a few papers provide a solution for estimating the population standard deviation of non-normally distributed data. In this thesis, some other distributions, which are commonly used in clinical studies, will be simulated to estimate the population standard deviation by using the methods mentioned above. The performance and the robustness of those methods for different sample sizes and different distribution parameters will be presented. Also, these methods will be evaluated on real-world datasets. This article will provide guidelines describing which methods perform best with non-normally distributed data.

Keywords: Meta-analysis; population standard deviation; sample range; Robustness

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At last but not least, I would like to say thank you to Miss. Jing Zhang for her continued accompany.

Notation

N : Population sample size

n: Group size

σ: Population standard deviation

σ^2: Population variance

SD: empirical standard deviation

i.i.d. : identically and independently distributed

RMSE: Root mean square error

Pdf: Probability Density Function

Cdf: Cumulative Distribution Function

Table of Contents

1. Introduction

In 1992, Guyatt[1], a professor in the Department of Clinical Epidemiology and Biostatistics at McMaster University, first introduced the term "evidence-based machine" (EBM). EBM is "the conscientious, explicit and judicious use of current best evidence in making decisions about the care of individual patients". The purpose of EBM is to combine the experience of clinicians, patient values, and the best available scientific information to guide clinical management decision-making. Meta- analysis is a statistical skill that Integrates the results from many different studies. Meta analysis is commonly used in EBM.

In order to integrate the results from different studies, researchers need a consistent format. When the outcome variable in a meta-analysis is continuous, the population standard deviation is always required. However, from many studies, the population standard deviation (denoted as σ below) is not reported, and sometimes only the range and the sample size are described. In that case, researchers need to estimate the σ to avoid studies from being excluded from the meta-analysis. It is quite necessary when there are not many studies available. Some effective methods have been proposed to estimate the σ based on the range and the sample size, but they assume the data is normally distributed. For example: *Tippett Method*[2], *Ramirez and Cox Method*[3], *Hozo et al Method*[4], *Rychtar and Taylor Method*[5], *Mantel Method*[6], *Sokal and Rohlf Method*[7] as well as *Chen and Tyler Method*[8]. The numerical details of all methods mentioned above will be shown in the methods section later in this thesis

Although the above methods were designed for normally distributed data, it may be possible to apply them to certain data with non-normal distributions. Some bias will be introduced, but it may be acceptable as long as the data has a similar probability density function as a normal distribution. For example, Weibull distribution, Gamma distribution, mixture distribution of normal distribution which mix two normally distributed samples with some mixing proportions. This thesis will discuss the performance and robustness of those methods when applied to non-normal data and provide a general guideline for using the methods in different situations.

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This thesis is organized as follows. The “Methods” section includes all the numerical details of the methods to estimating σ. In the “Results” section, we use R to simulate a large amount of data with different distributions and compare the performance among those methods. In “Real data analysis”, we use real world data to test the performance among previous methods. In “Discussion” section, we will conclude the result for the whole thesis and introduce the next steps of our future work.

2. Methods

I. Existing Methods for Estimating σ

The estimator of population standard deviation (denoted as σ) is $\frac{R}{\xi(n)}$, where R is the sample range of the dataset. $\xi(n)$ can be obtained by applying the following methods and n is the sample size of the dataset.

a) Tippett Method[2]

$$\xi_{Tippett}(n) = 2n \int_{-\infty}^{\infty} z[\Phi(z)]^{n-1}\phi(z)\ dz$$

Where $\phi(z) = \frac{1}{\sqrt{2\pi}}\ e^{-z^2/2}$ and $\Phi(z) = \int_{-\infty}^{z} \phi(z)\, dz$, z is defined as a real number.

b) Ramirez and Cox Method[3] (RC Method)

$$\xi_{RC}(n) = 3\sqrt{ln\, n} - 1.5.$$

c) Hozo et al Method[4] (Hozo Method[4])

$$\xi_{Hozo}(n) = \begin{cases} \sqrt{12}\ \ for\ n \le 15 \\ 4\ for\ \ 15 < n \le 70, \\ 6\ for\ 70 < n. \end{cases}$$

d) Rychtar and Taylor Method[5] (RT Method)

$$\xi_{RT}(n) = \frac{E(R)}{E(SD)} = \frac{\xi_{Tippett}(n)}{\sqrt{\frac{2}{n-1}}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}}$$

where $\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt$ and t is defined as a real number. E(R) and E(SD) are defined as the expected value of the range and the empirical standard deviation.

e) Mantel Method[6]

$$\xi_{Mantel}(n) = \sqrt{n}.$$

f) Sokal and Rohlf Method[7] (SR Method)

$$\xi_{SR}(n) = \begin{cases} 2, for\ n \leq 5 \\ 3, for\ 5 < n \leq 15 \\ 4, for\ 15 < n \leq 50 \\ 5, for\ 50 < n \leq 250 \\ 6, for\ 250 < n \leq 800 \\ 6.5, for\ 800 < n. \end{cases}$$

g) Chen and Tyler Method[8] (CT Method)

$$\xi_{CT}(n) \approx 2\Phi^{-1}\left(0.5264^{1/n}\right)$$

where $\Phi^{-1}(z)$ is the inverse function of $\Phi(z)$.

In 1925, Tippett[2] first proposed an unbiased estimator of σ. Theoretically, it is the best method to estimate σ. However, the result requires complicated numerical calculation or tabulation. In order to solve that, in 1951, Mantel[6] N provided a quick and simple way to estimate σ when the sample size is less than 15. As a general rule of thumb, this method sacrificed accuracy within an acceptable range for convenience. Later in 1987, Sokal RR & Rohlf FJ (denoted as Sokal & Rohlf[7] below) provided a table for estimating σ. It improves both accuracy and convenience compared to the Mantel Method[6]. In 1999, Chen and Tyler[8] used the inverse cumulative distribution function to approximate the variance of the extreme values which can then be used to approximate the variance of the sample range. Researchers can use the table created by Harter and Balakrishnan to obtain the first four moments of the sample range distribution, which can be used to derive the exact variance. It gives a quite accurate estimation. Even though Chen & Tyler method[8] still requires some numerical calculation, it is much simpler than Tippett's method[2] especially when the sample size is large (n > 500). In 2005, Hozo SP, Djulbegovic and Hozo I[4] (denoted as Hozo et al below) proposed a new table to estimate σ based on the Sokal & Rohlf method[7]. It is more designed for teachers' common practical uses. Then, in 2005, Ramírez and Cox created a new simple formula to estimate σ. Recently in 2020, the Rychtar and Taylor Method[5] provided an unbiased estimator of the sample standard deviation rather than the population standard deviation. However, there are some disadvantages of this method. First, it underestimates σ for small sample size. In addition to this, this method is only suitable when researchers are only interested in a particular sample instead of the whole population. Therefore, this method will not be discussed later in this thesis, but it is still a suitable method in some specific situation. As mentioned by Walter[9], researchers would like to estimate the SD for a particular tide during a particular super-moon. This phenomenon won't happen again in the future. Therefore, they do not need the population standard deviation. In that case, Rychtar and Taylor Method[5] would be the best choice.

II. Simulation Methods

In order to conduct simulation studies and compare the performance of the existing methods estimating σ, we need a simulation algorithm. Let X_1, X_2, ……, X_n be identically and independently distributed (i.i.d.) random variables. Let $X_{(1)} \le X_{(2)}$ …… $\le X_{(n)}$ be the order statistics obtained by arranging the proceeding random sample in increasing order of magnitude. $X_{(1)}$ denotes the minimum of the random sample whereas $X_{(n)}$ is the maximum.

There are two parameters to evaluate the performance of the methods: proportional bias and root mean square error. The proportional bias measures the residuals relative to the true σ while the RMSE measures the square root of the variance of the residuals. In this thesis, proportional bias will be the most critical parameter to evaluate performance. The term "bias" in this thesis means proportional bias instead of regular bias. The following thresholds for proportional bias will be adopted for all simulation methods: below 0.1 is best; between 0.1 and 0.2, the method is acceptable. However, the simulation will reject the method if its bias is larger than 0.2. Other users might adopt their own thresholds based on the real situation. Unlike proportional bias which is always between 0 and 1, RMSE has an extensive fluctuation range. As a result, it is hard to set a specific threshold for RMSE, but we can still use RMSE to compare the performance among different methods.

Three non-normal distributions will be considered: mixture distribution of normals, gamma distribution and Weibull distribution. Mixture distribution of normals is commonly used in the biology field when two species are mixed with some proportion. Gamma distribution is a two-parameter continuous probability distribution. Exponential distribution, Chi-square distribution and Erlang distribution are special cases of the Gamma distribution. The Weibull distribution has an almost bell- shape probability distribution curve when the shape parameter is greater than 1. Its properties would be close to normal distribution's properties. Therefore, the methods designed for normal distributions may have a good fit on Weibull distribution.

Methods for Mixture Distribution of Normal

Given a finite set of pdfs p_1(x), p_2(x), ……, p_n(x) or corresponding cdfs P_1(x), P_2(x), ……, P_n(x) and weights w_1, w_2, ……, w_n which satisfy w_i >0 and $\sum w_i = 1$. The mixture distribution can be written as:

$$F(x) = \sum_{i=1}^{n} w_i \, P_i(x)$$

$$f(x) = \sum_{i=1}^{n} w_i \, p_i(x)$$

In this thesis, we will only discuss the situation when n = 2 and p_1(x), p_2(x) follow the normal distribution. The mixture distribution of two normal distributions' pdf is:

$$f(x) = \ w_1 \, p_1(x) + \ (1 - w_1) p_2(x)$$

where p_1(x), p_2(x)is distributed as ~ $N(\mu_1, \sigma_1)$ and $N(\mu_2, \sigma_2)$, w_1 = p. Based on this pdf, we can derive the first and second moments of mixture distribution of normal as

$$E[X] = \int x \, [p \, . \, p_1(x) + (1-p) \, . \ p_2(x)] \, dx = p \ \int x \, . \ p_1(x) \, dx + \ (1-p) \ \int x \, . \, p_2(x) \, dx$$

$$= p\mu_1 + (1-p)\mu_2$$

$$E[X^2] = \int x^2 \, [p \, . \, p_1(x) + (1-p) \, . \ p_2(x)] \, dx = p \int x^2 \, . \ p_1(x) \, dx + \ (1-p) \int x^2 \, . \, p_2(x) \, dx$$

$$= p \ \ (\mu_1^2 + \ \sigma_1^2) \ \ + (1-p)(\mu_2^2 + \ \sigma_2^2)$$

Therefore,

$$\sigma^2 = Var[x] = E[X^2] - E[X]^2 = \ p(\mu_1^2 + \ \sigma_1^2) + (1-p)(\mu_2^2 + \ \sigma_2^2) - \ (p\mu_1 + (1-p)\mu_2)^2 \quad [1]$$

Normally, researchers test the performance by using bias which shows the difference between actual value and approximate value. However, unlike the simple normal distribution, the population standard deviation of mixed normal distribution keeps changing while we adjust its parameters. The proportional bias is the regular bias divided by the true population standard deviation value. By using proportional bias, we can easily compare all methods while the parameters are set differently. The term "bias" refers to a proportional bias in the following thesis.

Additionally, N (0,1) is the baseline distribution. The mixing proportion of the two distributions, the sample size of the simulated data as well as the μ and σ of the second normal distribution will be varied.

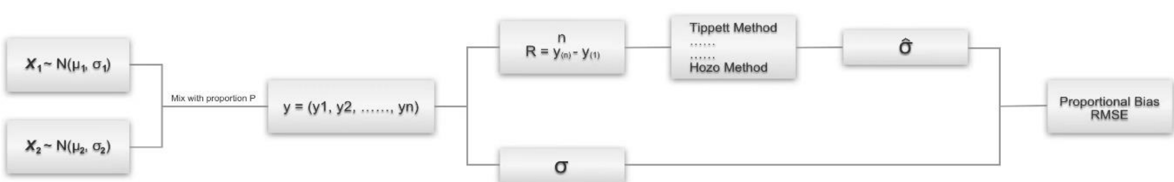


Figure 1 Simulation Algorithm for the Mixture Distribution of Normal

Figure1 shows an algorithm for mixture distribution of normal. First, we generate sample $\boldsymbol{X}_1$ = (X_{11}, X_{12}, ……, X_{1n}) are i.i.d. as N(μ_1, σ_1) and $\boldsymbol{X}_2$ = (X_{21}, X_{22}, ……, X_{2n}) are i.i.d. as N(μ_2, σ_2). Then we generate $\boldsymbol{M}_1$ = (M_{11}, M_{12}, ……, M_{1n}) are i.i.d. as Uniform(0, 1). We can set $y_i = X_{1i}$ if $M_i < p$ and $y_i = X_{2i}$ if $M_i \geq p$. Based on the parameter μ_1, σ_1, μ_2, σ_2 and p, we can obtain the σ^2 from equation [1]. According to $\boldsymbol{y}$ = (y_1, y_2, ……, y_n), we have R = $y_{(n)}$ - $y_{(1)}$ and the sample size n. By applying the methods in previous section, we can easily derive the $\widehat{\sigma}$ for each method. Thus, we can have $proportional\ bias = \frac{\widehat{\sigma}-\sigma}{\sigma}$ and $RMSE = \sqrt{\frac{(\widehat{\sigma}-\sigma)^2}{n}}$ for each method. In order to reduce sampling variation, this algorithm will be repeated 1000 times and use the average value of proportional bias and RMSE. More repetition will significantly increase the running time of the algorithm while only bring slightly or even no improvement of accuracy. 1000 repetition is enough to keep sampling variation within a low level.

Methods for Weibull Distribution

Two-parameter Weibull distribution is a continuous probability distribution with pdf:

$$f(x) = \begin{cases} \frac{c}{\lambda}(\frac{x}{\lambda})^{c-1}e^{-\left(\frac{x}{\lambda}\right)^c}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

Where c > 0 is the shape parameter and λ > 0 is the scale parameter. The population standard deviation for Weibull distribution is calculated by:

$$\sigma = \lambda\sqrt{\Gamma\left(1+\frac{2}{c}\right) - \Gamma(1+\frac{1}{c})^2}$$

Algorithm 2 produces the proportional bias and RMSE for Weibull distributed data by using different methods

1) generate sample $\boldsymbol{X} = (X_1, X_2, \ldots\ldots, X_n)$ are i.i.d. and follow Weibull (c, λ).
2) derive the σ based on the parameter c, λ.
3) applying the methods to drive the $\widehat{\sigma}$ based on $R = X_{(n)} - X_{(1)}$ and the sample size n.
4) calculating the proportional bias and RMSE for each method.
5) repeat previous steps 1000 times and use the average value of proportional bias and RMSE.

Methods for Gamma Distribution

Gamma distribution is a two-parameter family of continuous probability distributions with pdf:

$$f(x) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$$

With a shape parameter k and a scale parameter θ. The population standard deviation for gamma distribution is calculated by:

$$\sigma = \theta\sqrt{k}$$

There are many special distributions contained in the gamma distribution such as exponential distribution, chi-square distribution, Erlang distribution, etc.

In this thesis, we will only focus on the two most commonly used distributions: exponential distribution and chi-square distribution.

Gamma distribution is exponential distribution when shape parameter k = 1. The reciprocal of scale parameter is the parameter for exponential distribution. In addition, gamma distribution is chi-square distribution when scale parameter θ is constant at 2. The degree of freedom of chi-square distribution is half of shape parameter $\frac{k}{2}$.

Algorithm 3 produces the proportional bias and RMSE for gamma distributed data by using different methods

1) generate sample ***X*** = (X_1, X_2, ……, X_n) are i.i.d. and follow exponential (λ) or Chi-square (v).
2) derive the σ based on the parameter λ (or v)

3) applying the methods to drive the $\widehat{\sigma}$ based on $R = X_{(n)} - X_{(1)}$ and the sample size n.

4) calculating the proportional bias and RMSE for each method.

5) repeat previous steps 1000 times and use the average value of proportional bias and RMSE.

3. Results for σ

1) Mixture Distribution of Normal

Before we get into numerical results, we will set some rules for the parameter setting. In terms of the sample size, we will focus on the situation when the sample size is less or equal to 50. As the sample size is enlarged, the bias and RMSE will keep decreasing. A sample size of less or equal to 50 is sufficient for us to see the performance and trend of different methods. In real life applications, the difference of the means of two mixed normal distributions is less than 2. Therefore, in this thesis, we only consider the situation when the difference is less than 2. Similarly, we will focus on the situation when the difference of the standard deviations of two mixed normal distributions is less than 2.

