Toric Ideals of Finite Simple Graphs

Toric Ideals of Finite Simple Graphs

By Graham Keiper, B.Sc., M.Sc.

A Thesis Submitted to the School of Graduate Studies in the Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy

McMaster University © Copyright by G. Keiper August 22, 2022

McMaster University
Ph.D. Thesis (2022)
Hamilton, Ontario (Mathematics and Statistics)

TITLE: Toric Ideals of Finite Simple Graphs
AUTHOR: G. Keiper (McMaster University)
SUPERVISOR: Dr. Adam Van Tuyl
NUMBER OF PAGES: viii, 86

Abstract

This thesis deals with toric ideals associated with finite simple graphs. In particular we establish some results pertaining to the nature of the generators and syzygies of toric ideals associated with finite simple graphs.

The first result dealt with in this thesis expands upon work by Favacchio, Hofscheier, Keiper, and Van Tuyl [13] which states that for $G = H_1 \sqcup_\varphi H_2$, a graph obtained by gluing H_1 to H_2 along an induced subgraph, we can obtain I_G from I_{H_1} and I_{H_2} as follows: $I_G = (I_{H_1} + I_{H_2} : f^\infty)$ for a particular monomial f. Our contribution is to sharpen the result and show that f^∞ can be replaced by f^2.

The second result treated by this thesis pertains to graded Betti numbers of toric ideals of complete bipartite graphs. We show that by counting specific subgraphs one can explicitly compute a minimal set of generators for the ideal $I_{K_{n,m}}$ as well as minimal generating sets for the first two syzygy modules. Additionally we provide formulas for the graded Betti numbers $\beta_{i,j}(R/I_{K_{n,m}})$ where $i = 1, 2, 3$.

The final topic treated pertains to a relationship between the fundamental group the finite simple graph G, $\pi_1(G)$, and the associated toric ideal I_G. It was shown by Villareal [37] as well as Hibi and Ohsugi [26] that the generators of I_G correspond to the closed even walks of G thus linking algebraic properties to combinatorial ones. Therefore it is a natural question whether there is a relationship between the toric ideal I_G and the fundamental group $\pi_1(G)$ of the graph. We show, under the assumption that G is a bipartite graph with some additional assumptions, one can conceive of the set of binomials in I_G with coprime terms, $\mathcal{B}(I_G)$, as a group with an appropriately chosen operation $\star$ and establish a group isomorphism $(\mathcal{B}(I_G), \star) \cong \pi_1(G)/H$ where H is a normal subgroup. We exploit this relationship further to obtain information about the generators of I_G as well as bounds on the Betti numbers. We are also able to characterise all regular sequences and hence compute the depth of I_G. We also use the framework to prove that $I_G = (\langle \mathcal{G} \rangle : (e_1 \cdots e_m)^\infty)$ where $\mathcal{G}$ is a set of binomials which correspond to a generating set of $\pi_1(G)$.

Acknowledgements

This thesis is very far from a result of my unaided efforts. I owe a debt of gratitude to the many people who supported me throughout the process and without whom I certainly would not have succeeded.

First off I would like to thank my family for their support and love throughout my studies, in particular my father and mother, Roland Keiper and Lynda Quinn. I would like to thank my loving girlfriend Julia Bahns and her entire family in Germany who have treated me as one of their own. I would also like to thank Father Don Aitchison and David Ingram for mentoring me during my time at TCS. From Western University I would like to thank Matthias Franz as well as Jan Minac for supporting and encouraging me to continue my studies in mathematics.

More immediately, I would like to thank Tim Römer and the University of Osnabrück for hosting me during spring and summer of 2020, as well as for helpful discussions about problems related to this thesis. I would like to also thank Dinh Van Le and Alexandros Grodos for welcoming me and working with me. I would like to thank my collaborators Huy Tài Hà, Jonathan O'Rourke, Hasan Mahmood, Johannes Hofscheier, Giuseppe Favacchio, Federico Galetto, Miguel Eduardo Uribe Paczka, and Craig Kohne. To a large extent our joint work helped to form the direction as well as content of this thesis. From McMaster I would like to thank Andrew Nicas and Hans Boden for helpful discussions on how to connect concepts from commutative algebra to those of algebraic topology. I would also like to thank Antoine Deza for helpful feedback and guidance during committee meetings.

Most importantly I would like to my supervisor Adam Van Tuyl. It is no exaggeration to say that without him this thesis could not have been written. Adam you are exemplary as a mathematician as well as a human being. A student couldn't ask for a better advisor. I thank you for your patience with me, for your readiness to help with whatever issues arose during my five years with you.

Contents

Declaration of Authorship

I, G. Keiper, declare that this thesis titled, “Toric Ideals of Finite Simple Graphs ” and the work presented in it are my own. I confirm that:

- Chapter 1: Introduction
  - This chapter is entirely my own work
  - The purpose of this chapter is to place the thesis within the context of existing literature as well as state the main results of the thesis
- Chapter 2: Background
  - This chapter is my own work
  - The chapter consists of existing definitions as well as the results of others which are needed for the thesis
- Chapter 3: Improving a Result on The Splitting of Toric Ideals
  - This chapter is based upon joint work with Giuseppe Favacchio, Johannes Hofscheier, and Adam Van Tuyl [13]
  - Theorem 3.2.1 is my own contribution which extends previous work which is the purpose of this chapter
- Chapter 4: Syzygies of Toric Ideals of Complete Bipartite Graphs
  - This chapter is entirely my own work
  - In this chapter I provide novel results pertaining to the syzygies of $I_{K_{n,m}}$
- Chapter 5: Fundamental Group Background
  - This chapter is entirely my own work
  - The chapter is a collection and summary of existing results needed to understand Chapters 6 and 7
- Chapter 6: Fundamental Group and Toric Ideals
  - This chapter is entirely my own work
  - The chapter introduces some binary operations on binomials belonging to a toric ideal and establishes a link between the fundamental group of a graph and its toric ideal
- Chapter 7: Fundamental Group Applications
  - This chapter is entirely my own work

  - The chapter works out some of the consequences of the relationship between the toric ideal associated with finite simples graphs and the fundamental groups of such graphs

- Chapter 8: Conclusion
  - This chapter is entirely my own work

Chapter 1

Introduction

1.1 Basic Problem and A Motivating Example

Toric ideals appear in a surprisingly diverse number of areas ranging from algebraic geometry, where their study began [8], to algebraic statistics [10]. Geometrically toric ideals can be thought of as the ideals which correspond to varieties which have dense algebraic torus actions from which they get their name. Toric ideals (and hence toric varieties) can be approached in many different ways. They are of particular interest because they have features which allow us to exploit associated combinatorial structures to calculate important invariants. In fact they are often introduced in terms of discrete geometry (for example [34]). Given the fact that they are easier to work with than more general classes of varieties and ideals, it is hardly surprising that techniques such as toric degeneration (see [8]) exist which will re-frame problems about more general classes of varieties and ideals into problems about toric ideals. Thus it is of great interest to understand how toric ideals work.

Algebraically, toric ideals are often defined as follows:

Definition 1.1.1. For $\beta \in \mathbb{Z}^n$, let $\beta_+ \in \mathbb{Z}^n_+$ be given by the positive entries in β and $\beta_- \in \mathbb{Z}^n_+$ by the negative entries of β, so that $\beta = \beta_+ - \beta_-$. Let $A \in M_{m,n}(\mathbb{Z})$. Then A defines a $\mathbb{Z}$-linear map $A : \mathbb{Z}^n \to \mathbb{Z}^m$, given by $x \mapsto Ax$. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. The **toric ideal** of A is defined to be

$$I_A = \langle \boldsymbol{x}^{\beta_+} - \boldsymbol{x}^{\beta_-} \mid \beta \in \ker(A) \rangle \subseteq R,$$

where $x^\gamma = x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n}$ for any $\gamma \in \mathbb{Z}^n_+$.

An equivalent and simpler formulation is that **toric ideals are prime binomial ideals**. While there are many interesting aspects of toric ideals, this thesis is concerned with the basic question:

Question 1.1.2. *What can be said about the graded Betti numbers of toric ideals?*

One can view this question as fitting in with a long tradition of research into resolutions and syzygies of polynomial ideals stretching back to Hilbert. As stated the question is too broad. Therefore we will restrict attention to the case of toric ideals arising from finite simple graphs. We define toric ideals of finite simple graphs as follows:

Let $\mathbb{K}$ be a field of characteristic zero. Let G be a finite simple graph with vertex set $V(G) = \{x_1, \ldots, x_m\}$ and edge set

$$E(G) = \{e_1 = \{x_{i_{1,1}}, x_{i_{1,2}}\}, e_2 = \{x_{i_{2,1}}, x_{i_{2,2}}\}, \ldots, e_n = \{x_{i_{n,1}}, x_{i_{n,2}}\}\}.$$

Then the **toric ideal associated with** G, denoted I_G, is the kernel of the map $K[e_1, \ldots, e_n] \to \mathbb{K}[x_1, \ldots, x_m]$ defined by $e_j \mapsto x_{i_{j,1}} x_{i_{j,2}}$.

We note that the finite simple graph G may be represented by an **incidence matrix** A defined as:

$$a_{i,j} = \begin{cases} 0 & \text{if } x_i \notin e_j \\ 1 & \text{if } x_i \in e_j \end{cases} \tag{1.1.1}$$

When conceived in such a way, the definition coincides precisely with the previous definition, that is $I_G = I_A$, where A is the incidence matrix of G. It should be noted that under the standard grading, I_G is a homogenous ideal.

Throughout this thesis we will concern ourselves only with finite simple graphs which are topologically **connected**. The reason for this is that the algebraic invariants we are interested in, graded Betti numbers, can be obtained via a Künneth formula.

To convince the reader that toric ideals of finite simple graphs are a source of more than just interesting examples we provide the following example which, though not touched upon in this thesis, provides an example of a problem arising in the wild where knowledge of toric ideals is useful. For more details see [10], [35] or [20]. Starting with the pioneering paper of Diaconis and Sturmfels [10] the relatively young field of algebraic statistics makes use of toric ideals for what are called **Hierarchical Models**. We demonstrate how hierarchical models can quite naturally be related to toric ideals and in particular toric ideals of finite simple graphs.

Suppose that we wished to know whether there is a relationship between nationality and enjoying hamburgers. A standard way to display this information is to use what is called a **contingency table**.

	Canada	USA	UK	Total
Enjoys Hamburgers:	2	5	1	8
Does Not Enjoy Hamburgers:	1	3	2	6
Total:	3	8	3	14

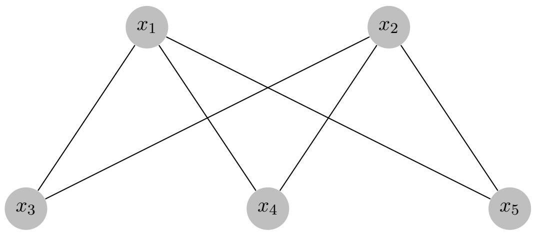
Since we are only considering two factors (nationality and hamburger enjoyment) such a table is called a 2-way table. If one wanted to determine whether or not there was a relationship between burger enjoyment and nationality one could start with the null hypothesis that the two factors are unrelated. To do this we calculate the expected values of the marginals assuming there is no relationship. This is done by assuming the totals are fixed and that no outcome is more favoured than any other. We obtain:

	Canada	USA	UK	Total
Enjoys Hamburgers:	1.71	4.57	1.71	8
Does Not Enjoy Hamburgers:	1.28	3.43	1.28	6
Total:	3	8	3	14

We would then like to be able to say whether the deviation of the observed outcome from the expected value is sufficiently large to reject the null hypothesis or not. A standard approach is to use **Fisher's exact test**. However this involves computing χ^2 statistics for every single table with the given totals. For very small examples this is practical. However it quickly becomes computationally infeasible. What can be done instead is to approximate the p-value using the **Markov chain Monte Carlo method** via the **Metropolis-Hastings Algorithm**. In order to implement the algorithm we require something called a **Markov basis**. In order to obtain a Markov basis we first re-frame the contingency table as a matrix. In our example the matrix can be conceived as follows:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

A Markov basis is then given by a generating set of the toric ideal, I_A, corresponding to A. Note that the matrix in the above example has a very particular form: every column contains exactly two entries of 1. We can therefore reinterpret it as an incidence matrix of a graph. The corresponding graph is $K_{2,3}$, as given below.



All 2-way models can be represented by matrices corresponding to incidence matrices and hence their toric ideals correspond to toric ideals of graphs. Hence if we can say something about the generators of toric ideals of finite simple graphs, we can say something about Markov bases for 2-way models.

Restricting our attention to toric ideals associated with graphs will allow us to make use of the combinatorics of the graph to answer questions about the associated toric ideal. Information obtained from these combinatorial methods about the graded Betti numbers will often allow us to obtain information about other invariants such as regularity, projective dimension, and the Hilbert function. For example, the combinatorics of G are useful for providing results in the spirit of Stillman's question on the relationship between projective dimension and number of generators of a homogeneous ideal generated by forms of bounded degree [28] for which we obtain partial results in Chapter 6. During the course of our investigation we came up with a related conjecture which states that the projective dimension is dependent only on the number of generators and the degrees are irrelevant in providing an upper bound (which is not the case in general).

Conjecture 1.1.3. *Let G be a finite simple graph. Let I_G be the associated toric ideal. Suppose that $\{g_1, \ldots, g_n\}$ is a minimal generating set of I_G, then $\beta_i(R/I_G) \leq \binom{n}{i}$.*

One should note that the study of toric ideals associated with finite simple graphs is in fact broader than it may seem at first glance. It was shown by Petrović and Thoma and Vladoiu [30] that much of the information we are interested in obtaining for a general toric ideal I can be encoded in the more general but related class of toric ideals associated with hypergraphs.

Though the initial motivation was to study the graded Betti numbers of I_G, during the course of our investigation we also investigated the relationship between the fundamental group of the graph, $\pi_1(G)$, and its toric ideal, I_G. The results we obtained pertaining to this relationship are presented in Chapter 5 and Chapter 6. In particular we make the relationship explicit by defining a group operation on binomials of I_G which is isomorphic to a quotient of the fundamental group. This relationship extends to a relationship between the first syzygies of I_G and the group relations of $\pi_1(G)$.

1.2 Results in the Literature

We will now go over some of the existing results pertaining to toric ideals of finite simple graphs in order to provide the context for this thesis.

1.2.1 Generating Sets in the Literature

The strong link between the combinatorics of the graph and the algebraic properties of the toric ideal is well illustrated by the work of Villareal [37] and Ohsugi and Hibi [24] which informs us that we may obtain a generating set of I_G from the set of closed even walks of G. This is achieved by associating to every closed even walk $\gamma = (e_1, e_2, \ldots, e_{2k})$ in G a binomial $f_\gamma = \prod_{2\nmid i} e_i - \prod_{2|i} e_i$ in I_G. The set of all such binomials associated with closed even walks is a generating set of the toric ideal.

One may note however that the set of closed even walks is infinite (for example one could simply repeat the same walk over and over. To obtain a finite generating set we can restrict our attention to **primitive binomials**.

We say that $x^{\alpha_+} - x^{\alpha_-} \in I_G$ is primitive if and only if there exists no $x^{\beta_+} - x^{\beta_-} \in I_G$ such that $x^{\beta_+} | x^{\alpha_+}$ and $x^{\beta_-} | x^{\alpha_-}$ (see p. 33 [34]).

The concept of primitive binomials has been used to define **primitive closed even walks** (closed even walks which correspond to primitive binomials) which also generate the toric ideal but of which there are only finitely many. It was further established by Ohsugi and Hibi that paths which are generators have specific forms namely

Proposition 1.2.1 (Proposition 2.2 [26]). *Let G be a finite connected graph. Then I_G is generated by f_Γ where Γ is one of the following even closed walks:*

i) Γ is an even cycle of G;

ii) $\Gamma = (C_1, C_2)$, where C_1 and C_2 are odd cycles of G having exactly one common vertex; or

iii) $\Gamma = (C_1, \Gamma_1, C_2, \Gamma_2)$, where C_1 and C_2 are odd cycles of G having no common vertex and where Γ_1 and Γ_2 are walks of G both of which combine a vertex v_1 of C_1 and a vertex v_2 of C_2.

We can see the utility of such formulations in the example given in the previous section. One could examine the graph underlying the model and quickly obtain a generating set (in this case the binomials corresponding to the 4-cycles of the underlying graph) and hence a Markov basis to be used in the Metropolis-Hastings Algorithm.

1.2.2 Gröbner Bases in the Literature

One of the interesting properties possessed by such generating sets is that they are intimately related to Gröbner bases. A foundational result is:

Proposition 1.2.2 (Proposition 8.1.9 [38]). *Let G be a finite simple graph. Then the set of primitive walks in G is a Gröbner basis with respect to any monomial ordering.*

Of course Gröbner bases have many applications; they allow us to formulate questions about ideals as questions about monomial ideals as well as implement a variety of useful algorithms. There has been lots of recent work on investigating certain classes of graphs which have particularly nice or interesting Gröbner bases, for example Tatakis and Thoma [36] or Hibi, Nishiyama, Ohsugi, and Shikama [21]. For example typically square-free initial ideals are more desirable, as well as Gröbner bases that have certain degree properties.

Of particular interest are toric ideals of finite simple graphs that arise from graphs for which every even n-cycle with $n \geq 6$ has a chord which are sometimes called **gap free graphs** in the literature. In such cases we know that the toric ideal is quadratically generated.

It was shown by D'Ali in [9] that all gap-free graphs have quadratic square-free Gröbner bases. Such cases are of interest, for example, because it is a necessary condition for a linear resolution. It was shown by Hibi, Takayuki, Nishiyama, Kenta, Ohsugi, Hidefumi, Shikama, Akihiro [21] that despite this not all toric ideals with quadratic generators even have a quadratic Gröbner bases. Such questions are of interest since having Gröbner bases with particular properties can often provide us with information about the ideal. For example if we wish to apply Stanley-Reisner theory we should aim to have squarefree initial ideals.

Another application of Gröbner bases to problems related to toric ideals of finite simple graphs was found by Galetto, Hofscheier, Keiper, Kohne, Van Tuyl and Paczka [15] where they used Gröbner bases to compute an initial ideal which they then showed to be equal to the graded Betti numbers of the underlying ideal. The family which they investigated is a particular class of ideals called robust toric ideals which more generally are of interest, see for example Boocher and Robeva [4].

1.3 Graded Betti Numbers and Invariants in the Literature

Returning to the question which motivated this thesis, it was noticed during our investigation that the graded Betti numbers of a toric ideal of a finite simple graph seemed to be bounded above by binomial coefficients, hence we conjectured that $\beta_{i,j}(I_G) \leq \binom{n}{i}$ where n is the number of generators contained in a minimal generating set. For general ideals it is untrue that the number of generators provides such a bound for the graded Betti numbers. It was shown by Bruns [5] that "'every' finite free resolution is a free resolution of an ideal generated by three elements". Fortunately for us the additional structure possessed by toric ideals of finite simple graphs allows us to say far more.

Some algebraic invariants which have been computed for toric ideals of finite graphs are as follows: Following Villareal [38], for a finite simple graph G, we define the set of all cycles in G to be the **cycle space** and denote it by $\mathcal{Z}(G)$.

Theorem 1.3.1. *Let G be a connected graph and let I_G be the toric ideal of the edge subring $\mathbb{K}[G]$. Then*

$$ht(I_G) = dim_F \mathcal{Z}(G).$$

Corollary 1.3.2. *If G is a connected graph with n vertices and $\mathbb{K}[E(G)]/I_G$ its edge subring, then*

$$dim(\mathbb{K}[E(G)]/I_G) = \begin{cases} n, & \text{if } G \text{ is not bipartite.} \\ n-1, & \text{otherwise.} \end{cases} \tag{1.3.1}$$

1.4 Graded Betti Numbers Novel Results

1.4.1 $K_{n,m}$ Results

It was shown by Campillo and Marijuan [6], Campillo and Pison [7], and Aramova and Herzog [1], that one can compute the multi-graded Betti numbers of a multi-homogeneous toric ideal by computing the reduced simplicial homology groups in a manner analogous to what can be done for monomial ideals. While a powerful theoretical result it can often be difficult to apply in practice. In Chapter 4 we utilise this result to explicitly compute the second and third graded Betti numbers for complete bipartite graphs $K_{n,m}$.

Theorem 1.4.1 (Theorem 4.4.1)**.**

$$\beta_{1,3}(I_{K_{n,m}}) = 2\left(\binom{m}{2}\binom{n}{3} + \binom{m}{3}\binom{n}{2}\right) + 4\binom{m}{3}\binom{n}{3}$$
$$\beta_{1,i}(I_{K_{n,m}}) = 0 \text{ for } i \neq 3$$

Theorem 1.4.2 (Theorem 4.5.1)**.**

$$\begin{aligned}\beta_{2,4}(I_{K_{n,m}}) &= 3\left(\binom{n}{4}\binom{m}{2} + \binom{n}{2}\binom{m}{4}\right) + 9\binom{n}{3}\binom{m}{3} \\ &\quad + 15\left(\binom{n}{4}\binom{m}{3} + \binom{n}{3}\binom{m}{4}\right) + 15\binom{n}{4}\binom{m}{4} \\ \beta_{2,i}(I_{K_{n,m}}) &= 0 \quad \text{for } i \neq 4\end{aligned}$$

We note that the number of generators is already implied by existing results to be $\binom{n}{2}\binom{m}{2}$, the number of 4-cycles.

For example D'Ali provided a characterisation in [9] of all torc ideals of finite simple graphs which posses linear resolutions. For some special cases the goal of computing the graded Betti numbers have already been achieved. For example Biermann and Van Tuyl computed an explicit formula for the case $K_{2,d}$, which is of interest given the results of D'Ali. Further work by Galetto, Hofscheier, K., Kohne, Paczka, and Van Tuyl [15] computed a special case by first computing the initial ideal and then showing that it was equivalent.

Further work has been done in this area by the author with respect to "splitting" ideals. This work originally inspired by the splitting of monomial ideals had as one of its main results

Theorem 1.4.3 (Theorem 4.5 [13])**.** *Let G_1 and G_2 be a splitting of a graph G along a path graph $P_l \cong H \subseteq G$ which we describe as a walk $h = (h_1, \ldots, h_l)$ from a vertex x_1 to a vertex x_2 such that any vertex of H distinct from the endpoints considered as a vertex inside G has degree 2. If G_1 is bipartite, then we obtain.*

$$I_G = (I_{G_1} + I_{G_2}) : \mathcal{E}(h)^\infty,$$

where $\mathcal{E}(h)$ is the product of even indexed edges of h as per Definition 2.3.12

During the course of our investigation it was discovered that this result does not require saturation. In fact it is enough to use $\mathcal{E}(h)^2$ in general and $\mathcal{E}(h)$ when both G_1 and G_2 are bipartite. We have thus included in this thesis the following modification:

Proposition 1.4.4 (Theorem 3.2.1)**.** *Let G_1 and G_2 be a splitting of a graph G along a path graph $P_l \cong H \subseteq G$ which we describe as a walk $h = (h_1, \ldots, h_l)$ from a vertex x_1 to a vertex x_2 such that any vertex of H distinct from the endpoints considered as a vertex inside G has degree 2. If G_1 is bipartite, then we obtain.*

$$I_G = (I_{G_1} + I_{G_2}) : \mathcal{E}(h)^2,$$

where $\mathcal{E}(h)$ is the product of even indexed edges of h as per Definition 2.3.12

Corollary 1.4.5 (Corollary 3.2.3)**.** *If both G_1 and G_2 are bipartite we can replace $\mathcal{E}(h)^2$ with $\mathcal{E}(h)$.*

1.5 Relationship between $\pi_1(G)$ and I_G

One new result contained in this thesis is establishing a close relationship between the fundamental group of a finite simple graph G, denoted $\pi_1(G)$, and the toric ideal, I_G associated with this graph. This is quite natural since as stated before, the generators of I_G, are associated with closed even walks in G as stated in Villareal [37] and Ohsugi and Hibi [24] above. We took this association one step further and associated generators of I_G to elements in the fundamental group to investigate whether the nature of $\pi_1(G)$ is related to I_G beyond the fact that both are generated by closed walks.

We note that $\pi_1(G)$ is generated by all closed walks of G. Therefore when recasting the problem in terms of the fundamental group of G, we are in fact interested in a particular subgroup of the fundamental group, which we call the **alternating fundamental group** and denote by $A(\pi_1(G, x_0))$. This subgroup is the normal subgroup of $\pi_1(G, x_0)$ consisting of loops of even length. We will frequently restrict attention to bipartite graphs where $\pi_1(G) = A(\pi_1(G))$ for convenience.

In the case of bipartite graphs our main result involves defining an appropriate group structure on the set of binomials whose terms contain no common factor in I_G, which we denote $(\mathrm{red}(\mathcal{B}(I_G)), \star)$ (where $\mathrm{red}(\mathcal{B}(I_G))$ is the set and $\star$ the operation), and then showing that it is isomorphic to a quotient of $\pi_1(G)$

Proposition 1.5.1 (Theorem 6.4.2)**.** *Let G be a finite simple bipartite graph with the syzygy-to-group-relation property. The following are true*

1. $\Psi : \pi_1(G, x_0) \to (\mathrm{red}(\mathcal{B}(I_G)), \star)$ *is a surjective group homomorphism.*
2. $\langle g[\pi_1(G, x_0), \pi_1(G, x_0)]g^{-1} : g \in \pi_1(G, x_0)\rangle = ker(\Psi)$.
3. $(\mathrm{red}(\mathcal{B}(I_G)), \star) \cong \pi_1(G, x_0)/ker(\Psi) \cong H_1(G)$ *the first homology group of G.*
4. $\pi_1(G, x_0)/ker(\Psi)$ *is independent of our choice of x_0.*

After having established this relationship we move forward to exploit the relationship to obtain novel results as well as providing new proofs of existing results. Among the new results we wish to highlight:

Theorem 1.5.2 (Theorem 7.3.1)**.** *Let G be a bipartite finite simple graph with the syzygy-to-group-relation property, where $E(G) = \{e_1, \ldots, e_m\}$. Let $\mathcal{G} = \{g_1, \ldots, g_n\}$ be binomials such that there exists a minimal generating set of the fundamental group of G at x_0 $\{\gamma_1, \ldots, \gamma_n\} \subseteq \pi_1(G, x_0)$ such that $\Psi(\gamma_i) = g_i$. Then $I_G = (\langle \mathcal{G} \rangle : (e_1 \cdots e_m)^\infty)$*

We can exploit results about fundamental groups to obtain results about toric ideals. One interesting example is an analogue of Van Kampen's Theorem allowing us to obtain information about the toric ideal of a graph from the toric ideals of its subgraphs.

Theorem 1.5.3 (Theorem 7.3.3)**.** *Let G be a finite simple bipartite graph with the syzygy-to-group-relation property. Let $H_1, H_2, \ldots, H_k$ be subgraphs such that $G = \bigcup_{i=1}^k H_i$ and $H_i \cap H_j$ is path connected for $i, j \in [k]$. Let*

$$H = \bigcup_{i \neq j} (E(H_i) \bigcap E(H_j))$$

Then

$$I_G = \left(\sum_{i=1}^k I_{H_i} : \left(\prod_{e_i \in H} e_i \right)^\infty \right)$$

Another application of this framework we wish to highlight is that one may use the fundamental group of the graph G to characterise all regular sequences in the toric ideal I_G which is shown in Chapter 6.

Theorem 1.5.4 (Theorem 7.2.1)**.** *Let G be a finite simple graph. Let $\mathcal{G} = \{g_1, \ldots, g_n\}$ be binomials which correspond to a minimal generating set $\{\gamma_1, \ldots, \gamma_n\}$ of the fundamental group $\pi_1(G)$, then $\mathcal{G}$ is a regular sequence.*

As a corollary:

Corollary 1.5.5. *Let G be a finite simple bipartite graph and I_G its toric ideal, then* $\operatorname{rank}(\pi_1(G)) = \operatorname{depth}(I, R)$.

Finally we apply the framework to obtain a series of results on the nature of the syzygies of I_G

Theorem 1.5.6. *Let G be a bipartite finite simple graph and let I_G be its toric ideal. Let $\langle g_1, \ldots, g_n, h_1, h_2, \ldots, h_m : r_1, \ldots, r_m \rangle$ be a representation of $A(\pi_1(G, x_0))$ where the generators correspond to generators of I_G, with the g_i corresponding to minimal generators in the fundamental group and the h_j are expressed in terms of the g_i in the m relations and the relations are minimal on these generators. Then the total number of minimal first syzygies is $\beta_1(I_G) \geq \binom{n}{2} + m$.*

1.6 Structure and Content Summary

We end this Chapter with rough outline of the structure of the thesis.

Chapter 1, which you have just read, introduces the nature of the problem and highlights some of the existing literature. It then describes how this thesis fits into the existing literature and highlights some of results we obtained.

Chapter 2 introduces most of the necessary background and notation that will be needed for the remainder of the thesis. We repeat some of the definitions mentioned in the introduction for the sake of completeness. We review the required graph theory and notation, the necessary commutative algebra and algebraic invariants which we wish to investigate. We again define toric ideals in general before defining toric ideals of finite simple graphs as well as providing the reader with some examples. The background Chapter is not exhaustive and further background will be introduced when needed including Chapter 4 which provides a brief account of results from algebraic toplogy pertaining to the fundamental group of a graph.

Chapter 3 will focus on an extension of a result obtained by Favaccio, Hofscheir, Keiper and Van Tuyl [13]. We recall the result in question and show how it can be refined.

Chapter 4 will focus on a technique using subgraphs to compute the first and second syzygies for complete bipartite graphs $K_{n,m}$. This result utilises the work of Aramova and Herzog [1] linking the graded Betti numbers to simplicial complexes.

Chapter 5 provides the necessary background on the fundamental group and algebraic topology needed to understand Chapter 6. We review the definition of the fundamental group and some basic facts about it. We review how the fundamental group applies to finite simple graphs specifically, including a well known algorithm for obtaining the generators.

In Chapter 6 we introduce the promised relationship between the fundamental group of a finite simple graph G and its toric ideal. This begins with a large number of definitions and notation.

In Chapter 7 we use the relationship established in Chapter 6 to formulate and prove some results related to generating sets, syzygies, projective dimension and regular sequences of toric ideals associated with finite simple graphs.

In Chapter 8 we conclude with a summary of the thesis and then list a number of open questions still to be dealt with as well as directions for future research.

Chapter 2

Background

In this Chapter we provide some basic definitions and notation which will be used repeatedly throughout this thesis. Further, some context and exposition is provided to aid the reader who is unfamiliar with this material. We introduce the relevant background on finite simple graphs which is central to this thesis. This includes definitions, notation and some well known, as well as lesser known, results which we will draw on.

We provide a brief review of the needed commutative algebra and homological invariants of ideals which will be of interest. These will not be presented in highest generality but rather as they appear in this thesis specifically.

