BOUNDARY VERSUS INTERIOR DEFECTS FOR A GINZBURG–LANDAU MODEL WITH TANGENTIAL ANCHORING CONDITIONS

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Abstract

In this thesis, we study six Ginzburg-Landau minimization problems in the context of two-dimensional nematic liquid crystals with the intention of finding conditions for the existence of boundary vortices. The first minimization problem consists of the standard Ginzburg-Landau energy on bounded, simply connected domains $\Omega \subset \mathbb{R}^2$ with boundary energy penalizing minimizers who stray from being parallel to some smooth $\mathbb{S}^1$-valued boundary function g of degree $\mathcal{D} \geq 1$. The second and third minimization problems consider the same Ginzburg-Landau energy but now with divergence and curl penalization in the interior and boundary function taken to be $g = \tau$, the positively oriented unit tangent vector to the boundary. The remaining three problems involve minimizing the same energies, but now over the set for which all functions are precisely parallel to the given boundary data (up to a set for which their norms can be zero). These six problems are classified under two categories called the *weak* and *strong* orthogonal problems.

In each of the six problems, we show that conditions exist for which sequences of minimizers converge to a limiting $\mathbb{S}^1$-valued vector field describing an equilibrium configuration for nematic material with defects. In some cases, energy estimates are obtained that show vortices belong to the boundary exclusively and the exact number of these vortices are known. A special case is also studied in the strong orthogonality setting. The analysis here suggests that geometries exist for which boundary vortices may be energetically preferable to interior vortices in the case where interior and boundary vortices have similar energy contributions.

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Chapter 1

Introduction

Throughout this thesis, the mathematics presented will be taken in the context of modeling the molecular order of nematic liquid crystal in two-dimensional space. However, it should be noted that the physical application of these results do not solely belong to liquid crystal phenomenon. Other interesting topics in physics such as the study of superconductors make use of the same mathematical framework. The goal of this introduction is to build up and discuss the necessary background material needed to introduce the main problems of this thesis. We begin by familiarizing the reader with some basics of nematic liquid crystal theory and then explore some of the related mathematical research done spanning over the last thirty years. From there, the core problem of this work is presented along with some results which form the basis of the thesis.

1.1 Nematic Liquid Crystals: What are they?

The term *liquid crystal* at first glance may be quite puzzling. From a traditional understanding of states of matter, one is taught in elementary science classes that for isotropic liquids, molecules are randomly oriented and are free to flow within the confines of their vessel. On the other hand, the opposite is true of solid crystals since molecules constituting crystalline structures are highly ordered in both an orientational sense (with respect to the axes of the molecules) and positional sense (with respect to their location in space). What is not usually presented is the fact that certain materials exist that can live within a state intermediate between solid and liquid called 'liquid crystal'. Roughly speaking, substances in the liquid crystal state retain some partial ordering of the molecules whether it be orientational or both orientational and positional (reminiscent of the solid state) but are simultaneously capable of diffusing throughout the container (reminiscent of the liquid state).

In the materials physics literature there are two well-known categories of substances that can achieve the liquid crystal state and these categories aim to partition based on the primary factor driving the self-ordering behaviour. One of these classes describe substances that greatly depend on the thermal energy of the system and has the label *thermotropic liquid crystals*. For materials in this grouping there exists a temperature range for which the molecules do not move fast enough to fully break the molecular order imposed by the solid state but also do not move slow enough to form a traditional solid. The other category of liquid crystals are labeled *lyotropic liquid crystals* and these substances begin self-ordering when introduced to a solvent. In this situation it is the interaction between the molecules of the solvent and material in question that forces partial ordering and the degree of ordering can depend on concentration [15]. For substances in any of these groups, molecular shape and structure play a fundamental role in how self-organization is accomplished. In this work we adopt a somewhat naive view of molecules that coincide with the simplest structure known to facilitate the liquid crystal state, namely, molecules that are long, rigid and rod-shaped.

Beyond the material classifications mentioned above, the liquid crystal state itself can be broken into different phases which are defined by the type and degree of ordering the molecules exhibit. For the purposes of this thesis and discussion we restrict our attention to the *nematic phase* for thermotropic liquid crystals. The term 'nematic' has its origin from the Greek word $\nu\eta\mu\alpha$ which translates to *thread*. The reason for this title choice comes from the characteristic string-like discontinuities observable in the nematic phase called *disclinations* [34]. A rather simplistic but sufficient description of nematic liquid crystals are those in which a degree of local, long-range orientational order is achieved throughout the material sample but positional order is not. To visualize this, one can imagine a small subset of the material sample where the molecules have a preferred direction of molecular alignment with respect to their long axis. However, the preferred direction observed at this local level can change as a function of position throughout the entire container. Classically, this direction of preferred molecular alignment at a point x in the material sample is commonly represented by a vector $\mathbf{d}(x)$ called the *director*. Since the magnitude of $\mathbf{d}$ has no physical meaning, the director is also taken to be of unit length. The director will be treated with slightly more detail in the following section.

1.2 Modeling Liquid Crystals

As with any model for physical systems, it is important to consider the mathematical structure used to represent the natural phenomenon one wishes to study. For nematic liquid crystal continuum theories, there are typically three general components that are used together to describe the system. These three components are:

1. a function within an appropriate function space X representing molecular orientation,

2. an energy for the system which takes into account the geometry of the space in which the liquid crystal fills. Usually this is given as an integral functional
$$\mathscr{F} : X \to \mathbb{R} \cup \{+\infty\}$$
where the integral is taken over the space containing the liquid crystal,

3. (a) a variational partial differential equation describing stationary points (or physically observable equilibrium configurations) of $\mathscr{F}$ whose solutions are elements of X, and

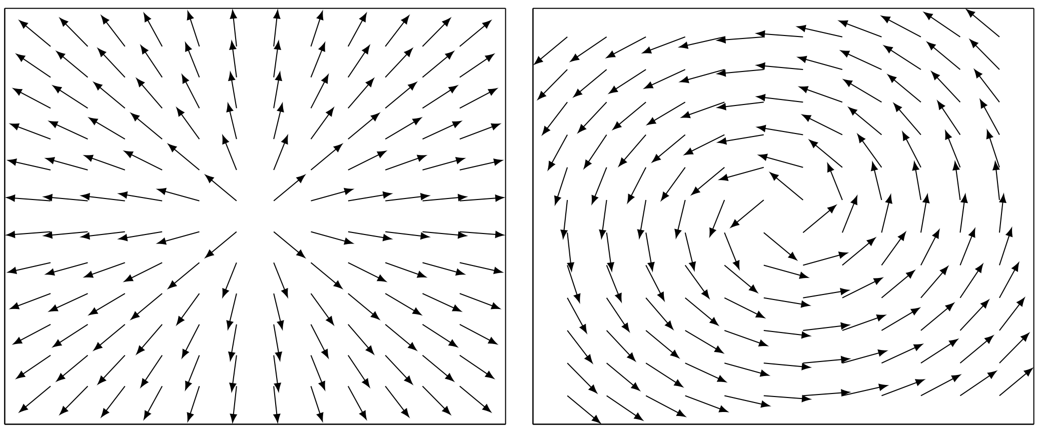
   (b) boundary conditions to be paired with the partial differential equation which either come naturally from the structure of the energy or are imposed to fit the desired situation to be studied.

The functional $\mathscr{F}$ from item 2 in this list can be constructed based on the physically observable details of the system. For example, we can ask what the relevant elastic distortions are and then try to quantify them appropriately. Items 3(a) and 3(b) follow from computing the first variation of $\mathscr{F}$. That is, calculating the Euler-Lagrange equations and natural boundary conditions associated to $\mathscr{F}$. Additional boundary conditions can sometimes be imposed if the given system requires it and the problem is well-posed. The most challenging aspect of modeling liquid crystal comes from item 1, finding the appropriate function space X to work with. The function space must strike a balance between being physically relevant and having appropriate regularity properties to satisfy the variational equations. In this thesis, we work within the context of a simple system governed by the *Oseen-Frank model* which we discuss below. A brief overview of a more refined representation called the *Landau-de Gennes model* can be found in Appendix A.

The Oseen-Frank Model

Consider a container filled with nematic liquid crystal in either two or three dimensions. The container can be represented as a bounded, simply-connected domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ with smooth boundary $\partial\Omega$. Let
$$\mathbf{d} : \Omega \to \mathbb{S}^{N-1}$$

be a unit vector field representing the average direction of molecular alignment at each point of Ω, i.e., the director for the nematic sample. In order to derive an energy with appropriate elastic distortions for a given molecular configuration, it is useful to think about how the molecules in a small neighbourhood of the sample might interact with one another. Does the molecular configuration tend to spread out near a point $x \in \Omega$? Perhaps there is a slight tendency for the molecules to circulate around some region. In three dimensions one could also even observe helical-like rotation about some axis. To quantify these different configurations, we can borrow some machinery from vector calculus. Specifying to the two-dimensional case for a moment, we consider two 'extreme' molecular configurations in the plane, namely, curl-free and divergence-free configurations as shown below in figure 1.1. In the first panel



(a) Curl-free director field (b) Divergence-free director field

Figure 1.1: Molecular splay and bend near the origin.

we see a molecular configuration that tends to spread out from the origin. This 'spreading' configuration will be referred to as molecular *splay* and can be quantified by the divergence of the director $\operatorname{div}\mathbf{d}$. The second represents a configuration in which the molecules are 'bending' around the origin. The degree of molecular *bend* can be gauged by the curl of the director $\operatorname{curl}\mathbf{d}$. In this setting, one can see that a combination of the divergence and curl of the director will capture quite a bit about the geometry associated to any configuration in two dimensions. After this crude visualization, it then makes sense to consider the two-dimensional Oseen-Frank energy [14]

$$\tilde{\mathscr{F}}_{OF}(\mathbf{d}) = \frac{1}{2}\int_{\Omega}(k_s(\operatorname{div}\mathbf{d})^2 + k_b(\operatorname{curl}\mathbf{d})^2)\,dx, \quad \mathbf{d} \in \mathbb{S}^1.$$

The quantities k_s and k_b are called the *splay modulus* and *bend modulus* respectively and are taken to be positive constants in this functional.

Remark 1.1. *A more realistic energy would consider the splay and bend moduli as functions of the space variables.*

In the three-dimensional version of the Oseen-Frank energy, two more energy densities can be introduced to account for the new intricacies made available by the extra spacial dimension. These densities involve a *twist* component and a *saddle-splay* component [41]. We omit writing the functional for the three-dimensional case since our focus for this work will be solely on a two-dimensional problem.

In the special case where the bend and splay moduli are equal

$$k = k_s = k_b$$

we obtain the identity

$$k_s(\operatorname{div}\mathbf{d})^2 + k_b(\operatorname{curl}\mathbf{d})^2 = k\left(|\nabla\mathbf{d}|^2 + 2\det(\nabla\mathbf{d})\right). \tag{1.2.1}$$

The identity (1.2.1) can be used to derive a much simpler form of the functional $\tilde{\mathscr{F}}_{OF}$ when an appropriate function space is considered. To obtain this simplification, let $\mathbf{g} \in \mathbb{S}^1$ be such that

$$X = \{\mathbf{d} \in H^1(\Omega;\mathbb{S}^1) : \mathbf{d} = \mathbf{g} \text{ on } \partial\Omega\}$$

is nonempty. Having $\mathbf{g}$ specified in this context amounts to knowing the molecular configuration of the nematic sample along the boundary of the container. Over the set X, the null Lagrangian $\det(\nabla\mathbf{d})$ integrates to a constant $C = C(\mathbf{g})$ depending only on $\mathbf{g}$. Therefore over the function space X the Oseen-Frank functional with equal bend and splay moduli takes the simplified form

$$\tilde{\mathscr{F}}_{OF}(\mathbf{d}) = \frac{k}{2}\int_\Omega |\nabla\mathbf{d}|^2\,dx + C(\mathbf{g}).$$

Assuming $k = 1$ and subtracting off C we arrive at the *one constant approximation* for the Oseen-Frank energy

$$\mathscr{F}_{OF}(\mathbf{d}) = \tilde{\mathscr{F}}_{OF}(\mathbf{d}) - C = \frac{1}{2}\int_\Omega |\nabla\mathbf{d}|^2\,dx.$$

With these simplifications in place, the new functional $\mathscr{F}_{OF}$ now represents a crude, generalized measure of molecular deviation away from the director $\mathbf{d}$.

Remark 1.2. *It should be noted that the boundary conditions imposed by the set X are not dealt with in this thesis. The intent of the above explanation is to be taken in a motivational context.*

Now that a function space and energy have been assigned we can begin looking for energy minimizing configurations. That is, solutions of the minimization problem $\inf_{\mathbf{d}\in X}\mathscr{F}_{OF}$. Upon taking the first variation of $\mathscr{F}_{OF}$ over X one obtains that directors corresponding to minimizing configurations are weak solutions of the Euler-Lagrange system

$$\begin{cases} -\Delta\mathbf{d} = |\nabla\mathbf{d}|^2\mathbf{d} & \text{in } \Omega, \\ \mathbf{d} = \mathbf{g} & \text{on } \partial\Omega, \end{cases}$$

which is the well-studied $\mathbb{S}^1$-valued harmonic map problem (see [18] for a light introduction). Therefore the minimizing configuration represented by the director $\mathbf{d}$ is the unique solution to this system and is a smooth $\mathbb{S}^1$-valued harmonic map.

1.3 Ginzburg-Landau Energy Minimizers

To give context for the main problems of this thesis, it will be useful for the reader to encounter some foundational background material related to some relaxed versions of the minimization problem addressed in the previous section. We begin this discussion with a brief overview of some influential work that was established in the early to mid 1990s which concentrate on a minimization problem with given Dirichlet boundary data. From there, we will move to more recent work that deals with different boundary conditions.

Nematic Minimizing Configurations with Strong Anchoring

Recall from above that the minimization class used for $\mathscr{F}_{OF}$ was the set

$$X = \{\mathbf{d} \in H^1(\Omega;\mathbb{S}^1) : \mathbf{d} = \mathbf{g} \text{ on } \partial\Omega\}.$$

The condition that $\mathbf{d} = \mathbf{g}$ along the container's boundary is called the *strong anchoring condition* in liquid crystal theory. In partial differential equation terminology, this simply coincides with a Dirichlet boundary condition. Recall also from Section 1.2 that it was assumed the strong anchoring set X was nonempty in order to derive the one constant approximation functional $\mathscr{F}_{OF}$. In general, this assumption cannot always be made unless more is known about the function $\mathbf{g}$ and the geometry of Ω. This problem is clearly highlighted by [11, Lemma 5]. In particular if $\partial\Omega = \mathbb{S}^1$, then

$$X \neq \varnothing \iff \deg(\mathbf{g};\partial\Omega) = 0.$$

In other words, the winding number of $\mathbf{g}$ along $\partial\Omega$ must be zero to even consider the minimization problem $\inf_{\mathbf{d}\in X}\mathscr{F}_{OF}$. This of course is a concern since many physically observable configurations where $\deg(\mathbf{g};\partial\Omega) \in \mathbb{Z}\setminus\{0\}$ cannot be analyzed. Luckily, there is a workaround for this setback and it forms the basis of the famous text of Bethuel, Brezis and Hélein [10]. To allow for non-zero degree boundary conditions, the authors of [10] extend X to the set

$$H^1_{\mathbf{g}}(\Omega;\mathbb{R}^2) = \{\mathbf{d} \in H^1(\Omega;\mathbb{R}^2) : \mathbf{d} = \mathbf{g}\}$$

and then introduce the term

$$\frac{1}{4\varepsilon^2}\int_\Omega (1-|\mathbf{d}|^2)^2\,dx \tag{1.3.1}$$

to the Oseen-Frank energy where $\varepsilon > 0$ is a small constant. The analysis done in this book then observes the behaviour of minimizers $\mathbf{d}_\varepsilon \in H^1_{\mathbf{g}}(\Omega;\mathbb{R}^2)$ for the Ginzburg-Landau functional

$$G_\varepsilon(\mathbf{d};\Omega) = \frac{1}{2}\int_\Omega \left(|\nabla \mathbf{d}|^2 + \frac{1}{2\varepsilon^2}(1-|\mathbf{d}|^2)^2\right)dx$$

with associated Euler-Lagrange equations

$$\begin{cases} -\Delta \mathbf{d}_\varepsilon = \dfrac{1}{\varepsilon^2}(1-|\mathbf{d}_\varepsilon|^2)\mathbf{d}_\varepsilon & \text{in } \Omega, \\ \quad \mathbf{d}_\varepsilon = \mathbf{g} & \text{on } \partial\Omega, \end{cases} \tag{1.3.2}$$

as the weighting $\varepsilon \to 0$. The point of including the new potential (1.3.1) is that for small $\varepsilon > 0$, the quantity $(1-|\mathbf{d}|^2)^2$ becomes heavily penalized in the energy and thus one would expect $|\mathbf{d}_\varepsilon| \to 1$ as $\varepsilon \to 0$ as a means to approximate the classical $\mathbb{S}^1$-valued director. By the main theorem of [10], this is exactly what happens and the precise statement is given below for convenience:

Theorem (Bethuel, Brezis and Hélein)**.** *Let $\Omega \subset \mathbb{R}^2$ be starshaped with smooth boundary and let $\mathbf{g} : \partial\Omega \to \mathbb{S}^1$ be smooth with degree $\mathcal{D} = \deg(\mathbf{g};\partial\Omega) > 0$. Then there is a subsequence $\varepsilon_n \to 0$, exactly $\mathcal{D}$ points $\{a_1,\dots,a_{\mathcal{D}}\} \subset \Omega$ and a smooth harmonic map $\mathbf{d}_0 : \Omega\setminus\{a_1,\dots,a_{\mathcal{D}}\} \to \mathbb{S}^1$ with $\mathbf{d}_0 = \mathbf{g}$ on $\partial\Omega$ such that*

$$\mathbf{d}_{\varepsilon_n} \to \mathbf{d}_0 \textit{ in } C^k_{loc}(\Omega\setminus\cup_i\{a_i\})\ \forall k \textit{ and in } C^{1,\alpha}(\overline{\Omega}\setminus\cup_i\{a_i\})\ \forall\alpha < 1.$$

Shortly after this was proven, Michael Struwe in [39] showed a similar result with relaxed assumptions on Ω. Here, Ω can now be multiply-connected and weak H^1-convergence of minimizers is achieved along a subsequence $\varepsilon_n \to 0$ to a smooth $\mathbb{S}^1$-valued harmonic map on $\Omega\setminus\{a_1,\dots,a_{\mathcal{D}}\}$.

The elements of the finite singular set $\{a_1,\dots,a_{\mathcal{D}}\}$ are called the *vortices* of the limiting map and their existence is a consequence of the extension of X to $H^1_{\mathbf{g}}(\Omega;\mathbb{R}^2)$. By relaxing the assumption $\mathbf{d}\in\mathbb{S}^1$, the topology associated to non-zero degree boundary conditions forces the existence of a non-empty zero set

$$\{x\in\Omega : |\mathbf{d}_\varepsilon| = 0\},$$

which we also call a 'vortex containing set' for $\mathbf{d}_\varepsilon$. The vortex containing set for $\mathbf{d}_\varepsilon$ is not completely understood, but one of the main results of [10, 39] show that it can at least be covered by a finite number of balls with radius of order ε, and the number of such balls is bounded independent of ε. Therefore, when $\varepsilon > 0$ is small, the vortex containing set must also be small. The method for constructing this cover actually depends on a larger set (which this ball covering also covers) that has historically been called the 'bad set'. Specifically, the bad set is defined to be

$$S_\varepsilon := \{x\in\Omega : |\mathbf{d}_\varepsilon| < 1/2\}$$

which in a loose sense, represents the set of points in Ω where $\mathbf{d}_\varepsilon$ begins to stray from acting like a classical director. The general idea is to use this set to define a 'good set' of points $\Omega\setminus S_\varepsilon$ where $\mathbf{d}_\varepsilon$ does not touch any of the problematic portions of the domain. To obtain the fact that the cardinality of this ball covering is uniformly bounded in ε, an important property of minimizers called *η-compactness* (or *η-ellipticity*) can be shown to hold, which basically states that for solutions of (1.3.2) satisfying the logarithmic energy bound over a region $\mathcal{R}\subset\Omega$

$$\frac{1}{2}\int_{\mathcal{R}}\left(|\nabla\mathbf{d}_\varepsilon|^2 + \frac{1}{2\varepsilon^2}\left(1-|\mathbf{d}_\varepsilon|^2\right)^2\right)dx \le \eta|\ln\varepsilon|$$

for a constant $\eta > 0$ (small enough) independent of ε, there is a smaller region $\tilde{\mathcal{R}}\subset\mathcal{R}$ where one has the pointwise bound

$$|\mathbf{d}_\varepsilon(x)| \ge \frac{1}{2}$$

for all $x\in\tilde{\mathcal{R}}$ and there is a constant $C > 0$ independent of ε where

$$\frac{1}{\varepsilon^2}\int_{\tilde{\mathcal{R}}}(1-|\mathbf{d}_\varepsilon|^2)^2\,dx \le C\eta.$$

In other words, the bad set S_ε and the region $\tilde{\mathcal{R}}$ are disjoint under these conditions.

Finally, it will be worth stating some fundamental results which stem from the analysis leading up to the main theorems of [10] and [39]. In this way, we can compare these classic findings to those analyzed here in this work for an extended Ginzburg-Landau functional with boundary energy.

Classic Results for Ginzburg-Landau Minimizers with Strong Anchoring

(SI) Solutions $\mathbf{d}_\varepsilon$ of (1.3.2) have the pointwise bounds

$$|\mathbf{d}_\varepsilon| \leq 1, \quad |\nabla \mathbf{d}_\varepsilon| \leq \frac{C}{\varepsilon}$$

for all $x \in \Omega$ where C is a constant independent of ε. [39, Lemma 2.2]

(SII) There is a subsequence of minimizers $\{\mathbf{d}_{\varepsilon_n}\} \subset H^1_{\mathbf{g}}(\Omega; \mathbb{R}^2)$ for G_ε and a constant $C > 0$ independent of ε_n such that

$$G_\varepsilon(\mathbf{d}_{\varepsilon_n}; \Omega) \leq \pi \mathcal{D} |\ln \varepsilon_n| + C.$$

Moreover, for certain $\sigma \in (0, \sigma_0)$, there are balls $B_\sigma(a_i)$ of radius σ centered at the vortices $\{a_i\}$ such that the energy on these balls satisfy

$$G_\varepsilon(u_{\varepsilon_n}; \cup_{i=1}^{\mathcal{D}} B_\sigma(a_i)) \geq \pi \mathcal{D} \ln \frac{\sigma}{\varepsilon_n} - C$$

where C is a constant independent of ε and σ. [10, Theorems III.1 & V.3],

(SIII) The limiting vortices $\{a_i\}_{i=1}^{\mathcal{D}}$ are always interior points. That is, $a_i \notin \partial\Omega$ for all $i = 1, \ldots, \mathcal{D}$. Moreover, if $\mathcal{C}(a_i)$ is a small circle enclosing the vortex $a_i \in \Omega$, then the limiting harmonic map $\mathbf{d}_0$ satisfies $\deg(\mathbf{d}_0; \mathcal{C}(a_i)) = 1$ for all $i = 1, \ldots, \mathcal{D}$. [10, Theorem VI.2]

Relaxing the Strong Anchoring Condition

One of the main benefits to working with the strong anchoring condition is the knowledge of boundary behaviour for minimizers $\mathbf{d}_\varepsilon \in H^1_{\mathbf{g}}(\Omega; \mathbb{R}^2)$ of G_ε. Indeed, the Dirichlet boundary problem as treated in most texts on partial differential equations usually signifies the simplest class of problems since it drastically reduces the complicated analysis needed near the boundary. A particular example of where the strong anchoring condition is useful in the context of liquid crystals comes from item (SIII) of the classic results list above, namely that the set of vortices $\{a_1, \ldots, a_{\mathcal{D}}\}$ never intersect the boundary $\partial\Omega$. This fact makes the process of finding the location of nematic point defects much easier.

An obvious question that arises from result (SIII) is:

In which situations are *there boundary vortices?*

It turns out that minimal alteration is needed to produce them. One way to do this is to drop the strong anchoring condition and then modify the energy functional so that boundary behaviour is accounted for. Over the last couple decades, there has been some advancement concerning this very question. In this section, we will provide two interesting problems that implement this relaxation, one of which heavily relates to the main focus of this thesis. The first problem comes from [3] where the authors seek minimizers for the Ginzburg-Landau functional

$$\mathscr{F}_{weak}(u;\Omega) = G_\varepsilon(u;\Omega) + \frac{K}{2\varepsilon^s}\int_{\partial\Omega} |u-g|^2\, ds$$

over the class $X = H^1(\Omega;\mathbb{R}^2)$ where $\Omega \subset \mathbb{R}^2$ can be taken to be a bounded, simply-connected domain with smooth boundary, g is a smooth $\mathbb{S}^1$-valued vector field on $\partial\Omega$ of degree $\mathcal{D} > 0$, $s \in (0,1]$ and $K > 0$ is a constant. Minimizers for $\mathscr{F}_{weak}$ satisfy the Euler-Lagrange system

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon & \text{in } \Omega, \\ \partial_n u_\varepsilon = -\dfrac{K}{\varepsilon^s}(u_\varepsilon - g) & \text{on } \partial\Omega. \end{cases}$$

The boundary behaviour imposed by this new energy term is called the *weak anchoring condition* and its function is to penalize minimizers whose magnitude and direction are far from g along $\partial\Omega$. As in the strong anchoring problems of [10] and [39], the authors of [3] study the convergence of minimizers u_ε for $\mathscr{F}_{weak}$ as $\varepsilon \to 0$. Many of the classical results for the strong anchoring condition hold in this case as well, such as the convergence along a subsequence $\varepsilon_n \to 0$ to a $\mathbb{S}^1$-valued harmonic map u_0 outside a singular set. The singular set in this case however may include points along $\partial\Omega$.

Since a minimizer u_ε need not be $\mathbb{S}^1$-valued on the boundary in this setting, it is reasonable to expect that the formation of boundary vortices is not impossible. What is found is that the exponent $s \in (0,1]$ included in the boundary integral plays a fundamental role in dictating the location of vortices. By constructing appropriate upper and lower bounds for $\mathscr{F}_{weak}$ it can be shown that there is a critical exponent $s = s_\star \in (0,1]$ such that for $0 < s < s_\star$ the limiting map u_0 has exactly $\mathcal{D}$ vortices, all of which are located on $\partial\Omega$. For $s_\star < s \leq 1$ the vortices for u_0 occur strictly within the interior Ω. At the critical exponent $s = s_\star$ the value of the constant K can dictate the location of vortices, where appropriately small K yields boundary vortices while appropriately large values of K give interior vortices.

The second problem we'd like to consider now is closer to that of what will be dealt with in this thesis. In his paper [32], Moser considers the free boundary data problem

$$\inf\left\{G_\varepsilon(u;\Omega)+\frac{1}{2\varepsilon^s}\int_{\partial\Omega}\langle u,n\rangle^2\,ds : u\in H^1(\Omega;\mathbb{R}^2)\right\} \tag{1.3.3}$$

where again $\Omega\subset\mathbb{R}^2$ is a bounded, simply connected domain with smooth boundary, $s\in(0,1]$ and n is the outward unit normal vector to $\partial\Omega$. The bilinear functional

$$\langle\cdot,\cdot\rangle:\mathbb{R}^N\to\mathbb{R}$$

denotes the standard innerproduct for vectors in $\mathbb{R}^N$ and solutions of (1.3.3) solve the system

$$\begin{cases}-\Delta u_\varepsilon=\dfrac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon & \text{in } \Omega,\\[2mm] \partial_n u_\varepsilon=-\dfrac{1}{\varepsilon^s}\langle u_\varepsilon,n\rangle n & \text{on } \partial\Omega.\end{cases}$$

Instead of penalizing minimizers which do not 'stay close' to some specified vector field, in this problem the additional boundary energy simply penalizes minimizers which stray from being parallel to τ, the positively oriented tangent vector to the boundary. In this way, Moser's problem is a slight weakening of what is expected of minimizers in [3] with $g=\tau$, in the sense that minimizers now only must be close to the *axis* associated to g and not g itself. Even so, as before from the discussion of [3], there is no $\mathbb{S}^1$-value constraint along $\partial\Omega$ for minimizers u_ε of Moser's problem and thus vortices may form there. To see some effects of this boundary energy in the context of liquid crystals, we refer the reader to the work of García-Cervera, Giorgi and Joo [20]. However, the work done in [20] will involve mixed boundary conditions on a square domain. Nonetheless, this reference nicely motivates the use of such a boundary energy.

Although the physical basis for [32] has its roots in ferromagnetic bodies, the model used in this work is equivalent to the relaxed Oseen-Frank setting for nematic material. It is found that the critical exponent for this problem is $s_\star=1$ in the sense that the limiting harmonic map u_0 has two boundary vortices when $0<s<1$. If $s=1$ then there are either exactly two boundary vortices or exactly one interior vortex. We note that in this case, the number of vortices for u_0 does *not* always coincide with the degree of the fixed boundary function as it did for [10] and [39] since $\mathcal{D}=\deg(n;\partial\Omega)=1$.

This concludes our discussion of Ginzburg-Landau minimizing problems and we may now give the main problems of this thesis.

1.4 Thesis Outline

Throughout this entire thesis, $\Omega \subset \mathbb{R}^2 \cong \mathbb{C}$ will denote a bounded, simply-connected domain with C^4-smooth boundary $\Gamma := \partial\Omega$. We associate to Γ a small tubular neighbourhood

$$\mathcal{N}_\Gamma = \text{small tubular neighbourhood of } \Gamma,$$

and we fix a function

$$g \in C^4(\mathcal{N}_\Gamma; \mathbb{S}^1) \cap C^4(\overline{\Omega}; \mathbb{R}^2)$$

with positive degree $\mathcal{D} = \deg(g; \Gamma) \in \mathbb{Z}_+$. The case of g having negative degree can be transformed to the former via complex conjugation [10]. The requirement that $g \in C^4(\mathcal{N}_\Gamma; \mathbb{S}^1)$ will allow us to perform orthogonal decompositions of functions with respect to the orthonormal basis $\{g(x), g^\perp(x)\}$ near Γ which will be needed later.

In this thesis, we consider two Ginzburg-Landau minimization problems in the context of modeling nematic liquid crystal in the relaxed Oseen-Frank setting. The primary focus in both of these problems is to understand when the formation of boundary vortices occur. The first main problem is a generalization of Moser's tangential problem (1.3.3), where we require the orientation of minimizers along Γ to be roughly parallel to a given function g of degree $\mathcal{D} \geq 1$. Specifically, we study

$$\begin{cases} \inf\limits_{u \in H^1(\Omega;\mathbb{R}^2)} G_\varepsilon^W(u), \\ G_\varepsilon^W(u) := G_\varepsilon(u) + \dfrac{W}{2\varepsilon^s} \displaystyle\int_\Gamma \langle u, g^\perp \rangle^2 \, ds, \end{cases} \tag{W.O.}$$

where $s \in (0, 1]$, $W \in (0, +\infty)$ and $g \in C^4(\mathcal{N}_\Gamma; \mathbb{S}^1) \cap C^4(\overline{\Omega}; \mathbb{R}^2)$ are fixed and we recall the notation

$$G_\varepsilon(u) := \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2 \right) dx = \int_\Omega e_\varepsilon(u) \, dx.$$

The minimizing problem (W.O.) will be called the *weak orthogonality problem* and we say the boundary integral

$$\frac{W}{2\varepsilon^s} \int_\Gamma \langle u, g^\perp \rangle^2 \, ds$$

enforces the *weak orthogonality condition*, analogous to the weak anchoring condition from Section 1.3 employed by the authors of [3]. Associated to

(W.O.) is a limiting case given by

$$\begin{cases} \inf\limits_{u\in\mathcal{H}(\Omega)} G_\varepsilon(u;\Omega), \\ \mathcal{H}(\Omega) = \{u \in H^1(\Omega;\mathbb{R}^2) : \langle u, g^\perp\rangle = 0 \text{ on } \Gamma\}, \end{cases} \tag{S.O.}$$

which we call the *strong orthogonality problem*. The restriction $\langle u, g^\perp\rangle = 0$ on Γ will be called the *strong orthogonality condition* and this carries the interpretation that $u = f(x)g$ on Γ where $f : \Gamma \to \mathbb{R}$ is a scalar function. The motivation for studying (S.O.) comes from a limiting observation related to (W.O.) and is rigorously justified in Chapter 2.

Remark 1.3. *Observe that in problem* (S.O.) *the functional G_ε can be replaced by G_ε^W for any fixed $W \in (0,+\infty)$ since $G_\varepsilon^W(v) = G_\varepsilon(v)$ for all $v \in \mathcal{H}(\Omega)$. The choice to use G_ε as opposed to G_ε^W merely emphasizes that the minimization problem is independent of W.*

The second main problem we consider is another variant of (1.3.3), but instead of generalizing the boundary energy, we study a generalization of the interior energy which adds additional penalization for either molecular bend or splay as introduced in Section 1.2. Consider the functional

$$\mathscr{F}_\varepsilon(u) := \frac{1}{2}\int_\Omega \left(\tilde{k}|\nabla u|^2 + h_{\tilde{k}}(u) + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right) dx$$

where $h_{\tilde{k}}(u)$ is either equal to $\kappa(\operatorname{div} u)^2$ or $\kappa(\operatorname{curl} u)^2$ and where $\tilde{k}$ and κ are known positive constants. The functional $\mathscr{F}_\varepsilon$, within the context we consider for this work, was originally used by Colbert-Kelly and Phillips in [14] where convergence of its H^1-minimizers with smooth Dirichlet boundary data was analyzed. Our goal here is to observe the strong anchoring relaxation of this problem through the lens of (1.3.3). Specifically, we consider the minimization problems

$$\begin{cases} \inf\limits_{u\in H^1(\Omega;\mathbb{R}^2)} \mathscr{F}_\varepsilon^W(u), \\ \mathscr{F}_\varepsilon^W(u) := \mathscr{F}_\varepsilon(u) + \dfrac{W}{2\varepsilon^s}\displaystyle\int_\Gamma \langle u, n\rangle^2\, ds, \end{cases} \tag{W.O.*}$$

where $s \in (0,1]$, $W \in (0,+\infty)$, n is the outward unit normal to Γ and

$$\begin{cases} \inf\limits_{u\in\mathcal{H}_\tau(\Omega)} \mathscr{F}_\varepsilon(u), \\ \mathcal{H}_\tau(\Omega) := \left\{u \in H^1(\Omega;\mathbb{R}^2) : \langle u, n\rangle = 0 \text{ on } \Gamma\right\}, \end{cases} \tag{S.O.*}$$

in an effort to study how molecular bend and splay contribute to the formation of boundary vortices in a relatively simple setting.

For all problems (W.O.), (W.O.*), (S.O.) and (S.O.*), we are interested in observing minimizers u_ε and their limiting map u_0 along a subsequence $\varepsilon_n \to 0$ near the vortex-containing set $\{x \in \overline{\Omega} : |u_\varepsilon| = 0\}$. Our main goal is to determine when boundary vortices occur and what the behaviour of minimizers are around boundary vortices when the parameter $\varepsilon > 0$ is small. Below, we give a brief overview of the content contained in Chapters 2-6 of this thesis. Cumulatively, these chapters can be strung together to prove our two main theorems, which we state after the chapter overview.

- In **Chapter 2**, we prove the existence of minimizing solutions u_ε for (W.O.) and (S.O.) and derive their associated Euler-Lagrange equations. At the end of the chapter, these equations are used to develop a Pohozaev identity that will be used several times throughout this work. Also in this chapter, we justify the claim that (S.O.) can be thought of as a limiting case of problem (W.O.) where the weighting $W \to +\infty$. From here, some pointwise estimates for u_ε and its gradient are given. Finally, an optimal upper bound for the energy of minimizers is found which is shown to be logarithmic in ε.

- In **Chapter 3**, our main goal is to prove an η-compactness result which is tailored for solutions of (W.O.) and (S.O.). We then use this property to show that a finite cover of balls exists (whose number is independent of ε) for the associated 'bad set' as discussed in Section 1.3. The η-compactness property depends on a lengthy integral estimate which is derived at the beginning of this chapter. At the end, we then show that a static ball covering can be constructed that covers the bad sets for all $\varepsilon > 0$ small enough along some subsequence.

- In **Chapter 4**, we define a notion of orientation for minimizers with respect to the boundary data g on Γ. Using this orientation, a topological integer D called *the boundary index* is defined which aims to describe the winding behaviour of minimizers around boundary vortices. From here, we take the time to analyze how this quantity relates to the degree of interior vortices for u_ε by proving two identities. The first of these identities is a global one and also includes how the degree $\mathcal{D}$ of g relates to the winding of u_ε around all vortices in the domain. The second identity provides the way in which interior degrees and boundary indices can be added together locally. These quantities are then utilized to develop a lower bound for the Dirichlet energy of u_ε on annuli. The final part of the chapter then uses this lower bound in combination with an intricate vortex ball expansion/fusion argument to obtain a global lower bound for the energy of u_ε outside the bad set.

- In **Chapter 5**, we show that the upper bound from Chapter 2 can be combined with the lower bound of Chapter 4 to produce a uniform bound for the energy of u_ε outside of the static bad set covering. This bound can then be used to extract a subsequence of minimizers which converge weakly in H^1 to an $\mathbb{S}^1$-valued harmonic map outside of a finite number of point singularities in $\overline{\Omega}$. We also show that for $\varepsilon > 0$ small enough, one can conclude that the degree and boundary index associated to each non-trivial vortex is equal to one. Moreover, in the case of solutions for (W.O.) we prove that when $0 < s < 1$, the only non-trivial vortices that occur are located on Γ and that there are precisely $2\mathcal{D}$ of them. For $s = 1$ and for solutions of (S.O.), it may be possible for both interior and boundary vortices to simultaneously exist. To analyze this further, we look at a specific example in the context of the strong orthogonality problem where the domain is taken to be the unit disc with tangential boundary data. In this example, we show that the renormalized energy associated to the asymptotic expansion of the energy for boundary vortices attains a smaller minimum value than that of the interior renormalized energy. This suggests that a pair of boundary vortices may be energetically preferable over a single interior vortex.

- In **Chapter 6**, we discuss and prove some modified estimates for (W.O.*) and (S.O.*) that were originally seen in the previous chapters for (W.O.) and (S.O.). In this case however, we make the critical assumption that solutions of (W.O.*) and (S.O.*) satisfy similar pointwise bounds as proven in Chapter 2. We show that conditions can be found that ensure the formation of boundary vortices in the case of curl penalization and conclude that the vortices of the limiting map have degree or boundary indices equal to one whenever uniform energy bounds can be obtained.

By combining the results of Chapters 2-5, we prove...

Theorem 1.1. *Suppose* $\{u_\varepsilon\}_{\varepsilon>0}$ *is a sequence of minimizers for either* (W.O.) *or* (S.O.) *with associated boundary function* g *having degree* $\mathcal{D} \geq 1$*. Then there is a subsequence* $\varepsilon_n \to 0$ *and a finite number of point singularities* $\Sigma \subset \overline{\Omega}$ *such that*

$$u_{\varepsilon_n} \rightharpoonup u_0 \quad \textit{weakly in } H^1_{loc}(\overline{\Omega} \setminus \Sigma; \mathbb{R}^2)$$

where $u_0 \in H^1(\Omega \setminus \Sigma; \mathbb{R}^2)$ *is an* $\mathbb{S}^1$*-valued harmonic map. Moreover, the degree and boundary index associated to each vortex for* u_0 *in* Σ *is equal to one and in the specific case where* u_0 *is a limiting map of solutions for* (W.O.)*, then* $\Sigma \subset \Gamma$ *with* $|\Sigma| = 2\mathcal{D}$ *whenever* $0 < s < 1$.

By Chapter 6 and elements of the previous chapters, we prove...

Theorem 1.2.

(a)

Assume $h_{\tilde{k}}(v) = \kappa(\operatorname{curl} v)^2$ *and suppose* $\{u_\varepsilon\}_{\varepsilon>0}$ *is a sequence of minimizers for either* (W.O.*) *or* (S.O.*) *such that there is a constant* $C_0 > 0$ *independent of* ε *for which*

$$|u_\varepsilon| \leq C_0, \quad |\nabla u_\varepsilon| \leq \frac{C_0}{\varepsilon}$$

for all $x \in \Omega$. *Then there is a subsequence* $\varepsilon_n \to 0$ *and a finite set of point singularities* Σ *of* $\overline{\Omega}$ *such that*

$$u_{\varepsilon_n} \rightharpoonup u_0 \quad \textit{weakly in } H^1_{loc}(\overline{\Omega} \setminus \Sigma; \mathbb{R}^2)$$

where $u_0 \in H^1(\Omega \setminus \Sigma; \mathbb{R}^2)$ *with* $|u_0| = 1$ *almost everywhere. The degree and boundary index associated to each vortex for* u_0 *in* Σ *is equal to one. In the specific case where* u_0 *is a limiting map of solutions for* (W.O.*), *we have* $\Sigma = \{q_1, q_2\} \subset \Gamma$ *whenever* $0 < s < 1$. *If* u_0 *is the limiting map of solutions for* (S.O.*) *or* (W.O.*) *with* $s = 1$, *then either* $\Sigma = \{p_1\} \subset \Omega$ *or* $\Sigma = \{q_1, q_2\} \subset \Gamma$.

(b)

Assume $h_{\tilde{k}}(v) = \kappa(\operatorname{div} v)^2$ *and suppose* $\{u_\varepsilon\}_{\varepsilon>0}$ *is a sequence of minimizers for* (S.O.*) *or* (W.O.*) *with* $s = 1$ *satisfying the given pointwise bounds from part (a). Then there is a subsequence* $\varepsilon_n \to 0$ *and a finite set of point singularities* Σ *of* $\overline{\Omega}$ *such that*

$$u_{\varepsilon_n} \rightharpoonup u_0 \quad \textit{weakly in } H^1_{loc}(\overline{\Omega} \setminus \Sigma; \mathbb{R}^2)$$

where $u_0 \in H^1(\Omega \setminus \Sigma; \mathbb{R}^2)$ *with* $|u_0| = 1$ *almost everywhere. The degree and boundary index associated to each vortex for* u_0 *in* Σ *is equal to one and either* $\Sigma = \{p_1\} \subset \Omega$ *or* $\Sigma = \{q_1, q_2\}$.

The final chapter of this thesis, **Chapter 7**, is dedicated to stating some new problems that have arisen from this work.

Chapter 2

Existence and Some Basic Properties of Minimizers

We begin our analysis by deriving some fundamental properties for solutions of (W.O.) and (S.O.).

2.1 Existence of Minimizing Solutions

The first obvious step in the analysis of our problem is to justify the existence of minimizing solutions. For the weak orthogonality functional G_ε^W, the Hilbert space $H^1(\Omega;\mathbb{R}^2)$ will suffice as an appropriate minimizing space to consider since the energy can be bounded below by the H^1-norm. The trace theorem for H^1 functions ensures all elements of this space belong to $L^2(\Gamma;\mathbb{R}^2)$ and therefore convergence with respect to the boundary energy can be dealt with easily. In the strong orthogonality problem we restrict to the subspace $\mathcal{H}(\Omega)$ as defined in (S.O.) to enforce the strong orthogonality condition along Γ. Thus, the existence of minimizers for (W.O.) and (S.O.) will follow easily by the direct method from the calculus of variations (see [16] for a nice introduction on this topic). That is, for each problem we will show that a minimizing sequence $\{u_n\}_{n=1}^\infty$ converges to some limit u_ε within the desired minimizing space.

Lemma 2.1 (Existence)**.** *For each $\varepsilon > 0$ and fixed $s \in (0, 1]$, $W \in (0, +\infty)$ in the weak orthogonality problem, there exists $u_\varepsilon \in H^1(\Omega;\mathbb{R}^2)$ such that*

$$G_\varepsilon^W(u_\varepsilon) = \inf_{v\in H^1(\Omega;\mathbb{R}^2)} G_\varepsilon^W(v).$$

In the strong orthogonality problem, for each $\varepsilon > 0$ there is $u_\varepsilon \in \mathcal{H}(\Omega)$ satisfying

$$G_\varepsilon(u_\varepsilon) = \inf_{v\in\mathcal{H}(\Omega)} G_\varepsilon(v).$$

Proof. Consider first the weak orthogonality problem and let

$$m := \inf_{v \in H^1(\Omega;\mathbb{R}^2)} G_\varepsilon^W(v) \geq 0.$$

Since we are ultimately interested in small values of $\varepsilon > 0$, we may assume $0 < \varepsilon \leq 1$. Given Ω is bounded, by Hölder's inequality

$$\|v\|_{L^2(\Omega;\mathbb{R}^2)} \leq 2^{1/4}|\Omega|^{1/4}\|v\|_{L^4(\Omega;\mathbb{R}^2)}$$

where $|\Omega|$ denotes the standard Lebesgue measure of Ω and

$$\begin{aligned}\int_\Omega (1-|v|^2)^2\,dx &\geq \|v\|^4_{L^4(\Omega;\mathbb{R}^2)} - 2\|v\|^2_{L^2(\Omega;\mathbb{R}^2)} + |\Omega| \\ &\geq \frac{1}{2|\Omega|}\|v\|^4_{L^2(\Omega;\mathbb{R}^2)} - 2\|v\|^2_{L^2(\Omega;\mathbb{R}^2)}.\end{aligned}$$

Choose $c > 0$ so that $\|v\|^4_{L^2(\Omega;\mathbb{R}^2)} \geq 2|\Omega|(3\|v\|^2_{L^2(\Omega;\mathbb{R}^2)} - c)$. Applying this to the above yields

$$\int_\Omega (1-|v|^2)^2\,dx \geq \|v\|^2_{L^2(\Omega;\mathbb{R}^2)} - c.$$

Now let $\{u_n\}_{n=1}^\infty \subset H^1(\Omega;\mathbb{R}^2)$ denote a minimizing sequence for G_ε^W. Taking c as above and n large enough,

$$m + 1 \geq G_\varepsilon^W(u_n) \geq \frac{1}{4}\|u_n\|^2_{H^1(\Omega;\mathbb{R}^2)} - c.$$

That is, minimizing sequences are uniformly bounded in the Hilbert space $H^1(\Omega;\mathbb{R}^2)$. Therefore there exists a subsequence $\{u_{n_j}\}$ and a function $u_\varepsilon \in H^1(\Omega;\mathbb{R}^2)$ so that $u_{n_j} \rightharpoonup u_\varepsilon$ weakly in H^1. By Sobolev embedding there is a further subsequence (still denoted $\{u_{n_j}\}$) such that $u_{n_j} \to u_\varepsilon$ pointwise almost everywhere in Ω. Then

$$\int_\Omega (1-|u_\varepsilon|^2)^2\,dx \leq \liminf_{j\to\infty} \int_\Omega (1-|u_{n_j}|^2)^2\,dx$$

by Fatou's lemma. Moreover, by Lemma D.3 we have

$$\int_\Gamma \langle u_\varepsilon, g^\perp\rangle^2\,ds \leq \liminf_{j\to\infty} \int_\Gamma \langle u_{n_j}, g^\perp\rangle^2\,ds$$

and therefore

$$G_\varepsilon^W(u_\varepsilon) \leq \liminf_{j\to\infty} G_\varepsilon^W(u_{n_j}) = m.$$

In the case of strong orthogonality, the minimizing sequence is taken to belong to $\mathcal{H}(\Omega)$. Existence follows as above except we now obtain by Lemma D.3

$$\int_\Gamma \langle u_\varepsilon, g^\perp\rangle^2 \, ds \le \liminf_{j\to\infty} \int_\Gamma \langle u_{n_j}, g^\perp\rangle^2 \, ds = 0$$

showing that $u_\varepsilon \in \mathcal{H}(\Omega)$. □

2.2 Euler-Lagrange Equations

In this section, we derive the Euler-Lagrange equations associated to the minimization problems (W.O.) and (S.O.).

(W.O.) Case

Let $u \in H^1(\Omega;\mathbb{R}^2)$ be a minimizer for G_ε^W and let $v \in H^1(\Omega;\mathbb{R}^2)$ be arbitrary. Taking the first variation of G_ε^W gives the equation

$$\begin{aligned}\frac{d}{dt} G_\varepsilon^W(u+tv)\Big|_{t=0} &= \int_\Omega \left(\sum_{i,j} u^i_{x_j} v^i_{x_j} - \frac{1}{\varepsilon^2}(1-|u|^2)\langle u, v\rangle \right) dx \\ &\quad + \frac{W}{\varepsilon^s}\int_\Gamma \langle \langle u, g^\perp\rangle g^\perp, v\rangle \, ds \\ &= 0\end{aligned}$$

for all $v \in H^1(\Omega;\mathbb{R}^2)$. To obtain the pointwise Euler-Lagrange equations, we assume u and v have sufficient regularity to apply integration by parts. Doing this yields

$$\int_\Omega \langle -\Delta u - \frac{1}{\varepsilon^2}(1-|u|^2)u, v\rangle \, dx + \int_\Gamma \langle \partial_n u + \frac{W}{\varepsilon^s}\langle u, g^\perp\rangle g^\perp, v\rangle \, ds = 0$$

and therefore minimizers satisfy the Euler-Lagrange system

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2}(1-|u|^2)u & \text{in } \Omega, \\ \partial_n u = -\dfrac{W}{\varepsilon^s}\langle u, g^\perp\rangle g^\perp & \text{on } \Gamma. \end{cases} \tag{2.2.1}$$

(S.O.) Case

Suppose $u \in \mathcal{H}(\Omega)$ is a minimizer for G_ε and let $v \in \mathcal{H}(\Omega)$. The first variation for G_ε is given by

$$\frac{d}{dt}G_\varepsilon(u+tv)\bigg|_{t=0} = \int_\Omega \left(\sum_{i,j} u^i_{x_j} v^i_{x_j} - \frac{1}{\varepsilon^2}(1-|u|^2)\langle u, v\rangle \right) dx = 0.$$

Integrating by parts, we are left with

$$\frac{d}{dt}G_\varepsilon(u+tv)\bigg|_{t=0} = \int_\Omega \langle -\Delta u - \frac{1}{\varepsilon^2}(1-|u|^2)u, v\rangle \, dx + \int_\Gamma \langle \partial_n u, v\rangle \, ds = 0.$$

To obtain the appropriate boundary conditions we write u and v on Γ using the orthonormal frame $\{g(x), g^\perp(x)\}$. For $x \in \mathcal{N}_\Gamma$ (the small tubular neighbourhood of Γ),

$$\begin{aligned} u &= \langle u, g\rangle g + \langle u, g^\perp\rangle g^\perp = u_\parallel g + u_\perp g^\perp, \\ \partial_n u &= \partial_n(u_\parallel g + u_\perp g^\perp) = u_\parallel \partial_n g + \partial_n u_\parallel g + u_\perp \partial_n g^\perp + \partial_n u_\perp g^\perp, \\ \partial_\tau u &= \partial_\tau(u_\parallel g + u_\perp g^\perp) = u_\parallel \partial_\tau g + \partial_\tau u_\parallel g + u_\perp \partial_\tau g^\perp + \partial_\tau u_\perp g^\perp. \end{aligned} \tag{2.2.2}$$

Then using the fact that $u_\perp = v_\perp = 0$ on Γ,

$$\begin{aligned} \int_\Gamma \langle \partial_n u, v\rangle \, ds &= \int_\Gamma \langle \partial_n(u_\parallel g + u_\perp g^\perp), v_\parallel g + v_\perp g^\perp\rangle \, ds \\ &= \int_\Gamma \langle u_\parallel \partial_n g + \partial_n u_\parallel g + \partial_n u_\perp g^\perp, v_\parallel g\rangle \, ds \\ &= \int_\Gamma \left(u_\parallel \langle \partial_n g, g\rangle + \partial_n u_\parallel \right) v_\parallel \, ds. \end{aligned}$$

Since the inner product of an $\mathbb{S}^1$-valued function with any of its directional derivatives is zero (see Lemma B.1) we have $\langle \partial_n g, g\rangle = 0$ and therefore by the first variation equation

$$\int_\Gamma \partial_n u_\parallel v_\parallel \, ds = 0$$

for all $v_\parallel$ and so $\partial_n u_\parallel = 0$. Thus, a minimizer for G_ε over $\mathcal{H}(\Omega)$ must satisfy the Euler-Lagrange system

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2}(1-|u|^2)u & \text{in } \Omega, \\ u_\perp = 0 & \text{on } \Gamma, \\ \partial_n u_\parallel = 0 & \text{on } \Gamma. \end{cases} \tag{2.2.3}$$

Remark 2.1. *Notice that* $u_\perp = 0$ *along* Γ *implies* $\partial_\tau u_\perp = 0$ *on* Γ.

2.3 Justification of the Strong Orthogonality Problem

It is clear by direct observation of the functional G_ε^W that for large values of W, minimizers are incentivized to decrease their projection along $g^\perp$ on Γ. This leads to the expectation that along some sequence $W_n \to +\infty$ for fixed $\varepsilon > 0$ we would find

$$\langle u_{\varepsilon,W_n}, g^\perp \rangle \to \langle u_{\varepsilon,\infty}, g^\perp \rangle = 0$$

with respect to some topology where $u_{\varepsilon,\infty} \in \mathcal{H}(\Omega)$ is a limiting function for the sequence $\{u_{\varepsilon,W_n}\}_{n=1}^\infty$. It turns out that such a limiting function exists and also corresponds to a $\mathcal{H}(\Omega)$-minimizer for G_ε. The following lemma allows us to view the strong orthogonality problem (S.O.) as a limiting case of the weak orthogonality problem (W.O.).

Lemma 2.2. *Let* $\{W_n\}_{n=1}^\infty$ *be an increasing sequence of real numbers such that* $W_n \to +\infty$ *as* $n \to \infty$ *and let* $\{u_{\varepsilon,W_n}\}_{n=1}^\infty$ *denote a sequence of* $H^1(\Omega;\mathbb{R}^2)$*-minimizers for* $G_\varepsilon^{W_n}$ *with* $\varepsilon > 0$ *and* $s \in (0,1]$ *fixed. Then there is a subsequence* $\{W_{n_j}\}_{j=1}^\infty$ *and a function* $u_{\varepsilon,\infty} \in \mathcal{H}(\Omega)$ *such that* $u_{\varepsilon,W_{n_j}} \rightharpoonup u_{\varepsilon,\infty}$ *weakly in* $H^1(\Omega;\mathbb{R}^2)$ *and* $u_{\varepsilon,\infty}$ *minimizes* G_ε *over* $\mathcal{H}(\Omega)$.

Proof. Fix any $W > 0$ and let $u_{\varepsilon,W} \in H^1(\Omega;\mathbb{R}^2)$ be a minimizer for G_ε^W. Let $\tilde{u}$ be a $\mathcal{H}(\Omega)$-minimizer for G_ε. Since $\mathcal{H}(\Omega) \subset H^1(\Omega;\mathbb{R}^2)$ and $\tilde{u}$ is independent of W we have

$$G_\varepsilon^W(u_{\varepsilon,W}) \le G_\varepsilon^W(\tilde{u}) = G_\varepsilon(\tilde{u}) = C \tag{2.3.1}$$

where C is a constant independent of W. As in the existence proof of Lemma 2.1, there is a constant $c > 0$ independent of W so that

$$\|u_{\varepsilon,W}\|_{H^1}^2 - c \le 4G_\varepsilon^W(u_{\varepsilon,W}).$$

Therefore

$$\|u_{\varepsilon,W}\|_{H^1} \le \tilde{C} \tag{2.3.2}$$

uniformly in W. Applying inequality (2.3.2) to the sequence $\{u_{\varepsilon,W_n}\}_{n=1}^\infty$ yields a subsequence $\{u_{\varepsilon,W_{n_j}}\}_{j=1}^\infty$ and a weak limit $u_{\varepsilon,\infty} \in H^1(\Omega;\mathbb{R}^2)$ such that $u_{\varepsilon,W_{n_j}} \rightharpoonup u_{\varepsilon,\infty}$ weakly in $H^1(\Omega;\mathbb{R}^2)$.