First of all, we equally mixed standard normal distribution *N*(0.1) and *N*(1,1) to form a mixed distribution of normals and investigate the effect on proportional bias and RMSE when we vary the sample size. According to figures 2 & 3, the mean and standard deviation of two normal distributions are fixed, as well as the mixing proportion (p = 0.5). After enlarging the sample size of the data, the bias of all the methods except Mantel[6] and SR methods converge to 0. In terms of SR method, it is a piecewise function, and its proportional bias keeps fluctuating within the range of -0.2 to 0.2. As a result, this method is acceptable for use. Hozo[4] shows a significant proportional bias for sample sizes less than 10. In terms of the RMSE, all methods' RMSE except the Mantel method[6] converge to 0 as the sample size enlarges. Hozo[4] and SR methods have slightly higher RMSE than other three methods.

For the figure 4 & 5, in order to see how the bias varies with the mixing proportion, we chose *N*(0,1) and *N*(0,5) with the fixed simulated sample size N = 50. The bias will always be symmetric around p = 0.5, so for extra clarity we plot values only between 0 and 0.5. As the proportion p goes towards 0 (or 1), the mixture distribution will converge to the single normal distribution. Correspondingly, as p converges to

0.5, the pdf of the mixture distribution of normal will diverge away from the normal distribution. However, RMSE has a different situation with proportion bias. The mantel method[6] still has a bad performance. Overall, RMSE for all methods have a small bump when the mixed proportion is between 0 and 0.1 and starts getting large when mixed proportion approaches to 0.5. As a result, all the methods are getting worse when the mixed proportion approaches 0.5, which is the situation when two normal distributions are equally mixed. Tippett[2] still shows the smallest bias for most of the cases whereas the Mantel method[6] does not perform well. Generally, when p is less than 0.15, all the methods have a relatively small proportional bias which are less than 0.2. However, researchers should be careful with the situation when the mix proportion is around 0.05 where there is a local maximum for RMSE. 0.15 would be the recommended mixed proportion for using those methods. Varying p will have a more significant effect on the bias and RMSE when the difference between the two normal distribution means is larger. Thus, we will discuss how big the difference between two normal distribution means is acceptable.

In the end, we take an analysis of varying the mean or variance of the second normal distribution. From the previous paragraph, it shows the largest bias at $p = 0.5$. So, we take the simulated sample size $N = 50$, mixed proportion $p = 0.5$ and vary the μ of the second normal distribution. Based on figures 6 & 7, when the difference of the mean is increasing, the absolute value of bias keeps increasing simultaneously. Those methods shows a good performance when the difference of mean is less than 2 with the proportional bias almost equal to zero. Additionally, except for the Mantel method[6], the RMSE is stable (approximately 0.2) when the μ of the second normal distribution is enlarged. Therefore, we highly recommend using those methods to estimate σ when the difference of mean is less than 2.

From figures 8 & 9, we can see that the difference of variance of two mixed normal distributions will not have much effect on the method's performance if we keep their mean equal. The RMSE shows a linearly increasing trend as the difference of σ increases, which means the robustness of all the methods is getting worse. However, even though the difference of two σ arrives 2, its RMSE is still very small. Therefore, we can confidently conclude that all methods except the Mantel method[6] have a good performance when the difference of σ is less than 2.

In the end, we will consider the situation when both μ and σ are different for two normal distributions. In practice, people barely use the case when both μ and σ of mixture distribution of normal have a relatively large difference. Therefore, we will slightly increase the μ to 0.5 of the second normal distribution and vary its σ. Comparing figures 8.5 & 9.5 with figures 8 & 9, we can see they show a very similar situation. As a result, we can conclude that all methods have a good performance when the differences of both μ and σ are within 2.

Overall, the Tippett[2] method is the most appropriate method for a mixture of normal distribution, whereas the Mantel[6] is the worst one. Even though Tippett[2] requires tabulation and numerical integration, it gives the least error. If we don't have the integration for Tippett[2] method, Ramirez and Cox method[3] is the second-best choice. We recommend using those methods when the difference of the mean of two mixed normal distributions is less than 2 while the values of two variance do not really matter. Also, smaller mixed proportions will bring a better performance. Larger differences of μ with smaller mixed proportion p may also have good performance and be acceptable to use, but it requires more investigation based on their own situation.

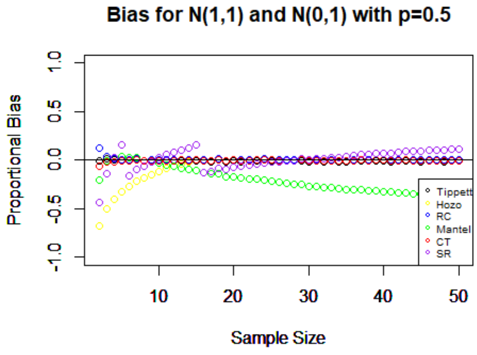


Figure 2

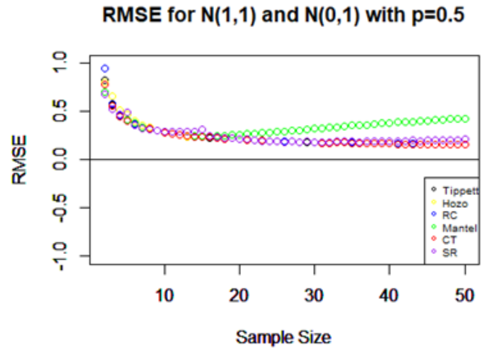


Figure 3

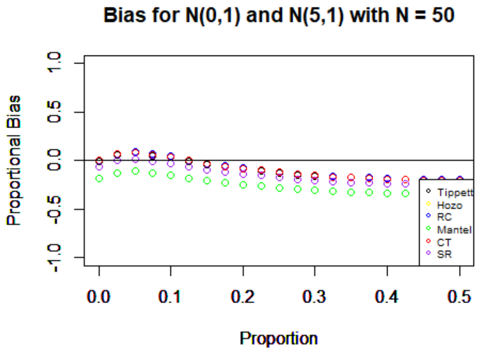


Figure 4

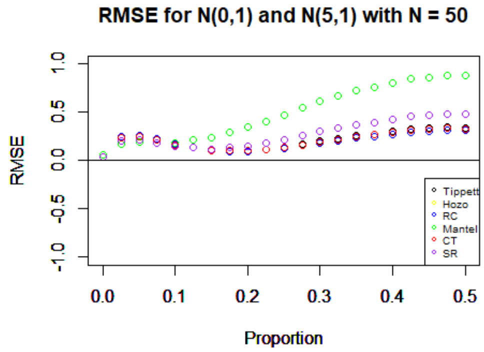


Figure 5

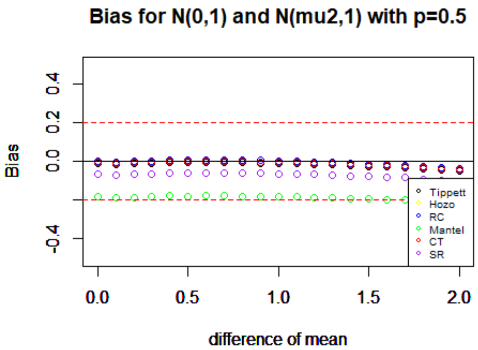


Figure 6

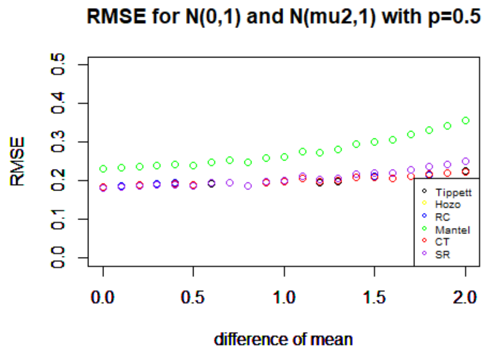


Figure 7

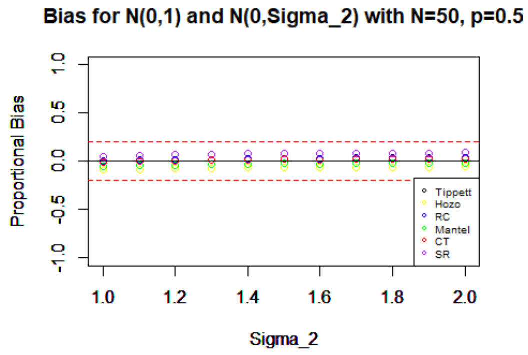


Figure 8

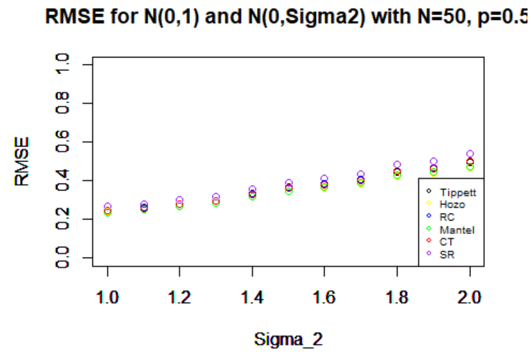


Figure 9

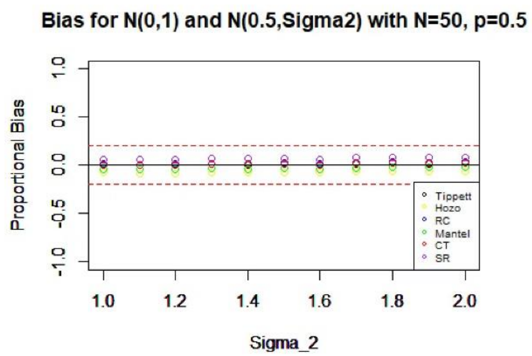


Figure 8.5

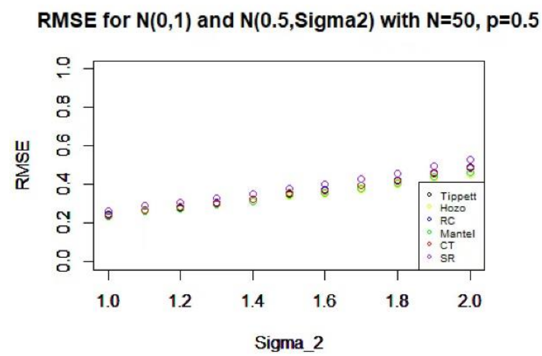


Figure 9.5

2) Weibull Distribution

There are two parameters – shape and scale parameters for Weibull Distribution. For the shape parameter k, the probability density function shows a curve similar to the exponential curve when k < 1 and shows a bell curve when k > 1. From *Johnson and Kotz*[10], when the shape parameter k = 3.6, the Weibull distribution is similar in shape to the normal and shows zero skewness. As a result, we will set Weibull with k = 3.6 as our baseline distribution.

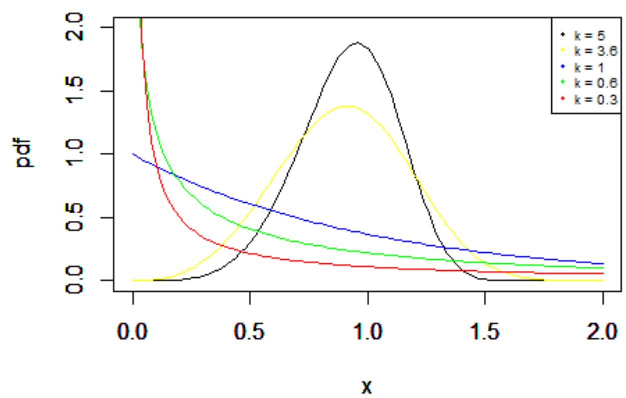


Figure 10: Probability Density Function for Weibull Distribution

In terms of scale parameter, From *Johnson and Kotz*[10], they obtain the order statistics function for standard Weibull distribution (set scale parameter to 1)by using linear transformation $\xi_0 + \lambda X'$ from Weibull distribution:

$$E[X_r] = \frac{n!}{(r-1)!\,(n-r)!}\Gamma(1+\frac{1}{c})\sum_{i=0}^{r-1}\frac{(-1)^i\binom{r-1}{i}}{(n-r+i+1)^{1+\frac{1}{c}}}$$

where $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$.

By using the order statistics, the minimum and maximum is

$$E[X_{(1)}] = n \cdot \Gamma\left(1 + \frac{1}{c}\right) \cdot \frac{1}{n^{(1+\frac{1}{c})}}$$

$$E[X_{(n)}] = n \cdot \Gamma(1 + \frac{1}{c}) \sum_{i=0}^{n-1} \frac{(-1)^i \binom{n-1}{i}}{(i+1)^{1+\frac{1}{c}}}$$

$$E[R] = n \cdot \Gamma\left(1 + \frac{1}{c}\right) \cdot [\sum_{i=0}^{n-1} \frac{(-1)^i \binom{n-1}{i}}{(i+1)^{1+\frac{1}{c}}} - \frac{1}{n^{\left(1+\frac{1}{c}\right)}}]$$

From *Johnson and Kotz*[10], the standard deviation of two-parameter Weibull distribution is:

$$\sigma = \lambda \sqrt{\Gamma\left(1 + \frac{2}{c}\right) - \Gamma(1 + \frac{1}{c})^2}$$

From the equation above, the range for transformed standard Weibull distribution is free of scale parameters which means the range for two-parameter Weibull distribution is proportional to the scale parameter. Then, the standard deviation of two-parameter Weibull distribution is also proportional to the scale parameter. Additionally, ξ for all methods is only depends on the sample size. Based on those, the estimated population standard deviation $\hat{\sigma}$ and its proportional bias are calculated by:

$$\hat{\sigma} = \frac{R}{\xi}$$

$$Proportional\ Bias = \frac{\sigma - \hat{\sigma}}{\sigma} = \frac{\lambda \sqrt{\Gamma\left(1 + \frac{2}{c}\right) - \Gamma(1 + \frac{1}{c})^2} - \frac{R}{\xi}}{\lambda \sqrt{\Gamma\left(1 + \frac{2}{c}\right) - \Gamma(1 + \frac{1}{c})^2}}$$

As a result, the proportional bias will be free of scale parameters, which means the scale parameter will not influence the performance of our methods. Therefore, we can easily set $\lambda = 1$ as the baseline distribution.

Similar to the mixture distribution of normal, we will focus on the situation when the sample size is less or equal to 50. In practice, any scale and shape parameter is possible. Therefore, we will use scale and shape parameters less or equal to 50 to see the trend of the methods' performance.

From Figures 11 & 12, only the bias of the Mantel method[6] keeps increasing as the sample size increases. Thus, the Mantel method[6] will not be recommended at any situation. In terms of the RC method and the Hozo method[4], both are piecewise functions. The proportional bias becomes significant at the end of each segment and sharply drops after each breakpoint, and it goes to 0 around the mid-point of each segment. For the remaining three methods – Tippett method[2], the RC method and the CT method, all of them show the best performance. Within those three methods, RC method does not require numerical integration or tabulation to estimate. However, the proportional bias of those three methods starts slightly diverging away from zero as the sample size becomes large. In terms of the RMSE, all the methods except Mantel method[6] show a decreasing trend as the sample size increases. SR and Hozo[4] have relatively higher RMSE compared with Tippett[2], RC and CT methods, but it is not a huge difference.

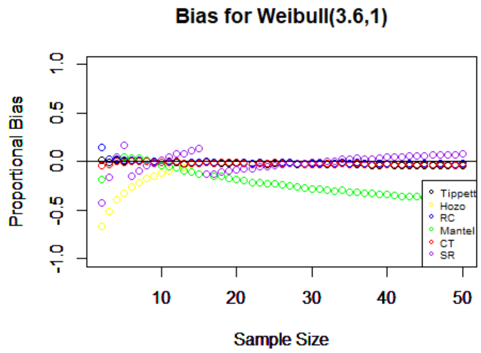


Figure 11

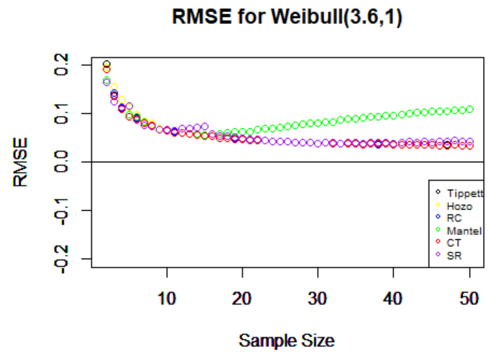


Figure 12

Overall, we recommend using Tippett[2] method when we have numerical integration or tabulation available to estimate. Otherwise, we recommend using the RC method.

