THIRD AND FOURTH ORDER RUNGE-KUTTA METHODS

THE NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS BY THIRD AND FOURTH ORDER RUNGE-KUTTA METHODS

By

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SCOPE AND CONTENTS: An examination of third and fourth order Runge-Kutta methods which can be utilized to solve various ordinary differential equations is considered.

Preface

C. Runge originally suggested the numerical methods of solving differential equations which will be examined, and were subsequently improved on by, to mention a few, K.Heun, and W. Kutta. The entirety of these methods have, as a result, been referred to as the Runge-Kutta methods for the numerical solution of differential equations.

The first section of the thesis consists of the derivation of third and fourth order Runge-Kutta methods and their respective truncation errors. Notation, definitions, and various concepts are introduced as needed in the various sections.

The numerical solutions of differential equations using third order Runge-Kutta methods are then discussed in the second section. Various formulae and relationships are derived here for third order methods. In all numerical tables that follow, the results were obtained using a Bendix Model G-15 Digital Computer.

In the third section, one considers fourth order Runge-Kutta methods for the numerical solution of ordinary differential equations. However, in addition to considerations of symmetry, reduction of operations and storage requirements, as examined in section two, one examines a Runge-Kutta method due to Blum which basically modifies a programming procedure.

Finally in the last section, one investigates methods due to A. Ralston which minimize a bound on the truncation error derived in the first section.

An appendix is also included containing various programs for the Bendix G-15D that have been needed throughout the sections.

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SECTION I

INTRODUCTION

To solve differential equations numerically is important for in solving practical problems such as those encountered in engineering, one may obtain ordinary differential equations which even though are linear or of simple form cannot be solved analytically. Furthermore, in some cases, the analytical solution obtained may be complex and to determine a value for the dependent variable for some value of the independent variable, a considerable amount of computation may be required. Thus, numerical methods for the solution of differential equations are desirable and necessary.

To solve numerically the first order differential equation

$$\frac{dy}{dx} = f(x,y) \qquad (1.00)$$

which satisfies the given initial condition $y(x_0) = y_0$, one basically wishes to determine the change in the dependent variable y (denoted by dy) which corresponds to an increment in the independent variable x (denoted by h). Starting with the initial values (x_o, y_o) and denoting the uniform increment in x by h, then at the (n+1)th calculation one obtains the numerical solution (x_{n+1}, y_{n+1}) given by $x_{n+1} = x_o + (n+1)h$ and

$$y_{n+1} = y_n + dy \qquad (1.10)$$

where $y_n = y(x_n)$ has been calculated previously and an expression for dy is desired.

For Runge-Kutta methods of order K, dy is defined as

$$dy = h\left[w_1 f(x_n,y_n) + w_2 f(x_n+m_2h, y_n+n_2h) + \dots + w_k f(x_n+m_kh, y_n+n_kh)\right] \quad (1.11)$$

where w_1, m_i, n_i, w_i, $i=2,\dots k$ are constants to be determined so that when (1.11) is expanded in a power series in h and used in (1.10), then the coefficients of like powers of h in the Taylor's series

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \dots \quad (1.12)$$

and in (1.11) must agree up to and including the power h^k. To simplify calculations, one writes (1.11) as

$$y_{n+1} - y_n = dy = w_1K_1 + w_2K_2 + \dots + w_kK_k \quad (1.20)$$

where K_i, $i=1,\dots k$ are given by

$$K_1 = hf(x_n,y_n) \quad (1.21)$$

$$K_2 = hf(x_n + a_oh,\ y_n + a_1K_1) \quad (1.22)$$

$$K_3 = hf(x_n + b_oh,\ y_n + b_1K_1 + b_2K_2) \quad (1.23)$$

$$K_4 = hf(x_n + c_oh,\ y_n + c_1K_1 + c_2K_2 + c_3K_3) \quad (1.24)$$

$$\vdots$$

$$K_k = hf(x_n + i_oh,\ y_n + i_1K_1 + \dots + i_{k-1}K_{k-1}) \quad (1.25)$$

where again the constants a_i, b_i, c_i, ... i_i, w_i, are to be determined.

One now considers the third order Runge-Kutta methods in which case only (1.21-1.23) are considered and (1.20) becomes

$$dy = y_{n+1} - y_n = w_1K_1 + w_2K_2 + w_3K_3 \tag{1.30}$$

To simplify notation, we will write y for y_n, x for x_n, and f for f(x,y) when no ambiguity occurs. Continuing the numerical solution of the differential equation from (x_n,y_n), one immediately obtains

$$K_1 = hf \tag{1.31}$$

In order to evaluate K_2 as a power series in h, one requires the Taylor expansion in two variables

$$f(x+p,y+q) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(p\frac{\partial}{\partial x} + q\frac{\partial}{\partial y}\right)^n f(x,y) \tag{1.32}$$

and the notation

$$f_{\underbrace{x\ldots\ldots x}_{i\text{ factors}}\underbrace{y\ldots\ldots y}_{j\text{ factors}}} = \frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j} \tag{1.33}$$

Then, in view of (1.31-1.33)

$$K_2 = hf + A_2h^2 + \tfrac{1}{2}A_3h^3 + \tfrac{1}{6}A_4h^4 + O(h^5) \tag{1.34}$$

where

$$A_{n+1} = D_a^n f = \left(a_o\frac{\partial}{\partial x} + a_1 f\frac{\partial}{\partial y}\right)^n f \qquad n=0,1,2,3 \tag{1.35}$$

Similarly as a power series in h, (1.23) becomes

$$K_3 = hf + B_2h^2 + \frac{1}{2}B_3h^3 + \frac{1}{6}B_4h^4 + O(h^5) \quad (1.36)$$

where

$$B_2 = D_bf \ , \qquad B_3 = D_b{}^2f + 2b_2f_yD_af \quad (1.37)$$

$$B_4 = D_b{}^3f + 3b_2f_yD_a{}^2f + 6b_2(D_bf_y)(D_af) \quad (1.38)$$

and

$$D_b{}^nf = (b_o\frac{\partial}{\partial x} + (b_1+b_2)f\frac{\partial}{\partial y})^nf \quad (1.39)$$

On multiplying (1.31), (1.34), (1.36) respectively by w_1, w_2, w_3 and adding, one determines

$$y_{n+1}-y_n = dy = C_1h + C_2h^2 + \frac{1}{2}C_3h^3 + \frac{1}{6}C_4h^4 + O(h^5) \quad (1.40)$$

where

$$C_1 = (w_1+w_2+w_3)f$$

$$C_2 = (w_2a_o + w_3b_o)f_x + (w_2a_1+w_3[b_1+b_2])ff_y$$

$$\begin{aligned} C_3 = &(w_2a_o{}^2 + w_3b_o{}^2)f_{xx} \\ &+ (2w_2a_oa_1 + 2w_3b_o[b_1+b_2])ff_{xy} \\ &+ (w_2a_1^2 + w_3[b_1+b_2]^2)f^2f_{yy} \\ &+ 2w_3b_2a_of_xf_y + 2w_3a_1b_2ff_y^2 \end{aligned}$$

$$\begin{aligned} C_4 = &(w_2a_o{}^3 + w_3b_o{}^3)f_{xxx} \\ &+ 3(w_2a_o^2a_1 + w_3b_o{}^2[b_1+b_2])ff_{xxy} \\ &+ 3(w_2a_oa_1^2 + w_3b_o[b_1+b_2]^2)f^2f_{xyy} \\ &+ (w_2a_1^3 + w_3[b_1+b_2]^3)f^3f_{yyy} \end{aligned}$$

$$+ 3w_3b_2f_yD_a^2f + 6w_3b_2(D_af)(D_bf_y)$$

Furthermore, one knows that

$$\frac{df}{dx} = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)f(x,y)$$

and in general

$$D^nf = \frac{d^nf}{dx^n} = \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)^n f \qquad (1.41)$$

Apply (1.41) to (1.12); then, the Taylor expansion becomes

$$y_{n+1}-y_n = hf + \frac{1}{2}T_2h^2 + \frac{1}{6}T_3h^3 + \frac{1}{24}T_4h^4 + \frac{1}{120}T_5h^5 + O(h^6) \qquad (1.42)$$

where

$$T_2 = Df \qquad T_3 = D^2f + f_yDf \qquad (1.43)$$

$$T_4 = D^3f + f_yD^2f + f_y^2Df + 3Df_yDf \qquad (1.44)$$

$$T_5 = D^4f + f_yD^3f + f_y^2D^2f + f_y^3Df$$

$$+ 7f_yDfDf_y + 3f_{yy}DfDf_y + 4D^2fDf_y + 6DfD^2f_y \qquad (1.45)$$

In order that (1.40) and (1.42) agree up to and including the power h^3, the following relations between the coefficients must hold

$$w_1 + w_2 + w_3 = 1 \qquad (1.50)$$

$$a_ow_2 + b_ow_3 = \frac{1}{2} \qquad (1.51)$$

$$a_1w_2 + (b_1 + b_2)w_3 = \frac{1}{2} \qquad (1.52)$$

$$a_o^2w_2 + b_o^2w_3 = 1/3 \qquad (1.53)$$

$$a_oa_1w_2 + b_o(b_1 + b_2)w_3 = 1/3 \qquad (1.54)$$

$$a_1^2w_2 + (b_1 + b_2)^2w_3 = 1/3 \qquad (1.55)$$

$$2a_ob_2w_3 = 1/3 \qquad (1.56)$$

$$2a_1 b_2 w_3 = 1/3 \qquad (1.57)$$

Immediately from (1.56) and (1.57),

$$a_o = a_1 \qquad (1/60)$$

and using this result in (1.51) and (1.52), one obtains

$$b_o = b_1 + b_2 \qquad (1.61)$$

In view of (1.60), and (1.61) the relationships (1.50-1.57) become

$$w_1 + w_2 + w_3 = 1 \qquad (1.62)$$

$$a_o w_2 + b_o w_3 = 1/2 \qquad (1.63)$$

$$a_o^2 w_2 + b_o^2 w_3 = 1/3 \qquad (1.64)$$

$$a_o b_2 w_3 = 1/6 \qquad (1.65)$$

The solution of 1.60-1.65 is considered in detail in section II. In view of (1.60) and (1.61)

$$C_4 = (a_o^3 w_2 + b_o^3 w_3) D^3 f + 3a_o^2 b_2 w3 f_y D^2 f$$
$$+6a_o b_o b_2 w_3 Df Df_y \qquad (1.70)$$

The error in our numerical solution consists of an expression

$$Eh^4 + O(h^5) \qquad (1.71)$$

where E depending on f(x,y) and its partial derivatives is evaluated as

$$E = 1/24T_4 - 1/6C_4$$
$$= (1/24 - [a_o^3 w_2 + b_o^3 w_3]/6) D^3 f$$
$$+ (1/24 - a_o^2 b_2 w_3/2) D^2 f\ f_y$$
$$+ (3/24 - a_o b_o b_2 w_3) Df\ Df_y +$$

$$1/24 f_y^2 Df + O(h^5) \tag{1.72}$$

For small h, $O(h^5)$ is insignificant. Thus, in (1.71), one calls Eh^4 the truncation error for the third order Runge-Kutta method. The third order truncation error is considered in detail in section IV.

Similarly, if one insists that the Taylor expansion (1.42) and the numerical solution (1.20-1.25) agree up to and including the power h^4, then (1.20) becomes

$$dy = w_1K_1 + w_2K_2 + w_3K_3 + w_4K_4 \tag{1.73}$$

Proceeding in a similar manner for the fourth order methods, one expands (1.21-1.24) in powers of h, multiplies by corresponding coefficients w_1, w_2, w_3, and w_4, and finally by equating respective coefficients of powers of h in (1.73) and (1.42), one obtains corresponding relationships for the constants of the fourth order methods. These relationships are found in section III.

The truncation error for the fourth order methods will consist of the following expression

$$Eh^5 + O(h^6) \tag{1.74}$$

where for small h, $O(h^6)$ is negligible and E, depending on f(x,y) and its partial derivatives is given by

$$\begin{aligned} E = {} & E_1D^4f + E_2f_yD^3f + E_3f_y^2D^2f + E_4f_y^3Df \\ & + E_5f_yDfDf_y + E_6f_{yy}DfDf_y + E_7D^2fDf_y + E_8DfD^2f_y \end{aligned} \tag{1.75}$$

where

$$E_1 = 1/120 - (a_o^4 w_2 + b_o^4 w_3 + w_4)/24$$

$$E_2 = 1/120 - (a_o^3 b_2 w_3 + [a_o^3 c_2 + b_o^3 c_3] w_4)/6$$

$$E_3 = 1/120 - a_o^2 b_2 c_3 w_4/2$$

$$E_4 = 1/120$$

$$E_5 = 7/120 - a_o(b_o + 1) b_2 c_3 w_4$$

$$E_6 = 1/40 - (a_o^2 b_2^2 w_3 + [a_o c_2 + b_o c_3]^2 w_4)/2$$

$$E_7 = 1/30 - (a_o^2 b_o b_2 w_3 + [a_o^2 c_2 + b_o^2 c_3] w_4)/2$$

$$E_8 = 1/20 - (a_o b_o^2 b_2 w_3 + [a_o c_2 + b_o c_3] w_4)/2$$

To obtain the above simplified expressions for E_i $i=1...8$, one has assumed $c_o=1$, $a_o=a_1$, $b_o=b_1+b_2$, and $c_o=c_1+c_2+c_3$ which are proven in section III. In section IV, one examines the fourth order truncation error and derives a number of fourth order substitution methods.