Finally we review concepts and definitions related toric ideals themselves, again mostly limited to the task at hand rather than in full generality.

Some further definitions and background will be added as needed in subsequent chapters (this being done to increase the ease of reading).

2.1 Finite Simple Graphs

Graphs are ubiquitous in mathematics. They have been studied for hundreds of years reaching at least as far back as the famed seven bridges of Königsberg posed and solved by Euler.

There is no reason apparent to restrict attention to finite simple graphs other than the fact finite simple graphs are what have been studied in much of the existing literature on toric ideals associated with graphs: for example [17], [23], [25] (and many, many more).

However, one should note that if we allow for other classes of graphs we can obtain perfectly cogent theories adjusting definitions in obvious ways. Since we are limiting ourselves to finite simple graphs, and since we will study them in great detail, we will provide all the relevant background here. The following information can be found in Wilson [39].

Definition 2.1.1. A **finite simple graph** G consists of two sets: a finite set of **vertices**

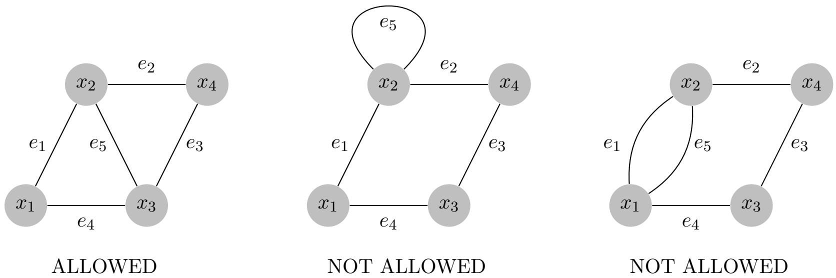
$$V(G) = \{x_1, \dots, x_r\}$$

which we call the **vertex set** and a finite set of **edges**

$$E(G) = \{e_1 = \{x_{1i}, x_{1j}\}, \ldots, e_n = \{x_{ni}, x_{nj}\}\}$$

called the **edge set**, which consists of non-ordered pairs of elements from the vertex set where $x_{li} \neq x_{lj}$ where each pair x_i, x_j can have at most one edge between them.

Example 2.1.2. Some examples and non-examples of finite simple graphs are as follows:



where the second graph has an illegal edge, e_5, from a vertex, x_2, to itself and the third graph has two edges, e_1 and e_5, which share the same two vertices, x_1 and x_2. Note that if we allowed either of these cases the graphs in question could not be described as abstract simplicial complexes.

Remark 2.1.3. Recall that an abstract finite simplicial complex on a finite set S is simply a subset of $\mathcal{P}(S)$, the power set of S, which is closed under inclusion. Thus one may note that a finite simple graph is not just a special case of a graph, but that it is also a finite simplicial complex. This observation is important in pointing the way to extending the association to ideals which cannot be associated to graphs, but can nonetheless be associated to simplicial complexes.

As it turns out we will be able to glean information about toric ideals associated with finite simple graphs from homomorphisms between their associated graphs. This being the case we review the definition of a graph homomorphism.

Definition 2.1.4. A **graph homomorphism** between a finite simple G and a finite simple graph H $\varphi : G \to H$ is defined as set map $V(\varphi) : V(G) \to V(H)$ such that if $\{x, y\} \in E(G)$ then $\{\varphi(x), \varphi(y)\} \in E(H)$.

Remark 2.1.5. Since we have restricted ourselves to finite simple graphs this imposes some conditions on graph homomorphisms. For example we see that vertices which share an edge cannot be mapped to each other since this would imply a loop, which is forbidden. We will make use of graph homomorphisms to define gluing operations in subsequent chapters.

While investigating toric ideals of finite simple graphs we will extensively rely on properties inherited by toric ideals of subgraphs, and in particular, toric ideals of what are called induced subgraphs, which we define below after providing a formal definition of a subgraph.

Definition 2.1.6. For a finite simple graph G we define a **subgraph** H **of** G, written $H \subseteq G$, to be a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ where $e \in E(H)$ is such that $e \subseteq V(H)$.

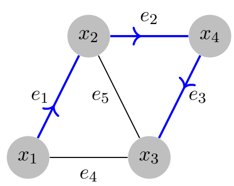
Definition 2.1.7. For a subgraph $H \subseteq G$, we say that H is an **induced subgraph** if $e \in E(H)$ for all $e = \{x_1, x_2\} \subseteq V(H)$.

As mentioned in the introduction, walks in finite simple graphs will play a key role. For instance they will tell us about the generators, but also the syzygies of the associated toric ideal. With this in mind, we define exactly what we mean by a walk and introduce some specialised terminology to describe special types of walks.

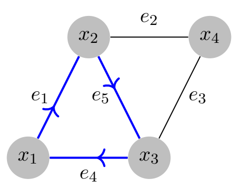
Definition 2.1.8. We define a **walk** w in a finite simple graph G to be a sequence of adjacent edges i.e., $(e_1, \dots, e_l)$ such that $e_i \cap e_{i+1} \neq \emptyset$. Alternatively a walk is a sequence of vertices $(x_1, \dots, x_{l+1})$ such that $\{x_i, x_{i+1}\} \in E(G)$. We call x_1 the initial vertex and x_{l+1} the terminal vertex. We call a walk **closed** if the initial and terminal vertices are equal, i.e., for $w = (x_1, \dots, x_{l+1})$ we have $x_1 = x_{l+1}$. We call a walk w **an even walk** if $l \equiv 0 (\mod 2)$

Example 2.1.9. We provide an example of two walks on the finite simple graph from Example 2.1.2.

I) (e_1, e_2, e_3) is a walk from x_1 to x_3 which can also be described by the vertex sequence (x_1, x_2, x_4, x_3).



II) (e_1, e_5, e_4) is a **closed** walk from x_1 to x_1 which could also be described by the vertex sequence (x_1, x_2, x_3, x_1).



Definition 2.1.10. Given a finite simple graph G and a walk $w = (e_1, e_2, \ldots, e_{l-1}, e_l)$, we define the **inverse** of a walk, denoted by w^{-1}, to be $w^{-1} = (e_l, e_{l-1}, \ldots, e_2, e_1)$.

Definition 2.1.11. For a finite simple graph G and walks $w_1 = (x_1, \ldots, x_n)$ and $w_2 = (y_1, \ldots, y_m)$ represented as sequences of vertices where $x_n = y_1$ we can define an operation which we call **concatenation** as $w_1w_2 := (x_1, \ldots, x_{n-1}, x_n = y_1, y_2, \ldots, y_m)$

Remark 2.1.12. In algebraic topology such a construction leads to the concept of a fundamental semi-group. One should note that the operation of concatenation can only be defined when the terminal vertex of the first walk equals the initial vertex of the second walk and so is not in general defined for arbitrary walks.

Definition 2.1.13. A closed walk $(x_1, x_2, \ldots, x_{n+1} = x_1)$ in which the vertices $x_1, \ldots, x_n \in V(G)$ are distinct is called an **n-cycle** and a **cycle when the number of edges is not specified.**

Definition 2.1.14. A **tree** is a graph which contains no cycles.

Remark 2.1.15. Trees themselves are an object of extensive study and are highly amenable to combinatorial and topological treatment. For example, topologically one can think of trees as finite simple graphs which are homotopy equivalent to a point. If a tree is connected, then combinatorially it is characterised by the property $|V(G)| - |E(G)| = 1$. We will make use of trees to determine the generators of the fundamental group of a graph.

Definition 2.1.16. A **maximal tree** of G is a tree which is a subgraph of G which is maximal with respect to inclusion, i.e., it is not contained in some other subgraph which is a tree.

Theorem 2.1.17. *(1A.1. [18]) Every connected finite simple graph G contains a maximal tree T. Further, every tree in G is contained in a maximal tree.*

2.2 Commutative Algebra and Homological Invariants

Since this thesis is primarily concerned with gleaning algebraic information from combinatorial information, we introduce the background in commutative algebra necessary to understand the results. We will focus our attention on multi-graded polynomial rings and their ideals. For a more complete exposition see, for example Eisenbud [12].

2.2.1 Graded Rings

Definition 2.2.1. Let R be a commutative ring with unity. Let S be a **semigroup** (a set with a closed binary operation and identity element). We say that R has an S**-grading** if there exists a decomposition $R = \bigoplus_{s \in S} R_s$ such that for any $s, t \in R$ we have $R_s R_t \subseteq R_{st}$. We refer to R_s as the s^{th} **graded component of R**.

An element $f \in R_s$ is called an s**-form** or **homogeneous element**. For a graded ring R we define a **homogeneous ideal** I to be an ideal such that there exists a generating set β such that if $g \in \beta$ then $g \in R_s$ for some $s \in S$, i.e., I is generated by homogeneous elements. A homogeneous ideal I inherits a grading from R and we define $I_s = I \cap R_s$, the s^{th} **graded component of I.** Finally given S-graded rings R and Q we define a graded homomorphism $\varphi : R \to Q$ to be a ring homomorphism such that $\varphi(R_s) \subseteq Q_s$. For an S-graded ring R we can similarly define an **S-graded module** M to be a module $M = \bigoplus_{s \in S} M_s$ such that $R_s M_t \subseteq M_{s+t}$.

Remark 2.2.2. We will be concerned entirely with the cases where $S = \mathbb{Z}_{\geq 0}$, $S = \mathbb{Z}$, $S = \mathbb{Z}^n_{\geq 0}$, and $S = \mathbb{Z}^n$ (all considered as semigroups under addition). In the latter two cases we will use the term **multigrading**. These cases are important because $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}^n_{\geq 0}$ is the semigroup for polynomials in a single variable and in n variables respectively. Similarly $\mathbb{Z}$ and $\mathbb{Z}^n$ correspond to the semi-groups for Laurent polynomials in one variable and n variables respectively.

Since we are concerned with commutative algebra we fix some notation as follows:

Definition 2.2.3.

1. $\mathbb{K}$ refers to a field of characteristic 0.
2. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring and $\alpha = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$. We then say that the **monomial** $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = x^\alpha$ and has **multidegree** α. For a monomial $x^\alpha \in R$, we define $\text{multideg}(x^\alpha) = \alpha$.
3. Let $R = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring and $\alpha = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$. We say that the monomial x^α has **total degree** (or simply **degree**) equal to $\sum_{i=1}^n a_i$. We write $\deg(x^\alpha) = \sum_{i=1}^n a_i$.
4. Let $x^\alpha \in \mathbb{K}[x_1, \dots, x_n]$ be a monomial where $\alpha = (a_1, \dots, a_n)$. The **radical of** x^α is $\sqrt{x^\alpha} = x^{(\min(a_1,1),\dots,\min(a_n,1))}$
5. A **binomial** $f \in R$ is an element which can be expressed as $f = x^{\alpha_1} \pm x^{\alpha_2}$. When $\deg(\alpha_1) = \deg(\alpha_2)$ we say that the binomial is a **homogeneous binomial**.

Unless otherwise specified the reader can now assume that the notation of Definition 2.2.3 applies. We will sometimes continue to write $\mathbb{K}[x_1, \dots, x_n]$ to emphasise when we are dealing with polynomial rings.

Remark 2.2.4. For an ideal $I \subseteq R$, we note that since the 0^{th}-graded piece of R is $\mathbb{K}$, that $\mathbb{K}$ acts on each graded component I_i, that is, $R_0 I_i = I_i$. Hence I_i can be considered as a $\mathbb{K}$-vector space. We also note that each $f \in R$ can be expressed uniquely as a sum of monomials when considered as a basis for R as a $\mathbb{K}$-vector space.

Definition 2.2.5. Let $I, J \subseteq R$ be ideals. We define the **quotient (or colon) ideal of I by J** to be

$$(I : J) = \{r \in R \mid rJ \subseteq I\}.$$

Definition 2.2.6. Let $I, J \subseteq R$ be ideals. We define the **saturation of I by J** to be

$$(I : J^{\infty}) = \bigcup_{n=1}^{\infty} (I : J^n).$$

2.2.2 Monomial Ideals, Binomial Ideals, and Gröbner Bases

We are interested in the intersection between combinatorics and commutative algebra. Having said a little bit about the combinatorics pertaining to graph theory we now review concepts that belong to commutative algebra.

Commutative algebra is in its simplest form is the study of commutative rings and related structures. Of particular interest are structures related to polynomial rings which can be linked to geometry. In our case we are interested specifically in monomial and binomial ideals of polynomial rings.

Definition 2.2.7. Fix a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$. A **monomial ideal** I is an ideal which is generated by monomials

$$I = (x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_k}).$$

A **binomial ideal** I is an ideal which is generated by binomials

$$I = (x^{\alpha_1} - x^{\beta_1}, x^{\alpha_2} - x^{\beta_2}, \ldots, x^{\alpha_k} - x^{\beta_k}).$$

Definition 2.2.8. For a ring $\mathbb{K}[x_1, \ldots, x_n]$ we define a **monomial order** to be be a total ordering $\geq$ on $\mathbb{Z}^n_{\geq 0}$ such that for all $\alpha \in \mathbb{Z}^n_{\geq 0}$, we have $(0, \ldots, 0) \leq \alpha$, and for $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$ if $\alpha \leq \beta$ then $\alpha + \gamma \leq \beta + \gamma$. This allows us to define an ordering on monomials in R by defining $x^{\alpha} \geq x^{\beta}$ when $\text{multideg}(x^{\alpha}) \geq \text{multideg}(x^{\beta})$.

Definition 2.2.9. Given a ring $\mathbb{K}[x_1, \ldots, x_n]$ and a monomial order $\geq$ and an arbitrary polynomial $f = c_1 x^{\alpha_1} + c_2 x^{\alpha_2} + \cdots + c_k x^{\alpha_k} \in \mathbb{K}[x_1, \ldots, x_n]$, we define the **leading term** of f to be $LT(f) = x^{\alpha_i}$ where $\alpha_i \geq \alpha_j$ for $j = 1, \ldots, k$. Given an ideal $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ we define the **leading term ideal**, $\text{LT}(I)$ to be the monomial ideal generated by the leading terms of elements in I. That is, $\text{LT}(I) = \langle \{\text{LT}(f) : f \in I\} \rangle$.

Remark 2.2.10. We note that since $\mathbb{K}[x_1, \ldots, x_n]$ is Noetherian, the ideals we will be dealing with and their initial ideals will be finitely generated.

Definition 2.2.11. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring with monomial order $\geq$, $I \subseteq R$ an ideal, and $\beta = \{g_1, \ldots, g_k\}$ such that $I = \langle \beta \rangle$. i.e. a generating set of I. If $\text{LT}(I) = \langle \{\text{LT}(g_1), \ldots, \text{LT}(g_k)\} \rangle$, then we call β a **Gröbner basis** of I.

Remark 2.2.12. Gröbner bases are of great importance generally since they allow a generalisation of the familiar Euclidean Algorithm for polynomials in a single variable to be extended to multivariable polynomial rings. This result is the basis for a wide range of useful algorithms and results. For our purposes they will be useful since they will allow us to obtain information about ideals of interest from their initial ideals, which being monomial ideals, are easier to work with.

2.2.3 Homological Invariants

We now introduce the needed background on various homological invariants which are of general interest in commutative algebra. The following definitions and concepts are taken from Mac Lane [22], Eisenbud [12] and Herzog and Hibi [19] where the interested reader can see the material layed out in greater detail.

Definition 2.2.13. For a given ring R and set S we define a free R-module F on the set S to be a set map $\iota : S \to F$ such that for any R-module M and any set map $f : S \to M$ there exists a unique R-module homomorphism $\bar{f} : F \to M$ such that the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \exists! \bar{f}} \\ & & M \end{array}$$

Remark 2.2.14. It is often simpler to note that the R-modules which satisfy these conditions are simply direct sums of R.

Definition 2.2.15. A **resolution** of an R-module M is an exact sequence:

$$\cdots \xrightarrow{\partial_{i+1}} E_i \xrightarrow{\partial_i} \cdots \xrightarrow{\partial_2} E_1 \xrightarrow{\partial_1} E_0 \xrightarrow{\epsilon} M \to 0$$

The map $\epsilon : E \to M$ is often referred to as the **augmentation map** and is viewed as a map between the chain complexes

$$\begin{array}{ccccccccccc} E : \cdots & \xrightarrow{\partial_i} & E_{i-1} & \xrightarrow{\partial_{i-1}} & \cdots & \xrightarrow{\partial_2} & E_1 & \xrightarrow{\partial_1} & E_0 & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \\ M : \cdots & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

If the modules E_i are free R-modules then we call the resolution **free**.

Remark 2.2.16. Definitions 2.2.13 and 2.2.15 can be recast replacing a module M with a graded module M and the R-module homomorphisms with graded R-module homomorphisms. In such cases we refer to **graded resolutions**.

For the following definitions and concepts we direct the interested reader to Peeva [27]

Definition 2.2.17. A graded free resolution

$$F : \cdots \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} F_0 \to M$$

of a graded finitely generated R-module M is **minimal** if $\partial_i(F_i) \subseteq (x_1, \ldots, x_n)F_{i-1}$ for all $i \geq 0$.

The following foundational results can be found in Peeva's *Graded Syzygies* [27] in the first chapter We have compressed several results into one for the readers benefit.

Theorem 2.2.18. *Let M be a graded finitely generated R-module.*

1. *There exists a minimal graded free resolution of M.*
2. *Up to an isomorphism, there exists a unique minimal graded free resolution of M.*
3. *A graded free resolution, which at each step of the resolution has a minimal number of generators, is minimal.*

When all put together this shows that the minimal graded free resolution of an R-module is a well defined invariant. It does not depend on the particular choice of resolution. We use this fact to develop the following definition.

Definition 2.2.19. Let I be a homogeneous ideal of the polynomial ring R. Associated with I is a **minimal graded free resolution** of the form:

$$0 \to \bigoplus_j R(-j)^{\beta_{l,j}(I)} \to \bigoplus_j R(-j)^{\beta_{l-1,j}(I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{0,j}(I)} \to I \to 0,$$

where $l \leq n$ and $R(-j)$ is the free R-module obtained by **shifting the degrees** of R by j (i.e., so that $R(-j)_a = R_{a-j}$). The number $\beta_{i,j}(I)$, the $(i,j)^{th}$ **graded Betti number** of I, equals the number of minimal generators of degree j in the i^{th} syzygy module of I. We define the i^{th} **total Betti number** of I to be $\beta_i(I) = \sum_j \beta_{i,j}(I)$.

Remark 2.2.20. Sometimes we will instead refer to the minimal graded free resolution of the quotient R/I in which case we have

$$0 \to \bigoplus_j R(-j)^{\beta_{l+1,j}(R/I)} \to \bigoplus_j R(-j)^{\beta_{l,j}(R/I)} \to \cdots \to \bigoplus_j R(-j)^{\beta_{1,j}(R/I)} \to R \to R/I \to 0,$$

where we have have

$$\beta_{0,j}(R/I) = \begin{cases} 1, & j = 0 \\ 0, & j > 0 \end{cases}$$

and $\beta_{i,j}(R/I) = \beta_{i-1,j}(I)$ for $i > 0$, that is, the homological degree has been shifted by one.

Definition 2.2.21. A useful way to to display the graded Betti numbers of a graded R-module is via a **Betti table**. This is a table in which the columns correspond to the homological dimension and the rows correspond to the degree of the generator given by the grading:

	0	1	2	$\cdots$	i	$\cdots$
total:	$\beta_0(I)$	$\beta_1(I)$	$\beta_2(I)$	$\cdots$	$\beta_i(I)$	$\cdots$
0:	$\beta_{0,0}(I)$	$\beta_{1,1}(I)$	$\beta_{2,2}(I)$	$\cdots$	$\beta_{i,i}(I)$	$\cdots$
1:	$\beta_{0,1}(I)$	$\beta_{1,2}(I)$	$\beta_{2,3}(I)$	$\cdots$	$\beta_{i,i+1}(I)$	$\cdots$
2:	$\beta_{0,2}(I)$	$\beta_{1,3}(I)$	$\beta_{2,4}(I)$	$\cdots$	$\beta_{i,i+2}(I)$	$\cdots$
3:	$\beta_{0,3}(I)$	$\beta_{1,4}(I)$	$\beta_{2,5}(I)$	$\cdots$	$\beta_{i,i+3}(I)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\cdots$
j:	$\beta_{0,j}(I)$	$\beta_{1,j+1}(I)$	$\beta_{2,j+2}(I)$	$\cdots$	$\beta_{i,i+j}(I)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Example 2.2.22. Let $R = \mathbb{K}[e_1, \ldots, e_6]$ and $I = \langle e_2e_4 - e_1e_5, e_3e_4 - e_1e_6, e_3e_5 - e_2e_6\rangle$. Then R/I has a minimal graded free resolution

$$0 \to R^2(-3) \xrightarrow{\begin{bmatrix} -e_3 & e_6 \\ e_2 & -e_5 \\ -e_1 & e_4 \end{bmatrix}} R^3(-2) \xrightarrow{\begin{bmatrix} e_2e_4 - e_1e_5 & e_3e_4 - e_1e_6 & e_3e_5 - e_2e_6 \end{bmatrix}} R \to R/I \to 0$$

and an associated Betti table

	0	1	2
total:	1	3	2
0:	1	-	-
1:	-	3	2

We can see immediately from the Betti table the regularity and the projective dimension (defined below). We also can see that the resolution is linear.

Having introduced the concept of minimal free resolutions and graded Betti numbers, it is natural to explore what these can be used for. The following definitions can be found in Eisenbud [12].

Definition 2.2.23. Let R be a commutative ring and M and R-module. We define an M**-regular sequence** to be a sequence of elements $(r_1, \ldots, r_n)$ where $r_i \in R$ such that r_i is not a zero divisor of $M/(r_1, \ldots, r_{i-1})M$. We define the length of a regular sequence to be the number of elements in the sequence.

Definition 2.2.24. Let R be a Noetherian ring, M an R-module, and $I \subset R$ and ideal. We define the **depth** of I in M, denoted $\operatorname{depth}_R(I, M)$ to be the supremum of the lengths of regular sequences consisting of elements of I.

Definition 2.2.25. For an ideal $I \in R$ we define its **projective dimension** to be $\operatorname{pdim}(I) := \max\{i : \beta_i(I) \neq 0\}$.

Definition 2.2.26.

1. For a ring R we define the **dimension** (or **Krull dimension**) as the supremum of the length of the lengths of chains of prime ideals in R. We denote it by $\dim R$. That is $\dim R = \sup\{l : P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_l,\ P_i \in \operatorname{Spec}(R)\}$.

2. Let I be an ideal of R. We define the **codimension** (or **height**) of I to be $\min\{\dim(R_P) : I \subseteq P,\ P \in \operatorname{Spec}(R)\}$ where R_P is the localization of R by P.

In this thesis we will sometimes obtain information about the graded Betti numbers of an ideal I using its leading term ideal $\operatorname{LT}(I)$ which is useful via the following proposition:

Proposition 2.2.27. *Let $I \subseteq R$ be an ideal. Let $\geq$ be a monomial order. Then $\beta_{i,j}(I) \leq \beta_{i,j}(\operatorname{LT}(I))$.*

Given that the Betti number of the leading term ideal $\operatorname{LT}(I)$ of an ideal I provide an upper bound for the graded Betti numbers of I we will also find it useful to have an upper bound for the graded Betti numbers of an initial ideal. This can be obtained via a non-minimal resolution called the Taylor Complex. For more information on the Taylor Complex see for example Herzog and Hibi [19].

Definition 2.2.28 (Definition 7.1 [19])**.** Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal with a minimal generating set $\{u_1, u_2, \ldots, u_s\}$. We define the **Taylor Complex**, $\mathbb{T}_{(u_1,\ldots,u_s)}$ as follows:

1) Let T_1 be the free R module with basis $e_1, \ldots, e_s$ corresponding to the generators of I

2) Let $T_i = \bigwedge^i T_1$ for $i = 0, \ldots, s$. We note that

$$\{e_F = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \ :\ F = \{j_1 < j_2 < \cdots < j_i\} \subseteq [s]\}$$

is a basis for T_i.

3) Let $\partial : T_i \to T_{i-1}$ be defined

$$\partial(e_F) = \sum_{i \in F} (-1)^{\sigma(F,i)} \frac{u_F}{u_{F\setminus\{i\}}} e_{F\setminus\{i\}}$$

where for $G \subseteq [n]$, u_G denotes the least common multiple of the monomials u_i with $i \in G$ and where $\sigma(F,i) = |\{j \in F \ :\ j < i\}|$

4) We define a grading such that the degree of each e_F is equal to $\deg(u_F)$

Theorem 2.2.29 (Theorem 7.1.1 [19])**.** *Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal with a minimal generating set $\{u_1, u_2, \ldots, u_s\}$. Then the Taylor complex $\mathbb{T}_{(u_1,\ldots,u_s)}$ is a graded free resolution of R/I.*

Now we can state the property of the Taylor complex which we wish to utilise in this thesis, that it provides an upper bound on the Betti numbers of a monomial ideal and hence leading term ideal (which are in turn upper bound of the underlying ideal).

Corollary 2.2.30 (Corollary 7.1.2 [19])**.** *Let $I \subseteq R = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal with a minimal generating set $\{u_1, u_2, \ldots, u_s\}$. Then $\beta_i(R/I) \leq \binom{s}{i}$ for $i = 1, \ldots, s$.*

2.2.4 Toric Ideals

Having introduced the relevant background we are now in a position to define the central object of study in this thesis, namely toric ideals. We begin with a general definition before moving to the more specific case of toric ideals associated with finite simple graphs. For the interested reader more information can be found in Herzog and Hibi [20]

Definition 2.2.31. Suppose we have two Laurent polynomial rings $\mathbb{K}[e_1, \ldots, e_n], \mathbb{K}[x_1, \ldots, x_r]$ for some field k. Let $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{Z}^r \backslash \{0\}$, where $a_i = (a_{i1}, a_{i2}, \ldots, a_{ir})$. We define the ring homomorphism

$$\varphi : \mathbb{K}[e_1, \ldots, e_n] \to \mathbb{K}[x_1, \ldots, x_r]$$

$$e_i \mapsto x^{a_i} = x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_r^{a_{ir}}$$

We call $\ker(\varphi)$ the **toric ideal associated with** $\mathcal{A}$. We assign the e_i the multidegree of its image i.e. a_i.

Remark 2.2.32. Clearly $\ker(\varphi)$ is prime as the image of φ is an integral domain.

Theorem 2.2.33 (Theorem 3.4)**.** *[20] Let $I \subseteq S$ be a prime binomial ideal. Then I is a toric ideal.*

Since toric ideals are prime binomial ideals, they possess some very useful properties which allow us to express elements of a toric ideal rather simply in terms of a generating set. Further they have Gröbner bases which can be verified in a more straightforward manner than is the case for general ideals.

Lemma 2.2.34 (Lemma 3.8)**.** *[20] Let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal generated by the binomials $f_1, \ldots, f_r$. Let $x^u - x^v$ be a binomial belonging to I. Then, there exists an expression*

$$x^u - x^v = \sum_{k=1}^{s} \epsilon_k x^{w_k} f_{i_k}$$

where $\epsilon \in \{\pm 1\}$, $w_k \in \mathbb{Z}^n_{\geq 0}$, and $1 \leq i_k \leq r$ for $k = 1, \ldots s$ and where $x^{w_p} f_{i_p} \neq x^{w_q} f_{i_q}$ for all $1 \leq p < q \leq s$

Theorem 2.2.35 (Theorem 3.11)**.** *[20] Let I be a binomial ideal of S and $\{g_1, \ldots, g_s\}$ a set of nonzero binomials in I. Then $\{g_1, \ldots, g_s\}$ is a Gröbner basis of I with respect to a monomial order $<$, if and only if for all binomials $0 \neq u - v \in I$, either u or v belongs to $(\mathrm{LT}_<(g_1), \ldots, \mathrm{LT}_<(g_s))$.*

We will however make use of more algebraically flavoured definitions involving ideals constructed from the kernel of integer matrices (alternatively $\mathbb{Z}$-linear maps) and kernels of maps from a polynomial ring to a mononmial ideal in another polynomial ring. In order to define toric ideals for integer matrices we need to introduce the following notation.

Definition 2.2.36. For $\beta \in \mathbb{Z}^n$, let $\beta_+ \in \mathbb{Z}_+^n$ given by the positive entries in β and $\beta_- \in \mathbb{Z}_+^n$ by the negative entries of β, so that

$$\beta = \beta_+ - \beta_-.$$

Definition 2.2.37. Let $A \in M_{m,n}(\mathbb{Z})$. Then A defines a $\mathbb{Z}$-linear map $A : \mathbb{Z}^n \to \mathbb{Z}^m$, given by $x \mapsto Ax$. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ where $\mathbb{K}$ is a field. The **toric ideal** of A is defined to be

$$I_A = \langle \boldsymbol{x}^{\beta_+} - \boldsymbol{x}^{\beta_-} \mid \beta \in \ker(A) \rangle \subseteq R.$$

Example 2.2.38. We provide an example illustrating the above definitions:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\ker(A) = \langle (1, -2, 1) \rangle$$

Then $\beta = (1, -2, 1)$ and so $\beta_+ = (1, 0, 1)$ and $\beta_- = (0, 2, 0)$.

Since our matrix has 3 columns let $R = \mathbb{K}[x_1, x_2, x_3]$, and so

$$I_A = \left\langle \boldsymbol{x}^{\beta_+} - \boldsymbol{x}^{\beta_-} \right\rangle = \left\langle x_1 x_3 - x_2^2 \right\rangle.$$

2.3 Toric Ideals of Finite Simple Graphs

We are now finally in a position to define the principal object studied in this thesis.