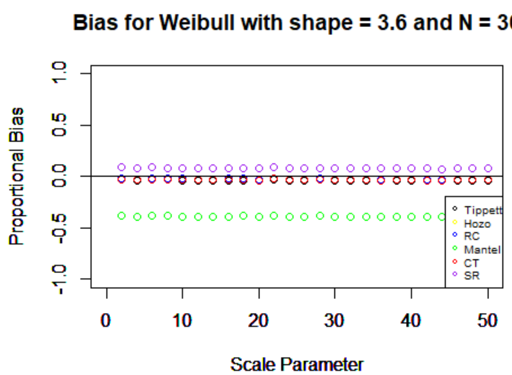


Figure 13

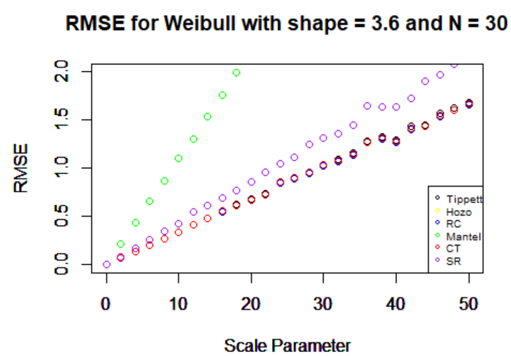


Figure 14

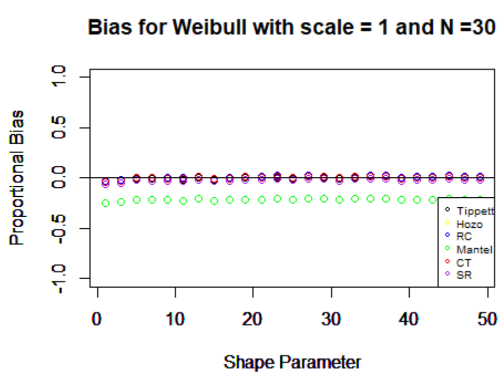


Figure 15

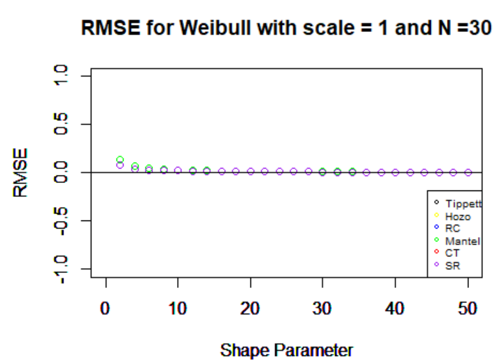


Figure 16

From figures 13 - 16, we will discuss the situation when we are varying the parameter of the Weibull distribution – λ and k. Even though we found that the scale parameter will not influence the proportional bias, we still need to investigate how RMSE change based on the scale parameter. The proportional bias barely changes while we are varying the λ and k. It means the choice of different shape and scale parameters will not significantly influence the performance estimation methods which confirms our result above. However, the RMSE shows a linearly increasing trend when we vary the scale parameter which means the robustness will decrease. The shape parameter does not show a significant effect on the RMSE. The RMSE converges to 0 when we change the shape parameter. As a result, we can still use the result for Weibull (3.6, 1) as a reference for all the situations, but researchers should be careful when the scale parameter is too large ($\lambda > 10$).

3) Gamma Distribution

We will discuss two special cases of gamma distribution in this chapter – exponential distribution and chi-square distribution.

- Exponential distribution

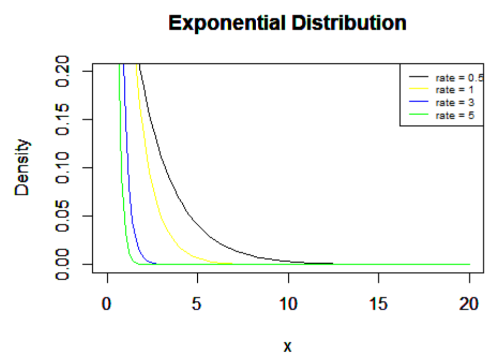


Figure 17: Probability Density Function for Exponential Distribution

Similar to the previous distribution, we will focus on the sample size less than or equal to 50. In terms of the parameter of the exponential distribution , from figure 17, we can see the probability density curve is getting to x- axis more quickly as the parameter increases. The effect of changing parameter become smaller when the parameter is enlarged. There are only four situation are included in this thesis: parameter equal to 0.5, 1, 3, 5.

For exponential distribution with parameter equal to θ, we have $\sigma = \theta$. From *Johnson and Kotz*[10], they obtain the order statistics function for exponential distribution:

$$E[X_{(1)}] = \frac{\theta}{n} \; and \; E[X_{(n)}] = \theta(1 + \frac{1}{2} + \frac{1}{3} \ldots.. + \frac{1}{n})$$

$$E[R] = E\left[X_{(n)}\right] - E\left[X_{(1)}\right] = \theta(1 + \frac{1}{2} + \frac{1}{3} \ldots \ldots + \frac{1}{n-1})$$

Therefore, the proportional bias can be derived:

$$Proportional\ Bias = \frac{\sigma - \hat{\sigma}}{\sigma} = \frac{\theta - \frac{\theta(1 + \frac{1}{2} + \frac{1}{3} \ldots \ldots + \frac{1}{n-1})}{\xi}}{\theta}$$

$$= 1 - \frac{(1 + \frac{1}{2} + \frac{1}{3} \ldots \ldots + \frac{1}{n-1})}{\xi}$$

Which is free of parameter θ. Therefore, the change of parameter will not influence the proportional bias. However, we still need to investigate the RMSE for each method.

According to figures 18 – 25, using different parameters in exponential distributions does not cause a massive difference in proportional bias which confirmed our result above. Tippett method[2] still the best performance and its proportional bias converges to 0 as the sample size increase. RC and CT methods also have a convergent trend as the sample size increases. In terms of the Hozo[4] and SR method, those two methods are piecewise functions instead of continuous functions. Therefore, the proportional bias of those two methods keeps bouncing within each segment and does not have a convergent trend from the plots. However, as the sample size is getting close to the mid-point of each segment, the proportional bias converges to 0. As a result, researchers can choose Hozo[4] or SR methods when the sample size is close to the mid-point of the segment. Mantel method[6] is not recommended in this situation. However, varying parameters does show a significant effect on RMSE. Overall, the RMSE descends as the parameter increases. For parameters greater than 3, the RMSE for all methods converges to 0.In terms of the exponential distribution with parameter 0.5, except Mantel method[6], the

RMSE of all other methods converge to around 0.85 when the sample size increases. Researchers should be careful when the parameter is less than 1.

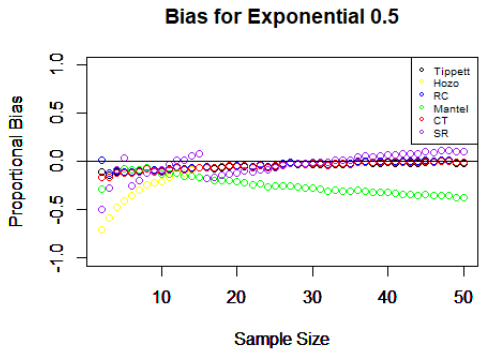


Figure 18

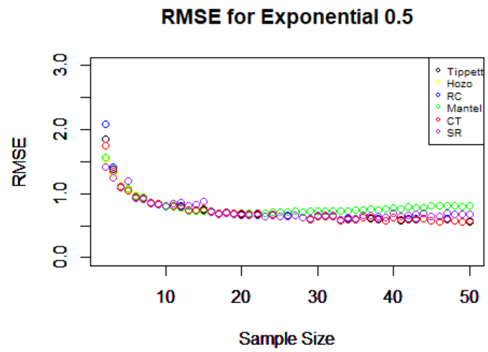


Figure 19

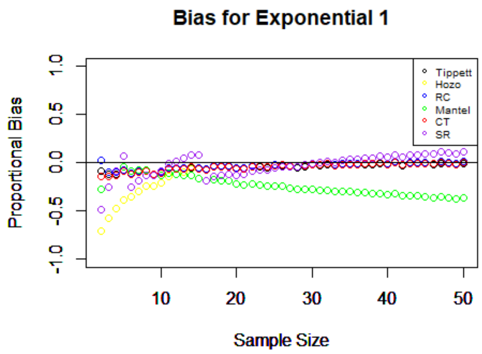


Figure 20

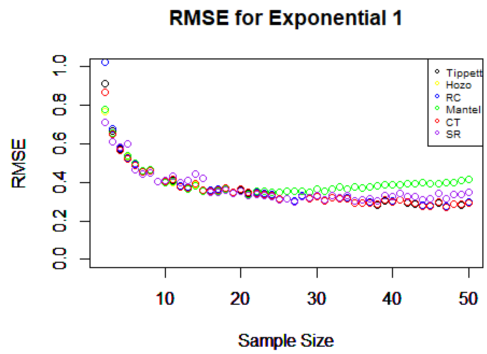


Figure 21

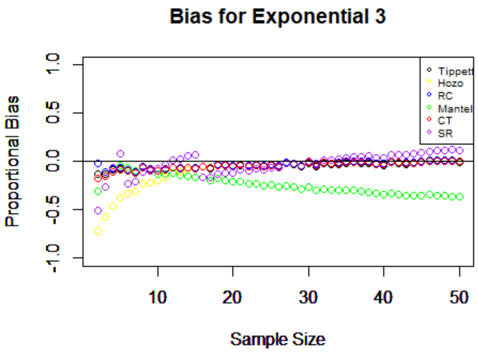


Figure 22

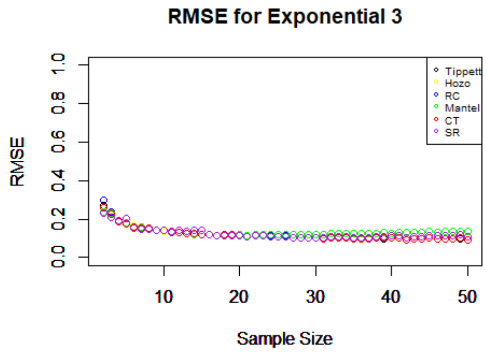


Figure 23

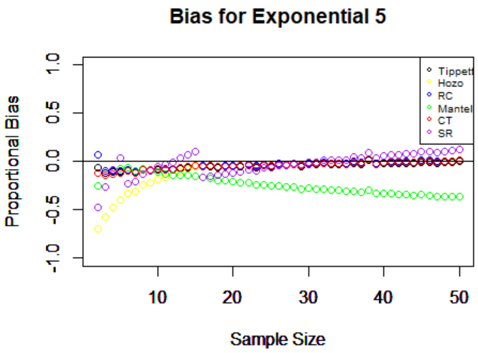


Figure 24

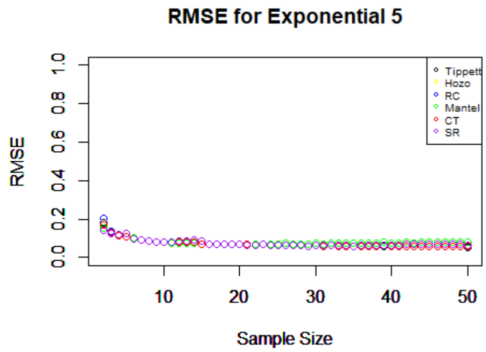


Figure 25

- Chi-square distribution

For the parameter chosen, we will focus on the sample size less or equal to 50 and choose 1, 5, 10 and 20 as the parameter for Chi-square distribution. The pdf of the chi-square distribution has an exponential shape when the parameter is 1 whereas its pdf curve has a bell shape when the parameter is greater than 3. The difference in pdf shape causes a slightly different bias situation between when the parameter equals 1 and when the parameter equals other values. We will use $\chi^2(j)$ to denote chi-squared distribution with a parameter equal to j and j denote as the parameter for chi-square distribution. When the sample size is less than 30, $\chi^2(1)$ has higher proportional bias than chi-square distribution with higher parameter. However, when the sample size is greater than 30, they show a similar situation.

For j greater or equal to 5, its pdf curve is closer to the normal distribution pdf curve. As a result, it has a relatively low proportional bias compared with the other situation. The Mantel[6] is the worst method as usual, and we will not consider this method. Except Hozo[4] and SR method, the other three methods have an extraordinary performance with their proportional bias almost equal to 0 for all sample sizes. Hence, Tippett[2], RC and the CT methods are the top choices.

Then, we will look at the RMSE for chi-square distribution. From figures 28, 30, 32 and 34 , the RMSE slightly increases when j increases. All methods except the Mantel[6] method show a similar situation. Mantel[6] will be rejected due to the relatively high RMSE. When j is large (> 10), SR start to have slightly higher RMSE than other methods when the sample size increasing. Therefore, SR will not be recommended for Chi- square distribution when the j is larger than 10.

Overall, the Mantel method[6] will not be considered. Tippett[2], RC, and the CT methods' RMSE show an increasing trend as sample size increases. Combined with the result for proportional bias, these three methods are the top choices for estimating σ. SR method will not be recommended when the parameter is larger than 10. Even though the plots above do not show a huge difference among these three methods, Tippett[2] method is more recommended because it is an unbiased estimator for normal situations. Researchers can still choose one of those three methods based on their own preferences and situation.

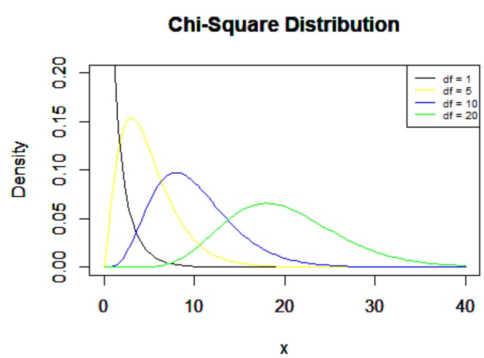


Figure 26: Probability Density Function for Chi-square Distribution

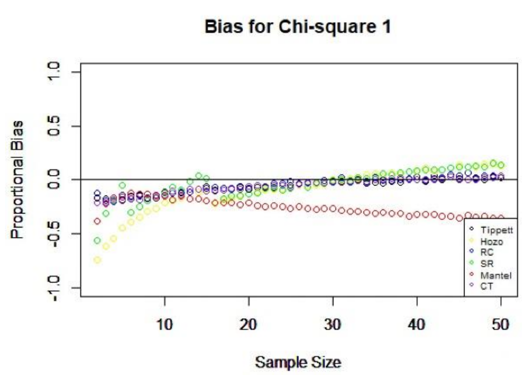


Figure 27

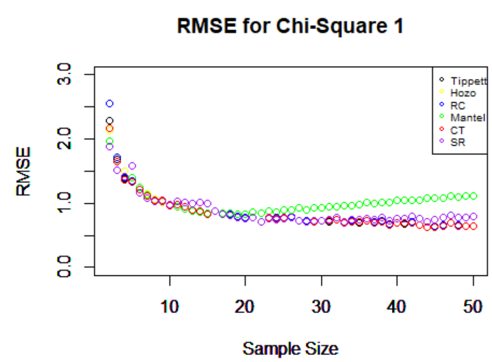


Figure 28

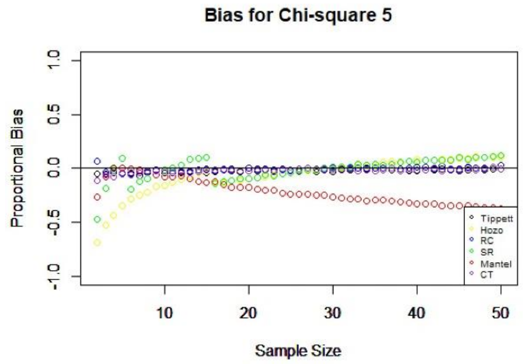


Figure 29

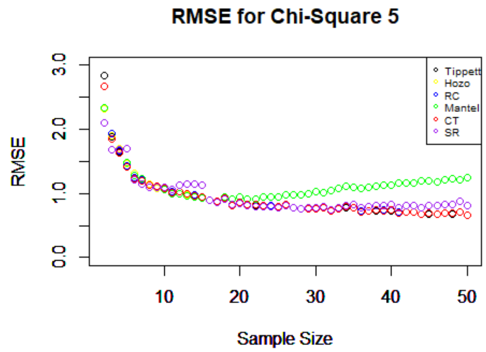


Figure 30

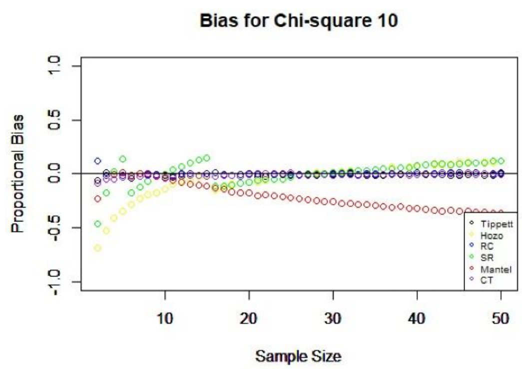


Figure 31

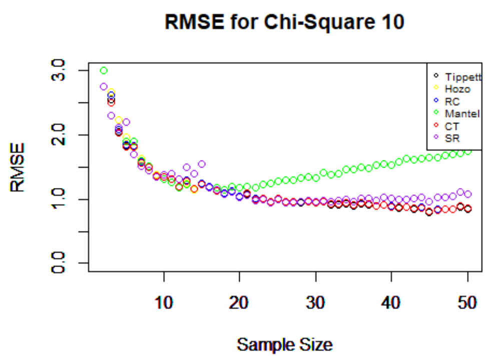


Figure 32

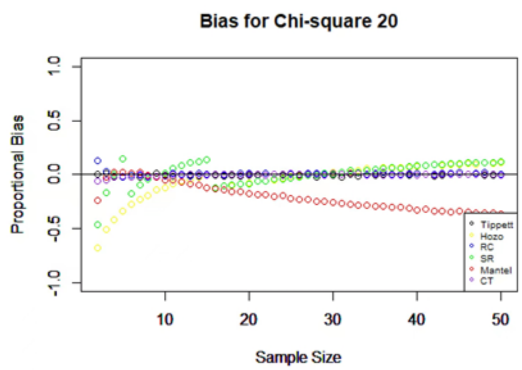


Figure 33

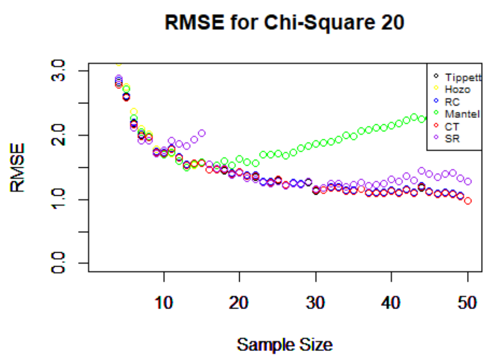


Figure 34

4. Result for σ^2

This section will repeat the algorithm from previous section to simulate the σ^2 instead of σ. The parameter selection rule for σ^2 follows exactly same as the rule for σ.