Third order Runge-Kutta methods are now considered in the following section.

SECTION II

THIRD ORDER RUNGE-KUTTA METHODS

Insisting that the Taylor expansion (1.42) and our numerical solution (1.40) agree up to and including the power h^3, one obtained the following relationships

$$w_1 + w_2 + w_3 = 1 \tag{2.10}$$

$$a_o w_2 + b_o w_3 = 1/2 \tag{2.11}$$

$$a_o^2 w_2 + b_o^2 w_3 = 1/3 \tag{2.12}$$

$$a_o b_2 w_3 = 1/6 \tag{2.13}$$

together with

$$a_o = a_1 \tag{2.14}$$

$$b_o = b_1 + b_2 \tag{2.15}$$

Since there are 6 equations and 8 constants, one evaluates the third order coefficients in terms of the parameters a_o and b_o as follows: (use a for a_o, b for b_o)

$$w_1 = 1 + \frac{2-3(a + b)}{6ab} \tag{2.20}$$

$$w_2 = \frac{3b - 2}{6a(b - a)} \tag{2.21}$$

$$w_3 = \frac{2 - 3a}{6b(b - a)} \tag{2.22}$$

$$a_1 = a \tag{2.23}$$

$$b_1 = \frac{3ab(1 - a) - b^2}{a(2 - 3a)} \tag{2.24}$$

$$b_2 = \frac{b(b - a)}{a(2 - 3a)} \qquad (2.25)$$

In view of (2.20-2.25), one has the restriction

$$ab(a - b)(2 - 3a) \neq 0 \qquad (2.26)$$

With only the preceding restrictions on the values of the parameters a,b, they may otherwise be arbitrarily chosen.

Prompted by reasons of convenience and symmetry, one may reduce the indeterminacy of the equations (2.10-2.15), by assuming $w_1 = w_2$; hence, by equating (2.20) and (2.21), the following quadratic equation in the variable b is obtained,

$$(6a-6)b^2 + (4-6a^2)b + (3a^2 - 2a) = 0 \qquad (2.30)$$

which will have real solutions if a satisfies the relationship

$$f(a) = 9a^4 - 18a^3 + 18a^2 - 12a + 4 \geq 0 \qquad (2.31)$$

It is easily seen using program 2-1 that $f(a) \geq 0$ for all values of a . However, in view of (2.20-2.25), the following values of the parameter

$$0 < a \leq 1 \qquad a \neq 2/3$$

will produce suitable values for b such that $w_1 = w_2$.

Similarly $w_1 = w_3$ produces the following quadratic in b

$$(6a - 3)b^2 + (2 - 6a^2)b + 6a^2 - 4a = 0 \qquad (2.32)$$

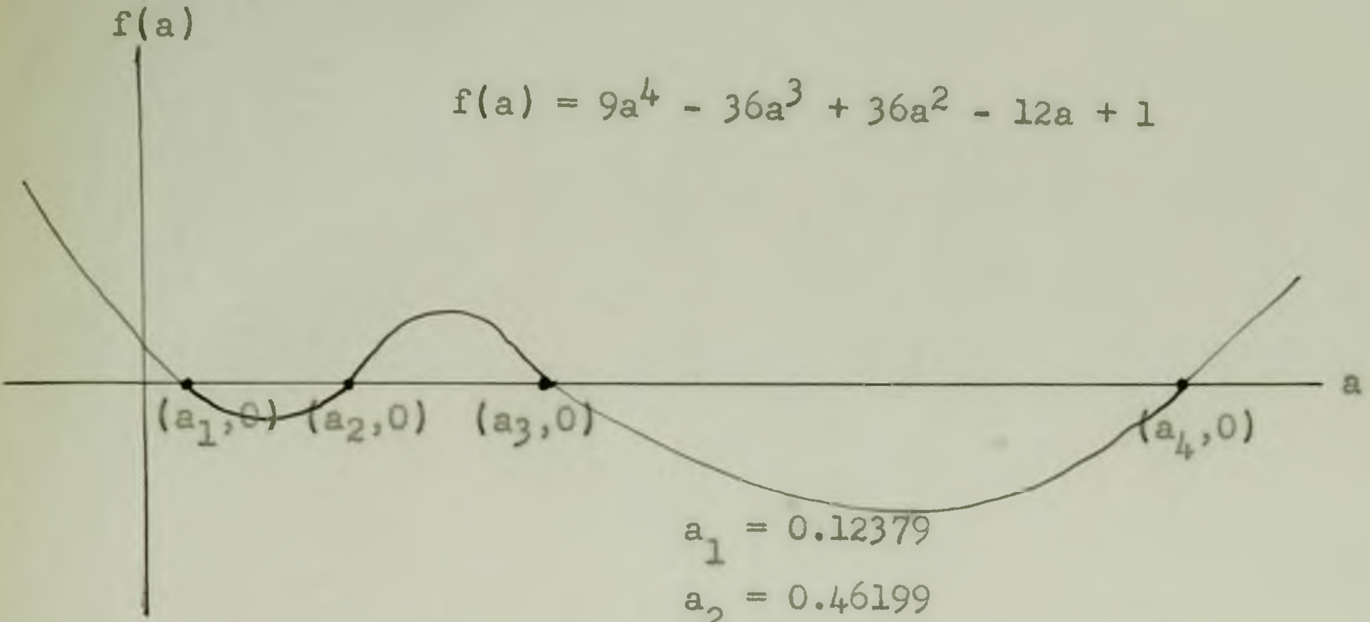
which will have real values for b if a satisfies

$$f(a) = 9a^4 - 36a^3 + 36a^2 - 12a + 1 \geq 0 \qquad (2.33)$$

With $f(a) = 0$, program 2-2 obtains four real roots

$$a = 0.12379,\ 0.46199,\ 0.75150,\ 2.6927$$

and graphically the function $f(a)$ behaves as follows:



$$a_1 = 0.12379$$
$$a_2 = 0.46199$$
$$a_3 = 0.75150$$
$$a_4 = 2.6927$$

In view of the above sketch and (2.20-2.26), the following values of a

$$0 < a \leq 0.12379 \qquad 0.46199 \leq a \leq 0.75150 \qquad a \neq 2/3$$

will determine suitable values for b such that $w_1 = w_3$.

With the assumption $w_2 = w_3$ one obtains the quadratic

$$3b^2 - 2b + 3a^2 - 2a = 0 \tag{2.34}$$

which will have real roots if

$$f(a) = 9a^2 - 6a - 1 \leq 0 \tag{2.35}$$

If $f(a) = 0$ in (2.35), then $a = -0.13807,\ 0.80474$ and by examining the graph of (2.35), one determines that

$$0 < a \leq 0.80474 \qquad a \neq 2/3$$

will yield suitable corresponding values for b in (2.34) such that $w_2 = w_3$ is satisfied.

On assuming $w_1 = w_2 = w_3$ as a further symmetry, one obtains an impossible solution.

If one now discards the symmetry requirements of the last paragraph, and instead investigates the possibility of reducing the number of calculations in the numerical solution, one obtains the following relationships between the parameters a and b:

Assumption	Relationship	
$w_1 = 0$	$b = \dfrac{2 - 3a}{3 - 6a}$	(2.40)
$w_2 = 0$	$b=2/3$, a arbitrary	(2.41)
$b_1 = 0$	$b = 3a - 3a^2$	(2.42)

It must be noted that an infinity of methods of reasonable accuracy can be devised by assigning values for a in any of (2.40-2.42). Furthermore, equating other coefficients to zero result in impossible solutions.

By combining relationships from (2.40-2.42), one obtains useful substitution processes. In the first place, from (2.41) and (2.42), the quadratic equation

$$9a^2 - 9a + 2 = 0 \qquad (2.43)$$

is obtained which determines the values

$$a = 1/3 \qquad b = 2/3$$

and hence yields a method in which $w_2 = 0$ and $b_1 = 0$

Secondly, from (2.40) and (2.42), a must satisfy the cubic equation

$$18a^3 - 27a^2 + 12a - 2 = 0 \qquad (2.44)$$

which has a real root $a = 0.89255$. Since $a = 0.89255$ determines $w_1 = 0$ and $b_1 = 0$, one obtains a numerical method, denoted by 3I6 in Table 2.2, which is an iterative procedure of the type

$$K_i = f(K_{i-1}) \qquad i=2,3 \qquad (2.45)$$

Furthermore, a reduced number of storage registers are required when the method is programmed. (see Appendix, program 2-3)

The combination of (2.40) and (2.41) yields no allowable solution.

Having determined methods which separately incorporate symmetry and minimization of calculations, one now determines methods which utilize both considerations. By combining symmetry and minimization restrictions from the following table

Table 2.1

Symmetry	Minimization
$S_1 : w_1 = w_2$	$M_1: w_1 = 0$
$S_2 : w_1 = w_3$	$M_2: w_2 = 0$
$S_3 : w_2 = w_3$	$M_3: b_1 = 0$
	$M_4: w_1=0 \; b_1=0$
	$M_5: w_2=0 \; b_1=0$

one obtains a number of additional methods.

Imposing the conditions S_1 and M_3, one obtains

$$54a^4 - 144a^3 + 144a^2 - 63a + 10 = 0$$

which has two real roots 0.40210, 0.66656, and hence two numerical methods having $w_1 = w_2$ and $b_1 = 0$ are determined.

Assuming S_2 and M_3, one determines the equation

$$54a^4 - 117a^3 + 90a^2 - 27a + 2 = 0$$

which has two real roots 0.10745 and 0.66655 giving two methods in which the conditions $w_1 = w_3$, and $b_1 = 0$ are satisfied.

Furthermore, S_3 and M_1 yield a cubic equation

$$18a^3 - 10a^2 + 15a - 2 = 0$$

having a real root 0.14352 .

On imposing S_3 and M_3, one establishes the equation

$$27a^3 - 54a^2 + 36a - 8 = 0$$

which realizes one real root 0.66000 and hence furnishes a method having $w_2 = w_3$ and $b_1 = 0$.

All other combinations from Table 2.1 yield impossible solutions.

Although the numerical solution of only a first order differential equation has been specifically mentioned, the third order Runge-Kutta methods derived are also applicable to systems of first order differential equations. If we are given a system of N first order differential equations

$$\frac{dy_i}{dx} = f_i(x,y_1,\ldots y_N) \qquad i=1,\ldots N \tag{2.46}$$

with initial conditions

$$y_i(x_o) = Y_i \qquad i=1,\ldots N$$

we may define

$$y_o = x \qquad Y_o = x_o \qquad f_o = 1$$

and hence our system (2.46) is written in the convenient form

$$\frac{dy_i}{dx} = f_i(y_o,y_1,\ldots,y_N) \qquad i=0,1,\ldots,N \tag{2.47}$$

$$y_i = Y_i \text{ at } x=x_o \qquad i=0,1,\ldots,N$$

The numerical solution of our system (2.47) using a third order method is then given by the following: for i=0,1,...,N

$$K_{i1} = hf_i(Y_o,Y_1,\ldots,Y_N) \tag{2.50}$$

$$K_{i2} = hf_i(Y_o+aK_{01},\ Y_1+aK_{11},\ldots,Y_N+aK_{N1}) \tag{2.51}$$

$$K_{i3} = hf_i(Y_o+bK_{01},\ Y_1+b_1K_{11}+b_2K_{12},\ldots Y_N+b_1K_{N1}+b_2K_{N2}) \tag{2.52}$$

where K_{i1} i=0,...,N is computed before K_{i2}; K_{i2} i=0,...,N before K_{i3} and the increment in Y_i i=0,...N is given by

$$Y_i + dY_i = Y_i + w_1K_{i1} + w_2K_{i2} + w_3K_{i3} \tag{2.53}$$

In view of (2.50-2.52) when the system (2.47) is solved on a digital computer, our third order substitution methods will require 3N + A storage registers where A is a constant of the program.

A Runge-Kutta third order procedure due to S. D. Conte and R. F. Reeves is now obtained which requires 2N + A storage registers for the solution of (2.47) rather than the usual 3N + A. To obtain 2N + A storage registers, one insists that the quantities,

$$\begin{aligned} &Y_i + w_1 K_{i1} \\ &Y_i + aK_{i1} \qquad\qquad i=1,\ldots N \\ &Y_i + b_1 K_{i1} \end{aligned}$$

from (2.51-2.53) be equal, and as a result, the identities

$$w_1 = a \tag{2.60}$$

$$b_2 = b - a \tag{2.61}$$

must be satisfied and will make the system (2.10-2.15) determinate.

It is easily verified that the condition

$$abb_2(2 - 3a) = 0 \tag{2.62}$$

is incompatible with (2.10-2.15), and (2.60-2.61).

