THE ASYMPTOTIC PLATEAU PROBLEM

ON THE ASYMPTOTIC PLATEAU PROBLEM IN HYPERBOLIC SPACE

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A Thesis Submitted to the School of Graduate Studies in Partial Fulfilment of the Requirements for the Degree Master of Science in Mathematics

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Master of Science (2022)
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TITLE: On the Asymptotic Plateau Problem in Hyperbolic Space
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NUMBER OF PAGES: ix, 41

Abstract

We are concerned with the existence of hypersurfaces in hyperbolic space whose principal curvatures $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ satisfy a prescribed curvature relation $f(\kappa) = \sigma \in (0, 1)$ and has a prescribed asymptotic boundary at infinity.

Under standard assumptions on the curvature function $f(\kappa)$, the problem has been extensively studied by Bo Guan, Joel Spruck and their collaborators in a series of papers [27, 25, 18, 16, 14, 15, 17]; a special case of the problem in which the curvature vector κ lies in the positive cone $K_n^+ = \{\kappa_i > 0 \ \forall\ i\}$ has been completely solved in [17] and the result is essentially optimal. In [14], by applying the same approach to the general case, they proved the existence of solutions only for $0 < \sigma_0 < \sigma < 1$ where σ_0 is some number between 0.3703 and 0.3704.

In this thesis, we follow their method and extend their result in [14] to hold for all $0 < \sigma < 1$, with the help of an additional assumption $\sum_{i=1}^n f_i \leq C$ on the curvature function. In particular, our theorem applies to the curvature quotient $f = \frac{H_k}{H_{k-1}}$ for all $1 \leq k \leq n$ in the k-th Garding cone, where H_k is the k-th normalized elementary symmetric polynomial.

Acknowledgement

My most sincere and foremost thanks will go to my thesis supervisor Dr. Siyuan Lu. Firstly, I would like to thank him for accepting me as his graduate student, teaching me knowledge and writing a recommendation letter for my Ph.D. applications. Secondly, I would like to thank him for suggesting the thesis topic, showing me the step-by-step derivations repeatedly and instructing me how to think like a geometric analyst. Finally and most importantly, I would like to thank him for sharing his perspectives on methodology, attitude and taste towards independent mathematical research of higher quality. Due to his implicit and explicit influence, I have decided to pursue a Ph.D. in mathematics.

I would like to thank Dr. Lia Bronsard for her humorous lectures on Functional Analysis and Partial Differential Equations, for being so kind to write a recommendation letter for my Ph.D. applications and for her being willing to serve on the examination committee of my thesis.

I would like to thank Dr. Eric Sawyer for his course in Real Analysis; the homework problems were super hard but indeed rewarding. I would also like to thank him for writing a recommendation letter for my Ph.D. applications and the encouragement during the Zoom meeting in which we discussed my applications.

I would like to thank Dr. McKenzie Wang for always being nice and friendly when I took his course in Smooth Manifolds and when I was being one of the teaching assistants for his course in Complex Analysis . I would also like to thank him for serving on my examination committee.

Finally, I would like to thank my parents for their love and support over more than twenty years, both mentally and financially.

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List of all Abbreviations and Symbols

- We use the Einstein summation convention i.e. $F^{ij}a_{ij}$ would mean

$$\sum_{i=1}^{n}\sum_{j=1}^{n}F^{ij}a_{ij}$$

  , unless otherwise noted.

- Throughout this thesis, all hypersurfaces in $\mathbb{H}^{n+1}$ are assumed to be connected and orientable. If Σ is a complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field of Σ is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_{+} \setminus \Sigma$, and the principal curvatures are calculated with respect to this normal vector field.

Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Chapter 1

Introduction

Fix $n \geq 2$. Let $\mathbb{H}^{n+1}$ be the hyperbolic space of dimension $n+1$ and let $\partial_\infty \mathbb{H}^{n+1}$ denote the ideal boundary of $\mathbb{H}^{n+1}$ at infinity.

The study of minimal hypersurfaces (i.e. hypersurfaces whose mean curvature vanishes everywhere) with prescribed asymptotic boundary Γ was first initiated by Anderson [2, 3], in which he proved the existence of a complete, absolutely area-minimizing locally integral n-current Σ in $\mathbb{H}^{n+1}$ whose support M has Γ as its asymptotic limit; the boundary regularity at infinity of his construction was then studied by Hardt and Lin in [19]. In particular, they found that near points of the boundary Γ, $M \cup \Gamma$ may be described as the graph of some function which is a solution to a Dirichlet problem and the PDE is degenerate along the part of boundary corresponding to Γ. In [24], Lin studied the Dirichlet problem and proved that the graph is as smooth as the boundary. Based on Lin's method, Tonegawa [31] extended their results to hypersurfaces with constant mean curvature and later the same problem was studied by Nelli-Spruck [25] and Guan-Spruck [18] using a different approach.

For hypersurfaces of constant Gauss curvature in hyperbolic space with prescribed asymptotic boundary at infinity, the problem was initiated by Labourie [23] in $\mathbb{H}^3$ and settled by Rosenberg-Spruck [27] in $\mathbb{H}^{n+1}$. It is then natural to consider the problem for more general curvature functions as in the works [16, 14, 15, 17] of Bo Guan, Joel Spruck and their collaborators.

Suppose that $f \in C^2(K) \cap C^0(\overline{K})$ is a symmetric function defined in an open symmetric convex cone $K \subseteq \mathbb{R}^n$ with vertex at the origin, containing the positive cone

$$K_n^+ = \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \ \forall\ i\} \subseteq K$$

. Given a disjoint collection of closed embedded smooth $(n-1)$-dimensional submanifolds $\Gamma = \{\Gamma_1, \ldots, \Gamma_m\} \subseteq \partial_\infty \mathbb{H}^{n+1}$ and a constant $0 < \sigma < 1$, we study the problem of finding a smooth complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying

$$\kappa(x) \in K \text{ and } f(\kappa(x)) = \sigma \text{ for all } x \in \Sigma \tag{1.1}$$

with the asymptotic boundary

$$\partial \Sigma = \Gamma \tag{1.2}$$

where $\kappa = (\kappa_1, \ldots, \kappa_n)$ denotes the vector of induced hyperbolic principal curvatures of Σ and for a hypersurface Σ satisfying the second requirement (1.2), we say it is asymptotic to Γ at infinity. We will call (1.1)-(1.2) the **asymptotic Plateau problem in hyperbolic space**.

The first requirement (1.1) in its most general form will be of

$$f(\kappa(x)) = \psi(x, \nu), \quad x \in \Sigma$$

for some well-defined positive function ψ of position x and the unit normal ν. Here we consider only the case that the right-hand side is a constant $\psi = \sigma \in (0, 1)$. We shall show later in chapter two that the right-hand side must be smaller than one for a solution to exist; see corollary 2.5.1. Note that (1.1) is a relation among the hypersurface's principal curvatures, so the equation will be referred to as the **curvature relation** satisfied by Σ. Two typical special cases that have been well-studied are hypersurfaces of constant mean curvature in which $f(\kappa) = \frac{1}{n}\sum_{i=1}^n \kappa_i$; and hypersurfaces of constant Gauss curvature in which $f(\kappa) = (\kappa_1 \cdots \kappa_n)^{\frac{1}{n}}$.

There are several equivalent models of the hyperbolic space $\mathbb{H}^{n+1}$, each of which is useful in certain contexts. In this thesis, we will use the **upper half-space model** for $\mathbb{H}^{n+1}$:

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds^2 = \frac{\sum_{i=1}^n dx_i^2}{x_{n+1}^2}$$

so that we can identify $\partial_\infty \mathbb{H}^{n+1}$ with $\mathbb{R}^n = \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$ and (1.2) may be understood in the Euclidean sense.

Since the hyperbolic metric is conformally equivalent to the Euclidean metric with a coefficient x_{n+1}^{-2} of conformality, the hyperbolic principal curvatures κ_i of Σ are related to its Euclidean κ_i^e principal curvatures by the following relation:

$$\kappa_i = x_{n+1}\kappa_i^e + \nu^{n+1}, \quad 1 \le i \le n \quad \text{at } (x, x_{n+1}) \in \Sigma \tag{1.3}$$

where ν is the Euclidean unit normal vector to Σ and $\nu^{n+1} = \nu \cdot e_{n+1}$. Consequently, a smooth hypersurface solution to (1.1)-(1.2) must be the graph of a smooth function u over some bounded domain $\Omega \subseteq \mathbb{R}^n$

$$\Sigma = \text{graph}(u) = \{(x, u(x)) \in \mathbb{H}^{n+1} : x \in \Omega\}$$

and the asymptotic boundary Γ must be the boundary of that domain i.e. $\Gamma = \partial\Omega$. The proofs of both (1.3) and that $\Sigma = \text{graph}(u)$ will be given in section 2.1.

Therefore, we can begin by assuming $\Sigma = \text{graph}(u)$ is a graphic hypersurface and for it to satisfy the two requirements (1.1)-(1.2), the function u must meet some conditions as well. It turns out that the curvature relation (1.1) can be written as a partial differential equation (PDE hereafter) in a local orthonormal frame and hence our geometric problem can thus be reduced to the following

Dirichlet problem for an implicitly defined fully non-linear second order elliptic PDE:

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \quad \text{in } \Omega \subseteq \mathbb{R}^n \tag{1.4}$$
$$u = 0 \quad \text{on } \partial\Omega = \Gamma \tag{1.5}$$

where the exact formula of G will be given in section 2.4. We shall seek solutions u with $\kappa[u] \in K$ where $\kappa[u] := \kappa[\text{graph}(u)]$ and call them the **admissible solutions**. Once we obtain such a solution u to this Dirichlet problem, its graph $\Sigma := \text{graph}(u)$ will be a solution to the asymptotic Plateau problem (1.1)-(1.2).

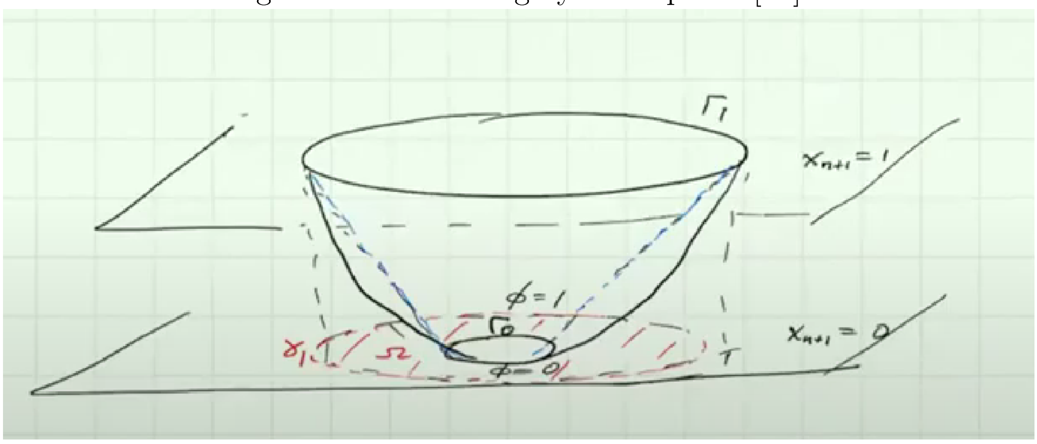
The following illustration of a similar problem considered in $\mathbb{R}^{n+1}$ would be helpful to understand our problem in $\mathbb{H}^{n+1}$.

Example 1. Let Γ_0 and Γ_1 be two strictly convex smooth closed codimension 2 hypersurfaces in parallel planes $\{x_{n+1} = 0\}$ and $\{x_{n+1} = 1\}$, respectively. Suppose the projection γ_1 of Γ_1 onto the lower plane $\{x_{n+1} = 0\}$ contains Γ_0. Does there exist a hypersurface Σ of constant Gauss curvature K_0 for K_0 sufficiently small? Intuitively, the answer is affirmative; see below for a drawing by Spruck [29]. Moreover, the hypersurface is the graph of some function u over the annulus Ω whose inner boundary is Γ_0 and outer boundary is γ_1. The function is a solution to the following Dirichlet problem

$$\det(u_{ij}) = K_0(1 + |\nabla u|^2)^{\frac{n+2}{2}}, \quad \text{in } \Omega$$
$$u = \phi, \quad \text{on } \partial\Omega$$

where $\phi = 1$ on γ_1 and $\phi = 0$ on Γ_0.

Figure 1.1: A drawing by Joel Spruck [29].



For more details, see [20, 13, 12].

Before we proceed to discuss our method of solution, we shall state the assumptions imposed on the curvature function $f(\kappa)$. First of all, f is assumed to satisfy the **fundamental structure conditions**:

$$f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \quad \text{for } \lambda \in K \text{ and } 1 \leq i \leq n \tag{1.6}$$

$$f \text{ is a concave function in } K \tag{1.7}$$

As shown in [6], the first condition will imply the PDE (1.4) is elliptic for admissible solutions and the second condition will make G concave with respect to D^2u.

We shall also assume

$$f > 0 \text{ in } K \text{ and } f = 0 \text{ on } \partial K \tag{1.8}$$

This third condition implies that the PDE (1.4) will be uniformly elliptic on compact subdomains of Ω for admissible solutions satisfying a priori bounds in the C^2 norm and therefore allows us to apply the Evans-Krylov interior estimate [10, 22] to derive the $C^{2,\alpha}$ and higher order estimates.

In addition, the following few mild conditions are imposed:

$$f \text{ is normalized: } f(1,\dots,1) = 1 \tag{1.9}$$

$$f \text{ is homogeneous of degree one: } f(t\kappa) = tf(\kappa) \text{ for } t \geq 0 \text{ and } \kappa \in K \tag{1.10}$$

$$\lim_{R\to\infty} f(\lambda_1,\dots,\lambda_{n-1},\lambda_n + R) \geq 1 + \varepsilon_0 \quad \text{uniformly in } B_{\delta_0}(\mathbf{1}) \tag{1.11}$$

for some fixed $\varepsilon_0 > 0$ and $\delta_0 > 0$.

All these assumptions though technical, they are satisfied by a large class of curvature functions, especially those of the most interest. For example, consider the k-th normalized elementary symmetric polynomial

$$H_k(\kappa_1,\dots,\kappa_n) := \frac{1}{\binom{n}{k}}\sigma_k(\kappa_1,\dots,\kappa_n) = \frac{1}{\binom{n}{k}}\sum_{1\leq j_1<j_2<\cdots<j_k\leq n} \kappa_{j_1}\kappa_{j_2}\cdots\kappa_{j_k}, \quad 1 \leq k \leq n$$
$$H_0 := 1$$

which are defined in the k-th Garding cone

$$K_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \quad \forall\, 1 \leq j \leq k\}$$

Remark 1. K_n is the positive cone and we denote it by K_n^+ to stress the positivity feature $\kappa_i > 0$.

Now, we point out that both the higher order mean curvature $H_k^{\frac{1}{k}}$ and the curvature quotient $\left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}, 1 \leq l < k \leq n$ satisfy all the conditions (1.6)-(1.11) in K_k. In particular, they include the following important special cases [1]:

$$H_1(\kappa) = \frac{1}{n}\sum_{i=1}^{n} \kappa_i \qquad \text{the mean curvature}$$
$$H_2(\kappa) = \frac{2}{n(n-1)}\sum_{i<j} \kappa_i\kappa_j \qquad \text{the scalar curvature}$$
$$H_n(\kappa) = \kappa_1\cdots\kappa_n \qquad \text{the Gauss curvature}$$

[1]The prescribed curvature equation that we intend to study is $H_k = \text{const}$; we impose the exponent $\frac{1}{k}$ in order for it to satisfy the homogeneity condition (1.10). Note that it is equivalent to study $H_k^{1/k} = \text{const}$ and the same reasoning applies to the curvature quotient.