1) Mixture Distribution of Normal

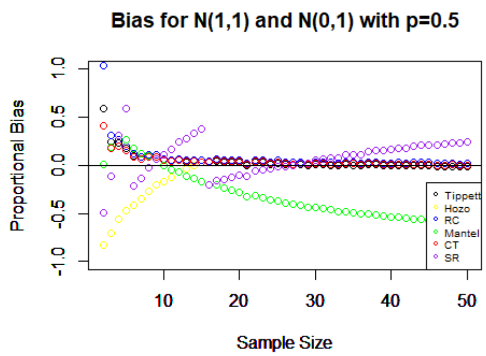


Figure 35

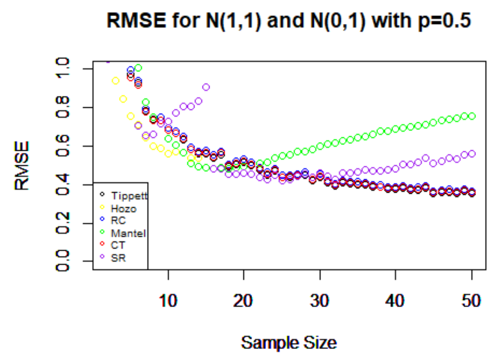
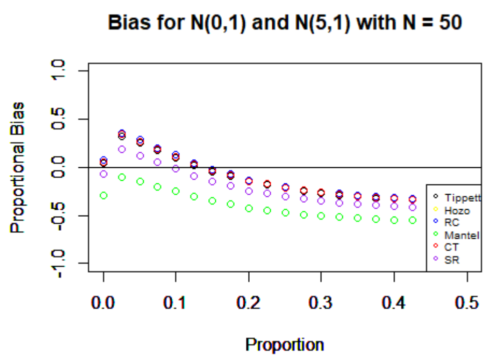


Figure 36

According to the figure 35 & 36, the mean and standard deviation of two normal distributions are fixed, as well as the mixing proportion (p = 0.5). It shows a similar situation as a result for σ, but the proportional bias and RMSE are relatively larger. After enlarging the sample size of the data to 50, the bias and RMSE of all the methods except Mantel[6] and SR methods show a similar result in that proportional bias converges to 0.



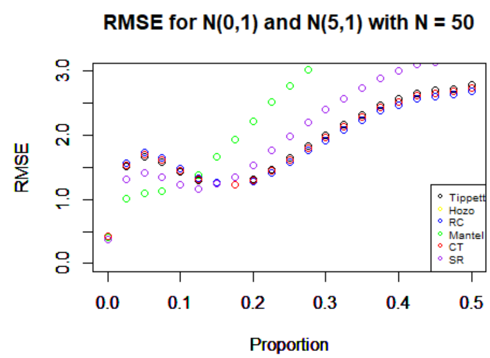


Figure 37 *Figure 38*

Figures 37 & 38 show how the bias and RMSE vary with the mixing proportion for N(0,1) and N(0,5) with the fixed simulated sample size N = 50. Same as the result for σ, the bias and RMSE will always be symmetric around p = 0.5, so for extra clarity we plot values only between 0 and 0.5. As the proportion p goes towards 0 (or 1), the mixture distribution will converge to the single normal distribution. Therefore, the proportional bias is approximately 0 at this point, but it has a small bump around p = 0.03. Then the bias shows a decreasing trend as the proportion goes toward 0.5. The RMSE also have a bump at p = 0.03 and it have a local minimum when p = 0.15. After that point, the RMSE shows an increasing trend. As a result, all the methods have poor performance and robustness when the mixed proportion approaches 0.5, which is when two normal distributions are equally mixed. Tippett[2] still shows the smallest bias for most of the cases whereas the Mantel method[6] does not perform well. Generally, when p is less than 0.15, all the methods have a relatively small proportional bias which are less than 0.2. However, researchers should be careful with the situation when the mix proportion is around 0.03 where is the local maximum for bias and RMSE. 0.1 to 0.15 would be the recommended mixed proportion for using those methods.

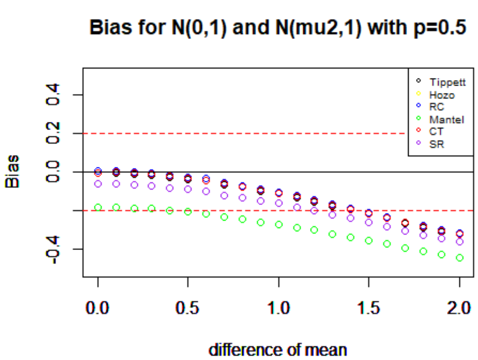


Figure 39

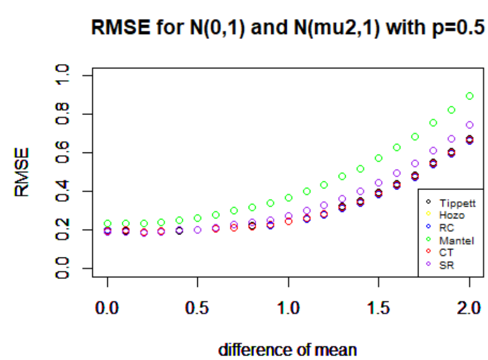


Figure 40

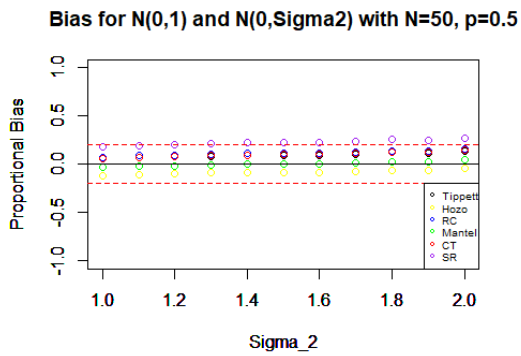


Figure 41

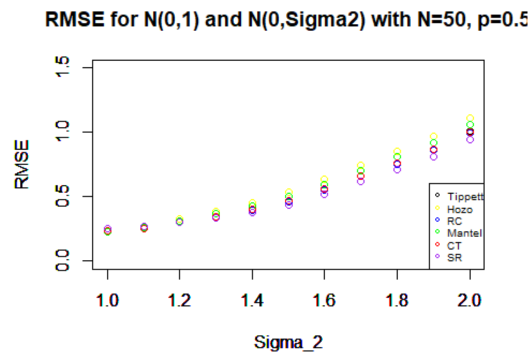


Figure 42

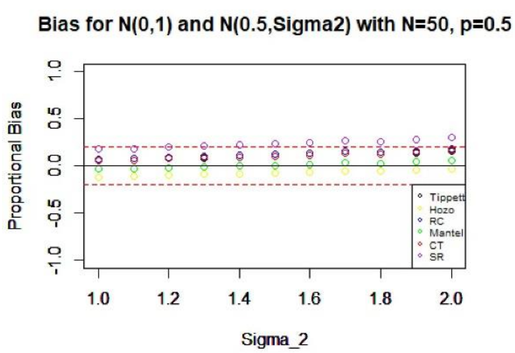


Figure 41.5

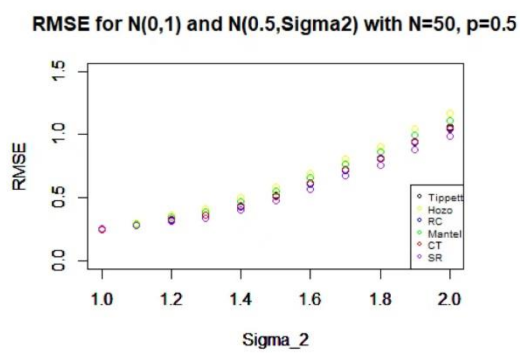


Figure 42.5

From figures 39 to 42, we take an analysis of varying the mean or variance of the second normal distribution. We set the simulated sample size N = 50, mixed proportion p =0.5 and vary the μ of the second normal distribution. According to figures 39 and 40, when the difference of the mean is increasing, the absolute value of proportional bias keeps increasing simultaneously. Except for the Mantel method[6], all the methods' absolute values of bias are less than 0.2 when the difference of mean is less than 1. Besides the proportional bias, all methods' RMSE shows a significant increase when the difference of mean is larger than 1. Therefore, we recommend using those methods to estimate σ when the difference of mean is less than 1. From the figures 41 & 42, changing the variance of two mixed normal distributions has a slight effect on the method's performance if we keep their mean equal. All methods have a small proportional bias while we keep their mean equal. The RMSE shows a linearly increasing trend as the difference of σ increases which means the robustness of all the methods is getting worse. All methods shows a really similar situation. From figures 41.5 & 42.5, we will discuss the situation when both μ and σ are different. Comparing those two figures with 41 & 42, we can see slight change of the mean will not influence the performance of all the methods.

Overall, the result for σ^2 is similar at the result for σ. Tippett method[2] is the most appropriate method for mixture normal distribution whereas the Mantel[6] is the worst one. Ramirez and Cox method[3] is the second-best choice. We recommend using those methods when the difference of the mean and the variance of two mixed normal distributions are both less than 1. Also, smaller mixed proportion will bring a better performance. Therefore, those methods are recommended when the mixing proportion is between 0.1 to 0.15. Larger difference of μ with smaller mixed proportion p may also have good performance and acceptable to use, but it requires more investigation based on their own situation.

2) Weibull Distribution

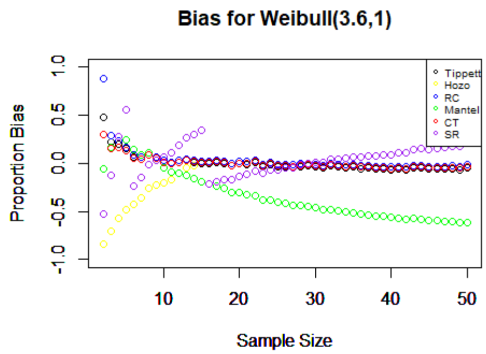


Figure 43

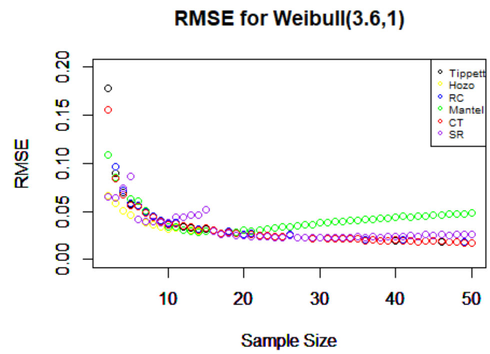


Figure 44

From the Figures 43 & 44, only the bias of Mantel method[6] keeps increasing as the sample size increases. Thus, the Mantel method[6] will not be recommended at any situation. In terms of RC method and Hozo[4] method, both are piecewise functions. The proportional bias becomes large at the end of each segment and sharply drops after each breakpoint, and it goes to 0 around the mid-point of each segment. For the remaining three methods – Tippett method[2], RC method and CT method, all of them show the best performance. Within those three methods, RC method does not require numerical integration or tabulation to estimate. However, the proportional bias of those three methods starts slightly diverging away from zero as the sample size becomes large. In terms of the RMSE, all the methods except Mantel method[6] show a decreasing trend as the sample size increases. SR and Hozo[4] method have relatively higher RMSE compared with Tippett[2], RC and CT methods, but it is not a huge difference.

Overall, we recommend using Tippett method[2] when we have numerical integration or tabulation available to estimate. Otherwise, we recommend using the RC method.

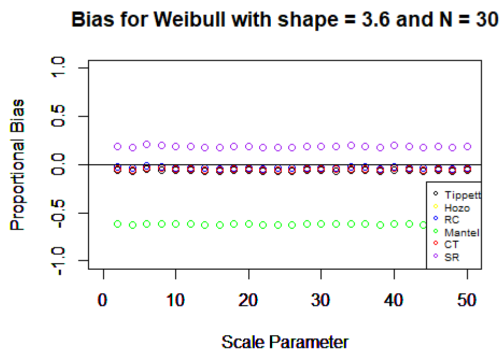


Figure 45

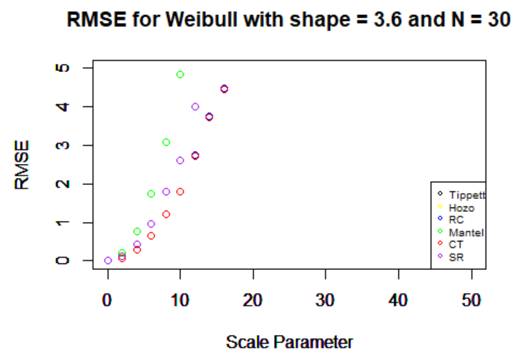


Figure 46

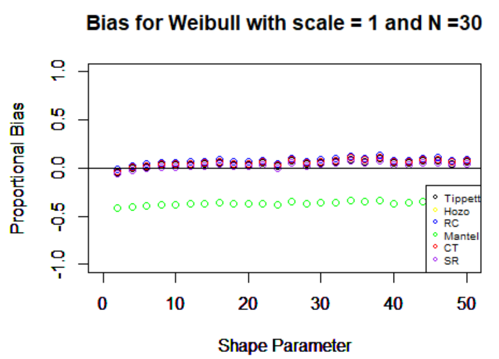


Figure 47

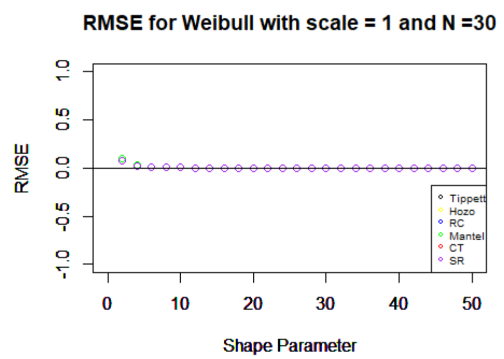
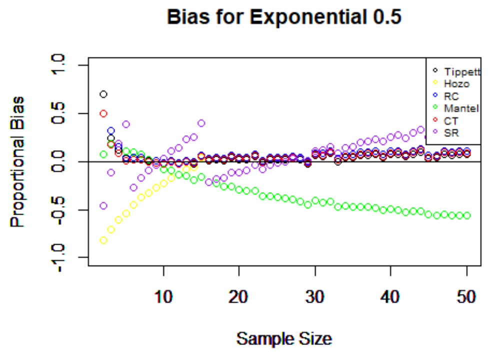


Figure 48

From figure 45 – 48, we will discuss the situation when we vary the parameters of the Weibull distribution – λ and k. The proportional bias barely changes while we are varying the λ and k. It means the choice of different shape and scale parameters will not significantly influence the performance estimation methods. However, the RMSE shows a significantly increasing trend when we are varying the scale parameter which means the robustness will decrease. The shape parameter does not show a significant effect on the RMSE. The RMSE converges to 0 when we are changing the shape parameter. As a result, we can still use the result for Weibull (3.6, 1) as a reference for all the situations, but researcher should be careful when the scale parameter is too large ((k > 8).