A solution of (2.10-2.15) together with (2.60-2.61) is obtained as follows: by eliminating b_2 from (2.61) and (2.13), one obtains

$$w_3 = \frac{1}{6a(b - a)} \tag{2.70}$$

From (2.70), and (2.11), one establishes

$$w_2 = \frac{3a(b - a) - b}{6a^2(b - a)} \qquad (2.71)$$

and from (2.70) and (2.12)

$$w_2 = \frac{2a(b - a) - b^2}{6a^3(b - a)} \qquad (2.72)$$

By the equality of (2.71) and (2.72), one obtains the identity

$$b = a(2 - 3a) \qquad (2.73)$$

In view of (2.73), (2.70) and (2.71), one utilizes the equation (2.10) to obtain

$$6a^3 - 6a^2 + 3a - 1 = 0 \qquad (2.74)$$

Since (2.74) has a non-zero real root 0.62654, the system (2.10-2.15) with the conditions (2.60-2.61) has a solution. Using in order (2.73), (2.71), (2.70), (2.61) and (2.60), one obtains the values of the remaining coefficients. (see 3I7, table 2.2).

As a result, the solution of (2.10-2.15) together with (2.60-2.61) establishes coefficients for a third order Runge-Kutta method which uses only $2N + A$ storage registers rather than the conventional number $3N + A$. Program 2-4 illustrates how the computation is arranged to require only $2N + A$ storage registers.

In solving (2.10-2.15) and obtaining the solution (2.20-2.25) which will be denoted by 3I , one required the assumption

$$ab(a - b)(2 - 3a) \neq 0 \qquad (2.26)$$

The question arises as to what solutions are possible for (2.10-2.15) if

$$ab(a - b)(2 - 3a) = 0$$

and on careful examination only the following possibilities may occur

$$b = 0, \quad a = b, \quad a = 2/3,$$

while

$$a = 0$$

yields an impossible solution.

Assuming b = 0, one simplifies (2.11) and (2.12) to obtain respectively

$$w_2 = \frac{1}{2a}$$

and

$$w_2 = \frac{1}{3a^2}$$

which establish

$$a = 2/3$$

If now a = 2/3 and b = 0 is applied to (2.10-2.15), one yields the following solution, denoted by 3II, with w_3 as parameter:

Method 3II

$w_1 = \frac{1}{4} - w_3$ $\qquad$ $a = 2/3$ $\qquad$ $b = 0$

$w_2 = 3/4$ $\qquad$ $a_1 = 2/3$ $\qquad$ $b_1 = -\frac{1}{4w_3}$

$w_3 =$ arbitrary $\qquad$ $b_2 = \frac{1}{4w_3}$

By requiring the conditions $w_1=w_3$, $w_1=w_2$, $w_2=w_3$, and $w_1=0$ to be satisfied for Method 3II, one obtains respectively the solutions 3II1,3II2, 3II3, and 3II4 in Table 2.2 .

Alternatively, if a=b then (2.11) and (2.12) become respectively

$$a(w_2 + w_3) = 1/2$$

and

$$a^2(w_2 + w_3) = 1/3$$

from which it is obvious that a=b=2/3. These values in (2.10-2.15) determine the coefficients for Method 3III which follows:

Method 3III

$w_1 = 1/4$ $\qquad$ $a = 2/3$ $\qquad$ $b = 2/3$

$w_2 = \frac{3}{4} - w_3$ $\qquad$ $a_1 = 2/3$ $\qquad$ $b_1 = \frac{2}{3} - \frac{1}{4w_3}$

$w_3 =$ arbitrary $\qquad$ $b_2 = \frac{1}{4w_3}$

Imposing the conditions in turn $w_1=w_3$, $w_1=w_2$, $w_2=w_3$ and $w_3=3/4$ on the coefficients of Method 3III, one determines the entries 3III1, 3III2, 3III3, and 3III4 in Table 2.2 page 21.

However, on assuming $a=2/3$, one obtains the coefficients of Method 3II.

The entries of the following Table 2.2 are used in the third order Runge-Kutta equations as follows:

$$K_1 = hf(x,y)$$

$$K_2 = hf(x + ah, y + a_1K_1)$$

$$K_3 = hf(x + bh, y + b_1K_1 + b_2K_2)$$

where $a=a_1$ and the increment in y is given by

$$dy = w_1K_1 + w_2K_2 + w_3K_3$$

Table 2.2

Method	w_1	w_2	w_3	a	b	b_1	b_2
3I1	1/6	1/6	2/3	1	1/2	-1/2	1
3I2	1/6	2/3	1/6	1/2	1	-1	2
3I3	0	4/7	3/7	1/4	5/6	-13/18	14/9
3I4	0	3/4	1/4	1/3	1	-1	2
3I5	1/4	0	3/4	1/3	2/3	0	2/3
3I6	0	0.35098	0.64902	0.89255	0.28871	0	0.28871
3I7	0.62654	0.85614	-0.48268	0.62654	0.075426	0.62654	-0.55111
3II1	1/8	3/4	1/8	2/3	0	-2	2
3II2	3/4	3/4	-1/2	2/3	0	-1/2	1/2
3II3	-1/2	3/4	3/4	2/3	0	-1/3	1/3
3II4	0	3/4	1/4	2/3	0	-1	1
3III1	1/4	1/2	1/4	2/3	2/3	-1/3	1
3III2	1/4	1/4	1/2	2/3	2/3	1/6	1/2
3III3	1/4	3/8	3/8	2/3	2/3	0	2/3
3III4	1/4	0	3/4	2/3	2/3	1/3	1/3

When presenting papers of this type, one always concludes a section with some numerical example, to illustrate the previous discussion. In choosing a suitable differential equation, one must be able to solve the equation analytically in order to compare results of the numerical solution with the analytic one. The following example illustrates the point. The linear differential equation

$$\frac{dy}{dx} = y(4 - 3/2\tan(3/2)x) \quad (0,1) \qquad (2.75)$$

has an analytic solution

$$y = e^{4x}\cos(3/2)x \qquad (2.76)$$

By computing the analytic value y(x) from (2.76), and the numerical value ye(x) from (2.75), one is able to obtain an estimate of the accuracy of the method by further computing the difference

$$|y(x) - ye(x)| = |E(x)| \qquad (2.77)$$

where E(x) denotes the value of the error. Program 2-5 was used to obtain the results of Table 2.3 ,page 23, which has been constructed for the increments h = 0.25, 0.2, 0.1, and 0.05 in order that the following observation be made: if the number of steps of calculation be increased (i.e. h is decreased) then the value of $|E(x)|$ decreases.

We now consider Runge-Kutta fourth order methods, in the next section.

Table 2.3

	Method	y(1)	ye(1)	|E(1)|
0(0.25)1	3I2	51..38620	51..70087	51..31467
	3I6	51..38620	51..28961	51..38330
	3II4	51..38620	51..19136	51..19484
	3III3	51..38620	51..19110	50..93902
0(0.2)1	3I2	51..38620	51..62236	51..23618
	3I6	51..38620	51..16248	51..22372
	3II4	51..38620	51..24835	51..13785
	3III3	51..38620	51..45826	50..72062
0(0.1)1	3I2	51..38620	51..43485	50..48669
	3I6	51..38620	51..35403	50..32156
	3II4	51..38620	51..35501	50..31172
	3III3	51..38620	51..40503	50..18849
0(0.05)1	3I2	51..38620	51..39245	49..62649
	3I6	51..38620	51..38236	49..38227
	3II4	51..38620	51..38118	49..50064
	3III3	51..38620	51..38908	49..28957

NOTE: Floating point notation is used.

SECTION III

FOURTH ORDER RUNGE-KUTTA METHODS

If (1.73) and the Taylor expansion (1.42) agree up to and including the power h^4, the equations (1.20-1.25) for the fourth order Runge-Kutta method are given by

$$K_1 = hf(x,y) \tag{3.11}$$

$$K_2 = hf(x + ah,\ y + a_1K_1) \tag{3.12}$$

$$K_3 = hf(x + bh,\ y + b_1K_1 + b_2K_2) \tag{3.13}$$

$$K_4 = hf(x + ch,\ y + c_1K_1 + c_2K_2 + c_3K_3) \tag{3.14}$$

$$dy = w_1K_1 + w_2K_2 + w_3K_3 + w_4K_4 \tag{3.15}$$

and after equating corresponding coefficients in (1.73) and (1.42), one first obtains the relationships

$$a = a_1 \tag{3.16}$$

$$b = b_1 + b_2 \tag{3.17}$$

$$c = c_1 + c_2 + c_3 \tag{3.18}$$

and then the equations

$$w_1 + w_2 + w_3 + w_4 = 1 \tag{3.21}$$

$$aw_2 + bw_3 + cw_4 = 1/2 \tag{3.22}$$

$$a^2w_2 + b^2w_3 + c^2w_4 = 1/3 \tag{3.23}$$

$$a^3w_2 + b^3w_3 + c^3w_4 = 1/4 \tag{3.24}$$

$$ab_2w_3 + w_4(ac_2 + bc_3) = 1/6 \tag{3.25}$$

$$a^2b_2w_3 + w_4(a^2c_2 + b^2c_3) = 1/12 \tag{3.26}$$

$$abb_2w_3 + w_4(ac_2 + bc_3)c = 1/8 \tag{3.27}$$

$$ab_2c_3w_4 = 1/24 \tag{3.28}$$

The condition $c = 1$ will now be derived for (3.21-3.28) By eliminating w_2 from (3.23) and (3.24), one obtains

$$(ab - b^3)w_3 + (ac - c^3)w_4 = \frac{1}{3}a - \frac{1}{4} \tag{3.31}$$

and from (3.22) and (3.23)

$$(ab - b^2)w_3 + (ac - c^2)w_4 = \frac{1}{2}a - \frac{1}{3} \tag{3.32}$$

Proceeding to eliminate w_4 from (3.31) and (3.32), one determines

$$b(a - b)(c - b)w_3 = \frac{1}{4} + \frac{1}{2}ac - \frac{1}{3}(a + c) \tag{3.33}$$

From (3.25) and (3.27),

$$ab_2(c - b)w_3 = \frac{1}{6}c - \frac{1}{8} \tag{3.34}$$

In view of (3.28), a, $b_2 \neq 0$ and it may also be easily shown that $(c - b)$ and w_3 are non-zero.

To eliminate b_2 from (3.34), one uses (3.25) and (3.26) to obtain first

$$c_3 = \frac{2a - 1}{12w_4 b(a - b)} \tag{3.35}$$

where $a-b,\ b \neq 0$.

Then using (3.35) in (3.28), one obtains

$$b_2 = \frac{b(a - b)}{2a(2a - 1)} \tag{3.36}$$

where $a \neq 1/2$. The following expression for w_3 is then obtained

$$b(a - b)(c - b)w_3 = \left(\frac{1}{3}c - \frac{1}{4}\right)(2a - 1) \tag{3.37}$$

Comparing (3.37) and (3.33), one obtains

$$ac = a$$

but since $a \neq 0$, the result follows that

$$c = 1 \qquad (3.38)$$

In view of (3.38), the equation (3.34) simplifies to

$$ab_2(1 - b)w_3 = 1/24$$

which implies that

$$b \neq 1 \qquad (3.39)$$

Having eight equations (3.21-3.28), and ten unknowns, one uses a,b, as parameters and obtains expressions for the remaining coefficients as follows: in view of (3.37), and (3.38)

$$w_3 = \frac{2a - 1}{12b(a - b)(1 - b)} \qquad (3.40)$$

and using this result in (3.32), one obtains

$$w_4 = \frac{1}{2} + \frac{2(a + b)-3}{12(1 - a)(1 - b)} \qquad (3.41)$$

From (3.21-3.28), the remaining coefficients are determined as

$$w_1 = \frac{1}{2} + \frac{1 - 2(a + b)}{12ab} \qquad (3.42)$$

$$w_2 = \frac{2b - 1}{12a(b - a)(1 - a)} \qquad (3.43)$$

$$b_2 = \frac{b(b - a)}{2a(1 - 2a)} \qquad (3.44)$$

$$c_2 = \frac{(1-a)(a+5b-2-4b^2)}{2a(b-a)(6ab-4[a+b]+ 3)} \qquad (3.45)$$

and a_1, b_1, c_1 are given by

$$a_1 = a \tag{3.46}$$

$$b_1 = b - b_2 \tag{3.47}$$

$$c_1 = 1 - c_2 - c_3 \tag{3.48}$$

The expressions (3.40-3.48) are subject to the restrictions

$$a \neq 1, \quad abc_3w_4 \neq 0, \quad a \neq 1/2, \quad a \neq b, \quad b \neq 1$$

and the solutions of (3.21-3.28) possible when these restrictions are removed will be examined at the end of this section.