Remark 2. The ultimate goal is to solve the asymptotic Plateau problem (1.1)-(1.2) for $f = H_k^{1/k}$ and $f = \left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}$ in the k-th Garding cone K_k.

The standard way to solve a fully non-linear elliptic PDE is through the method of continuity; see chapter 17 of [11]. However, as we shall in see in section 2.4, the PDE (1.4) is degenerate where $u = 0$ and we cannot apply the continuity method directly. Instead, following the method exhibited in the works [16, 14, 15, 17] of Bo Guan, Joel Spruck and their collaborators, we will study the Dirichlet problem with an approximate boundary condition:

$$G(D^2u, Du, u) = 0, \quad u > 0 \text{ in } \Omega \subseteq \mathbb{R}^n \tag{1.12}$$
$$u = \varepsilon, \quad \text{on } \partial\Omega \tag{1.13}$$

.

There are two main difficulties in applying the continuity method to the approximate Dirichlet problem (1.12)-(1.13). The first one is to show the second normal derivative is a priori bounded on the boundary i.e. $u_{nn} \leq C$ on $\partial\Omega$, which usually requires some geometrical assumptions about the domain Ω. In this thesis, as in [14], we will assume the domain Ω is **mean-convex** i.e. the Euclidean mean curvature of $\partial\Omega$ is non-negative $\mathcal{H}_{\partial\Omega} \geq 0$. The second problem is that since the PDE is fully non-linear elliptic, we need to obtain a global $C^{2,\alpha}$ estimate instead of a C^2 estimate, which can be accomplished either by the Evans-Krylov theorem [10, 22] or Calabi's third derivative estimate [8]. In this thesis, as in [16, 14, 15, 17], we will use the former method.

When $\varepsilon > 0$ is sufficiently small, it follows from the continuity method that the Dirichlet problem (1.12)-(1.13) is solvable for all $\sigma \in (0, 1)$.

Theorem 1.0.1 ([14])**.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth mean-convex domain i.e. $\mathcal{H}_{\partial\Omega} \geq 0$. If f satisfies (1.6)-(1.11), then for any $\sigma \in (0, 1)$ and $\varepsilon > 0$ sufficiently small, there exists a unique admissible solution $u^\varepsilon \in C^\infty(\overline{\Omega})$ of the Dirichlet problem (1.12)-(1.13) satisfying the following a priori estimates

$$\sqrt{1 + |Du^\varepsilon|^2} \leq \frac{1}{\sigma} + C\varepsilon, \quad u^\varepsilon|D^2u^\varepsilon| \leq C \quad \text{on } \partial\Omega \tag{1.14}$$
$$u^\varepsilon|D^2u^\varepsilon| \leq \frac{C}{\varepsilon^2} \quad \text{in } \Omega \tag{1.15}$$

where $C > 0$ is independent of ε.

We can then obtain a sequence of solutions u^ε from which we can extract a uniformly convergent subsequence u^{ε_k}, whose limit as $\varepsilon_k \to 0$ is a solution to the original Dirichlet problem (1.4)-(1.5). However, the estimate (1.15) does not allow us to pass to the limit. Instead, we shall obtain such an estimate $u^\varepsilon|D^2u^\varepsilon|^2 \leq C$ by proving a maximum principle for the largest hyperbolic principal curvature i.e. $\kappa_{\max} \leq C$.

The existence of solutions to the original Dirichlet problem can be proved as follows. Since $\kappa_{\max} \leq C$, the hyperbolic principal curvatures of an admissible solution u^ε are uniformly bounded above by a constant independent of ε. Also, since $f(\kappa[u^\varepsilon]) = \sigma$ in K and $f = 0$ on ∂K, it follows that the hyperbolic principal curvatures κ_i admit a uniform positive lower bound independent of ε on compact subdomains Ω' of Ω; see theorem 2.2.1 or lemma 2.5.1. Hence the PDE (1.12) is uniformly elliptic on Ω' for admissible solutions u^ε and by the interior estimates of Evans-Krylov [10, 22], we obtain uniform $C^{2,\alpha}$ estimates for admissible solutions on Ω'. Finally, the existence of solutions to (1.4)-(1.5) can be ensured by taking the limit $\varepsilon \to 0$. See chapter three for more details.

Using this method, Bo Guan, Joel Spruck, Marek Szapiel and Ling Xiao have completely solved the problem (1.1)-(1.2) when $K = K_n^+$ is the positive cone.

Theorem 1.0.2 ([16, 15, 17])**.** When $K = K_n^+$ is the positive cone and $\Gamma = \partial\Omega \in C^2$, the asymptotic Plateau problem (1.1)-(1.2) admits a solution for all $\sigma \in (0, 1)$.

In addition, if the boundary $\Gamma = \partial\Omega$ satisfies either of the following conditions

(i) $\Gamma = \partial\Omega$ is $C^{2,\alpha}$ mean-convex,

(ii) $\Gamma = \partial\Omega$ is C^2 and strictly Euclidean star-shaped about the origin, or

(iii) if the curvature function f satisfies (1.6)-(1.11) in K_n^+ and

$$\sum_{i=1}^{n} f_i > \sum_{i=1}^{n} \lambda_i^2 f_i \text{ in } K_n^+ \cap \{\lambda : 0 < f(\lambda) < 1\}$$

,

then the solution is unique.

Remark 3. The latest version of their theorem in [17] greatly improves their previous results [16, 15] by using a different test function in the proof of the maximum principle $\kappa_{\max} \leq C$; see theorem 1.3 in [17].

In [16, 15], the domain Ω was assumed to be at least C^3 but now it is sufficient for it to be only C^2. Moreover, the assumptions (1.10) and (1.11) on the curvature function can be removed. We also point out that in their theorem, the mean-convexity assumption $\mathcal{H}_{\partial\Omega}$ on the domain Ω is not needed for the existence of solutions.

The next task is certainly to solve the problem in the general cone K. However, the problem becomes so much harder to solve and so far they only obtained a partial result.

Theorem 1.0.3 ([14])**.** Suppose $\Gamma = \partial\Omega$ for some bounded smooth domain $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \geq 0$ and f satisfies (1.6)-(1.11). The asymptotic Plateau problem (1.1)-(1.2) is solvable in K for all $0 < \sigma_0 < \sigma < 1$, where σ_0 is some number between 0.3703 and 0.3704.

That is, the existence of a solution Σ with $\kappa[\Sigma] \in K$ for all $0 < \sigma < 1$ is still not guaranteed even when the condition $H_{\partial\Omega} \geq 0$ is imposed. The question of whether the problem (1.1)-(1.2) is solvable for all $\sigma \in (0, 1)$ with the general curvature constraint $\kappa[\Sigma] \in K$ and no geometrical assumptions about the domain Ω other than being smooth and bounded i.e. remove the condition $\mathcal{H}_{\partial\Omega} \geq 0$, remains unsettled. In particular, we expect the problem to be solvable in the k-th Garding cone for all $\sigma \in (0,1)$ when $f = H_k^{1/k}$ is the higher order mean curvature or when $f = \left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}$ is the curvature quotient for $1 \leq l < k \leq n$.

Most recently, Sui [30] studied the problem with the more general curvature relation

$$H_k^{1/k}(\kappa(x)) = \psi(x) \text{ and } \kappa \in K_n^+ \text{ for all } x \in \Sigma$$

and proved the existence of a smooth admissible solution in $\mathbb{H}^3$ under the assumption that a locally strictly convex subsolution exists. However, the solution to the most general case is still far from clear and one of the difficulties being that many inequalities may not work without the positivity condition $\kappa_i > 0$.

In this thesis, we note that the central reason why theorem 1.0.3 fails to hold for all $\sigma \in (0,1)$ is that their curvature estimate $\kappa_{\max} \leq C$ only holds for $\sigma \in (\sigma_0, 1)$. Therefore, the task reduces to improving the curvature estimate. Specifically, we prove

Theorem 1.0.4. Suppose $\Gamma = \partial\Omega$ for some bounded smooth domain $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \geq 0$ and the curvature function f satisfies

$$\text{there exists some } C > 0 \text{ such that } \sum_{i=1}^{n} f_i(\lambda) \leq C \text{ for all } \lambda \in K \tag{1.16}$$

in addition to (1.6)-(1.11) in the general cone K. Then the curvature estimate $\kappa_{\max} \leq C$ holds for all $\sigma \in (0,1)$ in K and hence there exists for all $\sigma \in (0,1)$, a smooth complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1.1)-(1.2) with uniformly bounded principal curvatures

$$|\ \kappa[\Sigma]\ | \leq C \text{ on } \Sigma$$

Moreover, Σ is the graph of a unique admissible solution $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ of the Dirichlet problem (1.4)-(1.5).

Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and

$$\sqrt{1+|Du|^2} \leq \frac{1}{\sigma}, \quad u|D^2u| \leq C \quad \text{in } \Omega,$$
$$\sqrt{1+|Du|^2} = \frac{1}{\sigma} \quad \text{on } \partial\Omega.$$

That is, we extend theorem 1.0.3 to hold for all $\sigma \in (0,1)$ with the aid of this additional assumption (1.16). Our improvement is based on an observation that, by examining the inequalities in their proof of the curvature estimate from a different perspective, it would turn out that the key issue is to estimate the sum $\sum_{i=1}^n f_i$ in (optimally a subset of) K and this is the only place we impose the assumption (1.16); all the other results in [14] remain intact.

Recall again that we desire the problem (1.1)-(1.2) to be solvable in the k-th Garding cone K_k for all $\sigma \in (0,1)$ when the curvature function is either the higher order mean curvature $f = H_k^{1/k}$ or their quotients $\left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}$, $1 \leq l < k \leq n$. Note that the cases that $(f, K) = (H_1, K_1)$ and $(f, K) = (H_n^{1/n}, K_n^+)$ have already been taken care of in [23, 27, 25, 18]; the results in [16, 15, 17] apply to general f in the positive cone K_n^+ (but not in the k-th cone K_k) and in particular to the quotients $\left(\frac{H_n}{H_l}\right)^{\frac{1}{n-l}}$, $1 \leq l < n$. In other words, the case $(f, K) = \left(\left(\frac{H_n}{H_l}\right)^{\frac{1}{n-l}}, K_n^+\right)$ has been resolved as well. For the remaining indices $1 < k < n$, the problem (1.1)-(1.2) still awaits solutions and new methods should be employed. We emphasize here that our theorem 1.0.4 applies to the particular quotient $\frac{H_k}{H_{k-1}}$ in the k-th Garding cone where $1 \leq k \leq n$, therefore it fills in one of the missing pieces to the "jigsaw puzzle".

Curvature Function	Cone	Solution
H_1	K_1	[25, 18]
$H_n^{1/n}$	K_n^+	[23, 27]
$\left(\frac{H_n}{H_l}\right)^{\frac{1}{n-l}}$, $1 \leq l < n$	K_n^+	[15, 17]
general f	K_n^+	[15, 17]
$\frac{H_k}{H_{k-1}}$, $1 \leq k \leq n$	K_k	theorem 1.0.4
$H_k^{1/k}$, $1 < k < n$	K_k	unknown
$\left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}$, $0 \leq l < k-1 < k < n$	K_k	unknown

Table 1.1: Current Progress on the Asymptotic Plateau Problem

The thesis is organized as follows. Chapter two is of a preliminary nature, providing necessary background for understanding the problem in concern and a list of facts we will use frequently in our proofs of the main results. While chapter three is devoted to prove theorem 1.0.4; we also prove the uniqueness of solutions for (1.12)-(1.13) in section 3.1, which immediately yields a global gradient estimate $\sqrt{1 + |Du|^2} \leq \frac{1}{\sigma}$. We note that the gradient estimate and the condition (1.11) are essential for the derivation of a boundary C^2 estimate i.e. $|D^2 u| \leq C$ on $\partial\Omega$, which will be proved in section 3.2. The centerpiece of this thesis is section 3.3, which contains the proof of a new curvature estimate $\kappa_{\max} \leq C$ relying on (1.16). The proofs of all these estimates are all heavily dependent on the auxiliary results listed in chapter two. Together with the boundary C^2 estimate, we obtain a global C^2 estimate and hence a $C^{2,\alpha}$ estimate as required by the method of continuity. This new curvature estimate improves theorem 1.0.3 and yields theorem 1.0.4; we emphasize that this is the only place we make changes to the results in [14].

Chapter 2

Preliminaries

We recall here some notions of Riemannian Geometry and introduce our notations along the way. All these materials can be found in a standard textbook such as [9].

A **Riemannian manifold** is a smooth manifold M equipped with a Riemannian metric g, which is a correspondence associating to each point p of M an inner product denoted by $\langle \cdot, \cdot \rangle_p$ or $g_p(\cdot, \cdot)$ on the tangent space T_pM and varying smoothly in the following sense: for any two smooth vector fields $X, Y \in \mathfrak{X}(M)$, the inner product $g_p(X_p, Y_p)$ is a smooth function of p.

An **affine connection** ∇ on M is a map $\mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ denoted by $(X, Y) \mapsto \nabla_X Y$ which satisfies

(i) $C^\infty(M)$-linearity in X: $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$.

(ii) $\mathfrak{X}(M)$-additive in Y: $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$.

(iii) the Leibniz rule: $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$ and smooth functions $f, g \in C^\infty(M)$. Every Riemannian manifold (M, g) admits a unique affine connection ∇ such that it is symmetric

$$\nabla_X Y - \nabla_Y X = XY - YX$$

and compatible with the Riemannian metric

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

. We call it the **Levi-Civita connection** on M. From now on, ∇ will always denote the Levi-Civita connection of some Riemannian manifold rather than an arbitrary affine connection.

The **curvature tensor** R of M is a correspondence associating to every pair of vector fields $X, Y \in \mathfrak{X}(M)$ a mapping $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X, Y)Z = \nabla_Y(\nabla_X Z) - \nabla_X(\nabla_Y Z) + \nabla_{[X,Y]} Z$$

Let $\{e_1, \dots, e_n\}$ be a local frame on M. We denote the **local representation** of the Riemannian metric by $g_{ij} = g(e_i, e_j)$ and its inverse by g^{ij}; we also use the following abbreviations:

$$\nabla_i := \nabla_{e_i}, \quad \nabla_{ij} := \nabla_i \nabla_j - \nabla_{\nabla_i e_j}$$
$$R_{ijkl} := \langle R(e_k, e_l)e_j, e_i \rangle, \quad R^i_{jkl} := g^{im} R_{mjkl}$$

and the **Christoffel symbols** Γ^k_{ij} are the unique coefficients such that $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$.

Moreover, for a smooth function v defined on M, we identify its gradient ∇v and Hessian $\nabla_{ij} v$ as the vector fields defined by

$$\langle (\nabla v)_p, w \rangle = dv_p(w), \quad p \in M \quad w \in T_p M$$
$$\nabla_{ij} v = \nabla_i (\nabla_j v) - \Gamma^k_{ij} \nabla_k v$$

Finally, we list a few important formulas in Riemannian geometry which will be used in section 3.3. Let X denote the position vector of M, ν the outer unit normal on M and $\{h_{ij}\}$ the second fundamental form of M, we have

$$\begin{aligned} X_j &= -h_{ij} \nu && \text{Gauss formula} \\ \nu_i &= h_{ij} e_j && \text{Weingarten equation} \\ h_{ijk} &= h_{ikj} && \text{Codazzi equation} \\ R_{ijkl} &= h_{ik} h_{jl} - h_{il} h_{jk} && \text{Gauss equation} \end{aligned}$$

where $h_{ijk} := \nabla_k h_{ij}$.