Next, we prove $u_{\varepsilon,\infty} \in \mathcal{H}(\Omega)$ by showing

$$\int_\Gamma \langle u_{\varepsilon,\infty}, g^\perp \rangle^2 \, ds = 0.$$

Suppose in order to derive a contradiction that

$$\liminf_{j\to\infty} \int_\Gamma \langle u_{\varepsilon,W_{n_j}}, g^\perp\rangle^2\, ds > 0. \tag{2.3.3}$$

Upon taking the limit infimum as $j \to \infty$ across inequality (2.3.1) we obtain

$$\liminf_{j\to\infty} \frac{W_{n_j}}{2\varepsilon^s} \int_\Gamma \langle u_{\varepsilon,W_{n_j}}, g^\perp\rangle^2\, ds \le \tilde{C}.$$

However, since $\{W_{n_j}\}_{j=1}^\infty$ is increasing, inequality (2.3.3) implies

$$\liminf_{j\to\infty} \frac{W_{n_j}}{2\varepsilon^s} \int_\Gamma \langle u_{\varepsilon,W_{n_j}}, g^\perp\rangle^2\, ds = +\infty$$

giving the desired contradiction. It now follows by Lemma D.3 (using the fact $g^\perp$ is continuous) that

$$\int_\Gamma \langle u_{\varepsilon,\infty}, g^\perp\rangle^2\, ds \le \liminf_{j\to\infty} \int_\Gamma \langle u_{\varepsilon,W_{n_j}}, g^\perp\rangle^2\, ds = 0$$

which shows $u_{\varepsilon,\infty} \in \mathcal{H}(\Omega)$. Finally, we prove $u_{\varepsilon,\infty}$ minimizes G_ε. Using the fact that $\tilde{u} \in \mathcal{H}(\Omega)$ is a known minimizer for G_ε, we trivially have

$$G_\varepsilon(\tilde{u}) \le G_\varepsilon(u_{\varepsilon,\infty}).$$

Upon relabeling subsequences as necessary, Lemma D.1, Fatou's lemma and the result above allows us to write

$$G_\varepsilon(u_{\varepsilon,\infty}) \le \liminf_{j\to\infty} G_\varepsilon^{W_{n_j}}(u_{\varepsilon,W_{n_j}}) \le C = G_\varepsilon(\tilde{u}).$$

Therefore $G_\varepsilon(u_{\varepsilon,\infty}) = G_\varepsilon(\tilde{u})$ and so $u_{\varepsilon,\infty} \in \mathcal{H}(\Omega)$ must be a minimizer for G_ε over $\mathcal{H}(\Omega)$. □

2.4 Pointwise Bounds for Minimizers and Their Gradients

In this section, we show that solutions to (2.2.1) and (2.2.3) continue to satisfy the classic pointwise bounds as originally obtained in [10, 39] for minimizers of G_ε in the Dirichlet case and [3, 32] for problems of weak anchoring-type. In short, Lemma 2.3 below confirms that in the context of Moser's problem (1.3.3), generalizing the energy to account for boundary functions of degree $\deg(g;\Gamma) \ge 1$ does not change the pointwise bounds.

Lemma 2.3. *Suppose u_ε is a solution of (2.2.1) or (2.2.3). Then*

$$|u_\varepsilon| \leq 1 \text{ on } \Omega \tag{2.4.1}$$

and there is a constant $C_0 > 0$ independent of ε for which

$$|\nabla u_\varepsilon| \leq \frac{C_0}{\varepsilon} \tag{2.4.2}$$

for all $x \in \Omega$.

The proof of both (2.4.1) and (2.4.2) follows [3, Lemma 3.2]. The main difference here occurs in the accounting for strong orthogonal boundary conditions. We also note that the techniques used below use the fact that solutions of the Euler-Lagrange equations are smooth. The reader can refer to Appendix C for a regularity statement and discussion on this topic.

Proof. We begin by justifying (2.4.1) using a maximum principle argument. Define the scalar function $V = |u|^2 - 1$ so that

$$\frac{1}{2}\nabla V = (u^1 u^1_{x_1} + u^2 u^2_{x_1}, u^1 u^1_{x_2} + u^2 u^2_{x_2}) = u \cdot \nabla u.$$

Then we have

$$\frac{1}{2}\Delta V = |\nabla u|^2 + \langle \Delta u, u\rangle = |\nabla u|^2 + \frac{1}{\varepsilon^2}V(V+1)$$

which of course readily implies

$$\frac{1}{\varepsilon^2}(V+1)V = \frac{1}{\varepsilon^2}|u|^2 V \leq \frac{1}{2}\Delta V$$

since $|\nabla u|^2$ is non-negative. Multiplying both sides of this inequality by

$$V_+ := \max\{V, 0\}$$

and integrating over Ω, we have $V_+ V = V_+^2$ and

$$0 \leq \int_\Omega |u|^2 V_+^2 \, dx \leq \frac{1}{2}\int_\Gamma V_+ \partial_n V \, ds - \frac{1}{2}\int_\Omega |\nabla V_+|^2 \, dx.$$

Focusing on the boundary integral, note that we may write for $x \in \Gamma$

$$\begin{aligned}\partial_n V &= \langle \nabla V, n \rangle \\ &= 2(u^1 u^1_{x_1} n^1 + u^2 u^2_{x_1} n^1 + u^1 u^1_{x_2} n^2 + u^2 u^2_{x_2} n^2) \\ &= 2(u^1 \langle \nabla u^1, n \rangle + u^2 \langle \nabla u^2, n \rangle) \\ &= 2\langle u, \partial_n u \rangle.\end{aligned}$$

If u is a solution of (2.2.1) then

$$\begin{aligned}\partial_n V &= 2\langle u, \partial_n u \rangle \\ &= -\frac{2W}{\varepsilon^s} \langle u, \langle u, g^\perp \rangle g^\perp \rangle \\ &= -\frac{2W}{\varepsilon^s} \langle u, g^\perp \rangle^2 \\ &\leq 0.\end{aligned}$$

If u is a solution of (2.2.3), we can write using the decomposition (2.2.2),

$$\begin{aligned}\partial_n V &= 2\langle u, \partial_n u \rangle \\ &= 2\langle u_\parallel g + u_\perp g^\perp, u_\parallel \partial_n g + \partial_n u_\parallel g + u_\perp \partial_n g^\perp + \partial_n u_\perp g^\perp \rangle.\end{aligned}$$

The boundary conditions $u_\perp = \partial_n u_\parallel = 0$ implies

$$\begin{aligned}\partial_n V &= 2\langle u_\parallel g, u_\parallel \partial_n g + \partial_n u_\perp g^\perp \rangle \\ &= 2(u_\parallel)^2 \langle g, \partial_n g \rangle \\ &= 0\end{aligned}$$

and therefore in either case

$$\frac{1}{2} \int_\Gamma V_+ \partial_n V \, ds \leq 0.$$

The string of inequalities now simplifies to

$$0 \leq \int_\Omega |u|^2 V_+^2 \, dx \leq -\frac{1}{2} \int_\Omega |\nabla V_+|^2 \, dx \implies V_+ \equiv 0$$

and so $|u| \leq 1$ in Ω.

To prove the gradient bound (2.4.2), suppose in order to derive a contradiction that there exists sequences $\varepsilon_k \to 0$ and $x_k \in \overline{\Omega}$ so that $t_k := |\nabla u_k(x_k)| = \|\nabla u_k\|_\infty$ satisfies $t_k \varepsilon_k \to \infty$ as $k \to \infty$. Let

$$v_k(x) := u_k\left(x_k + \frac{x}{t_k}\right)$$

which is defined whenever $y = x_k + x/t_k \in \Omega$. Likewise we define $h(x) := g(y)$ whenever $y \in \Gamma$. By the uniform bound (2.4.1) and the choice of scaling, we have

$$\|v_k\|_\infty = \|u_k\|_\infty \leq 1 \text{ and } |\nabla v_k(0)| = 1 \tag{2.4.3}$$

for all k. For each $i, j = 1, 2$ the chain rule gives

$$\frac{\partial^2 v_k^i}{\partial x_j^2} = \frac{1}{t_k^2}\frac{\partial^2 u_k^i}{\partial y_j^2}$$

so that

$$-\Delta v_k = \frac{1}{(t_k\varepsilon_k)^2}(1 - |v_k|^2)v_k, \qquad \text{for } x \in t_k[\Omega - x_k].$$

Therefore we obtain the uniform convergence

$$\|\Delta v_k\|_\infty \leq \frac{1}{(t_k\varepsilon_k)^2} \to 0 \tag{2.4.4}$$

as $k \to +\infty$. The limiting behaviour of the sequence $\{x_k\}$ gives two cases to consider. Suppose first that there is some subsequence so that $t_k \operatorname{dist}(x_k, \Gamma) \to +\infty$. Then the sequence of domains for v_k

$$t_k[\Omega - x_k] \to \mathbb{R}^2 \text{ as } k \to +\infty$$

and the regularity induced by the partial differential equation paired with a diagonal argument shows the existence of a limiting function v with $v_k \to v$ in C^k_{loc}. By (2.4.4) we have $\|\Delta v\|_\infty = 0$ and so v is a bounded harmonic function in all of $\mathbb{R}^2$. Therefore v is a constant function giving $|\nabla v| \equiv 0$ which contradicts the gradient bound (2.4.3).

Finally, suppose $t_k \operatorname{dist}(x_k, \Gamma)$ is bounded uniformly so that after a rotation and translation, the domains of v_k converge to the half-space

$$t_k[\Omega - x_k] \to \mathbb{R}^2_+ \text{ as } k \to +\infty.$$

For each k, the weak orthogonality problem becomes

$$\begin{cases} -\Delta v_k = \dfrac{1}{(t_k\varepsilon_k)^2}(1-|v_k|^2)v_k & \text{in } t_k[\Omega - x_k], \\ \partial_n v_k = -\dfrac{W}{t_k\varepsilon_k^s}\langle v_k, h^\perp\rangle h^\perp & \text{on } t_k[\Gamma - x_k], \end{cases} \tag{2.4.5}$$

while the strong orthogonality problem converts to

$$\begin{cases} -\Delta v_k = \dfrac{1}{(t_k\varepsilon_k)^2}(1-|v_k|^2)v_k & \text{in } t_k[\Omega - x_k], \\ \langle v_k, h^\perp\rangle = 0 & \text{on } t_k[\Gamma - x_k], \\ \partial_n\langle v_k, h\rangle = 0 & \text{on } t_k[\Gamma - x_k]. \end{cases} \tag{2.4.6}$$

As before, there is a bounded harmonic limit v defined on $\mathbb{R}^2_+$ with $v_k \to v$ in C^k_{loc} in both the weak and strong orthogonality case. Using (2.4.3) and the fact that $\|h\|_\infty \leq 1$, the normal derivative along the boundary for system (2.4.5) has the estimate

$$\|\partial_n v_k\|_\infty \leq \frac{W}{t_k\varepsilon_k^s} \to 0 \text{ as } k \to +\infty$$

and so the limiting harmonic map v satisfies the Neumann condition $\partial_n v = 0$ on $\partial\mathbb{R}^2_+$. Applying the reflection principle, v can be extended to a bounded harmonic function on all of $\mathbb{R}^2$ and now the same contradiction argument applies from before.

In the strong orthogonality problem, the boundary function h converges to a constant vector field on $\partial\mathbb{R}^2_+$ and the boundary conditions of (2.4.6) implies $\langle v, h^\perp\rangle = 0$ and $\partial_n\langle v, h\rangle = 0$ along $\partial\mathbb{R}^2_+$. Let $\tilde{h}$ denote the extension of the constant vector field h to all of $\overline{\mathbb{R}^2_+}$ and note that $\langle v, \tilde{h}\rangle$ is a harmonic scalar function defined on $\mathbb{R}^2_+$ since

$$\Delta\langle v, \tilde{h}\rangle = \langle \Delta v, \tilde{h}\rangle = 0.$$

This fact paired with the Neumann condition $\partial_n\langle v, h\rangle = \partial_n\langle v, \tilde{h}\rangle = 0$ allows us to use the reflection principle and Liouville's theorem to conclude $\langle v, h\rangle$ is a constant function on $\mathbb{R}^2$. Next, since $\langle v, h^\perp\rangle = 0$ on $\partial\mathbb{R}^2_+$ we have $\langle v, \tilde{h}\rangle = \pm|v|$ along the boundary. But $\langle v, h\rangle$ is constant, so we have $\langle v, \tilde{h}\rangle = \pm|v|$ on all of $\mathbb{R}^2$ and thus v is a constant vector field parallel to h. The contradiction argument can be used once more to conclude $\varepsilon|\nabla u_\varepsilon| \leq C_0$ where C_0 is a constant independent of ε. □

2.5 An Upper Bound for the Energy

In some sense, constructing an optimal upper bound for the energy of a minimizer requires some knowledge of what one expects to see. If a given energy functional is a component of some physical model, then one could look to experimental data and observe what realistic energy minimizing states look like. Another method of a purely mathematical nature would be to think about the simplest case of what might add some significant energy to the system. As mentioned in the introduction, it has been shown in [10] and [39] that the energy of minimizers $G_\varepsilon(v)$ with Dirichlet boundary data $v = g$ on Γ have an upper bound of the form

$$G_\varepsilon(v) \leq \pi\mathcal{D}|\ln\varepsilon| + C \tag{2.5.1}$$

where $\mathcal{D} = \deg(g;\Gamma)$ and C is a constant independent of ε. The idea used to construct this upper bound, vaguely speaking, comes from the realization that the boundary data causes 'tension' in the system (due to its non-zero winding around the boundary) and that some sort of compensation is needed in the interior to counteract it. From the point of view of degree theory, this counteractive interior winding should balance the winding contributed by g on Γ. The simplest vector field (satisfying an $\mathbb{S}^1$-value constraint to adhere to the director model) that comes to mind with non-trivial winding could be something of the form

$$F(x) = \frac{x}{|x|} \quad \text{or} \quad \frac{x^\perp}{|x|},$$

which correspond to the curl-free and divergence-free vector fields as shown in figure 1.1. Both of these vector fields are the profile of a vortex with degree equal to one, and by observing their energy over annuli with inner radius of order ε, one can easily calculate that this energy will be equal to $\pi|\ln\varepsilon| + c$ where c is a constant independent of ε. Thus, to counteract the winding of the boundary data, one can 'plant' $\mathcal{D}$ local vector fields that look like $F(x)$ around Ω and then try to smoothly connect them in some way that does not depend on ε. The total energy of this construction should therefore have a bound of the form (2.5.1). Of course, it is the process of connecting these vector fields independently of ε which makes the upper bound truly non-trivial.

In the case where vortices can appear on the boundary, we aim to use the same principles as above, namely, calculating the energy of a simple vortex when centered on Γ. However, a simple vortex on Γ should have energy close to half of a full vortex of degree one in the interior. In any case, this can be accomplished by creating an appropriate test function.

Proposition 2.4. *Let u_ε be a solution to the minimization problem* (W.O.) *with $\mathcal{D} = \deg(g;\Gamma) \geq 1$. There is a constant $C > 0$ independent of ε so that*

$$G_\varepsilon^W(u_\varepsilon;\Omega) \leq \pi s\mathcal{D}|\ln\varepsilon| + C.$$

If u_ε is a solution to (S.O.)*, there is a constant $C > 0$ independent of ε so that*

$$G_\varepsilon(u_\varepsilon;\Omega) \leq \pi\mathcal{D}|\ln\varepsilon| + C.$$

Before we begin the proof of this proposition, we define some set notation and introduce a local polar coordinate system near the boundary that will be used often throughout the rest of this work.

Let $R > 0$ and $x_0 \in \overline{\Omega}$. Define

$$\omega_R(x_0) := B_R(x_0) \cap \Omega \tag{2.5.2}$$

and if $x_0 \in \Gamma$, we also set

$$\Gamma_R(x_0) := \overline{\omega_R(x_0)} \cap \Gamma. \tag{2.5.3}$$

When $x_0 \in \Gamma$, we may define (as done in [3, 4]) a local polar coordinate system on $\omega_R(x_0)$ as follows. Let $\tau(x_0)$ denote the positively oriented unit tangent vector to Γ at x_0. Define the polar coordinates (r,θ) centered at x_0 so that θ is the angle measured from the ray defined by $\tau(x_0)$ and set $r = |x - x_0|$. By the smoothness of Γ, we may always choose R small enough so that

$$\omega_R(x_0) = \{(r,\theta) : \theta_1(r) < \theta < \theta_2(r),\ 0 < r < R\}$$

where $\theta_1(r)$ and $\theta_2(r)$ are smooth functions satisfying

$$|\theta_1(r)| \leq cr, \quad |\pi - \theta_2(r)| \leq cr \tag{2.5.4}$$

for some constant $c = c(\Gamma) \geq 0$. From this choice of coordinates we may also parametrize $\Gamma_R(x_0) \setminus \{x_0\}$ in two pieces, namely

$$\begin{aligned} \Gamma_R^+(x_0) &:= \{(r,\theta_1(r)) : 0 < r < R\}, \\ \Gamma_R^-(x_0) &:= \{(r,\theta_2(r)) : 0 < r < R\}. \end{aligned} \tag{2.5.5}$$

That is, $\Gamma_R^+(x_0)$ is the segment of $\Gamma_R(x_0)$ following $\tau(x)$ away from x_0 and $\Gamma_R^-(x_0)$ is the remaining segment with orientation flowing towards x_0.

Annular regions can be treated in the same way. For any $x_0 \in \overline{\Omega}$ set

$$A_{r_1,r_2}(x_0) := \omega_{r_2}(x_0) \setminus \overline{\omega_{r_1}(x_0)}, \quad 0 < r_1 < r_2. \tag{2.5.6}$$

When $x_0 \in \Gamma$ and $r_2 > 0$ is taken small enough, the intersection $\overline{A_{r_1,r_2}(x_0)} \cap \Gamma$ consists of two disjoint smooth arcs

$$\begin{aligned} \Gamma^+_{r_1,r_2}(x_0) &:= \{(r, \theta_1(r)) : r_1 < r < r_2\} \\ \Gamma^-_{r_1,r_2}(x_0) &:= \{(r, \theta_2(r)) : r_1 < r < r_2\} \end{aligned} \tag{2.5.7}$$

where $\theta_1(r)$ and $\theta_2(r)$ are as in (2.5.4). For notational convenience, we also set

$$\Gamma^\pm_{r_1,r_2}(x_0) := \overline{A_{r_1,r_2}(x_0)} \cap \Gamma = \Gamma^+_{r_1,r_2}(x_0) \cup \Gamma^-_{r_1,r_2}(x_0). \tag{2.5.8}$$

Lastly, we will require some notation to denote energies that are restricted to subsets Ω' of $\overline{\Omega}$. To do this, we set

$$\begin{aligned} G_\varepsilon(u; \Omega') &:= \frac{1}{2} \int_{\Omega'} \left(|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right) dx \\ G^W_\varepsilon(u; \Omega') &:= G_\varepsilon(u; \Omega') + \frac{W}{2\varepsilon^s} \int_{\Gamma \cap \overline{\Omega'}} \langle u, g^\perp \rangle^2 \, ds. \end{aligned}$$

Proof of Proposition 2.4.

(S.O.) Case

For strong orthogonality, the result is an easy consequence of [39, Lemma 2.1]. Let v_ε be a minimizer for G_ε over

$$H^1_g(\Omega) = \{v \in H^1(\Omega; \mathbb{R}^2) : v = g \text{ on } \Gamma\}$$

which we now use as a comparison function. The inclusion $H^1_g(\Omega) \subset \mathcal{H}(\Omega)$ implies $G_\varepsilon(u_\varepsilon) \leq G_\varepsilon(v_\varepsilon)$ and applying [39, Lemma 2.1] to $G_\varepsilon(v_\varepsilon)$ yields

$$G_\varepsilon(u_\varepsilon) \leq G_\varepsilon(v_\varepsilon) \leq \pi\mathcal{D}|\ln\varepsilon| + C.$$

(W.O.) Case

In the case of weak orthogonality, we construct a test function following the methods of [3, Lemma 3.1] and [27, Proposition 3.1]. First, we consider $2\mathcal{D}$ sets of the form $\omega_R(q_j)$ where $\{q_j\}_{j=1}^{2\mathcal{D}}$ are well-separated points on Γ and R is chosen so that

$$2\varepsilon^s < R < \frac{1}{2}|q_i - q_i|$$

for all indices $i \neq j$. On each of these sets, an $\mathbb{S}^1$-valued function $v^{(j)}$ is constructed that simulates a 'half-vortex' planted at q_j within some small annular region contained in $\omega_R(q_j)$. For such a configuration, one can obtain the bound

$$G_\varepsilon^W(v^{(j)};\omega_R(q_j)) \leq \frac{\pi}{2}\ln\varepsilon^{-s} + \text{ constant independent of } \varepsilon$$

as a means to incorporate the exponent $s \in (0,1]$. In particular, this is achieved by ensuring $v^{(j)}$ is equal to $\pm g$ on one side of q_j and then an approximate π rotation along $\partial B_R(q_j)\cap\Omega$ is made so that $v^{(j)}$ is equal to $\mp g$ on the other side of q_j. Combining this rotation with an appropriate cut-off function near q_j that incorporates the correct scaling will give the desired bound. Finally, we construct the remaining part of the test function via a harmonic extension to fill the remaining part of the domain.

Assume that the points $\{q_j\}_{j=1}^{2\mathcal{D}}$ are labeled such that q_{j+1} is the first point found by following the positively oriented tangent vector field along Γ starting from q_j and note that these points partition Γ into $2\mathcal{D}$ smooth segments C_j in the sense that

$$\Gamma = \bigcup_{j=1}^{2\mathcal{D}} C_j$$

with C_j being the curve connecting q_j and q_{j+1}. Next, let γ be a lifting of g on the curve $\Gamma_R(q_j)$, that is,

$$g = e^{i\gamma} \quad \text{on } \Gamma_R(q_j). \tag{2.5.9}$$

Define as in [3, 27] the functions

$$\begin{aligned}
h_1(r) &= \gamma\left(re^{i\theta_1(r)}\right) + (j-1)\pi,\\
h_2(r) &= \gamma\left(re^{i\theta_2(r)}\right) + j\pi,\\
\phi(r,\theta) &= \frac{h_2(r)-h_1(r)}{\theta_2(r)-\theta_1(r)}(\theta-\theta_1(r)) + h_1(r),
\end{aligned}$$

where $\theta_1(r)$ and $\theta_2(r)$ are as in (2.5.4). In this way we have

$$e^{i\phi(r,\theta)} = \begin{cases} g & \text{on } \Gamma_R^+(q_j) \text{ for } j \text{ odd},\\ -g & \text{on } \Gamma_R^-(q_j) \text{ for } j \text{ odd},\\ -g & \text{on } \Gamma_R^+(q_j) \text{ for } j \text{ even},\\ g & \text{on } \Gamma_R^-(q_j) \text{ for } j \text{ even}. \end{cases}$$

Choose a cut-off function $\eta_\varepsilon(r) \in C^\infty$ near q_j satisfying

$$\begin{cases} 0 \le \eta_\varepsilon(r) \le 1 & \text{for all } r, \\ \eta_\varepsilon(r) = 0 & \text{for } r < \varepsilon^s, \\ \eta_\varepsilon(r) = 1 & \text{for } r \ge 2\varepsilon^s, \\ |\eta_\varepsilon'(r)| \le \dfrac{c_0}{\varepsilon^s} & \text{for } \varepsilon^s < r < 2\varepsilon^s,\ c_0 \text{ a constant independent of } \varepsilon \end{cases}$$

and set

$$\psi(r,\theta) = \eta_\varepsilon(r)\phi(r,\theta) + (1 - \eta_\varepsilon(r))(\gamma(q_j) + (j-1)\pi)$$

so that we may define the test function $v_\varepsilon^{(j)}$ on $\omega_R(q_j)$ via

$$v_\varepsilon^{(j)}(r,\theta) = e^{i\psi(r,\theta)} = (\cos(\psi(r,\theta)), \sin(\psi(r,\theta))). \tag{2.5.10}$$

Remark 2.2. *By the construction of $\phi(r,\theta)$ and $\psi(r,\theta)$, note that the phase of $v_\varepsilon^{(j)}$ rotates by approximately π on $\partial B_r(q_j) \cap \Omega$ for all $2\varepsilon^s \le r \le R$ and then unwinds as r decreases from $2\varepsilon^s$ to ε^s. In this way, we simulate a half-vortex in the annular region $A_{2\varepsilon^s,R}(q_j)$.*

At this point we immediately have

$$\frac{1}{2\varepsilon^2} \int_{\omega_R(q_j)} (1 - |v_\varepsilon|^2)^2 \, dx = 0$$

since $|v_\varepsilon| = 1$ on $\omega_R(q_j)$. It is also easy to see that there exists a constant $c_1 \ge 0$ independent of ε where

$$\frac{W}{2\varepsilon^s} \int_{\Gamma_R(q_j)} \langle v_\varepsilon, g^\perp \rangle^2 \, ds \le c_1. \tag{2.5.11}$$

Indeed we have clearly have $|\Gamma_{2\varepsilon^s}(q_j)| \le C\varepsilon^s$ for C independent of ε and the cut-off function ensures $v_\varepsilon = \pm g$ on $\Gamma^\pm_{2\varepsilon^s,R}(q_j)$. Therefore by Cauchy-Schwarz

$$\begin{aligned} \frac{W}{2\varepsilon^s} \int_{\Gamma_R(q_j)} \langle v_\varepsilon, g^\perp \rangle^2 \, ds &= \frac{W}{2\varepsilon^s} \int_{\Gamma_{2\varepsilon^s}(q_j)} \langle v_\varepsilon, g^\perp \rangle^2 \, ds + \frac{W}{2\varepsilon^s} \int_{\Gamma^\pm_{2\varepsilon^s,R}(q_j)} \langle v_\varepsilon, g^\perp \rangle^2 \, ds \\ &\le \frac{W|v_\varepsilon|^2|g^\perp|^2}{2\varepsilon^s} \cdot C\varepsilon^s + 0 \\ &\le c_1. \end{aligned}$$

To estimate the energy on $\omega_R(q_j)$ it will be convenient to use polar coordinates:

$$G_\varepsilon(v_\varepsilon; \omega_R(q_j)) = \frac{1}{2}\int_0^R \int_{\theta_1(r)}^{\theta_2(r)} |\nabla v_\varepsilon|^2 r \, d\theta dr$$

and note that we may write

$$|\nabla v_\varepsilon|^2 = |\partial_r v_\varepsilon|^2 + \frac{1}{r^2}|\partial_\theta v_\varepsilon|^2 = (\partial_r \psi)^2 + \frac{1}{r^2}(\partial_\theta \psi)^2.$$

Dealing with the radial derivative first, observe

$$\begin{aligned} \partial_r \psi &= \eta_\varepsilon(r)\partial_r \phi + \eta_\varepsilon'(r)(\phi - (\gamma(q_j) + (j-1)\pi)), \\ |\partial_r v_\varepsilon|^2 = |\partial_r \psi|^2 &\le 2\left(|\eta_\varepsilon(r)|^2|\partial_r\phi|^2 + |\eta_\varepsilon'(r)|^2|\phi - (\gamma(q_j) + (j-1)\pi)|^2\right), \end{aligned}$$

where γ is as in (2.5.9). It is straightforward to show that a constant c_2 can be found for which

$$|\partial_r \phi|^2, \ |\phi - (\gamma(q_j) + (j-1)\pi)|^2 \le c_2$$

on $\omega_R(q_j) \setminus \{q_j\}$ where c_2 is independent of ε. Using this and the fact that $\eta_\varepsilon(r) = 0$ for $r < \varepsilon^s$ and $\eta_\varepsilon'(r) = 0$ for $r \in (0, R) \setminus (\varepsilon^s, 2\varepsilon^s)$, we have

$$\int_{A_{\varepsilon^s,R}(q_j)} |\partial_r \phi|^2 \, dx \le c_2 |\omega_R(q_j)|$$

and

$$\begin{aligned} \int_{A_{\varepsilon^s,2\varepsilon^s}(q_j)} |\eta_\varepsilon'(r)|^2 |\phi - (\gamma(q_j) + (j-1)\pi)|^2 \, dx &\le \frac{c_2 c_0^2}{\varepsilon^{2s}} |A_{\varepsilon^s,2\varepsilon^s}(q_j)| \\ &\le \frac{c_2 c_0^2}{\varepsilon^{2s}} \cdot 3\pi\varepsilon^{2s} \\ &= 3\pi c_2 c_0^2. \end{aligned}$$

Therefore there is a constant c_3 independent of ε for which

$$\int_{\omega_R(q_j)} |\partial_r v_\varepsilon|^2 \, dx \le c_3. \tag{2.5.12}$$

For the angular derivative:

$$\partial_\theta \psi = \eta_\varepsilon(r)\partial_\theta \phi = \eta_\varepsilon(r)\frac{h_2(r) - h_1(r)}{\theta_2(r) - \theta_1(r)},$$

$$|\partial_\theta v_\varepsilon|^2 = |\partial_\theta \psi|^2 = (\eta_\varepsilon(r))^2 \frac{(h_2(r) - h_1(r))^2}{(\theta_2(r) - \theta_1(r))^2}.$$

We can also derive a uniform bound for the angular energy over the set $\omega_{2\varepsilon^s}(q_j)$. To see this, observe by (2.5.4) and the smoothness of γ, we can find $c > 0$ so that

$$|h_2(r) - h_1(r)| \le \pi + cr, \quad \text{and} \quad |\theta_2(r) - \theta_1(r)| \ge \pi - cr.$$

Using these estimates and by properties of the cut-off function, a basic uniform estimate can be calculated as follows:

$$\begin{aligned}
\int_{\omega_{2\varepsilon^s}(q_j)} \frac{1}{r^2}|\partial_\theta v|^2\, dx &= \int_\varepsilon^{2\varepsilon^s} \int_{\theta_1(r)}^{\theta_2(r)} (\eta_\varepsilon(r))^2 \frac{(h_2(r) - h_1(r))^2}{r(\theta_2(r) - \theta_1(r))^2}\, d\theta dr \\
&= \int_\varepsilon^{2\varepsilon^s} (\eta_\varepsilon(r))^2 \frac{(h_2(r) - h_1(r))^2}{r(\theta_2(r) - \theta_1(r))}\, dr \\
&\le \int_{\varepsilon^s}^{2\varepsilon^s} \frac{(\pi + cr)^2}{r(\pi - cr)}\, dr \\
&\le \frac{(\pi + cR)^2}{(\pi - cR)} \int_{\varepsilon^s}^{2\varepsilon^s} \frac{1}{r}\, dr \\
&= \frac{(\pi + cR)^2}{(\pi - cR)} \ln(2).
\end{aligned}$$

Therefore, it must be the case that the primary energy contribution comes from within the annular region $A_{2\varepsilon^s,R}(q_j)$. Using the same estimates as above along with the properties of the cut-off function, we may write

$$\begin{aligned}
\int_{\omega_R(q_j)} \frac{1}{r^2}|\partial_\theta v_\varepsilon|^2\, dx &\le \int_{\varepsilon^s}^R \frac{(\pi + cr)^2}{r(\pi - cr)}\, dr \\
&= \int_{\varepsilon^s}^R \frac{\pi}{r}\, dr + \int_{\varepsilon^s}^R \frac{4c\pi}{\pi - cr}\, dr - \int_{\varepsilon^s}^R c\, dr \\
&\le \pi s|\ln \varepsilon| + c_4
\end{aligned}$$

where c_4 is independent of ε. With this and applying inequalities (2.5.11) and

(2.5.12), there is a constant c_5 independent of ε so that

$$G_\varepsilon^W(v_\varepsilon;\omega_R(q_j)) \le \frac{\pi}{2}s|\ln\varepsilon| + c_5. \tag{2.5.13}$$

This completes estimating the energy on $\omega_R(q_j)$.

Next, we must fill in the remaining piece of the domain

$$\tilde\Omega := \Omega \setminus \bigcup_{j=1}^{2\mathcal{D}} \omega_R(q_j)$$

with a test function V_ε so that the energy on $\tilde\Omega$ remains uniformly bounded in ε. Define the closed contour

$$\tilde\Gamma := \partial\tilde\Omega = \left(\Gamma \setminus \cup_{j=1}^{2\mathcal{D}}\Gamma_R(q_j)\right)\bigcup\left(\cup_{j=1}^{2\mathcal{D}}\partial B_R(q_j)\cap\Omega\right)$$

with orientation matching that of Γ where they coincide. In this way, observe that the circular arcs $\partial B_R(q_j)\cap\Omega$ are negatively oriented for each $j = 1,\dots,2\mathcal{D}$. With this in mind, we define boundary data $\tilde g : \tilde\Gamma \to \mathbb{S}^1$ by setting

$$\tilde g := \begin{cases} g & \text{on } \tilde\Gamma\cap C_j \text{ for } j \text{ odd} \\ -g & \text{on } \tilde\Gamma\cap C_j \text{ for } j \text{ even} \\ v_\varepsilon^{(j)} & \text{on } \partial B_R(q_j)\cap\Omega \text{ for each } j=1,\dots,2\mathcal{D}. \end{cases}$$

By the construction of $v_\varepsilon^{(j)}$ and the negative orientation associated with the arc $\partial B_R(q_j)\cap\Omega$, the phase of $\tilde g$ turns by approximately $-\pi$ on each $\partial B_R(q_j)\cap\Omega$, $j=1,\dots,2\mathcal{D}$ for a combined associated phase turn of $-2\pi\mathcal{D}$. The remaining pieces of the boundary data will contribute a phase of $2\pi\mathcal{D}$ to $\tilde g$ since both g and $-g$ are of degree $\mathcal{D}$. Therefore the net phase of $\tilde g$ around $\tilde\Gamma$ is zero, i.e.

$$\deg(\tilde g;\tilde\Gamma) = 0.$$

Now we may define the remaining test function on $\tilde\Omega$ by letting V_ε be the $\mathbb{S}^1$-valued harmonic extension of $\tilde g$ to $\tilde\Omega$. It is known that this extension has bounded energy and since V_ε is equal to $\pm g$ where $\tilde\Gamma$ and Γ coincide,

$$G_\varepsilon^W(V_\varepsilon;\tilde\Omega) = G_\varepsilon(V_\varepsilon;\tilde\Omega) = \frac{1}{2}\int_{\tilde\Omega}|\nabla V_\varepsilon|^2\,dx \le c_6 \tag{2.5.14}$$

for c_6 independent of ε. Defining

$$h_\varepsilon = \begin{cases} V_\varepsilon & \text{in } \tilde{\Omega} \\ v_\varepsilon^{(j)} & \text{in } \omega_R(q_j) \text{ for each } j = 1, \ldots, 2\mathcal{D}, \end{cases}$$

and using inequalities (2.5.13) and (2.5.14) we obtain

$$G_\varepsilon^W(h_\varepsilon) = \sum_{j=1}^{2\mathcal{D}} G_\varepsilon^W(v_\varepsilon^{(j)}; \omega_R(q_j)) + G_\varepsilon^W(V_\varepsilon; \tilde{\Omega}) \leq \pi s \mathcal{D} |\ln \varepsilon| + C$$

as desired. □

2.6 A Pohozaev-Type Identity

The final section of this chapter is dedicated to developing a Pohozaev-type identity for solutions of (2.2.1) and (2.2.3). As usual, Pohozaev identities typically utilize a vector field of the form

$$\psi \cdot \nabla u := (\langle \psi, \nabla u^1 \rangle, \langle \psi, \nabla u^2 \rangle) = \left(\psi^1 u^1_{x_1} + \psi^2 u^1_{x_2}, \psi^1 u^2_{x_1} + \psi^2 u^2_{x_2}\right)$$

where ψ can be chosen later to suit the needs of the situation. The following proposition is a fairly standard result (see [3, 39] for example), but we derive it here for completeness.

Proposition 2.5. *Let $\psi \in C^1(\omega_r; \mathbb{R}^2)$. If u is a solution of (2.2.1) or (2.2.3):*

$$\begin{aligned} \int_{\partial \omega_r} (e_\varepsilon(u) \langle \psi, n \rangle - \langle \partial_n u, \psi \cdot \nabla u \rangle) \, ds \\ = \int_{\omega_r} \left(e_\varepsilon(u) \operatorname{div} \psi - \sum_{j,l} \psi^l_{x_j} \langle \partial_{x_j} u, \partial_{x_l} u \rangle \right) dx. \end{aligned} \tag{2.6.1}$$

Proof.

We begin by taking the inner product on both sides of the PDE in (2.2.1) or (2.2.3) with the vector field $\psi \cdot \nabla u$ and integrating over ω_r:

$$-\int_{\omega_r} \langle \psi \cdot \nabla u, \Delta u \rangle \, dx = \int_{\omega_r} \frac{1}{\varepsilon^2} \langle u(1 - |u|^2), \psi \cdot \nabla u \rangle \, dx.$$

Labeling

(I) $-\int_{\omega_r} \langle \psi \cdot \nabla u, \Delta u \rangle \, dx,$

(II) $\displaystyle\int_{\omega_r} \frac{1}{\varepsilon^2}\langle u(1-|u|^2), \psi\cdot\nabla u\rangle\, dx,$

we consider each separately.

(I):

Applying integration by parts,

$$\begin{aligned}
-\int_{\omega_r}\langle \psi\cdot\nabla u, \Delta u\rangle\, dx &= -\sum_{i=1}^{2}\int_{\omega_r}(\psi^1 u^i_{x_1} + \psi^2 u^i_{x_2})\Delta u^i\, dx \\
&= \sum_{i=1}^{2}\left(\int_{\omega_r}\langle \nabla(\psi^1 u^i_{x_1} + \psi^2 u^i_{x_2}), \nabla u^i\rangle\, dx \right.\\
&\qquad\left. - \int_{\partial\omega_r}(\psi^1 u^i_{x_1} + \psi^2 u^i_{x_2})\partial_n u^i\, ds\right) \\
&= \sum_{i=1}^{2}\int_{\omega_r}\langle \nabla(\psi^1 u^i_{x_1} + \psi^2 u^i_{x_2}), \nabla u^i\rangle\, dx \\
&\qquad - \int_{\partial\omega_r}\langle \partial_n u, \psi\cdot\nabla u\rangle\, ds.
\end{aligned}$$

By the product rule for gradients we have

$$\nabla(\psi^1 u^i_{x_1} + \psi^2 u^i_{x_2}) = \psi^1\nabla u^i_{x_1} + u^i_{x_1}\nabla\psi^1 + \psi^2\nabla u^i_{x_2} + u^i_{x_2}\nabla\psi^2$$

and therefore

$$\begin{aligned}
\langle \nabla(\psi^1 u^i_{x_1} + \psi^2 u^i_{x_2}), \nabla u^i\rangle &= \psi^1 u^i_{x_1}u^i_{x_1x_1} + (u^i_{x_1})^2\psi^1_{x_1} + \psi^2 u^i_{x_1}u^i_{x_2x_1} + u^i_{x_1}u^i_{x_2}\psi^2_{x_1} \\
&\quad + \psi^1 u^i_{x_2}u^i_{x_1x_2} + u^i_{x_1}u^i_{x_2}\psi^1_{x_2} + \psi^2 u^i_{x_2}u^i_{x_2x_2} + (u^i_{x_2})^2\psi^2_{x_2}.
\end{aligned}$$

Collecting terms, we see that

$$\psi^1(u^i_{x_1}u^i_{x_1x_1} + u^i_{x_2}u^i_{x_1x_2}) + \psi^2(u^i_{x_1}u^i_{x_2x_1} + u^i_{x_2}u^i_{x_2x_2}) = \frac{1}{2}\langle \psi, \nabla\left(|\nabla u^i|^2\right)\rangle$$

and the remaining part can be written

$$\psi^1_{x_1}(u^i_{x_1})^2 + \psi^2_{x_1}u^i_{x_1}u^i_{x_2} + \psi^1_{x_2}u^i_{x_2}u^i_{x_1} + \psi^2_{x_2}(u^i_{x_2})^2 = \sum_{j,l}\psi^l_{x_j}u^i_{x_j}u^i_{x_l}.$$

Returning to the integral

$$\sum_{i=1}^{2}\int_{\omega_r}\langle\nabla(\psi^1 u^i_{x_1}+\psi^2 u^i_{x_2}),\nabla u^i\rangle\,dx = \sum_{i=1}^{2}\int_{\omega_r}\frac{1}{2}\langle\psi,\nabla\left(|\nabla u^i|^2\right)\rangle\,dx + \int_{\omega_r}\left(\sum_{j,l}\psi^l_{x_j}u^i_{x_j}u^i_{x_l}\right)dx$$
$$= \frac{1}{2}\int_{\omega_r}\langle\psi,\nabla\left(|\nabla u|^2\right)\rangle\,dx + \int_{\omega_r}\left(\sum_{j,l}\psi^l_{x_j}\langle\partial_{x_j}u,\partial_{x_l}u\rangle\right)dx.$$

Integrating the first term above by parts yields

$$\frac{1}{2}\int_{\omega_r}\langle\psi,\nabla\left(|\nabla u|^2\right)\rangle\,dx = \frac{1}{2}\int_{\partial\omega_r}|\nabla u|^2\langle\psi,n\rangle\,ds - \frac{1}{2}\int_{\omega_r}|\nabla u|^2\operatorname{div}\psi\,dx.$$

Therefore we can write

$$-\int_{\omega_r}\langle\psi\cdot\nabla u,\Delta u\rangle\,dx = \int_{\partial\omega_r}\left(\frac{1}{2}|\nabla u|^2\langle\psi,n\rangle - \langle\partial_n u,\psi\cdot\nabla u\rangle\right)ds + \int_{\omega_r}\left(\sum_{j,l}\psi^l_{x_j}\langle\partial_{x_j}u,\partial_{x_l}u\rangle - \frac{1}{2}|\nabla u|^2\operatorname{div}\psi\right)dx.$$

(II):

Observe that

$$\frac{1}{\varepsilon^2}\langle u(1-|u|^2),\psi\cdot\nabla u\rangle = \frac{1}{\varepsilon^2}\sum_{i=1}^{2}(1-|u|^2)(\psi^1 u^i u^i_{x_1}+\psi^2 u^i u^i_{x_2})$$
$$= \frac{1}{2\varepsilon^2}\sum_{i=1}^{2}(1-|u|^2)(\psi^1\partial_{x_1}(u^i)^2+\psi^2\partial_{x_2}(u^i)^2)$$
$$= \frac{1}{2\varepsilon^2}\sum_{i=1}^{2}(1-|u|^2)\langle\psi,\nabla|u^i|^2\rangle$$
$$= -\frac{1}{4\varepsilon^2}\langle\psi,\nabla(1-|u|^2)^2\rangle.$$

Integrating by parts, we then have

$$\begin{aligned}\int_{\omega_r} \frac{1}{\varepsilon^2}\langle u(1-|u|^2), \psi\cdot\nabla u\rangle\, dx &= -\int_{\omega_r} \frac{1}{4\varepsilon^2}\langle \psi, \nabla(1-|u|^2)^2\rangle\, dx \\ &= \int_{\omega_r} \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \operatorname{div}\psi\, dx \\ &\quad - \int_{\partial\omega_r} \frac{1}{4\varepsilon^2}(1-|u|^2)^2\langle \psi, n\rangle\, ds.\end{aligned}$$

Putting (I) and (II) together gives the result. □

Chapter 3

Bad Sets and η-Compactness

As minimizers of (W.O.) and (S.O.) are intended to represent approximate directors for nematic material satisfying certain boundary conditions, we are primarily concerned about the region of the domain where $|u_\varepsilon|$ is close to unity and where the phase of u_ε is close to the phase of $\pm g$ (modulo π) along Γ. However, as previously mentioned in the introduction, the boundary conditions used within this work can force $|u_\varepsilon|$ to be small and the phase of u_ε to be far from the phase of $\pm g$ along Γ (in the weak orthogonality case) in some subsets of the domain Ω. In these regions, a minimizer is not a suitable candidate for a physically relevant approximate director and thus it is important to determine how large this set is. In the literature, the set of points where u_ε is not behaving as we'd like is called the *bad set*. A bad set can be defined in many ways, and the cut-off for which we consider $|u_\varepsilon|$ to be small is essentially arbitrary. Indeed, since we expect that as $\varepsilon \to 0$ one finds $|u_\varepsilon| \to 1$ in the majority of the domain, for ε small enough, any real number $\ell \in (0,1)$ could be used to say that the set of points where $|u_\varepsilon| < \ell$ constitutes part of the bad set. Classically, the value of ℓ is chosen to be 1/2 (see [10] for example). Following in this way and in the sense of [3, 32], we define a bad set as follows.

Definition 3.1 (The Bad Set)**.** *For fixed $\varepsilon > 0$, the bad set for u_ε is defined by the collection of points*

$$S_\varepsilon := \left\{ x \in \overline{\Omega} : |u_\varepsilon(x)| < \frac{1}{2} \text{ or } |\langle u_\varepsilon(x), g^\perp(x)\rangle| > \frac{1}{4} \right\}.$$

In this chapter, we prove a property of minimizers called η-compactness which allows us to find an upper bound on the size of the bad set S_ε. Essentially, the property of η-compactness states that if the energy of u_ε over some region $\mathcal{R}_1 \subset \overline{\Omega}$ can be bounded by $\eta|\ln\varepsilon|$ where η is some appropriate (small) constant independent of ε, then there is a subset $\mathcal{R}_2 \subset \mathcal{R}_1$ such that $|u_\varepsilon(x)| \geq 1/2$ for all $x \in \mathcal{R}_2$ and $|\langle u_\varepsilon(x), g^\perp(x)\rangle| \leq 1/4$ for all $x \in \mathcal{R}_2 \cap \Gamma$. In other words, if the energy in some region is small enough, it can be shown that a smaller region within it cannot contain a vortex.

The η-compactness property can be used in conjunction with a covering argument to show that S_ε can be covered by a finite, disjoint family of balls whose number is bounded independent of ε. Moreover, each ball can be shown to have radii of order no larger than ε^s $(0 < s \leq 1)$. In this way, it is evident that as ε tends to zero, the measure of S_ε also tends to zero and thus the regions where u_ε does not behave like a classical director shrink as $\varepsilon \to 0$.

3.1 An Integral Estimate on Balls

The η-compactness property as mentioned at the beginning of this chapter requires that we analyze the energy of u_ε on ω_r as a function of radius r, assuming all else is fixed. It turns out that such functions are naturally occurring byproducts from integration by parts against appropriate vector fields. For $x_0 \in \overline{\Omega}$, define as in [3, 4, 39] the radius-dependent function

$$F(r) := F(r; x_0, u, \varepsilon) = r \int_{\partial B_r(x_0)\cap\Omega} e_\varepsilon(u)\, ds.$$

In the special case where $x_0 \in \Gamma$, we also define

$$F_\Gamma(r) := F(r) + \frac{Wr}{2\varepsilon^s} \sum_{x\in\partial\Gamma_r(x_0)} \langle u, g^\perp\rangle^2.$$

Although the energy bounds to be presented below in Lemma 3.2 appear unmotivated, it will be clear within the proof of the η-compactness property that they are exactly what is needed to obtain the result.

Lemma 3.2. *Let $x_0 \in \overline{\Omega}$. There exists constants $C > 0$ and $r_0 > 0$ such that for $\varepsilon \in (0, 1)$ and $r \in (0, r_0)$ we have:*

1. *If $x_0 \in \Omega$ and $\overline{\omega_r(x_0)} \cap \Gamma = \varnothing$,*

$$\frac{1}{4\varepsilon^2} \int_{\omega_r(x_0)} (1 - |u|^2)^2\, dx \leq r \int_{\omega_r(x_0)} \frac{1}{2}|\nabla u|^2 dx + F(r), \qquad (3.1.1)$$

2. *If $x_0 \in \Gamma$ and u satisfies the strong orthogonality condition,*

$$\frac{1}{4\varepsilon^2} \int_{\omega_r(x_0)} (1 - |u|^2)^2\, dx \leq C \left[r \int_{\omega_r(x_0)} \frac{1}{2}|\nabla u|^2\, dx + F(r) + \frac{r^2}{\varepsilon} \right], \quad (3.1.2)$$

3. If $x_0 \in \Gamma$ and u satisfies the weak orthogonality condition,

$$\begin{aligned} \frac{1}{4\varepsilon^2}\int_{\omega_r(x_0)} (1-|u|^2)^2\,dx + \frac{W}{2\varepsilon^s}\int_{\Gamma_r(x_0)} \langle u, g^\perp\rangle^2\,ds \\ \leq C\left[r\int_{\omega_r(x_0)} \frac{1}{2}|\nabla u|^2\,dx + F_\Gamma(r) + \frac{Wr^2}{\varepsilon^s}\right]. \end{aligned} \tag{3.1.3}$$

Proof. In what follows, the arguments used reflect those presented in [39] for the interior case and [3] for the boundary case. We also note that C will denote a generic positive constant independent of ε throughout this proof which is subject to change.

Step 1: $x_0 \in \Omega$

Assume $\omega_r = B_r(x_0) \subset \Omega$. Let n and τ represent the unit normal and tangent vectors to $\partial\omega_r$ respectively and define the vector field $X = x - x_0$. Of course, $|X| \leq r$ for all $x \in \omega_r$ with $\langle X, n\rangle = X_n = r$ on $\partial\omega_r$ and $\langle X, \tau\rangle = X_\tau = 0$ on $\partial\omega_r$. To obtain (3.1.1), consider the Pohosaev-type identity (2.6.1) and take $\psi = X$.

Estimates Along $\partial\omega_r$:

The lefthand side of (2.6.1) can be written as the sum of integrals $I_1 + I_2$ where

$$\begin{aligned} I_1 &= \int_{\partial\omega_r} \left\{\frac{1}{2}|\nabla u|^2 X_n - \langle \partial_n u, X\cdot\nabla u\rangle\right\} ds, \\ I_2 &= \frac{1}{4\varepsilon^2}\int_{\partial\omega_r} (1-|u|^2)^2 X_n\,ds. \end{aligned}$$

Since $X = rn$ on $\partial\omega_r$, we have that $X\cdot\nabla u = r\partial_n u$. The first integral has estimate

$$\begin{aligned} I_1 &= r\int_{\partial\omega_r} \left\{\frac{1}{2}|\nabla u|^2 - \langle \partial_n u, \partial_n u\rangle\right\} ds \\ &= r\int_{\partial\omega_r} \left\{\frac{1}{2}|\nabla u|^2 - |\partial_n u|^2\right\} ds \\ &\leq r\int_{\partial\omega_r} \frac{1}{2}|\nabla u|^2\,ds. \end{aligned}$$

The integral I_2 is easily seen to be

$$I_2 = \frac{r}{4\varepsilon^2}\int_{\partial\omega_r} (1-|u|^2)^2\,ds$$

and therefore

$$I_1 + I_2 \leq r \int_{\partial \omega_r} \frac{1}{2} \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right\} ds = F(r).$$

Estimates in ω_r:

The righthand side of (2.6.1) can be written

$$\int_{\omega_r} \left\{ e_\varepsilon(u) \operatorname{div} X - \sum_{j,l} X^l_{x_j} \langle \partial_{x_j} u, \partial_{x_l} u \rangle \right\} dx = J_1 + J_2$$

where

$$J_1 = \int_{\omega_r} \left\{ \frac{1}{2} |\nabla u|^2 \operatorname{div} X - \sum_{j,l} X^l_{x_j} \langle \partial_{x_j} u, \partial_{x_l} u \rangle \right\} dx,$$

$$J_2 = \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \operatorname{div} X \, dx.$$

Since $X^l_{x_j} = \delta_{jl}$ and $\operatorname{div} X = 2 > 2 - r$,

$$\begin{aligned} J_1 &\geq \int_{\omega_r} \left\{ |\nabla u|^2 - \frac{r}{2} |\nabla u|^2 - \sum_{j,l} \delta_{jl} \langle \partial_{x_j} u, \partial_{x_l} u \rangle \right\} dx \\ &= \int_{\omega_r} \left\{ -\frac{r}{2} |\nabla u|^2 + |\nabla u|^2 - |\nabla u|^2 \right\} dx \\ &= -r \int_{\omega_r} \frac{1}{2} |\nabla u|^2 \, dx. \end{aligned}$$

For J_2 we use $\operatorname{div} X > 1$ to get

$$J_2 \geq \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx.$$

Putting everything together,

$$\frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx - r \int_{\omega_r} \frac{1}{2} |\nabla u|^2 \, dx \leq J_1 + J_2 = I_1 + I_2 \leq F(r)$$

which proves inequality (3.1.1).

Step 2: $x_0 \in \Gamma$

Let $r_0 > 0$ be chosen small enough so that $\Gamma \cap B_r(x_0)$ consists of a single smooth arc satisfying $|\Gamma_r| \leq Cr$ for all $0 < r \leq r_0$ and that ω_r is strictly starshaped with respect to some point $x_1 \in \omega_r$ for all $0 < r \leq r_0$. As in [3] we let $\mathcal{N}$ be a $2r_0$-neighbourhood of Γ, and by taking r_0 smaller if necessary, it is known that there exists a vector field $X \in C^2(\mathcal{N};\mathbb{R}^2)$ satisfying

$$\langle X, n\rangle = X_n = 0 \quad \text{for all } x \in \Gamma_r, \tag{3.1.4}$$

$$|X - (x - x_0)| \leq C|x - x_0|^2 \quad \text{for all } x \in \omega_r, \tag{3.1.5}$$

$$|\partial_{x_i} X^j - \delta_{ij}| \leq C|x - x_0| \quad \text{for all } x \in \omega_r, \tag{3.1.6}$$

for a constant $C > 0$ and for any $x_0 \in \Gamma$. To obtain inequalities (3.1.2) and (3.1.3) we consider the Pohosaev-type identity (2.6.1) with $\psi = X$ and find estimates for several of its terms. Using $\partial\omega_r = \Gamma_r \cup (\partial B_r(x_0) \cap \Omega)$, it will be convenient to perform these estimates on Γ_r and $\partial B_r(x_0) \cap \Omega$ separately.