Having derived expressions for the coefficients of the numerical method in terms of the parameters a,b, one now examines various symmetries of the weights w_i i=1,2,3,4 and in so doing only the following cases of Table 3.1, are permissable. By evaluating the discriminent, where possible, of the quadratic equations in table 3.1, one obtains a range of values for "a" (or "b") for each of the symmetries SS_i, i=1,2,...11. For example, insisting that the quadratic equation of SS_1 have real roots, one establishes that a must satisfy

$$f(a) = 36a^6-120a^5+160a^4-116a^3+57a^2-20a+4 \geq 0$$

However, program 2-1 easily establishes that $f(a) \geq 0$ for all a. In view of the expressions (3.40-3.48) a suitable range for a would be

$$0 < a < \frac{1}{2} \qquad \frac{1}{2} < a < 1$$

Table 3.1

Method	Symmetry	Equation
SS_1	$w_1=w_2$	$(6a^2-8a+4)b^2+(-6a^3+6a^2+a-2)b+(2a^3-3a^2+a)=0$
SS_2	$w_1=w_3$	$(6b^2-8b+4)a^2+(-6b^3+6b^2+b-2)a+(2b^3-3b^2+b)=0$
SS_3	$w_1=w_4$	$(2-4b)a^2+(-4b^2+8b-3)a+(2b^2-3b+1)=0$
SS_4	$w_2=w_3$	$2a^2 - (1+2b)a + (2b^2-b)=0$
SS_5	$w_2=w_4$	$(6a^2-4a+2)b^2+(-6a^3+3a-3)b+(4a^3-3a^2+1)=0$
SS_6	$w_3=w_4$	$(6b^2-4b+2)a^2+(-6b^3+3b-3)a+(4b^3-3b^2+1)=0$
SS_7	$w_1=w_2=w_4$	$(3-6a)b^2+(6a^2)b-(3a^2-2a+1)=0$
SS_8	$w_1=w_3=w_4$	$(3-6a)b^2+(6a^2-2)b+(1-3a^2)=0$
SS_9	$w_1=w_2$, $w_3=w_4$	$(3-3a)b^2+(3a^2-2)b-(a-1)=0$
SS_{10}	$w_1=w_3$, $w_2=w_4$	$(3a)b^2+(-3a^2-1)b+(3a^2-2a+1)=0$
SS_{11}	$w_1=w_4$, $w_2=w_3$	$a+b=1$

To obtain the corresponding value of b for each a, one then uses the quadratic equation SS_1. The expressions (3.40-3.48) will then be used to determine the remaining coefficients for a numerical method in which $w_1 = w_2$. The following Table 3.2, page 30 exhibits the discriminent of the quadratic equations for the symmetries SS_i i=1,...10 of Table 3.1 and suggests a suitable range for either a or b whichever the case may be.

The symmetry consideration $w_1=w_4$, $w_2=w_3$, namely SS_{11} exhibits a simple relationship

$$a + b = 1$$

which simplifies (3.40-3.48) as

$$w_1 = \frac{1}{2} - \frac{1}{12ab} \qquad a_1 = a \qquad c_1 = \frac{2a^2(6b-1)+a(b-2)-b^2}{2a(6ab - 1)}$$

$$w_2 = \frac{1}{12ab} \qquad b_1 = b - \frac{b}{2a} \qquad c_2 = \frac{b(a - b)}{2a(6ab-1)}$$

$$w_3 = \frac{1}{12ab} \qquad b_2 = \frac{b}{2a} \qquad c_3 = \frac{a}{6ab - 1}$$

$$w_4 = \frac{1}{2} - \frac{1}{12ab}$$

A solution due to Kutta is obtained from the above when a=1/3 and b=2/3, (see Method 4I1, Table 3.6).

Table 3.2

Symmetry	Discriminent	Range
SS_1	$36a^6-120a^5+160a^4-116a^3+57a^2-20a+4$	$0 < a < 1/2$ $1/2 < a < 1$
SS_2	$36b^6-120b^5+160b^4-116b^3+57a^2-20b+4$	$0 < b < 1/2$ $1/2 < b < 1$
SS_3	$16b^4 - 32b^3 + 24b^2 - 8b + 1$	$0 < b < 1$
SS_4	$-12b^2 + 12b + 1$	$0 < b < 1/2$ $1/2 < b < 1$
SS_5	$36a^6-96a^5+100a^4-44a^3+9a^2-2a+1$	$0 < a < 1/2$ $1/2 < a < 1$
SS_6	$36b^6-96b^5+100b^4-44b^3+9b^2-2b+1$	$0 < b < 1/2$ $1/2 < b < 1$
SS_7	$3a^4 - 6a^3 + 7a^2 - 4a + 1$	$0 < a < 1/2$ $1/2 < a < 1$
SS_8	$9a^4 - 18a^3 + 3a^2 + 6a - 2$	$0.42265 < a < 0.57735$
SS_9	$9a^4 - 24a^2 + 24a - 8$	$0.89054 < a < 1$
SS_{10}	$9a^4 - 36a^3 + 30a^2 - 12a + 1$	$0 < a < 0.10946$

Proceeding in the same manner as section II, one now investigates the possibility of reducing the number of calculations involved in the various fourth order numerical methods. Hence by equating the various coefficients to zero, one obtains only the following relationships:

Table 3.3

Method	Assumption	Relationship
MM_1	$w_1=0$	$a = (2b - 1)/(6b - 2)$
MM_2	$w_2=0$	$b = 1/2$
MM_3	$b_1=0$	$4a^2 - 3a + b = 0$
MM_4	$c_2=0$	$4b^2 - 5b - a + 2 = 0$
MM_5	$c_1=0$	$(-12a^2+12a-4)b^2+(12a^2-15a+5)b +(-4a^2+6a-2) = 0$

By examining the discriminent of MM_3 and MM_4, one may easily show that in order to have real roots, the conditions

$$b \leq 9/16 \quad \text{for } MM_3$$

and

$$a \geq 7/16 \quad \text{for } MM_4$$

must respectively hold. Similarly, on examination of the discriminent of MM_5, it is established that for real values of b, a must satisfy

$$f(a) = -48a^4+120a^3-103a^2+42a-7 \geq 0$$

and using program 2-2, one establishes a suitable range for a as $0.44805 < a < 0.5$ and $0.5 < a < 1$.

It must be noted that other coefficients from the fourth order method equated to zero result in impossible solutions.

To reduce further the number of calculations, two coefficients may be equated to zero. For example,

$$w_1=0 \quad b_1=0$$

requires that a be a root of

$$24a^3-26a^2+8a-1 = 0 \tag{3.49}$$

and $a=0.68594$ satisfies (3.49). The following Table 3.4 lists the various possibilities that have a solution. The equation obtained and its root(s) x, $0 < x < 1$, are also tabulated.

Table 3.4

Method	Assumption	Equation	Root
MM_6	$w_1=b_1=0$	$24a^3-26a^2+8a-1 = 0$	$a=0.68594$
MM_7	$w_1=c_1=0$	$24b^3-36b^2+19b-3 = 0$	$b=0.27465$
MM_8	$w_2=b_1=0$	$8a^2 - 6a + 1 = 0$	$a=1/4$
MM_9	$b_1=c_1=0$	$96a^5-192a^4+158a^3-71a^2+17a-2 = 0$	$a=0.81215$
MM_{10}	$c_1=c_2=0$	$96a^5-288a^4+350a^3-211a^2+61a-6 = 0$	$a=0.18810$

Again it must be noted that all other pairs of coefficients equated to zero yield impossible solutions. For the simple value of the parameter in MM_8, the other coefficients have been calculated and are given by Method 4I2 Table 3.6

Having investigated all possible symmetry and minimum conditions individually, one may wish to incorporate both considerations into a fourth order numerical method. With this in mind, one examines the compatability of the minimum conditions with each of the symmetry possibilities.

For example, if one assumes $w_1=0$ (MM_1) and $w_2=w_3$ (SS_4), one requires that a be a root of the equation

$$36a^4-72a^3+48a^2-12a+1 = 0$$

which has no real roots in the interval (0,1). As a result, MM_1 and SS_4 are incompatible on the range (0,1).

On the other hand, assuming MM_1 and SS_5, one requires that

$$6a^3-10a^2+6a-1 = 0$$

and a=0.26530 is a root.

Furthermore, MM_1 and SS_6 produce the equation

$$6b^3-10b^2+6b-1 = 0$$

which exhibits a root b=0.26530. Hence, one obtains a fourth order numerical method which incorporates the assumptions MM_1 and SS_6. Continuing in this way, one shows that MM_1 is incompatible with the remaining symmetry conditions.

Similarly, one establishes that MM_2 and the symmetries SS_i i=1,...11 are incompatible.

On assuming MM_3 and SS_i i=1,...11, one laboriously obtains equations in all cases except for SS_{11}, SS_{11} having no solution with MM_3. The following Table 3.5 tabulates the equations obtained and their solutions.

Table 3.5

Method	Equation	Solution(s)
MM_3 SS_1	$48a^4-100a^3+84a^2-34a+5 = 0$	a=0.30934 a=0.79658
MM_3 SS_2	$192a^5-352a^4+268a^3-108a^2+22a-1 = 0$	a=0.061326
MM_3 SS_3	$64a^5-144a^4+128a^3-56a^2+12a-1 = 0$	a=0.25,0.51149 a=0.45299
MM_3 SS_4	$64a^5-144a^4+132a^3-64a^2+17a-2 = 0$	a=0.70279
MM_3 SS_5	$48a^5-68a^4+48a^3-22a^2+7a-1 = 0$	a=0.31373
MM_3 SS_6	$192a^6-416a^5+332a^4-120a^3+18a^2+a-1=0$	a=0.81948
MM_3 SS_7	$48a^4 - 60a^3 + 24a^2 - 1 = 0$	a=0.32159
MM_3 SS_8	$48a^4 - 60a^3 + 24a^2 - 4a + 1 = 0$	no solution
MM_3 SS_9	$48a^5-108a^4+90a^3-35a^2+7a-1 = 0$	a=0.89100
MM_3 SS_{10}	$48a5_{-}60a^4+16a^3+7a^2-5a+1 = 0$	a=0.67260

If it be found advantageous, one may proceed to examine

MM_4 and MM_5 separately with SS_i $i=1,\ldots 11$ to obtain other solutions.

To obtain solution (3.40-3.48) which we will denote as Method I, a number of restrictions were assumed for the equations (3.21-3.28). The question arises as to what solutions will be obtained for (3.21-3.28) if these restrictions are removed. Using (3.22-3.24) and b=1, one obtains an impossible solution. Furthermore in view of (3.28), one need only examine the solution of (3.21-3.28) when the restrictions $a\neq b$, $a\neq 1$, and $a\neq 1/2$ are removed.

Previously, one had determined c=1 for (3.21-3.28), but only after the assumptions $a\neq b$, $a\neq 1$, and $a\neq 1/2$ had been imposed. However, it may again be established that c=1 when one assumes in turn a=b, a=1, and a=1/2. For example, on assuming a=b in (3.21-3.28), one eliminates w_2 and w_3 from first (3.22) and (3.23), and then from (3.23) and (3.24) to obtain respectively

$$(ac - c^2)w_4 = \frac{1}{2}a - \frac{1}{3}$$

and

$$(ac^2 - c^3)w_4 = \frac{1}{3}a - \frac{1}{4}$$

By eliminating w_4, one determines the equation

$$(4-6a)c^2 + (6a^2-3)c + 3a-4a^2 = 0$$

From (3.25) and (3.26), it is immediately established that a=1/2; thus, the previous equation becomes

$$2c^2 - 3c + 1 = 0$$

which exhibits the roots c=1/2, and c=1. The value c=1/2 is impossible in view of (3.22-3.24) and hence c=1 as required. Similarly, a=1, and a=1/2 each determines c=1.

Turning to (3.21-3.28) and imposing a=b, one eliminates w_2, w_3 from (3.21) and (3.22) to obtain

$$w_4 = \frac{3a - 2}{6(a - 1)} \qquad (3.50)$$

Similarly, from (3.23) and (3.24)

$$w_4 = \frac{4a - 3}{12(a - 1)} \qquad (3.51)$$

Using (3.50) and (3.51), one obtains a=1/2 and with w_3 as parameter, the coefficients for the solution of (3.21-3.28) when a=b is given by

Method 4II

$w_1 = 1/6$	$a = 1/2$	$b = 1/2$	$c = 1$
$w_2 = 2/3 - w_3$	$a_1 = 1/2$	$b_1 = 1/2 - \frac{1}{6w_3}$	$c_1 = 0$
w_3 =arbitrary		$b_2 = \frac{1}{6w_3}$	$c_2 = 1 - 3w_3$
$w_4 = 1/6$			$c_3 = 3w_3$

For convenience of symmetry and the reduction of operations, one assumes in turn $w_1=w_2$, $w_1=w_3$, $w_2=w_3$ and $w_2=0$ which respectively determine the values $w_3= 1/2, 1/6, 1/3, 2/3$ and hence the methods 4II1, 4II2, 4II3, and 4II4 of Table 3.6. The assumption of other symmetries or the reduction of operations yield vales for w_3 which either duplicate the above methods or determine impossible solutions.