Definition 2.3.1. For a finite simple graph G with $V(G) = \{x_1, \ldots, x_r\}$ and $E(G) = \{e_1, \ldots, e_n\}$ and a field $\mathbb{K}$, define $\mathbb{K}[V(G)] = \mathbb{K}[x_1, \ldots, x_r]$ and $\mathbb{K}[E(G)] = \mathbb{K}[e_1, \ldots, e_n]$. With this notation we define the **toric ideal associated with a graph** G to be the kernel of the following map:

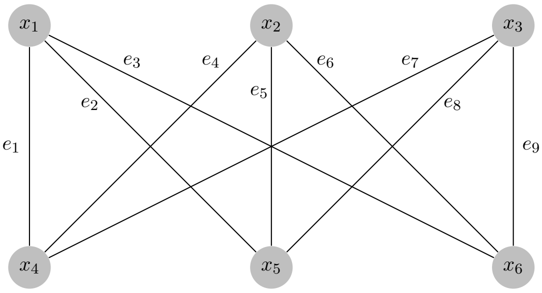
$$\varphi_G : \mathbb{K}[E(G)] \to \mathbb{K}[V(G)]$$

defined by

$$e_l \mapsto x_{li} x_{lj}$$

We denote the kernel of this map by $ker(\varphi_G) = I_G$

Example 2.3.2. It is illustrative to examine the toric ideal associated with the complete bipartite graph $K_{3,3}$, as given below



For this graph with this labelling, we have

$$\varphi_G : k[e_1, \ldots, e_9] \to k[x_1, \ldots, x_6]$$

where

$$\begin{array}{lll} e_1 \mapsto x_1x_4 & e_2 \mapsto x_1x_5 & e_3 \mapsto x_1x_6 \\ e_4 \mapsto x_2x_4 & e_5 \mapsto x_2x_5 & e_6 \mapsto x_2x_6 \\ e_7 \mapsto x_3x_4 & e_8 \mapsto x_3x_5 & e_9 \mapsto x_3x_6 \end{array}$$

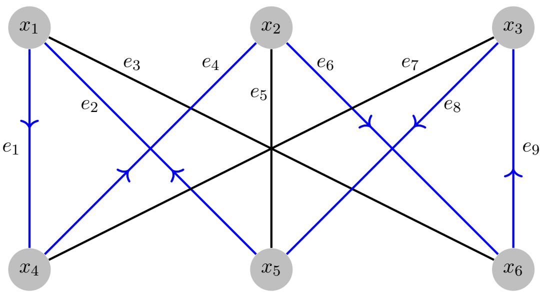
The toric ideal is the ideal

$$\begin{aligned} I_G = \langle & e_6e_8 - e_5e_9, e_3e_8 - e_2e_9, e_6e_7 - e_4e_9, \\ & e_5e_7 - e_4e_8, e_3e_7 - e_1e_9, e_2e_7 - e_1e_8, \\ & e_3e_5 - e_2e_6, e_3e_4 - e_1e_6, e_2e_4 - e_1e_5\rangle. \end{aligned}$$

It is well known that the generators of I_G correspond to closed even walks in G, and in particular, I_G is a homogeneous ideal generated by binomials. The following results can be found, for example, in Villareal [38].

Proposition 2.3.3. *I_G is generated by homogeneous binomials corresponding to the closed even walks of the graph G.*

Example 2.3.4. Let G be as in the previous example. This graph has the closed even walk $(e_1, e_4, e_6, e_9, e_8, e_2)$ which corresponds to the binomial $e_1e_6e_8 - e_4e_9e_2$. This walk is illustrated as:



By plugging in $e_1e_6e_8 - e_4e_9e_2$ into φ_G, we observe that it is indeed in the kernel of φ_G i.e. I_G

$$\begin{aligned}\varphi_G(e_1e_6e_8 - e_4e_9e_2) &= \varphi_G(e_1e_6e_8) - \varphi_G(e_4e_9e_2) \\ &= (x_1x_4)(x_2x_6)(x_3x_5) - (x_4x_6)(x_3x_6)(x_5x_1) \\ &= x_1x_2x_3x_4x_5x_6 - x_1x_2x_3x_4x_5x_6 \\ &= 0\end{aligned}$$

Knowing that a set of generators can be obtained from the combinatorics of the graph is a good start. However the reader may note that the set of closed even walks is infinite (one could for example repeat the same walk times as one wanted to create new walks) and so may be difficult to work with. In order to impose finiteness we will need to refine our notion of closed even walks to a subclass called primitive closed even walks.

Definition 2.3.5. Let I be a binomial ideal. A binomial $t^\alpha - t^\beta \in I$ is called **primitive** if there is no other binomial $t^\gamma - t^\delta \in I$ such that $t^\gamma | t^\alpha$ and $t^\delta | t^\beta$.

Remark 2.3.6. In our case of toric ideals associated with finite simple graphs primitive binomials correspond to closed even walks which cannot be obtained from the concatenation of two other closed even walks. All primitive binomials come from closed even walks in the graph.

Definition 2.3.7. The set of all primitive binomials is called a **Graver basis**. It is always finite for the toric ideal of a finite simple graph.

Proposition 2.3.8. *Let G be a finite simple graph. Then the set of primitive walks in G is a Gröbner basis with respect to any monomial ordering.*

Note that in particular this gives that the Graver basis is a **universal Gröbner basis**, that is a set which is a Gröbner basis for any possible monomial order.

Proposition 2.3.9. *Let G be a graph and let I_G be its toric ideal. If f is a polynomial in any reduced Gröbner bases of I_G then*

a) f is a primitive binomial and $f = f_w$ for some even closed walk of the graph G

b) If G is bipartite, then f is primitive and $f = f_w$ for some even cycle of the graph G

Proposition 2.3.10. *(Herzog and Hibi [19]) Let G be a bipartite graph. Then the following are equivalent:*

i) Every cycle of G of length ≥ 6 has a chord.

ii) I_G possesses a quadratic Gröbner basis.

Since we will often be going back and forth between edges considered as edges of a graph G and considered as variables of the ring $\mathbb{K}[E(G)]$ the following definitions have proven useful for stating and proving results.

Definition 2.3.11. For a given finite simple graph G we define a map, $\mathfrak{W}$, from the set of walks $\mathcal{W}(G)$ (not necessarily closed or even) to the ring $\mathbb{K}[E(G)]$ to be

$$\mathfrak{W} : \mathcal{W}(G) \to \mathbb{K}[E(G)]$$

$$(e_1, e_2, \ldots, e_k) \mapsto \prod_{i=1}^{k} e_i$$

Definition 2.3.12. For a given finite simple graph G we define a map, $\mathcal{E}$, from the set of walks $\mathcal{W}(G)$ (not necessarily closed or even) to the ring $\mathbb{K}[E(G)]$ to be

$$\mathcal{E} : \mathcal{W}(G) \to \mathbb{K}[E(G)]$$

$$(e_1, e_2, \ldots, e_k) \mapsto \prod_{2|i} e_i$$

Definition 2.3.13. For a given finite simple graph G we define a map, $\mathcal{O}$, from the set of walks $\mathcal{W}(G)$ (not necessarily closed or even) to the ring $\mathbb{K}[E(G)]$ to be

$$\mathcal{O} : \mathcal{W}(G) \to \mathbb{K}[E(G)]$$

$$(e_1, e_2, \ldots, e_k) \mapsto \prod_{2\nmid i} e_i$$

Remark 2.3.14. First note that both Definition 2.3.12 and Definition 2.3.13 depend on the order of the walk. We also draw attention to the useful fact that for a closed even walk w in a finite simple graph G, we have $f_w = \mathcal{O}(w) - \mathcal{E}(w) \in I_G$.

Chapter 3

Improving a Result on the Splitting of Toric Ideals of Graphs

3.1 Existing Result on Splitting Toric Ideals of Graphs

In this Chapter we will present an improved result related to a paper on splitting of toric ideals of finite simple graphs written jointly by G. Favaccio, J. Hofscheir, G. Keiper and A. Van Tuyl [13]. The purpose of the paper was to present a splitting result analogous to C. Francisco, T. Ha, A. Van Tuyl [14] but for toric ideals rather than monomial ideals. We wished to answer the following questions:

Question 3.1.1.

i) Given a toric ideal I when do we have $I = I_1 + I_2$ where I_1 and I_2 are both toric ideals?

ii) If $I = I_1 + I_2$ what is the relation between the Betti numbers of I and the Betti numbers of I_1 and I_2?

In the paper [13] we managed to achieve some results in these directions focusing specifically on toric ideals associated with finite simple graphs. In this chapter we will present some results which offer improvement over those stated in the paper. However we begin by providing some background.

For the reader's benefit we start by recalling some concepts defined in the aforementioned paper. One of the key concepts was that of "gluing" two graphs together. This allows us to conceptualise a given graph as two smaller graphs "glued" together. As we will see the choice of subgraphs to be glued together is in no way canonical nor is the gluing of two given subgraphs.

Definition 3.1.2. [13] Let G_1 and G_2 be two finite simple graphs. Let $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ be subgraphs. Let $\psi : H_1 \to H_2$ be a graph isomorphism. Then we define the gluing of G_1 and G_2 along ψ, denoted $G_1 \bigsqcup_\psi G_2$, by the quotient

$$V(G_1 \sqcup_\psi G_2) = \{V(G_1) \sqcup V(G_2) : x \in H_1,\ x \sim \psi(x)\}$$

$$E(G_1 \sqcup_\psi G_2) = \{E(G_1) \sqcup E(G_2) : e \in H_1,\ e \sim \psi(e)\}$$

Example 3.1.3. Let

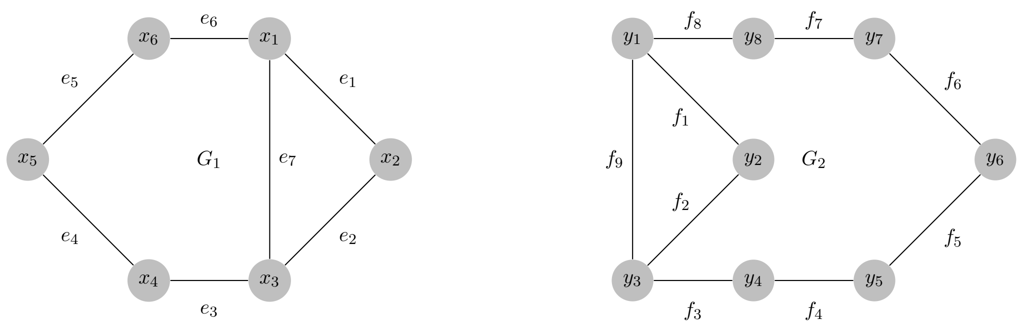
$$V(G_1) = \{x_1, x_2, \ldots, x_6\},\quad E(G_1) = \{e_i = \{x_i, x_{i+1}\} | 1 \le i \le 5\} \cup \{e_6 = \{x_6, x_1\}, e_7 = \{x_1, x_3\}\}$$

$$V(H_1) = \{x_1, x_2, x_3\}\ \ E(H_1) = \{e_1, e_2, e_7\}$$

and

$$V(G_2) = \{y_1, \ldots y_8\},\quad E(G_2) = \{f_i = \{y_i, y_{i+1}\} | 1 \le i \le 7\} \cup \{f_8 = \{y_8, y_1\}\}$$

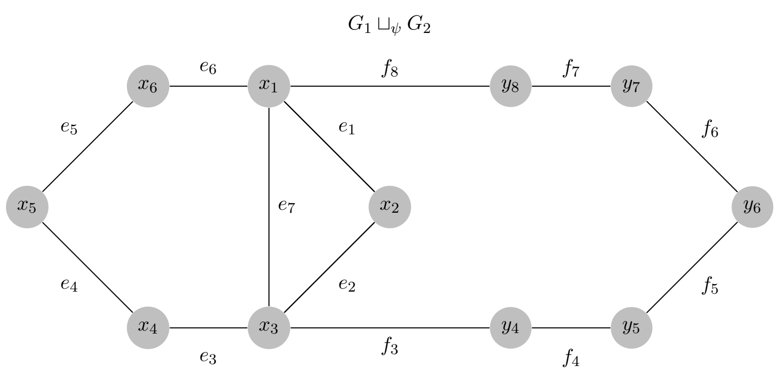
$$V(H_2) = \{y_1, y_2, y_3\},\quad E(H_2) = \{f_1, f_2, f_9\}$$



We define the graph isomorphism between H_1 and H_2 to be

$$\psi : H_1 \to H_2,\quad x_i \mapsto y_i \text{ for } i = 1, 2, 3$$

and obtain $G_1 \sqcup_\psi G_2$, where $V(G_1 \sqcup_\psi G_2) = \{x_1, \ldots, x_6, y_4, y_5, y_6, y_7, y_8\}$, $E(G_1 \sqcup_\psi G_2) = \{e_i = \{x_i, x_{i+1}\} | 1 \le i \le 5\} \cup \{e_6 = \{x_6, x_1\}, e_7 = \{x_1, x_3\}\} \cup \{f_i = \{x_i, x_{i+1}\} | 4 \le i \le 7\}$



Definition 3.1.4. [13] Let $G = (V(G), E(G))$ be a finite simple graph. Suppose there are

two subsets $W_1, W_2 \subseteq V(G)$ whose union gives $V(G)$, and denote the induced subgraph with vertex set W_i by G_i for $i = 1, 2$. Let $Y = W_1 \cap W_2$ and denote the corresponding induced subgraph by H. We say that *G_1 and G_2 form a splitting of G along H* if the graph obtained by removing the vertices Y from G yields two disconnected pieces.

We are now in a position to recall the result which we wish to expand on. We have included the proof of the original in order to make the exposition of the extension shorter and to give insight into the nature of the result.

Theorem 3.1.5 (Theorem 4.5 [13])**.** *Let G_1 and G_2 be a splitting of a graph G along a path graph $P_l \cong H \subseteq G$ which we describe as a walk $h = (h_1, \ldots, h_l)$ from a vertex x_1 to a vertex x_2 such that any vertex of H distinct from the endpoints considered as a vertex inside G has degree 2. If G_1 is bipartite, then we obtain.*

$$I_G = (I_{G_1} + I_{G_2}) : \mathcal{E}(h)^\infty,$$

where $\mathcal{E}(h)$ is the product of even indexed edges of h as per Definition 2.3.12

Remark 3.1.6. The fact that we have saturated with respect to only the even indexed edges of H is quite important. When $H = \{e_1\}$ there are no even indexed edges and we get the following corollary.

Corollary 3.1.7 (Corollary 4.8 [13])**.** *Let G be a graph, and suppose that G_1 and G_2 form a splitting of G along an edge e. If G_1 is bipartite, then $I_G = I_{G_1} + I_{G_2}$.*

3.2 Refinement of Result

The purpose of this chapter is to refine Theorem 1.4.3 to do away with the requirement that we take a saturation. We claim that we can replace f^∞ with f^2.

Theorem 3.2.1. *Let G_1 and G_2 be a splitting of a graph G along a path graph $P_l \cong H \subseteq G$ which we describe as a walk $h = (h_1, \ldots, h_l)$ from a vertex x_1 to a vertex x_2 such that any vertex of H distinct from the endpoints considered as a vertex inside G has degree 2. If G_1 is bipartite, then we obtain.*

$$I_G = (I_{G_1} + I_{G_2}) : \mathcal{E}(h)^2,$$

where $\mathcal{E}(h)$ is the product of even indexed edges of h as per Definition 2.3.12

Proof. Suppose that $\mathcal{E}^2 r \in I_{G_1} + I_{G_2}$. Since we have $I_{G_1} + I_{G_2} \subseteq I_G$ and I_G is prime it follows that $r \in I_G$ since $\mathcal{E}^2$ is a monomial. Hence $I_{G_1} + I_{G_2} : \mathcal{E}^2 \subseteq I_G$. For the converse inclusion we recall the setup of the proof of Theorem 3.1.5. Without loss of generality let the endpoints of H be called x_1 and x_2. By Proposition 2.3.8 we have that I_G is generated by binomials corresponding to primitive closed even walks p in G. Note that a primitive closed even walk p cannot contain a subpath in G_1 starting and ending at the same endpoint of H (otherwise, as G_1 is bipartite, this subpath would be even, and thus

p is a concatenation of closed even walks contradicting the fact that p was chosen to be primitive).

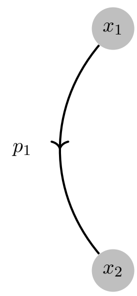
Let us label the edges in H by $h_1, \dots, h_l$ and the remaining edges in G_2 by $h_{l+1}, \dots, h_n$. Furthermore, label the edges in G_1 which are not contained in H by $e_1, \dots, e_m$. Using this notation, we can write a primitive closed even walk p as follows

$$p = (\underbrace{e_{i_{11}}, \dots, e_{i_{1r_1}}}_{:=p_1}, \overbrace{h_{j_{11}}, \dots, h_{j_{1s_1}}}^{:=q_1}, \underbrace{e_{i_{21}}, \dots, e_{i_{2r_2}}}_{:=p_2}, \overbrace{h_{j_{21}}, \dots, h_{j_{2s_2}}}^{:=q_2}, \dots,$$
$$\underbrace{e_{i_{u1}}, \dots, e_{i_{ur_u}}}_{:=p_u}, \overbrace{h_{j_{u1}}, \dots, h_{j_{us_u}}}^{:=q_u}). \quad (3.2.1)$$

We obtain subpaths $p_1, p_2, \dots, p_u$ that consist of edges of $E(G_1) \setminus E(G_2)$ as well as subpaths $q_1, q_2, \dots, q_u$ which consist of edges of $E(G_2)$ that begin at one of the endpoints of H, and end at the other endpoint. We note that since G_1 is bipartite we must have $r_i \equiv r_j (\bmod 2)$, if this were not the case we could find an odd cycle in G_1 since each p_i goes either from x_1 to x_2 or from x_2 to x_1 by the previous remark, and so either $p_i p_j$ or $p_i p_j^{-1}$ would be an odd cycle in G_1.

We show that $u \leq 2$ which follows from the fact that there are only a limited number of possible primitive closed even walks which are composed of edges belonging to both graphs.

Without loss of generality suppose that p starts at vertex x_1. The path goes first through the bipartite graph G_1 and cannot return to x_1 by our earlier remark without first passing through x_2. Therefore the subpath p_1 goes through $E(G_1) \backslash E(G_2)$ from x_1 to x_2.

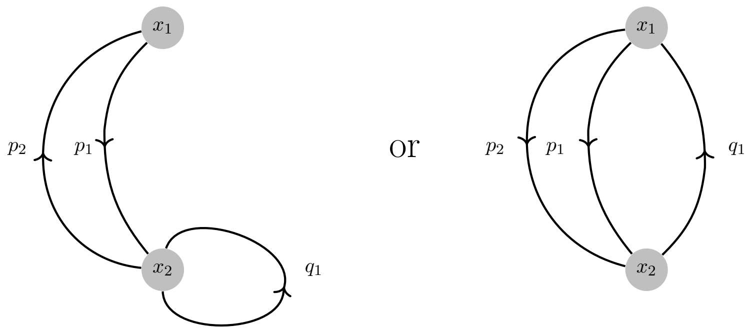


By definition of p_1 the path now must enter G_2. After entering G_2 the subpath q_1 must end either at x_1 or x_2.

If q_1 ends at x_2 we note that s_1, the length of q_1, must be odd or else p would not be primitive. Therefore it must be that s_1 is even. Since we do not yet have a closed even walk it follows that the path continues and the next subpath is p_2 which must go from x_2 to x_1. We call this case A.

If q_1 ends at x_1 and $r_1 \equiv s_1 (\bmod 2)$ then p_1q_1 is a closed even walk which is a subpath of p and since p was assumed to be primitive we must have $p = p_1q_1$ in which case $u = 1$.

If q_1 ends at x_1 and $r_1 \not\equiv s_1 (\bmod 2)$ then p_1q_1 is not a closed even subpath and so we must follow q_1 with p_2 which must go from x_1 to x_2 through G_1. We call this case B.



CASE A

CASE B

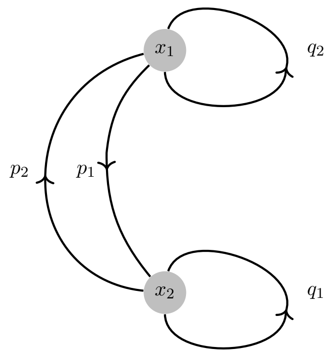
We now treat these two cases separately.

Suppose that we are in case B. Note that have $\mathcal{E}(p_1p_2^{-1})|\mathcal{E}(p)$ and $\mathcal{O}(p_1p_2^{-1})|\mathcal{O}(p)$ (this follows from the fact that r_1 and r_2 have the same parity and s_1 has a different parity). It follows that p is not primitive a contradiction.

Suppose instead we are in case A. Since q_1 has odd length and by our earlier observation that the lengths of p_1 and p_2 have the same parity we can see that $p_1q_1p_2$ is not a closed even walk. Therefore there must exist a further subpath q_2 which goes through G_2 and ends at either x_1 or x_2.

If q_2 ends at x_2 we see that $r_2 \not\equiv s_2 (\bmod 2)$ or else p_2q_2 is a closed even subpath. Therefore we assume that $s_2 \not\equiv r_2 (\bmod 2)$. However this also leads to a contradiction since now p_2q_2 has odd length, and thus $q_1p_2q_2$ is a closed even subpath contradicting the fact that p is primitive.

We must therefore conclude that q_2 is an odd lengthed path from x_1 to x_1 through G_2. We therefore have that $p_1q_1p_2q_2$ is a closed even walk and since p is primitive must have $p = p_1q_1p_2q_2$.



We conclude the proof by showing that our path path p in G is associated with a binomial f_p which is contained in $(I_{G_1} + I_{G_2}) : \mathcal{E}(h)^2$.

First note that if $u = 0$, then p is entirely contained in G_1 or G_2 and hence belongs to I_{G_1} or I_{G_2} and there is nothing to prove.

If there is at least one such path p_1, we proceed as follows. We denote the edges of the path graph H by the walk $h = (h_1, \dots, h_l)$ (ordered such that they form a path from x_1 to x_2).

Let $w_1 = p_1 h^{-1}$ and let α be the walk obtained by omitting p_1 from p.

$$\alpha = (\overbrace{h_{j_{11}}, \dots, h_{j_{1s_1}}}^{q_1}, \underbrace{e_{i_{21}}, \dots, e_{i_{2r_2}}}_{p_2}, \overbrace{h_{j_{21}}, \dots, h_{j_{2s_2}}}^{q_2}, \dots, \underbrace{e_{i_{u1}}, \dots, e_{i_{ur_u}}}_{p_u}, \overbrace{h_{j_{u1}}, \dots, h_{j_{us_u}}}^{q_u})$$

and define $p' = \alpha h$

Our goal is to now we express $\mathcal{E}(h) f_p$ as a linear combination of f_w and $f_{p'}$, that is,

$$\mathcal{E}(h) f_p = c_1 f_{w_1} + c_2 f_{p'}.$$

We see by construction that p' is a closed even walk which when decomposed as in (3.2.1) has one fewer paths through $E(G_1) \backslash E(G_2)$. We can then repeat our construction and express $f_{p'}$ as a linear combination of some $f_{w_2} \in I_{G_1}$ and $f_{p''}$.

$$\mathcal{E}(h) f_{p'} = c_3 f_{w_2} + c_4 f_{p''}$$

Since p has at most two subpaths through $E(G_1) \backslash E(G_2)$ we see that $f_{p''} \in I_{G_2}$. Thus we may obtain

$$\begin{aligned} \mathcal{E}(h)^2 f_p =& \mathcal{E}(h) c_1 f_{w_1} + c_2 c_3 f_{w_2} + c_2 c_4 f_{p''} \in (I_{G_1} + I_{G_2}) \\ \Rightarrow & f_p \in I_{G_1} + I_{G_2} : \mathcal{E}(h)^2 \end{aligned}$$

There are two cases depending on whether l is even or odd. If l is even, we can write

$$f_p = \mathcal{O}(p_1)\mathcal{O}(\alpha) - \mathcal{E}(p_1)\mathcal{E}(\alpha)$$

and

$$f_w = \mathcal{O}(p_1)\mathcal{O}(h^{-1}) - \mathcal{E}(p_1)\mathcal{E}(h^{-1}) \in I_{G_1}, \quad f_{p'} = \mathcal{O}(\alpha)\mathcal{O}(h) - \mathcal{E}(\alpha)\mathcal{E}(h).$$

We examine the following linear combination of f_w and $f_{p'}$:

$$\mathcal{O}(\alpha)f_w + \mathcal{E}(p_1)f_{p'} \tag{3.2.2}$$
$$= \mathcal{O}(\alpha)(\mathcal{O}(p_1)\mathcal{O}(h^{-1}) - \mathcal{E}(p_1)\mathcal{E}(h^{-1})) + \mathcal{E}(p_1)(\mathcal{O}(\alpha)\mathcal{O}(h) - \mathcal{E}(\alpha)\mathcal{E}(h)) \tag{3.2.3}$$
$$= \mathcal{O}(p_1)\mathcal{O}(\alpha)\mathcal{O}(h^{-1}) - \mathcal{E}(p_1)\mathcal{E}(\alpha)\mathcal{E}(h). \tag{3.2.4}$$

We note that since l is even we have $\mathcal{O}(h^{-1}) = \mathcal{E}(h)$ and so we obtain

$$\mathcal{E}(h)(\mathcal{O}(p_1)\mathcal{O}(\alpha) - \mathcal{E}(p_1)\mathcal{E}(\alpha)) = \mathcal{E}(h)f_p$$

as required.

Suppose instead that l is odd. Then we have

$$f_p = \mathcal{O}(p_1)\mathcal{E}(\alpha) - \mathcal{E}(p_1)\mathcal{O}(\alpha)$$

and

$$f_w = \mathcal{O}(p_1)\mathcal{E}(h^{-1}) - \mathcal{E}(p_1)\mathcal{O}(h^{-1}), \quad f_{p'} = \mathcal{O}(\alpha)\mathcal{E}(h) - \mathcal{E}(\alpha)\mathcal{O}(h)$$

$$\mathcal{E}(\alpha)f_w - \mathcal{E}(p_1)f_{p'} \tag{3.2.5}$$
$$= \mathcal{E}(\alpha)(\mathcal{O}(p_1)\mathcal{E}(h^{-1}) - \mathcal{E}(p_1)\mathcal{O}(h^{-1})) - \mathcal{E}(p_1)(\mathcal{O}(\alpha)\mathcal{E}(h) - \mathcal{E}(\alpha)\mathcal{O}(h)) \tag{3.2.6}$$
$$= \mathcal{E}(\alpha)\mathcal{O}(p_1)\mathcal{E}(h^{-1}) - \mathcal{E}(p_1)\mathcal{O}(\alpha)\mathcal{E}(h) \tag{3.2.7}$$

Since l is odd, we have $\mathcal{E}(h) = \mathcal{E}(h^{-1})$ and hence we have

$$\mathcal{E}(h)(\mathcal{O}(p_1)\mathcal{E}(\alpha) - \mathcal{E}(p_1)\mathcal{O}(\alpha)) = \mathcal{E}(h)f_p$$

as required.

□

Example 3.2.2. We provide a worked out example of an instance were we are required to use $\mathcal{E}(h)^2$ rather than $\mathcal{E}(h)$. Let G_1 be

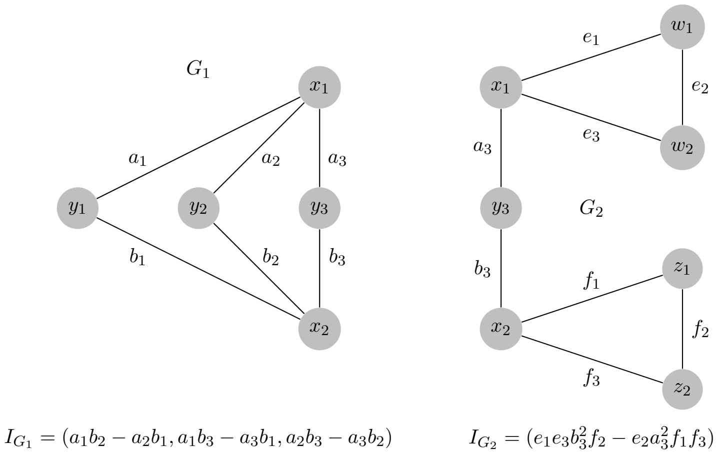
$$V(G_1) = \{x_1, x_2, y_1, y_2, y_3\}$$

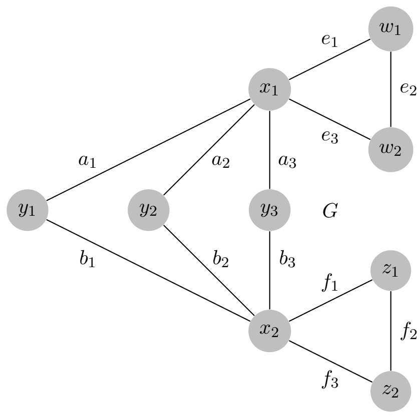
$$E(G_1) = \{a_1 = \{x_1, y_1\}, a_2 = \{x_1, y_2\}, a_3 = \{x_1, y_3\}, b_1 = \{x_2, y_1\}, b_2 = \{x_2, y_2\}, b_3 = \{x_2, y_3\}\}$$

$$V(G_2) = \{x_1, x_2, y_3, w_1, w_2, z_1, z_2\}$$

$$E(G_2) = \{e_1 = \{x_1, w_1\}, e_2 = \{w_1, w_2\}, e_3 = \{w_2, x_1\}, a_3 = \{x_1, y_3\}, b_3 = \{x_2, y_3\}, f_1 = \{x_2, z_1\}, f_2 = \{z_1, z_2\}, f_3 = \{z_2, x_2\}\}$$

$$V(G) = V(G_1) \cup V(G_2), \quad E(G) = E(G_1) \cup E(G_2)$$





$$I_G = (a_3b_2 - a_2b_3, a_3b_1 - a_1b_3, a_2b_1 - a_1b_2, b_3^2e_1e_3f_2 - a_3^2e_2f_1f_3, b_2b_3e_1e_3f_2 - a_2a_3e_2f_1f_3, b_1b_3e_1e_3f_2 - a_1a_3e_2f_1f_3, b_1^2e_1e_3f_2 - a_1^2e_2f_1f_3, b_2^2e_1e_3f_2 - a_2^2e_2f_1f_3, b_1b_2e_1e_3f_2 - a_1a_2e_2f_1f_3)$$

However we see that if we examine $(I_{G_1} + I_{G_2}) : (b_3)$ the last three generators are missing and we only have

$$(I_{G_1} + I_{G_2}) : (b_3) = (a_3b_2 - a_2b_3, a_3b_1 - a_1b_3, a_2b_1 - a_1b_2, b_3^2e_1e_3f_2 - a_3^2e_2f_1f_3, b_2b_3e_1e_3f_2 - a_2a_3e_2f_1f_3, b_1b_3e_1e_3f_2 - a_1a_3e_2f_1f_3)$$

It is precisely these three generators which are represented by closed even walks which pass through $E(G_1)\backslash E(G_2)$ twice and thus require b_3^2.

Corollary 3.2.3. *If both G_1 and G_2 are bipartite we can replace $\mathcal{E}(h)^2$ with $\mathcal{E}(h)$.*

Proof. Note that in the proof of Theorem 3.2.1 that $u = 2$ only in case A. Case A is only possible if G_2 is not bipartite, therefore when both G_1 and G_2 are bipartite we have $u = 1$ and hence $I_G = (I_{G_1} + I_{G_2}) : \mathcal{E}(h)$. □

Remark 3.2.4. Given that it was our goal to obtain such splitting results it makes sense that we focused on saturating with respect to $\mathcal{E}(h)$. In the proof of Theorem 3.2.1 we saw that because there are possibly two paths through G_1, p_1 and p_2 that we have to repeat our linear combination twice which pushed the degree up to two. However if we are no longer focused on achieving a direct sum (recall that we chose $\mathcal{E}$ because when $l = 1$ the product is empty and hence $I_G = I_{G_1} + I_{G_2}$) we see that we can replace $\mathcal{E}(h)^2$ with either of $\mathcal{O}(h)^2$ or $\mathcal{O}(h)\mathcal{E}(h) = \mathfrak{W}(h)$. We restate this in the corollary below.

