Some Applications of Open Effective Field Theories to Gravitating Quantum Systems

Some Applications of Open Effective Field Theories to Gravitating Quantum Systems

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Lay Abstract:

Open Effective Field Theories are a class of quantum theories in which a measured sector (the system) is used to make physical predictions with, while interacting with an unmeasured sector (the environment). In this thesis arguments are made that Open EFTs are useful for studying gravitating quantum systems, especially when there is an event horizon present (for example in gravitational fields like that of a black hole). Open EFTs are applied to simple toy problems in such settings to illustrate their usefulness.

Abstract:

Open Effective Field Theories are a class of Effective Field Theories (EFTs) built using ideas from open quantum systems in which a measured sector (the system) interacts with an unmeasured sector (the environment). It is argued that Open EFTs are useful tools for any situation in which a quantum system couples to a gravitational background with an event horizon. The main reason for this is that for any EFT of gravity one generically expects perturbation theory to breakdown at late times (when interactions with the background persist indefinitely). It is shown that the tools of Open EFTs allow one to resum late-time perturbative breakdowns in order to make reliable late time predictions (without resorting to solving the dynamics exactly). To build evidence of their usefulness to these types of gravitational problems, Open EFT approximation methods are applied to two toy models relating to black hole physics.

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Declaration of Academic Achievement:

I, Gregory Paul Kaplanek, declare that this thesis titled, "Some Applications of Open Effective Field Theories to Gravitating Quantum Systems" and the work presented in it are my own. Significant contributions from collaborators are as follows:

- **Chapter 2**: The contents of this chapter are published in the *Journal of High Energy Physics*, given in reference [1]. The idea for this work was due to Dr. Burgess, with the calculations and plots done by myself.

- **Chapter 3**: The contents of this chapter are published in the *Journal of High Energy Physics*, given in reference [2]. The idea for this work was due to Dr. Burgess and Dr. Holman. The paper was a collaborative effort between Dr. Burgess, Dr. Holman and myself, with my most visible contributions being the solutions to the hotspot model in both the perturbative and exact limits, and the calculations of the corresponding correlators.

- **Chapter 4**: The contents of this chapter have been published in the journal *Fortschritte der Physik*, given in reference [3]. The idea for this work was due to Dr. Burgess and Dr. Holman. The paper was a collaborative effort between Dr. Burgess, Dr. Holman and myself, with my most visible contributions being the calculations of correlators and decoherence using the Markovian reduced density matrix, as well as the calculations of the correlators resulting from the mean-field approximation.

Dedication:

For Mia and for Ana.

Chapter 1

Introduction

The goal of this thesis is to motivate the use of Open Effective Field Theories (EFTs) for applications on some gravitating quantum systems (especially when event horizons are present), underlining why they are useful for these specific problems. As argued below, calculations involving quantum systems weakly coupled to gravitational backgrounds are often plagued with late time breakdowns of perturbation theory — this is problematic since late times are often times precisely the regimes of interest in such calculations. The techniques discussed in this thesis are one way of reliably resumming these late time issues in perturbative theories (which EFTs always are for gravitational applications). Although this thesis focuses on applications of Open EFT methods to toy problems only, the idea is to emphasize their utility so that these be eventually applied to more realistic gravitational problems (in particular those involving EFTs of gravity).

1.1 Effective Field Theories

The building blocks of fundamental physics appear to be *quantum fields* — roughly speaking, Quantum Field Theory (QFT) is the quantum mechanics of fields (like the electromagnetic field), which are here operators which take on a different value at each point in the spacetime continuum [4, 5]. Excitations of quantum fields appear to us as elementary particles (such as electrons, photons and so on) which implies that particle number is not generically conserved in QFTs, which helps to explain why they are so ubiquitous in modern physical descriptions, from relativistic applications to many-body physics. It is often said that QFT is the most important unification of special relativity and the principles of quantum mechanics (for example important ingredients like cluster decomposition are satisfied by QFTs, and relativity of simultaneity implying the existence of antiparticles [4]).

Given the huge variety of particles and quantum phenomena seen at different energy scales, the discussion of QFTs naturally leads to the concept of *Effective Field Theories* [6]. So-called Wilsonian EFTs are built out of quantum fields, in settings where there exists a hierarchy of energy scales $E \ll M$ that can be exploited in order to yield a low-energy expansion of an action in powers of $E/M \ll 1$ — these are designed to make predictions about the low-energy behaviour of relativistic[1] quantum phenomena (occurring with energies $\sim E$) without needing all the details of the higher energy theory (or UV "completion", occurring at the heavy mass scale $\sim M$). For example, the most successful and general theory of the elementary particles is the Standard Model, which includes electroweak interactions. In this more UV-complete theory the weak nuclear force is described as being mediated by the W and Z gauge bosons. However in sufficiently low-energy processes Fermi's theory of weak interactions is much easier to use for calculations (an EFT where there are no gauge bosons at all *ie.* the heavy degrees of freedom W and Z have been integrated out).

To illustrate, consider the decay of a muon into an electron and two types of neutrinos. Electroweak theory predicts that the W boson interacts with the muon and muon neutrino, as well as the electron and electron neutrino (without the electron and muon interacting directly). What this means is that this electroweak (flavour-changing) process is mediated by a W-boson (with mass $M_W \sim 80\text{GeV}$), and with some effort the decay amplitude can be computed (explicitly depending on the mass M_W). Since the masses of the electron and muon ($m_e \sim 0.0005\text{GeV}$ and $m_\mu \sim 0.1\text{GeV}$) are extremely small compared to M_W, this decay amplitude can be expanded for large M_W (since all momenta will be small compared to M_W in the rest frame of the decaying muon) — after expansion, this decay amplitude scales like $1/M_W^2$ to leading order. Fermi's theory of weak interactions is the EFT which captures this leading-order behaviour, by considering an effective interaction in which the electron and muon interact directly (with this operator in the effective action being suppressed by a power of $1/M_W^2$, as expected from the generic description of an EFT as being a low-energy expansion in powers of $E/M \ll 1$). It turns out for these sorts of low-energy processes, Fermi's EFT of weak interactions is far more convenient for computing various decay and scattering amplitudes. From this perspective, EFTs are useful because they are *practical*.

The EFT framework allows for the systematic assessment of theoretical errors when computing

[1]It is worth emphasizing here, that EFTs are incredibly useful for studying non-relativistic problems as well (like condensed matter systems)

any observable, since one can in principle work to whatever arbitrary order in the low-energy expansion (in powers of $E/M \ll 1$, with more and more work with each increasing order). This is especially true when the effective interactions weakly couple the field content so that well-established *perturbative* techniques apply (many if not most EFTs are weakly interacting, with some exceptions like QCD describing the strong nuclear force).

Related to this, since the action for a Wilsonian EFT can have arbitrarily many operators in its action (*ie.* we can work to any power of $E/M \ll 1$ we'd like) this totally dismisses renormalizability as a criterion for the well-posedness of an EFT: they are (in general) notoriously non-renormalizable theories. Long ago, it was thought that a field theory should be renormalizable in order to be well-defined (meaning *all* divergences can be absorbed by a finite number of counter-terms), however EFT quickly does away with this outdated notion — the point is that an EFT is plenty predictive even though it is not always renormalizable (the reason for this being that to compute any physical observable to a particular order in the EFT expansion, one only needs a finite number of counter-terms, which is all that is needed in practice anyways). The aforementioned Fermi theory of weak interactions is an example of such an EFT, as it is technically non-renormalizable, and yet it makes plenty of predictions that helped pave the way for its more UV-complete counterpart (the Standard Model).

Underlying the framework of EFTs is the principle of *decoupling* — this principle dictates that we tend to have different physical descriptions depending on what length scale we are studying (for example the motion of water in your bathtub obeys equations which don't need to take into account the details of string theory — it is then said that the physics at these two scales is decoupled). Decoupling says that effective descriptions change depending on the length scales we probe: most of the details of small-distance phenomena seem to be irrelevant for the physical description of larger-distance phenomena, and it turns out that quantum fields seem to obey this general physical principle. Philosophically this is at base of the EFT programme.

The above discussion has focused so far on the ways EFTs are useful when its more UV-complete behaviour is *known.* There is however another way in which EFTs can be useful: they help inform us about high energy behaviour when UV completions are still mysterious to us. If we think we understand the symmetries and field content of a low-energy theory, we can straightforwardly work to higher and higher order in a low-energy EFT expansion. By studying physical observables and their dependence on the parameters in such an EFT (and then comparing this to experimental data) one can in principle glean information as to what kind of new physics kicks in at higher energies (in this way, this algorithm is very general as it is by construction agnostic about many of the details of the UV completion). The Standard Model EFT (SMEFT) community works along these lines — by thinking of the Standard Model as the low-energy limit of a fancier UV completion, one looks at higher dimension operators in its low-energy expansion in the hopes of finding deviations and inklings of new physics beyond the Standard Model. Another EFT of this type important to this thesis are EFTs of gravity.

EFTs of Gravity and Semiclassical Gravity

Given that the earlier discussion motivated EFTs through the marriage of special relativity and quantum mechanics (at least partially), it may first come as a surprise that Einstein's theory of gravity, General Relativity (GR) [7, 8], may be regarded as the lowest order term in an EFT expansion. In an *EFT of gravity* [6, 9] the field content is the gravitational metric $g_{\mu\nu}$, and the symmetry is general covariance (*aka.* diffeomorphism invariance of the theory under local changes in coordinates). One can build a low-energy Wilsonian EFT expansion out of various contractions of the Riemann curvature tensor $R_{\mu\nu\sigma\rho}[g]$ in an action of the form[2]

$$S = -\int \mathrm{d}^4x\, \sqrt{-g}\bigg[\Lambda + \frac{M_{\rm Pl}^2}{2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + \dots\bigg] \tag{1.1}$$

where $M_{\rm Pl} = (8\pi G)^{-1/2}$ is the Planck mass, Λ is the cosmological constant, and $R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$ and $R = R^{\mu}{}_{\mu}$ are standard definitions of the Ricci tensor and scalar. Depending on what the particular application of the EFT of gravity is, it is well-controlled (meaning all quantum corrections are small and the theoretical errors can be quantified) when understood as a low energy expansion in powers of E/M as earlier (where the heavy scale M may sometimes be the Plank mass $M_{\rm Pl}$, but this need not always be the case).

Just like the Fermi theory of weak interactions, GR is non-renormalizable (meaning, if one keeps the leading-order term in (1.1) all divergences in the theory cannot be absorbed by a finite number of counterterms — this turns out to be because the coupling $M_{\rm Pl}^2$ has dimensions of mass squared). However as emphasized earlier, renormalizability is *not* necessary for an EFT to be predictive. For this reason, the action (1.1) is interesting to study and helps to inform us of how gravitons (the particle excitations of $g_{\mu\nu}$) behave, and may help to provide clues about what more UV complete physics may look like.

Our universe is of course filled with matter, and so from this point of view we should be able to couple other kinds of quantum fields to gravity. In what follows, we explore two important

[2]If one is expanding to second order in curvature, in principle one should include $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ if one is being completely general about what scalars one can build out of two curvature terms — note however that this term is related to the other curvature squared terms in (1.1) through a topological invariant, and so can be eliminated.

examples where gravity couples to other forms of matter; the first where the quantum fluctuations of the metric are studied, and the latter where the metric behaves classically.

The first example where gravity is coupled to other kinds of quantum fields, is in the theory of inflation [11, 12, 13, 14, 15, 16] with action

$$S = -\int \mathrm{d}^4x\, \sqrt{-g}\, \left[\frac{M_{\rm Pl}^2}{2} R + \frac{1}{2} g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + V(\phi) + \dots\right] \tag{1.2}$$

where ϕ is a real scalar field (which drives the accelerated expansion of the early universe), and $V(\phi)$ is a potential in ϕ. In order to study scalar fluctuations as well as gravitons, one takes (1.2) and expands

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu} \tag{1.3}$$

about a de Sitter space background $\tilde{g}_{\mu\nu}$ (thought to be the large-scale geometry of our universe at the time of this writing), where $h_{\mu\nu}$ are the small gravitational fluctuations (whose excitations eventually give rise to gravitons). The dynamical variables in the above theory are then ϕ and h_{ij}, which have been used to describe the power spectrum of the Cosmic Microwave Background (CMB) radiation at very late times to a remarkable degree of accuracy [17, 18] (note however no quantum correlations have been detected in the CMB — we return to this point briefly in §5).

Moving on to the second example, when we constrain the EFT of gravity given in (1.1) to be in the *semiclassical gravity* regime, we are in a regime where the gravitational field is classical (keeping only the classical background $\tilde{g}_{\mu\nu}$ in the expansion $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$ and neglecting quantum effects) all while responding/backreacting to the presence of other quantum fields it is coupled to. A further approximation to this is *QFT in Curved Spacetime* (QFTCS) [19, 20, 21], where gravity is classical but the other quantum fields do not back-react onto the metric (meaning quantum processes like particle production to do with the other quantum fields are free to occur, but the gravitational background is assumed to behave as if no quantum fields are present at all). Semiclassical gravity and QFTCS gives rise to many interesting results, most notably for this thesis: the fact that for classical spacetimes $\tilde{g}_{\mu\nu}$ which contain event horizons, there arises a notion of *temperature* for any quantum fields that reside there. This is known to be true for free quantum field theories coupled to classical gravity of the form

$$S = -\int \mathrm{d}^4x\, \sqrt{-g}\, \left[\frac{1}{2}\tilde{g}_{\mu\nu}\partial^\mu\phi\partial^\nu\phi + \frac{1}{2}m^2\phi^2 + \dots\right] \tag{1.4}$$

where the "…" are meant to emphasize that there are higher-order interaction terms being neglected here (when thought of as some, here unspecified, low-energy EFT expansion). The fact that a notion temperature arises in all of the well-understood spacetimes with horizons

(Minkowski [24], de Sitter [25], and various black hole spacetimes) is partially what makes the later discussions of Open EFTs so appropriate in these settings.

Most famously, when $\tilde{g}_{\mu\nu}$ is Schwarzschild spacetime,

$$\mathrm{d}s^2 = \tilde{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\phi\mathrm{d}\phi^2 \ , \tag{1.5}$$

with $r_s = 2GM_{\mathrm{BH}}$ the Schwarschild radius (with M_{BH} the mass of the black hole) one encounters the *Hawking effect* [22, 23]. The quantum field ϕ behaves as if it is immersed in a thermal bath at Hawking temperature (for observers far from the black hole)

$$T_{\mathrm{H}} = \frac{1}{4\pi r_s} \ , \tag{1.6}$$

and furthermore the black hole radiates perfect blackbody radiation at the above temperature and therefore slowly loses mass and starts to disappear at late times. The process of black hole evaporation is of course only true in the semiclassical gravity regime (in this case, when $M_{\mathrm{Pl}}/M_{\mathrm{BH}} \ll 1$ so that quantum gravity effects are small[3]), which means that the description stops being semiclassical at some point when the black hole loses enough of its mass. As a result there has been much debate about the fate of black holes at very late times [26, 27, 28, 29, 30, 31]).

Before moving on to the next subsection we make two final remarks. First, that in both of the standard gravitational examples explored in the above (and indeed many others as well), *late times are regimes of great interest* that we would like to understand. Secondly, if we understand actions such as (1.4) to be the zeroth-order terms in a low-energy EFT expansion (since in (1.4) we ignore higher-order interactions of the quantum fields) then this means we are *implicitly saying we have perturbation theory under control* (including at late times). In the next section we argue that interactions in EFTs of gravity are settings where perturbation theory may in fact be suspect at late times.

1.2 Late Times, Secular Growth and Gravity

Consider an arbitrary quantum Hamiltonian

$$\hat{H} \ = \ \hat{H}_0 + g\hat{H}_{\mathrm{int}} \tag{1.7}$$

[3]The reason for this is that the relevant low-energy scale for power-counting the EFT is set by the typical size of a derivative in the background configuration $E^2 \sim \partial^2\tilde{g}$. For the Schwarzschild background (1.5) one finds $E^2 \sim r_s/r^3$, which means that the EFT expansion is organized in powers of $E/M_{\mathrm{Pl}} \sim 1/(r_s M_{\mathrm{Pl}}) \sim M_{\mathrm{Pl}}/M_{\mathrm{BH}} \ll 1$ for $r \sim r_s$ near the event horizon. Whenever $M_{\mathrm{Pl}}/M_{\mathrm{BH}} \ll 1$ this ensures that quantum loops associated with the metric are negligible [6].

where there is some dimensionless coupling $g \ll 1$ assumed small so that the interaction $\hat{H}_{\rm int}$ can be considered weak, and where the quantum evolution under the free Hamiltonian $\hat{H}_0$ is known. Solving for the dynamics in time-dependent perturbation theory generically leads to non-sensical predictions at late times for observables, known as *secular growth*. An observable $\mathcal{O}(t)$ computed in perturbation theory,

$$\mathcal{O}(t) \ = \ \sum_n g^n \mathcal{O}_n(t) \ , \tag{1.8}$$

is called secularly growing if $\mathcal{O}_n(t) \to \infty$ at late times $t \to \infty$ (for any $n \geq 1$). The reason for this breakdown of perturbation theory and why it can be so generally expected to occur is that fundamentally observables are computed using unitary operators of the form

$$\hat{U}(t) \ := \ e^{-i(\hat{H}_0 + g\hat{H}_{\rm int})t} \ . \tag{1.9}$$

In performing a perturbative expansion, the evolution operator effectively gets expanded in $g \ll 1$ such that $U(t) \simeq e^{-i\hat{H}_0 t}(1 - ig\hat{H}_{\rm int}t + \ldots)$ — what this means is that no matter how small the coupling g is, there always comes a time late enough in which it is a bad approximation to compute $\hat{U}(t)$ in powers of $g\hat{H}_{\rm int}t$.

As emphasized above, the EFT paradigm is generally a perturbative programme (especially when gravity is involved) and so one may naturally ask why so many calculations in particle physics seem to be free of secular growth issues: the answer is that this is because scattering calculations are one of the few types of quantum mechanical calculations where this is distinctly *not* an issue. In a typical scattering problem, the wavepackets of the initial components start off distantly separated at early times, and only interact for a brief moment as their wavefunctions overlap during the scattering event, and then finally the products freely fly off at late times (see Figure 1.1). This exception to the rule of secular growth helps to explain why the commonness of secular breakdowns are somewhat under-appreciated in the particle physics community.

There are however plenty of other communities where secular growth is a well-known obstruction to reliably computing late time behaviour. In particular, it is an issue whenever interactions persist indefinitely (so perturbatively small effects accumulate to eventually yield large divergent contributions): most notably in cases where particles interact with a medium, or *environment*. Perhaps the best-studied setting where this happens is in calculations involving thermal baths — when these thermal perturbative calculations are phrased in terms of EFTs, secular growth often manifests itself in momentum space as severe (*aka.* power-law as opposed to logarthmic) divergences, which translate to secular growth predictions for observables such as correlation functions (for example, in hard thermal loop QCD calculations for near-massless bosons [32, 33],

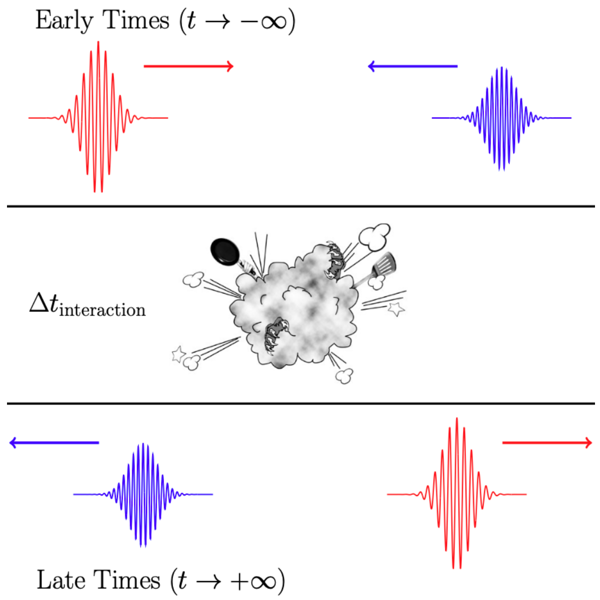

Figure 1.1: A cartoon of a scattering problem: non-interacting wavepackets are set up far apart from each other at $t \to -\infty$, interact for a brief moment $\Delta t_{\text{interaction}}$ as their wavefunctions overlap (with $\Delta t_{\text{interaction}}$ a short time interval over which perturbation theory can be trusted), and the products fly off again for late times $t \to +\infty$.

or the breakdown of mean-field methods when calculating nearby a critical point [34]).

Gravity as a Medium

It turns out that essentially any quantum system coupled to a gravitational field tends to suffer secular breakdowns of perturbation theory. The reason for this is not that there is something special about gravity (in this regard), but simply because the gravitational background is eternally present: for any quantum degrees of freedom coupled to the background, gravity acts as a medium for secular growth effects to manifest over. There are plenty of examples of this in the literature, most prominently in cosmological settings [35, 36, 37, 38, 39, 40, 41, 42], but has been shown to occur in Minkowski space [43, 44] as well as black hole spacetimes [45]. Since secular divergences are known to be commonplace in thermal field theories, this is perhaps unsurprising since all of these spacetimes have horizons and therefore a notion of temperature within them.

1.3 Late-Time Resummations

Just because weakly-coupled theories suffer secular growth problems does not mean we need to exactly solve the theory in order to make reliable predictions at late times. Through a *late-time* *resummation*, one can extend the validity of perturbation theory to gain access to late time behaviour (at least for certain regimes of parameter space).

The simplest example of this has to do with particle decay, where the number of particles $N(t)$ in the system obeys the evolution equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = -\Gamma N(t) \ . \tag{1.10}$$

The decay rate Γ is computed in perturbation theory, and so it may seem at first surprising that we can trust the solution

$$N(t) \simeq N(0)e^{-\Gamma t} \tag{1.11}$$

out to very late times where $\Gamma t \gg 1$. If one were to solve the equation naively in perturbation theory, all the equation tells you is that

$$N(t) \simeq N(0)[1 - \Gamma t + \ldots] \ , \tag{1.12}$$

which clearly suffers the problem of secular growth and stops begin useful the moment $\Gamma t \sim \mathcal{O}(1)$ or larger. Why then can we trust the exponential in (1.11)?

The saving grace here is the observation that the evolution equation $\frac{\mathrm{d}}{\mathrm{d}t}N(t) = -\Gamma N(t)$ is *time-local*. It does not depend explicitly on t (only implicitly through N), and therefore its perturbative solution can be derived *anywhere* along the time axis. For example, one need not solve equation (1.10) starting at $t = 0$, one can instead solve it at a different later time $t = t_j > 0$, where the perturbative solution is

$$N(t) = N(t_j)[1 - \Gamma(t - t_j) + \ldots] \tag{1.13}$$

where (similar to the above) the perturbative solution can be trusted for $\Gamma(t - t_j) \ll 1$. The above implies that we have a family of solutions, each which can be trusted for some very small window of times — if these windows overlap, then we can trust the solution over the *union* of all these windows, and for this reason it truly makes sense to trust the exponential decay law $N(t) \simeq N(0)e^{-\Gamma t}$, even though Γ is computed in perturbation theory. See Figure 1.2. If, for example, the decay rate is computed to second order in the interaction so that $\Gamma \sim \mathcal{O}(g)$, then it is said that the solution has been resummed to all orders in gt, while neglecting orders g^2t effects. This happily means we have access to much later times than the original perturbation series.

The above late-time resummation argument is in spirit a renormalization group-like argument as it is reminiscent of how perturbative expressions for the runnings of couplings (as a function

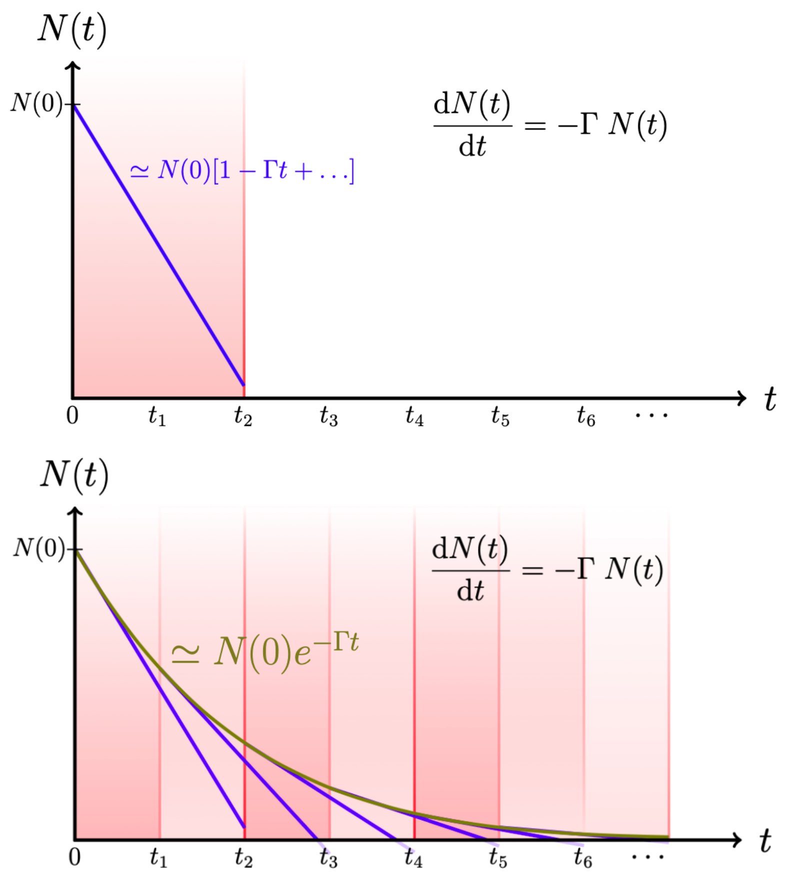

Figure 1.2: In the upper figure, perturbation theory gives a valid solution over a tiny interval $t \in [0, t_2]$ where $\Gamma t \ll 1$. In the lower figure, since the evolution equation is time-local, it does not matter which tiny interval we perturbatively solve the equation in. We can trust each of the perturbative solutions in the small interval where it is valid and stitch them all together to trust the resummed solution $N(t) \simeq N(0)e^{-\Gamma t}$ out to very late times.

of energy scale μ) get their domain of validity extended. For example in a standard course in quantum electrodynamics (QED), one often computes the running of the renormalized[4] fine structure constant $\alpha(\mu)$ in perturbation theory [6] for energy scales $\mu \geq m_e$ above the mass of the electron (where famously $\alpha_0 := \alpha(m_e) = e^2/(4\pi) \simeq 1/137$ with e the elementary charge). At one loop in QED, one finds

$$\alpha(\mu) \simeq \left[1 - \frac{\alpha}{3\pi} \log\left(\frac{\mu_0^2}{\mu^2}\right)\right] \alpha(\mu_0) \tag{1.14}$$

for any energy scale μ nearby a reference scale μ_0 such that $\left|\alpha \log(\mu^2/\mu_0^2)\right| \ll 1$. Since the reference scale μ_0 is arbitrary, one can derive the solution (1.14) for μ close to *any* applicable QED scale $\mu_0 \geq m_e$ — differentiating the above expression with respect to $\log(\mu^2)$ one then finds

[4]typically in the modified minimal subtraction renormalization scheme

that (1.14) implies

$$\frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\log(\mu^2)} = + \frac{\alpha^2(\mu)}{3\pi} \ , \tag{1.15}$$

which gives an answer for $\alpha(\mu)$ with a much larger domain of validity than (1.14) would naively imply, requiring the much weaker condition $\alpha \ll 1$. When (1.16) is solved with the initial condition $\alpha_0 = \alpha(m_e)$, one finds that

$$\alpha(\mu) = \left[\frac{1}{\alpha_0} - \frac{1}{3\pi} \log\left(\frac{\mu^2}{m_e^2} \right) \right]^{-1} \tag{1.16}$$

and it is here said that the renormalization group improved running implied by (1.16) is valid to all orders in $\alpha \log(\mu^2/m_e^2)$ (so long as $\alpha \ll 1$) — this is the same kind of resummation being performed at late times in the remainder of this work.

There are different ways to derive late-time resummations in the literature, however the method used in Open EFTs (and so pertinent to this thesis) comes from Markovianity arguments in the study of open quantum systems.

1.4 Open Quantum Systems

The framework of Open Quantum Systems (OQSs) [46, 47, 48, 49, 50, 51, 52, 53] is designed to study a sector A (called the open system) whose degrees of freedom we are interested in tracking, which interacts with some unobserved sector B (called the environment). Sectors A and B have their own respective Hilbert spaces $\mathcal{H}_{\text{System}}$ and $\mathcal{H}_{\text{Environment}}$ which means that the interaction is assumed to mix the two sectors with a Hamiltonian of the form

$$\hat{H} = \hat{H}_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \hat{H}_B + g\hat{H}_{\text{int}} \ . \tag{1.17}$$

For simplicity we assume in this section that the above Hamiltonian is time-independent. The operators $\hat{H}_{A,B}$ and $\mathbb{I}_{A,B}$ are free Hamiltonians and identities on the Hilbert spaces of the system and environment respectively, and the coupling g is usually assumed to be small so that the system can be studied perturbatively assuming that the operator

$$\hat{H}_0 := \hat{H}_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes \hat{H}_B \tag{1.18}$$

can be treated as a free Hamiltonian (under which the evolution of the system can be solved). The situation is depicted in Figure 1.3. As with any quantum system, the density matrix $\hat{\rho}(t)$ describing the state of both sectors A and B (an operator on $\mathcal{H}_{\text{System}} \otimes \mathcal{H}_{\text{Environment}}$) obeys the

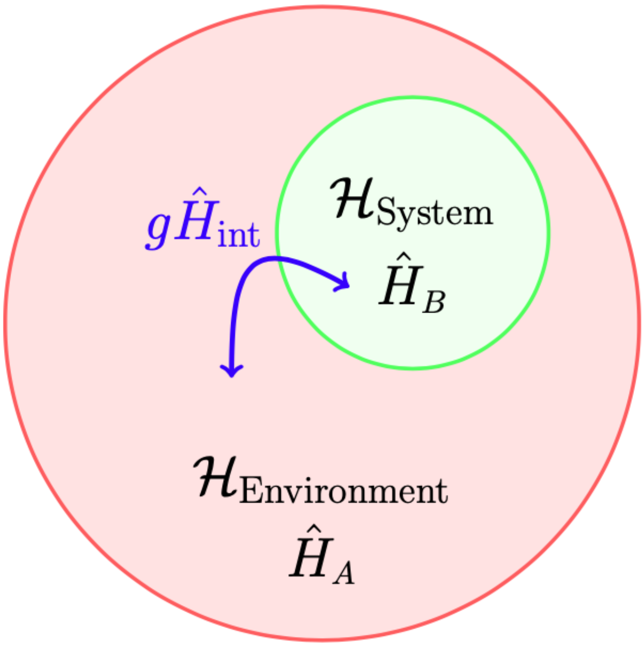

Figure 1.3: Schematic set-up for an open quantum system interacting with an environment.

Liouville equation

$$\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{H}_0 + g\hat{H}_{\rm int}, \hat{\rho}(t)] \ . \tag{1.19}$$

The unitary time evolution underlying the equation is guaranteed to preserve the essential properties of the density matrix: (i) $\hat{\rho}^\dagger = \hat{\rho}$, (ii) $\mathrm{Tr}[\hat{\rho}] = 1$ and (iii) $\hat{\rho} \geq 1$ (the latter meaning that the eigenvalues of $\hat{\rho}$ are non-negative). In particular the properties (ii) and (iii) together ensure that all probabilities associated with the density matrix are properly bounded between 0 and 1.

In the framework of OQSs, the goal is to understand what the accessible sector A is doing (without directly tracking the evolution of B), which means to track the reduced density matrix

$$\hat{\varrho}_A := \mathop{\mathrm{Tr}}_B[\hat{\rho}] \tag{1.20}$$

with all the degrees of freedom of sector B traced away. In principle, $\hat{\varrho}_A$ (an operator acting on $\mathcal{H}_{\rm System}$ alone) allows one to calculate whatever time-dependent observables desired in the system. What is intriguing about OQSs is that the time evolution is no longer unitary — since the interaction mixes the Hilbert spaces of the system and environment, if one tracks the evolution of $\hat{\varrho}_A$ alone then it is free to undergo dissipation (leakage of energy into the environment), as well as decoherence (loss of quantum coherence so that $\hat{\varrho}_A$ tends towards a classical statistical ensemble).

1.4.1 Master Equations

Typically the evolution of $\hat{\varrho}_A$ is understood using a *master equation*, which is derived from the underlying Liouville equation. In this subsection we briefly derive an elementary master equation, which is derived from the interaction picture version of equation (1.19)

$$\frac{\partial \hat{\rho}^I}{\partial t} \;=\; -ig[\hat{H}^I_{\rm int}(t), \hat{\rho}^I(t)] \;. \tag{1.21}$$

where the interaction picture variables are defined as

$$\begin{aligned} \hat{H}^I_{\rm int}(t) &:= e^{+i\hat{H}_0 t}\hat{H}_{\rm int}e^{-i\hat{H}_0 t} \\ \hat{\rho}^I(t) &:= e^{+i\hat{H}_0 t}\hat{\rho}(t)e^{-i\hat{H}_0 t} \end{aligned} \tag{1.22}$$

and similarly for the reduced density matrix

$$\hat{\varrho}^I_A(\tau) \;=\; e^{+i\hat{H}_A t}\hat{\varrho}_A(t)e^{-i\hat{H}_A t} \;. \tag{1.23}$$

The best would be if one could simply apply the partial trace operation Tr_B directly onto (1.21) so as to yield a master equation involving $\hat{\varrho}_A$ only. It turns out this is not so straightforward, and one needs a more useful form of the Liouville equation to make progress. To this end we note that there are other ways of writing the interaction picture Liouville equation, the first being the integral version of (1.21)

$$\hat{\rho}^I(t) \;=\; \hat{\rho}^I(0) - ig\int_0^t \mathrm{d}s\; [\hat{H}^I_{\rm int}(s), \hat{\rho}^I(s)] \;, \tag{1.24}$$

and another version which involves inserting (1.24) into the RHS of (1.21) giving

$$\frac{\partial \hat{\rho}^I}{\partial t} \;=\; -ig[\hat{H}^I_{\rm int}(t), \hat{\rho}^I(0)] - g^2\int_0^t \mathrm{d}s\; \big[\hat{H}^I_{\rm int}(t), [\hat{H}^I_{\rm int}(s), \hat{\rho}^I(s)]\big] \;. \tag{1.25}$$

Note that (1.21), (1.24) and (1.25) are all equivalent. However it turns out that the last equation is a useful starting point for developing a simple master equation if one additionally assumes that the initial state is uncorrelated

$$\hat{\rho}^I(0) = \hat{\varrho}^I_A(0) \otimes \hat{\mathfrak{B}} \tag{1.26}$$

where $\hat{\mathfrak{B}}$ is the initial state of the bath (assumed to be time-independent). By examining equation (1.25), one can easily deduce that any correlations for $t > 0$ must come in at $\mathcal{O}(g)$ or higher[5], and so under this assumption one performs the so-called Born approximation

$$\hat{\rho}^I(t) \simeq \hat{\varrho}^I_A(t) \otimes \hat{\mathfrak{B}} + \mathcal{O}(g) \;, \tag{1.27}$$

[5] Inserting $\rho(t) = \hat{\varrho}_A(t) \otimes \hat{\mathfrak{B}} + g\hat{\rho}_{\rm correlations}(t)$ into (1.25) shows that this ansatz is consistent.

in equation (1.25) giving

$$\frac{\partial \hat{\rho}^{I}}{\partial t} \simeq -ig[\hat{H}^{I}_{\text{int}}(t), \hat{\rho}^{I}(0)] - g^2 \int_0^t \mathrm{d}s \, \big[\hat{H}^{I}_{\text{int}}(t), [\hat{H}^{I}_{\text{int}}(s), \hat{\varrho}^{I}_{A}(s) \otimes \hat{\mathfrak{B}}]\big] \ . \tag{1.28}$$

If one then assumes that the interaction is of the form

$$g \, \hat{H}^{I}_{\text{int}}(t) = g \, \hat{S}^{I}(t) \otimes \hat{B}^{I}(t) \ , \tag{1.29}$$

then one can easily take the partial trace of (1.28) which gives rise to (after a change of integration variable $s \to t - s$)

$$\frac{\partial \hat{\varrho}^{I}_{A}}{\partial t} \simeq -g^2 \int_0^t \mathrm{d}s \, \Big(\langle \hat{B}^{I}(t) \hat{B}^{I}(t-s) \rangle_{\hat{\mathfrak{B}}} \big[\hat{S}^{I}(t), \hat{S}^{I}(t-s) \hat{\varrho}^{I}_{A}(t-s) \big] + \text{h.c.} \Big) \tag{1.30}$$

where for simplicity we assume that

$$\operatorname*{Tr}_{B} \Big(-ig[\hat{H}^{I}_{\text{int}}(t), \hat{\rho}^{I}(0)] \Big) = 0 \tag{1.31}$$

and we define the environment average

$$\langle \hat{O} \rangle_{\hat{\mathfrak{B}}} := \operatorname*{Tr}_{B}[\hat{O}\hat{\mathfrak{B}}] \ . \tag{1.32}$$

The master equation (1.30) is a simple master equation, which happens to agree with the leading-order term of its more sophisticated counterpart the *Nakajima-Zwanzig equation* [54, 55] used in the §2 and §4 of this thesis. However, as it stands equation (1.30) is not time-local and so an additional approximation is required if it is to help us gain access to late times.

1.4.2 Horizons, Open EFTs and Lindblad Equations

For any type of EFT of gravity which couples quantum objects to a gravitational background it seems that an OQS description strongly resembles the way in which an event horizon partitions the degrees of freedom in the problem. The horizon naturally splits up the system into an observed and an unobserved sector, which interact with one another (even though we can only make measurements on one side of the horizon — see Figure 1.4). Considering this, and the fact that gravitational backgrounds seem to introduce secular growth issues, it turns out that the techniques of OQSs are extremely useful for studying such EFTs of gravity.

In order to be considered an Open EFT however, there should be a hierarchy of scales available in order to simplify computations. It turns out that access to late times relies on whether there is a limit of master equations like (1.30) in which it becomes *time-local* on the right hand-side,

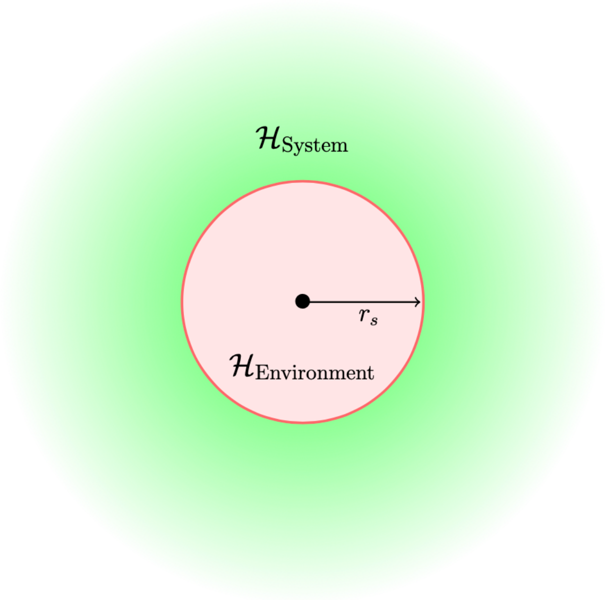

Figure 1.4: A cartoon of how an open quantum systems treatment may be applied to a regions of spacetime separated by an event horizon (in this case that of a black hole with Schwarzschild radius r_s).

such that

$$\frac{\partial \hat{\varrho}^I_A}{\partial t} \simeq -g^2 \int_0^\infty \mathrm{d}s \left(\langle \hat{B}^I(t)\hat{B}^I(t-s)\rangle_{\hat{\mathfrak{B}}} \left[\hat{S}^I(t), \hat{S}^I(t-s)\hat{\varrho}^I_A(t)\right] + \text{h.c.} \right) . \tag{1.33}$$

Here one expands (about $s = 0$) the operator

$$\varrho^I_A(s) \simeq \rho^I_A(t) - s\partial_t \varrho^I_A(t) + \dots \tag{1.34}$$

on the RHS of (1.33), under the assumption that there exists a hierarchy of scale between the correlation time of $\langle \hat{B}^I(t)\hat{B}^I(t-s)\rangle_{\hat{\mathfrak{B}}}$ and the size of $\partial_t \varrho^I_A(t)$ — the limit in which (1.33) is true is known as the *Markovian approximation*[6]. Note that in equation (1.33) we have additionally assumed that we probe times t much larger than the correlation time in the environment (so that the upper limit on the integration can be approximated by ∞).

The Markovian approximation coarse-grains the evolution along the time axis, since it loses the "memory" of the integration over s. What is so powerful about the Markovian approximation is that one can convert equations like (1.33) into so-called Lindblad form [56, 57, 58] (now written

[6]Whose namesake comes from the Markov chain stochastic process, a sequence of events in which the probability of an event occurring depends only on the state attained in the previous event.

in the Schrödinger picture)

$$\frac{\partial \hat{\rho}_A(t)}{\partial t} = -i[\hat{H}_A + g^2 \hat{H}_{\rm eff}, \varrho_A(t)] + g^2 \sum_{j,k} h_{jk} \left(\hat{A}_j \varrho_A(t) \hat{A}_k^\dagger - \frac{1}{2} \left\{ \hat{A}_k^\dagger \hat{A}_j, \varrho_A(t) \right\} \right) \tag{1.35}$$

for some effective Hamiltonian $\hat{H}_{\rm eff}$, and some family of operators $\hat{A}_j$ (usually closely related to operators $\hat{S}^I(t)$ in the original interaction), and the matrix $[h_{ij}]$ are called the Kossakowski coefficients (which in practice are the s-integrals in (1.33) over the correlations $\langle \hat{B}^I(t)\hat{B}^I(s)\rangle_{\hat{\mathfrak{B}}}$).

Most crucially, Lindblad equations like (1.35) are time-local and therefore reliably predict late time behaviour. The way this works is to notice that time-dependent perturbation theory would instead predict the behaviour (in the interaction picture, where $\hat{H}^I_{\rm eff}(t) := e^{+i\hat{H}_A t}\hat{H}_{\rm eff}e^{-i\hat{H}_A t}$ and $\hat{A}^I_j(t) := e^{+i\hat{H}_A t}\hat{A}_j e^{-i\hat{H}_A t}$)

$$\begin{aligned} \hat{\rho}^I_A(t) \simeq\ & -ig^2 \int_{t_0}^t {\rm d}s\ [\hat{H}^I_{\rm eff}(s), \varrho^I_A(t_0)] \\ & + g^2 \int_{t_0}^t {\rm d}s \sum_{j,k} h_{jk} \left(\hat{A}^I_j(s) \varrho^I_A(t_0) \hat{A}^{I\dagger}_k(s) - \frac{1}{2} \left\{ \hat{A}^{I\dagger}_k(s) \hat{A}^I_j(s), \varrho^I_A(t_0) \right\} \right) \end{aligned} \tag{1.36}$$

for any initial time $t_0 \geq 0$ (*c.f.* equation (1.14)), with this perturbative expression being valid for $g^2(t-t_0)/\xi \ll 1$ (with ξ some typical timescale associated with the environment). Since the solution is valid over any perturbatively small interval anywhere along the time axis (picking t_0 anywhere we like, not necessarily at $t_0 = 0$), this means we can trust the solutions over the *union* of all these tiny overlapping time intervals and one can therefore trust the integrated solution to (equivalent to (1.35))

$$\frac{\partial \hat{\rho}^I_A(t)}{\partial t} = -ig^2[\hat{H}^I_{\rm eff}(t), \varrho_A(t)] + g^2 \sum_{j,k} h_{jk} \left(\hat{A}^I_j(t) \varrho^I_A(t) \hat{A}^{I\dagger}_k - \frac{1}{2} \left\{ \hat{A}^{I\dagger}_k \hat{A}^I_j, \varrho^I_A(t) \right\} \right) \tag{1.37}$$

to all orders in $g^2 t/\xi$ (so long as $g \ll 1$). This is the same late-time resummation argument given in §1.3 and is the main tool of the Open EFT treatment considered in this thesis as it is fundamentally a hierarchy of scales argument (for other examples of Open EFTs used for gravitational problems see [61, 62, 63, 64, 65, 66, 67, 68]).

Finally we note that Lindblad equations like (1.35) are easily shown to preserve hermicity and unit traces of $\hat{\varrho}_A$. It is also a theorem that if the eigenvalues of the Kossakowski matrix $[h_{ij}]$ are non-negative then the Lindblad equation preserves positivity[7,8]. One of the lines of reasoning

[7]Technically, it preserves a stronger notion of positivity called complete positivity — by definition a finite dimensional operator $\hat{O}$ is completely positive if $\hat{O} \otimes \mathbb{I}_N$ it is positive for any identity operator $\mathbb{I}_N$. This notion of positivity is important in open quantum systems since it ensures states are positivity-preserving no matter what kind of system they may get entangled with.

[8]The proof that complete positivity is preserved by Lindblad equations with non-negative Kossakowski coefficients is rather technical (using the technology of so-called quantum dynamical semigroups), and so we refer the reader to [56, 57, 59, 60].

taken in the Open EFT approach in this thesis is that if one consistently tracks the size of theoretical errors when taking each approximation in the above then one must end up with a positivity-preserving Lindblad equation (since the underlying Liouville equation one starts with is positivity-preserving).

1.4.3 Qubits

For the rest of this section we make use of the earlier developments of this section so as to study the simple toy model of a qubit (or Unruh-DeWitt detector [24, 69]) which moves through some spacetime coupled to a quantum field (see Figure 1.5). We specify to the simple case in which the

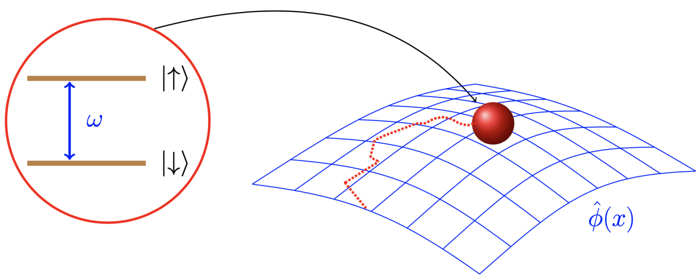

Figure 1.5: A cartoon of an Unruh-DeWitt detector: a qubit which traverses some trajectory through a curved spacetime, while interacting with a quantum field living there.

role of the environment is played by a real scalar field ϕ living in flat Minkowski space (assumed to be massless and non-interacting), and the qubit free Hamiltonian is

$$\hat{H}_{A}(\tau) = \frac{\omega}{2}\hat{\sigma}_3 \, \frac{\mathrm{d}\tau}{\mathrm{d}t} \tag{1.38}$$

where $\hat{\sigma}_j$ denote the standard 2×2 Pauli matrices. We pick the interaction between the open system and environment to be (in the interaction picture)

$$g\, \hat{H}^{I}_{\mathrm{int}}(\tau) = g\, \hat{S}^{I}(t) \otimes \hat{B}^{I}(t)\, \frac{\mathrm{d}\tau}{\mathrm{d}t} \tag{1.39}$$

with

$$\hat{S}^{I}(t) := \begin{bmatrix} 0 & e^{+i\omega\tau} \\ e^{-i\omega\tau} & 0 \end{bmatrix} \qquad \text{and} \qquad \hat{B}^{I}(t) := \phi^{I}[y^{\mu}(\tau)] \tag{1.40}$$

where ϕ^{I} is the interaction picture field and the trajectory of qubit is $y^{\mu}(\tau)$ (parametrized by the proper time τ as measured by the qubit). Note that both (1.38) and (1.39) include Jacobian factors $\mathrm{d}\tau/\mathrm{d}t$, which ensure that these operators generate translations in terms of the qubit's proper time (where t is the Minkowski time with which the scalar's free Hamiltonian generates time translations in). The trajectory is here chosen to be (for $a > 0$)

$$y^\mu(\tau) := \left(\tfrac{1}{a}\sinh(a\tau), \tfrac{1}{a}\cosh(a\tau), 0, 0\right) \ . \tag{1.41}$$

Observers moving along the above uniformly accelerated trajectory will famously experience the Minkowski vacuum $|\mathrm{M}\rangle$ to be a thermal state at Unruh temperature $T = a/(2\pi)$ [24, 70]. By assuming the environment is prepared in the Minkoswski vacuum $\hat{\mathfrak{B}} = |\mathrm{M}\rangle\langle\mathrm{M}|$, it turns out that (1.30) becomes the set of equations

$$\begin{aligned} \frac{\partial \varrho^{I}_{11}}{\partial\tau} &= g^2 \int_{-\tau}^{\tau} \mathrm{d}s\ \mathcal{W}(s) e^{-i\omega s} - 4g^2 \int_0^\tau \mathrm{d}s\ \mathrm{Re}[\mathcal{W}(s)] \cos(\omega s) \varrho^{I}_{11}(\tau - s)\ , \\ \frac{\partial \varrho^{I}_{12}}{\partial\tau} &= -2g^2 \int_0^\tau \mathrm{d}s\ \mathrm{Re}[\mathcal{W}(s)] e^{+i\omega s} \varrho^{I}_{12}(\tau - s) + 2g^2 e^{+2i\omega\tau} \int_0^\tau \mathrm{d}s\ \mathrm{Re}[\mathcal{W}(s)] e^{-i\omega s} \varrho^{I*}_{12}(\tau - s)\ , \end{aligned} \tag{1.42}$$

where (with the limit $\epsilon \to 0^+$ understood)

$$\mathcal{W}(\tau) \ = \ \mathrm{Tr}_B\left(\phi^{I}[y^\mu(\tau)]\phi^{I}[y^\mu(0)]\right) \ = \ -\frac{a^2}{16\pi^2 \sinh^2\left[\frac{a(\tau - i\epsilon)}{2}\right]} \ . \tag{1.43}$$

and where we have used

$$\hat{\varrho}^{I}_{A} \ = \ \begin{bmatrix} \varrho^{I}_{11} & \varrho^{I}_{12} \\ \varrho^{I}_{21} & \varrho^{I}_{22} \end{bmatrix} = \begin{bmatrix} \varrho^{I}_{11} & \varrho^{I}_{12} \\ \varrho^{I*}_{12} & 1 - \varrho^{I}_{11} \end{bmatrix} \ . \tag{1.44}$$

So as to take the Markovian approximation, we note that the correlator $\mathcal{W}(\tau)$ is peaked around $\tau = 0$ and correlations die away exponentially fast with timescale $\sim 1/a$. When the timescales associated with $\partial_\tau \varrho^{I}_{A}(\tau)$ are large compared to $1/a$ the Markovian approximation should apply. However, as emphasized in [71] (as well [1, 72, 90]) taking the Markovian approximation in (1.34) results in an equation of motion that additionally requires $\omega/a \ll 1$ (otherwise the subleading terms in the Taylor series (1.34) become too large, spoiling the approximation). One way of very easily attaining this (surprisingly stringent) Markovian limit is to take instead of (1.34) the Taylor series

$$\hat{\varrho}^{I}_{A}(\tau - s)\hat{S}^{I}(\tau - s) \simeq \hat{\varrho}^{I}_{A}(\tau)\hat{S}^{I}(\tau) - s\left(\partial_\tau \hat{\varrho}^{I}_{A}(\tau)\hat{S}^{I}(\tau) + \hat{\varrho}^{I}_{A}(\tau)\partial_\tau \hat{S}^{I}(\tau)\right) + \dots\ , \tag{1.45}$$

keeping the leading-order term so that (1.42) becomes (also assuming $\tau \gg 1/a$ so the limits on the integrals can be taken to be ∞)

$$\begin{aligned} \frac{\partial \varrho^{I}_{11}}{\partial\tau} &\simeq g^2 \int_{-\infty}^{\infty} \mathrm{d}s\ \mathcal{W}(s) - 4g^2 \int_0^\infty \mathrm{d}s\ \mathrm{Re}[\mathcal{W}(s)] \varrho^{I}_{11}(\tau)\ , \\ \frac{\partial \varrho^{I}_{12}}{\partial\tau} &\simeq -2g^2 \int_0^\tau \mathrm{d}s\ \mathrm{Re}[\mathcal{W}(s)] \varrho^{I}_{12}(\tau) + 2g^2 e^{+2i\omega\tau} \int_0^\tau \mathrm{d}s\ \mathrm{Re}[\mathcal{W}(s)] \varrho^{I*}_{12}(\tau)\ . \end{aligned} \tag{1.46}$$

After some integration, the above equations simplify to

$$\begin{aligned} \frac{\partial \varrho^I_{11}}{\partial \tau} &\simeq \frac{g^2 a}{4\pi^2}\bigg(1 - 2\varrho^I_{11}(\tau)\bigg)\ , \\ \frac{\partial \varrho^I_{12}}{\partial \tau} &\simeq \frac{g^2 a}{4\pi^2}\bigg(- \varrho^I_{12}(\tau) + e^{+2i\omega\tau}\varrho^{I*}_{12}(\tau)\bigg)\ . \end{aligned} \tag{1.47}$$

These time-local equations have solutions which can be trusted to all orders in $g^2 a\tau$ where[9]

$$\begin{aligned} \varrho^I_{11}(\tau) &\simeq \frac{1}{2} - \bigg[\frac{1}{2} - \varrho_{11}(0)\bigg]e^{-\frac{g^2 a\tau}{2\pi^2}}\ , \\ \varrho^I_{12}(\tau) &\simeq e^{-\frac{g^2 a\tau}{4\pi^2}}\bigg[\varrho_{12}(0) + \varrho^*_{12}(0)\frac{g^2 a}{4\pi^2\omega}(1 - e^{2i\omega\tau})\bigg]\ , \end{aligned} \tag{1.48}$$

which means that the late-time state attained by the qubit is

$$\lim_{\tau\to\infty}\varrho^I_A(\tau)\ \simeq\ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \tag{1.49}$$

which is a maximally mixed state (to be understood as the high-temperature limit of a thermal state at the Unruh temperature), with the off-diagonals vanishing in the late time limit diminishing all quantum coherence.

To connect the above solution to the earlier discussion of secular growth one notes that the solution $\varrho^I_{11}(\tau)$ found in time-dependent perturbation theory is simply

$$\varrho^I_{11}(\tau) \simeq \varrho_{11}(0) - \bigg(\varrho_{11}(0) - \frac{1}{2}\bigg)\frac{g^2 a\tau}{2\pi^2} + \ldots \qquad \text{(Perturbation Theory)} \tag{1.50}$$

which is easy seen to be the $g^2 a\tau \ll 1$ limit of (1.48). All this means is that perturbation theory correctly captures the departure from the initial state, but fails to work when $g^2 a\tau \gtrsim \mathcal{O}(1)$ (resulting in a secular growth breakdown at late times). Finally it is worth noting that one can easily place the equations (1.47) into Lindblad form, where

$$\frac{\partial \hat{\varrho}_A}{\partial \tau} \ =\ -i\left[\frac{\omega}{2}\hat{\sigma}_3, \hat{\varrho}_A\right] + g^2\sum_{j,k=1}^{3} h_{jk}\bigg(\hat{\sigma}_j\hat{\varrho}_A\hat{\sigma}^\dagger_k - \frac{1}{2}\Big\{\hat{\sigma}^\dagger_k\hat{\sigma}_j, \hat{\varrho}_A\Big\}\bigg) \tag{1.51}$$

where the Kossakowski matrix is in this case simply

$$[h_{jk}] = \begin{bmatrix} \frac{g^2 a}{2\pi^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\ , \tag{1.52}$$

which is easily seen to have non-negative eigenvalues. What this confirms is the logic that when the appropriate limit of the underlying equations is taken, all equations should be well-behaved physically (in this case, we see that the resulting equation of motion preserves positivity without requiring any further approximations, like the rotating-wave approximation — see also [74, 75]).

[9]In solving ϱ^I_{12} one additionally assumes that $\frac{g^2 a}{4\pi^2\omega} \ll 1$ so as to examine a simplifeid non-degenerate regime

1.5 Point Particle EFTs

Wilsonian effective field theories described in §1.1 are organized in terms of a hierarchy of energy scales. In the case that one studies a tiny compact object, in which the details of the short-distance (*aka.* high energy) physics is unknown to us, it is useful to organize an EFT in terms of a hierarchy of length scales. These types of effective theories are known as *point-particle EFTs* (PPEFTs). We briefly describe these since both §3 and §4 use ideas from PPEFTs.

In general PPEFTs can be useful for parametrizing UV physics over length scales of the same order *or* much larger than the size of the compact object (or point particle). Oftentimes these effective theories are applied to atoms with complicated substructure which ordinary QFT methods are hard-pressed to describe (this already becomes an issue for relatively small nuclei, with atomic numbers $Z \gtrsim 2$). For example, if one seeks to compute observables like energy shifts in non-trivial atoms (when finite nuclear size effects are difficult to derive from first principles), PPEFTs are useful in accounting for nuclear-size effects in a model-independent way (with systematic control over theoretical errors at every order in the EFT expansion) [76, 77, 78, 79, 80, 81, 82].