Estimates Along Γ_r:

By (3.1.4) we may write $X = \langle X, \tau\rangle\tau = X_\tau \tau$ where τ is the unit tangent vector to Γ_r and so $X \cdot \nabla u = X_\tau \partial_\tau u$ on Γ_r. Whether one is in the strong or weak orthogonality case the lefthand side of (2.6.1) reads

$$\int_{\Gamma_r} \{e_\varepsilon(u)X_n - \langle \partial_n u, X \cdot \nabla u\rangle\}\, ds = -\int_{\Gamma_r} \langle \partial_n u, X_\tau \partial_\tau u\rangle\, ds.$$

For the strong orthogonality condition, representation (2.2.2) for $\partial_n u$ and $\partial_\tau u$ is used along Γ_r with the known conditions

$$u_\perp = \partial_n u_\parallel = \partial_\tau u_\perp = 0$$

to obtain

$$\langle \partial_n u, X_\tau \partial_\tau u\rangle = X_\tau \langle u_\parallel \partial_n g + \partial_n u_\perp g^\perp, u_\parallel \partial_\tau g + \partial_\tau u_\parallel g\rangle.$$

Expanding the inner product,

$$X_\tau \left((u_\parallel)^2 \langle \partial_n g, \partial_\tau g\rangle + u_\parallel \partial_\tau u_\parallel \langle \partial_n g, g\rangle + u_\parallel \partial_n u_\perp \langle g^\perp, \partial_\tau g\rangle + \partial_n u_\perp \partial_\tau u_\parallel \langle g^\perp, g\rangle \right).$$

The expansion simplifies considerably using

$$\langle \partial_n g, g\rangle = \langle g^\perp, g\rangle = 0$$

(by Lemma B.1) so that

$$\langle \partial_n u, X_\tau \partial_\tau u\rangle = X_\tau ((u_\parallel)^2 \langle \partial_n g, \partial_\tau g\rangle + u_\parallel \partial_n u_\perp \langle g^\perp, \partial_\tau g\rangle).$$

Noting that since g is smooth, there is a constant C_g so that $\max\{|\nabla g|, |\nabla g|^2\} \leq C_g$ on the tubular neighbourhood $\mathcal{N}_\Gamma$. Moreover, by Lemma 2.3 we also have $|u_\parallel| \leq 1$, $|\nabla u| \leq C_0\varepsilon^{-1}$ on Ω. Using Cauchy-Schwarz,

$$|(u_\parallel)^2\langle\partial_n g, \partial_\tau g\rangle| \leq |\partial_n g||\partial_\tau g| \leq |\nabla g|^2 \leq C_g,$$
$$|u_\parallel \partial_n u_\perp \langle g^\perp, \partial_\tau g\rangle| \leq |\partial_n u_\perp||g^\perp||\partial_\tau g| \leq |\nabla u||\nabla g| \leq C_g C_0 \varepsilon^{-1}.$$

Therefore, there is a constant c for which

$$|\langle\partial_n u, X_\tau \partial_\tau u\rangle| \leq |X_\tau|\frac{c}{\varepsilon}.$$

Moreover since $|X_\tau| \leq Cr$ and $|\Gamma_r| \leq Cr$ we have another constant C (independent of ε) so that

$$\left|\int_{\Gamma_r} \langle\partial_n u, X\cdot\nabla u\rangle\, ds\right| \leq \int_{\Gamma_r} |X_\tau|\frac{c}{\varepsilon}\, ds \leq \frac{Cr^2}{\varepsilon}.$$

The weak orthogonality condition along Γ_r is slightly more delicate. In this scenario, we use representation (2.2.2) for $\partial_\tau u$ once more so that

$$\begin{aligned} -\int_{\Gamma_r} \langle\partial_n u, X_\tau\partial_\tau u\rangle\, ds &= \frac{W}{\varepsilon^s}\int_{\Gamma_r} \langle\langle u, g^\perp\rangle g^\perp, X_\tau\partial_\tau u\rangle\, ds \\ &= \frac{W}{\varepsilon^s}\int_{\Gamma_r} X_\tau\langle u_\perp g^\perp, u_\parallel\partial_\tau g + \partial_\tau u_\parallel g + u_\perp\partial_\tau g^\perp + \partial_\tau u_\perp g^\perp\rangle\, ds \\ &= \frac{W}{\varepsilon^s}\int_{\Gamma_r} X_\tau u_\perp\partial_\tau u_\perp\, ds + \frac{W}{\varepsilon^s}\int_{\Gamma_r} X_\tau\langle u_\perp g^\perp, u_\parallel\partial_\tau g\rangle\, ds. \end{aligned} \tag{3.1.7}$$

Integrating by parts in the first integral of (3.1.7),

$$\begin{aligned} \frac{W}{\varepsilon^s}\int_{\Gamma_r} X_\tau u_\perp\partial_\tau u_\perp\, ds &= \frac{W}{2\varepsilon^s}\int_{\Gamma_r} X_\tau\partial_\tau(u_\perp)^2\, ds \\ &= \frac{W}{2\varepsilon^s}\left(\langle u, g^\perp\rangle^2 X_\tau\big|_{x\in\partial\Gamma_r} - \int_{\Gamma_r}\langle u, g^\perp\rangle^2\partial_\tau X_\tau\, ds\right) \end{aligned}$$

By Proposition B.2 the tangential derivative of X satisfies

$$\partial_\tau X_\tau = 1 + f(X, \tau, DX, D\tau)$$

where $|f| \leq C|x - x_0| = Cr$ on Γ_r. Then

$$\begin{aligned} -\int_{\Gamma_r} \langle u, g^\perp \rangle^2 \partial_\tau X_\tau \, ds &= -\int_{\Gamma_r} \langle u, g^\perp \rangle^2 \, ds - \int_{\Gamma_r} \langle u, g^\perp \rangle^2 f \, ds \\ &\leq -\int_{\Gamma_r} \langle u, g^\perp \rangle^2 \, ds + Cr|u|^2 |g^\perp|^2 ||\Gamma_r| \\ &\leq -\int_{\Gamma_r} \langle u, g^\perp \rangle^2 \, ds + Cr^2. \end{aligned}$$

On the other hand by (3.1.5) and the reverse triangle inequality, we have along Γ_r

$$||X_\tau| \mp r| \leq Cr^2$$

so that for r small enough, $|X_\tau| \leq r$ on $\partial\Gamma_r(x_0)$ and so

$$\left| \langle u, g^\perp \rangle^2 X_\tau \big|_{x \in \partial\Gamma_r} \right| \leq r \sum_{x \in \partial\Gamma_r(x_0)} \langle u, g^\perp \rangle^2.$$

The second integral from (3.1.7) has the basic estimate

$$\left| \frac{W}{\varepsilon^s} \int_{\Gamma_r} X_\tau \langle u_\perp g^\perp, u_\parallel \partial_\tau g \rangle \, ds \right| \leq \frac{W}{\varepsilon^s} |\Gamma_r| |X_\tau| |u_\perp| |u_\parallel| |\partial_\tau g| \leq \frac{W}{\varepsilon^s} C(g) r^2.$$

Putting these estimates together,

$$-\int_{\Gamma_r} \langle \partial_n u, X_\tau \partial_\tau u \rangle \, ds \leq -\frac{W}{2\varepsilon^s} \int_{\Gamma_r} \langle u, g^\perp \rangle^2 \, ds + \frac{Wr}{2\varepsilon^s} \sum_{x \in \partial\Gamma_r(x_0)} \langle u, g^\perp \rangle^2 + \frac{CWr^2}{\varepsilon^s}$$

Estimates Along $\partial B_r(x_0) \cap \Omega$:

The lefthand side of (2.6.1) along $\partial B_r(x_0) \cap \Omega$ can be written as the sum of integrals $I_1 + I_2$ where

$$I_1 = \int_{\partial B_r(x_0) \cap \Omega} \left\{ \frac{1}{2} |\nabla u|^2 X_n - \langle \partial_n u, X \cdot \nabla u \rangle \right\} ds,$$

$$I_2 = \frac{1}{4\varepsilon^2} \int_{\partial B_r(x_0) \cap \Omega} (1 - |u|^2)^2 X_n \, ds.$$

Using the decomposition $X = X_n n + X_\tau \tau$ and noticing that

$$X \cdot \nabla u = (\langle X, \nabla u^1 \rangle, \langle X, \nabla u^2 \rangle) = X_n \partial_n u + X_\tau \partial_\tau u$$

we can write

$$\begin{aligned}-\langle \partial_n u, X \cdot \nabla u\rangle &= -X_n\langle \partial_n u, \partial_n u\rangle - X_\tau\langle \partial_n u, \partial_\tau u\rangle \\ &= -X_n|\partial_n u|^2 - X_\tau\langle \partial_n u, \partial_\tau u\rangle.\end{aligned}$$

Again by (3.1.5) it is easily estimated on $\partial B_r(x_0) \cap \Omega$ that $|X_n|$, $|X_\tau| \le Cr$. Using this paired with the identity $|\nabla u|^2 = |\partial_n u|^2 + |\partial_\tau u|^2$, Cauchy-Schwarz and Young's inequality:

$$\begin{aligned}I_1 &= \int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{1}{2}|\nabla u|^2 X_n - X_n|\partial_n u|^2 - X_\tau\langle \partial_n u, \partial_\tau u\rangle\right\} ds \\ &= \int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{1}{2}|\partial_\tau u|^2 X_n - \frac{1}{2}X_n|\partial_n u|^2 - X_\tau\langle \partial_n u, \partial_\tau u\rangle\right\} ds \\ &\le Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{1}{2}|\partial_\tau u|^2 + \frac{1}{2}|\partial_n u|^2 + \frac{1}{2}|\partial_n u|^2 + \frac{1}{2}|\partial_\tau u|^2\right\} ds \\ &= Cr\int_{\partial B_r(x_0)\cap\Omega} |\nabla u|^2\, ds.\end{aligned}$$

For I_2,

$$I_2 = \frac{1}{4\varepsilon^2}\int_{\partial B_r(x_0)\cap\Omega}(1-|u|^2)^2 X_n\, ds \le \frac{Cr}{4\varepsilon^2}\int_{\partial B_r(x_0)\cap\Omega}(1-|u|^2)^2\, ds.$$

Thus, for $C > 0$ large enough we have

$$I_1 + I_2 \le Cr\int_{\partial B_r(x_0)\cap\Omega} \frac{1}{2}\left\{|\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\} ds = CF(r)$$

and therefore

$$\begin{aligned}\int_{\partial\omega_r} \{e_\varepsilon(u)X_n - \langle \partial_n u, X\cdot\nabla u\rangle\}\, ds &= I_1 + I_2 - \int_{\Gamma_r}\langle \partial_n u, X\cdot \nabla u\rangle\, ds \\ &\le C\left[F(r) + \frac{r^2}{\varepsilon}\right]\end{aligned}$$

in the strong orthogonality case and

$$\int_{\partial\omega_r} \{e_\varepsilon(u)X_n - \langle \partial_n u, X\cdot\nabla u\rangle\}\,ds = I_1 + I_2 - \int_{\Gamma_r} \langle \partial_n u, X\cdot\nabla u\rangle\,ds$$
$$\leq -\frac{W}{2\varepsilon^s}\int_{\Gamma_r}\langle u, g^\perp\rangle^2\,ds + C\left[F_\Gamma(r) + \frac{Wr^2}{\varepsilon^s}\right]$$

in the weak orthogonality case.

Estimates in ω_r:

For the righthand side of (2.6.1) we write

$$\int_{\omega_r}\left\{e_\varepsilon(u)\operatorname{div}X - \sum_{j,l} X^l_{x_j}\langle\partial_{x_j}u, \partial_{x_l}u\rangle\right\}dx = J_1 + J_2$$

where

$$J_1 = \int_{\omega_r}\left\{\frac{1}{2}|\nabla u|^2\operatorname{div}X - \sum_{j,l} X^l_{x_j}\langle\partial_{x_j}u, \partial_{x_l}u\rangle\right\}dx,$$
$$J_2 = \frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\operatorname{div}X\,dx.$$

By (3.1.6) we use $|X^l_{x_j}| \leq \delta_{jl} + Cr$ on ω_r and using Cauchy-Schwarz and Young's inequality:

$$\begin{aligned}\sum_{j,l} X^l_{x_j}\langle\partial_{x_j}u, \partial_{x_l}u\rangle &\leq \sum_{j,l}|X^l_{x_j}||\langle\partial_{x_j}u, \partial_{x_l}u\rangle| \\ &\leq \sum_{j,l}(\delta_{jl}+Cr)\left(\frac{1}{2}|\partial_{x_j}u|^2 + \frac{1}{2}|\partial_{x_l}u|^2\right) \\ &= |\nabla u|^2 + 2Cr|\nabla u|^2.\end{aligned}$$

Now, since

$$\operatorname{div}X = X^1_{x_1} + X^2_{x_2} = 2 + (X^1_{x_1} - 1) + (X^2_{x_2} - 1) \geq 2 - 2Cr \qquad (3.1.8)$$

we have

$$\begin{aligned} J_1 &\geq \int_{\omega_r} \left\{ \frac{1}{2}|\nabla u|^2 \operatorname{div} X - |\nabla u|^2 - 2Cr|\nabla u|^2 \right\} dx \\ &\geq \int_{\omega_r} \left\{ |\nabla u|^2 - |\nabla u|^2 - Cr|\nabla u|^2 - 2Cr|\nabla u|^2 \right\} dx \\ &\geq -Cr \int_{\omega_r} |\nabla u|^2 \, dx. \end{aligned}$$

Finally, by choosing r_0 smaller if necessary, by (3.1.8) we have $\operatorname{div} X \geq 2 - 2Cr \geq 1$ which gives

$$J_2 = \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \operatorname{div} X \, dx \geq \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx.$$

Therefore we can find C large enough so that for the strong orthogonality condition

$$\begin{aligned} \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx - Cr \int_{\omega_r} \frac{1}{2}|\nabla u|^2 \, dx &\leq J_1 + J_2 \\ &= I_1 + I_2 \\ &\leq C \left[F(r) + \frac{r^2}{\varepsilon} \right] \end{aligned}$$

which completes the proof for inequality (3.1.2) and

$$\begin{aligned} &\frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx - Cr \int_{\omega_r} \frac{1}{2}|\nabla u|^2 \, dx \\ &\qquad \leq -\frac{W}{2\varepsilon^s} \int_{\Gamma_r} \langle u, g^\perp \rangle^2 \, ds + C \left[F_\Gamma(r) + \frac{W r^2}{\varepsilon^s} \right] \end{aligned}$$

for inequality (3.1.3) in the weak orthogonality case. □

3.2 η-Compactness

We are now in a position to state and prove the η-compactness property of minimizers. As a reminder, recall that the η-compactness property allows one to relate a certain logarithmic bound on the energy to the non-existence of vortices. Specifically, the idea here is that for two concentric balls, if the energy on the larger ball is small enough, then it is impossible for vortex to exist in the smaller ball.

Theorem 3.3 (η-Compactness). *Let $\frac{3}{4}s \leq \beta < \gamma < s \leq 1$. There exists constants η, $\tilde{C}$, $\varepsilon_0 > 0$ such that for any solution u_ε of the Euler–Lagrange equations* (2.2.1) *or* (2.2.3) *with $\varepsilon \in (0, \varepsilon_0)$, if $x_0 \in \overline{\Omega}$ and*

$$G_\varepsilon^W(u_\varepsilon; \omega_{2\varepsilon^\beta}(x_0)) \leq \eta |\ln \varepsilon|,$$

then

$$|u_\varepsilon| \geq \frac{1}{2} \quad \text{in } \omega_{\varepsilon^\gamma}(x_0), \tag{3.2.1}$$

$$|\langle u_\varepsilon, g^\perp \rangle| \leq \frac{1}{4} \quad \text{on } \Gamma \cap \overline{\omega_{\varepsilon^\gamma}(x_0)}, \tag{3.2.2}$$

$$\frac{1}{4\varepsilon^2} \int_{\omega_{\varepsilon^\gamma}(x_0)} (1 - |u_\varepsilon|^2)^2 \, dx + \frac{W}{2\varepsilon^s} \int_{\Gamma \cap \overline{\omega_{\varepsilon^\gamma}(x_0)}} \langle u_\varepsilon, g^\perp \rangle^2 \, ds \leq \tilde{C}\eta. \tag{3.2.3}$$

Remark 3.1. *In the specific case that u_ε is a solution to* (2.2.3)*, note that $G_\varepsilon^W(u_\varepsilon)$ is replaced by $G_\varepsilon(u_\varepsilon)$ and $s = 1$ in the statement of Theorem 3.3. Moreover, the bound* (3.2.2) *is trivially satisfied and* (3.2.3) *reduces to*

$$\frac{1}{4\varepsilon^2} \int_{\omega_{\varepsilon^\gamma}(x_0)} (1 - |u_\varepsilon|^2)^2 \, dx \leq \tilde{C}\eta.$$

Proof. The case where $x_0 \in \Omega$ and $\omega_{2\varepsilon^\beta}(x_0) \cap \Gamma = \varnothing$ follows exactly from [39, Lemma 2.3]. In the situation where $x_0 \in \Omega$ and $\omega_{2\varepsilon^\beta}(x_0) \cap \Gamma \neq \varnothing$, this can be reduced to the former case or the case where the ball is centered on the boundary [32]. Therefore, it is sufficient to prove the result for when $x_0 \in \Gamma$ and thus we proceed as in [3, 4]. Observe first that by the mean value theorem for integrals, there exists $r_\varepsilon \in (2\varepsilon^\gamma, 2\varepsilon^\beta)$ such that

$$\int_{2\varepsilon^\gamma}^{2\varepsilon^\beta} \frac{F_\Gamma(r)}{r} \, dr = F_\Gamma(r_\varepsilon) \int_{2\varepsilon^\gamma}^{2\varepsilon^\beta} \frac{dr}{r} = F_\Gamma(r_\varepsilon) \ln\left(\frac{\varepsilon^\beta}{\varepsilon^\gamma}\right) = F_\Gamma(r_\varepsilon)(\gamma - \beta)|\ln \varepsilon|.$$

Using this fact and the energy bound assumption,

$$\eta |\ln \varepsilon| \geq G_\varepsilon^W(u_\varepsilon; \omega_{2\varepsilon^\beta} \setminus \omega_{2\varepsilon^\gamma}) = \int_{2\varepsilon^\gamma}^{2\varepsilon^\beta} \frac{F_\Gamma(r)}{r} \, dr = F_\Gamma(r_\varepsilon)(\gamma - \beta)|\ln \varepsilon|$$

and therefore

$$F_\Gamma(r_\varepsilon) \leq \frac{\eta}{\gamma - \beta}.$$

In the strong orthogonality case, the same calculation can be refined by replacing $F_\Gamma(r)$ with $F(r)$ so that

$$F(r_\varepsilon) \leq \frac{\eta}{\gamma - \beta}.$$

If u is a weak orthogonal solution, we use inequality (3.1.3) to obtain

$$\begin{aligned}\frac{1}{4\varepsilon^2}\int_{\omega_{r_\varepsilon}(x_0)} (1-|u_\varepsilon|^2)^2\,dx + \frac{W}{2\varepsilon^s}\int_{\Gamma_{r_\varepsilon}(x_0)} \langle u_\varepsilon, g^\perp\rangle^2\,ds & \\ \leq C\left[r_\varepsilon \int_{\omega_{r_\varepsilon}(x_0)} \frac{1}{2}|\nabla u|^2\,dx + F_\Gamma(r_\varepsilon) + \frac{W r_\varepsilon^2}{\varepsilon^s}\right] & \\ \leq C\left[2\varepsilon^\beta \eta |\ln\varepsilon| + \frac{\eta}{\gamma-\beta} + 4W\varepsilon^{2\beta - s}\right] & \\ \leq C\left[2\varepsilon^{3s/4}\eta|\ln\varepsilon| + \frac{\eta}{\gamma-\beta} + 4W\sqrt{\varepsilon^s}\right]. &\end{aligned}$$

If u is a strong orthogonal solution, a similar bound is obtained by using inequality (3.1.2) from Lemma 3.2:

$$\begin{aligned}\frac{1}{4\varepsilon^2}\int_{\omega_{2\varepsilon^\gamma}(x_0)} (1-|u|^2)^2 dx &\leq \frac{1}{4\varepsilon^2}\int_{\omega_{r_\varepsilon}(x_0)} (1-|u|^2)^2 dx \\ &\leq C\left[r_\varepsilon \int_{\omega_{r_\varepsilon}(x_0)} \frac{1}{2}|\nabla u|^2\,dx + F(r_\varepsilon) + \frac{r_\varepsilon^2}{\varepsilon}\right] \\ &\leq C\left[2\varepsilon^\beta\eta|\ln\varepsilon| + \frac{\eta}{\gamma-\beta} + 4\varepsilon^{2\beta-1}\right] \\ &\leq C\left[2\varepsilon^{3/4}\eta|\ln\varepsilon| + \frac{\eta}{\gamma-\beta} + 4\sqrt{\varepsilon}\right].\end{aligned}$$

From the continuous extension of $w^{3s/4}|\ln w|$ to $[0,1]$, note that we have

$$\max_{0\leq w\leq 1} w^{3s/4}|\ln w| = \frac{4}{3se}.$$

Using this, we have the bound

$$2\varepsilon^{3s/4}\eta|\ln\varepsilon| + \frac{\eta}{\gamma-\beta} \leq C_1\eta, \quad C_1 := \frac{8}{3se} + \frac{1}{\gamma-\beta}.$$

Let $\varepsilon < \varepsilon_0$ where ε_0 is to be chosen later and assume $C \geq 1$. Then

$$\frac{1}{4\varepsilon^2}\int_{\omega_{2\varepsilon^\gamma}(x_0)}(1-|u|^2)^2dx + \frac{W}{2\varepsilon^s}\int_{\Gamma_{2\varepsilon^\gamma}(x_0)}\langle u_\varepsilon, g^\perp\rangle^2\,ds \leq CC_2\eta$$

where

$$C_2 = \begin{cases} C_1 + 4\sqrt{\varepsilon_0}\eta^{-1} & \text{if } u \text{ is a strong orthogonal solution,} \\ C_1 + 4\max\{W,1\}\sqrt{\varepsilon_0^s}\eta^{-1} & \text{if } u \text{ is a weak orthogonal solution.} \end{cases}$$

Defining $\tilde{C} := CC_2$ proves inequality (3.2.3).

To prove (3.2.1), we proceed by contradiction. Let $x_0 \in \overline{\Omega}$ and assume there is some $x_2 \in \omega_{\varepsilon^\gamma}(x_0)$ such that $|u(x_2)| < 1/2$. Then since $|\nabla u| \leq C_0\varepsilon^{-1}$, one can use the mean value theorem to obtain

$$|u(x) - u(x_2)| \leq |\nabla u||x - x_2| \leq \frac{C_0}{\varepsilon}|x - x_2|.$$

For $x \in \omega_{\varepsilon/4C_0}(x_2)$

$$|u(x) - u(x_2)| \leq \frac{C_0}{\varepsilon}\cdot\frac{\varepsilon}{4C_0} = \frac{1}{4} \implies |u(x)| \leq \frac{3}{4}.$$

Note that since $\varepsilon/4C_0 < \varepsilon^\gamma$ and $x_2 \in \omega_{\varepsilon^\gamma}(x_0)$, we have $\omega_{\varepsilon/4C_0}(x_2) \subset \omega_{2\varepsilon^\gamma}(x_0)$. Also, there is a constant $\alpha > 0$ for which

$$|\omega_r(x)| \geq \alpha r^2 \tag{3.2.4}$$

for all $x \in \overline{\Omega}$ and for all $r \leq 1$. Using this, inequality (3.2.3) and the lower estimate $(1-|u|^2)^2 \geq 49/2^8$ on $\omega_{\varepsilon/4C_0}(x_2)$

$$\begin{aligned} \tilde{C}\eta &\geq \frac{1}{4\varepsilon^2}\int_{\omega_{2\varepsilon^\gamma}(x_0)}(1-|u|^2)^2dx \\ &\geq \frac{1}{4\varepsilon^2}\int_{\omega_{\varepsilon/4C_0}(x_2)}(1-|u|^2)^2dx \\ &\geq \frac{49}{2^{10}\varepsilon^2}\cdot\alpha\left(\frac{\varepsilon^2}{4^2C_0^2}\right) \\ &\geq \frac{49\alpha}{2^{14}C_0^2}. \end{aligned}$$

Recall that $\tilde{C} = CC_2$. Assuming

$$\varepsilon_0 \leq \begin{cases} \frac{1}{4\cdot 4^2}\left(\frac{49\alpha}{2^{14}C_0^2 C}\right)^2 & \text{if } u \text{ is a strong orthogonal solution,} \\ \left(\frac{1}{4\cdot 4^2}\left(\frac{49\alpha}{2^{14}C_0^2 C \max\{W,1\}}\right)^2\right)^{1/s} & \text{if } u \text{ is a weak orthogonal solution,} \end{cases}$$

we have

$$\tilde{C}\eta = CC_2\eta \leq CC_1\eta + \frac{1}{2}\left(\frac{49\alpha}{2^{14}C_0^2}\right)$$

and therefore

$$CC_1\eta \geq \frac{1}{2}\left(\frac{49\alpha}{2^{14}C_0^2}\right) = \frac{49\alpha}{2^{15}C_0^2}.$$

Choosing η smaller than $49\alpha/(2^{15}CC_1C_0^2)$ yields the contradiction.

Finally, we prove (3.2.2). As noted in remark 3.1, if u_ε is a strong orthogonal solution then we trivially have $|\langle u_\varepsilon, g^\perp\rangle| = 0 < 1/4$ on all of Γ. Therefore we restrict our attention to weak orthogonal solutions for the remainder of this proof. Many of the steps here follow those found in [3, 4], but are altered to match the boundary conditions of this work.

Recall from the proof of Lemma 3.2 that for $x_0 \in \Gamma$, the radius bound r_0 was chosen small enough so that $\omega_r(x_0)$ could be assumed to be strictly starshaped around some $x_1 \in \omega_r(x_0)$. Taking $r = r_\varepsilon$, the starshape constraint allows us to write

$$\langle x - x_1, n\rangle \geq \frac{r_\varepsilon}{4} \quad \text{on } \partial\omega_{r_\varepsilon}(x_0)$$

where n is the unit normal vector to $\partial\omega_{r_\varepsilon}(x_0)$. We begin by setting $\psi = x - x_1$ in (2.6.1) and studying the integrand of the lefthand side of the identity. First, note that by using the orthogonal decompositions

$$\nabla u_\varepsilon^i = \langle \nabla u_\varepsilon^i, n\rangle n + \langle \nabla u_\varepsilon^i, \tau\rangle \tau, \qquad i = 1, 2,$$

the ith component of the vector field $\psi \cdot \nabla u_\varepsilon$ can be written

$$((x - x_1)\cdot \nabla u_\varepsilon)^i = \langle \nabla u_\varepsilon^i, n\rangle\langle x - x_1, n\rangle + \langle \nabla u_\varepsilon^i, \tau\rangle\langle x - x_1, \tau\rangle$$

which leads to

$$\langle \partial_n u_\varepsilon, (x - x_1)\cdot\nabla u_\varepsilon\rangle = |\partial_n u|^2\langle x - x_1, n\rangle + \langle \partial_n u_\varepsilon, \partial_\tau u_\varepsilon\rangle\langle x - x_1, \tau\rangle.$$

Next, using the lower bound from the starshape constraint, we have

$$\frac{1}{4\varepsilon^2}(1 - |u_\varepsilon|^2)^2\langle x - x_1, n\rangle \geq \frac{r_\varepsilon}{16\varepsilon^2}(1 - |u_\varepsilon|^2)^2 \geq 0$$

on all of $\partial\omega_{r_\varepsilon}(x_0)$. Using this and an orthogonal decomposition of the gradient once more,

$$e_\varepsilon(u_\varepsilon)\langle x - x_1, n\rangle \geq \frac{1}{2}(|\partial_n u_\varepsilon|^2 + |\partial_\tau u_\varepsilon|^2)\langle x - x_1, n\rangle$$

and therefore the lefthand side of (2.6.1) has the lower bound

$$\begin{aligned}\int_{\partial\omega_{r_\varepsilon}} &(e_\varepsilon(u_\varepsilon)\langle x - x_1, n\rangle - \langle \partial_n u_\varepsilon, (x - x_1)\cdot\nabla u_\varepsilon\rangle)\, ds \\ &\geq \int_{\partial\omega_{r_\varepsilon}} \left(\frac{1}{2}\langle x - x_1, n\rangle(|\partial_\tau u_\varepsilon|^2 - |\partial_n u_\varepsilon|^2) - \langle x - x_1, \tau\rangle\langle\partial_n u_\varepsilon, \partial_\tau u_\varepsilon\rangle\right) ds.\end{aligned}$$

Focusing on the righthand side of (2.6.1),

$$\operatorname{div}(x - x_1) = 2, \quad \partial_{x_j}\psi^l = \delta_{j,l}$$

and so it holds that

$$e_\varepsilon(u_\varepsilon)\operatorname{div}\psi - \sum_{j,l}\psi^l_{x_j}\langle\partial_{x_j}u, \partial_{x_l}u\rangle = \frac{1}{2\varepsilon^2}(1 - |u_\varepsilon|^2)^2.$$

Combining this with the lower bound on the boundary integral yields the inequality

$$\begin{aligned}\int_{\partial\omega_{r_\varepsilon}} &(\langle x - x_1, n\rangle|\partial_\tau u_\varepsilon|^2 - 2\langle x - x_1, \tau\rangle\langle\partial_n u_\varepsilon, \partial_\tau u_\varepsilon\rangle)\, ds \\ &\leq \int_{\partial\omega_{r_\varepsilon}}\langle x - x_1, n\rangle|\partial_n u_\varepsilon|^2\, ds + \frac{1}{\varepsilon^2}\int_{\omega_{r_\varepsilon}}(1 - |u_\varepsilon|^2)^2 dx.\end{aligned} \tag{3.2.5}$$

The second term of the integral on the lefthand side of (3.2.5) can be bounded easily by Cauchy-Schwarz, the Peter-Paul inequality and using the fact that $|x - x_1| \leq 2r_\varepsilon$,

$$\begin{aligned}
\left|\int_{\partial\omega_{r_\varepsilon}} 2\langle x-x_1,\tau\rangle\langle\partial_n u_\varepsilon,\partial_\tau u_\varepsilon\rangle\,ds\right| &\le \int_{\partial\omega_{r_\varepsilon}} 2|\langle x-x_1,\tau\rangle||\langle\partial_n u_\varepsilon,\partial_\tau u_\varepsilon\rangle|\,ds \\
&\le \int_{\partial\omega_{r_\varepsilon}} 2|x-x_1||\tau||\partial_n u_\varepsilon||\partial_\tau u_\varepsilon|\,ds \\
&\le \int_{\partial\omega_{r_\varepsilon}} 4r_\varepsilon\left(\frac{16|\partial_n u_\varepsilon|^2}{2}+\frac{|\partial_\tau u_\varepsilon|^2}{16\cdot 2}\right)ds \\
&= \int_{\partial\omega_{r_\varepsilon}} \left(32r_\varepsilon|\partial_n u_\varepsilon|^2+\frac{r_\varepsilon|\partial_\tau u_\varepsilon|^2}{8}\right)ds.
\end{aligned}$$

The starshape constraint allows us to bound the first integral on the lefthand side of (3.2.5) by

$$\int_{\partial\omega_{r_\varepsilon}} \langle x-x_1,n\rangle|\partial_\tau u_\varepsilon|^2 \ge \int_{\partial\omega_{r_\varepsilon}} \frac{r_\varepsilon|\partial_\tau u_\varepsilon|^2}{4}\,ds$$

and therefore the total lefthand side of (3.2.5) has the lower bound

$$\begin{aligned}
&\int_{\partial\omega_{r_\varepsilon}} \left(\langle x-x_1,n\rangle|\partial_\tau u_\varepsilon|^2-2\langle x-x_1,\tau\rangle\langle\partial_n u_\varepsilon,\partial_\tau u_\varepsilon\rangle\right)ds \\
&\qquad\qquad \ge \frac{r_\varepsilon}{8}\int_{\partial\omega_{r_\varepsilon}} |\partial_\tau u_\varepsilon|^2\,ds - 32r_\varepsilon\int_{\partial\omega_{r_\varepsilon}} |\partial_n u_\varepsilon|^2\,ds.
\end{aligned}$$

Using Cauchy-Schwarz again, the first integral on the righthand side of (3.2.5) has the simple bound

$$\begin{aligned}
\left|\int_{\partial\omega_{r_\varepsilon}} \langle x-x_1,n\rangle|\partial_n u_\varepsilon|^2\,ds\right| &\le \int_{\partial\omega_{r_\varepsilon}} |x-x_1||n||\partial_n u_\varepsilon|^2\,ds \\
&\le 2r_\varepsilon\int_{\partial\omega_{r_\varepsilon}} |\partial_n u_\varepsilon|^2\,ds.
\end{aligned}$$

Putting these bounds together, we arrive at the inequality

$$\int_{\partial\omega_{r_\varepsilon}} |\partial_\tau u_\varepsilon|^2\,ds \le 272\int_{\partial\omega_{r_\varepsilon}} |\partial_n u_\varepsilon|^2\,ds + \frac{8}{r_\varepsilon\varepsilon^2}\int_{\omega_{r_\varepsilon}} (1-|u_\varepsilon|^2)^2dx.$$

Decomposing the boundary $\partial\omega_{r_\varepsilon} = \Gamma_{r_\varepsilon} \cup (\partial B_{r_\varepsilon}(x_0) \cap \Omega)$ and using the known boundary condition from (2.2.1) we write

$$\begin{aligned}\int_{\partial\omega_{r_\varepsilon}} |\partial_n u_\varepsilon|^2\,ds &= \int_{\Gamma_{r_\varepsilon}} |\partial_n u_\varepsilon|^2\,ds + \int_{\partial B_{r_\varepsilon}(x_0)\cap\Omega} |\partial_n u_\varepsilon|^2\,ds \\ &= \frac{W^2}{\varepsilon^{2s}} \int_{\Gamma_{r_\varepsilon}} \langle u, g^\perp\rangle^2\,ds + \int_{\partial B_{r_\varepsilon}(x_0)\cap\Omega} |\partial_n u_\varepsilon|^2\,ds.\end{aligned}$$

Next, we use the crude estimate

$$\begin{aligned}\int_{\partial B_{r_\varepsilon}(x_0)\cap\Omega} |\partial_n u_\varepsilon|^2\,ds &\le \int_{\partial B_{r_\varepsilon}(x_0)\cap\Omega} |\nabla u_\varepsilon|^2\,ds \\ &\le \frac{r_\varepsilon}{r_\varepsilon}\int_{\partial B_{r_\varepsilon}(x_0)\cap\Omega} \left(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right) ds \\ &= \frac{2F(r_\varepsilon)}{r_\varepsilon} \\ &\le \frac{2F_\Gamma(r_\varepsilon)}{r_\varepsilon}\end{aligned}$$

and so

$$\begin{aligned}\int_{\partial\omega_{r_\varepsilon}} |\partial_\tau u_\varepsilon|^2\,ds \le \frac{272W^2}{\varepsilon^{2s}} &\int_{\Gamma_{r_\varepsilon}} \langle u, g^\perp\rangle^2\,ds \\ &+ \frac{8}{r_\varepsilon\varepsilon^2}\int_{\omega_{r_\varepsilon}} (1-|u_\varepsilon|^2)^2 dx + \frac{544F_\Gamma(r_\varepsilon)}{r_\varepsilon}.\end{aligned}$$

Recall from the beginning of this proof that $r_\varepsilon \in (2\varepsilon^\gamma, 2\varepsilon^\beta)$ is chosen such that

$$F_\Gamma(r_\varepsilon) \le \frac{\eta}{\gamma-\beta}.$$

Therefore we have the basic bound

$$\frac{544F_\Gamma(r_\varepsilon)}{r_\varepsilon} \le \frac{272\eta}{\gamma-\beta}\varepsilon^{-\gamma} \le \frac{272\eta}{\gamma-\beta}\varepsilon^{-s}.$$

Applying inequality (3.2.3) to the remaining terms of the integral (recall that this inequality holds for radius r_ε), we can find constants C' and C'' independent of ε and x_0 such that

$$\frac{272W^2}{\varepsilon^{2s}}\int_{\Gamma_{r_\varepsilon}} \langle u, g^\perp\rangle^2\,ds \le C'\eta\varepsilon^{-s}$$

and
$$\frac{8}{r_\varepsilon \varepsilon^2}\int_{\omega_{r_\varepsilon}}(1-|u_\varepsilon|^2)^2dx \leq \frac{C''\eta}{r_\varepsilon} \leq C''\eta\varepsilon^{-s}.$$

Hence,
$$\int_{\partial\omega_{r_\varepsilon}}|\partial_\tau u_\varepsilon|^2\,ds \leq C'\eta\varepsilon^{-s} + C''\eta\varepsilon^{-s} + \frac{272\eta}{\gamma-\beta}\varepsilon^{-s} \leq C\varepsilon^{-s}. \tag{3.2.6}$$

Next, we apply the Sobolev embedding theorem to u_ε on the one-dimensional set Γ_{r_ε} so that each component of u_ε satisfies
$$u^i_\varepsilon(x) - u^i_\varepsilon(y) = \int_y^x \partial_\tau u^i_\varepsilon\,ds, \quad \forall x,y\in\Gamma_{r_\varepsilon}.$$

By Hölder's inequality,
$$\begin{aligned}|u_\varepsilon(x)-u_\varepsilon(y)| &\leq \sum_{i=1}^2 |u^i_\varepsilon(x)-u^i_\varepsilon(y)|\\ &= \sum_{i=1}^2\left|\int_y^x \partial_\tau u^i_\varepsilon\,ds\right|\\ &\leq \sum_{i=1}^2\int_y^x|\partial_\tau u^i_\varepsilon|\,ds\\ &\leq \sum_{i=1}^2(\mathrm{dist}(x,y))^{1/2}\|\partial_\tau u^i_\varepsilon\|_{L^2(\Gamma_{r_\varepsilon})}\\ &\leq 2(\mathrm{dist}(x,y))^{1/2}\|\partial_\tau u_\varepsilon\|_{L^2(\Gamma_{r_\varepsilon})}.\end{aligned}$$

Therefore by (3.2.6) and the smoothness and compactness of Γ there is a constant C independent of ε and x_0 such that
$$|u_\varepsilon(x)-u_\varepsilon(y)| \leq C\sqrt{|x-y|}\varepsilon^{-s/2} \tag{3.2.7}$$
holding for all $x,y\in\Gamma_{r_\varepsilon}$. We are now in a position to apply the same contradiction type argument as for (3.2.1).

Suppose in order to derive a contradiction that there is some point $x_2\in\Gamma_{r_\varepsilon}$ such that $|\langle u_\varepsilon(x_2), g^\perp(x_2)\rangle| > 1/4$. By adding and subtracting $u(x)$ in the first component and $g^\perp(x)$ in the second component,

$$\begin{aligned}\langle u_\varepsilon(x_2), g^\perp(x_2)\rangle &= \langle u_\varepsilon(x_2) - u_\varepsilon(x), g^\perp(x_2)\rangle + \langle u_\varepsilon(x), g^\perp(x_2)\rangle \\ &= \langle u_\varepsilon(x_2) - u_\varepsilon(x), g^\perp(x_2)\rangle + \langle u_\varepsilon(x), g^\perp(x_2) - g^\perp(x)\rangle \\ &\quad + \langle u_\varepsilon(x), g^\perp(x)\rangle.\end{aligned}$$

Applying the triangle inequality, Cauchy-Schwarz and the uniform bounds $|u_\varepsilon|$, $|g^\perp| \leq 1$ wherever necessary,

$$|\langle u_\varepsilon(x_2), g^\perp(x_2)\rangle| \leq |\langle u_\varepsilon(x), g^\perp(x)\rangle| + |u_\varepsilon(x) - u_\varepsilon(x_2)| + |g^\perp(x) - g^\perp(x_2)|$$

and thus by the assumption $|\langle u_\varepsilon(x_2), g^\perp(x_2)\rangle| > 1/4$ we have the lower bound

$$|\langle u_\varepsilon(x), g^\perp(x)\rangle| > 1/4 - |u_\varepsilon(x) - u_\varepsilon(x_2)| - |g^\perp(x) - g^\perp(x_2)|.$$

Consider the ball of radius $\rho = \varepsilon^s/(16^2C^2)$ centered at x_2. Then by (3.2.7)

$$|u_\varepsilon(x) - u_\varepsilon(x_2)| \leq C\left(\frac{\varepsilon^{s/2}}{16C}\right)\varepsilon^{-s/2} = \frac{1}{16}$$

for all $x \in \Gamma_{r_\varepsilon} \cap B_\rho(x_2)$. By the smoothness of $g^\perp$ on Γ, there is a constant C' independent of ε and x_0 so that

$$|g^\perp(x) - g^\perp(x_2)| \leq C'\varepsilon^s$$

which again holds for all $x \in \Gamma_{r_\varepsilon} \cap B_\rho(x_2)$. Choosing ε_0 small enough so that $C'\varepsilon_0^s \leq 1/16$ we have

$$\begin{aligned}|\langle u_\varepsilon(x), g^\perp(x)\rangle| &> 1/4 - |u_\varepsilon(x) - u_\varepsilon(x_2)| - |g^\perp(x) - g^\perp(x_2)| \\ &\geq 1/4 - 1/16 - 1/16 \\ &= 1/8.\end{aligned}$$

Applying inequality (3.2.3) and noting $|\Gamma_{r_\varepsilon} \cap B_\rho(x_2)| \geq \rho$,

$$\tilde{C}\eta \geq \frac{W}{2\varepsilon^s}\int_{\Gamma_{r_\varepsilon}\cap B_\rho(x_2)} \langle u_\varepsilon, g^\perp\rangle^2\, ds > \frac{W}{2\varepsilon^s}\cdot\frac{\varepsilon^s}{64C^2}\cdot\left(\frac{1}{8}\right)^2 = \frac{W}{8192C^2}.$$

Therefore by choosing η small enough we arrive at a contradiction. □

3.3 Covering the Bad Set

As a consequence of Theorem 3.3, we can now show that the bad set

$$S_\varepsilon = \left\{ x \in \overline{\Omega} : |u_\varepsilon(x)| < \frac{1}{2} \text{ or } |\langle u_\varepsilon(x), g^\perp(x)\rangle| > \frac{1}{4} \right\}$$

can be covered by a finite, disjoint collection of balls with radius of order no larger than ε^s. Remarkably, it can also be shown that the number of such balls needed to cover S_ε is bounded independent of ε. In other words, for ε small, the set S_ε is small.

Proposition 3.4. *Suppose first that S_ε corresponds to the bad set for a minimizer of* (W.O.). *There exists $\tilde{N} \in \mathbb{N}$ depending only on Ω, a constant $\lambda > 1$ independent of ε and points $p_{\varepsilon,1}, \ldots, p_{\varepsilon,I_\varepsilon} \in S_\varepsilon \cap \Omega$, $q_{\varepsilon,1}, \ldots, q_{\varepsilon,J_\varepsilon} \in S_\varepsilon \cap \Gamma$ such that*

(i) $I_\varepsilon + J_\varepsilon \leq \tilde{N}$,

(ii) $S_\varepsilon \subset \bigcup_{i=1}^{I_\varepsilon} B_{\lambda\varepsilon}(p_{\varepsilon,i}) \cup \bigcup_{j=1}^{J_\varepsilon} B_{\lambda\varepsilon^s}(q_{\varepsilon,j})$,

(iii) $\{B_{\lambda\varepsilon}(p_{\varepsilon,i}), B_{\lambda\varepsilon^s}(q_{\varepsilon,j})\}_{1\leq i\leq I_\varepsilon, 1\leq j\leq J_\varepsilon}$ *are mutually disjoint with centers satisfying*

$$|p_{\varepsilon,i} - p_{\varepsilon,j}| > 8\lambda\varepsilon, \quad |q_{\varepsilon,i} - q_{\varepsilon,j}| > 8\lambda\varepsilon^s, \quad \textit{and} \quad |p_{\varepsilon,i} - q_{\varepsilon,j}| > 8\lambda\varepsilon^s,$$

(iv) $\overline{B_{\lambda\varepsilon}(p_{\varepsilon,i})} \cap \Gamma = \varnothing$ *for all* $i = 1, \ldots, I_\varepsilon$.

If S_ε is the bad set for a minimizer of (S.O.), *then the above holds with $s = 1$.*

Definition 3.5 (Bad Balls)**.** *Any ball that belongs to the covering*

$$\{B_{\lambda\varepsilon}(p_{\varepsilon,i}), B_{\lambda\varepsilon^s}(q_{\varepsilon,j})\}_{1\leq i\leq I_\varepsilon, 1\leq j\leq J_\varepsilon}$$

for S_ε from Proposition 3.4 will be generally referred to as a bad ball.

At this point, the specific boundary conditions related to (W.O.) and (S.O.) do not play a significant role in the proof of Proposition 3.4, other than the fact that bad balls centered on the boundary have a different radial scaling. The heart of the proof comes from the form of the local energy bound as seen in the context of Theorem 3.3. In fact, when compared to [3, 4], the structure of the η-compactness results is the same as the one proved in this work. Since Proposition 3.4 relies more on the structure of the energy bound, the proof is nearly identical to that found in [3, 4]. However, for the sake of completeness, we prove Proposition 3.4 in the case of strong orthogonality.

Proof. Consider the cover of S_ε given by $\{B_{2\varepsilon^\beta}(y)\}_{y\in S_\varepsilon}$. By Vitali's covering lemma, there exists a finite collection of points $y_1, y_2, \ldots, y_{N_\varepsilon} \in \overline{S_\varepsilon}$ such that $\{B_{2\varepsilon^\beta}(y_i)\}_{i=1}^{N_\varepsilon}$ are mutually disjoint and $\{B_{10\varepsilon^\beta}(y_i)\}_{i=1}^{N_\varepsilon}$ is a cover for S_ε. By Theorem 3.3 and the upper bound from Proposition 2.4, we have

$$N_\varepsilon \eta |\ln\varepsilon| \le \sum_{i=1}^{N_\varepsilon} G_\varepsilon(u_\varepsilon, \omega_{2\varepsilon^\beta}(y_i)) \le G_\varepsilon(u_\varepsilon) \le \mathcal{D}(\pi + C)||\ln\varepsilon|.$$

Therefore

$$N_\varepsilon \le \frac{\mathcal{D}(\pi + C)}{\eta}$$

and so $N_\varepsilon = N$ is bounded independent of ε.

Now by the mean value theorem for integrals, we again use the fact that there exists some $r_\varepsilon \in (\varepsilon^\gamma, \varepsilon^\beta)$ so that

$$G(u_\varepsilon; \omega_{\varepsilon^\beta}(y_i) \setminus \omega_{\varepsilon^\gamma}(y_i)) \ge (\gamma - \beta)|\ln\varepsilon| F(r_\varepsilon).$$

Applying Lemma 3.2 then gives

$$\begin{aligned}\frac{1}{4\varepsilon^2}\int_{\omega_\varepsilon(y_i)} (1-|u|^2)^2\,dx &\le \frac{1}{4\varepsilon^2}\int_{\omega_{r_\varepsilon}(y_i)} (1-|u|^2)^2\,dx \\ &\le C\left[\varepsilon^{3/4} G(u_\varepsilon, \omega_{r_\varepsilon}(y_i)) + \frac{G(u_\varepsilon; \omega_{\varepsilon^\beta}(y_i)\setminus\omega_{\varepsilon^\gamma}(y_i))}{(\gamma-\beta)|\ln\varepsilon|} + \sqrt{\varepsilon}\right] \\ &\le M\end{aligned}$$

where $M > 0$ is a constant independent of ε and $i = 1, \ldots, N$. Since $(1 - |u|^2)^2 \ge 9/16$ on $\omega_\varepsilon(y_i)$, the same idea used in the contradiction argument of Theorem 3.3 can be applied so that

$$\frac{1}{4\varepsilon^2}\int_{\omega_\varepsilon(y_i)} (1-|u|^2)^2\,dx \ge \frac{1}{4\varepsilon^2}\cdot\left(\frac{9}{16}\right)\cdot(\alpha\varepsilon^2) = \frac{9\alpha}{64}$$

independent of ε and $i = 1, \ldots, N$, where α is as in (3.2.4). Therefore

$$(I_\varepsilon + J_\varepsilon)\frac{9\alpha}{64} \le \sum_{i=1}^{N}\frac{1}{4\varepsilon^2}\int_{\omega_{r_\varepsilon}(y_i)} (1-|u|^2)^2\,dx \le MN.$$

Setting $\tilde{N} = \lceil(64MN/9\alpha)\rceil + 1$ finishes the proof for (i). Next, we employ Vitali's covering argument again but now on the collection of balls $\{B_\varepsilon(y)\}_{y\in S_\varepsilon}$. Then there is a finite set of points $p_{\varepsilon,1}, \ldots, p_{\varepsilon,I_\varepsilon} \in S_\varepsilon \cap \Omega$, $q_{\varepsilon,1}, \ldots, q_{\varepsilon,J_\varepsilon} \in S_\varepsilon \cap \Gamma$

so that $\{B_\varepsilon(p_{\varepsilon,i}), B_\varepsilon(q_{\varepsilon,j})\}_{1\le i\le I_\varepsilon, 1\le j\le J_\varepsilon}$ are mutually disjoint and

$$S_\varepsilon \subset \bigcup_{i=1}^{I_\varepsilon} B_{5\varepsilon}(p_{\varepsilon,i}) \cup \bigcup_{j=1}^{J_\varepsilon} B_{5\varepsilon}(q_{\varepsilon,j})$$

with I_ε and J_ε bounded independent of ε. If $\lambda = 5$ and the current ball covering satisfies conditions (ii)-(iv) we are done. In the case that condition (iii) is not satisfied, we apply the ball merging method presented in [10, Theorem IV.1]. Here, balls whose centers do not satisfy the minimum distance $8\lambda\varepsilon$ are merged by increasing $\lambda > 5$ and given modified centers. Since there are a finite number of balls, this merging process will terminate and (iii) will be satisfied after a finite number of steps with λ independent of ε. If all conditions are satisfied with the exception of (iv), then each ball whose closure intersects the boundary is encapsulated into its own boundary ball with center $y_i^* \in \overline{B_{\lambda\varepsilon}(p_{\varepsilon,i})} \cap \Gamma$ and radius $2\lambda\varepsilon$. If this new collection of balls satisfy (iii), we are done. If not, we apply the merging process again. Finally, it is worth noting that the merging process could produce interior balls which intersect the boundary. In this case, we proceed in the same way by including the boundary-intersecting interior ball into a boundary ball and then merging again. As before, since there are only a finite number of balls, this process will terminate in a finite number of steps until all interior balls have positive distance to the boundary Γ, or there are no interior balls left near Γ. □

While the finite ball covering of S_ε is a significant feat in determining where u_ε is nicely behaved, there is an associated complication given by the fact that the cover is not static. In other words, the centers of the bad balls, in general, move as $\varepsilon \to 0$. Thus, if one wants to study u_ε away from vortices, say on the set

$$\Omega \setminus \left\{ \bigcup_{i=1}^{I_\varepsilon} \overline{B_{\lambda\varepsilon}(p_{\varepsilon,i})} \cup \bigcup_{j=1}^{J_\varepsilon} \overline{B_{\lambda\varepsilon^s}(q_{\varepsilon,j})} \right\},$$

then one must also be aware of the fact that the domain is now moving when ε decreases. One way to rectify this issue is to observe that by the compactness of $\overline{\Omega}$, subsequences of the bad ball centers can be extracted which approach fixed points in $\overline{\Omega}$. Therefore by taking $\varepsilon > 0$ small enough, all bad balls will have congregated near one of the limiting points, and so a larger static ball can be used to encapsulate all nearby bad balls. Of course by doing this, we are potentially covering large portions of the domain where u_ε behaves nicely. Nonetheless, for ε small, these new fixed balls can also be taken to be small. In the next proposition, we construct a collection of fixed balls which act as a static cover for S_ε.

Proposition 3.6. *For any sequence of $\varepsilon \to 0$ there is a subsequence $\varepsilon_n \to 0$, a constant $\sigma_0 > 0$ and a finite collection of fixed points $\{p_1, \dots, p_I\} \subset \Omega$, $\{q_1, \dots, q_J\} \subset \Gamma$ such that for any $0 < \sigma < \sigma_0$ and for all $n \in \mathbb{N}$, the collection of sets*

$$\mathcal{S}_\sigma := \{B_\sigma(p_i)\}_{i=1}^I \cup \{B_{\sigma^s}(q_j)\}_{j=1}^J \tag{3.3.1}$$

are mutually disjoint and cover S_{ε_n}. The result holds for $s = 1$ in the case of strong orthogonality.

Once again, as Proposition 3.6 does not depend on the boundary conditions, this result can be shown as done in [3, 4]. The case of strong orthogonality is shown here with details filled in.

Proof. By Proposition 3.4, the sequence of cardinalities $\{I_\varepsilon, J_\varepsilon\}$ for the approximate vortices $\{p_{\varepsilon,i}, q_{\varepsilon,j}\}$ are uniformly bounded by $\tilde{N}$. By Balzano-Weierstrass there exists a subsequence ε_n so that $I_{\varepsilon_n} = \tilde{I}$, $J_{\varepsilon_n} = \tilde{J}$ are eventually constant for ε_n small enough which leaves us with the sets of accumulation points $\{p_{\varepsilon_n,1}, \dots, p_{\varepsilon_n,\tilde{I}}\}$ and $\{q_{\varepsilon_n,1}, \dots, q_{\varepsilon_n,\tilde{J}}\}$ along this subsequence. Upon taking further subsequences if necessary, Balzano-Weierstrass gives I distinct interior limits $\{p_i\}_{i=1}^I \subset \Omega$ and J distinct boundary limits $\{q_j\}_{j=1}^J \subset \Gamma$. Next, let $\{y_i\}_{i=1}^{I+J} = \{p_i\}_{i=1}^I \cup \{q_j\}_{j=1}^J$. Since these limit points are well-separated, we can define

$$\sigma_0 := \frac{1}{4} \min\{|y_i - y_j|,\ \mathrm{dist}(p_i, \Gamma) : i \neq j,\ i = 1, \dots, I,\ j = 1, \dots, J\}$$

so that

$$\mathcal{S}_\sigma := \{B_\sigma(p_i)\}_{i=1}^I \cup \{B_\sigma(q_j)\}_{j=1}^J$$

is a disjoint collection for any $0 < \sigma < \sigma_0$ with $\overline{B_\sigma(p_i)} \cap \Gamma = \varnothing$ for all $i = 1, \dots, I$. Finally, given that $0 < \sigma < \sigma_0$ is fixed (and independent of ε) the set $\mathcal{S}_\sigma$ covers S_{ε_n} for all ε_n small enough. □

Chapter 4

Lower Bounds for the Energy

In Chapter 3, we were concerned about ensuring that the set of points S_ε where u_ε does not behave like a classical director is small. Now that it is known such a set is small for $\varepsilon > 0$ small, we can focus on determining how minimizers behave when they are close to S_ε. Since the finite bad ball covering from definition 3.5 for S_ε has a known (and uncomplicated) geometry, it will be more convenient to consider the behaviour of u_ε near bad balls. To begin uncovering how one should observe u_ε around bad balls, we can look to the energy.

Suppose B is some bad ball. Then by Proposition 2.4,

$$G_\varepsilon^W(u_\varepsilon; B) \leq \pi\mathcal{D}|\ln\varepsilon| + C.$$

The first obvious insight is that the energy associated to a bad ball will grow no larger than logarithmically in ε. A more subtle insight comes from the factor $\mathcal{D}$, the degree (or winding number) of g along Γ, which highlights the fact that the tension created by the winding of the boundary data is related to the energy of the system. Thus, observing how minimizers wind around bad balls appears to be a step in the direction of quantifying the energy contribution of a vortex.

In this chapter, we show that the energy contribution of a single non-trivial defect is logarithmic in ε and that the intensity of the energy is attributed to the *square* of the winding behaviour of u_ε around the defect. The way in which we measure the winding will depend on the bad ball type being observed, i.e., interior bad balls or boundary bad balls. It is also shown that a relationship holds between the degree of the boundary data g and the cumulative winding behaviour of u_ε around the bad balls. This relationship will then be used to develop a global lower bound for the energy that matches the upper bound up to an additive constant.

4.1 Winding Quantifications

Our focus in this section will be to describe the way in which we quantify the winding of u_ε around each bad ball type (interior or boundary). For interior balls the topological *degree* of an $\mathbb{S}^1$-valued map, taking values in $\mathbb{Z}$, is well understood. We refer the reader to [38, Section 3.4.1] for a discussion on this topic.

A much harder problem is quantifying the winding of u_ε along the arc $\partial B \cap \Omega$ where B is a boundary bad ball. Indeed, since the approximate half-circle $\partial B \cap \Omega$ is not a closed contour, it is not obvious how to apply the notion of *degree* as used for interior balls. Moreover, due to the boundary conditions of the minimization problems (W.O.) and (S.O.), we are open to the possibility of u_ε making approximate $\mathbb{Z}\pi$ rotations about the arc as opposed to the standard $2\mathbb{Z}\pi$ rotations used in degree theory. In this way, there is a sense in which the winding or 'degree' of u_ε along a boundary bad ball is fractional. To avoid the issue of fractional degrees, we define a new topological quantity called the *boundary index* for u_ε along $\partial B \cap \Omega$ which will be based on rotations of u_ε modulo π (a 'half-turn' quantity) as opposed to rotations modulo 2π. To give the reader an intuitive view on how this quantity can be defined, we begin by analyzing solutions of (S.O.) since this case is easier to work with.

Let $B_{\lambda\varepsilon}(q_{\varepsilon,j})$ be some fixed boundary bad ball as defined in Definition 3.5. Fix $R > \lambda\varepsilon$ so that for all $\lambda\varepsilon \le r \le R$ the closure of $\omega_r(q_{\varepsilon,j})$ does not intersect the closure of any other bad ball. By definition of a bad ball, we have $|u_\varepsilon| \ge 1/2$ over the annulus $A_{\lambda\varepsilon,R}(q_{\varepsilon,j})$ and $\langle u_\varepsilon, g^\perp\rangle = 0$ on boundary portions $\Gamma^{\pm}_{\lambda\varepsilon,R}(q_{\varepsilon,j})$. Define the $\mathbb{S}^1$-valued function

$$v_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|} = e^{i\varphi}$$

locally on $A_{\lambda\varepsilon,R}(q_{\varepsilon,j})$ and let γ be a lifting of g on $\Gamma_R(q_{\varepsilon,j})$, that is, $g = e^{i\gamma(x)}$ for $x \in \Gamma_R(q_{\varepsilon,j})$. Since $\langle u_\varepsilon, g^\perp\rangle = 0$ on boundary portions $\Gamma^{\pm}_{\lambda\varepsilon,R}(q_{\varepsilon,j})$, it is easy to see that

$$\varphi - \gamma = 0 \mod \pi \quad \text{on } \Gamma^{\pm}_{\lambda\varepsilon,R}(q_{\varepsilon,j}). \tag{4.1.1}$$

Consider the contour given by $\partial B_r(q_{\varepsilon,j}) \cap \Omega$ and define

$$q^+_{\varepsilon,j} = q^+_{\varepsilon,j}(r) := \partial B_r(q_{\varepsilon,j}) \cap \Gamma^+_{\lambda\varepsilon^s,R}(q_{\varepsilon,j}),$$

$$q^-_{\varepsilon,j} = q^-_{\varepsilon,j}(r) := \partial B_r(q_{\varepsilon,j}) \cap \Gamma^-_{\lambda\varepsilon^s,R}(q_{\varepsilon,j}).$$

Suppose $\partial B_r(q_{\varepsilon,j}) \cap \Omega$ is oriented such $q^-_{\varepsilon,j}$ denotes the beginning of the curve. Then by observation (4.1.1) (the phase of v_ε and the phase of g differ by integer multiples of π), there is an integer $D_j \in \mathbb{Z}$ such that

$$\int_{\partial B_r \cap \Omega} \partial_\tau \varphi \, ds = \gamma(q_{\varepsilon,j}^+) - \gamma(q_{\varepsilon,j}^-) - D_j \pi.$$

The integer D_j is what will eventually be called *the boundary index* and it represents the net integer number of 'half-turns' v_ε makes along the arc $\partial B_r \cap \Omega$. However, there is a way in which the above work can be generalized to include weakly orthogonal solutions. Thus, before we finally give a formal definition for the boundary index, we proceed with this development.

Let $B_{\lambda\varepsilon^s}(q_{\varepsilon,j})$ be some fixed boundary bad ball as defined in Definition 3.5 with $s \in (0, 1]$. Fix $R > \lambda\varepsilon^s$ so that for all $\lambda\varepsilon^s \leq r \leq R$ the closure of $\omega_r(q_{\varepsilon,j})$ does not intersect the closure of any other bad ball. By definition of a bad ball, we have $|u_\varepsilon| \geq 1/2$ over the annulus $A_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$ and $|\langle u_\varepsilon, g^\perp\rangle| \leq 1/4$ on boundary portions $\Gamma^\pm_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$. Define the $\mathbb{S}^1$-valued function

$$v_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|} = e^{i\varphi}$$

locally on $A_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$. Let γ be a lifting of g on $\Gamma_R(q_{\varepsilon,j})$, that is, $g = e^{i\gamma(x)}$ for $x \in \Gamma_R(q_{\varepsilon,j})$. Using the bound $|\langle v_\varepsilon, g^\perp\rangle| \leq 1/2$ on boundary portions $\Gamma^\pm_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$, it is easy to calculate that this means the phase of v_ε and the phase of $\pm g$ are always within a relative angle of $\pi/6$. More precisely, there is a function $c_j(x)$ on $\Gamma^\pm_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$ satisfying $|c_j(x)| \leq \pi/6$ such that

$$\varphi - \gamma = c_j(x) \mod \pi \quad \text{on } \Gamma^\pm_{\lambda\varepsilon^s,R}(q_{\varepsilon,j}). \tag{4.1.2}$$

The correction function $c_j(x)$ is of course a piecewise function on $\Gamma^\pm_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$,

$$c_j(x) = \begin{cases} c_j^+(x) & \text{on } \Gamma^+_{\lambda\varepsilon^s,R}(q_{\varepsilon,j}), \\ c_j^-(x) & \text{on } \Gamma^-_{\lambda\varepsilon^s,R}(q_{\varepsilon,j}), \end{cases}$$

but the components $c_j^\pm$ should be chosen so that angle measurement is consistent. To do this, we must define a sense of orientation with respect to the boundary function g.