Corollary 3.2.5. *Let G_1 and G_2 be a splitting of a graph G along a path graph $P_l \cong H \subseteq G$ which we describe as a walk $h = (h_1, \ldots, h_l)$ from a vertex x_1 to a vertex x_2 such that any vertex of H distinct from the endpoints considered as a vertex inside G has degree 2. If G_1 is bipartite, then we obtain.*

$$I_G = (I_{G_1} + I_{G_2}) : \mathcal{E}(h)^2 = (I_{G_1} + I_{G_2}) : \mathcal{O}(h)^2 = (I_{G_1} + I_{G_2}) : \mathfrak{W}(h),$$

Proof. It will suffice to repeat the proof of theorem 3.2.1 up to the portion of the proof where we show that we can express $\mathcal{E}(h)f_p$ as a linear combination $\mathcal{E}(h)f_p = c_1 f_w + c_2 f_{p'}$. If we show that it is also possible to obtain a linear combination

$$\mathcal{O}(h)f_p = c_1 f_w + f_{p'}$$

then we can replace $\mathcal{E}(h)$ with $\mathcal{O}(h)$ in either of the two possible steps.

Again we break into cases were l is even and odd. If l is even, we can write

$$f_p = \mathcal{O}(p_1)\mathcal{O}(\alpha) - \mathcal{E}(p_1)\mathcal{E}(\alpha)$$

and

$$f_w = \mathcal{O}(p_1)\mathcal{O}(h^{-1}) - \mathcal{E}(p_1)\mathcal{E}(h^{-1}) \in I_{G_1}, \quad f_{p'} = \mathcal{O}(\alpha)\mathcal{O}(h) - \mathcal{E}(\alpha)\mathcal{E}(h)$$

We examine the following linear combination of f_w and $f_{p'}$:

$$\mathcal{E}(\alpha)f_w + \mathcal{O}(p_1)f_{p'} \tag{3.2.8}$$

$$= \mathcal{E}(\alpha)(\mathcal{O}(p_1)\mathcal{O}(h^{-1}) - \mathcal{E}(p_1)\mathcal{E}(h^{-1})) + \mathcal{O}(p_1)(\mathcal{O}(\alpha)\mathcal{O}(h) - \mathcal{E}(\alpha)\mathcal{E}(h)) \tag{3.2.9}$$

$$= \mathcal{O}(p_1)\mathcal{O}(\alpha)\mathcal{E}(h^{-1}) - \mathcal{E}(p_1)\mathcal{E}(\alpha)\mathcal{O}(h). \tag{3.2.10}$$

We note that since l is even we have $\mathcal{E}(h^{-1}) = \mathcal{O}(h)$ and so we obtain

$$\mathcal{O}(h)(\mathcal{O}(p_1)\mathcal{O}(\alpha) - \mathcal{E}(p_1)\mathcal{E}(\alpha)) = \mathcal{O}(h)f_p$$

as required.

Suppose instead that l is odd. Then we have

$$f_p = \mathcal{O}(p_1)\mathcal{E}(\alpha) - \mathcal{E}(p_1)\mathcal{O}(\alpha)$$

and

$$f_w = \mathcal{O}(p_1)\mathcal{E}(h^{-1}) - \mathcal{E}(p_1)\mathcal{O}(h^{-1}), \quad f_{p'} = \mathcal{O}(\alpha)\mathcal{E}(h) - \mathcal{E}(\alpha)\mathcal{O}(h).$$

Thus

$$\mathcal{O}(\alpha)f_w - \mathcal{O}(p_1)f_{p'} \tag{3.2.11}$$
$$= \mathcal{O}(\alpha)(\mathcal{O}(p_1)\mathcal{E}(h^{-1}) - \mathcal{E}(p_1)\mathcal{O}(h^{-1})) - \mathcal{O}(p_1)(\mathcal{O}(\alpha)\mathcal{E}(h) - \mathcal{E}(\alpha)\mathcal{O}(h)) \tag{3.2.12}$$
$$= \mathcal{O}(h)\mathcal{O}(p_1)\mathcal{E}(\alpha) - \mathcal{O}(h^{-1})\mathcal{E}(p_1)\mathcal{O}(\alpha). \tag{3.2.13}$$

Since l is odd, we have $\mathcal{E}(h) = \mathcal{E}(h^{-1})$ and hence we have

$$\mathcal{O}(h)(\mathcal{O}(p_1)\mathcal{E}(\alpha) - \mathcal{E}(p_1)\mathcal{O}(\alpha)) = \mathcal{O}(h)f_p$$

as required.

□

Remark 3.2.6. We will see that in some sense the results of this chapter are a special case of Theorem 7.3.3 in Chapter 7. In both results we see that we are expressing a toric ideal in terms of a sum of the toric ideals of subgraphs and then saturating with variables belonging to the intersection of these subgraphs.

It seems probable that there is still more to be investigated with respect to splittings of toric ideals of finite simple graphs. For example it would be nice to have an answer to the following questions:

Question 3.2.7.

1. *Can Theorem 3.2.1 be adapted to the case where both G_1 and G_2 are not necessarily bipartite?*
2. *Can Theorem 3.2.1 be reformulated for the case where H is not a path graph?*
3. *In Theorem 4.11 in [13] a formula was obtained for the graded Betti numbers in the case $H = \{e_1\}$. Can we obtain similar formulas for other subgraphs H?*

Chapter 4

Syzygies of Toric Ideals of Complete Bipartite Graphs

In this chapter we study the graded Betti numbers of complete bipartite graphs. Complete bipartite graphs are of interest and have been well investigated in the past due to their high level of symmetry. We will utilise techniques developed by Campillo and Marijuan [6], Campillo and Pison [7], and Aramova and Herzog [1] to show that the first and second syzygies are linear and the corresponding first and second graded and total Betti numbers can be computed by explicit formulae. This will be done by making use of particular subgraphs and showing how they imply the existence of certain syzygies whenever they are embedded as a subgraph within a larger graph.

4.1 Motivational Question

Given the interest in toric ideals associated with complete bipartite graphs it would be useful if we knew something about their graded Betti numbers. More specifically we are interested in the following question:

Question 4.1.1. *Let $G = K_{n,m}$. Are there explicit formulas for $\beta_{i,j}(I_G)$ for $i, j \geq 0$?*

When this question was posed it was hoped that by restricting attention to the family of complete bipartite graphs it would make finding an answer easier as these graphs have a high level of symmetry.

We approached this question by noting that the image of a closed even walk of a graph provide us with a subgraph. We have seen how such subgraphs provide information about the generators, namely that such subgraphs can be associated to generators. Further the paper of Biermann, O'Keefe and Van Tuyl [3] shows how we can bound regularity using subgraphs. During our investigation we discovered how in an analogous manner we can examine certain subgraphs and get information about syzygies.

In an attempt to answer Question 4.1.1 we will first define and fix notation for complete bipartite graphs $K_{n,m}$ as well as their toric ideals $I_{K_{n,m}}$. We will then introduce the necessary machinery to provide a partial answer to Question 4.1.1 before going to provide formulas allowing us to compute the first three graded Betti numbers for toric ideals of complete bipartite graphs.

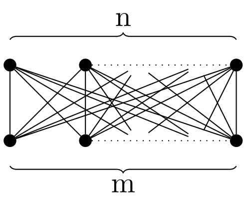
4.2 Minimal Generating sets of $I_{K_{n,m}}$

We start by recalling the definitions of complete bipartite graphs as well as fixing notation and labelling.

Definition 4.2.1. We define the **complete bipartite graph** on n and m vertices, denoted by $K_{n,m}$, to have the vertex and edge sets as follows: $V(K_{n,m}) = \{x_1, \dots, x_n, y_1, \dots, y_m\}$ $E(K_{n,m}) = \{e_{m(i-1)+j} = \{x_i, y_j\} : x_i, y_j \in V(K_{n,m})\}$

As mentioned in the introduction, complete bipartite graphs have been of interest, for example it was shown that the only toric ideals of finite simple graphs with linear resolutions are $K_{2,d}$. See Biermann, O'Keefe, and Van Tuyl [3] as well as D'Ali [9].

It will be useful to conceive of complete bipartite graphs $K_{n,m}$ as having the following arrangement in the plane:



where the top row of vertices are the $x_1, \dots, x_n$ and the bottom row are the $y_1, \dots y_m$. Hence when reading the labels of the edges from left to right from the top row of vertices we get $e_1, e_2, \dots, e_{nm}$.

The following lemma will be used in Proposition 4.2.3 to show that the set of binomials corresponding to 4-cycles is indeed a minimal generating set.

Lemma 4.2.2. *Let G be a finite simple graph and I_G its associated toric ideal with the standard grading (inherited from ϕ_G). Suppose that $f \in I_G$ is of degree 4 and that $f = \sum_{i=1} c_i g_i$ as in Lemma 2.2.34 where $f \neq g_i$ and no sum with fewer summands is equal to f. Then $f = c_1 g_1 + c_2 g_2$ where the variables making up g_1 and g_2 are a copy of the complete bipartite graph on 4 vertices, K_4.*

Proof. Let $f = e_1 e_3 - e_2 e_4 \in I_G$. Since there are no binomials of lower degree than 4 we conclude that the g_i must be degree 4 binomials and the coefficients c_i must be ± 1. Since the coefficients are ± 1 there must exist a g_i such that one of its terms is equal (up to ± 1) to $e_1 e_3$. This implies that the underlying vertices (which we will denote as x_1, x_2, x_3, and x_4) are also equal and so g_i is the binomial corresponding to a 4-cycle on the same 4 vertices (but with necessarily at least one different edge). The g_j which the other term of g_i cancels with must also have these 4 vertices. We see then that our attention is limited to 4-cycles contained in a copy of K_4. Let

$$V(K_4) = \{x_1, x_2, x_3, x_4\}$$

and

$$E(K_4) = \{e_1 = \{x_1, x_2\}, e_2 = \{x_2, x_3\}, e_3 = \{x_3, x_4\}, e_4 = \{x_4, x_1\}, e_5 = \{x_1, x_3\}, e_6 = \{x_2, x_4\}\}$$

We see that up to rotation the only 4-cycles are (e_1, e_2, e_3, e_4), (e_1, e_6, e_3, e_5), (e_5, e_2, e_6, e_4) which correspond to the binomials $e_1e_3 - e_2e_4$, $e_1e_3 - e_5e_6$ and $e_5e_6 - e_2e_4$. The linear combination must therefore be $e_1e_3 - e_2e_4 = (e_1e_3 - e_5e_6) + (e_5e_6 - e_2e_4)$. □

Proposition 4.2.3. *Let*

$$\mathcal{B}_{n,m} = \{F_{(i,k),(j,l)} = \{x_i, y_j\}\{x_k, y_l\} - \{y_j, x_k\}\{y_l, x_i\} : i, k \in [n],\ j, l \in [m],\ i < k,\ j < l\}$$

then $I_{K_{n,m}} = \langle \mathcal{B}_{n,m} \rangle$. Further $\mathcal{B}_{n,m}$ is a minimal generating set of $I_{K_{n,m}}$ which is a Gröbner basis under some monomial order.

Proof. We note that $K_{n,m}$ is such that every cycle of length ≥ 6 has a chord, hence by Proposition 2.3.10 we have that I_G possess a quadratic Gröbner basis under some monomial order. Since quadratic binomials in $I_{K_{n,m}}$ correspond to 4-cycles it follows that $I_{K_{n,m}}$ is generated by the set of binomials corresponding to all such 4-cycles. We can count the number of generators in this generating set by counting the number four cycles. This is simply the number of ways to select two vertices on the top and two vertices on the bottom, namely

$$\binom{m}{2}\binom{n}{2}$$

That $\mathcal{B}_{n,m}$ is in fact a minimal set of generators follows from the proof of Lemma 4.2.2 since there are no copies of K_4 and hence binomials corresponding to 4-cycles cannot be expressed in terms of other binomials. □

This provides us with the 0^{th} graded Betti numbers which correspond to the minimal generators of $I_{K_{n,m}}$.

Corollary 4.2.4.

$$\beta_{0,i}(I_{K_{n,m}}) = \begin{cases} \binom{m}{2}\binom{n}{2} & \text{if } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

4.3 Simplicial Methods for Toric Ideals

We begin with a review of some simplicial methods we will use to compute the graded Betti numbers of toric ideals of finite simple graphs.

Definition 4.3.1 (Theorem 7.9 [33])**.** Let G be a finite simple graph and I_G its toric ideal. Suppose that $\mathbb{K}[E(G)]$ has the induced grading. We define the set of all monomials in $\mathbb{K}[E(G)]$ of multidegree α to be the **fiber of** α. For $\alpha \in \mathbb{Z}_+^m$ we denote the fiber of α by C_α

Remark 4.3.2. Recall that we define the toric ideal of a finite graph as the kernel of the ring homomorphism $\varphi : \mathbb{K}[e_1, \ldots, e_n] \to \mathbb{K}[x_1, \ldots, x_m]$ where $e_i \mapsto \prod_{x_j \in e_i} x_j$, that is each edge is mapped to the product of its vertices. Thus if we take a monomial $x^\alpha \in \mathbb{K}[x_1, \ldots, x_m]$ we see that the fiber of α as defined above is simply $\varphi^{-1}(x^\alpha)$, hence the name fiber.

We now fix notation for the multidegrees we will need for the toric ideals $I_{K_{n,m}}$

Definition 4.3.3. For $I_{K_{n,m}}$ the toric ideal associated with the complete biparite graph on n and m vertices we define a multidegree

$$\alpha = (\alpha_{x_1}, \ldots, \alpha_{x_n}, \alpha_{y_1}, \ldots, \alpha_{y_m})$$

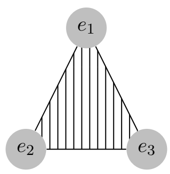
to be such that that α_{x_i} corresponds to the degree of x_i, $i = 1, \ldots, n$ and α_{y_j} corresponds to the degree of y_j, $j = 1, \ldots, m$ (recall that for e such that $\phi(e) = x_i y_j$ we say that e has multidegree $(0, \ldots, 0, \overbrace{1}^{i}, 0, \ldots, 0, \overbrace{1}^{n+j}, 0, \ldots, 0)$).

Definition 4.3.4 (Theorem 7.9 [33])**.** Let $\Gamma(\alpha)$ be the simplicial complex on vertices $e_1, \ldots, e_n$ whose faces are the radicals of monomials in C_α and all of their factors. We say that $\Gamma(\alpha)$ is generated by the radicals of the monomials in C_α

Example 4.3.5. Let $G = K_{3,3}$ the complete bipartite graph on 3 and 3 vertices.

$$\begin{aligned} V(G) &= \{x_1, x_2, x_3, y_1, y_2, y_3\} \\ E(G) &= \{e_1 = \{x_1, y_1\}, e_2 = \{x_1, y_2\}, e_3 = \{x_1, y_3\}, e_4 = \{x_2, y_1\}, e_5 = \{x_2, y_2\}, \\ &\qquad e_6 = \{x_2, y_3\}, e_7 = \{x_3, y_1\}, e_8 = \{x_3, y_2\}, e_9 = \{x_3, y_3\}\} \end{aligned}$$

Let our multidegree be $\alpha = (3, 0, 0, 1, 1, 1)$ where the first three entries are the degrees of x_i, $i = 1, 2, 3$ and the next three entries correspond to the degrees of y_j, $j = 1, 2, 3$. Then the fiber of Definition 4.3.1 is $C_\alpha = \{e_1e_2e_3\}$ and the complex of Definition 4.3.4, $\Gamma(\alpha)$, is simply the simplex $\Gamma(\alpha) = \{e_1e_2e_3, e_1e_2, e_1e_3, e_2e_3, e_1, e_2, e_3, \emptyset\}$:



The following theorem of Aramova and Herzog will be the key ingredient in establishing formulas for the graded Betti numbers of complete bipartite graphs.

Theorem 4.3.6 ([1]). *Let G be a finite simple graph and I_G its toric ideal. For a multidegree $\alpha \in \mathbb{N}^r$ and $i \geq 0$ we have*

$$\beta_{i,\alpha}(I_G) = \dim(\widetilde{H}_i\left(\Gamma(\alpha);\mathbb{K}\right))$$

Remark 4.3.7. We have restricted our attention to toric ideals associated with finite simple graphs however this result holds for arbitrary toric ideals.

4.4 The Minimal Syzygies and Presentations of $I_{K_{n,m}}$

We are now in a position to state one of the main results of this chapter, namely an explicit formula for $\beta_1(I_{K_{n,m}})$. After we have done so we will provide an explicit list of a minimal set of first syzygies for $I_{K_{n,m}}$. One should note that formulas for the graded Betti numbers of the entire resolution is known in the case of $I_{K_{2,d}}$, see for example [3].

Theorem 4.4.1.

$$\beta_{1,i}(I_{K_{n,m}}) = \begin{cases} 2\left(\binom{m}{2}\binom{n}{3} + \binom{m}{3}\binom{n}{2}\right) + 4\binom{m}{3}\binom{n}{3} & \text{if } i = 6 \\ 0 & \text{otherwise} \end{cases}$$

Proof. We will prove the formula by showing that

$$\beta_{1,\alpha}(I) = \dim(\widetilde{H}_1\left(\Gamma(\alpha);\mathbb{K}\right)) = 0$$

when $|\alpha| \neq 6$ and that

$$\sum_{|\alpha|=6} \dim(\widetilde{H}_1\left(\Gamma(\alpha);\mathbb{K}\right)) = 2\left(\binom{m}{2}\binom{n}{3} + \binom{m}{3}\binom{n}{2}\right) + 4\binom{m}{3}\binom{n}{3}$$

when $|\alpha| = 6$

We will show that we do not need to check every degree α when attempting to compute the $\beta_{1,\alpha}$. Using Proposition 4.2.3 we obtain a minimal generating set which is also a Gröbner basis under some monomial order. Therefore we can pick such a monomial order and we obtain a square free quadratic Gröbner basis. Recall from Proposition 2.2.27 we have

$$\beta_{i,j}\left(I_{K_{n,m}}\right) \leq \beta_{i,j}\left(\mathrm{LT}\left(I_{K_{n,m}}\right)\right)$$

where $\mathrm{LT}(I_{K_{n,m}})$ is the leading term ideal of $I_{K_{n,m}}$ under the given monomial ordering.

Our goal is to establish $\beta_{i,j}\left(\mathrm{LT}\left(I_{K_{n,m}}\right)\right) = 0$ for certain values of i and j which by the inequality above implies that $\beta_{i,j}(I_{K_{n,m}}) = 0$.

Recall that Corollary 2.2.30 tells us that the minimal free resolution of a monomial ideal is contained in the Taylor resolution of this ideal and hence the graded Betti numbers of the Taylor resolution are an upper bound for the graded Betti numbers of the monomial ideal. We note that $\mathrm{LT}(I_{K_{n,m}})$ is a monomial ideal generated by square free degree 2 monomials (degree 4 in the induced grading), hence the taylor resolution has syzygies which are bounded by degree 4 (degree 8 in the induced grading) and our resolution has syzygies which are also bounded by degree 2. Thus $\beta_{1,j}(I_{K_{n,m}}) = \beta_{1,j}(\mathrm{LT}(I_{K_{n,m}})) = 0$ when $j > 8$.

It follows that we only need to check for syzygies of degree 8 or less in the induced grading.

Up to permutation of the vertices of $K_{n,m}$ there are only 8 multidegrees of total degree 8 which have non-empty fibers (for example the preimage of x_1^8 is empty because each edge must correspond to an x_i and y_j and no product of edges maps to a single vertex). Therefore we can restrict our attention to $K_{4,4}$ because it contains all possible cases which can be realised as fibers noting that for larger graphs we could simply permute variables to obtain these cases and because of the symmetry of $K_{n,m}$ this changes nothing.

The fibers are as follows where the first 4 entries correspond the vertices x_1, x_2, x_3, x_4 (which we perceive as being on the top) and the last 4 correspond to the vertices y_1, y_2, y_3, y_4 (which we can perceive as being on the bottom), that is of the form: $(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$ Our 8 possible multidegrees are:

1. $\alpha_1 = (4, 0, 0, 0, 1, 1, 1, 1)$
2. $\alpha_2 = (3, 1, 0, 0, 1, 1, 1, 1)$
3. $\alpha_3 = (3, 1, 0, 0, 2, 1, 1, 0)$
4. $\alpha_4 = (2, 2, 0, 0, 2, 1, 1, 0)$
5. $\alpha_5 = (2, 2, 0, 0, 2, 2, 0, 0)$
6. $\alpha_6 = (2, 1, 1, 0, 1, 1, 1, 1)$
7. $\alpha_7 = (2, 1, 1, 0, 2, 1, 1, 0)$
8. $\alpha_8 = (1, 1, 1, 1, 1, 1, 1, 1)$

which correspond to the complexes

1. $\Gamma(\alpha_1) = \{e_1e_2e_3e_4\}$
2. $\Gamma(\alpha_2) = \{e_1e_2e_3e_8, e_1e_2e_4e_7, e_1e_3e_4e_6, e_2e_3e_4e_5\}$
3. $\Gamma(\alpha_3) = \{e_1e_2e_3e_5, e_1e_2e_7, e_1e_3e_6\}$
4. $\Gamma(\alpha_4) = \{e_1e_6e_7, e_2e_3e_5, e_1e_2e_5e_7, e_1e_3e_5e_6\}$
5. $\Gamma(\alpha_5) = \{e_1e_2e_5e_6\}$
6. $\Gamma(\alpha_6) = \{e_1e_2e_7e_{12}, e_1e_2e_8e_{11}, e_1e_3e_6e_{12}, e_1e_3e_8e_{10}, e_1e_4e_6e_{11}, e_1e_4e_7e_{10}\}$
7. $\Gamma(\alpha_7) = \{e_1e_6e_{11}, e_1e_7e_{10}, e_1e_2e_5e_{11}, e_1e_2e_7e_9, e_1e_3e_5e_{10}, e_1e_3e_6e_9, e_2e_3e_5e_9\}$

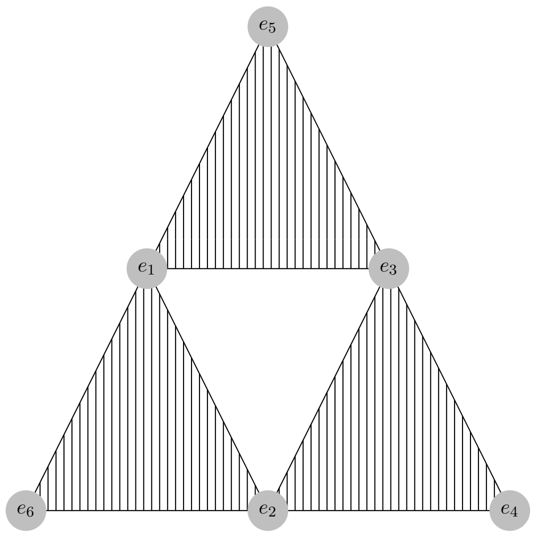
8. $\Gamma(\alpha_8) = \{e_ie_je_ke_l : i \in \{1,2,3,4\},\ j \in \{5,6,7,8\},\ k \in \{9,10,11,12\},$
$l \in \{13,14,15,16\},\ i \not\equiv j \not\equiv k \not\equiv l \pmod 4\}$

We then compute that $\widetilde{H}_1(\Gamma(\alpha_i)) = 0$ for $i = 1, \dots, 8$. Note that these computations can be performed using a variety of techniques, we chose to simply compute the reduced homology of these complexes using Macaulay2 [16]. Thus we can conclude that $\beta_{1,8}(I_{K_{4,4}}) = 0$.

Since for larger graphs of type $K_{n,m}$ we would simply be permuting these eight vertices (which makes no difference because of the symmetry of the $K_{n,m}$ means you are essentially examining the corresponding subgraph). We then can restrict our attention to syzygies of degree 1 (i.e. linear syzygies) (these have degree 2 in the induced grading). In this case there are even fewer combinations to check, namely:

1. (3,0,0,1,1,1)
2. (2,1,0,1,1,1)
3. (2,1,0,2,1,0)
4. (1,1,1,1,1,1)

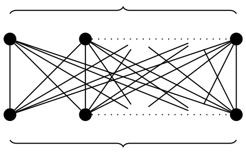
For illustrative purposes we examine a fiber which has nontrivial homology namely, $(2,1,0,1,1,1)$ which has fiber $\{e_1e_6e_2, e_5e_1e_3, e_3e_2e_4\}$ which in turn has corresponding complex:



By inspection we can see that

$$\dim_{\mathbb{K}} \left(\widetilde{H}_1(\Gamma(2,1,0,1,1,1); \mathbb{K}) \right) = 1$$

We now count up how many syzygies of this type exist in $K_{n,m}$. It is useful to recall our embedding of $K_{n,m}$ to aid in counting the syzygies.



Counting the syzygies corresponding to multidegrees $(2,1,0,1,1,1)$ up to permutation is really just the number of ways to arrange the 2 and 1 on the left in the degrees corresponding to vertices on the "top" and the three 1's on the right to the degrees corresponding to vertices on the "bottom". Thus we select two vertices from the top (the x vertices) and 3 from the bottom (the y vertices). Choose which of the ones on the top will have degree 2, there are two ways to do this hence

$$2\binom{n}{2}\binom{m}{3}$$

Now do the same selecting 3 from the top and 2 from the bottom. It follows that the total number of syzygies of this type is

$$2\left(\binom{n}{2}\binom{m}{3}+\binom{n}{3}\binom{m}{2}\right)$$

This is of course the first part of the given formula in Theorem 4.4.1 that we wish to prove.

Using Macaulay 2 we compute that the reduced homology of the remaining multidegrees:

$$\dim_{\mathbb{K}}(\widetilde{H}_1(\Gamma((3,0,0,1,1,1));\mathbb{K}))=0$$
$$\dim_{\mathbb{K}}(\widetilde{H}_1(\Gamma((2,1,0,2,1,0));\mathbb{K}))=0$$

and that

$$\dim_{\mathbb{K}}(\widetilde{H}_1(\Gamma((1,1,1,1,1,1));\mathbb{K}))=4$$

We note that the number of ways to obtain a mutlidegree $(1,1,1,1,1,1)$ is equal to the number of ways to select 3 x-vertices and 3 y-vertices and is hence

$$\binom{n}{3}\binom{m}{3}.$$

Since such multidegrees provide 4 syzygies the total contribution is

$$4\binom{n}{3}\binom{m}{3}.$$

Adding the syzygies obtained from multidegrees $(2,1,0,1,1,1)$ to those of $(1,1,1,1,1,1)$ we obtain

$$2\left(\binom{n}{2}\binom{m}{3}+\binom{n}{3}\binom{m}{2}\right)+4\binom{n}{3}\binom{m}{3}$$

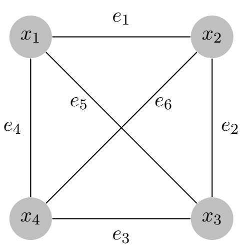
as required. □

We can obtain the following corollary immediately.

Corollary 4.4.2. *$I_{K_{n,m}}$ has a linear presentation.*

Having discovered a formula for computing the number of minimal syzygies of $I_{K_{n,m}}$ we might now ask whether we can go further and obtain explicit sets of minimal syzygies for $I_{K_{n,m}}$. In order to achieve this goal we refine a result of Biermann, O'Keefe and Van Tuyl [3]. In their paper they show how graded Betti numbers are related between induced subgraphs and graphs. However one may note that minimal generating sets are not canonical. Further we may even have that when the graded Betti numbers are in agreement it may not be the case that there is a copy of the resolution of I_H within I_G.

Example 4.4.3. One should note that in general generating sets are not unique and hence neither are minimal free resolutions (despite being isomorphic). For example consider K_4.



We have 3 potential minimal generating sets (up to a factor of ± 1) which are given by selecting any two of the following three elements: $e_1e_3 - e_2e_4$, $e_1e_3 - e_6e_5$, and $e_2e_4 - e_5e_6$.

In fact we note that this is the set of all closed even primitive walks and that by selecting any two we can obtain the third one as a sum or difference of the chosen two.

Therefore we see in such a case even at the 0^{th} homological degree of a minimal free resolution we have the generators of the free module mapping to different elements in the toric ideal depending on our choice of generators. Thus minimal free resolutions are not unique in such a case.

To remedy this we will now show that if we fix a minimal generating set of I_H and extend it to a minimal generating set of I_G we see that the resolution of I_H is in fact directly contained in the resolution of I_G.

Lemma 4.4.4 (Replacement Lemma for Subgraphs)**.** *Let G be a finite simple graph and $H \subseteq G$ an induced subgraph. Let $\iota : H \to G$ be the inclusion map and let $\tilde{\iota} : \mathbb{K}[E(H)] \to \mathbb{K}[E(G)]$ be the induced map on the corresponding polynomial rings. Then if $\{g_1, \dots, g_n\}$ is a minimal generating set of I_H, it can be extended to a minimal generating set $\{\tilde{\iota}(g_1), \dots, \tilde{\iota}(g_n), h_{n+1}, \dots, h_m\}$ for I_G.*

Proof. Let $\{g_1, \dots, g_n\}$ be a minimal generating set for I_H as above. Let $\{h_1, \dots, h_m\}$ be a minimal generating set of I_G. What we wish to do is extend the set $\{\tilde{\iota}(g_1), \dots, \tilde{\iota}(g_n)\}$ to a minimal generating set for I_G. Since $\{h_1, \dots, h_m\}$ is a minimal generating set for each $\tilde{\iota}(g_i)$ we may write $\tilde{g_i} = \sum_j c_{i,j} h_j$ with the minimal number of terms. By Proposition 2.2.34 the $c_{i,j}$ are monomials. We list a fixed choice of n such expressions. Up to relabelling suppose that $\{h_1, \dots, h_k\}$ are the generators which have a non-zero coefficient in at least one of the expressions.

We claim that $k = n$ and that we may replace $\{h_1, \dots, h_n\}$ with $\{\tilde{\iota}(g_1)), \dots, \tilde{\iota}(g_n))\}$ to form a minimal generating set $\{\tilde{\iota}(g_1), \dots, \tilde{\iota}(g_n), h_{n+1}, \dots, h_m\}$ for I_G. Since $\tilde{g_i}$ is a binomial this means that for a given monomial order we have $\mathrm{LT}(c_{i,j}h_j) | \mathrm{LT}(\tilde{\iota} g_i))$, hence the support of $\mathrm{LT}(c_{i,j}h_j)$ belongs to the support of $\mathrm{LT}(\tilde{g_i})$ and hence that the support of $\varphi(\mathrm{LT}(c_{i,j}h_j))$ belongs to the support of $\varphi(\mathrm{LT}(\tilde{\iota}(g_i))$ which in turn is contained in H. Since these binomials belong to our toric ideal it follows that the non-leading term's image under φ is a product of the same vertices. Since the non-leading term cancels with some other term, it follows that the binomial of the term it cancels with has the same support on the vertices.