Assuming now $a=1$, in (3.21-3.28), one eliminates w_2 and w_4 from (3.22) and (3.23) to obtain

$$w_3 = \frac{1}{6b(1-b)} \tag{3.52}$$

Similarly from (3.23) and (3.24), one determines

$$w_3 = \frac{1}{12b^2(1-b)} \tag{3.53}$$

which together with (3.52) establishes $b=1/2$. With w_4 as parameter, the remaining coefficients are given by

Method 4III

$w_1 = 1/6$	$a = 1$	$b = 1/2$	$c = 1$
$w_2 = 1/6 - w_4$	$a_1 = 1$	$b_1= 3/8$	$c_1= 1-\frac{1}{4w_4}$
$w_3 = 2/3$		$b_2= 1/8$	$c_2=-\frac{1}{12w_4}$
w_4 = arbitrary			$c_3=\frac{1}{3w_4}$

By imposing in turn the conditions $w_1=w_4$, $w_2=w_3$, $w_2=w_4$, $w_3=w_4$ and $c_1=0$ one obtains the respective values of the parameter

$w_4 = 1/6,\ 1/2,\ 1/12,\ 2/3,\ 1/4$ and hence determines respectively the coefficients 4III1, 4III2, 4III3, 4III4, and 4III5 of Table 3.6.

Similarly, one assumes $a=1/2$ in (3.21-3.28) and then eliminates w_2, w_4 from (3.21), (3.22) and (3.23) to obtain

$$b(b - 1)(b - 1/2)w_3 = 0 \tag{3.54}$$

It is easily shown that $w_3=0$ and $b=1$ yield impossible solutions while $b=1/2$ duplicates Method 4II. As a result, $b=0$ and this value determines the following method:

Method 4IV

$w_1 = 1/6 - w_3$	$a = 1/2$	$b = 0$	$c = 1$
$w_2 = 2/3$	$a_1 = 1/2$	$b_1 = -1/(12w_3)$	$c_1 = -1/2 - 6w_3$
$w_3 =$ arbitrary		$b_2 = 1/(12w_3)$	$c_2 = 3/2$
$w_4 = 1/6$			$c_3 = 6w_3$

Insisting in turn the conditions $w_1=w_2$, $w_1=w_3$, $w_2=w_3$, $w_3=w_4$ and $c_1=0$, one obtains the values $w_3= -1/2,\ 1/12,\ 2/3,\ 1/6$, and $-1/12$ and hence each of these values for w_3 respectively determine the coefficients 4IV1, 4IV2, 4IV3, 4IV4, and 4IV5 of Table 3.6.

Table 3.6 now follows.

Table 3.6

Method	w_1	w_2	w_3	w_4	a	b	b_1	b_2	c	c_1	c_2	c_3
4I1	1/8	3/8	3/8	1/8	1/3	2/3	-1/3	1	1	1	-1	1
4I2	1/6	0	2/3	1/6	1/4	1/2	0	1/2	1	1	-2	2
4II1	1/6	1/6	1/2	1/6	1/2	1/2	1/6	1/3	1	0	-1/2	3/2
4II2	1/6	1/2	1/6	1/6	1/2	1/2	-1/2	1	1	0	1/2	1/2
4II3	1/6	1/3	1/3	1/6	1/2	1/2	0	1/2	1	0	0	1
4II4	1/6	0	2/3	1/6	1/2	1/2	1/4	1/4	1	0	-1	2
4III1	1/6	0	2/3	1/6	1	1/2	3/8	1/8	1	-1/2	-1/2	2
4III2	1/6	-1/3	2/3	1/2	1	1/2	3/8	1/8	1	1/2	-1/6	2/3
4III3	1/6	1/12	2/3	1/12	1	1/2	3/8	1/8	1	-2	-1	4
4III4	1/6	-1/2	2/3	2/3	1	1/2	3/8	1/8	1	5/8	-1/8	1/2
4III5	1/6	-1/12	2/3	1/4	1	1/2	3/8	1/8	1	0	-1/3	4/3
4IV1	2/3	2/3	-1/2	1/6	1/2	0	1/6	-1/6	1	5/2	3/2	-3
4IV2	1/12	2/3	1/12	1/6	1/2	0	-1	1	1	-1	3/2	1/2
4IV3	-1/2	2/3	2/3	1/6	1/2	0	-1/8	1/8	1	-9/2	3/2	4
4IV4	0	2/3	1/6	1/6	1/2	0	-1/2	1/2	1	-3/2	3/2	1
4LV4	1/4	2/3	-1/12	1/6	1/2	0	1	-1	1	0	3/2	-1/2

In section II, we derived a Runge-Kutta third order procedure due to Conte and Reeves which reduced the number of storage registers required to solve (2.47) from 3N+A to 2N+A. A similar treatment of Runge-Kutta fourth order methods, namely the reduction of storage registers, is now considered.

To solve (2.47) using the fourth order method (3.11-3.15), one may easily see that 4N+A storage registers are required where again A is a constant of the program (see p.16). Although the simultaneous first order differential equations (2.47) could be treated in a similar manner, let us for the sake of simplification consider the solution of

$$\frac{dy}{dx} = f(x,y) \qquad y(x_o) = y_o \tag{1.00}$$

using (3.11-3.15). In view of applying (3.11-3.15) to (1.00) the maximum number of storage registers required, namely four, occurs at that stage of the numerical procedure when one stores the quantities

$$\begin{aligned} &y_o + b_1K_1 + b_2K_2 \\ &y_o + c_1K_1 + c_2K_2 \\ &y_o + w_1K_1 + w_2K_2 \end{aligned} \tag{3.55}$$

and K_3

As a result, if one is able to reduce the number of registers required at this stage of the calculation to three, then one never exceeds this number for the entire program.

Clearly, three registers will suffice if the quantities

(3.55) to be stored are linearly dependent. (3.55) will be linearly dependent if

$$\begin{vmatrix} 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \\ 1 & w_1 & w_2 \end{vmatrix} = 0 \qquad (3.56)$$

and will be referred to as the condition for minimum storage.

One examines the compatibility of our fourth order methods with (3.56).

S. Gill examined Method 4II together with condition (3.56) and obtained the following equation:

$$18w_3^2 - 12w_3 + 1 = 0 \qquad (3.57)$$

having roots

$$w_3 = \frac{1}{3}\left(1 \pm \frac{1}{\sqrt{2}}\right) \qquad (3.58)$$

The coefficients obtained using (3.58) for Method 4II are due to Gill and are given by Table 3.7 which follows.

Table 3.7

	Gill I	Gill II
w_1	1/6	1/6
w_2	$\frac{1}{3}(1-2^{-\frac{1}{2}})$	$\frac{1}{3}(1+2^{-\frac{1}{2}})$
w_3	$\frac{1}{3}(1+2^{-\frac{1}{2}})$	$\frac{1}{3}(1-2^{-\frac{1}{2}})$
w_4	1/6	1/6
a	1/2	1/2
b	1/2	1/2

	Gill I	Gill II
b_1	$-\frac{1}{2}+2^{-\frac{1}{2}}$	$-\frac{1}{2}-2^{-\frac{1}{2}}$
b_2	$1-2^{-\frac{1}{2}}$	$1+2^{-\frac{1}{2}}$
c	1	1
c_1	0	0
c_2	$-2^{-\frac{1}{2}}$	$2^{-\frac{1}{2}}$
c_3	$1+2^{-\frac{1}{2}}$	$1-2^{-\frac{1}{2}}$

Thus, the preceding modifications due to Gill choose intermediate points for (3.11-3.15) which minimize the number of storage registers required in the program to just three. In order to utilize the modification due to Gill, a scheme for Gill I which successively evaluates the quantities y_i, l_i, K_i i=1,2,3,4 is illustrated below. At the jth evaluation in our program, one continues the calculations as follows: going across, one has

$y_1 = y(x_j)$		$K_1=hf(x_j,y_1)$
$y_2 = y_1 + \frac{1}{2}K_1$	$l_2 = K_1$	$K_2=hf(x_j+\frac{1}{2}h,y_2)$
$y_3=y_2+(1-2^{-\frac{1}{2}})(K_2-l_2)$	$l_3=(2-2^{\frac{1}{2}})K_2+(-2+3\cdot 2^{-\frac{1}{2}})l_2$	$K_3=hf(x_j+\frac{1}{2}h,y_3)$
$y_4=y_3+(1+2^{-\frac{1}{2}})(K_3-l_3)$	$l_4=(2+2^{\frac{1}{2}})K_3+(-2-3\cdot 2^{-\frac{1}{2}})l_3$	$K_4=hf(x_j+h,y_4)$
$y_5=y_4+\frac{1}{6}K_4-\frac{1}{3}l_4 = y(x_{j+1})$		

and by replacing y_1 by y_5, then one again repeats the above calculations of y_i, l_i, and K_i. For an example of the above scheme see program 3-1.

Considering Method 4III in view of condition (3.56) one obtains no solution. Similarly, (3.56) is incompatible with Method 4IV.

Using (3.56) Gill has developed coefficients for two fourth order methods which reduce the number of storage registers required for the solution of (2.47) from (4N+A) to (3N+A). One now considers whether the number of storage registers can be reduced for fourth order methods for which the coefficients have been previously obtained. In one such case, Blum has considered coefficients 4II3 of Table 3.6 and modified the order of operations to obtain a sequence of calculations which require only 3N+A registers to solve (2.47).

The following modification due to Blum determines a saving of N storage registers by calculating the quantities p_i, q_i, r_i i=0,1,2,3, in that order. Let

$$(y_j)_N = (y_o, y_1, \ldots, y_N)$$

and define

$$(a)_N + (b)_N = (a+b)_N$$

The Blum procedure is then given horizontally by j=0,1...N

$$p_o = (y_j)_N \qquad q_o = y_j \qquad r_o = hf_j(p_o)$$

$$p_1 = p_o + (r_o/2)_N \qquad q_1 = r_o \qquad r_1 = hf_j(p_1)$$

$$p_2 = p_1 + (r_1/2 - q_1/2)_N \qquad q_2 = q_1/6 \qquad r_2 = hf_j(p_2) - r_1/2$$

$$p_3 = p_2 + (r_2)_N \qquad q_3 = q_2 - r_2 \qquad r_3 = hf_j(p_3) + 2r_2$$

$$p_4 = p_3 + (q_3 + r_3/6)_N$$

and the sequence of operations is repeated by replacing p_o by p_4. It is clear that the above process requires only 3N+A storage registers, but furthermore one now shows the above Blum modification is equivalent to the Method 4II3 of Table 3.6 page 39.

Using program notation for K_{ji} i=0,1,2,3 j=0,...N, one first notes that $K_{j0}=hf_j(p_o)=r_o$ and

$$p_1=p_o+(r_o/2)_N=(y_j + K_{j0}/2)_N$$

Furthermore

$$K_{j1}=hf_j((y_j+K_{j0}/2)_N) =hf_j(p_1)=r_1$$

and

$$q_1 = r_o = K_{j0}$$

which give

$$p_2=p_o+(r_o/2+r_1/2-q_1/2)_N=(y_j+K_{j1}/2)_N$$

and thus

$$K_{j2}=hf_j((y_j+K_{j1}/2)_N) = hf_j(p_2)$$

Also

$$r_2=K_{j2}-K_{j1}/2 \quad \text{and} \quad q_2=K_{j0}/6$$

and thus

$$p_3=(y_j+K_{j1}/2+K_{j2}-K_{j1}/2)_N=(y_j+K_{j2})_N$$

determines

$$K_{j3}=hf_j((y_j+K_{j2})_N)=hf_j(p_3)$$

from which it immediately follows that

$$r_3=K_{j3}+2(K_{j2}-K_{j1}/2)=K_{j3}+2K_{j2}-K_{j1}$$

and

$$q_3=q_2-r_2=K_{j0}/6-K_{j2}+K_{j1}/2 \quad ;$$

as a result

$$\begin{aligned}p_4&=p_3+(q_3+r_3/6)_N\\&=(y_j+K_{j2}+K_{j0}/6-K_{j2}+K_{j1}/2\\&\qquad+(K_{j3}+2K_{j2}-K_{j1})/6)_N\\&=(y_j+(K_{j0}+2K_{j1}+2K_{j2}+K_{j3})/6)_N\end{aligned}$$

which is the solution obtained by using Method 4II3 of Table 3.6 and hence verifies the equivalence. For a program incorporating the Blum modification see program 3-2 in the Appendix.

In order to compare third and fourth order Runge-Kutta methods, one uses the fourth order program 3-3 to solve the differential equation

$$\frac{dy}{dx} = y(4 - 3/2\tan(3/2)x) \text{ at } (0,1) \qquad (2.75)$$

having the analytic solution $y = e^{4x}\cos(3/2)x$. Results, using a third order method, have been obtained for (2.75) in Table 2.3; fourth order results now follow on page 46 for (2.75), given in Table 3.8.

In Table 3.8, one observes immediately the similarity of the results using 4II3 and Blum. This fact is not surprising for Blum only rearranged the computing order of Method 4II3 and used exactly the same coefficients in his numerical solution so that comparable results should be obtained.