The PPEFT framework need not however be constrained to describe atoms only though: for astrophysical objects these same techniques can be used to help understand the UV physics for compact objects like black holes. For example, PPEFTs have been used to quantify the size of potential UV physics near the event horizon of a black hole in [83, 84] (since quantum gravity effects are thought to modify near-horizon physics). As described later in this thesis, PPEFTs are in part used to develop a toy model of a black hole (which also contains Open EFT features), which is designed to apply over macroscopically large distances compared to the size of the black hole.

1.6 Outline of Thesis

The rest of this thesis is structured as follows: In §2 we study a qubit which hovers near the event horizon of a black hole, and we find its late-time thermal state. In §3 we introduce the hotspot model, which is meant to serve as a toy model of a black hole highlighting the OQS nature of a black hole's degrees of freedom (being split by the presence of an event horizon). This model is understood as a PPEFT, meant to serve as a predictive model only over distances much bigger than the size of the toy black hole. In §4 we study the hotspot model further, while applying two Open EFT techniques on the field describing the exterior of the toy black hole. Finally, we make some concluding remarks in §5.

Chapter 2

Hot Qubits on the Horizon

G. Kaplanek and C. P. Burgess,
"Qubits on the Horizon: Decoherence and Thermalization near Black Holes,"
JHEP **01** (2021), 098
doi:10.1007/JHEP01(2021)098
[arXiv:2007.05984 [hep-th]].

2.1 Preface

We begin by studying the so-called Unruh-deWitt detector model (encountered earlier in §1.4.3), in which a particle detector (*ie.* a qubit in this case) is coupled to a real scalar field ϕ, which in turn is coupled to a classical Schwarzschild spacetime. The qubit is here forced to hover just outside the event horizon of the black hole.

A free real scalar field is the simplest type of quantum field that can interact with a gravitational background. Despite this the mathematics involved in coupling a scalar to Schwarzschild space are immensely complicated — computing correlation functions like $\langle \phi\phi \rangle$ are in general very complicated mode sums [85], which can at best be only approximated by asymptotic forms. A large portion of the work in this chapter has to do with controlling the near-horizon limit of a class of correlation functions (obeying the *Hadamard* form coincidence limit) — in particular for static observers sitting outside the horizon, correlations simplify drastically due to the enormous redshifts factors that contribute near the horizon of the black hole. It is shown that the late time limit $\Delta t / r_s \gg 1$ (with Δt the Schwarzschild coordinate time) is accessible for the remarkably simple self-correlations found along the trajectory of the qubit (with all corrections being parametrically small and controllable).

Armed with these simplified correlations, the Nakajima-Zwanzig master equation is used to take the Markovian approximation using the hierarchy of scales between the Hawking temperature ($\sim r_s$ the correlation time in the environment) and the size of the variation of the qubit's reduced density matrix. One finds the expected result that the qubit thermalizes to the Hawking temperature at very late times (because of the equivalence principle of GR there is also a close relationship here with the uniformly accelerated trajectory studied in [71] and §1.4.3)

The major theme of this paper is control of theoretical error in the late time regime of the qubit: both the complicated correlations of Schwarzschild space, and the Markovian limit of the associated qubit state take a remarkably simple form once one is careful to stay within the described approximation scheme. Another point of emphasis is just how tightly the late time resummation constrains the parameter space here — by being careful to remain within the domain of validity of the Markovian approximation one finds that only small qubit gaps (compared to the Hawking temperature) admit a Markovian description.

2.2 Paper

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Qubits on the horizon: decoherence and thermalization near black holes

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Abstract: We examine the late-time evolution of a qubit (or Unruh-De Witt detector) that hovers very near to the event horizon of a Schwarzschild black hole, while interacting with a free quantum scalar field. The calculation is carried out perturbatively in the dimensionless qubit/field coupling g, but rather than computing the qubit excitation rate due to field interactions (as is often done), we instead use Open EFT techniques to compute the late-time evolution to all orders in g^2t/r_s (while neglecting order g^4t/r_s effects) where $r_s = 2GM$ is the Schwarzschild radius. We show that for qubits sufficiently close to the horizon the late-time evolution takes a simple universal form that depends only on the near-horizon geometry, assuming only that the quantum field is prepared in a Hadamard-type state (such as the Hartle-Hawking or Unruh vacua). When the redshifted energy difference, ω_∞, between the two qubit states (as measured by a distant observer looking at the detector) satisfies $\omega_\infty r_s \ll 1$ this universal evolution becomes Markovian and describes an exponential approach to equilibrium with the Hawking radiation, with the off-diagonal and diagonal components of the qubit density matrix relaxing to equilibrium with different characteristic times, both of order r_s/g^2.

Keywords: Effective Field Theories, Black Holes, Renormalization Group, Renormalization Regularization and Renormalons

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1 Introduction and summary

Making reliable predictions can be difficult at the best of times. But reliably predicting behaviour at very late times is notoriously hard. What makes it difficult is the inevitable breakdown of perturbative methods that happens at late times; a huge handicap given that perturbative methods dominate a theorist's intellectual toolbag.

Perturbative methods break down for a simple reason: if a Hamiltonian can be written $H = H_0 + gH_1$ for some small dimensionless parameter g, then there is always a time beyond which the time-evolution operator $U(t, t_0) = \exp[-iH(t-t_0)]$ is not well-described by perturbing in g. The time where this breakdown occurs scales as an inverse power of g, but is eventually exceeded no matter how small g might happen to be. Like death and taxes, perturbative failure is just a matter of time.

This might not be bothersome if such late times were never of interest. However many important physical processes occur on long time-scales like these. For example, even when individual photons interact weakly with individual atoms, phenomena like refraction and reflection (where 100% of photons scatter in one direction or another) occur on time-scales long enough to invalidate perturbing in electromagnetic interactions. Thermalization is another phenomenon whose time-scales are very large and scale inversely with coupling strengths like g.

In this paper we explore similar late-time issues for interacting quantum systems moving in gravitational fields. That similar phenomena must exist — particularly in the presence of horizons — is clear given the thermal nature of quantum fields in these spacetimes [1–6]. Test probes should be expected to thermalize in such environments, and any description of this process should share all of the late-time complications that thermalization calculations always have [7–14]. In this paper we show this is true for quantum systems exterior to a Schwarzschild black hole, extending our own earlier work that does so for spacetimes with Rindler [15, 16] and de Sitter horizons [17].

The reason for doing so is not because this kind of thermalization is soon likely to be observed. On the contrary, it is worth doing because the tools used are informative in their own right. In particular, they show how standard techniques used to describe late-time behaviour in optics and thermal physics apply equally well in gravitational settings [18–28]. This makes them potentially relevant to late-time puzzles known to occur in gravity, such as the problem of secular growth in cosmological spacetimes [29–57] (for reviews see [58, 59]) and to problems like information loss [60] or 'firewall' problems [61, 62] in black-hole physics (for reviews see [63, 64]).

In this paper we compute the late-time evolution of a two-level quantum system (i.e. a qubit or Unruh detector [5, 65, 66]) that hovers at fixed radius $r = r_0$ above the event horizon of a Schwarzschild black hole while interacting with a quantum scalar field. We do so perturbatively in the dimensionless coupling strength g with which the qubit interacts with the quantum field. We show that if ω_∞ is the redshifted splitting of the two qubit energy levels, and if $\omega_\infty\, r_s \ll 1$ where $r_s = 2GM$ is the usual Schwarzschild radius, and if the scalar-field mass also satisfies $m\, r_s \ll 1$, then such a qubit has universal late-time behaviour (for $t \gtrsim r_s/g^2$) provided that it sits sufficiently close to the event horizon: $0 < r_0 - r_s \ll r_s$.

Not surprisingly, this universal evolution describes the evolution of the qubit towards an asymptotic thermal state whose temperature equals the Hawking temperature $T = T_H := (4\pi r_s)^{-1}$. Perhaps more surprisingly we show that this approach to equilibrium is also very robust, occurring exponentially with two different thermalization time-scales proportional to

$$\xi = \frac{4\pi \tanh\left(2\pi r_s\, \omega_\infty\right)}{g^2 \omega_\infty} \simeq \frac{8\pi^2 r_s}{g^2} + \cdots \quad \text{since } \omega_\infty r_s \ll 1\,. \tag{1.1}$$

This evolution is robust in the sense that it depends only on the qubit/field coupling strength, g, and on the background geometry for any quantum state whose Wightman function has the standard 'Hadamard' form [67–69] at small field separations: i.e. it satisfies (3.13), reproduced here as

$$G_\Omega(x, x') = \frac{1}{8\pi^2\, \sigma(x, x')} + \cdots\,, \tag{1.2}$$

where $\sigma(x, x')$ is half the square of the geodesic distance between spacetime points x and x'. In particular, eq. (1.1) applies equally well if the quantum field is prepared in either the Hartle-Hawking or Unruh vacua, and is independent of the scalar-field mass in the mass range $m\, r_s \ll 1$.

It has been known for some time that Hadamard behaviour suffices for deriving the steady-state Hawking flux around Schwarzschild black holes [70], and our results extend this conclusion to the approach to equilibrium for quantum probes. We remark in passing that our results differ from early — and some recent — calculations of Unruh detectors in Schwarzschild geometries [71–76], which often compute qubit excitation rates, finding results that differ when the field is prepared in different states (such as the Hartle-Hawking or Unruh vacua). These calculations usually compute the rate with which a qubit is excited out of its ground state, as opposed to the qubit's late-time approach to its asymptotic thermal state (as is computed here). Although the excitation rate can be accessed perturbatively in g, more effort is required to obtain the approach to equilibrium since the time-scale involved is of order r_s/g^2.

We are able to make reliable predictions using arguments of Open Effective Field Theories (Open EFTs) [24, 26, 28]. As is explained in more detail in [16], these recast techniques from elsewhere in physics into an effective field theory language that is easily adapted to gravitational systems. In essence these arguments have a renormalization-group like structure: one sets up a differential evolution equation for the object of interest (in this case the reduced density matrix for the qubit) whose domain of validity is larger than the integrated evolution from which it is derived. That is, one explicitly evolves the system using perturbation theory starting from an arbitrary initial time, t_0. Although perturbative evolution can only be used to evolve a limited way forward in time, say from t_0 to t_1, within this window the result can be differentiated with respect to time to derive a differential evolution equation.

If this evolution equation itself makes no specific reference to t_0 then the same construction could equally well be used to derive the same evolution equation starting at t_1, with perturbative validity out to t_2, and again starting at t_2 and so on. Whenever this can be done the solutions to the differential evolution equation can be valid on the union of each of these derivation intervals. If $g \ll 1$ is the small perturbative expansion parameter then this process ends up resumming all orders in $g^2 t$, say, but neglecting contributions in the evolution[1] that are of order $g^4 t$. As a result the solutions found this way can be trusted even when $t \sim \mathcal{O}(r_s/g^2)$.

One reason to explore the simple qubit systems considered here is to make this construction very explicit, making it easier to understand. The starting point for the argument is the Nakijima-Zwanzig equation [77, 78], which is a general evolution equation for the reduced 2×2 density matrix, $\boldsymbol{\varrho}(t)$, of the qubit. It is obtained by tracing over the Liouville equation describing the evolution of the full qubit/field system, and then eliminating that part of the density matrix that describes the non-qubit degrees of freedom. The result is an integro-differential evolution equation that is useful because it refers only to the qubit's reduced density matrix and not to the other degrees of freedom, which appear only implicitly through correlation functions of $H_{\rm int}$. Although the Nakajima-Zwanzig equation does not in itself automatically allow perturbative time-evolution to be extended out to very late times, it provides a useful starting point for identifying situations where this can be done.

[1]Order $g^4 t$ evolution is similarly predicted using a more accurate evolution equation, and so on.

As is true for most effective field theories, relative s implicity comes only when there is a hierarchy of scales that can be exploited. The important hierarchy arises in this case if the field correlation function $\langle H_{\rm int}(t)H_{\rm int}(t')\rangle$ falls off to zero for $|t-t'| > \zeta$, for some characteristic time-scale ζ. In this case the useful hierarchy arises when exploring time-evolution over much longer time-scales $\Delta t \gg \zeta$. Access to late times can happen if the Nakajima-Zwanzig equation remains sufficiently simple once expanded in powers of this ratio $\zeta/\Delta t$.

The qubit example studied here shows in detail how this can happen: the leading terms in the Nakajima-Zwanzig equation become Markovian, in the sense that $\partial_t \boldsymbol{\varrho}(t)$ depends only on $\boldsymbol{\varrho}(t)$ and not on the details of its past history prior to time t. Markovian behaviour of this form emerges for qubits near a black hole once $\Delta t \gg r_s$ (at least this is true when the redshifted energy difference ω_∞ between the two qubit energy levels — as seen by a static observer looking at the qubit far from the black hole — satisfies $\omega_\infty r_s \ll 1$), Evolution to all orders in $g^2 t$ is then described by a Lindblad equation [79, 80]. (Some implications of Lindblad evolution in Schwarzschild geometries are also explored in [81–87].) By deriving the Lindblad equation as a limit of the Nakajima-Zwanzig equation for this system, we are able to assess its domain of validity.

This paper. The rest of this paper is organized as follows. The next section, section 2, sets up the system whose late-time near-horizon evolution is to be computed. In particular section 2 defines our qubit/quantum-field system for static spacetimes, and then briefly explores the properties of qubit trajectories that hover at fixed positions just above a Schwarzschild black hole.

Section 3 follows this with a brief description of how reduced density matrices are evolved in open systems, describing the Nakajima-Zwanzig equation whose solutions govern the qubit's late-time behaviour. Since at lowest nontrivial order the quantum field enters into the qubit evolution only through its Wightman function, we also summarize in section 3 the near-horizon form for this function for field states that satisfy the Hadamard form for small separations.

Finally, section 4 shows how the near-horizon limit of the Wightman function allows the Nakajima-Zwanzig equation to be approximated by a Markov process, describing the late-time exponential decay towards a Hawking-temperature thermal state. The time-scale for this approach to equilibrium is computed for qubits asymptotically close to the horizon, and found to be universal in the sense that it is determined only by qubit properties and the black-hole geometry. Provided $m\, r_s \ll 1$ this rate is largely independent of the details of the quantum field, and assumes only that it is prepared in a Hadamard state. In particular the approach to equilibrium is the same when the field is prepared in either a Hartle-Hawking or Unruh state.

2 Qubits in Schwarzschild

This section sets up the framework — a qubit/field system and the spacetime through which the qubit moves — that is used to perform the calculations to follow.

2.1 The qubit/scalar system

The system whose evolution we follow consists of a real massive scalar field $\phi(x)$ coupled to a single two-level qubit through the action $S = S_B + S_Q + S_{\rm int}$, where S_B describes a self-interacting quantum field

$$S_B = -\frac{1}{2}\int \mathrm{d}^4x\,\sqrt{-g}\Big[g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + (m^2+\xi R)\phi^2 + \frac{\lambda}{12}\,\phi^4\Big]\,, \tag{2.1}$$

within a background metric[2] given as in (2.20) or (2.22). For this paper we neglect self-interactions ($\lambda = 0$), though we briefly comment in the conclusions on how things can change in their presence. The coupling ξ plays no role because Schwarzschild is Ricci flat, and for reasons to be clear below the mass m is assumed to satisfy $m\,r_s \ll 1$.

The free qubit action is given by[3]

$$S_Q = \int \mathrm{d}^4x \int \mathrm{d}s\left[\frac{i}{2}\,\mathcal{Z}\,\mathfrak{z}^i\,\dot{\mathfrak{z}}_i - \sqrt{-\dot{y}^2}\left(\omega_0 - \frac{i\hat{\omega}}{4}\,\epsilon^{ijk}u_i\,\mathfrak{z}_j\mathfrak{z}_k\right)\right]\delta^4[x-y(s)]\,, \tag{2.2}$$

where $\mathfrak{z}_i(s)$ are classical Grassman variables (with $i = 1, 2, 3$) with $\dot{\mathfrak{z}}_i := \mathrm{d}\mathfrak{z}_i/\mathrm{d}s$ and $\mathfrak{z}^i := \delta^{ij}\mathfrak{z}_j$. The quantities $\mathcal{Z}$, ω_0 and $\hat{\omega}$ and u^i are real parameters (with $u_iu^i = 1$), with $\mathcal{Z}$ eventually absorbed into the $\mathfrak{z}_i$ to obtain a convenient normalization that simplifies later formulae.[4] The integral over d^4x is trivially done using the delta-function, and reveals that the integration is over a specific timelike world line $x^\mu = y^\mu(s)$, along which the qubit moves through the ambient spacetime.[5] Here s is a parameter along this world line, and the quantity

$$\dot{y}^2 := g_{\mu\nu}[y(s)]\,\dot{y}^\mu\,\dot{y}^\nu \tag{2.3}$$

is what is required to ensure that S_Q is invariant under reparameterizations of s. It is usually convenient to fix this freedom by choosing proper time, τ, along the curve as the parameter, in which case $\dot{y}^2 = -1$.

Interactions beween $\mathfrak{z}$ and ϕ are assumed to take the form

$$S_{\rm int} = \frac{i\hat{g}}{2}\int \mathrm{d}^4x\int \mathrm{d}\tau\;\phi\,\epsilon^{ijk}n_i\mathfrak{z}_j\mathfrak{z}_k\;\delta^4[x-y(\tau)] = \frac{i\hat{g}}{2}\int \mathrm{d}\tau\;\phi[y(\tau)]\,\epsilon^{ijk}n_i\mathfrak{z}_j(\tau)\mathfrak{z}_k(\tau)\,, \tag{2.4}$$

where $\hat{g}$ and n^i are real coupling constants, with $n_in^i = 1$, and our analysis is ultimately performed perturbatively in $\hat{g}$.

[2] Nothing precludes also quantizing the fluctuations of the metric about the given background, using standard EFT arguments [88–91], though for simplicity we do not do so since we do not expect this not to alter our main point. In principle, dropping metric fluctuations can justified quantitatively by working with $N \gg 1$ scalar fields and computing in the leading large-N limit.

[3] Terms linear in $\mathfrak{z}_i$ do not appear in this action because we require the classical Grassmann action to be Grassman-even.

[4] Hats appear on couplings like $\hat{\omega}$ and $\hat{g}$ to distinguish them from the corresponding quantities once appropriate powers of $\mathcal{Z}$ have been absorbed into $\mathfrak{z}_i$.

[5] For real systems $y^\mu(s)$ is itself a dynamical variable to be quantized, but for simplicity we ignore this complication here (treating the qubit trajectory as being specified), since it does not affect our later discussions.

Quantization. Working in the interaction picture, quantization of $\mathfrak{z}_i$ and ϕ is performed as if they did not interact, with interactions included in powers of g (and λ) once time-evolution is evaluated.

For $\mathfrak{z}_i$ this quantization goes through as usual [92], keeping in mind this is a constrained system. To see why, recall that the canonical momenta are given by

$$\mathfrak{p}^i := \frac{\delta S_Q}{\delta \dot{\mathfrak{z}}_i} = \frac{i\mathcal{Z}}{2}\,\mathfrak{z}^i\,. \tag{2.5}$$

Because this cannot be solved for $\dot{\mathfrak{z}}_i$ as a function of the $\mathfrak{p}^j$ it is instead regarded as a constraint: $\mathfrak{p}^i - \frac{i}{2}\mathcal{Z}\,\mathfrak{z}^i = 0$. The qubit hamiltonian (generating evolution in proper time τ) becomes

$$\mathfrak{h} = \mathfrak{p}^i\dot{\mathfrak{z}}_i - \left[\frac{i}{2}\,\mathfrak{z}^i\,\dot{\mathfrak{z}}_i - \left(\omega_0 - \frac{i\hat{\omega}}{4}\epsilon^{ijk}u_i\,\mathfrak{z}_j\mathfrak{z}_k\right)\right] = \omega_0 - \frac{i\hat{\omega}}{4}\epsilon^{ijk}u_i\,\mathfrak{z}_j\mathfrak{z}_k\,. \tag{2.6}$$

The canonical quantization conditions turn out to imply that the anticommutator of $\mathfrak{p}^i$ and $\mathfrak{z}_j$ is proportional to δ^i_j, and the parameter $\mathcal{Z}$ can be chosen to ensure that the quantum version of the Grassmann condition becomes

$$\Big\{\mathfrak{z}_i\,,\mathfrak{z}_j\Big\} = 2\,\delta_{ij}\,. \tag{2.7}$$

The space of quantum states furnishes a representation of this algebra, and for a 2-level qubit this representation is two-dimensional. The required operator representations for the $\mathfrak{z}_i$ therefore are the Pauli matrices

$$\mathfrak{z}_1 = \boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \mathfrak{z}_2 = \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \mathfrak{z}_3 = \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{2.8}$$

With this choice, properties of the Pauli matrices ensure that $\epsilon^{ijk}\mathfrak{z}_j\mathfrak{z}_k = 2i\mathfrak{z}^i$, and so defining coordinates so that $\mathbf{u}\cdot\boldsymbol{\sigma} = \boldsymbol{\sigma}_3$ then allows (2.6) to be written explicitly as

$$\mathfrak{h} = \omega_0\,\mathbb{I} + \frac{\omega}{2}\,\boldsymbol{\sigma}_3\,, \tag{2.9}$$

where $\mathbb{I}$ is the 2×2 unit matrix, and hats are dropped on ω when variables are normalized so that (2.7) holds. Eq. (2.9) reveals the free-qubit energy eigenvalues to be $\omega_0 \pm \frac{1}{2}\,\omega$ and so ω_0 is their mean energy while ω (which we take to be positive) gives their level splitting (as measured by an observer whose time is the qubit's proper time, τ).

With this representation for $\mathfrak{z}_i$ the qubit/field interaction (2.4) becomes

$$S_{\rm int} = -g\int \mathrm{d}\tau\;\phi[y(\tau)]\,n^i\boldsymbol{\sigma}_i = -g\int \mathrm{d}\tau\;\phi[y(\tau)]\,\boldsymbol{\sigma}_1\,, \tag{2.10}$$

where the second equality specializes to the case where n^i is perpendicular to u^i (and so can be chosen to lie along the '1' axis).

In the absence of couplings ($\lambda = g = 0$) the scalar field is quantized in the usual fashion for a static curved space [6]. The interaction-picture field equation is

$$\big(-\Box + m^2 + \xi R\big)\phi = 0\,, \tag{2.11}$$

and so for a static geometry one expands

$$\phi(x) = \sum_n \Big[u_n(x)\, \mathfrak{a}_n + u_n^*(x)\, \mathfrak{a}_n^* \Big] \,, \tag{2.12}$$

where u_n simultaneously satisfies $\mathfrak{L}_t u_n = -i\omega_n u_n$ and eq. (2.11), where $\mathfrak{L}_t$ is the symmetry generator in the timelike direction along which the metric is static. Canonical commutation relations imply the creation and annihilation operators satisfy $[\mathfrak{a}_n \,, \mathfrak{a}_m^*] = \delta_{mn}$. As mentioned earlier, for applications to Schwarzschild our interest is in small masses,[6] $m\, r_s \ll 1$, and Ricci-flatness makes the term ξR drop out of subsequent discussion.

The scalar field hamiltonian (including self-interactions) is easily computed in the presence of any spacetime metric of the form

$$\mathrm{d}s^2 = -f\, \mathrm{d}t^2 + \gamma_{ab}\, \mathrm{d}x^a\, \mathrm{d}x^b \tag{2.13}$$

where $g_{at} = 0$ and both $g_{tt} = -f$ and the spatial metric $g_{ab} = \gamma_{ab}$ are t-independent. The hamiltonian density (for the generator of evolution in t) is then given by

$$\mathcal{H} = \Pi \partial_t \phi - \mathcal{L} \tag{2.14}$$

where $\mathcal{L}$ is the lagrangian density from (2.1) and the canonical momentum is

$$\Pi := \frac{\delta S_B}{\delta \partial_t \phi} = \sqrt{\frac{\gamma}{f}}\, \partial_t \phi \,, \tag{2.15}$$

where (2.1) is again used, together with the metric (2.13), to evaluate the derivative. Therefore the scalar-field hamiltonian density, $\mathcal{H}$, becomes

$$\mathcal{H} = \sqrt{f\gamma}\, \left[\frac{(\partial_t \phi)^2}{2f} + \frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2}(m^2 + \xi R)\phi^2 + \frac{\lambda}{4!}\phi^4 \right] . \tag{2.16}$$

Total energy. We can now assemble everything to identify the total field/qubit hamiltonian, leading to the following sum:

$$H = H_0 + H_{\mathrm{int}} \tag{2.17}$$

where the 'free' hamiltonian is the sum of the two free hamiltonians constructed above

$$H_0 = \mathcal{H} \otimes \mathbb{I} + \mathcal{I} \otimes \mathfrak{h}\, \frac{\mathrm{d}\tau}{\mathrm{d}t} \,. \tag{2.18}$$

Here $\mathcal{I}$ is the unit operator in the scalar-field state-space, while $\mathcal{H}$ and $\mathfrak{h}$ are as given in eqs. (2.16) and (2.9). The factor $\mathrm{d}\tau/\mathrm{d}t$ is required because $\mathfrak{h}$ generates translations in proper time τ while H_0 (and $\mathcal{H}$) are defined to generate translations in t.

Since the interaction Lagrangian does not involve time derivatives its contribution to the hamiltonian is simple to write down, starting from (2.10)

$$H_{\mathrm{int}} = g\, \phi[y(\tau)] \otimes n^i \boldsymbol{\sigma}_i\, \frac{\mathrm{d}\tau}{\mathrm{d}t} = g\, \phi[y(\tau)] \otimes \boldsymbol{\sigma}_1\, \frac{\mathrm{d}\tau}{\mathrm{d}t} \,. \tag{2.19}$$

[6]Physically, once the scalar mass becomes much bigger than the temperature its states become exponentially rarely occupied, leading one to expect them to decouple from qubit evolution. This expectation can be very explicitly verified for simple systems such as an accelerating qubit moving through flat spacetime [16].

In what follows we compute the implications of this interaction out to second order in g. The nature of this perturbation theory depends on the relative size of ω and the $\mathcal{O}(g^2)$ corrections to the qubit energy levels, and for simplicity we work in the regime where these corrections are much smaller than the qubit's zeroth-order level splitting, a restriction that eventually leads to the parameter conditions summarized in table 1.

2.2 Near-horizon geometry

Our interest lies in the near-horizon limit of the exterior of a spinless black hole, defined in Schwarzschild coordinates by

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\,, \tag{2.20}$$

where $r > r_s$, where $r = r_s := 2GM$ defines the event horizon. These coordinates are useful because they fall into the category defined in (2.13), with t being the static time on which the metric does not depend, and so $\mathfrak{L}_t = \partial_t$. The (Kretchsmann) curvature invariant for this geometry is $R_{\rho\sigma\tau\kappa}R^{\rho\sigma\tau\kappa} = 12r_s^2/r^6$, and so is nonsingular at $r = r_s$.

Schwarzschild coordinates famously break down at the horizon, in the vicinity of which Kruskal coordinates, (T, X, θ, ϕ), defined by

$$\begin{aligned} T &= \sqrt{\frac{r}{r_s} - 1}\;\exp\left(\frac{r}{2r_s}\right)\sinh\left(\frac{t}{2r_s}\right) \\ X &= \sqrt{\frac{r}{r_s} - 1}\;\exp\left(\frac{r}{2r_s}\right)\cosh\left(\frac{t}{2r_s}\right)\,, \end{aligned} \tag{2.21}$$

are more useful. In terms of these the line element becomes

$$\mathrm{d}s^2 = \frac{4r_s^3}{r}\,e^{-r/r_s}\left(-\mathrm{d}T^2 + \mathrm{d}X^2\right) + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \tag{2.22}$$

where now $r = r(X, T)$ is the implicit function of X and T given by solving

$$X^2 - T^2 = \left(\frac{r}{r_s} - 1\right)e^{r/r_s}\,, \tag{2.23}$$

and so is given by

$$r(X, T) = r_s\Big[1 + \mathcal{W}(z)\Big] \quad \text{where} \quad z := \frac{1}{e}(X^2 - T^2)\,, \tag{2.24}$$

and $\mathcal{W}$ is the Lambert W function defined by $\mathcal{W}(z)\exp[\mathcal{W}(z)] = z$ (which has a unique real solution for $z > 0$ and two real branches for $-e^{-1} < z < 0$).

Although these coordinates are well-behaved at the horizon, the spatial geometry at fixed T is T-dependent. This is a reflection of the coordinate-independent statement that the metric is only static outside of the horizon (where it can be rewritten as (2.20)). Inside the horizon the metric's symmetry directions are all spacelike.

Hovering world-lines. The qubits whose late-time evolution we follow are chosen to hover at fixed $r = r_0 > r_s$ above the event horizon. This is clearly not a geodesic and so the qubit can be maintained along this trajectory through the action of some non-gravitational force, whose detailed nature need not concern us here.

Consider two events on this world-line that are distinguished by two values, τ_1 and τ_2, of the qubit's proper time; separated by $\Delta\tau := \tau_2 - \tau_1 > 0$. These two events are separated by a redshifted time Δt for hovering observers situated at spatial infinity, with (2.20) implying that

$$\Delta\tau := \int_{t_1}^{t_2} \mathrm{d}t \sqrt{-g_{\mu\nu}\frac{\mathrm{d}y_0^\mu}{\mathrm{d}t}\frac{\mathrm{d}\dot{y}_0^\nu}{\mathrm{d}t}} = \Delta t\sqrt{1-\frac{r_s}{r_0}}\,, \tag{2.25}$$

where $y_0^\mu(t)$ denotes the curve along which only t varies, with r, θ and ϕ all fixed. Notice, for future use, that $\Delta\tau \ll \Delta t$ when $0 < r_0 - r_s \ll r_s$.

Both of these intervals differ from the geodesic separation of these two points,

$$\Delta s := \int_{t_1}^{t_2} \mathrm{d}t \sqrt{-g_{\mu\nu}\frac{\mathrm{d}y_g^\mu}{\mathrm{d}t}\frac{\mathrm{d}\dot{y}_g^\nu}{\mathrm{d}t}}\,, \tag{2.26}$$

where the subscript 'g' indicates that integration is evaluated along the geodesic $y_g^\mu(t)$ that satisfies

$$\ddot{y}_g^\mu + \Gamma^\mu_{\nu\lambda}\dot{y}_g^\nu\dot{y}_g^\lambda = 0 \tag{2.27}$$

as well as $\theta(t) = \theta_0$, $\phi(t) = \phi_0$ for all t while $r(t_1) = r(t_2) = r_0$. Because this is a geodesic it must describe the longest time interval as measured along any timelike curve that connects the two events, so $\Delta s > \Delta\tau$. Such a geodesic is possible if $\mathrm{d}r/\mathrm{d}t(t_1) > 0$ is chosen appropriately, since then any freely falling body initially moves radially away from the horizon before eventually turning back and falling into the black hole.

In what follows it proves more convenient to work with the Synge world function, $\sigma(x_1, x_2)$, defined for timelike geodesics by [93–95]

$$\sigma(x_1, x_2) = -\frac{1}{2}\,(\Delta s)^2\,, \tag{2.28}$$

since this has an integral form that is easier to manipulate (see appendix A).

For later purposes we are interested in a limit that simultaneously has a small invariant interval, $\Delta s \ll r_s$, but corresponds to late times $\Delta t \gg r_s$. We remark in passing that the above formulae show that both of these can be simultaneously true provided we pick $r_0 > r_s$ sufficiently close to the horizon so that

$$1 - \frac{r_s}{r_0} \ll 1\,. \tag{2.29}$$

Being close to the horizon suffices because the curve that hovers at fixed $r = r_s$ *is* a geodesic, although it is a null geodesic — for which $\Delta s = 0$ — rather than a timelike one.

3 Time evolution in open systems

We return now to the evolution of the qubit that hovers just above the horizon while interacting with the quantum field. Our interest is in how this qubit responds to the fluctuations of the quantum field, and in how this response becomes universal in the late-time limit very near the horizon.

Since it is only the qubit's behaviour that is to be predicted, it is convenient to trace out the scalar field from the system density matrix, and work instead only with the qubit's 2×2 reduced density matrix, defined as

$$\boldsymbol{\varrho}(t) := \underset{\phi}{\mathrm{Tr}}\left[\rho(t)\right] \tag{3.1}$$

where $\rho(t)$ is the total — i.e. the combined field/qubit — density matrix, and the trace is over the scalar-field part of the Hilbert space.

When needed we assume the field and qubit to be initially uncorrelated,

$$\rho_0 := \rho(t_{\mathrm{i}}) = \Omega \otimes \boldsymbol{\varrho_0} \tag{3.2}$$

where $\boldsymbol{\varrho_0}$ defines the initial qubit state and Ω is the density matrix for the quantum field. Three commonly made choices for Ω might be the Hartle-Hawking state, $\Omega_H := |\mathbb{H}\rangle\langle\mathbb{H}|$, the Unruh state $\Omega_U := |\mathbb{U}\rangle\langle\mathbb{U}|$ or the Boulware state, $\Omega_B := |\mathbb{B}\rangle\langle\mathbb{B}|$. These are all pure states that are candidate vacua for the field, with $|\mathbb{H}\rangle$ corresponding to the vacuum in the presence of a black hole that is in equilibrium with a bath of radiation prepared at the Hawking temperature, while $|\mathbb{U}\rangle$ is the late-time vacuum for a black hole that forms in isolation.

Time evolution for $\boldsymbol{\varrho}$ is in principle determined by the evolution of the full system's density matrix, which in the interaction picture satisfies

$$\partial_t \rho(t) = -i\Big[V(t)\,,\rho(t)\Big]\,, \tag{3.3}$$

where $V(t) := e^{iH_0 t} H_{\mathrm{int}} e^{-iH_0 t}$. Eq. (3.3) has a standard perturbative solution

$$\rho(t) = \rho_0 - i\int_{t_{\mathrm{i}}}^{t} \mathrm{d}s\Big[V(s)\,\rho_0\Big] - \frac{1}{2}\int_{t_{\mathrm{i}}}^{t}\mathrm{d}s_1\int_{t_{\mathrm{i}}}^{s_1}\mathrm{d}s_2\,\Big[V(s_2)\,,\Big[V(s_1)\,\rho_0\Big]\Big] + \cdots\,, \tag{3.4}$$

given the initial condition $\rho(t_{\mathrm{i}}) = \rho_0$.

As is discussed at great length elsewhere — see for example [16, 17, 28] and references therein — there are two major obstacles to using eqs. (3.3) or (3.4) to predict $\boldsymbol{\varrho}(\tau)$ at late times.

1. At first sight one could trace (3.3) over the scalar-field sector to obtain $\partial_t\boldsymbol{\varrho}$, but the result is hard to solve for $\boldsymbol{\varrho}(\tau)$, because the dependence of the right-hand side on $\boldsymbol{\varrho}$ is only implicitly given through its dependence on the full density matrix ρ.

2. The solution (3.4) does not have this same difficulty because in his equation we may use $\rho_0 = \Omega \otimes \boldsymbol{\varrho_0}$. The problem with (3.4) is that the perturbative approximation on which it relies systematically breaks down at very late times (in the present example this breakdown occurs at times of order $t \sim r_s/g^2$).

The Nakajima-Zwanzig equation [77, 78] provides the solution to problem (1) above, and this is useful because the result shows how to find solutions that are not afflicted by problem (2), in that they allow reliable perturbative predictions even when t is so large that $g^2 t$ cannot be neglected relative to r_s.

3.1 The Nakajima-Zwanzig equation

The logic of the Nakajima-Zwanzig equation is to project the full density matrix onto the reduced density matrix and its complement:

$$\boldsymbol{\varrho} = \mathcal{P}(\rho) \quad \text{and} \quad \Xi := \mathcal{Q}(\rho) \tag{3.5}$$

for some projection operator $\mathcal{P}^2 = \mathcal{P}$ and the second definition uses $\mathcal{Q} := 1 - \mathcal{P} = \mathcal{Q}^2$. Since the time-evolution equation (3.3) for ρ is a linear equation it can be turned into a pair of coupled linear evolution equations for the two quantities $\boldsymbol{\varrho}$ and Ξ. Eliminating Ξ from this system gives the Nakajima-Zwanzig equation: an evolution equation that involves only $\boldsymbol{\varrho}$, but is nonlocal in time due to the elimination of Ξ. Because this is essentially a linear problem, it can be solved in great generality [77, 78].

As applied to the current example, following identical steps as given in [16, 17] leads to the following result at second order in the coupling g:

$$\begin{aligned}\frac{\partial \boldsymbol{\varrho}^I(\tau)}{\partial \tau} \simeq g^2 \int_0^\tau \mathrm{d}s\, \Big(& G_\Omega(\tau,s)\, \big[\mathfrak{m}^I(s)\,\boldsymbol{\varrho}^I(s), \mathfrak{m}^I(\tau)\big] \\ & + G^*_\Omega(\tau,s)\big[\mathfrak{m}^I(\tau), \boldsymbol{\varrho}^I(s)\,\mathfrak{m}^I(s)\big]\Big) - i\left[\frac{\delta\omega}{2}\boldsymbol{\sigma_3}, \boldsymbol{\varrho}^I(\tau)\right],\end{aligned} \tag{3.6}$$

where $\boldsymbol{\varrho}^I(\tau) = e^{+i\mathfrak{h}\tau}\boldsymbol{\varrho}(\tau)\, e^{-i\mathfrak{h}\tau}$ is the reduced density matrix in the interaction-picture representation and so similarly $\mathfrak{m}^I(\tau) := e^{+i\mathfrak{h}\tau}\boldsymbol{\sigma_1} e^{-i\mathfrak{h}\tau}$ (and conventions generally follow [16, 17]). At this point several features of (3.6) bear explanation.

- First, notice that eq. (3.6) gives the evolution of $\boldsymbol{\varrho}$ as a function of proper time along the qubit trajectory, and does so despite its derivation starting from the Liouville equation (3.3), which is phrased in terms of the geometry's static time coordinate, t. This occurs in detail because of the time-dilation factors $\mathrm{d}\tau/\mathrm{d}t$ that appear in the Hamiltonian in eqs. (2.18) and (2.19).

- Second, in expression (3.6) the quantity $G_\Omega(\tau,s)$ represents the scalar-field Wightman functions
$$G_\Omega(\tau,s) := \operatorname*{Tr}_\phi\Big(\Omega\ \phi[y(\tau)]\phi[y(s)]\Big)\,, \tag{3.7}$$
evaluated at two places along the qubit trajectory, $y^\mu(s)$, and we use the property $G_\Omega(s,\tau) = G^*_\Omega(\tau,s)$ for the Wightman function of a real scalars. These are fairly complicated functions when evaluated in Schwarzschild spacetime and are usually given implicitly in terms of a sum over mode functions [65, 71, 81, 96–102]. In what follows we choose a near-horizon trajectory along which they take a simple form.

- Third, the final term in (3.6) comes from a counter-term interaction, obtained by replacing $\omega \to \omega_{\text{bare}} = \omega + \delta\omega$ in $\mathfrak{h}$, with $\delta\omega$ regarded as being $\mathcal{O}(g^2)$. This counter-term is required because the qubit/field interaction shifts the inter-level energy spacing, and so makes the parameter ω appearing in $\mathfrak{h}$ no longer equal to this spacing. If the parameter in $\mathfrak{h}$ is therefore instead called $\omega_{\rm bare}$ then ω remains the physical spacing of qubit levels if $\delta\omega$ is chosen to cancel the order g^2 qubit energy shift.[7]

- Finally, notice that (3.6) would agree with the time derivative of (3.4) if in the right-hand side one were to replace $\varrho^I(s)$ with its initial condition ϱ^I_0. Furthermore, such a replacement at face value seems to be compulsory, because the difference between $\varrho^I(s)$ and ϱ^I_0 is higher order in g. It is this assumption that $\varrho^I(s)$ and ϱ^I_0 are only perturbatively different that breaks down at very late times, and when it does it is (3.6) that is the more reliable equation.

Since $\boldsymbol{\varrho}$ is a Hermitian 2×2 matrix with unit trace, its elements $\varrho_{21} = \varrho^*_{12}$ and $\varrho_{22} = 1 - \varrho_{11}$ can be eliminated from (3.6) to leave the following two decoupled evolution equations for the remaining two variables ϱ_{11} and ϱ_{12}:

$$\begin{aligned} \frac{\partial \varrho^I_{11}}{\partial \tau} = g^2 \int_{-\tau}^{\tau} {\rm d}s\ e^{-i\omega s} G_\Omega(\tau, \tau - s) \\ -4g^2 \int_0^\tau {\rm d}s\ {\rm Re}\big[G_\Omega(\tau, \tau - s)\big] \cos(\omega s) \varrho^I_{11}(\tau - s)\,, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \frac{\partial \varrho^I_{12}}{\partial \tau} = -i\delta\omega\ \varrho^I_{12}(\tau) - 2g^2 \int_0^\tau {\rm d}s\ {\rm Re}\big[G_\Omega(\tau, \tau - s)\big] e^{+i\omega s} \varrho^I_{12}(\tau - s) \\ +2g^2 e^{+2i\omega\tau} \int_0^\tau {\rm d}s\ {\rm Re}\big[G_\Omega(\tau, \tau - s)\big] e^{-i\omega s} \varrho^{I*}_{12}(\tau - s)\,. \end{aligned} \tag{3.9}$$

These two equations perform a change of integration variables $s \to \tau - s$ relative to (3.6), since the result takes a particularly simple form when the Wightman functions are translation invariant in τ. These are the main equations on which the remainder of the paper rely.

We note in passing that it can happen that the appearance in the above equations of the oscillatory factors $e^{\pm i\omega s}$ and $e^{i\omega\tau}$ can complicate the construction of their solutions. Such terms can be removed from an ordinary differential equation by standard changes of dependent variable, which in the present instance amount to returning to the Schrödinger picture. The result in Schrödinger picture is

$$\begin{aligned} \frac{\partial \varrho_{11}}{\partial \tau} = g^2 \int_{-\tau}^{\tau} {\rm d}s\ e^{-i\omega s} G_\Omega(\tau, \tau - s) \\ -4g^2 \int_0^\tau {\rm d}s\ {\rm Re}\big[G_\Omega(\tau, \tau - s)\big] \cos(\omega s) \varrho^I_{11}(\tau - s)\,, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \frac{\partial \varrho_{12}}{\partial \tau} = -i(\omega + \delta\omega)\varrho_{12}(\tau) - 2g^2 \int_0^\tau {\rm d}s\ {\rm Re}\big[G_\Omega(\tau, \tau - s)\big] \varrho_{12}(\tau - s) \\ +2g^2 \int_0^\tau {\rm d}s\ {\rm Re}\big[G_\Omega(\tau, \tau - s)\big] \varrho^*_{12}(\tau - s)\,. \end{aligned} \tag{3.11}$$

[7]A bonus of this definition is that $\delta\omega$ also automatically cancels an ultraviolet divergence that arises in the computed energy shift.

3.2 Near-horizon Wightman function

So far the description of qubit evolution has been quite general, with little said about the specific field state Ω or about the details of the qubit trajectory. Application of this formalism to a qubit near a black hole requires filling in some of this detail, starting with some information about the scalar-field Wightman function in a Schwarzschild geometry.

Hadamard correlation functions. As mentioned earlier, for generic trajectories the scalar field Wightman function can be quite complicated, even for comparatively simple states like the Hartle-Hawking, Unruh or Boulware vacua [65, 71, 96–102]. One of our central points is that the late-time evolution very close to the horizon does not depend on which of these choices for field state is made, with universal predictions relying only on the state being 'Hadamard', in the sense that the Wightman correlation function

$$G_\Omega(x,x') := \underset{\phi}{\mathrm{Tr}}\Big[\Omega\, \phi(x)\, \phi(x')\Big] \tag{3.12}$$

has — in four spacetime dimensions — the following limit as $x \to x'$ [69, 103–106]:

$$G_\Omega(x,x') = \frac{1}{8\pi^2}\left\{ \frac{\Delta^{1/2}(x,x')}{\sigma_\epsilon(x,x')} + V(x,x') \log\left|\frac{\sigma_\epsilon(x,x')}{L^2}\right| + W_\Omega(x,x') \right\}\,, \tag{3.13}$$

with

$$\sigma_\epsilon(x,x') := \sigma(x,x') + 2i\epsilon[\mathcal{T}(x) - \mathcal{T}(x')] + \epsilon^2\,, \tag{3.14}$$

and $\sigma(x,x')$ the so-called Synge world function [68, 93] that is equal to half the square of the geodetic length between x and x' (see appendix A). Here $\mathcal{T}$ is any future-increasing function of time, and $\epsilon \to 0^+$ a small-distance regulator with dimensions of length that appears in the above formula so that $G_\Omega(x,x')$ satisfies the correct temporal boundary conditions.

The quantities $\Delta(x,x')$, $V(x,x')$ and $W_\Omega(x,x')$ are biscalar functions that are symmetric in $x \leftrightarrow x'$, and regular in the limit that $x \to x'$. The renormalization length scale $L > 0$ is introduced on dimensional grounds, and different values for L can be absorbed into the precise definition of $W_\Omega(x,x')$. The subscript Ω on W_Ω is meant to emphasize that its detailed form depends on the state Ω [107]. The same is *not* true of the functions $\Delta(x,x')$ and $V(x,x')$, which are universal in the sense that they depend only on the geometry of the spacetime (and — in the case of $V(x,x')$ — on parameters like the mass of the field).

What this says is that the leading part of the coincident limit of $G_\Omega(x,x')$ is universal in curved space, and shares in particular the singularity structure also found in flat space. The Hadamard form expresses the physical condition common to all effective field theories [28] that states that the details of very high-energy field modes are irrelevant provided because for slowly changing backgrounds they are prepared within their adiabatic vacuum. This amounts to a quantum variant of the principle of equivalence: modes with wavelengths much shorter than the local radius of curvature do not 'know' that they are in curved space.

Because they depend only on local properties, there is a general procedure for computing the geometric functions $V(x,x')$ and $\Delta(x,x')$ in the coincident limit, for which

$\sigma(x,x') \to 0$ [68, 108]. For a real massive scalar field evaluated on a Ricci-flat spacetime (like the Schwarzschild geometry) they have the form

$$\Delta^{1/2}(x,x') = 1 + \frac{1}{360} R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} R_{\alpha\lambda\beta\rho} \sigma^{\mu}\sigma^{\nu}\sigma^{\lambda}\sigma^{\rho} + \mathcal{O}(\sigma^{5/2}) \tag{3.15}$$
$$V(x,x') = \left(\frac{m^2}{2} - \frac{1}{360} R^{\rho\sigma\tau}{}_{\mu} R_{\rho\sigma\tau\nu} \sigma^{\mu}\sigma^{\nu} \right) + \left(\frac{m^4}{16} + \frac{1}{1440} R_{\rho\sigma\tau\kappa} R^{\rho\sigma\tau\kappa} \right) \sigma_{\mu}\sigma^{\mu} + \mathcal{O}(\sigma^{3/2}),$$

where $\partial_\mu \sigma = \sigma_\mu$ and $\sigma^\mu = g^{\mu\nu}\sigma_\nu$ obey the relation $\sigma_\mu\sigma^\mu = 2\sigma$. Terms written $\mathcal{O}(\sigma^{3/2})$ are those containing three or more factors of σ_μ. For massive fields it is conventional to choose the form of $W_\Omega(x,x')$ so that $L^2 = 2/m^2$, so that

$$G_\Omega(x,x') \simeq \frac{1}{8\pi^2} \left\{ \frac{1}{\sigma_\epsilon(x,x')} + \left(\frac{m^2}{2} + \dots \right) \log \left| \frac{m^2 \sigma_\epsilon(x,x')}{2} \right| + \cdots \right\}. \tag{3.16}$$

Of the vacuum states described above, the Hartle-Hawking [109] and Unruh vacua [110] are both Hadamard states, and so share the same values for $\Delta(x,x')$ and $V(x,x')$ but not for $W_\Omega(x,x')$). The Boulware vacuum is not, however, as can be seen from its singular form for the stress-energy tensor at the horizon [102].

In practice the leading behaviour suffices for our purposes, which means we may use $\Delta(x,x') \simeq 1$ and drop $V(x,x')$ in the applications to follow, leaving the result

$$G_\Omega(x,x') \simeq \frac{1}{8\pi^2 \left[\sigma(x,x') - i\epsilon[\mathcal{T}(x) - \mathcal{T}(y)] + \epsilon^2 \right]} + \cdots, \tag{3.17}$$

that applies when $|\sigma(x,x')|$ is much smaller than both r_s^2 and m^{-2}.

Evaluation for qubits hovering near the horizon. What is special about the small-$\sigma(x,x')$ limit is that it applies not just as $x \to x'$, but also when x and x' are generic points situated sufficiently close to a null geodesic. Small $\sigma(x,x')$ should apply in particular for any two points hovering at a fixed position $(r,\theta,\phi) = (r_0,\theta_0,\phi_0)$ just outside the Schwarzschild event horizon, with $\sigma(x,x') \to 0$ as $r_0 \to r_s$.

The function $\sigma(x,x')$ is evaluated in this limit in appendix A for points on such a hovering trajectory as a function of their separation Δt in Schwarzschild time, with the result [75, 86, 87])

$$\sigma(x,x') = -8r_s^2 \left(1 - \frac{r_s}{r_0}\right) \sinh^2\left(\frac{\Delta t}{4r_s}\right) + \mathcal{O}\left(\frac{\sigma^2}{r_s^2}\right) \tag{3.18}$$

in the limit $\sigma(x,x') \to 0$. It is important that (3.18) remains valid even if $\Delta t \gg r_s$, provided that r_0 is chosen close enough to r_s to ensure that $|\sigma(x,x')| \ll r_s^2$. (The validity of this approximation in the regime $\Delta t \gg r_s$ is verified numerically in appendix A.)

For separations for which (3.18) applies, eq. (3.17) states that the Wightman function for any Hadamard state has the form

$$G_\Omega(t+\Delta t, t) \simeq -\frac{1}{64\pi^2 r_s^2 \left(1 - \frac{r_s}{r_0}\right) \Big(\sinh\left[\Delta t/(4r_s)\right] - i\epsilon/(4r_s) \Big)^2} + \cdots. \tag{3.19}$$

4 Universal late-time near-horizon evolution

This section ties everything together to obtain a closed-form expression for the two universal thermalization time-scales that arise for qubits hovering asymptotically close to the horizon. The result is surprisingly simple because of an apparently paradoxical result: the simplicity occurs because in the near-horizon limit one can exploit the Wightman function's small-$\sigma(x,x')$ Hadamard form (3.13). This seems paradoxical because thermalization occurs in the limit of very long time separations, $\Delta t \gg r_s$. The coexistence of these two limits is possible only because of the enormous time-dilation that relates static clocks running very near the horizon and those far from the black hole; two near-horizon events separated by a small geodesic separation can look to a distant observer like they are separated by very large times.

4.1 The near-horizon Nakajima-Zwanzig equation

The starting point is the interaction-picture Nakajima-Zwanzig equations (3.8) and (3.9) for the qubit state $\boldsymbol{\varrho}^I(\tau)$. At order g^2 this gives

$$\frac{\partial \varrho^I_{11}}{\partial \tau} = g^2 \int_{-\tau}^{\tau} \mathrm{d}s\; e^{-i\omega s}\, \mathfrak{F}\,(\gamma s) - 4g^2 \int_0^{\tau} \mathrm{d}s\, \mathrm{Re}\,[\mathfrak{F}\,(\gamma s)]\; \cos(\omega s)\, \varrho^I_{11}(\tau - s)\,, \tag{4.1}$$

and

$$\begin{aligned}\frac{\partial \varrho^I_{12}}{\partial \tau} &= -i\delta\omega\; \varrho^I_{12}(\tau) - 2g^2 \int_0^{\tau} \mathrm{d}s\; \mathrm{Re}\,[\mathfrak{F}\,(\gamma s)]\, e^{+i\omega s} \varrho^I_{12}(\tau - s) \\ &\quad + 2g^2 e^{+2i\omega\tau} \int_0^{\tau} \mathrm{d}s\; \mathrm{Re}\,[\mathfrak{F}\,(\gamma s)]\, e^{-i\omega s} \varrho^{I*}_{12}(\tau - s)\,,\end{aligned} \tag{4.2}$$

where

$$\mathfrak{F}(\Delta t) := G_\Omega(t + \Delta t, t) = \mathfrak{F}(\gamma \Delta\tau) \quad \text{with} \quad \gamma := \frac{1}{\sqrt{1 - r_s/r_0}}\,. \tag{4.3}$$

Our later interest is in late times as seen by an observer far from the black hole, so changing coordinates $\tau = t/\gamma$ gives

$$\frac{\partial \varrho^I_{11}}{\partial t} = g^2 \int_{-t}^{t} \mathrm{d}s\; e^{-i\omega_\infty s} \overline{\mathfrak{F}}(s) - 4g^2 \int_0^{t} \mathrm{d}s\; \mathrm{Re}\left[\,\overline{\mathfrak{F}}(s)\right] \cos(\omega_\infty s) \varrho^I_{11}(t - s)\,, \tag{4.4}$$

and

$$\begin{aligned}\frac{\partial \varrho^I_{12}}{\partial t} &= -i\delta\omega_\infty\; \varrho^I_{12}(t) - 2g^2 \int_0^{t} \mathrm{d}s\; \mathrm{Re}\left[\,\overline{\mathfrak{F}}(s)\right] e^{+i\omega_\infty s} \varrho^I_{12}(t - s) \\ &\quad + 2g^2 e^{+2i\omega_\infty t} \int_0^{t} \mathrm{d}s\; \mathrm{Re}\left[\,\overline{\mathfrak{F}}(s)\right] e^{-i\omega_\infty s} \varrho^{I*}_{12}(t - s)\,,\end{aligned} \tag{4.5}$$

where for convenience we define the redshifted qubit gap as seen by observers looking at the qubit far from the black hole

$$\omega_\infty := \omega \sqrt{1 - \frac{r_s}{r_0}}\,, \tag{4.6}$$

and perform a similar scaling of the $\mathcal{O}(g^2)$ counter-term $\delta\omega_\infty := \delta\omega(1 - r_s/r_0)$. Finally, $\overline{\mathfrak{F}}$ denotes the scaled Wightman function

$$\overline{\mathfrak{F}}(t) := \left(1 - \frac{r_s}{r_0}\right) \mathfrak{F}(t)\,. \tag{4.7}$$

In the small-$\sigma(x,x')$ limit inspection of (3.19) shows that $\overline{\mathfrak{F}}$ has the simple asymptotic form

$$\overline{\mathfrak{F}}(\Delta t) \simeq -\frac{1}{64\pi^2 r_s^2 \left(\sinh\left[\Delta t/(4r_s)\right] - i\epsilon/(4r_s)\right)^2}\,, \tag{4.8}$$

which is identical to the analogous result for the massless Rindler correlation function found in [16] once one replaces $r_s \to 1/(2a)$. Recall from appendix A that this asymptotic form for $\overline{\mathfrak{F}}(t)$ is valid so long as $|\sigma(x,x')| \ll r_s^2$ and so applies when

$$\frac{\Delta t}{r_s} \ll \left|2\log\left(\frac{1 - r_s/r_0}{4}\right)\right|\,, \tag{4.9}$$

and so in particular there is always an $r_0 > r_s$ sufficiently close to the horizon for which this is satisfied, no matter how large $\Delta t/r_s$ happens to be.

4.2 The late-time Markovian approximation

From here on the story evolves much as it did in the Rindler example considered in [16], by virtue of the similarity between (4.8) and its counterpart for an accelerated qubit in flat spacetime.

In particular eqs. (4.4) and (4.5) greatly simplify when $\boldsymbol{\varrho}^I$ is slowly varying (compared with the light-crossing time of the black hole, r_s) and we focus on $t \gg r_s$, because in this case the sharply peaked form for $\overline{\mathfrak{F}}(t)$ allows the upper integration limit to be taken to infinity, and implies that a Taylor expansion in the integrand of $\varrho_{ij}(t-s)$ in powers of s converges very quickly.