We can continue inductively to conclude that all binomials have the same vertex support which belongs to H. Since we assumed that H is an induced subgraph it follows that it contains all edges between it vertices that are also in G, and hence all of $h_1, \dots, h_k$ belong to I_H. It follows that since $\{h_1, \dots, h_k\}$ generates $\{\tilde{\iota}(g_1), \dots, \tilde{\iota}(g_n)\}$ it generates $\tilde{\iota}(I_H)$. It follows that that $n \le k$. Since $\{\tilde{\iota}(g_1), \dots, \tilde{\iota}(g_n)\}$ is a minimal generating set we see that it can also generate any of the h_i and hence $\{\tilde{\iota}(g_1), \dots, \tilde{\iota}(g_n), h_{k+1}, \dots, h_m\}$ is a generating set of I_G. Since we assumed $\{h_1, \dots, h_m\}$ to be minimal it follows that $k = n$ and we are done. ☐

Remark 4.4.5. An alternative proof can be found in Reyes, Tatakis and Thoma [31]

Proposition 4.4.6. *Let G be a finite simple graph, $H \subseteq G$ be an induced sub-graph and let $\iota : H \to G$ be the inclusion map. Let $\tilde{\iota} : \mathbb{K}[E(H)] \to \mathbb{K}[E(G)]$ be the induced map on polynomial rings. Let $F(I_H)_i$ be a minimal free resolution of I_H over a generating set $\{g_1, \dots, g_n\}$ and $F(I_G)_i$ be a minimal free resolution of I_G over a generating set $\{\tilde{\iota}(g_1, \dots \tilde{\iota}(g_n), h_{n+1}, \dots, h_m\}$. We can consider both I_H and I_G as $\mathbb{K}[E(G)]$-modules with $E(G) \backslash E(H)$ acting trivially on I_H. Then the injective map $\tilde{\iota}$ induces an injective chain* *map on the minimal free resolutions. It follows then that the minimal syzygies of I_H are minimal syzygies of I_G for the given choice of generating sets.*

Proof. First we note that the minimal free resolution $F(I_H)$ is projective and hence we can lift $\tilde{\iota}$ to a chain map from $F(I_H)$ to $F(I_G)$.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & F_2(I_H) & \xrightarrow{\partial_2'} & F_1(I_H) & \xrightarrow{\partial_1'} & F_0(I_H) & \xrightarrow{\partial_0'} & I_H \\ & \downarrow \tilde{\iota}_2 & & \downarrow \tilde{\iota}_1 & & \downarrow \tilde{\iota}_0 & & \downarrow \tilde{\iota} \\ \cdots \longrightarrow & F_2(I_G) & \xrightarrow{\partial_2} & F_1(I_G) & \xrightarrow{\partial_1} & F_0(I_G) & \xrightarrow{\partial_0} & I_G \end{array}$$

We show that this chain map is injective inductively. It is clear that $\tilde{\iota}$ is injective. Thus suppose $\tilde{\iota}_l$ is injective. We wish to show that $\iota_{l+1}^{\sim}$ is injective. Suppose then that $\iota_{l+1}^{\sim}(S) = 0$ for some syzygy $S = c_1S_1 + \cdots c_tS_t \in F_{l+1}(I_H)$ where $S_1, \ldots S_t$ are a minimal generating set of $F_{l+1}(I_H)$. By induction we know that the generators of $F_l(I_H)$ are contained in the generators of $F_l(I_G)$ and hence the relations between them in $F_l(I_H)$ will equal the relations between them in $F_l(I_G)$ (or else $\tilde{\iota}_l$ would fail to be injective). We then see that $\tilde{\iota}_l(c_1S_1 + \cdots c_tS_t) = \tilde{\iota}(c_1)\tilde{\iota}_l(S_1) + \cdots + \tilde{\iota}(c_t)\tilde{\iota}_l(S_t) = 0$ if and only if $c_1S_1 + \cdots + c_tS_t = 0$, and hence $\tilde{\iota}_{l+1}$ is injective as required. □

Definition 4.4.7. We define the following three families of graph homomorphisms $A = \left\{\varphi_{A^{(n,m)}_{(i_1,i_2,j_1,j_2,j_3)}}\right\}$, $B = \left\{\varphi_{B^{(n,m)}_{(i_1,i_2,i_3,j_1,j_2)}}\right\}$, $C = \left\{\varphi_{C^{(n,m)}_{(i_1,i_2,i_3,j_1,j_2,j_3)}}\right\}$ as follows:

$\varphi_{A^{(n,m)}_{(i_1,i_2,j_1,j_2,j_3)}} : K_{2,3} \mapsto K_{n,m}$ is a graph homomorphism

$$x_1 \mapsto x_{i_1} \quad x_2 \mapsto x_{i_2} \quad y_1 \mapsto y_{j_1} \quad y_2 \mapsto y_{j_2} \quad y_3 \mapsto y_{j_3}$$

such that $i_1 < i_2$ and $j_1 < j_2 < j_3$.

$\varphi_{B^{(n,m)}_{(i_1,i_2,i_3,j_1,j_2)}} : K_{3,2} \mapsto K_{n,m}$ is a graph homomorphism

$$x_1 \mapsto x_{i_1} \quad x_2 \mapsto x_{i_2} \quad x_3 \mapsto x_{i_3} \quad y_1 \mapsto y_{j_1} \quad y_2 \mapsto y_{j_2}$$

such that $i_1 < i_2 < i_3$ and $j_1 < j_2$

$\varphi_{C^{(n,m)}_{(i_1,i_2,i_3,j_1,j_2,j_3)}} : K_{3,3} \mapsto K_{n,m}$ is a graph homomorphism

$$x_1 \mapsto x_{i_1} \quad x_2 \mapsto x_{i_2} \quad x_3 \mapsto x_{i_3} \quad y_1 \mapsto y_{j_1} \quad y_2 \mapsto y_{j_2} \quad y_3 \mapsto y_{j_3}$$

such that $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$

Remark 4.4.8. We first note any map in family A can be thought of as being defined by two maps $\varphi_1 : [2] \to [n]$ and $\varphi_2 : [3] \to [m]$ such that $\varphi_{A^{(n,m)}_{(i_1,i_2,j_1,j_2,j_3)}}(x_i) = x_{\varphi_1(i)}$ and $\varphi_{A^{(n,m)}_{(i_1,i_2,j_1,j_2,j_3)}}(y_i) = y_{\varphi_2(i)}$. Using these maps we can write the action on the map on the edges e_r as

$$\varphi(e_r) = \varphi(e_{3(i-1)+j}) = e_{m(\varphi_1(i)-1)+\varphi_2(j)}$$

where $r = 3(i-1)+j$ is obtained via the divison algorithm. We can write the action on the generators $F_{(i,k),(j,l)}$ as

$$\varphi(F_{(i,k),(j,l)}) = F_{(\varphi_1(i),\varphi_1(k)),(\varphi_2(j),\varphi_2(l))}$$

The map acts in the exact same manner on the free module corresponding to these generators in the first step of the resolution.

We can define similar maps φ_1 and φ_2 for the maps in the other two families.

We also note that $|A| = \binom{n}{2}\binom{m}{3}$, $|B| = \binom{n}{3}\binom{m}{2}$ and $|C| = \binom{n}{3}\binom{m}{3}$. We will show in the following proposition how these account for the first syzygies of $I_{K_{n,m}}$

Proposition 4.4.9. *Let $G = K_{n.m}$ and consider the toric ideal $I_{K_{n,m}}$. Let*

$$\bigoplus_{\{(i,k),(j,l):F_{(i,k),(j,l)}\in\mathcal{B}_{2,3}\}} R^A_{(i,k),(j,l)}$$

be the first step of the minimal graded free resolution of $I_{2,3}$ with respect to the minimal generators $\mathcal{B}_{2,3}$. Let

$$\bigoplus_{\{(i,k),(j,l):F_{(i,k),(j,l)}\in\mathcal{B}_{3,2}\}} R^B_{(i,k),(j,l)}$$

be the first step of the minimal graded free resolution of $I_{3,2}$ with respect to the minimal generators $\mathcal{B}_{3,2}$. Let

$$\bigoplus_{\{(i,k),(j,l):F_{(i,k),(j,l)}\in\mathcal{B}_{3,3}\}} R^C_{(i,k),(j,l)}$$

be the first step of the graded minimal free resolution of $I_{3,3}$ with respect to the minimal generators $\mathcal{B}_{3,3}$. We have that

- $\Lambda_1 = e_3R^A_{(1,2),(1,2)} - e_2R^A_{(1,2),(1,3)} + e_1R^A_{(1,2),(2,3)}$
- $\Lambda_2 = -e_6R^A_{(1,2),(1,2)} + e_5R^A_{(1,2),(1,3)} - e_4R^A_{(1,2),(2,3)}$

are a set of minimal syzygies for $I_{2,3}$

- $\Xi_1 = e_5R^B_{(1,2),(1,2)} - e_3R^B_{(1,3),(1,2)} + e_1R^B_{(2,3),(1,2)}$
- $\Xi_2 = e_6R^B_{(1,2),(1,2)} - e_4R^B_{(1,3),(1,2)} + e_2R^B_{(2,3),(1,2)}$

are a set of minimal syzygies for $I_{3,2}$

- $\Pi_1 = e_7R^C_{(1,2),(2,3)} + e_6R^C_{(1,3),(1,2)} - e_5R^C_{(2,3),(1,2)} + e_1R^C_{(2,3),(2,3)}$

- $\Pi_2 = -e_8R^C_{(1,2),(1,3)} + e_5R^C_{(1,3),(1,3)} - e_3R^C_{(2,3),(1,3)} - e_1R^C_{(2,3),(2,3)}$

- $\Pi_3 = -e_9R^C_{(1,2),(1,2)} + e_6R^C_{(1,3),(1,2)} - e_2R^C_{(2,3),(1,3)} + e_1RC_{(2,3),(2,3)}$

- $\Pi_4 = -e_9R^C_{(1,2),(1,2)} + e_8R^C_{(1,2),(1,3)} - e_4R^C_{(1,3),(2,3)} + e_1R^C_{(2,3),(2,3)}$

are a set of minimal syzygies for $I_{3,3}$.

Then a minimal set of syzygies of $I_{K_{n,m}}$ *is given by*

$$\{\varphi_A(\Lambda_{k_a}), \varphi_B(\Xi_{k_b}), \varphi_C(\Pi_{k_c})|\ \varphi_A \in A,\ \varphi_B \in B,\ \varphi_C \in C,\ k_a, k_b \in [2],\ k_c \in [4]\}$$

Proof. One should first note that since the generators of $K_{n,m}$ always correspond to the set of all 4-cycles by Proposition 4.2.3, since we see the families φ_A, φ_B, and φ_C take 4-cycles to 4-cycles they will take generators $I_{2,3}$, $I_{3,2}$ and $I_{3,3}$ to generators of $I_{K_{n,m}}$. We also note that φ_A, φ_B, and φ_C have images which are induced subgraphs, it follows that for a given multidegree α which is supported on those vertices that the simplicial complex of α for the toric ideal of the induced subgraph and the simplicial complex of α for the toric ideal of $K_{n,m}$ will be identical. Since Λ_k, Ξ_k, Π_k all correspond the minimal syzygies of $I_{2,3}$, $I_{3,2}$ and $I_{3,3}$ respectively then their images are still syzygies which have the correct multidegree and hence they correspond to the minimal syzygies of $I_{K_{n,m}}$. ☐

Corollary 4.4.10. *The syzygy matrix of* $I_{K_{n,m}}$ *consists of columns which have either three or four nonzero entries all of which are variables of* $\mathbb{K}[E(G)]$ *(that is degree one monomials).*

Proof. This follows immediately. ☐

4.5 Second Syzygies of $I_{K_{n,m}}$

We employ a similar technique as in Theorem 4.4.1 in order to obtain a formula for $\beta_2(I_{K_{n,m}})$. This technique could in principle be applied to compute formulas for arbitrarily high syzygies, however, as we will see, it becomes increasingly unwieldy.

Proposition 4.5.1.

$$\beta_{2,i}(I_{K_{n,m}}) = \begin{cases} 3\left(\binom{n}{4}\binom{m}{2} + \binom{n}{2}\binom{m}{4}\right) + 9\binom{n}{3}\binom{m}{3} + 15\left(\binom{n}{4}\binom{m}{3} + \binom{n}{3}\binom{m}{4}\right) + 15\binom{n}{4}\binom{m}{4} & \textit{if } i = 8 \\ 0 & \textit{otherwise} \end{cases}$$

Proof. As in the proof of Theorem 4.4.1 we will show that we do not need to check every degree α when attempting to compute the $\beta_{2,\alpha}$.

By Proposition 4.2.3 we have a minimal generating set which is also a Gröbner basis under some monomial order. Therefore we can pick such a monomial order and we obtain a square free quadratic Gröbner basis. Recall from Proposition 2.2.27 we have

$$\beta_{i,j}\left(I_{K_{n,m}}\right) \leq \beta_{i,j}\left(\mathrm{LT}\left(I_{K_{n,m}}\right)\right)$$

where $\mathrm{LT}(I_{K_{n,m}})$ is the leading term ideal of $I_{K_{n,m}}$ under the given monomial ordering.

Our goal is to establish $\beta_{i,j}\left(\mathrm{LT}\left(I_{K_{n,m}}\right)\right) = 0$ for certain values of i and j which by the inequality above implies that $\beta_{i,j}(I_{K_{n,m}}) = 0$.

Recall that Corollary 2.2.30 tells us that the minimal free resolution of a monomial ideal is contained in the Taylor resolution of this ideal and hence the graded Betti numbers of the Taylor resolution are an upper bound for the graded Betti numbers of the monomial ideal. We note that $\mathrm{LT}(I_{K_{n,m}})$ is a monomial ideal generated by square free degree 2 monomials (degree 4 in the induced grading), hence the taylor resolution has syzygies which are bounded by degree 4 (degree 8 in the induced grading) and second syzygies which are bounded by degree 6 (degree 12 in the induced grading) and hence the resolution of $I_{K_{n,m}}$ has second syzygies which are also bounded by degree 12. Thus $\beta_{2,j}(I_{K_{n,m}}) = \beta_{2,j}(\mathrm{LT}(I_{K_{n,m}})) = 0$ when $j > 12$.

Hence we need only check degrees 8, 10 and 12.

Once again not all multi-degrees have non-empty fibers, the non-empty multi-degrees will be the ones which correspond to each x-vertex being match by a y-vertex. Thus for degree 10 up to a permutation of the vertices of $K_{n,m}$ we must check the following multi-degrees:

$A = \{(i_1, i_2, i_3, i_4, i_5, j_1, j_2, j_3, j_4, j_5) : i_k, j_k \in \mathbb{Z}_{>0},$
$i_5 \leq i_4 \leq i_3 \leq i_2 \leq i_1 \geq j_1 \geq j_2 \geq j_3 \geq j_4 \geq j_5,\ \sum_{k=1}^{5} i_k = 5 = \sum_{k=1}^{5} j_k\}$

We then verify using Macaualy2 (or some other method) that for $\alpha \in A$ we have $\dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha); \mathbb{K})) = 0$

Similarly for degree 12 up to a permutation of the vertices of $K_{n,m}$ we have multi-degrees:

$B = \{(i_1, i_2, i_3, i_4, i_5, i_6, j_1, j_2, j_3, j_4, j_5, j_6) : i_k, j_k \in \mathbb{Z}_{>0},\quad i_6 \leq i_5 \leq i_4 \leq i_3 \leq i_2 \leq i_1 \geq j_1 \geq j_2 \geq j_3 \geq j_4 \geq j_5 \geq j_6,\ \sum_{k=1}^{6} i_k = 6 = \sum_{k=1}^{6} j_k\}$

and similarly verify using Macaualy2 (or some other method) that for $\alpha \in B$ we have $\dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha); \mathbb{K})) = 0$

Note that one can save a great deal of time by computing on Macaulay2 that $I_{K_{6,6}}$ has no total degree 12 second syzygies and that $I_{K_{5,5}}$ has no degree 10 second syzygies which implies that the complexes in question also have trivial second homology.

We now will use Macaulay2 to investigate the possible multi-degrees which have total degree equal to 8 (under the induced grading). We investigate the complexes corresponding to

$\alpha_1 = (4, 0, 0, 0, 1, 1, 1, 1),\quad \Gamma(\alpha_1) = \{e_1 e_2 e_3 e_4\},\quad \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_1); \mathbb{K})) = 0$

$\alpha_2 = (4,0,0,0,2,1,1,0),\ \Gamma(\alpha_2) = \{e_1e_2e_3\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_2);\mathbb{K})) = 0$

$\alpha_3 = (4,0,0,0,2,2,0,0),\ \Gamma(\alpha_3) = \{e_1e_2e_3\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_3);\mathbb{K})) = 0$

$\alpha_4 = (4,0,0,0,3,1,0,0),\ \Gamma(\alpha_4) = \{e_1e_5e_9\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_4);\mathbb{K})) = 0$

$\alpha_5 = (4,0,0,0,4,0,0,0),\ \Gamma(\alpha_5) = \{e_1e_5\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_5);\mathbb{K})) = 0$

$\alpha_6 = (3,1,0,0,1,1,1,1),\ \Gamma(\alpha_6) = \{e_1e_2e_3e_8, e_1e_2e_4e_7, e_1e_3e_4e_6, e_2e_3e_4e_5\}$

$\dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_6);\mathbb{K})) = 1$

$\alpha_7 = (3,1,0,0,2,1,1,0),\ \Gamma(\alpha_7) = \{e_1e_2e_7, e_1e_3e_6, e_1e_2e_3e_5\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_7);\mathbb{K})) = 0$

$\alpha_8 = (3,1,0,0,2,2,0,0),\ \Gamma(\alpha_8) = \{e_1e_2e_5, e_1e_2e_6\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_8);\mathbb{K})) = 0$

$\alpha_9 = (3,1,0,0,3,1,0,0),\ \Gamma(\alpha_9) = \{e_1e_6, e_1e_2e_5\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_9);\mathbb{K})) = 0$

$\alpha_{10} = (2,2,0,0,1,1,1,1),\ \Gamma(\alpha_{10}) = \{e_1e_2e_7e_8, e_1e_4e_6e_7, e_2e_3e_5e_8, e_2e_4e_5e_7, e_3e_4e_5e_6\}$

$\dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_{10});\mathbb{K})) = 1$

$\alpha_{11} = (2,2,0,0,2,1,1,0),\ \Gamma(\alpha_{11}) = \{e_1e_6e_7, e_1e_2e_5e_7, e_2e_3e_5\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_{11});\mathbb{K})) = 0$

$\alpha_{12} = (2,2,0,0,2,2,0,0),\ \Gamma(\alpha_{12}) = \{e_1e_6, e_1e_2e_5e_6, e_2e_5\},\ \dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_{12});\mathbb{K})) = 0$

$\alpha_{13} = (2,1,1,0,1,1,1,1)$

$$\Gamma(\alpha_{13}) = \{e_1e_2e_7e_{12}, e_1e_2e_8e_{11}, e_1e_3e_6e_{12}, e_1e_3e_8e_{10}, e_1e_4e_6e_{11}, \\ e_1e_4e_7e_{10}, e_2e_3e_5e_{12}, e_2e_3e_8e_9, e_2e_4e_5e_{11}, e_2e_4e_7e_9, e_3e_4e_5e_{10}, e_3e_4e_6e_9\}$$

$\dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_{13});\mathbb{K})) = 5$

$\alpha_{14} = (2,1,1,0,2,1,1,0)$

$\Gamma(\alpha_{14}) = \{e_1e_6e_{11}, e_1e_7e_{10}, e_1e_2e_5e_{11}, e_1e_2e_7e_9, e_1e_3e_5e_{10}, e_1e_3e_6e_9, e_2e_3e_5e_9\}$

$\dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_{14});\mathbb{K})) = 1$

$\alpha_{15} = (1,1,1,1,1,1,1,1)$

$\Gamma(\alpha_{15}) = \{e_ie_je_ke_l : i \in \{1,2,3,4\},\ j \in \{5,6,7,8\},\ k \in \{9,10,11,12\},\ l \in \{13,14,15,16\},\ i \not\equiv j \not\equiv k \not\equiv l \pmod 4\}$

$\dim_{\mathbb{K}}(\widetilde{H}_2(\Gamma(\alpha_{15});\mathbb{K})) = 15$

We see then that only $\alpha_6 = (3,1,0,0,1,1,1,1)$, $\alpha_{10} = (2,2,0,0,1,1,1,1)$, $\alpha_{13} = (2,1,1,0,1,1,1,1)$, $\alpha_{14} = (2,1,1,0,2,1,1,0)$, $\alpha_{15} = (1,1,1,1,1,1,1,1)$ have non-trivial second homology. In order to prove our formula we must count the number of ways we can obtain such a multidegrees.

First examining α_6 we note that we must select 2 vertices from top and 4 from bottom thus $\binom{n}{2}\binom{m}{4}$. We then see that there are two choices for weighting the top two vertices, one must receive weight 3 and one must receive weight 1. Thus we have $2\binom{n}{2}\binom{m}{4}$. Finally we could have selected 2 vertices from the top and 4 from the bottom at the beginning thus there are $2(\binom{n}{2}\binom{m}{4}+\binom{n}{4}\binom{m}{2})$ ways to obtain this mutlidegree. Since $\dim_{\mathbb{K}}(\tilde{H}_2(\Gamma(\alpha_6);\mathbb{K}))=1$ it follows that α_6 adds $2(\binom{n}{2}\binom{m}{4}+\binom{n}{4}\binom{m}{2})$ syzygies.

Next consider α_{10}. We again note that we must pick either 2 vertices from the top and 4 from the bottom or 2 from the bottom and 4 from the top. Thus $\binom{n}{2}\binom{m}{4}+\binom{n}{4}\binom{m}{2}$. Since there is no way to permute the vertex weights and $\dim_{\mathbb{K}}(\tilde{H}_2(\Gamma(\alpha_6);\mathbb{K}))=1$, it follows that α_{10} contributes $\binom{n}{2}\binom{m}{4}+\binom{n}{4}\binom{m}{2}$ syzygies.

Next consider α_{13}. We choose 3 vertices on the top and 4 on the bottom (or vice versa) giving $\binom{n}{3}\binom{m}{4}+\binom{n}{4}\binom{m}{3}$. We see that the degree 2 vertex can be chosen to be any of the 3 vertices on its side hence $3(\binom{n}{3}\binom{m}{4}+\binom{n}{4}\binom{m}{3})$. Finally we note that $\dim_{\mathbb{K}}(\tilde{H}_2(\Gamma(\alpha_{13});\mathbb{K}))=5$, hence each copy of the complex will add 5 syzygies. Therefore we conculde that α_{13} contributes $15(\binom{n}{3}\binom{m}{4}+\binom{n}{4}\binom{m}{3})$ syzygies.

Next consider α_{14}. We must pick 3 vertices from the top and 3 vertices from the bottom, hence $\binom{n}{3}\binom{m}{3}$. we note that on both the top and bottom there are three choices (and so 9 in total) as to which vertex will get degree 2, so we then have $9\binom{n}{3}\binom{m}{3}$. Then we note that $\dim_{\mathbb{K}}(\tilde{H}_2(\Gamma(\alpha_{14});\mathbb{K}))=1$ and so we conlcude that α_{14} contributes $9\binom{n}{3}\binom{m}{3}$ syzygies.

Finally we consider α_{15}. We must pick 4 vertices from the top and 4 vertices from the bottom, hence $\binom{n}{4}\binom{m}{4}$. We note that since all vertices must have degree 1 there are no ways to permute them. Finally we note that $\dim_{\mathbb{K}}(\tilde{H}_2(\Gamma(\alpha_{15});\mathbb{K}))=15$. Therefore we conclude that α_{15} contributes $15\binom{n}{4}\binom{m}{4}$ syzygies.

Since we have shown that no other degree types result in complexes with non-trivial second homology it follows that these are all of the syzygies and adding them together we obtain

$$3\left(\binom{n}{4}\binom{m}{2}+\binom{n}{2}\binom{m}{4}\right)+9\binom{n}{3}\binom{m}{3}+15\left(\binom{n}{4}\binom{m}{3}+\binom{n}{3}\binom{m}{4}\right)+15\binom{n}{4}\binom{m}{4}$$

as required. □

To conclude this chapter we note that one could likely apply similar techniques to compute the Betti numbers of other classes of graphs. It should be mentioned that though the proof uses simplicial methods the original intuition came from noting that certain syzygies came from certain types of subgraphs and all that needed to be done was count up the number of these subgraphs and multiply by the number of syzygies they introduced.

We then hope that it may be possible to give a description of all syzygies according to what subgraphs are present in a graph, breaking a larger problem into the sum of smaller ones with known results. We should note that in the specific cases we treated here, namely $\beta_0(I_{K_{n,m}})$, $\beta_1(I_{K_{n,m}})$ and $\beta_2(I_{K_{n,m}})$ are the only cases where the resolution is linear, once one investigates $\beta_3(I_{K_{n,m}})$ there are nonlinear syzygies.

Chapter 5

Fundamental Group Background

One of our goals is to introduce results linking toric ideals of finite simple graphs, I_G, to the fundamental groups of connected finite simple graphs, $\pi_1(G, x_0)$. The connection is based on the observation that a closed even walk w in G corresponds to both an element of the fundamental group $\pi_1(G, x_0)$ as well as a binomial f_w of the toric ideal I_G. In this chapter the necessary background in algebraic topology and group theory will be supplied to understand subsequent chapters involving the relationship between the fundamental group of a finite simple graph and the toric ideal of a finite simple graph. We begin with recalling some facts about free groups and free products of groups before moving on to the fundamental group and how these pertain to finite simple graphs.

5.1 Group Theory Concepts

Since the topic of the next two chapters of this thesis is the relationship between toric ideals of connected finite simple graphs and the fundamental group of these graphs, we should review what is known about the fundamental groups of finite simple graphs. One important fact is that the fundamental group of a finite simple graph is always a free group on a finite set of generators. We therefore recall the definition of free groups and free products of groups which will play a key role in subsequent chapters.

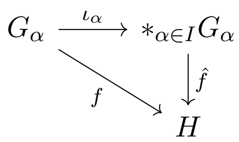
Definition 5.1.1. Given a set S we can define a free group on S denoted $\mathcal{F}(S)$ to be the unique group such that for any group G and set map $f : S \to G$, we have a unique group homomorphism

$$\begin{array}{ccc} S & \xrightarrow{\iota} & \mathcal{F}(S) \\ & \underset{f}{\searrow} & \downarrow \hat{f} \\ & & G \end{array}$$

Remark 5.1.2. Since the definition is abstract we note that in practice this works as follows: Let $S = \{a, b, c, d\}$, then $\mathcal{F}(S) = \langle a, b, c, d \rangle$ is the group generated by four elements which have no relations among them. Elements of $\mathcal{F}(S)$ are simply the finite "words" created from the letters a, b, c, d and their inverses. The group operation is simply the concatenation of words. For example $abcad * bdcd = abcadbdcd$. Clearly the identity element is the empty word since $g* = g$. Further every element has an inverse given by the word with the inverse of each of the letters in the reverse order. For example $abbda$ is an element which has an inverse $a^{-1}d^{-1}b^{-1}b^{-1}a^{-1}$.

We will also require the notion of a free product of groups in order to express Van Kampen's Theorem and apply an analogue to the case of toric ideals of finite simple graphs in Chapter 7.

Definition 5.1.3. Given groups G_α indexed by some set I we define their **free product** denoted by $*_{\alpha \in I} G_\alpha$ equipped with maps $\iota_\alpha : G_\alpha \to *_{\alpha \in I} G_\alpha$ to be the unique group such that for any group H and homomorphism $f : G_\alpha \to H$ there exits a unique homomorphism $\hat{f} : *_{\alpha \in I} G_\alpha \to H$ such that $f = \hat{f} \circ \iota_\alpha$ which is expressed by the following commutative diagram



Remark 5.1.4. If we want we can interpret the free group on a set as the free product of the infinite cyclic group generated by each element of the set S.

We now introduce the notion of a commutator subgroup. This will be made of use of to abelianize groups and to relate the fundamental group of a graph to the first homology group of the graph.

Definition 5.1.5. Let G be a group. Let $g, h \in G$. We define the **commutator of** g **and** h, denoted $[g, h]$, to be $[g, h] = ghg^{-1}h^{-1}$. Similarly we define the **commutator subgroup**, which we denote by $[G, G]$, to be $[G, G] = \{ghg^{-1}h^{-1} : g, h \in G\}$.

Remark 5.1.6. $[G, G]$ is always a normal subgroup of G. Further $G/[G, G]$ is always an abelian group and is called the **abelianization of** G.

Theorem 5.1.7 (Nielsen–Schreier Theorem [2])**.** *Every subgroup of a free group is free.*

Theorem 5.1.8 ([32] 2.3.6)**.** *Let F be a free group on $\{a_1, \ldots, a_n\}$. Then its abelianization $F/[F, F]$ has a presentation $\left\langle a_1, \ldots, a_n : a_i a_j a_i^{-1} a_j^{-1} \right\rangle$.*

5.2 Fundamental Group Background

We now introduce the definitions needed to understand the fundamental group construction following the treatment found in [18]:

Definition 5.2.1. Let X be a topological space and let $I = [0, 1]$ be the standard unit interval in $\mathbb{R}$. We call a continuous function $f : I \to X$ a **path** in X. We call $f(0)$ the initial point and $f(1)$ the terminal point.

Definition 5.2.2. We call a topological space X **path connected** if for every $x, y \in X$ there exists a path $f : I \to X$ such that $f(0) = x$ and $f(1) = y$.

Definition 5.2.3. Let X and Y be topological spaces. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions from X to Y. We call f and g **homotopic** and write $f \simeq g$ if there exists a continuous function $F : X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. We call such a function F a **homotopy between** f **and** g.

Definition 5.2.4. Given a subspace $A \subseteq X$ and a homotopy $F : X \times I \to Y$, we say that F is a **homotopy relative to** A if $F : A \times I \to Y$ is independent of I, that is $F(a, t) = F(a, s)$ for all $t, s \in I$. If $F(x, 0) = f$ and $F(x, 1) = g(x)$ we write $f \simeq g \text{ rel}(A)$.

Definition 5.2.5. We call $f : X \to Y$ and $g : Y \to X$ a **homotopy equivalence** if $f \circ g \simeq id_X$ and $g \circ f \simeq id_Y$. In such a case we say that X and Y are homotopy equivalent.

Definition 5.2.6. Given paths $f : I \to X$ and $g : I \to X$ where $f(1) = g(0)$ we define the **composition of paths** $f * g : I \to X$ to be the path

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

We are in a position to define the fundamental group of a space. The relationship between the fundamental of a finite simple graph and the toric ideal associated with a finite simple graph will be the focus of the remainder of the thesis.

Definition 5.2.7. Given a topological space X and a point $x_0 \in X$ we define **the fundamental group of** X **at basepoint** x_0 denoted $\pi_1(X, x_0)$ to be the set of all paths $f : I \to X$ such that $f(0) = f(1) = x_0$ up to homotopy $\text{rel}\{0, 1\}$. We write the homotopy class of f as $[f]$. We endow the set of equivalence classes with the group operation defined as follows: For $[f], [g] \in \pi_1(X, x_0)$ we have $[f] * [g] = [f * g]$ where $f * g$ is the path composition defined in 6.1.5.