3) Gamma Distribution

- Exponential Distribution



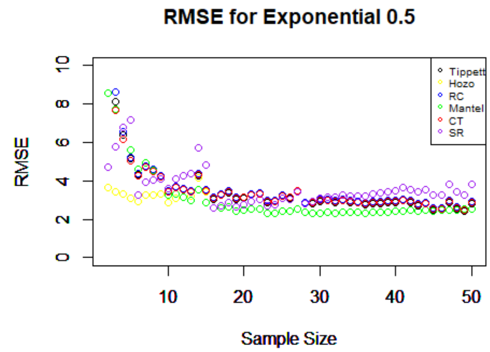


Figure 49 & 50

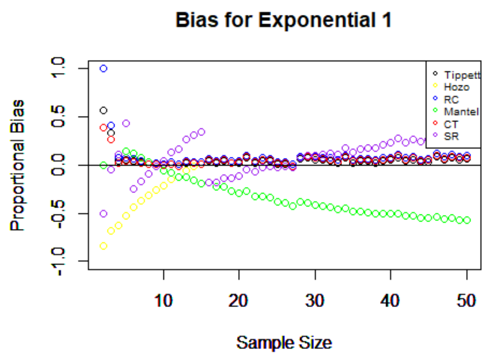


Figure 51

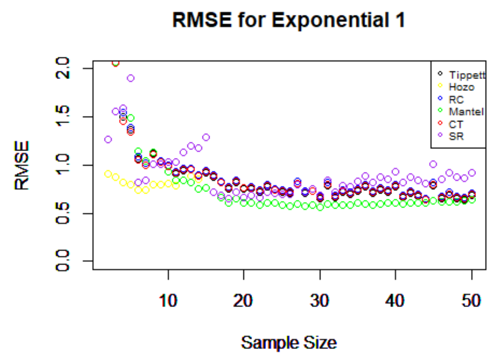


Figure 52

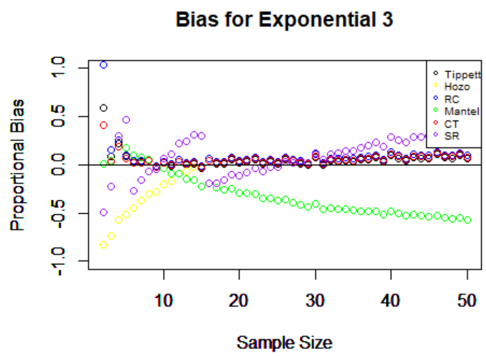


Figure 53

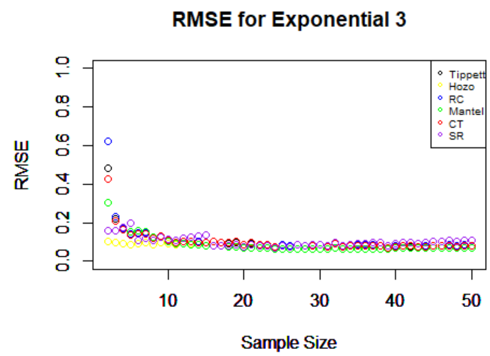


Figure 54

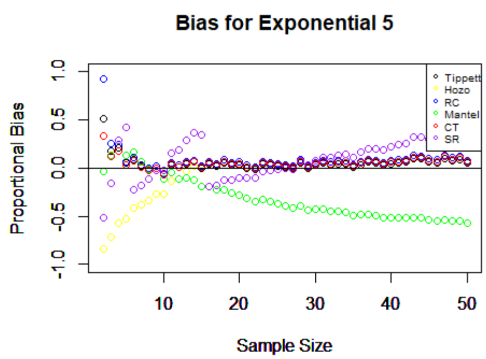


Figure 55

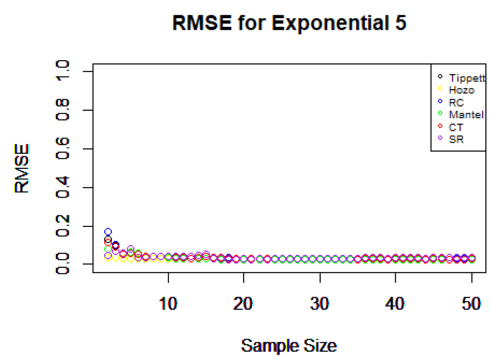


Figure 56

First, we look at the proportional bias for the estimation of σ^2. Comparing figures 49, 51, 53, 55, the change of parameter barely influences the proportional bias of all methods. Among all methods, Tippett[2], RC and CT methods are the best methods. Their proportional bias goes below 0.1 when the sample size is enlarged. Within those three methods at large sample size, Tippett method[2] has the lowest proportional bias (approximately 0.05) while RC and CT have a similar proportional bias (approximately 0.08). For Hozo[4] and SR methods, because they are piecewise functions, we only recommend using them when the sample size is near the mid-point of each segment. Mantel[6] will not be accepted at any situation. According to figures 50, 52, 54, 56, RMSE significantly decreases when we enlarge the parameter. The RMSE dramatically increases when the parameter drops below 1 which means all methods have poor robustness at that situation. Overall, for the exponential distribution, we recommend Tippett method[2] if integration or tabulation is available, otherwise RC and CT are also a good option. However, researchers should be careful and do more tests when the parameter is less than 1.

- Chi-square Distribution

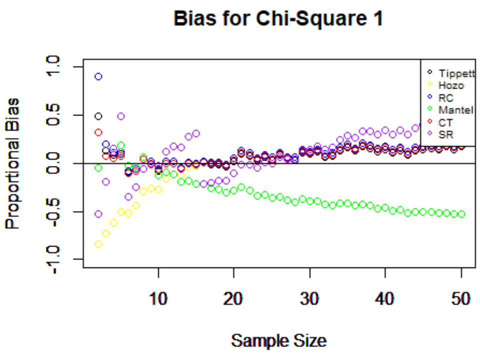


Figure 57

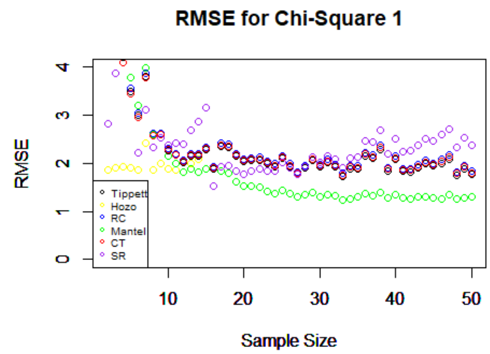


Figure 58

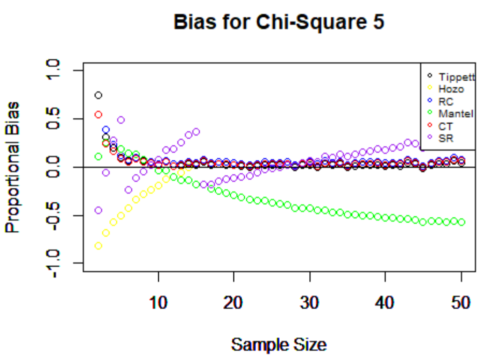


Figure 59

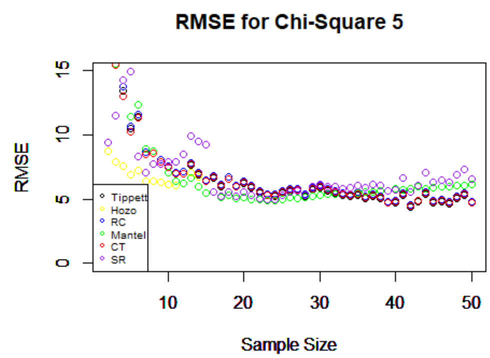
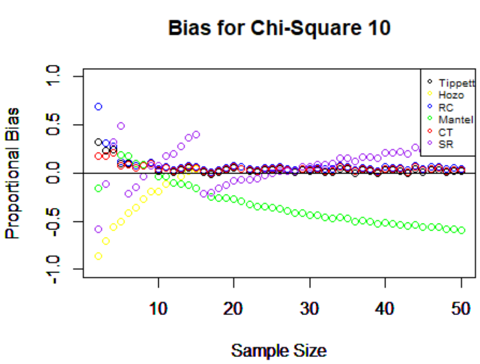


Figure 60



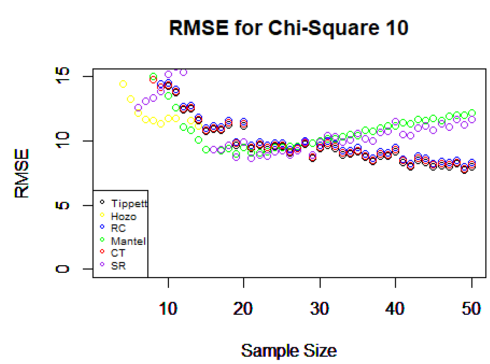
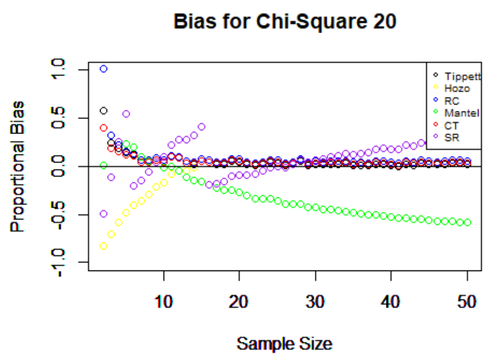


Figure 61 *Figure 62*



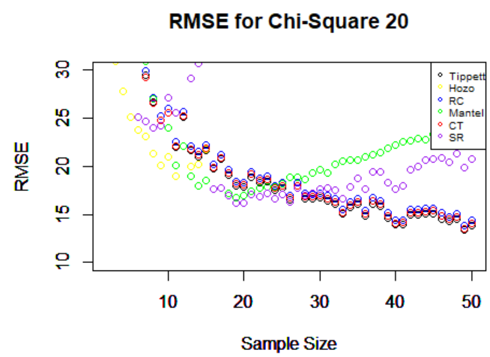


Figure 63 *Figure 64*

Chi-Squared distribution shows a different situation as exponential distribution. The proportional bias generally decreases, and the bias curve becomes smoother when we increase the parameter for chi-squared distribution. However, the RMSE significantly increases when the parameter increases. Therefore, it is hard to keep low bias and good robustness at the same time. We recommend choosing a parameter less than 5. Even though we have a slightly higher bias, we significantly improve the robustness. However, researchers can still choose another parameter based on their own situation. Comparing the methods, Tippett method[2] still has the lowest proportional bias and RMSE. Therefore, it is the best method for Chi- squared distribution if integration or tabulation is available. After that, RC and CT are the second-best options – their proportional bias and RMSE are slightly higher than Tippett[2]. Similar to exponential distribution, we only recommend using Hozo[4] or SR method when the sample size near the mid-point of each segment and we do not recommend the Mantel method[6] for any situation.

5. Real Data Analysis

In the previous section, we randomly generated a set of data with known distribution. However, data does not always exactly follow a known distribution in real world. Therefore, in this section, we compare the performance of the different methods using real-life data. Two datasets are chosen from Macdonald's website[11] – Plasma glucose levels data[13] and Pearson's crab data[12]. There are several reasons why we chose these two datasets. Firstly, those two datasets can be fitted with either Weibull distribution, gamma distribution or mixture of normal distribution. Secondly, full details for those two datasets are missing for some reason. Only the range and the sample size are provided. It satisfied the situation why we need to use those methods to estimate σ or σ^2.

Plasma Glucose Levels Data[13]

This dataset contains the plasma glucose concentrations for the population in Western Samoa and Nauru respectively and was collected in August and September 1978 and January 1982.The original study found that the frequency distribution of plasma glucose concentration in certain population has two distinct sub-groups – a non-diabetic sub-group and a hyperglycaemic sub-group. These two groups show a double peak in the best-fit frequency distribution. The point where two curves intersects indicate a plasma glucose level at which diabetes could be diagnosed. Raper provided a bimodality situation and fit the data by using a bimodal log normal distribution. Then, McDonald[11] worked on the sub-group Western Samoa females with ages between 45 – 54 with the sample size equal to 89. This thesis will use this sub-group data to test the performance of different methods for estimating the population standard deviation.

There are two approaches will be used for the Plasma Glucose Levels dataset[13]. For approach 1, we apply different methods for the whole dataset. we obtain the overall σ by using the overall range and the whole sample size. For approach 2, we first estimate the σ for each subgroup by using the sample size and the range for each subgroup. Then, we calculate the overall σ based on the two subgroups' σ.

For approach 1, the full dataset cannot be found online. Therefore, the exact range and population standard deviation is unknown , and only the histogram is given. However, we can still estimate the range of the data. First, from figure 66, we can roughly get the minimum is 50 while the maximum is 430. We can take the range of the data to be 380.

From McDonald's result[11], he used Mixture of N (102.1, 21.9) and N(262.6,56.4) with mixed proportion equal to 0.83 and 0.17 respectively to fit the dataset. This model fit assumes a constant coefficient of variation. He attempted to find an unconstrained fit, but it failed to converge. Therefore, based on this mixture of normal model, the population standard deviation for the whole dataset is approximate 67.538. The estimated σ and $\sigma^2$by different method will be compared with this value and the proportional bias for different methods is shown in the table 1. Compare the result from table 1 with the previous section's result, Mantel method[6] is still the worst method. Hozo[4] method shows the best performance with the least proportional bias for σ and σ^2. The rest of methods has similar situation that proportional bias is around 1.4 for σ and around 0.3 for σ^2.

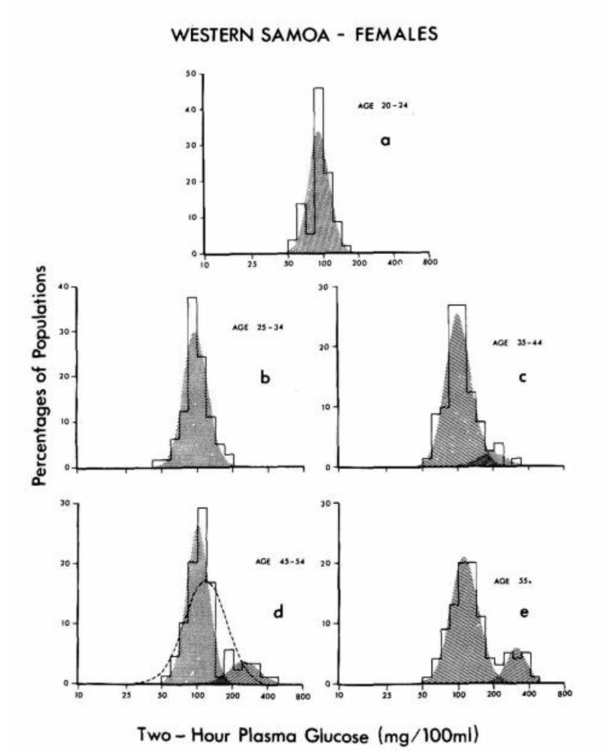


Figure 65: histograms for Female Plasma Glucose Levels Data[13]

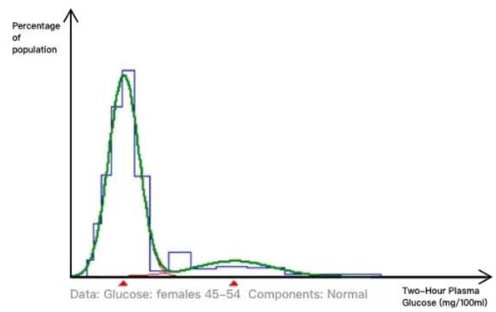


Figure 66: histogram the sub-group Western Samoa females with age between 45 – 54

For approach 2, since we fit the data by mixture distribution of normals, we will use the estimation methods to estimate standard deviations for each normal distribution and use those two standard deviations to get the σ for the whole dataset. According to McDonald's work[11], he fit the dataset by using mixture of N (102.1, 21.9) and N(262.6,56.4) with mixed proportion equal to 0.83 and 0.17 respectively and total sample size equal to 89. Therefore, we can approximate N_1= 73.87 and N_2= 15.13. For the accuracy, we will keep the decimals for the subgroup sample size. In terms of the ranges of the two subgroups, we can roughly estimate them according to figure 66 by eye with R_1 = 200 – 50 = 150 and R_2 = 430 – 100 = 330. Therefore, the standard deviation for each subgroup can be estimated based on the subgroup's range and sample size. Then, we use the two subgroups' standard deviation to derive the overall σ. The proportional bias for estimated overall σ and $\sigma^2$by different methods is shown in the table 1. In terms of the proportional bias result for approach 2, except for the Mantel method[6], all methods shows a similar result as approach 1. The proportional bias for Mantel method[6] dramatically decreased when we applied approach 2.

In conclusion, except the Mantel method[6], all other methods show close results with a proportional bias of around 0.14. According to the thresholds for simulation study in the previous section, the proportional bias is less than 0.2 which is acceptable for our research. Surprisingly, Hozo[4] method has a very small proportional bias whereas Mantel method[6] is still the worst. As a result, the robustness of those methods except Mantel[6] is acceptable. Comparing two different approaches, both of them show a similar situation. It is hard to tell which method is better. In terms of approach 2, we applied the methods on two normal distribution instead of mixture distribution of normals. Those methods have a best performance on normally distributed data. As a result, the proportional bias for approach 2 would be lower relative to approach 1. However, we can only estimate the range of each subgroup by eye when we are applying approach 2. It will cause a large bias when the division of two subgroup is obscure. Therefore, we recommend using approach 1 when the divide of two subgroup is obscure and using method 2 otherwise.