In the following sections, we list auxiliary results that will be used frequently and implicitly in chapter three. Note that most results are provided with detailed proofs except those requiring tedious proofs, for which we shall only briefly sketch the proofs and refer the reader to sources of their full proofs.

Throughout this thesis, all hypersurfaces in $\mathbb{H}^{n+1}$ are assumed to be connected and orientable. If Σ is a complete hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary at infinity, then the normal vector field of Σ is chosen to be the one pointing to the unique unbounded region in $\mathbb{R}^{n+1}_+ \setminus \Sigma$, and the principal curvatures are calculated with respect to this normal vector field.

2.1 Σ as a Graph

In this section, we derive the simple relation (1.3) between Euclidean principal curvatures and hyperblic principal curvatures; as a consequence, we establish the fact that Σ is the graph of some function over a domain $\Omega \subseteq \mathbb{R}^n$.

Let Σ be a hypersurface in $\mathbb{H}^{n+1}$. We shall use g and ∇ to denote the induced hyperbolic metric and Levi-Civita connection on Σ, respectively. As Σ is also a submanifold of $\mathbb{R}^{n+1}$, we shall distinguish a geometric quantity with respect to the Euclidean metric by adding a 'tilde' over the corresponding hyperbolic quantity. For example, $\tilde{g}$ denotes the induced metric on Σ from $\mathbb{R}^{n+1}$ and $\tilde{\nabla}$ is its Levi-Civita connection.

Suppose Σ is locally represented as the graph of some function $u \in C^2(\Omega), u > 0$ over Ω:

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in \Omega\}$$

. The coordinate vector fields on Σ and the hyperbolic unit normal are

$$X_i = e_i + u_i e_{n+1}, \quad \mathbf{n} = u\nu$$

where

$$\nu = \left(\frac{-Du}{w}, \frac{1}{w}\right) = \frac{-u_i e_i + e_{n+1}}{w}$$

is the Euclidean unit normal and $w = \sqrt{1 + |Du|^2}$.

The first fundamental form is then by definition

$$g_{ij} = \langle X_i, X_j \rangle = \frac{1}{u^2}(\delta_{ij} + u_i u_j) = \frac{\tilde{g}_{ij}}{u^2}$$

The Christoffel symbol for the hyperbolic metric is

$$\begin{aligned}
\Gamma_{ij}^k &= \frac{1}{2}\sum_m (\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij}) g^{mk} \\
&= \frac{1}{x_{n+1}}(-\delta_{jk}\delta_{i,n+1} - \delta_{ik}\delta_{j,n+1} + \delta_{ij}\delta_{k,n+1})
\end{aligned}$$

and it is related to the Euclidean version by

$$\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k - \frac{u_i \delta_{kj} + u_j \delta_{ik} - \tilde{g}^{kl} u_l \tilde{g}_{ij}}{u}$$

Let ∇ be the Riemannian connection of $\mathbb{H}^{n+1}$. Then

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k = \left(\frac{\delta_{ij}}{x_{n+1}} + u_{ij} - \frac{u_i u_j}{x_{n+1}}\right) e_{n+1} - \frac{u_j e_i + u_i e_j}{x_{n+1}}$$

and the second fundamental form is

$$h_{ij} = \langle \nabla_{X_i} X_j, \mathbf{n} \rangle = \frac{\delta_{ij} + u_i u_j + u u_{ij}}{u^2 w} = \frac{\tilde{h}_{ij}}{u} + \frac{\nu^{n+1}}{u^2} \tilde{g}_{ij} \tag{2.1}$$

The hyperbolic principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ of Σ are the roots of

$$\det(h_{ij} - \kappa g_{ij}) = \frac{1}{u^n} \det\left(\tilde{h}_{ij} - \frac{1}{u}\left(\kappa - \frac{1}{w}\right)\tilde{g}_{ij}\right) = 0$$

Therefore,

$$\kappa_i = u\tilde{\kappa}_i + \nu^{n+1} = u\tilde{\kappa}_i + \frac{1}{w} \tag{2.2}$$

and we can use this relation to prove

Theorem 2.1.1 ([16])**.** If Σ is a complete C^2 hypersurface in $\mathbb{H}^{n+1}$ with compact asymptotic boundary $\partial\Sigma \subseteq \{x_{n+1} = \varepsilon\}$ for some $\varepsilon > 0$, then Σ is the graph of some function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ over a bounded domain $\Omega \subseteq \{x_{n+1} = \varepsilon\}$:

$$\Sigma = \{(x, u(x)) \in \mathbb{R}^{n+1} : x \in \Omega\}$$

such that $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. Moreover, $\partial\Sigma = \partial\Omega$.

Here we say Σ has compact asymptotic boundary if $\partial\Sigma \subseteq \partial_\infty \mathbb{H}^{n+1}$ is compact with respect to the Euclidean metric in $\mathbb{R}^n$.

Proof. Let T be the set of $t \geq \varepsilon$ such that $\Sigma_t := \Sigma \cap \{x_{n+1} \geq t\}$ is a vertical graph and let t_0 denote the minimum of T.

Suppose to the contrary that $t_0 > c$. Then there must exist a point $p \in \partial\Sigma_{t_0}$ such that $\nu^{n+1}(p) = 0$ i.e. the normal vector to Σ at p is horizontal. It follows from (2.2) that $\tilde{\kappa}_i = \frac{\kappa_i}{t_0} > 0$ for all $1 \leq i \leq n$ at p.

However, if P is the plane through p spanned by $\mathbf{e}$ and $\nu(p)$, then $\Sigma \cap P$ is a curve having non-positive curvature at p. This is a contradiction and so $t_0 = c$.

See lemma 2.1 in [30] for a slightly different proof. □

2.2 Properties of the Curvature Function

We first recall the **Euler's theorem** on homogeneous functions, which will be used most frequently throughout the text but without saying so.

Lemma 2.2.1 (Euler's theorem)**.** Let $\Omega \subseteq \mathbb{R}^n$ be open. Suppose $f \in C^1(\Omega)$ is positively homogeneous of degree k in Ω. Then

$$kf(x) = \nabla f(x) \cdot x = \sum_{i=1}^{n} x_i f_i(x) \quad \text{for all } x \in \Omega$$

Proof. Define $g(t) := f(tx)$. Note that if we see g as $g(t) = t^k f(x)$ by exploiting the homogeneity condition, then

$$g'(t) = kt^{k-1} f(x)$$

. On the other hand, if we see g as $g(t) = f(tx)$ and apply the chain rule, then

$$g'(t) = \nabla f(tx) \cdot x$$

. Hence we have $kt^{k-1} f(x) = \nabla f(tx) \cdot x$ and setting $t = 1$ yields the result. □

Lemma 2.2.2. Suppose the smooth symmetric function f is concave, normalized and homogeneous of degree one on an open symmetric convex set K containing K_n^+. Then for all $\lambda \in K$ we have

$$f(\lambda) \leq \frac{1}{n} \sum_{i=1}^{n} \lambda_i \quad \text{and} \quad \sum_{i=1}^{n} f_i(\lambda) \geq 1 \tag{2.3}$$

Proof. For the first inequality, note that since f is concave we have for all $x, y \in K$ that

$$f(x) - f(y) \leq \sum_{i=1}^{n} f_i(y) \cdot (x - y)$$

Also since f is homogeneous of degree one, we have by the Euler's theorem

$$\nabla f(\lambda) \cdot \lambda = f(\lambda)$$

Applying this formula to $\lambda = \mathbf{1} = (1, 1, \ldots, 1)$, we have $\sum_{i=1}^{n} f_i(\mathbf{1}) = f(\mathbf{1}) = 1$. Note further that as f is symmetric, $f_i(\mathbf{1}) = f_j(\mathbf{1})$ for all $1 \leq i, j \leq n$. This can be seen from the definition of partial derivative. It follows that $f_i(\mathbf{1}) = \frac{1}{n}$.

Therefore, we have

$$\begin{aligned} f(\lambda) &\leq f(\mathbf{1}) + \sum_{i=1}^{n} f_i(\mathbf{1})(\lambda_i - 1) \\ &= f(\mathbf{1}) + \sum_{i=1}^{n} f_i(\mathbf{1})\lambda_i - \sum_{i=1}^{n} f_i(\mathbf{1}) \\ &= \frac{1}{n} \sum_{i=1}^{n} \lambda_i \end{aligned}$$

This proves the first inequality.

For the second inequality, we apply Euler's theorem and concavity again

$$\begin{aligned} \sum_{i=1}^{n} f_i(\lambda) &= \sum_{i=1}^{n} f_i(\lambda) + [f(\lambda) - f(\lambda)] \\ &= f(\lambda) + \sum_{i=1}^{n} f_i(\lambda) - \nabla f(\lambda) \cdot \lambda \\ &= f(\lambda) + \sum_{i=1}^{n} f_i(\lambda)(1 - \lambda_i) \\ &\geq f(\mathbf{1}) = 1 \end{aligned}$$

□

Lemma 2.2.3. Suppose f satisfies (1.6)-1.10) in K and let $\lambda \in K$. If $\lambda_r < 0$ for some $1 \leq r \leq n$, then

$$\sum_{i \neq r} f_i \lambda_i^2 \geq \frac{1}{n-1}(2f \cdot |\lambda_r| + f_r \lambda_r^2)$$

$$\sum_{i \neq r} f_i \lambda_i^2 \geq \frac{1}{n} \sum_{i=1}^{n} f_i \lambda_i^2$$

Proof. Since $\lambda_r < 0$, we have $\min_{1\le i\le n} \lambda_i \le \lambda_r < 0$. Also, since $\sum_{i=1}^n f_i\lambda_i = f > 0$ and $f_i > 0$ for all $1 \le i \le n$, there must be at least one $\lambda_s > 0$ and hence $\max_{1\le i\le n} \lambda_i > 0$. We may now assume $\lambda = (\lambda_1, \ldots, \lambda_n)$ is ordered as

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$$

with $\lambda_1 > 0$ and $\lambda_n < 0$.

Since f is homogeneous of degree one, we have by Euler's theorem that $\sum_{i=1}^n f_i(\lambda)\lambda_i = f(\lambda)$ for all $\lambda \in K$. Hence

$$\sum_{i\ne n} f_i\lambda_i = f + f_n|\lambda_n|$$

If we apply the Cauchy-Schwartz inequality to the product $\sum_{i\ne n} f_i \cdot \sum_{i\ne n} f_i\lambda_i^2$, we obtain

$$\left(\sum_{i\ne n} f_i\right)\cdot\left(\sum_{i\ne n} f_i\lambda_i^2\right) \ge \left(\sum_{i\ne n}\sqrt{f_i}\cdot\sqrt{f_i}\lambda_i\right)^2 = \left(\sum_{i\ne n} f_i\lambda_i\right)^2$$
$$= (f + f_n|\lambda_n|)^2 = f^2 + 2ff_n|\lambda_n| + f_n^2\lambda_n^2$$

By concavity of f in K, we have that $f_n \ge f_i$ for all $1 \le i \le n$ and so $f_n\lambda_n^2 \ge f_r\lambda_r^2$.

It is then easy to deduce that

$$\left(\sum_{i\ne n} f_i\right)\cdot\left(\sum_{i\ne n} f_i\lambda_i^2\right) \le (n-1)f_n\cdot\sum_{i\ne n} f_i\lambda_i^2$$

Therefore, we have obtained

$$f^2 + 2ff_n|\lambda_n| + f_n^2\lambda_n^2 \le (n-1)f_n\cdot\sum_{i\ne n} f_i\lambda_i^2$$

Finally, since $|\lambda_n| \ge |\lambda_r|$ and $f_n\lambda_n^2 \ge f_r\lambda_r^2$, it follows that

$$\sum_{i\ne r} f_i\lambda_i^2 \ge \sum_{i\ne n} f_i\lambda_i^2 \ge \frac{(f^2/f_n) + 2f|\lambda_n| + f_n\lambda_n^2}{n-1} \ge \frac{2f|\lambda_n| + f_n\lambda_n^2}{n-1}$$
$$\ge \frac{2f|\lambda_r| + f_r\lambda_r^2}{n-1}$$

This proves the first inequality. □

Theorem 2.2.1 ([6]). Suppose Σ is a smooth complete hypersurface in $\mathbb{H}^{n+1}$ satisfying $f(\kappa[\Sigma]) = \sigma$ in K and the curvature function $f(\kappa)$ satisfies (1.7) and (1.8) in K. Then there exists some $\delta > 0$ such that

$$\sum_{i=1}^n \kappa_i \ge \delta > 0 \quad \text{in } \{\kappa \in K : f(\kappa) \ge \sigma\}$$

Remark 4. The same conclusion holds in the set $\{\kappa : f(\kappa) \geq \inf \psi\}$ for the general curvature relation $f(\kappa) = \psi(x, \nu)$.

Proof. The set $\{\kappa : f(\kappa) \geq \sigma\}$ is closed, convex and symmetric. The unique closest point in this set to the origin is therefore of the form $(b, \ldots, b)$ for some $b > 0$ and we can take $\delta = nb$. $\square$

Theorem 2.2.2. Both the higher order mean curvature $H_k^{\frac{1}{k}}$ and the curvature quotient $\left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}}$ satisfy all the assumptions (1.6)-(1.11) in the k-th Garding cone K_k.

Proof. For the fundamental structure conditions (1.6)-(1.7), see theorem 2.16 in [28] and section 2 in [32]; these are common knowledge in literature. The condition (1.8) follows immediately from the definition of the k-th Garding cone:

$$K_k := \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \quad \forall\ 1 \leq j \leq k\}$$

. Similarly, the conditions (1.9) and (1.10) are straightforward to verify from the definition of the k-th elementary symmetric polynomial:

$$H_k(\kappa) := \frac{1}{\binom{n}{k}} \sigma_k(\kappa_1, \ldots, \kappa_n) = \frac{1}{\binom{n}{k}} \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \kappa_{j_1} \kappa_{j_2} \cdots \kappa_{j_k}, \quad 1 \leq k \leq n$$

While for (1.11), we have by direct computation that

$$\lim_{R \to \infty} f(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n + R) = \begin{cases} \infty, & f = H_k^{\frac{1}{k}} \\ \left(\frac{k}{l}\right)^{\frac{1}{k-l}}, & f = \left(\frac{H_k}{H_l}\right)^{\frac{1}{k-l}} \end{cases}$$

$\square$

2.3 The Curvature Relation

In this section, we prove some useful formulas with the help of (1.1); they will be used frequently without comments in subsequent parts of the thesis.

Let $\mathcal{S}$ be the vector space of $n \times n$ symmetric matrices. For the open symmetric convex cone $K \subseteq \mathbb{R}^n$ with $K_n^+ \subseteq K$, we set

$$S_K := \{A \in \mathcal{S} : \lambda(A) \in K\}$$

where $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$ denotes eigenvalues of A.

For a function F defined in $\mathcal{S}_K$, we denote

$$F^{ij}(A) := \frac{\partial F}{\partial a_{ij}}(A), \quad F^{ij,kl}(A) := \frac{\partial^2 F}{\partial a_{ij} \partial a_{kl}}(A), \quad A = \{a_{ij}\} \in \mathcal{S}_K$$

.

We are interested in the case that F depends only on the eigenvalues of A i.e.

$$F(A) = f(\lambda(A))$$

.

Lemma 2.3.1 ([28])**.** Suppose f is a smooth symmetric function satisfying the fundamental structure conditions (1.6)-(1.7) in K and $F(A) = f(\lambda(A))$. Then for all $A \in S_K$,

(i) F is smooth and $\{F^{ij}\}$ is symmetric.