Definition 4.1. *We say that u is* positively oriented *(p.o.) with respect to g at $x \in \Gamma$ provided $\langle u(x), g(x)\rangle > 0$ and* negatively oriented *(n.o.) at $x \in \Gamma$ with respect to g if $\langle u(x), g(x)\rangle < 0$.*

Geometrically speaking, Definition 4.1 amounts to u lying within a local double cone with vertex $x \in \Gamma$ and axis of symmetry defined by $g(x)$. In particular, let $(\tilde\rho, \tilde\theta)$ denote the polar coordinate system where $\tilde\rho > 0$ is the radial distance from $x \in \Gamma$ and $\tilde{\theta} \in [-\pi, \pi]$ is the measured angle between $u(x)$ and $g(x)$, with positive rotational orientation counterclockwise away from g. Then we define

$$\mathcal{C}(\alpha) = \mathcal{C}^+(\alpha) \cup \mathcal{C}^-(\alpha)$$

where $\alpha \in [0, \pi/2)$ and

$$\begin{aligned}\mathcal{C}^+(\alpha) &= \{(\tilde{\rho}, \tilde{\theta}) \in (0, +\infty] \times [-\alpha, \alpha]\},\\ \mathcal{C}^-(\alpha) &= \{(\tilde{\rho}, \tilde{\theta}) \in (0, +\infty] \times ([-\pi, \alpha - \pi] \cup [\pi - \alpha, \pi])\}.\end{aligned}$$

Then u is p.o. at $x \in \Gamma$ if there is $\alpha \in [0, \pi/2)$ such that $u(x) \in \mathcal{C}^+(\alpha)$ and n.o. at $x \in \Gamma$ if there is $\alpha \in [0, \pi/2)$ such that $u(x) \in \mathcal{C}^-(\alpha)$. Within this context, it is clear that if $u \notin \mathcal{C}(\alpha)$ for any $\alpha \in [0, \pi/2)$ then $u(x)$ is orthogonal to $g(x)$.

Remark 4.1. *When restricting attention to the strong orthogonality condition, if $|u(x)| > 0$ for $x \in \Gamma$ then $u(x)$ always belongs to $\mathcal{C}(0)$.*

Returning to the function $c_j(x)$, if v_ε is positively oriented on a component of $\Gamma^\pm_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$, then the corresponding branch of the piecewise function will be the relative angle between v_ε and the axis defined by g in $\mathcal{C}^+(\pi/6)$ with positive orientation dictated by a counterclockwise rotation relative to g. If v_ε is negatively oriented on a component of $\Gamma^\pm_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$, then the corresponding branch of the piecewise function will be the relative angle between v_ε and the axis defined by $-g$ in $\mathcal{C}^-(\pi/6)$ with positive orientation dictated by a counterclockwise rotation relative to $-g$.

Remark 4.2. *In the special case where u_ε is a strong orthogonal solution, $c_j^\pm = 0$ on all of $\Gamma^\pm_{\lambda\varepsilon,R}(q_{\varepsilon,j})$.*

Consider the contour given by $\partial B_r(q_{\varepsilon,j}) \cap \Omega$ where $\lambda\varepsilon^s \leq r \leq R$ and let

$$\begin{aligned} q^+_{\varepsilon,j} &= q^+_{\varepsilon,j}(r) := \partial B_r(q_{\varepsilon,j}) \cap \Gamma^+_{\lambda\varepsilon^s,R}(q_{\varepsilon,j}),\\ q^-_{\varepsilon,j} &= q^-_{\varepsilon,j}(r) := \partial B_r(q_{\varepsilon,j}) \cap \Gamma^-_{\lambda\varepsilon^s,R}(q_{\varepsilon,j}). \end{aligned} \tag{4.1.3}$$

Suppose that the orientation of $\partial B_r(q_{\varepsilon,j}) \cap \Omega$ is such that $q^-_{\varepsilon,j}$ indicates the beginning of the curve. Then by observation (4.1.2) and the fact that $|c_j| \leq \pi/6$, we have as in the strong orthogonality case

$$\int_{\partial B_r \cap \Omega} \partial_\tau \varphi \, ds = (\gamma(q^+_{\varepsilon,j}) + c^+_j(q^+_{\varepsilon,j})) - (\gamma(q^-_{\varepsilon,j}) + c^-_j(q^-_{\varepsilon,j})) - D_j \pi$$

where $D_j \in \mathbb{Z}$ is the approximate integer number of 'half-turns' v_ε makes along the arc $\partial B_r(q_{\varepsilon,j}) \cap \Omega$. We can now rigorously define the boundary index for u_ε.

Definition 4.2 (Boundary Index)**.** *Let $A_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$ be an annular region surrounding a boundary bad ball with center $q_{\varepsilon,j}$ and with $R > \lambda\varepsilon^s$ chosen small enough such that $\overline{A_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})}$ does not intersect the closure of any other bad ball. Let φ be the lifting of the normalization of u_ε in $A_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$, γ the lifting of g on $\Gamma_R(q_{\varepsilon,j})$ and let c_j be the correction function as defined by* (4.1.2)*. Then the* <u>*boundary index*</u> *of u_ε on $\partial B_r(q_{\varepsilon,j}) \cap \Omega$ for $\lambda\varepsilon^s \leq r \leq R$ is defined by the integer*

$$D_j := \frac{1}{\pi}\left[(\gamma(q^+_{\varepsilon,j}) + c^+_j(q^+_{\varepsilon,j})) - (\gamma(q^-_{\varepsilon,j}) + c^-_j(q^-_{\varepsilon,j})) - \int_{\partial B_r(q_{\varepsilon,j})\cap\Omega} \partial_\tau\varphi\, ds\right]$$

where $\{q^-_{\varepsilon,j}, q^+_{\varepsilon,j}\}$ are as in (4.1.3) *and $\partial B_r(q_{\varepsilon,j}) \cap \Omega$ has orientation such that $q^-_{\varepsilon,j}$ indicates the beginning of the curve. We will often use the notation*

$$D_j := \operatorname{ind}(u_\varepsilon; \partial B_r(q_{\varepsilon,j}) \cap \Omega)$$

for when the arc over which D_j is calculated needs to be stated explicitly.

As in the case for the degree of an interior ball, it can shown that D_j is independent of the chosen radius $r \in [\lambda\varepsilon^s, R]$.

Lemma 4.3. *Let $A_{\lambda\varepsilon^s,R}(q_{\varepsilon,j})$ be as in Definition 4.2. Then the associated boundary index $D_j = \operatorname{ind}(u_\varepsilon, \partial B_r(q_{\varepsilon,j}) \cap \Omega)$ is independent of the radius $r \in [\lambda\varepsilon^s, R]$.*

Proof. We omit the subscript ε in this proof and sometimes the q_j dependence for space. Let $\lambda\varepsilon^s \leq a < b \leq R$ and consider the closed contour defined by the boundary of the annulus $A_{a,b}(q_j)$ with decomposition

$$\partial A_{a,b}(q_j) = \Gamma^-_{a,b}(q_j) \cup C_a(q_j) \cup \Gamma^+_{a,b}(q_j) \cup C_b(q_j)$$

where we have employed the short-form notation for the circular arcs

$$C_a = C_a(q_j) = \partial B_a(q_j) \cap \Omega, \quad C_b = C_b(q_j) = \partial B_b(q_j) \cap \Omega.$$

Here, we take the orientation of $\Gamma^\pm_{a,b}(q_j)$ to coincide with the positive orientation of Γ. The circular arcs C_a and C_b are to be oriented as in Definition 4.2. Define the boundary indices

$$D_a = \operatorname{ind}(u_\varepsilon; \partial B_a(q_j) \cap \Omega), \quad D_b = \operatorname{ind}(u_\varepsilon; \partial B_b(q_j) \cap \Omega).$$

Since $|u_\varepsilon| \geq 1/2$ inside $A_{a,b}(q_j)$, the degree of u_ε along the positively oriented boundary $\partial A_{a,b}(q_j)$ is zero and thus

$$\int_{\partial A_{a,b}(q_j)} \partial_\tau\varphi\, ds = \int_{\Gamma^\pm_{a,b}} \partial_\tau\varphi\, ds + \int_{C_a} \partial_\tau\varphi\, ds - \int_{C_b} \partial_\tau\varphi\, ds = 0.$$

Along the boundary segments $\Gamma^{\pm}_{a,b}(q_j)$, we can write

$$\partial_\tau \varphi = \begin{cases} \partial_\tau \gamma + \partial_\tau c_j^+ & \text{on } \Gamma^+_{a,b}(q_j), \\ \partial_\tau \gamma + \partial_\tau c_j^- & \text{on } \Gamma^-_{a,b}(q_j), \end{cases}$$

which gives

$$\begin{aligned} \int_{\Gamma^-_{a,b}} \partial_\tau \varphi \, ds &= (\gamma(q^-(a)) + c_j^-(q^-(a))) - (\gamma(q^-(b)) + c_j^-(q^-(b))), \\ \int_{\Gamma^+_{a,b}} \partial_\tau \varphi \, ds &= (\gamma(q^+(b)) + c_j^+(q^+(b))) - (\gamma(q^+(a)) + c_j^+(q^+(a))). \end{aligned} \tag{4.1.4}$$

On the other hand, by Definition 4.2

$$\begin{aligned} \int_{C_a} \partial_\tau \varphi \, ds &= (\gamma(q^+(a)) + c_j^+(q^+(a)) - (\gamma(q^-(a)) + c_j^-(q^-(a))) - D_a \pi \\ -\int_{C_b} \partial_\tau \varphi \, ds &= (\gamma(q^-(b)) + c_j^-(q^-(b))) - (\gamma(q^+(b)) + c_j^+(q^+(b)) + D_b \pi. \end{aligned} \tag{4.1.5}$$

Adding (4.1.4) and (4.1.5) together,

$$\int_{\partial A_{a,b}(q_j)} \partial_\tau \varphi \, ds = -D_a \pi + D_b \pi = 0$$

which of course readily gives $D_a = D_b$. □

4.2 The Energy Contribution of a Defect

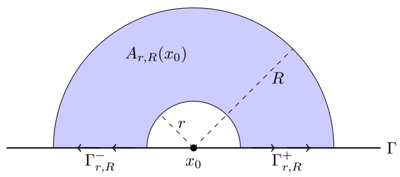
Next, we develop a lower bound for the Dirichlet energy of u_ε on an annulus using our notion of interior degree and boundary index. We begin by recognizing that when $x_0 \in \Gamma$, there are four possible orientations for u on $\Gamma^{\pm}_{r,R}(x_0)$:

(a) u is p.o. on $\Gamma^+_{r,R}$ and n.o. on $\Gamma^-_{r,R}$,

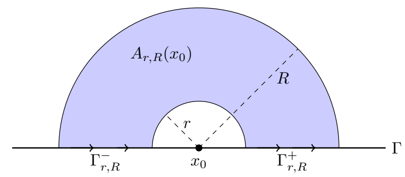
(b) u is p.o. on $\Gamma^+_{r,R}$ and on $\Gamma^-_{r,R}$,

(c) u is n.o. on $\Gamma^+_{r,R}$ and p.o. on $\Gamma^-_{r,R}$,

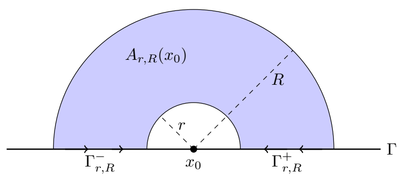
(d) u is n.o. on $\Gamma^+_{r,R}$ and on $\Gamma^-_{r,R}$.



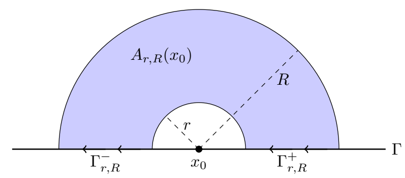
(a) u is p.o. on $\Gamma^+_{r,R}$ and n.o. on $\Gamma^-_{r,R}$



(b) u is p.o. on $\Gamma^+_{r,R}$ and on $\Gamma^-_{r,R}$



(c) u is n.o. on $\Gamma^+_{r,R}$ and p.o. on $\Gamma^-_{r,R}$



(d) u is n.o. on $\Gamma^+_{r,R}$ and on $\Gamma^-_{r,R}$

Figure 4.1: The four possible orientations for u on $\Gamma^\pm_{r,R}$ assuming Γ is flat and $g = \tau$, the positively oriented unit tangent vector to Γ.

These four orientations are shown above in figure 4.1 in the case of a flat boundary and strong orthogonality with tangential boundary data. With this in mind, we can define a polar representation for u on $A_{r,R}(x_0)$, $x_0 \in \overline{\Omega}$. Recall the local polar coordinate system developed in Section 2.5 by Definitions (2.5.2)-(2.5.8). Let $\gamma(x)$ be such that $g(x) = e^{i\gamma(x)}$ along $\Gamma_R(x_0)$ with $\gamma_0 = \gamma(x_0)$ provided $x_0 \in \Gamma$. Using this, we set

$$u(\rho, \theta) = f(\rho, \theta)e^{i\psi(\rho,\theta)} \text{ on } A_{r,R}(x_0)$$

where

$$\psi(\rho, \theta) = \begin{cases} d\theta + \phi(\rho, \theta) & \text{if } B_R(x_0) \subset \Omega \\ D\theta + \gamma_0 + \phi(\rho, \theta) & \text{if } x_0 \in \Gamma \text{ and } u \text{ is p.o. on } \Gamma^+_{r,R}, \\ D\theta + \gamma_0 + \phi(\rho, \theta) + \pi & \text{if } x_0 \in \Gamma \text{ and } u \text{ is n.o. on } \Gamma^+_{r,R}. \end{cases} \quad (4.2.1)$$

Here, ϕ is a smooth single-valued correction function defined on $A_{r,R}(x_0)$ and the integers $d, D \in \mathbb{Z}$ are the associated degree and boundary index for u depending on the center of the annulus. Defining the polar form of u in this way allows the boundary index D to determine the orientation of u along $\Gamma^-_{r,R}$. Indeed, when R is taken to be appropriately small and $D \in 2\mathbb{Z}$, the phase difference across $\Gamma^\pm_{r,R}$ will be approximately an even multiple of π. Therefore the orientation of u is preserved along $\Gamma^\pm_{r,R}$ (corresponding to cases (b) and (d)). On the other hand, if $D \in 2\mathbb{Z} + 1$ the phase difference across $\Gamma^\pm_{r,R}$ will be approximately an odd multiple of π which forces the orientation of u to be opposite on either side of x_0 (corresponding to cases (a) and (c)).

Returning to the correction function ϕ momentarily, in estimates to come it will be important to appropriately bound the magnitude of ϕ along Γ. The following proposition indicates that $|\phi|$ may be bounded in the following way:

Proposition 4.4. *Let ϕ be as defined in (4.2.1). Then there exists a constant $C_1 > 0$ for which $|\phi(\rho, \theta(\rho))| \leq C_1(|\langle u, g^\perp \rangle| + \rho)$ on $\Gamma^{\pm}_{r,R}$ provided $|u| \geq 1/2$ and $|\langle u, g^\perp \rangle| \leq 1/4$.*

Proposition 4.4 is claimed in [32] (for the case where $g^\perp = n$) but is not shown explicitly. For completeness and to ensure it is still true for arbitrary boundary data g, we prove it here.

Proof. By definition of the inner product

$$|\langle u, g^\perp \rangle| = |u||\cos(\psi - (\gamma - \pi/2))| = |u||\sin(\psi - \gamma)|$$

Using the assumptions $|\langle u, g^\perp \rangle| \leq 1/4$ and $|u| \geq 1/2$ we have

$$\frac{1}{2}|\sin(\psi - \gamma)| \leq |\langle u, g^\perp \rangle| \leq \frac{1}{4} \implies -\frac{1}{2} \leq \sin(\psi - \gamma) \leq \frac{1}{2}.$$

Therefore we easily see that $u \in \mathcal{C}(\pi/6)$ for all $x \in \Gamma^{\pm}_{r,R}$. In order to show the desired inequality, we consider the possible four cases of orientation for u separately.

Case (a) - u is positively oriented on $\Gamma^{+}_{r,R}$ and negatively oriented on $\Gamma^{-}_{r,R}$:

Along $\Gamma^{+}_{r,R}$ we have

$$\psi - \gamma = D\theta + \gamma_0 - \gamma + \phi = \xi$$

with $D \in 2\mathbb{Z} + 1$ and $\xi \in [-\pi/6, \pi/6]$. Applying the triangle inequality

$$|\phi| \leq |\xi| + |D\theta| + |\gamma_0 - \gamma| \leq \frac{\pi}{6} + C\rho.$$

On the curve $\Gamma^{-}_{r,R}$ the phase difference satisfies $D\theta + \gamma_0 - \gamma + \phi = D\pi + \xi$. A similar estimate gives

$$|\phi| \leq |\xi| + |D||\pi - \theta| + |\gamma_0 - \gamma| \leq \frac{\pi}{6} + C\rho.$$

Case (b) - u is positively oriented on $\Gamma^{+}_{r,R}$ and on $\Gamma^{-}_{r,R}$:

In this scenario, $\psi - \gamma = D\theta + \gamma_0 - \gamma + \phi = \xi$ along $\Gamma^{+}_{r,R}$ and $D\theta + \gamma_0 - \gamma + \phi = D\pi + \xi$ on $\Gamma^{-}_{r,R}$ but now $D \in 2\mathbb{Z}$. The estimate $|\phi| \leq \pi/6 + C\rho$ is obtained identically to that of case 1.

Case (c) - u is negatively oriented on $\Gamma^+_{r,R}$ and positively oriented $\Gamma^-_{r,R}$:

On $\Gamma^+_{r,R}$ we have

$$\psi - \gamma = D\theta + \gamma_0 - \gamma + \phi + \pi = \pi + \xi$$

and on $\Gamma^-_{r,R}$ the phase difference satisfies $D\theta + \gamma_0 - \gamma + \phi + \pi = D\pi + \pi + \xi$ where $D \in 2\mathbb{Z} + 1$. The same estimates are applied here as in case 1 to obtain $|\phi| \leq \pi/6 + C\rho$.

Case (d) - u is negatively oriented on $\Gamma^+_{r,R}$ and on $\Gamma^-_{r,R}$:

In this final case, the estimates from case 3 on $\Gamma^\pm_{r,R}$ still hold but now with $D \in 2\mathbb{Z}$ to preserve orientation.

Remark 4.3. *The case where $\xi = 0$ in each of the four cases above corresponds to the strong orthogonality condition.*

If needed, we may assume r and R are chosen small enough such that $C\rho \leq \pi/12$, for example, so that

$$|\phi| \leq \frac{\pi}{6} + C\rho \leq \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}$$

on $\Gamma^\pm_{r,R}$. Next, we return to the inner product

$$|\langle u, g^\perp \rangle| = |u||\sin(\psi - \gamma)| \geq \frac{1}{2}|\sin(\psi - \gamma)|$$

and observe by the addition formula for sines

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b),$$

the π-translation invariance

$$|\sin(D\theta+\gamma_0+\gamma)| = |\sin(D\theta+\gamma_0+\gamma+\pi)|, \ |\cos(D\theta+\gamma_0+\gamma)| = |\cos(D\theta+\gamma_0+\gamma+\pi)|$$

and the reverse triangle inequality, we have

$$\begin{aligned} |\langle u, g^\perp \rangle| &\geq \frac{1}{2}|\sin(D\theta + \gamma_0 - \gamma + \phi)| \\ &= \frac{1}{2}|\sin(\phi)||\cos(D\theta + \gamma_0 - \gamma)| - \frac{1}{2}|\cos(\phi)||\sin(D\theta + \gamma_0 - \gamma)| \end{aligned}$$

on $\Gamma^\pm_{r,R}$ for any of the four orientation scenarios. By the smoothness of Γ and γ, we know for r and R small enough,

$$|\sin(D\theta + \gamma_0 - \gamma)|, \ |1 - |\cos(D\theta + \gamma_0 - \gamma)|| \leq C\rho.$$

Therefore we may assume $|\cos(D\theta + \gamma_0 - \gamma)| \geq 1/2$ on $\Gamma^{\pm}_{r,R}$ and we obtain

$$|\langle u, g^{\perp}\rangle| \geq \frac{1}{4}|\sin(\phi)| - \frac{1}{2}C\rho.$$

Finally, since $|\phi| \leq \pi/4$ we have the estimate

$$|\sin(\phi)| \geq \frac{1}{2}|\phi|$$

which leads to the inequality

$$|\langle u, g^{\perp}\rangle| \geq \frac{1}{8}|\phi| - \frac{1}{2}C\rho.$$

Arranging for $|\phi|$, we can find a universal constant C so that

$$|\phi| \leq 8(|\langle u, g^{\perp}\rangle| + \frac{C}{2}\rho) \leq C(|\langle u, g^{\perp}\rangle| + \rho)$$

giving the desired inequality. □

Let us now estimate the Dirichlet energy for u_ε about $x_0 \in \overline{\Omega}$.

Theorem 4.5. *Suppose $x_0 \in \overline{\Omega}$ and assume that $1/2 \leq |u| \leq 1$ in $A_{r,R}(x_0)$ and $|\langle u, g^{\perp}\rangle| \leq 1/4$ on $\Gamma^{\pm}_{R}$. Additionally, suppose that there is some number K such that*

$$G(u;\Omega) \leq K|\ln\varepsilon| + K,$$
$$\frac{1}{\varepsilon^2}\int_{\omega_{\varepsilon^\gamma}(x_0)} (1-|u|^2)^2\,dx + \frac{1}{\varepsilon^s}\int_{\Gamma_{\varepsilon^\gamma}} \langle u, g^{\perp}\rangle^2\,ds \leq K$$

where ε^γ is as in Theorem 3.3. There exists a constant C depending only on Ω, γ and K such that:

(i) If $B_R(x_0) \subset \Omega$, $\varepsilon \leq r < R \leq r_0$ and $d \neq 0$,

$$\int_{A_{r,R}(x_0)} |\nabla u|^2\,dx \geq 2d^2\pi \ln\left(\frac{R}{r}\right) - C. \tag{4.2.2}$$

(ii) If $x_0 \in \Gamma$, $\varepsilon^s \leq r < R \leq r_0$ and $D \neq 0$,

$$\int_{A_{r,R}(x_0)} |\nabla u|^2\,dx \geq D^2\pi \ln\left(\frac{R}{r}\right) - C. \tag{4.2.3}$$

Remark 4.4. *The hypothesis of this theorem includes both the weak and strong orthogonality problems. Substituting $\langle u, g^{\perp}\rangle = 0$ and $s = 1$ into the above yields a version of the theorem tailored to the strong orthogonality problem.*

Proof. As in previous proofs, C will denote a constant independent of ε throughout and is subject to change. We begin with the polar representation $u(\rho,\theta) = f(\rho,\theta)e^{i\psi(\rho,\theta)}$ centered at $x_0 \in \overline{\Omega}$. If $B_R(x_0) \subset \Omega$, the Dirichlet energy of u has lower estimate

$$\begin{aligned}\int_{A_{r,R}(x_0)} |\nabla u|^2\,dx &= \int_{A_{r,R}(x_0)} \left(f^2|\nabla\psi|^2 + |\nabla f|^2\right) dx\\ &\geq \int_{A_{r,R}(x_0)} f^2|\nabla d\theta + \nabla\phi|^2\,dx\\ &= \int_{A_{r,R}(x_0)} \frac{d^2f^2}{\rho^2}\,dx + \int_{A_{r,R}(x_0)} \frac{2df^2}{\rho^2}\partial_\theta\phi\,dx + \int_{A_{r,R}(x_0)} f^2|\nabla\phi|^2\,dx\\ &= I_1 + I_2 + I_3.\end{aligned}$$

The same lower estimate is obtained for when $x_0 \in \Gamma$ but with d replaced by D. We consider each integral separately for all three cases:

<u>I_1 (i)</u>

The proof of this case is identical to Struwe's arguments from [39, Proposition 3.4] and [40, Proposition 3.4']. However, we do give the full calculations here to fill in the details.

$$\begin{aligned}\int_{A_{r,R}(x_0)} \frac{d^2f^2}{\rho^2}\,dx &= \int_{A_{r,R}(x_0)} \frac{d^2f^2}{\rho^2}\,dx + \int_{A_{r,R}(x_0)} \frac{d^2}{\rho^2}\,dx - \int_{A_{r,R}(x_0)} \frac{d^2}{\rho^2}\,dx\\ &= \int_{A_{r,R}(x_0)} \frac{d^2}{\rho^2}\,dx - \int_{A_{r,R}(x_0)} \frac{d^2(1-f^2)}{\rho^2}\,dx\\ &= 2d^2\pi\ln\left(\frac{R}{r}\right) - \int_{A_{r,R}(x_0)} \frac{d^2(1-f^2)}{\rho^2}\,dx\\ &= 2d^2\pi\ln\left(\frac{R}{r}\right) - I_4.\end{aligned}$$

The integral I_4 needs to be treated differently depending on the sizes of r and R. In particular, if $r < \varepsilon^\gamma < R$ then

$$I_4 = \int_{A_{r,\varepsilon^\gamma}(x_0)} \frac{d^2(1-f^2)}{\rho^2}\,dx + \int_{A_{\varepsilon^\gamma,R}(x_0)} \frac{d^2(1-f^2)}{\rho^2}\,dx = I_5 + I_6.$$

Estimating I_5 via Cauchy-Schwarz,

$$\begin{aligned} I_5 &= d^2 \int_{A_{r,\varepsilon^\gamma}(x_0)} \frac{1-f^2}{\rho^2}\,dx \\ &\le d^2 \left(\int_{A_{r,\varepsilon^\gamma}(x_0)} \frac{1}{\rho^4}\,dx \right)^{1/2} \left(\int_{A_{r,\varepsilon^\gamma}(x_0)} (1-f^2)^2\,dx \right)^{1/2}. \end{aligned}$$

For the first integral in the product, we compute

$$\int_{A_{r,\varepsilon^\gamma}(x_0)} \frac{1}{\rho^4}\,dx = \int_0^{2\pi} \int_r^{\varepsilon^\gamma} \frac{1}{\rho^4}\,\rho\,d\rho\,d\theta = \pi \left(\frac{1}{r^2} - \frac{1}{\varepsilon^{2\gamma}} \right) \le \frac{\pi}{\varepsilon^2}.$$

Therefore

$$\begin{aligned} I_5 &\le \frac{d^2\sqrt{\pi}}{\varepsilon} \left(\int_{A_{r,\varepsilon^\gamma}(x_0)} (1-f^2)^2\,dx \right)^{1/2} \\ &\le d^2\sqrt{\pi} \left(\frac{1}{\varepsilon^2} \int_{\omega_{\varepsilon^\gamma}(x_0)} (1-f^2)^2\,dx \right)^{1/2} \\ &\le d^2\sqrt{\pi K}. \end{aligned}$$

For integral I_6, the same procedure as above applies but now we use $\varepsilon^\gamma \le \rho$ to obtain

$$I_6 \le \frac{d^2\sqrt{\pi}}{\varepsilon^\gamma} \left(\int_{A_{\varepsilon^\gamma,R}(x_0)} (1-f^2)^2\,dx \right)^{1/2} \le d^2\sqrt{\pi} \left(\frac{1}{\varepsilon^{2\gamma}} \int_\Omega (1-f^2)^2\,dx \right)^{1/2}.$$

Now since $\gamma < 1$,

$$\frac{1}{\varepsilon^{2\gamma}} \int_\Omega (1-f^2)^2\,dx \le 4\varepsilon^{2(1-\gamma)} G(u;\Omega) \le 4\varepsilon^{2(1-\gamma)} K(1+|\ln\varepsilon|) \le C(K,\gamma)$$

and therefore we have the bound $I_6 \le d^2\sqrt{\pi C}$. Now if $R \le \varepsilon^\gamma$ we need only the estimate for I_5 since $|I_4| \le |I_5|$. In the case where $\varepsilon^\gamma \le r$, then the estimate for I_6 is only needed since $|I_4| \leq |I_6|$. In any case, we have the lower bound

$$\begin{aligned} I_1 &= 2d^2\pi \ln\left(\frac{R}{r}\right) - I_4 \\ &\geq 2d^2\pi \ln\left(\frac{R}{r}\right) - |I_5| - |I_6| \\ &\geq 2d^2\pi \ln\left(\frac{R}{r}\right) - C. \end{aligned}$$

Note that for future calculations we may assume $r < \varepsilon^\gamma < R$.

I_1 (ii)

In this scenario the proof is more closely related to the ideas of Moser [32, Proposition 5.6]. The annular region can be described by the set

$$A_{r,R}(x_0) = \{r \leq \rho \leq R,\ \theta_1(\rho) \leq \theta \leq \theta_2(\rho)\}$$

where $\theta_2 - \theta_1 \geq \pi - C\rho$. We have

$$\begin{aligned} \int_{A_{r,R}(x_0)} \frac{D^2 f^2}{\rho^2}\, dx &= \int_{A_{r,R}(x_0)} \frac{D^2}{\rho^2}\, dx - \int_{A_{r,R}(x_0)} \frac{D^2(1-f^2)}{\rho^2}\, dx \\ &= D^2 \int_r^R \int_{\theta_1}^{\theta_2} \frac{1}{\rho}\, d\theta\, d\rho - I_4 \\ &= D^2 \int_r^R \frac{\theta_2 - \theta_1}{\rho}\, d\rho - I_4 \\ &\geq D^2 \int_r^R \frac{\pi}{\rho}\, d\rho - D^2 \int_r^R \frac{C\rho}{\rho}\, d\rho - I_4 \\ &= D^2 \pi \ln\left(\frac{R}{r}\right) - D^2 C(R-r) - I_4. \end{aligned}$$

As in case (i) we use

$$I_4 = \int_{A_{r,\varepsilon^\gamma}(x_0)} \frac{D^2(1-f^2)}{\rho^2}\, dx + \int_{A_{\varepsilon^\gamma,R}(x_0)} \frac{D^2(1-f^2)}{\rho^2}\, dx = I_5 + I_6.$$

Proceeding as before

$$I_5 \leq D^2 \left(\int_{A_{r,\varepsilon^\gamma}(x_0)} \frac{1}{\rho^4}\, dx \right)^{1/2} \left(\int_{A_{r,\varepsilon^\gamma}(x_0)} (1-f^2)^2\, dx \right)^{1/2}$$

and since $\theta_2 - \theta_1 \leq 2\pi$, $r \geq \varepsilon^s \geq \varepsilon$

$$\int_{A_{r,\varepsilon^\gamma}(x_0)} \frac{1}{\rho^4}\,dx = \int_r^{\varepsilon^\gamma}\int_{\theta_1}^{\theta_2} \frac{1}{\rho^3}\,d\theta\,d\rho \leq \int_r^{\varepsilon^\gamma} \frac{2\pi}{\rho^3}\,d\rho \leq \frac{\pi}{\varepsilon^2}.$$

Then $I_5 \leq D^2\sqrt{\pi K}$ is obtained by the same estimate from case (i). Without any modifications to the arguments used before for I_6, we still have $I_6 \leq D^2\sqrt{\pi C}$. Therefore

$$I_1 \geq D^2\pi \ln\left(\frac{R}{r}\right) - C$$

<u>I_2 (i)</u>

$$\begin{aligned}
\int_{A_{r,R}(x_0)} \frac{2df^2}{\rho^2}\partial_\theta\phi\,dx &= \int_{A_{r,R}(x_0)} \frac{2d(f^2-1)}{\rho^2}\partial_\theta\phi\,dx - \int_{A_{r,R}(x_0)} \frac{2d}{\rho^2}\partial_\theta\phi\,dx \\
&= \int_{A_{r,R}(x_0)} \frac{2d(f^2-1)}{\rho^2}\partial_\theta\phi\,dx - \int_r^R\int_0^{2\pi} \frac{2d}{\rho}\partial_\theta\phi\,d\theta\,d\rho \\
&= \int_{A_{r,R}(x_0)} \frac{2d(f^2-1)}{\rho^2}\partial_\theta\phi\,dx - \int_r^R \frac{2d(\phi(\rho,2\pi)-\phi(\rho,0))}{\rho}\,d\rho \\
&= \int_{A_{r,R}(x_0)} \frac{2d(f^2-1)}{\rho^2}\partial_\theta\phi\,dx.
\end{aligned}$$

Applying Young's inequality

$$\begin{aligned}
|I_2| &\leq 2\int_{A_{r,R}(x_0)} \frac{|d||1-f^2|}{\rho}\left|\frac{1}{\rho}\partial_\theta\phi\right|\,dx \\
&\leq 4\int_{A_{r,R}(x_0)} \frac{d^2|1-f^2|^2}{\rho^2}\,dx + \frac{1}{4}\int_{A_{r,R}(x_0)} |\nabla\phi|^2\,dx.
\end{aligned}$$

Using the fact that $|1-f^2|^2 \leq |1-f^2|$,

$$|I_2| \leq 4I_4 + \frac{1}{4}\int_{A_{r,R}(x_0)} |\nabla\phi|^2\,dx \leq C + \frac{1}{4}\int_{A_{r,R}(x_0)} |\nabla\phi|^2\,dx.$$

$\underline{I_2 \text{ (ii)}}$

In this estimate we utilize the bound

$$|\phi(\rho,\theta_2)-\phi(\rho,\theta_1)| \le 2C\left(\sum_{x\in\partial\Gamma^{\pm}_{\rho}} |u_\perp(\rho,\theta_i(\rho))| + \rho\right)$$

on $\Gamma^{\pm}_{r,R}$ which follows directly from Proposition 4.4.

$$\begin{aligned}
\int_{A_{r,R}(x_0)} \frac{2Df^2}{\rho^2}\partial_\theta\phi\,dx &= \int_{A_{r,R}(x_0)} \frac{2D(f^2-1)}{\rho^2}\partial_\theta\phi\,dx - \int_{A_{r,R}(x_0)} \frac{2D}{\rho^2}\partial_\theta\phi\,dx \\
&= \int_{A_{r,R}(x_0)} \frac{2D(f^2-1)}{\rho^2}\partial_\theta\phi\,dx - \int_r^R \int_{\theta_1}^{\theta_2} \frac{2D}{\rho}\partial_\theta\phi\,d\theta\,d\rho \\
&= \int_{A_{r,R}(x_0)} \frac{2D(f^2-1)}{\rho^2}\partial_\theta\phi\,dx - \int_r^R \frac{2D(\phi(\rho,\theta_2)-\phi(\rho,\theta_1))}{\rho}\,d\rho \\
&\le \int_{A_{r,R}(x_0)} \frac{2D(f^2-1)}{\rho^2}\partial_\theta\phi\,dx \\
&\qquad + 4|D|C\int_{\Gamma^{\pm}_{r,R}} \frac{|\langle u, g^\perp\rangle|}{\rho}\,d\rho + \int_r^R \frac{4|D|C\rho}{\rho}\,d\rho \\
&= \int_{A_{r,R}(x_0)} \frac{2D(f^2-1)}{\rho^2}\partial_\theta\phi\,dx + 4|D|C\int_{\Gamma^{\pm}_{r,R}} \frac{|\langle u, g^\perp\rangle|}{\rho}\,d\rho + C.
\end{aligned}$$

The same methods used in case (i) can now be implemented for the first integral in the sum above:

$$\begin{aligned}
\int_{A_{r,R}(x_0)} \frac{2D(f^2-1)}{\rho^2}\partial_\theta\phi\,dx &\le 4I_4 + \frac{1}{4}\int_{A_{r,R}(x_0)} |\nabla\phi|^2\,dx \\
&\le C + \frac{1}{4}\int_{A_{r,R}(x_0)} |\nabla\phi|^2\,dx.
\end{aligned}$$

The second integral is treated as similarily, starting with breaking up the interval $(r,R) = (r,\varepsilon^\gamma] \cup (\varepsilon^\gamma, R)$:

$$\int_{\Gamma^{\pm}_{r,R}} \frac{|\langle u, g^\perp\rangle|}{\rho}\,d\rho = \int_{\Gamma^{\pm}_{r,\varepsilon^\gamma}} \frac{|\langle u, g^\perp\rangle|}{\rho}\,d\rho + \int_{\Gamma^{\pm}_{\varepsilon^\gamma,R}} \frac{|\langle u, g^\perp\rangle|}{\rho}\,d\rho.$$

For $\rho \in (r, \varepsilon^\gamma]$,

$$\begin{aligned}
\int_{\Gamma^\pm_{r,\varepsilon^\gamma}} \frac{|\langle u, g^\perp\rangle|}{\rho}\,d\rho &\leq \left(\int_{\Gamma^\pm_{r,\varepsilon^\gamma}} \frac{1}{\rho^2}\,d\rho\right)^{1/2} \left(\int_{\Gamma^\pm_{r,\varepsilon^\gamma}} \langle u, g^\perp\rangle^2\,d\rho\right)^{1/2} \\
&\leq \frac{\sqrt{2}}{\varepsilon^{s/2}} \left(\int_{\Gamma_{\varepsilon^\gamma}} \langle u, g^\perp\rangle^2\,ds\right)^{1/2} \\
&= \sqrt{2}\left(\frac{1}{\varepsilon^s}\int_{\Gamma_{\varepsilon^\gamma}} \langle u, g^\perp\rangle^2\,ds\right)^{1/2} \\
&\leq \sqrt{2K}.
\end{aligned}$$

On the remaining interval $\rho \in (\varepsilon^\gamma, R)$,

$$\begin{aligned}
\int_{\Gamma^\pm_{\varepsilon^\gamma,R}} \frac{|\langle u, g^\perp\rangle|}{\rho}\,d\rho &\leq \left(\int_{\Gamma^\pm_{\varepsilon^\gamma,R}} \frac{1}{\rho^2}\,d\rho\right)^{1/2} \left(\int_{\Gamma^\pm_{\varepsilon^\gamma,R}} \langle u, g^\perp\rangle^2\,d\rho\right)^{1/2} \\
&\leq \frac{\sqrt{2}}{\varepsilon^{\gamma/2}} \left(\int_{\Gamma_{\varepsilon^\gamma}} \langle u, g^\perp\rangle^2\,ds\right)^{1/2} \\
&= \sqrt{2}\left(\frac{1}{\varepsilon^\gamma}\int_{\Gamma} \langle u, g^\perp\rangle^2\,ds\right)^{1/2} \\
&= \sqrt{2}\left(\varepsilon^{s-\gamma}\frac{1}{\varepsilon^s}\int_{\Gamma} \langle u, g^\perp\rangle^2\,ds\right)^{1/2} \\
&\leq 2(K\varepsilon^{s-\gamma}(1+|\ln\varepsilon|))^{1/2} \\
&\leq C(K,\gamma).
\end{aligned}$$

Therefore

$$|I_2| \leq C + \frac{1}{4}\int_{A_{r,R}(x_0)} |\nabla\phi|^2\,dx.$$

<u>I_3 (i) & (ii)</u>

By assumption, $|f| \geq 1/2$ and so

$$I_3 = \int_{A_{r,R}(x_0)} f^2|\nabla\phi|^2\,dx \geq \frac{1}{4}\int_{A_{r,R}(x_0)} |\nabla\phi|^2\,dx.$$

In all cases, we estimate using

$$\int_{A_{r,R}(x_0)} |\nabla u|^2\,dx \geq I_1 - |I_2| + I_3.$$

For $B_R(x_0) \subset \Omega$

$$\int_{A_{r,R}(x_0)} |\nabla u|^2\,dx \geq 2d^2\pi \ln\left(\frac{R}{r}\right) - C(\Omega,\gamma,K)$$

and for $x_0 \in \Gamma$

$$\int_{A_{r,R}(x_0)} |\nabla u|^2\,dx \geq D^2\pi \ln\left(\frac{R}{r}\right) - C(\Omega,\gamma,K).$$

□

4.3 Winding Identities and Ball Grouping

Given the definitions of how we quantify the winding of u_ε along the boundary of bad balls, it is not completely obvious at first glance how the degrees of interior bad balls, the boundary indices of the boundary bad balls and $\mathcal{D} = \deg(g;\Gamma)$ are related. Indeed, the degree counts full 2π-rotations of u_ε along curves while the boundary index counts only π-rotations. To get a better handle on how these topological quantities are related, we proceed to view u_ε in a punctured domain.

Consider a minimizer u_ε of either (W.O.) or (S.O.) with small $\varepsilon > 0$ fixed and define the punctured domain

$$\tilde{\Omega} := \Omega \setminus \left\{ \bigcup_{i=1}^{I_\varepsilon} \overline{B_{\lambda\varepsilon}(p_{\varepsilon,i})} \cup \bigcup_{j=1}^{J_\varepsilon} \overline{B_{\lambda\varepsilon^s}(q_{\varepsilon,j})} \right\}. \tag{4.3.1}$$

That is, $\tilde{\Omega}$ is the domain for u_ε with the closures of the bad balls removed. For notational convenience in this section, we will suppress the ε subscripts and set $r = \lambda\varepsilon$, $\rho = \lambda\varepsilon^s$. Let $\tilde{\Gamma}$ be the boundary portions of $\tilde{\Omega}$ defined by

$$\tilde{\Gamma} := \Gamma \setminus \left\{ \bigcup_{j=1}^{J} \overline{B_\rho(q_j)} \right\}. \tag{4.3.2}$$

Assuming that the boundary bad ball centers $\{q_j\}_{j=1}^J \subset \Gamma$ are ordered with respect to the counterclockwise orientation around Γ, we also define

$$\tilde{\Gamma}_j := \text{connected component of } \tilde{\Gamma} \text{ linking } q_j^+(\rho) \text{ and } q_{j+1}^-(\rho) \tag{4.3.3}$$

where $q_j^+(\rho)$ and $q_{j+1}^-(\rho)$ are as in (4.1.3). In this way, we have the decomposition

$$\tilde{\Gamma} = \bigcup_{j=1}^{J} \tilde{\Gamma}_j. \tag{4.3.4}$$

Remark 4.5. *With this labeling convention, it should be understood that q_1 and q_{J+1} correspond to the same point.*

Before we give a proof of the relationship between the degrees and boundary indices, it is possible to visualize and informally justify what one should expect to see. We begin by noting that the boundary of the punctured domain $\tilde{\Omega}$ has three fundamental 'types' of boundary components. There are the circles $\partial B_r(p_i)$ from the interior bad balls, the arcs $\partial B_\rho(q_j) \cap \Omega$ from the boundary bad balls, and the curve segments $\tilde{\Gamma}$ connecting the boundary bad ball arcs. By the definition of bad balls, it is ensured that u_ε does not vanish on any of these curves. Our analysis begins on the closed contour

$$C = \tilde{\Gamma} \cup \bigcup_{j=1}^{J} (\partial B_\rho(q_j) \cap \Omega)$$

which is taken to be positively oriented, in the sense that the orientation of C matches the orientation of Γ where they coincide. Given that $|\langle u_\varepsilon, g^\perp \rangle| \leq 1/4$ on $\tilde{\Gamma}$, it is known that along each segment $\tilde{\Gamma}_j$, the phase of u_ε stays relatively close to the phase of $\pm g$ (modulo π) depending on the orientation of u_ε on that component. Moreover, since $\varepsilon > 0$ is assumed to be small, the boundary arcs $\partial B_\rho(q_j) \cap \Omega$ are small compared to $\tilde{\Gamma}$ and thus the turning behaviour of u_ε on C should be primarily governed by the turning behaviour of g. The identity $\deg(g; \Gamma) = \deg(-g; \Gamma)$ also suggests that this turning behaviour of u_ε will not depend on its orientation with respect to g on any given component $\tilde{\Gamma}_j$. Therefore, it appears as though u_ε will have a net phase of approximately $2\pi\mathcal{D}$ along $\tilde{\Gamma}$. Along the arcs, the boundary index associated to each gives the approximate net number of π-rotations made by u_ε. By the definition of the boundary index, each arc should contribute approximately $-D_j\pi$ to the net phase of u_ε on C. Thus, the total net phase of u_ε on C will be the sum

$$2\pi\mathcal{D} - \pi \sum_{j=1}^{J} D_j.$$

By degree theory, this number should be equal to the sum of the net phases

$2\pi d_i = 2\pi \deg(u_\varepsilon; \partial B_r(p_i))$ of u_ε on the circles $\partial B_r(p_i)$. Therefore, one should see

$$2\pi\mathcal{D} - \pi \sum_{j=1}^{J} D_j = 2\pi \sum_{i=1}^{I} d_i \implies \mathcal{D} = \sum_{i=1}^{I} d_i + \frac{1}{2}\sum_{j=1}^{J} D_j.$$

This vague visualization turns out to give the correct identity relating the winding behaviour of u_ε to the winding of g on Γ. Moreover, the identity holds independent of u_ε being a weak or strong orthogonal solution.

Proposition 4.6 (Winding Identity)**.** *Let u_ε be a solution of either* (W.O.) *or* (S.O.) *with associated bad ball covering $\{B_{\lambda\varepsilon}(p_{\varepsilon,i}), B_{\lambda\varepsilon^s}(q_{\varepsilon,j})\}_{1\le i\le I_\varepsilon, 1\le j\le J_\varepsilon}$. Let*

$$d_i = \deg(u_\varepsilon; \partial B_{\lambda\varepsilon}(p_{\varepsilon,i})),$$
$$D_j = \operatorname{ind}(u_\varepsilon; \partial B_{\lambda\varepsilon^s}(q_{\varepsilon,j}) \cap \Omega),$$

be the degrees and boundary indices for u_ε about its interior and boundary bad balls respectively. Then

$$\mathcal{D} = \sum_{i=1}^{I_\varepsilon} d_i + \frac{1}{2}\sum_{j=1}^{J_\varepsilon} D_j. \tag{4.3.5}$$

Proof. Continuing with the short-form notation, we let $r = \lambda\varepsilon$ and $\rho = \lambda\varepsilon^s$. Suppose the boundary bad ball centers $\{q_j\}_{j=1}^{J_\varepsilon} \subset \Gamma$ are ordered with respect to the counterclockwise orientation around Γ and let $\tilde{\Omega}$, $\tilde{\Gamma}$ and $\tilde{\Gamma}_j$ be as in (4.3.1), (4.3.2) and (4.3.3) respectively. We also use the notation

$$C_j = \partial B_\rho(q_j) \cap \Omega$$

to represent the arcs created by the boundary bad balls. Since $|u_\varepsilon| \ge 1/2$ on the punctured domain $\tilde{\Omega}$, we may define the normalization

$$v_\varepsilon := \frac{u_\varepsilon}{|u_\varepsilon|} \in H^1(\tilde{\Omega}; \mathbb{S}^1).$$

Recall that we may write v_ε locally in $\tilde{\Omega}$ via the lifting φ,

$$v_\varepsilon = e^{i\varphi},$$

and likewise, we can write g locally on Γ via the lifting γ so that $g = e^{i\gamma}$.

Remark 4.6. *It is worth pointing out that even though φ can be defined only* *locally in* $\tilde{\Omega}$*, since* $\nabla v_\varepsilon = \nabla e^{i\varphi} = ie^{i\varphi}\nabla\varphi$*, we have*

$$\nabla\varphi = \overline{iv_\varepsilon}\nabla v_\varepsilon$$

which indicates that $\nabla\varphi$ *is actually globally defined on* $\tilde{\Omega}$*.*

Along each component $\tilde{\Gamma}_j$, we have an observation similar to that of (4.1.2), namely that for every $j = 1, \ldots, J_\varepsilon$,

$$\varphi - \gamma = c_j(x) \mod \pi \text{ on } \tilde{\Gamma}_j$$

where c_j is a correction function for the phase of u_ε satisfying $|c_j| \le \pi/6$ when u_ε is a weak orthogonal solution. If u_ε is a strong orthogonal solution, one can take $c_j = 0$ for all $j = 1, \ldots, J_\varepsilon$ for the remainder of this proof. Note that to accommodate for this slightly new notation for the correction function, Definition 4.2 for the boundary index of u_ε on $\partial B_\rho(q_j) \cap \Omega$ can be rewritten as

$$D_j = \frac{1}{\pi}\left[(\gamma(q_j^+) + c_j(q_j^+)) - (\gamma(q_j^-) + c_{j-1}(q_j^-)) - \int_{C_j} \partial_\tau\varphi \, ds\right]$$

where we are using

$$\begin{aligned} q_j^+ &= \overline{(B_\rho(q_j) \cap \Omega)} \cap \tilde{\Gamma}_j, \\ q_j^- &= \overline{(B_\rho(q_j) \cap \Omega)} \cap \tilde{\Gamma}_{j-1}, \end{aligned}$$

and the convention $c_0 = c_J$ on $\tilde{\Gamma}_J$.

Since v_ε is $\mathbb{S}^1$-valued on $\tilde{\Omega}$, the total degree of v_ε along the boundary $\partial\tilde{\Omega}$ is zero and thus

$$\begin{aligned} \int_{\partial\tilde{\Omega}} \partial_\tau\varphi \, ds &= \int_{\tilde{\Gamma}} \partial_\tau\varphi \, ds + \sum_{j=1}^{J_\varepsilon} \int_{C_j} \partial_\tau\varphi \, ds - \sum_{i=1}^{I_\varepsilon} \int_{\partial B_r(p_i)} \partial_\tau\varphi \, ds \\ &= I_1 + I_2 - 2\pi \sum_{i=1}^{I_\varepsilon} d_i \\ &= 0. \end{aligned} \tag{4.3.6}$$

Along the connected components $\tilde{\Gamma}_j$, the gradient of φ satisfies

$$\nabla\varphi = \nabla\gamma + \nabla c_j$$

and therefore integral I_1 of (4.3.6) can be written

$$I_1 = \int_\Gamma \partial_\tau \gamma \, ds - \sum_{j=1}^{J_\varepsilon} \int_{\Gamma_\rho(q_j)} \partial_\tau \gamma \, ds + \sum_{j=1}^{J_\varepsilon} \int_{\tilde{\Gamma}_j} \partial_\tau c_j \, ds$$

where decomposition (4.3.4) was used for $\tilde{\Gamma}$ in the last integral. Computing each integral leads to the sums

$$I_1 = 2\pi\mathcal{D} - \sum_{j=1}^{J_\varepsilon}(\gamma(q_j^+) - \gamma(q_j^-)) + \sum_{j=1}^{J_\varepsilon}(c_j(q_{j+1}^-) - c_j(q_j^+)).$$

Using the definition of D_j, integral I_2 from (4.3.6) can be written

$$\begin{aligned} I_2 &= \sum_{j=1}^{J_\varepsilon}[(\gamma(q_j^+) + c_j(q_j^+)) - (\gamma(q_j^-) + c_{j-1}(q_j^-))] - \pi \sum_{j=1}^{J_\varepsilon} D_j \\ &= \sum_{j=1}^{J_\varepsilon}(\gamma(q_j^+) - \gamma(q_j^-)) + \sum_{j=1}^{J_\varepsilon}(c_j(q_j^+) - c_{j-1}(q_j^-)) - \pi \sum_{j=1}^{J_\varepsilon} D_j. \end{aligned}$$

Summing I_1 and I_2 then leads to

$$\begin{aligned} I_1 + I_2 &= 2\pi\mathcal{D} + \sum_{j=1}^{J_\varepsilon}(c_j(q_{j+1}^-) - c_j(q_j^+)) + \sum_{j=1}^{J_\varepsilon}(c_j(q_j^+) - c_{j-1}(q_j^-)) - \pi \sum_{j=1}^{J_\varepsilon} D_j \\ &= 2\pi\mathcal{D} + \sum_{j=1}^{J_\varepsilon}(c_j(q_{j+1}^-) - c_{j-1}(q_j^-)) - \pi \sum_{j=1}^{J_\varepsilon} D_j \\ &= 2\pi\mathcal{D} - \pi \sum_{j=1}^{J_\varepsilon} D_j. \end{aligned}$$

Substituting this back into (4.3.6) and dividing through by 2π gives the desired identity. □

It is clear that equation (4.3.5) should be viewed as a *global* identity, in the sense that it relates all degrees and boundary indices of u_ε to one another via the degree of g on Γ. However, this raises the question as to whether or not there is a sense in which interior degrees and boundary indices can be added *locally*. It is well-known from standard degree theory that this can be done easily for groupings of interior bad balls. More precisely, suppose there is some

$x_0 \in \Omega$ and $R > 0$ such that the ball $\mathscr{B}_R(x_0)$ satisfies

$$\overline{\mathscr{B}_R(x_0)} \cap \Gamma = \varnothing, \quad \bigcup_{i \in \mathcal{I}} B_{\lambda\varepsilon}(p_{\varepsilon,i}) \subset \mathscr{B}_R(x_0)$$

where $\mathcal{I}$ is a nonempty set of indices for interior bad balls. We assume also that $R > 0$ is chosen so that $\overline{\mathscr{B}_R(x_0)}$ does not intersect the closure of any other bad ball (of interior or boundary type). Then it is easy to show

$$\deg(u_\varepsilon; \partial\mathscr{B}_R(x_0)) = \sum_{i \in \mathcal{I}} \deg(u_\varepsilon; \partial B_{\lambda\varepsilon}(p_{\varepsilon,i})).$$

In the lemma to follow, we prove that a similar result can be shown for when $\mathscr{B}_R$ is taken to be a ball centered at some point located on the boundary. However, in this case we allow for $\mathscr{B}_R$ to contain both interior and boundary bad balls. In this way, the lemma provides the sense in which degrees and boundary indices can be added locally.