After choosing $\delta\omega$ to cancel the field-induced shift in qubit energy — which means picking

$$\delta\omega_\infty = \delta\omega\left(1 - \frac{r_s}{r_0}\right) = -g^2\mathcal{D}_S\,, \tag{4.10}$$

with $\mathcal{D}_S$ defined by (4.14) below — these steps lead (at face value) to the following approximate evolution equations (see [16] for details)

$$\frac{\partial\varrho^I_{11}(t)}{\partial t} \simeq \frac{2g^2\mathcal{C}_S}{e^{4\pi r_s\omega_\infty} + 1} - 2g^2\mathcal{C}_S\,\varrho^I_{11}(t)\,, \tag{4.11}$$

and

$$\frac{\partial\varrho^I_{12}(t)}{\partial t} \simeq -g^2\mathcal{C}_S\,\varrho^I_{12}(t) + g^2(\mathcal{C}_S - i\mathcal{D}_S)\,e^{+2i\omega_\infty t}\varrho^{I*}_{12}(t)\,, \tag{4.12}$$

in which the quantities $\mathcal{C}_S$ and $\mathcal{D}_S$ are defined by

$$\mathcal{C}_S = 2\int_0^\infty \mathrm{d}s\,\mathrm{Re}[\overline{\mathfrak{F}}(s)]\cos(\omega_\infty s) \;\simeq\; \frac{\omega_\infty\coth\left(2\pi r_s\omega_\infty\right)}{4\pi} \tag{4.13}$$

	$g^2\mathcal{C}_S/\omega_\infty \ll 1$	$g^2\mathcal{D}_S/\omega_\infty \ll 1$
$r_s\omega_\infty \ll 1$	$\frac{g^2}{8\pi^2 r_s\omega_\infty}$	$\frac{g^2}{2\pi^2}\log\left[\epsilon/(2r_s)\right]$
$r_s\omega_\infty \gg 1$	$\frac{g^2}{4\pi}$	$\frac{g^2}{2\pi^2}\log(e^\gamma\omega_\infty\epsilon)$

Table 1. The large- and small-$\omega_\infty r_s$ asymptotic forms for the two quantities that must be small to work with nondegenerate perturbation theory (see [16]).

and[8]

$$\mathcal{D}_S = 2\int_0^\infty \mathrm{d}s\ \mathrm{Re}[\overline{\mathfrak{F}}(s)]\sin(\omega_\infty s) \ \simeq\ \frac{\omega_\infty}{2\pi^2}\log\left(\frac{e^\gamma\epsilon}{2r_s}\right) + \frac{\omega_\infty}{2\pi^2}\mathrm{Re}\left[\psi^{(0)}(-2ir_s\omega_\infty)\right]\,. \tag{4.14}$$

where $\psi^{(0)}(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function [112].

Control over approximations. The words 'at face value' are added above eqs. (4.11) and (4.12) because the term involving $\mathcal{D}_S$ must actually be dropped in the above if we are consistent. The reasons for this lie in the size of the deviations from the leading approximation, and the assumptions that must be made in order to neglect them. We briefly summarize the issues, following closely the discussion in [16, 17]. A side effect of this observation — together with the energy shift (4.10) — is the elimination of all singular dependence[9] in the limit $\epsilon \to 0$ that enters through eq. (4.14).

There are two kinds of approximations to consider — one convenient and one essential. The issue of convenience concerns the relative size of the qubit splitting ω and the generic size of field-driven corrections to this splitting. Assuming ω_∞ to be much larger than the corrections to ω induced by the interactions with the field simplifies calculations by allowing use of non-degenerate methods. In terms of the functions $\mathcal{C}_S$ and $\mathcal{D}_S$ this condition requires

$$\frac{g^2\mathcal{C}_S}{\omega_\infty} \ll 1 \quad \text{and} \quad \frac{g^2\mathcal{D}_S}{\omega_\infty} \ll 1\,. \tag{4.15}$$

Table 1 displays the asymptotic form for these two quantities in the limit of large and small $\omega_\infty r_s$, showing that they require $\omega_\infty r_s$ not to be taken smaller than $g^2/4\pi$.

The essential approximation is the one that makes the Markovian evolution dominate the Nakajima-Zwanzig evolution. To see what this involves, recall that the Markovian approximation is derived from the Nakajima-Zwanzig equation by Taylor expanding $\varrho^I_{ij}(t-s) \simeq \varrho^I_{ij}(t) - s\dot{\varrho}^I_{ij}(t) + \cdots$ inside the integrands of equations (4.4) and (4.5),

$$g^2\int_0^t \mathrm{d}s\ f(s)\varrho^I_{ij}(t-s) \simeq g^2\int_0^\infty \mathrm{d}s\ f(s)\big[\varrho^I_{ij}(t) - s\dot{\varrho}^I_{ij}(t) + \ldots\big] \tag{4.16}$$

[8] A flat-space analog of $\mathcal{D}_S$ is computed in [16] for generic field masses $m \neq 0$ and with the replacement $a \to 2/r_s$ (using the different notation Δ_M there). Equation (4.14) follows as the $m \to 0^+$ limit of this function (this same function is evaluated in [111]).

[9] For the purposes of estimating the size of different contributions we take ϵ here to be much smaller than other scales, but not infinitely small so that logarithms of ϵ cannot overwhelm powers of g^2.

	$g^2 \frac{\mathrm{d}\mathcal{C}_S}{\mathrm{d}\omega_\infty} \ll 1$	$g^2 \frac{\mathrm{d}\mathcal{D}_S}{\mathrm{d}\omega_\infty} \ll 1$	$\frac{\omega_\infty}{\mathcal{C}_S} \frac{\mathrm{d}\mathcal{C}_S}{\mathrm{d}\omega_\infty} \ll 1$	$\frac{\omega_\infty}{\mathcal{C}_S} \frac{\mathrm{d}\mathcal{D}_S}{\mathrm{d}\omega_\infty} \ll 1$
$r_s\omega_\infty \ll 1$	$g^2 r_s \omega_\infty/3$	$\frac{g^2}{2\pi^2} \log[\epsilon/(2r_s)]$	$8\pi^2 r_s^2 \omega_\infty^2/3$	$4r_s\omega_\infty \log[\epsilon/(2r_s)]$
$r_s\omega_\infty \gg 1$	$\frac{g^2}{4\pi}$	$\frac{g^2}{2\pi^2} \log(e^\gamma \omega_\infty \epsilon)$	1	$\frac{2}{\pi} \log(e^\gamma \omega_\infty \epsilon)$

Table 2. The large- and small-$\omega_\infty r_s$ asymptotic forms for the four quantities that must be small to believe the Markovian approximation to the Nakajima-Zwanzig equation (see [16]). Notice that $r_s\omega_\infty \gg 1$ is incompatible with Markovian evolution.

where $t \gg r_s$ is used to take the upper limit of integration to infinity (given the exponential falloff of $f(s)$ for $s \gg r_s$). The size of the $s\dot{\varrho}^I_{ij}(t)$ term characterizes the size of deviations from the Markovian limit, and we evaluate it to understand what demands are made on the free parameters of the model by the requirement that these be small. Physically this amounts to requiring the evolution time-scale of ϱ^I_{ij} to be large compared with the domain of support of the rest of the integrand.

The quantitative conditions are obtained self-consistently, by evaluating $\dot{\varrho}^I_{ij}$ assuming the time dependence is given by (4.11) and (4.12), whose integration implies

$$\varrho^I_{11}(t) = \frac{1}{e^{4\pi r_s \omega_\infty} + 1} + \left[\varrho_{11}(0) - \frac{1}{e^{4\pi r_s\omega_\infty} + 1}\right] e^{-2g^2 \mathcal{C}_S\, t}\,, \tag{4.17}$$

and

$$\varrho^I_{12}(t) = e^{-g^2\mathcal{C}_S\, t} \left[\varrho_{12}(0) + \varrho^*_{12}(0) \left(\frac{g^2 \mathcal{D}_S}{2\omega_\infty} + i \frac{g^2\mathcal{C}_S}{2\omega_\infty}\right)(1 - e^{2i\omega_\infty t})\right]\,. \tag{4.18}$$

Differentiating this to find $\dot{\varrho}^I_{ij}$ then allows the $\dot{\varrho}^I$ term to be computed in equations like (4.16), and requiring the result to be negligible relative to the leading term for all of the integrals appearing in eqs. (4.4) and (4.5) requires the following four quantities all to be negligible:

$$g^2 \frac{\mathrm{d}\mathcal{C}_S}{\mathrm{d}\omega_\infty} \ll 1\,, \quad g^2 \frac{\mathrm{d}\mathcal{D}_S}{\mathrm{d}\omega_\infty} \ll 1\,, \quad \frac{\omega_\infty}{\mathcal{C}_S}\frac{\mathrm{d}\mathcal{C}_S}{\mathrm{d}\omega_\infty} \ll 1 \quad \text{and} \quad \frac{\omega_\infty}{\mathcal{C}_S}\frac{\mathrm{d}\mathcal{D}_S}{\mathrm{d}\omega_\infty} \ll 1\,. \tag{4.19}$$

The first two of these are required whenever the derivative in $\dot{\varrho}_{ij}$ is of order $g^2\mathcal{C}_S$ while the second two arise when it is order ω_∞. [Differentiation with respect to ω_∞ arises from use of identities like $s\cos(\omega_\infty s) = (\mathrm{d}/\mathrm{d}\omega_\infty)\sin(\omega_\infty s)$ in equations like (4.16).]

Table 2 displays the asymptotic behaviour for these four quantities in the limits where $r_s\omega_\infty$ is very large or very small. This table makes clear in particular that only $r_s\omega_\infty \ll 1$ is consistent in the Markovian regime, since otherwise the bounds $\omega_\infty \mathcal{C}'_S/\mathcal{C}_S \ll 1$ and $\omega_\infty\mathcal{D}'_S/\mathcal{C}_S \ll 1$ necessarily fail (because the $e^{i\omega_\infty t}$ oscillations are too rapid).

The asymptotic forms of table 2 say more than just this, however. From them we also notice that $r_s\omega_\infty \ll 1$ implies[10] $\mathcal{D}_S/\omega \sim \mathcal{D}'_S$ and so

$$\frac{g^2\mathcal{D}_S}{\omega_\infty} \simeq g^2 \frac{\mathrm{d}\mathcal{D}_S}{\mathrm{d}\omega_\infty} = \left(\frac{g^2\mathcal{C}_S}{\omega_\infty}\right) \times \left(\frac{\omega_\infty}{\mathcal{C}_S}\frac{\mathrm{d}\mathcal{D}_S}{\mathrm{d}\omega_\infty}\right) \ll \frac{g^2\mathcal{C}_S}{\omega_\infty}\,, \tag{4.20}$$

[10] Using $\psi^{(0)}(z) \simeq \frac{1}{z} - \gamma + \frac{\pi^2 z}{6} - \zeta(3) z^2 + \dots$ for $|z| \ll 1$ (with ζ the Riemann zeta function) [112], $\mathcal{D}_S \simeq \frac{\omega_\infty}{2\pi^2}(\log(\frac{\epsilon}{2r_s}) + 4\zeta(3)(r_s\omega_\infty)^2 + \mathcal{O}[(r_s\omega_\infty)^4])$ and $\mathcal{D}'_S \simeq \frac{1}{2\pi^2}(\log(\frac{\epsilon}{2r_s}) + 12\zeta(3)(r_s\omega_\infty)^2 + \mathcal{O}[(r_s\omega_\infty)^4])$.

which implies that the $g^2\mathcal{D}_S/\omega_\infty$ term appearing in the solution (4.18) is negligible relative to the $g^2\mathcal{C}_S/\omega_\infty$ term. This means that the $g^2\mathcal{D}_S$ terms in the Markovian evolution equations (4.12) can be neglected, allowing the Markovian evolution instead to be written as (4.11) and

$$\frac{\partial \varrho^I_{12}(t)}{\partial t} \simeq -g^2\mathcal{C}_S\, \varrho^I_{12}(t) + g^2\mathcal{C}_S\, e^{+2i\omega_\infty t}\varrho^{I*}_{12}(t)\,. \tag{4.21}$$

In particular the divergent quantity $\mathcal{D}_S$ plays no role in the Markovian limit, apart from shifting the qubit energy levels in the way that is renormalized into the definition of ω. Following the steps discussed at great length in [16, 17]) shows that these equations preserve positivity of $\boldsymbol{\varrho}(t)$ to $\mathcal{O}(g^2)$ in the Markovian limit, with no additional approximations necessary.

The solutions in the Markovian regime therefore become

$$\varrho^I_{11}(t) = \frac{1}{e^{4\pi r_s\omega_\infty}+1} + \left[\varrho_{11}(0) - \frac{1}{e^{4\pi r_s\omega_\infty}+1}\right] e^{-2t/\xi}\,, \tag{4.22}$$

and

$$\varrho^I_{12}(t) = e^{-t/\xi}\left[\varrho_{12}(0) + i\varrho^*_{12}(0)\frac{g^2\mathcal{C}_S}{2\omega_\infty}(1-e^{2i\omega_\infty t})\right]\,, \tag{4.23}$$

where

$$\xi \;:=\; \frac{1}{g^2\mathcal{C}_S} \;=\; \frac{4\pi\tanh(2\pi r_s\omega_\infty)}{g^2\omega_\infty} \;\simeq\; \frac{8\pi^2 r_s}{g^2} \tag{4.24}$$

and the last line follows since the Markovian approximation demands $\omega_\infty r_s \ll 1$. These solutions describe the exponential decay towards a thermal distribution (with temperature $T = 1/(4\pi r_s) = T_H$ that equals the Hawking temperature), doing so with the characteristic time-scale $\xi \simeq 8\pi^2 r_s/g^2$. Notice that the approach to equilibrium takes place twice as fast for the diagonal components of $\boldsymbol{\varrho}$ compared to its off-diagonal parts.

We remark in passing that it is also possible to solve the Nakajima-Zwanzig equation at late times using weaker assumptions than those that lead to the above Markovian solutions, using methods similar to those used in [17] (see also [111], where a non-Markovian solution for an accelerated qubit is derived by method of Laplace transforms). The utility of such a solution is less interesting here since the Markovian condition $r_s\omega_\infty = r_s\omega\sqrt{1-r_s/r_0} \ll 1$ is satisfied for any qubit of fixed rest-frame energy splitting that hovers sufficiently close to the black-hole horizon.

Frame independence of the Markovian limit. Since the solution (4.22) and (4.23) refers to the redshifted time Δt defined in (2.25) one may wonder about the physical meaning of tracking a time coordinate as measured by a hovering observer far from the black hole horizon, and whether the discussion of perturbativity and Markovianity applies only in such a frame.

What matters is that there is a hierarchy of scales between the late times of interest, $\Delta t(r)$, (such as the equilibration time) and the correlation time of the environment, $\tau_c(r)$ (in this case, the local Hawking temperature). These can equally well be compared at the position of the qubit or by an observer at infinity. Although gravitational redshift changes both Δt and τ_c (and this is why they depend on r), this redshift cancels in their ratio;

observers at all radii agree that $\Delta t(r) \gg \tau_c(r)$. Both a local observer travelling with the qubit and one hovering at spatial infinity agree on the heirarchies of scale

$$\frac{\Delta t}{r_s} \sim T_H \Delta t \gg 1 \qquad \iff \qquad \frac{\Delta \tau}{r_s\sqrt{1 - r_s/r_0}} \sim T(r_0)\Delta\tau \gg 1 \tag{4.25}$$

where $T_H = (4\pi r_s)^{-1}$ is the Hawking temperature, and $T(r_0) = T_H/\sqrt{1 - r_s/r_0}$ is the local temperature. This means that there is nothing special about tracking the qubit evolution in terms of the redshifted time Δt, and in fact an observer at any radius will agree on a hierachy of scales as in (4.25).

5 Conclusions

In summary, this paper shows how Open EFT methods can lend themselves to late-time resummation in more general gravitational systems than the cosmological examples previously explored. As in the examples of [16, 17] simplicity arises near the horizon at late times, even when the underlying geometry tends to makes quantum mechanical calculations difficult. Standard tools for open quantum systems give relatively easy access to times of the order r_s/g^2, at least in the specific instance of an Unruh-DeWitt detector placed very close to a Schwarzschild horizon and interacting with a quantum field. The resulting evolution describes qubit thermalization with the expected Hawking radiation, asymptoting to the Hawking temperature $T_H = (4\pi r_s)^{-1}$. The time-scale for thermalizing a hovering qubit can be computed, and in the very-near-horizon limit takes a universal form that relies only on properties of the near-horizon geometry given only the relatively weak assumption that the quantum field is prepared in a vacuum state of Hadamard form (including in particular the Hartle-Hawking and Unruh states).

What makes the late-time evolution easy to resum is its Markovian nature over Schwarzschild times that are long compared with r_s. Autocorrelations of the field in a Hadamard state then fall off very robustly for qubits hovering very near the horizon, effectively washing out the past entanglement history. As one might expect from the equivalence principle, the qubit behaviour becomes equivalent to that of a qubit accelerating through flat space in the limit of infinite acceleration. It is the large acceleration (and blueshift) experienced by the qubit which ensures that the quantum field mass eventually becomes negligible in the near-horizon limit, explaining why the mass largely drops out of our result. As a consequence, the Markovian evolution seems likely to be very robust, at least asymptotically close to the horizon (provided $r_s\omega_\infty \ll 1$, so that the qubit states are not too split to allow thermal excitation).

The absence of mass dependence (in the $m\, r_s \ll 1$ limit) also carries information about dependence on the scalar self-coupling, λ. Scalar self-couplings are known to give rise to secular effects for accelerated observers even in flat space [15] (see also [113] for other evidence for secular growth in black-hole geometries), where it is known that they can also be resummed at late times. For the Rindler problem late-time resummation amounts to re-organizing perturbation theory using a small shifted mass, $\delta m^2 \sim \lambda a^2$, similar to the development of small temperature-dependent masses in thermal environments [15].

Similarity with the Rindler problem makes it is very plausible that a similar resummation can be obtained near the Schwarzschild horizon by shifting the scalar mass by an amount $\delta m^2 \sim \lambda/r_s^2$, making the m-independence of near-horizon qubit evolution likely also to imply the same for λ-dependence, at least when λ is small and times are late.

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A The Synge world function

This appendix derives some of the features of the Synge world-function that are used in the main text.

Definitions. To this end consider two points, x and x', that are connected by a timelike geodesic Γ. If λ is an affine parameterization of Γ then it is described by the curve $y^\mu(\lambda)$ along which

$$\ddot{y}^\mu + {\Gamma^\mu}_{\nu\sigma}\dot{y}^\nu\dot{y}^\sigma = 0 \tag{A.1}$$

is obeyed for all λ, with $\dot{y}^\mu := \mathrm{d}y^\mu/\mathrm{d}\lambda$. The fact that Γ connects x and x' is expressed as the boundary conditions $y(\lambda_\mathrm{i}) = x'$ and $y(\lambda_\mathrm{f}) = x$.

For such a geodesic the Synge world function, $\sigma(x, x')$, is defined by [93–95]

$$\sigma(x, x') := \frac{1}{2}(\lambda_\mathrm{f} - \lambda_\mathrm{i}) \int_{\lambda_\mathrm{i}}^{\lambda_\mathrm{f}} \mathrm{d}\lambda \; g_{\mu\nu}\dot{y}^\mu\dot{y}^\nu \,, \tag{A.2}$$

where the integral is performed along the geodesic Γ. This integral is actually quite easy to evaluate because the geodesic equation (A.1) implies that the quantity

$$\zeta := g_{\mu\nu}\dot{y}^\mu\dot{y}^\nu \tag{A.3}$$

is independent of λ along Γ, and so $\sigma(x, x') = \frac{1}{2}\,\zeta\,(\lambda_\mathrm{f} - \lambda_\mathrm{i})^2$. For timelike curves ζ is negative, and if in that case the parameter is chosen to be proper time along the geodesic — i.e. if $\lambda = \tau$ — then $\zeta = -1$ and $\lambda_\mathrm{f} - \lambda_\mathrm{i} = \Delta s$, establishing that

$$\sigma(x, x') = -\frac{1}{2}(\Delta s)^2, \tag{A.4}$$

as used in the main text.

Expansion as $x \to x'$. The dependence of $\sigma(x, x')$ on the geometry can be made explicit in the limit $x \to x'$. This is most easily done using (A.2) and specializing the evaluation of ζ to the point x' (as can be freely done since ζ is independent of λ), leading to

$$\sigma(x, x') = \frac{1}{2}(\lambda_\mathrm{f} - \lambda_\mathrm{i})^2 \, g'_{\mu\nu} \dot{y}^\mu(\lambda_\mathrm{i}) \dot{y}^\nu(\lambda_\mathrm{i}) \,, \tag{A.5}$$

where here (and below) a prime on a field like $g'_{\mu\nu}$ indicates that it is evaluated at x'.

Expanding $y^\mu(\lambda_\mathrm{f})$ in powers of $\lambda_\mathrm{f} - \lambda_\mathrm{i}$ gives

$$y^\mu(\lambda_\mathrm{f}) = y^\mu(\lambda_\mathrm{i}) + (\lambda_\mathrm{f} - \lambda_\mathrm{i})\, \dot{y}^\mu(\lambda_\mathrm{i}) + \frac{1}{2}(\lambda_\mathrm{f} - \lambda_\mathrm{i})^2\, \ddot{y}^\mu(\lambda_\mathrm{i}) + \frac{1}{6}(\lambda_\mathrm{f} - \lambda_\mathrm{i})^3\, \dddot{y}^\mu(\lambda_\mathrm{i}) + \dots, \tag{A.6}$$

in which we use the boundary conditions $y(\lambda_\mathrm{i}) = x'$ and $y(\lambda_\mathrm{f}) = x$, as well as eliminating $\ddot{y}^\mu$ using the geodesic equation, leading to

$$\begin{aligned} x^\mu - x'^\mu \;=\; & (\lambda_\mathrm{f} - \lambda_\mathrm{i})\, \dot{y}^\mu(\lambda_\mathrm{i}) - \frac{1}{2}(\lambda_\mathrm{f} - \lambda_\mathrm{i})^2 \Gamma^{\mu\prime}_{\rho\nu}\, \dot{y}^\rho(\lambda_\mathrm{i}) \dot{y}^\nu(\lambda_\mathrm{i}) \\ & - \frac{1}{6}(\lambda_\mathrm{f} - \lambda_\mathrm{i})^3 \Big(\partial_\rho \Gamma^{\mu\prime}_{\nu\sigma} - 2\Gamma^{\mu\prime}_{\rho\eta}\Gamma^{\eta\prime}_{\nu\sigma} \Big)\, \dot{y}^\rho(\lambda_\mathrm{i}) \dot{y}^\nu(\lambda_\mathrm{i}) \dot{y}^\sigma(\lambda_\mathrm{i}) + \dots . \end{aligned} \tag{A.7}$$

Inverting the above gives a series expansion for $(\lambda_\mathrm{f} - \lambda_\mathrm{i})\dot{y}^\mu(\lambda_\mathrm{i})$ in powers of $x - x'$:

$$\begin{aligned} (\lambda_\mathrm{f} - \lambda_\mathrm{i}) \dot{y}^\mu(\lambda_\mathrm{i}) \;=\; & (x - x')^\mu + \frac{1}{2}\, \Gamma^{\mu\prime}_{\lambda\nu} (x - x')^\lambda (x - x')^\nu \\ & + \frac{1}{6} \Big(\partial_\lambda \Gamma^{\mu\prime}_{\nu\sigma} + \Gamma^{\mu\prime}_{\lambda\eta}\Gamma^{\eta\prime}_{\nu\sigma} \Big) (x - x')^\lambda (x - x')^\nu (x - x')^\sigma + \dots , \end{aligned} \tag{A.8}$$

which, when used in (A.5), gives [93, 95]

$$\sigma(x, x') = \frac{1}{2} g'_{\mu\nu}\, (x - x')^\mu (x - x')^\nu \;+\; \frac{1}{4} g'_{\mu\nu,\sigma}\, (x - x')^\mu (x - x')^\nu (x - x')^\sigma + \cdots . \tag{A.9}$$

One can continue in this way to any fixed order.[11]

Expansion for fixed r in Schwarzschild. We next evaluate (A.9) for the special case x and x' lie along a trajectory at fixed $r = r_0$ (and θ and ϕ) in the Schwarzschild geometry. Choosing x' to correspond to $t_\mathrm{i} = 0$ and x to be $t_\mathrm{f} = \Delta t$, we have (in Kruskal coordinates)

$$\begin{aligned} T - T' &= \sqrt{\frac{r_0}{r_s} - 1}\; \exp\left(\frac{r_0}{2r_s}\right) \sinh\left(\frac{\Delta t}{2r_s}\right) \\ X - X' &= \sqrt{\frac{r_0}{r_s} - 1}\; \exp\left(\frac{r_0}{2r_s}\right) \left[\cosh\left(\frac{\Delta t}{2r_s}\right) - 1\right] . \end{aligned} \tag{A.10}$$

So using (2.22)

$$-\, g'_{TT} = g'_{XX} = \frac{4r_s^3}{r_0}\, e^{-r_0/r_s} \,, \tag{A.11}$$

[11] Although neither the coefficients nor $x - x'$ in this expansion are covariant, the final result is (transforming as a bi-scalar). A more explicitly covariant expression can be found by expanding in a more covariant variable, but expression (A.9) suffices for our present purposes.

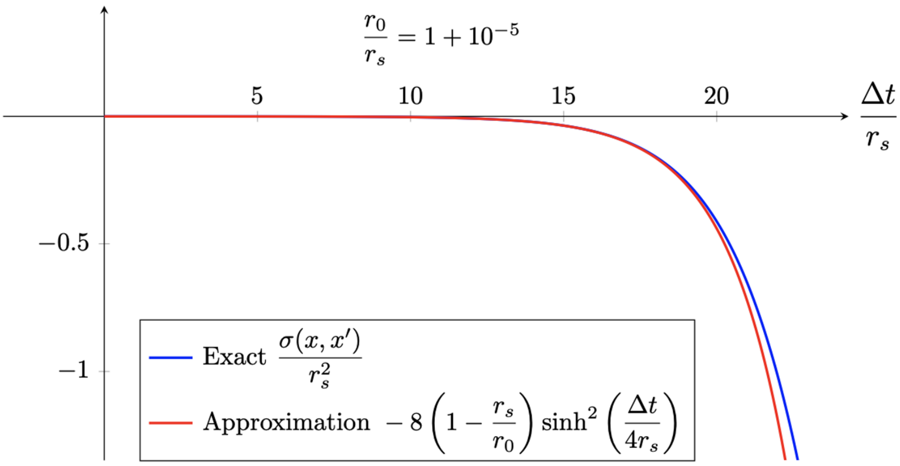

Figure 1. Numerical comparison of the Synge world-function and the asymptotic expressions (A.12) and (A.15), showing how (A.12) enjoys the broader domain of validity. This plot assumes $r_0/r_s = 1 + 10^{-5}$.

the leading-order term in (A.9) is [75]

$$\begin{aligned} \sigma(x,x') &= \frac{1}{2}\, g'_{XX}\left[-(T-T')^2 + (X-X')^2\right] + \mathcal{O}[(x-x')^3] \\ &= -8r_s^2\left(1-\frac{r_s}{r_0}\right)\sinh^2\left(\frac{\Delta t}{4r_s}\right) + \mathcal{O}[(x-x')^3]\,. \end{aligned} \tag{A.12}$$

which uses the identity $\sinh^2 a - (\cosh a - 1)^2 = 4\sinh^2(a/2)$.

Evaluating the sub-leading terms in the series shows that corrections are of order

$$\mathcal{O}\left[(x-x')^3\right]] = \mathcal{O}\left\{r_s^2\left[\left(1-\frac{r_s}{r_0}\right)\sinh^2\left(\frac{\Delta t}{4r_s}\right)\right]^2\right\} = \mathcal{O}\left[\frac{\sigma^2(x,x')}{r_s^2}\right]\,, \tag{A.13}$$

showing that (A.12) is a good approximation so long as $|\sigma(x,x')| \ll r_s^2$, or

$$\left(1-\frac{r_s}{r_0}\right)\sinh^2\left(\frac{\Delta t}{4r_s}\right) \ll 1\,. \tag{A.14}$$

Notice that this can remain valid even when $\Delta t/r_s \gg 1$ so long as r_0 is sufficiently close to r_s that (A.14) remains satisfied.

Performing the same calculation using Schwarzschild coordinates instead gives

$$\sigma(x.x') \simeq -\frac{1}{2}\left(1-\frac{r_s}{r_0}\right)(\Delta t)^2 + \dots\,. \tag{A.15}$$

Although this agrees with (A.12) for $\Delta t \ll r_s$, the domain of validity of (A.9) turns out to be larger, applying even when $\Delta t/r_s$ is *not* small. This can be seen numerically in σ, as is shown in figures 1 and 2. Also shown in these figures is how the domain of validity of (A.12) can be extended out to extremely large values of $\Delta t/r_s$ simply by choosing r_0 to be ever-closer to the horizon itself.

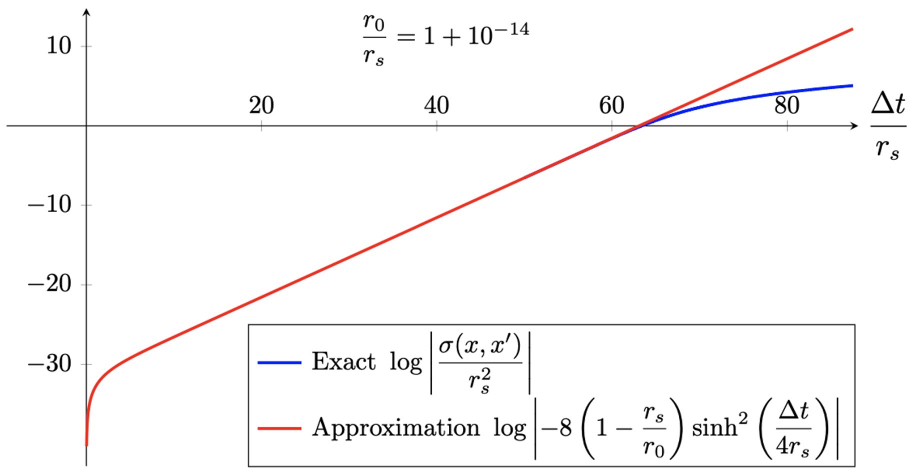

Figure 2. Numerical comparison of the Synge world-function and the asymptotic expressions (A.12) and (A.15), showing how (A.12) enjoys the broader domain of validity. This plot assumes $r_0/r_s = 1 + 10^{-14}$.

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Chapter 3

Hotspots as Toy Black Holes

G. Kaplanek, C. P. Burgess and R. Holman,

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3.1 Preface

As described in §1, any time a quantum system is coupled to spacetime with an event horizon, it seems that an open quantum systems description is useful. One of the most interesting settings to apply Open EFT techniques are then of course in black hole spacetimes. What complicates this is that ordinary black holes are compact objects with horizons localized around the black hole, which always end up involving complicated mathematics. This tends to obfuscate the OQS features that are present in such calculations.

For this reason the following paper introduces the exactly solvable hotspot model — a toy model of a black hole designed to emphasize the open quantum features held by a black hole, so as to provide a simpler playground to test Open EFT techniques. The basic set-up for the model is for the external region of the black hole (outside the event horizon) to be modelled by a single scalar field ϕ, and the internal region of the black hole (*ie.* inside the horizon) to be modelled by a large amount of hot/thermal scalar fields χ^a which cannot be directly probed in the external region. Since black holes have an enormous amount of degrees of freedom hidden behind their event horizons, we assume there are $N \gg 1$ scalars χ^a. The internal and external degrees of freedom only interact in a compact region of space, which we take here to be some infinitesimally small point (the "hotspot").

Because the interaction occurs at a single point, PPEFT techniques are used to describe this model which means that the hotspot model can only describe distances macroscopically far away from the black hole when calculating physically relevant quantities for the external region of the black hole. Since the interaction is taken to be Gaussian, the model is exactly solved by calculating the 2-point correlation functions for the model. Notably, since the interaction region is assumed to be a single infinitesimally-small point, one encounters "Coulomb-like" singularities (seen in other contexts as well [86, 87, 88, 89]) when computing physical observables which are later renormalized using PPEFT techniques.

3.2 Paper

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Influence through mixing: hotspots as benchmarks for basic black-hole behaviour

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Abstract: Effective theories are being developed for fields outside black holes, often with an unusual open-system feel due to the influence of large number of degrees of freedom that lie out of reach beyond the horizon. What is often difficult when interpreting such theories is the absence of comparisons to simpler systems that share these features. We propose here such a simple model, involving a single external scalar field that mixes in a limited region of space with a 'hotspot' containing a large number of hot internal degrees of freedom. Since the model is at heart gaussian it can be solved explicitly, and we do so for the mode functions and correlation functions for the external field once the hotspot fields are traced out. We compare with calculations that work perturbatively in the mixing parameter, and by doing so can precisely identify its domain of validity. We also show how renormalization-group EFT methods can allow some perturbative contributions to be resummed beyond leading order, verifying the result using the exact expression.

Keywords: Black Holes, Effective Field Theories, Nonperturbative Effects, Renormalization Group

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Contents

1 Introduction

At long last the detection of gravitational waves [1] has made near-horizon black-hole physics an experimental science, and this is very likely to deepen our understanding of General Relativity (GR) and/or end its hundred-year reign as the paradigm of choice when describing gravity. With the advent of measurements — eventually precision measurements — it behooves theorists to raise their game when quantifying the kinds of physics one might hope to see in this new regime. And this they are doing; both by pushing the accuracy of GR gravitational-wave predictions, and by exploring more systematically the predictions of alternatives theories when gravitational fields are strong (for reviews see [2–5]).

Effective field theories (EFTs) are usually important tools for this kind of work, because they allow predictions for physics on observable length scales that are robust to changes in the details of what goes on at smaller scales [6–10]. This is useful both when these smaller scales are understood and when they are not. Although EFT methods have a long history, their use is even now still being developed for black hole applications [11–27]; a development that has been slowed both by the relative novelty of EFT applications to gravity in general [10, 28–32] and by some of the novel aspects of black hole physics in particular, since these differ from more garden-variety applications of EFT techniques.

One issue — though not the only one [25–27] — that complicates developing EFT methods for black-hole behaviour is their open and thermal nature, since the entanglement and decoherence that such physics can involve is not captured by traditional Wilsonian EFT tools. Such differences have led some to ask whether an effective description of extra-horizon physics might involve unusual features (such as nonlocality) or otherwise evade the arguments that usually preclude these phenomena from arising in a Wilsonian context [33–43].

What usually helps when developing EFT tools are concrete systems for which both UV and IR sectors are well-understood and within which the EFT description can be assessed by comparing to other methods. These kinds of comparisons are not yet available for black holes, and the search for effective descriptions of black-hole physics are the poorer for it. The purpose of this paper is to help fix this situation by providing a simple black-hole proxy that can help fill this void. On one hand the model should be simple enough to solve, but on the other hand share enough black hole properties to be informative about some of their putative EFT descriptions.

The model we propose — inspired by similar models in condensed matter systems [44–46] — has a large number of degrees of freedom with a thermal character and no gap; to which an external field couples only in a small region of space; what we call here for brevity a 'hotspot'. We model the thermal degrees of freedom as a collection of N massless scalar fields — χ^a with $a = 1, \cdots, N$ — that are initially prepared in a thermal bath. These fields are meant to model the black hole's interior. We take these fields to 'interact' with the external massless scalar field ϕ, which is a proxy for the black hole's exterior. The word interact appears in quotes because ϕ only couples to the χ^a through a bilinear mixing term of the form

$$\mathcal{L}_{\text{mix}} = -g_a\, \chi^a\, \phi\,, \tag{1.1}$$

and so the entire theory remains gaussian and can be solved in great detail.

So far this just describes a field mixing with a thermal bath. To make it more black-hole-like we imagine these two sectors only mix in a small localized region of space, and not interacting — even gravitationally — otherwise. In order to do this we imagine space at a given time to come with two spatial sheets, $\mathcal{R}_+$ and $\mathcal{R}_-$, with ϕ living only on $\mathcal{R}_+$ and χ^a living on $\mathcal{R}_-$. These two branches only intersect on a small spherical ball, $\mathcal{S}_\xi$, of radius ξ, that plays the role of the black hole itself (see figure 1).

In principle gravity can be included in this model, and does not generate couplings between the two sectors away from their overlap on $\mathcal{S}_\xi$ (and this is why we take $\mathcal{R}_\pm$ to be disjoint). We do not pursue this gravitational coupling further in this paper, focussing instead on how the field ϕ responds to the presence of the localized hotspot built from the thermal fields χ^a. As a result our model does not capture the causal nature of the horizon or the exponential redshifts that arise in its vicinity for real black holes.

Broadly speaking there are two types of black-hole EFTs that are usually pursued, and both can have counterparts in our hotspot model. The main variant is one that is appropriate to gravitational wave emission, and applies on length scales $\lambda \gg r_s$ that are much larger than the black hole's size (see figure 2). In this 'world-line' or 'point-particle' EFT the closest distance to the black hole that can be directly resolved corresponds to a cutoff that has size $\epsilon \gg r_s$ and so the black hole dynamics is described by its center-of-mass coordinate; it is regarded as a point mass moving along a trajectory in spacetime. The response of the black hole to applied 'bulk' fields (and the back-reaction of the black hole back onto these fields) is described by an action defined as a functional of the bulk fields integrated along the black hole's one-dimensional world line. This type of EFT is obtained in the hotspot example by taking the radius ξ of the interaction sphere $\mathcal{S}_\xi$ to be much

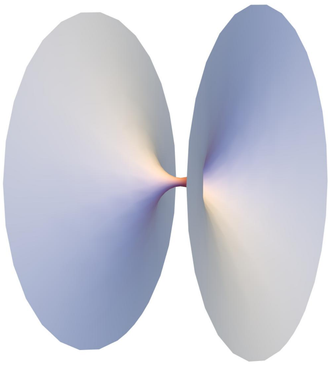


Figure 1: A cartoon of the two spatial branches, $\mathcal{R}_+$ and $\mathcal{R}_-$, in which the field ϕ and the N fields χ^a repsectively live. The two types of fields only couple to one another in the localized throat region, which can be taken to be a small sphere of radius ξ (or effectively a point in the limit that ξ is much smaller than all other scales of interest).

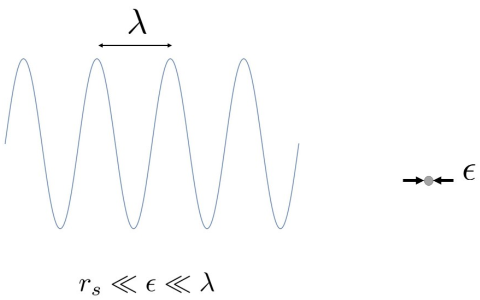


Figure 2: An EFT regime appropriate for small black holes, for which the UV cutoff scale is much smaller than the length of any low-energy probe, $\epsilon \ll \lambda$, but much larger than the horizon scale $\epsilon \gg r_s$.

smaller than all other scales: $\xi \ll \epsilon \ll \lambda$.

The puzzle for this EFT is how it should capture the enormous number of degrees of freedom that are internal to the black hole, its perfect absorber properties and the Hawking radiation that comes with it. In [12, 47–52] these are modelled by 'integrating in' a large number of degrees of freedom, and in the hotspot model it is the χ^a fields that play this role. The drawback of this approach is the model-dependence that enters when choosing these extra degrees of freedom. Although the fluctuation-dissipation theorem implies that predictions in linear response do not depend on these details, it remains open the extent to which other predictions do, and if so whether the same might be true for low-energy black hole properties. Although the extra degrees of freedom can again be integrated out, they are not the traditional massive states of the usual Wilsonian treatments, and so can

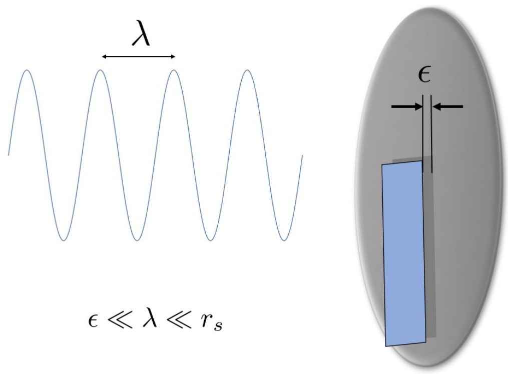


Figure 3: A 'membrane' EFT regime for which hypothetical UV physics modifies near-horizon properties, for which the cutoff scale is much smaller than the length of any low-energy probe, which is in turn much smaller than the horizon scale $\epsilon \ll \lambda \ll r_s$.

lead to actions with unusual properties including some forms of nonlocality. In companion papers [53, 54] we use the hotspot model to explore some of these properties in an effort to ascertain the rules for such an EFT, and the extent to which locality and ordinary Wilsonian reasoning breaks down.

The second class of black hole models to which our hotspot setup can be relevant are those for which probe scales, λ, and the UV cutoff length, ϵ, are both much smaller than the horizon size, but where an effective description — whether of conventional [55–59] or more exotic [60–70] physics — is envisaged to apply sufficiently near the event horizon (see figure 3). The beginnings of an EFT treatment of this kind of physics are developed in [71, 72], and involves an effective 3-dimensional action defined on a membrane that shrink-wraps the world-tube a distance ϵ from the black hole event horizon. EFT methods underline that the microscopic length ϵ is a regulator scale and so drops out of all physical predictions (as regulators always do), and this makes the EFT framework particularly useful for understanding the physical significance[1] of the length-scales involved in these types of models. This type of EFT can be studied within the hotspot framework by allowing the radius ξ of the interaction sphere $\mathcal{S}_\xi$ remain larger than the cutoff scale: $\epsilon \ll \lambda \ll \xi$. We do not pursue this variant further in this paper.

The remainder of this paper sets up the hotspot framework and derives the equations that govern how ϕ responds to the hotspot (topics that dominate the discussion of section 2). Along the way we also make some preliminary explorations of its physical implications (with more to follow in [53, 54]). We find in particular the following noteworthy properties.

- The field equations satisfied by the Heisenberg-picture field ϕ are solved explicitly under the assumption that the hotspot couplings g_a of (1.1) turn on suddenly at time

[1] In particular, the relevant physical scale involves both couplings and the intrinsic UV length scale, and so for weak coupling is often smaller than are the physical length scales of any micro-physics that may be involved [71, 72].

$t = 0$ and remain constant thereafter. The result is first computed perturbatively in the hotspot coupling g_a in section 3, and then as an exact expression in section 5. Using the mode expansion of (2.25) and (2.26) our perturbative solution for the mode functions appearing in ϕ is given in (3.1) and (3.2), while the exact result is given in (5.10) and (5.11). The quantity $\tilde{g}$ appearing in these expressions is defined by $\tilde{g}^2 := \sum_a g_a^2 = Ng^2$.

Using the Heisenberg picture allows us to work in position space where we can follow the passage of the initial transient wave (generated by the turn-on of the couplings) as well as watch how the ϕ field settles down at later times in the on-going presence of the hotspot coupling. Computing both exact and perturbative results allows us to identify precisely which small dimensionless parameter controls the perturbative expansion.

- The Heisenberg-picture evolution is used to compute the Wightman function $\mathcal{W}(t, \mathbf{x}; t', \mathbf{x}') = \langle \phi(t, \mathbf{x})\, \phi(t', \mathbf{x}') \rangle$ for the external field, assuming the ϕ field starts in its vacuum at $t = 0^-$ and the χ^a fields are prepared in a thermal state. The perturbative result is given in (3.10) while the exact expression is in (5.14), (5.17) and (5.20). These results are computed for arbitrary spacetime separations for the fields, but we also obtain specific formulae for the regime where $t > |\mathbf{x}|$ and $t' > |\mathbf{x}'|$, but $t - t'$ is otherwise arbitrary.

  This result has a thermal character (in the sense that its temperature-dependent part satisfies a detailed-balance relation — the Kubo-Martin-Schwinger (or KMS) condition [73, 74] — though does so in a way that depends on the distance from the hotspot.

- Section 4 detours to explore the consequences of supplementing the basic hotspot interaction of (1.1) with a self-coupling, also localized at the hotspot, having the form
$$\mathcal{L}_\lambda = -\frac{\lambda}{2}\, \phi^2\,. \tag{1.2}$$
  Including this coupling is not simply an intellectual exercise because its presence is often required to renormalize divergences that arise because fields like ϕ diverge at the hotspot position once couplings are turned on there. As is well-known from other contexts [76–87] having fields divergence at the position of a source like this is fairly generic — the simplest example being the Coulomb potential diverging at the position of a source charge. From an EFT perspective the presence of couplings like λ is often compulsory, because the requirement that UV divergences drop out of physical observables causes the couplings to run and $\lambda = 0$ need not be a fixed point of this renormalization-group (RG) flow.

  Section 4 computes the renormalization-group evolution implied for the coupling λ in the hotspot model, along the way showing how this can be used to resum contributions to all orders in $\lambda(\epsilon)/4\pi\epsilon$ (where ϵ is a near-hotspot regularization scale) along the lines explored in [80–82]. The λ-dependence of the Wightman function is also given in the general expressions quoted above, and comparison with the exact result — cf. eqs. (5.14), (5.17) and (5.20) — verifies how the RG resummation captures the λ-dependence of the full expression.

Finally, section 6 briefly summarizes some of our conclusions and discusses some directions for future work. Many of the calculational details are given in a collection of appendices.

2 Modelling the hotspot

This section sets up the benchmark model whose properties we study. We do so using the language of open systems, with degrees of freedom divided up into an observable system and an 'environment' — a proxy for the black hole interior — whose properties are never measured.

2.1 Hotspot definition

For the observable sector we choose a single real scalar field, $\phi(x)$, and take it to live in a spatial region, $\mathcal{R}_+$, of infinite extent. The environment is given by N real scalar fields, χ^a with $a = 1, \cdots, N$, that reside in a different spatial region $\mathcal{R}_-$. While one or both of $\mathcal{R}_+, \mathcal{R}_-$ could in principle be curved, we take them to be flat for simplicity. We also take all of these fields to be massless.

We suppose that the fields interact with one another locally and only do so on a relatively small codimension-1 2-sphere, $\mathcal{S}_\xi$, of radius ξ which is the only place where $\mathcal{R}_+$ and $\mathcal{R}_-$ actually touch one another (see figure 1). In practice this means that both $\mathcal{R}_+$ and $\mathcal{R}_-$ have a small sphere excised from the origin (for all time) and the surface of this sphere is identified in the two spaces.

Our interest for much of this paper is in scales much larger than ξ and so consider the idealization of taking $\xi \to 0$, in which case $\mathcal{S}_\xi$ reduces to a single point of contact between $\mathcal{R}_+$ and $\mathcal{R}_-$, which we take to be the origin $\mathbf{x} = \mathbf{0}$ of both $\mathcal{R}_\pm$. In this limit the couplings of ϕ to χ^a are captured by an effective action localized at $\mathbf{x} = 0$.

2.1.1 Action and Hamiltonian

The action that defines the model is therefore taken to be $S = S_+ + S_- + S_{\rm int}$ where the kinematics of ϕ and χ^a are described by

$$S_+ = -\frac{1}{2}\int_{\mathcal{R}_+} \mathrm{d}^4x \; \partial_\mu\phi\, \partial^\mu\phi \quad \text{and} \quad S_- = -\frac{1}{2}\int_{\mathcal{R}_-} \mathrm{d}^4x \; \delta_{ab}\, \partial_\mu\chi^a\partial^\mu\chi^b \,. \tag{2.1}$$

Our later interest is usually in the case where the χ^a couplings do not break the $O(N)$ symmetry of their kinetic terms.

The lowest-dimension interaction (mixing, really) that involves ϕ on the interaction surface is given by

$$S_{\rm int} = -\int_{\mathcal{S}^t_\xi} \mathrm{d}^4x \; \left[G_a\, \chi^a \phi + \frac{G_\phi}{2}\, \phi^2 \right] \,, \tag{2.2}$$

in which the integration is over the world-tube, $\mathcal{S}^t_\xi$ swept out by the surface $\mathcal{S}_\xi$ over time. The Einstein summation convention applies, so there is an implied sum over a. The couplings G_a and G_ϕ here have dimension mass: $[G_a] = [G_\phi] = +1$.

In the limit $\xi \to 0$ the 2-sphere $\mathcal{S}_\xi$ degenerates to a point and this interaction becomes

$$S_{\rm int} \simeq -\int \mathrm{d}t\;\left[g_a\,\chi^a(t,\mathbf{0})\,\phi(t,\mathbf{0}) + \frac{\lambda}{2}\,\phi^2(t,\mathbf{0})\right]\,, \tag{2.3}$$

where the integration is over the proper time of the interaction point $\mathbf{x} = 0$ in both $\mathcal{R}_+$ and $\mathcal{R}_-$. The couplings appearing here are $g_a = 4\pi\xi^2 G_a$ and $\lambda = 4\pi\xi^2 G_\phi$ and have dimensions $[g_a] = [\lambda] = -1$. Although the coupling λ might seem unnecessary, in later sections we see how it can be generated by the presence of the couplings g_a.

In what follows we allow the couplings g_a and λ to depend on time, and in particular will use this time dependence to turn on suddenly the interaction between the fields at $t = 0$. Doing so allows us both to study transient effects associated with the couplings turning on as well as late-time effects after the transients have passed.

The quantization of this model follows closely the treatment of a field coupled to a central qubit given in [46]. The canonical momenta for this problem are

$$\mathfrak{p} := \partial_t\phi \qquad \text{and} \qquad \Pi_a := \delta_{ab}\,\partial_t\chi^b\,, \tag{2.4}$$

and quantization proceeds by demanding these satisfy the equal-time commutation relations

$$\Big[\phi(t,\mathbf{x}),\mathfrak{p}(t,\mathbf{y})\Big] = i\delta^3(\mathbf{x}-\mathbf{y}) \quad \text{and} \quad \Big[\chi^a(t,\mathbf{x}),\Pi_b(t,\mathbf{y})\Big] = i\delta^a_b\delta^3(\mathbf{x}-\mathbf{y})\,. \tag{2.5}$$

The free Hamiltonian is $H_0 := \mathcal{H}_+ \otimes \mathcal{I}_- + \mathcal{I}_+ \otimes \mathcal{H}_-$, where $\mathcal{H}_\pm$ and $\mathcal{I}_\pm$ are the Hamiltonian and identity operators acting separately within the ϕ- and χ-sectors of the Hilbert space. Explicitly

$$\mathcal{H}_+ := \frac{1}{2}\int_{\mathcal{R}_+}\mathrm{d}^3\mathbf{x}\,\left[\mathfrak{p}^2 + (\boldsymbol{\nabla}\phi)^2\right] \quad \text{and} \quad \mathcal{H}_- := \frac{1}{2}\int_{\mathcal{R}_-}\mathrm{d}^3x\,\left[\delta^{ab}\Pi_a\Pi_b + \delta_{ab}\boldsymbol{\nabla}\chi^a\cdot\boldsymbol{\nabla}\chi^b\right]. \tag{2.6}$$

The interaction Hamiltonian (in the limit of a point-like interaction surface) is similarly

$$H_{\rm int} = g_a\,\phi(t,\mathbf{0}) \otimes \chi^a(t,\mathbf{0}) + \frac{\lambda}{2}\,\phi^2(t,\mathbf{0}) \otimes \mathcal{I}_-\;. \tag{2.7}$$

2.1.2 Initial conditions and the sudden approximation

For later calculations we assume the state of the total system at $t = 0$ to be of the form

$$\rho(0) = \rho_+ \otimes \rho_-\,, \tag{2.8}$$

for separate density matrices $\rho_\pm$ in the two sectors. In general, interactions introduce correlations and so do not preserve this factorized form, and it is for this reason that we imagine the couplings between ϕ and χ^a to be initially absent, being turned on suddenly with

$$g_a(t) = \Theta(t)\,g_a\,, \tag{2.9}$$

where $\Theta(t)$ is the Heaviside step function. This allows us to prepare initially uncorrelated states and then observe how the joint system reacts to the onset of coupling.

In practice we choose the ϕ sector initially to be in its vacuum,

$$\rho_+ = |\text{vac}\rangle\langle\text{vac}| \tag{2.10}$$

where $|\text{vac}\rangle$ is the standard Minkowski vacuum defined by $\mathfrak{a}_{\mathbf{p}}|\text{vac}\rangle = 0$. With eventual comparison to black holes in mind we take the χ^a sector to be in a thermal state,

$$\rho_- = \varrho_\beta := \frac{e^{-\beta\mathcal{H}_-}}{\text{Tr}'[e^{-\beta\mathcal{H}_-}]}\,, \tag{2.11}$$

with inverse temperature $\beta > 0$. The prime on the trace indicates that it is only taken over the χ sector.

2.2 Time evolution in different pictures

Our goal is to solve for the time-evolution of the ϕ-sector of the system and because S_{int} is bilinear in ϕ and χ^a the system's evolution can be evaluated in quite some generality. An exact solution is in particular given in section 5, after first detouring in section 3 to describe an approximate solution that is evaluated perturbatively in the following combination of hotspot couplings

$$\tilde{g}^2 := \delta^{ab} g_a g_b = N g^2\,, \tag{2.12}$$

where the second equality specializes to the case where all couplings are equal.

Although not required when solving the model, a large-N limit can be defined wherein the coupling $\tilde{g}$ is held fixed (and need not itself be particularly small) as $N \to \infty$. This limit is briefly discussed in section 3.3, where it is shown that the behaviour of the χ^a fields becomes particularly simple since they become oblivious to the presence of the ϕ field. The large-N limit is not used elsewhere in this paper, besides in section 3.3.

2.2.1 Interaction picture

For perturbative evaluation we first diagonalize the free Hamiltonian. This is done in the usual way, by writing (with time-dependence as appropriate for the interaction picture)

$$\phi(x) = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}}\Big[e^{ip\cdot x}\mathfrak{a}_{\mathbf{p}} + e^{-ip\cdot x}\mathfrak{a}^*_{\mathbf{p}}\Big] \quad \text{and} \quad \chi^a(x) = \int \frac{\mathrm{d}^3p}{\sqrt{(2\pi)^2 2E_p}}\Big[e^{ip\cdot x}\mathfrak{b}^a_{\mathbf{p}} + e^{-ip\cdot x}\mathfrak{b}^{a*}_{\mathbf{p}}\Big] \tag{2.13}$$

where $p\cdot x := p_\mu x^\mu = -E_p t + \mathbf{p}\cdot\mathbf{x}$ with $E_p = |\mathbf{p}|$, and the canonical commutation relations imply the usual creation- and annihilation-operator algebra: $[\mathfrak{a}_{\mathbf{p}}, \mathfrak{a}_{\mathbf{q}}] = [\mathfrak{b}^a_{\mathbf{p}}, \mathfrak{b}^b_{\mathbf{q}}] = 0$ (together with their adjoints) and $[\mathfrak{a}_{\mathbf{p}}, \mathfrak{a}^*_{\mathbf{q}}] = \delta^3(\mathbf{p}-\mathbf{q})$ while $[\mathfrak{b}^a_{\mathbf{p}}, \mathfrak{b}^{b*}_{\mathbf{q}}] = \delta^{ab}\delta^3(\mathbf{p}-\mathbf{q})$. This diagonalizes $\mathcal{H}_\pm$:

$$\mathcal{H}_+ = \int \mathrm{d}^3p\, \frac{E_p}{2}\Big[\mathfrak{a}_{\mathbf{p}}\mathfrak{a}^*_{\mathbf{p}} + \mathfrak{a}^*_{\mathbf{p}}\mathfrak{a}_{\mathbf{p}}\Big] \quad \text{and} \quad \mathcal{H}_- = \int \mathrm{d}^3p\, \frac{E_p}{2}\delta_{ab}\Big[\mathfrak{b}^a_{\mathbf{p}}\mathfrak{b}^{b*}_{\mathbf{p}} + \mathfrak{b}^{a*}_{\mathbf{p}}\mathfrak{b}^b_{\mathbf{p}}\Big]\,. \tag{2.14}$$

The interaction-picture interaction Hamiltonian in the pointlike limit ($\xi \to 0$) similarly becomes

$$\begin{aligned} H_{\rm int}(t) &= g_a \phi(t, \mathbf{0}) \otimes \chi^a(t, \mathbf{0}) + \frac{\lambda}{2}\phi^2(t, \mathbf{0}) \otimes \mathcal{I}_- \\ &= \int \frac{\mathrm{d}^3 p\, \mathrm{d}^3 q}{2(2\pi)^3\sqrt{E_p E_q}} \bigg[g_a \big(\mathfrak{a}_{\mathbf{p}} e^{-iE_p t} + \mathfrak{a}^*_{\mathbf{p}} e^{+iE_p t}\big) \otimes \big(\mathfrak{b}^a_{\mathbf{q}} e^{-iE_q t} + \mathfrak{b}^{a*}_{\mathbf{q}} e^{+iE_q t}\big) \\ &\qquad + \frac{\lambda}{2}\big(\mathfrak{a}_{\mathbf{p}} e^{-iE_p t} + \mathfrak{a}^*_{\mathbf{p}} e^{+iE_p t}\big)\big(\mathfrak{a}_{\mathbf{q}} e^{-iE_q t} + \mathfrak{a}^*_{\mathbf{q}} e^{+iE_q t}\big) \otimes \mathcal{I}_- \bigg] . \end{aligned} \tag{2.15}$$

Matrix elements of this can be used in standard fashion to compute the evolution of the system's state.

2.2.2 Heisenberg picture

Later sections solve explicitly for time evolution, and do so by solving how the fields evolve in Heisenberg picture, including the effects of the couplings in $H_{\rm int}$. To this end it is worth briefly setting up the Heisenberg picture quantities and in particular exposing differences from the interaction-picture description given above.