; when A is diagonal,

(ii) $F^{ij} = f_i\delta_{ij}$ and so the linearized operator $L = F^{ij}\partial_{ij}$ is elliptic.

(iii) we have

$$F^{ij}(A)a_{ij} = \sum_i f_i(\lambda(A))\lambda_i(A) \tag{2.4}$$

$$F^{ij}(A)a_{ik}a_{kj} = \sum_i f_i(\lambda(A))\lambda_i^2(A) \tag{2.5}$$

where the Einstein summation convention is being used.

(iv) F is concave i.e. $F^{ij,kl}(A)\xi_{ij}\xi_{kl} \leq 0$ for all $\xi \in \mathcal{S}$ and $A \in S_K$.

Proof.

(i) Smoothness of F follows from the smoothness of f and symmetry of F follows from the symmetry of A.

(ii) Since $F^{ij} = \sum_k f_{\lambda_k}\frac{\partial\lambda_k}{\partial a_{ij}}$, we need to compute $\frac{\partial\lambda_k}{\partial a_{ij}}$. Consider a variation $\tilde{A}_{ij} := A_{ij} + \varepsilon$. If $j < i$ then

$$\det(\tilde{A} - \lambda I) = \prod_{k\neq i,j}(\lambda_k - \lambda)(\lambda^2 - (\lambda_i + \lambda_j) + \lambda_i\lambda_j - \varepsilon^2)$$

. It follows that

$$\tilde{\lambda}_k = \lambda_k, \quad \text{if } k \neq i, j$$

$$\tilde{\lambda}_i = \frac{\lambda_i + \lambda_j}{2} + \sqrt{\left(\frac{\lambda_i - \lambda_j}{2}\right)^2 + \varepsilon^2}$$

$$\tilde{\lambda}_j = \frac{\lambda_i + \lambda_j}{2} - \sqrt{\left(\frac{\lambda_i - \lambda_j}{2}\right)^2 + \varepsilon^2}$$

i.e. $\tilde{\lambda}_i = \lambda_i + O(\varepsilon^2)$ and $\tilde{\lambda}_j = \lambda_j + O(\varepsilon^2)$. Hence $\frac{\partial\lambda_k}{\partial a_{ij}} = 0$ if $k \neq i, j$.

If $i = j$, then $\tilde{\lambda}_k = \lambda_k$ for $k \neq i$ and $\tilde{\lambda}_i = \lambda_i + \varepsilon$. Thus in all cases, we have $\frac{\partial\lambda_k}{\partial a_{ij}} = \delta_{ki}\delta_{ij}$ and

$$F^{ij} = \sum_k f_{\lambda_k}\frac{\partial\lambda_k}{\partial a_{ij}} = f_i\delta_{ij}$$

(iii) We just do direct computation.

$$F^{ij}a_{ij} = f_i\delta_{ij}a_{ij} = f_i a_{ii} = f_i\lambda_i$$
$$F^{ij}a_{ik}a_{kj} = f_i\delta_{ij}a_{ik}a_{kj} = f_i\delta_{ij}a_{ij}^2 = f_i a_{ii}^2 = f_i\lambda_i^2$$

(iv) Let $A, B \in S_K$. We want to show

$$F^{ij}(A)(B-A)_{ij} \geq F(B) - F(A)$$

. Assume the eigenvalues $\{\lambda_i\}$ of A and eigenvalues $\{\mu_i\}$ of B are arranged so that

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$$
$$f_{\lambda_1} \geq f_{\lambda_2} \geq \cdots \geq f_{\lambda_n}$$

. By some linear algebra, we have

$$F^{ij}B_{ij} \geq f_{\lambda_1}\mu_1 + \cdots + f_{\lambda_n}\mu_n$$

and hence

$$F^{ij}(B-A)_{ij} = F^{ij}B_{ij} - \sum f_{\lambda_i}\lambda_i \geq \sum f_{\lambda_i}(\mu_i - \lambda_i) \geq f(\mu) - f(\lambda) = F(B) - F(A)$$

by the concavity of f.

□

Lemma 2.3.2 ([28]). When $A = \{a_{ij}\} \in S_K$ is diagonal with simple eigenvalues, we have

(i) $(f_i - f_j)(\kappa_i - \kappa_j) \leq 0$.

(ii) $F^{ij,kl}a_{ij}a_{kl} = F^{ij}a_{ii}a_{jj} + \sum_{i\neq j}\frac{f_i - f_j}{\kappa_i - \kappa_j}(a_{ij})^2$.

Remark 5. It follows from (i) that

$$\frac{f_i - f_j}{\lambda_i - \lambda_j} \leq 0$$

. This fact is used in our proof of curvature estimate, when ensuring the positivity of a particular term; see section 3.3.

Proof.

(i) Suppose $\lambda_i > \lambda_j$ and let λ^* denote the vector obtained from λ by interchanging λ_i and λ_j. By symmetry and convexity of K, the ray

$$\lambda^* + t(\lambda_i - \lambda_j)(e_i - e_j), \quad 0 \leq t \leq 1$$

is in K.

Since f is symmetric and concave, the graph of

$$t \mapsto f(\lambda^* + t(\lambda_i - \lambda_j)(e_i - e_j))$$

is symmetric and concave about its maximum, which occurs at $t = \frac{1}{2}$. Hence,

$$(\lambda_i - \lambda_j)(f_i - f_j) \leq 0$$

(ii) Note that

$$F^{ij,kl} = \sum_r f_r \frac{\partial^2 \lambda_r}{\partial a_{ij} \partial a_{kl}} + \sum_{r,s} f_{rs} \frac{\partial \lambda_r}{\partial a_{ij}} \frac{\partial \lambda_s}{\partial a_{kl}}$$

From our previous calculations, we see that

$$\sum_{r,s} f_{rs} \frac{\partial \lambda_r}{\partial a_{ij}} \frac{\partial \lambda_s}{\partial a_{kl}} = f_{ik} \delta_{ij} \delta_{kl}$$

. Similarly, the second derivative $\frac{\partial^2 \lambda_r}{\partial a_{ij} \partial a_{kl}}$ is non-zero only when $(i,j) = (k,l)$ and $i \neq j$, in which case we have

$$\frac{\partial^2 \lambda_i}{\partial a_{ij}^2} = \frac{1}{\lambda_i - \lambda_j}, \quad \frac{\partial^2 \lambda_j}{\partial a_{ij}^2} = -\frac{1}{\lambda_i - \lambda_j}, \quad \text{if } \lambda_i > \lambda_j$$

. Hence,

$$F^{ij,kl} = f_{ik} \delta_{ij} \delta_{kl} + \frac{f_i - f_j}{\lambda_i - \lambda_j} (1 - \delta_{ij}) \delta_{kl}^{ij}$$

where $\delta_{kl}^{ij} = 1$ if $(i,j) = (k,l)$ and zero otherwise.

□

Lemma 2.3.3. Let u be the height function of Σ and $v \in C^2(\Sigma)$, we have

(i) $\nabla_{ij} \frac{1}{u} = \frac{1}{u}(g_{ij} - \nu^{n+1} h_{ij})$

(ii) $\nabla_{ij} \frac{v}{u} = v \nabla_{ij} \frac{1}{u} + \frac{1}{u} \tilde{\nabla}_{ij} v - \frac{1}{u^2} \tilde{g}^{kl} u_k v_l \tilde{g}_{ij}$.

(iii) $(\nu^{n+1})_i = -\tilde{h}_{ij} \tilde{g}^{jk} u_k$.

(iv) $\tilde{\nabla}_{ij} \nu^{n+1} = -\tilde{g}^{kl} (\nu^{n+1} \tilde{h}_{il} \tilde{h}_{kj} + u_l \tilde{\nabla}_k \tilde{h}_{ij})$

Proof. We first recall how we define the Hessian of $v \in C^2(\Sigma)$.

$$\nabla_{ij} v := \nabla_i (\nabla_j v) - \Gamma_{ij}^k \nabla_k v = \tilde{\nabla}_{ij} v + \frac{1}{u} (u_i v_j + u_j v_i - \tilde{g}^{kl} u_k v_l \tilde{g}_{ij})$$

(i) If we substitute $v = u$ into the Hessian, we get

$$\nabla_{ij} u = \tilde{\nabla}_{ij} u + \frac{2 u_i u_j}{u} - \frac{1}{u} \tilde{g}^{kl} u_k u_l \tilde{g}_{ij}$$

and

$$\begin{aligned}
\nabla_{ij} \frac{1}{u} &= -\frac{1}{u^2} \tilde{\nabla}_{ij} u + \frac{1}{u^3} \tilde{g}^{kl} u_k u_l \tilde{g}_{ij} \\
&= -\frac{1}{u^2} \tilde{h}_{ij} \nu^{n+1} + \frac{1}{u^3} [1 - (\nu^{n+1})^2] \tilde{g}_{ij} \\
&= -\frac{\nu^{n+1}}{u} h_{ij} + \frac{(\nu^{n+1})^2}{u^3} \tilde{g}_{ij} + \frac{1}{u^3} [1 - (\nu^{n+1})^2] \tilde{g}_{ij} \\
&= \frac{1}{u} (g_{ij} - \nu^{n+1} h_{ij})
\end{aligned}$$

where we have used

$$\begin{aligned}\tilde{g}^{kl}u_k u_l &= |\tilde{\nabla} u|^2 = 1 - (\nu^{n+1})^2 \\ \tilde{\nabla}_{ij} u &= \tilde{h}_{ij}\nu^{n+1} \\ \tilde{g}_{ij} &= u^2\delta_{ij}\end{aligned}$$

along with (2.2).

(ii) This follows from the same computation as in (i), just start with the definition of Hessian.

(iii) This is the Weingarten formula as stated in the beginning of this chapter.

(iv) This is the Codazzi equation as stated in the beginning of this chapter.

□

Lemma 2.3.4 ([15])**.** If Σ is a smooth hypersurface in $\mathbb{H}^{n+1}$ satisfying (1.1), then in a local orthonormal frame we have

$$\begin{aligned}F^{ij}\nabla_{ij}\frac{1}{u} &= -\frac{\sigma\nu^{n+1}}{u} + \frac{1}{u}\sum_{i=1}^{n} f_i \\ F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} &= \frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum_{i=1}^{n} f_i\kappa_i^2\end{aligned}$$

Proof. For the first identity,

$$\begin{aligned}F^{ij}\nabla_{ij}\frac{1}{u} &= \frac{1}{u}F^{ij}(g_{ij} - \nu^{n+1}h_{ij}) && \text{by lemma 2.3.3 (i)} \\ &= \frac{1}{u}\left(\sum f_i\cdot 1 - \nu^{n+1}\sum f_i\kappa_i\right) && \text{by (2.4)} \\ &= -\frac{\sigma\nu^{n+1}}{u} + \frac{1}{u}\sum f_i && \text{by lemma 2.2.1}\end{aligned}$$

To prove the second identity, we first expand $F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u}$ by lemma 2.3.3 (ii)

$$F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u} = \nu^{n+1}F^{ij}\nabla_{ij}\frac{1}{u} + \frac{1}{u}F^{ij}\tilde{\nabla}_{ij}\nu^{n+1} - \frac{1}{u^2}F^{ij}\tilde{g}^{kl}u_k(\nu^{n+1})_l\tilde{g}_{ij} \tag{2.6}$$

and we compute each term as follows. The first term follows from the identity we just proved:

$$\nu^{n+1}F^{ij}\nabla_{ij}\frac{1}{u} = \frac{\nu^{n+1}}{u}\left(\sum f_i - \nu^{n+1}\sigma\right) \tag{2.7}$$

. For the second term, we use lemma 2.3.3 (iv) to get

$$\frac{1}{u}F^{ij}\tilde{\nabla}_{ij}\nu^{n+1} = \frac{1}{u}F^{ij}(-\tilde{g}^{kl}(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij}))$$

and we need to evaluate each term in the bracket. For the first one, we do direct computation

$$\begin{aligned}
F^{ij}\tilde{g}^{kl}\tilde{h}_{il}\tilde{h}_{kj} &= \frac{1}{u^2}F^{ij}\tilde{h}_{ik}\tilde{h}_{kj} && \text{since } \tilde{g}^{kl} = \frac{\delta_{kl}}{u^2} \\
&= F^{ij}\left(h_{ik} - \frac{\nu^{n+1}}{u^2}\tilde{g}_{ik}\right)\left(h_{kj} - \frac{\nu^{n+1}}{u^2}\tilde{g}_{kj}\right) && \text{by (2.1)} \\
&= F^{ij}(h_{ik}h_{kj} - 2\nu^{n+1}h_{ij} + (\nu^{n+1})^2\delta_{ij}) && \text{since } \tilde{g}_{ij} = u^2\delta_{ij} \\
&= \sum_{i=1}^{n} f_i\kappa_i^2 - 2\nu^{n+1}\sum_{i=1}^{n} f_i\kappa_i + (\nu^{n+1})^2\sum_{i=1}^{n} f_i && \text{by (2.4)-(2.5)} \\
&= \sum_{i=1}^{n} f_i\kappa_i^2 - 2\nu^{n+1}\sigma + (\nu^{n+1})^2\sum_{i=1}^{n} f_i && \text{by lemma 2.2.1 and (1.1)}
\end{aligned}$$

For the second one, we proceed as follows. By (2.2) and (1.1), we have

$$f(u\tilde{\kappa}_1 + \nu^{n+1}, \ldots, u\tilde{\kappa}_n + \nu^{n+1}) = \sigma$$

, or equivalently

$$F(\{\tilde{g}^{ik}(u\tilde{h}_{kj} + \nu^{n+1}\tilde{g}_{kj})\}) = \sigma$$

We differentiate it to obtain

$$F^{ij}(u\tilde{\nabla}_k\tilde{h}_{ij} + u_k\tilde{h}_{ij} + (\nu^{n+1})_k u^2\delta_{ij}) = 0$$

and so

$$\begin{aligned}
F^{ij}\tilde{\nabla}_k\tilde{h}_{ij} &= -u_kF^{ij}\frac{\tilde{h}_{ij}}{u} - (\nu^{n+1})_k uF^{ij}\delta_{ij} \\
&= -u_kF^{ij}\left(h_{ij} - \frac{\nu^{n+1}}{u^2}\tilde{g}_{ij}\right) - (\nu^{n+1})_k uF^{ij}\delta_{ij} && \text{by (2.1)} \\
&= -u_kF^{ij}h_{ij} + u_k\nu^{n+1}F^{ij}\delta_{ij} - (\nu^{n+1})_k uF^{ij}\delta_{ij} && \text{since } \tilde{g}_{ij} = u^2\delta_{ij} \\
&= -u_k\sum f_i\kappa_i + u_k\nu^{n+1}\sum f_i - (\nu^{n+1})_k u\sum f_i && \text{by (2.4)} \\
&= -u_k\sigma + u_k\nu^{n+1}\sum f_i - (\nu^{n+1})_k u\sum f_i && \text{by lemma (2.2.1)}
\end{aligned}$$