We will now introduce a well established and very useful tool, the Van Kampen Theorem, which allows one to gain information about the fundamental group of a space from the fundamental group of its subspaces provided they satisfy certain conditions on their intersections.

Theorem 5.2.8 (Van Kampen's Theorem [18])**.** *Let X be the union of path connected open sets U_i indexed by some set J such that $x_0 \in \bigcap_{i\in J}^n U_i$. Further suppose that $U_i \cap U_k$ is path connected for all $i, k \in J$. Define $\iota_{ik} : \pi_1(U_i \cap U_k) \to U_i$ to be the map induced by the inclusion $U_i \cap U_k \hookrightarrow U_i$. Further let $j_i : \pi_1(U_i) \to \pi_1(X)$ be the maps induced by the inclusions $U_i \hookrightarrow X$. Then the homomorphism induced by the j_i is*

$$\Psi : *_{i\in J}\pi_1(U_i) \to \pi_1(X)$$

and it is a surjective homomorphism. If we further require that $U_i \cap U_k \cap U_l$ is also path connected, then the kernel of Ψ is given by the normal subgroup generated by $\iota_{ik}(\gamma)\iota_{kl}(\gamma)^{-1}$.

The following fact is needed for Theorem 6.4.2 in Chapter 7.

Theorem 5.2.9 (Theorem 2A.1 [18]). *By regarding loops as singular 1-cycles, we obtain a homomorphism $h : \pi_1(X, x_0) \to H_1(X)$. If X is path connected, then h is surjective and has kernel the commutator subgroup of $\pi_1(X, x_0)$, so h induces an isomorphism from the abelianization of $\pi_1(X)$ onto $H_1(X)$*

5.3 Fundamental Group of Finite Simple Graphs

Theorem 5.3.1 (Proposition 1A.2 [18]). *For a connected finite simple graph G with a maximal spanning tree T we have that $\pi_1(G, x_0)$ is a free group with a basis given by the loops γ_i corresponding to the edges $e_i \in E(G)\backslash E(T)$.*

Definition 5.3.2. Let G be a finite simple graph. Let $w = (e_1, \ldots, e_n)$ be a path. We define a length function $l(w) = n$. This map has the property that $l(w_1 \circ \cdots \circ w_k) = \sum_{i=1}^{k} l(w_i)$, $l(g) = l(g^{-1})$

Lemma 5.3.3. *For $\gamma \in \pi_1(G, x_0)$ there is a unique representative by a path of minimal length $(x_0, x_1, x_2, \ldots, x_n, x_0)$. Further all other representatives are of the form*

$$(x_0, \ldots, x_i, y_1, y_2, \ldots, y_{m-1}, y_m, y_{m-1}, \ldots, y_1, x_i, x_{i+1}, \ldots, x_n, x_0)$$

Proof. If this is not true then at some point it takes a different path and does not return along the same path. i.e. $(x_0, x_1, \ldots, x_i, y_1, \ldots, y_n, x_0)$ but then

$$(x_i, y_1, \ldots, y_n, x_0, x_n, x_{n-1}, \ldots, x_{i+1}, x_i)$$

must be trivial, which is a contradiction. □

It will often be easier to work with particular representatives of loops. With this in mind we now provide a definition which allows us to specify particularly useful representatives.

Definition 5.3.4. For $\gamma \in \pi_1(G, x_0)$ we denote its unique reduced representative as outlined in Lemma 5.3.3 to be $r(\gamma)$.

Remark 5.3.5. It is important to note that the above representative is merely homotopic to γ and not relatively homotopic i.e. we do not require the basepoint to be fixed.

Remark 5.3.6. If we consider the free group generated by the edges of a graph G, one could think of the situation as being equivalent to that of assigning each edge in the walk a direction which determines whether it is the edge or its inverse (this is given by the walk, the initial vertex is the terminal vertex of the previous edge) and then taking the reduced word in the free group.

Chapter 6

Fundamental Group and Toric Ideals

In this chapter we introduce a framework for relating the fundamental group of a finite simple graph G, $\pi_1(G, x_0)$ to the toric ideal associated with G, I_G. The idea for attempting to relate the two algebraic structures comes from simply noting that a given closed even walk w in a finite simple graph G correspond to an element in the fundamental group $\pi_1(G), x_0$ as well as a binomial f_w in the associated toric ideal I_G. In order to make use of this insight this chapter will provide us with the necessary machinery to exploit this relationship to obtain results in chapter 7. We begin by introducing a subgroup of $\pi_1(G, x_0)$ which we call the alternating fundamental group. We then define operations on the binomials of I_G which in some sense correspond to the operations of closed even walks in G and establish various properties of these operations. Finally we establish a of homomorphism from the fundamental group $\pi_1(G, x_0)$ to a group which has as its elements the binomials of I_G in Theorem 6.4.2.

6.1 The Alternating Fundamental Group

We wish to establish a relationship between the fundamental group of a graph and its toric ideal.

Notation 6.1.1. Defying typical convention, for the entirety of this chapter we introduce the convention that $x^{\alpha_+} - x^{\alpha_-}$ represents any binomial as opposed to binomials coming from integer vectors, hence allowing the terms to possibly not be coprime.

Definition 6.1.2. Given a connected finite simple graph G and a vertex x_0 we define the **alternating fundamental group** of G with basepoint x_0 to be the set of elements of $\pi_1(G, x_0)$ which can be represented by a path of even length. We denote this set of elements by $A(\pi_1(G, x_0))$.

Remark 6.1.3. $A(\pi_1(G, x_0))$ is well defined since for α, β walks in G such that $\alpha \simeq \beta$ by Lemma 5.3.3 they differ only by some number of paths traced and retraced. However this adds an even number of edges to the path and so the parity is independent on the choice of representative.

Proposition 6.1.4. *$A(\pi_1(G,x_0)) \trianglelefteq \pi_1(G,x_0)$ is a normal subgroup.*

Proof. Let $\gamma \in A(\pi_1(G,x_0))$. By Remark 6.1.3 we know that parity is independent of the choice of representative, hence we pick the minimal representative, say γ. Let $g \in \pi_1(G,x_0)$, then by Definition 5.3.2 we have

$$l(g\gamma g^{-1}) = l(g) + l(\gamma) + l(g^{-1}) = l(\gamma) + 2l(g)$$

and so $g\gamma g^{-1} \in A(\pi_1(G,x_0))$ as required. □

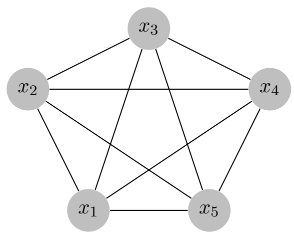
Proposition 6.1.5. *Let G be a finite simple graph. The following hold:*

1. *Every closed even walk w in G has a representative in $A(\pi_1(G,x_0))$*
2. *Given two representatives of elements in the fundamental group $\gamma_1 = (e_1,\dots,e_k)$ and $\gamma_2 = (f_1,\dots,f_l)$ (thought of as walks), the resulting walk $(e_1,\dots,e_k,f_1,\dots,f_l)$ is a representative of $\gamma_1\gamma_2$*

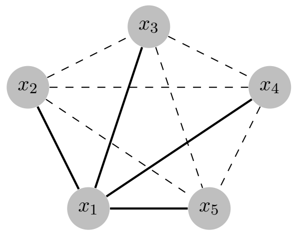
Proof. These follow directly from the definitions. □

Example 6.1.6. We provide on example of a graph G for which we compute a minimal generating set of $\pi_1(G,x_0)$ using Theorem 5.3.1. We then modify it to obtain a minimal generating set for the alternating fundamental group $A(\pi_1(G,x_0))$. Let $G = K_5$ which we label as follows:

$$V(G) = \{x_1,x_2,x_3,x_4,x_5\}$$
$$E(G) = \{e_1 = \{x_1,x_2\}, e_2 = \{x_1,x_3\}, e_3 = \{x_1,x_4\}, e_4 = \{x_1,x_5\}, e_5 = \{x_2,x_3\},$$
$$e_6 = \{x_2,x_4\}, e_7 = \{x_2,x_5\}, e_8 = \{x_3,x_4\}, e_9 = \{x_3,x_5\}, e_{10} = \{x_4,x_5\}\}$$



We select the maximal tree T given by $V(T) = \{x_1,\dots,x_5\}$, $E(T) = \{e_1,e_2,e_3,e_4\}$ and use the algorithm in 5.3.1.



We note that there are six edges $\{e_5, e_6, e_7, e_8, e_9, e_{10}\}$ which had to be removed from G in order to obtain T. Adding each one of these edges we introduce the following 3-cycles where $\gamma_1 = (e_1, e_5, e_2)$, $\gamma_2 = (e_1, e_6, e_3)$, $\gamma_3 = (e_1, e_7, e_4)$, $\gamma_4 = (e_2, e_8, e_3)$, $\gamma_5 = (e_2, e_9, e_4)$, $\gamma_6 = (e_3, e_{10}, e_4)$. From Theorem 5.3.1 we have $\pi_1(G, x_1) = \langle\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\rangle$ is a minimal presentation of $\pi_1(G, x_1)$.

In order to obtain the alternating fundamental group we could take a generating set $\{\gamma_i\gamma_j : i, j \in [6]\}$. We note that this is indeed a generating set since $\pi_1(G, x_1)$ is a free group thus for an element $\gamma \in A(\pi_1(G, x_1)) \subseteq \pi_1(G, x_1)$ there is a unique reduced word in terms of the γ_i $i = 1, \ldots 6$ such that $\gamma = \prod \gamma_{j_1}$. Since γ is even there must be an even number of the odd length walks γ_i $i = 1, \ldots 6$ and by grouping the γ_i into pairs we achieve the desired decomposition.

Remark 6.1.7. We can see from this relatively simple example that there are some difficulties associated with the alternating fundamental group as we have defined it. First we can note that for $\gamma_1 = (e_1, e_5, e_2)$, we have $\gamma_1^2 \in A(\pi_1(G, x_0))$. However, as we noted in the introduction of this chapter our goal is to associate elements of the alternating fundamental group with binomials in the toric ideal. In this specific case we see that $f_{\gamma_1^2} = 0$ and that an element which is a generator of $A(\pi_1(G, x_0))$ offers no real analogue in I_G. In the final chapter of this thesis we will speculate as to how the situation can potentially be rescued, however for the remainder of the thesis unless otherwise stated we will assume that we are working with bipartite finite simple graphs in which case we have $\pi_1(G, x_0) = A(\pi_1(G, x_0))$ which avoids these difficulties entirely.

Remark 6.1.8. For the remainder of the chapter we will restrict our attention to graphs G which are path connected. This being the case we will not have to worry about which basepoint we select since for such the graphs the fundamental groups at different basepoints are isomorphic.

6.2 Operations on Binomials

We will now introduce a number of definitions with the ultimate goal of allowing us to construct a group operation on the set of binomials in a toric ideal. These constructions are valid for any toric ideal. Our interest in this group is that we will ultimately construct a homomophism from the fundamental group of a graph to this group consisting of binomials in the toric ideal.

Definition 6.2.1. Let I be a binomial ideal. We define $\mathcal{B}(I)$ to be the set of binomials in I.

Definition 6.2.2. Let I be an ideal of a ring R. We define the non-commutative binary operation $\circ : \mathcal{B}(I) \times \mathcal{B}(I) \to \mathcal{B}(I)$ as follows: For $g_i = x^{\alpha_+} - x^{\alpha_-}$, $g_j = x^{\beta_+} - x^{\beta_-} \in \mathcal{B}(I)$

$$(x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-}) = \left(\frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\alpha_-}}\right) x^{\alpha_+} - \left(\frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\beta_+}}\right) x^{\beta_-}$$
$$= b^1_{i,j} g_i + b^2_{i,j} g_j.$$

We call $b^1_{i,j} = \frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\alpha_-}}$ and $b^2_{i,j} = \frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\beta_+}}$ the ***cancelation coefficients*** of g_1 and g_2. It is sometimes computationally useful to note that the exponents of the binomials are as follows:

$$(x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-}) = x^{\rho_+} - x^{\rho_-}$$
$$\rho^i_+ = \max(\alpha^i_-, \beta^i_+) - \alpha^i_- + \alpha^i_+$$
$$\rho^i_- = \max(\alpha^i_-, \beta^i_+) - \beta^i_+ + \beta^i_-$$

Remark 6.2.3. We note that the operation $\circ$ is not commutative.

Definition 6.2.4. We define the **reduction** of a binomial $x^{\alpha_+} - x^{\alpha_-} \in I_G$ to be

$$\operatorname{red}(x^{\alpha_+} - x^{\alpha_-}) = \frac{x^{\alpha_+} - x^{\alpha_-}}{\gcd(x^{\alpha_+}, x^{\alpha_-})}$$

Remark 6.2.5. Note that $\operatorname{red}(x^{\alpha_+} - x^{\alpha_-}) = 0$ implies that $x^{\alpha_+} - x^{\alpha_-} = 0$

Definition 6.2.6. We introduce the convention that $0 \in I_G$ is represented as a binomial by $1 - 1$.

Lemma 6.2.7. *Let $\gamma \in A(\pi_1(G, x_0))$. Then for any two walks w and v representing γ we have* $\operatorname{red}(f_w) = \operatorname{red}(f_v)$.

Proof. Let $w = (e_1, e_2, \dots, e_{2l})$ and $v = (f_1, \dots, f_{2k})$. By Definition 5.3.3 we have that w and v can both be reduced to $r(w) = r(v)$, a subsequence of w and v obtained by removing segments which are paths followed by their inverses i.e of the form pp^{-1}. We therefore must show that for some when we have a closed even walk $\gamma = \gamma_1 pp^{-1}\gamma_2$ then $\operatorname{red}(f_{\gamma_1 pp^{-1}\gamma_2}) = \operatorname{red}(f_{\gamma_1\gamma_2})$. Let a closed even walk be a concatenation of paths $\gamma = \gamma_1 pp^{-1}\gamma_2$. We note that $pp^{-1} = (g_1, \dots, g_j, g_j, \dots, g_1)$ and so

$$f_{\gamma_1 pp^{-1}\gamma_2} = g_1 \cdots g_k \mathcal{O}(\gamma_1)\mathcal{O}(\gamma_2) - g_1 \cdots g_k \mathcal{E}(\gamma_1)\mathcal{E}(\gamma_2)$$

when $l(\gamma_1)$ is even and

$$f_{\gamma_1 pp^{-1}\gamma_2} = g_1 \cdots g_k \mathcal{O}(\gamma_1)\mathcal{E}(\gamma_2) - g_1 \cdots g_k \mathcal{E}(\gamma_1)\mathcal{O}(\gamma_2)$$

when $l(\gamma_1)$ is odd. It follows that $g_1 \cdots g_k$ is removed by reduction, i.e. $\text{red}(f_{\gamma_1 pp^{-1}\gamma_2}) = \text{red}(f_{\gamma_1\gamma_2})$. □

The following operator captures some of the essential nature of $\circ$ but will greatly simplify some statements and their proofs.

Definition 6.2.8. We define a binary operation $\odot : \mathcal{B}(I) \times \mathcal{B}(I) \to \mathcal{B}(I)$ as follows: For $x^{\alpha_+} - x^{\alpha_-},\ x^{\beta_+} - x^{\beta_-} \in \mathcal{B}(I_G)$

$$(x^{\alpha_+} - x^{\alpha_-}) \odot (x^{\beta_+} - x^{\beta_-}) = x^{\beta_+}(x^{\alpha_+} - x^{\alpha_-}) - x^{\alpha_-}(x^{\beta_+} - x^{\beta_-}) = x^{\alpha_+}x^{\beta_-} - x^{\alpha_-}x^{\beta_+}$$

being a linear combination of elements in I_G It follows that the resulting element lies in I_G and since it is a binomial is in fact a binary operation on $\mathcal{B}(I_G)$.

Remark 6.2.9. The utility of this definition comes from the fact that if we conceive of $x^{\alpha_+} - x^{\alpha_-}$ and $x^{\beta_+} - x^{\beta_-}$ as corresponding to closed even walks then $(x^{\alpha_+} - x^{\alpha_-}) \odot (x^{\beta_+} - x^{\beta_-})$ corresponds to the concatenation of these walks which provides us with a representative of there "product" in the fundamental group which we make explicit in the following proposition.

Lemma 6.2.10. *$\odot : \mathcal{B}(I) \times \mathcal{B}(I) \to \mathcal{B}(I)$ is commutative, that is, for $x^{\alpha_+} - x^{\alpha_-}, x^{\beta_+} - x^{\beta_-} \in \mathcal{B}(I)$ we have $x^{\alpha_+} - x^{\alpha_-} \odot x^{\beta_+} - x^{\beta_-} = x^{\beta_+} - x^{\beta_-} \odot x^{\alpha_+} - x^{\alpha_-}$*

Proof. This follows immediately from the definition. □

Proposition 6.2.11. *Let $\gamma_1 = (e_1, \ldots, e_{2n})$ and $\gamma_2 = (f_1, \ldots, f_{2m})$ be representatives of elements in $A(\pi_1(G, x_0))$. Then the concatenation of paths is*

$$\gamma_1\gamma_2 = (e_1, \ldots, e_{2n}, f_1, \ldots, f_{2m})$$

The binomial this corresponds to is

$$\mathcal{O}(\gamma_1)\mathcal{O}(\gamma_2) - \mathcal{E}(\gamma_1)\mathcal{E}(\gamma_2) = f_{\gamma_1} \odot f_{\gamma_2} = f_{\gamma_1\gamma_2}.$$

Proof. This follows directly from the definitions. Note that as usual this is dependent on our choice of representatives for γ_1 and γ_2. □

The following proposition shows how these two operations are the same up to a monomial coefficient:

Proposition 6.2.12. *Let $g_i = x^{\alpha_+} - x^{\alpha_-}$ and $g_j = x^{\beta_+} - x^{\beta_-}$. The operations $\circ$ and $\odot$ are equivalent up to reduction:*

$$\text{red}((x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-})) = \text{red}((x^{\alpha_+} - x^{\alpha_-}) \odot (x^{\beta_+} - x^{\beta_-}))$$

Proof. Note that $\text{lcm}\left(b^1_{i,j}x^{\alpha_+}, b^2_{i,j}x^{\beta_-}\right)$ divides $\text{lcm}\left(x^{\alpha_+}x^{\beta_+}, x^{\alpha_-}x^{\beta_-}\right)$ and so the results follows. □

6.3 Properties of Binomial Operations

Having defined the binary operations we will be utilising, we now establish some of their properties.

Lemma 6.3.1. *Let I be a toric ideal of the ring R. The following statements hold:*

1. *$\mathcal{B}(I)$ is closed under $\circ$ and* red.
2. *$\circ$ is associative i.e.*

$$\Big((x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-})\Big) \circ (x^{\delta_+} - x^{\delta_-}) = (x^{\alpha_+} - x^{\alpha_-}) \circ \Big((x^{\beta_+} - x^{\beta_-}) \circ (x^{\delta_+} - x^{\delta_-})\Big).$$

3. $\operatorname{red}(\operatorname{red}(x^{\alpha_+} - x^{\alpha_-})) = \operatorname{red}(x^{\alpha_+} - x^{\alpha_-})$.
4. $\operatorname{red}(\operatorname{red}((x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-}))) = \operatorname{red}(\operatorname{red}(x^{\alpha_+} - x^{\alpha_-}) \circ \operatorname{red}(x^{\beta_+} - x^{\beta_-}))$.
5. *For $k \in \mathbb{K}[E(G)]$ we have*

$$k \cdot ((x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-})) = (k \cdot (x^{\alpha_+} - x^{\alpha_-})) \circ (k \cdot (x^{\beta_+} - x^{\beta_-})).$$

6. $0 \circ (x^{\alpha_+} - x^{\alpha_-}) = (x^{\alpha_+} - x^{\alpha_-}) \circ 0 = x^{\alpha_+} - x^{\alpha_-}$.
7. $(x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\alpha_-} - x^{\alpha_+}) = 0$.

Proof. 1. By definition we have:

$$\begin{aligned}(x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-}) &= \left(\frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\alpha_-}}\right) x^{\alpha_+} - \left(\frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\beta_+}}\right) x^{\beta_-} \\ &= \left(\frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\alpha_-}}\right) (x^{\alpha_+} - x^{\alpha_-}) - \left(\frac{\operatorname{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\beta_+}}\right) (x^{\beta_+} - x^{\beta_-})\end{aligned}$$

Therefore $(x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-})$ can be expressed as a linear combination of $(x^{\alpha_+} - x^{\alpha_-})$ and $(x^{\beta_+} - x^{\beta_-})$ hence belongs to I. Since the cancellation coefficients are always monomials, it further follows that $(x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-}) \in \mathcal{B}(I)$. To see that $\operatorname{red}(x^{\alpha_+} - x^{\alpha_-}) \in \mathcal{B}(I)$ we note that I is prime and that I contains no monomials.

2. Suppose that

$$\begin{aligned}\Big((x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\beta_+} - x^{\beta_-})\Big) \circ (x^{\delta_+} - x^{\delta_-}) &= (x^{\rho_+} - x^{\rho_-}) \circ (x^{\delta_+} - x^{\delta_-}) \\ &= x^{\lambda_+} - x^{\lambda_-}\end{aligned} \tag{6.3.1}$$

$$\begin{aligned}(x^{\alpha_+} - x^{\alpha_-}) \circ \Big((x^{\beta_+} - x^{\beta_-}) \circ (x^{\delta_+} - x^{\delta_-})\Big) &= (x^{\alpha_+} - x^{\alpha_-}) \circ (x^{\theta_+} - x^{\theta_-}) \\ &= x^{\kappa_+} - x^{\kappa_-}\end{aligned} \tag{6.3.2}$$

Then we must show that $\lambda_+ = \kappa_+$ and $\lambda_- = \kappa_-$. By definition we have:

$$\lambda_+^i = \max(\rho_-^i, \delta_+^i) - \rho_-^i + \rho_+^i$$
$$\lambda_-^i = \max(\rho_-^i, \delta_+^i) - \delta_+^i + \delta_-^i$$

and

$$\rho_+^i = \max(\alpha_-^i, \beta_+^i) - \alpha_-^i + \alpha_+^i$$
$$\rho_-^i = \max(\alpha_-^i, \beta_+^i) - \beta_+^i + \beta_-^i.$$

Hence

$$\begin{aligned}\lambda_+ &= \max(\max(\alpha_-^i, \beta_+^i) - \beta_+^i + \beta_-^i, \delta_+^i) - (\max(\alpha_-^i, \beta_+^i) - \beta_+^i + \beta_-^i) + (\max(\alpha_-^i, \beta_+^i) - \alpha_-^i + \alpha_+^i) \\ &= \max(\max(\alpha_-^i, \beta_+^i) - \beta_+^i + \beta_-^i, \delta_+^i) + \beta_+^i - \beta_-^i + \alpha_+^i - \alpha_-^i \\ &= \max(\alpha_-^i, \max(\beta_-^i, \delta_+^i) - \beta_-^i + \beta_+^i) - \alpha_-^i + \alpha_+^i\end{aligned}$$

$$\lambda_- = \max(\max(\alpha_-^i, \beta_+^i) - \beta_+^i + \beta_-^i, \delta_+^i) - \delta_+^i + \delta_-^i$$

Similarly

$$\kappa_+ = \max(\alpha_-^i, \theta_+^i) - \alpha_-^i + \alpha_+^i$$
$$\kappa_- = \max(\alpha_-^i, \theta_+^i) - \theta_+^i + \theta_-^i$$

and

$$\theta_+ = \max(\beta_-^i, \delta_+^i) - \beta_-^i + \beta_+^i$$
$$\theta_- = \max(\beta_-^i, \delta_+^i) - \delta_+^i + \delta_-^i.$$

Hence

$$\kappa_+ = \max(\alpha_-^i, \max(\beta_-^i, \delta_+^i) - \beta_-^i + \beta_+^i) - \alpha_-^i + \alpha_+^i$$

$$\begin{aligned}\kappa_- &= \max(\alpha_-^i, \max(\beta_-^i, \delta_+^i) - \beta_-^i + \beta_+^i) - (\max(\beta_-^i, \delta_+^i) - \beta_-^i + \beta_+^i) + \max(\beta_-^i, \delta_+^i) - \delta_+^i + \delta_-^i \\ &= \max(\alpha_-^i, \max(\beta_-^i, \delta_+^i) - \beta_-^i + \beta_+^i) + \beta_-^i - \beta_+^i - \delta_+^i + \delta_-^i \\ &= \max(\max(\alpha_-^i, \beta_+^i) - \beta_+^i + \beta_-^i, \delta_+^i) - \delta_+^i + \delta_-^i\end{aligned}$$

Hence we see that $\lambda_+ = \kappa_+$ and $\lambda_- = \kappa_-$ and so the operation is associative.

3) This is immediate from the fact that the terms of $\text{red}(x^{\alpha_+} - x^{\alpha_-})$ are coprime.

4) We will show that

$$\text{red}\left((x^{\alpha_+} - x^{\alpha_-}) \odot \left(x^{\beta_+} - x^{\beta_-}\right)\right) = \text{red}\left(\text{red}\left(x^{\alpha_+} - x^{\alpha_-}\right) \odot \text{red}\left(x^{\beta_+} - x^{\beta_-}\right)\right)$$

and then appeal to 6.2.12 to obtain

$$\text{red}\left((x^{\alpha_+} - x^{\alpha_-}) \circ \left(x^{\beta_+} - x^{\beta_-}\right)\right) = \text{red}\left(\text{red}\left(x^{\alpha_+} - x^{\alpha_-}\right) \circ \text{red}\left(x^{\beta_+} - x^{\beta_-}\right)\right)$$

By definition

$$\text{red}\left((x^{\alpha_+} - x^{\alpha_-}) \odot \left(x^{\beta_+} - x^{\beta_-}\right)\right) = \frac{x^{\alpha_+}x^{\beta_+} - x^{\alpha_-}x^{\beta_-}}{\gcd(x^{\alpha_+}x^{\beta_+}, x^{\alpha_-}x^{\beta_-})}.$$

Further

$$\text{red}\,(x^{\alpha_+} - x^{\alpha_-}) \odot \text{red}\left(x^{\beta_+} - x^{\beta_-}\right) = \frac{x^{\alpha_+}x^{\beta_+} - x^{\alpha_-}x^{\beta_-}}{\gcd(x^{\alpha_+}, x^{\alpha_-})\gcd(x^{\beta_+}, x^{\beta_-})}$$

Since $\gcd(x^{\alpha_+}, x^{\alpha_-})\gcd(x^{\beta_+}, x^{\beta_-})$ is clearly a common divisor of $x^{\alpha_+}x^{\beta_+}$ and $x^{\alpha_-}x^{\beta_-}$, it follows it divides $\gcd(x^{\alpha_+}x^{\beta_+}, x^{\alpha_-}x^{\beta_-})$ and that the two binomials will be equal after reduction.

5)

$$\begin{aligned}
(k\cdot(x^{\alpha_+} - x^{\alpha_-}))\circ(k\cdot(x^{\beta_+} - x^{\beta_-})) &= (kx^{\alpha_+} - kx^{\alpha_-})\circ(kx^{\beta_+} - kx^{\beta_-})) \\
&= \left(\frac{\text{lcm}(kx^{\alpha_-}, kx^{\beta_+})}{kx^{\alpha_-}}\right)kx^{\alpha_+} - \left(\frac{\text{lcm}(kx^{\alpha_-}, kx^{\beta_+})}{kx^{\beta_+}}\right)kx^{\beta_-} \\
&= \left(\frac{k\,\text{lcm}(x^{\alpha_-}, x^{\beta_+})}{kx^{\alpha_-}}\right)kx^{\alpha_+} - \left(\frac{k\,\text{lcm}(x^{\alpha_-}, x^{\beta_+})}{kx^{\beta_+}}\right)kx^{\beta_-} \\
&= k\cdot((x^{\alpha_+} - x^{\alpha_-})\circ(x^{\beta_+} - x^{\beta_-}))
\end{aligned}$$

6) The proof is a straightforward application of the definitions:

$$\begin{aligned}
(1-1)\circ(x^{\alpha_+} - x^{\alpha_-}) &= \left(\frac{\text{lcm}(1, x^{\alpha_+})}{1}\right)1 - \left(\frac{\text{lcm}(1, x^{\alpha_+})}{x^{\alpha_+}}\right)x^{\alpha_-} \\
&= x^{\alpha_+} - x^{\alpha_-}
\end{aligned}$$

with the proof of the reverse order being nearly identical.

7) Again we prove the theorem for the operator $\odot$ and then note that $\text{red}(0) = 0$ and so the result hold for $\circ$.

$$(x^{\alpha_+} - x^{\alpha_-})\odot(x^{\alpha_-} - x^{\alpha_+}) = x^{\alpha_+}x^{\alpha_-} - x^{\alpha_-}x^{\alpha_+} = 0$$

□

Since we have now established that the operation $\circ$ is associative we can extend our notion of cancellation coefficients introduced in Definition 6.2.2. This notion of cancellation coefficients will be very important in proving some results about syzygies of toric ideals.

Definition 6.3.2. Since we now have that $\circ$ is associative we extend the notation of **cancellation coefficients** beyond the binary case. Given n binomials

$$g_1 = (x^{\alpha^1_+} - x^{\alpha^1_-}),\ g_2 = (x^{\alpha^2_+} - x^{\alpha^2_-}), \ldots,\ g_n = (x^{\alpha^n_+} - x^{\alpha^n_-}) \in \mathcal{B}(I)$$

we have

$$\begin{aligned}
&(x^{\alpha^1_+} - x^{\alpha^1_-}) \circ (x^{\alpha^2_+} - x^{\alpha^2_-}) \circ \cdots \circ (x^{\alpha^n_+} - x^{\alpha^n_-})\\
=&b^1_{(1,\ldots,n)}(x^{\alpha^1_+} - x^{\alpha^1_-}) + b^2_{(1,\ldots,n)}(x^{\alpha^2_+} - x^{\alpha^2_-}) + \cdots + b^n_{(1,\ldots,n)}(x^{\alpha^n_+} - x^{\alpha^n_-})\\
=&b^1_{(1,\ldots,n)}x^{\alpha^1_+} - b^n_{(1,\ldots,n)}x^{\alpha^n_-}
\end{aligned}$$

We note that associativity is required for the cancellation coefficients $b^i_{(1,\ldots,n)}$ to be well defined.

In some sense the cancellation coefficients we have introduced in the above definition are the building blocks for all other cancellations, we make this clear in the following lemma.