Keeping in mind that we later entertain time-dependent couplings, $g_a(t)$, the full time-evolution operator $U(t, t')$ can be defined as the solution to $\partial_t U(t, t_0) = -iH(t)U(t, t_0)$ that satisfies $\mathrm{U}(t = t_0) = \mathcal{I}$. This leads to the usual time-ordered form

$$U(t, t_0) = \mathcal{T} \exp\left(-i\int_{t_0}^t \mathrm{d}s\ H(s)\right) . \tag{2.16}$$

It is this transformation that is used to construct time-dependent Heisenberg-picture operators, $A_H(t)$, from Schrödinger-picture operators, A_S, using:

$$A_H(t) = U^*(t, 0) A_S \mathrm{U}(t, 0) . \tag{2.17}$$

We assume here that the two pictures agree at $t = 0$.

A virtue of transforming to the Heisenberg picture that the state does not evolve at all. In Heisenberg picture it is the field operators that carry the burden of any time evolution when computing correlation functions or transition amplitudes. This means that the factorized form (2.8) for ρ can also be used at later times, ensuring that χ^a-sector expectation values can always be taken using the thermal state (2.11).

Eq. (2.17) implies in particular that the Heisenberg picture field operators are given by

$$\phi_H(t, \mathbf{x}) := U^*(t, 0)\big[\phi_S(\mathbf{x}) \otimes \mathcal{I}_-\big]U(t, 0) \quad \text{and} \quad \chi^a_H(t, \mathbf{x}) := U^*(t, 0)\big[\mathcal{I}_+ \otimes \chi^a_S(\mathbf{x})\big]U(t, 0) , \tag{2.18}$$

and similarly for their conjugate momenta. An important conceptual point about this definition is that the presence of the interaction term in H implies that the Heisenberg field operators do not only act separately on the two sectors of the Hilbert space. In particular, expansion of $\phi_H(t, \mathbf{x})$ in terms of creation and annihilation operators involve *both* $\mathfrak{a}_{\mathbf{p}}$ and $\mathfrak{b}^a_{\mathbf{p}}$, as does the expansion of the $\chi^a_H(t, \mathbf{x})$.

In later sections the time-evolution of the fields ϕ_H and χ^a_H is determined by explicitly integrating their Heisenberg-picture field equations. These express the differential version of (2.17),

$$\partial_t A_H(t) = -iU^*(t,0)[A_S, H_S(t)]\mathrm{U}(t,0) = -i[A_H(t), H_H(t)] \; . \tag{2.19}$$

To work out the implications of (2.19) for the field operators explicitly we first record the following Schrödinger-picture commutators with the full Hamiltonian

$$-\,i\Big[\phi_S(\mathbf{x}), H_S(t)\Big] = \mathfrak{p}_S(\mathbf{x})\,, \qquad -i\Big[\chi^a_S(\mathbf{x}), H^S(t)\Big] = \Pi^a_S(\mathbf{x})\,, \tag{2.20}$$

$$-\,i\Big[\Pi^a_S(\mathbf{x}), H_S(t)\Big] = \nabla^2\chi^a_S(\mathbf{x}) - g_a\delta^3(\mathbf{x})\;\phi_S(\mathbf{0})$$

and

$$-\,i\Big[\mathfrak{p}_S(\mathbf{x}), H_S(t)\Big] = \nabla^2\phi_S(\mathbf{x}) - \delta^3(\mathbf{x})\Big(g_a\;\chi^a_S(\mathbf{0}) + \lambda\phi_S(\mathbf{0})\Big)\,. \tag{2.21}$$

Using these in (2.19) yields the equations of motion[2]

$$\left(-\partial_t^2 + \nabla^2\right)\phi_H(t,\mathbf{x}) = \delta^3(\mathbf{x})\Big[\lambda\phi_H(t,\mathbf{0}) + g_a\chi^a_H(t,\mathbf{0})\Big] \tag{2.22}$$

and

$$\delta_{ab}\left(-\partial_t^2 + \nabla^2\right)\chi^b_H\,(t,\mathbf{x}) = \delta^3\,(\mathbf{x})\;\;g_a\phi_H\,(t,\mathbf{0})\;. \tag{2.23}$$

These equations can be solved because they are linear in all of the fields, a consequence of H_{int} describing more of a mixing between ϕ and χ^a than an honest-to-God interaction. It is convenient to do so by first expanding the fields in terms of mode functions and then using the field equations to set up a coupled series of linear differential equations. That is, writing

$$\phi_H(t,\mathbf{x}) = \phi(t,\mathbf{x}) + \hat{\phi}(t,\mathbf{x}) \quad \text{and} \quad \chi^a_H(t,\mathbf{x}) = \chi^a(t,\mathbf{x}) + \hat{\chi}^a(t,\mathbf{x})\,, \tag{2.24}$$

with ϕ and χ^a being the interaction-picture fields given by (2.13), then the deviations from the interaction picture are

$$\hat{\phi}\,(t,\mathbf{x}) = \int\frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3\,2E_p}}\Big\{\Big[S_{\mathbf{p}}\,(t,\mathbf{x})\,\mathfrak{a}_{\mathbf{p}} + S^*_{\mathbf{p}}\,(t,\mathbf{x})\,\mathfrak{a}^*_{\mathbf{p}}\Big]\otimes\mathcal{I}_- + \mathcal{I}_+\otimes\,\delta_{ab}\Big[s^a_{\mathbf{p}}\,(t,\mathbf{x})\;\mathfrak{b}^b_{\mathbf{p}} + s^{a*}_{\mathbf{p}}\,(t,\mathbf{x})\;\mathfrak{b}^{b*}_{\mathbf{p}}\Big]\Big\} \tag{2.25}$$

and

$$\hat{\chi}^a(t,\mathbf{x}) = \int\frac{\mathrm{d}^3p}{\sqrt{(2\pi)^3 2E_p}}\Big\{\Big[R^a_{\mathbf{p}}(t,\mathbf{x})\,\mathfrak{a}_{\mathbf{p}} + R^a_{\mathbf{p}}(t,\mathbf{x})^*\,\mathfrak{a}^*_{\mathbf{p}}\Big]\otimes\mathcal{I}_- + \mathcal{I}_+\otimes\,\delta_{bc}\Big[r^{ab}_{\mathbf{p}}(t,\mathbf{x})\mathfrak{b}^c_{\mathbf{p}} + r^{ab*}_{\mathbf{p}}(t,\mathbf{x})\mathfrak{b}^{c*}_{\mathbf{p}}\Big]\Big\} \tag{2.26}$$

where the to-be-determined mode functions $\{S_{\mathbf{p}}, s^a_{\mathbf{p}}, R^a_{\mathbf{p}}, r^{ab}_{\mathbf{p}}\}$ vanish in the absence of H_{int}.

[2] Use of (2.19) assumes no further time-dependence arises through a time-dependence of couplings after they are initially turned on, which amounts to assuming the 'sudden' approximation when turning on couplings at $t = 0$.

Inserting (2.25) and (2.26) into the Heisenberg equations of motion (2.22) and (2.23) leads to the following set of coupled equations for the mode functions $\{S_{\mathbf{p}}, s^a_{\mathbf{p}}, R^a_{\mathbf{p}}, r^{ab}_{\mathbf{p}}\}$:

$$\begin{aligned}\left(-\partial_t^2 + \nabla^2\right) S_{\mathbf{p}}(t,\mathbf{x}) &= \left[\lambda\left(e^{-iE_pt} + S_{\mathbf{p}}(t,\mathbf{0})\right) + g_b R^b_{\mathbf{p}}(t,\mathbf{0})\right]\delta^3(\mathbf{x}) \\ \left(-\partial_t^2 + \nabla^2\right) s^a_{\mathbf{p}}(t,\mathbf{x}) &= \left[\lambda\, s^a_{\mathbf{p}}(t,\mathbf{0}) + \delta^{ab} g_b\, e^{-iE_pt} + g_b\, r^{ba}_{\mathbf{p}}(t,\mathbf{0})\right]\delta^3(\mathbf{x}) \\ \delta_{ab}\left(-\partial_t^2 + \nabla^2\right) R^b_{\mathbf{p}}(t,\mathbf{x}) &= g_a\left[e^{-iE_pt} + S_{\mathbf{p}}(t,\mathbf{0})\right]\delta^3(\mathbf{x}) \\ \delta_{ac}\left(-\partial_t^2 + \nabla^2\right) r^{cb}_{\mathbf{p}}(t,\mathbf{x}) &= g_a s^b_{\mathbf{p}}(t,\mathbf{0})\,\delta^3(\mathbf{x})\,.\end{aligned} \tag{2.27}$$

2.3 Integrating out χ^a

We wish to understand how the ϕ field responds to the presence of the hotspot, and we do so under the assumption that no measurements directly involve the fields χ^a. Because no χ^a measurements are made the χ^a mode functions can be solved as functions of the ϕ mode functions to obtain a reduced set of equation to solve.

To see how this works explicitly consider preparing the ϕ field in its vacuum and then suddenly turn on hotspot couplings at $t = 0$. This should generate a flurry of transient behaviour before the ϕ field settles down at late times into a new adiabatic vacuum whose properties we wish to compute. To this end write $g_a(t) = g_a\,\Theta(t)$ and $\lambda(t) = \lambda\,\Theta(t)$, and so the time-dependence of eqs. (2.27) can be made more explicit:

$$\begin{aligned}\left(-\partial_t^2 + \nabla^2\right) S_{\mathbf{p}}\left(t,\mathbf{x}\right) &= \Theta\left(t\right)\left[\lambda\left[e^{-iE_pt} + S_{\mathbf{p}}\left(t,\mathbf{0}\right)\right] + g_a R^a_{\mathbf{p}}\left(t,\mathbf{0}\right)\right]\delta^3\left(\mathbf{x}\right) \\ \delta_{ab}\left(-\partial_t^2 + \nabla^2\right) R^b_{\mathbf{p}}\left(t,\mathbf{x}\right) &= \Theta\left(t\right)\, g_a\left[e^{-iE_pt} + S_{\mathbf{p}}\left(t,\mathbf{0}\right)\right]\delta^3\left(\mathbf{x}\right)\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\left(-\partial_t^2 + \nabla^2\right) s^a_{\mathbf{p}}\left(t,\mathbf{x}\right) &= \Theta\left(t\right)\left[\lambda\, s^a_{\mathbf{p}}\left(t,\mathbf{0}\right) + \delta^{ab} g_b\, e^{-iE_pt} + g_b\, r^{ba}_{\mathbf{p}}\left(t,\mathbf{0}\right)\right]\delta^3\left(\mathbf{x}\right) \\ \left(-\partial_t^2 + \nabla^2\right) r^{ab}_{\mathbf{p}}\left(t,\mathbf{x}\right) &= \Theta\left(t\right)\,\delta^{ac} g_c s^b_{\mathbf{p}}\left(t,\mathbf{0}\right)\,\delta^3\left(\mathbf{x}\right)\,.\end{aligned} \tag{2.29}$$

These are to be solved subject to the initial conditions

$$\begin{aligned}& S_{\mathbf{p}}(0,\mathbf{x}) = \partial_t S_{\mathbf{p}}(0,\mathbf{x}) = s^a_{\mathbf{p}}(0,\mathbf{x}) = \partial_t s^a_{\mathbf{p}}(0,\mathbf{x}) = 0 \\ \text{and}\quad & R^a_{\mathbf{p}}(0,\mathbf{x}) = \partial_t R^a_{\mathbf{p}}(0,\mathbf{x}) = r^{ab}_{\mathbf{p}}(0,\mathbf{x}) = \partial_t r^{ab}_{\mathbf{p}}(0,\mathbf{x}) = 0\,.\end{aligned} \tag{2.30}$$

2.3.1 Solving the χ^a equations

The mode functions associated with χ^a can be eliminated from the coupled equations (2.28) and (2.29) with initial conditions (2.30) by using the retarded propagator

$$G_R(t,\mathbf{x};t',\mathbf{y}) = \frac{\Theta(t-t')}{4\pi|\mathbf{x}-\mathbf{y}|}\,\delta\Big[(t-t') - |\mathbf{x}-\mathbf{y}|\Big]\,, \tag{2.31}$$

that satisfies the equation of motion

$$\left(-\partial_t^2 + \nabla^2\right) G_R(t,\mathbf{x};t',\mathbf{y}) = -\delta(t-t')\delta^3(\mathbf{x}-\mathbf{y})\,. \tag{2.32}$$

In terms of this the formal solutions for $R^a_{\mathbf{p}}$ and $r^{ab}_{\mathbf{p}}$ (the mode functions appearing in χ^a) are

$$\begin{aligned} R^a_{\mathbf{p}}(t,\mathbf{x}) &= -\delta^{ab} g_b \int_0^\infty \mathrm{d}s\; G_R(t,\mathbf{x};\tau,\mathbf{0}) \Big[e^{-iE_p \tau} + S_{\mathbf{p}}(\tau,\mathbf{0}) \Big] \\ &= -\delta^{ab} g_b \frac{\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|} \Big[e^{-iE_p(t-|\mathbf{x}|)} + S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) \Big] \end{aligned} \tag{2.33}$$

and

$$r^{ab}_{\mathbf{p}}(t,\mathbf{x}) = -\delta^{ac} g_c \frac{\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|} s^b_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) \,. \tag{2.34}$$

These solutions have support only in the forward lightcone of the event where the couplings turn on, and there give the mode functions at a distance $r = |\mathbf{x}|$ from the hotspot in terms of their values at the hotspot position, but as a function of the retarded time $t_r := t - r$ and with an amplitude that is suppressed by a power of $1/r$.

Using these solutions to eliminate $R^a_{\mathbf{p}}$ and $r^{ab}_{\mathbf{p}}$ from (2.28) and (2.29) leaves a coupled set of equations involving only the mode functions appearing in ϕ:

$$\begin{aligned} \Big(-\partial_t^2 + \nabla^2\Big) S_{\mathbf{p}}(t,\mathbf{x}) = \Theta(t)\bigg(& \lambda \big[e^{-iE_p t} + S_{\mathbf{p}}(t,\mathbf{0}) \big] \\ & - \frac{\tilde{g}^2 \Theta(t-|\mathbf{y}|)}{4\pi|\mathbf{y}|} \Big[e^{-iE_p(t-|\mathbf{y}|)} + S_{\mathbf{p}}(t-|\mathbf{y}|,\mathbf{0}) \Big] \bigg|_{|\mathbf{y}|=0} \bigg) \delta^3(\mathbf{x}) \end{aligned} \tag{2.35}$$

as well as

$$\begin{aligned} \Big(-\partial_t^2 + \nabla^2\Big) s^a_{\mathbf{p}}(t,\mathbf{x}) = \Theta(t)\bigg[& \lambda\, s^a_{\mathbf{p}}(t,\mathbf{0}) + \frac{\tilde{g}}{\sqrt{N}}\, e^{-iE_p t} \\ & - \frac{\tilde{g}^2 \Theta(t-|\mathbf{y}|)}{4\pi|\mathbf{y}|} s^a_{\mathbf{p}}(t-|\mathbf{y}|,\mathbf{0}) \bigg|_{|\mathbf{y}|=0} \bigg) \delta^3(\mathbf{x}) \end{aligned} \tag{2.36}$$

where we specialize to the case where all of the g_a's have the same size, and use (2.12) to write $g_a = \tilde{g}/\sqrt{N}$ for all a. The factor of $\sqrt{N}$ is extracted here for convenience because it cancels an explicit factor of N that comes from the summation over the index 'a' in (2.35).

Eqs. (2.35) and (2.36) reveal a characteristic 'Coloumb' singularity as $|\mathbf{y}| \to 0$, which at face value appears to threaten any program to solve (2.35) and (2.36) iteratively as a series in $\tilde{g}^2$ and λ. In what follows, this divergence at $|\mathbf{y}| = 0$ is regularized by instead evaluating $\mathbf{y}$ at the microscopically small scale $|\mathbf{y}| = \epsilon$. This divergence problem is a general issue that arises when exploring effective field theories describing compact sources, where the domain of validity of the low-energy/long-wavelength theory does not allow sufficient spatial resolution to resolve the source's structure; it is generic that external fields diverge at the position of a compact source.

But the example of the Coulomb field for a small charge distribution also suggests that evaluating the $1/r$ divergence at $r = 0$ is really an artefact of trying to extrapolate to zero an external solution that is not actually appropriate in the microscopic theory within which the source's structure can be resolved. A general EFT treatment of these issues is possible [10, 80–84] (and tested in detail calculating nuclear finite-size effects in atoms [85–87]), and shows how all such divergences get renormalized by the effective couplings in the action — such as the coupling λ of hotspot action (2.3) — that describes the source's low-energy properties (as we also see in detail below).

2.3.2 Renormalization of λ and ϵ-regularization

Regulating the field equations on the microscopic surface $|\mathbf{y}| = \epsilon$ allows (2.35) and (2.36) to be rewritten

$$\left(-\partial_t^2+\nabla^2\right) S_{\mathbf{p}}(t,\mathbf{x}) = \left(\Theta(t)\lambda\left[e^{-iE_pt}+S_{\mathbf{p}}(t,\mathbf{0})\right] - \frac{\tilde{g}^2\Theta(t-\epsilon)}{4\pi\epsilon}\left[e^{-iE_p(t-\epsilon)}+S_{\mathbf{p}}(t-\epsilon,\mathbf{0})\right]\right)\delta^3(\mathbf{x}) \tag{2.37}$$

and

$$\left(-\partial_t^2 + \nabla^2\right) s^a_{\mathbf{p}}(t,\mathbf{x}) = \left(\Theta(t)\left[\lambda\, s^a_{\mathbf{p}}(t,\mathbf{0}) + \frac{\tilde{g}}{\sqrt{N}}\, e^{-iE_pt}\right] - \frac{\tilde{g}^2\Theta(t-\epsilon)}{4\pi\epsilon} s^a_{\mathbf{p}}(t-\epsilon,\mathbf{0})\right)\delta^3(\mathbf{x}) \tag{2.38}$$

where we use $\Theta(t)\Theta(t-\epsilon) = \Theta(t-\epsilon)$ since $\epsilon > 0$.

These equations can also be formally integrated using the retarded propagator (2.31) to give

$$\begin{aligned} S_{\mathbf{p}}(t,\mathbf{x}) &= -\frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\left(e^{-iE_p(t-|\mathbf{x}|)} + S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\right) \\ &\quad + \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|-\epsilon)}{16\pi^2|\mathbf{x}|\epsilon}\left[e^{-iE_p(t-|\mathbf{x}|-\epsilon)} + S_{\mathbf{p}}(t-|\mathbf{x}|-\epsilon,\mathbf{0})\right] \\ &\simeq -\left(\frac{\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\left[\lambda - \frac{\tilde{g}^2}{4\pi\epsilon}\right] + \frac{\tilde{g}^2\delta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\right)\left[e^{-iE_p(t-|\mathbf{x}|)} + S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\right] \\ &\quad - \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\left[(-iE_p)e^{-iE_p(t-|\mathbf{x}|)} + \partial_t S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\right] \end{aligned} \tag{2.39}$$

and

$$\begin{aligned} s^a_{\mathbf{p}}(t,\mathbf{x}) &= -\frac{\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\left[\lambda\, s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) + \frac{\tilde{g}}{\sqrt{N}}\, e^{-iE_p(t-|\mathbf{x}|)}\right] + \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|-\epsilon)}{16\pi^2|\mathbf{x}|\epsilon} s^a_{\mathbf{p}}(t-|\mathbf{x}|-\epsilon,\mathbf{0}) \\ &\simeq -\left(\frac{\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\left[\lambda - \frac{\tilde{g}^2}{4\pi\epsilon}\right] - \frac{\tilde{g}^2\delta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\right) s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) \\ &\quad - \frac{\tilde{g}\Theta(t-|\mathbf{x}|)}{4\pi\sqrt{N}|\mathbf{x}|} e^{-iE_p(t-|\mathbf{x}|)} - \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\partial_t s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) \end{aligned} \tag{2.40}$$

where the approximate equalities exploit the fact that ϵ is a microscopic quantity to expand each of the last terms in powers of ϵ, and dropping terms that are $\mathcal{O}(\epsilon)$. Note that this expansion in ϵ implicitly assumes that $E_p\epsilon \ll 1$ for all modes when expanding the exponential function.

Although the $1/\epsilon$ term diverges, this divergence can be absorbed by redefining

$$\lambda_R := \lambda - \frac{\tilde{g}^2}{4\pi\epsilon}\,, \tag{2.41}$$

showing that the divergence renormalizes the ϕ self-coupling λ. Dropping the subscript 'R' for notational simplicity, eqs. (2.39) and (2.40) become[3]

$$S_{\mathbf{p}}(t,\mathbf{x}) = -\frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\Big(e^{-iE_p(t-|\mathbf{x}|)} + S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\Big) - \frac{\tilde{g}^2\delta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|} - \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\Big[(-iE_p)e^{-iE_p(t-|\mathbf{x}|)} + \partial_t S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\Big] \tag{2.42}$$

and

$$s^a_{\mathbf{p}}(t,\mathbf{x}) = -\frac{\tilde{g}\,\Theta(t-|\mathbf{x}|)}{4\pi\sqrt{N}|\mathbf{x}|}\,e^{-iE_p(t-|\mathbf{x}|)} - \frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) - \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\partial_t s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\,. \tag{2.43}$$

These are the equations that are to be solved in the next sections to determine the mode functions for ϕ, and from these also determine its response to the hotspot.

3 Perturbative response

This section provides one of the points of comparison for the exact results of section 5. Here we solve eqs. (2.42) and (2.43) iteratively in $\tilde{g}$ and λ, and use the lowest order solutions to determine perturbatively how the fields evolve in time.

3.1 Mode functions

The iterative solution to (2.42) and (2.43) gives the perturbative result

$$S_{\mathbf{p}}(t,\mathbf{x}) \simeq -\left(\lambda - \frac{i\tilde{g}^2E_p}{4\pi}\right)\frac{\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}e^{-iE_p(t-|\mathbf{x}|)} - \frac{\tilde{g}^2\delta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|} \qquad \text{(perturbative)} \tag{3.1}$$

and

$$s^a_{\mathbf{p}}(t,\mathbf{x}) \simeq -\frac{\tilde{g}\,\Theta(t-|\mathbf{x}|)}{4\pi\sqrt{N}|\mathbf{x}|}\,e^{-iE_p(t-|\mathbf{x}|)} \qquad \text{(perturbative)} \tag{3.2}$$

to leading nontrivial order in λ and $\tilde{g}$. The real part of the perturbative solution (3.2) is shown in figure 4, which shows how the result is nonzero only after the passage of the wave-front that radiates out from the turn-on event at $t = \mathbf{x} = 0$.

Using these mode functions in the expansion for ϕ, the leading-order perturbative limit of the Heisenberg-picture fields truncated at order $\tilde{g}^2$ can be written

$$\phi_H(t,\mathbf{x}) \simeq \left(\phi(t,\mathbf{x}) - \frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\phi(t-|\mathbf{x}|,\mathbf{0}) - \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\mathfrak{p}(t-|\mathbf{x}|,\mathbf{0}) - \frac{\tilde{g}^2\delta(t-|\mathbf{x}|)}{16\pi^2|\mathbf{x}|}\phi(0,\mathbf{0})\right)\otimes\mathcal{I}_- - \frac{\tilde{g}\,\Theta(t-|\mathbf{x}|)}{4\pi\sqrt{N}|\mathbf{x}|}\sum_{a=1}^{N}\mathcal{I}_+\otimes\chi^a(t-\mathbf{x},\mathbf{0}) \tag{3.3}$$

where (as above) ϕ and χ are the interaction-picture fields given in (2.13) and $\mathfrak{p} = \partial_t\phi$ is the canonical momentum defined in (2.4). The Heaviside step functions show how $\phi_H(t,\mathbf{x})$ does not respond to the turn-on of the hotspot couplings at $t = 0$ until after the transient wave reaches the particular point $\mathbf{x}$, after which mode interference occurs. The sum over a is written explicitly in (3.3) to underline the necessity of keeping this term, even in the large-N limit despite the factor of $1/\sqrt{N}$.

[3] In arriving at (2.42) and (2.43) we simplify the terms which come multiplied by $\delta(t-|\mathbf{x}|)$ by using the initial conditions to eliminate $S_{\mathbf{p}}(0,\mathbf{0}) = s^a_{\mathbf{p}}(0,\mathbf{0}) = 0$ in the final result.

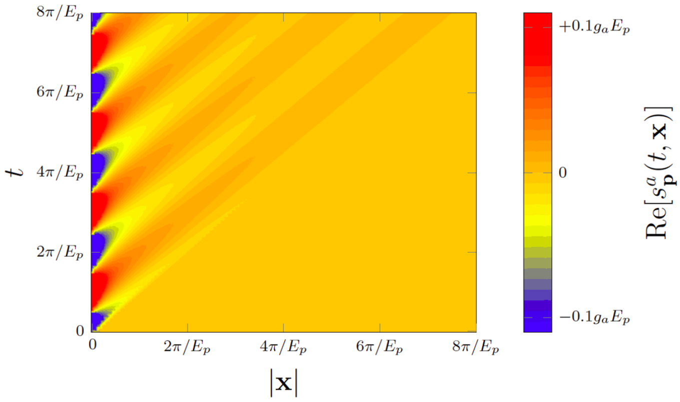


Figure 4: $\mathrm{Re}[s^a_{\mathbf{p}}(t,\mathbf{x})]$ from (3.2) *vs* t and $|\mathbf{x}|$, showing the wave-front emanating from $t = \mathbf{x} = 0$, the growth for small $|\mathbf{x}|$ and the oscillatory behaviour with wavelength set by E_p. (Colour online.).

3.2 Two-point ϕ correlator

The physical implications of the field evolution just calculated gets communicated to observables through field correlators, and because the model considered here is gaussian the two-point function carries all of this information. For observers situated in $\mathcal{R}_+$ only the correlators of the field ϕ can be accessed, and so we therefore next compute the two-point correlator,

$$W_\beta(t,\mathbf{x};t',\mathbf{x}') := \mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t',\mathbf{x}')\rho_0\Big] = \frac{1}{Z_\beta}\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t',\mathbf{x}')\big(|\mathrm{vac}\rangle\langle\mathrm{vac}|\otimes e^{-\beta\mathcal{H}_-}\big)\Big], \tag{3.4}$$

where ρ_0 denotes the system's state, for which we use the state given in (2.8), (2.10) and (2.11). $Z_\beta := \mathrm{Tr}'\left[e^{-\beta\mathcal{H}_-}\right]$ is the partition function for the N thermal χ^a fields.

Using the perturbative solution for ϕ_H given in (3.3) allows the leading-order in $\tilde{g}^2$ and λ contribution to be written in terms of the free correlation functions,

$$\begin{aligned}
&W_\beta\left(t,\mathbf{x};t',\mathbf{x}'\right)\\
&\quad\simeq \langle\mathrm{vac}|\phi\left(t,\mathbf{x}\right)\phi\left(t',\mathbf{x}'\right)|\mathrm{vac}\rangle\\
&\quad-\frac{\lambda\Theta\left(t-|\mathbf{x}|\right)}{4\pi|\mathbf{x}|}\langle\mathrm{vac}|\phi\left(t-|\mathbf{x}|,\mathbf{0}\right)\phi\left(t',\mathbf{x}'\right)|\mathrm{vac}\rangle-\frac{\lambda\Theta\left(t'-|\mathbf{x}'|\right)}{4\pi|\mathbf{x}'|}\langle\mathrm{vac}|\phi\left(t,\mathbf{x}\right)\phi\left(t'-|\mathbf{x}'|,\mathbf{0}\right)|\mathrm{vac}\rangle\\
&\quad+\frac{\tilde{g}^2\Theta\left(t-|\mathbf{x}|\right)\Theta\left(t'-|\mathbf{x}'|\right)}{16\pi^2|\mathbf{x}||\mathbf{x}'|}\mathrm{Tr}'\left[\chi^a\left(t-|\mathbf{x}|,\mathbf{0}\right)\chi^b\left(t'-|\mathbf{x}'|,\mathbf{0}\right)\varrho_\beta\right]\\
&\quad-\frac{\tilde{g}^2\Theta\left(t-|\mathbf{x}|\right)}{16\pi^2|\mathbf{x}|}\langle\mathrm{vac}|\mathfrak{p}\left(t-|\mathbf{x}|,\mathbf{0}\right)\phi\left(t',\mathbf{x}'\right)|\mathrm{vac}\rangle-\frac{\tilde{g}^2\Theta\left(t'-|\mathbf{x}'|\right)}{16\pi^2|\mathbf{x}'|}\langle\mathrm{vac}|\phi\left(t,\mathbf{x}\right)\mathfrak{p}\left(t'-|\mathbf{x}'|,\mathbf{0}\right)|\mathrm{vac}\rangle\\
&\quad-\frac{\tilde{g}^2\delta\left(t-|\mathbf{x}|\right)}{16\pi^2|\mathbf{x}|}\langle\mathrm{vac}|\phi\left(0,\mathbf{0}\right)\phi\left(t',\mathbf{x}'\right)|\mathrm{vac}\rangle-\frac{\tilde{g}^2\delta\left(t'-|\mathbf{x}'|\right)}{16\pi^2|\mathbf{x}'|}\langle\mathrm{vac}|\phi\left(t,\mathbf{x}\right)\phi\left(0,\mathbf{0}\right)|\mathrm{vac}\rangle\,.
\end{aligned} \tag{3.5}$$

This can be simplified using the following explicit forms for the free correlators

$$\langle \text{vac}| \phi\,(t,\mathbf{x})\,\phi\,(t',\mathbf{x}')\,|\text{vac}\rangle = \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]}\,, \tag{3.6}$$

$$\langle \text{vac}| \mathfrak{p}\,(t,\mathbf{x})\,\phi\,(t',\mathbf{x}')\,|\text{vac}\rangle = \frac{t-t'}{2\pi^2\Big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\Big]^2}\,, \tag{3.7}$$

and

$$\langle \text{vac}|\phi\,(t,\mathbf{x})\,\mathfrak{p}\,(t',\mathbf{x}')\,|\text{vac}\rangle = -\langle \text{vac}|\mathfrak{p}(t,\mathbf{x})\phi(t',\mathbf{x}')|\text{vac}\rangle = \frac{-t+t'}{2\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]^2}\,, \tag{3.8}$$

leading to

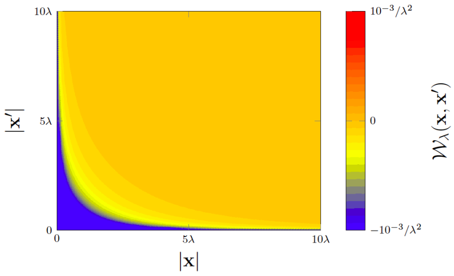
$$\begin{aligned} W_\beta(t,\mathbf{x};t',\mathbf{x}') &\\ \simeq \; & \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]} \\ &+\frac{\lambda}{16\pi^3}\bigg(\frac{\Theta(t-|\mathbf{x}|)}{|\mathbf{x}|}\frac{1}{(t-t'-|\mathbf{x}|-i\delta)^2-|\mathbf{x}'|^2}+\frac{\Theta(t'-|\mathbf{x}'|)}{|\mathbf{x}'|}\frac{1}{(t-t'+|\mathbf{x}'|-i\delta)^2-|\mathbf{x}|^2}\bigg) \\ &-\frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)\Theta(t'-|\mathbf{x}'|)}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\Big[\frac{\pi}{\beta}(t-|\mathbf{x}|-t'+|\mathbf{x}'|-i\delta)\Big]} \\ &+\frac{\tilde{g}^2}{32\pi^4}\bigg(-\frac{\Theta(t-|\mathbf{x}|)}{|\mathbf{x}|}\frac{t-t'-|\mathbf{x}|}{\big[(t-t'-|\mathbf{x}|-i\delta)^2-|\mathbf{x}'|^2\big]^2}+\frac{\Theta(t'-|\mathbf{x}'|)}{|\mathbf{x}'|}\frac{t-t'+|\mathbf{x}'|}{\big[(t-t'+|\mathbf{x}'|-i\delta)^2-|\mathbf{x}|^2\big]^2}\bigg) \\ &+\frac{\tilde{g}^2}{64\pi^4}\bigg(\frac{\delta(t-|\mathbf{x}|)}{|\mathbf{x}|\big[-(t'+i\delta)^2-|\mathbf{x}'|^2\big]}+\frac{\delta(t'-|\mathbf{x}'|)}{|\mathbf{x}'|\big[-(t-i\delta)^2-|\mathbf{x}|^2\big]}\bigg)\,, \end{aligned} \tag{3.9}$$

where the inverse temperature $\beta = 1/T$ arises from the thermal average in the χ sector.

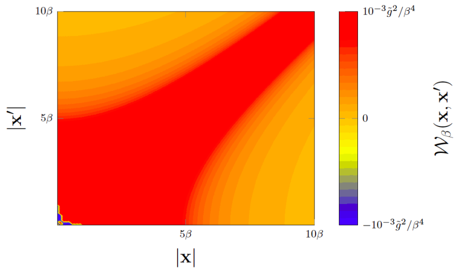
Of particular interest is the form of this result after the passage of the transients, with both t and t' chosen to lie in the future light cone of the switch-on event (i.e. $t > |\mathbf{x}|$ and $t' > |\mathbf{x}'|$). In this region the above expression becomes

$$\begin{aligned} W_\beta\,(t,\mathbf{x};t',\mathbf{x}') &\\ \simeq \; & \frac{1}{4\pi^2\Big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\Big]}+\frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\left[\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2-(|\mathbf{x}+|\mathbf{x}'|)^2}\right] \\ &-\frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\Big[\frac{\pi}{\beta}\,(t-|\mathbf{x}|-t'+|\mathbf{x}'|-i\delta)\Big]} \\ &+\frac{\tilde{g}^2}{32\pi^4}\left(-\frac{1}{|\mathbf{x}|}\frac{t-t'-|\mathbf{x}|}{\Big[(t-t'-|\mathbf{x}|-i\delta)^2-|\mathbf{x}'|^2\Big]^2}+\frac{1}{|\mathbf{x}'|}\frac{t-t'+|\mathbf{x}'|}{\Big[(t-t'+|\mathbf{x}'|-i\delta)^2-|\mathbf{x}|^2\Big]^2}\right) \end{aligned} \tag{3.10}$$

As expected, in this limit the $\mathcal{O}(\lambda)$ and $\mathcal{O}(\tilde{g}^2)$ terms break translation invariance, though time-translation invariance is restored once the transients due to the coupling turn-on have passed. Rotations about the position of the hotspot remain a symmetry. Apart from a



(a) λ-dependent term.



(b) $\tilde{g}^2$-dependent term.

Figure 5: The equal-time $t = t'$ limits of the λ-dependent term and $\tilde{g}^2$-dependent term of the Wightman function given in eq. (3.11). (Colour online.).

global $1/r$ fall-off the thermal $\mathcal{O}(\tilde{g}^2)$ term depends only on the retarded times $t_r = t - |\mathbf{x}|$ and $t'_r = t' - |\mathbf{y}|$, with correlations that die exponentially once $t_r - t'_r \gg \beta$. By contrast, the temperature-independent $\mathcal{O}(\tilde{g}^2)$ term — and the $\mathcal{O}(\lambda)$ contributions — preserve the power-law fall-off for large $t - t'$, but modify its amplitude in a way that becomes less important further from the hotspot.

For some applications it is the equal-time correlator evaluated with $t = t'$ that is of interest (at late times $t > |\mathbf{x}|, |\mathbf{x}'|$). In this case the above simplifies to[4]

$$W_\beta(t,\mathbf{x};t,\mathbf{x}') \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|(|\mathbf{x}|+|\mathbf{x}'|)} - \frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(|\mathbf{x}|-|\mathbf{x}'|)\right]} + \frac{\tilde{g}^2}{16\pi^4(|\mathbf{x}|^2-|\mathbf{x}'|^2)^2} \tag{3.11}$$

Notice (3.11) is real-valued, as should be the case for unitary time-evolution. The λ- and the $\tilde{g}^2$-dependent terms of (3.11) are plotted in figure 5

3.3 Two-point χ^a correlator in the large-N limit

Although the $\langle\phi\phi\rangle$ correlator does not simplify in the large-N limit, the same is not true for $\langle\chi\chi\rangle$ correlations. This can be seen by inserting the perturbative formulae (3.1) and (3.2) for $S_{\mathbf{p}}$ and $s^a_{\mathbf{p}}$ into the implicit solutions (2.33) and (2.34) for the χ^a-mode functions $R^a_{\mathbf{p}}$ and $r^{ab}_{\mathbf{p}}$, which gives

$$R^a_{\mathbf{p}}(t,\mathbf{x}) \simeq -\frac{\tilde{g}}{\sqrt{N}} \cdot \frac{\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|} e^{-iE_p(t-|\mathbf{x}|)} + \dots \tag{3.12}$$

and

$$r^{ab}_{\mathbf{p}}(t,\mathbf{x}) = \frac{\tilde{g}^2}{N} \cdot \frac{\Theta(t-|\mathbf{x}|-\epsilon)}{16\pi^2|\mathbf{x}|\epsilon} e^{-iE_p(t-|\mathbf{x}|-\epsilon)} \dots \tag{3.13}$$

[4]Note that this formula has no $i\delta$'s left in it — the reason for this is that any poles located at $|\mathbf{x}|+|\mathbf{x}'|$ can safely have $\delta \to 0^+$ taken. For the remaining poles at $|\mathbf{x}|-|\mathbf{x}'|$, we use the identity $\frac{1}{(z\pm i\delta)^2} = \frac{1}{z^2} \pm i\pi\delta'(z)$ and notice that a cancellation occurs.

to leading order in $\tilde{g}^2$ and λ. The solution (3.13) contains a $1/\epsilon$ divergence, which can be absorbed into the coupling for a self-interaction proportional to $\mathcal{I}_+ \otimes \delta_{ab}\chi^a\chi^b$ (although for brevity we do not do so here).

Eqs. (3.12) and (3.13) show that the mixing of χ^a with ϕ is suppressed by powers of $1/N$, and so become negligible in the large-N limit. The same suppression does not occur in the $\langle\phi\phi\rangle$ correlator because the explicit $1/N$ suppression is compensated by the sum over a and b in the combination $g_a g_b \langle \chi^a\, \chi^b\rangle$. It follows that these correlators satisfy

$$\mathrm{Tr}[\chi^a_H(t,\mathbf{x})\chi^b_H(t',\mathbf{x}')\rho_0] = \mathrm{Tr}'\left[\chi^a(t,\mathbf{x})\chi^b(t',\mathbf{x}')\varrho_\beta\right] + \mathcal{O}(1/N) \tag{3.14}$$

and so in the limit $N \gg 1$ are simply the thermal correlation functions for free fields, as if the ϕ field did not exist.

For completeness we quote here the explicit form for this free thermal correlator, with details of the calculation given in section A.1. The result evaluated at spacetime points $x = (t, \mathbf{x})$ and $x' = (t', \mathbf{x}')$ is[5]

$$\begin{aligned}\langle\chi^a(x)\chi^b(x')\rangle_\beta &:= \mathrm{Tr}'\left[\chi^a(t,\mathbf{x})\chi^b(t',\mathbf{x}')\varrho_\beta\right] \qquad (3.15)\\ &= \frac{\delta^{ab}}{8\pi\beta|\mathbf{x}-\mathbf{x}'|}\left\{\coth\left[\frac{\pi}{\beta}(t-t'+|\mathbf{x}-\mathbf{x}'|-i\delta)\right]-\coth\left[\frac{\pi}{\beta}(t-t'-|\mathbf{x}-\mathbf{x}'|-i\delta)\right]\right\},\end{aligned}$$

in agreement with standard formulae [92]. In this expression the limit $\delta \to 0^+$ is to be taken at the end of the calculation. Notice that eq. (3.15) obeys the required reality property (for real scalars)

$$\langle\chi^a(y)\chi^b(x)\rangle_\beta = \Big[\langle\chi^b(x)\chi^a(y)\rangle_\beta\Big]^*, \tag{3.16}$$

and at zero temperature ($\beta \to \infty$) goes over to

$$\langle\chi^a(x)\chi^b(x')\rangle_\beta \to \frac{\delta^{ab}}{4\pi^2[-(t-t'-i\delta)^2 + |\mathbf{x}-\mathbf{x}'|^2]}, \tag{3.17}$$

as it should.

4 RG Improvement and resumming the λ expansion

This section studies the dependence of hotspot physics on the self-coupling λ, in particular exploring how limiting it is to treat its implications perturbatively. Although the validity of expansions in λ might seem to be a tangential issue if one's focus is on the thermal coupling $\tilde{g}$, it really is not. As discussed earlier, the response of a field like ϕ to the hotspot typically diverges at the hotspot position — cf. for example equations (2.39) and (2.40) — and these divergences are ultimately handled by being renormalized into couplings like λ, as in (2.41). As a consequence of this renormalization couplings like λ run in the renormalization-group

[5]Note the given $i\delta$-prescription is only valid for real time arguments. Given that the N environment fields are assumed to be prepared in a thermal state, this correlation function must obey the Kubo-Martin-Schwinger (KMS) condition $\langle\chi^a(t-i\beta,\mathbf{x})\chi^b(0,\mathbf{0})\rangle_\beta = \langle\chi^a(t,\mathbf{x})\chi^a(0,\mathbf{0})\rangle^*_\beta$, which assumes a shift in imaginary time — for the correct $i\delta$-prescription in this case, see (A.19) in appendix A.1 (which agrees with the above prescription for real time arguements).

sense, and (as we show here, following [80, 81]) this can make it inconsistent to set them to zero at all scales.

This section derives precisely how the coupling λ runs, and along the way shows that the dimensionless expansion parameter that justifies treating λ perturbatively turns out to be $\lambda/4\pi\epsilon$, where ϵ is the very small regularization length scale used to regulate the divergences (as in eqs. (2.39) and (2.40)). Physically, both λ and ϵ might reasonably be expected to be of order the size ξ of the compact hotspot; a length scale that has been assumed to be much smaller than the other scales of physical interest. If perturbative calculations actually require $\lambda \ll 4\pi\epsilon$ then they may not be that useful, since λ would have to be much smaller even than this already very microscopic scale. The renormalization-group arguments presented here show how perturbative predictions can be extended to the regime $\lambda \gtrsim 4\pi\epsilon$, providing results that can be compared to the exact calculations to follow in section 5.

4.1 Effective interactions and boundary condition

To better understand the effects of λ beyond perturbation theory this section temporarily turns off the coupling $\tilde{g}$ in order to eliminate unnecessary distractions. Non-perturbative information is then extracted by leaving λ nonzero for all time and exploring more systematically how it modifies the dynamics of the ϕ field. A natural framework for this is the language of point-particle (or world-line) EFTs, since these systematically incorporate the effects of small objects on their surroundings, organized in powers of ka (where a is the object's size and k is the momentum of a typical probe). In practice we therefore work completely in the $\mathcal{R}_+$ sector, following closely the logic of [10, 80, 81], with the bulk field interacting only with the contact interaction

$$H_{\rm int}(t) = \frac{\lambda}{2} \int {\rm d}^3x \; \phi^2(t,\mathbf{x})\, \delta^3(\mathbf{x}) = \frac{\lambda}{2}\, \phi^2(t,\mathbf{0})\,, \tag{4.1}$$

with λ independent of time.

The implications of λ are incorporated by identifying the mode functions that are appropriate in the presence of this interaction. Since the Heisenberg equation of motion in this case — cf. equation (2.22) — is

$$\left(-\partial_t^2 + \nabla^2\right) \phi\,(t,\mathbf{x}) = \lambda\, \phi\,(t,\mathbf{0})\; \delta^3(\mathbf{x})\,, \tag{4.2}$$

this is also the equation satisfied by each mode function, $u_{\omega\ell m}(t,\mathbf{x})$, in an expansion (cf. equation (2.13)) like

$$\phi(t,\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \int {\rm d}\omega \left[u_{\omega\ell m}(t,\mathbf{x}) \mathfrak{a}_{\omega\ell m} + u^*_{\omega\ell m}(t,\mathbf{x}) \mathfrak{a}^*_{\omega\ell m} \right]. \tag{4.3}$$

Once the λ-dependence of these mode functions is identified by solving (4.2), the implications for the Wightman function are obtained from formulae like

$$\langle {\rm vac}|\, \phi(t,\mathbf{x})\phi(s,\mathbf{y})\, |{\rm vac}\rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} \int {\rm d}\omega\; u_{\omega\ell m}(t,\mathbf{x}) u^*_{\omega\ell m}(s,\mathbf{y})\,. \tag{4.4}$$

Here $|\mathrm{vac}\rangle$ satisfies $\mathfrak{a}_{\omega\ell m}\,|\mathrm{vac}\rangle = 0$, and $\mathfrak{a}_{\omega\ell m}$ satisfies the standard commutation relations $[\mathfrak{a}_{\omega\ell m}, \mathfrak{a}^*_{\tilde{\omega}\tilde{\ell}\tilde{m}}] = \delta(\omega - \tilde{\omega})\delta_{\ell\tilde{\ell}}\delta_{m\tilde{m}}$, and the mode functions $u_{\omega\ell m}$ are assumed to be properly normalized.

The main observation is that the dependence of $u_{\omega\ell m}$ on λ can be inferred by integrating its equation of motion

$$\left(-\partial_t^2 + \nabla^2\right) u_{\omega\ell m}(t, \mathbf{x}) = \lambda\, u_{\omega\ell m}(t, \mathbf{x})\, \delta^3(\mathbf{x})\,, \tag{4.5}$$

over a tiny sphere $B_\epsilon := \{\ \mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| \leq \epsilon\ \}$ of radius $\epsilon > 0$ centred around the origin. Following standard steps [80–82, 84] this integration leads to a λ-dependent boundary condition near the hotspot, of the form

$$4\pi\epsilon^2 \left(\frac{\partial u_{\omega\ell m}(t, \mathbf{x})}{\partial r}\right)_{r=\epsilon} = \lambda\ u_{\omega\ell m}(t, \mathbf{x})\Big|_{r=\epsilon}. \tag{4.6}$$

That is, for $r > \epsilon$ mode functions simply satisfy the Klein-Gordon equation

$$\left(-\partial_t^2 + \nabla^2_{\mathbf{x}}\right) u_{\mathbf{p}}(t, \mathbf{x}) = 0\,, \tag{4.7}$$

and only learn about the coupling λ through its appearance in the boundary condition (4.6).

Concretely, expanding the solution in terms of spherical harmonics,

$$u_{\omega\ell m}(t, \mathbf{x}) = e^{-i\omega t} R_{\omega\ell}(r) Y_{\ell m}(\theta, \phi), \tag{4.8}$$

the radial solutions are spherical Bessel functions

$$R_{\omega\ell}(r) = C_\ell(\omega) j_\ell(\omega r) + D_\ell(\omega) y_\ell(\omega r)\,, \tag{4.9}$$

where $C_\ell(\omega)$ and $D_\ell(\omega)$ are integration constants, whose ratio is determined by the boundary condition (4.6) and so is λ-dependent. Explicitly, the boundary condition (4.6) implies

$$4\pi\epsilon^2\, \partial_r R_{\omega\ell}(\epsilon) = \lambda\, R_{\omega\ell}(\epsilon) \tag{4.10}$$

Substituting the solution (4.9) into (4.10), and using the Bessel function identity

$$\partial_r f_\ell(\omega r) = \frac{\ell}{r}\, f_\ell(\omega r) - \omega f_{\ell+1}(\omega r) \tag{4.11}$$

(that holds for both $f_\ell = j_\ell$ and $f_\ell = y_\ell$), shows that the boundary condition (4.10) becomes

$$\frac{\lambda}{4\pi\epsilon} = \Big(r\,\partial_r \ln R_{\omega\ell}\Big)_{r=\epsilon} = \frac{\ell j_\ell(\omega\epsilon) - \omega\epsilon j_{\ell+1}(\omega\epsilon) + (D_\ell/C_\ell)[\ell y_\ell(\omega\epsilon) - \omega\epsilon y_{\ell+1}(\omega\epsilon)]}{j_\ell(\omega\epsilon) + (D_\ell/C_\ell)\, y_\ell(\omega\epsilon)}\,, \tag{4.12}$$

and this, once solved, leads to the following solution for the λ-dependence of D_ℓ/C_ℓ

$$\frac{D_\ell(\omega)}{C_\ell(\omega)} = -\,\frac{[(\lambda/4\pi\epsilon) - \ell] j_\ell(\omega\epsilon) + \omega\epsilon j_{\ell+1}(\omega\epsilon)}{[(\lambda/4\pi\epsilon) - \ell] y_\ell(\omega\epsilon) + \omega\epsilon y_{\ell+1}(\omega\epsilon)}\,. \tag{4.13}$$

These expressions simplify in the limit of practical interest, where $\omega\epsilon \ll 1$. In this limit we may use the expansions

$$j_\ell\,(z) = \frac{\sqrt{\pi}\; z^\ell}{2^{\ell+1}\Gamma\left(\ell + \frac{3}{2}\right)} + \mathcal{O}\left(z^{\ell+2}\right) \quad \text{and} \quad y_\ell\,(z) = -\frac{2^\ell\,\Gamma\left(\frac{1}{2} + \ell\right)}{\sqrt{\pi}\; z^{\ell+1}} + \mathcal{O}\left(z^{-\ell+1}\right)\,, \tag{4.14}$$

to find that (4.13) becomes

$$\frac{D_\ell\,(\omega)}{C_\ell\,(\omega)} \simeq \frac{\pi}{\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)}\left[\frac{(\lambda/4\pi\epsilon)-\ell}{(\lambda/4\pi\epsilon)+\ell+1}\right]\left(\frac{\omega\epsilon}{2}\right)^{2\ell+1}. \tag{4.15}$$

The coefficient here can be simplified using

$$\frac{\pi}{\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{3}{2}\right)} = \frac{2^{4\ell+2}\,[\ell!]^2}{2\,(2\ell+1)\,[(2\ell)!]^2}\,. \tag{4.16}$$

The inverse of (4.15) — or equivalently, the small $\omega\epsilon$ limit of (4.12) — similarly becomes

$$\frac{\lambda}{4\pi\epsilon} \simeq \frac{\ell\,(\omega\epsilon/2)^{2\ell+1}+(\ell+1)\,X_\ell\,(\omega)}{(\omega\epsilon/2)^{2\ell+1}-X_\ell\,(\omega)} \tag{4.17}$$

where

$$X_\ell(\omega) := \frac{1}{\pi}\,\Gamma\left(\ell+\frac{3}{2}\right)\Gamma\left(\ell+\frac{1}{2}\right)\frac{D_\ell(\omega)}{C_\ell(\omega)}\,. \tag{4.18}$$

Whether the numerator and denominator of thse last expressions can be further expanded depends on how the quantities $(\lambda/4\pi\epsilon)-\ell$ and D_ℓ/C_ℓ behave when $\omega\epsilon \ll 1$.

4.2 Renormalization group and the interpretation of ϵ

There are two ways to read the above boundary conditions. The naive way is as given in (4.13) or (4.15): they give $D_\ell(\omega)/C_\ell(\omega)$ as an explicit function of ℓ and the two dimensionless variables $\lambda/4\pi\epsilon$ and $\omega\epsilon$. What is bothersome about this interpretation is that it makes D_ℓ/C_ℓ depend not only on the coupling λ, but also on the arbitrary regularization scale ϵ.

But if D_ℓ/C_ℓ depends on ϵ then so also will the physical observables that are built from it. Normally regularization dependence in a calculation drops out of physical quantities because it gets renormalized into a redefinition of the couplings. Or, equivalently, it is cancelled by an implicit regularization dependence that is hidden within couplings like λ.

4.2.1 Running of λ

This observation suggests a different way to interpret the above boundary condition [80, 81]. This alternative reading insists physical quantities cannot depend on arbitrary regularization scales, and because of this neither can D_ℓ/C_ℓ. In this case expressions like (4.12) or (4.17) should be reinterpreted as making explicit how $\lambda=\lambda(\epsilon)$ must depend on ϵ in order to ensure that D_ℓ/C_ℓ remains ϵ-independent. That is to say, in this interpretation (4.17) is an RG equation for the coupling $\lambda(\epsilon)$.

To see what the evolution implied by (4.17) means more explicitly, it is worth expressing it in differential form. As explored in detail in appendix B, this can be put into a universal form by defining the new variable $v(\epsilon)$ using

$$\frac{\lambda}{2\pi\epsilon} = (2\ell+1)\,v-1\,, \tag{4.19}$$

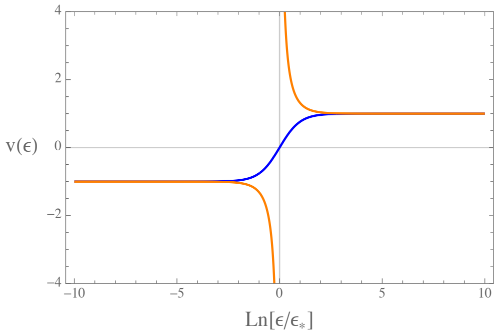


Figure 6: The two categories of RG flow described by the evolution equation (4.20). This figure plots the universal variable $v(\epsilon)$ against $\ln(\epsilon/\epsilon_\star)$. An example is shown with both $|v| > 1$ and $v < 1$, and the plot shows the sign of $|v| - 1$ is invariant because $v = \pm 1$ are fixed points. (figure taken from [87].).

for which differentiation of (4.17) becomes

$$\epsilon \, \frac{\mathrm{d}v}{\mathrm{d}\epsilon} = \left(\ell + \frac{1}{2}\right)(1 - v^2)\,. \tag{4.20}$$

As is easily verified, the solution of (4.20) subject to the initial condition $v(\epsilon_0) = v_0$ is given by

$$v(\epsilon) = \frac{(v_0 + 1)(\epsilon/\epsilon_0)^{2\ell+1} + (v_0 - 1)}{(v_0 + 1)(\epsilon/\epsilon_0)^{2\ell+1} - (v_0 - 1)} \tag{4.21}$$

and this agrees with (4.17) once (4.19) is used, with integration constant v_0 determining the combination D_ℓ / C_ℓ. These generically describe evolution from $v = -1$ to $v = +1$ as ϵ ranges from 0 to ∞. A plot of two representative solutions to (4.21) is given in figure 6.

Of course there is nothing wrong with simply regarding the boundary condition as specifying D_ℓ / C_ℓ once a coupling λ_0 is specified using a specific choice of regularization scale ϵ_0. What the RG interpretation tells us is that once this choice is made, we are completely free to use any other regularization scale, ϵ_1 instead, provided that we also change the value of the coupling to λ_1 where both pairs (ϵ_0, λ_0) and (ϵ_1, λ_1) lie on the same RG trajectory $\lambda(\epsilon)$ defined by (4.17) (or, equivalently by (4.19) and (4.21)). It is only when the coupling and regulator are changed in this correlated way that physical quantities remain unchanged.

4.2.2 RG-invariant characterization of coupling strength

Because physical observables depend only on the coupling *trajectories* it is more informative to specify the strength of the coupling by labelling the coupling trajectories using a more convenient RG-invariant parameterization, rather than simply by specifying its value $\lambda_0 = \lambda(\epsilon_0)$ for a specific (but arbitrary) regularizations scale ϵ_0. This section follows [80, 81] and identifies a particular choice of RG-invariant parameterization that is convenient because (unlike the value λ_0, say) the parameters are simply related to physical quantities.

To this end the first observation is that the evolution equation (4.20) has two fixed points, $v = \pm 1$, along which v is ϵ-independent. Trajectories that *do* evolve therefore cannot cross $v = \pm 1$ and so fall into two distinct categories, distinguished by

$$\eta_\star := \text{sign}\left(v^2 - 1\right) . \tag{4.22}$$

$\eta_\star$ is an RG-invariant quantity inasmuch as the sign of $v^2 - 1$ does not depend on ϵ for any $v(\epsilon)$ satisfying (4.20).

Any specific curve can be uniquely characterized in an RG-invariant way by specifying both $\eta_\star$ and the new variable $\epsilon_\star$, defined as the place where the curve passes through zero (if $\eta_\star = -1$) or where it diverges (if $\eta_\star = +1$). Using this definition the general solution (4.21) simplifies to

$$v(\epsilon) = \frac{(\epsilon/\epsilon_\star)^{2\ell+1} + \eta_\star}{(\epsilon/\epsilon_\star)^{2\ell+1} - \eta_\star} \tag{4.23}$$

and this shows that the pair $(\eta_\star, \epsilon_\star)$ are related to any specific choice of initial condition (ϵ_0, v_0) by $\eta_\star = \text{sign}(v_0^2 - 1)$ and

$$\left(\frac{\epsilon_\star}{\epsilon_0}\right)^{2\ell+1} = \eta_\star\left(\frac{v_0 - 1}{v_0 + 1}\right) = \left|\frac{v_0 - 1}{v_0 + 1}\right| . \tag{4.24}$$

What makes these variables convenient is that $\epsilon_\star$ provides an invariant length scale that is shared by all representatives (ϵ, v) or (ϵ, λ) along a particular RG trajectory. It is consequently this length scale — and *not* ϵ_0 or λ_0, say — that is physical and so whose size characterizes the values of physical observables. This is shown in detail in [80–84, 86, 87], where cross sections and energy shifts in many examples are evaluated and found to be simply related to $\epsilon_\star$.