The second term in (2.6) then evaluates to

$$\begin{aligned}
\frac{1}{u}F^{ij}\tilde{\nabla}_{ij}\nu^{n+1} &= \frac{1}{u}F^{ij}(-\tilde{g}^{kl}(\nu^{n+1}\tilde{h}_{il}\tilde{h}_{kj} + u_l\tilde{\nabla}_k\tilde{h}_{ij})) && \text{by lemma 2.3.3 (iv)} \\
&= -\frac{\nu^{n+1}}{u}F^{ij}\tilde{g}^{kl}\tilde{h}_{il}\tilde{h}_{kj} - \frac{1}{u}\tilde{g}^{kl}u_lF^{ij}\tilde{\nabla}_k\tilde{h}_{ij} \\
&= -\frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2 + 2(\nu^{n+1})^2\frac{\sigma}{u} - \frac{(\nu^{n+1})^3}{u}\sum f_i \\
&\quad + \sigma\frac{|\tilde{\nabla}u|^2}{u} - \frac{|\tilde{\nabla}u|^2}{u}\nu^{n+1}\sum f_i + \frac{u_k}{u^2}(\nu^{n+1})_k\sum f_i
\end{aligned}$$

and by using $|\tilde{\nabla} u|^2 = 1 - (\nu^{n+1})^2$, we have

$$\frac{1}{u} F^{ij} \tilde{\nabla}_{ij} \nu^{n+1} = -\frac{\nu^{n+1}}{u} \sum f_i \kappa_i^2 + [1 + (\nu^{n+1})^2] \frac{\sigma}{u} - \frac{\nu^{n+1}}{u} \sum f_i + \frac{u_k}{u^2} (\nu^{n+1})_k \sum f_i \tag{2.8}$$

For the third term in (2.6), we do not need to expand it too much further, because it will be cancelled out by the last term in (2.8):

$$-\frac{1}{u^2} F^{ij} \tilde{g}^{kl} u_k (\nu^{n+1})_l \tilde{g}_{ij} = -\frac{u_k}{u^2} (\nu^{n+1})_k \sum f_i$$

Finally, we substitute everything that we have computed so far back into (2.6):

$$\begin{aligned}
F^{ij} \nabla_{ij} \frac{\nu^{n+1}}{u} &= \nu^{n+1} F^{ij} \nabla_{ij} \frac{1}{u} + \frac{1}{u} F^{ij} \tilde{\nabla}_{ij} \nu^{n+1} \\
&\quad - \frac{1}{u^2} F^{ij} \tilde{g}^{kl} u_k (\nu^{n+1})_l \tilde{g}_{ij} \qquad \text{by lemma 2.3.3 (ii)} \\
&= \frac{\nu^{n+1}}{u} \left(\sum f_i - \nu^{n+1} \sigma \right) - \frac{\nu^{n+1}}{u} \sum f_i \kappa_i^2 \\
&\quad + [1 + (\nu^{n+1})^2] \frac{\sigma}{u} - \frac{\nu^{n+1}}{u} \sum f_i \\
&= -\frac{\nu^{n+1}}{u} \sum f_i \kappa_i^2 + \frac{\sigma}{u}
\end{aligned}$$

□

2.4 The Differential Operator G

In this section, we show the conversion of the curvature relation $f(\kappa[\Sigma]) = \sigma$ into the PDE (1.4)

$$G(D^2 u, Du, u) = \sigma$$

and prove a few properties associated with the operator G.

According to [4], the Euclidean principal curvatures κ^e are the eigenvalues of the symmetric matrix

$$a_{ij}^e := \frac{1}{w} \gamma^{ik} u_{kl} \gamma^{lj} \tag{2.9}$$

where

$$\gamma^{ij} = \delta_{ij} - \frac{u_i u_j}{w(1+w)} \text{ with inverse } \gamma_{ij} = \delta_{ij} + \frac{u_i u_j}{1+w} \text{ and } \gamma_{ik} \gamma_{kj} = g_{ij}^e$$

By the relation (2.2), the hyperbolic principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ are the eigenvalues of the symmetric matrix

$$a_{ij}[u] := \frac{\delta_{ij} + u \gamma^{ik} u_{kl} \gamma^{lj}}{w} \tag{2.10}$$

From now on, we confine ourself to consider the symmetric matrix $A[u] := \{a_{ij}[u]\}$ and the matrix operator $F(A) = f(\lambda(A))$. The differential operator G in equation (1.4) is given by

$$G(D^2u, Du, u) = F(A[u])$$

. We shall now prove a few properties of the operator G which will be used in section 3.1.

Lemma 2.4.1. If we define

$$G^{ij} := \frac{\partial G}{\partial u_{ij}}, \quad G^i := \frac{\partial G}{\partial u_i}, \quad G_u := \frac{\partial G}{\partial u}$$

, then

(i) $G^{ij} = \frac{u}{w} F^{kl}\gamma^{ki}\gamma^{jl}$ and so G is degenerate where $u = 0$.

(ii) $G_u = G - \nu^{n+1}\sum f_i$.

(iii) $|G^s| \leq \frac{\sigma}{w} + \frac{2}{w}\sum f_i + 2\sum f_i|\kappa_i|$.

Proof. We prove by direct computation.

(i) Since

$$G^{ij} := \frac{\partial G}{\partial u_{ij}} = \frac{\partial F}{\partial u_{ij}} = \frac{\partial F}{\partial a_{kl}}\frac{\partial a_{kl}}{\partial u_{ij}} = \frac{u}{w}F^{kl}\gamma^{ki}\gamma^{jl} \qquad \text{by (2.10)}$$

, it follows that G is degenerate where $u = 0$.

(ii) we compute

$$\begin{aligned} G_u :&= \frac{\partial G}{\partial u} = \frac{\partial F}{\partial u} = \frac{\partial F}{\partial a_{ij}}\frac{\partial a_{ij}}{\partial u} = F^{ij}\frac{1}{w}\gamma^{ik}u_{kl}\gamma^{lj} && \text{by (2.10)} \\ &= F^{ij}a^e_{ij} && \text{by (2.9)} \\ &= \sum_{i=1}^n f_i\kappa^e_i && \text{by (2.4)} \\ &= \sum_{i=1}^n f_i\frac{\kappa_i - \nu^{n+1}}{u} && \text{by (1.3)} \\ &= \frac{1}{u}f(\kappa) - \frac{\nu^{n+1}}{u}\sum_{i=1}^n && \text{by lemma 2.2.1} \end{aligned}$$

In other words,

$$uG_u = G - \nu^{n+1}\sum_{i=1}^n f_i \tag{2.11}$$

(iii) By direct computation, we have

$$G^s := \frac{\partial G}{\partial u_s} = \frac{\partial F}{\partial a_{ij}} \frac{\partial a_{ij}}{\partial u_s}$$
$$= -\frac{u_s}{w^2} \sum f_i \kappa_i - \frac{2}{w} F^{ij} a_{ik} \left(\frac{w u_k \gamma^{sj} + u_j \gamma^{ks}}{1+w} \right) + \frac{2}{w^2} F^{ij} u_i \gamma^{sj}$$

and the result follows.

□

The eigenvalues of $\{G^{ij}\}$ are related to the eigenvalues of $\{F^{ij}\}$ (which are f_i's) by

Lemma 2.4.2 ([14]). Let $0 < \mu_1 \le \mu_2 \le \cdots \le \mu_n$ denote the eigenvalues of $\{G^{ij}\}$ and let $w = \sqrt{1 + |Du|^2}$. Then

$$w\mu_k \le u f_k \le w^3 \mu_k, \quad 1 \le k \le n$$

Proof. For $\xi \in \mathbb{R}^n$, we have from lemma 2.4.1 that

$$uF^{ij}\xi_i\xi_j = wG^{kl}\gamma_{ik}\gamma_{lj}\xi_i\xi_j = wG^{kl}\xi_k^{'}\xi_l^{'}$$

where

$$\xi_i^{'} := \gamma_{ik}\xi_k = \xi_i + \frac{(\xi \cdot Du)u_i}{1+w}$$

.

Note that

$$|\xi|^2 \le |\xi^{'}|^2 = |\xi|^2 + |\xi \cdot Du|^2 \le w^2 |\xi|^2$$

. Since both $\{G^{ij}\}$ and $\{F^{ij}\}$ are positive definite, the result follows from the min-max characterization of eigenvalues. □

We also prove a few properties for the linearized operator $\mathcal{L}$ of G:

$$\mathcal{L} := G^{ij}\partial_i\partial_j + G^i\partial_i + G_u$$

Lemma 2.4.3 ([14]). Suppose the curvature function f satisfies the fundamental structure condition (1.6)-(1.7), the normality condition (1.9) and the homogeneity condition (1.10) in K. Then

(i) $L\left(1 - \frac{\varepsilon}{u}\right) \le -\frac{(1-\sigma)\varepsilon}{u^2 w} \sum f_i$ in Ω, where

$$L := \mathcal{L} - \frac{2}{w^2} F^{ij} a_{ik} u_k \partial_j$$

(ii) $\mathcal{L}(x_i u_j - x_j u_i) = 0$ and $\mathcal{L}u_i = 0$ for all $1 \le i, j \le n$.

Proof.

For (i), just note that by computation

$$\mathcal{L}\left(1 - \frac{\varepsilon}{u}\right) \le -\frac{(1-\sigma)\varepsilon}{u^2 w} \sum f_i - \frac{2\varepsilon}{u^2 w^2} F^{ij} a_{ik} u_k u_j$$

.

For (ii), see [7]. □

2.5 Height Estimates and the Asymptotic Angle Condition

Throughout this section, unless otherwise stated, let Σ is a hypersurface in $\mathbb{H}^{n+1}$ with $\partial\Sigma \subseteq P(\varepsilon) := \{x_{n+1} = \varepsilon\}$ and let $\Omega \subseteq \mathbb{R}^n \times \{0\}$ be the region such that its vertical lift $\Omega^\varepsilon := \{(x, \varepsilon) : x \in \Omega\}$ to $P(\varepsilon)$ is bounded by $\partial\Sigma$ and $\mathbb{R}^n \backslash \Omega$ is connected and unbounded. It is allowable that Ω has several connected components.

Lemma 2.5.1 ([14])**.** Let B_1 be a ball of radius R centered at $a = (a', -\sigma R) \in \mathbb{R}^{n+1}$ and B_2 be a ball of radius of R centered at $b = (b', \sigma R) \in \mathbb{R}^{n+1}$, where $\sigma \in (0, 1)$. We have

(i) $\Sigma \cap \{x_{n+1} < \varepsilon\} = \emptyset$.

(ii) If $\partial\Sigma \subseteq B_1$, then $\Sigma \subseteq B_1$.

(iii) If $B_1 \cap P(\varepsilon) \subseteq \Omega^\varepsilon$, then $B_1 \cap \Sigma = \emptyset$.

(iv) If $B_2 \cap \Omega^\varepsilon = \emptyset$, then $B_2 \cap \Sigma = \emptyset$.

Remark 6. This lemma yields the C^0 estimate for admissible solutions u to our Dirchlet problem and a uniform positive lower bound for the hyperbolic principal curvatures of graph(u). There are other methods to prove these results but they are all similar in nature.

Proof.

(i) Let $c := \min_\Sigma x_{n+1}$ and suppose to the contrary that $0 < c < \varepsilon$. Then the horosphere $P(c)$ satisfies $f(\kappa) = 1$, since all such horospheres have principal curvature $\mathbf{1}$. So $P(c)$ lies below Σ and has an interior contact point, which violates the maximum principle. Hence $c = \varepsilon$.

(ii) We expand B_1 continuously untill it contains Σ and reverse the process. Consider $S_1 := \partial B_1 \cap \mathbb{H}^{n+1}$. Note that $\kappa_i[S_1] = \sigma$. Since both Σ and S_1 satisfy $f(\kappa) = \sigma$, there cannot be a first contact.

(iii) We shrink B_1 until it is inside Σ and when we expand it there cannot be a first contact as in (ii).

(iv) Suppose $B_2 \cap \Sigma \neq$. We may shrink B_2 unitil it lies below $P(\varepsilon)$ and so it is outside Σ. Now reverse the process, if there were a first interior contact, then the outward normal to Σ at this contact point is the inward normal to $S_2 := \partial B_2 \cap \mathbb{H}^{n+1}$. Since $\kappa_i[S_2] = \sigma$ with respect to its inward normal and $f(\kappa[S_2]) = \sigma$, this violates the maximum principle.

□

Now we show the upward unit normal ν^{n+1} tends to a fixed asymptotic angle on approach to the boundary.

Lemma 2.5.2 ([14]). Suppose f satisfies (1.6), (1.9) and (1.10) in K. If $\kappa[\Sigma] \in K$ and

$$\sigma_2 \leq f(\kappa[\Sigma]) \leq \sigma_1$$

for some $0 \leq \sigma_2 \leq \sigma_1 \leq 1$, then we have for $\partial\Sigma \in C^2$ and $\varepsilon > 0$ sufficiently small,

$$\sigma_2 - \frac{\varepsilon\sqrt{1-\sigma_2^2}}{r_2} - \frac{\varepsilon^2(1+\sigma_2)}{r_2^2} < \nu^{n+1} < \sigma_1 + \frac{\varepsilon\sqrt{1-\sigma_1^2}}{r_1} + \frac{\varepsilon^2(1-\sigma_1)}{r_1^2} \quad \text{on } \partial\Sigma$$

where r_1 and r_2 are the maximal radii of interior and exterior spheres to $\partial\Omega$, respectively. In particular, when $\sigma_1 = \sigma_2 = \sigma$ we have $\nu^{n+1} \to \sigma$ on $\partial\Sigma$ as $\varepsilon \to 0$.

Proof. We assume $r_2 < \infty$ and fix $x_0 \in \partial\Omega$. Let e_1 denote the outward pointing unit normal to $\partial\Omega$ at x_0. Define

$$a_1 := (x_0 - r_1 e_1, -R_1\sigma_1), \quad a_2 := (x_0 + r_2 e_1, R_2\sigma_2)$$

where R_1, R_2 satisfy

$$R_1^2 = r_1^2 + (R_1\sigma_1 + \varepsilon)^2, \quad R_2^2 = r_2^2 + (R_2\sigma_2 - \varepsilon)^2 \tag{2.12}$$

; and let $B_1 := B_{R_1}(a_1)$ and $B_2 := B_{R_2}(a_2)$.

Then $B_1 \cap P(\varepsilon)$ is a ball of radius r_1 internally tangent to $\partial\Omega^\varepsilon$ at x_0 and $B_2 \cap P(\varepsilon)$ is a ball of radius r_2 externally tangent to $\partial\Omega^\varepsilon$ at x_0. Therefore by lemma 2.5.1 (iii) and (iv), we have $B_1 \cap \Sigma = \emptyset$ and $B_2 \cap \Sigma = \emptyset$ i.e.

$$-\frac{u - \sigma_2 R_2}{R_2} < \nu^{n+1} < \frac{u + \sigma_1 R_1}{R_1} \quad \text{at } x_0 \in \partial\Omega$$

Since $u = \varepsilon$ on $\partial\Omega$, we have

$$\sigma_2 - \frac{\varepsilon}{R_2} < \nu^{n+1} < \frac{\varepsilon}{R_1} + \sigma_1 \quad \text{at } x_0 \in \partial\Omega$$

and from (2.12),

$$\frac{1}{R_1} = \frac{\sqrt{(1-\sigma_1)^2 r_1^2 + \varepsilon^2} - \varepsilon\sigma_1}{r_1^2 + \varepsilon^2} < \frac{\sqrt{1-\sigma_1^2}}{r_1} + \frac{\varepsilon(1-\sigma_1)}{r_1^2}$$
$$\frac{1}{R_2} = \frac{\sqrt{(1-\sigma_2)^2 r_2^2 + \varepsilon^2} + \varepsilon\sigma_2}{r_2^2 + \varepsilon^2} < \frac{\sqrt{1-\sigma_2^2}}{r_2} + \frac{\varepsilon(1+\sigma_2)}{r_2^2}$$

.

Hence the result. □

We have the following two important consequences, the first of which will not be used in our proofs of the main results but it is helpful for us to understand the problem.