Lemma 6.3.3. *Given binomials*

$$(x^{\alpha^1_+} - x^{\alpha^1_-}), (x^{\alpha^2_+} - x^{\alpha^2_-}), \ldots, (x^{\alpha^n_+} - x^{\alpha^n_-}) \in \mathcal{B}(I)$$

and monomial coefficients $c_1, \ldots c_n$ where $c_i x^{\alpha^i_-} = c_{i+1}x^{\alpha^{i+1}_+}$ for $i = 1, \ldots, n-1$ then

$$\sum_{i=1}^{n} c_i(x^{\alpha^i_+} - x^{\alpha^i_-}) = c_1 x^{\alpha^1_+} - c_n x^{\alpha^n_-}$$

and there exists a monomial $k \in R$ such that $c_i = k \cdot b^i_{(1,\ldots,n)}$

Proof. We prove this by induction. Start with n=2 given $(x^{\alpha^1_+} - x^{\alpha^1_-}), (x^{\alpha^2_+} - x^{\alpha^2_-}) \in \mathcal{B}(I)$ then by definition

$$\left(\frac{\operatorname{lcm}(x^{\alpha^1_-}, x^{\alpha^2_+})}{x^{\alpha^1_-}}\right)\left(x^{\alpha^1_+} - x^{\alpha^1_-}\right) + \left(\frac{\operatorname{lcm}(x^{\alpha^1_-}, x^{\alpha^2_+})}{x^{\alpha^2_+}}\right)\left(x^{\alpha^2_+} - x^{\alpha^2_-}\right).$$

Clearly any c_1 and c_2 such that $c_1 x^{\alpha^1_-} = c_2 x^{\alpha^2_+}$ will be divided by $b^1_{1,2}$ and $b^2_{1,2}$ respectively.

Suppose that this holds for $n = k$.

Now suppose that $n = k+1$. Then we have $\sum_{i=1}^{k+1} c_i \left(x^{\alpha^i_+} - x^{\alpha^i_-}\right)$ where $c_i x^{\alpha^i_-} = c_{i+1}x^{\alpha^{i+1}_+}$ for $i = 1, \ldots, k$. Clearly $c_i x^{\alpha^i_-} = c_{i+1}x^{\alpha^{i+1}_+}$ holds for $i = 1, \ldots, k-1$ as well and hence $\sum_{i=1}^{k} c_i \left(x^{\alpha_+^i} - x^{\alpha_-^i}\right)$ satisfies the induction hypothesis. Thus

$$\sum_{i=1}^{k+1} c_i \left(x^{\alpha_+^i} - x^{\alpha_-^i}\right) = \sum_{i=1}^{k} c_i \left(x^{\alpha_+^i} - x^{\alpha_-^i}\right) + c_{k+1}\left(x^{\alpha_+^{k+1}} - x^{\alpha_-^{k+1}}\right)$$
$$= \sum_{i=1}^{k} k \cdot b^i_{(1,\dots,k)} \left(x^{\alpha_+^i} - x^{\alpha_-^i}\right) + c_{k+1}\left(x^{\alpha_+^{k+1}} - x^{\alpha_-^{k+1}}\right)$$

Hence $k \cdot b^k_{(1,\dots,k)} x^{\alpha_-^k} = c_{k+1} x^{\alpha_+^{k+1}} = k' \cdot \operatorname{lcm}\left(b^k_{(1,\dots,k)} x^{\alpha_-^k}, x^{\alpha_+^{k+1}}\right)$. By definition we have

$$b^i_{(1,\dots,k+1)} = \frac{\operatorname{lcm}\left(b^k_{(1,\dots,k)} x^{\alpha_-^k}, x^{\alpha_+^{k+1}}\right)}{b^k_{(1,\dots,k)} x^{\alpha_-^k}} b^i_{(1,\dots k)}$$

for $i = 1, \dots k$ and

$$b^{k+1}_{(1,\dots k+1)} = \frac{\operatorname{lcm}\left(b^k_{(1,\dots,k)} x^{\alpha_-^k}, x^{\alpha_+^{k+1}}\right)}{x^{\alpha_+^{k+1}}}$$

Hence

$$k' \cdot \operatorname{lcm}\left(b^k_{(1,\dots,k)} x^{\alpha_-^k}, x^{\alpha_+^{k+1}}\right) = k' \cdot b^k_{(1,\dots,k+1)} x^{\alpha_-^k} = k' \cdot b^{k+1}_{(1,\dots,k+1)} x^{\alpha_+^{k+1}}$$

We have

$$k \cdot b^k_{(1,\dots,k)} = k' \cdot b^k_{(1,\dots k+1)}$$

since $b^k_{(1,\dots k)} | b^k_{(1,\dots,k+1)}$ we have $k = k' \cdot \frac{b^k_{(1,\dots k+1)}}{b^k_{(1,\dots,k)}}$ and thus

$$k \cdot b^i_{(1,\dots k)} = k' \cdot \frac{b^k_{(1,\dots k+1)}}{b^k_{(1,\dots,k)}} b^i_{(1,\dots,k+1)} \frac{b^k_{(1,\dots,k)} x^{\alpha_-^k}}{\operatorname{lcm}\left(b^k_{(1,\dots,k)} x^{\alpha_-^k}, x^{\alpha_+^{k+1}}\right)}$$
$$= k' \cdot \frac{\left(\frac{\operatorname{lcm}\left(b^k_{(1,\dots,k)} x^{\alpha_-^k}, x^{\alpha_+^{k+1}}\right)}{x^{\alpha_-^k}}\right)}{b^k_{(1,\dots,k)}} b^i_{(1,\dots,k+1)} \frac{b^k_{(1,\dots,k)} x^{\alpha_-^k}}{\operatorname{lcm}\left(b^k_{(1,\dots,k)} x^{\alpha_-^k}, x^{\alpha_+^{k+1}}\right)}$$
$$= k' \cdot b^i_{(1,\dots,k+1)}$$

Hence $c_i = k' \cdot b^i_{(1,\dots,k+1)}$ for $i = 1, \dots, k+1$ as required.

□

Remark 6.3.4. Note that monomial coefficients cover all interesting cases since non monomial coefficients will simply be sums of systems of coefficients coming from the monomial case.

Definition 6.3.5. In the special case where binomials

$$(x^{\alpha_+^1} - x^{\alpha_-^1}), (x^{\alpha_+^2} - x^{\alpha_-^2}), \ldots, (x^{\alpha_+^n} - x^{\alpha_-^n}) \in \mathcal{B}(I)$$

with monomial coefficients $c_1, \ldots c_n$ such that $c_i x^{\alpha_-^i} = c_{i+1} x^{\alpha_+^{i+1}}$ for $i = 1, \ldots, n-1$ and

$$\sum_{i=1}^{n} c_i(x^{\alpha_+^i} - x^{\alpha_-^i}) = c_1 x^{\alpha_+^1} - c_n x^{\alpha_-^n} = 0$$

we call the ordered sequence $((x^{\alpha_+^1} - x^{\alpha_-^1}), (x^{\alpha_+^2} - x^{\alpha_-^2}), \ldots, (x^{\alpha_+^n} - x^{\alpha_-^n}))$ a **cancellation-sequence**

Remark 6.3.6. If we took the $x^{\alpha_+^i} - x^{\alpha_-^i}$ to be generators this is simply a syzygy. Note that if $(g_1, \ldots, g_n)$ is a cycle-sequence and $\sigma \in D_n$ (the group of symmetries of an n-gon) then $(g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(n)})$ is also a cycle sequence.

Corollary 6.3.7. *Let* $\mathrm{red}(\mathcal{B}(I_G))$ *be the set of reduced binomials in* I_G *and* 0*. With the binary operation* $\star : \mathrm{red}(\mathcal{B}(I_G)) \times \mathrm{red}(\mathcal{B}(I_G)) \to \mathrm{red}(\mathcal{B}(I_G))$ *defined by* $f \star g = \mathrm{red}(f \circ g)$ $\mathrm{red}(\mathcal{B}(I_G))$ *is a finitely generated torsion free abelian group.*

Proof. That $(\mathrm{red}(\mathcal{B}(I)), \star)$ is an abelian group follows from 6.3.1. To see that it is torsion free note that for $x^{\alpha_+} - x^{\alpha_-} \in \mathrm{red}(\mathcal{B}(I))$, we have

$$\underbrace{(x^{\alpha_+} - x^{\alpha_-}) \star (x^{\alpha_+} - x^{\alpha_-}) \star \cdots \star (x^{\alpha_+} - x^{\alpha_-})}_{n} = 0$$

if and only if

$$\underbrace{(x^{\alpha_+} - x^{\alpha_-}) \odot (x^{\alpha_+} - x^{\alpha_-}) \odot \cdots \odot (x^{\alpha_+} - x^{\alpha_-})}_{n} = 0$$

which occurs only when $0 = (x^{\alpha_+} - x^{\alpha_-})$. □

Definition 6.3.8. Define

$$\Psi : \pi_1(G, x_0) \to \mathrm{red}(\mathcal{B}(I))$$

as

$$\gamma \mapsto \mathrm{red}(f_\gamma)$$

Remark 6.3.9. This map is well defined by Lemma 6.2.7.

Proposition 6.3.10. *The map* Ψ *has the following properties:*

1. $\Psi(\gamma_1\gamma_2) = \mathrm{red}(\Psi(\gamma_1) \odot \Psi(\gamma_2)) = \mathrm{red}(\Psi(\gamma_1) \circ \Psi(\gamma_2)) = \Psi(\gamma_1) \star \Psi(\gamma_2)$
2. $\Psi(\gamma^{-1}) = -\Psi(\gamma)$

Proof. 1) Let $\gamma_1 = (e_1, \ldots, e_{2n})$ and $\gamma_2 = (f_1, \ldots, f_{2m})$. We have that $\gamma_1\gamma_2 = (e_1, \ldots, e_{2n}, f_1, \ldots, f_{2m})$ corresponds to $f_{\gamma_1} \odot f_{\gamma_2}$ by Proposition 6.2.11, hence

$$\begin{aligned}\Psi(\gamma_1\gamma_2) = \mathrm{red}(f_{\gamma_1\gamma_2}) = \mathrm{red}\left(f_{\gamma_1} \odot f_{\gamma_2}\right) &= \mathrm{red}(f_{\gamma_1} \circ f_{\gamma_2}) \\ &= \mathrm{red}(\mathrm{red}(f_{\gamma_1} \circ f_{\gamma_2})) \\ &= \mathrm{red}(\mathrm{red}(f_{\gamma_1}) \circ \mathrm{red}(f_{\gamma_2})) \\ &= \mathrm{red}(\Psi(\gamma_1) \circ \Psi(\gamma_2)) \\ &= \Psi(\gamma_1) \star \Psi(\gamma_2).\end{aligned}$$

2) Note that in the fundamental group the inverse of an element is given by reversing the direction of the path. This changes the even indices to odd indices and vice versa in the corresponding binomial, which under our map causes the sign to be changed. □

Corollary 6.3.11. $\Psi(id_{\pi_1(G,x_0)}) = 0$.

Proof. By Proposition 6.3.10 2) we have $\Psi(id) = -\Psi(id)$, hence $\Psi(id) = 0$. □

Corollary 6.3.12. *For $\gamma_1, \ldots, \gamma_k \in A(\pi_1(G, x_0))$ if $\gamma_1\gamma_2 \cdots \gamma_k = id$ then*

$$\mathrm{red}(\Psi(\gamma_1) \circ \cdots \circ \Psi(\gamma_k)) = 0 = \Psi(\gamma_1) \circ \cdots \circ \Psi(\gamma_k)$$

This corresponds by 6.3.3 to some system of coefficients

$$\sum_{i=1}^{k} b^i_{(1,\ldots,k)} \Psi(\gamma_i) = 0$$

Hence a syzygy.

Remark 6.3.13. Let $\gamma_1, \ldots, \gamma_k \in A(\pi_1(G, x_0))$ such that $\gamma_1\gamma_2 \cdots \gamma_k = id$ and $\sigma \in S_k$. We note that by Lemma 6.2.10 $\odot$ is commutative and hence

$$\begin{aligned}0 = \Psi(\gamma_1) \circ \cdots \circ \Psi(\gamma_k) &= \Psi(\gamma_1) \odot \cdots \odot \Psi(\gamma_k) \\ &= \Psi(\gamma_{\sigma(1)}) \odot \cdots \odot \Psi(\gamma_{\sigma(k)}) \\ &= \Psi(\gamma_{\sigma(1)}) \circ \cdots \circ \Psi(\gamma_{\sigma(k)})\end{aligned}$$

and thus

$$\sum_{i=1}^{k} b^{\sigma(i)}_{(\sigma(1),\ldots,\sigma(k))} \Psi(\gamma_{\sigma(i)}) = 0$$

hence for a given relation in the homology group there could be many syzygies on the ideal side.

We would like to show that all syzygies arise essentially in this way. We will do this by starting with the case that all coefficients are monomials and breaking the syzygy into cycle-sequences and then noting that the cases where the coefficients are not monomials can be broken into a sum where the coefficients are monomials via expansion.

We describe an algorithm for breaking a syzygy into a sum of cycle sequences:

Algorithm 6.3.14. Let $S := c_1g_1 + \cdots c_ng_n$ be a syzygy of I_G, that is, $c_1g_1 + \cdots c_ng_n = 0$ where g_i are from a set of minimal generators of I_G and $c_i \in R$. We can always reduce to the case where c_i are monomials by having generators g_i appear more than once. We can break the syzygy S into cycle-sequences in the following way:

1. Fix a monomial ordering $\geq$.
2. Starting at g_1 we look at the second term of g_1, $T_2(g_1)$. Since we have a syzygy it follows that there is some binomial g_{i_2} such that $c_1T_2(g_1) = c_{i_2}T_{k_2}(g_{i_2})$ where $k_2 \in \{1, 2\}$.
3. We then look for a term of some binomial g_{i_3} such that $c_{i_2}T_{k_2+1(mod\ 2)}(g_{i_2}) = c_{i_3}T_{k_3}(g_{i_3})$. Such a term again must exist because S is a syzygy.
4. We continue this process until we can only continue by cancelling with $c_1T_1(g_1)$.
5. We then make the list $(g_1, g_{i_2}, g_{i_3}, \ldots, g_{i_l})$ which we call a **cancellation sequence**.
6. We then repeat the algorithm on a generator which was not included in the cycle and repeat the process.
7. We will then obtain a decomposition of the syzygy in terms of cancellation cycles $\{\mathcal{C}_1 = (g_1, \ldots, g_l), \ldots, \mathcal{C}_k\}$

6.4 Homomorphism Theorem

The following definition was introduced because an error was spotted in a lemma required for Theorem 6.4.2. We have stated the result of the lemma as a definition. We hope to show in the future that all finite simple bipartite graphs satisfy the following definition.

Definition 6.4.1. Let G be a connected finite simple bipartite graph. We say that G has the **syzygy-to-group-relation property** if for every collection of $g_1, \ldots, g_m \in I_G$ of reduced binomials corresponding to closed even walks $w_1, \ldots, w_m$ in G and such that $g_1 \circ \cdots \circ g_m = 0$ there exist $\gamma_1, \ldots, \gamma_m \in \pi_1(G, x_0)$ such that $\Psi(\gamma_i) = g_i$ and $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_m} = id$.

Theorem 6.4.2. *Let G be a connected finite simple bipartite graph that has the syzygy-to-group-relation property. The following are true:*

1. $\Psi : \pi_1(G, x_0) \to (\text{red}(\mathcal{B}(I)), \star)$ *is a surjective group homomorphism.*
2. $\langle g[\pi_1(G, x_0), \pi_1(G, x_0)]g^{-1} : g \in \pi_1(G, x_0)\rangle = \ker(\Psi)$.
3. $\pi_1(G, x_0)/\ker(\Psi) \cong H_1(G)$ *the first homology group of G.*
4. $\pi_1(G, x_0)/\ker(\Psi)$ *is independent of our choice of x_0.*

Proof. 1) That this map is surjective follows immediately from the fact that the toric ideal of a graph is generated by the closed even walks of the graph. That this is a group homomorphism follows from Lemma 6.3.10.

2) Clearly $[\pi_1(G), \pi_1(G)] \subseteq \Psi^{-1}(0)$, therefore we need only show the reverse inclusion.

This being the case we examine

$$\gamma \in \pi_1(G)/\left\langle g[\pi_1(G, x_0), \pi_1(G, x_0)]g^{-1} : g \in \pi_1(G, x_0)\right\rangle$$

where $\Psi(\gamma) = 0$ and show that in this case $\gamma = id$

It is a fact that since the fundamental group is a free group, if we take its ableianization it has the same generators and rank but with commutator relations.

Suppose $\Psi(\gamma) = 0$. We can write any element of the fundamental group uniquely as a reduced word in term of the groups minimal generators $\{\gamma_1, \ldots, \gamma_n\}$. Therefore suppose that $\gamma = \gamma_{i_1} \cdots \gamma_{i_m}$. We then have

$$\Psi(\gamma_{i_1} \cdots \gamma_{i_m}) = 0$$

and by Proposition 6.3.10 we have

$$\text{red}(\Psi(\gamma_1) \circ \Psi(\gamma_2) \circ \cdots \circ \Psi(\gamma_n)) = 0.$$

Hence $\Psi(\gamma_1) \circ \Psi(\gamma_2) \circ \cdots \circ \Psi(\gamma_n) = 0$ by Definition 6.4.1 this means there exist cycles such that some arrangements of the elements are the identity i.e. $\gamma_1 \cdots \gamma_n = id$. However note that the elements we have selected are a free generating set and hence the only relations on them are the commutativity relations hence this element belongs to the commutator.

3) This follows immediately from Theorem 5.2.9.

4) Since the homology group is not dependent on basepoint it follows that we need not pay attention to it when examining the toric ideal of a graph.

□

Corollary 6.4.3. *Let G be a finite simple bipartite graph with the syzygy-to-group-relation property. Let I_G be its toric ideal. Let $\mathcal{H} = \{g_1, \ldots, g_m\}$ be a minimal generating set of I_G. Then $\mathcal{H}$ contains a subset $\{g_{i_1}, \ldots, g_{i_k}\}$ such that there exist $\gamma_{i_1}, \ldots, \gamma_{i_k} \in \pi_1(G, x_0)$ where $\Psi(\gamma_{i_j}) = g_{i_j}$ and $\{\gamma_{i_1}, \ldots, \gamma_{i_k}\}$ is a minimal generating set of $\pi_1(G, x_0)$.*

Proof. Suppose $\gamma \in \pi_1(G, x_0)$, then γ can be represented as a closed even walk in G which in turn corresponds to a binomial f_γ. Since $\mathcal{H}$ generates I_G it follows that f_γ is a linear combination of elements of $\mathcal{H}$, $f = \sum_i^k c_i g_i$. We then have $f - \sum_i^k c_i g_i = 0$ and use the fact that G has the syzygy-to-group-relation property. □

Chapter 7

Fundamental Group Applications

Now that we know that all syzygies correspond to relations in the alternating fundamental group, we would like to be able to say something about a minimal generating set for the syzygies. The next theorem allows us to obtain a lower bound on the number of syzygies required using information from the fundamental group.

7.1 Bounds on Number of Syzygies

Theorem 7.1.1. *Let G be a bipartite finite simple graph with the syzygy-to-group-relation property. Let $\langle g_1, \ldots, g_n, h_1, h_2, \ldots, h_m : r_1, \ldots, r_m \rangle$ be a representation of $\pi_1(G, x_0)$ where the generators correspond to generators of I_G, with the g_i corresponding to a set of minimal generators γ_i in the fundamental group and the h_j are expressed in terms of the g_i in the m relations and the relations are minimal on these generators. Then the total number of minimal first syzygies is $\beta_2(R/I_G) \geq \binom{n}{2} + m$.*

Proof. Since we assumed G has the syzygy-to-group-relation property (definition 6.4.1) we know that syzygies in the toric ideal correspond to relations in the fundamental group, we see that it is necessary that the relations corresponding to the syzygies generates all relations in the group. Thus by theorem 6.4.2 we will need the $\binom{n}{2}$ syzygies coming from the commutativity relations and at least a further m syzygies which correspond to the m relations in $\langle g_1, \ldots, g_n, h_1, \ldots, h_m : r_1, \ldots, r_m \rangle$. □

We can also obtain a crude upper bound via the following proposition:

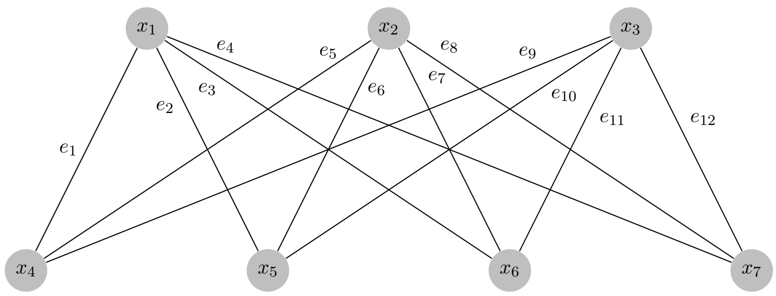
Theorem 7.1.2. *Let G be a bipartite finite simple graph with the syzygy-to-group-relation property. Let $\langle g_1, \ldots, g_n, h_1, h_2, \ldots, h_m : r_1, \ldots, r_m \rangle$ be a representation of $\pi_1(G, x_0)$ where the generators correspond to generators of I_G, with the g_i corresponding to a set of minimal generators γ_i in the fundamental group and the h_j are expressed in terms of the g_i in the m relations and the relations are minimal on these generators. Let $\mathrm{num}(r_i)$ be the number of generators involved in the relation r_i. Then*

$$\beta_2(R/I_G) \leq \binom{n}{2} + \sum_{i=1}^{m} (\mathrm{num}(r_i) - 1)!$$

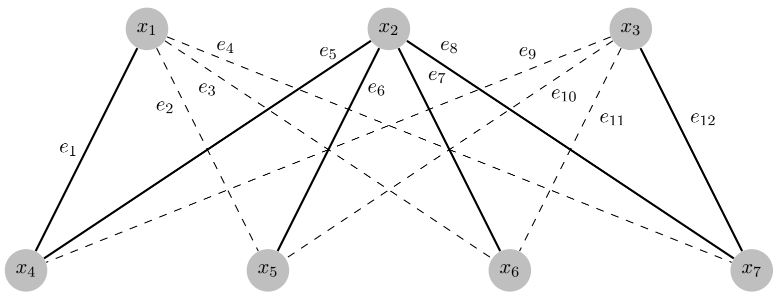
Proof. The reason we take $(\mathrm{num}(r_i)-1)!$ is because for each relation $r_i = h_i g_{i_k}^{-1} \cdots g_{i_1}^{-1}$ we need to take every possible circular permutation since $f_{h_i} \circ f_{g_{i_k}^{-1}} \circ \cdots \circ f_{g_{i_1}^{-1}}$ may have different cancellation coefficients and some of these may not be generated by the others.

A given syzygy must correspond to a relation in the fundamental group by the syzygy-to-group-relation property. However this is only up to a permutation of the order of the elements which make up the relation. It is therefore possible that different permutations of this relation require different coefficients and hence give different syzygies. However there could not be more syzygies than this number since this would imply the existence of a relation in the fundamental group not generated by either the commutator relations or the m relations from $\langle g_1, \ldots, g_n, h_1, \ldots, h_m : r_1, \ldots, r_m \rangle$ □

Example 7.1.3. We give an example of this in practice. Consider $K_{3,4}$. In the previous section we showed a formula for the first syzygies, so we can compare our bounds with the actual total Betti numbers.



Recall that we can obtain the generators of the fundamental group by looking at a maximal spanning tree



So we see that there are 6 generators which correspond to the dashed edges. They correspond to

$$g_1 = e_5e_2 - e_1e_6 \qquad g_2 = e_5e_3 - e_1e_7 \qquad g_3 = e_5e_4 - e_1e_8$$
$$g_4 = e_8e_9 - e_{12}e_5 \qquad g_5 = e_8e_{10} - e_{12}e_6 \qquad g_6 = e_8e_{11} - e_{12}e_7$$

There are $\binom{3}{2}\binom{4}{2} = 18$ total generators corresponding to all four cycles

$$\begin{array}{ccc} \mathfrak{g}_7 = e_5e_{10} - e_9e_6 & \mathfrak{g}_8 = e_5e_{11} - e_9e_7 & \mathfrak{g}_9 = e_8e_3 - e_4e_7 \\ \mathfrak{g}_{10} = e_8e_2 - e_4e_6 & \mathfrak{g}_{11} = e_6e_3 - e_2e_7 & \mathfrak{g}_{12} = e_6e_{11} - e_{10}e_7 \\ \mathfrak{g}_{13} = e_1e_{10} - e_9e_2 & \mathfrak{g}_{14} = e_1e_{11} - e_9e_3 & \mathfrak{g}_{15} = e_1e_{12} - e_9e_4 \\ \mathfrak{g}_{16} = e_2e_{11} - e_{10}e_3 & \mathfrak{g}_{17} = e_2e_{12} - e_{10}e_4 & \mathfrak{g}_{18} = e_3e_{12} - e_{11}e_4 \end{array}$$

$$g_7 = g_4^{-1}g_5$$

$$\begin{aligned} e_6(e_8e_9 - e_{12}e_5) + e_5(e_{12}e_6 - e_8e_{10}) &= e_6e_8e_9 - e_5e_8e_{10} \\ &= e_8(e_6e_9 - e_5e_{10}) \end{aligned}$$

$$g_8 = g_4^{-1}g_6$$

$$\begin{aligned} e_7(e_8e_9 - e_{12}e_5) + e_5(e_{12}e_7 - e_8e_{11}) &= e_7e_8e_9 - e_5e_8e_{11} \\ &= e_8(e_7e_9 - e_5e_{11}) \end{aligned}$$

$$g_9 = g_2g_3^{-1}$$

$$\begin{aligned} e_4(e_1e_7 - e_5e_3) + e_3(e_5e_4 - e_1e_8) &= e_4e_1e_7 - e_3e_1e_8 \\ &= e_1(e_4e_7 - e_3e_8) \end{aligned}$$

$$g_{10} = g_1g_3^{-1}$$

$$\begin{aligned} e_8(e_5e_2 - e_1e_6) + e_6(e_1e_8 - e_5e_4) &= e_8e_5e_2 - e_6e_5e_4 \\ &= e_5(e_8e_2 - e_6e_4) \end{aligned}$$

$$g_{11} = g_2^{-1}g_1$$

$$\begin{aligned} e_7(e_5e_2 - e_1e_6) + e_6(e_1e_7 - e_5e_3) &= e_7e_5e_2 - e_6e_5e_3 \\ &= e_5(e_7e_2 - e_6e_3) \end{aligned}$$

$$g_{12} = g_6g_5^{-1}$$

$$\begin{aligned} e_{10}(e_{12}e_7 - e_8e_{11}) + e_{11}(e_8e_{10} - e_{12}e_6) &= e_{10}e_{12}e_7 - e_{11}e_{12}e_6 \\ &= e_{12}(e_{10}e_7 - e_{11}e_6) \end{aligned}$$

$$g_{13} = g_4^{-1}g_1^{-1}g_5$$

$$\begin{aligned} e_2(e_8e_9 - e_{12}e_5) + e_{12}(e_5e_2 - e_1e_6) + e_1(e_{12}e_6 - e_8e_{10}) &= e_2e_8e_9 - e_1e_8e_{10} \\ &= e_8(e_2e_9 - e_1e_{10}) \end{aligned}$$

$$g_{14} = g_6g_4g_2$$

$$\begin{aligned} e_1(e_8e_{11} - e_{12}e_7) + e_{12}(e_1e_7 - e_5e_3) + e_3(e_{12}e_5 - e_8e_9) &= e_1e_8e_{11} - e_3e_8e_9 \\ &= e_8(e_1e_{11} - e_8e_9) \end{aligned}$$

$$g_{15} = g_3g_4$$

$$\begin{aligned} e_9(e_5e_4 - e_1e_8) + e_1(e_8e_9 - e_{12}e_5) &= e_9e_5e_4 - e_1e_{12}e_5 \\ &= e_5(e_9e_4 - e_1e_{12}) \end{aligned}$$

$$g_{16} = g_1^{-1}g_2g_6^{-1}g_5$$

$$\begin{aligned} e_3e_{12}e_{10}(e_1e_6 - e_5e_2) + e_2e_{12}e_{10}(e_5e_3 - e_1e_7) + e_1e_2e_{10}(e_{12}e_7 - e_8e_{11}) + e_{11}e_1e_2(e_8e_{10} - e_{12}e_6) \\ = e_3e_{12}e_{10}e_1e_6 - e_{11}e_1e_2e_{12}e_6 \\ = e_{12}e_6e_1(e_3e_{10} - e_{11}e_2) \end{aligned}$$

$$g_{17} = g_1^{-1}g_5g_3$$

$$\begin{aligned} e_{10}(e_5e_4 - e_1e_8) + e_1(e_8e_{10} - e_{12}e_6) + e_{12}(e_1e_6 - e_5e_2) &= e_{10}e_5e_4 - e_{12}e_5e_2 \\ &= e_5(e_{10}e_4 - e_{12}e_2) \end{aligned}$$

$$g_{18} = g_2g_6^{-1}g_3^{-1}$$

$$\begin{aligned} e_{12}(e_5e_3 - e_1e_7) + e_1(e_{12}e_7 - e_8e_{11}) + e_{11}(e_1e_8 - e_5e_4) &= e_{12}e_5e_3 - e_{11}e_5e_4 \\ &= e_5(e_{12}e_3 - e_{11}e_4) \end{aligned}$$

Since there are 6 generators in the fundamental group and 12 additional generators, we should have $\binom{6}{2} + 12 = 27$ as a lower bound on the number of syzygies by Theorem 7.1.1. To compute the upper bound note that we have 6 generators for the fundamental group. Then we have 7 relations which involve 3 generators. We have 4 relations which involve 4 generators. Finally we have 1 relation which involves 5 generators. Hence by Theorem 7.1.2 we should have an upper bound of $\binom{6}{2} + 7*(2!) + 4*(3!) + 1*(4!) = 77$. The formula from the previous section says there will be

$$2\left(\binom{3}{2}\binom{4}{3} + \binom{3}{3}\binom{4}{2}\right) + 4\binom{3}{3}\binom{4}{3} = 52$$

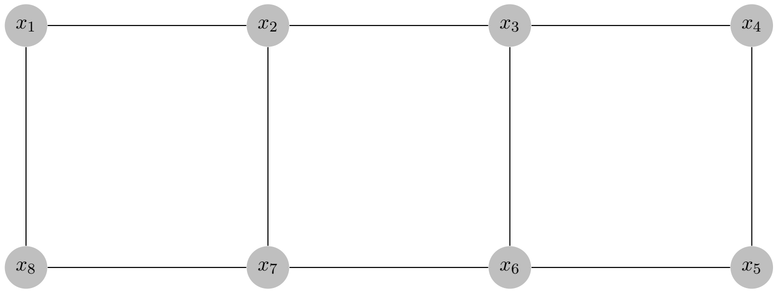
syzygies. Since $27 \leq 52 \leq 77$ we see that neither of these bounds is particularly tight.