Lemma 4.7. *Let $\mathcal{I}$ and $\mathcal{J}$ be sets of indices for a collection of interior bad balls $\{B_{\lambda\varepsilon}(p_{\varepsilon,i})\}_{i\in\mathcal{I}}$ and boundary ball balls $\{B_{\lambda\varepsilon^s}(q_{\varepsilon,j})\}_{j\in\mathcal{J}}$ respectively. The index sets may or may not be empty. Suppose there is a point $y_0 \in \Gamma$ and radius $R > 0$ such that the ball $\mathscr{B}_R(y_0)$ satisfies*

$$\left(\bigcup_{i \in \mathcal{I}} B_{\lambda\varepsilon}(p_{\varepsilon,i}) \cup \bigcup_{j \in \mathcal{J}} B_{\lambda\varepsilon^s}(q_{\varepsilon,j}) \right) \subset \mathscr{B}_R(y_0)$$

and $\overline{\mathscr{B}_R(y_0)}$ does not intersect the closure of any other bad ball (of interior or boundary type). Then if $\mathscr{D} = \operatorname{ind}(u_\varepsilon; \partial\mathscr{B}_R(y_0) \cap \Omega)$, $d_i = \deg(u_\varepsilon; \partial B_{\lambda\varepsilon}(p_{\varepsilon,i}))$ and $D_j = \operatorname{ind}(u_\varepsilon; \partial B_{\lambda\varepsilon^s}(q_{\varepsilon,j}) \cap \Omega)$,

$$\mathscr{D} = \sum_{j \in \mathcal{J}} D_j + 2 \sum_{i \in \mathcal{I}} d_i. \tag{4.3.7}$$

Proof. Due to the geometry of the problem, if $\mathcal{J}$ is nonempty, we may assume the centers of the boundary bad balls $\{q_{\varepsilon,j}\}_{j\in\mathcal{J}}$ are ordered with respect to the positive orientation of Γ. Suppose $|\mathcal{J}| = J$ and assume $R > 0$ is chosen large enough such that the ordered endpoints $\{y_0^-, y_0^+\}$ of the arc $\partial\mathscr{B}_R(y_0) \cap \Omega$ satisfy

$$|y_0^- - q_1^-| > 0, \quad |y_0^+ - q_J^+| > 0.$$

Note that this adjustment can be made since all bad balls have been shown to have positive distance between them. Define the punctured half-ball

$$\tilde{\mathscr{B}}_R(y_0) := \mathscr{B}_R(y_0) \setminus \left\{ \bigcup_{i\in\mathcal{I}} \overline{B_{\lambda\varepsilon}(p_{\varepsilon,i})} \cup \bigcup_{j\in\mathcal{J}} \overline{B_{\lambda\varepsilon^s}(q_{\varepsilon,j})} \right\}$$

and let

$$\tilde{\Gamma} := \left(\Gamma \cap \overline{\mathscr{B}_R(y_0)}\right) \setminus \left\{ \bigcup_{j=1}^{J} \overline{B_{\lambda\varepsilon^s}(q_j)} \right\}.$$

With this, we can define

$$\tilde{\Gamma}_j = \begin{cases} \text{connected component of } \tilde{\Gamma} \text{ linking } y_0^- \text{ and } q_1^- & \text{if } j = 0, \\ \text{connected component of } \tilde{\Gamma} \text{ linking } q_j^+ \text{ and } q_{j+1}^- & \text{if } 1 \le j \le J-1, \\ \text{connected component of } \tilde{\Gamma} \text{ linking } q_J^+ \text{ and } y_0^+ & \text{if } j = J, \end{cases}$$

so that $\tilde{\Gamma} = \cup_{j=0}^{J} \tilde{\Gamma}_j$. Finally, we also define notation for the circular arcs

$$C_j := \partial B_{\lambda\varepsilon^s}(q_j) \cap \Omega, \quad \tilde{C} = \partial \tilde{\mathscr{B}}_R(y_0) \cap \Omega.$$

As we've done before, we let $v_\varepsilon = e^{i\varphi} \in H^1(\tilde{\mathscr{B}}_R(y_0); \mathbb{S}^1)$ be the local representation of the normalization of u_ε on $\tilde{\mathscr{B}}_R(y_0)$. The total degree of v_ε on $\partial\tilde{\mathscr{B}}_R(y_0)$ is zero and so

$$\begin{aligned} \int_{\partial\tilde{\mathscr{B}}_R(y_0)} \partial_\tau \varphi \, ds &= \sum_{j=0}^{J} \int_{\tilde{\Gamma}_j} \partial_\tau \varphi \, ds + \sum_{j=1}^{J} \int_{C_j} \partial \varphi \, ds \\ &\quad - \int_{\tilde{C}} \partial_\tau \varphi \, ds - \sum_{i\in\mathcal{I}} \int_{\partial B_{\lambda\varepsilon}(p_i)} \partial_\tau \varphi \, ds \\ &= \sum_{j=0}^{J} \int_{\tilde{\Gamma}_j} \partial_\tau \varphi \, ds + \sum_{j=1}^{J} \int_{C_j} \partial \varphi \, ds - \int_{\tilde{C}} \partial_\tau \varphi \, ds - \sum_{i\in\mathcal{I}} 2\pi d_i \\ &= I_1 + I_2 - I_3 - \sum_{i\in\mathcal{I}} 2\pi d_i \\ &= 0. \end{aligned}$$

The sum of integrals represented by I_1 is treated the same way as done in Proposition 4.6:

$$\begin{aligned} I_1 = (\gamma(q_1^-) + c_0(q_1^-)) - (\gamma(y_0^-) + c_0(y_0^-)) + (\gamma(y_0^+) + c_J(y_0^+)) \\ - (\gamma(q_J^+) + c_J(q_J^+)) + \sum_{j=1}^{J-1} [(\gamma(q_{j+1}^-) + c_j(q_{j+1}^-)) - (\gamma(q_j^+) + c_j(q_j^+))] \end{aligned}$$

where c_j is the corresponding correction function for the relative phase of v_ε with respect to $\pm g$ on $\tilde{\Gamma}_j$ and γ is the lifting of $g = e^{i\gamma}$ on $\Gamma_R(y_0)$. For I_2, we employ the definition of boundary index to obtain

$$I_2 = \sum_{j=1}^{J}[(\gamma(q_j^+) + c_j(q_j^+)) - (\gamma(q_j^-) + c_{j-1}(q_j^-))] - \pi \sum_{j=1}^{J} D_j.$$

With $\mathscr{D}$ representing the boundary index for u_ε on $\partial\mathscr{B}_R(y_0)\cap\Omega$, the remaining integral I_3 is written

$$-I_3 = \pi\mathscr{D} + (\gamma(y_0^-) + c_0(y_0^-)) - (\gamma(y_0^+) + c_J(y_0^+)).$$

Upon summing, it is easy to see that the phase components cancel and we are left with

$$I_1 + I_2 - I_3 = \pi\mathscr{D} - \pi \sum_{j=1}^{J} D_j.$$

Substituting this back into the original sum gives

$$\pi\mathscr{D} - \pi \sum_{j=1}^{J} D_j - \sum_{i\in\mathcal{I}} 2\pi d_i = 0$$

and therefore dividing through by π leads to (4.3.7). □

4.4 A Global Lower Bound for the Energy

In this next lemma, we employ the local result of Theorem 4.5 to develop a lower bound for the Dirichlet energy on $\mathcal{S}_\sigma$ as defined in (3.3.1), which in turn can be used to find a uniform upper bound for the energy on the set

$$\Omega_\sigma := \Omega \setminus \mathcal{S}_\sigma. \tag{4.4.1}$$

Before we begin, note that since the set $\mathcal{S}_\sigma$ is a cover for S_{ε_n} we have $|u_{\varepsilon_n}| \geq 1/2$ on $\partial B_\sigma(p_i)$ for all $i = 1, \ldots, I$. Therefore we may define

$$d_i := \deg(u_{\varepsilon_n}; \partial B_\sigma(p_i)).$$

Similarly, since $|u_{\varepsilon_n}| \geq 1/2$ on $\partial B_\sigma(q_j) \cap \Omega$ and $|\langle u_{\varepsilon_n}, g^\perp\rangle| \leq 1/4$ on $\Gamma_\sigma(q_j) \cap \partial B_\sigma(q_j)$ for all $j = 1, \ldots, J$, the function u_{ε_n} has a boundary index

$$D_j := \operatorname{ind}(u_{\varepsilon_n}; \partial B_\sigma(q_j) \cap \Omega).$$

Lemma 4.8. *Suppose ε_n is the subsequence taken in Proposition 3.6. There exists a constant C, independent of ε_n and σ such that:*

$$G_\varepsilon(u_{\varepsilon_n}; B_\sigma(p_i)) \geq \pi|d_i| \ln\left(\frac{\sigma}{\varepsilon_n}\right) - C, \quad i = 1, \ldots, I, \tag{4.4.2}$$

$$G_\varepsilon(u_{\varepsilon_n}; B_\sigma(q_j) \cap \Omega) \geq \frac{\pi}{2}|D_j| \ln\left(\frac{\sigma^s}{\varepsilon_n^s}\right) - C, \quad j = 1, \ldots, J. \tag{4.4.3}$$

The heart of the proof for this lemma comes from a result developed by Sandier [37] (and could also be done as in Jerrard [25]) which employs properties of the logarithmic lower bound seen in Theorem 4.5. In short, the method involves a two-step dynamic approach where balls containing subsets of S_ε are grown and merged in such a way where the energy on these balls can be estimated from below while maintaining the natural scale ε. Moreover, this method is independent of u_ε being a minimizer.

One of the main differences between this work and Sandier are details surrounding the boundary. The work done in [37] assumes Dirichlet boundary conditions and thus one does not obtain boundary vortices there. In the present case, boundary vortices are plausible and thus some extra care needs to be taken when one performs the expansion/fusion argument of Sandier. Much of the heavy lifting required for these new considerations are nicely explained in [5] and [4]. In these papers, there is an added layer of complexity since the interior bad balls have radii of order ε while the boundary bad balls have radii of order ε^s for $s \in (0, 1]$ as in this work. For the purposes of completeness, we present a proof of a relaxed variation of [4, Lemma 7.1] where we only need to consider bad balls of order ε for both the interior and boundary to account for the strong orthogonality condition.

Proof. For ε_n small enough, we may assume by Proposition 3.4 that each bad ball center $p_{\varepsilon_n,i}$ and $q_{\varepsilon_n,j}$ are within a distance of $\sigma/8$ of their respective limits. We begin the proof inside interior σ-balls.

Fix $i = 1, \ldots, I$ and consider $B_\sigma(p_i)$ which contains $K \geq 1$ disjoint bad interior balls of radius $\lambda\varepsilon$. Define for each bad ball B^k, $k = 1, \ldots, K$, a time varying radius $R_k(t) = t\lambda\varepsilon$ so that $R_k(1)$ is the original radius of B^k. Observe also that each B^k carries a well-defined degree $\tilde{d}_k = \deg(u_\varepsilon; \partial B^k)$ since $|u_\varepsilon| \geq 1/2$ on ∂B^k.

<u>Step 1 - Initial Bad Ball Expansion:</u>

We now observe the process of growing the radii of each ball $B^k(1)$ continuously in time and estimating the energy over the union of these expanded balls from below. Since all $B^k(1)$ are well-separated, there is a closed interval $[1, t_1]$ such that $\overline{B^k(t)} \cap \overline{B^{k'}(t)} = \varnothing$ for all $t \in [1, t_1)$, $\forall k \neq k'$ and $\overline{B^k(t_1)} \cap \overline{B^k(t_1)} \neq \varnothing$ for at least some ball pair $k \neq k'$. That is, the right endpoint $t = t_1$ corresponds to the first instance in time where at least two of the growing balls touch. By Theorem 4.5, the energy over the collection of balls can be estimated from below using the contained annuli $\cup_{k=1}^{K} A_{R_k(1),R_k(t)}(p_{\varepsilon,k}) = \cup_{k=1}^{K} B^k(t) \setminus \overline{B^k(1)}$

$$G(u_\varepsilon; \cup_{k=1}^{K} B^k(t)) \geq G(u_\varepsilon; \cup_{k=1}^{K} B^k(t) \setminus \overline{B^k(1)}) \geq \pi \sum_{k=1}^{K} \tilde{d}_k^{\,2} \ln\left(\frac{R_k(t)}{R_k(1)}\right) - C.$$

But since $R_k(t)/R_k(1) = t$ for all $k = 1, \ldots, K$ and

$$|d_i| = \left|\sum_{k=1}^{K} \tilde{d}_k\right| \leq \sum_{k=1}^{K} \tilde{d}_k^{\,2}$$

we have

$$G(u_\varepsilon; \cup_{k=1}^{K} B^k(t)) \geq \pi |d_i| \ln(t) - C$$

for all $t \in [1, t_1]$.

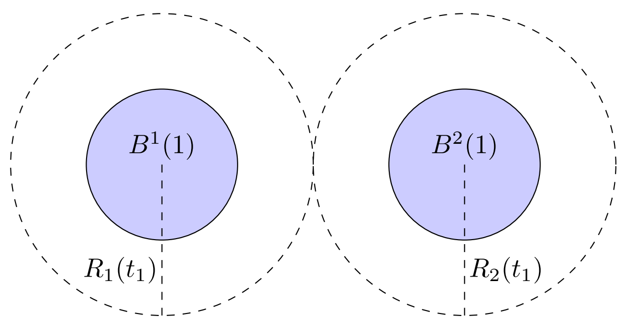


Figure 4.2: Expansion of two disjoint interior bad balls that collide at time $t = t_1$.

Step 2 - Bad Ball Fusion:

Assume at time $t = t_1$ that $M \geq 1$ clusters of M_ℓ ($\ell = 1, \ldots, M$) balls have touched. Then for each cluster there is a ball $\tilde{B}^\ell(t_1)$ of radius $\tilde{R}^\ell(t_1) = \sum_{k=1}^{M_\ell} R_k(t_1)$ which contains all M_ℓ touching balls. If any of the balls $\overline{\tilde{B}^\ell(t_1)}$ intersects the closure of any other balls (including fused balls from other clusters), we add their radii together and continue this process until all balls are well-separated. Redefine M to be the number of resulting merged/fused balls created from this process, M_ℓ the total number of original balls $B^k(t_1)$ contained in $\tilde{B}^\ell(t_1)$ and $\tilde{R}^\ell(t_1)$ the sum of their radii. The energy on $\tilde{B}^\ell(t_1)$ can be easily estimated by

$$\begin{aligned} G(u_\varepsilon; \tilde{B}^\ell(t_1)) &\geq \sum_{k=1}^{M_\ell} G(u_\varepsilon; B^k(t_1)) \\ &\geq \sum_{k=1}^{M_\ell} \tilde{d}_k^{\,2} \pi \ln(t_1) - C \\ &\geq |\deg(u_\varepsilon; \partial \tilde{B}^\ell(t_1))| \pi \ln(t_1) - C. \end{aligned}$$

Next, we utilize a fundamental observation which allows us to rewrite the lower bound on $\tilde{B}^\ell(t_1)$ above in terms of the new radius $\tilde{R}^\ell(t_1)$ and a defined "seed size" which in some sense preserves the scaling of the radii of the original disjoint balls from step 1. Using the addendo property of equal ratios

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \implies \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_1 + a_2}{b_1 + b_2}$$

we may write for every $k = 1, \ldots, M_\ell$,

$$t_1 = \frac{R_k(t_1)}{R_k(1)} = \frac{\tilde{R}^\ell(t_1)}{\tilde{r}^\ell(t_1)}$$

where $\tilde{r}^\ell(t_1) = \sum_{k=1}^{M_\ell} R_k(1) = \mathcal{O}(\varepsilon)$ and so

$$G(u_\varepsilon; \cup_{\ell=1}^M \tilde{B}^\ell(t_1)) \geq \sum_{\ell=1}^{M} |\deg(u_\varepsilon; \partial \tilde{B}^\ell(t_1))| \pi \ln\left(\frac{\tilde{R}^\ell(t_1)}{\tilde{r}^\ell(t_1)}\right) - C.$$

It is worth noting here that the seed size $\tilde{r}^\ell(t_1)$ does not correspond to an inner radius of some annulus. It is only some value of order ε that preserves the ratio in the lower bound. Now let $K_* = K - \sum_{\ell=1}^M M_\ell$ which corresponds to the number of original balls $B^k(t_1)$ that have not been included in the fusion process. The total energy on the new collection of balls $\{B^k(t_1)\}_{k=1}^{K_*} \cup \{\tilde{B}^\ell(t_1)\}_{\ell=1}^M$ is then given by

$$\begin{aligned} G\left(u_\varepsilon; \bigcup_{k=1}^{K_*} B^k(t_1) \cup \bigcup_{\ell=1}^{M} \tilde{B}^\ell(t_1)\right) &\geq \left(\sum_{k=1}^{K_*} |\tilde{d}_k| + \sum_{\ell=1}^{M} |\deg(u_\varepsilon; \partial \tilde{B}^\ell(t_1))|\right) \pi \ln(t_1) - C \\ &\geq \left(\left|\sum_{k=1}^{K_*} \tilde{d}_k + \sum_{\ell=1}^{M} \deg(u_\varepsilon; \partial \tilde{B}^\ell(t_1))\right|\right) \pi \ln(t_1) - C \\ &= \pi |d_i| \ln(t_1) - C. \end{aligned}$$

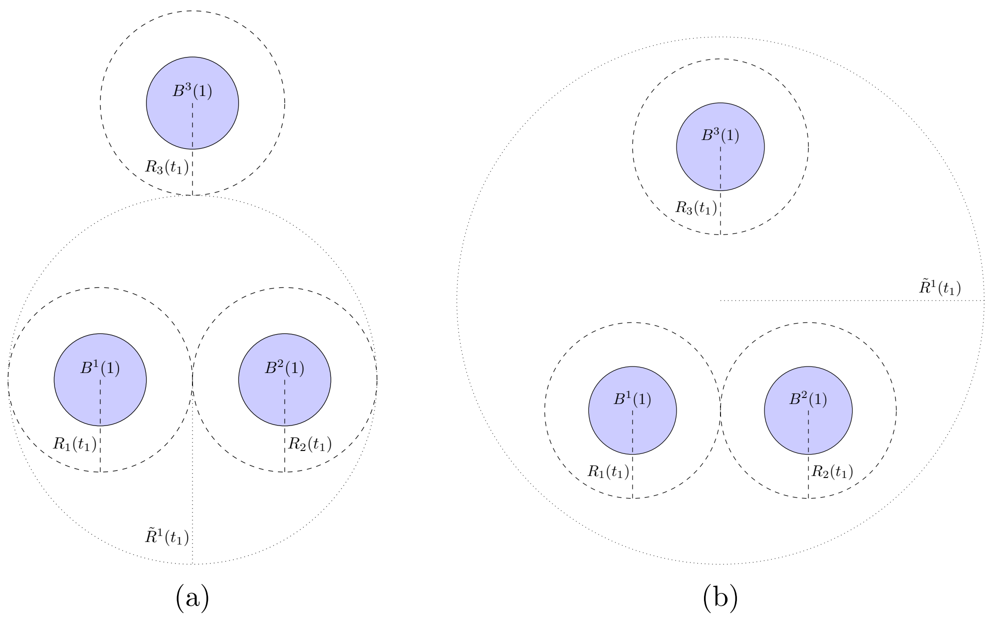


Figure 4.3: (a) A ball of radius $\tilde{R}^1(t_1)$ is created as a result of merging the expanded balls $B^1(t_1)$ and $B^2(t_1)$. This new disc interacts with another expanded bad ball $B^3(t_1)$. (b) The merged ball is refined to include all three expanded balls.

Step 3 - Repeat as needed:

The expansion process can now be performed again on the new collection of balls from above

$$\{B^k(t_1)\}_{k=1}^{K_*} \cup \{\tilde{B}^\ell(t_1)\}_{\ell=1}^{M}.$$

Let $t = t_2 > t_1$ be the first instance in time for which at least two balls from the above set make contact. By Theorem 4.5 the energy on the annular region $\tilde{B}^\ell(t) \setminus \overline{\tilde{B}^\ell(t_1)}$ has lower estimate

$$\begin{aligned} G(u_\varepsilon; \tilde{B}^\ell(t) \setminus \tilde{B}^\ell(t_1)) &\geq |\deg(u_\varepsilon; \partial\tilde{B}^\ell(t_1))|\pi \ln\left(\frac{\tilde{R}^\ell(t)}{\tilde{R}^\ell(t_1)}\right) - C \\ &= |\deg(u_\varepsilon; \partial\tilde{B}^\ell(t_1))|\pi \ln\left(\frac{t}{t_1}\right) - C \end{aligned}$$

and therefore we may estimate the energy on $\tilde{B}^\ell(t)$ for $t \in (t_1, t_2)$ by

$$\begin{aligned} G(u_\varepsilon; \tilde{B}^\ell(t)) &\geq G(u_\varepsilon; \tilde{B}^\ell(t) \setminus \overline{\tilde{B}^\ell(t_1)}) + G(u_\varepsilon; \tilde{B}^\ell(t_1)) \\ &\geq |\deg(u_\varepsilon; \partial\tilde{B}^\ell(t_1))|\pi \ln\left(\frac{t}{t_1}\right) + |\deg(u_\varepsilon; \partial\tilde{B}^\ell(t_1))|\pi \ln(t_1) - C \\ &= \pi|\deg(u_\varepsilon; \partial\tilde{B}^\ell(t_1))| \ln(t) - C. \end{aligned}$$

Therefore the energy over the set of expanded balls $\{B^k(t)\}_{k=1}^{K_*} \cup \{\tilde{B}^\ell(t)\}_{\ell=1}^{M}$ for $t \in (t_1, t_2)$ can be bounded below via

$$\begin{aligned} G\left(u_\varepsilon; \bigcup_{k=1}^{K_*} B^k(t) \cup \bigcup_{\ell=1}^{M} \tilde{B}^\ell(t)\right) &\geq \left(\sum_{k=1}^{K_*} |\tilde{d}_k| + \sum_{\ell=1}^{M} |\deg(u_\varepsilon; \partial\tilde{B}^\ell(t_1))|\right)\pi \ln(t) - C \\ &\geq \pi|d_i| \ln(t) - C \end{aligned}$$

as before. For ε small enough, we may continue the expansion/fusion process until we are left with a single ball $B(t_*) \subset B_\sigma(p_i)$ with radius $R(t_*)$ and associated seed size $r = \mathcal{O}(\varepsilon)$ where t_* is the time such that $R(t_*) = \sigma/2$. This process terminates in a finite number of steps since there are only a finite number of bad balls. Then

$$\begin{aligned} G(u_\varepsilon; B_\sigma(p_i)) &\geq G(u_\varepsilon; B(t_*)) \\ &\geq \pi|d_i| \ln(t_*) - C \\ &= \pi|d_i| \ln\left(\frac{R}{r}\right) - C \\ &\geq \pi|d_i| \ln\left(\frac{\sigma}{\varepsilon}\right) - C. \end{aligned}$$

Now we describe the same process for balls $B_\sigma(q_j) \cap \Omega$ centered on the boundary. In what follows, for the sake of space, we interpret $B_\sigma(q_j)$ to mean $B_\sigma(q_j) \cap \Omega$. Fix $j = 1, \ldots, J$ and let $\mathcal{I}$ denote the set of indices for which B^i, $i \in \mathcal{I}$ are the interior balls of radius $\lambda\varepsilon$ contained in $B_\sigma(q_j)$. Similarly, let $\mathcal{K}$ be the set of indices for which $\mathcal{B}^k$, $k \in \mathcal{K}$ are the boundary balls of radius $\lambda\varepsilon$ contained in $B_\sigma(q_j)$. As before, defined is a time varying radius $R_i(t) = t\lambda\varepsilon$ so that $R_i(1)$ is the original radius of B^i. Similarly we define $\mathcal{R}_k(t) = t\lambda\varepsilon$ for the balls $\mathcal{B}^k$. For each B^i there is a well-defined degree $\tilde{d}_i = \deg(u_\varepsilon; \partial B^i)$ and for each $\mathcal{B}^k$ a boundary index $\tilde{D}_k = \mathrm{ind}(u_\varepsilon; \partial\mathcal{B}^k)$ since $|u_\varepsilon| \geq 1/2$ on $\partial B^i \cup \partial\mathcal{B}^k$. The boundary index D_j associated to $B_\sigma(q_j)$ can be calculated (by Lemma

4.7)

$$D_j = \sum_{k\in\mathcal{K}} \tilde{D}_k + 2\sum_{i\in\mathcal{I}} \tilde{d}_i.$$

Step 1 - Initial Bad Ball Expansion:

During the initial expansion, there is a time $t = t_1 > 1$ such that either:

(i) at least two of the bad balls make contact and $\overline{B^i(t_1)} \cap \Gamma = \varnothing$ for all $i \in \mathcal{I}$,

(ii) all bad balls are well-separated but $\overline{B^i(t_1)} \cap \Gamma \neq \varnothing$ for at least one $i \in \mathcal{I}$,

(iii) at least two of the bad balls make contact and there is at least one $i \in \mathcal{I}$ such that $\overline{B^i(t_1)} \cap \Gamma \neq \varnothing$. We assume here that either $\overline{B^i(t_1)}$ is disjoint from the cluster of bad balls which make contact, or the time $t = t_1$ corresponds to a "double collision", where $\overline{B^i(t_1)}$ simultaneously contacts Γ and a bad ball cluster. In other words, the distance from the center of $B^i(t_1)$ to the cluster and the distance from the center to the boundary are equal.

In any of these cases, consider the annular regions

$$\left(\bigcup_{i\in\mathcal{I}} B^i(t) \setminus \overline{B^i(1)}\right) \cup \left(\bigcup_{k\in\mathcal{K}} \mathcal{B}^k(t) \setminus \overline{\mathcal{B}^k(1)}\right)$$

for $t \in [1, t_1]$. Using the fact that $R_i(t)/R_i(1) = \mathcal{R}_k(t)/\mathcal{R}_k(1) = t$ for all $i \in \mathcal{I},\ k \in \mathcal{K}$ we can apply Theorem 4.5 to obtain

$$\begin{aligned} G\left(u_\varepsilon; \bigcup_{i\in\mathcal{I}} B^i(t) \cup \bigcup_{k\in\mathcal{K}} \mathcal{B}^k(t)\right) &\geq G\left(u_\varepsilon; \bigcup_{i\in\mathcal{I}} B^i(t) \setminus \overline{B^i(1)}\right) + G\left(u_\varepsilon; \bigcup_{k\in\mathcal{K}} \mathcal{B}^k(t) \setminus \overline{\mathcal{B}^k(1)}\right) \\ &\geq \pi \sum_{i\in\mathcal{I}} \tilde{d}_i^2 \ln\left(\frac{R_i(t)}{R_i(1)}\right) + \frac{\pi}{2}\sum_{k\in\mathcal{K}} \tilde{D}_k^2 \ln\left(\frac{\mathcal{R}_k(t)}{\mathcal{R}_k(1)}\right) - C \\ &= \frac{\pi}{2}\left(\sum_{k\in\mathcal{K}} \tilde{D}_k^2 + 2\sum_{i\in\mathcal{I}} \tilde{d}_i^2\right) \ln(t) - C \end{aligned}$$

for all $t \in [1, t_1]$. The sum of the squares of the interior degrees and boundary indices can also be estimated from below,

$$\sum_{k\in\mathcal{K}} \tilde{D}_k^2 + 2\sum_{i\in\mathcal{I}} \tilde{d}_i^2 \geq \left|\sum_{k\in\mathcal{K}} \tilde{D}_k + 2\sum_{i\in\mathcal{I}} \tilde{d}_i\right| = |D_j|$$

which gives the lower bound

$$G\left(u_\varepsilon; \bigcup_{i\in\mathcal{I}} B^i(t) \cup \bigcup_{k\in\mathcal{K}} \mathcal{B}^k(t)\right) \geq \frac{\pi}{2}|D_j|\ln(t) - C.$$

Step 2 - Bad Ball Fusion:

The merging process near the boundary is more complicated than that of the interior case. To begin, we consider some base scenarios related to cases (i)-(iii) listed in step 1 and estimate the energy from below over balls that encapsulate the original bad balls at the time of first collision $t = t_1$. After this, an argument is made to show that after bad ball fusion, a lower bound can be found that is written in terms of D_j, the boundary index associated to $B_\sigma(q_j)$.

Assume first at time $t = t_1$ there is a cluster of $M \geq 2$ interior balls that have touched for which the closures of each do not make contact with the closures of boundary balls or $\partial\Omega$. Then as before in step 2 of the interior case, there is a ball $\overline{\tilde{B}(t_1)}$ of radius $\tilde{R}(t_1) = \sum_{i=1}^M R_i(t_1)$ which contains all M touching balls. If $\tilde{B}(t_1)$ intersects the closure of any other interior ball (that also does not make contact with $\partial\Omega$ or boundary balls), we add their radii together and continue this process until the closure of $\tilde{B}(t_1)$ has empty intersection with the closures of all other interior balls. Assume at this point that $\tilde{B}(t_1)$ has empty intersection with $\partial\Omega$ or any other boundary ball. Redefine M to be the total number of original balls $B^i(t_1)$ contained in $\tilde{B}(t_1)$ and let $\tilde{R}(t_1)$ be the sum of their radii. Then if $\tilde{r}(t_1) = \sum_{i=1}^M R_i(1)$, step 2 of the interior case gives

$$\begin{aligned} G(u_\varepsilon; \tilde{B}(t_1)) &\geq \sum_{i=1}^M G(u_\varepsilon; B^i(t_1)) \\ &\geq \sum_{i=1}^M \tilde{d}_i^2 \pi \ln(t_1) - C \\ &\geq |\deg(u_\varepsilon; \partial\tilde{B}(t_1))| \pi \ln(t_1) - C \\ &= |\deg(u_\varepsilon; \partial\tilde{B}(t_1))| \pi \ln\left(\frac{\tilde{R}(t_1)}{\tilde{r}(t_1)}\right) - C. \end{aligned}$$

Next, we deal with boundary ball collision. Suppose there is a cluster of $K \geq 2$ boundary balls $\mathcal{B}^k(t_1)$, $k = 1, \ldots, K$ that touch at time $t = t_1$ for which the closures of each do not make contact with the closures of interior balls. Then we encapsulate them into another boundary ball $\tilde{\mathcal{B}}(t_1)$ of radius $\tilde{\mathcal{R}} = \sum_{k=1}^K \mathcal{R}_k(t_1)$ with associated seed size $\tilde{\mathbf{r}}(t_1) = \sum_{k=1}^K \mathcal{R}_k(1) = \mathcal{O}(\varepsilon)$. The boundary index of u_ε on this new boundary ball is calculated by

$$\mathrm{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1)) = \sum_{k=1}^{K} \tilde{D}_k.$$

If the closure of $\tilde{\mathcal{B}}(t_1)$ touches the closures of any other boundary ball, we continue to enlarge $\tilde{\mathcal{B}}(t_1)$ until it no longer touches the remaining boundary balls. As before, it is assumed at this point that $\overline{\tilde{\mathcal{B}}(t_1)}$ does not intersect the closures of any interior balls. We redefine K, $\tilde{\mathcal{R}}(t_1)$ and $\tilde{\mathbf{r}}(t_1)$ accordingly. Since $\mathcal{R}_k(t_1)/\mathcal{R}(1) = \tilde{\mathcal{R}}(t_1)/\tilde{\mathbf{r}}(t_1) = t_1$ for all $k = 1, \ldots, K$ by the addendo property of equal ratios, we arrive at an energy lower bound over $\tilde{\mathcal{B}}(t_1)$ that can be estimated by

$$\begin{aligned} G(u_\varepsilon; \tilde{\mathcal{B}}(t_1)) &\geq \sum_{k=1}^{K} G(u_\varepsilon; \mathcal{B}^k(t_1)) \\ &\geq \frac{\pi}{2} \sum_{k=1}^{K} \tilde{D}_k^2 \ln(t_1) - C \\ &\geq \frac{\pi}{2} |\,\mathrm{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1))| \ln(t_1) - C \\ &= \frac{\pi}{2} |\,\mathrm{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1))| \ln\left(\frac{\tilde{\mathcal{R}}(t_1)}{\tilde{\mathbf{r}}(t_1)}\right) - C. \end{aligned}$$

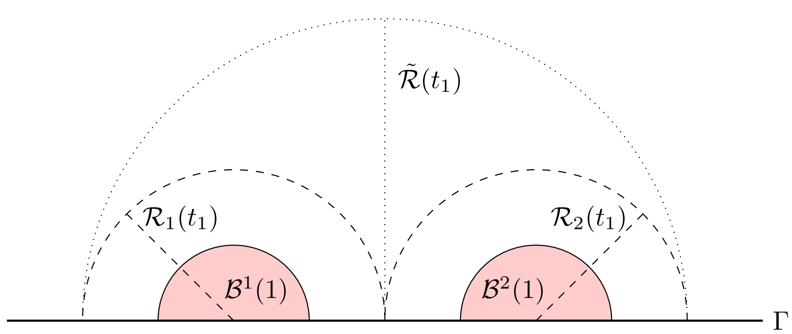


Figure 4.4: The merging of two boundary bad balls $\mathcal{B}^1(t_1)$ and $\mathcal{B}^2(t_1)$.

Next, we describe the process of a boundary ball colliding with an interior ball. Suppose there is a cluster of M interior balls $B^i(t_1)$ and K boundary balls $\mathcal{B}^k(t_1)$ that make contact at time $t = t_1$. Then a boundary ball $\tilde{\mathcal{B}}(t_1)$ is taken to enclose all M interior balls and K boundary balls. If $B^i(t_1)$ have radii

$R_i(t_1)$ and $\mathcal{B}^k(t_1)$ have radii $\mathcal{R}_k(t_1)$, we may assume that $\tilde{\mathcal{B}}(t_1)$ has radius

$$\tilde{\mathcal{R}}(t_1) = \sum_{k=1}^{K} \mathcal{R}_k(t_1) + 2\sum_{i=1}^{M} R_i(t_1)$$

and associated seed size

$$\tilde{\mathbf{r}}(t_1) = \sum_{k=1}^{K} \mathcal{R}_k(1) + 2\sum_{i=1}^{M} R_i(1).$$

If $\overline{\tilde{\mathcal{B}}(t_1)}$ intersects the closures of any other interior or boundary balls, we enlarge it until it no longer touches the remaining interior or boundary balls. As in the previous cases, all notation associated to $\tilde{\mathcal{B}}(t_1)$ is appropriately redefined to accommodate for this enlargement. The boundary index for $\tilde{\mathcal{B}}(t_1)$ can be calculated by the sum

$$\operatorname{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1)) = \sum_{k=1}^{K} \tilde{D}_k + 2\sum_{i=1}^{M} \tilde{d}_i$$

and note

$$|\operatorname{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1))| = \left|\sum_{k=1}^{K} \tilde{D}_k + 2\sum_{i=1}^{M} \tilde{d}_i\right| \le \sum_{k=1}^{K} \tilde{D}_k^2 + 2\sum_{i=1}^{M} \tilde{d}_i^2.$$

The energy over $\tilde{\mathcal{B}}(t_1)$ can be estimated from below similar to before

$$\begin{aligned} G(u_\varepsilon; \tilde{\mathcal{B}}(t_1)) &\ge \sum_{k=1}^{K} G(u_\varepsilon; \mathcal{B}^k(t_1)) + \sum_{i=1}^{M} G(u_\varepsilon; B^i(t_1)) \\ &\ge \frac{\pi}{2}\sum_{k=1}^{K} \tilde{D}_k^2 \ln(t_1) + \pi\sum_{i=1}^{M} \tilde{d}_i^2 \ln(t_1) - C \\ &= \frac{\pi}{2}\left(\sum_{k=1}^{K} \tilde{D}_k^2 + 2\sum_{i=1}^{M} \tilde{d}_i^2\right) \ln(t_1) - C \\ &\ge \frac{\pi}{2}|\operatorname{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1))| \ln(t_1) - C. \end{aligned}$$

Due to the way we are able to define the seed size for $\tilde{\mathcal{B}}(t_1)$,

$$\mathcal{R}_k(t_1)/\mathcal{R}_k(1) = 2R_i(t_1)/2R_i(1) = \tilde{\mathcal{R}}(t_1)/\tilde{\mathbf{r}}(t_1) = t_1$$

for all $i = 1, \ldots, M$ and $k = 1, \ldots, K$. That is, the ratios can still be preserved under the construction of this new boundary ball $\tilde{\mathcal{B}}(t_1)$. Therefore the lower bound on $\tilde{\mathcal{B}}(t_1)$ can be written in terms of its radius

$$G(u_\varepsilon; \tilde{\mathcal{B}}(t_1)) \geq \frac{\pi}{2} |\operatorname{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1))| \ln\left(\frac{\tilde{\mathcal{R}}(t_1)}{\tilde{\mathbf{r}}(t_1)}\right) - C.$$

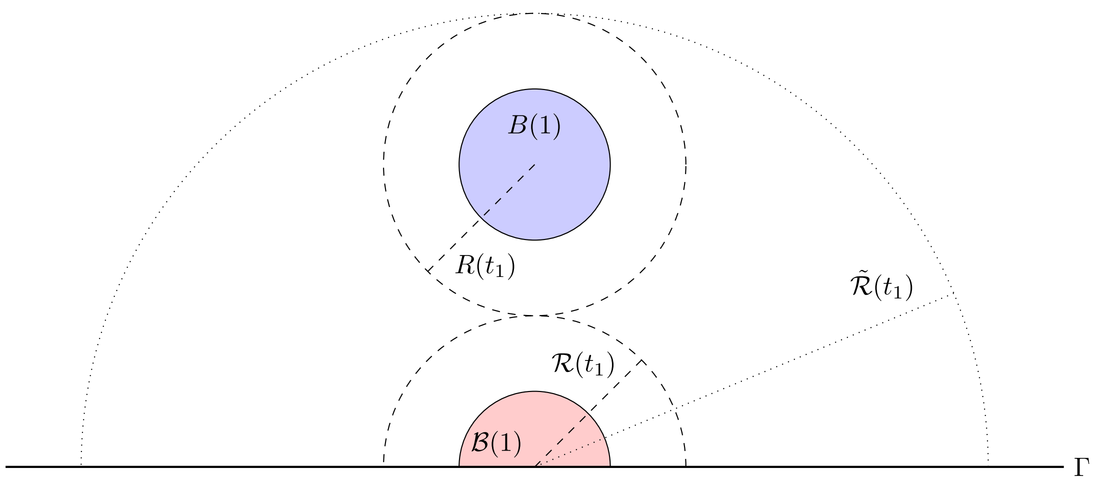


Figure 4.5: Merging of an expanded interior ball and expanded boundary ball.

Finally, we describe the process of an interior ball of degree $\tilde{d}$ colliding with $\partial\Omega$. If some interior ball $B(t_1)$ of radius $R(t_1)$ and associated seed size $r(t_1)$ makes contact with $\partial\Omega$ at time $t = t_1$, then we take some point $y \in \overline{B(t_1)} \cap \partial\Omega$ to define the center of a new boundary ball $\tilde{\mathcal{B}}(t_1)$ with radius $\tilde{\mathcal{R}}(t_1) = 2R(t_1)$ and seed size $\tilde{\mathbf{r}}(t_1) = 2r(t_1)$ that encloses $B(t_1)$. With this choice of seed size, the ratio $R(t_1)/r(t_1) = \tilde{\mathcal{R}}(t_1)/\tilde{\mathbf{r}}(t_1) = t_1$ is preserved. The boundary index for $\tilde{\mathcal{B}}(t_1)$ is

$$\operatorname{ind}(u_\varepsilon; \partial\tilde{\mathcal{B}}(t_1)) = 2\tilde{d}$$

by Lemma 4.7 and the energy on $\tilde{\mathcal{B}}(t_1)$ can be estimate from below by

$$\begin{aligned} G(u_\varepsilon;\tilde{\mathcal{B}}(t_1)) &\geq G(u_\varepsilon;B(t_1)) \\ &\geq \pi|\tilde{d}|\ln(t_1)-C \\ &= \frac{\pi}{2}|2\tilde{d}|\ln(t_1)-C \\ &= \frac{\pi}{2}|\operatorname{ind}(u_\varepsilon;\partial\tilde{\mathcal{B}}(t_1))|\ln(t_1)-C \\ &= \frac{\pi}{2}|\operatorname{ind}(u_\varepsilon;\partial\tilde{\mathcal{B}}(t_1))|\ln\left(\frac{\tilde{\mathcal{R}}(t_1)}{\tilde{\mathbf{r}}(t_1)}\right)-C. \end{aligned}$$

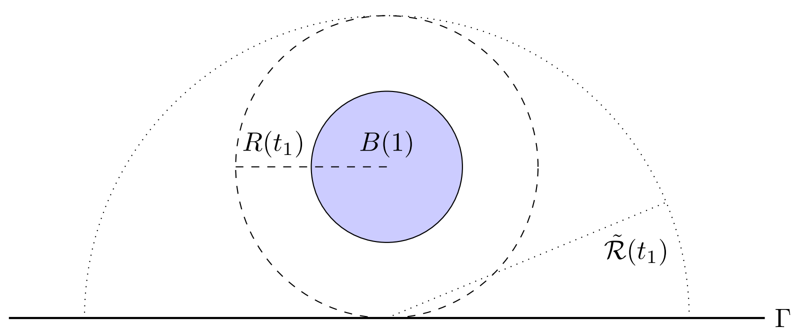


Figure 4.6: Introducing an artificial half-ball to include boundary-colliding interior balls.

Now that the base calculations and scenarios have been dealt with, the merging process can be fully explained. After the first expansion step, suppose at time $t = t_1$ there are

- M ball clusters comprised entirely of colliding interior balls (the closure of some of these balls may intersect $\partial\Omega$),
- K ball clusters comprised entirely of colliding boundary balls,
- N ball clusters comprised of interior-boundary balls (the closure of some of these interior balls may intersect $\partial\Omega$),
- P remaining disjoint interior balls (the closure of some of these balls may intersect $\partial\Omega$),
- Q remaining disjoint boundary balls.

If any of the M clusters of interior balls or P disjoint interior balls make contact with the boundary, we enclose the cluster and disjoint ball in a half-ball using the method described above. Each of the remaining interior ball clusters then get encapsulated into their own larger ball with associated energy preserving seed size. If any of these new balls touch $\partial\Omega$ but remain disjoint from boundary balls, we perform the half-ball containment again. If one of the interior clusters makes contact with another interior cluster or disjoint interior balls but remains disjoint from boundary balls, we enclose these in another interior ball with associated energy preserving seed size. This process continues until either all interior balls are isolated from the boundary and boundary balls, make contact with at least one boundary ball, or there are no isolated interior balls/clusters remaining. In any case, no interior ball and or associated cluster touches $\partial\Omega$ and we let W denote the number of boundary balls created from an interior ball coming into contact with $\partial\Omega$. The only objects left to consider are isolated boundary balls and their associated clusters, and interior-boundary ball clusters. We can then apply the appropriate half-ball containment method as needed until the closures of all clusters are disjoint. The integers M, K, N, P, Q and W are then redefined to match the quantities remaining after the merging process. It is also worth noting again that this process finishes in a finite number of steps since there are only a finite number of balls contained in $B_\sigma(q_j)$. Let

- $\{\tilde{B}^m(t_1)\}_{m=1}^M$ denote the set of interior balls from the fusion process,
- $\{\tilde{\mathcal{B}}^k(t_1)\}_{k=1}^K$ denote the set of boundary balls created from the process of pure boundary ball fusion,
- $\{\hat{\mathcal{B}}^n(t_1)\}_{n=1}^N$ denote the set of boundary balls created from the process of interior-boundary ball fusion,
- $\{\dot{\mathcal{B}}^w(t_1)\}_{w=1}^W$ denote the set of boundary balls created from interior balls interacting with $\partial\Omega$.

Using the base calculations from above, we estimate the energy from below over the region

$$\mathcal{C}_{t_1} := \bigcup_{m=1}^M \tilde{B}^m(t_1) \cup \bigcup_{k=1}^K \tilde{\mathcal{B}}^k(t_1) \cup \bigcup_{n=1}^N \hat{\mathcal{B}}^n(t_1) \cup \bigcup_{w=1}^W \dot{\mathcal{B}}^w(t_1) \cup \bigcup_{p=1}^P B^p(t_1) \cup \bigcup_{q=1}^Q \mathcal{B}^q(t_1)$$

where the last two unions are the remaining interior and boundary balls that were not included in the fusion process.

$$G(u_\varepsilon;\mathcal{C}_{t_1}) \geq \sum_{m=1}^{M} G(u_\varepsilon;\tilde{B}^m(t_1)) + \sum_{k=1}^{K} G(u_\varepsilon;\tilde{\mathcal{B}}^k(t_1)) + \sum_{n=1}^{N} G(u_\varepsilon;\hat{\mathcal{B}}^k(t_1)) + \sum_{w=1}^{W} G(u_\varepsilon;\dot{\mathcal{B}}^k(t_1)) + \sum_{p=1}^{P} G(u_\varepsilon;B^p(t_1)) + \sum_{q=1}^{Q} G(u_\varepsilon;\mathcal{B}^q(t_1)).$$

Applying the previous calculation to each ball type, the righthand side yields the bound

$$G(u_\varepsilon;\mathcal{C}_{t_1}) \geq \frac{\pi}{2}\left(2\sum_{m=1}^{M} |\deg(u_\varepsilon;\partial\tilde{B}^m(t_1))| + \sum_{k=1}^{K} |\operatorname{ind}(u_\varepsilon;\partial\tilde{\mathcal{B}}^k(t_1))| + \sum_{n=1}^{N} |\operatorname{ind}(u_\varepsilon;\partial\hat{\mathcal{B}}^k(t_1))| + \sum_{w=1}^{W} |\operatorname{ind}(u_\varepsilon;\partial\dot{\mathcal{B}}^k(t_1))| + 2\sum_{p=1}^{P} |\tilde{d}_p| + \sum_{q=1}^{Q} |\tilde{D}_q|\right)\ln(t_1) - C.$$

Since all of these balls are contained in $B_\sigma(q_j)$, the triangle inequality gives that the sum of degrees and boundary indices above is an upper bound for $|D_j|$. Therefore

$$G(u_\varepsilon;\mathcal{C}_{t_1}) \geq \frac{\pi}{2}|D_j|\ln(t_1) - C.$$

Step 3 - Repeat as needed:

The expansion process can now be performed again on the new collection of balls $\mathcal{C}_{t_1}$. Let $t = t_2 > t_1$ be the first instance in time for which at least one of the cases (i)-(iii) from step 1 are observed. The trick for obtaining the improved lower bound on

$$\mathcal{C}_t := \bigcup_{m=1}^{M} \tilde{B}^m(t) \cup \bigcup_{k=1}^{K} \tilde{\mathcal{B}}^k(t) \cup \bigcup_{n=1}^{N} \hat{\mathcal{B}}^n(t) \cup \bigcup_{w=1}^{W} \dot{\mathcal{B}}^w(t) \cup \bigcup_{p=1}^{P} B^p(t) \cup \bigcup_{q=1}^{Q} \mathcal{B}^q(t)$$

for $t \in [t_1, t_2]$ is identical to the interior case. That is, we consider the annular regions $\mathcal{C}_t \setminus \overline{\mathcal{C}_{t_1}}$ and apply Theorem 4.5 to obtain

$$G(u_\varepsilon;\mathcal{C}_t \setminus \overline{\mathcal{C}_{t_1}}) \geq \frac{\pi}{2}|D_j|\ln\left(\frac{t}{t_1}\right) - C$$

for all $t \in [t_1, t_2]$. Then

$$\begin{aligned} G(u_\varepsilon; \mathcal{C}_t) &\geq G(u_\varepsilon; \mathcal{C}_t \setminus \overline{\mathcal{C}_{t_1}}) + G(u_\varepsilon; \mathcal{C}_{t_1}) \\ &\geq \frac{\pi}{2}|D_j| \ln\left(\frac{t}{t_1}\right) + \frac{\pi}{2}|D_j| \ln(t_1) - C \\ &= \frac{\pi}{2}|D_j| \ln(t) - C \end{aligned}$$

for all $t \in [t_1, t_2]$. For ε small enough, we may continue the expansion/fusion process until we are left with a single half-ball $\mathcal{B}(t_*) \subset B_\sigma(q_j)$ with radius $\mathcal{R}(t_*)$ and associated seed size $\mathbf{r} = \mathcal{O}(\varepsilon)$ where t_* is the time such that $\mathcal{R}(t_*) = \sigma/2$. This process terminates in a finite number of steps since there are only a finite number of bad balls. Then

$$\begin{aligned} G(u_\varepsilon; B_\sigma(q_j)) &\geq G(u_\varepsilon; \mathcal{B}(t_*)) \\ &\geq \frac{\pi}{2}|D_j| \ln(t_*) - C \\ &= \frac{\pi}{2}|D_j| \ln\left(\frac{\mathcal{R}}{\mathbf{r}}\right) - C \\ &\geq \frac{\pi}{2}|D_j| \ln\left(\frac{\sigma}{\varepsilon}\right) - C. \end{aligned}$$

which finishes the proof. $\square$

Chapter 5

Convergence and Locating Vortices

In this chapter, we conclude our discussion on solutions of (W.O.) and (S.O.) by showing the existence of a limiting $\mathbb{S}^1$-valued harmonic map u_0 which is obtained along a subsequence $\varepsilon_n \to 0$. In the context of liquid crystals, u_0 can be interpreted as the 'classic' director to which we were previously introduced to in Section 1.1. However, the map will have a finite number of point singularities. The heavy lifting for this convergence result is primarily taken by the global lower bound found in the previous chapter. In fact, there are several easily obtained corollaries derived from Lemma 4.8 which paves the way for u_0. We will begin by proving these corollaries, but we note that many of the techniques follow the standard procedure originally developed in [10, Chapter VI] and [39].

5.1 Convergence Along a Subsequence

By Lemma 4.8, we find that for appropriately taken subsequences, a lower bound on the bad set cover $\mathcal{S}_\sigma$ is found to be

$$G_\varepsilon^W(u_{\varepsilon_n}; \mathcal{S}_\sigma) \geq \pi \left(\sum_{i=1}^{I} |d_i| + \frac{s}{2} \sum_{j=1}^{J} |D_j| \right) |\ln \varepsilon_n| - C \tag{5.1.1}$$

where $C > 0$ is a constant independent of ε. It is this bound that will be pivotal in obtaining our results.

Remark 5.1. *As noted in [4, Remark 7.3], it is clear from* (5.1.1) *that the* σ*-balls which constitute* $\mathcal{S}_\sigma$ *satisfying*

$$\deg(u_\varepsilon; \partial B_\sigma(p_i)) = 0 \quad \textit{or} \quad \operatorname{ind}(u_\varepsilon; \partial B_{\sigma^s}(q_j) \cap \Omega) = 0$$

do not have any meaningful contribution to the energy, and thus can be seen to belong to the set where u_{ε_n} converges. With this, we may omit all such balls with trivial degree and boundary indices.

Definition 5.1. *Let*

$$\Sigma := \{p_1, \ldots, p_I\} \cup \{q_1, \ldots, q_J\}$$

denote the collection of non-trivial vortex centers from $\mathcal{S}_\sigma$.

The first corollary we prove shows that the degrees d_i and boundary indices D_j can be taken to be independent of ε along subsequences. Moreover, we can show that all degrees and boundary indices associated to $\mathcal{S}_\sigma$ must be of the same sign.

Corollary 5.2. *Let $s \in (0, 1]$. Then along a subsequence, the degrees d_i, $i = 1, \ldots I$, and the boundary indices D_j, $j = 1, \ldots, J$, are constant in ε. Moreover, each d_i and D_j are positive.*

Proof. By pairing inequality (5.1.1) with the upper bound of Proposition 2.4, we obtain a constant $C > 0$ independent of ε such that

$$\sum_{i=1}^{I} |d_i| + \frac{s}{2} \sum_{j=1}^{J} |D_j| \leq s\mathcal{D} + \frac{C}{|\ln \varepsilon_0|}$$

and thus each d_i, D_j is bounded uniformly in ε. Upon taking subsequences, we may assume each d_i and D_j to be constant in ε. To show that each integer in the sum is positive, we utilize identity (4.3.5) to get

$$\sum_{i=1}^{I} |d_i| + \frac{s}{2} \sum_{j=1}^{J} |D_j| \geq s \left| \sum_{i=1}^{I} d_i + \frac{1}{2} \sum_{j=1}^{J} D_j \right| = s|\mathcal{D}| = s\mathcal{D}.$$

On the other hand, we have again by the pairing of inequality (5.1.1) and the upper bound of Proposition 2.4, a constant C independent of ε such that

$$\sum_{i=1}^{I} |d_i| + \frac{s}{2} \sum_{j=1}^{J} |D_j| \leq s\mathcal{D} + \frac{C}{|\ln \varepsilon_n|}.$$

Since each d_i and D_j are independent of ε, we take $\varepsilon_n \to 0$ to obtain the identity

$$\sum_{i=1}^{I} |d_i| + \frac{s}{2} \sum_{j=1}^{J} |D_j| = \sum_{i=1}^{I} d_i + \frac{s}{2} \sum_{j=1}^{J} D_j$$

and therefore d_i,$D_j \geq 0$ for each $i = 1, \dots, I$ and $j = 1, \dots, J$. Since the centers of σ-balls with trivial degrees or boundary indices have been ommited, this improves to d_i,$D_j > 0$ for each $i = 1, \dots, I$ and $j = 1, \dots, J$. □

We now show that the full energy can be uniformly bounded independent of ε provided we are outside the bad set cover $\mathcal{S}_\sigma$. In particular, it is shown that the energy is uniformly bounded in ε on Ω_σ as defined in (4.4.1).

Corollary 5.3. *For any $\sigma \in (0, \sigma_0)$, there exists a constant C independent of ε and σ such that*

$$G^W_{\varepsilon_n}(u_{\varepsilon_n}; \Omega_\sigma) \leq \pi s \mathcal{D} |\ln \sigma| + C.$$

Proof. By Lemma 4.8, Proposition 2.4 and using the fact that $\Omega = \Omega_\sigma \cup \mathcal{S}_\sigma$,

$$\begin{aligned} G^W_{\varepsilon_n}(u_{\varepsilon_n}; \Omega_\sigma) &= G^W_{\varepsilon_n}(u_{\varepsilon_n}; \Omega) - G^W_{\varepsilon_n}(u_{\varepsilon_n}; \mathcal{S}_\sigma) \\ &\leq \pi s \mathcal{D} |\ln \varepsilon_n| - \pi s \mathcal{D} \ln \frac{\sigma}{\varepsilon_n} + C \\ &= \pi s \mathcal{D} |\ln \sigma| + C \end{aligned}$$

for C independent of ε and σ. □

Corollary 5.4. *It holds that*

$$d_i = D_j = 1$$

for all $i = 1, \dots, I$ and $j = 1, \dots, J$.

Proof. Let $\sigma^s < R < \sigma_0^s$. Then

$$\mathcal{A} := \bigcup_{i=1}^{I} A_{\sigma,R}(p_i) \cup \bigcup_{j=1}^{J} A_{\sigma^s,R}(q_j) \subset \Omega_\sigma$$

for all $\sigma^s < R$ and the anuli in the union above are mutually disjoint. Applying Theorem 4.5 to each of the annuli, there is a constant C depending only on R such that

$$\frac{1}{2} \int_{\mathcal{A}} |\nabla u_\varepsilon|^2 \, dx \geq \pi \left(\sum_{i=1}^{I} d_i^2 + \frac{s}{2} \sum_{j=1}^{J} D_j^2 \right) |\ln \sigma| - C.$$

By the upper bound of Corollary 5.3, there is a constant C, depending only on R such that

$$\sum_{i=1}^{I} (d_i^2 - d_i) + \frac{s}{2} \sum_{j=1}^{J} (D_j^2 - D_j) \leq \frac{C}{|\ln \sigma|}.$$

Fixing R, we may choose σ small enough so that, $C/|\ln\sigma| < s/4$. Therefore

$$d_i^2 - d_i < \frac{s}{4} \leq \frac{1}{4}$$

for each $i = 1, \dots, I$ and

$$D_j^2 - D_j < \frac{1}{2}$$

for each $j = 1, \dots, J$. Since d_i and D_j are positive integers, it must be that $d_i = D_j = 1$ for all indices. □

Within the proof of Corollary 5.2, we showed

$$\sum_{i=1}^{I} d_i + \frac{s}{2}\sum_{j=1}^{J} D_j = s\mathcal{D}. \tag{5.1.2}$$

This equation gives yet another interesting fact about the behaviour of solutions u_ε of (W.O.). Namely, that for $0 < s < 1$, one will always have precisely $2\mathcal{D}$ non-trivial boundary vortices and zero non-trivial interior vortices.

Corollary 5.5. *If $0 < s < 1$ in the weak orthogonality problem, then $\Sigma \subset \Gamma$ and $|\Sigma| = 2\mathcal{D}$.*

Proof. By (5.1.2),

$$s\mathcal{D} = \sum_{i=1}^{I} d_i + \frac{s}{2}\sum_{j=1}^{J} D_j = (1-s)\sum_{i=1}^{I} d_i + s\mathcal{D}$$

and therefore it must be that $I = 0$. Since each $D_j = 1$ by Corollary 5.4, we have again by (5.1.2)

$$\frac{1}{2}\sum_{j=1}^{J} D_j = \frac{J}{2} = \mathcal{D}$$

giving $J = I + J = |\Sigma| = 2\mathcal{D}$ as desired. □

In the proof of Lemma 4.8, recall that the lower bound was developed for the Dirichlet energy for u_ε and did not depend on the boundary energy nor the norm penalizing term in the interior. Due to this, we actually have a uniform bound on these terms in the energy.

Corollary 5.6. *There exists a constant C independent of ε such that*

$$\frac{1}{4\varepsilon_n^2}\int_\Omega \left(1 - |u_{\varepsilon_n}|^2\right)^2 dx + \frac{W}{2\varepsilon_n^s}\int_\Gamma \langle u_{\varepsilon_n}, g^\perp\rangle^2 \, ds \leq C.$$

Proof. By Lemma 4.8, Proposition 2.4 and Corollary 5.2, we have

$$\begin{aligned}\frac{1}{4\varepsilon_n^2}\int_\Omega \left(1-|u_{\varepsilon_n}|^2\right)^2 dx + \frac{W}{2\varepsilon_n^s}\int_\Gamma \langle u_{\varepsilon_n}, g^\perp\rangle^2\, ds &= G_{\varepsilon_n}^W(u_{\varepsilon_n}) - \frac{1}{2}\int_\Omega |\nabla u_{\varepsilon_n}|^2\, ds \\ &\le G_{\varepsilon_n}^W(u_{\varepsilon_n}) - \frac{1}{2}\int_{\mathcal{S}_\sigma} |\nabla u_{\varepsilon_n}|^2\, ds \\ &\le \pi s\mathcal{D}|\ln\varepsilon_n| - \pi s\mathcal{D}|\ln\varepsilon_n| + C \\ &= C\end{aligned}$$

for C independent of ε. ☐

By stringing together the above corollaries and following the final remarks of [39], Lemma 5.7 below is a direct result of following the procedure outlined by [10, Chapter VI].

Lemma 5.7. *There exists* $u_0 \in H^1(\Omega\setminus\Sigma;\mathbb{R}^2)$ *such that along a subsequence* $\varepsilon_n \to 0$ *we have*

$$u_{\varepsilon_n} \rightharpoonup u_0 \quad \text{weakly in } H^1_{loc}(\overline{\Omega}\setminus\Sigma;\mathbb{R}^2)$$

for solutions of (W.O.) *or* (S.O.). *Moreover,* u_0 *is an* $\mathbb{S}^1$*-valued harmonic map satisfying*

$$\deg(u_0;\partial B_r(p_i)) = d_i = 1, \quad \operatorname{ind}(u_0;\partial B_r(q_j)\cap\Omega) = D_j = 1$$

for all $i = 1,\dots,I$, $j = 1,\dots,J$.

Thus, by combining Lemma 5.7 and the above corollaries, we have proved Theorem 1.1.

5.2 An Example of Strong Orthogonality on a Disc

In view of equation (5.1.2), we saw that for values of $s \in (0,1)$, weakly orthogonal solutions have exactly $2\mathcal{D}$ non-trivial boundary vortices and zero non-trivial interior vortices. In the special case where $s = 1$ (whether u_ε is a solution of (W.O.) or (S.O.)), equation (5.1.2) does not provide any immediate information regarding the allocation of interior and boundary vortices. For example, if the boundary function g has $\mathcal{D} = \deg(g;\Gamma) = 1$, then equation (5.1.2) does not see any difference between the energy associated to a single interior vortex with degree $d = 1$, or two boundary vortices with associated boundary indices $D = 1$. Therefore, it is possible that the geometry of Ω plays a role in determining whether interior or boundary vortices are energetically preferable.

One of the most pivotal results coming from the pioneering work of [10] was the discovery and development of a special real-valued function called the *renormalized energy*, which answers questions about the *precise* location of the vortices for the limiting harmonic map u_0 in Ω. In deriving an asymptotic expansion for the energy of minimizers, the renormalized energy function is a natural byproduct of this process and is found to be dependent only on point configurations of Ω. In particular, it is shown in [10] (in the case of Dirichlet boundary data) that the renormalized energy $W : \Omega^{\mathcal{D}} \to \mathbb{R}$ is minimized when the configuration input is equal to the $\mathcal{D}$-tuple of vortices $(a_1, \ldots, a_{\mathcal{D}})$ for u_0.