Corollary 2.5.1. Suppose f satisfies (1.6) and (1.9) in K. If Σ is a solution to the asymptotic Plateau problem (1.1)-(1.2), then $\sigma < 1$.

Proof. By lemma 2.5.1(i), we have $u > \varepsilon$ in Σ and so

$$\kappa[u] := \kappa[\mathrm{graph}(u)] = \kappa[\Sigma] < \kappa[P(\varepsilon)]$$

. Note that $\kappa[P(\varepsilon)] = \mathbf{1}$ by (2.2), thus we have

$$\begin{aligned}
\sigma &= f(\kappa[\Sigma]) && \text{by (1.1)}\\
&< f(\kappa[P(\varepsilon)]) && \text{by (1.6)}\\
&= f(1,\ldots,1) && \text{by (2.2)}\\
&= 1 && \text{by (1.9)}
\end{aligned}$$

□

Remark 7. The conclusion still holds even if we replace σ by a general right-hand side $\psi(x,\nu)$ in (1.1).

The following second consequence will be used in section 3.1.

Corollary 2.5.2. Suppose Σ is a smooth hypersurface in $\mathbb{H}^{n+1}$ satisfying (1.1). If Σ is globally a graph of some function u over some domain $\Omega \subseteq \mathbb{R}^n$:

$$\Sigma = \{(x, u(x)) \in \mathbb{H}^{n+1} : x \in \Omega\}$$

, then

$$F^{ij}\nabla_{ij}\frac{\sigma - \nu^{n+1}}{u} \geq 0$$

and so

$$\frac{\sigma - \nu^{n+1}}{u} \leq \sup_{\partial\Sigma} \frac{\sigma - \nu^{n+1}}{u} \quad \text{on } \Sigma$$

.

Moreover, if $u = \varepsilon > 0$ on $\partial\Omega$, then there exists some $\varepsilon_0 > 0$ depending on $\partial\Omega$ such that

$$\varepsilon \leq \varepsilon_0 \Rightarrow \frac{\sigma - \nu^{n+1}}{u} \leq \frac{\sqrt{1-\sigma^2}}{r} + \frac{\varepsilon(1+\sigma)}{r^2} \quad \text{in } \overline{\Omega}$$

where r is the maximal radius of exterior sphere tangent to $\partial\Omega$.

Proof. By lemma 2.3.4, we have

$$\begin{aligned}
F^{ij}\nabla_{ij}\frac{\sigma - \nu^{n+1}}{u} &= \sigma \cdot F^{ij}\nabla_{ij}\frac{1}{u} - F^{ij}\nabla_{ij}\frac{\nu^{n+1}}{u}\\
&= \left(-\frac{\sigma^2\nu^{n+1}}{u} + \frac{\sigma}{u}\sum f_i\right) - \left(\frac{\sigma}{u} - \frac{\nu^{n+1}}{u}\sum f_i\kappa_i^2\right)\\
&= \frac{\sigma}{u}\left(\sum f_i - 1\right) + \frac{\nu^{n+1}}{u}\left(\sum \kappa_i^2 f_i - \sigma^2\right)
\end{aligned}$$

Now, by the Jensen or the Cauchy-Schwartz inequality, we have

$$\sum \kappa_i^2 f_i \geq \frac{(\sum \kappa_i f_i)^2}{\sum f_i} = \frac{\sigma^2}{\sum f_i}$$

and therefore

$$\begin{aligned}
F^{ij}\nabla_{ij}\frac{\sigma-\nu^{n+1}}{u} &= \frac{\sigma}{u}\left(\sum f_i - 1\right) + \frac{\nu^{n+1}}{u}\left(\sum \kappa_i^2 f_i - \sigma^2\right)\\
&\geq \frac{\sigma}{u}\left(\sum f_i - 1\right) + \frac{\nu^{n+1}}{u}\left(\frac{\sigma^2}{\sum f_i} - \sigma^2\right)\\
&= \frac{\sigma}{u}\sum f_i\left(1-\frac{1}{\sum f_i}\right) - \frac{\sigma^2}{u}\nu^{n+1}\left(1-\frac{1}{\sum f_i}\right)\\
&> \frac{\sigma}{u}\left(1-\frac{1}{\sum f_i}\right) - \frac{\sigma^2}{u}\left(1-\frac{1}{\sum f_i}\right)\\
&= \frac{\sigma(1-\sigma)}{u}\left(1-\frac{1}{\sum f_i}\right)\\
&> 0
\end{aligned}$$

where we have used the facts that $\sum f_i \geq 1$ and that $\nu^{n+1} < 1$.

By the usual maximum principle, we have

$$\sup_{\Sigma}\frac{\sigma-\nu^{n+1}}{u} \leq \sup_{\partial\Sigma}\frac{\sigma-\nu^{n+1}}{u}$$

From this and the asymptotic angle condition in lemma 2.5.2, we can deduce

$$\begin{aligned}
\sup_{\Sigma}\frac{\sigma-\nu^{n+1}}{u} &\leq \sup_{\partial\Sigma}\frac{\sigma-\nu^{n+1}}{u}\\
&\leq \frac{\sqrt{1-\sigma^2}}{r} + \frac{\varepsilon(1+\sigma)}{r^2}
\end{aligned}$$

and the proof is complete. □

Chapter 3

The Dirichlet Problem

In this chapter, we derive a maximum principle $\kappa_{\max} \leq C$ for the largest hyperbolic principal curvature of admissible solutions, which holds for all $\sigma \in (0,1)$ (as compared to theorem 6.1 in [14]) and along with the well-established gradient estimate and boundary C^2 estimate in [14], it will lead to the existence of solutions for (1.4)-(1.5) and hence yields our theorem 1.0.4.

Before we proceed, we shall describe the method of solution by Guan-Spruck in [14] which we entirely follow here. As we have mentioned in chapter one and demonstrated in section 2.4, the PDE (1.4) is degenerate where $u = 0$ so we shall consider the Dirichlet problem with an approximate boundary condition:

$$G(D^2u, Du, u) = \sigma, \quad u > 0 \quad \text{in } \Omega \tag{3.1}$$

$$u = \varepsilon \quad \text{on } \partial\Omega \tag{3.2}$$

whose existence of solutions can be ensured by the method of continuity, as illustrated in [5]. More specifically, we consider a family of Dirichlet problem indexed by $0 \leq t \leq 1$:

$$G(D^2u^t, Du^t, u^t) = t\sigma + (1-t) \quad \text{in } \Omega \tag{3.3}$$

$$u^t = \varepsilon \quad \text{on } \partial\Omega \tag{3.4}$$

and the set

$$S = \{t \in [0,1] : (3.3)\text{-}(3.4) \text{ is solvable}\}$$

. The philosophy of the continuity method goes as follows. We shall show that S is a non-empty, open and closed subset of the unit interval $[0,1]$. Since $[0,1]$ is connected, it must follow that $S = [0,1]$ i.e. the Dirichlet problem (3.3)-(3.4) is solvable for all $0 \leq t \leq 1$. In particular, the problem is solvable for $t = 1$ and hence the Dirichlet problem of our interest (3.1)-(3.2) admits a solution.

To show S is non-empty, it is usually easy to find a solution for the case $t = 0$. In this problem, the constant function $u^0 \equiv \varepsilon$ serves as one such solution. Indeed, when u is a constant, the symmetric matrix in (2.10) is just the identity $\{\delta_{ij}\}$ and so

$$G[u^0] = F(A[u^0]) = F(\delta_{ij}) = f(\lambda(\delta_{ij})) = f(1,\ldots,1) = 1$$

by (1.9).

On the other hand, while the openness of S follows from the Implicit Function Theorem (see chapter 17 in [11]), the closedness requires a global a priori $C^{2,\alpha}$ estimate $|u|_{2,\alpha} \leq C$ for some constant $C > 0$ independent of t where

$$|u|_{2,\alpha} := \max_{\overline{\Omega}} |u| + \sum_i \max_{\overline{\Omega}} |u_i| + \sum_{i,j} \max_{\overline{\Omega}} |u_{ij}| + \sum_{i,j} \sup_{x,y\in\overline{\Omega},x\neq y} \frac{|u_{ij}(x) - u_{ij}(y)|}{|x-y|^\alpha}$$

. According to the standard elliptic theory [26] or the L_p theory in [1], in order to derive the $C^{2,\alpha}$ estimate, it suffices by concavity (1.7) to show a C^2 estimate $|D^2u| \leq C$; this has been perfectly done by Guan-Spruck in [14] by exploiting the geometry of the domain Ω and taking advantage of condition (1.11), so that we have

Theorem 3.0.1 ([14])**.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded smooth mean-convex domain i.e. $\mathcal{H}_{\partial\Omega} \geq 0$. If f satisfies (1.6)-(1.11), then for any $\sigma \in (0,1)$ and $\varepsilon > 0$ sufficiently small, there exists a unique admissible solution $u^\varepsilon \in C^\infty(\overline{\Omega})$ of the Dirichlet problem (1.12)-(1.13) satisfying the following a priori estimates

$$\sqrt{1+|Du^\varepsilon|^2} \leq \frac{1}{\sigma} + C\varepsilon, \quad u^\varepsilon|D^2u^\varepsilon| \leq C \quad \text{on } \partial\Omega \tag{3.5}$$

$$u^\varepsilon|D^2u^\varepsilon| \leq \frac{C}{\varepsilon^2} \quad \text{in } \Omega \tag{3.6}$$

where $C > 0$ is independent of ε.

Remark 8. Note that we also have uniqueness for the Dirichlet problem (3.1)-(3.2).

Note that the estimate (3.6) does not allow us to take the limit $\varepsilon \to 0$. In other words, it is sufficient for (3.6) to ensure that (3.1)-(3.2) admits a solution but it cannot be used to guarantee a solution to the original Dirichlet problem (1.4)-(1.5), when take the limit $\varepsilon \to 0$. We are then motivated to prove a new C^2 estimate for admissible solutions u^ε which is independent of ε. We will do so by estimating the principal curvatures $\kappa = (\kappa_1, \ldots, \kappa_n)$ of $\Sigma_\varepsilon = \text{graph}(u^\varepsilon)$; here we are concerned only with an upper bound because a uniform positive lower bound has been obtained in theorem 2.2.1 and lemma 2.5.1. Once we have obtained the curvature estimate, it follows that (3.1) is uniformly elliptic on compact subdomains of Ω and we have $|u^\varepsilon|_{2,\alpha} \leq C$ on compact subdomains of Ω by the Evans-Krylov regularity theory [10, 22]. Letting $\varepsilon \to 0$ will give us a solution to (1.4)-(1.5). Therefore, we can prove theorem (1.0.4) which we restate here for convenience.

Theorem 3.0.2. Suppose $\Gamma = \partial\Omega$ for some bounded smooth domain $\Omega \subseteq \mathbb{R}^n$ with $\mathcal{H}_{\partial\Omega} \geq 0$ and the curvature function f satisfies (1.16) in addition to (1.6)-(1.11) in the general cone K. Then the curvature estimate $\kappa_{\max} \leq C$ holds for all $\sigma \in (0,1)$ in K and hence there exists for all $\sigma \in (0,1)$, a smooth complete hypersurface Σ in $\mathbb{H}^{n+1}$ satisfying (1.1)-(1.2) with uniformly bounded principal curvatures

$$|\ \kappa[\Sigma]\ | \leq C \text{ on } \Sigma$$

Moreover, Σ is the graph of a unique admissible solution $u \in C^\infty(\Omega) \cap C^1(\overline{\Omega})$ of the Dirichlet problem (1.4)-(1.5). Furthermore, $u^2 \in C^\infty(\Omega) \cap C^{1,1}(\overline{\Omega})$ and

$$\begin{aligned} \sqrt{1+|Du|^2} &\leq \frac{1}{\sigma}, \quad u|D^2u| \leq C \quad \text{in } \Omega, \\ \sqrt{1+|Du|^2} &= \frac{1}{\sigma} \quad \text{on } \partial\Omega. \end{aligned}$$

Finally, we emphasize that our result applies to the particular curvature quotient $f = \frac{H_k}{H_{k-1}}$, for $1 \leq k \leq n$.

Corollary 3.0.1. For the curvature function $f = \frac{H_k}{H_{k-1}}$, the asymptotic Plateau problem (1.1)-(1.2) admits a solution in the k-th Garding cone K_k for all $\sigma \in (0,1)$.

Proof. We only need to verify that $f = \frac{H_k}{H_{k-1}}$ satisfies (1.16) in K_k. Indeed,

$$\begin{aligned} \sum_{i=1}^n f_i &= \sum_{i=1}^n \frac{\frac{\partial H_k}{\partial \lambda_i} H_{k-1} - H_k \frac{\partial H_{k-1}}{\partial \lambda_i}}{H_{k-1}^2} \\ &\leq \sum_{i=1}^n \frac{\frac{\partial H_k}{\partial \lambda_i} H_{k-1}}{H_{k-1}^2} \\ &= \frac{(n-k+1)H_{k-1}H_{k-1}}{H_{k-1}^2} \\ &= n-k+1 \end{aligned}$$

for all $\lambda \in K_k$. $\square$

Now, in what follows, we first prove that $G_u < 0$ in $\overline{\Omega}$ which implies uniqueness of solutions for (3.1)-(3.2) and a gradient estimate $\sqrt{1+|Du|^2} \leq \frac{1}{\sigma}$ as an immediate consequence; this is where we need the mean-convexity assumption $\mathcal{H}_{\partial\Omega} \geq 0$. Next, we employ the gradient estimate and the asymptotic behavior condition (1.11) to derive a boundary C^2 estimate for admissible solutions; the method we use here is highly based on the one developed by Caffarelli-Nirenberg-Spruck in [5, 6, 7], which is somewhat complicated but also truly delicate. We might skip a few tedious computational steps in order to make the proof clearer. Finally, we end the chapter with a detailed estimation for the hyperbolic principal curvatures of admissible solutions, which along with the boundary C^2 estimate derived in section 3.2, produces a global C^2 estimate and hence a $C^{2,\alpha}$ estimate.

3.1 Gradient Estimate and Uniqueness of Solutions

Theorem 3.1.1 ([14, 17])**.** Let $0 < \sigma < 1$ and $\Omega \subseteq \mathbb{R}^n$ be mean-convex i.e. $\mathcal{H}_{\partial\Omega} \geq 0$. Suppose $u \in C^2(\overline{\Omega})$ is a solution to (3.1)-(3.2) for some $\varepsilon > 0$. Then $G_u < 0$ in $\overline{\Omega}$. Consequently, the linearized operator of G satisfies the maximum principle and has trivial kernel. That is, (3.1)-(3.2) admits at most one solution. Moreover, we have the gradient estimate $\frac{1}{\sqrt{1+|Du|^2}} = \nu^{n+1} > \sigma$ in $\overline{\Omega}$.

Proof. Since
$$uG_u = G - \nu^{n+1} \sum f_i \leq \sigma - \nu^{n+1}$$
by (3.1), (2.11) and lemma 2.2.2, we have $G_u \leq \eta$ for $\eta := \frac{\sigma - \nu^{n+1}}{u}$. Hence, it suffices to show $\eta < 0$ in $\overline{\Omega}$.

According to corollary 2.5.2, η achieves its maximum on $\partial\Omega$ and we may assume the maximum occurs at $0 \in \partial\Omega$. We then choose coordinates $(x_1, \dots, x_n)$ at 0 such that x_n is the interior unit normal to $\partial\Omega$ at 0. Then at 0,
$$\begin{aligned} u_\alpha &= 0, \quad 1 \leq \alpha < n \\ u_n &> 0 \\ u_{nn} &\leq 0 \end{aligned}$$
In addition, we have
$$\eta_n = \frac{u_n}{u}\frac{u_{nn}}{w^3} - \frac{u_n}{u}\eta < 0 \Longrightarrow \frac{u_{nn}}{w^3} < \eta$$
.