To see explicitly why this is so, we write $\lambda(\epsilon)$ in terms of $\epsilon_\star$ by combining (4.19) and (4.23) to get

$$\frac{\lambda}{2\pi\epsilon} + 1 = (2\ell + 1)v(\epsilon) = (2\ell + 1)\,\frac{(\epsilon/\epsilon_\star)^{2\ell+1} + \eta_\star}{(\epsilon/\epsilon_\star)^{2\ell+1} - \eta_\star}\,, \tag{4.25}$$

and for later purposes also record its inverse (cf. eqs. (4.19) and (4.24))

$$\left(\frac{\epsilon_\star}{\epsilon}\right)^{2\ell+1} = \left|\frac{v - 1}{v + 1}\right| = \left|\frac{[\lambda/(4\pi\epsilon)] - \ell}{[\lambda/(4\pi\epsilon)] + \ell + 1}\right| . \tag{4.26}$$

eq. (4.25) can be used in (4.15) to determine the integration constant ratio D_ℓ/C_ℓ from which physical quantities are ultimately determined. This exercise gives

$$\frac{D_\ell(\omega)}{C_\ell(\omega)} = \frac{\pi\eta_\star}{\Gamma(\ell + \frac{3}{2})\,\Gamma(\ell + \frac{1}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1}\,, \tag{4.27}$$

verifying that the explicit dependence on ϵ and λ combines into the invariant combinations $\eta_\star$ and $\epsilon_\star$. In particular, it is the dimensionless quantity $\omega\epsilon_\star$ that controls the size of any physical response, and (4.27) shows quantitatively in this language how small angular momenta ℓ are preferred when $\omega\epsilon_\star \ll 1$.

In practical examples $\epsilon_\star$ is set by the size of the underlying object (in this case the hotspot) times the appropriate coupling that controls the interactions through which it is probed. For example, when a similar analysis is applied to describing the effects of finite nuclear size on the energy levels in pionic atoms, one finds $\epsilon_\star \sim R_N$ is of order the nuclear radius [82]. But the same analysis when describing nuclear-size effects on Hydrogen energy levels finds $\epsilon_\star \sim \alpha R_N$, with α the fine-structure constant [80, 81].

By comparison, if boundary conditions must be imposed for an effective theory outside the nucleus then $\epsilon > R_N$. Concrete examples like these show that a small source probed by a weakly coupled field tends to produce $\epsilon_\star \ll \epsilon$, if ϵ is regarded to be typical linear size of the compact object.

4.2.3 Resumming all orders in $\lambda/4\pi\epsilon$

It is instructive to explore the connection between $\lambda(\epsilon)$ and $\epsilon_\star$ explicitly in the weak-coupling limit, by expanding (4.25) in powers of $\epsilon_\star/\epsilon$. In this limit (4.23) simplifies to $v(\epsilon) \simeq 1 + 2\eta_\star\left(\epsilon_\star/\epsilon\right)^{2\ell+1} + \cdots$, and so

$$\frac{\lambda}{2\pi\epsilon} \simeq 2\ell + 2\eta_\star(2\ell+1)\left(\frac{\epsilon_\star}{\epsilon}\right)^{2\ell+1} . \tag{4.28}$$

For the $\ell = 0$ mode in particular λ becomes ϵ-independent in the perturbative limit, with

$$\lambda \simeq 4\pi\eta_\star\epsilon_\star \qquad (\text{for } \ell = 0 \text{ and } \epsilon_\star \ll \epsilon) . \tag{4.29}$$

This shows that for s-wave processes, expressions for physical quantities as functions of $\omega\epsilon_\star$ (for mode frequency ω) can be turned into corresponding expressions as functions of $\lambda\omega$, by using (4.29). Provided powers of $\epsilon_\star/\epsilon$ are negligible these expressions need have no dependence on regulators like ϵ, making any discussion of RG evolution completely unnecessary.

But what happens if ϵ is now decreased and $\lambda(\epsilon)$ adjusted along a particular RG flow to a point where $\epsilon_\star/\epsilon$ is no longer negligible and $\lambda/4\pi\epsilon$ is no longer small? The answer for the physical observable as a function of $\omega\epsilon_\star$ does not change at all, because physics depends only on which RG trajectory one lives, and not on the particular point one sits along this trajectory. All that changes as ϵ and λ are varied is that expression (4.29) can no longer be used to trade $\epsilon_\star$ for λ; instead one must go back to the full result (4.25) when doing so.

This observation provides a way to resum all orders in $\lambda/4\pi\epsilon$ while holding quantities like $\omega\epsilon_\star$ fixed. Suppose one computes an observable as a function of two dimensionless quantities $O = O[\omega\epsilon, \lambda/4\pi\epsilon]$, and does so perturbatively in $\lambda/4\pi\epsilon$. The result can be turned into an expression $O = O[\omega\epsilon, \epsilon_\star/\epsilon]$ by trading λ for $\epsilon_\star$ using (4.29). But we know that ϵ is just a calculational artefact that is not actually in the physical result, which must therefore really only be a function of the one variable $\omega\epsilon_\star$.

The result for the same observable elsewhere on the RG trajectory, where $\lambda/4\pi\epsilon$ and $\epsilon_\star/\epsilon$ are *not* small, is given by the same expression $O = O[\omega\epsilon_\star]$ since it does not depend at all on ϵ. Expressing this result in terms of $O = O[\omega\epsilon, \lambda/4\pi\epsilon]$ using (4.25) then gives an explicit resummation of the observable to all orders in $\lambda(\epsilon)/4\pi\epsilon$.

4.3 Resummation of the two-point function

The above reasoning can be applied to the perturbative calculation of ϕ-field response given above, allowing results that are derived to lowest nontrivial order in $\lambda/4\pi\epsilon$ to be promoted into expressions that work to all orders in this variable.

4.3.1 *s*-wave resummation

We derive the resummed results here, and then check (in this section) that they capture a full mode sum using the boundary condition (4.6). In later sections we also verify that the results found for large $\lambda/4\pi\epsilon$ in this way also agree with the exact expressions derived in section 5.

The starting point is the perturbative expression (3.10), that is given again here in the special case $\tilde{g} = 0$:

$$W_\beta(t,\mathbf{x};t',\mathbf{x}') \simeq \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]} + \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\left[\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2-(|\mathbf{x}+|\mathbf{x}'|)^2}\right]. \tag{4.30}$$

Anticipating the dominance of the $\ell = 0$ mode (as is appropriate in applications for which $\epsilon_\star \ll |\mathbf{x}|, |\mathbf{x}'|$ say, see next section), we may trade λ in this expression for $\epsilon_\star$ using (4.29) to find

$$W_\beta(t,\mathbf{x};t',\mathbf{x}') \simeq \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]} + \frac{\eta_\star\epsilon_\star}{4\pi^2|\mathbf{x}||\mathbf{x}'|}\left[\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2-(|\mathbf{x}+|\mathbf{x}'|)^2}\right]. \tag{4.31}$$

But an expression with broader validity than (4.30) can be obtained from (4.31) by using in this result the more general $\ell = 0$ relation giving $\epsilon_\star$ in terms of λ given in (4.26), leading to

$$\begin{aligned}\langle\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle \simeq \frac{1}{4\pi^2}&\left[\frac{1}{-(t-t'-i\eta)^2+|\mathbf{x}-\mathbf{x}'|^2}\right] \\ &+\frac{1}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\left|\frac{\lambda}{(\lambda/4\pi\epsilon)+1}\right|\left[\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2-(|\mathbf{x}|+|\mathbf{x}'|)^2}\right].\end{aligned} \tag{4.32}$$

This clearly agrees with (4.30) for small $\lambda/4\pi\epsilon$, but its validity is now extended to include the regime $\lambda \gtrsim 4\pi\epsilon$ provided only that $\ell = 0$ modes dominate when computing the hotspot influence. The conditions under which this is true are explored more fully in the next section, which verifies (4.32) starting directly from a mode-sum using modes that satisfy the boundary condition (4.6).

4.3.2 Mode-sum calculation

We next recompute (4.32) by evaluating the ϕ-field correlator as an exact function of λ, by calculating the sum over mode-functions whose λ-dependence is acquired through the boundary condition (4.6). As described above, this boundary condition fixes the ratio of integration constants D_ℓ/C_ℓ to be given in terms of $\epsilon_\star$ as in (4.27), repeated here for conenience:

$$\frac{D_\ell(\omega)}{C_\ell(\omega)} = \frac{\pi\eta_\star}{\Gamma(\ell+\frac{3}{2})\,\Gamma(\ell+\frac{1}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1}. \tag{4.33}$$

Mode normalization. The integration constants C_ℓ and D_ℓ are determined separately by combining (4.33) with mode-function normalization, which requires

$$\langle u_{\omega\ell m}, u_{\tilde{\omega}\tilde{\ell}\tilde{m}}\rangle = \delta(\omega-\tilde{\omega})\delta_{\ell\tilde{\ell}}\delta_{m\tilde{m}}\,, \quad \langle u_{\omega\ell m}, u^*_{\tilde{\omega}\tilde{\ell}\tilde{m}}\rangle = 0\,, \quad \langle u^*_{\omega\ell m}, u^*_{\tilde{\omega}\tilde{\ell}\tilde{m}}\rangle = -\delta(\omega-\tilde{\omega})\delta_{\ell\tilde{\ell}}\delta_{m\tilde{m}} \tag{4.34}$$

where the angle brackets denote the Klein-Gordon inner product

$$\langle F, G\rangle := i\int \mathrm{d}^3x\,\Big(F(t,\mathbf{x})\partial_t G^*(t,\mathbf{x}) - \partial_t F(t,\mathbf{x})G^*(t,\mathbf{x})\Big)\,. \tag{4.35}$$

As is easily verified, this inner product is time-independent when evaluated for any solutions F, G to the Klein-Gordon equation, and this remains true even in the presence of the modified boundary condition (4.6), whenever the effective coupling λ is real. To see why notice that this boundary condition implies the radial flux density of Klein-Gordon probability at $r=\epsilon$ is

$$J_r(\epsilon) \propto \Big[F(t,\mathbf{x})\partial_r G^*(t,\mathbf{x}) - \partial_r F(t,\mathbf{x})G^*(t,\mathbf{x})\Big]_{r=\epsilon} = (\lambda^*-\lambda)\Big[F(t,\mathbf{x})G^*(t,\mathbf{x})\Big]_{r=\epsilon} \tag{4.36}$$

and so vanishes for real λ.

The normalization integrals are computed in appendix C.1, leading to the following $\epsilon_\star$-dependent results for C_ℓ and D_ℓ separately

$$C_{\ell m}(\omega) = \sqrt{\frac{\omega}{\pi}}\left\{1 + \left[\frac{\pi}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1}\right]^2\right\}^{-1/2}\,, \tag{4.37}$$

and

$$D_{\ell m}(\omega) = \frac{\pi\eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1}\sqrt{\frac{\omega}{\pi}}\left\{1 + \left[\frac{\pi}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1}\right]^2\right\}^{-1/2}\,. \tag{4.38}$$

With these choices the mode functions and field operators satisfy the required boundary condition at $r=\epsilon$, and this completely determines their dependence on $\epsilon_\star$.

The main approximation made in deriving (4.37) and (4.38) is to assume that $\omega\epsilon$ is small enough to allow the replacement of $j_\ell(\omega\epsilon)$ and $y_\ell(\omega\epsilon)$ with their leading asymptotic forms; that is by using (4.15) instead of (4.13). Since the Bessel functions are explicitly known this approximation can be improved to any desired order in $\omega\epsilon$, by upgrading condition (4.33) using a more accurate representation of the Bessel functions.

Mode sum. We are now in a position to compute the Wightman function in terms of a mode sum, using the above $\epsilon_\star$-dependent form for the modes,

$$u_{\omega\ell m}(t,\mathbf{x}) = C_{\ell m}(\omega)e^{-i\omega t}\left[j_\ell(\omega r) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega r)\right]Y_{\ell m}(\theta,\phi)\,, \tag{4.39}$$

that properly matching at the boundary condition at $r=\epsilon$ (dropping subdominant terms in $\omega\epsilon$). As argued above, all explicit dependence on ϵ drops out of this exression once evaluated at $|\mathbf{x}| = \epsilon$, cancelling between any explicit dependence and the ϵ-dependence implicit without the coupling λ, leaving a dependence only on the RG-invariant quantity $\epsilon_\star$. In particular, eqs. (4.37) and (4.39) do *not* assume validity of the weak-coupling limit $\epsilon_\star \ll \epsilon$ (or equivalently $\lambda/4\pi\epsilon$ need not be much smaller than unity).

The Wightman function is given in terms of these modes by

$$\begin{aligned}\langle \phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\int_0^\infty \mathrm{d}\omega\, u_{\omega\ell m}(t,\mathbf{x})u^*_{\omega\ell m}(t',\mathbf{x}') \qquad (4.40)\\ &= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\int_0^\infty \mathrm{d}\omega\, e^{-i\omega(t-t')}|C_{\ell m}(\omega)|^2 \left[j_\ell(\omega|\mathbf{x}|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}|)\right]\\ &\quad \times \left[j_\ell(\omega|\mathbf{x}'|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}'|)\right] Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi')\ .\end{aligned}$$

Performing the sum over m (see appendix C.2 for details) leads to the intermediate expression

$$\begin{aligned}\langle \phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle &= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\int_0^\infty \mathrm{d}\omega\, e^{-i\omega(t-t')}|C_{\ell 0}(\omega)|^2 \left[j_\ell(\omega|\mathbf{x}|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}|)\right]\\ &\quad \times \left[j_\ell(\omega|\mathbf{x}'|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}'|)\right] P_\ell(\cos\theta)\ . \qquad (4.41)\end{aligned}$$

The terms in this sum involving two factors of j_ℓ reproduce the standard vacuum Minkowski Wightman function in the absence of the hotspot source.

Leading order in $\omega\epsilon_\star$. To make further progress we assume $\omega\epsilon_\star \ll 1$, which is a natural limit for modes with energies much smaller than the UV scale $1/\epsilon_\star$. In this case because $\omega\epsilon_\star$ appears raised to the power $2\ell+1$ it follows that the leading regime comes purely from the s-wave partial wave with $\ell = 0$. Using

$$j_0(\omega|\mathbf{x}|) = \frac{\sin(x)}{x} \qquad \text{and} \qquad y_0(x) = -\frac{\cos(x)}{x}\ , \qquad (4.42)$$

the contribution up to leading (linear) nontrivial order in $\omega\epsilon_\star$ can be simplified to

$$\begin{aligned}\langle \phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle &\simeq \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}\omega\, e^{-i\omega(t-t')}\bigg\{ \omega\sum_{\ell=0}^{\infty}(2\ell+1)j_\ell(\omega|\mathbf{x}|)j_\ell(\omega|\mathbf{x}'|)P_\ell(\cos\theta)\\ &\quad - \frac{(4\pi\eta_\star\epsilon_\star)}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\Big[\sin(\omega|\mathbf{x}|)\cos(\omega|\mathbf{x}'|)+\cos(\omega|\mathbf{x}|)\sin(\omega|\mathbf{x}'|)\Big] + \mathcal{O}(\omega^2\epsilon_\star^2)\bigg\}.\\ &\simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty \mathrm{d}\omega e^{-i\omega(t-t')}\bigg\{\sin\left(\omega|\mathbf{x}-\mathbf{x}'|\right) - \frac{(4\pi\eta_\star\epsilon_\star)}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\sin\Big[\omega(|\mathbf{x}|+|\mathbf{x}'|)\Big] + \mathcal{O}(\omega^2\epsilon_\star^2)\bigg\}\ .\end{aligned} \qquad (4.43)$$

Evaluating the remaining integrals then gives

$$\langle \phi\,(t,\mathbf{x})\,\phi\,(t',\mathbf{x}')\rangle \simeq \frac{1}{4\pi^2}\left[\frac{1}{-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2}\right] + \frac{(4\pi\eta_\star\epsilon_\star)}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\left[\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2-(|\mathbf{x}|+|\mathbf{x}'|)^2}\right], \tag{4.44}$$

where $\delta = 0^+$ is the usual positive infinitesimal that is taken to zero at the end of the calculation.

Notice that (4.44) precisely agrees with the result found in (4.31), and so guarantees that the result (4.32) is found once the combination $\eta_\star\epsilon_\star$ is traded for $\lambda(\epsilon)$ using (4.26). Higher orders in $\omega\epsilon_\star$ can be included systematically by including higher partial waves and by working to higher order in the small $\omega\epsilon$ expansion of the mode-functions.

5 Exact two-point correlator

In this section we evaluate the ϕ mode functions without perturbing in λ and $\tilde{g}$, and sum these modes to obtain the exact $\langle\phi\phi\rangle$ Wightman function.

5.1 Mode functions

The first step is to find the mode functions in a way that does not rely on couplings being small. To this end we must solve equations (2.42) and (2.43), which are repeated here for ease of reference, for the mode functions $S_{\mathbf{p}}$ and $s^a_{\mathbf{p}}$:

$$S_{\mathbf{p}}(t,\mathbf{x}) = -\frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\Big(e^{-iE_p(t-|\mathbf{x}|)} + S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\Big) - \frac{\delta^{ab}g_ag_b}{16\pi^2|\mathbf{x}|}\,\delta(t-|\mathbf{x}|) - \frac{\delta^{ab}g_ag_b}{16\pi^2|\mathbf{x}|}\,\Theta(t-|\mathbf{x}|)\Big[(-iE_p)e^{-iE_p(t-|\mathbf{x}|)} + \partial_t S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\Big] \tag{5.1}$$

and

$$s^a_{\mathbf{p}}(t,\mathbf{x}) = -\frac{\delta^{ab}g_b}{4\pi|\mathbf{x}|}\,\Theta(t-|\mathbf{x}|)\,e^{-iE_p(t-|\mathbf{x}|)} - \frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\,s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) - \frac{\delta^{bc}g_bg_c}{16\pi^2|\mathbf{x}|}\,\Theta(t-|\mathbf{x}|)\,\partial_t s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) \tag{5.2}$$

subject to the initial conditions (2.30).

Clearly, whenever $t < |\mathbf{x}|$ the right-hand sides of these equations vanish and so they imply that $S_{\mathbf{p}}(t,\mathbf{x}) = s^a_{\mathbf{p}}(t,\mathbf{x}) = 0$, as required by causality. In the opposite case $t > |\mathbf{x}|$ they instead become

$$S_{\mathbf{p}}(t,\mathbf{x}) = -\frac{\lambda}{4\pi|\mathbf{x}|}\big[e^{-iE_p(t-|\mathbf{x}|)} + S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\big] - \frac{\delta^{ab}g_ag_b}{16\pi^2|\mathbf{x}|}\Big[(-iE_p)e^{-iE_p(t-|\mathbf{x}|)} + \partial_t S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0})\Big] \qquad \text{when } t > |\mathbf{x}| \tag{5.3}$$

and

$$s^a_{\mathbf{p}}(t,\mathbf{x}) = -\frac{\delta^{ab}g_b}{4\pi|\mathbf{x}|}\, e^{-iE_p(t-|\mathbf{x}|)} - \frac{\lambda}{4\pi|\mathbf{x}|} s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) - \frac{\delta^{bc}g_b g_c}{16\pi^2|\mathbf{x}|}\, \partial_t s^a_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) \qquad \text{when } t > |\mathbf{x}|\,. \tag{5.4}$$

As suggested by the perturbative case, solutions to these equations unsurprisingly divergence at the location of the compact source $\mathbf{x} = \mathbf{0}$, which we regulate as before by replacing $\mathbf{x} = 0$ with $|\mathbf{x}| = \epsilon$: $S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) = S_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{y})\big|_{|\mathbf{y}|=\epsilon}$ and $\partial_t S^{(0)}_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) = \partial_t S^{(0)}_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{y})\big|_{|\mathbf{y}|=\epsilon}$ and so on.

To solve (5.3) and (5.4) (in the special case where $g_a = \tilde{g}/\sqrt{N}$ for all a) we make the *ansätze*

$$S_{\mathbf{p}}(t,\mathbf{x}) = F(|\mathbf{x}|)e^{-iE_p(t-|\mathbf{x}|)} \qquad \text{when } t > |\mathbf{x}| \tag{5.5}$$

and

$$s^a_{\mathbf{p}}(t,\mathbf{x}) = G(|\mathbf{x}|)e^{-iE_p(t-|\mathbf{x}|)} \quad (\text{for all } a) \qquad \text{when } t > |\mathbf{x}|\,, \tag{5.6}$$

and so $S^{(0)}_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) := F(\epsilon)\, e^{-iE_p(t-|\mathbf{x}|-\epsilon)}$ and $\partial_t S^{(0)}_{\mathbf{p}}(t-|\mathbf{x}|,\mathbf{0}) := F(\epsilon)\,(-iE_p)e^{-iE_p(t-|\mathbf{x}|-\epsilon)}$ and similarly for $s^a_{\mathbf{p}}$. These *ansätze* solve (5.3) and (5.4) provided F and G satisfy

$$F(|\mathbf{x}|) = \left(-\frac{\lambda}{4\pi|\mathbf{x}|} + \frac{i\tilde{g}^2E_p}{16\pi^2|\mathbf{x}|}\right)\left[1 + F(\epsilon)\, e^{+iE_p\epsilon}\right] \tag{5.7}$$

and

$$G(|\mathbf{x}|) = -\frac{\tilde{g}}{4\pi\sqrt{N}|\mathbf{x}|} + \left(-\frac{\lambda}{4\pi|\mathbf{x}|} + \frac{i\tilde{g}^2E_p}{16\pi^2|\mathbf{x}|}\right) G(\epsilon)\, e^{+iE_p\epsilon} \tag{5.8}$$

whose solutions are

$$F(|\mathbf{x}|) = \frac{-\dfrac{\lambda}{4\pi|\mathbf{x}|} + \dfrac{i\tilde{g}^2E_p}{16\pi^2|\mathbf{x}|}}{1 - \left(-\dfrac{\lambda}{4\pi\epsilon} + \dfrac{i\tilde{g}^2E_p}{16\pi^2\epsilon}\right)e^{+iE_p\epsilon}} \qquad \text{and} \qquad G(|\mathbf{x}|) = \frac{-\dfrac{\tilde{g}}{4\pi\sqrt{N}|\mathbf{x}|}e^{-iE_p(t-|\mathbf{x}|)}}{1 - \left(-\dfrac{\lambda}{4\pi\epsilon} + \dfrac{i\tilde{g}^2E_p}{16\pi^2\epsilon}\right)e^{+iE_p\epsilon}}\,. \tag{5.9}$$

Recalling that the derivation of equations (2.42) and (2.43) assume $\epsilon E_p \ll 1$ — see the discussion below equations (2.39) and (2.40) — we can take $e^{iE_p\epsilon} \simeq 1$ without loss, giving the mode functions

$$S_{\mathbf{p}}(t,\mathbf{x}) \simeq \frac{-\dfrac{\lambda}{4\pi|\mathbf{x}|} + \dfrac{i\tilde{g}^2E_p}{16\pi^2|\mathbf{x}|}}{1 + \dfrac{\lambda}{4\pi\epsilon} - \dfrac{i\tilde{g}^2E_p}{16\pi^2\epsilon}}e^{-iE_p(t-|\mathbf{x}|)} \qquad \text{when } t > |\mathbf{x}| \tag{5.10}$$

and

$$s^a_{\mathbf{p}}(t,\mathbf{x}) \simeq \frac{-\dfrac{\tilde{g}}{4\pi\sqrt{N}|\mathbf{x}|}e^{-iE_p(t-|\mathbf{x}|)}}{1 + \dfrac{\lambda}{4\pi\epsilon} - \dfrac{i\tilde{g}^2E_p}{16\pi^2\epsilon}}e^{-iE_p(t-|\mathbf{x}|)} \qquad \text{when } t > |\mathbf{x}|\,. \tag{5.11}$$

Comparing (5.10) and (5.11) to the perturbative solutions (3.1) and (3.2) in the regime $t > |\mathbf{x}|$, it is clear that the perturbative expression are valid only when

$$\left| \frac{\lambda}{4\pi\epsilon} - \frac{i\tilde{g}^2 E_p}{16\pi^2\epsilon} \right| \ll 1 \ . \tag{5.12}$$

For this to be small for all $\mathbf{p}$ requires each term to separately be small. If Λ is a bulk cutoff, so that $E_p < \Lambda$ for all $\mathbf{p}$ (which in principle is logically distinct from the UV cutoff $1/\epsilon$ associated with proximity to the hotspot) then at face value the perturbative limit requires

$$\frac{\lambda}{4\pi\epsilon} \ll 1 \qquad \text{and} \qquad \frac{\Lambda\tilde{g}^2}{16\pi^2\epsilon} \ll 1 \ . \tag{5.13}$$

5.2 Performing the mode sum

Given the mode functions in (5.10) and (5.11) the Wightman function of (3.4) can be evaluated as a mode sum. For two points $(t,\mathbf{x})$ and $(t',\mathbf{x}')$ satisfying $t > |\mathbf{x}|$ and $t' > |\mathbf{x}'|$, the result becomes

$$\begin{aligned} W_\beta\left(t,\mathbf{x};t',\mathbf{x}'\right) &= \frac{1}{Z_\beta}\mathrm{Tr}\left[\phi_H\left(t,\mathbf{x}\right)\phi_H\left(t',\mathbf{x}'\right)\left(|\mathrm{vac}\rangle\langle\mathrm{vac}|\otimes e^{-\beta\mathcal{H}_-}\right)\right] \\ &=: \mathscr{S}\left(t,\mathbf{x};t,\mathbf{x}'\right) + \mathscr{E}_\beta\left(t,\mathbf{x};t,\mathbf{x}'\right) \end{aligned} \tag{5.14}$$

where the functions $\mathscr{S}$ and $\mathscr{E}_\beta$ are defined by

$$\mathscr{S}(t,\mathbf{x};t',\mathbf{x}') = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p}\left(e^{-iE_pt+i\mathbf{p}\cdot\mathbf{x}} + S_{\mathbf{p}}(t,\mathbf{x})\right)\left(e^{+iE_pt'-i\mathbf{p}\cdot\mathbf{x}'} + S^*_{\mathbf{p}}(t',\mathbf{x}')\right) \tag{5.15}$$

and

$$\begin{aligned} \mathscr{E}_\beta(t,\mathbf{x};t',\mathbf{x}') = \frac{1}{Z_\beta}\sum_{a,b=1}^{N}\int\frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2E_p}}\int\frac{\mathrm{d}^3\mathbf{k}}{\sqrt{(2\pi)^3 2E_k}}\Big(&s^a_{\mathbf{p}}(t,\mathbf{x})s^{b*}_{\mathbf{k}}(t',\mathbf{x}')\mathrm{Tr}'\left[\mathfrak{b}^a_{\mathbf{p}}\mathfrak{b}^{b*}_{\mathbf{k}}e^{-\beta\mathcal{H}_-}\right] \\ &+ s^{a*}_{\mathbf{p}}(t,\mathbf{x})s^b_{\mathbf{k}}(t',\mathbf{x}')\mathrm{Tr}'\left[\mathfrak{b}^{a*}_{\mathbf{p}}\mathfrak{b}^b_{\mathbf{k}}e^{-\beta\mathcal{H}_-}\right]\Big) \ . \end{aligned} \tag{5.16}$$

These mode sums are performed explicitly in appendix D, giving the following result for $\mathscr{S}$

$$\begin{aligned} \mathscr{S}(t,\mathbf{x};t',\mathbf{x}') =& \frac{1}{4\pi^2\left[-(t-t'-i\delta)^2 + |\mathbf{x}-\mathbf{x}'|^2\right]} \\ &+ \frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\Big[I_-(t-t'+|\mathbf{x}|+|\mathbf{x}'|,c) - I_-(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) \\ &\qquad - I_+(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) + I_+(t-t'-|\mathbf{x}|-|\mathbf{x}'|,c)\Big] \\ &+ \frac{\epsilon}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\left[-\frac{1}{t-t'+|\mathbf{x}|+|\mathbf{x}'|-i\delta} + \frac{1}{t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta}\right] \\ &- \frac{32\pi^2\epsilon^4(1+\frac{\lambda}{2\pi\epsilon})}{\tilde{g}^4|\mathbf{x}||\mathbf{x}'|}\Big[I_-(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) + I_+(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c)\Big] \\ &- \frac{\epsilon^2}{4\pi^2|\mathbf{x}||\mathbf{x}'|(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)^2} \end{aligned} \tag{5.17}$$

where the parameter c is the following combination of couplings and ϵ,

$$c := \frac{16\pi^2\epsilon}{\tilde{g}^2}\left(1+\frac{\lambda}{4\pi\epsilon}\right)\,. \tag{5.18}$$

The functions $I_{\mp}(\tau)$ are defined by

$$I_{\mp}(\tau) = e^{\pm c\tau} E_1\big(\pm c[\tau - i\delta]\big) \quad \text{where} \quad E_1(z) := \int_z^{\infty} \mathrm{d}u\;\frac{e^{-u}}{u}\,, \tag{5.19}$$

is the E_n-function with $n = 1$ (closely related to the exponential integral function) and the limit $\delta \to 0^+$ is (as usual) understood.

The temperature-dependent contribution similarly evaluates to

$$\mathscr{E}_\beta(t,\mathbf{x};t,\mathbf{x}') = \frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\left[\Phi\left(e^{-\frac{2\pi(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)}{\beta}},1,\frac{c\beta}{2\pi}\right)+\Phi\left(e^{+\frac{2\pi(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)}{\beta}},1,\frac{c\beta}{2\pi}\right)\right]-\frac{2\pi}{c\beta}\Bigg] \tag{5.20}$$

where $\Phi(z,s,a)$ is the Lerch transcendent, defined by the series $\Phi(z,s,a) = \sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s}$ for complex numbers in the unit disc (with $|z| < 1$), and by analytic contribution elsewhere in the complex plane. Asymptotic forms for the functions $I_{\pm}$ and Φ are given in appendix D.1.1.

These expressions pass all of the smell tests. In particular, the full correlation function $W_\beta = \mathscr{S} + \mathscr{E}_\beta$ reduces to the perturbative correlation function quoted in (3.10) in the appropriate perturbative limit. The perturbative expression in powers of $\tilde{g}$ is obtained from the asymptotic form when both $c\tau \gg 1$ in $\mathscr{S}$ (as shown explicitly in section D.1.1) and $c\beta \gg 1$ in $\mathscr{E}_\beta$ (see section D.2.1). The expression found in this limit agrees with (4.32), obtained earlier by resumming the perturbative result to all orders in $\lambda/(4\pi\epsilon)$. Further taking $\lambda/(4\pi\epsilon) \ll 1$ in this expression then reproduces exactly the perturbative correlation function (3.10) found previously.

Finally, this result has a thermal character in the sense that $\mathscr{E}_\beta$ satisfies the Kubo-Martin-Schwinger (KMS) condition [73, 74], as we show explicitly in appendix D.2.3.

6 Conclusions

Black hole physics is a puzzle wrapped in an enigma hidden by a horizon, and ongoing studies of information loss show that Hawking radiation is the discovery that keeps on giving. Although Hawking radiation in principle occurs in the weak-field regime where calculation control should be good, reliable and explicit calculations of corrections to Hawking radiation are relatively rare (partly due to the extremely late times involved). Yet resolving issues such as the existence (or not) of firewalls, possible loss of locality and the like are crucial towards gaining an understanding of what a theory of quantum gravity should ultimately look like.

In this paper we construct a Caldeira-Leggett type [45] toy model of a hot compact relativistic object that captures some of the features of black holes, and so can be used as a benchmark against which real calculations can be compared. The model is simple enough to be solved explicitly, but complicated enough to capture some of the open-system effects believed to be important for black holes. Although the toy model cannot in itself resolve the thorniest puzzles associated with horizons, it can show which features are shared with more mundane systems that are hot and relatively small.

There are several directions in which this model might fruitfully be explored. In a companion work [53], we apply to it several approximate Open-EFT techniques designed to probe late-time evolution, to better identify their domains of validity and whether they can illuminate the extent to which the open nature of the hotspot causes a breakdown of local descriptions of the physics of the field ϕ living in $\mathcal{R}_+$. A second companion [54] explores the behaviour of an Unruh detector (or qubit) that couples to the field ϕ in the vicinity of the hotspot, to determine the extent to which it thermalizes as a function of its couplings and its distance from the hotspot.

Other useful directions might explore the regime where the radius ξ of the interaction sphere $\mathcal{S}_\xi$ is not small, and so parallels the EFT discussion of [71]. By tuning the physics at the interface between the spaces $\mathcal{R}_\pm$ one might hope to mock up the entanglement between modes inside and outside the horizon, and provide a simple analog of the matching calculations often used to extract black hole phenomenology from scattering amplitudes. The model can be further developed to include redshifting and the geometrical effects of gravitational fields in regions $\mathcal{R}_\pm$. We offer up the hotspot model in the hopes that such comparisons and extensions will prove instructive.

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A Thermal correlation functions

This appendix computes the thermal correlation functions used in the main text. This is partly done as a confidence-building exercise to verify the techniques used elsewhere.

A.1 Free thermal correlation function

The first correlation to compute is the standard two-point function for a free thermal field. Although the main text works with N fields for this correlation function it suffices to work with only one of the N copies and evaluate

$$\langle \chi(t,\mathbf{x})\chi(0,\mathbf{0})\rangle_\beta = \frac{1}{Z}\,\mathrm{Tr}\Big[\chi(t,\mathbf{x})\chi(0,\mathbf{0})\,e^{-\beta\mathcal{H}}\Big]\,, \tag{A.1}$$

where translation invariance in $\mathcal{R}_-$ is used to set one of the fields to $(t', \mathbf{x}') = (0, \mathbf{0})$, the relevant free-particle part of the Hamiltonian $\mathcal{H}_-$ is denoted $\mathcal{H}$ and Z denotes the partition function

$$\begin{aligned} Z := \mathrm{Tr}[e^{-\beta\mathcal{H}}] &= \sum_{N=0}^{\infty} \sum_{\{n_p\}_N} \langle \{n_p\}_N | \, e^{-\beta\mathcal{H}} \, |\{n_p\}_N\rangle \\ &= \prod_{\mathbf{p}} \sum_{n_p=0}^{\infty} e^{-\beta E_p n_p} = \prod_{\mathbf{p}} \frac{1}{1 - e^{-\beta E_p}} =: \prod_{\mathbf{p}} Z_p \, . \end{aligned} \tag{A.2}$$

This last expression temporarily switches for convenience to discretely normalized momenta, as would be appropriate when the spatial volume $\mathcal{V}$ is finite, and takes the eigenvalues of $\mathcal{H}$ to be $\sum_p n_p E_p$. The field expansion for this normalization of states is

$$\chi(t, \mathbf{x}) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2E_p\mathcal{V}}} \left[b_{\mathbf{p}} \, e^{-iE_p t + i\mathbf{p}\cdot\mathbf{x}} + b^*_{\mathbf{p}} \, e^{iE_p t - i\mathbf{p}\cdot\mathbf{x}} \right] \tag{A.3}$$

where b_p is the discretely normalized destruction operator, and the conversion between discrete and continuum normalization is given by

$$b_{\mathbf{p}} = \left[\frac{(2\pi)^3}{\mathcal{V}} \right]^{1/2} \mathfrak{b}_{\mathbf{p}} \quad \text{and} \quad \sum_{\mathbf{p}} = \frac{\mathcal{V}}{(2\pi)^3} \int \mathrm{d}^3 p \, , \tag{A.4}$$

and so on.

Inserting (A.3) into (A.1) allows it to be evaluated in the occupation-number basis, leading to

$$\begin{aligned} \langle \chi(t, \mathbf{x}) \chi(0, \mathbf{0}) \rangle_\beta &= \frac{1}{2\mathcal{V}Z} \sum_{\mathbf{kq}} \frac{\mathrm{Tr}\left[\left(e^{-iE_k t + i\mathbf{k}\cdot\mathbf{x}} b_{\mathbf{k}} + e^{+iE_k t - i\mathbf{k}\cdot\mathbf{x}} b^*_{\mathbf{k}} \right) \left(b_{\mathbf{q}} + b^*_{\mathbf{q}} \right) e^{-\beta\mathcal{H}} \right]}{\sqrt{E_k E_q}} \\ &= \frac{1}{2\mathcal{V}Z} \sum_{\mathbf{k}} \frac{\mathrm{Tr}\left[\left(e^{-iE_k t + i\mathbf{k}\cdot\mathbf{x}} b_{\mathbf{k}} + e^{+iE_k t - i\mathbf{k}\cdot\mathbf{x}} b^*_{\mathbf{k}} \right) (b_{\mathbf{k}} + b^*_{\mathbf{k}}) \, e^{-\beta\mathcal{H}} \right]}{E_k} \\ &= \frac{1}{2\mathcal{V}Z} \sum_{\mathbf{k}} \frac{e^{-iE_k t + i\mathbf{k}\cdot\mathbf{x}} \mathrm{Tr}\left[b_{\mathbf{k}} b^*_{\mathbf{k}} e^{-\beta\mathcal{H}} \right] + e^{+iE_k t - i\mathbf{k}\cdot\mathbf{x}} \mathrm{Tr}\left[b^*_{\mathbf{k}} b_{\mathbf{k}} e^{-\beta\mathcal{H}} \right]}{E_k} \, , \end{aligned} \tag{A.5}$$

and so, using $b_{\mathbf{k}} b^*_{\mathbf{k}} = b^*_{\mathbf{k}} b_{\mathbf{k}} + 1$, this becomes

$$\begin{aligned} \langle \chi(t, \mathbf{x}) \chi(0, \mathbf{0}) \rangle_\beta &= \frac{1}{2\mathcal{V}} \sum_{\mathbf{k}} \frac{e^{-iE_k t + i\mathbf{k}\cdot\mathbf{x}}}{E_k} + \frac{1}{\mathcal{V}Z} \sum_{\mathbf{k}} \frac{\cos(E_k t - \mathbf{k}\cdot\mathbf{x})}{E_k} \, \mathrm{Tr}\left[b^*_{\mathbf{k}} b_{\mathbf{k}} \, e^{-\beta\mathcal{H}} \right] \\ &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{e^{-iE_k t + i\mathbf{k}\cdot\mathbf{x}}}{2E_k} + \frac{1}{\mathcal{V}Z} \sum_{\mathbf{k}} \frac{\cos(E_k t - \mathbf{k}\cdot\mathbf{x})}{E_k} \, \mathrm{Tr}\left[b^*_{\mathbf{k}} b_{\mathbf{k}} \, e^{-\beta\mathcal{H}} \right] \end{aligned} \tag{A.6}$$

The required trace is a standard manipulation

$$\begin{aligned} \frac{1}{Z} \mathrm{Tr}\left[b^*_{\mathbf{k}} b_{\mathbf{k}} e^{-\beta\mathcal{H}} \right] &= \frac{1}{Z} \sum_{N=0}^{\infty} \sum_{\{n_p\}_N} \langle \{n_p\}_N | \, b^*_{\mathbf{k}} b_{\mathbf{k}} \, e^{-\beta\sum_{\mathbf{p}} E_p n_p} \, |\{n_p\}_N\rangle \\ &= \prod_{\mathbf{p}} \frac{1}{Z_p} \sum_{n_p=0}^{\infty} n_p \, e^{-\beta E_p n_p} = \prod_{\mathbf{p}} \frac{1}{e^{\beta E_p} - 1} \, , \end{aligned} \tag{A.7}$$

and so

$$\langle\chi(t,\mathbf{x})\chi(0,\mathbf{0})\rangle_\beta = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{-iE_kt+i\mathbf{k}\cdot\mathbf{x}}}{2E_k} + \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\cos(E_kt-\mathbf{k}\cdot\mathbf{x})}{E_k\,(e^{\beta E_k}-1)}\,. \tag{A.8}$$

It remains to perform the integrals. The first evaluates to the vacuum Wightman function while the second is the thermal correction. That is

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{e^{-iE_kt+i\mathbf{k}\cdot\mathbf{x}}}{2E_k} = \frac{1}{4\pi^2\left[-(t-i\delta)^2+|\mathbf{x}|^2\right]}\,, \tag{A.9}$$

where the limit $\delta\to0^+$ is understood, and the geometric series

$$\frac{1}{e^{\beta k}-1} = \sum_{n=1}^{\infty} e^{-n\beta k}\,, \tag{A.10}$$

allows the remaining term to be written

$$\begin{aligned}
\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\cos(E_kt-\mathbf{k}\cdot\mathbf{x})}{E_k\,(e^{\beta E_k}-1)} &= \frac{1}{4\pi^2|\mathbf{x}|}\int_0^\infty \mathrm{d}k\,\frac{e^{-ikt}+e^{+ikt}}{(e^{\beta k}-1)}\sin(k|\mathbf{x}|)\\
&= \frac{1}{4\pi^2|\mathbf{x}|}\sum_{n=1}^{\infty}\int_0^\infty \mathrm{d}k\,\left(e^{-ikt}+e^{+ikt}\right)e^{-n\beta k}\sin(k|\mathbf{x}|)\\
&= \frac{1}{4\pi^2|\mathbf{x}|}\sum_{n=1}^{\infty}\int_0^\infty \mathrm{d}k\,\left\{e^{-n\beta k}\sin\left[k(t+|\mathbf{x}|)\right]-e^{-n\beta k}\sin\left[k(t-|\mathbf{x}|)\right]\right\}\\
&= \frac{1}{8\pi\beta|\mathbf{x}|}\cdot\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(t+|\mathbf{x}|)/\beta}{n^2+[(t+|\mathbf{x}|)/\beta]^2}-\frac{(t-|\mathbf{x}|)/\beta}{n^2+[(t-|\mathbf{x}|)/\beta]^2}\right].
\end{aligned} \tag{A.11}$$

This final sum can be performed using identity (1.421.4) from [116], which for $z\in\mathbb{R}$ states

$$\coth(\pi z) = \frac{1}{\pi z}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{z}{n^2+z^2}, \tag{A.12}$$

and so

$$\begin{aligned}
\langle\chi(t,\mathbf{x})\chi(0,\mathbf{0})\rangle_\beta &= \frac{1}{8\pi^2|\mathbf{x}|}\left(\frac{1}{t+|\mathbf{x}|-i\delta}-\frac{1}{t-|\mathbf{x}|-i\delta}-\frac{1}{t+|\mathbf{x}|}+\frac{1}{t-|\mathbf{x}|}\right)\\
&\quad+\frac{\coth\left[\pi(t+|\mathbf{x}|)/\beta\right]-\coth\left[\pi(t-|\mathbf{x}|)/\beta\right]}{8\pi\beta|\mathbf{x}|}\,.
\end{aligned} \tag{A.13}$$

In the limit $\delta\to0^+$ (for real t and $\mathbf{x}$) the real parts of the second line cancel leaving

$$\langle\chi(t,\mathbf{x})\chi(0,\mathbf{0})\rangle_\beta = \frac{\coth\Big[\pi(t+|\mathbf{x}|-i\delta)/\beta\Big]-\coth\Big[\pi(t-|\mathbf{x}|-i\delta)/\beta\Big]}{8\pi\beta|\mathbf{x}|} \tag{A.14}$$

which is the result quoted in (3.15) of the main text (after restoring the arguments t' and $\mathbf{x}'$ of the second field using translational invariance).

A.1.1 The KMS condition

A subtlety with the $i\delta$-prescription in the above formula (A.14) is that it is only correct for *real* t and $|\mathbf{x}|$. This matters because the thermal correlation function is supposed to obey the Kubo-Martin-Schwinger (KMS) condition given by

$$\langle \chi(t - i\beta, \mathbf{x})\chi(t', \mathbf{x}')\rangle_\beta = \langle \chi(t', \mathbf{x}')\chi(t, \mathbf{x})\rangle_\beta = \langle \chi(t, \mathbf{x})\chi(t', \mathbf{x}')\rangle^*_\beta \,, \tag{A.15}$$

which (A.14) apparently does not satisfy because of the $i\delta$-prescription used there.

Later in appendix D.2.3 we prove that the function $\mathscr{E}_\beta$ obeys a KMS-type condition, and so for later use we here flesh out the argument for why the KMS is explicitly obeyed for the free thermal correlator $\langle \chi(t, \mathbf{x})\chi(t', \mathbf{x}')\rangle_\beta$. To see how this works, go back to (A.8) (with arguments t' and $\mathbf{x}'$ reinstated) which says

$$\langle \chi(t, \mathbf{x})\chi(t', \mathbf{x})\rangle_\beta = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{-iE_k(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2E_k\,(1 - e^{-\beta E_k})} + \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{+iE_k(t-t')-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2E_k\,(e^{\beta E_k} - 1)} \tag{A.16}$$

after use of the identity $1 + (e^{\beta E_k} - 1)^{-1} = (1 - e^{-\beta E_k})^{-1}$. Evaluating this with $t \to t - i\beta$ gives[6]

$$\begin{aligned}
\langle \chi(t-i\beta, \mathbf{x})\chi(t', \mathbf{x}')\rangle_\beta &= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{-\beta E_k}e^{-iE_k(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2E_k\,(1-e^{-\beta E_k})} + \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{+\beta E_k}e^{+iE_k(t-t')-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2E_k\,(e^{\beta E_k}-1)} \\
&= \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{-iE_k(t-t')+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2E_k\,(e^{\beta E_k}-1)} + \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{e^{+iE_k(t-t')-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{2E_k\,(1-e^{-\beta E_k})} \\
&= \langle \chi(t', \mathbf{x}')\chi(t, \mathbf{x})\rangle_\beta \,,
\end{aligned} \tag{A.17}$$

as required by the KMS condition (A.15).

The validity of the KMS condition can be made manifest in position space if we re-write (3.15) with an $i\delta$-prescription that is both consistent with the KMS condition and reduces to (3.15) in the limit of real t. To do this notice the identity

$$\coth(a + b) - \coth(a - b) = -\frac{\sinh(2b)}{\sinh(a + b)\sinh(a - b)} \qquad \text{for any } a, b \in \mathbb{C}\,, \tag{A.18}$$

which allows the correlation function to be written in the KMS-consistent form (cf. formula (A.13))

$$\langle \chi(t, \mathbf{x})\chi(t', \mathbf{x}')\rangle_\beta = -\frac{\sinh\left(\frac{2\pi|\mathbf{x}-\mathbf{x}'|}{\beta}\right)}{8\pi\beta|\mathbf{x}-\mathbf{x}'|\left[\sinh\left(\frac{\pi(t-t'+|\mathbf{x}-\mathbf{x}'|)}{\beta}\right) - \frac{i\pi\delta}{\beta}\right]\left[\sinh\left(\frac{\pi(t-t'-|\mathbf{x}-\mathbf{x}'|)}{\beta}\right) - \frac{i\pi\delta}{\beta}\right]}\,. \tag{A.19}$$

[6]Notice that replacing $t \to t + ib$ for some imaginary part b in the above makes the integral converge for all $-\beta < b < 0$, since $(e^{\beta E_k} - 1)^{-1} \simeq e^{-\beta E_k}$ and $(1 - e^{-\beta E_k})^{-1} \simeq 1 + e^{-\beta E_k}$ for large momenta $\beta E_k \gg 1$. This means that the correlator is a complex-analytic function of time $t \in \mathbb{C}$ in this strip [117] (with $-\beta < \mathrm{Im}[t] < 0$).

B RG evolution

This appendix summarizes some parts of the renormalization evolution not made explicit in the main text, closely following the discussion in appendix F of [87].

B.1 Universal evolution

The boundary conditions of the main text provide examples where the effective couplings are found to satisfy equations of the form

$$g(\epsilon) = \frac{A\rho_\epsilon^{2\zeta} + B}{C\rho_\epsilon^{2\zeta} + D}\,, \tag{B.1}$$

where g is a representative coupling and ϵ appears on the right-hand side through the variable ρ_ϵ. In the example of (4.17), for instance, we have $g = \lambda/4\pi\epsilon$ while $\rho_\epsilon = \frac{1}{2}\,\omega\epsilon$, the power is $\zeta = \ell + \frac{1}{2}$ and the parameters A, B, C and D are given explicitly by

$$A = \ell\,, \quad B = (\ell+1)\,X_\ell(\omega)\,, \quad C = 1 \quad \text{and} \quad D = -X_\ell(\omega)\,, \tag{B.2}$$

where

$$X_\ell\,(\omega) := \frac{1}{\pi}\,\Gamma\left(\ell+\tfrac{3}{2}\right)\,\Gamma\left(\ell+\tfrac{1}{2}\right)\,\frac{D_\ell\,(\omega)}{C_\ell(\omega)}\,. \tag{B.3}$$

For later use, eq. (B.1) also inverts to give

$$\rho_\epsilon^{2\zeta} = \frac{B - Dg}{Cg - A}\,. \tag{B.4}$$

The goal is to derive a universal differential version of this evolution (see, for example [80–84] for more details). To start this off directly differentiate (B.1) holding A, B, C, D fixed, leading to

$$\epsilon\,\frac{\mathrm{d}g}{\mathrm{d}\epsilon} = 2\zeta\left[\frac{AD - BC}{(C\rho_\epsilon^{2\zeta} + D)^2}\right]\rho_\epsilon^{2\zeta} = 2\zeta\left[\frac{(Cg - A)(B - Dg)}{AD - BC}\right]\,, \tag{B.5}$$

where the second equality uses (B.4) to trade $\rho_\epsilon^{2\zeta}$ for g. This evolution equation has fixed points at $g = g_*$, where

$$g_* = \frac{A}{C} \quad \text{or} \quad g_* = \frac{B}{D}\,, \tag{B.6}$$

which can also be seen as the $\rho_\epsilon \to 0$ and $\rho_\epsilon \to \infty$ limits of (B.1).

Equation (B.5) can be put into a standard form by redefining g to ensure that $g_* = \pm 1$. To this end write

$$g(\epsilon) = u(\epsilon) + \frac{1}{2}\left(\frac{A}{C} + \frac{B}{D}\right)\,, \tag{B.7}$$

in terms of which the fixed points are

$$u_* = \pm\frac{1}{2}\left(\frac{A}{C} - \frac{B}{D}\right) = \pm\left(\frac{AD - BC}{2CD}\right)\,, \tag{B.8}$$

and (B.5) becomes

$$\epsilon\,\frac{\mathrm{d}u}{\mathrm{d}\epsilon} = -\frac{2\zeta CD}{AD - BC}\left[u - \left(\frac{AD - BC}{2CD}\right)\right]\left[u + \left(\frac{AD - BC}{2CD}\right)\right]\,. \tag{B.9}$$

Finally rescale

$$u = \left[\frac{AD - BC}{2CD}\right] v \tag{B.10}$$

to see that

$$\epsilon \frac{\mathrm{d}v}{\mathrm{d}\epsilon} = \zeta(1 - v^2) \tag{B.11}$$

is an automatic consequence of (B.1) once one defines

$$g = u + \frac{AD + BC}{2CD} = \frac{1}{2}\left(\frac{A}{C} - \frac{B}{D}\right) v + \frac{1}{2}\left(\frac{A}{C} + \frac{B}{D}\right) . \tag{B.12}$$

These expressions emphasize that although the positions of the fixed points for g depend on the ratios A/C and B/D, the speed of evolution along the RG flow depends only on ζ. Indeed the general solution to (B.11) is

$$v(\epsilon) = \frac{(v_0 + 1)(\epsilon/\epsilon_0)^{2\zeta} + (v_0 - 1)}{(v_0 + 1)(\epsilon/\epsilon_0)^{2\zeta} - (v_0 - 1)} \tag{B.13}$$

where the integration constant is chosen to ensure $v(\epsilon_0) = v_0$. For $\zeta > 0$ this describes a universal flow that runs from $v = -1$ to $v = +1$ as ϵ flows from 0 to ∞.

Since the trajectories given in (B.13) cannot cross the lines $v = \pm 1$ for any finite nonzero ϵ there are two categories of flow, distinguished by the flow-invariant sign of $|v| - 1$ (see figure 6). That is, if $|v_0| - 1$ is negative (positive) for any $0 < \epsilon_0 < \infty$, then $|v(\epsilon)| - 1$ is negative (positive) for all $0 < \epsilon < \infty$. Every trajectory is therefore uniquely characterized by a pair of numbers. These can equally well be chosen to be the pair (ϵ_0, v_0) that specifies an initial condition $v_0 = v(\epsilon_0)$, or it can be taken to be the pair $(\epsilon_\star, y_\star)$ where $y_\star = \mathrm{sign}(|v| - 1) = \pm 1$ distinguishes the two classes of trajectories, and $\epsilon_\star$ is defined as the value of ϵ for which $v(\epsilon_\star) = 0$ (if $y_\star = -1$) or the value for which $v(\epsilon_\star) = \infty$ (if $y_\star = +1$). The parameterization using $(\epsilon_\star, y_\star)$ is useful because physical observables turn out to have particularly transparent expressions in terms of these variables.

For the specific cases given in (B.2) the fixed points are

$$\frac{A}{C} = \ell \quad \text{and} \quad \frac{B}{D} = -\ell - 1 \tag{B.14}$$

and λ is related to the universal scaling variable v by (B.12), which becomes

$$\frac{\lambda}{4\pi\epsilon} = \frac{1}{2}\Big[(2\ell + 1)\, v - 1\Big] \quad \text{and so} \quad v = \frac{1}{2\ell + 1}\left(\frac{\lambda}{2\pi\epsilon} + 1\right) , \tag{B.15}$$

as used in the main text.

C Mode properties

This appendix evaluates several properties associated with the modes in the presence of a $\frac{1}{2}\,\lambda\,\phi^2(\mathbf{0})$ interaction localized at the hotspot. The first subsection computes their normalization constants and the second evaluates the mode sums required for the Wightman function (in an approximate limit).

C.1 Mode normalization

This appendix computes the $\epsilon_\star$-dependence of the integration constants C_ℓ and D_ℓ, by requiring the mode functions to be properly normalized. As discussed in the main text, we do so using the standard Klein-Gordon inner product (4.35), since the reality of λ ensures this remains time-independent even with the λ-dependent boundary conditions of section 4.