On comparing Table 2.3 and Table 3.8 for the same increments h, one observes that the fourth order methods are more accurate but more computing time is required. However although computing time for fourth order methods exceeds that for third order methods, the reduction in error so obtained is well worth the expense of computing time. The following illustrates the point in question. Using the same

Table 3.8

	Method	y(1)	ye(1)	E(1)
0(0.25)1	4I1	51..38620	51..31352	50..72676
	4II3	51..38620	51..24901	51..13719
	Gill I	51..38620	51..26200	51..12420
	Blum	51..38620	51..24901	51..13719
0(0.2)1	4I1	51..38620	51..32893	50..57272
	4II3	51..38620	51..30551	50..80693
	Gill I	51..38620	51..30690	50..79296
	Blum	51..38620	51..30551	50..80691
0(0.125)1	4I1	51..38620	51..36575	50..20447
	4II3	51..38620	51..36407	50..22129
	Gill I	51..38620	51..36254	50..23660
	Blum	51..38620	51..36407	50..22129
0(0.1)1	4I1	51..38620	51..37551	50..10677
	4II3	51..38620	51..37523	50..10949
	Gill I	51..38620	51..37427	50..11916
	Blum	51..38620	51..37523	50..10948
0(o.05)1	4I1	51..38620	51..38522	48..96169
	4II3	51..38620	51..38527	48..91400
	Gill I	51..38620	51..38516	49..10212
	Blum	51..38620	51..38527	48..91171
0(0.04)1	4I1	51..38620	51..38580	48..40398
	4II3	51..38620	51..38582	48..38528
	Gill I	51..38620	51..38577	48..42763
	Blum	51..38620	51..38581	48..38681

increment h=0.05, one obtained

Method	Error	Time
3II2	50..53424	3 min, 26 sec
4II3	48..91400	4 min, 40 sec

However, to obtain the error of Method 3II2 using 4II3, one required a time of 2 min, 24 sec when h=0.1. On the other hand, if one uses Method 4II3 for the same computing time as for Method 3II2, namely 3 min, 26 sec, then one obtains an error of the magnitude 49..21168, an increment of h=0.0625 being used. This value of the error 49..21168 is seen to be a significant improvement over that of Method 3II2.

Other results obtained supported the above observation that for the same amount of computing time, the fourth order methods reduced the error more than using the third order numerical methods.

There is no immediate purpose in solvingother differential equations at this point; hence further examples will be given at the end of the next section.

SECTION IV

RUNGE-KUTTA METHODS WITH MINIMUM ERROR BOUNDS

One thus far has derived Runge-Kutta methods with respect to symmetry of coefficients, reduction of operations, and minimization of the number of storage registers. With rapid computers presently available, it may be argued that the reduction of operations to save time is unimportant. Furthermore, minimizing the number of storage registers may again seem insignificant as modern computers have been so manufactured to supply an indefinite number of storage locations.

As a result, one now examines another criterion in deriving Runge-Kutta numerical methods which is to obtain methods with the least error. Among the infinity of third and fourth order Runge-Kutta methods available, there must exist one unique set of coefficients for each order which minimize the truncation error as derived previously (see Section I, equations (1.72), and (1.75) page 7)

Using the above point of view , A. Ralston has obtained a set of coefficients respectively for order three and four which satisfy the requirement that a bound on the truncation errors of order three and four is minimized.

To obtain a bound on the truncation errors, we require the following notation: for a region about the numerical solution (x_n, y_n) of (1.00), define the constants M, and L, by

$$\left| f(x,y) \right| < M \tag{4.00}$$

$$\left| \frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right| < L^{i+j}/M^{j-1} \tag{4.01}$$

Then, using the expression (1.72) for the third order truncation error, and the above notation, one obtains the following bound on E for the Runge-Kutta third order methods:

$$|E| < (8|e_1| + |e_2| + |2e_2 + e_3| + |e_1 + e_3| + 2|e_3| + 2|e_4|)ML^3 \tag{4.10}$$

where

$$e_1 = 1/24 - [2(a+b) - 3ab]/36 \tag{4.11}$$

$$e_2 = 1/24 - a/12 \tag{4.12}$$

$$e_3 = 1/8 - b/6 \tag{4/13}$$

$$e_4 = 1/24 \tag{4.14}$$

Using program 4-1, one easily establishes that (4.10) will be minimized when $a=1/2$, $b=3/4$ in which case (4.10) becomes

$$|E| < 1/9ML^3 \tag{4.20}$$

and the equations of our numerical solution become

$$K_1 = hf(x,y) \tag{4.30}$$

$$K_2 = hf(x+h/2,\ y+K_1/2) \tag{4.31}$$

$$K_3 = hf(x+3h/4,\ y+3K_2/4) \tag{4.32}$$

and the increment in y is given by

$$dy = (2K_1 + 3K_2 + 4K_3)/9 \tag{4.33}$$

As a basis of comparison, the coefficient e in

$$|E| < eML^3 \tag{4.34}$$

is computed for the third order methods previously obtained. For example, using the coefficients of Method 3II, one obtains the value

$$e = 2/3$$

and hence for all numerical solutions derived from Method 3II, the error obtained is bounded by the expression

$$|E| < \frac{2}{3}ML^3$$

Similarly, using the coefficients of Method 3III, one obtains the bound on the error to be

$$|E| < \frac{1}{4}ML^3$$

In calculating the error bounds for coefficients derived from Method 3I, one uses the corresponding value of a and b in expressions (4.11-4.13) and together with (4.10) is able to evaluate e in (4.34). The values of e for two solutions of Method 3I are given. If a=1/3 and b=2/3, one obtains a solution due to Heun (Method 3I5) for which the error is given as

$$|E| < \frac{25}{108}ML^3$$

and similarly if a=1/2 and b=1, one determines the error bound

$$|E| < \frac{1}{4}ML^3$$

for a method referred to as Simpson's one-third rule (see Method 3I2, page 21).

Thus a minimum value of the bound (4.10) occurs when a=1/2 and b=3/4, and theoretically, the third order Runge-Kutta method obtained when a=1/2 and b=3/4 will give the least error. The following examples illustrate this result.

Using the best results of Table 2.3 and the results obtained using Ralston's third order coefficients, one obtains Table 4.1

Table 4.1

	Method	y(1)	ye(1)	$\lvert E(1) \rvert$
0(0.25)1	3III3	51..38620	51..48010	50..93902
	Ralst,	51..38620	51..45124	50..65040
0(0.2)1	3III3	51..38620	51..45826	50..72062
	Ralst.	51..38620	51..43947	50..53268
0(0.1)1	3III3	51..38620	51..40503	50..18849
	Ralst	51..38620	51..40186	50..15679
0(0.05)1	3III3	51..38620	51..38908	49..28957
	Ralst.	51..38620	51..38875	49..25715

From the above Table 4.1, Ralston's coefficients give the best numerical solution. The following differential equation

$$\frac{dy}{dx} = -\frac{2xy}{x^2 + 1} \qquad \text{at} \quad (0,5)$$

which exhibits an analytic solution

$$y = \frac{5}{x^2 + 1}$$

again is best solved using Ralston's coefficients as seen by Table 4.2 which follows:

Table 4.2

	Method	y(1)	ye(1)	$\lvert E(1) \rvert$
0(0.5)1	3I2	51..25000	51..25232	49..23163
	3I5	51..25000	51..25159	49..15907
	3III3	51..25000	51..24909	48..91248
	Ralst.	51..25000	51..25090	48..90256
0(0.25)1	3I2	51..25000	51..25022	48.21935
	3I5	51..25000	51..25014	48..14267
	3III3	51..25000	51..24987	48..13428
	Ralst.	51..25000	51..25006	47..59509
0(0.2)1	3L2	51..25000	51..25011	48..10605
	3I5	51..25000	51..25007	47..67520
	3III3	51..25000	51..24993	47..69427
	Ralst.	51..25000	51..25003	47..25940
0(0.125)1	3I2	51..25000	51..25002	47..24414
	3I5	51..25000	51..25001	47..14496
	3III3	51..25000	51..24998	47..17166
	Ralst.	51..25000	51..25001	46..53406
0(0.1)1	3I2	51..25000	51..25001	47..11826
	3I5	51..25000	51..25001	46..72479
	3III3	51..25000	51..24999	46..87738
	Ralst.	51..25000	51..25000	46..30518

It may be remarked that the preceding examples have been chosen to illustrate favourably the derived results. This is indeed so; that the method is not infallible is seen by the results of Table 4.3 for the differential equation

$$\frac{dy}{dx} = y\tan x + 2e^{x} \quad \text{at} \quad (0,0)$$

having an analytic solution

$$y = e^{x}(1 + \tan x) - \sec x$$

From Table 4.3, it is obvious that the Method 3I2 yields less error than that using Ralston's third order method. However, although in some cases, as for example the preceding differential equation, it may appear that Ralston's method is not the best one, it must be said that in solving a differential equation for which the analytical solution is unknown, one would rather use a method which theoretically yields the smallest error rather than some other numerical method.

Continuing with Ralston's third order method, one would predict that a method which works best for first order differential equations will also work best for systems of first order differential equations. As a result, a number of such systems were considered and the results obtained were favourable. One such example is presented here. Program 4-2 was used to obtain the results. The second order equation considered, which is easily written as a system of first order

Table 4.3

	Method	y(1)	ye(1)	|E(1)|
0(0.25)1	3I2	51..51009	51..51079	48..69580
	3I5	51..51009	51..50776	49..23346
	3III3	51..51009	51..50879	49..13023
	3II4	51..51009	51..50559	49..45036
	Ralst.	51..51009	51..50884	49..12543
0(0.2)1	3I2	51..51009	51..51048	48..38300
	3I5	51..51009	51..50881	49..12894
	3III3	51..51009	51..50939	48..70496
	3II4	51..51009	51..50758	49..25154
	Ralst.	51..51009	51..50941	48..68054
0(0.125)1	3I2	51..51009	51..51020	48..10452
	3I5	51..51009	51..50974	48..35095
	3III3	51..51009	51..50991	48..18539
	3II4	51..51009	51..50939	48..69962
	Ralst.	51..51009	51..50991	48..18005
0(0.1)1	3I2	51..51009	51..51015	47..52643
	3I5	51..51009	51..50991	48..18845
	3III3	51..51009	51..51000	47..99182
	3II4	51..51009	51..50972	48..37613
	Ralst.	51..51009	51..51000	47..96893
0(0.05)1	3I2	51..51009	51..51010	46..45776
	3I5	51..51009	51..51007	47..25940
	3III3	51..51009	51..51008	47..14496
	3II4	51..51009	51..51005	47..51880
	Ralst.	51..51009	51..51008	47..13733

differential equations, was

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 10e^{-3x}$$

satisfying the initial conditions

$$x = 0,\ y = 4,\ \frac{dy}{dx} = 0$$

and having the analytic solution

$$y = e^{-2x}(13\sin x - \cos x) + 5e^{-3x} \qquad ;$$

Although for h=0.9 in Table 4.4 for the above differential equation Ralston's method may not seem the best, its superiority becomes evident as h is decreased.

Table 4.4 now follows, after which the truncation error for Runge-Kutta methods of order four will be considered.

Table 4.4

	Method	y(3.6)	ye(3.6)	|E(3.6)|
0(0.9)3.6	3I2	-48..35234	-50..71700	50..71348
	3II4	-48..35234	-50..71648	50..71296
	3III4	-48..35234	-50..71649	50..71297
	Ralst.	-48..35234	-50..79608	50..79255
0(0.6)3.6	3I2	-48..35234	-49..29099	49..25576
	3II4	-48..35234	-49..29449	49..25926
	3III4	-48..35234	-49..29449	49..25926
	Ralst.	-48..35234	-49..29334	49..25811
0(0.4)3.6	3I2	-48..35234	-48..74432	48..39198
	3II4	-48..35234	-48..75515	48..40280
	3III4	-48..35234	-48..75516	48..40281
	Ralst.	-48..35234	-48..72851	48..37616
0(0.3)3.6	3I2	-48..35234	-48..48562	48..13427
	3II4	-48..35234	-48..49046	48..13812
	3III4	-48..35234	-48..49046	48..13812
	Ralst.	-48..35234	-48..48045	48..12811
0(0.2)3.6	3I2	-48..35234	-48..38574	47..33391
	3II4	-48..35234	-48..38670	47..34355
	3III4	-48..35234	-48..38671	47..34363
	Ralst.	-48..35234	-48..38417	47..31824

For the fourth order methods of Section III, one utilizes the notation (4.00-4.01) and laboriously derives a bound on the fourth order truncation error (1.75) as

$$|E| < (16|e_1| + 4|e_2| + |e_2+3e_3| + |2e_2+3e_3| + |e_2+e_3| + |e_3| + 8|e_4| + |e_5| + |2e_5+e_7| + |e_5+e_6+e_7| + |e_6| + |2e_6+e_7| + |e_7| + 2|e_8|)ML^4 \qquad (4.40)$$

where

$$e_1 = [(a^3-a^4)w_2 + (b^3-b^4)w_3]/24 - 1/480 \qquad (4.41)$$

$$e_2 = ab_2w_3(1-b^2)/2 - 1/30 \qquad (4.42)$$

$$e_3 = 1/120 - [a^3b_2w_3 + (a^3c_2+b^3c_3)w_4]/6 \qquad (4.43)$$

$$e_4 = a^2b_2w_3(1-b)/2 - 1/120 \qquad (4.44)$$

$$e_5 = 1/120 - a/48 \qquad (4.45)$$

$$e_6 = 1/40 - [a^2b_2^2w_3 + (ac_2+bc_3)^2w_4]/2 \qquad (4.46)$$

$$e_7 = 7/120 - (1+b)/24 \qquad (4.47)$$

$$e_8 = 1/120 \qquad (4.48)$$

Using program 4-3, one determines that the values

$$a = 0.4 \qquad b = 0.45574$$

will minimize the bound (4.40) on the fourth order truncation error and this bound will be given by

$$|E| < 5.46 \times 10^{-2} ML^4$$

By using inturn the coefficients 4II, 4III, 4IV in (4.40) one obtains respectively the values $w_3=5/3$, $w_4=10/51$, and $w_3=-5/78$ which will minimize the bound (4.40) on the truncation error. The bounds for Methods 4II, 4III, 4IV are given respectively by

$$|E| < 7.22\times10^{-2}ML^4$$

$$|E| < 19.72\times10^{-2}ML^4$$

$$|E| < 17.64\times10^{-2}ML^4 \quad ;$$

as a result, the best bound on the fourth order truncation error occurs for Method 4I.