On the other hand, by lemma 2.2.2,
$$\sigma = f(\kappa) \leq \frac{1}{n} \sum \kappa_i$$
i.e. the hyperbolic mean curvature $H(\Sigma)$ of Σ is greater than or equal to σ. Thus, by (2.2), the Euclidean mean curvature $H^e(\Sigma)$ satisfies
$$\frac{1}{n}\frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right) u_{ij} = H^e(\Sigma) = \frac{H(\Sigma) - \nu^{n+1}}{u} \geq \frac{\sigma - \nu^{n+1}}{u} = \eta$$
. Since
$$\sum_{\alpha<n} u_{\alpha\alpha} = -u_n(n-1)\mathcal{H}_{\partial\Omega}$$
, restricting $H^e(\Sigma) \geq \eta$ to $\partial\Omega$, we get
$$n \cdot \eta \leq \frac{1}{w}\left(\sum_{\alpha<n} u_{\alpha\alpha} + \frac{u_{nn}}{w^2}\right) = -(n-1)\frac{u_n}{w}\mathcal{H}_{\partial\Omega} + \frac{u_{nn}}{w^3} \leq \frac{u_{nn}}{w^3} < \eta$$
. Hence $(n-1)\eta < 0$ and the gradient estimate follows immediately
$$\eta < 0 \Longrightarrow \nu^{n+1} > \sigma \Longrightarrow \sqrt{1 + |Du|^2} < \frac{1}{\sigma} \quad \text{in } \overline{\Omega}$$
. The proof is now complete. □

3.2 Boundary Second Derivative Estimate

Theorem 3.2.1 ([14]). Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain with $\mathcal{H}_{\partial\Omega} \geq 0$. If $u \in C^3(\overline{\Omega})$ is an admissible solution to the Dirchlet problem (3.1)-(3.2), then for $\varepsilon > 0$ sufficiently small
$$u|D^2 u| \leq C \quad \text{on } \partial\Omega$$

Proof. Consider an arbitrary point on $\partial\Omega$, which we may assume to be the origin of $\mathbb{R}^n$ and choose the coordinates so that the unit vector in the the positive x_n-axis is the interior normal to $\partial\Omega$ at the origin. There exists a uniform constant $r > 0$ such that $\partial\Omega \cap B_r(0)$ can be represented as a graph:

$$x_n = \rho(x') = \frac{1}{2}\sum_{\alpha,\beta<n} B_{\alpha\beta}x_\alpha x_\beta + O(|x'|^3), \quad x' := (x_1, \ldots, x_{n-1})$$

We may assume $r \geq \varepsilon$. Since $u = \varepsilon$ on $\partial\Omega$, we have $u(x', \rho(x')) = \varepsilon$ on $\partial\Omega \cap B_r(0)$ and so

$$u_{\alpha\beta}(0) = -u_n(0)B_{\alpha\beta}, \quad \alpha, \beta < n$$

. Hence,

$$|u_{\alpha\beta}(0)| \leq C|Du(0)|, \quad \alpha, \beta < n$$

where $C > 0$ depends on the Euclidean mean curvature of $\partial\Omega$.

Now consider for each $\alpha < n$ the operator

$$T_\alpha := \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(x_\beta\partial_n - x_n\partial_\beta)$$

. By lemma 2.4.3 and the boundary condition $u = \varepsilon$ on $\partial\Omega$, we have

$$\begin{aligned} &\mathcal{L}(T_\alpha u) = 0 \\ &|T_\alpha u| + \frac{1}{2}\sum_{l<n} u_l^2 \leq C && \text{in } \Omega \cap B_\varepsilon(0) \\ &|T_\alpha u| + \frac{1}{2}\sum_{l<n} u_l^2 \leq C|x|^2 && \text{on } \partial\Omega \cap B_\varepsilon(0) \end{aligned}$$

from which we can define

$$\phi := \pm T_\alpha u + \frac{1}{2}\sum_{l<n} u_l^2 - \frac{C}{\varepsilon^2}|x|^2$$

so that $\phi \leq 0$ on $\partial[\Omega \cap B_\varepsilon(0)]$. It follows from lemma 2.4.2 and lemma 2.4.1 that

$$\mathcal{L}\phi \geq \sum_{l<n} G^{ij}u_{li}u_{lj} - \frac{C}{\varepsilon}\left(\sum f_i + \sum f_i|\kappa_i|\right)$$

. According to [21], for each point at $\Omega \cap B_\varepsilon(0)$ there exists an index r such that

$$\sum_{l<n} G^{ij}u_{li}u_{lj} \geq \frac{c_0}{2u}\left(\sum_{i\neq r} f_i\kappa_i^2 - \frac{2}{w^2}\sum f_i\right)$$

. Thus,

$$L\phi \geq -C_1\left(G^{ij}\phi_i\phi_j + \frac{1}{\varepsilon}\sum f_i\right)$$

.

Now define $h := (e^{C_1\phi} - 1) - A\left(1 - \frac{\varepsilon}{u}\right)$ with A large enough. By lemma 2.4.3, we have

$$\begin{aligned} h &\leq 0 \quad \text{on } \partial[\Omega \cap B_\varepsilon(0)] \\ Lh &\geq 0 \quad \text{in } \Omega \cap B_\varepsilon(0) \end{aligned}$$

. It follows from the usual maximum principle that $h \leq 0$ in $\Omega \cap B_\varepsilon(0)$. Since $h(0) = 0$, we have $h_n(0) \leq 0$ and the mixed normal-tangential derivative estimate

$$|u_{\alpha n}(0)| \leq \frac{A}{C_1\varepsilon} u_n(0)$$

It remains to estimate the pure normal second derivative. We may assume $\{u_{\alpha\beta}\}$ is diagonal, $1 \leq \alpha, \beta < n$. Since $u_\alpha(0) = 0$ for $\alpha < n$, we have from (2.10) that

$$A[u] = \frac{1}{w} \begin{bmatrix} 1 + uu_{11} & 0 & \cdots & uu_{1n}/w \\ 0 & 1 + uu_{22} & \cdots & uu_{2n}/w \\ \vdots & \vdots & \ddots & \vdots \\ uu_{n1}/w & uu_{n2}/w & \cdots & 1 + uu_{nn}/w^2 \end{bmatrix}$$

By lemma 1.2 in [6], if $\varepsilon u_{nn}(0)$ is very large, then the eigenvalues of $A[u]$ are given asymptotically by

$$\begin{aligned} \lambda_\alpha &= \frac{1}{w}[(1 + \varepsilon u_{\alpha\alpha}(0)] + o(1), \quad \alpha < n \\ \lambda_n &= \frac{\varepsilon u_{nn}(0)}{w^3}\left[1 + O\left(\frac{1}{\varepsilon u_{nn}(0)}\right)\right] \end{aligned}$$

. However, from the double tangential derivative estimate $|u_{\alpha\beta}(0)| \leq C|Du(0)|$, the gradient estimate in section 3.1 and condition (1.11), we see that for $\varepsilon > 0$ sufficiently small

$$\sigma \geq \frac{1}{w}\left(1 + \frac{\varepsilon}{2}\right) \geq \sigma\left(1 + \frac{\varepsilon}{2}\right) > \sigma$$

which is a contradiction.

The proof is now complete. □

3.3 Curvature Estimate

We prove the following maximum principle for the largest principal curvature of Σ by following the same method in [16, 14, 15, 17], which improves theorem 6.1 in [14] and hence extends theorem 1.0.3 to hold for all $\sigma \in (0, 1)$.

Theorem 3.3.1. Suppose f satisfies (1.16) in addition to (1.6)-(1.11) and $\sigma \in (0, 1)$. Let $\Sigma = \mathrm{graph}(u)$ be a smooth graph in $\mathbb{H}^{n+1}$ satisfying $f(\kappa) = \sigma$ and $\partial_\infty \Sigma \subseteq \partial_\infty \mathbb{H}^{n+1}$.

For $x \in \Sigma$, define the largest principal curvature of Σ at x by

$$\kappa_{\max}(x) := \max_{1 \leq i \leq n} \kappa_i(x)$$

and let $a > 0$ be a constant such that

$$\nu^{n+1} \geq 2a > 0 \quad \text{on } \Sigma$$

. We have

$$\max_{\Sigma} \frac{\kappa_{\max}}{\nu^{n+1} - a} \leq \max \left\{ C, \max_{\partial\Sigma} \frac{\kappa_{\max}}{\nu^{n+1} - a} \right\}$$

Remark 9. The constant $a > 0$ arises from the gradient estimate. Specifically, since $\nu^{n+1} \geq C > 0$, we simply choose $a > 0$ to be the number such that $0 < 2a \leq C$. See section 3.1.

Proof. Denote

$$M_0 := \sup_{\Sigma} \frac{\kappa_{\max}}{\nu^{n+1} - a}$$

Suppose the maximum M_0 is attained at an interior point $x_0 \in \Sigma$. We choose a local orthonormal frame around x_0 such that $h_{ij}(x_0) = \kappa_i(x_0)\delta_{ij}$ and for notational convenience, we may assume $\kappa_1(x_0) = \kappa_{\max}(x_0)$. In what follows, we will suppress notation to not write out x_0 but keep in mind all the calculations are done at x_0. Note that we may also assume $\kappa_1 > \nu^{n+1} > 0$ otherwise we would have $\kappa_1 \leq \nu^{n+1} \leq 1$ already.

Now since $\frac{h_{11}}{\nu^{n+1} - a}$ has a local maximum at x_0, we have

$$\frac{h_{11i}}{h_{11}} - \frac{\nabla_i \nu^{n+1}}{\nu^{n+1} - a} = 0 \tag{3.7}$$

$$\frac{h_{11ii}}{h_{11}} - \frac{\nabla_{ii} \nu^{n+1}}{\nu^{n+1} - a} \leq 0 \tag{3.8}$$

We multiply (3.8) by $h_{11}F^{ii} = \kappa_1 F^{ii}$ (and summing over i),

$$F^{ii} h_{11ii} - \frac{\kappa_1}{\nu^{n+1} - a} F^{ii} \nabla_{ii} \nu^{n+1} \leq 0 \tag{3.9}$$

Our strategy goes as follows. We will estimate each term in (3.9) and conclude that if κ_1 is bigger than some certain constant then the quantity in (3.9) will be strictly positive which is a contradiction. Hence κ_1 should be smaller than that constant and we obtain the desired upper bound for κ_1.

By differentiating the equation $F(h_{ij}) = \sigma$ twice and using the formulas introduced at the beginning of chapter two, we find that the first term in (3.9) is

$$F^{ii} h_{11ii} = -F^{ij,kl} h_{ij1} h_{kl1} + \sigma(1 + \kappa_1^2) - \kappa_1 \left(\sum f_i + \sum \kappa_i^2 f_i \right)$$

. Also, by lemma 2.3.4 the second term in (3.9) is

$$\frac{\kappa_1}{\nu^{n+1}-a}F^{ii}\nabla_{ii}\nu^{n+1} = 2\kappa_1 F^{ii}\frac{u_i}{u}\frac{\nabla_i\nu^{n+1}}{\nu^{n+1}-a} + \sigma\kappa_1\frac{1+(\nu^{n+1})^2}{\nu^{n+1}-a} - \frac{\kappa_1\nu^{n+1}}{\nu^{n+1}-a}\left(\sum f_i + \sum \kappa_i^2 f_i\right)$$

So (3.9) becomes

$$0 \geq -F^{ij,rs}h_{ij1}h_{rs1} + \sigma\left[1+\kappa_1^2 - \frac{1+(\nu^{n+1})^2}{\nu^{n+1}-a}\kappa_1\right] \tag{3.10}$$

$$+\left(\sum f_i + \sum \kappa_i^2 f_i\right)\frac{a\kappa_1}{\nu^{n+1}-a} \tag{3.11}$$

$$-\frac{2\kappa_1}{\nu^{n+1}-a}\sum_{i=1}^n f_i\frac{u_i}{u}\nabla_i\nu^{n+1} \tag{3.12}$$

Next, for the first term in (3.10) we apply lemma 2.3.2 to obtain

$$-F^{ij,kl}h_{ij1}h_{kl1} \geq \sum_{i\neq j}\frac{f_i-f_j}{\kappa_j-\kappa_i}h_{ij1}^2 \geq 2\sum_{i\geq 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}h_{i11}^2$$

Before we proceed, we derive

$$\begin{aligned}
\nabla_i\nu^{n+1} &= -h_{ij}^e(g^e)^{jk}u_k && \text{by (2.18) in [15]}\\
&= \left(\frac{\nu^{n+1}}{u}g_{ij}^e - uh_{ij}\right)(g^e)^{jk}u_k && \text{by (2.1)}\\
&= \left(\frac{\nu^{n+1}}{u}\delta_{ik} - uh_{ij}\frac{\delta_{jk}}{u^2}\right)u_k\\
&= \frac{\nu^{n+1}-\kappa_i}{u}\cdot\delta_{ik}u_k\\
&= \frac{u_i}{u}(\nu^{n+1}-\kappa_i)
\end{aligned}$$

It follows from (3.7) that

$$h_{i11} = \kappa_1\frac{\nabla_i\nu^{n+1}}{\nu^{n+1}-a} = \frac{\kappa_1}{\nu^{n+1}-a}\frac{u_i}{u}(\nu^{n+1}-\kappa_i)$$

and so the first term in (3.10) becomes

$$-F^{ij,kl}h_{ij1}h_{kl1} \geq 2\sum_{i\geq 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}h_{i11}^2 = 2\kappa_1^2\sum_{i\geq 2}\frac{f_i-f_1}{\kappa_1-\kappa_i}\frac{u_i^2}{u^2}\left(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}\right)^2$$

. Similarly, the last term in (3.12) becomes

$$\begin{aligned}
-\frac{2\kappa_1}{\nu^{n+1}-a}\sum_{i=1}^n f_i\frac{u_i}{u}\nabla_i\nu^{n+1} &= -\frac{2\kappa_1}{\nu^{n+1}-a}\sum_{i=1}^n f_i\frac{u_i}{u}\cdot\frac{u_i}{u}(\nu^{n+1}-\kappa_i)\\
&= 2\kappa_1\sum_{i=1}^n f_i\frac{u_i^2}{u^2}\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}
\end{aligned}$$

Therefore, we have

$$0 \geq \sigma \left[1 + \kappa_1^2 - \frac{1 + (\nu^{n+1})^2}{\nu^{n+1} - a}\kappa_1\right] + \frac{a\kappa_1}{\nu^{n+1} - a}\left(\sum f_i + \sum \kappa_i^2 f_i\right) \tag{3.13}$$

$$+ 2\kappa_1^2 \sum_{i\geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i}\frac{u_i^2}{u^2}\left(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a}\right)^2 + 2\kappa_1 \cdot \sum_{i=1}^{n} f_i \frac{u_i^2}{u^2}\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \tag{3.14}$$

Recall that our strategy is to make the right hand side strictly positive; more specifically, we will show the right hand side is bigger than $\sigma(\kappa_1^2 - C_1\kappa_1 + C_2)$ for some positive constants C_1, C_2 depending on a. Then, if κ_1 is large enough, then this quadratic expression will be strictly positive, which is a contradiction. Hence κ_1 must have an upper bound.

There are both positive and negative terms in this inequality and we shall estimate each negative term to show that their negativity won't affect the positivity of the overall sum.