Our mode functions have the form

$$u_{\omega\ell m}(t,\mathbf{x}) = e^{-i\omega t}\big[C_\ell(\omega) j_\ell(\omega r) + D_\ell(\omega) y_\ell(\omega r)\big] Y_{\ell m}(\theta,\phi), \tag{C.1}$$

where we have already seen that the boundary condition implies the ratio D_ℓ/C_ℓ is given by (4.15) or (4.33). Inserting this into the Klein-Gordon inner product (4.35) yields

$$\begin{aligned}\langle u_{\omega\ell m}, u_{\tilde\omega\tilde\ell\tilde m}\rangle &= i\int \mathrm{d}^3x\ \Big(u^*_{\omega\ell m}(t,\mathbf{x})\dot u_{\tilde\omega\tilde\ell\tilde m}(t,\mathbf{x}) - \dot u^*_{\omega\ell m}(t,\mathbf{x}) u_{\tilde\omega\tilde\ell\tilde m}(t,\mathbf{x})\Big)\\ &= C^*_\ell(\omega) C_{\tilde\ell}(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int \mathrm{d}^3x\ j_\ell(\omega r)Y_{\ell m}(\theta,\phi) j_{\tilde\ell}(\tilde\omega r)Y_{\ell m}(\theta,\phi)\\ &\quad + C^*_\ell(\omega) D_{\tilde\ell}(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int \mathrm{d}^3x\ j_\ell(\omega r)Y_{\ell m}(\theta,\phi) y_{\tilde\ell}(\tilde\omega r)Y_{\ell m}(\theta,\phi)\\ &\quad + D^*_\ell(\omega) C_{\tilde\ell}(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int \mathrm{d}^3x\ y_\ell(\omega r)Y_{\ell m}(\theta,\phi) j_{\tilde\ell}(\tilde\omega r)Y_{\ell m}(\theta,\phi)\\ &\quad + D^*_\ell(\omega) D_{\tilde\ell}(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int \mathrm{d}^3x\ y_\ell(\omega r)Y_{\ell m}(\theta,\phi) y_{\tilde\ell}(\tilde\omega r)Y_{\ell m}(\theta,\phi)\end{aligned} \tag{C.2}$$

Using orthonormality of the spherical harmonics $Y_{\ell m}(\theta,\phi)$

$$\int \mathrm{d}^2\Omega\ Y_{\ell m}(\theta,\phi) Y_{\tilde\ell\tilde m}(\theta,\phi) = \delta_{\ell\tilde\ell}\,\delta_{m\tilde m}\,, \tag{C.3}$$

allows the above to be written as

$$\begin{aligned}\langle u_{\omega\ell m}, u_{\tilde\omega\tilde\ell\tilde m}\rangle = \delta_{\ell\tilde\ell}\,\delta_{m\tilde m}\Big[&C^*_\ell(\omega)C_\ell(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int_0^\infty \mathrm{d}r\ r^2 j_\ell(\omega r) j_\ell(\tilde\omega r)\\ &+ C^*_\ell(\omega)D_\ell(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int_0^\infty \mathrm{d}r\ r^2 j_\ell(\omega r) y_\ell(\tilde\omega r)\\ &+ D^*_\ell(\omega)C_\ell(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int_0^\infty \mathrm{d}r\ r^2 y_\ell(\omega r) j_\ell(\tilde\omega r)\\ &+ D^*_\ell(\omega)D_\ell(\tilde\omega)(\omega+\tilde\omega)e^{+i(\omega-\tilde\omega)t}\int_0^\infty \mathrm{d}r\ r^2 y_\ell(\omega r) y_\ell(\tilde\omega r)\Big]\,.\end{aligned} \tag{C.4}$$

The j-j and the y-y terms can be evaluated using the orthonormality relation for spherical Bessel functions,

$$\int_0^\infty \mathrm{d}r\ r^2 j_\ell(\omega r) j_\ell(\tilde\omega r) = \int_0^\infty \mathrm{d}r\ r^2 y_\ell(\omega r) y_\ell(\tilde\omega r) = \frac{\pi}{2\omega^2}\delta(\omega-\tilde\omega)\,, \tag{C.5}$$

while the cross-terms are evaluated in appendix C.1.1, giving

$$\int_0^\infty \mathrm{d}r\ r^2 j_\ell(\omega r) y_\ell(\tilde\omega r) = \frac{(\omega/\tilde\omega)^\ell}{\tilde\omega(\tilde\omega^2-\omega^2)}\ . \tag{C.6}$$

The above manipulations lead to the expression

$$\begin{aligned}\langle u_{\omega\ell m}, u_{\tilde{\omega}\tilde{\ell}\tilde{m}}\rangle = \delta_{\ell\tilde{\ell}}\delta_{m\tilde{m}}\Big[& C^*_\ell(\omega)C_\ell(\tilde{\omega})(\omega+\tilde{\omega})e^{+i(\omega-\tilde{\omega})t}\frac{\pi}{2\omega^2}\delta(\omega-\tilde{\omega}) \\ & + C^*_\ell(\omega)D_\ell(\tilde{\omega})(\omega+\tilde{\omega})e^{+i(\omega-\tilde{\omega})t}\frac{(\omega/\tilde{\omega})^\ell}{\tilde{\omega}(\tilde{\omega}^2-\omega^2)} \\ & + D^*_\ell(\omega)C_\ell(\tilde{\omega})(\omega+\tilde{\omega})e^{+i(\omega-\tilde{\omega})t}\frac{(\tilde{\omega}/\omega)^\ell}{\omega(\omega^2-\tilde{\omega}^2)} \\ & + D^*_\ell(\omega)D_\ell(\tilde{\omega})(\omega+\tilde{\omega})e^{+i(\omega-\tilde{\omega})t}\frac{\pi}{2\omega^2}\delta(\omega-\tilde{\omega})\Big]\,. \end{aligned} \tag{C.7}$$

Of these, the terms with δ-functions are simplified if we take $\tilde{\omega}\to\omega$, and after some simplification on the cross-terms the above becomes

$$\begin{aligned}\langle u_{\omega\ell m}, u_{\tilde{\omega}\tilde{\ell}\tilde{m}}\rangle = \delta_{\ell\tilde{\ell}}\,\delta_{m\tilde{m}}\Bigg(& \frac{\pi}{\omega}(|C_\ell(\omega)|^2+|D_\ell(\omega)|^2)\delta(\omega-\tilde{\omega}) \\ & + \frac{e^{+i(\omega-\tilde{\omega})t}}{\omega-\tilde{\omega}}\Big[-C^*_\ell(\omega)D_\ell(\tilde{\omega})\frac{\omega^\ell}{\tilde{\omega}^{\ell+1}} + D^*_\ell(\omega)C_\ell(\tilde{\omega})\frac{\tilde{\omega}^\ell}{\omega^{\ell+1}}\Big]\Bigg)\,. \end{aligned} \tag{C.8}$$

The second line of this last equation seems suspicious because it is time-dependent and the Klein-Gordon inner product should not be when evaluated on a solution to the Klein-Gordon equation. However, this has not yet accounted for the relation between C_ℓ and D_ℓ that follows from the boundary condition, which states

$$\frac{D_{\ell m}(\omega)}{C_{\ell m}(\omega)} \simeq \frac{\pi\eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\cdot\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1}\,. \tag{C.9}$$

Using this, the square bracket in (C.8) becomes

$$\begin{aligned} & \left[-C^*_\ell(\omega)\,D_\ell(\tilde{\omega})\,\frac{\omega^\ell}{\tilde{\omega}^{\ell+1}} + D^*_\ell(\omega)\,C_\ell(\tilde{\omega})\,\frac{\tilde{\omega}^\ell}{\omega^{\ell+1}}\right] \\ & \qquad = C^*_\ell(\omega)\,C_\ell(\tilde{\omega})\left[-\frac{D_\ell(\tilde{\omega})}{C_\ell(\tilde{\omega})}\frac{\omega^\ell}{\tilde{\omega}^{\ell+1}} + \frac{D^*_\ell(\omega)}{C^*_\ell(\omega)}\frac{\tilde{\omega}^\ell}{\omega^{\ell+1}}\right] \\ & \qquad = \frac{\pi\eta_\star C^*_{\ell m}(\omega)\,C_{\ell m}(\tilde{\omega})}{\Gamma\left(\ell+\frac{1}{2}\right)\Gamma\left(\ell+\frac{3}{2}\right)}\left[-\left(\frac{\tilde{\omega}\epsilon_\star}{2}\right)^{2\ell+1}\frac{\omega^\ell}{\tilde{\omega}^{\ell+1}} + \left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1}\frac{\tilde{\omega}^\ell}{\omega^{\ell+1}}\right] \\ & \qquad = 0\,, \end{aligned} \tag{C.10}$$

so, as expected, the boundary conditions ensure the time-independence of the inner product.

The final result then is

$$\begin{aligned}\langle u_{\omega\ell m}, u_{\tilde{\omega}\tilde{\ell}\tilde{m}}\rangle & = \delta_{\ell\tilde{\ell}}\,\delta_{m\tilde{m}}\,\frac{\pi}{\omega}(|C_{\ell m}(\omega)|^2+|D_{\ell m}(\omega)|^2)\delta(\omega-\tilde{\omega}) \\ & = \delta_{\ell\tilde{\ell}}\,\delta_{m\tilde{m}}\,\frac{\pi}{\omega}\left\{1+\left[\frac{\pi}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})}\right]^2\left(\frac{\omega\epsilon_\star}{2}\right)^{4\ell+2}\right\}|C_{\ell m}(\omega)|^2\delta(\omega-\tilde{\omega})\,, \end{aligned} \tag{C.11}$$

which uses $\eta_\star^2 = 1$. Proper normalization then implies

$$C_{\ell m}(\omega) = \sqrt{\frac{\omega}{\pi}} \left\{ 1 + \left[\frac{\pi}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})} \left(\frac{\omega \epsilon_\star}{2} \right)^{2\ell+1} \right]^2 \right\}^{-1/2}, \tag{C.12}$$

and

$$D_{\ell m}(\omega) = \frac{\pi \eta_\star}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})} \left(\frac{\omega \epsilon_\star}{2} \right)^{2\ell+1} \sqrt{\frac{\omega}{\pi}} \left\{ 1 + \left[\frac{\pi}{\Gamma(\ell+\frac{1}{2})\Gamma(\ell+\frac{3}{2})} \left(\frac{\omega \epsilon_\star}{2} \right)^{2\ell+1} \right]^2 \right\}^{-1/2} \tag{C.13}$$

as claimed in the main text. It is straightforward to similarly check the other relations $\langle u_{\omega \ell m}, u_{\tilde{\omega}\tilde{\ell}\tilde{m}} \rangle = \delta_{\ell\tilde{\ell}}\, \delta_{m\tilde{m}}\, \delta(\omega - \tilde{\omega})$ and $\langle u_{\omega \ell m}, u^*_{\tilde{\omega}\tilde{\ell}\tilde{m}} \rangle = 0$.

C.1.1 Evaluating the $j_\ell \cdot y_\ell$ product integral

We next compute the integral that appears in (C.6) above, when calculating the cross terms when normalizing the mode functions. First we use $j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell+\frac{1}{2}}(z)$ to write

$$\begin{aligned} \int_0^\infty \mathrm{d}r\ r^2 j_\ell(ar) y_\ell(br) &= (-1)^{\ell+1} \int_0^\infty \mathrm{d}r\ r^2 j_\ell(ar) j_{-\ell-1}(br) \\ &= \frac{(-1)^{\ell+1}\pi}{2\sqrt{ab}} \int_0^\infty \mathrm{d}r\ r J_{\ell+\frac{1}{2}}(ar) J_{-\ell-\frac{1}{2}}(br) \\ &= \frac{(-1)^{\ell+1}\pi}{2\sqrt{ab}} \lim_{\epsilon\to 0^+} \int_0^\infty \mathrm{d}r\ r^{1-\epsilon} J_{\ell+\frac{1}{2}}(ar) J_{-\ell-\frac{1}{2}}(br)\,. \end{aligned} \tag{C.14}$$

From here we must use the formula (10.22.56) from [118] where

$$\int_0^\infty \mathrm{d}r\ r^{-\lambda} J_\mu(ar) J_\nu(br) = \frac{a^\mu \Gamma\left(\frac{\nu+\mu-\lambda+1}{2}\right)\ {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\mu-\nu-\lambda+1}{2}, \mu+1; \frac{a^2}{b^2}\right)}{2^\lambda b^{\mu-\lambda+1} \Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right) \Gamma(\mu+1)} \tag{C.15}$$

which assumes that $0 < a < b$ and $\mathrm{Re}[\mu+\nu+1] > \mathrm{Re}[\lambda] > -1$. In the case that $0 < a < b$ and picking some tiny $\epsilon > 0$ so that $\lambda = \epsilon - 1$ as well as $\mu = \ell + \frac{1}{2}$ and $\nu = -\ell - \frac{1}{2}$ we get

$$\int_0^\infty \mathrm{d}r\ r^2 j_\ell(ar) y_\ell(br) = \frac{(-1)^{\ell+1}\pi}{2\sqrt{ab}} \lim_{\epsilon\to 0^+} \left\{ \frac{2\cos(\pi\ell)}{\pi} \cdot \frac{a^{\frac{1}{2}+\ell} b^{-\frac{1}{2}-\ell}}{a^2 - b^2} + \mathcal{O}(\epsilon) \right\} \tag{C.16}$$

Noting that $\cos(\pi\ell) = (-1)^\ell$ and taking the limit $\epsilon \to 0^+$ gives

$$\int_0^\infty \mathrm{d}r\ r^2 j_\ell(ar) y_\ell(br) = \frac{(a/b)^\ell}{b(b^2 - a^2)} \qquad \text{when } 0 < a < b \tag{C.17}$$

which is only true for the case $a < b$. For the other case $a > b$ we switch the positions of the Bessel functions in the formula giving us (we need to simultaneously swap a and b, as well as $\ell + \frac{1}{2}$ and $-\ell - \frac{1}{2}$)

$$\begin{aligned} \int_0^\infty \mathrm{d}r\ r^2 j_\ell(ar) y_\ell(br) &= (-1)^{\ell+1} \int_0^\infty \mathrm{d}r\ r^2 j_{-\ell-1}(br) j_\ell(ar) \\ &= \frac{(-1)^{\ell+1}\pi}{2\sqrt{ab}} \lim_{\epsilon\to 0^+} \int_0^\infty \mathrm{d}r\ r^{1-\epsilon} J_{-\ell-\frac{1}{2}}(br) J_{\ell+\frac{1}{2}}(ar) \\ &= \frac{(a/b)^\ell}{b(b^2 - a^2)} \qquad \text{when } 0 < b < a\,. \end{aligned} \tag{C.18}$$

Combining gives the result for any $a > 0$ and $b > 0$

$$\int_0^\infty \mathrm{d}r\; r^2 j_\ell(ar) y_\ell(br) = \frac{(a/b)^\ell}{b(b^2-a^2)} \tag{C.19}$$

as quoted in (C.6).

C.2 Mode sum

This appendix evaluates the mode sum encountered in the main text when computing the Wightman function for ϕ in the presence of the localized $\lambda\phi^2(t,\mathbf{0})$ hotspot interaction. As argued in the main text, the Wightman function is given by the mode sum

$$\begin{aligned}
&\langle \phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle \\
&\quad= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\int_0^\infty \mathrm{d}\omega\; u_{\omega\ell m}(t,\mathbf{x})u^*_{\omega\ell m}(t',\mathbf{x}') \qquad\qquad (C.20)\\
&\quad= \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{+\ell}\int_0^\infty \mathrm{d}\omega\; e^{-i\omega(t-t')}|C_{\ell m}(\omega)|^2\left[j_\ell(\omega|\mathbf{x}|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac12)\Gamma(\ell+\frac32)}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}|)\right]\\
&\qquad\qquad \times\left[j_\ell(\omega|\mathbf{x}'|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac12)\Gamma(\ell+\frac32)}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}'|)\right] Y_{\ell m}(\theta,\phi)Y^*_{\ell m}(\theta',\phi')\;.
\end{aligned}$$

C.2.1 Evaluating the sums

Next exploit spherical symmetry about the origin to rotate our coordinate axes so that the $\mathbf{x}'$ direction is the 3-axis of polar coordinates, in which case we can set $\theta' = 0$ (not specifying ϕ'). Noting the identity (14.4.30) from [118] we can write

$$Y_{\ell m}(0,\phi') = \delta_{m0}\sqrt{\frac{2\ell+1}{4\pi}}\,, \tag{C.21}$$

along with $Y_{\ell 0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$ where P_ℓ is the Legendre polynomial of degree ℓ. This gives

$$\begin{aligned}
&\langle \phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle \\
&\quad= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\int_0^\infty \mathrm{d}\omega\; e^{-i\omega(t-t')}|C_{\ell 0}(\omega)|^2\big[j_\ell(\omega|\mathbf{x}|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac12)\Gamma(\ell+\frac32)}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}|)\big]\\
&\qquad\qquad \times\big[j_\ell(\omega|\mathbf{x}'|) + \frac{\pi\eta_\star}{\Gamma(\ell+\frac12)\Gamma(\ell+\frac32)}\left(\frac{\omega\epsilon_\star}{2}\right)^{2\ell+1} y_\ell(\omega|\mathbf{x}'|)\big] P_\ell(\cos\theta)\;. \qquad (C.22)
\end{aligned}$$

Notice also in passing that the Gamma-matrix identities $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n) = (n-1)!$ and $\Gamma(n+\frac12) = 2^{1-2n}\sqrt{\pi}\,\Gamma(2n)/\Gamma(n)$ imply

$$\frac{\pi}{\Gamma(\ell+\frac12)\Gamma(\ell+\frac32)} = \frac{2\pi}{(2\ell+1)\Gamma(\ell+\frac12)^2} = \frac{2^{4\ell+2}\Gamma(\ell)^2}{8(2\ell+1)\Gamma(2\ell)^2} = \frac{2^{4\ell+2}[\ell!]^2}{2(2\ell+1)[(2\ell)!]^2}\,. \tag{C.23}$$

Perturbative limit. For further progress assume $\omega\epsilon_\star \ll 1$ and seek only the leading $\epsilon_\star$-dependent contribution. Because of the factor $(\omega\epsilon_\star)^{2\ell+1}$ this allows restricting to $\ell = 0$ in the $\epsilon_\star$-dependent term. In this regime we can approximate the mode sum

$$\begin{aligned}\langle\phi(t,\mathbf{x})\,\phi(t',\mathbf{x}')\rangle \simeq &\sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi}\int_0^{\infty}\mathrm{d}\omega\, e^{-i\omega(t-t')}\left(\frac{\omega}{\pi}\right)j_\ell(\omega|\mathbf{x}|)\,j_\ell(\omega|\mathbf{x}'|)\,P_\ell(\cos\theta) \\ &+\frac{1}{4\pi}\int_0^{\infty}\mathrm{d}\omega\, e^{-i\omega(t-t')}\left(\frac{\omega}{\pi}\right)j_0(\omega|\mathbf{x}|)\,\frac{\pi\eta_\star}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}\left(\frac{\omega\epsilon_\star}{2}\right)y_0(\omega|\mathbf{x}'|)\,P_0(\cos\theta) \\ &+\frac{1}{4\pi}\int_0^{\infty}\mathrm{d}\omega\, e^{-i\omega(t-t')}\left(\frac{\omega}{\pi}\right)\frac{\pi\eta_\star}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}\left(\frac{\omega\epsilon_\star}{2}\right)y_0(\omega|\mathbf{x}|)\,j_0(\omega|\mathbf{x}'|)\,P_0(\cos\theta) \\ &+\mathcal{O}\left(\omega^2\epsilon_\star^2\right).\end{aligned} \tag{C.24}$$

where the s-wave normalization simplifies to $|C_{\ell m}(\omega)|^2 \simeq \frac{\omega}{\pi}[1+\mathcal{O}(\omega^2\epsilon_\star^2)]$. Next use the explicit form for the low-order spherical Bessel functions,

$$j_0(x) = \frac{\sin(x)}{x} \qquad \text{and} \qquad y_0(x) = -\frac{\cos(x)}{x}\,, \tag{C.25}$$

and along with $\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}) = \frac{\pi}{2}$ and $P_0(x) = 1$ to get

$$\begin{aligned}\langle\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle \simeq &\frac{1}{4\pi^2}\int_0^{\infty}\mathrm{d}\omega\; e^{-i\omega(t-t')}\bigg\{\omega\sum_{\ell=0}^{\infty}(2\ell+1)j_\ell(\omega|\mathbf{x}|)j_\ell(\omega|\mathbf{x}'|)P_\ell(\cos\theta) \\ &-\frac{(4\pi\eta_\star\epsilon_\star)}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\int_0^{\infty}\mathrm{d}\omega\; e^{-i\omega(t-t')}\Big[\sin(\omega|\mathbf{x}|)\cos(\omega|\mathbf{x}'|)+\cos(\omega|\mathbf{x}|)\sin(\omega|\mathbf{x}'|)\Big]\bigg\}\end{aligned} \tag{C.26}$$

which drops $(\omega\epsilon_\star)^2$ terms. The ℓ sum is performed using (10.60.2) from [118] which says

$$\sum_{\ell=0}^{\infty}(2\ell+1)j_\ell(u)j_\ell(v)P_\ell(\cos\alpha) \;=\; \frac{\sin\sqrt{u^2+v^2-2uv\cos(\alpha)}}{\sqrt{u^2+v^2-2uv\cos(\alpha)}} \tag{C.27}$$

and so

$$\begin{aligned}&\langle\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\rangle \\ &\simeq \frac{1}{4\pi^2}\int_0^{\infty}\mathrm{d}\omega\; e^{-i\omega(t-t')}\omega\cdot\frac{\sin\left(\omega\sqrt{|\mathbf{x}|^2+|\mathbf{x}'|^2-2|\mathbf{x}||\mathbf{x}'|\cos(\theta)}\right)}{\omega\sqrt{|\mathbf{x}|^2+|\mathbf{x}'|^2-2|\mathbf{x}||\mathbf{x}'|\cos(\theta)}} \\ &\qquad -\frac{(4\pi\eta_\star\epsilon_\star)}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\int_0^{\infty}\mathrm{d}\omega\; e^{-i\omega(t-t')}\Big[\sin(\omega|\mathbf{x}|)\cos(\omega|\mathbf{x}'|)+\cos(\omega|\mathbf{x}|)\sin(\omega|\mathbf{x}'|)\Big] \\ &= \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^{\infty}\mathrm{d}\omega\; e^{-i\omega(t-t')}\sin(\omega|\mathbf{x}-\mathbf{x}'|) \\ &\qquad -\frac{(4\pi\eta_\star\epsilon_\star)}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\int_0^{\infty}\mathrm{d}\omega\; e^{-i\omega(t-t')}\sin\Big[\omega(|\mathbf{x}|+|\mathbf{x}'|)\Big]\,.\end{aligned} \tag{C.28}$$

The above result uses the alignment of the coordinates so that $\mathbf{x}'$ points along the 3-axis to write

$$|\mathbf{x}|^2+|\mathbf{x}'|^2-2|\mathbf{x}||\mathbf{x}'|\cos(\theta) = |\mathbf{x}-\mathbf{x}'|^2\,. \tag{C.29}$$

The frequency integral finally is

$$\int_0^\infty \mathrm{d}\omega\; e^{-iT\omega}\sin(X\omega) = \frac{i}{2}\int_{-\infty}^{\infty}\mathrm{d}\omega\;\Theta(\omega)\Big[e^{-i(T+X)\omega} + e^{-i(T-X)\omega}\Big] = \frac{1}{2}\left(\frac{1}{T+X-i\delta} - \frac{1}{T-X-i\delta}\right) \tag{C.30}$$

where δ is, as usual, the positive infinitesimal that arises in the Fourier transform of the Heaviside step function. In this way the mode sum evaluates to the result quoted in the main text:

$$\langle \phi\,(t,\mathbf{x})\,\phi\,(t',\mathbf{x}')\rangle \simeq \frac{1}{4\pi^2}\cdot\frac{1}{-\,(t-t'-i\delta)^2 + |\mathbf{x}-\mathbf{x}'|^2} - \frac{(4\pi\eta_\star\epsilon_\star)}{32\pi^3|\mathbf{x}||\mathbf{x}'|}\left[\frac{1}{t-t'+|\mathbf{x}|+|\mathbf{x}'|-i\delta} - \frac{1}{t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta}\right]. \tag{C.31}$$

D Mode sum for the exact two-point correlator

In this appendix we explicitly evaluate the mode sums for the functions $\mathscr{S}(t,\mathbf{x};t',\mathbf{x}')$ and $\mathscr{E}_\beta(t,\mathbf{x};t',\mathbf{x}')$ defined in (5.15) and (5.16), giving us a non-perturbative expression for the Wightman function $W_\beta(t,\mathbf{x};t',\mathbf{x}') = \mathscr{S}(t,\mathbf{x};t',\mathbf{x}') + \mathscr{E}_\beta(t,\mathbf{x};t',\mathbf{x}')$.

D.1 The temperature-independent contribution, $\mathscr{S}(t,\mathbf{x};t',\mathbf{x}')$

Using the explicit form for the mode function $S_{\mathbf{p}}(t,\mathbf{x})$ given in (5.10), the function $\mathscr{S}(t,\mathbf{x};t,\mathbf{x}')$ defined in (5.15) simplifies to

$$\mathscr{S}(t,\mathbf{x};t',\mathbf{x}') = \frac{1}{4\pi^2}\cdot\frac{1}{-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2} + \mathscr{S}_1(t,\mathbf{x};t,\mathbf{x}') + \mathscr{S}_2(t,\mathbf{x};t,\mathbf{x}') + \mathscr{S}_3(t,\mathbf{x};t,\mathbf{x}') \tag{D.1}$$

with the definitions

$$\mathscr{S}_1(t,\mathbf{x};t',\mathbf{x}') := \frac{1}{|\mathbf{x}'|}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p}\; e^{-iE_p(t-t'+|\mathbf{x}'|)+i\mathbf{p}\cdot\mathbf{x}}\,\frac{-\frac{\lambda}{4\pi} - \frac{i\tilde{g}^2E_p}{16\pi^2}}{1+\frac{\lambda}{4\pi\epsilon}+\frac{i\tilde{g}^2E_p}{16\pi^2\epsilon}} \tag{D.2}$$

and

$$\mathscr{S}_2(t,\mathbf{x};t',\mathbf{x}') := \frac{1}{|\mathbf{x}|}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p}\; e^{-iE_p(t-t'-|\mathbf{x}|)-i\mathbf{p}\cdot\mathbf{x}'}\,\frac{-\frac{\lambda}{4\pi} + \frac{i\tilde{g}^2E_p}{16\pi^2}}{1+\frac{\lambda}{4\pi\epsilon}-\frac{i\tilde{g}^2E_p}{16\pi^2\epsilon}} \tag{D.3}$$

as well as

$$\mathscr{S}_3(t,\mathbf{x};t',\mathbf{x}') := \frac{1}{|\mathbf{x}||\mathbf{x}'|}\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p} e^{-iE_p(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\frac{\left(\frac{\lambda}{4\pi}\right)^2+\left(\frac{\tilde{g}^2E_p}{16\pi^2}\right)^2}{\left(1+\frac{\lambda}{4\pi\epsilon}\right)^2+\left(\frac{\tilde{g}^2E_p}{16\pi^2\epsilon}\right)^2}\,. \tag{D.4}$$

Integrating the momentum angles away in spherical coordinates and simplifying turns the above into

$$\mathscr{S}_1(t,\mathbf{x};t',\mathbf{x}') = \frac{\epsilon}{4\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; e^{-ip(t-t'+|\mathbf{x}'|)}\sin(|\mathbf{x}|p)\frac{-4\pi\lambda/\tilde{g}^2 - ip}{c+ip} \tag{D.5}$$

and

$$\mathscr{S}_2(t,\mathbf{x};t',\mathbf{x}') = \frac{\epsilon}{4\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; e^{-ip(t-t'-|\mathbf{x}|)}\sin(|\mathbf{x}'|p)\frac{-4\pi\lambda/\tilde{g}^2+ip}{c-ip} \tag{D.6}$$

as well as

$$\mathscr{S}_3(t,\mathbf{x};t',\mathbf{x}') = \frac{\epsilon^2}{4\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; pe^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\frac{(4\pi\lambda/\tilde{g}^2)^2+p^2}{c^2+p^2} \tag{D.7}$$

where we define the constant

$$c := \frac{16\pi^2\epsilon}{\tilde{g}^2} + \frac{4\pi\lambda}{\tilde{g}^2}\;. \tag{D.8}$$

It is more convenient to arrange the integrals as

$$\begin{aligned}\mathscr{S}_1(t,\mathbf{x};t',\mathbf{x}') &= \frac{i\epsilon}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\left(e^{-ip(t-t'+|\mathbf{x}|+|\mathbf{x}'|)}-e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\right)\left[\frac{i(4\pi\lambda/\tilde{g}^2-c)}{p-ic}-1\right]\\ &= \frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\left(e^{-ip(t-t'+|\mathbf{x}|+|\mathbf{x}'|)}-e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\right)\frac{1}{p-ic}\\ &\quad -\frac{i\epsilon}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\left(e^{-ip(t-t'+|\mathbf{x}|+|\mathbf{x}'|)}-e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\right)\end{aligned} \tag{D.9}$$

and

$$\begin{aligned}\mathscr{S}_2(t,\mathbf{x};t',\mathbf{x}') &= -\frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\left(e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}-e^{-ip(t-t'-|\mathbf{x}|-|\mathbf{x}'|)}\right)\frac{1}{p+ic}\\ &\quad -\frac{i\epsilon}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\left(e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}-e^{-ip(t-t'-|\mathbf{x}|-|\mathbf{x}'|)}\right)\end{aligned} \tag{D.10}$$

as well as

$$\begin{aligned}\mathscr{S}_3(t,\mathbf{x};t',\mathbf{x}') &= \frac{\epsilon^2}{4\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\left[p-\frac{[c^2-(4\pi\lambda/\tilde{g}^2)^2]p}{c^2+p^2}\right]\\ &= \frac{\epsilon^2}{4\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; pe^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\\ &\quad -\frac{\epsilon^2[c^2-(4\pi\lambda/\tilde{g}^2)^2]}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}\left[\frac{1}{p-ic}+\frac{1}{p+ic}\right]\;.\end{aligned} \tag{D.11}$$

From here we note the elementary integrals (where the limit $\delta\to 0^+$ is understood)

$$\int_{-\infty}^\infty \mathrm{d}p\;\Theta(p)e^{-i\tau p} \;=\; \frac{-i}{\tau-i\delta} \qquad\text{and}\qquad \int_{-\infty}^\infty \mathrm{d}p\;\Theta(p)pe^{-i\tau p} \;=\; \frac{-1}{(\tau-i\delta)^2}\,, \tag{D.12}$$

as well as the integrals

$$I_\mp(\tau,c) = \int_0^\infty \mathrm{d}p\;\frac{e^{-ip\tau}}{p\mp ic}\;. \tag{D.13}$$

We defer the calculation of the integrals to section D.1.2, where the result (D.30) is given by

$$I_\mp(\tau,c) = e^{\pm c\tau}E_1\big(\pm c(\tau-i\delta)\big)\,, \tag{D.14}$$

where $E_1(z) := \int_z^\infty \mathrm{d}u\ \frac{e^{-u}}{u}$ is the so-called exponential E_n-function with $n = 1$, and where the limit $\delta \to 0^+$ is understood (note that we have $E_1^*(z) = E_1(z^*)$ is satisfied, and so $I_-(\tau) = I_+(-\tau)$ remains true).

With the above we find

$$\begin{aligned}\mathscr{S}_1(t,\mathbf{x};t',\mathbf{x}') = \frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\Big[I_-(t-t'+|\mathbf{x}|+|\mathbf{x}'|,c) - I_-(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c)\Big] \\ - \frac{\epsilon}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\left[\frac{1}{t-t'+|\mathbf{x}|+|\mathbf{x}'|-i\delta} - \frac{1}{t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta}\right]\end{aligned} \tag{D.15}$$

and

$$\begin{aligned}\mathscr{S}_2(t,\mathbf{x};t',\mathbf{x}') = -\frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\Big[I_+(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) - I_+(t-t'-|\mathbf{x}|-|\mathbf{x}'|,c)\Big] \\ - \frac{\epsilon}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\left[\frac{1}{t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta} - \frac{1}{t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta}\right]\end{aligned} \tag{D.16}$$

as well as

$$\begin{aligned}\mathscr{S}_3(t,\mathbf{x};t',\mathbf{x}') = -\frac{\epsilon^2}{4\pi^2|\mathbf{x}||\mathbf{x}'|}\cdot\frac{1}{(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)^2} \\ - \frac{32\pi^2\epsilon^4(1+\frac{\lambda}{2\pi\epsilon})}{\tilde{g}^4|\mathbf{x}||\mathbf{x}'|}\Big[I_-(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) + I_+(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c)\Big]\ .\end{aligned} \tag{D.17}$$

Finally, putting the above all together into the sum (D.1) gives

$$\begin{aligned}\mathscr{S}(t,\mathbf{x};t',\mathbf{x}') = &\ \frac{1}{4\pi^2\left[-(t-t'-i\delta)^2 + |\mathbf{x}-\mathbf{x}'|^2\right]} \\ &+ \frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\Big[I_-(t-t'+|\mathbf{x}|+|\mathbf{x}'|,c) - I_-(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) \\ &\qquad - I_+(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) + I_+(t-t'-|\mathbf{x}|-|\mathbf{x}'|,c)\Big] \\ &+ \frac{\epsilon}{8\pi^2|\mathbf{x}||\mathbf{x}'|}\left[-\frac{1}{t-t'+|\mathbf{x}|+|\mathbf{x}'|-i\delta} + \frac{1}{t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta}\right] \\ &- \frac{32\pi^2\epsilon^4(1+\frac{\lambda}{2\pi\epsilon})}{\tilde{g}^4|\mathbf{x}||\mathbf{x}'|}\Big[I_-(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c) + I_+(t-t'-|\mathbf{x}|+|\mathbf{x}'|,c)\Big] \\ &- \frac{\epsilon^2}{4\pi^2|\mathbf{x}||\mathbf{x}'|(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)^2}\end{aligned} \tag{D.18}$$

which is the result quoted in (5.17).

D.1.1 Perturbative limit of $\mathscr{S}$

From here we wish to consider the perturbative limit of the above, which is taken by assuming that

$$c\tau \ = \ \left(\frac{16\pi^2\epsilon}{\tilde{g}^2} + \frac{4\pi\lambda}{\tilde{g}^2}\right)\tau \gg 1\ . \tag{D.19}$$

Note that the function $E_1(z)$ has the following asymptotic series

$$E_1(z) \;\simeq\; e^{-z}\left[\frac{1}{z} - \frac{1}{z^2} + \mathcal{O}(z^{-3})\right] \qquad \text{for } |z| \gg 1 \tag{D.20}$$

which implies that the functions $I_{\mp}(\tau, c)$ have the following asymptotic series for $|c\tau| \gg 1$

$$I_{\mp}(\tau, c) \;\simeq\; \pm\frac{1}{c(\tau - i\delta)} - \frac{1}{c^2(\tau - i\delta)^2} + \mathcal{O}(|c\tau|^{-3}) \qquad \text{for } |c\tau| \gg 1\ . \tag{D.21}$$

We also note in passing that for any $z \in \mathbb{C}$ (with $|\text{Arg}(z)| < \pi$ so not directly on the branch cut) has the series expansion

$$E_1(z) \simeq -\gamma - \log(z) - \sum_{k=1}^{\infty} \frac{(-z)^k}{k \cdot k!} \tag{D.22}$$

which is a convergent sum for any $z \in \mathbb{C}$ but is particularly useful when $|z| \ll 1$. This means that for $|c\tau| \ll 1$ we have

$$I_{\mp}(\tau, c) \;\simeq\; -\gamma - \log\left(c(\tau - i\delta)\right) + \mathcal{O}(c\tau) \qquad |c\tau| \ll 1\,, \tag{D.23}$$

where this limit will clearly suffer from secular growth problems once $c\tau$ is no longer small.

Taking the $|c\tau| \gg 1$ limit of the expression $\mathscr{S}$ here (dropping $\mathcal{O}(|c\tau|^{-3})$ contributions), and simplifying after using $c = (16\pi^2\epsilon + 4\pi\lambda)/\tilde{g}^2$ yields

$$\begin{aligned}
&\mathscr{S}\,(t,\mathbf{x};t',\mathbf{x}') \\
&\simeq \frac{1}{4\pi^2\left[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\right]} \\
&\quad+\frac{1}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\cdot\frac{\lambda}{1+\frac{\lambda}{4\pi\epsilon}}\cdot\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2-(|\mathbf{x}|+|\mathbf{x}'|)^2} \\
&\quad+\frac{\tilde{g}^2}{32\pi^4\left(1+\frac{\lambda}{4\pi\epsilon}\right)^2}\left[-\frac{1}{|\mathbf{x}|}\frac{t-t'-|\mathbf{x}|}{\left[(t-t'-|\mathbf{x}|-i\delta)^2-|\mathbf{x}'|^2\right]^2}+\frac{1}{|\mathbf{x}'|}\frac{t-t'+|\mathbf{x}'|}{\left[(t-t'+|\mathbf{x}'|-i\delta)^2-|\mathbf{x}|^2\right]^2}\right] \\
&\quad-\frac{1}{64\pi^4|\mathbf{x}||\mathbf{x}'|}\cdot\frac{\lambda^2}{\left(1+\frac{\lambda}{4\pi\epsilon}\right)^2}\cdot\frac{1}{(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)^2}\ .
\end{aligned}\tag{D.24}$$

For perturbatively small λ (meaning $\lambda/(4\pi\epsilon)\ll 1$) the above turns into

$$\begin{aligned}
\mathscr{S}\,(t,\mathbf{x};t',\mathbf{x}') \simeq\; &\frac{1}{4\pi^2\left[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\right]} \\
&+\frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\cdot\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2-(|\mathbf{x}|+|\mathbf{x}'|)^2} \\
&+\frac{\tilde{g}^2}{32\pi^4}\left[-\frac{1}{|\mathbf{x}|}\frac{t-t'-|\mathbf{x}|}{\left[(t-t'-|\mathbf{x}|-i\delta)^2-|\mathbf{x}'|^2\right]^2}+\frac{1}{|\mathbf{x}'|}\frac{t-t'+|\mathbf{x}'|}{\left[(t-t'+|\mathbf{x}'|-i\delta)^2-|\mathbf{x}|^2\right]^2}\right]
\end{aligned}\tag{D.25}$$

where we neglect $\mathcal{O}(\lambda^2)$ contributions. Notice that this exactly matches the temperature-independent contribution to the correlator in the perturbative limit (see (3.10)).

D.1.2 Evaluating the integrals $I_{\mp}(\tau, c)$

Here we evaluate the integrals $I_{\mp}(\tau,c)$ defined in (D.13). Since $I_-(\tau,c) = I_+^*(-\tau,c)$, it suffices to compute $I_-(\tau,c)$ here. Splitting apart $I_-(\tau,c)$ into real and imaginary parts gives

$$I_-(\tau) = \int_0^\infty \mathrm{d}p \; \frac{p\cos(p\tau) + c\sin(p\tau)}{p^2+c^2} + i\int_0^\infty \mathrm{d}p \; \frac{c\cos(p\tau) + p\sin(p\tau)}{p^2+c^2} \; . \tag{D.26}$$

Assuming that $c > 0$ and $\tau \in \mathbb{R}$ and using formulas (3.723.1)-(3.723.4) from [116] the above can be easily evaluated to give

$$I_-(\tau,c) = -e^{c\tau}\mathrm{Ei}(-c\tau) + i\pi e^{c\tau}\Theta(-\tau) \tag{D.27}$$

where $\mathrm{Ei}(x) := -\int_{-x}^\infty \mathrm{d}u\; \frac{e^{-u}}{u}$ is the exponential integral function. Since $I_-(\tau,c) = I_+^*(-\tau,c)$ this immediately implies that

$$I_+(\tau,c) = -e^{-c\tau}\mathrm{Ei}(c\tau) - i\pi e^{-c\tau}\Theta(\tau) \tag{D.28}$$

The above formulae can be simplified into a more useful form by relating it to the function $E_1(z) := \int_z^\infty \mathrm{d}u\; \frac{e^{-u}}{u}$ (the so-called exponential E_n-function with $n = 1$, related to the exponential integral function for $x > 0$ by $E_1(x) = -\mathrm{Ei}(-x)$). Obviously E_1 closely related to Ei — however for complex arguments, the definition of Ei becomes somewhat ambiguous due to branch points at 0 and ∞, and so E_1 is better defined for this reason.

Noting the behaviour of the function $E_1(z)$ nearby its branch cut (along the negative real axis) where

$$\lim_{\eta\to0^+} E_1(-x \pm i\delta) \;=\; -\mathrm{Ei}(x) \mp i\pi \qquad \text{for } x > 0\,, \tag{D.29}$$

the functions $I_{\mp}(\tau,c)$ can be written in the more useful form

$$\begin{aligned} I_-(\tau,c) &= e^{c\tau}E_1\big(c(\tau - i\delta)\big) \\ I_+(\tau,c) &= e^{-c\tau}E_1\big(-c(\tau - i\delta)\big) \end{aligned} \tag{D.30}$$

where the limit $\delta \to 0^+$ is understood as usual (note that we have $E_1^*(z) = E_1(z^*)$ is satisfied, and so $I_-(\tau) = I_+(-\tau)$ remains true).

D.2 The temperature-dependent contribution, $\mathscr{E}_\beta(t, \mathbf{x}; t', \mathbf{x}')$

In order to evaluate the trace, we put the system in a box (as done in appendix A.1 for the free thermal correlation function). Performing the trace, and then reverting back to the continuum limit, yields the mode sum

$$\mathscr{E}_\beta\left(t,\mathbf{x};t',\mathbf{x}'\right) = \sum_{a=1}^N \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\,2E_p}\left[s^a_{\mathbf{p}}\left(t,\mathbf{x}\right)s^{a*}_{\mathbf{p}}\left(t',\mathbf{x}'\right) + \frac{2\mathrm{Re}\left[s^a_{\mathbf{p}}\left(t,\mathbf{x}\right)s^{a*}_{\mathbf{p}}\left(t',\mathbf{x}'\right)\right]}{e^{\beta E_p}-1}\right] . \tag{D.31}$$

Using the explicit form of the mode function $s^a_{\mathbf{p}}$ given in (5.11) the function $\mathscr{E}_\beta(t,\mathbf{x};t',\mathbf{x}')$ turns into the integral

$$\mathscr{E}_\beta\left(t,\mathbf{x};t',\mathbf{x}'\right) = \frac{\tilde{g}^2}{16\pi^2|\mathbf{x}||\mathbf{x}'|}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3\,2E_p}\left[\frac{e^{-iE_p(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}}{\left(\frac{\tilde{g}^2E_p}{16\pi^2\epsilon}\right)^2+\left(1+\frac{\lambda}{4\pi\epsilon}\right)^2}+\frac{2\cos\left[E_p\left(t-|\mathbf{x}|-t'+|\mathbf{x}'|\right)\right]}{\left[\left(\frac{\tilde{g}^2E_p}{16\pi^2\epsilon}\right)^2+\left(1+\frac{\lambda}{4\pi\epsilon}\right)^2\right]\left[e^{\beta E_p}-1\right]}\right]. \tag{D.32}$$

Integrating the angles away and simplifying the above yields

$$\begin{aligned}\mathscr{E}_\beta\left(t,\mathbf{x};t',\mathbf{x}'\right) &= \frac{4\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\frac{p}{p^2+c^2}\left[\,e^{-ip(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}+\frac{2\cos\left[p\left(t-|\mathbf{x}|-t'+|\mathbf{x}'|\right)\right]}{e^{\beta p}-1}\,\right] \qquad \text{(D.33)}\\ &= \frac{4\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}I\left(t-|\mathbf{x}|-t'+|\mathbf{x}'|,c,\beta\right)\end{aligned}$$

where c is the constant (5.18) consisting of the couplings and ϵ defined by

$$c \;:=\; \frac{1+\frac{\lambda}{4\pi\epsilon}}{\frac{\tilde{g}^2}{16\pi^2\epsilon}} \;=\; \frac{16\pi^2\epsilon}{\tilde{g}^2}+\frac{4\pi\lambda}{\tilde{g}^2}\,, \tag{D.34}$$

and we define the integral

$$I(\tau,c,\beta) = \int_0^\infty \mathrm{d}p\;\frac{p}{p^2+c^2}\left[\,e^{-i\tau p}+\frac{2\cos(\tau p)}{e^{\beta p}-1}\,\right], \tag{D.35}$$

which we evaluate here for $c,\beta>0$ and $\tau\in\mathbb{R}$. To compute I, use $\int_0^\infty \mathrm{d}q\;e^{-cq}\sin(pq) = \frac{p}{p^2+c^2}$ giving

$$I(\tau,c,\beta) = \int_0^\infty \mathrm{d}q\int_0^\infty \mathrm{d}p\;e^{-cq}\sin(pq)\left[\,e^{-i\tau p}+\frac{2\cos(\tau p)}{e^{\beta p}-1}\,\right]. \tag{D.36}$$

Next we rearrange the above into the form

$$I(\tau,c,\beta) = \int_0^\infty \mathrm{d}q\;e^{-cq}\left[\int_0^\infty \mathrm{d}p\;\frac{i(e^{-i(\tau+q)p}-e^{-i(\tau-q)p})}{2}+\int_0^\infty \mathrm{d}p\;\frac{\sin\left((\tau+q)p\right)-\sin\left((\tau-q)p\right)}{e^{\beta p}-1}\right]. \tag{D.37}$$

We note the elementary result (with the limit $\delta\to0^+$ understood)

$$\int_{-\infty}^{\infty}\Theta(x)e^{-iyx} \;=\; \int_0^\infty \mathrm{d}x\;e^{-iyx} \;=\; \frac{-i}{y-i\delta} \tag{D.38}$$

as well as formula (3.911.2) from [116] (valid for $a\neq0$ and $\mathrm{Re}[\beta]>0$)

$$\int_0^\infty \mathrm{d}x\;\frac{\sin(ax)}{e^{\beta x}-1} \;=\; \frac{\pi}{2\beta}\coth\left(\frac{\pi a}{\beta}\right)-\frac{1}{2a}\,. \tag{D.39}$$

With these formulae the integral (D.37) becomes

$$\begin{aligned}I(\tau,c,\beta) = \frac{1}{2}\int_0^\infty \mathrm{d}q\;e^{-cq}\bigg[&\frac{1}{\tau+q-i\delta}-\frac{1}{\tau-q-i\delta} \qquad \text{(D.40)}\\ &+\frac{\pi}{\beta}\coth\left(\frac{\pi(\tau+q)}{\beta}\right)-\frac{1}{\tau+q}-\frac{\pi}{\beta}\coth\left(\frac{\pi(\tau-q)}{\beta}\right)+\frac{1}{\tau-q}\bigg].\end{aligned}$$

Using the result of the Sochocki-Plemelj theorem

$$\frac{1}{x \pm i\delta} \;=\; \frac{1}{x} \mp i\pi\delta(x)\,, \tag{D.41}$$

the integral (D.40) simplifies to

$$I(\tau, c, \beta) = \frac{1}{2}\int_0^\infty \mathrm{d}q\; e^{-cq}\left[i\pi\delta(\tau+q) - i\pi\delta(\tau-q) + \frac{\pi}{\beta}\coth\left(\frac{\pi(\tau+q)}{\beta}\right) - \frac{\pi}{\beta}\coth\left(\frac{\pi(\tau-q)}{\beta}\right)\right]. \tag{D.42}$$

Applying the Sochocki-Plemelj theorem yet again to the above yields

$$I(\tau, c, \beta) = \frac{\pi}{2\beta}\int_0^\infty \mathrm{d}q\; e^{-cq}\left[\coth\left(\frac{\pi(q+[\tau-i\delta])}{\beta}\right) + \coth\left(\frac{\pi(q-[\tau-i\delta])}{\beta}\right)\right] \tag{D.43}$$

where the behaviour $\coth(z) \simeq 1/z$ near $z \simeq 0$ allows use of formula (D.41) (and so justifying the $i\delta$-prescription in the arguments of the $\coth(\cdot)$ functions). From here we note the identity

$$\coth\left(\frac{z}{2}\right) = \frac{2}{1-e^{-z}} - 1\,, \tag{D.44}$$

and make the change of integration variable to $Q = 2\pi q/\beta$ giving

$$\begin{aligned} I(\tau, c, \beta) &= \frac{1}{4}\int_0^\infty \mathrm{d}Q\; e^{-\frac{c\beta}{2\pi}Q}\left[\coth\left(\frac{Q + 2\pi[\tau-i\delta]/\beta)}{2}\right) + \coth\left(\frac{Q - 2\pi[\tau-i\delta]/\beta)}{2}\right)\right] \\ &= \frac{1}{2}\int_0^\infty \mathrm{d}Q\; e^{-\frac{c\beta}{2\pi}Q}\left[\frac{1}{1-e^{-2\pi(\tau-i\delta)/\beta}e^{-Q}} + \frac{1}{1-e^{+2\pi(\tau-i\delta)/\beta}e^{-Q}} - 1\right] \\ &= \frac{1}{2}\Phi\left(e^{-\frac{2\pi(\tau-i\delta)}{\beta}}, 1, \frac{c\beta}{2\pi}\right) + \frac{1}{2}\Phi\left(e^{+\frac{2\pi(\tau-i\delta)}{\beta}}, 1, \frac{c\beta}{2\pi}\right) - \frac{\pi}{c\beta}\,, \end{aligned} \tag{D.45}$$

where we use the integral representation (see formula (25.14.5) in [118])

$$\Phi(z, s, a) \;=\; \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}x\; \frac{x^{s-1}e^{-ax}}{1-ze^{-x}} \qquad \text{valid for } \mathrm{Re}[s] > 0,\; \mathrm{Re}[a] > 0 \;\&\; z \in \mathbb{C}\setminus[1,\infty) \tag{D.46}$$

where $\Phi(z, s, a)$ is the Lerch Transcendent, usually defined by the series (see formula (25.14.1) in [118])

$$\Phi(z, s, a) \;=\; \sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s} \qquad \text{valid for } |z| < 1\,. \tag{D.47}$$

For other values of $z \in \mathbb{C}$ (*ie.* not inside the unit disc) the function Φ is defined via analytic continuation in the complex plane. At the end of the day, using the above formula (D.45) in (D.33) yields

$$\mathscr{E}_\beta(t, \mathbf{x}; t', \mathbf{x}') = \frac{2\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\left[\Phi\left(e^{-\frac{2\pi(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)}{\beta}}, 1, \frac{c\beta}{2\pi}\right) + \Phi\left(e^{+\frac{2\pi(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)}{\beta}}, 1, \frac{c\beta}{2\pi}\right) - \frac{2\pi}{c\beta}\right] \tag{D.48}$$

which is the result quoted in (5.20) in the main text.

D.2.1 Perturbative limit of $\mathscr{E}_\beta$

Here we take the perturbative limit of $\mathscr{E}_\beta$ which turns out to be the limit in which

$$\frac{c\beta}{2\pi} \gg 1 \,. \tag{D.49}$$

We first note the asymptotic series of the Lerch transcendent for any large a

$$\Phi(z,s,a) \simeq \frac{1}{1-z}\frac{1}{a^s} + \sum_{n=1}^{N-1} \frac{(-1)^n\Gamma(s+n)}{n!\,\Gamma(s)} \cdot \frac{\mathrm{Li}_{-n}(z)}{a^{s+n}} + \mathcal{O}(a^{-s-N}) \qquad \text{for } a \gg 1 \tag{D.50}$$

for fixed $s \in \mathbb{C}$ and fixed $z \in \mathbb{C} \setminus [1,\infty)$, where $\mathrm{Li}_{-n}(z) = (z\partial_z)^n \frac{z}{1-z}$ are polylogarithm functions of negative integer order. Using $\mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}$ and $\mathrm{Li}_{-2}(z) = \frac{z+z^2}{(1-z)^3}$ as well as $\mathrm{Li}_{-3}(z) = \frac{z+4z^2+z^3}{(1-z)^4}$ we find that for $c\beta/(2\pi) \gg 1$ we have

$$\Phi\left(z,1,\frac{c\beta}{2\pi}\right) \simeq \frac{1}{1-z}\frac{2\pi}{c\beta} - \frac{z}{(1-z)^2}\left(\frac{2\pi}{c\beta}\right)^2 + \frac{z+z^2}{(1-z)^3}\left(\frac{2\pi}{c\beta}\right)^3 + \mathcal{O}((c\beta)^{-4}) \tag{D.51}$$

$$\Phi\left(\frac{1}{z},1,\frac{c\beta}{2\pi}\right) \simeq \left[1 - \frac{1}{1-z}\right]\frac{2\pi}{c\beta} - \frac{z}{(1-z)^2}\left(\frac{2\pi}{c\beta}\right)^2 - \frac{z+z^2}{(1-z)^3}\left(\frac{2\pi}{c\beta}\right)^3 + \mathcal{O}((c\beta)^{-4})$$

Which means that (half of) the sum of these two functions has the asymptotics

$$\frac{1}{2}\Phi\left(z,1,\frac{c\beta}{2\pi}\right) + \frac{1}{2}\Phi\left(\frac{1}{z},1,\frac{c\beta}{2\pi}\right) \simeq \frac{\pi}{c\beta} - \frac{z}{(1-z)^2}\left(\frac{2\pi}{c\beta}\right)^2 + \mathcal{O}((c\beta)^{-4})\,, \tag{D.52}$$

which when used for the function $I(\tau,c,\beta)$ given in (D.45) yields

$$I(\tau,c,\beta) \simeq -\frac{e^{-\frac{2\pi(\tau-i\delta)}{\beta}}}{\left[1-e^{-\frac{2\pi(\tau-i\delta)}{\beta}}\right]^2}\left(\frac{2\pi}{c\beta}\right)^2 + \mathcal{O}((c\beta)^{-4}) = -\frac{\pi^2}{c^2\beta^2}\mathrm{csch}^2\left(\frac{\pi[\tau-i\delta]}{\beta}\right) + \mathcal{O}((c\beta)^{-4}) \tag{D.53}$$

Using this and $c = \frac{16\pi^2\epsilon}{\tilde{g}^2}\left(1+\frac{\lambda}{4\pi\epsilon}\right)$, at the end of the day we find that $\mathscr{E}_\beta$ in the perturbative limit is

$$\mathscr{E}_\beta(t,\mathbf{x};t',\mathbf{x}') \simeq -\frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\left(1+\frac{\lambda}{4\pi\epsilon}\right)^2}\mathrm{csch}^2\left(\frac{\pi[t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta]}{\beta}\right) + \dots$$

which (at leading-order) is exactly the expected temperature-dependent contribution to the perturbative result when $\lambda/(4\pi\epsilon) \ll 1$ (see formula (3.10), which neglects $\mathcal{O}(\lambda^2)$ contributions).

D.2.2 $I(\tau,c,\beta)$ in the limit $\delta \to 0^+$

Because $\Phi(z,s,a)$ has a branch cut along $z \in [1,\infty)$, the limit $\delta \to 0^+$ of the expression (D.45) is somewhat tricky to take. For completeness we take this limit here: to this end, revert back to the integral form (D.42) and integrate the δ-functions explicitly to get

$$I(\tau,c,\beta) = \frac{\pi}{2\beta}\int_0^\infty \mathrm{d}q\; e^{-cq}\left[\coth\left(\frac{\pi(q+\tau)}{\beta}\right) + \coth\left(\frac{\pi(q-\tau)}{\beta}\right)\right] + \frac{i\pi}{2}\left[e^{+c\tau}\Theta(-\tau) - e^{-c\tau}\Theta(\tau)\right] \tag{D.54}$$

The remaining integrals over q are now ill-defined for general τ (the reason for this being that for any $\tau \in \mathbb{R}$ one of the two $\coth(\cdot)$ functions gets integrated over a singularity at $q = \tau$). Since the remaining integral is clearly symmetric under $\tau \to -\tau$, we assume for now that $\tau > 0$. For $\tau > 0$ the first integral is easier to compute, where we change the integration variable to $u = \exp\big(-2\pi(q+\tau)/\beta\big)$ where

$$\begin{aligned} \frac{\pi}{2\beta}\int_0^\infty \mathrm{d}q\; e^{-cq}\coth\left(\frac{\pi(q+\tau)}{\beta}\right) &= \frac{1}{4}e^{c\tau}\int_0^{\exp(-2\pi\tau/\beta)}\mathrm{d}u\; u^{\frac{c\beta}{2\pi}-1}\bigg[\frac{2}{1-u}-1\bigg] \\ &= \frac{1}{2}e^{c\tau}\mathrm{B}\Big(e^{-\frac{2\pi\tau}{\beta}};\frac{c\beta}{2\pi},0\Big) - \frac{\pi}{2c\beta} \end{aligned} \tag{D.55}$$

where $\mathrm{B}(z;a,b) = \int_0^z \mathrm{d}u\; u^{a-1}(1-u)^{b-1}$ is the incomplete Beta function. For the second integral, the procedure is similar with the added complication that the integrand gets integrated over the root at $q = -\tau$ (since we assume here that $\tau > 0$). We find in much the same way that

$$\frac{\pi}{2\beta}\int_0^\infty \mathrm{d}q\; e^{-cq}\coth\left(\frac{\pi(q-\tau)}{\beta}\right) = \frac{e^{-c\tau}}{2}\int_0^{\exp(+2\pi\tau/\beta)}\mathrm{d}u\; u^{\frac{c\beta}{2\pi}-1}(1-u)^{-1} - \frac{\pi}{2c\beta}\,, \tag{D.56}$$

however since the upper limit on the integral is greater than 1 (since $\tau > 0$ is assumed), the above integrand gets integrated over a singularity at $u = 1$. Interpreting the above integral as a Cauchy Principal value turns the above into

$$\frac{\pi}{2\beta}\int_0^\infty \mathrm{d}q\; e^{-cq}\coth\left(\frac{\pi(q-\tau)}{\beta}\right) = \frac{e^{-c\tau}}{2}\lim_{\eta\to0^+}\bigg[\int_0^{1-\eta}+\int_{1+\eta}^{\exp(+2\pi\tau/\beta)}\bigg]\mathrm{d}u\; u^{\frac{c\beta}{2\pi}-1}(1-u)^{-1} - \frac{\pi}{2c\beta}\,. \tag{D.57}$$

The first integral is easily seen to evaluate to $\mathrm{B}\big(1-\delta;\frac{c\beta}{2\pi},0\big)$, while the second integral requires a variable change $u = 1/v$ giving

$$\begin{aligned} \int_{1+\eta}^{\exp(+2\pi\tau/\beta)}\mathrm{d}u\; u^{\frac{c\beta}{2\pi}-1}(1-u)^{-1} &= \int_{1/(1+\eta)}^{\exp(-2\pi\tau/\beta)}\mathrm{d}v\; v^{-\frac{c\beta}{2\pi}}(1-v)^{-1} \\ &= \bigg[\int_0^{\exp(-2\pi\tau/\beta)} - \int_0^{1/(1+\eta)}\bigg]\mathrm{d}v\; v^{-\frac{c\beta}{2\pi}}(1-v)^{-1} \\ &= \mathrm{B}\Big(e^{-\frac{2\pi\tau}{\beta}};1-\frac{c\beta}{2\pi},0\Big) - \mathrm{B}\Big(\frac{1}{1+\eta};1-\frac{c\beta}{2\pi},0\Big) \end{aligned} \tag{D.58}$$

which then implies that

$$\begin{aligned} \frac{\pi}{2\beta}\int_0^\infty \mathrm{d}q\, e^{-cq}\coth\left(\frac{\pi(q-\tau)}{\beta}\right) = \frac{e^{-c\tau}}{2}\bigg[&\mathrm{B}\Big(e^{-\frac{2\pi\tau}{\beta}};1-\frac{c\beta}{2\pi},0\Big) \\ &+\lim_{\eta\to0^+}\Big\{\mathrm{B}\Big(1-\eta;\frac{c\beta}{2\pi},0\Big) - \mathrm{B}\Big(\frac{1}{1+\eta};1-\frac{c\beta}{2\pi},0\Big)\Big\}\bigg] - \frac{\pi}{2c\beta}\,. \end{aligned} \tag{D.59}$$

The limit can be taken noting $\mathrm{B}(z;a,0) \simeq -\log(1-z) - \psi^{(0)}(a) - \gamma + \mathcal{O}(z)$ for $z \to 1^-$ (with $\psi^{(0)}(z) := \Gamma'(z)/\Gamma(z)$ the digamma function) giving

$$\frac{\pi}{2\beta}\int_0^\infty \mathrm{d}q\; e^{-cq}\coth\left(\frac{\pi(q-\tau)}{\beta}\right) = \frac{e^{-c\tau}}{2}\bigg[\mathrm{B}\Big(e^{-\frac{2\pi\tau}{\beta}};1-\frac{c\beta}{2\pi},0\Big) + \pi\cot\left(\frac{\beta c}{2}\right)\bigg] - \frac{\pi}{2c\beta}\,.$$

where the identity $\psi^{(0)}(1-a) - \psi^{(0)}(a) = \pi\cot(\pi a)$ has been used. Putting the above all together (extending the domain to $\tau > 0$ for the above integrals) leaves

$$I(\tau,c,\beta) = \frac{e^{c|\tau|}}{2}\mathrm{B}\Big(e^{-\frac{2\pi|\tau|}{\beta}};\frac{c\beta}{2\pi},0\Big) + \frac{e^{-c|\tau|}}{2}\Big[\mathrm{B}\Big(e^{-\frac{2\pi|\tau|}{\beta}};1-\frac{c\beta}{2\pi},0\Big) + \pi\cot\Big(\frac{c\beta}{2}\Big)\Big] - \frac{\pi}{c\beta} - \frac{i\pi}{2}\mathrm{sign}(\tau)e^{-c|\tau|}. \tag{D.60}$$

To write the above formula in a slightly more convenient manner, we note formula (8.17.20) from [118] which implies that

$$B(z;1-a,0) = B(z;-a,0) + \frac{z^{-a}}{a}\,, \tag{D.61}$$

and so allows us to write the above formula as

$$I(\tau,c,\beta) = \frac{e^{c|\tau|}}{2}\mathrm{B}\Big(e^{-\frac{2\pi|\tau|}{\beta}};\frac{c\beta}{2\pi},0\Big) + \frac{e^{-c|\tau|}}{2}\Big[\mathrm{B}\Big(e^{-\frac{2\pi|\tau|}{\beta}};-\frac{c\beta}{2\pi},0\Big) + \pi\cot\Big(\frac{c\beta}{2}\Big)\Big] - \frac{i\pi}{2}\mathrm{sign}(\tau)e^{-c|\tau|}. \tag{D.62}$$

The beta function can only be related to the Lerch transcendent for arguments $|z| < 1$, which explains why the limit $\delta \to 0^+$ is not straightforward from the representation (D.45).