Table 1 Proportional Bias Result for Plasma Glucose Levels Data

Methods	**approach 1 Bias**		**approach 2 Bias**	
	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma^2}$	$\boldsymbol{\sigma}$	$\boldsymbol{\sigma^2}$
Tippett	0.141	0.302	0.143	-0.306
Hozo	-0.062	-0.120	0.077	-0.161
Ramirez & Cox	0.159	0.343	0.148	-0.319
Mantel	-0.404	-0.645	0.057	-0.117
Chen &Taylor	0.149	0.320	0.145	-0.312
Sokal & Rohlf	0.125	0.266	0.100	-0.211

Karl Pearson’s Crab Data[12]

This section will test the Karl Pearson crab data[12] from 1894. The dataset gives the ratio of “forehead” breadth to body length for 1000 crabs sampled at Naples by Weldon. This dataset does not show individual values, but instead provides the number of crabs in terms of categorical frequency. The abscissae of the dataset are the ratio of “forehead” to the body-length with one unit of abscissa is 0.004 of body-length. The first abscissa corresponds to 0.580 – 0.583 of the forehead to body-length ratio. The ordinates represent the number of individual crabs corresponding to each set of ratios of forehead to body-length. There are 29 set of abscissae, and we are using the mid-point method to estimate the range of the dataset. Therefore, the approximate range of this dataset is (29-1)*0.004-0.001 = 0.111.

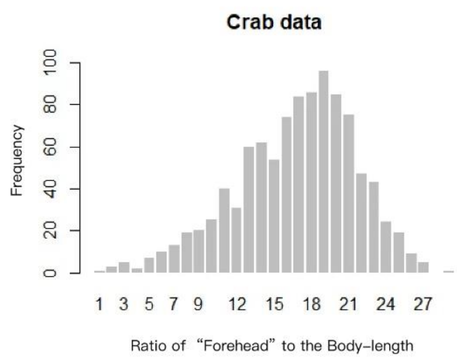


Figure 67: Histogram for Karl Pearson's Crab Data[12]

For the population standard deviation of this dataset, Weldon did not have the precise value. However, Peter Macdonald[11] provided two methods to fit the distribution of the crab dataset by using mixture distribution of normal and Weibull distribution. Based on his result, for mixture distribution of normal, we will be using N(0.63, 0.02) and N(0.65, 0.012) with mixing proportion equal 0.5. As a result, the estimated population standard deviation is 0.019022 for mixture distribution of normal. For the Weibull distribution, we will set mean equal to 0.6443 and the population standard deviation equal to 0.0207.

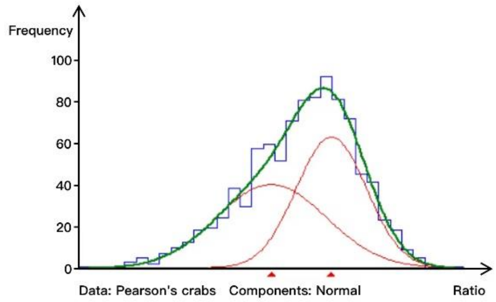


Figure 68: Fit the Data by Mixture Distribution of Normals

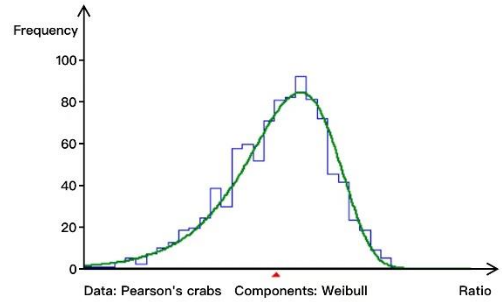


Figure 69: Fit the Data by Weibull Distribution

Table 2 provides the proportional bias of different methods. Similar to the Plasma glucose levels data[13], all other methods show a relatively low proportional bias (less than 0.2) except Mantel method[6] (around 0.8). The Tippet method shows a very small proportional bias when we fit the data by mixture distribution of normal. When we fit the data by Weibull distribution, the Tippet method still has good performance (proportional bias around 0.1), but it does not have a considerable lead compared with the mixture distribution of normal situation. Hozo method[4] has the second smallest proportional bias for mixture distribution of normal and the smallest proportional bias for Weibull distribution. For Ramirez & Cox[3], Chen &Taylor[8] and Sokal & Rohlf method[7], those three methods show a similar proportional bias for either mixture distribution of normal or Weibull distribution.

From McDonald's result[11], Weibull model has a smaller p – value (0.012) than the p-value of mixture distribution of normal (0.57). P-value is an important parameter in statistics which indicate how well a model explains the data. Less p -value means this model has a better performance to explain the data. Therefore, Weibull distribution is a better fit for this data and the method's performance for Weibull distribution would be more critical. Overall, Tippet and Hozo methods[4] will be the top choice whereas we will not consider the Mantel method[6] for any situation.

Table 2 Bias Results for Karl Pearson's Crab Data[12]

Method	***Mixed Normal***	***Weibull***
Tippett	-0.013	-0.112
Hozo	-0.028	-0.106
Ramirez & Cox	-0.086	-0.160
Mantel	-0.816	-0.830
Chen &Taylor	-0.094	-0.167
Sokal & Rohlf	-0.102	-0.175

Real Data Analysis Discussion

Tippett[2] method still shows a good performance for either real data analysis or data simulation. It confirmed that Tippet would be the best choice for estimating σ. Hozo method[4] has a quite small proportional bias in our two real data examples whereas it does not have such a dominant lead in data simulation. Only two real data examples are hard to reflect the exact performance of Hozo method[4], but It would still be the second-best choice. For Ramirez & Cox[3], Chen &Taylor[8] and Sokal & Rohlf[7] method, those three methods always show a similar performance for either situation. Therefore, researchers can decide which method to use based on their own preferences. In the end, Mantel method[6] will not be considered in any situation. To be clear, this is just general guidance for estimating σ. We still suggest readers can do more research based on their own data if possible.

6. Discussion

- ## Estimation of σ or σ^2 for non- normally distributed data

Among all the methods, Tippett method[2] is the best method to estimate σ or σ^2 for non-normally distributed data. It shows the smallest proportional bias and RMSE compared with other methods, but it requires numerical integration or tabulation to get the result. The Tippett method[2] is complicated for the researcher to use especially when the sample size is larger than 500. As a result, Ramirez and Cox method[3] and Chen & Tyler method[8] are the second-best methods. These two methods slightly sacrifice precision for convenience. In terms of Hozo et al method[4] and Sokal & Rohlf[7] method, both are piecewise functions and their bias keeps fluctuating. Therefore, we only recommend using them when the sample size is close to the mid-point of each segment. Mantel method[6] will not be taken in any situation.

Unlike for normal distributions, those methods do not always have good performance for non-normal distributions. We recommend using those methods for the following circumstance:

For the mixture distribution of normal, if we want to estimate σ, the difference of the mean of two mixed normal distributions should less than 4 while the values of two variance do not really matter. Smaller mixing proportion will cause a better performance. If we would like to estimate σ^2, the difference of the mean and the variance of two mixed normal distributions are both less than 2. The recommended range for mixing proportion is from 0.1 to 0.15.

For the Weibull distribution, the shape parameter and scale parameter barely influence the proportional bias, but the RMSE keeps increasing when the scale parameter is enlarged. Therefore, the scale parameter should be less than 10 for estimating σ and less than 8 for estimating σ^2.

For Exponential distribution, the parameter barely affects the proportional bias, but the RMSE becomes large when we have small parameters. Consequently, the parameter for exponential distribution should be larger than 1 for either estimating σ or σ^2.

For chi-squared distribution, parameter do not have much effect on either proportional bias or RMSE when we are estimating σ. Any parameter of chi-squared distribution can be taken for estimating σ. However, for estimating σ^2, its proportional bias slightly decreases and its RMSE significantly increases when the parameter is enlarged. As a result, the parameter should be less than for when we are estimating σ^2.

Above are the general guidebooks for the estimation of σ and σ^2 based on my own threshold. However, researchers can still choose other method based on its own preference and situation.

- Future work

In this thesis, we use the estimation methods designed for normal distribution to estimate σ and σ^2 for non-normal distributed data. Even though those methods still have a good performance with small proportional bias and RMSE, we still want an unbiased estimator for non-normal distributed data. Due to the lack of time, the unbiased estimator has not been found yet, but I will try to find it out in the future. Beside that, there are many challenges and uncharted things in statistical area await our exploration. I will make continuous effort to do more research in statistics area.

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Appendix

All numerical result summary from section 3 & 4 will be concluded in this section.

- **Mixture Distribution of Normal**

Sample	Tippett		Hozo		RC		Mantel		CT		SR	
Size N	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.010	0.393	-0.335	0.449	-0.001	0.396	0.030	0.412	-0.018	0.389	0.152	0.494
10	-0.019	0.276	-0.129	0.279	-0.011	0.278	-0.046	0.271	-0.018	0.276	0.006	0.284
15	-0.009	0.234	-0.007	0.235	0.001	0.237	-0.111	0.240	-0.005	0.235	0.147	0.322
20	-0.010	0.212	-0.076	0.212	0.001	0.215	-0.173	0.257	-0.005	0.213	-0.076	0.212
25	-0.011	0.197	-0.028	0.195	0.001	0.199	-0.222	0.291	-0.005	0.198	-0.028	0.195
30	-0.007	0.177	0.014	0.182	0.006	0.180	-0.259	0.315	-0.001	0.178	0.014	0.182
35	-0.009	0.167	0.044	0.179	0.004	0.169	-0.294	0.356	-0.003	0.168	0.044	0.179
40	-0.011	0.161	0.069	0.188	0.003	0.163	-0.324	0.381	-0.004	0.162	0.069	0.188
45	-0.011	0.162	0.092	0.202	0.003	0.163	-0.349	0.408	-0.004	0.162	0.092	0.202
50	-0.012	0.156	0.111	0.212	0.002	0.157	-0.372	0.428	-0.005	0.156	0.111	0.212

The numerical result of Estimation σ for N(1,1) and N(0,1) with p = 0.5

Proportio	Tippett		Hozo		RC		Mantel		CT		SR	
n P	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0	0.003	0.190	-0.052	0.187	0.015	0.193	-0.173	0.233	0.009	0.192	-0.052	0.187
0.05	0.080	0.520	0.020	0.477	0.092	0.531	-0.110	0.442	0.085	0.525	0.020	0.477
0.1	0.027	0.412	-0.030	0.389	0.039	0.420	-0.153	0.433	0.033	0.416	-0.030	0.389
0.15	-0.032	0.318	-0.086	0.343	-0.021	0.317	-0.202	0.489	-0.027	0.317	-0.086	0.343
0.2	-0.081	0.324	-0.132	0.393	-0.070	0.313	-0.242	0.592	-0.076	0.319	-0.132	0.393
0.25	-0.125	0.385	-0.173	0.474	-0.115	0.368	-0.278	0.695	-0.120	0.377	-0.173	0.474
0.3	-0.154	0.466	-0.201	0.563	-0.144	0.447	-0.302	0.792	-0.149	0.457	-0.201	0.563
0.35	-0.179	0.514	-0.224	0.617	-0.169	0.493	-0.323	0.854	-0.174	0.504	-0.224	0.617
0.4	-0.192	0.553	-0.237	0.658	-0.183	0.531	-0.334	0.899	-0.188	0.543	-0.237	0.658
0.45	-0.204	0.583	-0.248	0.691	-0.194	0.561	-0.343	0.933	-0.200	0.573	-0.248	0.691
0.5	-0.205	0.596	-0.249	0.703	-0.196	0.574	-0.345	0.945	-0.201	0.587	-0.249	0.703

The numerical result of Estimation σ for N(5,1) and N(0,1) with N = 50

Difference of Mean	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0	-0.006	0.198	-0.061	0.194	0.006	0.202	-0.181	0.236	-0.001	0.200	-0.061	0.194
1	-0.010	0.211	-0.064	0.208	0.002	0.214	-0.183	0.262	-0.004	0.212	-0.064	0.208
2	-0.049	0.233	-0.101	0.252	-0.037	0.231	-0.216	0.352	-0.044	0.232	-0.101	0.252
3	-0.110	0.294	-0.159	0.351	-0.099	0.284	-0.266	0.510	-0.105	0.289	-0.159	0.351
4	-0.163	0.416	-0.209	0.502	-0.153	0.399	-0.310	0.706	-0.158	0.408	-0.209	0.502
5	-0.206	0.587	-0.250	0.695	-0.197	0.565	-0.345	0.939	-0.202	0.577	-0.250	0.695
6	-0.241	0.777	-0.283	0.903	-0.232	0.751	-0.374	1.183	-0.237	0.765	-0.283	0.903
7	-0.269	0.997	-0.309	1.139	-0.260	0.967	-0.397	1.451	-0.265	0.983	-0.309	1.139
8	-0.289	1.201	-0.329	1.360	-0.281	1.168	-0.414	1.707	-0.286	1.186	-0.329	1.360
9	-0.306	1.428	-0.344	1.601	-0.298	1.391	-0.428	1.980	-0.302	1.412	-0.344	1.601
10	-0.321	1.645	-0.359	1.834	-0.313	1.605	-0.440	2.246	-0.318	1.627	-0.359	1.834

The numerical result of Estimation σ for $N(0,1)$ and $N(\mu_2,1)$ with $p = 0.5$ and $N = 50$

Difference of Sigma	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0	-0.002	0.212	-0.118	0.218	0.009	0.215	-0.118	0.218	0.002	0.213	-0.118	0.218
0.4	0.021	0.284	-0.098	0.278	0.032	0.289	-0.098	0.278	0.026	0.286	-0.098	0.278
0.8	0.044	0.401	-0.078	0.371	0.055	0.407	-0.078	0.371	0.049	0.403	-0.078	0.371
1.2	0.062	0.517	-0.062	0.459	0.073	0.526	-0.062	0.459	0.066	0.521	-0.062	0.459
1.6	0.077	0.618	-0.049	0.539	0.089	0.629	-0.049	0.539	0.082	0.622	-0.049	0.539
2	0.092	0.731	-0.036	0.629	0.103	0.745	-0.036	0.629	0.096	0.737	-0.036	0.629
2.4	0.099	0.852	-0.030	0.723	0.110	0.869	-0.030	0.723	0.103	0.858	-0.030	0.723
2.8	0.106	0.963	-0.023	0.816	0.118	0.983	-0.023	0.723	0.111	0.971	-0.023	0.816
3	0.104	1.009	-0.025	0.859	0.116	1.029	-0.025	0.859	0.109	1.017	-0.025	0.859

The numerical result of Estimation σ for $N(0,1)$ and $N(0,\sigma_2)$ with $p = 0.5$ and $N = 50$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.145	1.005	-0.484	0.752	0.165	1.027	0.239	1.110	0.126	0.985	0.549	1.502
10	0.045	0.662	-0.175	0.559	0.063	0.675	-0.010	0.623	0.048	0.664	0.100	0.707
15	0.044	0.540	0.049	0.543	0.066	0.554	-0.161	0.477	0.052	0.545	0.399	0.874
20	0.019	0.502	-0.111	0.457	0.043	0.516	-0.289	0.499	0.029	0.508	-0.111	0.457
25	0.012	0.449	-0.023	0.432	0.037	0.464	-0.375	0.532	0.023	0.455	-0.023	0.432
30	-0.007	0.428	0.036	0.454	0.019	0.443	-0.448	0.588	0.005	0.435	0.036	0.454
35	-0.003	0.410	0.106	0.482	0.024	0.424	-0.494	0.640	0.010	0.416	0.106	0.482
40	-0.006	0.375	0.161	0.502	0.022	0.391	-0.536	0.673	0.008	0.382	0.161	0.502
45	-0.012	0.376	0.204	0.546	0.017	0.390	-0.572	0.718	0.002	0.382	0.204	0.546
50	-0.016	0.357	0.244	0.573	0.013	0.371	-0.602	0.751	-0.003	0.363	0.244	0.573