There is a negative term inside the square bracket in the first line (3.13) and we observe that

$$\sigma\left[1 + \kappa_1^2 - \frac{1 + (\nu^{n+1})^2}{\nu^{n+1} - a}\kappa_1\right] \geq \sigma\left[1 + \kappa_1^2 - \frac{2}{a}\kappa_1\right]$$

which can be made positive by assuming $\kappa_1 \geq \frac{2}{a}$.

For the second line (3.14), the first term is positive due to lemma 2.3.2 but the second term can be potentially negative if $\kappa_i < \nu^{n+1}$. The problem being we are not sure about how negative they are. Let us break it into cases. When κ_i is largely negative, we may add those terms to the sum $\sum_{i=1}^{n} \kappa_i^2 f_i$ in (3.13) so that we could use the quadratic term to absorb the negativity:

$$\begin{aligned}
&\frac{a\kappa_1}{\nu^{n+1} - a}\sum_{i=1}^{n} \kappa_i^2 f_i + 2\kappa_1 \sum_{i=1}^{n} f_i \frac{u_i^2}{u^2}\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \\
\geq &\frac{\kappa_1}{\nu^{n+1} - a}\sum_{\kappa_i < \nu^{n+1}} f_i \cdot (a\kappa_i^2 + 2\kappa_i - 2\nu^{n+1}) \\
\geq &\frac{\kappa_1}{\nu^{n+1} - a}\sum_{\kappa_i < \nu^{n+1}} f_i \cdot (a\kappa_i^2 + 2\kappa_i - 2)
\end{aligned}$$

where we have used $\sum_{i=1}^{n} \frac{u_i^2}{u^2} = 1 - (\nu^{n+1}) \leq 1$.

Observe that

$$a\kappa_i^2 + 2\kappa_i - 2 \geq 0 \quad \text{if } \kappa_i \leq \frac{-1 - \sqrt{1 + 2a}}{a} \text{ or } \kappa_i \geq \frac{-1 + \sqrt{1 + 2a}}{a}$$

that is, we only need to worry about those κ_i's with $-\eta < \kappa_i < \nu^{n+1}$, where

$$\eta := \frac{1 + \sqrt{1 + 2a}}{a}$$

In particular, it suffices to consider the summands in the second sum of (3.14) with $-\eta < \kappa_i < \nu^{n+1}$.

$$\begin{aligned} 2\kappa_1 \sum_{i=1}^{n} f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} &\geq \frac{2\kappa_1}{\nu^{n+1} - a} \sum_{\kappa_i < \nu^{n+1}} f_i \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1}) \\ &= \frac{2\kappa_1}{\nu^{n+1} - a} \left(\sum_{\kappa_i \leq -\eta} + \sum_{-\eta < \kappa_i < \nu^{n+1}} \right) f_i \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1}) \end{aligned}$$

Now we can rewrite the inequality (3.13)-(3.14) as

$$0 \geq \sigma \left[1 + \kappa_1^2 - \frac{2}{a} \kappa_1 \right] + \frac{a\kappa_1}{\nu^{n+1} - a} \sum_{i=1}^{n} f_i \tag{3.15}$$

$$+ \frac{a\kappa_1}{\nu^{n+1} - a} \sum_{i=1}^{n} \kappa_i^2 f_i + 2\kappa_1 \sum_{\kappa_i \leq -\eta} f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \tag{3.16}$$

$$+ 2\kappa_1^2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} \left(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \right)^2 + 2\kappa_1 \cdot \sum_{-\eta < \kappa_i < \nu^{n+1}} f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \tag{3.17}$$

with the first two lines being positive. It remains to ensure the third line is positive. Note that the first term is positive by lemma 2.3.2; the trouble term is the second one. Consider

$$\begin{aligned} J &= \{ i : -\eta < \kappa_i < \nu^{n+1},\ \theta f_i < f_1 \} \\ L &= \{ i : -\eta < \kappa_i < \nu^{n+1},\ \theta f_i \geq f_1 \} \end{aligned}$$

where $\theta \in (0, 1)$ is to be determined. We have

$$2\kappa_1 \cdot \sum_{-\eta < \kappa_i < \nu^{n+1}} f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} = 2\kappa_1 \cdot \left(\sum_{i \in J} + \sum_{i \in L} \right) f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a}$$

For the J-sum, since $\sum_{i=1}^{n} \frac{u_i^2}{u^2} = 1 - (\nu^{n+1})^2 \leq 1$, we have

$$\frac{2\kappa_1}{\nu^{n+1} - a} \sum_{i \in J} f_i \frac{u_i^2}{u^2} (\kappa_i - \nu^{n+1}) \geq \frac{2\kappa_1}{\nu^{n+1} - a} \sum_{i \in J} f_i (\kappa_i - \nu^{n+1}) \tag{3.18}$$

$$> -\frac{2\kappa_1}{\nu^{n+1} - a} (\eta + 1) \sum_{i \in J} f_i \tag{3.19}$$

For the L-sum, we need to employ the first term in (3.17).

$$\begin{aligned} & 2\kappa_1^2 \sum_{i \geq 2} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} \left(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \right)^2 + 2\kappa_1 \cdot \sum_{i \in L} f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \\ \geq & 2\kappa_1^2 \sum_{i \in L} \frac{f_i - f_1}{\kappa_1 - \kappa_i} \frac{u_i^2}{u^2} \left(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \right)^2 + 2\kappa_1 \cdot \sum_{i \in L} f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \\ \geq & 2\kappa_1 (1 - \theta) \sum_{i \in L} \frac{\kappa_1}{\kappa_1 - \kappa_i} f_i \frac{u_i^2}{u^2} \left(\frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \right)^2 + 2\kappa_1 \cdot \sum_{i \in L} f_i \frac{u_i^2}{u^2} \frac{\kappa_i - \nu^{n+1}}{\nu^{n+1} - a} \end{aligned}$$

We note that

$$\frac{\kappa_1}{\kappa_1-\kappa_i} = 1+\frac{\kappa_i}{\kappa_1-\kappa_i} > 1-\frac{\eta}{1+\eta}, \quad \text{if } -\eta<\kappa_i<\nu^{n+1}$$

i.e. $\frac{\kappa_1}{\kappa_1-\kappa_i} > 1-\mu$ for some $0<\mu:=\frac{\eta}{1+\eta}<1$. We then have a coefficient of $(1-\theta)(1-\mu) = 1-\mu-\theta+\mu\theta > 1-(\mu+\theta) > 0$ by choosing $0<\theta<1-\mu$.

$$2\kappa_1(1-\theta)\sum_{i\in L}\frac{\kappa_1}{\kappa_1-\kappa_i}f_i\frac{u_i^2}{u^2}\left(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}\right)^2 + 2\kappa_1\cdot\sum_{i\in L}f_i\frac{u_i^2}{u^2}\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a} \tag{3.20}$$

$$\geq 2\kappa_1(1-\mu-\theta)\sum_{i\in L}f_i\frac{u_i^2}{u^2}\left(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}\right)^2 + 2\kappa_1\cdot\sum_{i\in L}f_i\frac{u_i^2}{u^2}\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a} \tag{3.21}$$

$$=2\kappa_1\sum_{i\in L}f_i\frac{u_i^2}{u^2}\left[\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}\left(\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}+1\right)\underbrace{-(\mu+\theta)\frac{\kappa_i-\nu^{n+1}}{\nu^{n+1}-a}}_{>0}\right] \tag{3.22}$$

$$\geq 2\kappa_1\sum_{i\in L}f_i\frac{u_i^2}{u^2}\frac{(\kappa_i-\nu^{n+1})(\kappa_i-a)}{(\nu^{n+1}-a)^2} \tag{3.23}$$

$$\geq 2\kappa_1\sum_{i\in L,\kappa_i>a}f_i\frac{(\kappa_i-\nu^{n+1})(\kappa_i-a)}{(\nu^{n+1}-a)^2} \tag{3.24}$$

$$\geq -2\kappa_1\sum_{i\in L,\kappa_i>a}f_i \tag{3.25}$$

Finally, adding up (3.19) and (3.25)

$$\begin{aligned}
&-\frac{2\kappa_1}{\nu^{n+1}-a}(\eta+1)\sum_{i\in J}f_i - 2\kappa_1\sum_{i\in L,\kappa_i>a}f_i \\
\geq& -2\kappa_1\frac{\eta+1}{\nu^{n+1}-a}\sum_{-\eta<\kappa_i<\nu^{n+1}}f_i \\
\geq& -C\kappa_1 \quad \text{by (1.16)}
\end{aligned}$$

The right-hand side of (3.15)-(3.17) becomes

$$0 \geq \sigma(1+\kappa_1^2-C\kappa_1)$$

for some $C>0$ depending only on a and the proof is complete.

□

Together with the boundary C^2 estimate in section 3.2, we obtain a global C^2 estimate

$$|u|_{C^2(\overline{\Omega})} \leq C$$

and hence a $C^{2,\alpha}$ estimate for admissble solutions. According to the method of continuity, we have a sequence of solutions u^ε to (3.1)-(3.2). Letting $\varepsilon\to 0$ yields theorem 1.0.4.

Bibliography

[1] S. Agmon, A. Douglis, and L. Nirenberg. “Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I.” In: *Comm. Pure Appl. Math.* 12 (1959), pp. 623–727.

[2] Michael T. Anderson. “Complete minimal varieties in hyperbolic space.” In: *Invent. Math.* 69.3 (1982), pp. 477–494.

[3] Michael T. Anderson. “Complete minimal hypersurfaces in hyperbolic n-manifolds.” In: *Comment. Math. Helv.* 58.2 (1983), pp. 264–290.

[4] L. Caffarelli, L. Nirenberg, and J. Spruck. “Nonlinear second order elliptic equations. IV. Starshaped compact Weingarten hypersurfaces.” In: *Current topics in partial differential equations.* Ed. by Y. Ohya et al. Kinokuniya, 1986, pp. 1–26.

[5] Luis Caffarelli, Louis Nirenberg, and Joel Spruck. “The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation.” In: *Comm. Pure Appl. Math.* 37.no. 3 (1984), pp. 369–402.

[6] Luis Caffarelli, Louis Nirenberg, and Joel Spruck. “The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian.” In: *Acta Mathematica* 155.no. 3-4 (1985), pp. 261–301.

[7] Luis Caffarelli, Louis Nirenberg, and Joel Spruck. “Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces.” In: *Comm. Pure Appl. Math.* 41.1 (1988), pp. 47–70.

[8] Eugenio Calabi. “Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens.” In: *Michigan Math. J.* 5 (1958), pp. 105–126.

[9] Manfredo do Carmo. *Riemannian geometry.* Translated from the second Portuguese edition by Francis Flaherty. Birkhäuser Boston, Inc., 1992.

[10] Lawrence C. Evans. “Classical solutions of fully nonlinear, convex, second-order elliptic equations.” In: *Comm. Pure Appl. Math.* 35.no. 3 (1982), pp. 333–363.

[11] David Gilbarg and Neil Trudinger. *Elliptic partial differential equations of second order.* reprint of the 1998 edition. Springer-Verlag, 2001.

[12] Bo Guan. “The Dirichlet problem for Monge-Ampère equations in non-convex domains and spacelike hypersurfaces of constant Gauss curvature.” In: *Trans. Amer. Math. Soc.* 350.12 (1998), pp. 4955–4971.

[13] Bo Guan and Joel Spruck. "Boundary-value problems on Sn for surfaces of constant Gauss curvature." In: *Ann. of Math.* 138.3 (1993), pp. 601–624.

[14] Bo Guan and Joel Spruck. "Hypersurfaces of constant curvature in hyperbolic space. II." In: *Journal of European Mathematical Society* 12.no. 3 (2010), pp. 797–817.

[15] Bo Guan and Joel Spruck. "Convex hypersurfaces of constant curvature in hyperbolic space." In: *Surveys in geometric analysis and relativity.* Ed. by Hubert L. Bray and William P. Minicozzi II. International Press, 2011, pp. 241–257.

[16] Bo Guan, Joel Spruck, and Marek Szapiel. "Hypersurfaces of constant curvature in hyperbolic space. I." In: *Journal of Geometric Analysis* 19.no. 4 (2009), pp. 772–795.

[17] Bo Guan, Joel Spruck, and Ling Xiao. "Interior curvature estimates and the asymptotic plateau problem in hyperbolic space." In: *Journal of Differential Geometry* 96.no. 2 (2014), pp. 201–222.

[18] Joel Guan Bo; Spruck. "Hypersurfaces of constant mean curvature in hyperbolic space with prescribed asymptotic boundary at infinity." In: *Amer. J. Math.* 122.5 (2000), pp. 1039–1060.

[19] Robert Hardt and Fang-Hua Lin. "Regularity at infinity for area-minimizing hypersurfaces in hyperbolic space." In: *Invent. Math.* 88.1 (1987), pp. 217–224.

[20] David Hoffman, Harold Rosenberg, and Joel Spruck. "Boundary value problems for surfaces of constant Gauss curvature." In: *Comm. Pure Appl. Math.* 45.8 (1992), pp. 1051–1062.

[21] Nina M. Ivochkina, Mi Lin, and Neil S. Trudinger. "The Dirichlet problem for the prescribed curvature quotient equations with general boundary values." In: *Geometric analysis and the calculus of variations.* Ed. by Jurgen Jost. International Press, 1996, pp. 125–141.

[22] N. V. Krylov. "Boundedly inhomogeneous elliptic and parabolic equations in a domain. (Russian)". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 47.no. 1 (1983), pp. 75–108.

[23] François Labourie. "Problème de Minkowski et surfaces à courbure constante dans les variétés hyperboliques. (French) [Minkowski problem and surfaces of constant curvature in hyperbolic manifolds]". In: *Bull. Soc. Math. France* 119.3 (1991), pp. 307–325.

[24] Fang-Hua Lin. "On the Dirichlet problem for minimal graphs in hyperbolic space." In: *Invent. Math.* 96.3 (1989), pp. 593–612.

[25] Barbara Nelli and Joel Spruck. "On the existence and uniqueness of constant mean curvature hypersurfaces in hyperbolic space." In: *Geometric analysis and the calculus of variations.* Ed. by Jurgen Jost. International Press, 1996, pp. 253–266.

[26] L. Nirenberg. "On a generalization of quasi-conformal mappings and its application to elliptic partial differential equations." In: *Contributions to the theory of partial differential equations*. Ed. by Lipman Bers, Salomon Trust, and Fritz John. Princeton University Press, 1954, pp. 95–100.

[27] Harold Rosenberg and Joel Spruck. "On the existence of convex hypersurfaces of constant Gauss curvature in hyperbolic space." In: *J. Differential Geom.* 40.no. 2 (1994), pp. 379–409.

[28] Joel Spruck. "Geometric aspects of the theory of fully nonlinear elliptic equations." In: *Global theory of minimal surfaces*. Ed. by David Hoffman. Amer. Math. Soc., 2005, pp. 283–309.

[29] Joel Spruck. *A Personal Tribute to Louis Nirenberg*. The 35th Annual Geometry Festival held by the Stony Brook University. 2021. URL: http://www.math.stonybrook.edu/geomfest21/abstracts.html.

[30] Zhenan Sui. "Convex hypersurfaces with prescribed scalar curvature and asymptotic boundary in hyperbolic space." In: *Calc. Var. Partial Differential Equations* 60.no. 1 (2021), Paper No. 45, 29 pp.

[31] Yoshihiro Tonegawa. "Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space." In: *Math. Z.* 221.4 (1996), pp. 591–615.

[32] Xu-Jia. Wang. "The k-Hessian equation." In: *Geometric analysis and PDEs*. Ed. by Sun-Yung Alice Chang, Antonio Ambrosetti, and Andrea Malchiodi. Springer, 2009, pp. 177–252.