D.2.3 KMS-like condition for $\mathscr{E}_\beta$

Here we show that the function $\mathscr{E}_\beta$ is thermal, in the sense that it obeys a KMS-like condition (as does the free thermal correlator of appendix A.1.1) where

$$\mathscr{E}_\beta(t-i\beta,\mathbf{x};t',\mathbf{x}') = \mathscr{E}_\beta(t',\mathbf{x}';t,\mathbf{x}) \tag{D.63}$$

cf. equation (A.15). The proof for this follows almost identically as the proof given in appendix A.1.1, save for the fact that $\mathscr{E}_\beta$ enjoys a time-translation invariance only when $t > |\mathbf{x}|$ and $t' > |\mathbf{x}'|$ — in this limit, we have the representation (D.33)

$$\mathscr{E}_\beta(t,\mathbf{x};t',\mathbf{x}') = \frac{4\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\frac{p}{p^2+c^2}\left[\frac{e^{-ip(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}}{1-e^{-\beta p}} + \frac{e^{+ip(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}}{e^{\beta p}-1}\right], \tag{D.64}$$

after using the identity $1 + (e^{\beta p}-1)^{-1} = (1-e^{-\beta p})^{-1}$. The above function is a complex analytic function of time for in the strip where $-\beta < \mathrm{Im}[t] < 0$ — we then clearly have

$$\mathscr{E}_\beta(t-i\beta,\mathbf{x};t',\mathbf{x}') = \frac{4\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\frac{p}{p^2+c^2}\left[\frac{e^{-\beta p}e^{-ip(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}}{1-e^{-\beta p}} + \frac{e^{+\beta p}e^{+ip(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}}{e^{\beta p}-1}\right] \tag{D.65}$$

$$= \frac{4\epsilon^2}{\tilde{g}^2|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\frac{p}{p^2+c^2}\left[\frac{e^{-ip(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}}{e^{\beta p}-1} + \frac{e^{+ip(t-|\mathbf{x}|-t'+|\mathbf{x}'|)}}{1-e^{-\beta p}}\right] \tag{D.66}$$

$$= \mathscr{E}_\beta(t',\mathbf{x}';t,\mathbf{x}) \tag{D.67}$$

which shows that (D.63) holds true. Note however that the full correlation function $W_\beta = \mathscr{S} + \mathscr{E}_\beta$ is not explicitly thermal since the temperature-independent contribution $\mathscr{S}$ obviously fails to satisfy a KMS-like condition analogous to (D.63). However, in a limit where $\mathscr{E}_\beta$ dominates over $\mathscr{S}$ (provided $t > |\mathbf{x}|$ and $t' > |\mathbf{x}'|$) one should expect to see thermality manifest itself more directly (this is explored using an Unruh-Dewitt detector model in [53], where it is shown that a stationary qubit outside the hotspot thermalizes whilst only interacting with ϕ).

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Chapter 4

Hotspot Open EFT Approximations

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4.1 Preface

The previous chapter is essential to the current chapter. Where the earlier chapter solves the hotspot model in full detail, the current chapter uses these exact solutions as a point of comparison when applying Open EFT techniques to the hotspot model.

So far in this thesis, much of the discussions of Open EFT techniques have been in reference to very simple problems involving particle decays and qubits (note that in [90] a qubit is shown to thermalize when sitting next to the hotspot) — the work presented here is so far closest to what one would wish to study in the Open EFT framework: what happens to the fields themselves at late times? In this case to what degree can one solve for the open system sector's evolution alone? It is shown here that the same arguments apply for fields as they do for qubits in so far as making reliable late-time predictions. The Nakajima-Zwanzig equation is developed here for the exterior field ϕ and assumed to be Markovian in the limit that the energy of the field modes are much smaller than the inverse temperature set by the hot interior fields χ^a. The Gaussian components of the reduced density matrix are here solved for in the field-basis, in order to make two points.

First, the equal-time correlator $\langle \phi(t)\phi(t) \rangle$ is computed by integrating over the Markovian solution found in the third section of the paper — interestingly, this correlator only agrees with the exact solution in a very specific limit of the perturbative regime (namely where the energy of allowed modes of the field are far smaller than the temperature of the hotspot interior). Secondly, the timescale for decoherence of the reduced density matrix is found to be cutoff dependent due to nature of the Markovian approximation taken.

In the last section an attempt is made to study non-locality of the hotspot in analogy to the non-locality that is thought to be exhibited by more realistic black holes. This is done by taking the (so far not discussed) *mean-field approximation* in which the (interaction picture) unitary evolution operator is split apart according to

$$\hat{V}(t) = \overline{\hat{V}}(t) \otimes \mathbb{I}_B + \hat{\mathcal{V}}(t) \tag{4.1}$$

where the mean-field evolution operator $\overline{\hat{V}}(t)$ is defined by the average $\overline{\hat{V}}(t) := \langle \hat{V}(t) \rangle_B$ over the environment's initial state, and the diffuse evolution operator is defined as the difference $\hat{\mathcal{V}}(t) := \hat{V}(t) - \overline{\hat{V}}(t) \otimes \mathbb{I}_B$. Writing $\overline{\hat{V}}(t)$ as the exponentiation of a mean-field Hamiltonian $\overline{\hat{H}}_{\text{int}}$ results in a *non-local Hamiltonian* in this framework, and is a good approximation for the evolution of the system in the limit where the diffuse contributions are small. This technique is known to be fruitful in a variety of settings where quantum systems propagate through mediums (like photons propagating through glass [6], or neutrinos interacting with the interior of the Sun [91]). It turns out that the mean-field approximation agrees with the solved physics of §3 only in the limit that the ϕ and χ^a become decoupled (*ie.* when the hotspot interaction is negligible).

The work presented here shows that there are regimes where Open EFT techniques agree with limits of the exact answer, but the Open EFT approximations are highly restrictive on the parameter space where things do align with the known answers.

4.2 Paper

Quantum Hotspots: Mean Fields, Open EFTs, Nonlocality and Decoherence Near Black Holes

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Abstract: Effective theories describing black hole exteriors resemble open quantum systems inasmuch as many unmeasurable degrees of freedom beyond the horizon interact with those we can see. A solvable Caldeira-Leggett type model of a quantum field that mixes with many unmeasured thermal degrees of freedom on a shared surface was proposed in `arXiv:2106.09854` to provide a benchmark against which more complete black hole calculations might be compared. We here use this model to test two types of field-theoretic approximation schemes that also lend themselves to describing black hole behaviour: Open EFT techniques (as applied to the fields themselves, rather than Unruh-DeWitt detectors) and mean-field methods. Mean-field methods are of interest because the effective Hamiltonians to which they lead can be nonlocal; a possible source for the nonlocality that is sometimes entertained as being possible for black holes in the near-horizon regime. Open EFTs compute the evolution of the field state, allowing discussion of thermalization and decoherence even when these occur at such late times that perturbative methods fail (as they often do). Applying both of these methods to a solvable system identifies their domains of validity and shows how their predictions relate to more garden-variety perturbative tools.

Contents

1 Introduction and Discussion

Quantum fields in black-hole backgrounds have long been known to behave in surprising ways [1, 2], even at energies well below whatever new physics ultimately describes gravity at its most foundational level. Although the nature of the approximations being made when studying these effects was initially puzzling, this quantum-gravitational interplay has been integrated into the broader framework of theoretical physics within the formalism of effective field theories (EFTs) [3, 4] (for reviews see [5–9]).

In the meantime black hole physics grew up, with the discovery of gravitational waves [10] confronting EFT calculations [11–21] of black hole properties with experimental measurements. This has stimulated much work and has underlined some of the unique challenges posed when working with black holes in an EFT framework. One of these challenges asks how the EFT should handle the large numbers of gapless and dissipative degrees of freedom [12, 22] associated with the black hole's entropy.

It would be useful to compare black-hole calculations with similar ones for well-understood solvable systems that share as many of these features as possible, and to this end ref. [23] proposed a solvable

Caldeira-Leggett style [24, 25] model in which an 'external' massless quantum field ϕ interacts with many unseen gapless thermal fields, but only on a surface meant as a poor man's model of the event horizon. Following [23], in what follows we call such a hot localized source a 'hotspot'.

In this paper we use this model to explore two types of approximate tools that both lend themselves to black-hole applications and capture different aspects of black-hole exceptionalism. (A companion paper [26] computes the response an Unruh-DeWitt (qubit) detector that couples to the external field ϕ at a fixed distance from the hotspot.) Applying approximate tools to a solvable model allows explicit identification of their domain of validity, which can be useful for applications to more realistic systems for which a full solution is not known.

The two approximations explored here are late-time Open-EFT methods[1] and mean-field expansions. Ref. [23] solves the system dynamics exactly, but does so within the Heisenberg picture and so obscures how the ϕ field state evolves once couplings to the hotspot are switched on. Open-EFT methods are designed to extract this state evolution, in principle allowing access to questions such as whether (and how quickly) the hotspot decoheres the external ϕ field. Furthermore it does so with a domain of validity that allows it to treat phenomena (like thermalization) that occur at times sufficiently late that naive perturbative methods generically fail.

Mean-field methods provide a framework within which an effective Hamiltonian description is possible even while including open-system effects (see for example [9] for practical examples treated in the same framework used here). Such a Hamiltonian description need not be guaranteed for generic systems. Furthermore, the effective mean-field Hamiltonian is often nonlocal and/or non-Hermitian, making it natural to ask whether mean-field methods might provide relatively mundane origins for exotic non-Wilsonian behaviour in the vicinity of black hole horizons; exotic behaviour that is often speculated to exist near black holes [44–50].

Our arguments and results are laid out as follows. §2 starts by briefly reviewing the hotspot model given in [23] and summarizes the results computed there that are relevant for later comparisons. The model's main variables are a massless field, ϕ, (meant to represent observable degrees of freedom on the near side of the horizon) plus a thermal bath (also with massless fields, χ^a) meant to model the dissipative effects of beyond-the-horizon physics. These fields 'interact' locally by mixing only on a surface meant to represent the horizon itself (though the interaction surface is not an actual geometrical local horizon).

Once the model is set up §3 formulates the evolution of the external field ϕ after tracing out the thermal degrees of freedom. This section does so by deriving eq. (3.22); a Nakajima-Zwanzig evolution equation [51, 52] – a common open-systems tool – for the reduced density matrix of the field ϕ. We find a limit in which this equation takes a Markovian form – (3.29) – and we solve for the field's density matrix in this Markovian limit, doing so perturbatively in the system couplings and working in the field basis where the solution is a Gaussian state. Using this result we compute the equal-time correlations $\langle \phi(t, \mathbf{x}) \phi(t, \mathbf{y}) \rangle$ – with the result (3.74) – and identify the domain of validity of Markovian methods by comparing this to the correlator given in §2.

§3 closes by computing a measure of the ϕ state's purity after the χ^a fields are integrated out, and computes the decoherence rate as a function of the ϕ-χ coupling parameters and hotspot temperature. For each mode the rate of departure from an initially pure free-vacuum state is controlled by the time scale $\tilde{g}^2/\beta$ where β is the inverse hotspot temperature and $\tilde{g}$ is a measure of the ϕ-χ couplings.

[1] Refs. [26–29] also explore the use of Open EFT techniques, but do so for the much simpler case where late-time predictions are only sought for an Unruh-DeWitt qubit detector [30, 31], rather than for the entire ϕ field. (See [32–41, 43] for related discussions of field decoherence in cosmology.)

Finally, §4 defines what a mean-field Hamiltonian is, and what this definition means – *c.f.* eq. (4.15) – for the field ϕ in the hotspot model. The result is in general nonlocal on the interaction surface and in time (with characteristic nonlocality scales given by the surface radius, ξ, and the inverse temperature, β), but is typically local in the radial 'off-horizon' direction. This mean-field Hamiltonian is used to compute once again the system correlation function $\langle \phi(t, \mathbf{x})\phi(s, \mathbf{y})\rangle$ for comparison with the results of §2, showing that validity of mean-field methods requires some couplings (like the ϕ self-coupling λ) to dominate the temperature-dependent combination $\tilde{g}^2/\beta$.

Taken together, the respective approximate methods described in §3 and §4 hold in complementary regimes of parameter space, both of which are subsets of the broader domain of perturbative methods found in [23].

2 The hotspot reloaded

This section briefly summarizes the hotspot-model setup as given in [23]. The fields involved consist of an observable sector — a single real massless scalar field, ϕ — and an unmeasured disspative and gapless environment — N real massless scalar fields, χ^a prepared in a thermal state. These two systems live in different spatial regions ($\mathcal{R}_\pm$) that only intersect in a small localized domain: a sphere $\mathcal{S}_\xi$ of radius ξ enclosing the origin that is identified for the two spaces (see Fig. 1). We choose here not to follow the gravitational back-reaction of these fields, and so treat $\mathcal{R}_\pm$ as independent flat spatial slices. A surface $\mathcal{S}_{\xi\pm}$ encircles the origin within each of these spaces, with $\mathcal{S}_{\xi\pm}$ identified to obtain the interaction surface $\mathcal{S}_\xi$. The χ^a fields are meant to represent unmeasured degrees of freedom internal to the black hole with which external fields can interact.

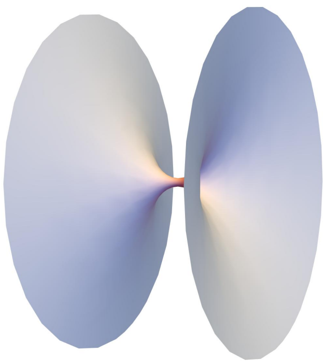

Figure 1. The two spatial branches, $\mathcal{R}_+$ and $\mathcal{R}_-$, in which the field ϕ and the N fields χ^a repsectively live. In practice the two regions are idealized as flat (though curvature can in principle also be included) with the spherical interaction region $\mathcal{S}_\xi$ identified. The two types of fields only couple to one another within $\mathcal{S}_\xi$ (which effectively becomes the world-line of a point in the special case that ξ is much smaller than all other scales of interest). Figure taken from [23].

Interactions on $\mathcal{S}_\xi$ are taken to be bilinear mixings of the fields, of the form $\mathcal{L}_{\rm int} = -g_a\, \chi^a\, \phi$, plus possible quadratic self-interactions of the fields as required by any renormalization-group flows.[2] With black holes in mind we consider observables that depend only on the ϕ field and do not directly measure any of the χ^a's.

The action for the model is

$$S_{UV} = -\frac{1}{2}\int_{\mathcal{R}^t_+} \mathrm{d}^4x\, \partial_\mu\phi\, \partial^\mu\phi - \frac{1}{2}\int_{\mathcal{R}^t_-} \mathrm{d}^4x\, \delta_{ab}\, \partial_\mu\chi^a\partial^\mu\chi^b - \int_{\mathcal{S}^t_\xi} \mathrm{d}^3x\, \left[G_a\, \chi^a\phi + \frac{G_\phi}{2}\,\phi^2\right], \tag{2.1}$$

where $\mathcal{R}^t_\pm$ denote spacetime regions whose spatial slices are $\mathcal{R}_\pm$, and the integration for the interaction is over the world-tube, $\mathcal{S}^t_\xi$ swept out by the surface $\mathcal{S}_\xi$ over time. The couplings G_a and G_ϕ have dimension mass, and G_ϕ is included because it can be required to renormalize divergences that arise due to the presence of the ϕ-χ mixing G_a. When performing calculations we usually specialize to the case where $G_a = G$ is a-independent and define the combination

$$\widetilde{G}^2 := \delta^{ab}G_aG_b = NG^2\,. \tag{2.2}$$

As described in [23] this hotspot model comes in two versions, depending on whether or not the radius ξ is regarded to be an ultraviolet scale.[3] When ξ is a UV scale the 2-sphere $\mathcal{S}_\xi$ degenerates to a point and the above description gets replaced by an effective action organized in powers of ξ. The leading ϕ-dependent interactions in this effective theory become

$$S_{\rm int-eff} \simeq -\int \mathrm{d}t\, \left[g_a\, \chi^a(t,\mathbf{0})\, \phi(t,\mathbf{0}) + \frac{\lambda}{2}\,\phi^2(t,\mathbf{0})\right], \tag{2.3}$$

where the integration is over the proper time of the interaction point. At leading order the effective couplings λ and g_a are related to the couplings in (2.1) by

$$g_a = 4\pi\xi^2 G_a \quad \text{and} \quad \lambda = 4\pi\xi^2 G_\phi \tag{2.4}$$

and so have dimensions of length. In this limit it is the quantity

$$\tilde{g}^2 = \delta^{ab}\, g_a g_b = N g^2 \tag{2.5}$$

that plays the same role as did (2.2) when ξ was not small.

2.1 Time evolution

The Hamiltonian for this system can be written $H = H_0 + H_{\rm int}$ where $H_0 := \mathcal{H}_+ \otimes \mathcal{I}_- + \mathcal{I}_+ \otimes \mathcal{H}_-$ is the free Hamiltonian, with $\mathcal{H}_\pm$ and $\mathcal{I}_\pm$ the Hamiltonian and identity operators acting separately within the ϕ- and χ-sectors of the Hilbert space:

$$\mathcal{H}_+ := \frac{1}{2}\int_{\mathcal{R}_+} \mathrm{d}^3\mathbf{x}\, \left[\mathfrak{p}^2 + \left(\boldsymbol{\nabla}\phi\right)^2\right] \quad \text{and} \quad \mathcal{H}_- := \frac{1}{2}\int_{\mathcal{R}_-} \mathrm{d}^3x\, \left[\delta^{ab}\Pi_a\Pi_b + \delta_{ab}\boldsymbol{\nabla}\chi^a \cdot \boldsymbol{\nabla}\chi^b\right], \tag{2.6}$$

and the canonical momenta are defined by $\mathfrak{p} := \partial_t\phi$ and $\Pi_a := \delta_{ab}\,\partial_t\chi^b$. The interaction Hamiltonian is similarly

$$H_{\rm int} = \int_{\mathcal{S}_\xi} \mathrm{d}^2x\, \left[G_a\,\phi\otimes\chi^a + \frac{G_\phi}{2}\,\phi^2\otimes\mathcal{I}_-\right], \tag{2.7}$$

[2]Such flows arise due to renormalizations of the Coulomb-like divergences that appear even at the classical level near the interaction surface [53–57].

[3]The same distinction also arises for black-hole EFTs, for which the black hole event horizon can either be regarded as being shrunk to a point – as in world-line point-particle EFTs [11–21] – or can be regarded as being macroscopic [58, 59]. This latter type of EFT can be used to summarize situations where it is ordinary GR [60–64] or exotic physics [65–73] whose UV near-horizon physics is being summarized.

which in the point-hotspot limit reduces to

$$H_{\rm int}(t) \simeq g_a\, \phi(t,\mathbf{0}) \otimes \chi^a(t,\mathbf{0}) + \frac{\lambda}{2}\, \phi^2(t,\mathbf{0}) \otimes \mathcal{I}_- \,. \tag{2.8}$$

Reference [23] solves explicitly for the time-evolution for this model within the Heisenberg picture, with most of the explicit results given for the point-hotspot limit (for which $\xi \to 0$). The remainder of this section quotes a few of the results found for later comparisons.

2.1.1 System state

In the Heisenberg picture states do not evolve with time, and correspond to the initial state of the system in the Schrödinger or interaction pictures. Correlation functions in [23] are computed assuming the ϕ and χ^a fields are initially uncorrelated, with

$$\rho_0 = \rho_+ \otimes \rho_- \,, \tag{2.9}$$

where the ϕ sector is in its standard Minkowski vacuum, $|{\rm vac}\rangle$, and the χ^a sector is in a thermal state, ϱ_β, with temperature $T = 1/\beta$:

$$\rho_+ = |{\rm vac}\rangle \langle {\rm vac}| \quad \text{and} \quad \rho_- = \varrho_\beta := \frac{e^{-\beta \mathcal{H}_-}}{{\rm Tr}_-[e^{-\beta\mathcal{H}_-}]}\,. \tag{2.10}$$

Here $\mathcal{H}_-$ is the χ^a-sector bulk Hamiltonian and the subscript '$-$' on the trace indicates that it is only taken over the χ sector.

2.1.2 Operator evolution

In Heisenberg picture the entire burden of time evolution falls on the field operators, which evolve according to

$$\phi_{H}(t,\mathbf{x}) := U^{-1}(t,0)\big[\phi_{S}(\mathbf{x}) \otimes \mathcal{I}_-\big] U(t,0) \quad \text{and} \quad \chi^a_{H}(t,\mathbf{x}) := U^{-1}(t,0)\big[\mathcal{I}_+ \otimes \chi^a_{S}(\mathbf{x})\big] U(t,0)\,, \tag{2.11}$$

where the full time-evolution operator is $U(t,t') = \mathcal{T} \exp\left(-i \int_{t'}^t {\rm d}s\; H(s)\right)$ and $\mathcal{T}$ denotes time-ordering.

In differential form, a generic Heisenberg-picture operator $A_{H}(t)$ satisfies $\partial_t A_{H}(t) = -i\Big[A_{H}(t), H_{H}(t)\Big]$, which in particular implies the fields ϕ_{H} and χ^a_{H} satisfy the equations of motion that follow from the action (2.1): $\Box\phi = \Box\chi^a = 0$ for all points exterior to $\mathcal{S}_\xi$. For points on $\mathcal{S}_\xi$ the presence of the interaction implies the equations of motion instead impose the boundary condition

$$\partial_r \phi\Big|_{r\to\xi} = \Big(G_a \chi^a + G_\phi\, \phi\Big)_{r\to\xi}\,, \tag{2.12}$$

(and similarly for χ^a). In the $\xi \to 0$ limit of a point source both the bulk equation and boundary condition are efficiently summarized by the equations

$$(-\partial_t^2 + \nabla^2)\phi_{H}(t,\mathbf{x}) = \delta^3(\mathbf{x}) \Big[\lambda \phi_{H}(t,\mathbf{0}) + g_a \chi^a_{H}(t,\mathbf{0})\Big] \tag{2.13}$$

and

$$\delta_{ab}(-\partial_t^2 + \nabla^2)\chi^b_{H}(t,\mathbf{x}) = \delta^3(\mathbf{x})\; g_a \phi_{H}(t,\mathbf{0})\,. \tag{2.14}$$

It is these last equations that are solved explicitly in [23], with the result used to compute the time-evolution of the system's correlation functions. These integrations are performed assuming the couplings G_a and G_ϕ (or g_a and λ) turn on suddenly at $t = 0$ and remain time-independent thereafter. These solutions generically diverge as $r \to \xi$ (or $\mathbf{r} \to 0$ in the point-hotspot limit) and because of this these equation must be regulated, such as by evaluating the boundary conditions at a small distance $\xi + \epsilon$ from the singular point. The divergences associated with taking $\epsilon \to 0$ are ultimately absorbed into renormalizations of couplings like G_ϕ [53–57]

2.2 Correlation functions

The main result of [23] is the calculation of the correlation functions for the fields, whose results we now quote for future use.

2.2.1 ϕ correlation functions

The ϕ-field Wightman function is defined by

$$W_\beta(t, \mathbf{x}; t', \mathbf{x}') := \mathrm{Tr}\Big[\phi_H(t, \mathbf{x})\phi_H(t', \mathbf{x}')\rho_0\Big]\,, \tag{2.15}$$

where ρ_0 is the state given in (2.9), with explicit formulae given in the small-hotspot limit ($\xi \to 0$).

The result computed to leading nontrivial order in $\tilde{g}^2$ and λ turns out to be given by

$$\begin{aligned} W_\beta(t, \mathbf{x}; t', \mathbf{x}') \simeq\ & \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2 + |\mathbf{x}-\mathbf{x}'|^2\big]} \\ & + \frac{\lambda}{16\pi^3}\bigg(\frac{\Theta(t-|\mathbf{x}|)}{|\mathbf{x}|}\frac{1}{(t-t'-|\mathbf{x}|-i\delta)^2 - |\mathbf{x}'|^2} + \frac{\Theta(t'-|\mathbf{x}'|)}{|\mathbf{x}'|}\frac{1}{(t-t'+|\mathbf{x}'|-i\delta)^2 - |\mathbf{x}|^2}\bigg) \\ & - \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)\Theta(t'-|\mathbf{x}'|)}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\Big[\frac{\pi}{\beta}(t-|\mathbf{x}|-t'+|\mathbf{x}'|-i\delta)\Big]} \\ & + \frac{\tilde{g}^2}{32\pi^4}\bigg(-\frac{\Theta(t-|\mathbf{x}|)}{|\mathbf{x}|}\frac{t-t'-|\mathbf{x}|}{\big[(t-t'-|\mathbf{x}|-i\delta)^2 - |\mathbf{x}'|^2\big]^2} + \frac{\Theta(t'-|\mathbf{x}'|)}{|\mathbf{x}'|}\frac{t-t'+|\mathbf{x}'|}{\big[(t-t'+|\mathbf{x}'|-i\delta)^2 - |\mathbf{x}|^2\big]^2}\bigg) \\ & + \frac{\tilde{g}^2}{64\pi^4}\bigg(\frac{\delta(t-|\mathbf{x}|)}{|\mathbf{x}|\big[-(t'+i\delta)^2 - |\mathbf{x}'|^2\big]} + \frac{\delta(t'-|\mathbf{x}'|)}{|\mathbf{x}'|\big[-(t-i\delta)^2 - |\mathbf{x}|^2\big]}\bigg) \qquad \text{(perturbative)}\,, \end{aligned} \tag{2.16}$$

where the delta functions and step functions describe the transients due to the switch-on of couplings at $t = |\mathbf{x}| = 0$. The infinitesimal $\delta \to 0^+$ is taken to zero at the end of the calculation.

Of most interest here is the form for the correlation function inside the future light-cone of this switch-on (*i.e.* for $t > |\mathbf{x}|$ and $t' > |\mathbf{x}'|$), for which this perturbative expression becomes

$$\begin{aligned} W_\beta(t, \mathbf{x}; t', \mathbf{x}') \simeq\ & \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2 + |\mathbf{x}-\mathbf{x}'|^2\big]} + \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\bigg[\frac{|\mathbf{x}|+|\mathbf{x}'|}{(t-t'-i\delta)^2 - (|\mathbf{x}+|\mathbf{x}'|)^2}\bigg] \\ & - \frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\Big[\frac{\pi}{\beta}(t-|\mathbf{x}|-t'+|\mathbf{x}'|-i\delta)\Big]} \\ & + \frac{\tilde{g}^2}{32\pi^4}\bigg(-\frac{1}{|\mathbf{x}|}\frac{t-t'-|\mathbf{x}|}{\big[(t-t'-|\mathbf{x}|-i\delta)^2 - |\mathbf{x}'|^2\big]^2} + \frac{1}{|\mathbf{x}'|}\frac{t-t'+|\mathbf{x}'|}{\big[(t-t'+|\mathbf{x}'|-i\delta)^2 - |\mathbf{x}|^2\big]^2}\bigg) \\ & \text{(inside light cone, perturbative)} \end{aligned} \tag{2.17}$$

The exact correlator is also computed in [23] and the result is compared there to this perturbative limit, verifying that the full dependence on λ agrees with RG-improved resummations using point-particle EFT boundary-condition based methods [57]. Because the full result is not needed in what follows it is not repeated here, beyond observing that the perturbative result emerges from the full one once it is expanded in powers of

$$\frac{\tilde{g}^2}{16\pi^2\epsilon\tau}\left(1+\frac{\lambda}{4\pi\epsilon}\right)^{-1} \ll 1 \quad \text{and} \quad \frac{\tilde{g}^2}{16\pi^2\epsilon\beta}\left(1+\frac{\lambda}{4\pi\epsilon}\right)^{-1} \ll 1\,. \tag{2.18}$$

For later comparisons it is useful to focus on the equal-time special case of these formulae, for which $t' = t$. Of particular interest when comparing with other approximation schemes is the equal-time limit of the perturbative result (2.17), which is

$$\begin{aligned} W_\beta(t,\mathbf{x};t,\mathbf{x}') \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} &- \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|(|\mathbf{x}|+|\mathbf{x}'|)} \\ &- \frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(|\mathbf{x}|-|\mathbf{x}'|)\right]} + \frac{\tilde{g}^2}{16\pi^4(|\mathbf{x}|^2-|\mathbf{x}'|^2)^2}\,. \end{aligned} \tag{2.19}$$

(inside light cone, perturbative)

2.2.2 χ^a correlation functions

Reference [23] also gives the explicit form for the χ^a free-field thermal correlator. Evaluated at spacetime points $x = (t, \mathbf{x})$ and $x' = (t', \mathbf{x}')$ the result (at large N) is

$$\begin{aligned} \langle \chi^a(t,\mathbf{x})\chi^b(t',\mathbf{x}')\rangle_\beta = \frac{\delta^{ab}}{8\pi\beta|\mathbf{x}-\mathbf{x}'|}\bigg\{&\coth\left[\frac{\pi}{\beta}(t-t'+|\mathbf{x}-\mathbf{x}'|-i\delta)\right] \\ &-\coth\left[\frac{\pi}{\beta}(t-t'-|\mathbf{x}-\mathbf{x}'|-i\delta)\right]\bigg\}\,, \end{aligned} \tag{2.20}$$

in agreement with standard formulae [74]. In this expression the limit $\delta \to 0^+$ is to be taken at the end of the calculation.

3 Open EFT late-time field evolution

The solution to the hotspot problem given in [23] is provided in the Heisenberg picture, and a drawback of this picture is that it obscures how the system's state evolves. This is unfortunate because it makes it difficult to compare with much of the literature on open quantum systems, which is often phrased in terms of the system's reduced density matrix (see for example [9]).

In this section we aim to make this comparison more transparent, by solving the hotspot problem using the Schrödinger-picture (in practice we use the interaction picture once we resort to perturbative methods), computing in particular the reduced density matrix $\sigma_S(t)$

$$\sigma_S(t) := \mathrm{Tr}_-[\rho_S(t)]\,, \tag{3.1}$$

for the ϕ-field (defined as the partial trace of the full Schrödinger-picture density matrix $\rho_S(t)$ over the χ^a sector).

The cumbersome nature of the Schrödinger picture for field theories prevented us from solving exactly for $\sigma_S(t)$ even within the hotspot model (though we have no reason to believe that this cannot be done). So we instead compute this evolution perturbatively in the hotspot couplings, also restricting for simplicity to the case of a point-like hotspot ($\xi \to 0$). A major drawback of using perturbation theory, however, is that perturbative methods intrinsically break down at late times, seeming to put beyond reach a reliable calculation of phenomena like decoherence or thermalization. We therefore also adapt Open EFT techniques [9, 27–29, 41, 75–77] to verify that they allow the perturbative result to be resummed to extend the perturbative domain of validity to very late times.[4] Doing so also allows the testing of these tools in this relatively unfamiliar quantum-field setting.

3.1 Open EFT evolution

In principle the time evolution of the system's state is given within the Interaction picture by the Liouville equation,

$$\frac{\partial \rho}{\partial t} \;=\; -i\big[H_{\rm int}(t), \rho(t)\big] \tag{3.2}$$

where $H_{\rm int}(t)$ is given by eq. (2.7), and where $\rho(t)$ is the interaction-picture state

$$\rho(t) = e^{+iH_0 t}\rho_S(t)\, e^{-iH_0 t} \tag{3.3}$$

where $H_0 = \mathcal{H}_+ \otimes \mathcal{I}_- + \mathcal{I}_+ \otimes \mathcal{H}_-$ is the free Hamiltonian for the combined system. It is easy to see from the above that the Schrödinger-picture reduced density matrix $\sigma_S(t)$ is related to the interaction-picture reduced density matrix $\sigma(t)$ through the relation

$$\sigma(t) = e^{+i\mathcal{H}_+ t}\sigma_S(t)\, e^{-i\mathcal{H}_+ t}\ . \tag{3.4}$$

When solving these equations we assume the uncorrelated initial state $\rho(0) = \rho_0 = \rho_+ \otimes \rho_-$ given in (2.9) where $\rho_+ = |\mathrm{vac}\rangle\langle\mathrm{vac}|$ and $\rho_- = \varrho_\beta$ is the thermal configuration for the χ sector.

3.1.1 Nakajima-Zwanzig equation

In principle the evolution of $\sigma(t)$ is given by taking the trace of (3.2) over all unmeasured degrees of freedom (in this case the fields χ^a). In perturbation theory one usually first formally solves (3.2) and then takes the trace of the result, leading to

$$\sigma(t) = \sigma(0) - i\int_0^t \mathrm{d}s\ \mathrm{Tr}_-\Big[H_{\rm int}(s)\,,\rho_0\Big] + (-i)^2\int_0^t \mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\ \mathrm{Tr}_-\Big[H_{\rm int}(s_1)\,,\Big[H_{\rm int}(s_2)\,,\rho_0\Big]\Big] + \cdots \tag{3.5}$$

where Tr_- denotes the partial trace only over the χ^a sector. The drawback of this expression is the relatively complicated dependence of its right-hand side on the full system's state. Because the right-hand side refers explicitly to the initial state ρ_0 successive terms in the series generically grow without bound for large t, causing the perturbative approximation to fail at late times and precluding accessing issues like thermalization and late-time decoherence.

The better route for late-time purposes is to take the trace of the differential relation (3.2) and to eliminate from this the dependence of the right-hand side on any unmeasured degrees of freedom (for a review of the steps given below see for example [9]). The good news is that because of the linearity of (3.2) this can be done in great generality, with the resulting evolution equation for $\sigma(t)$ known as the Nakajima-Zwanzig equation [51, 52]. Although this is a textbook derivation, we now describe it in some detail since it is not often applied to quantum fields (as we do here) in the relativity literature.

The logic of the derivation proceeds as follows. One first defines the super-operator $\mathscr{P}$ acting on operators in the Hilbert space by

$$\mathscr{P}(\mathcal{O}) \;=\; \mathrm{Tr}_-\left[\mathcal{O}\right]\otimes\rho_-\ , \tag{3.6}$$

[4] Open EFT techniques were also applied to the hotspot in ref. [26], but only to obtain the late-time thermalization behaviour of an Unruh-DeWitt detector [30, 31] that sits at rest displaced from the hotspot.

where $\rho_- = \varrho_\beta$ is the time-independent initial thermal density matrix for the fields χ^a. This is a projection super-operator because it satisfies $\mathscr{P}^2 = \mathscr{P}$, as therefore must its complement $\mathscr{R} = 1 - \mathscr{P}$. The definition (3.6) is defined so that it projects the full density matrix $\rho(t)$ onto the reduced density matrix $\sigma(t)$,

$$\mathscr{P}\big[\rho(t)\big] = \sigma(t) \otimes \rho_- \ , \tag{3.7}$$

and so $\mathscr{R}[\rho(t)]$ can be regarded as describing all of the unmeasured parts of the full density matrix.

Our goal is therefore to rewrite the Liouville equation as a coupled set of evolution equations for the mutually exclusive quantities $\mathscr{P}\big[\rho(t)\big]$ and $\mathscr{R}\big[\rho(t)\big]$, and then eliminate $\mathscr{R}\big[\rho(t)\big]$ from these by solving its equation as a function of $\mathscr{P}\big[\rho(t)\big]$. To this end write the interaction-picture Liouville equation (3.2) in terms of a linear Liouville super-operator,

$$\partial_t \rho = \mathscr{L}_t(\rho) \qquad \text{where} \qquad \mathscr{L}_t(\rho) = -i[H_{\rm int}(t), \rho\,] \ , \tag{3.8}$$

and project the result using the operators $\mathscr{P}$ and $\mathscr{R}$. Since $\mathscr{P} + \mathscr{R} = 1$ this leads to

$$\mathscr{P}(\partial_t \rho) = \mathscr{P}\mathscr{L}_t\mathscr{P}(\rho) + \mathscr{P}\mathscr{L}_t\mathscr{R}(\rho) \tag{3.9}$$

and

$$\mathscr{R}(\partial_t \rho) = \mathscr{R}\mathscr{L}_t\mathscr{P}(\rho) + \mathscr{R}\mathscr{L}_t\mathscr{R}(\rho)\,. \tag{3.10}$$

The unmeasured degrees of freedom are eliminated by formally solving eq. (3.10):

$$\mathscr{R}[\rho(t)] = \mathcal{G}(t,0)\mathscr{R}[\rho(0)] + \int_0^t \mathrm{d}s\ \mathcal{G}(t,s)\mathscr{R}\mathscr{L}_t\mathscr{P}[\rho(s)] \tag{3.11}$$

with

$$\mathcal{G}(t,s) = 1 + \sum_{n=1}^{\infty} \int_s^t \mathrm{d}s_1\ \cdots \int_s^{s_{n-1}} \mathrm{d}s_n\ \mathscr{R}\mathscr{L}_{s_1} \cdots \mathscr{R}\mathscr{L}_{s_n}\,, \tag{3.12}$$

and inserting the result into (3.9). This yields the desired self-contained evolution equation for $\mathscr{P}[\rho(t)]$:

$$\mathscr{P}[\partial_t\rho(t)] = \mathscr{P}\mathscr{L}_t\mathscr{P}[\rho(t)] + \mathscr{P}\mathscr{L}_t\mathcal{G}(t,0)\mathscr{R}[\rho(0)] + \int_0^t \mathrm{d}s\ \mathcal{K}(t,s)[\rho(s)] \tag{3.13}$$

with kernel

$$\mathcal{K}(t,s) = \mathscr{P}\mathscr{L}_t\mathcal{G}(t,s)\mathscr{R}\mathscr{L}_s\mathscr{P}\,. \tag{3.14}$$

For uncorrelated initial states of the form $\rho(0) = \rho_+ \otimes \rho_-$ the second term on the right-hand side of (3.13) vanishes because $\mathscr{P}\big[\rho(0)\big] = \rho(0)$ and so $\mathscr{R}\big[\rho(0)\big] = 0$.

In what follows we wish to use (3.13) but work only to second order in $H_{\rm int}(t)$, which means expanding out the kernel $\mathcal{K}(t,s)$ to second order in $\mathscr{L}_t$. At this order we can therefore take $\mathcal{G}(t,s) \simeq 1$ in $\mathcal{K}(t,s)$, which becomes

$$\mathcal{K}(t,s) \simeq \mathscr{P}\mathscr{L}_t\mathscr{R}\mathscr{L}_s\mathscr{P}\,, \tag{3.15}$$

and so (3.13) simplifies to

$$\mathscr{P}[\partial_t\rho(t)] \simeq \mathscr{P}\mathscr{L}_t\mathscr{P}[\rho(t)] + \int_0^t \mathrm{d}s\ \mathscr{P}\mathscr{L}_t\mathscr{R}\mathscr{L}_s\mathscr{P}[\rho(s)]\,. \tag{3.16}$$

Writing this out explicitly using the definitions of $\mathscr{P}$, $\mathscr{R}$ and $\mathscr{L}_t$ then gives the more explicit form

$$\begin{aligned}\frac{\partial\sigma}{\partial t} \simeq{}& -i\ \mathrm{Tr}_-\Big\{\Big[H_{\rm int}(t), \sigma(t)\otimes\rho_-\Big]\Big\} \\ & - \int_0^t \mathrm{d}s\ \mathrm{Tr}_-\bigg\{\bigg[H_{\rm int}(t)\,,\ \Big[H_{\rm int}(s)\,,\ \sigma(s)\otimes\rho_-\Big] - \mathrm{Tr}_-\Big(\Big[H_{\rm int}(s), \sigma(s)\otimes\rho_-\Big]\Big)\otimes\rho_-\bigg]\bigg\}\,.\end{aligned} \tag{3.17}$$

To apply this expression to the hotspot fields expand $H_{\rm int}(t)$ in terms of a basis of operators with the factorized form

$$H_{\rm int}(t) = \mathcal{A}^{A}(t) \otimes \mathcal{B}_{A}(t)\,, \tag{3.18}$$

where $\mathcal{A}^A$ acts only in the ϕ sector and $\mathcal{B}_A$ acts only in the χ^a sector. With this choice (3.16) simplifies to become

$$\begin{aligned}\frac{\partial \sigma}{\partial t} \simeq &-i\,\Big[\mathcal{A}_a(t)\,,\,\sigma(t)\Big]\langle\!\langle\,\mathcal{B}^a(t)\,\rangle\!\rangle \\ &+\int_0^t {\rm d}s\,\Big(\Big[\mathcal{A}^a(s)\,\sigma(s)\,,\mathcal{A}^b(t)\Big]\langle\!\langle\,\mathcal{B}_b(t)\mathcal{B}_a(s)\,\rangle\!\rangle + \Big[\mathcal{A}^b(t)\,,\sigma(s)\mathcal{A}^a(s)\Big]\langle\!\langle\,\mathcal{B}_a(s)\,\mathcal{B}_b(t)\,\rangle\!\rangle\Big) \\ &-\int_0^t {\rm d}s\,\Big(\Big[\mathcal{A}^a(s)\sigma(s)\,,\mathcal{A}^b(t)\Big] + \Big[\mathcal{A}^b(t)\,,\sigma(s)\mathcal{A}^a(s)\Big]\Big]\Big)\langle\!\langle\,\mathcal{B}_b(t)\,\rangle\!\rangle\langle\!\langle\,\mathcal{B}_a(s)\,\rangle\!\rangle\,.\end{aligned} \tag{3.19}$$

where $\langle\!\langle\,\mathcal{O}\,\rangle\!\rangle := {\rm Tr}_-\,[\rho_-\,\mathcal{O}] = {\rm Tr}_-\,[\varrho_\beta \mathcal{O}]$ is the thermal trace for operators acting purely in the χ sector.

For the point-hotspot system the interaction Hamiltonian given in (2.8) has the form $H_{\rm int}(t) = g_a\phi(t,\mathbf{0})\otimes\chi^a(t,\mathbf{0}) + \frac{1}{2}\lambda\phi^2(t,\mathbf{0})\otimes\mathcal{I}_-$ (in the Interaction picture) and so using $\langle\!\langle\,\chi^a\,\rangle\!\rangle = 0$ the second-order Nakajima-Zwanzig equation reduces to

$$\begin{aligned}\frac{\partial \sigma}{\partial t} \simeq -\frac{i\lambda}{2}\Big[\phi^2(t,\mathbf{0}),\sigma(t)\Big] + g_a g_b\int_0^t {\rm d}s\,\Big(&\langle\!\langle\,\chi^b(t,\mathbf{0})\chi^a(s,\mathbf{0})\,\rangle\!\rangle\Big[\phi(s,\mathbf{0})\,\sigma(s)\,,\phi(t,\mathbf{0})\Big] \\ &+\langle\!\langle\,\chi^a(s,\mathbf{0})\,\chi^b(t,\mathbf{0})\,\rangle\!\rangle\Big[\phi(t,\mathbf{0})\,,\sigma(s)\phi(s,\mathbf{0})\Big]\Big)\end{aligned} \tag{3.20}$$

We write the thermal correlation function for two χ fields as

$$g_a g_b\langle\!\langle\,\chi^b(t,\mathbf{0})\chi^a(s,\mathbf{0})\,\rangle\!\rangle =: \tilde{g}^2\mathscr{W}(t-s) = -\frac{\tilde{g}^2}{4\beta^2}\,{\rm csch}^2\left[\frac{\pi}{\beta}(t-s-i\delta)\right] \tag{3.21}$$

where $\delta = 0^+$ goes to zero at the end of any calculation and the first equality defines the function $\mathscr{W}(t)$ with $\tilde{g}^2 := \delta^{ab}\,g_a g_b$ as given in (2.5), while the second equality uses the explicit form for the $\mathbf{x}' \to \mathbf{x}$ limit of the correlation function given in (2.20).

Finally, after a change of integration variables $s \to t-s$ we arrive at the form for the Nakajima-Zwanzig equation whose properties are explored below:

$$\begin{aligned}\frac{\partial \sigma}{\partial t} \simeq &-\frac{i\lambda}{2}\Big[\phi^2(t,\mathbf{0}),\sigma(t)\Big] \\ &+g^2\int_0^t {\rm d}s\,\Big(\mathscr{W}(s)\Big[\phi(t-s,\mathbf{0})\,\sigma(t-s)\,,\phi(t,\mathbf{0})\Big] + \mathscr{W}^*(s)\Big[\phi(t,\mathbf{0})\,,\sigma(t-s)\phi(t-s,\mathbf{0})\Big]\Big)\,.\end{aligned} \tag{3.22}$$

3.1.2 Markovian limit

Since the correlation function $\mathscr{W}(s)$ is sharply peaked about $s = 0$ and falls off exponentially fast like $\mathscr{W}(s) \propto e^{-2\pi s/\beta}$ for $s \gg \beta$, the integral simplifies if the rest of the integrand varies more slowly in the region where $\mathscr{W}$ varies quickly. In such a case the integral is well-approximated by expanding the rest of the integrand in powers of s, using

$$\phi(t-s)\sigma(t-s) \simeq \phi(t,\mathbf{0})\,\sigma(t) - s\big[\partial_t\phi(t,\mathbf{0})\,\sigma(t) + \phi(t,\mathbf{0})\partial_t\sigma(t)\big] + \ldots \tag{3.23}$$

beneath the integral sign in (3.22). Notice that this assumes *both* ϕ and σ vary slowly, and so its justification requires both that $\sigma(t)$ should be slowly varying and that we work in an effective description that keeps only those modes of ϕ whose energies satisfy $E \ll 1/\beta$. This becomes relevant when choosing momentum cutoffs for later integrals, since this Markovian derivation requires $\Lambda \ll 1/\beta$. Part of the discussion to follow aims to identify more precisely the region of parameter space for which this Markovian approximation is valid (which we find by asking when the subleading terms in the series (3.23) are small).

Keeping only the leading-order term in the Taylor-series (3.23) yields the approximate equation of motion

$$\frac{\partial \sigma}{\partial t} \simeq -\frac{i\lambda}{2}\Big[\phi^2(t,\mathbf{0}),\sigma(t)\Big] + \tilde{g}^2\mathscr{C}\Big[\phi(t,\mathbf{0})\,\sigma(t)\,,\phi(t,\mathbf{0})\Big] + \tilde{g}^2\mathscr{C}^*\Big[\phi(t,\mathbf{0})\,,\sigma(t)\phi(t,\mathbf{0})\Big]\,. \tag{3.24}$$

where the coefficient is given by

$$\mathscr{C}(t) := \int_0^t \mathrm{d}s\, \mathscr{W}(s) \tag{3.25}$$

and the approximate equality requires $t \gg \beta$ so that the integration includes the strong peaking of $\mathscr{W}(s)$ (with exponential fall-off) noted above. Although the coefficient $\mathscr{C}(t)$ is in principle a function of t, in practice the narrowly peaked form of $\mathscr{W}(t)$ ensures it approaches a constant exponentially quickly once $t \gg \beta$. The value of this constant can be evaluated explicitly by taking the upper integration limit to infinity and evaluating the resulting integral using expression (3.21) for $\mathscr{W}(t)$:

$$\mathscr{C}(t) \simeq \mathscr{C}_\infty = \int_0^\infty \mathrm{d}s\, \mathscr{W}(s) = -\frac{1}{4\beta^2}\int_0^\infty \frac{\mathrm{d}s}{\sinh^2\left[\frac{\pi}{\beta}(s-i\delta)\right]} \simeq \frac{1}{4\pi\beta} - \frac{i}{4\pi^2\beta}\left[\frac{\beta}{\delta} + \mathcal{O}\left(\tfrac{\delta}{\beta}\right)\right]\,, \tag{3.26}$$

for $t \gg \beta$. The divergence as $\delta \to 0$ is a reflection of the divergence of the integrand as $s \to 0$.

It is sometimes useful to convert (3.24) to the Schrödinger picture, as is done by noting that

$$e^{-i\mathcal{H}_+ t}\frac{\partial \sigma}{\partial t}e^{+i\mathcal{H}_+ t} = \frac{\partial \sigma_S}{\partial t} + i[\mathcal{H}_+,\sigma_S(t)]\,. \tag{3.27}$$

which follows from the relation (3.4), giving

$$\frac{\partial \sigma_S}{\partial t} \simeq -i\Big[\mathcal{H}_+ + \tfrac{\lambda}{2}\phi_S^2(\mathbf{0}),\sigma_S(t)\Big] + \tilde{g}^2\mathscr{C}\Big[\phi_S(\mathbf{0})\,\sigma_S(t),\phi_S(\mathbf{0})\Big] + \tilde{g}^2\mathscr{C}^*\Big[\phi_S(\mathbf{0})\,,\sigma_S(t)\phi_S(\mathbf{0})\Big]\,. \tag{3.28}$$

Using this in (3.28) finally gives

$$\begin{aligned}\frac{\partial \sigma_S}{\partial t} &\simeq -i\Big[\mathcal{H}_+ + \tfrac{\lambda}{2}\phi_S^2(\mathbf{0}),\sigma_S(t)\Big] \\ &\quad + \frac{\tilde{g}^2}{4\pi}\left(\frac{1}{\beta} - \frac{i}{\pi\delta}\right)\Big[\phi_S(\mathbf{0})\,\sigma_S(t)\,,\phi_S(\mathbf{0})\Big] + \frac{\tilde{g}^2}{4\pi}\left(\frac{1}{\beta} + \frac{i}{\pi\delta}\right)\Big[\phi_S(\mathbf{0})\,,\sigma_S(t)\,\phi_S(\mathbf{0})\Big] \\ &= -i\Big[\mathcal{H}_+ + \tfrac{\lambda_{\rm ren}}{2}\phi_S^2(\mathbf{0}),\sigma_S(t)\Big] + \frac{\tilde{g}^2}{4\pi\beta}\Big(\Big[\phi_S(\mathbf{0})\,\sigma_S(t)\,,\phi_S(\mathbf{0})\Big] + \Big[\phi_S(\mathbf{0})\,,\sigma_S(t)\,\phi_S(\mathbf{0})\Big]\Big)\,,\end{aligned} \tag{3.29}$$

which shows that the divergence can be absorbed into the renormalization

$$\lambda_{\rm ren} \;:=\; \lambda - \frac{\tilde{g}^2}{2\pi^2\delta}\,. \tag{3.30}$$

3.1.3 Evolution equation in a field basis

It is easiest to solve an equation like (3.29) in a basis that diagonalizes the interaction Hamiltonian, and in this instance this suggests using a basis of field eigenstates defined as the basis that diagonalizes the Schrödinger-picture field operator $\phi(0,\mathbf{x}) = \phi_S(\mathbf{x})$:

$$\phi_S(\mathbf{x})\,|\varphi(\cdot)\rangle = \varphi(\mathbf{x})\,|\varphi(\cdot)\rangle \tag{3.31}$$

where the eigenvalue $\varphi(\mathbf{x})$ is a real-valued function of position.

The wave-functional of the free vacuum $|\text{vac}\rangle$ in this reprsentation is given by a gaussian [78]

$$\begin{aligned} \langle\varphi(\cdot)|\,\text{vac}\rangle \;&=\; \sqrt{\mathcal{N}_0}\;\exp\left[-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int d^3\mathbf{y}\;\mathcal{E}(\mathbf{x},\mathbf{y})\varphi(\mathbf{x})\varphi(\mathbf{y})\right] \\ &=\; \sqrt{\mathcal{N}_0}\;\exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\;E_p\;\varphi_{\mathbf{p}}\varphi_{-\mathbf{p}}\right] \end{aligned} \tag{3.32}$$

where the kernel $\mathcal{E}(\mathbf{x},\mathbf{y})$ is given by

$$\mathcal{E}(\mathbf{x},\mathbf{y}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\;E_p\;e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,, \tag{3.33}$$

where $E_p := |\mathbf{p}|$ and the normalization factor $\mathcal{N}_0$ is determined by using the normalization condition $\langle\text{vac}|\text{vac}\rangle = 1$.

In the field basis the reduced density matrix is a time-dependent functional of the basis field configurations $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$, with components

$$\sigma_S[\varphi_1,\varphi_2;t] := \langle\varphi_1(\cdot)|\sigma_S(t)|\varphi_2(\cdot)\rangle \tag{3.34}$$

which the Markovian equation (3.29) implies must satisfy

$$\begin{aligned} \frac{\partial\sigma_S[\varphi_1,\varphi_2;t]}{\partial t} \;\simeq\; &-i\,\langle\varphi_1(\cdot)|\,\Big[\mathcal{H}_+,\sigma_S(t)\Big]\,|\varphi_2(\cdot)\rangle \\ &-\left[\frac{\tilde{g}^2}{4\pi\beta}\big(\varphi_1(\mathbf{0})-\varphi_2(\mathbf{0})\big)^2+\frac{i\lambda_{\text{ren}}}{2}\big(\varphi_1^2(\mathbf{0})-\varphi_2^2(\mathbf{0})\big)\right]\sigma_S[\varphi_1,\varphi_2;t]\;. \end{aligned} \tag{3.35}$$

We henceforth drop the subscript 'ren' on the renormalized coupling parameter λ. Evaluating the commutator term using

$$\langle\varphi(\cdot)|\,\mathcal{H}_+\,|\Psi\rangle \;=\; \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\bigg[-\frac{\delta^2}{\delta\varphi(\mathbf{x})^2}+\big|\boldsymbol{\nabla}\varphi(\mathbf{x})\big|^2\bigg]\langle\varphi(\cdot)|\Psi\rangle\;, \tag{3.36}$$

for any state $|\Psi\rangle$, the equation of motion for σ_S finally becomes

$$\begin{aligned} \frac{\partial\sigma_S[\varphi_1,\varphi_2;t]}{\partial t} \simeq &-\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\bigg[-\frac{\delta^2}{\delta\varphi_1(\mathbf{x})^2}+\big|\boldsymbol{\nabla}\varphi_1(\mathbf{x})\big|^2+\frac{\delta^2}{\delta\varphi_2(\mathbf{x})^2}-\big|\boldsymbol{\nabla}\varphi_2(\mathbf{x})\big|^2\bigg]\sigma_S[\varphi_1,\varphi_2;t] \\ &-\frac{\tilde{g}^2}{4\pi\beta}\big(\varphi_1(\mathbf{0})-\varphi_2(\mathbf{0})\big)^2\sigma_S[\varphi_1,\varphi_2;t]-\frac{i\lambda}{2}\big(\varphi_1^2(\mathbf{0})-\varphi_2^2(\mathbf{0})\big)\sigma_S[\varphi_1,\varphi_2;t]\;. \end{aligned} \tag{3.37}$$

As is easily verified, when $\lambda = \tilde{g} = 0$ this equation has as a solution

$$\begin{aligned} \sigma_S(t,\varphi_1,\varphi_2) \;&=\; \langle\varphi_1(\cdot)|\text{vac}\rangle\,\langle\text{vac}|\varphi_2(\cdot)\rangle \\ &= \mathcal{N}_0\exp\bigg(-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\;\mathcal{E}(\mathbf{x}-\mathbf{y})\big[\varphi_1(\mathbf{x})\varphi_1(\mathbf{y})+\varphi_2(\mathbf{x})\varphi_2(\mathbf{y})\big]\bigg) \end{aligned} \tag{3.38}$$

where the second equality uses (3.32) and the kernel $\mathcal{E}(\mathbf{x}-\mathbf{y})$ is as defined in equation (3.33).