When a=0.4 and b=0.45574, the Runge-Kutta fourth order equations will be given by

$$K_1 = hf(x,y)$$
$$K_2 = hf(x + 0.4h,\ y + 0.4K_1)$$
$$K_3 = hf(x + 0.45574h,\ y+0.29698K_1+0.15876K_2)$$
$$K_4 = hf(x + h,\ y+0.21810K_1-3.0509K_2+3.8329K_3)$$
$$dy = 0.17476K_1-0.55148K_2+1.2055K_3+0.17118K_4$$

and will be denoted as the Ralston I method. Before illustrating the method numerically, one may wish to compute the value of e defined by

$$|E| < eML^4$$

for fourth order coefficients that have been obtained previously. For example, if one uses (4.40) and coefficients 4I1 and 4I2 of Table 3.6, one obtains respectively the error bounds

$$|E| < 9.91\times10^{-2}ML^4$$

and

$$|E| < 11.93\times10^{-2}ML^4 \quad .$$

If on the other hand, one uses the coefficients Gill I on page 41, then the error bound becomes

$$|E| < 8.83\times10^{-2}ML^4$$

Recalling the coefficients for the relationship $a+b=1$ (page 29), one uses them in (4.40) and obtains a numerical method denoted as Ralston II having rather simple coefficients. When $a=2/5$, (4.40) is a minimum and the equations for the Ralston II method are given by

$$K_1 = hf(x,y)$$
$$K_2 = hf(x + 2/5h,\ y + 2/5K_1)$$
$$K_3 = hf(x + 3/5h,\ y - 3/20K_1 + 3/4K_2)$$
$$K_4 = hf(x + h,\ y + 19/44K_1 - 15/44K_2 + 10/11K_3)$$
$$dy = (11K_1 + 25K_2 + 25K_3 + 11K_4)/72$$

for which the error bound is given by

$$|E| < 7.70 \times 10^{-2} ML^4$$

Numerical examples now follow. Program 3-3 was used to obtain the results.

For the differential equation of Table 3.8, one obtains the following values using Ralston I coefficients.

Table 4.5

	y(1)	ye(1)	|E(1)|
0(0.25)1	51..38620	51..33559	50..50606
0(0.2)1	51..38620	51..35761	50..28590
0(0.125)1	51..38620	51..37732	49..86841
0(0.1)1	51..38620	51..38162	49..45670
0(0.05)1	51..38620	51..38574	48..43983
0(0.04)1	51..38620	51..38601	48..19035

Comparing Tables 3.8, and 4.5, one notes that Ralston I method has the smallest error. Another example which is favourable is the differential equation

$$\frac{dy}{dx} = \frac{4y}{1 + x} \quad \text{at } (0,1) \tag{4.49}$$

having an analytic solution $y = (1 + x)^4$. The results for the differential equation are given in Table 4.6

On examining Table 4.6, one again notes the similarity of results for Method 4II3 and Blum (see 45th page). Furthermore, using Ralston I coefficients, one obtains, as desired, the least error.

As mentioned beforehand, one choses examples to best illustrate the theory. However, for the examples computed using fourth order numerical methods, the majority of the differential equations indicated the least error when Ralston I coefficients were used. Thus in solving a differential equation of which no analytic solution is known, one would clearly use a method which theoretically minimizes the error. Furthermore when fourth order methods were applied to systems of first order differential equations, Ralston's method was favourable in that the least error was obtained.

Table 4.6 now follows for the differential equation (4.49).

Table 4.6

	Method	y(1)	ye(1)	| E(1)|
0(0.25)1	4I1	52..160000	52..15939	49..60928
	4II3	52..16000	52..15937	49..63003
	Blum	52..16000	52..15937	49..63034
	Gill	52..16000	52..15937	49..63080
	Ralst I	52..16000	52..15946	49..53955
	Ralst II	52..16000	52..15938	49..62271
0(0.2)1	4I1	52..16000	52..15972	49..28717
	4II3	52..16000	52..15971	49..29709
	Blum	52..16000	52..15971	49..29694
	Gill	52..16000	52..15994	49..29861
	Ralst I	52..16000	52..15975	49..25314
	Ralst II	52..16000	52..15971	49..29358
0(0.1)1	4I1	52..16000	52..15995	48..55084
	4II3	52..16000	52..15995	48..57068
	Blum	52..16000	52..15995	48..57068
	Gill	52..16000	52..15994	48..58289
	Ralst I	52..16000	52..15995	48..47455
	Ralst II	52..16000	52..15995	48..56458
0(0.05)1	4I1	52..16000	52..15998	48..26550
	4II3	52..16000	52..15998	48..27618
	Blum	52..16000	52..15998	48..28229
	Gill	52..16000	52..15998	48..27855
	Ralst I	52..16000	52..15998	48..22736
	Ralst II	52..16000	52..15998	48..27161

APPENDIX

PROGRAM 2-1

```
1.  TITLE  GRAPH POLY DEG 6
2.  BEGIN
3.  A1: CARR(1)
4.  A=KEYBD
5.  B=KEYBD
6.  C=KEYBD
7   D=KEYBD
8.  E=KEYBD
9.  F=KEYBD
10. G=KEYBD
11. CARR(1)
12. X1=KEYBD
13. H=KEYBD
14. X2=KEYBD
15. CARR(2)
16  FOR X=X1(H)X2 BEGIN
17. PRINT(FL)=X
18. YE=A*(ABS X)↑6+B*(ABS X)↑4*X
      +C*(ABS X)↑4+D*(ABS X)↑2*X
      +E*(ABS X)↑2+F*X+G
19. PRINT(FL)=YE
20. CARR(1) END
21. BELLS(1)
22. GO TO A1
23. END
```

PROGRAM 2-2

```
1.  TITLE ROOT POLY DEG 6
2.  LIBRARY SIN(0101000)
            COS(0168000)
3.  FUNCTION (AA,BB,CC,DD,EE,FF,
                GG,XX=KK)
4.  BEGIN
5.  KK=AA*(ABS XX)↑6+BB*(ABS XX)↑4*XX
      +CC*(ABS XX)↑4+DD*(ABS XX)↑2*XX
      +EE*(ABS XX)↑2+FF*XX+GG
6.  RETURN
7.  END
8.  BEGIN
9.  CARR(1)
10. A=KEYBD
11. B=KEYBD
12, C=KEYBD
13. D=KEYBD
14. E=KEYBD
15. F=KEYBD
16. G=KEYBD
17. CARR(1)
18. START: X1=KEYBD
19. X2=KEYBD
20. CARR(1)
21. CALC: FF(A,B,C,D,E,F,G,X1=FX1)
22. PRINT(FL)=FX1
23. FF (A,B,C,D,E,F,G,X2=FX2)
24. PRINT(FL)=FX2
25. R=(X1*FX2-X2*FX1)/(FX2-FX1)
26. PRINT(FL)=R
27. I=KEYBD
28. CARR(1)
29. IF I=0 BEGIN
30. X1=R
3L. GO TO CALC END
32. GO TO START
33. END
```

PROGRAM 2-3

```
TITLE RUNGE KUTTA 3RD ORDER ITERATIVE
LIBRARY SIN (0101000), COS (0168000), ARCTN (0164000)
DATA A(9,9), XX(1), K(1)
SUBSCRIPTS (I,J), M
FUNCTION FF (HH,XX,YY=KK)
BEGIN
KK=HH*(XX+YY)
RETURN
END
BEGIN
CARR(1)
N=KEYBD
X1=KEYBD
X2=KEYBD
Y1=KEYBD
CARR(1)
Y=Y1
NN=N-1
NP=NN*N
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
STOP
READ(P)XX
A[I,J]=XX[0] END END
CARR(1)
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
IF A[I,J]=0 BEGIN
GO TO FINISH END
H=A[I,J]
PRINT(FL)=H
X3=X2-H
FOR X=X1(H)X3 BEGIN
XV=X
YV=Y
FF (H,XV,YV=K[0])
XV=X+0.89255*H
YV=Y+0.89255*K[0]
FF (H,XV,YV=K[0])
DY=0.35098*K[0]
XV=X+0.28871*H
YV=Y+0.28871*K[0]
FF (H,XV,YV=K[0])
DY=DY+0.64902*K[0]
Y=Y+DY END
PRINT(FL)=X
PRINT(FL)=Y
YE=2*EXP X-X-1
PRINT(FL)=YE
YET=Y-YE
PRINT(FL)=YET
Y=Y1
CARR(3) END END
FINISH: BELLS(2)
END
```

PROGRAM 2-4

```
TITLE CONTE REEVES 3RD ORDER 2N+A
LIBRARY SIN (0101000), COS (0168000), ARCTN (0168000)
DATA A(9,9), XX(1), K(1)
SUBSCRIPTS (I,J), M
FUNCTION FF (HH,XX,YY=KK)
BEGIN
KK=HH*(XX+YY)
RETURN
END
BEGIN
CARR(1)
N=KEYBD
X1=KEYBD
X2=KEYBD
Y1=KEYBD
CARR(1)
Y=Y1
NN=N-1
NP=NN*N
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
STOP
READ(P)XX
A[I,J]=XX[0] END END
CARR(1)
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
IF A[I,J]=0 BEGIN
GO TO FINISH END
H=A[I,J]
PRINT(FL)=H
X3=X2-H
FOR X=X1(H)X3 BEGIN
XV=X
YV=Y
FF (H,XV,YV=K[0])
XV=X+0.62654*H
YV=Y+0.62654*K[0]
FF (H,XV,YV=K[0])
Y=YV
XV=X+0.075426*H
YV=Y-0.55111*K[0]
Y=Y+0.85614*K[0]
FF (H,XV,YV=K[0])
Y=Y-0.48268*K[0]  END
PRINT(FL)=X
PRINT(FL)=Y
YE=2*EXP X-X-1
PRINT(FL)=YE
YET=Y-YE
PRINT(FL)=YET
Y=Y1
CARR(3) END END
FINISH: BELLS(2)
END
```

PROGRAM 2-5

```
TITLE RUNGE KUTTA THIRD ORDER GENERAL READ
LIBRARY SIN (0101000), COS (0168000), ARCTN (0164000)
DATA P(5), PP(5), S(4), A(9,9), Q(5), K(3), AA(5), BB(5), CC(4),
     XX(1)
SUBSCRIPTS M, (I,J)
FUNCTION FF (HH,XX,YY=KK)
BEGIN
KK=HH*YY*(4-1.5*SIN (1.5*XX)/COS (1.5*XX))
RETURN
END
BEGIN
CARR(1)
N=KEYBD
X1=KEYBD
X2=KEYBD
Y=KEYBD
CARR(1)
RR=0
SS=KEYBD
TT=KEYBD
CARR(1)
NN=N-1
NP=NN*N
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
STOP
TABS(1)
READ(P)XX
A[I,J]=XX[0] END END
BELLS(2)
STOP
CARR(1)
A1:READ(P)AA
READ(P)BB
READ(P)CC
FOR M=0(1)4 BEGIN
P[M]=AA[M]
PRINT(FL)=P[M] END
CARR(1)
FOR M=0(1)4 BEGIN
PP[M]=BB[M]
PRINT(FL)=PP[M] END
CARR(1)
FOR M=0(1)3 BEGIN
S[M]=CC[M]
PRINT(FL)=S[M] END
CARR(3)
FOR M=0(1)4
Q[M]=P[M]/PP[M]
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
IF A[I,J]=0 BEGIN
GO TO FINISH END
H=A[I,J]
PRINT(FL)=H
X3=X2-H
FOR X=X1(H)X3 BEGIN
FF (H,X,Y=K[0])
XV=X+Q[1]*H
YV=Y+Q[1]*K[0]
FF (H,XV,YV=K[1])
XV=X+Q[2]*H
YV=Y+Q[3]*K[0]+Q[4]*K[1]
FF (H,XV,YV=K[2])
T=0
FOR M=0(1)2
T=T+S[M]*K[M]
DY=T/S[3]
Y=Y+DY END
PRINT(FL)=X
YE=EXP (4*X)*COS (1.5*X)
PRINT(FL)=YE
PRINT(FL)=Y
YET=YE-Y
PRINT(FL)=YET
Y=1
IF TT=10 BEGIN
STOP END
CARR(3) END END
FINISH: RR=RR+1
CARR(5)
IF RR<SS BEGIN
CARR(3)
GO TO A1 END
BELLS(2)
END
```