The numerical result of Estimation σ^2 for $N(1,1)$ and $N(0,1)$ with $p = 0.5$

Proportion	Tippett		Hozo		RC		Mantel		CT		SR	
P	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0	0.029	0.397	-0.082	0.359	0.053	0.410	-0.300	0.394	0.040	0.403	-0.082	0.359
0.05	0.296	1.608	0.156	1.372	0.327	1.664	-0.119	1.077	0.310	1.632	0.156	1.372
0.1	0.096	1.353	-0.022	1.174	0.122	1.405	-0.255	1.213	0.108	1.375	-0.022	1.174
0.15	-0.047	1.236	-0.150	1.259	-0.025	1.253	-0.352	1.696	-0.037	1.243	-0.150	1.259
0.2	-0.150	1.371	-0.241	1.592	-0.130	1.340	-0.422	2.264	-0.141	1.356	-0.241	1.592
0.25	-0.234	1.659	-0.316	1.998	-0.216	1.595	-0.479	2.795	-0.226	1.630	-0.316	1.998
0.3	-0.280	2.028	-0.358	2.418	-0.263	1.949	-0.511	3.272	-0.273	1.993	-0.358	2.418
0.35	-0.324	2.356	-0.397	2.772	-0.309	2.269	-0.541	3.656	-0.317	2.317	-0.397	2.772
0.4	-0.347	2.605	-0.417	3.037	-0.331	2.514	-0.556	3.940	-0.340	2.565	-0.417	3.037
0.45	-0.361	2.791	-0.430	3.228	-0.346	2.698	-0.566	4.135	-0.354	2.750	-0.430	3.228
0.5	-0.363	2.822	-0.431	3.265	-0.348	2.728	-0.567	4.180	-0.356	2.781	-0.431	3.265

The numerical result of Estimation σ^2 for N(5,1) and N(0,1) with N = 50

Difference	Tippett		Hozo		RC		Mantel		CT		SR	
of Mean	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0	0.042	0.410	-0.070	0.375	0.067	0.422	-0.292	0.412	0.053	0.415	-0.070	0.375
1	0.014	0.466	-0.095	0.425	0.038	0.480	-0.311	0.487	0.025	0.472	-0.095	0.425
2	-0.070	0.629	-0.170	0.637	-0.048	0.636	-0.368	0.833	-0.061	0.632	-0.170	0.637
3	-0.193	0.934	-0.280	1.090	-0.174	0.909	-0.451	1.526	-0.185	0.923	-0.280	1.090
4	-0.290	1.645	-0.367	1.950	-0.274	1.584	-0.518	2.625	-0.283	1.618	-0.367	1.950
5	-0.368	2.754	-0.436	3.200	-0.353	2.659	-0.570	4.127	-0.361	2.712	-0.436	3.200
6	-0.421	4.280	-0.483	4.870	-0.407	4.153	-0.606	6.064	-0.414	4.224	-0.483	4.870
7	-0.461	6.173	-0.519	6.913	-0.448	6.013	-0.634	8.398	-0.455	6.102	-0.519	6.913
8	-0.495	8.392	-0.549	9.302	-0.483	8.194	-0.657	11.114	-0.490	8.305	-0.549	9.302
9	-0.518	11.045	-0.570	12.128	-0.507	10.808	-0.672	14.282	-0.513	10.940	-0.570	12.128
10	-0.538	14.003	-0.588	15.282	-0.528	13.724	-0.686	17.818	-0.534	13.880	-0.588	15.282

The numerical result of Estimation σ^2 for N(0,1) and N(μ_2,1) with p = 0.5 and N = 50

Difference	Tippett		Hozo		RC		Mantel		CT		SR	
of Sigma	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
0	0.035	0.448	-0.193	0.389	0.057	0.461	-0.193	0.389	0.044	0.453	-0.193	0.389
0.4	0.078	0.776	-0.160	0.631	0.100	0.799	-0.160	0.631	0.087	0.785	-0.160	0.631
0.8	0.144	1.317	-0.108	1.011	0.168	1.357	-0.108	1.011	0.154	1.333	-0.108	1.011
1.2	0.182	2.032	-0.079	1.508	0.207	2.095	-0.079	1.508	0.192	2.057	-0.079	1.508
1.6	0.217	2.938	-0.051	2.165	0.243	3.028	-0.051	2.165	0.227	2.973	-0.051	2.165
2	0.267	3.759	-0.012	2.763	0.293	3.875	-0.012	2.763	0.277	3.805	-0.012	2.763
2.4	0.293	5.191	0.008	3.769	0.320	5.350	0.008	3.769	0.304	5.253	0.008	3.769
2.8	0.306	6.380	0.018	4.604	0.333	6.578	0.018	4.604	0.316	6.457	0.018	4.604
3	0.318	7.195	0.028	5.219	0.346	7.415	0.028	5.219	0.329	7.281	0.028	5.219

The numerical result of Estimation σ^2 for N(0,1) and N(0,σ_2) with p = 0.5 and N = 50

- **Weibull Distribution**

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.013	0.099	-0.320	0.111	0.021	0.100	0.053	0.104	0.004	0.098	0.178	0.126
10	-0.006	0.066	-0.117	0.068	0.002	0.066	-0.033	0.065	-0.005	0.066	0.019	0.068
15	-0.013	0.055	-0.011	0.055	-0.003	0.055	-0.115	0.058	-0.009	0.055	0.142	0.076
20	-0.007	0.047	-0.072	0.048	0.005	0.047	-0.170	0.062	-0.002	0.047	-0.072	0.048
25	-0.024	0.042	-0.041	0.042	-0.012	0.042	-0.233	0.071	-0.018	0.042	-0.041	0.042
30	-0.031	0.040	-0.011	0.040	-0.019	0.040	-0.278	0.081	-0.026	0.040	-0.011	0.040
35	-0.034	0.038	0.018	0.039	-0.021	0.037	-0.312	0.090	-0.028	0.037	0.018	0.039
40	-0.035	0.036	0.043	0.040	-0.022	0.036	-0.341	0.097	-0.029	0.036	0.043	0.040
45	-0.032	0.035	0.068	0.042	-0.018	0.035	-0.363	0.104	-0.026	0.035	0.068	0.042
50	-0.032	0.036	0.088	0.045	-0.018	0.036	-0.384	0.110	-0.026	0.036	0.088	0.045

The numerical result of Estimation σ for Weibull(3.6,1)

Scale Paramete	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
2	-0.040	0.066	0.079	0.083	-0.026	0.065	-0.389	0.220	-0.034	0.066	0.079	0.083
4	-0.037	0.134	0.083	0.169	-0.023	0.131	-0.387	0.439	-0.031	0.132	0.083	0.169
6	-0.037	0.208	0.082	0.256	-0.023	0.204	-0.388	0.663	-0.031	0.206	0.082	0.256
8	-0.045	0.275	0.074	0.342	-0.031	0.270	-0.392	0.881	-0.038	0.272	0.074	0.342
10	-0.043	0.340	0.077	0.415	-0.029	0.333	-0.391	1.107	-0.036	0.336	0.077	0.415
20	-0.043	0.697	0.076	0.862	-0.029	0.685	-0.391	2.207	-0.037	0.690	0.076	0.862
30	-0.038	1.003	0.082	1.230	-0.024	0.980	-0.388	3.317	-0.032	0.990	0.082	1.230
40	-0.036	1.395	0.084	1.722	-0.022	1.371	-0.387	4.416	-0.029	1.381	0.084	1.722
50	-0.037	1.748	0.082	2.093	-0.023	1.710	-0.388	5.561	-0.031	1.728	0.082	2.093

The numerical result of Estimation σ for Weibull Distribution with constant shape parameter 3.6

Shape Paramete	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
2	-0.038	0.080	-0.046	0.080	-0.026	0.080	-0.252	0.133	-0.032	0.080	-0.046	0.080
4	-0.018	0.040	-0.027	0.039	-0.006	0.040	-0.237	0.067	-0.012	0.040	-0.027	0.039
6	-0.009	0.031	-0.017	0.031	0.004	0.031	-0.229	0.048	-0.003	0.031	-0.017	0.031
8	-0.016	0.025	-0.025	0.025	-0.003	0.026	-0.235	0.038	-0.010	0.025	-0.025	0.025
10	-0.007	0.022	-0.007	0.022	0.006	0.022	-0.228	0.031	-0.001	0.022	-0.016	0.022
20	0.003	0.013	-0.006	0.013	0.015	0.013	-0.221	0.017	0.008	0.013	-0.006	0.013
30	0.009	0.010	0.000	0.010	0.022	0.010	-0.216	0.012	0.015	0.010	0.000	0.010
40	0.008	0.007	-0.001	0.007	0.021	0.007	-0.216	0.009	0.014	0.007	-0.001	0.007
50	0.004	0.006	-0.005	0.006	0.017	0.006	-0.219	0.007	0.010	0.006	-0.005	0.006

The numerical result of Estimation σ for Weibull Distribution with constant scale parameter 1

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.129	0.059	-0.491	0.046	0.134	0.061	0.196	0.066	0.107	0.058	-0.202	0.089
10	0.040	0.039	-0.179	0.033	0.057	0.040	-0.015	0.037	0.043	0.039	0.094	0.042
15	0.020	0.030	0.025	0.031	0.041	0.031	-0.180	0.028	0.028	0.031	0.366	0.050
20	-0.010	0.027	-0.137	0.026	0.013	0.027	-0.309	0.030	0.000	0.027	-0.137	0.026
25	-0.019	0.023	-0.052	0.022	0.006	0.023	-0.393	0.033	-0.007	0.023	-0.052	0.022
30	-0.028	0.022	0.014	0.022	-0.003	0.022	-0.459	0.038	-0.016	0.022	0.014	0.022
35	-0.046	0.020	0.058	0.023	-0.020	0.021	-0.516	0.041	-0.034	0.021	0.058	0.023
40	-0.039	0.019	0.122	0.024	-0.012	0.020	-0.551	0.044	-0.026	0.020	0.122	0.024
45	-0.054	0.019	0.153	0.026	-0.027	0.020	-0.590	0.046	-0.041	0.019	0.153	0.026
50	-0.064	0.018	0.184	0.026	-0.036	0.018	-0.621	0.049	-0.051	0.018	0.184	0.026

The numerical result of Estimation σ^2 for Weibull(3.6,1)

Scale Paramete	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
2	-0.057	0.075	0.193	0.107	-0.029	0.075	-0.618	0.195	-0.044	0.075	0.193	0.107
4	-0.055	0.288	0.194	0.412	-0.028	0.288	-0.618	0.778	-0.042	0.287	0.194	0.412
6	-0.072	0.639	0.173	0.971	-0.045	0.645	-0.625	1.730	-0.059	0.640	0.173	0.971
8	-0.056	1.145	0.194	1.724	-0.028	1.155	-0.618	3.081	-0.043	1.148	0.194	1.724
10	-0.061	1.844	0.188	2.661	-0.033	1.851	-0.620	4.854	-0.048	1.844	0.188	2.661
20	-0.063	7.243	0.185	10.510	-0.035	7.264	-0.621	19.402	-0.050	7.239	0.185	10.510
30	-0.049	17.167	0.203	24.798	-0.021	17.286	-0.615	43.634	-0.036	17.195	0.203	24.798
40	-0.061	28.833	0.188	42.575	-0.033	28.999	-0.620	77.331	-0.048	28.859	0.188	42.575
50	-0.057	46.308	0.192	67.356	-0.029	46.564	-0.618	121.121	-0.044	46.348	0.192	67.356

The numerical result of Estimation σ^2 for Weibull Distribution with constant shape parameter 3.6

Shape Paramete	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
2	-0.040	0.081	-0.058	0.079	-0.016	0.083	-0.420	0.100	-0.029	0.081	-0.058	0.079
4	-0.021	0.019	-0.039	0.019	0.003	0.020	-0.409	0.029	-0.010	0.019	-0.039	0.019
6	0.028	0.011	0.010	0.011	0.054	0.011	-0.378	0.014	0.040	0.011	0.010	0.011
8	0.024	0.008	0.006	0.008	0.050	0.008	-0.381	0.009	0.036	0.008	0.006	0.008
10	0.009	0.005	-0.009	0.005	0.035	0.005	-0.390	0.006	0.021	0.005	-0.009	0.005
20	0.072	0.002	0.053	0.002	0.099	0.002	-0.352	0.002	0.085	0.002	0.053	0.002
30	0.079	0.001	0.059	0.001	0.106	0.001	-0.348	0.001	0.091	0.001	0.059	0.001
40	0.035	0.001	0.016	0.000	0.061	0.001	-0.375	0.000	0.047	0.001	0.016	0.000
50	0.097	0.000	0.077	0.000	0.124	0.000	-0.337	0.000	0.110	0.000	0.077	0.000

The numerical result of Estimation σ^2 for Weibull Distribution with constant scale parameter 1

- **Exponential Distribution**

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.117	1.030	-0.407	1.034	-0.110	1.036	-0.082	1.060	-0.125	1.024	0.026	1.185
10	-0.087	0.788	-0.189	0.789	-0.080	0.790	-0.112	0.783	-0.086	0.788	-0.064	0.796
15	-0.049	0.739	-0.047	0.739	-0.039	0.743	-0.147	0.727	-0.045	0.740	0.101	0.858
20	-0.047	0.681	-0.110	0.673	-0.036	0.685	-0.204	0.706	-0.043	0.683	-0.110	0.673
25	-0.043	0.666	-0.060	0.659	-0.031	0.672	-0.248	0.711	-0.038	0.668	-0.060	0.659
30	-0.020	0.639	0.001	0.651	-0.007	0.646	-0.269	0.725	-0.014	0.642	0.001	0.651
35	-0.021	0.612	0.032	0.647	-0.007	0.619	-0.303	0.744	-0.014	0.615	0.032	0.647
40	-0.010	0.596	0.069	0.655	0.004	0.603	-0.324	0.770	-0.004	0.599	0.069	0.655
45	-0.023	0.551	0.079	0.636	-0.009	0.559	-0.357	0.785	-0.016	0.555	0.079	0.636
50	-0.001	0.574	0.123	0.687	0.013	0.582	-0.365	0.820	0.005	0.577	0.123	0.687

The numerical result of Estimation σ for Exp(0.5)

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.109	0.535	-0.402	0.532	-0.101	0.538	-0.073	0.550	-0.116	0.532	0.036	0.611
10	-0.084	0.410	-0.186	0.406	-0.076	0.411	-0.108	0.406	-0.082	0.410	-0.060	0.414
15	-0.063	0.386	-0.060	0.387	-0.053	0.388	-0.160	0.379	-0.059	0.387	0.085	0.446
20	-0.054	0.334	-0.117	0.329	-0.043	0.336	-0.210	0.346	-0.049	0.335	-0.117	0.329
25	-0.050	0.318	-0.067	0.316	-0.039	0.321	-0.253	0.353	-0.045	0.319	-0.067	0.316
30	-0.024	0.292	-0.003	0.297	-0.011	0.295	-0.272	0.353	-0.018	0.294	-0.003	0.297
35	-0.030	0.313	0.021	0.327	-0.017	0.316	-0.309	0.386	-0.024	0.314	0.021	0.327
40	-0.019	0.307	0.060	0.339	-0.005	0.311	-0.330	0.383	-0.012	0.309	0.060	0.339
45	-0.020	0.306	0.082	0.352	-0.006	0.311	-0.355	0.399	-0.013	0.308	0.082	0.352
50	-0.010	0.289	0.113	0.339	0.004	0.292	-0.370	0.420	-0.003	0.290	0.113	0.339

The numerical result of Estimation σ for Exp(1)

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.094	0.170	-0.392	0.169	-0.086	0.171	-0.058	0.176	-0.102	0.169	0.054	0.198
10	-0.085	0.136	-0.187	0.134	-0.078	0.137	-0.110	0.135	-0.084	0.136	-0.062	0.138
15	-0.081	0.125	-0.079	0.125	-0.072	0.125	-0.177	0.123	-0.078	0.125	0.063	0.143
20	-0.049	0.111	-0.112	0.109	-0.039	0.112	-0.206	0.115	-0.045	0.111	-0.112	0.109
25	-0.048	0.105	-0.064	0.104	-0.036	0.106	-0.252	0.116	-0.042	0.105	-0.064	0.104
30	-0.010	0.106	0.011	0.108	0.003	0.107	-0.262	0.120	-0.004	0.107	0.011	0.108
35	-0.023	0.103	0.029	0.109	-0.010	0.104	-0.304	0.123	-0.017	0.103	0.029	0.109
40	0.005	0.098	0.086	0.106	0.019	0.099	-0.313	0.131	0.011	0.098	0.086	0.106
45	-0.013	0.098	0.090	0.112	0.001	0.099	-0.350	0.133	-0.006	0.098	0.090	0.112
50	-0.007	0.096	0.116	0.116	0.007	0.098	-0.368	0.136	0.000	0.097	0.116	0.116

The numerical result of Estimation σ for Exp(3)