To study the critical case where $s = 1$ more closely, we restrict our attention to strong orthogonal solutions on the unit disc with tangential forcing on the boundary. In other words, we take $\Omega = B_1(0)$ and set $g = \tau$, the positively oriented tangent vector to Γ. In doing this, the simple geometry of Ω will provide us with the ability to find a closed-form solution for W and thus, brings us closer to determining energetically preferable vortex configurations in the critical case where equation (5.1.2) is not of any use.

Since $\mathcal{D} = \deg(\tau; \Gamma) = 1$, equation (5.1.2) in combination with Corollary 5.4 implies that there are only two possibilities for defect locations. The first possibility is to have one and only one defect located in the interior of Ω, while the second is to have exactly two boundary defects. Each of these potential configurations are considered below.

Case 1: One Interior Defect

Let p denote the defect point in Ω. As there are no defects along $\Gamma = \partial B_1(0)$ we may assume $u_0 = \tau$ on Γ. Following in the style of [3, Section 6] and [36] we observe the conjugate function Φ_p which satisfies the PDE

$$\begin{cases} \Delta \Phi_p = 2\pi \delta_p(x) & \text{in } \Omega, \\ \dfrac{\partial \Phi_p}{\partial n} = g \times \partial_\tau g & \text{on } \Gamma. \end{cases}$$

The renormalized energy in this case takes the form

$$W(p) = \lim_{\rho \to 0} \left(\frac{1}{2} \int_{\Omega_\rho} |\nabla \Phi_p|^2 \, dx - \pi \ln \left(\frac{1}{\rho} \right) \right)$$

where $\Omega_\rho = \Omega \setminus B_\rho(p)$ and the energy has associated asymptotic expansion

$$G_\varepsilon(u_\varepsilon; \Omega) = \pi |\ln \varepsilon| + W(p) + Q_\Omega + o(1) \tag{5.2.1}$$

where Q_Ω is the vortex core energy associated to p as defined in [10, Lemma

IX.1]. Following [10, Theorem VIII.6], it can be shown that the renormalized energy W satisfies

$$\min_{p\in\Omega} W(p) = W(0) = 0 \tag{5.2.2}$$

and therefore $u_0 \in H^1_{loc}(\Omega \setminus \{0\}; \mathbb{S}^1)$.

Case 2: Two Boundary Defects

In this scenario, let q_1 and q_2 denote the defect points along Γ and consider the solution Φ_q to

$$\begin{cases} \Delta\Phi_q = 0 & \text{in } \Omega, \\ \dfrac{\partial\Phi_q}{\partial n} = g \times \partial_\tau g - \pi(\delta_{q_1}(x) + \delta_{q_2}(x)) & \text{on } \Gamma. \end{cases}$$

The associated renormalized energy and asymptotic expansion maintain the same form as in the interior case, namely

$$W(q_1, q_2) = \lim_{\rho\to 0} \left(\frac{1}{2} \int_{\Omega_\rho} |\nabla\Phi_q|^2 \, dx - \pi \ln\left(\frac{1}{\rho}\right) \right),$$

$$G(u_\varepsilon; \Omega) = \pi|\ln\varepsilon| + W(q_1, q_2) + 2Q_\Gamma + o(1), \tag{5.2.3}$$

except now Q_Γ represents the vortex core energy for a boundary defect. An explicit solution to the PDE for Φ_q is

$$\Phi_q(x) = \ln|x - q_1| + \ln|x - q_2|.$$

To see this, note that Φ_q is the sum of scaled fundamental solutions to the Laplacian with defect points located on the boundary. Thus, Φ_q is harmonic in Ω. On the other hand, observe that since $g = \tau$ we have $g \times \partial_\tau g = 1$ and

$$\begin{aligned} \frac{\partial\Phi_q}{\partial n} &= \langle \nabla\Phi_q, n\rangle \\ &= \sum_{j=1}^{2} \frac{1}{|x - q_j|^2} \langle x - q_j, x\rangle \\ &= \sum_{j=1}^{2} \frac{|x|^2 + |q_j|^2 - 2\langle x, q_j\rangle}{2|x - q_j|^2} \\ &= \sum_{j=1}^{2} \frac{|x - q_j|^2}{2|x - q_j|^2} \\ &= 1 \end{aligned}$$

on $\Gamma \setminus \{q_1, q_2\}$.

We now derive an expression for $W(q_1, q_2)$. Using integration by parts,

$$\int_{\Omega_\rho} |\nabla \Phi_q|^2 \, dx = \sum_{j=1}^{2} \int_{\partial B_\rho(q_j)\cap\Omega} \Phi_q \frac{\partial \Phi_q}{\partial n_{q_j}} ds + \int_{\Gamma\setminus(\Gamma_\rho(q_1)\cup\Gamma_\rho(q_2))} \Phi_q \frac{\partial \Phi_q}{\partial n} ds$$

where

$$n_{q_j} = -\frac{x - q_j}{|x - q_j|}$$

are the outward unit normal vectors to the respective curves $\partial B_\rho(q_j) \cap \Omega$, $j = 1, 2$. On each of these curves,

$$\begin{aligned}
\frac{\partial \Phi_q}{\partial n_{q_j}} &= \langle \nabla \Phi_q, n_{q_j} \rangle \\
&= -\frac{1}{|x - q_j|^2} \left\langle x - q_j, \frac{x - q_j}{|x - q_j|} \right\rangle - \frac{1}{|x - q_{j'}|^2} \left\langle x - q_{j'}, \frac{x - q_j}{|x - q_j|} \right\rangle \\
&= -\frac{1}{|x - q_j|} - \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_j||x - q_{j'}|^2}
\end{aligned}$$

where $j \neq j'$. Multiplying $\partial \Phi_q / \partial n_{q_j}$ by $\ln |x - q_j|$, on $\partial B_\rho(q_j) \cap \Omega$ we obtain

$$\begin{aligned}
-\left(\frac{1}{|x - q_j|} + \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_j||x - q_{j'}|^2} \right) \ln |x - q_j| = {} & \frac{\ln\left(\frac{1}{\rho}\right)}{\rho} \\
& + \frac{\ln\left(\frac{1}{\rho}\right)}{\rho} \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_{j'}|^2}
\end{aligned}$$

and

$$\frac{1}{2} \int_{\partial B_\rho(q_j)\cap\Omega} \frac{\ln\left(\frac{1}{\rho}\right)}{\rho} ds = \frac{\ln\left(\frac{1}{\rho}\right)}{2\rho} |\partial B_\rho(q_j) \cap \Omega| = \frac{\ln\left(\frac{1}{\rho}\right)}{2\rho} \Theta(\rho)\rho = \frac{\Theta(\rho)}{2} \ln\left(\frac{1}{\rho}\right)$$

where $\Theta(\rho) \to \pi$ as $\rho \to 0$. By Cauchy-Schwarz,

$$\left| \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_{j'}|^2} \right| \leq \frac{|x - q_{j'}|\rho}{|x - q_{j'}|^2} = \frac{\rho}{|x - q_{j'}|} \leq c\rho$$

where $c = \min_{x \in \overline{B_{\rho_0}(q_j) \cap \Omega}} |x - q_{j'}|$ for some fixed $\rho_0 > 0$ and therefore

$$\begin{aligned}
\frac{\ln\left(\frac{1}{\rho}\right)}{2\rho} \left| \int_{\partial B_\rho(q_j) \cap \Omega} \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_{j'}|^2} \, ds \right| &\le \frac{\ln\left(\frac{1}{\rho}\right)}{2\rho} c\rho |\partial B_\rho(q_j) \cap \Omega| \\
&= \frac{c}{2} \Theta(\rho) \rho \ln\left(\frac{1}{\rho}\right).
\end{aligned}$$

In this way, we conclude

$$\lim_{\rho \to 0} \int_{\partial B_\rho(q_j) \cap \Omega} -\frac{\langle x - q_{j'}, x - q_j \rangle \ln |x - q_j|}{|x - q_j||x - q_{j'}|^2} \, ds = 0.$$

Multiplying $\partial \Phi_q / \partial n_{q_j}$ by $\ln |x - q_{j'}|$, on $\partial B_\rho(q_j) \cap \Omega$ we get

$$\begin{aligned}
-\left(\frac{1}{|x - q_j|} + \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_j||x - q_{j'}|^2} \right) \ln |x - q_{j'}| = &-\frac{\ln |x - q_{j'}|}{\rho} \\
&- \frac{\ln |x - q_{j'}|}{\rho} \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_{j'}|^2}.
\end{aligned}$$

Integrating the first term, observe by the uniform continuity of $\ln |x - q_{j'}|$ on $\overline{B_{\rho_0}(q_j) \cap \Omega}$ that for all $\epsilon > 0$ there is $0 < \rho < \rho_0$ such that

$$|\ln |x - q_{j'}| - \ln |q_j - q_{j'}|| < \epsilon$$

for all $x \in \partial B_\rho(q_j) \cap \Omega$. Therefore

$$\begin{aligned}
\frac{1}{2} \int_{\partial B_\rho(q_j) \cap \Omega} -\frac{\ln |x - q_{j'}|}{\rho} \, ds = &-\frac{1}{2\rho} \int_{\partial B_\rho(q_j) \cap \Omega} (\ln |x - q_{j'}| - \ln |q_j - q_{j'}|) \, ds \\
&- \frac{1}{2\rho} \int_{\partial B_\rho(q_j) \cap \Omega} \ln |q_j - q_{j'}| \, ds \\
= &-\frac{1}{2\rho} \int_{\partial B_\rho(q_j) \cap \Omega} (\ln |x - q_{j'}| - \ln |q_j - q_{j'}|) \, ds \\
&- \frac{\Theta(\rho)}{2} \ln |q_j - q_{j'}|
\end{aligned}$$

and

$$\left| \frac{1}{2\rho} \int_{\partial B_\rho(q_j) \cap \Omega} (\ln |x - q_{j'}| - \ln |q_j - q_{j'}|) \, ds \right| \le \frac{\Theta(\rho)}{2} \epsilon \le \pi \epsilon.$$

Since this estimate holds for arbitrary $\epsilon > 0$ we conclude

$$\lim_{\rho \to 0} -\frac{1}{\rho} \int_{\partial B_\rho(q_j) \cap \Omega} (\ln |x - q_{j'}| - \ln |q_j - q_{j'}|) \, ds = 0$$

and so

$$\lim_{\rho \to 0} \frac{1}{2} \int_{\partial B_\rho(q_j) \cap \Omega} -\frac{\ln |x - q_{j'}|}{\rho} \, ds = -\frac{\pi}{2} \ln |q_j - q_{j'}|.$$

Applying Cauchy-Schwarz to the remaining term,

$$\left| \frac{\ln |x - q_{j'}|}{\rho} \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_{j'}|^2} \right| \leq \frac{\ln |x - q_{j'}|}{|x - q_{j'}|} \leq \tilde{c}$$

where $\tilde{c} = \max_{x \in \overline{B_{\rho_0}(q_j) \cap \Omega}} \ln |x - q_{j'}| / |x - q_{j'}|$. Then

$$\left| \frac{1}{2} \int_{\partial B_\rho(q_j) \cap \Omega} \frac{\ln |x - q_{j'}|}{\rho} \frac{\langle x - q_{j'}, x - q_j \rangle}{|x - q_{j'}|^2} \, ds \right| \leq \frac{\tilde{c} \rho \Theta(\rho)}{2} \to 0 \text{ as } \rho \to 0.$$

Putting these limits together,

$$\lim_{\rho \to 0} \left(\sum_{j=1}^{2} \frac{1}{2} \int_{\partial B_\rho(q_j) \cap \Omega} \Phi_q \frac{\partial \Phi_q}{\partial n_{q_j}} ds - \pi \ln \left(\frac{1}{\rho} \right) \right) = -\pi \ln |q_1 - q_2|.$$

Finally, we show that

$$\lim_{\rho \to 0} \int_{\Gamma \setminus (\Gamma_\rho(q_1) \cup \Gamma_\rho(q_2))} \Phi_q \frac{\partial \Phi_q}{\partial n} ds = \lim_{\rho \to 0} \int_{\Gamma \setminus (\Gamma_\rho(q_1) \cup \Gamma_\rho(q_2))} \Phi_q \, ds = 0$$

to conclude $W(q_1, q_2) = -\pi \ln |q_1 - q_2|$. First, observe that $\Phi_q \in L^1(\Gamma)$ and in fact

$$\int_\Gamma \Phi_q \, ds = 0.$$

To see this, we note that by symmetry it is enough to consider $q_j = (1, 0)$ for example. Then on $\mathbb{S}^1$

$$\begin{aligned}
\int_\Gamma \ln |x - q_j| \, ds &= \int_0^{2\pi} \ln \left(\sqrt{(\cos(\theta) - 1)^2 + \sin^2(\theta)} \right) d\theta \\
&= \int_0^{\pi} \ln (2 - 2\cos(\theta)) \, d\theta \\
&= 0.
\end{aligned}$$

Remark 5.2. *The last equality above comes from the formula*

$$\int_0^\pi \ln(a \pm b\cos(t))\, dt = \pi \ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right), \quad a \geq b$$

which can be found in a table of integrals (see [26, Appendix E] for example.)

Next, let

$$\chi_\rho = \chi(\Gamma \setminus (\Gamma_\rho(q_1) \cup \Gamma_\rho(q_2)))$$

denote the characteristic function for the set $\Gamma \setminus (\Gamma_\rho(q_1) \cup \Gamma_\rho(q_2))$ so that

$$\int_{\Gamma\setminus(\Gamma_\rho(q_1)\cup\Gamma_\rho(q_2))} \Phi_q\, ds = \int_\Gamma \Phi_q \chi_\rho\, ds.$$

Since $|\Phi_q\chi_\rho| \leq |\Phi_q|$ for all $\rho > 0$ and $\Phi_q\chi_\rho \to \Phi_q$ pointwise almost everywhere on Γ as $\rho \to 0$, the Lebesgue Dominated Convergence theorem yields

$$\lim_{\rho\to 0} \int_{\Gamma\setminus(\Gamma_\rho(q_1)\cup\Gamma_\rho(q_2))} \Phi_q\, ds = \int_\Gamma \Phi_q\, ds = 0.$$

Therefore the renormalized energy in the case of boundary vortices takes the form

$$W(q_1, q_2) = -\pi \ln |q_1 - q_2|.$$

Thus, $W(q_1, q_2)$ is minimized whenever q_1 and q_2 are antipodal. In particular, recalling (5.2.2) allows us to conclude

$$\min_{q_1,q_2\in\Gamma} W(q_1, q_2) = -\pi \ln |q_1 - (-q_1)| = -\pi \ln |2q_1| < 0 = \min_{p\in\Omega} W(p)$$

which suggests that a pair of boundary vortices may be the energetically favourable configuration. To come to that conclusion one would also need to take into account the constants Q_Ω and Q_Γ appearing in the asymptotic expansions (5.2.1) and (5.2.3) of the energy, in the case of interior and boundary vortices. These constants are associated to the core energy of each defect, that is, the energy associated with the defect profile function at length scale ε. For an interior vortex, the value of Q_Ω may be calculated from the equivariant solution (see [38], for example). For a boundary defect, if we blow-up at scale ε and pass to a formal limit we expect to converge to a problem in the half-plane, with tangential anchoring conditions. The unique solution in that case should be the same symmetric vortex, with exactly half of the core energy of an interior vortex, and so we conjecture that the core energy is $Q_\Gamma = \frac{1}{2}Q_\Omega$. Making this argument rigorous will involve some estimates such as those of [5]. This is discussed in more detail in the context of open problems in Section 7.1.

Chapter 6

Accounting for Molecular Bend and Splay

In the introduction of this work, we derived the one-constant approximation to the Oseen-Frank energy from the functional

$$\tilde{\mathscr{F}}_{OF}(\mathbf{d}) = \frac{1}{2}\int_{\Omega}(k_s(\operatorname{div}\mathbf{d})^2 + k_b(\operatorname{curl}\mathbf{d})^2)\,dx, \quad \mathbf{d} \in \mathbb{S}^1,$$

via identity (1.2.1) where we assumed the bend modulus k_b and splay modulus k_s for the nematic sample were equal to some constant $k > 0$. As noted in remark 1.1, a physically realistic model would ideally regard k_b and k_s as functions of the spacial variables, however, this greatly complicates any mathematical analysis to be done. One way to split the difference between these two extremes is to consider a situation in which the moduli are non-equal constants. That is, k_s and k_b are each positive constants but satisfy

$$k_b > k_s > 0 \quad \text{or} \quad k_s > k_b > 0.$$

To transform this problem into something that looks more familiar, it is easy to verify (see Proposition B.3) that for constant, unequal moduli, the integrand of $\tilde{\mathscr{F}}_{OF}$ can be written similarly to identity (1.2.1). Indeed, assume $k_s \neq k_b$ and define the quantities

$$\tilde{k} := \min\{k_s, k_b\}, \quad \hat{k} := \max\{k_s, k_b\} \quad \text{and} \quad \kappa := \hat{k} - \tilde{k}.$$

Then

$$k_s(\operatorname{div}\mathbf{d})^2 + k_b(\operatorname{curl}\mathbf{d})^2 = \tilde{k}|\nabla\mathbf{d}|^2 + h_{\tilde{k}}(\mathbf{d}) + 2\tilde{k}\det(\nabla\mathbf{d})$$

where

$$h_{\tilde{k}}(\mathbf{d}) := \begin{cases} \kappa(\operatorname{div}\mathbf{d})^2 & \text{if } \tilde{k} = k_b, \\ \kappa(\operatorname{curl}\mathbf{d})^2 & \text{if } \tilde{k} = k_s. \end{cases}$$

Through this identity, Colbert-Kelly and Philips [14] study minimizers of $\tilde{\mathscr{F}}_{OF}$ with $\mathbb{S}^1$-relaxation and Dirichlet boundary data $g \in C^3(\Gamma; \mathbb{S}^1)$, in particular,

$$\begin{cases} \inf\limits_{u \in H^1_g(\Omega;\mathbb{R}^2)} \mathscr{F}_\varepsilon(u), \\ \mathscr{F}_\varepsilon(u) := \dfrac{1}{2} \displaystyle\int_\Omega \left(\tilde{k}|\nabla u|^2 + h_{\tilde{k}}(u) + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right) dx. \end{cases}$$

Remark 6.1. *As expected, we see that for* $k_b > k_s$ *there is a penalization on the molecular bend of the nematic sample and similarly we observe a splay penalization in the case* $k_s > k_b$.

As shown in [14], it is seen that H^1_g-minimizers for $\mathscr{F}_\varepsilon$ still yield strictly interior vortices with the addition of these new energy terms. An interesting question to consider then, is to ask how these new energies interact with a system that permits boundary vortices. In this chapter, we analyze this question by returning to Moser's original minimization problem (1.3.3) and altering the interior energy accordingly but keeping the boundary energy to simplify the problem.

Let n denote the outward unit normal vector to Γ and τ the positively oriented unit tangent tangent vector to Γ. The main focus of this chapter is to study the behaviour of minimizers for the following problems:

$$\begin{cases} \inf\limits_{u \in H^1(\Omega;\mathbb{R}^2)} \mathscr{F}^W_\varepsilon(u), \\ \mathscr{F}^W_\varepsilon(u) := \mathscr{F}_\varepsilon(u) + \dfrac{W}{2\varepsilon^s} \displaystyle\int_\Gamma \langle u, n\rangle^2 \, ds, \end{cases} \tag{W.O.*}$$

where $s \in (0, 1]$, $W \in (0, +\infty)$ and

$$\begin{cases} \inf\limits_{u \in \mathcal{H}_\tau(\Omega)} \mathscr{F}_\varepsilon(u), \\ \mathcal{H}_\tau(\Omega) := \big\{ u \in H^1(\Omega; \mathbb{R}^2) : \langle u, n\rangle = 0 \text{ on } \Gamma \big\}. \end{cases} \tag{S.O.*}$$

The minimization problem (W.O.*) carries the same interpretation as (W.O.), except now we have specified $g^\perp = n$ and have included the additional elastic terms to penalize molecular bend and splay accordingly. As done in the case of (S.O.), we will interpret (S.O.*) as a limiting problem for (W.O.*) as the orthogonality weighting W tends to $+\infty$ for fixed $\varepsilon > 0$ and $s \in (0, 1]$.

6.1 Existence & Euler-Lagrange Equations

Just as in the existence proof for (W.O.) and (S.O.) (see Lemma 2.1), the existence of minimizers for (W.O.*) and (S.O.*) can be obtained via the direct method from the calculus of variations. The only difference here is the accommodations needed in the weak H^1-convergence for minimizing sequences involving the term $h_{\tilde{k}}(u)$, which follows from the known weak convergence of the gradient term.

The Euler-Lagrange equations for problems (W.O.*) and (S.O.*), for the most part, are fundamentally different to those found in the previous chapters. In deriving the Euler-Lagrange systems below, we will find that the associated system of PDEs for $\mathscr{F}_\varepsilon$ and $\mathscr{F}_\varepsilon^W$ are coupled in the vector components of u_ε via second order derivatives, which is certainly not the case for (W.O.) and (S.O.). This fact, unfortunately, will force us into imposing an additional hypothesis on minimizers in order to continue fruitfully. This hypothesis will be given in Section 6.3. Nevertheless, we begin by deriving the Euler-Lagrange system for $\mathscr{F}_\varepsilon$ and $\mathscr{F}_\varepsilon^W$.

(W.O.*) Case

$\underline{\tilde{k} = k_b}$

Let $v \in H^1(\Omega;\mathbb{R}^2)$. Finding the first variation of $\mathscr{F}_\varepsilon^W$, we get

$$\begin{aligned}\frac{d}{dt}\mathscr{F}_\varepsilon^W(u+tv)\bigg|_{t=0} &= \int_\Omega \left(\tilde{k}\sum_{i,j} u^i_{x_j}v^i_{x_j} + \kappa(\operatorname{div} u)(\operatorname{div} v) - \frac{1}{\varepsilon^2}(1-|u|^2)\langle u, v\rangle\right)dx \\ &\quad + \frac{W}{\varepsilon^s}\int_\Gamma \langle\langle u, n\rangle n, v\rangle\, ds \\ &= 0\end{aligned}$$

for all $v \in H^1(\Omega;\mathbb{R}^2)$. Assuming sufficient regularity for integrating by parts, the first term becomes

$$\int_\Omega \tilde{k}\sum_{i,j} u^i_{x_j}v^i_{x_j}\, dx = \int_\Gamma \tilde{k}\langle\partial_n u, v\rangle\, ds - \int_\Omega \tilde{k}\langle\Delta u, v\rangle\, dx.$$

Using integration by parts again on the second term,

$$\int_\Omega \kappa(\operatorname{div} u)(\operatorname{div} v)\, dx = \int_\Gamma \kappa(\operatorname{div} u)\langle n, v\rangle\, ds - \int_\Omega \kappa\langle\nabla \operatorname{div} u, v\rangle\, dx.$$

Putting these together, we obtain

$$-\int_\Omega \langle \tilde{k}\Delta u + \kappa\nabla \operatorname{div} u + \varepsilon^{-2}(1-|u|^2)u, v\rangle\, dx \\ + \int_\Gamma \langle \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n + W\varepsilon^{-s}\langle u, n\rangle n, v\rangle\, ds = 0$$

for all $v \in H^1(\Omega;\mathbb{R}^2)$ and so u satisfies

$$\begin{cases} -\tilde{k}\Delta u - \kappa\nabla \operatorname{div} u = \frac{1}{\varepsilon^2}u(1-|u|^2) & \text{in } \Omega, \\ \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n = -\frac{W}{\varepsilon^s}\langle u, n\rangle n & \text{on } \Gamma. \end{cases} \tag{6.1.1}$$

$\underline{\tilde{k} = k_s}$

The calculations in this case are identical to the former besides the curl term in the integrand of $\mathscr{F}_\varepsilon^W$. The first variation has the form

$$\begin{aligned} \frac{d}{dt}\mathscr{F}_\varepsilon^W(u+tv)\Big|_{t=0} &= \int_\Omega \left(\tilde{k}\sum_{i,j} u^i_{x_j}v^i_{x_j} + \kappa(\operatorname{curl} u)(\operatorname{curl} v) - \frac{1}{\varepsilon^2}(1-|u|^2)\langle u, v\rangle\right) dx \\ &\quad + \frac{W}{\varepsilon^s}\int_\Gamma \langle\langle u, n\rangle n, v\rangle\, ds \\ &= 0 \end{aligned}$$

holding for all $v \in H^1(\Omega;\mathbb{R}^2)$. Expanding the $\operatorname{curl} u$ term,

$$\int_\Omega \kappa(\operatorname{curl} u)(\operatorname{curl} v)\, dx = \int_\Omega \kappa(\operatorname{curl} u)v^2_{x_1}\, dx - \int_\Omega \kappa(\operatorname{curl} u)v^1_{x_2}\, dx = I_1 - I_2.$$

Assuming once more sufficient regularity, integrating by parts gives

$$\begin{aligned} \kappa^{-1}I_1 &= \int_\Gamma (\operatorname{curl} u)v^2 n^1\, ds - \int_\Omega \partial_{x_1}(\operatorname{curl} u)v^2\, dx, \\ \kappa^{-1}I_2 &= \int_\Gamma (\operatorname{curl} u)v^1 n^2\, ds - \int_\Omega \partial_{x_2}(\operatorname{curl} u)v^1\, dx. \end{aligned}$$

Using the notation

$$\nabla^\perp f := (-\partial_{x_2}f, \partial_{x_1}f)$$

for the skew-gradient of a real-valued function f, we can write

$$\begin{aligned}\kappa^{-1}(I_1 - I_2) &= \int_\Gamma (-(\operatorname{curl} u)n^2 v^1 + (\operatorname{curl} u)n^1 v^2)\, ds \\ &\quad + \int_\Omega (\partial_{x_2}(\operatorname{curl} u)v^1 - \partial_{x_1}(\operatorname{curl} u)v^2)\, dx \\ &= \int_\Gamma \langle (\operatorname{curl} u)n^\perp, v\rangle\, ds - \int_\Omega \langle \nabla^\perp \operatorname{curl} u, v\rangle\, dx \\ &= \int_\Gamma \langle (\operatorname{curl} u)\tau, v\rangle\, ds - \int_\Omega \langle \nabla^\perp \operatorname{curl} u, v\rangle\, dx.\end{aligned}$$

Therefore

$$\begin{aligned}-\int_\Omega \langle \tilde{k}\Delta u + \kappa \nabla^\perp \operatorname{curl} u + \varepsilon^{-2}(1-|u|^2)u, v\rangle\, dx \\ + \int_\Gamma \langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau + W\varepsilon^{-s}\langle u, n\rangle n, v\rangle\, ds = 0\end{aligned}$$

for all $v \in H^1(\Omega;\mathbb{R}^2)$ and so u satisfies

$$\begin{cases} -\tilde{k}\Delta u - \kappa\nabla^\perp \operatorname{curl} u = \frac{1}{\varepsilon^2}u(1-|u|^2) & \text{in } \Omega, \\ \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau = -\frac{W}{\varepsilon^s}\langle u, n\rangle n & \text{on } \Gamma. \end{cases} \tag{6.1.2}$$

(S.O.*) Case

Although there is no boundary energy to consider here, it is important to decompose our functions near the boundary in such a way that yields an informative set of equations. We can proceed as in Chapter 2 where the orthonormal frame $\{g(x), g^\perp(x)\}$ was used. In this case, the appropriate coordinates can be obtained using the Frenet frame $\{n, \tau\}$ for which the Frenet-Serret formulas hold:

$$\begin{cases} \partial_\tau n = K\tau \\ \partial_\tau \tau = -Kn \end{cases}$$

where $K = K(x)$ is the curvature of the boundary at $x \in \Gamma$. Setting

$$u_n := \langle u, n\rangle, \quad u_\tau := \langle u, \tau\rangle$$

and using the fact that $u_n = \partial_\tau u_n = 0$ on Γ, the decomposition of u on Γ and its normal and tangential derivatives can be written

$$
\begin{aligned}
u &= u_n n + u_\tau \tau, \\
\partial_n u &= \partial_n u_n n + u_\tau \partial_n \tau + \partial_n u_\tau \tau, \qquad (6.1.3)\\
\partial_\tau u &= -K u_\tau n + \partial_\tau u_\tau \tau.
\end{aligned}
$$

$\underline{\tilde{k} = k_b}$

Upon finding the first variation of $\mathscr{F}_\varepsilon$,

$$
\int_\Omega \left(\tilde{k} \sum_{i,j} u^i_{x_j} v^i_{x_j} + \kappa(\operatorname{div} u)(\operatorname{div} v) - \frac{1}{\varepsilon^2}(1-|u|^2)\langle u, v\rangle \right) dx = 0
$$

for all $v \in H^1(\Omega; \mathbb{R}^2)$. After integrating by parts, we obtain

$$
\begin{aligned}
-\int_\Omega \langle \tilde{k}\Delta u + \kappa \nabla \operatorname{div} u + \varepsilon^{-2}(1-|u|^2)u, v\rangle \, dx \\
+ \int_\Gamma \langle \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, v\rangle \, ds = 0.
\end{aligned}
$$

Restricting now to $v \in \mathcal{H}_\tau(\Omega)$ and employing (6.1.3), the boundary integral satisfies

$$
\begin{aligned}
0 &= \int_\Gamma \langle \tilde{k}(\partial_n u_n n + u_\tau \partial_n \tau + \partial_n u_\tau \tau) + \kappa(\operatorname{div} u)n, v_\tau \tau\rangle \, ds \\
&= \int_\Gamma \tilde{k}(u_\tau v_\tau \langle \partial_n \tau, \tau\rangle + \partial_n u_\tau v_\tau) \, ds \\
&= \int_\Gamma \tilde{k} \partial_n u_\tau v_\tau \, ds
\end{aligned}
$$

for all v_τ. Therefore, the Euler-Lagrange system is

$$
\begin{cases}
-\tilde{k}\Delta u - \kappa \nabla \operatorname{div} u = \dfrac{1}{\varepsilon^2} u(1-|u|^2) & \text{in } \Omega, \\
u_n = 0 & \text{on } \Gamma, \\
\partial_n u_\tau = 0 & \text{on } \Gamma.
\end{cases} \qquad (6.1.4)
$$

$\underline{\tilde{k} = k_s}$

Judging by all other strongly orthogonal Euler-Lagrange systems presented in this thesis, one would guess that the system in the case of $\tilde{k} = k_s$ would follow similarly. However, it turns out that this is not the case, since the $(\operatorname{curl} u)\tau$ term that will appear along the boundary is parallel to the test functions contained in $\mathcal{H}_\tau(\Omega)$ when observed along Γ. Thus, we obtain slightly different boundary conditions here, but these will be shown to not cause any issues later on. As above, we find the first variation of $\mathscr{F}_\varepsilon$ and integrate by parts to obtain

$$-\int_\Omega \langle \tilde{k}\Delta u + \kappa\nabla^\perp \operatorname{curl} u + \varepsilon^{-2}(1-|u|^2)u, v\rangle\, dx \\ + \int_\Gamma \langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, v\rangle\, ds = 0.$$

Restricting v to $\mathcal{H}_\tau(\Omega)$ allows us to rewrite the boundary integral as

$$\begin{aligned} 0 &= \int_\Gamma \langle \tilde{k}(\partial_n u_n n + u_\tau \partial_n \tau + \partial_n u_\tau \tau) + \kappa(\operatorname{curl} u)\tau, v_\tau \tau\rangle\, ds \\ &= \int_\Gamma (\tilde{k}\partial_n u_\tau + \kappa \operatorname{curl} u) v_\tau\, ds \end{aligned}$$

for all v_τ and therefore the Euler-Lagrange equations are

$$\begin{cases} -\tilde{k}\Delta u - \kappa\nabla^\perp \operatorname{curl} u = \dfrac{1}{\varepsilon^2}u(1-|u|^2) & \text{in } \Omega, \\ u_n = 0 & \text{on } \Gamma, \\ \tilde{k}\partial_n u_\tau + \kappa \operatorname{curl} u = 0 & \text{on } \Gamma. \end{cases} \tag{6.1.5}$$

When compared to the regularity discussion in appendix C of the Euler-Lagrange systems (2.2.1) and (2.2.3), the equations (6.1.1), (6.1.2), (6.1.4) and (6.1.5) would need to be treated much differently due to the derivative coupling in the PDEs. This coupling, unfortunately, drastically complicates the regularity analysis which we do not provide here. The main complication arises through the coupling in boundary conditions. Indeed, if one merely has smooth Dirichlet boundary data, then smoothness of solutions up to the boundary can be obtained easily through standard results [21]. Although the coupled systems are more complicated, it is important to recognize that the equations still satisfy a nice elliptic structure which allows one to conclude higher regularity of its weak solutions. In particular, we note that the PDEs defined on Ω satisfy the *Legendre–Hadamard condition*, which is fundamental in obtaining the higher regularity. We refer the reader to Appendix C for a small discussion on this subject.

6.2 Upper Bounds for the Energies

In setting $g^\perp = n$ (or equivalently, assuming $g = \tau$), this gives us the opportunity to study the consequence of accounting for molecular bend and splay near Γ in a relatively simple setting. Indeed, we expect that since u_ε is approximately parallel to τ in the weak orthogonality case (W.O.*) and exactly parallel to τ in the strong orthogonality problem (S.O.*) when restricted to Γ, the curl of u_ε near a simple boundary vortex will not account for much energy since a simple boundary vortex with near tangential behaviour on Γ will resemble a curl-free vector field as shown in figure 1.1. The divergence of such a vector field on the other hand, will contribute almost as much energy as the full gradient. Due to this difference, it can be shown that the upper bound on $\mathscr{F}_\varepsilon^W$ will in general depend on $s \in (0, 1]$ and the bend/splay penalization determined by $\tilde{k}$.

Proposition 6.1. *Suppose u_ε is a solution to* (W.O.*)*. Then there is a constant $C > 0$ independent of ε such that*

$$\mathscr{F}_\varepsilon^W(u_\varepsilon) \leq \begin{cases} \tilde{k}\pi s|\ln\varepsilon| + C & \text{if } \tilde{k} = k_s, \\ \pi\min\{\tilde{k}, \hat{k}s\}|\ln\varepsilon| + C & \text{if } \tilde{k} = k_b, \end{cases}$$

where $\tilde{k} = \min\{k_s, k_b\}$ and $\hat{k} = \max\{k_s, k_b\}$. If u_ε is a solution to (S.O.*)*, then there is a constant $C > 0$ independent of ε such that*

$$\mathscr{F}_\varepsilon(u_\varepsilon) \leq \tilde{k}\pi|\ln\varepsilon| + C.$$

Proof.

(S.O.*) Case

The proof for (S.O.*) solutions follows a similar trick that was used for (S.O.) solutions in Proposition 2.4. For both cases $\tilde{k} = k_b$ and $\tilde{k} = k_s$, by [14] there exists a minimizer $v_{\tilde{k}}$ for $\mathscr{F}_\varepsilon$ over the function space

$$H^1_\tau(\Omega) = \{v \in H^1(\Omega; \mathbb{R}^2) : v = \tau \text{ on } \Gamma\}.$$

The function $v_{\tilde{k}}$ can now be used as a comparison function. Since the inclusion $H^1_\tau(\Omega) \subset \mathcal{H}_\tau(\Omega)$ implies that $\mathscr{F}_\varepsilon(u_\varepsilon) \leq \mathscr{F}_\varepsilon(v_{\tilde{k}})$ and noting $\deg(\tau; \Gamma) = 1$, applying [14, Proposition 2.1] to $\mathscr{F}_\varepsilon(v_{\tilde{k}})$ yields

$$\mathscr{F}_\varepsilon(u_\varepsilon) \leq \mathscr{F}_\varepsilon(v_{\tilde{k}}) \leq \tilde{k}\pi|\ln\varepsilon| + C.$$

(W.O.*) Case

In this case, we use the identical comparison function constructed from the proof of Proposition 2.4 in the weak orthogonality problem. Due to the high notational demand this proof requires, we will write an abridged version here with emphasis given on the parts that require modification due to the new energy.

Consider two sets of the form $\omega_R(q_j)$ where $\{q_j\}_{j=1}^2$ are well-separated points on Γ and R is chosen so that

$$2\varepsilon^s < R < \frac{1}{2}|q_1 - q_2|.$$

On each of these sets, an $\mathbb{S}^1$-valued function $v^{(j)}$ is constructed that simulates a 'half-vortex' planted at q_j within some small annular region contained in $\omega_R(q_j)$. Let

$$\Gamma = \bigcup_{j=1}^{2} C_j$$

by a decomposition of Γ with C_1 being the curve connecting q_1 and q_2 following the positive orientation of Γ and C_2 the remaining curve. Next, let γ be a lifting of τ on the curve $\Gamma_R(q_j)$, that is $\tau = e^{i\gamma}$ on $\Gamma_R(q_j)$. As before, we use the method from [27] to define the functions

$$\begin{aligned}
h_1(r) &= \gamma\left(re^{i\theta_1(r)}\right) + (j-1)\pi, \\
h_2(r) &= \gamma\left(re^{i\theta_2(r)}\right) + j\pi, \\
\phi(r,\theta) &= \frac{h_2(r) - h_1(r)}{\theta_2(r) - \theta_1(r)}(\theta - \theta_1(r)) + h_1(r),
\end{aligned}$$

where $\theta_1(r)$ and $\theta_2(r)$ are as in (2.5.4). In this way we have

$$e^{i\phi(r,\theta)} = \begin{cases} \tau & \text{on } \Gamma_R^+(q_1), \\ -\tau & \text{on } \Gamma_R^-(q_1), \\ -\tau & \text{on } \Gamma_R^+(q_2), \\ \tau & \text{on } \Gamma_R^-(q_2). \end{cases}$$

Choose a cut-off function $\eta_\varepsilon(r) \in C^\infty$ near q_j satisfying

$$\begin{cases} 0 \le \eta_\varepsilon(r) \le 1 & \text{for all } r, \\ \eta_\varepsilon(r) = 0 & \text{for } r < \varepsilon^s, \\ \eta_\varepsilon(r) = 1 & \text{for } r \ge 2\varepsilon^s, \\ |\eta_\varepsilon'(r)| \le \dfrac{c_0}{\varepsilon^s} & \text{for } \varepsilon^s < r < 2\varepsilon^s,\ c_0 \text{ a constant independent of } \varepsilon \end{cases}$$

and set

$$\psi(r,\theta) = \eta_\varepsilon(r)\phi(r,\theta) + (1-\eta_\varepsilon(r))(\gamma(q_j) + (j-1)\pi)$$

so that we may define the test function $v_\varepsilon^{(j)}$ on $\omega_R(q_j)$ via

$$v_\varepsilon^{(j)}(r,\theta) = e^{i\psi(r,\theta)} = (\cos(\psi(r,\theta)), \sin(\psi(r,\theta))) \tag{6.2.1}$$

and therefore

$$\frac{1}{2\varepsilon^2}\int_{\omega_R(q_j)} (1-|v_\varepsilon|^2)^2\,dx = 0.$$

Since there is no difference in boundary energy compared to that of G_ε^W, we know from the proof of Proposition 2.4 that there is $c_1 \geq 0$ independent of ε where

$$\frac{W}{2\varepsilon^s}\int_{\Gamma_R(q_j)} \langle v_\varepsilon, n\rangle^2\,ds \leq c_1. \tag{6.2.2}$$

To estimate the energy on $\omega_R(q_j)$ it will be convenient to use polar coordinates:

$$\mathscr{F}_\varepsilon(v_\varepsilon;\omega_R(q_j)) = \frac{1}{2}\int_0^R \int_{\theta_1(r)}^{\theta_2(r)} (\tilde{k}|\nabla v_\varepsilon|^2 + h_{\tilde{k}}(v_\varepsilon))r\,d\theta dr$$

and note that we may write

$$\begin{aligned}
|\nabla v_\varepsilon|^2 &= |\partial_r v_\varepsilon|^2 + \frac{1}{r^2}|\partial_\theta v_\varepsilon|^2 = (\partial_r\psi)^2 + \frac{1}{r^2}(\partial_\theta\psi)^2,\\
(\operatorname{div} v_\varepsilon)^2 &= \sin^2(\theta-\psi)(\partial_r\psi)^2 + \frac{2}{r}\sin(\theta-\psi)\cos(\theta-\psi)\partial_r\psi\partial_\theta\psi\\
&\quad + \frac{1}{r^2}\cos^2(\theta-\psi)(\partial_\theta\psi)^2,\\
(\operatorname{curl} v_\varepsilon)^2 &= \cos^2(\theta-\psi)(\partial_r\psi)^2 - \frac{2}{r}\sin(\theta-\psi)\cos(\theta-\psi)\partial_r\psi\partial_\theta\psi\\
&\quad + \frac{1}{r^2}\sin^2(\theta-\psi)(\partial_\theta\psi)^2.
\end{aligned} \tag{6.2.3}$$

Before we specify to the individual cases for the value of $\tilde{k}$, we note that as before, the radial derivative of v_ε has the uniform bound

$$\int_{\omega_R(q_j)} |\partial_r v_\varepsilon|^2\,dx = \int_{\omega_R(q_j)} |\partial_r\psi|^2\,dx \leq c_3. \tag{6.2.4}$$

The square of the angular derivative is written

$$|\partial_\theta v_\varepsilon|^2 = |\partial_\theta \psi|^2 = (\eta_\varepsilon(r))^2 \frac{(h_2(r) - h_1(r))^2}{(\theta_2(r) - \theta_1(r))^2}.$$

By the smoothness of $|\partial_\theta v_\varepsilon|^2$ and the work provided in the proof of Proposition 2.4, there is a constant $c_4 > 0$ independent of ε such that

$$\int_{\omega_R(q_j)} \frac{1}{r}|\partial_\theta v|^2\, dx, \int_{\omega_{2\varepsilon^s}(q_j)} \frac{1}{r^2}|\partial_\theta v|^2\, dx \le c_4. \tag{6.2.5}$$

Therefore, using the fact that the cross terms found in equations (6.2.3) for $(\operatorname{div} v_\varepsilon)^2$ and $(\operatorname{curl} v_\varepsilon)^2$ satisfy

$$\begin{aligned}\left| \pm \int_{\omega_R(q_j)} \frac{2}{r} \sin(\theta - \psi)\cos(\theta - \psi)\partial_r\psi\partial_\theta\psi\, dx \right| &\le \int_{\omega_R(q_j)} (|\partial_r \psi|^2 + |\partial_\theta\psi|^2)\, drd\theta \\ &\le c_3 + c_4,\end{aligned}$$

it must be the case that the primary energy contribution comes from the $(1/r^2)|\partial_\theta v_\varepsilon|^2$ components of the energies within the annular region $A_{2\varepsilon^s,R}(q_j)$. For either $\tilde{k} = k_b$ or $\tilde{k} = k_s$, by the calculations of Proposition 2.4, the angular component from the Dirichlet energy will have the estimate

$$\tilde{k} \int_{\omega_R(q_j)} \frac{1}{r^2}|\partial_\theta v_\varepsilon|^2\, dx \le \tilde{k}\pi s|\ln\varepsilon| + c_5 \tag{6.2.6}$$

where c_5 is independent of ε. The only difference in calculation now comes from integrating the third term in the expressions of $(\operatorname{div} v_\varepsilon)^2$ and $(\operatorname{curl} v_\varepsilon)^2$ found in (6.2.3).

$\underline{\tilde{k} = k_s}$

Using the estimates above, we have

$$\begin{aligned}\int_{\omega_R(q_j)} (\operatorname{curl} v_\varepsilon)^2\, dx &\le \int_{A_{2\varepsilon^s,R}(q_j)} \frac{1}{r^2} \sin^2(\theta - \psi)(\partial_\theta\psi)^2\, dx + 4(c_3 + c_4) \\ &= \int_{2\varepsilon^s}^{R} \int_{\theta_1(r)}^{\theta_2(r)} \frac{1}{r} \sin^2(\theta - \psi)(\partial_\theta\psi)^2\, d\theta dr + 4(c_3 + c_4).\end{aligned}$$

Next, we analyze the difference $\theta - \psi$. Since $2\varepsilon^s \leq r \leq R$, we have

$$\begin{aligned}
\theta - \psi &= \theta - \left(\frac{h_2(r) - h_1(r)}{\theta_2(r) - \theta_1(r)} (\theta - \theta_1(r)) + h_1(r) \right) \\
&= \left(1 - \frac{h_2(r) - h_1(r)}{\theta_2(r) - \theta_1(r)} \right) \theta + \frac{h_2(r) - h_1(r)}{\theta_2(r) - \theta_1(r)} \theta_1(r) - \gamma\left(re^{i\theta_1(r)}\right) - (j-1)\pi.
\end{aligned}$$

Now, since γ in this scenario is a lifting of τ and $\tau(q_j)$ gives the reference for the angular measurement θ, we have that for small values of r, $\gamma(r) \leq \tilde{c}r$ where $\tilde{c}$ is a constant independent of ε. Moreover, since

$$\lim_{r \to 0^+} \frac{h_2(r) - h_1(r)}{\theta_2(r) - \theta_1(r)} = 1$$

and $|\theta_1(r)| \leq cr$, we can find a constant $\hat{c}$ such that

$$|\theta - \psi| \leq \begin{cases} \hat{c}r & \text{if } j = 1, \\ \pi + \hat{c}r & \text{if } j = 2. \end{cases}$$

Thus, for $R > 0$ chosen small enough, we have

$$\sin^2(\theta - \psi) = \sin^2(|\theta - \psi|) \leq kr$$

for $k > 0$ independent of ε and each $j = 1, 2$. Applying this to the curl estimate yields

$$\begin{aligned}
\int_{\omega_R(q_j)} (\operatorname{curl} v_\varepsilon)^2 \, dx &\leq \int_{2\varepsilon^s}^R \int_{\theta_1(r)}^{\theta_2(r)} \frac{1}{r} \sin^2(\theta - \psi)(\partial_\theta \psi)^2 \, d\theta dr + 4(c_3 + c_4) \\
&\leq k \int_{A_{2\varepsilon^s, R}(q_j)} \frac{1}{r} (\partial_\theta \psi)^2 \, dx + 4(c_3 + c_4) \\
&\leq kc_4 + 4(c_3 + c_4)
\end{aligned}$$

and so $\kappa \int_{\omega_R(q_j)} (\operatorname{curl} v_\varepsilon)^2 \, dx$ is uniformly bounded. With this and applying inequalities (6.2.2), (6.2.4) and (6.2.6), there is a constant c_6 independent of ε so that

$$\mathscr{F}_\varepsilon^W(v_\varepsilon; \omega_R(q_j)) \leq \frac{\tilde{k}\pi}{2} s |\ln \varepsilon| + c_6. \tag{6.2.7}$$

This completes estimating the energy on $\omega_R(q_j)$.

Next, we must fill in the remaining piece of the domain

$$\tilde{\Omega} := \Omega \setminus \bigcup_{j=1}^{2} \overline{\omega_R(q_j)}$$

with a test function V_ε so that the energy on $\tilde{\Omega}$ remains uniformly bounded in ε. Define the closed contour

$$\tilde{\Gamma} := \partial\tilde{\Omega} = \big(\Gamma \setminus \cup_{j=1}^{2}\Gamma_R(q_j)\big) \bigcup \big(\cup_{j=1}^{2}\partial B_R(q_j) \cap \Omega\big)$$

with orientation matching that of Γ where they coincide. In this way, observe that the circular arcs $\partial B_R(q_j) \cap \Omega$ are negatively oriented for each $j = 1, 2$. With this in mind, we define boundary data $\tilde{g} : \tilde{\Gamma} \to \mathbb{S}^1$ by setting

$$\tilde{g} := \begin{cases} \tau & \text{on } \tilde{\Gamma} \cap C_1 \\ -\tau & \text{on } \tilde{\Gamma} \cap C_2 \\ v_\varepsilon^{(j)} & \text{on } \partial B_R(q_j) \cap \Omega \text{ for each } j = 1, 2. \end{cases}$$

By the construction of $v_\varepsilon^{(j)}$ and the negative orientation associated with the arc $\partial B_R(q_j) \cap \Omega$, the phase of $\tilde{g}$ turns by approximately $-\pi$ on each $\partial B_R(q_j) \cap \Omega$, $j = 1, 2$ for a combined associated phase turn of -2π. The remaining pieces of the boundary data will contribute a phase of 2π to $\tilde{g}$ since both $\deg(\tau; \Gamma) = \deg(-\tau; \Gamma) = 1$. Therefore $\deg(\tilde{g}; \tilde{\Gamma}) = 0$ and so we may define the remaining test function on $\tilde{\Omega}$ by letting V_ε be the $\mathbb{S}^1$-valued harmonic extension of $\tilde{g}$ to $\tilde{\Omega}$. It is known that this extension has bounded energy and since V_ε is equal to $\pm\tau$ where $\tilde{\Gamma}$ and Γ coincide,

$$\begin{aligned} \mathscr{F}_\varepsilon^W(V_\varepsilon; \tilde{\Omega}) &= \mathscr{F}_\varepsilon(V_\varepsilon; \tilde{\Omega}) \\ &= \frac{1}{2}\int_{\tilde{\Omega}} \tilde{k}|\nabla V_\varepsilon|^2 + \kappa(\operatorname{curl} V_\varepsilon)^2\, dx \\ &\leq \frac{\tilde{k} + 2\kappa}{2}\int_{\tilde{\Omega}} |\nabla V_\varepsilon|^2\, dx \\ &\leq c_7 \end{aligned} \tag{6.2.8}$$

for c_7 independent of ε. Defining

$$H_\varepsilon = \begin{cases} V_\varepsilon & \text{in } \tilde{\Omega} \\ v_\varepsilon^{(j)} & \text{in } \omega_R(q_j) \text{ for each } j = 1, 2 \end{cases}$$

and using inequalities (6.2.7) and (6.2.8) we obtain

$$\mathscr{F}_\varepsilon^W(H_\varepsilon) = \sum_{j=1}^{2} \mathscr{F}_\varepsilon^W(v_\varepsilon^{(j)}; \omega_R(q_j)) + \mathscr{F}_\varepsilon^W(V_\varepsilon; \tilde{\Omega}) \leq \tilde{k}\pi s |\ln \varepsilon| + C$$

as desired.

$\underline{\tilde{k} = k_b}$

Using (6.2.3), we note that

$$(\operatorname{div} v_\varepsilon)^2 \leq (\operatorname{div} v_\varepsilon)^2 + (\operatorname{curl} v_\varepsilon)^2 = |\nabla v_\varepsilon|^2.$$

Then since $\tilde{k} + \kappa = \hat{k}$, the energy has the simple estimate

$$\begin{aligned}
\mathscr{F}_\varepsilon(v_\varepsilon; \omega_R(q_j)) &= \frac{1}{2} \int_{\omega_R(q_j)} (\tilde{k}|\nabla v_\varepsilon|^2 + \kappa(\operatorname{div} v_\varepsilon)^2)\, dx \\
&\leq \frac{1}{2} \int_{\omega_R(q_j)} (\tilde{k}|\nabla v_\varepsilon|^2 + \kappa|\nabla v_\varepsilon|^2)\, dx \\
&= \frac{\hat{k}}{2} \int_{\omega_R(q_j)} |\nabla v_\varepsilon|^2\, dx.
\end{aligned}$$

Now that we are back to the setting of Proposition 2.4, we have the estimate

$$\mathscr{F}_\varepsilon^W(u_\varepsilon) \leq \hat{k}\pi s |\ln \varepsilon| + C.$$

We may also consider the same comparison function v_{k_b} as in the case of (S.O.*). Since $v_{k_b} \in \mathcal{H}_\tau(\Omega)$, we obtain

$$\mathscr{F}_\varepsilon^W(u_\varepsilon) \leq \mathscr{F}_\varepsilon^W(v_{k_b}) = \mathscr{F}_\varepsilon(v_{k_b}) \leq \tilde{k}\pi |\ln \varepsilon| + C.$$

Therefore, we have

$$\mathscr{F}_\varepsilon^W(u_\varepsilon) \leq \pi \min\{\tilde{k}, \hat{k}s\} |\ln \varepsilon| + C$$

which completes the proof. □

6.3 An Assumption on Families of Minimizers

In Section 2.4, we proved that solutions of (2.2.1) and (2.2.3) have the uniform bound $|u_\varepsilon| \leq 1$ and their gradients satisfied $\varepsilon|\nabla u_\varepsilon| \leq C_0$ for some constant $C_0 > 0$ independent of ε. These properties were vital in obtaining the η-compactness result of Section 3.2.

Recall that the uniform bound $|u_\varepsilon| \leq 1$ was obtained through a maximum principle argument. Due to the second order derivative coupling in the PDEs associated to $\mathscr{F}_\varepsilon$ and $\mathscr{F}_\varepsilon^W$ of this section, the maximum principle cannot be applied here and thus it is not known whether $|u_\varepsilon|$ is uniformly bounded. However, it is still reasonable to expect that solutions to (W.O.*) and (S.O.*) will be uniformly bounded for ε small enough due to the norm penalization in the energy. Current results from the literature indicate that even in the simplest case where one is given smooth $\mathbb{S}^1$-valued Dirichlet boundary data, it is still not known if $|u_\varepsilon| \leq 1$. On the other hand, it *is* known that the Dirichlet problem does yield uniformly bounded minimizers, but the bounding constant is unknown. This result is expressed in [14, Proposition 2.2] and is based on a L^4 a priori estimate (see [8, Lemma 3.1] for details).

Recall also that the gradient bound of Section 2.4 was obtained by a rescaling which gave a limiting harmonic map. Once again, this argument does not directly apply in this section since we are not guaranteed a limiting harmonic function after a similar rescaling. On the other hand, it is still reasonable to assume that the ellipticity structure of the PDEs may be utilized to achieve a similar gradient bound. When given Dirichlet boundary data, it is shown in [14, Proposition 2.2] that the same bounding constant for $|u_\varepsilon|$ can be used for the gradient estimate. Once again, this heavily uses the Dirichlet condition and will not apply here.

With this being stated, we admit that there is no easy way to show that the above bounds apply to solutions of (W.O.*) or (S.O.*). However, for the sake of moving forward, we will assume a similar conclusion as [14, Proposition 2.2] for the remainder of this section in order to obtain η-compactness.

Assumption. *Suppose $\{u_\varepsilon\}_{\varepsilon>0}$ is a family of solutions for either* (W.O.*) *or* (S.O.*). *Then it will be assumed that there is a constant $C_0 > 0$ independent of ε such that*

$$|u_\varepsilon| \leq C_0, \quad |\nabla u_\varepsilon| \leq \frac{C_0}{\varepsilon}$$

for all $x \in \Omega$.

Definition 6.2. *Let Λ denote the class of functions satisfying the assumption above.*

6.4 Pohozaev Identities and Related Estimates

We begin this section by deriving Pohozaev identities for solutions of (W.O.*) or (S.O.*). These identities are then used to prove estimates analogous to the ones found in Lemma 3.2. We define a short-form notation for the integrand of $\mathscr{F}_\varepsilon$ by writing

$$e_\varepsilon^{\tilde{k}}(u) := \frac{\tilde{k}}{2}|\nabla u|^2 + \frac{1}{2}h_{\tilde{k}}(u) + \frac{1}{4\varepsilon^2}\left(1-|u|^2\right)^2.$$

Proposition 6.3. *Let $\psi \in C^\infty(\Omega;\mathbb{R}^2)$. If u is a solution of* (6.1.1) *or* (6.1.4)*, then*

$$\begin{aligned}\int_{\partial\omega_r} &\left\{e_\varepsilon^{k_b}(u)\langle\psi,n\rangle - \langle k_b\partial_n u + \kappa(\operatorname{div} u)n, \psi\cdot\nabla u\rangle\right\} ds \\ &= \int_{\omega_r}\left\{e_\varepsilon^{k_b}(u)\operatorname{div}\psi - k_b\sum_{j,l}\psi^l_{x_j}\langle\partial_{x_j}u,\partial_{x_l}u\rangle - \kappa(\operatorname{div} u)\sum_{i=1}^{2}\langle\partial_{x_i}\psi,\nabla u^i\rangle\right\} dx.\end{aligned} \tag{6.4.1}$$

If u is a solution of (6.1.2) *or* (6.1.5)*, then*

$$\begin{aligned}\int_{\partial\omega_r} &\left\{e_\varepsilon^{k_s}(u)\langle\psi,n\rangle - \langle k_s\partial_n u + \kappa(\operatorname{curl} u)\tau, \psi\cdot\nabla u\rangle\right\} ds \\ &= \int_{\omega_r}\left\{e_\varepsilon^{k_s}(u)\operatorname{div}\psi - k_s\sum_{j,l}\psi^l_{x_j}\langle\partial_{x_j}u,\partial_{x_l}u\rangle - \kappa(\operatorname{curl} u)\sum_{i=1}^{2}(-1)^{i-1}\langle\partial_{x_i}\psi,\nabla u^{3-i}\rangle\right\} dx.\end{aligned} \tag{6.4.2}$$

Proof.