We may be interested in some special cases

Corollary 7.1.4. *When the generators of a toric ideal of a finite simple graph correspond in a one to one manner with the generators of the fundamental group, with say n generators, then $\beta_2(R/I_G) = \binom{n}{2}$.*

Proof. The fact that in such a case we have $\beta_2(I_G) = \binom{n}{2}$ follows immediately from our two bounds since we would have $\binom{n}{2} + 0 \leq \beta_2(I_G) \leq \binom{n}{2} + 0$ since $m = 0$ and $\sum_{i=1}^{m}(\text{num}(r_i) - 1)!$ is an empty sum. □

Example 7.1.5. Consider the family of graphs G_t given by adjoining t 4-cycles sequentially via gluing along a single edge. For example G_3 is the graph



One can clearly see that in such a case the generators of the toric ideal correspond to elements of the fundamental group.

7.2 Regular Sequences and Depth

Theorem 7.2.1. *Let G be a bipartite finite simple graph with the syzygy-to-group-relation property. Let $\mathcal{G} = \{g_1, \ldots, g_n\}$ be binomials which correspond via Ψ to a minimal generating set $\{\gamma_1, \ldots, \gamma_n\}$ of the fundamental group $\pi_1(G)$ (by this we mean $\Psi(\gamma_i) = g_i$). Then $\mathcal{G}$ is a regular sequence.*

Proof. Suppose that g_i is a zero divisor of $R/(g_1, \ldots, g_{i-1})$. Then $rg_i = c_1 g_1 + \cdots + c_{i-1} g_{i-1}$. Hence there exists some way in the fundamental group to express the element γ_i in terms of $\{\gamma_1, \ldots, \gamma_{i-1}\}$, i.e $\gamma_i = \gamma_{i_1} \cdots \gamma_{i_m}$. However this contradicts the fact that $\{\gamma_1, \ldots, \gamma_n\}$ is a minimal generating set. □

Remark 7.2.2. We see in the proof that there was nothing particularly important about being a generating set for $\pi_1(G)$, but rather that $\{\gamma_1, \ldots, \gamma_n\}$ are independent in $\pi_1(G)$. Hence we note that any such independent set corresponds to a regular sequence.

Remark 7.2.3. Using the above we can recover a result of Villarreal noting that $\text{rank}(\pi_1(G)) = \dim_F \mathcal{Z}(G)$. This follows since the definition of the cycle space given in [37] is equivalent to the homology group of the graph considered as a simplicial complex. The rank of the homology group is equal to the rank of the fundamental group.

Corollary 7.2.4. *Let G be a bipartite finite simple graph with the syzygy-to-group-relation property and let I_G its toric ideal. Then we have* $\mathrm{rank}(\pi_1(G)) = depth(I_G, R)$.

Proof. Let $\{h_1, \dots h_k\}$ be a regular sequence of binomials in I_G (still need to justify why it is enough to focus on binomials). Each h_i corresponds to some monomial multiple of a binomial which corresponds to an element in the fundamental group. The assumption that h_i is not a zero divisor of $I_G/ < h_1, \dots, h_{i-1} >$ is equivalent to assuming that the h_i cannot generate each other in the fundamental group. Since the fundamental group of a graph is free, this means there is an invariant number of generators which is in this case n. It follows that $k \leq n$, and so $\mathrm{depth}(I, R)$ □

Corollary 7.2.5. *If I_G is generated by a minimal generating set of $\pi_1(G)$ then it is a complete intersection.*

7.3 Saturation Results for Toric Ideals

Theorem 7.3.1. *Let G be a bipartite finite simple graph with the syzygy-to-group-relation property, where $E(G) = \{e_1, \dots, e_m\}$. Let $\mathcal{G} = \{g_1, \dots, g_n\}$ be binomials such that there exists a minimal generating set of the fundamental group of G at x_0 $\{\gamma_1, \dots, \gamma_n\} \subseteq \pi_1(G, x_0)$ such that $\Psi(\gamma_i) = g_i$. Then $I_G = (\langle \mathcal{G} \rangle : (e_1 \cdots e_m)^\infty)$*

Proof. We know that I_G is generated by the set of primitive closed even walks from Proposition 2.3.3. It follows that we simply need to show that for an arbitrary primitive closed even walk w of G that $f_w \in (\langle \mathcal{G} \rangle : \langle e_1 \cdots e_m \rangle^\infty)$. Let $w = (e_{j_1}, \dots, e_{j_{2k}})$ be an arbitrary primitive closed even walk of G. Suppose that x is a vertex which is part of w. Since G is connected there exists a path σ from x_0 to x. Define $\gamma = \sigma w \sigma^{-1}$. Note that $\Psi(\gamma) = f_w$. Since $\{\gamma_1, \dots \gamma_n\}$ generates $\pi_1(G)$ we must have $\gamma = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}$ where $\gamma_{i_j} \in \{\gamma_1, \dots, \gamma_n\}$. Thus we have $f_w = \Psi(\gamma) = \mathrm{red}(f_\gamma) = \mathrm{red}(f_{\gamma_{i_1}} \circ f_{\gamma_{i_2}} \circ \cdots \circ f_{\gamma_{i_k}})$ which implies that $m f_w = f_{\gamma_{i_1}} \circ f_{\gamma_{i_2}} \circ \cdots \circ f_{\gamma_{i_k}}$ where m is some monomial. It follows then that $f_w \in (\langle \mathcal{G} \rangle : (e_1 \cdots e_m)^\infty)$ and hence $I_G = (\langle \mathcal{G} \rangle : (e_1 \cdots e_m)^\infty)$ □

Lemma 7.3.2. *Let γ_1 and γ_2 be primitive closed even walks of a finite simple graph G.*

$$\Lambda = \frac{f_{\gamma_1} \circ f_{\gamma_2}}{\mathrm{red}(f_{\gamma_1} \circ f_{\gamma_2})}$$

Then $\mathrm{supp}(\Lambda) \subseteq (E(\gamma_1) \cap E(\gamma_2))$ *edges which belong to both γ_1 and γ_2*

Proof. Let $f_{\gamma_1} = x^{\alpha_+} - x^{\alpha_-}$, and $f_{\gamma_2} = x^{\beta_+} - x^{\beta_-}$. Recall that the cancellation coefficients are defined

$$b_{1,2}^1 = \frac{\mathrm{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\alpha_-}}, \quad b_{1,2}^2 = \frac{\mathrm{lcm}(x^{\alpha_-}, x^{\beta_+})}{x^{\beta_+}}$$

Since we made the assumption that γ_1 and γ_2 are primitive it follows that $\gcd(x^{\alpha_+}, x^{\alpha_-}) = \gcd(x^{\beta_+}, x^{\beta_-}) = 1$.

Suppose $e \mid f_{\gamma_1} \circ f_{\gamma_2}$ for some $e \in E(G)$. We will show that $e \mid x^{\alpha+}$ and $e \mid x^{\beta-}$ and hence $\mathrm{supp}(\Lambda) \subseteq (E(\gamma_1) \cap E(\gamma_2))$.

By assumption we have $e \mid b^1_{1,2}x^{\alpha+}$ and $e \mid b^2_{1,2}x^{\beta-}$. Since e is prime it must divide either $b^1_{1,2}$ or $x^{\alpha+}$. If e divides $b^1_{1,2}$ it cannot divide $x^{\beta-}$ since $\gcd(b^1_{1,2}, x^{\beta-}) = 1$. e also cannot divide $b^2_{1,2}$ since $\gcd(b^1_{1,2}, b^2_{1,2}) = 1$. Therefore $e \mid x^{\alpha-}$. Since $\gcd(x^{\alpha-}, b^2_{1,2}) = 1$ we must therefore conclude that $e \mid x^{\alpha+}$ and $e \mid x^{\beta-}$.

It follows that $e \in (E(\gamma_1) \cap E(\gamma_2))$ and since every monomial coefficient is simply a product of variables the result follows. □

We are now in a position to prove an analogue of Van Kampen's theorem for toric ideals of finite simple graphs.

Theorem 7.3.3. *Let G be a finite simple bipartite graph with the syzygy-to-group-relation property. Let $H_1, H_2, \ldots, H_k$ be subgraphs such that $G = \bigcup_{i=1}^k H_i$ and $H_i \cap H_j$ is path connected for $i, j \in [k]$. Let*

$$H = \bigcup_{i \neq j} (E(H_i) \bigcap E(H_j))$$

Then

$$I_G = \left(\sum_{i=1}^{k} I_{H_i} : \left(\prod_{e_i \in H} e_i \right)^{\infty} \right)$$

Proof. We will show that every binomial corresponding to a primitive closed even walk is contained in $\left(\sum_{i=1}^k I_{H_i} : \left(\prod_{e_i \in H} e_i\right)^{\infty}\right)$. Let γ be a closed even primitive walk in G.

By Van Kampen's theorem (Theorem 5.2.8) applied to a finite simple graph we know that a loop in G may be expressed as a product of loops contained in the H_i, thus we have $\gamma = \gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_l}$ where $\gamma_{j_k} \in \pi_1(H_i, x_0)$ for some i. WOLG we may assume that these loops correspond to primitive closed even walks. We note that

$$f_\gamma = \Psi(\gamma_{j_1}\gamma_{j_2}\cdots\gamma_{j_l}) = \mathrm{red}(f_{\gamma_{j_1}} \circ f_{\gamma_{j_2}} \circ \cdots f_{\gamma_{j_l}})$$

This implies that $I_G = \left(\sum_{i=1}^k I_{H_i} : \langle E(G) \rangle^{\infty}\right)$. In order to show that we only need to saturate by the product of edges contained in the intersection we appeal to Lemma 7.3.2 and note that the monomial coefficients added by each operation can only come from edges common to both primitive walks, that is, edges belonging to H. □

Remark 7.3.4. Such a result is rather similar to the results of Chapter 3. A natural question to ask then is whether we can determine the smallest ideal which we can saturate with and further what the smallest power we are required to take is.

7.4 Open Questions

Given that this approach utilising the fundamental group and the relations between its elements to study toric ideals of graphs is somewhat novel there are many questions which naturally arise which have yet to be answered. For example

Question 7.4.1. *In this chapter we restricted our attention to bipartite graphs. One notes for example that $A(\pi_1(G))$ is not a free group when G is not bipartite. Further the toric ideals would also no longer be Cohen-Macualay. Do the results obtained in this chapter have analogues for non-bipartite graphs?*

We also focused primarily on total Betti numbers and simply the existence of certain generators and relations, however it is natural to ask

Question 7.4.2. *If one takes the degree of the generators into account; can we utilise the fundamental group to yield information about the degree of generators and syzygies? Can we in particular obtain results pertaining to the regularity?*

Conjecture 7.4.3. *Let G be a finite simple bipartite graph. Then R/I_G is a complete intersection if and only if the rank of the fundamental group is equal to the number of generators of I_G.*

We end with an alternative proof of a special case of a well known theorem found in [38] (Proposition 8.1.9) which was attributed as a result of Lemma 4.6 [34]

Proposition 7.4.4. *Let G be a finite simple bipartite graph with the syzygy-to-relation property. Then the set binomials corresponding to closed even walks are a universal Gröbner basis.*

Proof. For a finite simple bipartite graph the closed even walks of are elements of the fundamental group. The S-polynomial of any two elements in corresponds to the group operation between the corresponding two elements in the fundamental group. Since the group operation is closed it means that the S-polynomial corresponds to a closed even walk and is part of our generating set. Since it is also in the set regardless of the term order it divides it's own leading term. Thus we have a universal Gröbner basis. □

Chapter 8

Conclusion

We conclude by summarising what was done in this thesis as well as pointing to directions for future research.

8.1 Summary

We have managed to sharpen a previous result in [13] about splitting a graph into subgraphs and determining the generators of the graph in question from the generators of the subgraphs. The existing result stated that we needed to saturate with respect to a given monomial, $\mathcal{E}(h)$. It could have been possible that one would need to take the colon ideal with a very large power of $\mathcal{E}(h)$ since there is no upper bound. What we have now shown is that we can simply take $\mathcal{E}(h)^2$ which is rather simpler. We have also shown that in certain cases $\mathcal{E}(h)$ is enough.

In a similar vein we have utilised subgraphs to count and explicitly describe the generators as well as first syzygies $I_{K_{n,m}}$, as well as counting the number of second syzygies for this family of toric ideals. We managed to show that such toric ideals have linear presentations. We presented a method which demonstrates that a minimal generating set of a toric ideal of an induced subgraph can always be extended to a minimal generating set of the toric ideal of the whole graph. Further we were able to show that the resolution of toric ideals of induced subgraphs show up in the resolution of a toric ideal of a graph in an exact way allowing us to explicitly state what the higher syzygies will be. This extends on work by Biermann, O'Keefe and Van Tuyl [3] which demonstrated that the graded Betti numbers of toric ideals of induced subgraphs are a lower bound for the graded Betti numbers of toric ideals of the whole graph. We also show how these computations can be used to compute formulas for the graded Betti numbers. In fact, such an approach could be greatly extended. All that one needs to do is check a finite number of multidegrees which are determined by bounds on the total degrees (which one can obtain from the Betti numbers of an initial ideal under some ordering).

We also introduced a relationship between toric ideals associated with finite simple graphs and their fundamental groups and homology groups. It would be possible for one to view the results pertaining to the fundamental group from the perspective that images of loops in the fundamental group specify subgraphs. In some sense we are using the fundamental group to study subgraphs defined by elements in the fundamental group and how they interact with each other. We should also note that as this thesis is currently written the task is not yet complete as we had to assume the syzygies-to-group-relation property in order to achieve our results. We suspect that all finite simple bipartite graphs have this property and hope to prove this in the coming months. Further questions could be asked about how this could be extended to nonbipartite graphs.

We should note that our original goal of determining the graded Betti numbers of toric ideals of finite simple graphs remains wide open. Most importantly during the course of this thesis we came to believe that the following conjecture is true:

Conjecture 8.1.1 (Conjecture 1.1.3)**.** *Let G be a finite simple graph. Let I_G be the associated toric ideal. Suppose that $\{g_1, \ldots, g_n\}$ is a minimal generating set of I_G, then $\beta_i(R/I_G) \leq \binom{n}{i}$.*

Further we note that most of the results pertain to the first syzygies. However we are still quite interested in the original question which asks about minimal free resolutions in their entirety:

Question 8.1.2. *Let G be a finite simple graph. Let I_G be the associated toric ideal. Can we determine specific formulae for $\beta_i(I_G)$?*

8.2 Future Directions

It is clear that research into toric ideals of finite simple graphs has not yet been exhausted. In this section we outline a few directions that we believe may yield new results.

8.2.1 Non-Biparite Graphs

In Chapter 6 in the remark after Example 6.1.6 we noted that there are some difficulties associated with how we have chosen to associate the toric ideal associated with a graph and a subgroup of the fundamental group of this graph. We propose the alternate choice of subgroup of $\pi_1(G)$ which may be useful to study the non-bipartite case.

Definition 8.2.1. Let G be a finite simple graph. Let $\mathcal{G} = \{\gamma_1, \ldots, \gamma_k\}$ be a set of elements of $\pi_1(G)$ such that $\langle \mathcal{G} \rangle$ is a minimal presentation. We define a subgroup $\mathcal{A}(\pi_1(G))$ as follows: Let $S = \{\gamma_1, \ldots, \gamma_k\} \cap A(\pi_1(G))$. We note that elements of S correspond to closed even walks in G. Let $B = \mathcal{G} \backslash S$. If $B \neq \emptyset$ and $|B| = n$, then we relabel the generators so that $\gamma_i \in B$ for $i \leq n$ and $\gamma_i \in S$ for $i \geq n$.

i) If $B = \emptyset$ we define $\mathcal{A}(\pi_1(G)) = \langle \mathcal{G} \rangle$ (note here that $\mathcal{G} = S$ and G is bipartite.)

ii) If $|B| = 1$ we define $\mathcal{A}(\pi_1(G)) = \langle S \rangle$.

iii) If $|B| = n \geq 2$ we define $\mathcal{A}(\pi_1(G)) = \left\langle S \cup \{\gamma_1^{-1}\gamma_j \mid j = 2, \ldots, n\} \right\rangle$

Remark 8.2.2. We note that this subgroup does not depend on the indexing of the elements of B since we have $(\gamma_1^{-1}\gamma_i)^{-1}(\gamma_1\gamma_j) = \gamma_i^{-1}\gamma_j$ and hence γ_1^{-1} could be substituted for any of the $\gamma_i \in B$.

In some sense this subgroup has more of the properties that we desire. For example none of the generators are mapped to zero under the map $\Psi : \mathcal{A}(\pi_1(G, x_0)) \to (\mathcal{B}(I), \star)$ of 6.4.2. It also has a rank which is equal to the height of I_G in both the case that G is bipartite and not bipartite thus leading the way to extend Theorem 7.2.1 to the non-bipartite case.

8.2.2 Simplicial Complexes

One must note that in the existing literature the idea of associating a toric ideal to a finite simple graph has been extended to the case of finite simple hypergraphs. Work in this area has been done by Petrovic and Stasi [29].

Definition 8.2.3. We define an n-dimensional hypergraph to be a set of vertices and facets such that each facet is uniquely associated to exactly $n+1$ distinct vertices. That is for a hypergraph G we have vertex set $V(G) = \{x_1, \dots x_k\}$ and facet set $F(G) = \{f_1 = \{x_{1_1}, x_{1_2}, \dots, x_{1_{n+1}}\}, \dots, f_l = \{x_{l_1}, \dots, x_{l_{n+1}}\}\}$

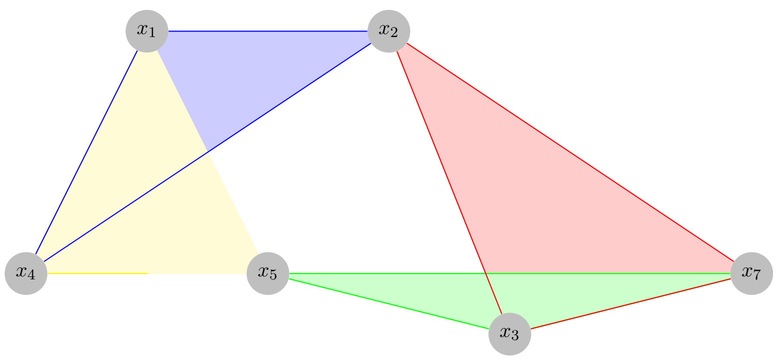
Definition 8.2.4 ([29])**.** Let G be an n-dimensional hypergraph as in Definition 8.2.3. Then the toric ideal associated to G, denoted I_G, is defined by the kernel of the map

$$\varphi : \mathbb{K}[F(G)] \to \mathbb{K}[V(G)]$$

$$f_i \mapsto x_{i_1} x_{i_2} \cdots x_{i_{n+1}}$$

We provide two examples which allow us to understand the uses and limits of this definition.

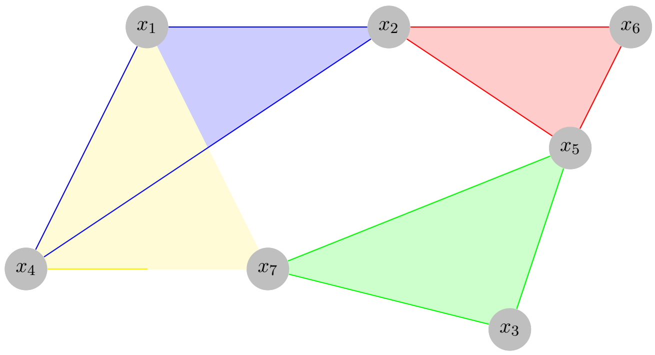
Example 8.2.5. Let G be the 2-dimensional hypergraph defined by $V(G) = \{x_1, \dots, x_6\}$, $F(G) = \{f_1 = \{x_1, x_2, x_4\},\ f_2 = \{x_1, x_4, x_5\},\ f_3 = \{x_2, x_3, x_7\},\ f_4 = \{x_5, x_3, x_7\}\}$.



Then we have $\varphi : \mathbb{K}[F(G)] \to \mathbb{K}[V(G)]$ $f_1 \mapsto x_1x_2x_4$ $f_2 \mapsto x_1x_4x_5$ $f_3 \mapsto x_2x_3x_7$ $f_4 \mapsto x_5x_3x_7$ Thus $\ker(\varphi) = (f_1f_3 - f_2f_4)$.

Remark 8.2.6. We may speculate based on this example that the relationship between elements in the toric ideal associated with the hypergraph and the first homotopy group remains with the single generator corresponding to the single loop. However the next example which is homotopic to the first and also contains an even loop has trivial toric ideal.

Example 8.2.7. Let G be the 2-dimensional hypergraph defined by $V(G) = \{x_1, \dots, x_7\}$ and $F(G) = \{f_1 = \{x_1, x_2, x_4\}, f_2 = \{x_1, x_4, x_7\}, f_3 = \{x_2, x_5, x_6\}, f_4 = \{x_5, x_3, x_7\}\}$



Here we have $\varphi : \mathbb{K}[F(G)] \to \mathbb{K}[V(G)]$ defined by $f_1 \mapsto x_1x_2x_4$ $f_2 \mapsto x_1x_4x_7$ $f_3 \mapsto x_2x_5x_6$ $f_4 \mapsto x_5x_3x_7$ which has a trivial kernel despite having 2 open four cycles

From these examples one can see that under such a formulation one loses the correspondence with algebraic topology. We may have instead expected that in the same way the generators of the toric ideal corresponded to elements in the first homology group we may have here a correspondence with higher homology groups.

Such an association can be maintained if we instead choose to define toric ideals for simplicial complexes by mapping each n-dimensional simplex to the product of its $n-1$-dimensional faces.

Definition 8.2.8. Let X be a finite simplicial complex. We define a family of maps $\{\varphi_n\}$ as follows. $\varphi_n : \mathbb{K}[F(X_n)] \to \mathbb{K}[F(X_{n-1})]$ where each n^{th} dimensional face is mapped to the product of it's $n-1$-dimensional faces.

Example 8.2.9.

$$X_0 = \{x_1, \dots, x_6\}$$

$$\begin{aligned} X_1 = \{e_1 = \{x_1, x_2\}, \quad & e_2 = \{x_2, x_3\}, e_3 = \{x_3, x_4\}, \quad & e_4 = \{x_4, x_1\}, \\ e_5 = \{x_1, x_5\}, \quad & e_6 = \{x_2, x_5\}, e_7 = \{x_3, x_5\}, \quad & e_8 = \{x_4, x_5\}, \\ e_9 = \{x_1, x_6\}, \quad & e_{10} = \{x_2, x_6\}, e_{11} = \{x_3, x_6\}, \quad & e_{12} = \{x_4, x_6\}\} \end{aligned}$$

$$X_2 = \{f_1 = \{e_1, e_5, e_6\}, \quad f_2 = \{e_2, e_6, e_7\}, f_3 = \{e_3, e_7, e_8\}, \quad f_4 = \{e_4, e_8, e_5\},$$
$$f_5 = \{e_1, e_9, e_{10}\}, \quad f_6 = \{e_2, e_{10}, e_{11}\}, f_7 = \{e_3, e_{11}, e_{12}\}, \quad f_8 = \{e_4, e_{12}, e_9\}\}$$

We have here a simplicial complex. In analogy with the case of graphs we may define a maps

$$\varphi_1 : \mathbb{K}[e_1, \ldots, e_{12}] \to \mathbb{K}[x_1, \ldots, x_6]$$

$$e_1 \mapsto x_1x_2$$
$$\vdots$$
$$e_{12} \mapsto x_4x_6$$

$$\varphi_2 : \mathbb{K}[f_1, \ldots, f_6] \to \mathbb{K}[e_1, \ldots, e_{12}]$$

$$f_1 \mapsto e_1e_5e_6$$
$$\vdots$$
$$f_8 \mapsto e_4e_{12}e_9$$

We may then examine the kernel in the usual way and see that we get

$$\ker(\varphi_2) = \langle f_1f_3f_5f_7 - f_2f_4f_6f_8 \rangle$$

We see that this element corresponds to an element in the homology group.

8.2.3 Gröbner Bases

Another direction for future research is investigating Gröbner bases of toric ideals of finite graphs. One of the results of [15] studied a family of graphs for which the graded Betti numbers of the initial ideal under a given monomial order were equal to the graded Betti number of the underlying ideal. This raises the question as to when such a relationship exists for general toric ideals associated with finite simple graphs since it would allow for similar techniques to be utilised to computed the graded Betti numbers for such ideals.

A natural place to begin would be the investigation of Gröbner bases of toric ideals associated with planar bipartite graphs. Some work has already been done in this direction by [11]. However there are still many unanswered questions in this direction. For example

Question 8.2.10. *Can one devise a monomial ordering which results in a Gröbner basis consisting of a minimal number of elements (across all possible monomial orderings)?*

We could also use bipartite planar graphs as a stepping stone towards proving (or disproving) Conjecture 1.1.3

Optimistically one could ask whether we could find a closed formula allowing us to compute the graded Betti number for such a class since we already know that all complete intersections belong to this class.

Question 8.2.11. *Can we find a formula allowing us to compute the graded Betti number of toric ideals of bipartite planar graphs using combinatorial information from the underlying graph?*

Bibliography

[1] A. Aramova and J. Herzog. Koszul cycles and Eliahou-Kervaire type resolutions. *J. Algebra* 181(2) (1996), 347–370.

[2] R. Baer and F. Levi. Freie Produkte und ihre Untergruppen. *Compositio Math.* 3 (1936), 391–398.

[3] J. Biermann, A. O'Keefe, and A. Van Tuyl. Bounds on the regularity of toric ideals of graphs. *Adv. in Appl. Math.* 85 (2017), 84–102.

[4] A. Boocher and E. Robeva. Robust toric ideals. *J. Symbolic Comput.* 68(part 1) (2015), 254–264.

[5] W. Bruns. "Jede" endliche freie Auflösung ist freie Auflösung eines von drei Elementen erzeugten Ideals. *J. Algebra* 39(2) (1976), 429–439.

[6] A. Campillo and C. Marijuan. Higher order relations for a numerical semigroup. *Sém. Théor. Nombres Bordeaux (2)* 3(2) (1991), 249–260.

[7] A. Campillo and P. Pisón. L'idéal d'un semi-groupe de type fini. *C. R. Acad. Sci. Paris Sér. I Math.* 316(12) (1993), 1303–1306.

[8] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric varieties*. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011.

[9] A. D'Alı. Toric ideals associated with gap-free graphs. *J. Pure Appl. Algebra* 219(9) (2015), 3862–3872.

[10] P. Diaconis and B. Sturmfels. Algebraic algorithms for sampling from conditional distributions. *Ann. Statist.* 26(1) (1998), 363–397.

[11] L. R. Doering and T. Gunston. Algebras arising from bipartite planar graphs. *Comm. Algebra* 24(11) (1996), 3589–3598.

[12] D. Eisenbud. *Commutative algebra*. Vol. 150. Graduate Texts in Mathematics. With a view toward algebraic geometry. Springer-Verlag, New York, 1995.

[13] G. Favacchio, J. Hofscheier, G. Keiper, and A. Van Tuyl. Splittings of toric ideals. *J. Algebra* 574 (2021), 409–433.

[14] C. A. Francisco, H. T. Hà, and A. Van Tuyl. Splittings of monomial ideals. *Proc. Amer. Math. Soc.* 137(10) (2009), 3271–3282.

[15] F. Galetto, J. Hofscheier, G. Keiper, C. Kohne, A. Van Tuyl, and M. E. U. Paczka. Betti numbers of toric ideals of graphs: a case study. *J. Algebra Appl.* 18(12) (2019), 1950226, 14.

[16] D. R. Grayson and M. E. Stillman. *Macaulay2, a software system for research in algebraic geometry.* Available at http://www.math.uiuc.edu/Macaulay2/.

[17] Z. Greif and J. McCullough. Green-Lazarsfeld condition for toric edge ideals of bipartite graphs. *J. Algebra* 562 (2020), 1–27.

[18] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[19] J. Herzog and T. Hibi. *Monomial ideals.* Vol. 260. Graduate Texts in Mathematics. Springer-Verlag London, Ltd., London, 2011.

[20] J. Herzog, T. Hibi, and H. Ohsugi. *Binomial ideals.* Vol. 279. Graduate Texts in Mathematics. Springer, Cham, 2018.

[21] T. Hibi, K. Nishiyama, H. Ohsugi, and A. Shikama. Many toric ideals generated by quadratic binomials possess no quadratic Gröbner bases. *J. Algebra* 408 (2014), 138–146.

[22] S. Mac Lane. *Homology.* Classics in Mathematics. Reprint of the 1975 edition. Springer-Verlag, Berlin, 1995.

[23] R. Nandi and R. Nanduri. Betti numbers of toric algebras of certain bipartite graphs. *J. Algebra Appl.* 18(12) (2019), 1950231, 18.

[24] H. Ohsugi and T. Hibi. Normal polytopes arising from finite graphs. *J. Algebra* 207(2) (1998), 409–426.

[25] H. Ohsugi and T. Hibi. Toric ideals generated by quadratic binomials. *J. Algebra* 218(2) (1999), 509–527.

[26] H. Ohsugi and T. Hibi. Toric ideals and their circuits. *J. Commut. Algebra* 5(2) (2013), 309–322.

[27] I. Peeva. *Graded syzygies.* Vol. 14. Algebra and Applications. Springer-Verlag London, Ltd., London, 2011.

[28] I. Peeva and M. Stillman. Open problems on syzygies and Hilbert functions. *J. Commut. Algebra* 1 (2009).

[29] S. Petrović and D. Stasi. Toric algebra of hypergraphs. *J. Algebraic Combin.* 39(1) (2014), 187–208.

[30] S. Petrović, A. Thoma, and M. Vladoiu. Hypergraph encodings of arbitrary toric ideals. *J. Combin. Theory Ser. A* 166 (2019), 11–41.

[31] E. Reyes, C. Tatakis, and A. Thoma. Minimal generators of toric ideals of graphs. *Advances in Applied Mathematics* 48(1) (2012), 64–78.

[32] D. J. S. Robinson. *A course in the theory of groups.* Second. Vol. 80. Graduate Texts in Mathematics. Springer-Verlag, New York, 1996.

[33] R. P. Stanley. *Combinatorics and commutative algebra.* Second. Vol. 41. Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1996.

[34] B. Sturmfels. *Gröbner bases and convex polytopes.* Vol. 8. University Lecture Series. American Mathematical Society, Providence, RI, 1996.

[35] S. Sullivant. *Algebraic statistics*. Vol. 194. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2018.

[36] C. Tatakis and A. Thoma. On the universal Gröbner bases of toric ideals of graphs. *J. Combin. Theory Ser. A* 118(5) (2011), 1540–1548.

[37] R. H. Villarreal. Rees algebras of edge ideals. *Comm. Algebra* 23(9) (1995), 3513–3524.

[38] R. H. Villarreal. *Monomial algebras*. Second. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2015.

[39] R. J. Wilson. *Introduction to graph theory*. Fourth edition. Longman, Harlow, 1996.