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.114	0.106	-0.405	0.105	-0.106	0.107	-0.078	0.110	-0.121	0.106	0.031	0.122
10	-0.116	0.083	-0.214	0.081	-0.108	0.083	-0.139	0.082	-0.115	0.083	-0.093	0.084
15	-0.061	0.078	-0.059	0.078	-0.051	0.078	-0.158	0.076	-0.057	0.078	0.087	0.091
20	-0.041	0.068	-0.105	0.067	-0.030	0.068	-0.199	0.070	-0.037	0.068	-0.105	0.067
25	-0.038	0.067	-0.055	0.066	-0.026	0.067	-0.244	0.072	-0.033	0.067	-0.055	0.066
30	-0.035	0.064	-0.014	0.065	-0.022	0.065	-0.280	0.073	-0.029	0.065	-0.014	0.065
35	-0.038	0.057	0.031	0.060	-0.014	0.058	-0.313	0.074	-0.021	0.058	0.013	0.060
40	-0.016	0.060	0.063	0.066	-0.003	0.061	-0.328	0.077	-0.010	0.060	0.063	0.066
45	-0.009	0.058	0.094	0.067	0.005	0.059	-0.348	0.079	-0.002	0.058	0.094	0.067
50	-0.013	0.057	0.110	0.069	0.001	0.058	-0.372	0.081	-0.006	0.057	0.110	0.069

The numerical result of Estimation σ for Exp(5)

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.048	5.305	-0.528	3.113	0.066	5.403	0.134	5.776	0.030	5.211	0.417	7.424
10	0.020	3.769	-0.195	3.092	0.037	3.832	-0.034	3.576	0.023	3.778	0.074	3.972
15	0.062	3.651	0.067	3.667	0.084	3.728	-0.147	3.020	0.070	3.679	0.422	5.103
20	0.033	3.101	-0.099	2.742	0.057	3.176	-0.279	2.458	0.044	3.133	-0.099	2.742
25	0.054	3.271	0.018	3.151	0.081	3.363	-0.349	2.432	0.066	3.313	0.018	3.151
30	0.055	2.923	0.101	3.065	0.083	3.009	-0.413	2.331	0.068	2.962	0.101	3.065
35	0.080	2.807	0.198	3.155	0.109	2.889	-0.452	2.406	0.093	2.845	0.198	3.155
40	0.092	2.859	0.286	3.445	0.122	2.948	-0.498	2.441	0.106	2.900	0.286	3.445
45	0.072	2.700	0.307	3.493	0.103	2.792	-0.535	2.438	0.087	2.743	0.307	3.493
50	0.078	2.921	0.364	3.957	0.110	3.022	-0.564	2.534	0.093	2.968	0.364	3.957

The numerical result of Estimation σ^2 for Exp(0.5)

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.021	1.700	-0.540	0.918	0.039	1.731	0.105	1.846	0.004	1.671	0.381	2.346
10	0.011	1.009	-0.202	0.819	0.028	1.026	-0.043	0.956	0.013	1.011	0.064	1.064
15	-0.057	0.892	-0.053	0.896	-0.038	0.910	-0.243	0.745	-0.050	0.899	0.262	1.240
20	0.061	0.809	-0.075	0.714	0.085	0.829	-0.260	0.634	0.071	0.818	-0.075	0.714
25	-0.022	0.814	-0.055	0.799	0.003	0.836	-0.395	0.615	-0.010	0.824	-0.055	0.799
30	0.006	0.723	0.049	0.759	0.032	0.745	-0.440	0.576	0.018	0.733	0.049	0.759
35	0.026	0.710	0.138	0.802	0.054	0.731	-0.480	0.590	0.039	0.720	0.138	0.802
40	0.046	0.659	0.221	0.798	0.076	0.680	-0.512	0.598	0.060	0.669	0.221	0.798
45	0.038	0.685	0.265	0.880	0.068	0.708	-0.550	0.618	0.052	0.696	0.265	0.880
50	0.066	0.680	0.348	0.923	0.097	0.703	-0.569	0.632	0.081	0.691	0.348	0.923

The numerical result of Estimation σ^2 for Exp(1)

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.057	0.128	-0.523	0.084	0.076	0.130	0.144	0.139	0.040	0.126	0.430	0.178
10	0.036	0.109	-0.182	0.089	0.053	0.110	-0.019	0.103	0.039	0.109	0.090	0.114
15	0.020	0.087	0.025	0.088	0.041	0.089	-0.180	0.074	0.028	0.088	0.366	0.123
20	0.052	0.096	-0.083	0.085	0.076	0.098	-0.267	0.075	0.062	0.097	-0.083	0.085
25	-0.003	0.084	-0.037	0.084	0.022	0.084	-0.384	0.084	0.008	0.084	-0.037	0.081
30	0.092	0.079	0.139	0.083	0.121	0.081	-0.392	0.063	0.106	0.080	0.139	0.083
35	0.023	0.081	0.135	0.092	0.051	0.084	-0.481	0.066	0.036	0.083	0.135	0.092
40	0.013	0.076	0.183	0.092	0.042	0.078	-0.527	0.066	0.027	0.077	0.183	0.092
45	0.077	0.068	0.313	0.087	0.108	0.070	-0.533	0.069	0.092	0.069	0.313	0.087
50	0.106	0.078	0.398	0.106	0.138	0.080	-0.553	0.070	0.121	0.079	0.398	0.106

The numerical result of Estimation σ^2 for Exp(3)

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.076	0.051	-0.515	0.031	0.094	0.052	0.164	0.056	0.058	0.051	0.455	0.072
10	-0.035	0.034	-0.238	0.029	-0.019	0.035	-0.086	0.033	-0.033	0.034	0.015	0.036
15	0.011	0.034	0.016	0.035	0.032	0.035	-0.187	0.029	0.019	0.035	0.355	0.048
20	0.003	0.031	-0.126	0.028	0.026	0.032	-0.301	0.025	0.013	0.031	-0.126	0.028
25	0.005	0.032	-0.030	0.031	0.030	0.033	-0.379	0.024	0.016	0.032	-0.030	0.031
30	0.020	0.029	0.064	0.031	0.047	0.030	-0.432	0.023	0.033	0.030	0.064	0.031
35	0.019	0.029	0.131	0.032	0.047	0.029	-0.483	0.024	0.032	0.029	0.131	0.032
40	0.081	0.029	0.262	0.035	0.111	0.030	-0.495	0.024	0.095	0.029	0.262	0.035
45	0.087	0.025	0.324	0.032	0.118	0.026	-0.529	0.024	0.102	0.025	0.324	0.032
50	0.038	0.028	0.312	0.038	0.068	0.029	-0.580	0.025	0.052	0.029	0.312	0.038

The numerical result of Estimation σ^2 for Exp(5)

- **Chi- Square Distribution**

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.202	0.898	-0.464	0.857	-0.195	0.903	-0.170	0.920	-0.209	0.894	-0.072	1.005
10	-0.136	0.712	-0.232	0.696	-0.129	0.715	-0.159	0.705	-0.135	0.712	-0.114	0.721
15	-0.096	0.644	-0.094	0.645	-0.087	0.648	-0.190	0.627	-0.093	0.645	0.046	0.729
20	-0.072	0.599	-0.133	0.580	-0.061	0.604	-0.225	0.581	-0.067	0.601	-0.133	0.580
25	-0.042	0.573	-0.059	0.566	-0.030	0.578	-0.247	0.567	-0.037	0.575	-0.059	0.566
30	-0.046	0.550	-0.026	0.559	-0.034	0.555	-0.289	0.574	-0.040	0.552	-0.026	0.559
35	-0.004	0.534	0.049	0.562	0.009	0.540	-0.291	0.578	0.002	0.537	0.049	0.562
40	-0.001	0.544	0.080	0.600	0.013	0.552	-0.317	0.577	0.006	0.547	0.080	0.600
45	0.011	0.488	0.116	0.559	0.025	0.496	-0.335	0.580	0.018	0.492	0.116	0.559
50	0.020	0.515	0.146	0.620	0.034	0.525	-0.351	0.585	0.027	0.520	0.146	0.620

The numerical result of Estimation σ for $\chi^2(1)$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.063	1.332	-0.371	1.432	-0.055	1.342	-0.025	1.380	-0.071	1.324	0.090	1.587
10	-0.044	1.114	-0.150	1.068	-0.036	1.122	-0.069	1.092	-0.043	1.115	-0.019	1.141
15	-0.016	0.931	-0.014	0.933	-0.006	0.940	-0.118	0.908	-0.012	0.934	0.139	1.171
20	-0.023	0.852	-0.087	0.836	-0.011	0.860	-0.184	0.910	-0.018	0.855	-0.087	0.836
25	-0.022	0.775	-0.039	0.770	-0.009	0.781	-0.231	0.960	-0.016	0.778	-0.039	0.770
30	-0.019	0.773	0.002	0.785	-0.006	0.780	-0.268	1.036	-0.013	0.776	0.002	0.785
35	-0.009	0.731	0.044	0.783	0.004	0.741	-0.294	1.064	-0.003	0.735	0.044	0.783
40	-0.012	0.738	0.068	0.812	0.002	0.745	-0.325	1.166	-0.005	0.741	0.068	0.812
45	-0.013	0.707	0.090	0.824	0.002	0.715	-0.350	1.207	-0.006	0.710	0.090	0.824
50	-0.008	0.696	0.115	0.875	0.006	0.707	-0.369	1.235	-0.002	0.701	0.115	0.875

The numerical result of Estimation σ for $\chi^2(5)$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.029	1.833	-0.348	2.029	-0.020	1.844	0.010	1.893	-0.037	1.822	0.130	2.167
10	-0.023	1.310	-0.132	1.324	-0.015	1.316	-0.049	1.295	-0.022	1.310	0.002	1.333
15	-0.015	1.135	-0.012	1.136	-0.005	1.144	-0.117	1.149	-0.011	1.138	0.140	1.442
20	-0.011	1.062	-0.077	1.069	0.000	1.069	-0.174	1.220	-0.006	1.064	-0.077	1.069
25	-0.003	0.971	-0.020	0.962	0.010	0.981	-0.216	1.258	0.003	0.975	-0.020	0.962
30	-0.008	0.956	0.013	0.979	0.005	0.969	-0.260	1.353	-0.002	0.961	0.013	0.979
35	-0.006	0.926	0.047	1.000	0.008	0.940	-0.292	1.458	0.001	0.932	0.047	1.000
40	-0.014	0.861	0.066	0.971	0.000	0.870	-0.326	1.575	-0.007	0.865	0.066	0.971
45	-0.006	0.861	0.097	1.029	0.008	0.871	-0.346	1.666	0.000	0.865	0.097	1.029
50	-0.008	0.794	0.116	0.985	0.007	0.799	-0.369	1.769	-0.001	0.796	0.116	0.985

The numerical result of Estimation σ for $\chi^2(10)$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	-0.002	2.427	-0.330	2.707	0.007	2.445	0.038	2.523	-0.010	2.410	0.161	2.953
10	0.001	1.836	-0.111	1.808	0.010	1.849	-0.026	1.803	0.002	1.838	0.027	1.883
15	-0.022	1.538	-0.020	1.540	-0.012	1.551	-0.123	1.559	-0.018	1.542	0.132	1.989
20	-0.010	1.377	-0.076	1.358	0.001	1.393	-0.173	1.564	-0.005	1.384	-0.076	1.358
25	-0.013	1.268	-0.030	1.255	0.000	1.283	-0.224	1.710	-0.007	1.274	-0.030	1.255
30	-0.003	1.200	0.018	1.239	0.010	1.222	-0.256	1.805	0.003	1.210	0.018	1.239
35	0.000	1.099	0.053	1.176	0.013	1.109	-0.288	2.045	0.006	1.103	0.053	1.176
40	-0.001	1.112	0.079	1.291	0.013	1.129	-0.317	2.163	0.005	1.119	0.079	1.291
45	-0.001	1.074	0.102	1.352	0.013	1.092	-0.343	2.278	0.005	1.082	0.102	1.352
50	-0.003	1.101	0.122	1.481	0.012	1.123	-0.366	2.392	0.004	1.110	0.122	1.481

The numerical result of Estimation σ for $\chi^2(20)$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.035	3.489	-0.534	1.910	0.053	3.551	0.119	3.781	0.017	3.431	0.399	4.778
10	-0.004	3.130	-0.214	2.467	0.012	3.185	-0.057	2.956	-0.002	3.138	0.048	3.305
15	-0.038	2.349	-0.033	2.360	-0.018	2.399	-0.227	1.912	-0.030	2.368	0.289	3.247
20	0.005	2.180	-0.124	1.901	0.028	2.234	-0.299	1.599	0.015	2.203	-0.124	1.901
25	0.026	2.057	-0.009	1.985	0.052	2.111	-0.366	1.456	0.038	2.082	-0.009	1.985
30	0.103	1.924	0.150	2.017	0.132	1.980	-0.387	1.321	0.116	1.950	0.150	2.017
35	0.094	1.770	0.214	1.998	0.124	1.825	-0.445	1.265	0.108	1.796	0.214	1.998
40	0.140	1.743	0.331	2.099	0.172	1.800	-0.468	1.265	0.155	1.770	0.331	2.099
45	0.109	2.120	0.351	2.711	0.141	2.194	-0.520	1.295	0.124	2.154	0.351	2.711
50	0.129	1.858	0.428	2.495	0.162	1.924	-0.543	1.296	0.145	1.889	0.428	2.495

The numerical result of Estimation σ^2 for $\chi^2(1)$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.107	12.333	-0.501	7.223	0.127	12.574	0.198	13.488	0.089	12.104	0.498	17.544
10	0.047	8.353	-0.174	6.761	0.064	8.507	-0.009	7.888	0.049	8.375	0.102	8.847
15	0.000	6.489	0.005	6.520	0.021	6.634	-0.196	5.495	0.008	6.543	0.340	9.460
20	-0.005	5.767	-0.132	5.144	0.018	5.913	-0.306	4.946	0.005	5.830	-0.132	5.144
25	0.033	5.146	-0.003	4.978	0.059	5.283	-0.362	4.951	0.045	5.207	-0.003	4.978
30	0.021	5.588	0.066	5.861	0.048	5.751	-0.432	5.327	0.034	5.663	0.066	5.861
35	0.018	5.085	0.130	5.797	0.046	5.244	-0.483	5.456	0.032	5.158	0.130	5.797
40	0.037	4.938	0.210	6.111	0.066	5.104	-0.516	5.661	0.050	5.015	0.210	6.111
45	0.057	5.145	0.288	6.780	0.087	5.321	-0.542	5.946	0.071	5.226	0.288	6.780
50	0.010	4.899	0.277	6.887	0.039	5.069	-0.591	6.156	0.024	4.977	0.277	6.887

The numerical result of Estimation σ^2 for $\chi^2(5)$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.109	19.167	-0.500	13.233	0.128	19.542	0.200	20.977	0.091	18.816	0.500	27.579
10	0.053	13.227	-0.169	10.928	0.071	13.478	-0.002	12.484	0.056	13.262	0.109	14.045
15	0.026	11.443	0.031	11.499	0.047	11.706	-0.175	9.798	0.034	11.540	0.375	17.098
20	0.038	10.353	-0.095	9.173	0.062	10.642	-0.276	9.021	0.048	10.477	-0.095	9.173
25	0.061	8.925	0.024	8.625	0.087	9.177	-0.345	9.289	0.073	9.037	0.024	8.625
30	0.011	9.783	0.055	10.274	0.038	10.074	-0.437	10.238	0.023	9.916	0.055	10.274
35	0.028	8.531	0.140	9.974	0.056	8.807	-0.479	10.724	0.041	8.657	0.140	9.974
40	0.018	9.226	0.188	11.553	0.046	9.552	-0.525	11.144	0.031	9.375	0.188	11.553
45	0.010	8.099	0.231	10.888	0.039	8.366	-0.562	11.786	0.024	8.220	0.231	10.888
50	0.029	8.132	0.302	11.915	0.060	8.434	-0.583	12.111	0.044	8.269	0.302	11.915

The numerical result of Estimation σ^2 *for* $\chi^2(10)$

Sample Size N	Tippett		Hozo		RC		Mantel		CT		SR	
	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
5	0.138	38.342	-0.487	25.828	0.158	39.122	0.232	42.105	0.119	37.610	0.539	55.750
10	0.040	24.555	-0.179	20.272	0.057	25.044	-0.015	23.121	0.042	24.623	0.094	26.152
15	0.041	20.823	0.046	20.931	0.063	21.329	-0.163	17.915	0.049	21.009	0.395	31.963
20	0.055	17.827	-0.081	16.264	0.079	18.282	-0.264	17.168	0.065	18.019	-0.081	16.264
25	0.048	17.495	0.012	16.846	0.075	18.036	-0.352	18.040	0.060	17.736	0.012	16.846
30	0.018	16.250	0.062	17.083	0.044	16.738	-0.434	19.718	0.030	16.470	0.062	17.083
35	0.032	14.886	0.157	17.201	0.060	15.363	-0.486	20.916	0.045	15.101	0.157	17.201
40	-0.003	15.171	0.164	19.454	0.025	15.735	-0.535	21.892	0.010	15.427	0.164	19.454
45	0.029	13.680	0.254	19.016	0.059	14.150	-0.554	23.286	0.043	13.890	0.254	19.016
50	0.037	15.379	0.312	22.750	0.068	15.950	-0.580	24.210	0.052	15.638	0.312	22.750

The numerical result of Estimation σ^2 *for* $\chi^2(20)$