Suppose first that u is a solution of (6.1.1) or (6.1.4). We begin by taking the inner product on both sides of the interior PDE with the vector field $\psi\cdot\nabla u$ and integrating over ω_r:

$$\begin{aligned}-\tilde{k}\int_{\omega_r}\langle\psi\cdot\nabla u,\Delta u\rangle\,dx - \kappa\int_{\omega_r}&\langle\psi\cdot\nabla u,\nabla\operatorname{div} u\rangle\,dx \\ &= \int_{\omega_r}\frac{1}{\varepsilon^2}\langle u(1-|u|^2),\psi\cdot\nabla u\rangle\,dx.\end{aligned}$$

Since the terms

$$-\tilde{k}\int_{\omega_r}\langle\psi\cdot\nabla u,\Delta u\rangle\,dx \quad\text{and}\quad \int_{\omega_r}\frac{1}{\varepsilon^2}\langle u(1-|u|^2),\psi\cdot\nabla u\rangle\,dx$$

have already been dealt with in Proposition 2.5, we focus only on the divergence term. Applying integration by parts,

$$-\int_{\omega_r} \langle \psi \cdot \nabla u, \nabla \operatorname{div} u \rangle \, dx = \int_{\omega_r} (\operatorname{div} u)(\operatorname{div}(\psi \cdot \nabla u)) \, dx - \int_{\partial\omega_r} (\operatorname{div} u) \langle \psi \cdot \nabla u, n \rangle \, ds.$$

Next, we compute $\operatorname{div}(\psi \cdot \nabla u)$.

$$\begin{aligned}
\operatorname{div}(\psi \cdot \nabla u) &= \partial_{x_1} \left(\psi^1 u^1_{x_1} + \psi^2 u^1_{x_2} \right) + \partial_{x_2} \left(\psi^1 u^2_{x_1} + \psi^2 u^2_{x_2} \right) \\
&= \psi^1 u^1_{x_1 x_1} + \psi^1_{x_1} u^1_{x_1} + \psi^2 u^1_{x_2 x_1} + \psi^2_{x_1} u^1_{x_2} \\
&\quad + \psi^1 u^2_{x_1 x_2} + \psi^1_{x_2} u^2_{x_1} + \psi^2 u^2_{x_2 x_2} + \psi^2_{x_2} u^2_{x_2} \\
&= \psi^1 (u^1_{x_1 x_1} + u^2_{x_1 x_2}) + \psi^2 (u^1_{x_2 x_1} + u^2_{x_2 x_2}) + \sum_{j,l} \psi^l_{x_j} u^j_{x_l} \\
&= \psi^1 \partial_{x_1} \operatorname{div} u + \psi^2 \partial_{x_2} \operatorname{div} u + \sum_{i=1}^{2} \langle \partial_{x_i} \psi, \nabla u^i \rangle \\
&= \langle \psi, \nabla \operatorname{div} u \rangle + \sum_{i=1}^{2} \langle \partial_{x_i} \psi, \nabla u^i \rangle.
\end{aligned}$$

Next, since

$$(\operatorname{div} u) \langle \psi, \nabla \operatorname{div} u \rangle = \langle \psi, (\operatorname{div} u) \nabla \operatorname{div} u \rangle = \frac{1}{2} \langle \psi, \nabla (\operatorname{div} u)^2 \rangle$$

we have

$$\begin{aligned}
\int_{\omega_r} (\operatorname{div} u) \operatorname{div}(\psi \cdot \nabla u) \, dx &= \int_{\omega_r} (\operatorname{div} u) \langle \psi, \nabla \operatorname{div} u \rangle \, dx \\
&\quad + \int_{\omega_r} \left((\operatorname{div} u) \sum_{i=1}^{2} \langle \partial_{x_i} \psi, \nabla u^i \rangle \right) dx \\
&= \frac{1}{2} \int_{\omega_r} \langle \psi, \nabla (\operatorname{div} u)^2 \rangle \, dx \\
&\quad + \int_{\omega_r} \left((\operatorname{div} u) \sum_{i=1}^{2} \langle \partial_{x_i} \psi, \nabla u^i \rangle \right) dx.
\end{aligned}$$

The first integral in the sum above can be written

$$\frac{1}{2}\int_{\omega_r}\langle\psi,\nabla(\operatorname{div}u)^2\rangle\,dx=\frac{1}{2}\int_{\partial\omega_r}(\operatorname{div}u)^2\langle\psi,n\rangle\,ds-\frac{1}{2}\int_{\omega_r}(\operatorname{div}u)^2\operatorname{div}\psi\,dx$$

and therefore

$$\begin{aligned}-\kappa\int_{\omega_r}\langle\psi\cdot\nabla u,\nabla\operatorname{div}u\rangle\,dx&=\int_{\partial\omega_r}\Big(\frac{\kappa}{2}(\operatorname{div}u)^2\langle\psi,n\rangle-\kappa(\operatorname{div}u)\langle\psi\cdot\nabla u,n\rangle\Big)\,ds\\&\quad+\int_{\omega_r}\left(\kappa(\operatorname{div}u)\sum_{i=1}^{2}\langle\partial_{x_i}\psi,\nabla u^i\rangle-\frac{\kappa}{2}(\operatorname{div}u)^2\operatorname{div}\psi\right)dx.\end{aligned}$$

Putting all of the integrals together gives (6.4.1). Suppose now that u is a solution of (6.1.2) or (6.1.5). As before, we find

$$\begin{aligned}-\tilde{k}\int_{\omega_r}\langle\psi\cdot\nabla u,\Delta u\rangle\,dx-\kappa\int_{\omega_r}\langle\psi\cdot\nabla u,\nabla^{\perp}\operatorname{curl}u\rangle\,dx&\\=\int_{\omega_r}\frac{1}{\varepsilon^2}\langle u(1-|u|^2),\psi\cdot\nabla u\rangle\,dx.&\end{aligned}$$

As above, we focus only on the curl term since the remaining integrals were covered in the proof of Proposition 2.5. Integrating by parts yields

$$\begin{aligned}-\int_{\omega_r}\langle\psi\cdot\nabla u,\nabla^{\perp}\operatorname{curl}u\rangle\,dx&=\int_{\omega_r}(\operatorname{curl}u)(\operatorname{curl}(\psi\cdot\nabla u))\,dx\\&\quad-\int_{\partial\omega_r}(\operatorname{curl}u)\langle\tau,\psi\cdot\nabla u\rangle\,ds.\end{aligned}$$

Calculating $\operatorname{curl}(\psi\cdot\nabla u)$ we get

$$\begin{aligned}\operatorname{curl}(\psi\cdot\nabla u)&=\partial_{x_1}(\psi^1u^2_{x_1}+\psi^2u^2_{x_2})-\partial_{x_2}(\psi^1u^1_{x_1}+\psi^2u^1_{x_2})\\&=\psi^1u^2_{x_1x_1}+\psi^1_{x_1}u^2_{x_1}+\psi^2u^2_{x_2x_1}+\psi^2_{x_1}u^2_{x_2}\\&\quad-\psi^1u^1_{x_1x_2}-\psi^1_{x_2}u^1_{x_1}-\psi^2u^1_{x_2x_2}-\psi^2_{x_2}u^1_{x_2}\\&=\psi^1(u^2_{x_1x_1}-u^1_{x_1x_2})+\psi^2(u^2_{x_2x_1}-u^1_{x_2x_2})\\&\quad+\langle\partial_{x_1}\psi,\nabla u^2\rangle-\langle\partial_{x_2}\psi,\nabla u^1\rangle\\&=\psi^1\partial_{x_1}(\operatorname{curl}u)+\psi^2\partial_{x_2}(\operatorname{curl}u)+\sum_{i=1}^{2}(-1)^{i-1}\langle\partial_{x_i}\psi,\nabla u^{3-i}\rangle\\&=\langle\psi,\nabla\operatorname{curl}u\rangle+\sum_{i=1}^{2}(-1)^{i-1}\langle\partial_{x_i}\psi,\nabla u^{3-i}\rangle.\end{aligned}$$

Next, since

$$(\operatorname{curl} u)\langle \psi, \nabla \operatorname{curl} u\rangle = \langle \psi, (\operatorname{curl} u)\nabla \operatorname{curl} u\rangle = \frac{1}{2}\langle \psi, \nabla (\operatorname{curl} u)^2\rangle$$

we have

$$\begin{aligned}
\int_{\omega_r} (\operatorname{curl} u)(\operatorname{curl}(\psi\cdot\nabla u))\,dx &= \int_{\omega_r} \langle \psi, (\operatorname{curl} u)\nabla \operatorname{curl} u\rangle\,dx \\
&\quad + \int_{\omega_r}\left((\operatorname{curl} u)\sum_{i=1}^{2}(-1)^{i-1}\langle \partial_{x_i}\psi, \nabla u^{3-i}\rangle\right)dx \\
&= \frac{1}{2}\int_{\omega_r}\langle \psi, \nabla(\operatorname{curl} u)^2\rangle\,dx \\
&\quad + \int_{\omega_r}\left((\operatorname{curl} u)\sum_{i=1}^{2}(-1)^{i-1}\langle \partial_{x_i}\psi, \nabla u^{3-i}\rangle\right)dx.
\end{aligned}$$

Integrating the first integral in the sum above by parts,

$$\frac{1}{2}\int_{\omega_r}\langle \psi, \nabla(\operatorname{curl} u)^2\rangle\,dx = \frac{1}{2}\int_{\partial\omega_r}(\operatorname{curl} u)^2\langle \psi, n\rangle\,ds - \frac{1}{2}\int_{\omega_r}(\operatorname{curl} u)^2\operatorname{div}\psi\,dx.$$

Therefore

$$\begin{aligned}
-\kappa\int_{\omega_r}\langle \psi\cdot\nabla u, \nabla^{\perp}\operatorname{curl} u\rangle\,dx &= \int_{\partial\omega_r}\left(\frac{\kappa}{2}(\operatorname{curl} u)^2\langle \psi, n\rangle - \kappa(\operatorname{curl} u)\langle \tau, \psi\cdot\nabla u\rangle\right)ds \\
&+ \int_{\omega_r}\left(\kappa(\operatorname{curl} u)\sum_{i=1}^{2}(-1)^{i-1}\langle \partial_{x_i}\psi, \nabla u^{3-i}\rangle - \frac{\kappa}{2}(\operatorname{curl} u)^2\operatorname{div}\psi\right)dx
\end{aligned}$$

which completes the proof upon combining with the remaining integrals. □

In what follows, we prove the analogue of Lemma 3.2 for solutions of this section. It is at this point where we must employ the boundedness assumption made in Section 6.3. As before, we recall that when $x_0 \in \Gamma$, we use the decomposition

$$\partial\omega_r(x_0) = \Gamma_r(x_0) \cup (\partial B_r(x_0)\cap\Omega).$$

Define as in [3, 4, 39] for $x_0 \in \overline{\Omega}$, the radius-dependent function

$$F^{\tilde{k}}(r) := F^{\tilde{k}}(r; x_0, u, \varepsilon) = r\int_{\partial B_r(x_0)\cap\Omega} e_{\varepsilon}^{\tilde{k}}(u)\,ds.$$

When $x_0 \in \Gamma$, we also define

$$F_\Gamma^{\tilde{k}}(r) := F^{\tilde{k}}(r) + \frac{Wr}{2\varepsilon^s} \sum_{x \in \partial\Gamma_r(x_0)} \langle u, n\rangle^2.$$

Lemma 6.4. *Let $x_0 \in \overline{\Omega}$ and assume $u \in \Lambda$. There exists constants $C(\tilde{k}, \kappa) > 0$ and $r_0(\tilde{k}, \kappa) > 0$ such that for $\varepsilon \in (0,1)$ and $r \in (0, r_0)$ we have*

(1) If $x_0 \in \Omega$ and $\overline{\omega_r(x_0)} \cap \Gamma = \varnothing$,

$$\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx \le C\left[r\int_{\omega_r}\frac{1}{2}\left(\tilde{k}|\nabla u|^2 + h_{\tilde{k}}(u)\right)dx + F^{\tilde{k}}(r)\right], \quad (6.4.3)$$

(2) If $x_0 \in \Gamma$ and u satisfies the strong orthogonality condition,

$$\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx \le C\left[r\int_{\omega_r}\frac{1}{2}\left(\tilde{k}|\nabla u|^2 + h_{\tilde{k}}(u)\right)dx + F^{\tilde{k}}(r) + \frac{r^2}{\varepsilon}\right], \quad (6.4.4)$$

(3) If $x_0 \in \Gamma$ and u satisfies the weak orthogonality condition,

$$\begin{aligned}\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx &+ \frac{W}{2\varepsilon^s}\int_{\Gamma_r}\langle u, n\rangle^2\,ds \\ &\le C\left[r\int_{\omega_r}\frac{1}{2}\left(\tilde{k}|\nabla u|^2 + h_{\tilde{k}}(u)\right)dx + F_\Gamma^{\tilde{k}}(r) + \frac{Wr^2}{\varepsilon^s}\right].\end{aligned} \quad (6.4.5)$$

Proof.

Step 1: $x_0 \in \Omega$

Assume first that $\tilde{k} = k_b$ and $\omega_r = B_r(x_0) \subset \Omega$. Let n and τ represent the unit normal and tangent vectors to $\partial\omega_r = \partial B_r(x_0)$ respectively and define the vector field $X = x - x_0$. Then, $|X| \le r$ for all $x \in \omega_r$ with $X_n = \langle X, n\rangle = r$ on $\partial\omega_r$ and $X_\tau = \langle X, \tau\rangle = 0$ on $\partial\omega_r$. To obtain (6.4.3), consider the Pohosaev identity (6.4.1) and take $\psi = X$.

Estimates Along $\partial\omega_r$:

The lefthand side of (6.4.1) from Proposition 6.3 can be written as the sum of integrals $I_1 + I_2 + I_3$ where

$$I_1 = \int_{\partial\omega_r} \left\{ \frac{\tilde{k}}{2} |\nabla u|^2 \langle X, n\rangle - \langle \tilde{k}\partial_n u, X \cdot \nabla u\rangle \right\} ds,$$

$$I_2 = \int_{\partial\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{div} u)^2 \langle X, n\rangle - \langle \kappa(\operatorname{div} u)n, X \cdot \nabla u\rangle \right\} ds,$$

$$I_3 = \frac{1}{4\varepsilon^2} \int_{\partial\omega_r} (1 - |u|^2)^2 \langle X, n\rangle \, ds.$$

Since $X = rn$ on $\partial\omega_r$, we have that $X \cdot \nabla u = r\partial_n u$. The first integral has estimate

$$\begin{aligned} I_1 &= r \int_{\partial\omega_r} \left\{ \frac{\tilde{k}}{2} |\nabla u|^2 - \tilde{k} \langle \partial_n u, \partial_n u\rangle \right\} ds \\ &= r \int_{\partial\omega_r} \left\{ \frac{\tilde{k}}{2} |\nabla u|^2 - \tilde{k} |\partial_n u|^2 \right\} ds \\ &\leq r \int_{\partial\omega_r} \frac{\tilde{k}}{2} |\nabla u|^2 \, ds. \end{aligned}$$

For I_2, we use Cauchy-Schwarz:

$$\begin{aligned} I_2 &= r \int_{\partial\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{div} u)^2 - \kappa \langle (\operatorname{div} u)n, \partial_n u\rangle \right\} ds \\ &\leq r \int_{\partial\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{div} u)^2 + \kappa |(\operatorname{div} u)n| |\partial_n u| \right\} ds \\ &\leq r \int_{\partial\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{div} u)^2 + \frac{\kappa}{2} (\operatorname{div} u)^2 + \frac{\kappa}{2} |\partial_n u|^2 \right\} ds \\ &\leq Cr \int_{\partial\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{div} u)^2 + \tilde{k} |\nabla u|^2 \right\} ds. \end{aligned}$$

The integral I_3 is easily seen to be

$$I_3 = \frac{r}{4\varepsilon^2} \int_{\partial\omega_r} (1 - |u|^2)^2 \, ds$$

and therefore there is a constant $C > 0$ so that

$$I_1 + I_2 + I_3 \leq Cr \int_{\partial\omega_r} \frac{1}{2} \left\{ \tilde{k} |\nabla u|^2 + \kappa (\operatorname{div} u)^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right\} ds = CF^{\tilde{k}}(r).$$

Estimates in ω_r:

The righthand side of (6.4.1) can be written as the sum of three integrals $J_1 + J_2 + J_3$ where

$$J_1 = \int_{\omega_r} \left\{ \frac{\tilde{k}}{2} |\nabla u|^2 \operatorname{div} X - \tilde{k} \sum_{j,l} X^l_{x_j} \langle \partial_{x_j} u, \partial_{x_l} u \rangle \right\} dx,$$

$$J_2 = \int_{\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{div} u)^2 \operatorname{div} X - \kappa (\operatorname{div} u) \sum_{i=1}^{2} \langle \partial_{x_i} X, \nabla u^i \rangle \right\} dx,$$

$$J_3 = \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \operatorname{div} X \, dx.$$

Since $X^l_{x_j} = \delta_{jl}$ and $\operatorname{div} X = 2 > 2 - r$,

$$J_1 \geq \int_{\omega_r} \left\{ \tilde{k} |\nabla u|^2 - \frac{\tilde{k} r}{2} |\nabla u|^2 - \tilde{k} \sum_{j,l} \delta_{jl} \langle \partial_{x_j} u, \partial_{x_l} u \rangle \right\} dx$$

$$= r \int_{\omega_r} \left\{ -\frac{\tilde{k} r}{2} |\nabla u|^2 + \tilde{k} |\nabla u|^2 - \tilde{k} |\nabla u|^2 \right\} dx$$

$$= -r \int_{\omega_r} \frac{\tilde{k}}{2} |\nabla u|^2 \, dx.$$

Similarly for J_2, we use $\operatorname{div} X > 2 - r$ and the fact that $\sum_{i=1}^{2} \langle \partial_{x_i} X, \nabla u^i \rangle = \sum_{i=1}^{2} u^i_{x_i} = \operatorname{div} u$,

$$J_2 \geq \int_{\omega_r} \left\{ \kappa (\operatorname{div} u)^2 - \frac{\kappa r}{2} (\operatorname{div} u)^2 - \kappa (\operatorname{div} u) \sum_{i=1}^{2} \langle \partial_{x_i} X, \nabla u^i \rangle \right\} dx,$$

$$= \int_{\omega_r} \left\{ -\frac{\kappa r}{2} (\operatorname{div} u)^2 + \kappa (\operatorname{div} u)^2 - \kappa (\operatorname{div} u)^2 \right\} dx,$$

$$= -r \int_{\omega_r} \frac{\kappa}{2} (\operatorname{div} u)^2 \, dx.$$

Lastly, for J_3 we use $\operatorname{div} X > 1$ to get

$$J_3 \geq \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx.$$

Putting everything together,

$$\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx - r\int_{\omega_r}\left\{\frac{\tilde{k}}{2}|\nabla u|^2+\frac{\kappa}{2}(\operatorname{div}u)^2\right\}dx \le J_1+J_2+J_3$$
$$\le CF^{\tilde{k}}(r)$$

which proves inequality (6.4.3) for $\tilde{k}=k_b$.

To obtain (6.4.3) for $\tilde{k}=k_s$, we continue to use the conditions mentioned in the preamble of this step but now take $\psi = X$ in equation (6.4.2).

Estimates Along $\partial\omega_r$:

The lefthand side of (6.4.2) can be written as the sum of integrals $I_4+I_5+I_6$ where

$$I_4=\int_{\partial\omega_r}\left\{\frac{\tilde{k}}{2}|\nabla u|^2\langle X,n\rangle - \langle \tilde{k}\partial_n u, X\cdot\nabla u\rangle\right\}ds,$$

$$I_5=\int_{\partial\omega_r}\left\{\frac{\kappa}{2}(\operatorname{curl}u)^2\langle X,n\rangle - \langle \kappa(\operatorname{curl}u)\tau, X\cdot\nabla u\rangle\right\}ds,$$

$$I_6=\frac{1}{4\varepsilon^2}\int_{\partial\omega_r}(1-|u|^2)^2\langle X,n\rangle\,ds.$$

Integrals I_4 and I_6 are handled identically to that of I_1 and I_3 above and so we estimate I_5 only. Using Cauchy-Schwarz:

$$\begin{aligned}I_5 &= r\int_{\partial\omega_r}\left\{\frac{\kappa}{2}(\operatorname{curl}u)^2 - \kappa\langle(\operatorname{curl}u)\tau,\partial_n u\rangle\right\}ds\\
&\le r\int_{\partial\omega_r}\left\{\frac{\kappa}{2}(\operatorname{curl}u)^2 + \kappa|(\operatorname{curl}u)\tau||\partial_n u|\right\}ds\\
&\le r\int_{\partial\omega_r}\left\{\frac{\kappa}{2}(\operatorname{curl}u)^2 + \frac{\kappa}{2}(\operatorname{curl}u)^2+\frac{\kappa}{2}|\partial_n u|^2\right\}ds\\
&\le Cr\int_{\partial\omega_r}\left\{\frac{\kappa}{2}(\operatorname{curl}u)^2 + \tilde{k}|\nabla u|^2\right\}ds.\end{aligned}$$

Again we can find $C>0$ large enough so that

$$I_4+I_5+I_6 \le Cr\int_{\partial\omega_r}\frac{1}{2}\left\{\tilde{k}|\nabla u|^2+\kappa(\operatorname{curl}u)^2+\frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\}ds = CF^{\tilde{k}}(r).$$

Estimates in ω_r:

The righthand side of (6.4.2) can be written as the sum $J_4 + J_5 + J_6$ where

$$J_4 = \int_{\omega_r} \left\{ \frac{\tilde{k}}{2} |\nabla u|^2 \operatorname{div} X - \tilde{k} \sum_{j,l} X^l_{x_j} \langle \partial_{x_j} u, \partial_{x_l} u \rangle \right\} dx,$$

$$J_5 = \int_{\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{curl} u)^2 \operatorname{div} X - \kappa(\operatorname{curl} u) \sum_{i=1}^{2} (-1)^{i-1} \langle \partial_{x_i} X, \nabla u^{3-i} \rangle \right\} dx,$$

$$J_6 = \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \operatorname{div} X \, dx.$$

As before, the integrals J_4 and J_6 are estimated exactly like J_1 and J_3 respectively. For J_5 observe that

$$\sum_{i=1}^{2} (-1)^{i-1} \langle \partial_{x_i} X, \nabla u^{3-i} \rangle = \sum_{i=1}^{2} (-1)^{i-1} u^{3-i}_{x_i} = u^2_{x_1} - u^1_{x_2} = \operatorname{curl} u.$$

This paired with the fact that $\operatorname{div} X > 2 - r$ we obtain

$$\begin{aligned} J_5 &\geq \int_{\omega_r} \left\{ \kappa(\operatorname{curl} u)^2 - \frac{\kappa r}{2} (\operatorname{curl} u)^2 - \kappa(\operatorname{curl} u) \sum_{i=1}^{2} (-1)^{i-1} \langle \partial_{x_i} X, \nabla u^{3-i} \rangle \right\} dx \\ &= \int_{\omega_r} \left\{ -\frac{\kappa r}{2} (\operatorname{curl} u)^2 + \kappa(\operatorname{curl} u)^2 - \kappa(\operatorname{curl} u)^2 \right\} dx \\ &= -r \int_{\omega_r} \frac{\kappa}{2} (\operatorname{curl} u)^2 \, dx \end{aligned}$$

and therefore

$$\begin{aligned} \frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx - r \int_{\omega_r} \left\{ \frac{\tilde{k}}{2} |\nabla u|^2 + \frac{\kappa}{2} (\operatorname{curl} u)^2 \right\} dx &\leq J_4 + J_5 + J_6 \\ &\leq C F^{\tilde{k}}(r). \end{aligned}$$

Step 2: $x_0 \in \Gamma$

Once again we assume first that $\tilde{k} = k_b$. Let $r_0 > 0$ be chosen small enough so that $\Gamma \cap B_r(x_0)$ consists of a single smooth arc satisfying $|\Gamma_r| \leq Cr$ for all $0 < r \leq r_0$ and that ω_r is strictly starshaped with respect to some point $x_1 \in \omega_r$ for all $0 < r \leq r_0$. As in the proof of Lemma 3.2, we let $X \in C^2(\mathcal{N}; \mathbb{R}^2)$ to be the vector field satisfying the conditions (3.1.4), (3.1.5) and

(3.1.6). To obtain inequalities (6.4.4) and (6.4.5), we consider the Pohosaev identity (6.4.1) with $\psi = X$ and find estimates for several of its terms. Using $\partial\omega_r = \Gamma_r \cup (\partial B_r(x_0) \cap \Omega)$, it will be convenient to perform these estimates on Γ_r and $\partial B_r(x_0) \cap \Omega$ separately.

Estimates Along Γ_r:

(W.O.*) Case

By (3.1.4) we may write $X = \langle X, \tau\rangle\tau$ where τ is the unit tangent vector to Γ_r and so $X \cdot \nabla u = X_\tau \partial_\tau u$ on Γ_r. Using the natural boundary condition from (6.1.1), the lefthand side of (6.4.1) is

$$-\int_{\Gamma_r} \langle \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, X_\tau \partial_\tau u\rangle \, ds = \frac{W}{\varepsilon^s}\int_{\Gamma_r} \langle u_n n, X_\tau \partial_\tau u\rangle \, ds.$$

The estimate process needed for the righthand side of this equation is identical to that done in the proof of Lemma 3.2, with $g^\perp = n$. Therefore, there is a constant C independent of ε such that

$$-\int_{\Gamma_r} \langle \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, X_\tau \partial_\tau u\rangle \, ds \le -\frac{W}{2\varepsilon^s}\int_{\Gamma_r} \langle u, n\rangle^2 \, ds + \frac{Wr}{2\varepsilon^s}\sum_{x\in\partial\Gamma_r(x_0)} \langle u, n\rangle^2 + \frac{CWr^2}{\varepsilon^s}.$$

Note that since $\langle X, n\rangle = 0$ for all $x \in \Gamma_r$, the remaining boundary integrals along Γ_r in (6.4.1) are zero.

(S.O.*) Case

For strong orthogonal functions, we require a closer look at the inner product

$$\langle \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, X_\tau \partial_\tau u\rangle = \langle \tilde{k}\partial_n u, X_\tau \partial_\tau u\rangle + \langle \kappa(\operatorname{div} u)n, X_\tau \partial_\tau u\rangle$$

using the representations (6.1.3) along with the boundary conditions from (6.1.4). For the first term, the analysis done in Lemma 3.2 combined with the fact that $u \in \Lambda$ gives a constant c_1 independent of ε such that

$$|\langle \tilde{k}\partial_n u, X_\tau \partial_\tau u\rangle| \le |X_\tau|\frac{c_1}{\varepsilon}.$$

For the second term,

$$\langle \kappa(\operatorname{div} u)n, X_\tau \partial_\tau u\rangle = X_\tau\langle \kappa(\operatorname{div} u)n, -Ku_\tau n + \partial_\tau u_\tau \tau\rangle = -\kappa K X_\tau (\operatorname{div} u)u_\tau$$

where we recall $K = K(x)$ is the curvature function for Γ, which is uniformly bounded independent of ε. Since $u \in \Lambda$, we have the bounds

$$|u_\tau| \leq C_0, \quad \text{and} \quad |\operatorname{div} u| \leq |u^1_{x_1}| + |u^2_{x_2}| \leq \frac{2C_0}{\varepsilon}$$

and therefore there is a constant c_2 independent of ε such that

$$|\kappa K X_\tau (\operatorname{div} u) u_\tau| \leq |X_\tau| \frac{c_2}{\varepsilon}.$$

Putting these estimates together, there is a constant c independent of ε where

$$|\langle \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, X_\tau \partial_\tau u\rangle| \leq |X_\tau| \frac{c}{\varepsilon}.$$

Now, given $|X_\tau| \leq Cr$ and $|\Gamma_r| \leq Cr$ we have another constant C (independent of ε) so that

$$\left| \int_{\Gamma_r} \langle \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, X \cdot \nabla u\rangle \, ds \right| \leq \int_{\Gamma_r} |X_\tau| \frac{c}{\varepsilon} \, ds \leq \frac{Cr^2}{\varepsilon}.$$

Estimates Along $\partial B_r(x_0) \cap \Omega$:

The lefthand side of (6.4.1) along $\partial B_r(x_0) \cap \Omega$ can be written as the sum of integrals $I_1 + I_2 + I_3$ where

$$I_1 = \int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{\tilde{k}}{2} |\nabla u|^2 \langle X, n\rangle - \langle \tilde{k}\partial_n u, X \cdot \nabla u\rangle \right\} ds,$$

$$I_2 = \int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{\kappa}{2} (\operatorname{div} u)^2 \langle X, n\rangle - \langle \kappa(\operatorname{div} u)n, X \cdot \nabla u\rangle \right\} ds,$$

$$I_3 = \frac{1}{4\varepsilon^2} \int_{\partial B_r(x_0)\cap\Omega} (1 - |u|^2)^2 \langle X, n\rangle \, ds.$$

Using $X = \langle X, n\rangle n + \langle X, \tau\rangle \tau$ notice that

$$X \cdot \nabla u = (\langle X, \nabla u^1\rangle, \langle X, \nabla u^2\rangle) = \langle X, n\rangle \partial_n u + \langle X, \tau\rangle \partial_\tau u$$

which allows us to write

$$\begin{aligned} -\langle \tilde{k}\partial_n u, X \cdot \nabla u\rangle &= -\tilde{k}\langle X, n\rangle \langle \partial_n u, \partial_n u\rangle - \tilde{k}\langle X, \tau\rangle \langle \partial_n u, \partial_\tau u\rangle \\ &= -\tilde{k}\langle X, n\rangle |\partial_n u|^2 - \tilde{k}\langle X, \tau\rangle \langle \partial_n u, \partial_\tau u\rangle. \end{aligned}$$

Again by (3.1.5) it is easily estimated on $\partial B_r(x_0) \cap \Omega$ that $|\langle X, n\rangle|$, $|\langle X, \tau\rangle| \leq$

Cr. Using this paired with the identity $|\nabla u|^2 = |\partial_n u|^2 + |\partial_\tau u|^2$, Cauchy-Schwarz and Young's inequality:

$$\begin{aligned} I_1 &= \int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{\tilde{k}}{2}|\nabla u|^2\langle X, n\rangle - \tilde{k}\langle X, n\rangle|\partial_n u|^2 - \tilde{k}\langle X, \tau\rangle\langle \partial_n u, \partial_\tau u\rangle \right\} ds \\ &= \int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{\tilde{k}}{2}|\partial_\tau u|^2\langle X, n\rangle - \frac{\tilde{k}}{2}\langle X, n\rangle|\partial_n u|^2 - \tilde{k}\langle X, \tau\rangle\langle \partial_n u, \partial_\tau u\rangle \right\} ds \\ &\leq \tilde{k}Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{1}{2}|\partial_\tau u|^2 + \frac{1}{2}|\partial_n u|^2 + \frac{1}{2}|\partial_n u|^2 + \frac{1}{2}|\partial_\tau u|^2 \right\} ds \\ &= Cr\int_{\partial B_r(x_0)\cap\Omega} \tilde{k}|\nabla u|^2\, ds. \end{aligned}$$

To estimate I_2, we once again write $X = \langle X, n\rangle n + \langle X, \tau\rangle\tau$ so that

$$-\langle \kappa(\operatorname{div} u)n, X\cdot\nabla u\rangle = -\kappa\langle X, n\rangle\langle(\operatorname{div} u)n, \partial_n u\rangle - \kappa\langle X, \tau\rangle\langle(\operatorname{div} u)n, \partial_\tau u\rangle.$$

Using the same methods for the estimation of I_1, we have the rough bound

$$\begin{aligned} I_2 &\leq \kappa Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{1}{2}(\operatorname{div} u)^2 + |(\operatorname{div} u)n||\partial_n u| + |(\operatorname{div} u)n||\partial_\tau u| \right\} ds \\ &\leq \kappa Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{1}{2}(\operatorname{div} u)^2 + \frac{1}{2}(\operatorname{div} u)^2 + \frac{1}{2}|\partial_n u|^2 + \frac{1}{2}(\operatorname{div} u)^2 + \frac{1}{2}|\partial_\tau u|^2 \right\} ds \\ &= Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{ \frac{3\kappa}{2}(\operatorname{div} u)^2 + \frac{\kappa}{2}|\nabla u|^2 \right\} ds. \end{aligned}$$

Lastly,

$$I_3 = \frac{1}{4\varepsilon^2}\int_{\partial B_r(x_0)\cap\Omega} (1-|u|^2)^2\langle X, n\rangle\, ds \leq \frac{Cr}{4\varepsilon^2}\int_{\partial B_r(x_0)\cap\Omega} (1-|u|^2)^2\, ds.$$

Thus, for $C > 0$ large enough we have

$$\begin{aligned} I_1 + I_2 + I_3 &\leq Cr\int_{\partial B_r(x_0)\cap\Omega} \frac{1}{2}\left\{ \tilde{k}|\nabla u|^2 + \kappa(\operatorname{div} u)^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \right\} ds \\ &= CF^{\tilde{k}}(r) \end{aligned}$$

and therefore

$$\begin{aligned}\int_{\partial\omega_r}\left\{e_\varepsilon^{\tilde{k}}(u)\langle X,n\rangle-\langle\tilde{k}\partial_n u+\kappa(\operatorname{div}u)n,X\cdot\nabla u\rangle\right\}ds\\ =\sum_{j=1}^{3}I_j-\int_{\Gamma_r}\langle\tilde{k}\partial_n u+\kappa(\operatorname{div}u)n,X\cdot\nabla u\rangle\,ds\\ \le C\left[F_\Gamma^{\tilde{k}}(r)+\frac{Wr^2}{\varepsilon^s}\right]-\frac{W}{2\varepsilon^s}\int_{\Gamma_r}\langle u,n\rangle^2\,ds\end{aligned}$$

when u is a solution of (6.1.1) and

$$\int_{\partial\omega_r}\left\{e_\varepsilon^{\tilde{k}}(u)\langle X,n\rangle-\langle\tilde{k}\partial_n u+\kappa(\operatorname{div}u)n,X\cdot\nabla u\rangle\right\}ds\le C\left[F^{\tilde{k}}(r)+\frac{r^2}{\varepsilon}\right]$$

when u is a solution of (6.1.4).

Estimates in ω_r:

The righthand side of (6.4.1) can be written as the sum of integrals $J_1+J_2+J_3$ where

$$J_1=\int_{\omega_r}\left\{\frac{\tilde{k}}{2}|\nabla u|^2\operatorname{div}X-\tilde{k}\sum_{j,l}X_{x_j}^l\langle\partial_{x_j}u,\partial_{x_l}u\rangle\right\}dx,$$

$$J_2=\int_{\omega_r}\left\{\frac{\kappa}{2}(\operatorname{div}u)^2\operatorname{div}X-\kappa(\operatorname{div}u)\sum_{i=1}^{2}\langle\partial_{x_i}X,\nabla u^i\rangle\right\}dx,$$

$$J_3=\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\operatorname{div}X\,dx,$$

and note that the following interior estimates do not depend on the orthogonality condition associated to u. Now, since J_1 and J_3 are identical to their Lemma 3.2 counterparts, we immediately have

$$J_1\ge -Cr\int_{\omega_r}\frac{\tilde{k}}{2}|\nabla u|^2\,dx$$

and

$$J_3\ge\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx.$$

Focusing on the last term in J_2, by adding and subtracting $\kappa(\operatorname{div} u)^2$, we get

$$\begin{aligned}
\kappa(\operatorname{div} u)\sum_{i=1}^{2}\langle \partial_{x_i}X, \nabla u^i\rangle &= \pm\kappa(\operatorname{div} u)^2 + \kappa\sum_{j,l}\left(X^l_{x_j}u^1_{x_1}u^j_{x_l} + X^l_{x_j}u^j_{x_l}u^2_{x_2}\right)\\
&= \kappa(\operatorname{div} u)^2 + \kappa\sum_{i=1}^{2}\left[(X^i_{x_i}-1)(u^i_{x_i})^2 + (X^i_{x_i}-1)u^1_{x_1}u^2_{x_2}\right]\\
&\quad + \kappa\sum_{\substack{j,l\\ j\neq l}}\left(X^l_{x_j}u^1_{x_1}u^j_{x_l} + X^l_{x_j}u^j_{x_l}u^2_{x_2}\right)\\
&\leq \kappa(\operatorname{div} u)^2 + \kappa\sum_{i=1}^{2}\left[|X^i_{x_i}-1||u^i_{x_i}|^2 + |X^i_{x_i}-1||u^1_{x_1}||u^2_{x_2}|\right]\\
&\quad + \kappa\sum_{\substack{j,l\\ j\neq l}}\left(|X^l_{x_j}||u^1_{x_1}||u^j_{x_l}| + |X^l_{x_j}||u^j_{x_l}||u^2_{x_2}|\right).
\end{aligned}$$

As before, $|X^l_{x_j}-\delta_{jl}|\leq Cr$ on ω_r by (3.1.6) and applying Young's inequality to each of the pairs $|u^1_{x_1}||u^l_{x_j}|$ and $|u^l_{x_j}||u^2_{x_2}|$ there exists $C>0$ such that

$$\kappa(\operatorname{div} u)\sum_{i=1}^{2}\langle \partial_{x_i}X, \nabla u^i\rangle \leq \kappa(\operatorname{div} u)^2 + C\kappa r|\nabla u|^2.$$

Now, since

$$\operatorname{div} X = X^1_{x_1} + X^2_{x_2} = 2 + (X^1_{x_1}-1) + (X^2_{x_2}-1) \geq 2 - 2Cr \qquad (6.4.6)$$

we write

$$\begin{aligned}
J_2 &\geq \int_{\omega_r}\left\{\frac{\kappa}{2}(\operatorname{div} u)^2\operatorname{div} X - \kappa(\operatorname{div} u)^2 - C\kappa r|\nabla u|^2\right\}dx,\\
&\geq \int_{\omega_r}\left\{\kappa(\operatorname{div} u)^2 - \kappa(\operatorname{div} u)^2 - \kappa Cr(\operatorname{div} u)^2 - C\kappa r|\nabla u|^2\right\}dx\\
&\geq -Cr\int_{\omega_r}\kappa(\operatorname{div} u)^2\,dx - Cr\int_{\omega_r}\tilde{k}|\nabla u|^2\,dx.
\end{aligned}$$

Therefore we can find C large enough so that

$$\begin{aligned}\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx - Cr\int_{\omega_r}\frac{\kappa}{2}(\operatorname{div} u)^2\,dx - Cr\int_{\omega_r}\frac{\tilde{k}}{2}|\nabla u|^2\,dx \\ \leq J_1 + J_2 + J_3 \\ \leq C\left[F_\Gamma^{\tilde{k}}(r) + \frac{Wr^2}{\varepsilon^s}\right] - \frac{W}{2\varepsilon^s}\int_{\Gamma_r}\langle u, n\rangle^2\,ds\end{aligned}$$

when u is a solution of (6.1.1) and

$$\begin{aligned}\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx - Cr\int_{\omega_r}\frac{\kappa}{2}(\operatorname{div} u)^2\,dx - Cr\int_{\omega_r}\frac{\tilde{k}}{2}|\nabla u|^2\,dx \\ \leq C\left[F^{\tilde{k}}(r) + \frac{r^2}{\varepsilon}\right]\end{aligned}$$

when u is a solution of (6.1.4). This completes the proof for the case when $\tilde{k} = k_b$. Inequalities (6.4.4) and (6.4.5) for $\tilde{k} = k_s$ are handled in a similar way. To see this, we still assume all conditions given in the preamble of this step but now take $\psi = X$ in equation (6.4.2).

Estimates Along Γ_r:

(W.O.*) Case

The estimates here are identical to what we just witnessed for divergence penalization. In particular,

$$\begin{aligned}-\int_{\Gamma_r}\langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X_\tau\partial_\tau u\rangle\,ds \leq -\frac{W}{2\varepsilon^s}\int_{\Gamma_r}\langle u, n\rangle^2\,ds \\ + \frac{Wr}{2\varepsilon^s}\sum_{x\in\partial\Gamma_r(x_0)}\langle u, n\rangle^2 + \frac{CWr^2}{\varepsilon^s}.\end{aligned}$$

(S.O.*) Case

As before, we decompose and analyze the inner product

$$\langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X_\tau\partial_\tau u\rangle.$$

Using (6.1.3) and the strong boundary data from (6.1.5),

$$\begin{aligned}\tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau &= \tilde{k}\partial_n u_n n + \tilde{k}u_\tau\partial_n\tau + (\tilde{k}\partial_n u_\tau + \kappa(\operatorname{curl} u))\tau \\ &= \tilde{k}\partial_n u_n n + \tilde{k}u_\tau\partial_n\tau.\end{aligned}$$

Using (6.1.3) once more,

$$\begin{aligned}\langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X_\tau \partial_\tau u\rangle &= X_\tau\langle \tilde{k}\partial_n u_n n + \tilde{k}u_\tau \partial_n \tau, -Ku_\tau n + \partial_\tau u_\tau \tau\rangle \\ &= -X_\tau K\tilde{k}u_\tau \partial_n u_n - X_\tau K\tilde{k}(u_\tau)^2\langle \partial_n \tau, n\rangle.\end{aligned}$$

Since $u \in \Lambda$, as before there is c independent of ε such that

$$|\langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X_\tau \partial_\tau u\rangle| \le |X_\tau|\frac{c}{\varepsilon}$$

and therefore

$$\left|\int_{\Gamma_r} \langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X\cdot\nabla u\rangle\, ds\right| \le \int_{\Gamma_r} |X_\tau|\frac{c}{\varepsilon}\, ds \le \frac{Cr^2}{\varepsilon}$$

for $C > 0$ independent of ε.

Estimates Along $\partial B_r(x_0)\cap\Omega$:

The lefthand side of (6.4.2) along $\partial B_r(x_0)\cap\Omega$ can be written $I_4+I_5+I_6$ where

$$\begin{aligned} I_4 &= \int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{\tilde{k}}{2}|\nabla u|^2\langle X, n\rangle - \langle \tilde{k}\partial_n u, X\cdot\nabla u\rangle\right\} ds, \\ I_5 &= \int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{\kappa}{2}(\operatorname{curl} u)^2\langle X, n\rangle - \langle \kappa(\operatorname{curl} u)\tau, X\cdot\nabla u\rangle\right\} ds, \\ I_6 &= \frac{1}{4\varepsilon^2}\int_{\partial B_r(x_0)\cap\Omega} (1-|u|^2)^2\langle X, n\rangle\, ds. \end{aligned}$$

Since the estimates for I_4 and I_6 are identical to the estimates of I_1 and I_3 above, only I_5 is considered. Writing $X = \langle X, n\rangle n + \langle X, \tau\rangle\tau$, the second term in I_5 can be written

$$-\langle \kappa(\operatorname{curl} u)\tau, X\cdot\nabla u\rangle = -\kappa\langle X, n\rangle\langle(\operatorname{curl} u)\tau, \partial_n u\rangle - \kappa\langle X, \tau\rangle\langle(\operatorname{curl} u)\tau, \partial_\tau u\rangle.$$

Using (3.1.5), Cauchy-Schwarz and Young's inequality:

$$\begin{aligned} I_5 &\le \kappa Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{1}{2}(\operatorname{curl} u)^2 + |(\operatorname{curl} u)\tau||\partial_n u| + |(\operatorname{curl} u)\tau||\partial_\tau u|\right\} ds \\ &\le \kappa Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{1}{2}(\operatorname{curl} u)^2 + \frac{1}{2}(\operatorname{curl} u)^2 + \frac{1}{2}|\partial_n u|^2 + \frac{1}{2}(\operatorname{curl} u)^2 + \frac{1}{2}|\partial_\tau u|^2\right\} ds \\ &= Cr\int_{\partial B_r(x_0)\cap\Omega} \left\{\frac{3\kappa}{2}(\operatorname{curl} u)^2 + \frac{\kappa}{2}|\nabla u|^2\right\} ds. \end{aligned}$$

Thus, for $C > 0$ large enough we have

$$\begin{aligned} I_4 + I_5 + I_6 &\le Cr \int_{\partial B_r(x_0)\cap\Omega} \frac{1}{2}\left\{\tilde{k}|\nabla u|^2 + \kappa(\operatorname{curl} u)^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\right\} ds \\ &= CF^{\tilde{k}}(r) \end{aligned}$$

and therefore

$$\begin{aligned} &\int_{\partial\omega_r} \left\{e_\varepsilon^{\tilde{k}}(u)\langle X, n\rangle - \langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X\cdot\nabla u\rangle\right\} ds \\ &\qquad\qquad = \sum_{j=4}^{6} I_j - \int_{\Gamma_r} \langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X\cdot\nabla u\rangle\, ds \\ &\qquad\qquad \le C\left[F_\Gamma^{\tilde{k}}(r) + \frac{Wr^2}{\varepsilon^s}\right] - \frac{W}{2\varepsilon^s}\int_{\Gamma_r} \langle u, n\rangle^2\, ds. \end{aligned}$$

when u is a solution of (6.1.2) and

$$\int_{\partial\omega_r} \left\{e_\varepsilon^{\tilde{k}}(u)\langle X, n\rangle - \langle \tilde{k}\partial_n u + \kappa(\operatorname{curl} u)\tau, X\cdot\nabla u\rangle\right\} ds \le C\left[F^{\tilde{k}}(r) + \frac{r^2}{\varepsilon}\right]$$

when u is a solution of (6.1.5).

<u>Estimates in ω_r:</u>

The righthand side of (6.4.2) can be written as the sum of three integrals $J_4 + J_5 + J_6$ where

$$\begin{aligned} J_4 &= \int_{\omega_r} \left\{\frac{\tilde{k}}{2}|\nabla u|^2 \operatorname{div} X - \tilde{k}\sum_{j,l} X^l_{x_j}\langle \partial_{x_j} u, \partial_{x_l} u\rangle\right\} dx, \\ J_5 &= \int_{\omega_r} \left\{\frac{\kappa}{2}(\operatorname{curl} u)^2 \operatorname{div} X - \kappa(\operatorname{curl} u)\sum_{i=1}^{2}(-1)^{i-1}\langle \partial_{x_i} X, \nabla u^{3-i}\rangle\right\} dx, \\ J_6 &= \frac{1}{4\varepsilon^2}\int_{\omega_r} (1-|u|^2)^2 \operatorname{div} X\, dx. \end{aligned}$$

The integrals J_4 and J_6 are estimated precisely like J_1 and J_3 so J_5 is the only integral that needs to be treated. Focusing on the last term in J_5, we add and subtract $(\operatorname{curl} u)^2$ to obtain

$$
\begin{aligned}
(\operatorname{curl} u) \sum_{i=1}^{2} (-1)^{i-1} \langle \partial_{x_i} X, \nabla u^{3-i} \rangle \\
&= (\operatorname{curl} u)^2 + \sum_{i=1}^{2} \left[(X^i_{x_i} - 1)(u^{3-i}_{x_i})^2 - (X^i_{x_i} - 1) u^1_{x_2} u^2_{x_1} \right] \\
&\quad + X^2_{x_1} u^2_{x_1} u^2_{x_2} + X^1_{x_2} u^1_{x_1} u^1_{x_2} - X^1_{x_2} u^1_{x_1} u^2_{x_1} - X^2_{x_1} u^1_{x_2} u^2_{x_2} \\
&\le (\operatorname{curl} u)^2 + \sum_{i=1}^{2} \left[|X^i_{x_i} - 1| |u^{3-i}_{x_i}|^2 + |X^i_{x_i} - 1| |u^1_{x_2}| |u^2_{x_1}| \right] \\
&\quad + |X^2_{x_1}| |u^2_{x_1}| |u^2_{x_2}| + |X^1_{x_2}| |u^1_{x_1}| |u^1_{x_2}| + |X^1_{x_2}| |u^1_{x_1}| |u^2_{x_1}| \\
&\quad\quad + |X^2_{x_1}| |u^1_{x_2}| |u^2_{x_2}|.
\end{aligned}
$$

By (3.1.6) we have $|X^l_{x_j} - \delta_{jl}| \le Cr$ on ω_r and applying Young's inequality to each of the derivative pairs, there exists $C > 0$ such that

$$
\kappa(\operatorname{curl} u) \sum_{i=1}^{2} (-1)^{i-1} \langle \partial_{x_i} X, \nabla u^{3-i} \rangle \le \kappa(\operatorname{curl} u)^2 + C\kappa r |\nabla u|^2.
$$

With this estimate and utilizing (6.4.6) once more,

$$
\begin{aligned}
J_5 &\ge \int_{\omega_r} \left\{ \frac{\kappa}{2} (\operatorname{curl} u)^2 \operatorname{div} X - \kappa(\operatorname{curl} u)^2 - C\kappa r |\nabla u|^2 \right\} dx \\
&\ge \int_{\omega_r} \left\{ \kappa(\operatorname{curl} u)^2 - \kappa(\operatorname{curl} u)^2 - C\kappa r (\operatorname{curl} u)^2 - C\kappa r |\nabla u|^2 \right\} dx \\
&\ge -Cr \int_{\omega_r} \kappa(\operatorname{curl} u)^2 \, dx - Cr \int_{\omega_r} \tilde{k} |\nabla u|^2 \, dx.
\end{aligned}
$$

Therefore by taking $C > 0$ large enough

$$
\begin{aligned}
\frac{1}{4\varepsilon^2} \int_{\omega_r} (1 - |u|^2)^2 \, dx - Cr \int_{\omega_r} \frac{\kappa}{2} (\operatorname{curl} u)^2 \, dx - Cr \int_{\omega_r} \frac{\tilde{k}}{2} |\nabla u|^2 \, dx \\
&\le J_4 + J_5 + J_6 \\
&\le C \left[F^{\tilde{k}}_{\Gamma}(r) + \frac{W r^2}{\varepsilon^s} \right] - \frac{W}{2\varepsilon^s} \int_{\Gamma_r} \langle u, n \rangle^2 \, ds
\end{aligned}
$$

when u is a solution of (6.1.2) and

$$\frac{1}{4\varepsilon^2}\int_{\omega_r}(1-|u|^2)^2\,dx - Cr\int_{\omega_r}\frac{\kappa}{2}(\operatorname{curl} u)^2\,dx - Cr\int_{\omega_r}\frac{\tilde{k}}{2}|\nabla u|^2\,dx \leq C\left[F^{\tilde{k}}(r)+\frac{r^2}{\varepsilon}\right]$$

when u is a solution of (6.1.5) which finishes the proof. □

6.5 η-Compactness & Final Results

Upon reflection, it is convenient to notice that from Section 3.3 and onwards, all of the arguments used to obtain Theorem 1.1 (bad set coverings and lower bounds) did not explicitly depend on the boundary conditions given in the Euler-Lagrange systems. That is, our explicit use of the boundary conditions given in these systems ended with the proof of Theorem 3.3 for η-compactness. Thus, once we show η-compactness for $\mathscr{F}_\varepsilon$ and $\mathscr{F}_\varepsilon^W$, we will almost be finished with our analysis.

Theorem 6.5 (η-Compactness)**.** *Let $\frac{3}{4}s \leq \beta < \gamma < s \leq 1$. There exists constants η, $\tilde{C}$, $\varepsilon_0 > 0$ such that for any solution $u_\varepsilon \in \Lambda$ of* (6.1.1), (6.1.2), (6.1.4) *or* (6.1.5) *with $\varepsilon \in (0,\varepsilon_0)$, if $x_0 \in \overline{\Omega}$ and*

$$\mathscr{F}_\varepsilon^W(u_\varepsilon;\omega_{2\varepsilon^\beta}(x_0)) \leq \eta|\ln\varepsilon|,$$

then

$$|u_\varepsilon| \geq \frac{1}{2} \quad \text{in } \omega_{\varepsilon^\gamma}(x_0), \tag{6.5.1}$$

$$|\langle u_\varepsilon, n\rangle| \leq \frac{1}{4} \quad \text{on } \Gamma\cap\overline{\omega_{\varepsilon^\gamma}(x_0)}, \tag{6.5.2}$$

$$\frac{1}{4\varepsilon^2}\int_{\omega_{\varepsilon^\gamma}(x_0)}(1-|u_\varepsilon|^2)^2\,dx + \frac{W}{2\varepsilon^s}\int_{\Gamma\cap\overline{\omega_{\varepsilon^\gamma}(x_0)}}\langle u_\varepsilon, n\rangle^2\,ds \leq \tilde{C}\eta. \tag{6.5.3}$$

Remark 6.2. *In the specific case that u_ε is a solution to* (6.1.4) *or* (6.1.5)*, note that $\mathscr{F}_\varepsilon^W(u_\varepsilon)$ is replaced by $\mathscr{F}_\varepsilon(u_\varepsilon)$ and $s=1$ in the statement of Theorem 6.5. Moreover, the bound* (6.5.2) *is trivially satisfied and* (6.5.3) *reduces to*

$$\frac{1}{4\varepsilon^2}\int_{\omega_{\varepsilon^\gamma}(x_0)}(1-|u_\varepsilon|^2)^2\,dx \leq \tilde{C}\eta.$$

Proof. Due to the structure of the estimates provided in Lemma 6.4, the arguments needed for conditions (6.5.1) and (6.5.3) are identical to that of theorem

3.3. Thus, it only remains to show condition (6.5.2) for Λ-solutions of (6.1.1) and (6.1.2).

It is enough to show that the tangential derivative of u satisfies the bound

$$\int_{\partial\omega_{r_\varepsilon}} |\partial_\tau u_\varepsilon|^2\, ds \leq C\varepsilon^{-s}$$

for a constant C independent of ε. The rest of the proof will follow from this. Suppose first u solves (6.1.1). Recall from the proof of Lemma 6.4 that for $x_0 \in \Gamma$, the radius bound r_0 was chosen small enough so that $\omega_r(x_0)$ could be assumed to be strictly starshaped around some $x_1 \in \omega_r(x_0)$. Taking $r = r_\varepsilon$, the starshape constraint allows us to write

$$\langle x - x_1, n\rangle \geq \frac{r_\varepsilon}{4} \quad \text{on } \partial\omega_{r_\varepsilon}(x_0)$$

where n is the unit normal vector to $\partial\omega_{r_\varepsilon}(x_0)$. We begin by setting $\psi = x - x_1$ in (6.4.1) and observing the integrand

$$e_\varepsilon^{\tilde{k}}(u)\langle\psi, n\rangle - \langle\tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, \psi\cdot\nabla u\rangle.$$

Setting $\Psi = \tilde{k}\partial_n u + \kappa(\operatorname{div} u)n$, we calculate

$$\langle\Psi, \psi\cdot\nabla u\rangle = \langle\Psi, \partial_n u\rangle\langle x - x_1, n\rangle + \langle\Psi, \partial_\tau u\rangle\langle x - x_1, \tau\rangle.$$

Using an orthogonal decomposition for the gradient and the star-shape condition,

$$e_\varepsilon^{\tilde{k}}(u)\langle\psi, n\rangle \geq \frac{\tilde{k}}{2}|\partial_n u|^2\langle x - x_1, n\rangle + \frac{\tilde{k}}{2}|\partial_\tau u|^2\langle x - x_1, n\rangle \geq \frac{\tilde{k}r_\varepsilon}{8}|\partial_\tau u|^2 + \frac{\tilde{k}r_\varepsilon}{8}|\partial_n u|^2.$$

Going back to the previous term, applying Cauchy-Schwarz and the Peter-Paul inequality,

$$\begin{aligned}|\langle\Psi, \psi\cdot\nabla u\rangle| &\leq |\Psi||\partial_n u||x - x_1||n| + |\Psi||\partial_\tau u||x - x_1||\tau| \\ &\leq 2r_\varepsilon|\Psi||\partial_n u| + 2r_\varepsilon|\Psi||\partial_\tau u| \\ &\leq \frac{8r_\varepsilon}{\tilde{k}}|\Psi|^2 + \frac{\tilde{k}r_\varepsilon}{8}|\partial_n u|^2 + \frac{16r_\varepsilon}{\tilde{k}}|\Psi|^2 + \frac{\tilde{k}r_\varepsilon}{16}|\partial_\tau u|^2.\end{aligned}$$

Thus, we have the lower bound

$$e_\varepsilon^{\tilde{k}}(u)\langle\psi, n\rangle - \langle\tilde{k}\partial_n u + \kappa(\operatorname{div} u)n, \psi\cdot\nabla u\rangle \geq \frac{\tilde{k}r_\varepsilon}{16}|\partial_\tau u|^2 - \frac{24r_\varepsilon}{\tilde{k}}|\Psi|^2. \quad (6.5.4)$$

Observing the righthand side of (6.4.1), we know

$$\operatorname{div}(x-x_1)=2, \quad \partial_{x_j}\psi^l=\delta_{j,l} \tag{6.5.5}$$

and so it holds that

$$e_\varepsilon^{\tilde{k}}(u)\operatorname{div}\psi-\tilde{k}\sum_{j,l}\psi^l_{x_j}\langle\partial_{x_j}u,\partial_{x_l}u\rangle-\kappa(\operatorname{div}u)\sum_{i=1}^{2}\langle\partial_{x_i}\psi,\nabla u^i\rangle=\frac{1}{2\varepsilon^2}(1-|u|^2)^2.$$

With this and the lower estimate (6.5.4), there is a constant C independent of ε such that

$$\int_{\partial\omega_{r_\varepsilon}}|\partial_\tau u|^2\,ds\le C\left[\int_{\partial\omega_{r_\varepsilon}}|\Psi|^2\,ds+\int_{\omega_{r_\varepsilon}}\frac{1}{4r_\varepsilon\varepsilon^2}(1-|u|^2)^2\,dx\right].$$

Decomposing the boundary $\partial\omega_{r_\varepsilon}=\Gamma_{r_\varepsilon}\cup(\partial B_{r_\varepsilon}(x_0)\cap\Omega)$ and using the known boundary condition, we have

$$|\Psi|^2=\frac{W^2}{\varepsilon^{2s}}\langle u,n\rangle^2$$

on the boundary portion Γ_{r_ε}. On $\partial B_{r_\varepsilon}(x_0)\cap\Omega$, we have the crude estimate

$$\begin{aligned}\int_{\partial B_{r_\varepsilon(x_0)}\cap\Omega}|\Psi|^2\,ds&\le\int_{\partial B_{r_\varepsilon(x_0)}\cap\Omega}\left(2\tilde{k}^2|\nabla u|^2+2\kappa^2(\operatorname{div}u)^2+\frac{1}{\varepsilon^2}(1-|u|^2)^2\right)ds\\&\le\tilde{c}\frac{F_\Gamma^{\tilde{k}}(r_\varepsilon)}{r_\varepsilon}\end{aligned}$$

where $\tilde{c}$ is a constant independent of ε. The desired bound is obtained from these observations as in the proof of Theorem 3.3. Now suppose u solves (6.1.2) and redefine $\Psi=\tilde{k}\partial_n u+\kappa(\operatorname{curl}u)\tau$. Setting $\psi=x-x_1$ in (6.4.2), the lefthand side integrand reads

$$e_\varepsilon^{\tilde{k}}(u)\langle x-x_1,n\rangle-\langle\Psi,(x-x_1)\cdot\nabla u\rangle.$$

As before, we calculate

$$\langle\Psi,(x-x_1)\cdot\nabla u\rangle=\langle\Psi,\partial_n u\rangle\langle x-x_1,n\rangle+\langle\Psi,\partial_\tau u\rangle\langle x-x_1,\tau\rangle$$

and we have lower bound

$$e_\varepsilon^{\tilde{k}}(u)\langle x-x_1,n\rangle\ge\frac{\tilde{k}r_\varepsilon}{8}|\partial_\tau u|^2+\frac{\tilde{k}r_\varepsilon}{8}|\partial_n u|^2.$$

Observing the integrand on the righthand side of (6.4.2), (6.5.5) allows us to write

$$e_\varepsilon^{\tilde{k}}(u)\operatorname{div}\psi - \tilde{k}\sum_{j,l}\psi_{x_j}^l\langle\partial_{x_j}u,\partial_{x_l}u\rangle - \kappa(\operatorname{curl}u)\sum_{i=1}^{2}(-1)^{i-1}\langle\partial_{x_i}\psi,\nabla u^{3-i}\rangle$$
$$= \frac{1}{2\varepsilon^2}(1-|u|^2)^2$$

and thus we have as before a constant $C>0$ independent of ε such that

$$\int_{\partial\omega_{r_\varepsilon}}|\partial_\tau u|^2\,ds \le C\left[\int_{\partial\omega_{r_\varepsilon}}|\Psi|^2\,ds + \int_{\omega_{r_\varepsilon}}\frac{1}{4r_\varepsilon\varepsilon^2}(1-|u|^2)^2\,dx\right].$$

The rest follows as before by replacing $\kappa(\operatorname{div}u)n$ with $\kappa(\operatorname{curl}u)\tau$. □

Now that η-compactness is taken care of, it is clear from the analysis for G_ε and G_ε^W in the earlier chapters that we also obtain a finite bad ball covering of the bad set S_ε. Moreover, if we define

$$G_\varepsilon^{W,\tilde{k}}(v) = \mathscr{F}_\varepsilon^W(v) - h_{\tilde{k}}(v),$$

then it is easy to see that the lower bounds for $G_\varepsilon^{W,\tilde{k}}$ are lower bounds for $\mathscr{F}_\varepsilon^W$ since $\mathscr{F}_\varepsilon^W(v) \ge G_\varepsilon^{W,\tilde{k}}(v)$ for all $v\in H^1(\Omega;\mathbb{R}^2)$ (and $\mathcal{H}_\tau(\Omega)$). Since the Dirichlet energy component of $G_\varepsilon^{W,\tilde{k}}$ differs only by a multiple of $\tilde{k}$ when compared to G_ε^W, all lower bounds from the previous chapters can be adjusted through multiplication by $\tilde{k}$ which is the correct adjustment needed for $\mathscr{F}_\varepsilon^W$. In particular, we have:

Corollary 6.6. *Suppose $x_0\in\overline{\Omega}$ and assume that $u\in\Lambda$ with $|u|\ge 1/2$ in $A_{r,R}(x_0)$ and $|\langle u,n\rangle|\le 1/4$ on $\Gamma_R^\pm$. Additionally, suppose that there is some number K such that*

$$\mathscr{F}_\varepsilon^W(u;\Omega)\le K|\ln\varepsilon| + K,$$
$$\frac{1}{\varepsilon^2}\int_{\omega_{\varepsilon^\gamma}(x_0)}(1-|u|^2)^2\,dx + \frac{1}{\varepsilon^s}\int_{\Gamma_{\varepsilon^\gamma}}\langle u,n\rangle^2\,ds \le K.$$

There exists a constant C depending only on Ω, γ, $\tilde{k}$, κ and K such that:

(i) If $B_R(x_0)\subset\Omega$, $\varepsilon\le r<R\le r_0$ and $d\ne 0$,

$$\frac{\tilde{k}}{2}\int_{A_{r,R}(x_0)}|\nabla u|^2\,dx \ge \tilde{k}d^2\pi\ln\left(\frac{R}{r}\right) - C. \tag{6.5.6}$$

(ii) If $x_0 \in \Gamma$, $\varepsilon^s \le r < R \le r_0$ and $D \neq 0$,

$$\frac{\tilde{k}}{2}\int_{A_{r,R}(x_0)} |\nabla u|^2\,dx \ge \frac{\tilde{k}}{2} D^2 \pi \ln\left(\frac{R}{r}\right) - C. \tag{6.5.7}$$

where d and D are the associated degree and boundary index for u.