PROGRAM 3-1

```
TITLE GILL I FOURTH ORDER
LIBRARY SIN (0101000), COS (0168000), ARCTN (0164000)
DATA A(9,9), XX(1)
SUBSCRIPTS (I,J), M
FUNCTION FF (HH,XX,YY=KK)
BEGIN
KK=HH*(XX+YY)
RETURN
END
BEGIN
CARR(1)
N=KEYBD
X1=KEYBD
X2=KEYBD
Y1=KEYBD
TT=KEYBD
CARR(1)
Y=Y1
NN=N-1
NP=NN*N
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
STOP
CARR(1)
READ(P)XX
A[I,J]=XX[0] END END
CARR(1)
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
IF A[I,J]=0 BEGIN
GO TO FINISH END
H=A[I,J]
PRINT(FL)=H
X3=X2-H
FOR X=X1(H)X3 BEGIN
XXX=X
YY=Y
FF (H,XXX,YY=KK)
XXX=X+H/2
YY=YY+KK/2
QQ=KK
FF (H,XXX,YY=KK)
XXX=X+H/2
YY=YY+(1-1/SQRT 2)*(KK-QQ)
QQ=(2-SQRT 2)*KK+(3/SQRT 2-2)*QQ
FF (H,XXX,YY=KK)
XXX=X+H
YY=YY+(1+1/SQRT 2)*(KK-QQ)
QQ=(2+SQRT 2)*KK+(-2-3/SQRT 2)*QQ
FF(H,XXX,YY=KK)
Y=YY+KK/6-QQ/3 END
PRINT(FL)=X
PRINT(FL)=Y
YE=2*EXP X-X-1
PRINT(FL)=YE
YET=YE-Y
PRINT(FL)=YET
Y=Y1
IF TT=10 BEGIN
STOP END
CARR(3) END END
FINISH: BELLS(5)
END
```

PROGRAM 3-2

```
TITLE BLUM MODIFICATION ORDER FOUR
LIBRARY SIN (0101000), COS (0168000), ARCTN (0164000)
DATA A(9,9), XX(1)
SUBSCRIPTS (I,J),M
FUNCTION FF (HH,XX,YY=KK)
BEGIN
KK=HH*(XX+YY)
RETURN
END
BEGIN
CARR(1)
N=KEYBD
X1=KEYBD
X2=KEYBD
Y1=KEYBD
TT=KEYBD
CARR(1)
F0=1
Y=Y1
NN=N-1
NP=NN*N
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
STOP
TABS(1)
READ(P)XX
A[I,J]=XX[0] END END
CARR(1)
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
IF A[I,J]=0 BEGIN
GO TO FINISH END
H=A[I,J]
PRINT(FL)=H
X3=X2-H
FOR X=X1(H)X3 BEGIN
AX=X
AY=Y
FF (H,AX,AY=VV)
BX=X
BY=Y
CX=H*F0
CY=VV
AX=AX+CX/2
AY=AY+CY/2
FF (H,AX,AY=VV)
BX=CX
BY=CY
CX=H*F0
CY=VV
AX=AX+CX/2-BX/2
AY=AY+CY/2-BY/2
FF (H,AX,AY=VV)
BX=BX/6
BY=BY/6
CX=H*F0-CX/2
CY=VV-CY/2
AX=AX+CX
AY=AY+CY
FF (H,AX,AY=VV)
BX=BX-CX
BY=BY-CY
CX=H*F0+2*CX
CY=VV+2*CY
Y=AY+BY+CY/6 END
PRINT(FL)=X
PRINT(FL)=Y
YE=2*EXP X-X-1
PRINT(FL)=YE
YET=Y-YE
PRINT(FL)=YET
IF TT=10 BEGIN
STOP END
Y=Y1
CARR(3) END END
FINISH: BELLS(5)
END
```

PROGRAM 3-3


```
TITLE RUNGE KUTTA ORDER FOUR GENERAL READ
LIBRARY SIN (0101000), COS (0168000), ARCTN (0164000)
DATA P(9),PP(9),S(5),A(9,9),Q(9),K(4),AA(9),BB(9),CC(5),XX(1)
SUBSCRIPTS (I,J),M
FUNCTION FF (HH,XX,YY=KK)
BEGIN
KK=HH*YY*(4-1.5*SIN (1.5*XX)/COS (1.5*XX))
RETURN
END
BEGIN
CARR(1)
N=KEYBD
X1=KEYBD
X2=KEYBD
Y=KEYBD
CARR(1)
RR=0
SS=KEYBD
TT=KEYBD
CARR(1)
NN=N-1
NP=NN*N
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
STOP
TABS(1)
READ(P)XX
A[I,J]=XX[0] END END
BELLS(2)
STOP
CARR(1)
A1:READ(P)AA
READ(P)BB
READ(P)CC
FOR M=0(1)8 BEGIN
P[M]=AA[M]
PRINT(FL)=P[M] END
CARR(1)
FOR M=0(1)8 BEGIN
PP[M]=BB[M]
PRINT(FL)=PP[M] END
CARR(1)
FOR M=0(1)4 BEGIN
S[M]=CC[M]
PRINT(FL)=S[M] END
CARR(3)
FOR M=0(1)8
Q[M]=P[M]/PP[M]
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
IF A[I,J]=0 BEGIN
GO TO FINISH END
H=A[I,J]
PRINT(FL)=H
X3=X2-H
FOR X=X1(H)X3 BEGIN
FF (H,X,Y=K[0])
XV=X+Q[1]*H
YV=Y+Q[1]*K[0]
FF (H,XV,YV=K[1])
XV=X+Q[2]*H
YV=Y+Q[3]*K[0]+Q[4]*K[1]
FF (H,XV,YV=K[2])
XV=X+Q[5]*H
YV=Y+Q[6]*K[0]+Q[7]*K[1]+Q[8]*K[2]
FF (H,XV,YV=K[3])
T=0
FOR M=0(1)3
T=T+S[M]*K[M]
DY=T/S[4]
Y=Y+DY END
PRINT(FL)=X
PRINT(FL)=Y
YE=EXP (4*X)*COS (1.5*X)
PRINT(FL)=YE
YET=Y-YE
PRINT(FL)=YET
Y=1
IF TT=10 BEGIN
STOP END
CARR(3) END END
FINISH: RR=RR+1
CARR(5)
IF RR<SS BEGIN
CARR(3)
GO TO A1 END
BELLS(2)
END
```

PROGRAM 4-1

```
1.  TITLE RALSTON COEFFICIENTS ORDER 3
2.  BEGIN
3.  X1=KEYBD
4.  H=KEYBD
5.  X2=KEYBD
6.  XX1=KEYBD
7.  HH=KEYBD
8.  XX2=KEYBD
9.  CARR(1)
10. FOR A=X1(H)X2 BEGIN
11. FOR B=XX1(HH)XX2 BEGIN
12. PRINT(FL)=A
13. PRINT(FL)=B
14. FNL=ABS (1/3-2*(2*A+2*B-3*A*B)/9)
          +ABS (1/6-B/3)
          +ABS (.5-2*A/3)
15. PRINT(FL)=FNL
16. CARR(2) END END
17. END
```

PROGRAM 4-2

```
1.  TITLE RK THIRD ORDER SYSTEM
2.  LIBRARY SIN (0101000),
            COS (0168000),
          ARCTN (0164000)
3.  DATA P(5),PP(5),S(4),A(9,9),XX(1)
         Q(5),K(3),AA(5),BB(5),CC(5),
4.  SUBSCRIPTS M,(I,J)
5.  FUNCTION FF(H,X,Y,Z=K)
6.  BEGIN
7.  K=H*Z
8.  RETURN
9.  END
10. FUNCTION GG (H,X,Y,Z=K)
11. BEGIN
12. K=H*(10*EXP (-3*X)-5*Y-4*Z)
13. RETURN
14. END
15. BEGIN
16. CARR(1)
17. H=KEYBD
18. X1=KEYBD
19. X2=KEYBD
20. Y=KEYBD
21. Z=KEYBD
22. CARR(1)
23. RR=0
24. SS=KEYBD
25. CARR(1)
26. NN=N-1
27. NP=NN*N
28. FOR I=0(1)NN BEGIN
29. FOR J=0(N)NP BEGIN
30. STOP
31. TABS(1)
32. READ(P)XX
33. A[I,J]=XX[0] END END
34. BELLS(2)
35. STOP
36. CARR(1)
37. A1:READ(P)AA
```

```
READ(P)BB
READ(P)CC
FOR M=0(1)4 BEGIN
P[M]=AA[M]
PRINT(FL)=P[M] END
CARR(1)
FOR M=0(1)4 BEGIN
PP[M]=SS[M]
PRINT(FL)=PP[M] END
CARR(1)
FOR M=0(1)3 BEGIN
S[M]=CC[M]
PRINT(FL)=S[M] END
CARR(3)
FOR M=0(1)4
Q[M]=P[M]/PP[M]
FOR I=0(1)NN BEGIN
FOR J=0(N)NP BEGIN
IF A[I,J]=0 BEGIN
GO TO FINISH END
H=A[I,J]
PRINT(FL)=H
X3=X2-H
FOR X=X1(H)X3 BEGIN
FF (H,X,Y,Z=K[0])
GG (H,X,Y,Z=KK[0])
XV=X+Q[1]*H
YV=Y+Q[1]*K[0]
ZV=Z+Q[1]*KK[0]
FF (H,XV,YV,ZV=K[1])
GG (H,XV,YV,ZV=KK[1])
XV=X+Q[2]*H
YV=Y+Q[3]*K[0]+Q[4]*K[1]
ZV=Z+Q[3]*KK[0]+Q[4]*KK[1]
FF (H,XV,YV,ZV=K[2])
GG (H,XV,YV,ZV=KK[2])
T=0
FOR M=0(1)2
T=T+S[M]*K[M]
DY=T/S[3]
TT=0
FOR M=0(1)2
TT=TT+S[M]*KK[M]
DZ=TT/S[3]
Y=Y+DY
Z=Z+DZ END
PRINT(FL)=X
PRINT(FL)=Y
YE=EXP (-2*X)*(13*SIN X-COS X)+5*EXP (-3*X)
PRINT(FL)=YE
YET=Y-YE
PRINT(FL)=YET
Y=4
Z=0
CARR(3) END END
FINISH:RR=RR+1
CARR(5)
IF RR<SS BEGIN
CARR(3)
GO TO A1 END
BELLS(2)
END
```

PROGRAM 4-3


```
TITLE RALSTON COEFFICIENTS ORDER 4
BEGIN
X1=KEYBD
H=KEYBD
X2=KEYBD
XX1=KEYBD
HH=KEYBD
XX2=KEYBD
CARR(1)
T1=KEYBD
T2=KEYBD
FOR A=X1(H)X2 BEBIN
FOR B=XX1(HH)XX2 BEGIN
PRINT(FL)=A
PRINT(FL)=B
W1=0.5+(1-2*(A+B))/(12*A*B)
W2=(2*B-1)/(12*A*(B-A)*(1-A))
W3=(2*A-1)/(12*B*(A-B)*(1-B))
W4=0.5+(2*(A+B)-3)/(12*(1-A)*(1-B))
B2=B*(B-A)/(2*A*(1-2*A))
C2=(1-A)*(A+5*B-2-4*B↑2)/(2*A*(B-A)*(6*A*B-4*(A+B)+3))
C3=(2*A-1)*91-A)*(1-B)/(B*(A-B)*(6*A*B-4*(A+B)+3))
IF T1=1 BEGIN
B1=B-B2
C1=1-C2-C3
CARR(1)
PRINT(FL)=W1
PRINT(FL)=W2
PRINT(FL)=W3
PRINT(FL)=W4
PRINT(FL)=B1
PRINT(FL)=B2
CARR(1)
PRINT(FL)=C1
PRINT(FL)=C2
PRINT(FL)=C3
CARR(1) END
IF T2=2 BEGIN
E1=((A-A↑3)*W2+(B-B↑3)*W3)/24-1/80
E2=A*B2*W3*(1-B↑2)/2-1/30
E3=1/120-(A↑3*B2*W3+(A↑3*C2+B↑3*C3)*W4)/6
E4=A↑2*B2*W3*(1-B)/2-1/120
E5=1/120-A/48
E6=1/40-(A↑2*B2↑2*W3+(A*C2+B*C3)↑2*W4)/2
E7=7/120-(1+B)/24
E8=1/120
EE=16*ABS E1+4*ABS E2+ABS (E2+3*E3)+ABS (2*E2+3*E3)+ABS (E2+E3)
   +ABS E3+8*ABS E4+ABS E5+ABS (2*E5+E7)+ABS (E5+E6+E7)+ABS E6
   +ABS (2*E6+E7)+ABS E7+ABS E8*2
PRINT(FL)=EE
CARR(1) END END END
END
```

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