Using the fact that $\mathcal{D} = \deg(\tau;\Gamma) = 1$, we also have:

Corollary 6.7. *Let $\{p_1, \dots, p_I\}$ and $\{q_1, \dots, q_J\}$ denote the centers of interior balls and boundary balls respectively for the cover $\mathcal{S}_\sigma$ for S_{ε_n} with associated degrees d_i and boundary indices D_j. Then*

$$\mathscr{F}^W_\varepsilon(u_{\varepsilon_n};\mathcal{S}_\sigma) \ge \tilde{k}\pi\left(\sum_{i=1}^I d_i + \frac{s}{2}\sum_{j=1}^J D_j\right)\ln\left(\frac{\sigma}{\varepsilon_n}\right) - C \ge \tilde{k}\pi s \ln\left(\frac{\sigma}{\varepsilon_n}\right) - C$$

where C is a constant independent of ε and σ.

At this point, we are ready to discuss the little remaining results needed to prove Theorem 1.2. As in the proof of Theorem 1.1, we can finish with a string of easily obtained corollaries to conclude. In fact, most of the work has already been shown in Corollaries 5.2 – 5.6. All that is needed is to specify in which situations these corollaries apply to our new functionals $\mathscr{F}_\varepsilon$ and $\mathscr{F}^W_\varepsilon$.

The main observation to consider is that Corollaries 5.2 – 5.6 worked on the basis that the upper bound and lower bound for G^W_ε both had the factor $\pi s\mathcal{D}$ in front of their logarithmic terms. Unfortunately, we do not have this luxury in all of the cases for $\mathscr{F}_\varepsilon$ and $\mathscr{F}^W_\varepsilon$. However, it is true for all strong and weak solutions (for all values of $s \in (0,1]$) in the case that $\tilde{k} = k_s$ (curl penalization). Thus, Corollaries 5.2 – 5.6 also apply to solutions in this case where we take $g^\perp = n$ and replace G^W_ε with $\mathscr{F}^W_\varepsilon$. The main difference here is that Corollary 5.6 would need to be slightly modified as follows:

Corollary 6.8. *If $\tilde{k} = k_s$, there exists a constant C independent of ε such that*

$$\frac{1}{4\varepsilon_n^2}\int_\Omega (1-|u_\varepsilon|^2)^2\,dx + \frac{\kappa}{2}\int_\Omega (\operatorname{curl} u_{\varepsilon_n})^2\,dx + \frac{W}{2\varepsilon_n^s}\int_\Gamma \langle u_{\varepsilon_n}, n\rangle^2\,ds \le C.$$

We also have an analogue of Lemma 5.7 in the case that $\tilde{k} = k_s$. Let Σ be as in Definition 5.1.

Lemma 6.9. *Suppose $\tilde{k} = k_s$. There exists $u_0 \in H^1(\Omega \setminus \Sigma; \mathbb{R}^2)$ such that along a subsequence $\varepsilon_n \to 0$ we have*

$$u_{\varepsilon_n} \rightharpoonup u_0 \quad \textit{weakly in } H^1_{loc}(\overline{\Omega} \setminus \Sigma; \mathbb{R}^2)$$

for solutions of (W.O.*) *or* (S.O.*). *Moreover, $|u_0| = 1$ almost everywhere and*

$$\deg(u_0; \partial B_r(p_i)) = d_i = 1, \quad \operatorname{ind}(u_0; \partial B_r(q_j) \cap \Omega) = D_j = 1$$

for all $i = 1, \ldots, I$, $j = 1, \ldots, J$.

The proof of this lemma still follows the uniform bounding techniques of [39] and [10, Chapter VI]. However, we are not claiming in this case that u_0 is an $\mathbb{S}^1$-valued harmonic map. The estimates required to show this heavily depend on the structure of the Ginzburg-Landau equations for G_ε^W (see [9]) which we simply do not have in this case. At most, we know that the limiting director should be equal to one almost everywhere in the domain. This fact is easily seen by Corollary 6.8 since the uniform bound will imply $|u_{\varepsilon_n}| \to 1$ in $L^2(\Omega)$ and thus $|u_0| = 1$ almost everywhere. Combining the above corollaries with Lemma 6.9 proves Theorem 1.2 (a).

To tackle the problem when $\tilde{k} = k_b$, we observe that the logarithmic coefficient of the upper bound (Proposition 6.1) for $\mathscr{F}_\varepsilon^W$ (or $\mathscr{F}_\varepsilon$) matches the logarithmic coefficient of our lower bound only in the case of strong orthogonality or weak orthogonality when $s = 1$. The reason for this 'miss-match' is because it is most likely that the lower bound is not optimal for divergence penalization. We refer the reader to Section 7.3 for a small discussion on this topic. Nevertheless, Corollaries 5.2 – 5.6 apply to solutions in these special cases with $g^\perp = n$ and G_ε^W replaced by $\mathscr{F}_\varepsilon^W$. Again, the main difference comes in comparing Corollary 5.6 to this case. The modification is as follows:

Corollary 6.10. *If $\tilde{k} = k_b$, there exists a constant C independent of ε such that*

$$\frac{1}{4\varepsilon_n^2} \int_\Omega (1 - |u_\varepsilon|^2)^2 \, dx + \frac{\kappa}{2} \int_\Omega (\operatorname{div} u_{\varepsilon_n})^2 \, dx + \frac{W}{2\varepsilon_n^s} \int_\Gamma \langle u_{\varepsilon_n}, n \rangle^2 \, ds \le C$$

provided $\{u_{\varepsilon_n}\}$ is a sequence of minimizers for (S.O.*) *or* (W.O.*) *with $s = 1$.*

Finally, we also have

Lemma 6.11. *Suppose $\tilde{k} = k_b$. There exists $u_0 \in H^1(\Omega \setminus \Sigma; \mathbb{R}^2)$ such that along a subsequence $\varepsilon_n \to 0$ we have*

$$u_{\varepsilon_n} \rightharpoonup u_0 \quad \textit{weakly in } H^1_{loc}(\overline{\Omega} \setminus \Sigma; \mathbb{R}^2)$$

for solutions of (S.O.*) *or* (W.O.*) *with* $s = 1$. *Moreover,* $|u_0| = 1$ *almost everywhere and*

$$\deg(u_0; \partial B_r(p_i)) = d_i = 1, \quad \mathrm{ind}(u_0; \partial B_r(q_j) \cap \Omega) = D_j = 1$$

for all $i = 1, \ldots, I$, $j = 1, \ldots, J$.

This finishes the proof of Theorem 1.2 (b).

Chapter 7

Future Problems

Throughout this work, we have come across several questions that provoked some interest for future problems. In this chapter, we provide three problems that would be interesting to tackle in projects to come.

7.1 The Core Energy of a Boundary Vortex

In Section 5.2, we showed that the renormalized energy associated to boundary vortices on a unit disk with tangential forcing in the strong orthogonality setting yielded a smaller minimum when compared to the renormalized energy for the interior vortex. While this is a good first step, the main drawback in Section 5.2 was that we could not precisely state that boundary vortices are energetically preferable since the relationship between the core energy for an interior vortex Q_Ω and the core energy for a boundary vortex Q_Γ is unknown. As stated in Section 5.2, we suspect that $Q_\Gamma = \frac{1}{2}Q_\Omega$, since a simple boundary vortex after a blow-up should look like a full interior vortex 'cut in half'. To somewhat justify this, we give a possible first step in what might be a fruitful exploration.

Suppose $\{q_\varepsilon\}$ is a sequence of approximate boundary vortices for u_ε which converge to $q_0 \in \Gamma$. Define

$$v^\varepsilon(x) = u_\varepsilon(q_\varepsilon + \varepsilon x)$$

so that we can perform an ε blow-up around the points q_ε. With this change of variables, observing v^ε through a blow-up near the boundary as $\varepsilon \to 0$, one might be able to show that after a translation and rotation, the existence of a limiting function v that satisfies

$$\begin{cases} -\Delta v = (1-|v|^2)v & \text{in } \mathbb{R}^2_+, \\ v^2 = 0 & \text{on } [x_2 = 0], \\ \partial_n v^1 = 0 & \text{on } [x_2 = 0]. \end{cases}$$

Since the boundary index for u_ε is $D = 1$, reflecting v across the x_1-axis should yield a solution v_* to the Ginzburg-Landau equation $-\Delta v_* = (1-|v_*|^2)v_*$ in $\mathbb{R}^2$ with a vortex of degree $d = 1$. Then, one could employ [13, Theorem 1] to show

$$\frac{1}{2}\int_{\mathbb{R}^2}(1-|v_*|^2)^2\,dx = \pi$$

which implies v_* is the unique degree one radial solution. One could then maybe take an approach close to that of [38, Proposition 3.11] to conclude $Q_\Gamma = \frac{1}{2}Q_\Omega$.

Remark 7.1. *Of course, it is not necessary that one requires $Q_\Gamma = \frac{1}{2}Q_\Omega$ in order to show boundary vortices are energetically preferable. Perhaps it would be easier to show $Q_\Gamma \leq \frac{1}{2}Q_\Omega$, which would also yield the result.*

7.2 Uniformly Bounded Minimizers for $\mathscr{F}_\varepsilon^W$

One of the most surprising issues of this thesis comes from the inability to show that minimizers of $\mathscr{F}_\varepsilon$ or $\mathscr{F}_\varepsilon^W$ satisfy a uniform bound. Several attempts have been made to do this, which include regularity techniques as in [14] and perimeter estimation methods for bad sets as done in [2]. It is not yet clear how to show that minimizers truly belong to the bounded class Λ as defined in Definition 6.2.

7.3 Lower Bounds For Splay Penalization

At the end of Section 6.5, we showed that the lower bound for $\mathscr{F}_\varepsilon$ and $\mathscr{F}_\varepsilon^W$ was optimal in the case of curl penalization. However, it turns out that this is not the case for divergence penalization. In some sense, this was expected. Indeed, since the boundary data forces a curl-free type vector field near the boundary, there is not much energy contribution coming from the curl term of the energy. Thus, the lower bound derived strictly from the Dirichlet energy is enough. On the other hand, since boundary vortices will most likely look curl-free, this means that the divergence term in the energy (when $\tilde{k} = k_b$) will contribute a significant amount of energy. Assuming that the energy contribution would be close to that of the gradient, we would expect to find that on an annulus

$A_{r,R}$ centered on the boundary,

$$\kappa \int_{A_{r,R}} (\operatorname{div} u)^2 \, dx \geq \pi \kappa D^2 \ln\left(\frac{R}{r}\right) - C$$

for a constant $C > 0$ independent of ε.

Remark 7.2. *Upon using the polar representation for u as done in the proof of Theorem 4.5, it is clear that finding such a lower bound for divergence would incredibly complicated.*

Adding this to the estimate from the gradient we already have and noting that $\tilde{k} + \kappa = \hat{k}$, one should expect that boundary vortices have an energy estimate

$$\int_{A_{r,R}} (\tilde{k}|\nabla u|^2 + \kappa(\operatorname{div} u)^2) \, dx \geq \pi \hat{k} D^2 \ln\left(\frac{R}{r}\right) - C.$$

Away from the boundary, vortices in the interior can potentially orient themselves to have a divergence-free profile. Thus, the interior estimates we currently have would not need change. Therefore, we suspect that the optimal lower bound on the bad set cover $\mathcal{S}_\sigma$ is

$$\mathscr{F}_\varepsilon^W(u_\varepsilon; \mathcal{S}_\sigma) \geq \pi \left(\tilde{k} \sum_{i=1}^{I} d_i + \frac{\hat{k}s}{2} \sum_{j=1}^{J} D_j \right) \ln\left(\frac{\sigma}{\varepsilon}\right) - C.$$

Pairing this with the known upper bound $\mathscr{F}_\varepsilon^W(u_\varepsilon) \leq \pi \min\{k, \hat{k}s\} |\ln \varepsilon| + C$, it is easy to see that convergence of minimizers would be achieved for all cases as opposed to just strong orthogonality and weak orthogonality with $s = 1$. With this, we believe further investigation into deriving a lower bound for the divergence term would be useful.

Appendix A

The Landau-de Gennes Model

In Section 1.2, we claimed that a more refined model for nematic liquid crystal is given by the Landau-de Gennes theory. We will give a brief overview of this model here.

The Landau-de Gennes Model

A large drawback of the Oseen-Frank model is the inherent direction associated with the director $\mathbf{d} \in \mathbb{S}^{N-1}$. Typically, the head and tail of the long rod-like molecules comprising the nematic sample are indistinguishable and therefore $\mathbf{d}$ and $-\mathbf{d}$ should represent the same molecular configuration. This is certainly cause for concern since energy minimizing states could correspond to configurations which are not orientable (see [6] and [7] for more detail). One way around this issue was introduced by Pierre-Gilles de Gennes (see [17] for an excellent reference on this material) by constructing a mathematical framework that accounts for the molecular head-tail symmetry. The brilliant idea here is to not represent the molecular order by $\mathbb{S}^{N-1}$-valued functions but by objects that can be identified with elements of real projective space $\mathbb{R}P^{N-1}$. These objects proposed by de Gennes are called *Q-tensor order parameters* and take the form of matrix-valued functions

$$Q : \Omega \subset \mathbb{R}^N \to \mathcal{S}$$

where $\mathcal{S}$ is the space of real $N \times N$ symmetric, traceless matrices

$$\mathcal{S} = \left\{ Q \in M_N(\mathbb{R}) : Q = Q^T, \ \mathrm{tr}(Q) = 0 \right\}.$$

The Q-tensor order parameter is related to the calculated second moment matrix A associated to a molecular orientation distribution law for the system, which is given by a probability density function. This second moment matrix is capable of easily describing *uniaxial* nematics (having only one preferred direction of alignment), *biaxial* nematics (having more than one preferred direction of molecular alignment [30]) and the isotropic state (no molecular ordering). An important aspect of the Landau-de Gennes theory is understanding the form of A in the isotropic case. Observe that in the isotropic state the configuration of molecules are equally distributed over all orientations and thus the probability density for finding a molecule in a particular direction is given by a constant function. Upon calculating the second moment matrix associated to the constant probability density function one finds the isotropic state is a constant multiple of the identity matrix [41]. Specifically, one obtains in dimension N

$$A_{\text{iso}} = \frac{1}{N} I$$

where I denotes the $N \times N$ identity matrix. To obtain the desired $\mathcal{S}$-valued Q-tensor order parameter, we begin looking at the space of matrices formally written

$$Q = A - A_{\text{iso}}.$$

Hence, the order parameter for the Landau-de Gennes model is a measure of deviation from the isotropic state. Therefore molecular orientations with associated constant probability distributions are given by Q coinciding with the $N \times N$ zero matrix $Q = \mathbf{0}_N$.

A Q-tensor describes the uniaxial, biaxial and isotropic phases based on its eigenvalues. In the case where $N = 3$, we say a Q-tensor is

- *biaxial* when all three of its eigenvalues are distinct,
- *uniaxial* when two of its eigenvalues are equal and non-zero (i.e., there are two distinct eigenvalues for Q),
- *isotropic* when all eigenvalues are equal.

The definitions for uniaxial and isotropic Q-tensors also apply to the case where $N = 2$. However it should be noted that the notion of biaxial Q-tensors does not exist in two dimensions. This is due to the constraint $\mathrm{tr}(Q) = 0$ which leaves only the option for eigenvalues of equal magnitude and opposite sign or a repeated zero eigenvalue. In dimension two or three, the Spectral theorem allows one to write uniaxial Q-tensors in the special form

$$Q = s\left(\mathbf{d} \otimes \mathbf{d} - \frac{1}{N} I\right), \quad \mathbf{d} \in \mathbb{S}^{N-1} \tag{A.0.1}$$

where s is a non-zero scalar [3, 30]. In this way the eigenvector $\mathbf{d}$ acts as the director for the nematic and the equality

$$\mathbf{d} \otimes \mathbf{d} = (-\mathbf{d}) \otimes (-\mathbf{d})$$

ensures that the Q-tensor encodes the identification $\mathbf{d} \sim -\mathbf{d}$. Therefore, in a sense the Q-tensor strips $\mathbf{d}$ of its orientation and allows one to work only with the *axis* defined by the director.

A typical minimization problem in the Landau-de Gennes model is of the form $\inf_{Q \in X} \mathscr{F}_{LdG}$ where

$$\mathscr{F}_{LdG}(Q) = \int_{\Omega} \left(\frac{1}{2} |\nabla Q|^2 + \frac{1}{L} f_B(Q) \right) dx, \quad X = H^1(\Omega; \mathcal{S}).$$

The Dirichlet energy for Q in this setting mimics that of the one constant approximation to the Oseen-Frank energy, $L > 0$ is a constant and

$$f_B(Q) = -\frac{a}{2} \operatorname{tr}(Q^2) - \frac{b}{3} \operatorname{tr}(Q^3) + \frac{c}{4} (\operatorname{tr}(Q^2))^2 - d \tag{A.0.2}$$

is a bulk potential derived from a 4th order Taylor expansion about the isotropic state $Q = \mathbf{0}$ [33] which penalizes non-uniaxial Q-tensors. The positive constants a, b, and c are temperature dependent and d is a constant chosen so that $\min f_B = 0$. In fact this minimum is achieved on a special set of uniaxial Q-tensors [31] which are of the form

$$Q = s_* \left(\mathbf{d} \otimes \mathbf{d} - \frac{1}{N} I \right), \quad s_* = \begin{cases} \frac{a\sqrt{2}}{c} & \text{if } N = 2 \\ \frac{b + \sqrt{b^2 + 24ac}}{4c} & \text{if } N = 3 \end{cases}.$$

Appendix B

Miscellaneous Results and Equations

The following lemma is used often when we encounter the terms of the form $\langle g, \partial_n g\rangle$ where g is a smooth $\mathbb{S}^1$-valued function.

Lemma B.1. *Let $f \in C^1(\Omega; \mathbb{S}^1)$ and suppose $v \in \mathbb{S}^1$. Then*

$$\langle f, D_v f\rangle = 0$$

where $D_v f$ denotes the directional derivative of f in direction v.

Proof. Since f is $\mathbb{S}^1$-valued, $|f|^2 = (f^1)^2 + (f^2)^2 = 1$ and so

$$\frac{1}{2}\nabla(|f|^2) = f^1\nabla f^1 + f^2\nabla f^2 = (0,0).$$

On the other hand,

$$D_v f = (\langle \nabla f^1, v\rangle, \langle \nabla f^2, v\rangle)$$

implying

$$\begin{aligned}\langle f, D_v f\rangle &= \langle f^1\nabla f^1, v\rangle + \langle f^2\nabla f^2, v\rangle \\ &= \langle f^1\nabla f^1 + f^2\nabla f^2, v\rangle \\ &= 0\end{aligned}$$

for any $v \in \mathbb{S}^1$. □

Proposition B.2. *Let $X \in C^2(\mathcal{N}; \mathbb{R}^2)$ be the vector field of Lemma 3.2 satisfying conditions (3.1.4) and (3.1.5). Then*

$$\partial_\tau X_\tau = 1 + f(X, \tau, DX, D\tau)$$

on $\Gamma_r(x_0)$ where $|f| \le C|x - x_0| = Cr$.

Proof. We begin by expanding the tangential derivative and adding and subtracting $|\tau|^2$:

$$\begin{aligned}
\partial_\tau X_\tau &= \langle \nabla \langle X, \tau \rangle, \tau \rangle \\
&= |\tau|^2 + \langle \nabla \langle X, \tau \rangle, \tau \rangle - |\tau|^2 \\
&= 1 + \sum_{i=1}^{2} \left(X^i \langle \nabla \tau^i, \tau \rangle + (X^i_{x_i} - 1)(\tau^i)^2 \right) + (X^1_{x_2} + X^2_{x_1})\tau^1\tau^2 \\
&= 1 + f(X, \tau, DX, D\tau).
\end{aligned}$$

Using the conditions of (3.1.5) we have $|X^i| \le |X_\tau| \le Cr$ and $|X^i_{x_j} - \delta_{ij}| \le Cr$. Moreover, $|\tau^1\tau^2| \le 1$ and $|\langle \nabla \tau^i, \tau \rangle| \le C$ where C is independent of ε and x_0. Thus each term of f is bounded by a constant times r and so $|f| \le Cr$. □

Proposition B.3. *Suppose k_s and k_b are positive, unequal constants. Define $\tilde{k} = \min\{k_s, k_b\}$ and $\kappa = \max\{k_s, k_b\} - \tilde{k}$. Then*

$$k_s(\operatorname{div} u)^2 + k_b(\operatorname{curl} u)^2 = \tilde{k}|\nabla u|^2 + \kappa(\operatorname{div} u)^2 + 2\tilde{k}\det(\nabla u) \quad \textit{for} \quad \tilde{k} = k_b$$

$$k_s(\operatorname{div} u)^2 + k_b(\operatorname{curl} u)^2 = \tilde{k}|\nabla u|^2 + \kappa(\operatorname{curl} u)^2 + 2\tilde{k}\det(\nabla u) \quad \textit{for} \quad \tilde{k} = k_s.$$

Proof.

<u>Case 1: $\tilde{k} = k_b$</u>

$$\begin{aligned}
k_s(\operatorname{div} u)^2 + k_b(\operatorname{curl} u)^2 &= k_s(u^1_{x_1} + u^2_{x_2})^2 + k_b(u^2_{x_1} - u^1_{x_2})^2 \\
&= k_s(u^1_{x_1})^2 + 2k_s u^1_{x_1} u^2_{x_2} + k_s(u^2_{x_2})^2 + k_b(u^1_{x_2})^2 \\
&\quad - 2k_b u^1_{x_2} u^2_{x_1} + k_b(u^2_{x_1})^2 \\
&= \underbrace{k_s(u^1_{x_1})^2 + k_s(u^2_{x_2})^2 + k_b(u^1_{x_2})^2 + k_b(u^2_{x_1})^2}_{A} \\
&\quad + \underbrace{2k_s u^1_{x_1} u^2_{x_2} - 2k_b u^1_{x_2} u^2_{x_1}}_{B}.
\end{aligned}$$

In A, we add and subtract the terms $k_b(u^1_{x_1})^2$ and $k_b(u^2_{x_2})^2$ to get

$$\begin{aligned}
A &= k_s(u^1_{x_1})^2 - k_b(u^1_{x_1})^2 + k_b(u^1_{x_1})^2 + k_s(u^2_{x_2})^2 - k_b(u^2_{x_2})^2 \\
&\quad + k_b(u^2_{x_2})^2 + k_b(u^1_{x_2})^2 + k_b(u^2_{x_1})^2 \\
&= (k_s - k_b)(u^1_{x_1})^2 + k_b(u^1_{x_1})^2 + (k_s - k_b)(u^2_{x_2})^2 + k_b(u^2_{x_2})^2 + k_b(u^1_{x_2})^2 + k_b(u^2_{x_1})^2 \\
&= k_b\left[(u^1_{x_1})^2 + (u^1_{x_2})^2 + (u^2_{x_1})^2 + (u^2_{x_2})^2\right] + (k_s - k_b)(u^1_{x_1})^2 + (k_s - k_b)(u^2_{x_2})^2 \\
&= k_b|\nabla u|^2 + (k_s - k_b)(u^1_{x_1})^2 + (k_s - k_b)(u^2_{x_2})^2.
\end{aligned}$$

In B, we add and subtract the term $2k_b u^1_{x_1} u^2_{x_2}$,

$$\begin{aligned}
B &= 2k_s u^1_{x_1} u^2_{x_2} - 2k_b u^1_{x_2} u^2_{x_1} \\
&= 2k_s u^1_{x_1} u^2_{x_2} - 2k_b u^1_{x_1} u^2_{x_2} + 2k_b u^1_{x_1} u^2_{x_2} - 2k_b u^1_{x_2} u^2_{x_1} \\
&= 2(k_s - k_b) u^1_{x_1} u^2_{x_2} + 2k_b\left(u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1}\right) \\
&= 2(k_s - k_b) u^1_{x_1} u^2_{x_2} + 2k_b \det(\nabla u).
\end{aligned}$$

Putting A and B back together,

$$\begin{aligned}
k_s(\operatorname{div} u)^2 + k_b(\operatorname{curl} u)^2 &= A + B \\
&= k_b|\nabla u|^2 + (k_s - k_b)(\operatorname{div} u)^2 + 2k_b \det(\nabla u).
\end{aligned}$$

<u>Case 2: $\tilde{k} = k_s$</u>

We still have

$$\begin{aligned}
k_s(\operatorname{div} u)^2 + k_b(\operatorname{curl} u)^2 &= \underbrace{k_s(u^1_{x_1})^2 + k_s(u^2_{x_2})^2 + k_b(u^1_{x_2})^2 + k_b(u^2_{x_1})^2}_{A} \\
&\quad + \underbrace{2k_s u^1_{x_1} u^2_{x_2} - 2k_b u^1_{x_2} u^2_{x_1}}_{B}.
\end{aligned}$$

In A, we now add and subtract the terms $k_s(u^1_{x_2})^2$ and $k_s(u^2_{x_1})^2$.

$$\begin{aligned}
A &= k_s(u^1_{x_1})^2 + k_s(u^2_{x_2})^2 + k_b(u^1_{x_2})^2 - k_s(u^1_{x_2})^2 + k_s(u^1_{x_2})^2 \\
&\quad + k_b(u^2_{x_1})^2 - k_s(u^2_{x_1})^2 + k_s(u^2_{x_1})^2 \\
&= k_s(u^1_{x_1})^2 + k_s(u^2_{x_2})^2 + (k_b - k_s)(u^1_{x_2})^2 + k_s(u^1_{x_2})^2 + (k_b - k_s)(u^2_{x_1})^2 + k_s(u^2_{x_1})^2 \\
&= k_s\left[(u^1_{x_1})^2 + k_s(u^2_{x_1})^2 + k_s(u^1_{x_2})^2 + k_s(u^2_{x_2})^2\right] + (k_b - k_s)(u^1_{x_2})^2 + (k_b - k_s)(u^2_{x_1})^2 \\
&= k_s|\nabla u|^2 + (k_b - k_s)(u^1_{x_2})^2 + (k_b - k_s)(u^2_{x_1})^2.
\end{aligned}$$

In B, we add and subtract the term $2k_s u^1_{x_2} u^2_{x_1}$,

$$\begin{aligned}
B &= 2k_s u^1_{x_1} u^2_{x_2} - 2k_b u^1_{x_2} u^2_{x_1} \\
&= 2k_s u^1_{x_1} u^2_{x_2} - 2k_s u^1_{x_2} u^2_{x_1} + 2k_s u^1_{x_2} u^2_{x_1} - 2k_b u^1_{x_2} u^2_{x_1} \\
&= 2k_s (u^1_{x_1} u^2_{x_2} - u^1_{x_2} u^2_{x_1}) + 2(k_s - k_b) u^1_{x_2} u^2_{x_1} \\
&= 2k_s \det(\nabla u) + 2(k_s - k_b) u^1_{x_2} u^2_{x_1} \\
&= -2(k_b - k_s) u^1_{x_2} u^2_{x_1} + 2k_s \det(\nabla u).
\end{aligned}$$

Adding A and B back together,

$$\begin{aligned}
k_s(\operatorname{div} u)^2 + k_b(\operatorname{curl} u)^2 &= A + B \\
&= k_s|\nabla u|^2 + (k_b - k_s)(\operatorname{curl} u)^2 + 2k_s \det(\nabla u).
\end{aligned}$$

□

Appendix C

Regularity of Minimizers

In most references involving minimizers of the Ginzburg-Landau functional, a technical discussion of solution regularity is often omitted since related results are viewed as standard knowledge in elliptic PDE theory. For the sake of completion, this section will be dedicated to roughly explaining the necessary arguments needed to deduce additional smoothness of weak solutions to the Euler-Lagrange equations. We begin by dealing with equations (2.2.1) and (2.2.3). It will be convenient to begin with a result for which the regularity of our weak solutions will be built upon. The following lemma is taken from a series of detailed lecture notes produced by Professor Giovanni Leoni (Carnegie Mellon) on his website. Access to these notes can be found through [28]. Similar results can be found in [24, Chapter 2].

Lemma C.1. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with C^2 boundary. Let $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$ be such that compatibility condition holds*

$$\int_\Omega f\,dx = \int_{\partial\Omega} g\,ds$$

and let $u \in H^1(\Omega)$ be a weak solution of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_n u = g & \text{on } \partial\Omega. \end{cases}$$

Then $u \in H^2(\Omega)$ and we have the estimate

$$\|\nabla^2 u\|_{L^2} \leq C(\Omega)\left(\|f\|_{L^2} + \|g\|_{H^{1/2}} + \|u\|_{H^1}\right).$$

With this, we aim to explain how one obtains the following lemma.

Lemma C.2. *Let u_ε denote a weak solution for* (2.2.1) *or* (2.2.3)*. Assume* Γ *is a C^4-smooth curve and that*

$$g \in C^4(\mathcal{N}_\Gamma; \mathbb{S}^1) \cap C^4(\overline{\Omega}; \mathbb{R}^2)$$

where $\mathcal{N}_\Gamma$ is a small tubular neighbourhood of Γ. Then it holds

$$u_\varepsilon \in C^\infty(\Omega; \mathbb{R}^2) \cap C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)$$

for all $\alpha \in (0, 1)$.

Remark C.1. *The use of the neighbourhood $\mathcal{N}_\Gamma$ is mainly needed for strong orthogonal solutions. The requirement that g be $\mathbb{S}^1$-valued in a neighbourhood of the boundary will be used for a decomposition of u near Γ using the basis $\{g, g^\perp\}$.*

Although it will not be shown, the main driver of the results used here rely on the method of translations for elliptic operators (see [12, 23, 35], for example). To use this method, it is often required that the boundary data and boundary curve have smoothness order matching that of the desired Sobolev regularity order for the weak solution in question. By this, we mean that if one would like to show that weak solutions belong to $H^k(\Omega; \mathbb{R}^2)$, then we should require C^k-smoothness on the boundary data and the boundary curve . Thus, it is apparent from our hypothesis that we would like to show weak solutions belong to $H^4(\Omega; \mathbb{R}^2)$. Once this is done, the Sobolev embedding theorem can then be used to obtain solution inclusion in the Hölder space $C^{2,\alpha}(\overline{\Omega}; \mathbb{R}^2)$.

In the weak orthogonality problem (W.O.), the derived Euler-Lagrange equations (2.2.1) take the form of a system of semi-linear PDEs with Robin boundary conditions. To deal with questions of regularity, it is insightful to recast (2.2.1) in terms of a linear PDE with Neumann boundary conditions. The ability to do this comes from the fact that weak H^1 solutions for the weak orthogonality problem are known to exist by Lemma 2.1. To see how this operates, let u_ε be a weak H^1 solution for (2.2.1) and define the functions

$$f_\varepsilon(x) := \frac{1}{\varepsilon^2}(1 - |u_\varepsilon(x)|^2)u_\varepsilon(x), \quad h_\varepsilon(x) = -\frac{W}{\varepsilon^s}\langle u_\varepsilon(x), g^\perp\rangle g^\perp.$$

Note that since it is assumed $g \in C^4(\mathcal{N}_\Gamma; \mathbb{S}^1) \cap C^4(\overline{\Omega}; \mathbb{R}^2)$, both f_ε and h_ε are defined on all of Ω. The idea now is to consider the non-homogeneous Neumann problem for the Poisson equation

$$\begin{cases} -\Delta v = f_\varepsilon & \text{in } \Omega, \\ \partial_n v = h_\varepsilon & \text{on } \Gamma, \end{cases} \tag{C.0.1}$$

and deduce $H^2(\Omega;\mathbb{R}^2)$ regularity for the weak solution $v \in H^1(\Omega;\mathbb{R}^2)$ through standard elliptic estimates. Using the fact that $v = u_\varepsilon$ is a known weak solution for (2.2.1), and therefore (C.0.1), the general results for v transfer to u_ε and then a bootstrapping argument can be implemented to further increase the regularity for u_ε. By Lemma C.1 it is required that three conditions for (C.0.1) are checked in order to achieve $v \in H^2(\Omega;\mathbb{R}^2)$:

1. $f_\varepsilon \in L^2(\Omega;\mathbb{R}^2)$,
2. $h_\varepsilon \in H^{1/2}(\Gamma;\mathbb{R}^2)$,
3. the components f_ε^j and h_ε^j satisfy the compatibility condition
$$\int_\Omega f_\varepsilon^j \, dx = \int_\Gamma h_\varepsilon^j \, ds.$$
for each $j = 1, 2$.

The first condition is a consequence of the Sobolev embedding theorem in two dimensions. Since $u_\varepsilon \in L^p(\Omega;\mathbb{R}^2)$ for all $p \in [1,\infty)$, the scalar function $(1 - |u_\varepsilon|^2)$ can be shown to belong to $L^p(\Omega)$ for all $p \in [1,\infty)$ using the boundedness of Ω and Minkowski's inequality. Young's inequality can then be applied to obtain $f_\varepsilon \in L^2(\Omega;\mathbb{R}^2)$ since

$$\begin{aligned}\int_\Omega |f_\varepsilon|^2 \, dx &= \frac{1}{\varepsilon^4}\int_\Omega |1 - |u_\varepsilon|^2|^2 |u_\varepsilon|^2 \, dx \\ &\leq \frac{1}{2\varepsilon^4}\int_\Omega \left(|1 - |u_\varepsilon|^2|^4 + |u_\varepsilon|^4\right) dx \\ &< +\infty.\end{aligned}$$

The $H^{1/2}$ trace condition for h_ε can be shown by proving $h_\varepsilon \in H^1(\Omega;\mathbb{R}^2)$. It is easy to see already $h_\varepsilon \in L^2(\Omega;\mathbb{R}^2)$ using the boundedness of g and Cauchy-Schwarz:

$$\begin{aligned}\int_\Omega |h_\varepsilon|^2 \, dx &= \int_\Omega |\langle u_\varepsilon, g^\perp\rangle|^2 |g^\perp|^2 \, dx \\ &\leq \int_\Omega |u_\varepsilon|^2 |g^\perp|^4 \, dx \\ &\leq \||g^\perp|^4\|_\infty \|u_\varepsilon\|_{L^2}^2 \\ &< +\infty.\end{aligned}$$

To deduce the form of the first order weak derivatives of h_ε, let $\varphi \in C_0^1(\Omega)$

and note that

$$-\frac{\varepsilon^s}{W} h_\varepsilon^j = \langle u_\varepsilon, g^\perp \rangle (g^\perp)^j = \sum_{i=1}^{2} u_\varepsilon^i (g^\perp)^i (g^\perp)^j.$$

Seeing that the components of $g^\perp$ are four-times continuously differentiable, the product $(g^\perp)^i (g^\perp)^j \varphi \in C_0^1(\Omega)$ and so for fixed $i, j, k = 1, 2$ the definition of weak derivative states

$$\int_\Omega (\partial_{x_k} u_\varepsilon^i)(g^\perp)^i (g^\perp)^j \varphi \, dx = -\int_\Omega u_\varepsilon^i \partial_{x_k}((g^\perp)^i (g^\perp)^j \varphi) \, dx.$$

The derivative on the righthand side can be expanded

$$\partial_{x_k}((g^\perp)^i (g^\perp)^j \varphi) = (\partial_{x_k}(g^\perp)^i)(g^\perp)^j \varphi + (g^\perp)^i (\partial_{x_k}(g^\perp)^j)\varphi + (g^\perp)^i (g^\perp)^j \partial_{x_k}\varphi$$

and then collecting like terms relative to φ gives

$$\int_\Omega ((\partial_{x_k} u_\varepsilon^i)(g^\perp)^i (g^\perp)^j + u_\varepsilon^i (\partial_{x_k}(g^\perp)^i)(g^\perp)^j + u_\varepsilon^i (g^\perp)^i (\partial_{x_k}(g^\perp)^j))\varphi \, dx$$
$$= -\int_\Omega u_\varepsilon^i (g^\perp)^i (g^\perp)^j \partial_{x_k}\varphi \, dx$$

for all $\varphi \in C_0^1(\Omega)$. Therefore the weak derivatives for the components of h_ε have the form

$$\partial_{x_k} h_\varepsilon^j = -\frac{W}{\varepsilon^s} \sum_{i=1}^{2} ((\partial_{x_k} u_\varepsilon^i)(g^\perp)^i (g^\perp)^j + u_\varepsilon^i (\partial_{x_k}(g^\perp)^i)(g^\perp)^j + u_\varepsilon^i (g^\perp)^i (\partial_{x_k}(g^\perp)^j)).$$

To show $\partial_{x_k} h_\varepsilon^j \in L^2(\Omega)$, we begin by applying the triangle inequality repeatedly to obtain the bound

$$|\partial_{x_k} h_\varepsilon^j| \leq \frac{W \|g^\perp\|_{C^1}^2}{\varepsilon^s} \sum_{i=1}^{2} (|\partial_{x_k} u_\varepsilon^i| + 2|u_\varepsilon^i|).$$

Squaring both sides of this inequality, applying Young's inequality on the cross-terms and integrating leads to the estimate

$$\int_\Omega |\partial_{x_k} h_\varepsilon^j|^2 \, dx \leq \frac{C W^2 \|g^\perp\|_{C^1}^4}{\varepsilon^{2s}} \sum_{i=1}^{2} (\|u_\varepsilon^i\|_{L^2}^2 + \|\partial_{x_k} u_\varepsilon^i\|_{L^2}^2) < +\infty$$

where C is a constant independent of u_ε^i and $\partial_{x_k} u_\varepsilon^i$ for $i = 1, 2$. Since these estimates hold for all $j, k = 1, 2$, we have $h_\varepsilon \in H^1(\Omega; \mathbb{R}^2)$ and so the trace of

h_ε belongs to $H^{1/2}(\Gamma;\mathbb{R}^2)$.

The third and final point to check is the component-wise compatibility condition. This fact follows immediately from the known existence of a solution to the Neumann problem (C.0.1). However, the compatibility condition can be easily derived via direct observation of the weak formulation of the Euler-Lagrange equations:

$$\int_\Omega \left(\sum_{i,j} u^i_{x_j}\varphi^i_{x_j} - \frac{1}{\varepsilon^2}(1-|u|^2)\langle u,\varphi\rangle \right) dx + \frac{W}{\varepsilon^s}\int_\Gamma \langle\langle u, g^\perp\rangle g^\perp, \varphi\rangle\, ds = 0$$

holding for all $\varphi \in H^1(\Omega;\mathbb{R}^2)$. Setting $\varphi = (1,0)$ produces

$$\int_\Omega f^1_\varepsilon\, dx = \int_\Gamma h^1_\varepsilon\, ds$$

while setting $\varphi = (0,1)$ gives the same equation for the second components of f_ε and h_ε.

Now that $H^2(\Omega;\mathbb{R}^2)$ regularity for u_ε has been established, one can show increased regularity for f_ε and h_ε. By the Sobolev embedding theorem in two dimensions, $H^2(\Omega;\mathbb{R}^2) \subset L^\infty(\Omega;\mathbb{R}^2)$ and therefore $u_\varepsilon \in H^1(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$. By [12, Proposition 9.4], the product rule can be applied on the components of f_ε to obtain $f_\varepsilon \in H^1(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$. Similarly, h_ε can be shown to belong to $H^2(\Omega)$. At this point, we can begin bootstrapping to continue the process until we have reached $u_\varepsilon \in H^4(\Omega;\mathbb{R}^2)$. As mentioned above, the Sobolev embedding theorem can then be used to obtain $u_\varepsilon \in C^{2,\alpha}(\overline{\Omega};\mathbb{R}^2)$. When away from the boundary, the bootstrapping process can continue indefinitely since there is no longer a differentiability cap enforced by the boundary data. In this case, the interior regularity $u_\varepsilon \in C^\infty(\Omega;\mathbb{R}^2)$ is achieved.

Remark C.2. *Generally speaking, the work for interior regularity, in practice, is done before regularity estimates up to the boundary.*

For the strong orthogonality problem (S.O.), the Euler-Lagrange equations (2.2.3) are composed of the same interior semi-linear PDEs as dealt with before, but now we are given two scalar conditions on the projections of u_ε with respect to g and $g^\perp$ along the boundary Γ. By the same bootstrapping argument, the interior regularity estimates give $u_\varepsilon \in C^\infty(\Omega;\mathbb{R}^2)$. To obtain regularity up to the boundary, we can look at u_ε on the neighbourhood $\mathcal{N}_\Gamma$ where we have the decomposition

$$u_\varepsilon = u_\parallel g + u_\perp g^\perp.$$

Using this, we calculate $u_\parallel$ is a weak solution of the scalar equation

$$\Delta u_\parallel = \langle u, \Delta g\rangle + 2\sum_{i=1}^{2}\langle \nabla u^i, \nabla g^i\rangle - \frac{1}{\varepsilon^2}(1-|u|^2)u_\parallel = f_\varepsilon^\parallel(u)$$

satisfying the Neumann condition $\partial_n u_\parallel = 0$ on the boundary Γ. Similarly, $u_\perp$ is a weak solution of the equation

$$\Delta u_\perp = \langle u, \Delta g^\perp\rangle + 2\sum_{i=1}^{2}\langle \nabla u^i, \nabla (g^\perp)^i\rangle - \frac{1}{\varepsilon^2}(1-|u|^2)u_\perp = f_\varepsilon^\perp(u)$$

satisfying the Dirichlet condition $u_\perp = 0$ on Γ. Since both $f_\varepsilon^\parallel(u)$, $f_\varepsilon^\perp(u) \in L^2(\mathcal{N}_\Gamma)$, we can apply similar elliptic regularity estimates to obtain the desired regularity.

In Section 6.1, it is shown that the Euler–Lagrange equations associated to the minimization problem for $\mathscr{F}_\varepsilon$ and $\mathscr{F}_\varepsilon^W$ form a system of partial differential equations which are coupled in the second order derivatives. Understanding the underlying structure of this system is deeply important for studying the regularity of its weak solutions. Due to the domain and range dimensions of our problem, we restrict ourselves to observing coupled linear systems involving two equations in two variables.

Let $u \in H^1(\Omega;\mathbb{R}^2)$ be a weak solution of the system

$$\begin{cases} -\displaystyle\sum_{k,m,j=1}^{2} \partial_{x_m} A_{1,j}^{k,m}\partial_{x_k} u^j = f_1 \\ -\displaystyle\sum_{k,m,j=1}^{2} \partial_{x_m} A_{2,j}^{k,m}\partial_{x_k} u^j = f_2 \end{cases} \tag{C.0.2}$$

where it is assumed $f_1, f_2 \in L^2(\Omega)$ and $A_{i,j}^{k,m}$ are constants for all index parings. Using the definition presented in [22] and restricting to the case of system (C.0.2), we say that the *Legendre–Hadamard condition* is satisfied by the matrix of coefficients $(A_{i,j}^{k,m})$ if there exists a constant $\gamma > 0$ so that

$$\sum_{k,m,i,j=1}^{2} A_{i,j}^{k,m}\xi_k\xi_m\eta^i\eta^j \geq \gamma|\xi|^2|\eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^2.$$

Proposition C.3. *The matrix of coefficients* $(A_{i,j}^{k,m})$ *for systems* (6.1.1), (6.1.2), (6.1.4) *and* (6.1.5) *satisfy the Legendre–Hadamard condition.*

Proof.

The general form for the divergence penalized systems ($\tilde{k} = k_b$) is

$$\begin{cases} -(\partial_{x_1} k_s \partial_{x_1} u^1 + \partial_{x_2} k_b \partial_{x_2} u^1 + \partial_{x_1}(k_s - k_b)\partial_{x_2} u^2) = \frac{1}{\varepsilon^2} u^1(1 - |u|^2) \\ -(\partial_{x_2}(k_s - k_b)\partial_{x_1} u^1 + \partial_{x_1} k_b \partial_{x_1} u^2 + \partial_{x_2} k_s \partial_{x_2} u^2) = \frac{1}{\varepsilon^2} u^2(1 - |u|^2) \end{cases}.$$

The elements of the matrix of coefficients are

$$A^{1,1}_{1,1} = A^{2,2}_{2,2} = k_s, \quad A^{2,2}_{1,1} = A^{1,1}_{2,2} = k_b, \quad A^{2,1}_{1,2} = A^{1,2}_{2,1} = k_s - k_b$$

and the remaining constants are zero. Let $\xi, \eta \in \mathbb{R}^2$ and consider the sum

$$\sum_{k,m,i,j=1}^{2} A^{k,m}_{i,j} \xi_k \xi_m \eta^i \eta^j = k_s(\xi_1\eta^1)^2 + k_b(\xi_2\eta^1)^2 + k_b(\xi_1\eta^2)^2 + k_s(\xi_2\eta^2)^2 + 2(k_s - k_b)\xi_1\xi_2\eta^1\eta^2.$$

Adding and subtracting the terms $k_b(\xi_1\eta^1)^2$ and $k_b(\xi_2\eta^2)^2$ we obtain

$$\sum_{k,m,i,j=1}^{2} A^{k,m}_{i,j} \xi_k \xi_m \eta^i \eta^j = k_b|\xi|^2|\eta|^2 + (k_s - k_b)(\xi_1\eta^1 + \xi_2\eta^2)^2 \geq k_b|\xi|^2|\eta|^2.$$

For the curl penalized systems ($\tilde{k} = k_s$), we can write

$$\begin{cases} -(\partial_{x_1} k_s \partial_{x_1} u^1 + \partial_{x_2} k_b \partial_{x_2} u^1 - \partial_{x_2}(k_b - k_s)\partial_{x_1} u^2) = \frac{1}{\varepsilon^2} u^1(1 - |u|^2) \\ -(-\partial_{x_1}(k_b - k_s)\partial_{x_2} u^1 + \partial_{x_1} k_b \partial_{x_1} u^2 + \partial_{x_2} k_s \partial_{x_2} u^2) = \frac{1}{\varepsilon^2} u^2(1 - |u|^2) \end{cases}$$

giving

$$A^{1,1}_{1,1} = A^{2,2}_{2,2} = k_s, \quad A^{2,2}_{1,1} = A^{1,1}_{2,2} = k_b, \quad A^{1,2}_{1,2} = A^{2,1}_{2,1} = -(k_b - k_s)$$

where remaining constants are zero. Now

$$\sum_{k,m,i,j=1}^{2} A^{k,m}_{i,j} \xi_k \xi_m \eta^i \eta^j = k_s(\xi_1\eta^1)^2 + k_b(\xi_2\eta^1)^2 + k_b(\xi_1\eta^2)^2 + k_s(\xi_2\eta^2)^2 - 2(k_b - k_s)\xi_1\xi_2\eta^1\eta^2.$$

Adding and subtracting the terms $k_s(\xi_2\eta^1)^2$ and $k_s(\xi_1\eta^2)^2$ we obtain

$$\sum_{k,m,i,j=1}^{2} A_{i,j}^{k,m}\xi_k\xi_m\eta^i\eta^j = k_s|\xi|^2|\eta|^2 + (k_b - k_s)(\xi_1\eta^2 - \xi_2\eta^1)^2 \geq k_s|\xi|^2|\eta|^2.$$

□

The structure provided by the Legendre–Hadamard condition gives several useful regularity results. To give an example, we quote a simplified version of [22, Theorem 4.11] for completeness

Theorem C.4. *Let $u \in H^1(\Omega;\mathbb{R}^2)$ be a weak solution to system (C.0.2) where $A_{i,j}^{k,m}$ satisfies the Legendre–Hadamard condition and for some integer $k \geq 0$ we have $f_i \in H^k(\Omega)$. Then $u \in H_{loc}^{k+2}(\Omega;\mathbb{R}^2)$ and for every $S_0 \Subset \Omega$ there is a constant C depending on k, Ω, S_0 and the $A_{i,j}^{k,m}$'s such that*

$$\|D^{k+2}u\|_{L^2(S_0)} \leq C\left(\|u\|_{L^2(\Omega)} + \|f\|_{H^k(\Omega)}\right).$$

Appendix D

Boundary Integral Convergence

The following three lemmas are used together several times throughout this thesis in order to prove convergences results along $\Gamma = \partial\Omega$.

The first lemma is given in the context of Hilbert spaces, but the result can be generalized to weak convergence in L^p spaces $p \in [1, \infty)$ as seen in [19, Theorem 1].

Lemma D.1. *Suppose H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and let $\{h_n\}_{n=1}^{\infty} \subset H$. If $h_n \rightharpoonup h$ weakly in H then*

$$\|h\|_H \leq \liminf_{n\to\infty} \|h_n\|_H.$$

That is, the norm $\|\cdot\|_H$ is sequentially weakly lower semicontinuous.

Proof. By Cauchy-Schwarz

$$|\langle h, h_n \rangle_H| \leq \|h\|_H \|h_n\|_H.$$

Since $h_n \rightharpoonup h$ weakly in H we have $\langle h, h_n \rangle_H \to \|h\|_H^2$ as $n \to \infty$. Then

$$\liminf_{n\to\infty} |\langle h, h_n \rangle_H| \leq \liminf_{n\to\infty} \|h\|_H \|h_n\|_H \implies \|h\|_H \leq \liminf_{n\to\infty} \|h_n\|_H.$$

□

In the next lemma, we show that the trace operator $T : H^1(\Omega) \to L^2(\partial\Omega)$ preserves weak convergence. In fact, this result is true of any bounded linear functional on a Hilbert space. Please refer to [1, 29] for an excellent treatment of trace theory.

Lemma D.2. *Let T be the trace operator on $H^1(\Omega)$ and let $\{u_n\}_{n=1}^{\infty} \subset H^1(\Omega)$ with $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$. Then $Tu_n \rightharpoonup Tu$ weakly in $L^2(\partial\Omega)$.*

Proof. Fix $\varphi \in L^2(\partial\Omega)$ and define the bounded linear functional $\ell : H^1(\Omega) \to \mathbb{R}$

$$\ell(u) = \int_{\partial\Omega} (Tu)\varphi \, ds.$$

Since $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ we have $\ell(u_n) \to \ell(u)$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} \ell(u - u_n) = \lim_{n\to\infty} \int_{\partial\Omega} (Tu - Tu_n)\varphi \, ds = 0.$$

Since $\varphi \in L^2(\partial\Omega)$ was chosen arbitrarily, the result holds for all $\varphi \in L^2(\partial\Omega)$ and thus $Tu_n \rightharpoonup Tu$ weakly in $L^2(\partial\Omega)$. □

Lemma D.3. *Let Ω be a smooth bounded domain and consider a sequence $\{u_n\}_{n=1}^{\infty} \subset H^1(\Omega; \mathbb{R}^2)$ such that $u_n \rightharpoonup u$ weakly in H^1 for some $u \in H^1(\Omega; \mathbb{R}^2)$. Then*

$$\int_{\partial\Omega} \langle u, g\rangle^2 \, ds \le \liminf_{n\to\infty} \int_{\partial\Omega} \langle u_n, g\rangle^2 \, ds$$

for any $g \in C(\partial\Omega; \mathbb{R}^2)$.

Proof. By Lemma D.2 it is known $u_n \rightharpoonup u$ weakly in $L^2(\partial\Omega; \mathbb{R}^2)$. Thus, for $g \in C(\partial\Omega; \mathbb{R}^2) \subset L^2(\partial\Omega; \mathbb{R}^2)$ we have the convergence

$$\lim_{n\to\infty} \int_{\partial\Omega} \langle u_n - u, g\rangle \, ds = 0.$$

Next, let $\varphi \in L^2(\partial\Omega)$ be arbitrary and note that $\varphi g \in L^2(\partial\Omega; \mathbb{R}^2)$, $\langle u, g\rangle \in L^2(\partial\Omega)$. Then

$$\int_{\partial\Omega} \left(\langle u_n, g\rangle - \langle u, g\rangle\right) \varphi \, ds = \int_{\partial\Omega} \langle u_n - u, \varphi g\rangle \, ds \to 0$$

as $n \to \infty$ by the weak convergence of $\{u_n\}_{n=1}^{\infty}$ in $L^2(\partial\Omega; \mathbb{R}^2)$. Therefore $\langle u_n, g\rangle \rightharpoonup \langle u, g\rangle$ weakly in $L^2(\partial\Omega)$ and applying Lemma D.1 to the sequence $\{\langle u_n, g\rangle\}_{n=1}^{\infty}$ in the Hilbert space $L^2(\partial\Omega)$ yields the result. □

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