3.2 Solutions for the reduced density matrix

We next solve eq. (3.37) for the ϕ-sector density matrix in the presence of the hotspot interactions.

3.2.1 Gaussian ansatz

Keeping in mind that the hotspot 'interactions' are all bilinear in the fields we seek solutions to (3.37) subject to the more general Gaussian ansatz

$$\sigma_S[\varphi_1, \varphi_2; t] = \mathcal{N}(t) \exp\bigg(-\frac{1}{2} \int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{y} \; \Big\{ \mathcal{A}_1(\mathbf{x}, \mathbf{y}; t)\, \varphi_1(\mathbf{x})\varphi_1(\mathbf{y}) + \mathcal{A}_2(\mathbf{x}, \mathbf{y}; t)\, \varphi_2(\mathbf{x})\varphi_2(\mathbf{y}) + 2\mathcal{B}(\mathbf{x}, \mathbf{y}; t)\, \varphi_1(\mathbf{x})\varphi_2(\mathbf{y}) \Big\} \bigg) , \tag{3.39}$$

with the kernels $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{B}$ to be determined. Note that we can without loss of generality assume the symmetry

$$\mathcal{A}_j(\mathbf{x}, \mathbf{y}; t) = \mathcal{A}_j(\mathbf{y}, \mathbf{x}; t) \; , \tag{3.40}$$

and that hermiticity of the reduced density matrix – $\sigma_S^*[\varphi_1, \varphi_2; t] = \sigma_S[\varphi_2, \varphi_1; t]$ – implies $\mathcal{N}^*(t) = \mathcal{N}(t)$ and

$$\mathcal{A}_1^*(\mathbf{x}, \mathbf{y}; t) = \mathcal{A}_2(\mathbf{x}, \mathbf{y}; t) \quad \text{and} \quad \mathcal{B}^*(\mathbf{x}, \mathbf{y}; t) = \mathcal{B}(\mathbf{y}, \mathbf{x}; t)\, . \tag{3.41}$$

Notice that $\sigma_S = |\Psi\rangle\langle\Psi|$ for a gaussian pure state $\langle \varphi(\cdot) | \Psi \rangle \propto \exp\left[-\frac{1}{2} K(\mathbf{x}, \mathbf{y})\, \varphi(\mathbf{x})\varphi(\mathbf{y}) \right]$ only if

$$\mathcal{A}_1(\mathbf{x}, \mathbf{y}) = \mathcal{A}_2^*(\mathbf{x}, \mathbf{y}) = K(\mathbf{x}, \mathbf{y}) \quad \text{and} \quad \mathcal{B}(\mathbf{x}, \mathbf{y}) = 0\, . \tag{3.42}$$

with the free vacuum (3.38) corresponding to the choice $K(\mathbf{x}, \mathbf{y}) = \mathcal{E}(\mathbf{x} - \mathbf{y})$. Since the first of these is an automatic consequence of (3.41), this shows that $\mathcal{A}_1$ and $\mathcal{A}_2$ can be regarded as the deformations of the ground state away from the free result due to the interactions, while having $\mathcal{B} \neq 0$ corresponds to the interaction causing the initially pure state to become mixed.

The kernels are obtained by plugging the ansatz (3.39) into (3.37) and equating the coefficients of the different functional forms on both sides of the equation. The details are worked out in Appendix A.1, with the results simply quoted here. Equating the coefficients of terms independent of φ_i implies

$$\frac{1}{\mathcal{N}} \frac{\partial \mathcal{N}}{\partial t} = -\frac{i}{2} \int \mathrm{d}^3\mathbf{x} \Big[\mathcal{A}_1(\mathbf{x}, \mathbf{x}; t) - \mathcal{A}_2(\mathbf{x}, \mathbf{x}; t) \Big] . \tag{3.43}$$

This expression can also be derived from the condition that $\mathrm{Tr}_+ \sigma_S(t) = 1$ for all times. The coefficient of $\varphi_1(\mathbf{x})\varphi_1(\mathbf{y})$ similarly gives

$$\frac{\partial \mathcal{A}_1(\mathbf{x}, \mathbf{y}; t)}{\partial t} = -i\nabla^2_{\mathbf{x}} \delta^3(\mathbf{x} - \mathbf{y}) + \left(\frac{\tilde{g}^2}{2\pi\beta} + i\lambda \right) \delta^3(\mathbf{x})\delta^3(\mathbf{y}) + \int \mathrm{d}^3\mathbf{z} \; \Big[-i\mathcal{A}_1(\mathbf{z}, \mathbf{x}; t)\mathcal{A}_1(\mathbf{z}, \mathbf{y}; t) + i\mathcal{B}(\mathbf{x}, \mathbf{z}; t)\mathcal{B}(\mathbf{y}, \mathbf{z}; t) \Big] , \tag{3.44}$$

while the coefficient of $\varphi_2(\mathbf{x})\varphi_2(\mathbf{y})$ leads to

$$\frac{\partial \mathcal{A}_2(\mathbf{x}, \mathbf{y}; t)}{\partial t} = i\nabla^2_{\mathbf{x}} \delta^3(\mathbf{x} - \mathbf{y}) + \left(\frac{\tilde{g}^2}{2\pi\beta} - i\lambda \right) \delta^3(\mathbf{x})\delta^3(\mathbf{y}) + \int \mathrm{d}^3\mathbf{z} \; \Big[i\mathcal{A}_2(\mathbf{x}, \mathbf{z}; t)\mathcal{A}_2(\mathbf{y}, \mathbf{z}; t) - i\mathcal{B}(\mathbf{z}, \mathbf{x}; t)\mathcal{B}(\mathbf{z}, \mathbf{y}; t) \Big] . \tag{3.45}$$

Finally, the coefficient of $\varphi_1(\mathbf{x})\varphi_2(\mathbf{y})$ gives

$$\frac{\partial \mathcal{B}(\mathbf{x}, \mathbf{y}; t)}{\partial t} = -\frac{\tilde{g}^2}{2\pi\beta} \delta^3(\mathbf{x})\delta^3(\mathbf{y}) + \int \mathrm{d}^3\mathbf{z} \; \Big[-i\mathcal{A}_1(\mathbf{z}, \mathbf{x}; t)\mathcal{B}(\mathbf{z}, \mathbf{y}; t) + i\mathcal{B}(\mathbf{x}, \mathbf{z}; t)\mathcal{A}_2(\mathbf{z}, \mathbf{y}; t) \Big] . \tag{3.46}$$

The implications of these equations are simpler to see in momentum space, so we define

$$\mathcal{A}_j(\mathbf{x},\mathbf{y};t) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\; A_j(\mathbf{k},\mathbf{q};t)\; e^{+i\mathbf{k}\cdot\mathbf{x}}\, e^{-i\mathbf{q}\cdot\mathbf{y}} \tag{3.47}$$

$$\text{and}\quad \mathcal{B}(\mathbf{x},\mathbf{y};t) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\; B(\mathbf{k},\mathbf{q};t)\; e^{+i\mathbf{k}\cdot\mathbf{x}}\, e^{-i\mathbf{q}\cdot\mathbf{y}}\,,$$

for which the symmetry $\mathcal{A}_j(\mathbf{x},\mathbf{y};t) = \mathcal{A}_j(\mathbf{y},\mathbf{x};t)$ of (3.40) implies

$$A_j(\mathbf{k},\mathbf{q};t) \;=\; A_j(-\mathbf{q},-\mathbf{k};t)\;. \tag{3.48}$$

In terms of these equation (3.43) becomes

$$\frac{1}{\mathcal{N}}\frac{\partial\mathcal{N}}{\partial t} \;=\; -\frac{i}{2}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Big[A_1(\mathbf{k},\mathbf{k};t) - A_2(\mathbf{k},\mathbf{k};t)\Big] \tag{3.49}$$

while equations (3.44) through (3.46) become

$$\begin{aligned}\frac{\partial A_1(\mathbf{k},\mathbf{q};t)}{\partial t} &= i(2\pi)^3|\mathbf{k}|^2\delta^3(\mathbf{k}-\mathbf{q}) + \left(\frac{\tilde{g}^2}{2\pi\beta} + i\lambda\right) \\ &\quad + \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\Big[-iA_1(\mathbf{k},\mathbf{p};t)A_1(\mathbf{p},\mathbf{q};t) + iB(\mathbf{k},\mathbf{p};t)B(-\mathbf{q},-\mathbf{p};t)\Big]\,,\end{aligned} \tag{3.50}$$

$$\begin{aligned}\frac{\partial A_2(\mathbf{k},\mathbf{q};t)}{\partial t} &= -i(2\pi)^3|\mathbf{k}|^2\delta^3(\mathbf{k}-\mathbf{q}) + \left(\frac{\tilde{g}^2}{2\pi\beta} - i\lambda\right) \\ &\quad + \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\Big[iA_2(\mathbf{k},\mathbf{p};t)A_2(\mathbf{p},\mathbf{q};t) - iB(-\mathbf{p},-\mathbf{k};t)B(\mathbf{p},\mathbf{q};t)\Big]\end{aligned} \tag{3.51}$$

and

$$\frac{\partial B(\mathbf{p},\mathbf{q};t)}{\partial t} = -\frac{\tilde{g}^2}{2\pi\beta} + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Big[-iA_1(\mathbf{p},\mathbf{k};t)B(\mathbf{k},\mathbf{q};t) + iB(\mathbf{p},\mathbf{k};t)A_2(\mathbf{k},\mathbf{q};t)\Big]\,. \tag{3.52}$$

These are the equations we solve in the next few sections. Notice in particular that (3.46) or (3.52) implies $\tilde{g}^2/\beta \neq 0$ is an obstruction to $B(\mathbf{q},\mathbf{x},t) = 0$ being a solution.

3.2.2 Perturbative solution

As is easily verified, for $\lambda = \tilde{g} = 0$ these above equations are solved by

$$B(\mathbf{p},\mathbf{q},t) = 0 \quad\text{and}\quad A_j(\mathbf{k},\mathbf{q}) \;=\; (2\pi)^3|\mathbf{k}|\;\delta^3(\mathbf{k}-\mathbf{q})\,, \tag{3.53}$$

corresponding to the vacuum solution of (3.38). We next seek solutions that are perturbatively close to this vacuum solution, as should be possible for small $\tilde{g}$ and λ.

To this end we write

$$A_j(\mathbf{k},\mathbf{q};t) = (2\pi)^3|\mathbf{k}|\;\delta^3(\mathbf{k}-\mathbf{q}) + \mathfrak{a}_j(\mathbf{k},\mathbf{q};t) + \dots \quad\text{and}\quad B(\mathbf{k},\mathbf{q};t) = \mathfrak{b}(\mathbf{k},\mathbf{q};t) + \dots\,, \tag{3.54}$$

and linearize eqs. (3.49) through (3.52) in the perturbations $\mathfrak{a}_j$ and $\mathfrak{b}$. The resulting evolution equations decouple, to become

$$\frac{\partial\mathfrak{a}_1(\mathbf{k},\mathbf{q};t)}{\partial t} = \frac{\tilde{g}^2}{2\pi\beta} + i\lambda - i(|\mathbf{k}|+|\mathbf{q}|)\,\mathfrak{a}_1(\mathbf{k},\mathbf{q};t)\,, \tag{3.55}$$

$$\frac{\partial \mathfrak{a}_2(\mathbf{k},\mathbf{q};t)}{\partial t} = \frac{\tilde{g}^2}{2\pi\beta} - i\lambda + i(|\mathbf{k}|+|\mathbf{q}|)\,\mathfrak{a}_2(\mathbf{k},\mathbf{q};t) \tag{3.56}$$

and

$$\frac{\partial \mathfrak{b}(\mathbf{k},\mathbf{q};t)}{\partial t} = -\frac{\tilde{g}^2}{2\pi\beta} - i\big(|\mathbf{k}|-|\mathbf{q}|\big)\,\mathfrak{b}(\mathbf{k},\mathbf{q};t)\,. \tag{3.57}$$

These are to be solved subject to the initial conditions

$$\mathfrak{a}_1(\mathbf{k},\mathbf{q};0) = \mathfrak{a}_2(\mathbf{k},\mathbf{q};0) = \mathfrak{b}(\mathbf{k},\mathbf{q};0) = 0\,, \tag{3.58}$$

since the scalar ϕ is starts off in its vacuum state.

The solutions to these initial-value problems are given by

$$\mathfrak{a}_1(\mathbf{k},\mathbf{q};t) = \left(\lambda - \frac{i\tilde{g}^2}{2\pi\beta}\right)\frac{1-e^{-i\big(|\mathbf{k}|+|\mathbf{q}|\big)t}}{|\mathbf{k}|+|\mathbf{q}|}\,, \tag{3.59}$$

$$\mathfrak{a}_2(\mathbf{k},\mathbf{q};t) = \left(\lambda + \frac{i\tilde{g}^2}{2\pi\beta}\right)\frac{1-e^{+i\big(|\mathbf{k}|+|\mathbf{q}|\big)t}}{|\mathbf{k}|+|\mathbf{q}|} \tag{3.60}$$

and

$$\mathfrak{b}(\mathbf{k},\mathbf{q};t) = \left(\frac{i\tilde{g}^2}{2\pi\beta}\right)\frac{1-e^{-i(|\mathbf{k}|-|\mathbf{q}|)t}}{|\mathbf{k}|-|\mathbf{q}|}\,. \tag{3.61}$$

In the limit $\mathbf{k}=\mathbf{q}$ (or in the limit of small t) this last solution simplifies to

$$\mathfrak{b}(\mathbf{k},\mathbf{q};t) \to -\frac{\tilde{g}^2 t}{2\pi\beta} \qquad \text{when} \qquad \mathbf{k}\to\mathbf{q}\,. \tag{3.62}$$

As remarked earlier — and explored in more detail in §3.4 — nonzero $\mathfrak{b}$ is a signature of σ_S becoming a mixed state, and so (3.61) shows that this only happens for nonzero $\tilde{g}^2/\beta$. Furthermore when $\tilde{g}^2/\beta$ is nonzero there can be no static solution with $\partial_t\mathfrak{b}=0$, (and in particular no solution with $\mathfrak{b}=0$) and $|\mathfrak{b}(\mathbf{k},\mathbf{k};t)|$ monotonically increases. The coupling λ, by contrast, just deforms the ground state but leaves it pure.

The normalization $\mathcal{N}(t)$ is found in a similar way. Using the above solution in (3.49) allows it to be written

$$\begin{aligned}\frac{\partial_t\mathcal{N}}{\mathcal{N}} &= -\frac{i}{2}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\Big[\mathfrak{a}_1(\mathbf{k},\mathbf{k};t)-\mathfrak{a}_2(\mathbf{k},\mathbf{k};t)\Big]\\ &= \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3|\mathbf{k}|}\left[\lambda\cos(|\mathbf{k}|t)\sin(|\mathbf{k}|t) - \frac{\tilde{g}^2}{2\pi\beta}\sin^2(|\mathbf{k}|t)\right].\end{aligned} \tag{3.63}$$

The integral on the right-hand side diverges in the ultraviolet, which we regulate using a momentum cutoff $|\mathbf{k}|<\Lambda$, leading to the result

$$\mathcal{N}(t) = \exp\left(C_0 - \frac{\tilde{g}^2\Lambda^2 t}{16\pi^3\beta} - \frac{\lambda}{16\pi^2 t}\sin(2\Lambda t) + \frac{\tilde{g}^2}{16\pi^3\beta t}\sin^2(\Lambda t)\right)\,, \tag{3.64}$$

where C_0 is the integration constant. In terms of the initial condition $\mathcal{N}(0)=\mathcal{N}_0$, where $\mathcal{N}_0$ is the normalization of the free-vacuum state, we have

$$\mathcal{N}_0 = \exp\left(C_0 - \frac{\lambda\Lambda}{8\pi^2}\right)\,, \tag{3.65}$$

and so

$$\mathcal{N}(t) = \mathcal{N}_0\,\exp\left\{\frac{\Lambda}{8\pi^2}\left[-\frac{\tilde{g}^2\Lambda t}{2\pi\beta}\left(1-\frac{\sin^2(\Lambda t)}{(\Lambda t)^2}\right) + \lambda\left(1-\frac{\sin(2\Lambda t)}{2\Lambda t}\right)\right]\right\}\,. \tag{3.66}$$

The significance of the divergences in the time-dependence of $\mathcal{N}(t)$ is discussed further in §3.4.

3.3 Equal-time ϕ-correlator

This section uses the reduced density matrix for ϕ computed in the previous section to calculate the $\langle\phi\,\phi\rangle$ two-point function. Comparison of the result with the Wightman function given in §2 provides a check on the domain of validity of the Nakajima-Zwanzig late-time evolution.

The correlator evaluated at $t = t'$ is convenient because it has a simple representation in terms of the reduced density matrix. This is because the only evolution operators that appear are the ones that convert between Heisenberg, interaction and Schrödinger pictures:

$$\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\,\phi_H(t,\mathbf{x}')\,\rho_0\Big] = \mathrm{Tr}_+\Big[\phi(t,\mathbf{x})\,\phi(t,\mathbf{x}')\,\sigma(t)\Big] = \mathrm{Tr}_+\Big[\phi_S(\mathbf{x})\,\phi_S(\mathbf{x}')\,\sigma_S(t)\Big]\,, \tag{3.67}$$

where Tr_+ in the second two terms denotes a trace only over the ϕ sector of the Hilbert space.

The calculation based on the field-representation of the Schrödinger-picture reduced density matrix evaluates the trace using a partition of unity written as a functional integral over the field eigenstates,

$$\begin{aligned}\mathrm{Tr}_+\Big[\phi_S(\mathbf{x})\phi_S(\mathbf{x}')\sigma_S(t)\Big] &= \int \mathcal{D}\varphi\ \langle\varphi(\cdot)|\,\phi_S(\mathbf{x})\,\phi_S(\mathbf{x}')\,\sigma_S(t)|\varphi(\cdot)\rangle \\ &= \int \mathcal{D}\varphi\ \varphi(\mathbf{x})\,\varphi(\mathbf{x}')\,\sigma_S[\varphi,\varphi;t]\,,\end{aligned} \tag{3.68}$$

with $\sigma_S[\varphi_1,\varphi_2;t]$ given in terms of the kernels $\mathcal{A}_j$ and $\mathcal{B}$ as in (3.39). Our focus here is in particular on the λ and $\tilde{g}$ dependent parts of the density matrix, since Appendix A.3 verifies that the above functional integral correctly reproduces the free-field Wightman function inasmuch as it shows that in the limit $\lambda = \tilde{g} = 0$ eq. (3.68) reproduces the usual expression

$$\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\,|\mathrm{vac}\rangle = \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2}\,. \tag{3.69}$$

Eq. (3.68) shows that only the diagonal part of the reduced density matrix is required to compute the equal-time Wightman function. Explicitly, this is given by

$$\sigma_S[\varphi,\varphi;t] = \mathcal{N}(t)\exp\bigg(-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{x}'\ \mathcal{M}(\mathbf{x},\mathbf{x}';t)\varphi(\mathbf{x})\varphi(\mathbf{x}')\bigg)\,, \tag{3.70}$$

where

$$\mathcal{M}(\mathbf{x},\mathbf{x}';t) := 2\mathrm{Re}\Big[\mathcal{A}_1(\mathbf{x},\mathbf{x}';t) + \mathcal{B}(\mathbf{x},\mathbf{y};t)\Big]\,, \tag{3.71}$$

and we use the symmetry (3.41). Evaluating the gaussian integrals then implies

$$\begin{aligned}\mathrm{Tr}_+[\phi_S(\mathbf{x})\phi_S(\mathbf{x}')\sigma_S(t)] &= \mathcal{N}(t)\int \mathcal{D}\varphi\ \varphi(\mathbf{x})\varphi(\mathbf{x}')\ e^{-\frac{1}{2}\int \mathrm{d}^3\mathbf{z}_1\int \mathrm{d}^3\mathbf{z}_2\ \mathcal{M}(\mathbf{z}_1,\mathbf{z}_2;t)\varphi(\mathbf{z}_1)\varphi(\mathbf{z}_2)} \\ &= \mathcal{M}^{-1}(\mathbf{x},\mathbf{x}';t)\,,\end{aligned} \tag{3.72}$$

where $\mathcal{M}^{-1}(\mathbf{x},\mathbf{x}';t)$ is the inverse of $\mathcal{M}(\mathbf{x},\mathbf{x}';t)$, in the sense that $\int \mathrm{d}^3\mathbf{z}\ \mathcal{M}^{-1}(\mathbf{x},\mathbf{z};t)\mathcal{M}(\mathbf{z},\mathbf{x}';t) = \delta^3(\mathbf{x}-\mathbf{x}')$. The components $\mathcal{M}^{-1}(\mathbf{x},\mathbf{x}')$ are computed explicitly in Appendix A.2 using (3.71) and the perturbative solutions for $\mathcal{A}_j$ and $\mathcal{B}$ given earlier. Using the result in (3.72) then gives

$$\begin{aligned}\mathrm{Tr}_+[\phi_S(\mathbf{x})\phi_S(\mathbf{x}')\sigma_S(t)] = \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} &- \frac{\lambda}{16\pi^3(|\mathbf{x}|^2-|\mathbf{x}'|^2)}\bigg[\frac{1}{|\mathbf{x}'|}\,\Theta(t-|\mathbf{x}'|) - \frac{1}{|\mathbf{x}|}\,\Theta(t-|\mathbf{x}|)\bigg] \\ &+ \frac{\tilde{g}^2}{32\pi^3\beta|\mathbf{x}||\mathbf{x}'|}\,\Theta(t-|\mathbf{x}|)\,\delta(|\mathbf{x}|-|\mathbf{x}'|)\,.\end{aligned} \tag{3.73}$$

Specializing to the forward light cone of the switch-on of couplings — *i.e.* to $t > |\mathbf{x}|$ and $t > |\mathbf{x}'|$ — the above becomes

$$\mathrm{Tr}_+[\phi_S(\mathbf{x})\phi_S(\mathbf{x}')\sigma_S(t)] \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} - \frac{\lambda}{16\pi^3|\mathbf{x}|\,|\mathbf{x}'|(|\mathbf{x}|+|\mathbf{x}'|)} + \frac{\tilde{g}^2}{32\pi^3\beta|\mathbf{x}||\mathbf{x}'|}\,\delta(|\mathbf{x}|-|\mathbf{x}'|)\,. \tag{3.74}$$

Comparison with Heisenberg picture

Expression (3.74) is to be compared with the Wightman function computed in Heisenberg picture, whose expansion at linear order in λ and $\tilde{g}^2$ is quoted in (2.17) and whose equal-time limit is given in (2.19), reproduced for convenience here:

$$W_\beta(t,\mathbf{x};t,\mathbf{x}') \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|(|\mathbf{x}|+|\mathbf{x}'|)} - \frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(|\mathbf{x}|-|\mathbf{x}'|)\right]} + \frac{\tilde{g}^2}{16\pi^4(|\mathbf{x}|^2-|\mathbf{x}'|^2)^2}\,. \tag{3.75}$$

Although the free and the λ-dependent terms here agree with those in (3.74), the same is in general not true for the $\tilde{g}^2$-dependent terms. This need not be a problem since they should only be expected to agree within the domain of validity of both, and (3.74) is derived under more restrictive assumptions.

Recall in particular that the derivation within the Schrödinger picture approximated the Nakajima-Zwanzig equation (3.22) using a Markovian limit in which $\sigma(t-s)\phi(t-s,\mathbf{0}) \simeq \sigma(t)\phi(t,\mathbf{0})$ is used under the integral sign. Doing this assumes both σ_s and ϕ vary very slowly on the time-scale β for which the thermal χ^a correlator was sharply peaked (see the discussion surrounding eq. (3.23)). This is only valid if the UV cutoff, Λ, for the ϕ-field modes satisfies $\beta\Lambda \ll 1$, since only in this EFT is the field ϕ sufficiently slowly varying.

Within this regime the term involving the hyperbolic function has a microscopic width and so approaches a delta function. To see this explicitly it is easier to work in momentum space, in which case (3.75) is given by the following mode sum

$$W_\beta(t,\mathbf{x};t,\mathbf{x}') \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\sin(p|\mathbf{x}-\mathbf{x}'|) - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; e^{-ip|\mathbf{x}'|}\sin\left[p(|\mathbf{x}|+|\mathbf{x}'|)\right] + \frac{\tilde{g}^2}{64\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; p\left\{\cos\left[p(|\mathbf{x}|+|\mathbf{x}'|)\right] + \frac{2\cos\left[p(|\mathbf{x}|-|\mathbf{x}'|)\right]}{e^{\beta p}-1}\right\}. \tag{3.76}$$

Because $p < \Lambda \ll \beta^{-1}$ we can expand the temperature-dependent last term using $\beta p \ll 1$,

$$\frac{\tilde{g}^2}{64\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; p\left\{\cos\left[p(|\mathbf{x}|+|\mathbf{x}'|)\right] + \left[\frac{2}{\beta p} - 1 + \frac{\beta p}{6} + \mathcal{O}(p^3\beta^3)\right]\cos\left[p(|\mathbf{x}|-|\mathbf{x}'|)\right]\right\}, \tag{3.77}$$

and perform the momentum integrals term-by-term, giving

$$\frac{\tilde{g}^2}{64\pi^4\beta|\mathbf{x}||\mathbf{x}'|}\left\{-\frac{1}{(|\mathbf{x}|+|\mathbf{x}'|)^2} + \left[\frac{2\pi}{\beta}\delta(|\mathbf{x}|-|\mathbf{x}'|) + \frac{1}{(|\mathbf{x}|-|\mathbf{x}'|)^2} - \frac{\beta}{6}\delta''(|\mathbf{x}|-|\mathbf{x}'|) + \dots\right]\right\}. \tag{3.78}$$

Using this in the Wightman function gives

$$W_\beta(t,\mathbf{x};t,\mathbf{x}') \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|\big)} + \frac{\tilde{g}^2}{32\pi^3\beta|\mathbf{x}||\mathbf{x}'|}\,\delta\big(|\mathbf{x}|-|\mathbf{x}'|\big) + \frac{\tilde{g}^2}{16\pi^4\big(|\mathbf{x}|^2-|\mathbf{x}'|^2\big)^2} + \mathcal{O}(\beta)\,. \tag{3.79}$$

Notice that the leading term in this expansion indeed matches the Schrodinger-picture calculation. This comparison reveals more explicitly the long-wavelength domain of validity inherent in the Markovian limit,[5] revealing it to depend on both an expansion in powers of $\tilde{g}^2$ and on taking long wavelengths compared to the thermal length scale β.

3.4 Decoherence

Earlier sections explore the behaviour of the ϕ-field state by computing its reduced density matrix once the hotspot fields χ^a are all integrated out. It was there argued that the kernel $\mathcal{B}(\mathbf{x}, \mathbf{x}'; t)$ — or equivalently $\mathfrak{b}(\mathbf{k}, \mathbf{q}; t)$ — provides a diagnostic of whether the state is pure or mixed (because pure states seem to require $\mathcal{B} = \mathfrak{b} = 0$.)

That is what makes eqs. (3.57) and (3.61) so interesting; they quantify the rate with which decoherence accumulates when hotspot couplings are turned on with ϕ prepared in its vacuum state. These equations show in particular the growth of decoherence at early times has a momentum-independent universal rate, set by $\tilde{g}^2/(2\pi\beta)$. Since all momenta start to decohere with the same rate, the accumulated decoherence should be dominated by the highest momenta, for which there is the most available phase space. It is for this reason that measures of integrated decoherence – such as the purity diagnostic computed explicitly below – tend to diverge in the ultraviolet, at least so long as the cutoff remains below the characteristic hotspot temperature, as assumed for the Markovian approximation above.

3.4.1 Purity

To pin down the decoherence process more precisely it is useful to have a practical diagnostic for state purity. A standard choice for this is often $\mathrm{Tr}_+[\sigma_S^2]$ since this is unity if and only if the state is pure (in which case $\sigma_S^2 = \sigma_S$).

This trace can be computed as a function of the state kernels $\mathcal{A}_j$ and $\mathcal{B}$, as follows:

$$
\begin{aligned}
\mathrm{Tr}[\sigma_S^2] &= \int \mathcal{D}\varphi_1 \int \mathcal{D}\varphi_2 \; \langle\varphi_1|\,\sigma_S(t)\,|\varphi_2\rangle \, \langle\varphi_2|\,\sigma_S(t)\,|\varphi_1\rangle \\
&= \mathcal{N}(t)^2 \int \mathcal{D}\varphi_1 \int \mathcal{D}\varphi_2 \; \exp\left\{ -\frac{1}{2} \int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{y} \left[2\,\mathrm{Re}\Big[\mathcal{A}_1(\mathbf{x},\mathbf{y};t)\Big] \Big(\varphi_1(\mathbf{x})\varphi_1(\mathbf{y}) + \varphi_2(\mathbf{x})\varphi_2(\mathbf{y})\Big) \right.\right. \\
&\qquad\qquad \left.\left. + 4\,\mathrm{Re}\Big[\mathcal{B}(\mathbf{x},\mathbf{y};t)\Big]\,\varphi_1(\mathbf{x})\varphi_2(\mathbf{y}) \right] \right\} \qquad (3.80) \\
&= \mathcal{N}(t)^2 \left\{ \det\left[\frac{1}{2\pi} \begin{pmatrix} 2\,\mathrm{Re}[\mathcal{A}_1(t)] & 2\,\mathrm{Re}[\mathcal{B}(t)] \\ 2\,\mathrm{Re}[\mathcal{B}(t)] & 2\,\mathrm{Re}[\mathcal{A}_1(t)] \end{pmatrix} \right] \right\}^{-1/2} .
\end{aligned}
$$

In the last line we write quantities $\mathcal{B}(\mathbf{x}, \mathbf{y})$ as matrices, with rows and columns labelled by position. In this notation the matrix $\mathrm{Re}[\mathcal{B}(t)]$ is symmetric, since the identity $\mathcal{B}^*(\mathbf{x}, \mathbf{y}; t) = \mathcal{B}(\mathbf{y}, \mathbf{x}; t)$ means

$$
\mathrm{Re}[\mathcal{B}](\mathbf{x},\mathbf{y};t) = \frac{\mathcal{B}(\mathbf{x},\mathbf{y};t) + \mathcal{B}^*(\mathbf{x},\mathbf{y};t)}{2} = \frac{\mathcal{B}(\mathbf{x},\mathbf{y};t) + \mathcal{B}(\mathbf{y},\mathbf{x};t)}{2}\,, \qquad (3.81)
$$

a fact that has been used in writing the last line of (3.80).

This result can be further simplified using the following identity[6] that applies for any two square matrices X and Y:

$$
\det\begin{pmatrix} X & Y \\ Y & X \end{pmatrix} = \det(X - Y)\,\det(X + Y)\,. \qquad (3.82)
$$

[5] At least in the way it is derived here. A Markovian limit with a broader domain of validity might also be possible, such as if only $\sigma_S(t - s)$ is expanded in powers of s without also expanding $\phi(t - s)$. Examples along these lines are seen in some simpler examples involving qubits interacting with fields [28].

[6] To see this, take the determinant of both sides of $\left[\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right]\left[\begin{smallmatrix} X & Y \\ Y & X \end{smallmatrix}\right]\left[\begin{smallmatrix} I & -I \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} X+Y & 0 \\ Y & X-Y \end{smallmatrix}\right]$.

This leads to

$$\mathrm{Tr}[\sigma_S^2] = \mathcal{N}(t)^2 \left\{ \det\left[\frac{1}{\pi}\Big(\mathrm{Re}[\mathcal{A}_1(t)] - \mathrm{Re}[\mathcal{B}(t)]\Big)\right] \det\left[\frac{1}{\pi}\Big(\mathrm{Re}[\mathcal{A}_1(t)] + \mathrm{Re}[\mathcal{B}(t)]\Big)\right]\right\}^{-1/2} \tag{3.83}$$

$$= \frac{\det\left[\frac{1}{\pi}\Big(\mathrm{Re}[\mathcal{A}_1] + \mathcal{B}(t)\Big)\right]}{\sqrt{\det\left[\frac{1}{\pi}\Big(\mathrm{Re}[\mathcal{A}_1(t)] - \mathrm{Re}[\mathcal{B}(t)]\Big)\right] \det\left[\frac{1}{\pi}\Big(\mathrm{Re}[\mathcal{A}_1(t)] + \mathrm{Re}[\mathcal{B}(t)]\Big)\right]}} \tag{3.84}$$

which uses

$$\mathcal{N}(t)^2 = \det\left[\frac{1}{\pi}\Big(\mathrm{Re}[\mathcal{A}_1] + \mathcal{B}(t)\Big)\right] = \det \mathcal{M}(t)\,. \tag{3.85}$$

Notice that this last relation verifies that $\mathcal{B}(\mathbf{x}, \mathbf{y}) = 0$ implies $\mathrm{Tr}[\sigma_S^2] = 1$, so $\mathcal{B} = 0$ is sufficient to ensure that σ_S describes a pure state.

These expressions can be made even more explicit by switching to momentum space and evaluating them using the above perturbative solution near the free vacuum. In this case repeated use of the identity $\det = \exp \mathrm{Tr} \log$ leads to

$$\begin{aligned} \mathrm{Tr}[\sigma_S(t)^2] &= \exp\left[\int \mathrm{d}^3\mathbf{k}\; \frac{\pi}{|\mathbf{k}|} \cdot \frac{2\tilde{g}^2 \mathfrak{b}(\mathbf{k}, \mathbf{k}; t)}{(2\pi)^4} + \mathcal{O}(\tilde{g}^4)\right] \\ &= \exp\left[-\frac{\tilde{g}^2 t}{16\pi^4\beta} \int \frac{\mathrm{d}^3\mathbf{k}}{|\mathbf{k}|} + \mathcal{O}(\tilde{g}^4)\right] \\ &= \exp\left[-\frac{\tilde{g}^2\Lambda^2 t}{8\pi^3\beta} + \mathcal{O}(\tilde{g}^4)\right]\,, \end{aligned} \tag{3.86}$$

revealing the divergence described above. These ultimately arise because the Markovian derivation required $\Lambda \ll 1/\beta$. Presumably the same calculation would not have diverged if Λ could have been taken larger than the temperature, though we have not succeeded yet in capturing this evolution in a fuller calculation.

Notice that the purity starts out at 1 since the initial vacuum state is pure, and then drops monotonically as time passes. A strictly perturbative calculation would only have been able to capture the leading contribution in powers of $\tilde{g}^2$, and so would have given

$$\mathrm{Tr}[\sigma_S(t)^2] \simeq 1 - \frac{\tilde{g}^2\Lambda^2 t}{8\pi^3\beta} + \mathcal{O}(\tilde{g}^4) \qquad \text{(perturbative)}\,, \tag{3.87}$$

with the linear secular growth with t eventually causing the perturbative calculation to break down. It is the deviation of Nakajima-Zwanzig equation from straight-up perturbation theory that allows the resummation of the secularly growing terms to all orders in $\tilde{g}^2 t$ into the exponential form visible in (3.86), along the lines also seen in simpler examples (such as in [27–29, 76]). This allows us to see that it approaches 0 for late times, corresponding to a maximally mixed state (in D-dimensional quantum mechanics, a maximally mixed state has purity $1/D$ and in the present instance $D \to \infty$).

4 Mean-field methods

One of our goals is to explore the nature of nonlocality in open systems, in hopes that this can shed light on whether nonlocality can also arise near horizons for black holes. Nonlocality in this context traditionally means the extent to which the effective action (or Hamiltonian) is not simply the integral of a Hamiltonian density that depends only on fields and their derivatives at a single point (a property normally held by Wilsonian actions as a consequence of cluster decomposition and microcausality [78]).

Locality is a question for which the language of open quantum systems used above is less well adapted because it frames its predictions in terms of the evolution of the reduced density matrix that is obtained after tracing out any dependence on unmeasured sectors of the Hilbert space, as given by the Nakajima-Zwanzig equation [51, 52]. Although entanglement and information exchange ensure this equation naturally involves nonlocality (particularly in time), the absence of an effective Hamiltonian or action in this approach makes its lessons relatively obscure.

To help understand the issues that are at play this section sets up the mean-field approximation for the hotspot system, since the mean-field limit provides a natural definition of an effective hamiltonian (and effective action) that can provide a good approximation to the system's evolution in some circumstances even for open systems [9]. Part of the purpose is to identify the kinds of nonlocality that can emerge for the mean-field Hamiltonian, but at the same time also to identify precisely when the mean-field description is a good approximation to the full dynamics. The hotspot provides a relatively simple test laboratory for exploring these ideas.

4.1 Definitions

The essence of the mean-field approximation is that averages in the unmeasured (or 'environment') part of the open system dominate the fluctuations in this sector, allowing an informative expansion in powers of small deviations from the mean. There is some freedom in how to set up this expansion when working beyond the leading order, which we first summarize before computing how things look specifically in the hotspot example.

We use, as before, the 'double-bracket' notation

$$\langle\!\langle\, \mathcal{O}\,\rangle\!\rangle := \mathrm{Tr}_{-}\Big[\rho_\beta\, \mathcal{O}\Big] \quad \text{where} \quad \rho_\beta := \mathcal{I}_{+}\otimes \varrho_\beta \tag{4.1}$$

with ϱ_β being the thermal state appearing in (2.10) and the partial trace running only over the environmental sector of the Hilbert space (which in the hotspot example is the sector spanned at the initial time by the χ^a fields). For example, for $\mathcal{O} = \sum_n \mathcal{A}_n \otimes \mathcal{B}_n$ expanded in terms of a basis of operators acting in the ϕ and χ^a sectors,

$$\langle\!\langle\, \mathcal{O}\,\rangle\!\rangle = \sum_n \mathcal{A}_n \;\mathrm{Tr}_{-}\Big[\varrho_\beta\, \mathcal{B}_n\Big] = \sum_n \mathcal{A}_n \left\langle \mathcal{B}_n \right\rangle_\beta . \tag{4.2}$$

In particular, $\langle\!\langle\, \mathcal{O}\,\rangle\!\rangle$ is an operator that acts only in the measured sector (*i.e.* the ϕ sector of the hotspot example).

4.1.1 Mean-field evolution

For a perturbative analysis it is worth specializing to the interaction picture, for which expectation values for observables evolve according to

$$\mathcal{A}(t) := \mathrm{Tr}\Big[\rho_S(t)\,\mathcal{O}_S\Big] = \mathrm{Tr}\Big[\rho_I(t)\mathcal{O}_I(t)\Big] = \mathrm{Tr}\Big[\rho_I(0)V^\star(t)\mathcal{O}_I(t)V(t)\Big]\,, \tag{4.3}$$

where $V(t) = U_0^\star(t)U(t)$ is the interaction-picture evolution operator for the state $\rho_I(t)$. In what follows we drop the subscript 'I' for interaction-picture quantities.

For the observables that only measure the ϕ sector the Schrödinger picture operator has the factorized form $\mathcal{O}_S = \mathcal{O}_{S+}\otimes\mathcal{I}_-$. This factorization remains true in the interaction picture provided that the free part of the Hamiltonian does not couple the two sectors to one another. That is, if $H = H_0 + H_{\rm int}$ where $H_0 = \mathcal{H}_+ \otimes \mathcal{I}_- + \mathcal{I}_+ \otimes \mathcal{H}_-$, then

$$U_0(t) := \exp\Big[-iH_0 t\Big] = U_+(t) \otimes U_-(t)\,, \tag{4.4}$$

and so in the interaction picture

$$\mathcal{O}(t) := U_0^\star(t)\, \mathcal{O}_s\, U_0(t) = \mathcal{O}_+(t) \otimes \mathcal{I}_- \tag{4.5}$$

with $\mathcal{O}_+(t) = U_+^\star(t)\, \mathcal{O}_{s+}\, U_+(t)$.

Now comes the main point: in the mean-field approximation the interaction-picture evolution operator is well-approximated within the observable sector by the operator

$$\begin{aligned} \overline{V}(t) \;&:=\; \langle\!\langle\, V(t)\, \rangle\!\rangle \\ &=\; I - i\int_0^t \mathrm{d}\tau\, \langle\!\langle\, H_{\rm int}(\tau)\, \rangle\!\rangle + \frac{1}{2}(-i)^2 \int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2\, \langle\!\langle\, H_{\rm int}(\tau_1)\, H_{\rm int}(\tau_2)\, \rangle\!\rangle + \cdots\,, \end{aligned} \tag{4.6}$$

where $H_{\rm int}(t)$ denotes the usual interaction-picture interaction Hamiltonian $H_{\rm int}(t) = U_0^\star(t) H_{\rm int} U_0(t)$. Writing the full evolution operator as a mean-field part plus the rest (with the rest here called the 'diffuse' evolution, in analogy to optics)

$$V(t) =: \overline{V}(t) \otimes \mathcal{I}_- + \mathcal{V}(t)\,, \tag{4.7}$$

the mean-field approximation is a good one when contributions of the diffuse evolution operator, $\mathcal{V}(t)$ are parametrically small.

The point of defining the mean-field evolution using $\overline{V}(t)$ (as opposed to simply averaging the Hamiltonian, say) is that this ensures that for any observable of the form (4.5) there is no cross-interference – to all orders in perturbation theory – between the mean field evolution and diffuse evolution. That is, computing the expectation of $\mathcal{O}(t)$ using (4.7) gives

$$\mathrm{Tr}\Big[\rho(t)\,\mathcal{O}(t)\Big] = \mathrm{Tr}\Big[V(t)\rho(0)V^\star(t)\,\mathcal{O}(t)\Big] = \mathrm{Tr}_+\Big[\overline{V}(t)\rho_+\overline{V}^\star(t)\,\mathcal{O}_+(t)\Big] + \mathrm{Tr}\Big[\mathcal{V}(t)\rho(0)\,\mathcal{V}^\star(t)\mathcal{O}(t)\Big]\,, \tag{4.8}$$

with the first trace being only over the measured (ϕ) sector, and we assume the uncorrelated initial conditions (2.9) and (2.10). Notice the absence of any cross terms involving both $\overline{V}$ and $\mathcal{V}$, which vanish because (4.6) and (4.7) together imply $\langle\!\langle\, \mathcal{V}(t)\, \rangle\!\rangle = 0$.

4.1.2 Mean-field Hamiltonian

Given the mean-field evolution, the mean-field interaction Hamiltonian, $\overline{H}_{\rm int}(t)$, is defined as the operator that generates $\overline{V}(t)$, or equivalently is related to $\overline{V}(t)$ by the usual iterative expression

$$\overline{V}(t) = I - i\int_0^t \mathrm{d}\tau\, \overline{H}_{\rm int}(\tau) + \frac{1}{2}(-i)^2\int_0^t \mathrm{d}\tau_1 \int_0^{\tau_1} \mathrm{d}\tau_2\, \overline{H}_{\rm int}(\tau_1)\, \overline{H}_{\rm int}(\tau_2) + \cdots\,. \tag{4.9}$$

Comparing this with (4.6) we read off

$$\overline{H}_{\rm int}(t) = \langle\!\langle\, H_{\rm int}(t)\, \rangle\!\rangle - i\int_0^t \mathrm{d}\tau\, \langle\!\langle\, \delta H_{\rm int}(t)\, \delta H_{\rm int}(\tau)\, \rangle\!\rangle + \cdots\,, \tag{4.10}$$

where the ellipses represent terms at least third order in $\delta H_{\rm int}$ and

$$\delta H_{\rm int}(t) := H_{\rm int}(t) - \langle\!\langle\, H_{\rm int}(t)\, \rangle\!\rangle\,. \tag{4.11}$$

Notice that starting at second order in $\delta H_{\rm int}$ the mean-field Hamiltonian need not be hermitian, since $\overline{H}_{\rm int}(t) = \mathfrak{H}(t) + i\,\mathfrak{I}(t)$ with

$$\mathfrak{H}(t) \;=\; \langle\!\langle\, H_{\rm int}(t) \,\rangle\!\rangle - \frac{i}{2}\int_0^t {\rm d}\tau \;\langle\!\langle\, \big[\delta H_{\rm int}(t)\,,\delta H_{\rm int}(\tau)\big] \,\rangle\!\rangle + \cdots$$
$$\text{and}\quad \mathfrak{I}(r) \;=\; -\frac{1}{2}\int_0^t {\rm d}\tau \;\langle\!\langle\, \big\{\delta H_{\rm int}(t)\,,\delta H_{\rm int}(\tau)\big\} \,\rangle\!\rangle + \cdots\,. \tag{4.12}$$

The failure of unitarity for $\overline{V}(t)$ that arises in this way merely reflects the relatively artificial nature of the mean-field/diffuse split, since 'diffuse' interactions with fluctuations in the environment can deplete probability from the mean-field description.

These diffuse interactions are themselves described perturbatively by

$$\mathcal{V}(t) \simeq -i\int_0^t {\rm d}\tau \;\delta H_{\rm int}(\tau) + \cdots\,, \tag{4.13}$$

where ellipses now represent terms second-order in $H_{\rm int}$. (Recall that knowing $\mathcal{V}$ at linear order suffices to compute observables to second order because expressions like (4.8) depend quadratically on $\mathcal{V}$.) It is only the total evolution (with mean-field and diffuse contributions combined) that must be unitary, and this can be expressed as a generalization of the optical theorem, relating the imaginary part of $\overline{H}_{\rm int}$ to the rate of diffuse scattering [9]. This makes the relative size of the imaginary and real parts of $\overline{H}_{\rm int}$ a proxy for the relative importance of diffuse and mean-field evolution.

4.1.3 Domain of validity

Broadly speaking, mean-field descriptions arise as good approximations for real systems in two common ways, depending on whether or not $\langle\!\langle\, H_{\rm int} \,\rangle\!\rangle$ is zero. The simplest case is when $\langle\!\langle\, H_{\rm int} \,\rangle\!\rangle \neq 0$, because then perturbation theory alone can justify the mean-field approximation. This can be seen because the leading (linear) contribution in powers of $H_{\rm int}$ is necessarily a mean-field result (because $\mathcal{V}$ first enters expressions like (4.8) at second order). Neutrino interactions with matter inside the Sun or Earth provide practical examples of this type, for which the mean-field description follows as a consequence of the extreme feebleness of the weak interactions [79, 80].

Things are more subtle when $\langle\!\langle\, H_{\rm int} \,\rangle\!\rangle = 0$, however, because then the leading contribution of $\overline{V}(t)$ to eq. (4.8) arises at the same order in $\delta H_{\rm int}$ as does the diffuse contribution $\mathcal{V}(t)$. This is typically what happens for the interactions of photons with transparent dielectric materials, for example, and in this case perturbation theory in $\delta H_{\rm int}$ itself is insufficient for mean-field methods to dominate.[7] This is also the regime appropriate to the hotspot, and in what follows we identify possible control parameters for mean-field methods by comparing for the hotspot the relative size of the real and imaginary parts of $\overline{H}_{\rm int}$.

4.2 Application to the hotspot

The above is made concrete by specializing to the specific hotspot interactions of the previous sections. Consider first the case of non-negligible hotspot size, ξ, where the underlying coupling has the form given in (2.7):

$$H_{\rm int}(t) = \int_{S_\xi} {\rm d}^2 x\; G_a\, \chi^a(t,\mathbf{x})\,\phi(t,\mathbf{x})\,, \tag{4.14}$$

[7] For photons it is instead a large-N argument based on coherence that justifies mean-field methods (see *e.g.* [9]).

with S_ξ the 2-sphere of radius ξ centred on the origin. For the purposes of the present discusion the coupling $G_\phi\,\phi^2$ can be regarded as part of the unperturbed Hamiltonian because it does not couple the ϕ and χ^a sectors to one another.

4.2.1 Nonlocal mean-field Hamiltonian

Since $\langle \chi^a \rangle = 0$ in the thermal bath the mean of the interaction Hamiltonian vanishes, $\langle\!\langle\, H_{\rm int}(t)\,\rangle\!\rangle = 0$, and so the leading term in (4.10) is the second-order contribution, giving

$$\overline{H}_{\rm int}(t) \simeq -i\widetilde{G}^2 \int_0^t {\rm d}\tau \int_{S_\xi} {\rm d}^2x \int_{S_\xi} {\rm d}^2x'\; \mathscr{W}_\beta(t,\mathbf{x};\tau,\mathbf{x}')\,\phi(t,\mathbf{x})\,\phi(\tau,\mathbf{x}')\,, \tag{4.15}$$

where (as before) $\langle\!\langle\, \chi^a(t,\mathbf{x})\,\chi^b(\tau,\mathbf{x}')\,\rangle\!\rangle = \delta^{ab}\,\mathscr{W}_\beta(t,\mathbf{x};\tau,\mathbf{x}')$ and $\widetilde{G}^2 := G_aG_b\,\delta^{ab} = NG^2$ is the coupling after summing over environmental fields. Evaluating the correlator using (2.20) gives the explicit form

$$\begin{aligned}\mathscr{W}_\beta(t,\mathbf{x};\tau,\mathbf{x}') \;=\; \frac{1}{8\pi\beta|\mathbf{x}-\mathbf{x}'|}\bigg\{&\coth\left[\frac{\pi}{\beta}\,(t-\tau+|\mathbf{x}-\mathbf{x}'|-i\delta)\right] \\ &- \coth\left[\frac{\pi}{\beta}\,(t-\tau-|\mathbf{x}-\mathbf{x}'|-i\delta)\right]\bigg\}\,,\end{aligned} \tag{4.16}$$

Expression (4.15) for the mean-field interaction Hamiltonian is explicitly nonlocal, in two different ways. First, it is nonlocal in space because the correlator $\langle \chi^a(t,\mathbf{x})\,\chi^b(t',\mathbf{x}')\rangle$ has support for arbitrary pairs of points $\mathbf{x}$ and $\mathbf{x}'$ on the localized interaction region $\mathcal{S}_\xi$. No interactions at all (local or nonlocal) arise in $\overline{H}_{\rm int}(t)$ away from $\mathcal{S}_\xi$ as a consequence of the absence of χ^a fields anywhere in $\mathcal{R}_+$ away from the interaction surface. This does not preclude the field ϕ from acquiring nontrivial autocorrelations away from $\mathcal{S}_\xi$ in response to these interactions, however, such as those seen in (2.16). The spatial nonlocality has a relatively simple form in the limit where the times of interest are more widely separated than the light-crossing time for the hotspot itself: $|t-\tau| \geq |\mathbf{x}-\mathbf{x}'| \gg 2\xi$. In this case eq. (4.16) shows that the effective interaction (4.15) becomes

$$\overline{H}_{\rm int}(t) \simeq \frac{i\tilde{g}^2}{4\beta^2}\int_0^t {\rm d}\tau\;\hat{\phi}_{\ell=0}(t,\xi)\,\hat{\phi}_{\ell=0}(\tau,\xi)\;{\rm csch}^2\left[\frac{\pi}{\beta}\,(t-\tau-i\delta)\right]\,, \tag{4.17}$$

which uses the relation (2.4) (*i.e.* $g_a = 4\pi\xi^2 G_a$) and defines the projector onto the $\ell = 0$ spherical harmonic of the field $\phi(\mathbf{x},t)$

$$\hat{\phi}_{\ell=0}(t,\xi) := \frac{1}{4\pi}\int_0^{4\pi} {\rm d}^2\Omega\;\phi(t,r=\xi,\theta,\varphi)\,, \tag{4.18}$$

with the integration ${\rm d}^2\Omega$ being over 4π solid angle. This form, local in angular-momentum space (and so nonlocal in position space), is also seen in other applications [81].

The second source of nonlocality is in time, although this dies off exponentially quickly once $\pi|t-\tau| \gg \beta$. This nonlocality is only consistent with the approximation that led to (4.17) if $\beta \gg \xi$ since otherwise it is impossible to satisfy both $|t-\tau| \lesssim \beta/\pi$ and $|t-\tau| \gg 2\xi$. It should be noticed in this context that in the black hole analogy $\beta = 4\pi\xi$ and so choosing $|t-\tau| \gg 2\xi$ would also imply $|t-\tau| \gg \beta/\pi$.

Both of these sources of nonlocality have their roots in the fluctutation of χ^a and so exist only in regions where the χ^a's have support. This is why all of the nonlocality mentioned above is restricted to the world-tube swept out by the interaction surface $\mathcal{S}_\xi$, from the point of view of an external observer in $\mathcal{R}_+$.

The leading contribution to the diffuse evolution describing deviations from the mean-field limit is given by (4.13), which (because $\langle\!\langle\, H_{\rm int} \,\rangle\!\rangle = 0$ in the present instance), becomes

$$\mathcal{V}(t) \simeq -i \int_0^t {\rm d}\tau \; H_{\rm int}(\tau) = -i \int_0^t {\rm d}\tau \; \int_{S_\xi} {\rm d}^2 x \; G_a \, \chi^a(\tau, \mathbf{x}) \, \phi(\tau, \mathbf{x}) \,. \tag{4.19}$$

As mentioned above, second-order contributions from this generically compete with first-order contributions from (4.15) since both arise at second order in $H_{\rm int}$.

4.2.2 The local limit

The above mean-field interaction simplifies and becomes approximately local in the special case that ξ and β are both microscopic scales, showing how locality ultimately re-emerges in the long-wavelength limit. To examine this limit explicitly we use small ξ to integrate out the hotspot's spatial size, leading to the effective point-like coupling given in (2.8). Using this to compute the mean-field Hamiltonian then gives

$$\overline{H}_{\rm int}(t) = \frac{\lambda}{2} \, \phi^2(t, \mathbf{0}) - i \tilde{g}^2 \phi(t, \mathbf{0}) \int_0^t {\rm d}s \; \mathscr{W}_\beta(s) \, \phi(t - s, \mathbf{0}) \,, \tag{4.20}$$

where (as before) $\langle \chi^a(t, \mathbf{0}) \, \chi^b(t-s, \mathbf{0}) \rangle = \delta^{ab} \, \mathscr{W}_\beta(s)$, with

$$\mathscr{W}_\beta(s) := - \frac{1}{4\beta^2} \; {\rm csch}^2 \left[\frac{\pi}{\beta}(s - i\delta) \right] . \tag{4.21}$$

Two simplifications follow from the observation that $\mathscr{W}_\beta(s)$ is peaked exponentially sharply around $s = 0$, with width of order β/π. First, the upper integration limit can be taken to infinity at the expense of errors that are $\sim e^{-2\pi t/\beta}$ and so are exponentially small in the regime $t \gg \beta/\pi$. Second, for fields varying on scales long compared with β we can expand $\phi(t-s, \mathbf{0}) \simeq \phi(t, \mathbf{0}) - s \, \partial_t \phi(t, \mathbf{0}) + \cdots$ inside the integrand to get

$$\overline{H}_{\rm int}(t) = \frac{\lambda}{2} \, \phi^2(t, \mathbf{0}) + \mathcal{A} \, \phi^2(t, \mathbf{0}) + \mathcal{B} \, \phi \, \partial_t \phi(t, \mathbf{0}) + \dots \,, \tag{4.22}$$

where

$$\mathcal{A} = -i \tilde{g}^2 \int_0^\infty {\rm d}s \; \mathscr{W}_\beta(s) \quad \text{and} \quad \mathcal{B} = i \tilde{g}^2 \int_0^\infty {\rm d}s \; s \, \mathscr{W}_\beta(s) \,, \tag{4.23}$$

and so on.

These reveal the coefficient $\mathcal{A}$ to be a renormalization of the effective coupling λ, while the coefficient $\mathcal{B}$ multiplies a new effective interaction proportional to $\phi \, \partial_t \phi$. This last interaction plays no role in the physics to follow because (depending on the operator ordering) it either involves the commutator of ϕ with its canonical momentum (and so is a divergent contribution to an irrelevant field-independent piece in $\overline{H}_{\rm int}$) or it involves a total time derivative, $\partial_t \phi^2$ (and so can be eliminated using an appropriate canonical transformation).

The integrals giving the coefficients $\mathcal{A}$ and $\mathcal{B}$ can be evaluated explicitly to give

$$\mathcal{A} = - \tilde{g}^2 \left[\frac{1}{4\pi^2 \delta} + \frac{i}{4\pi \beta} \right] + \mathcal{O}(\delta) \quad \text{and} \quad \mathcal{B} = i \tilde{g}^2 \left[\frac{1}{4\pi^2} \log\left(\frac{2\pi\delta}{\beta} \right) - \frac{i}{8\pi} \right] + \mathcal{O}(\delta) \,, \tag{4.24}$$

where the infinitesimal δ is meant to be taken to zero. The divergences arise because $\mathscr{W}_\beta \sim s^{-2}$ as $s \to 0$, and so can be regarded as being ultraviolet in origin. They arise here as divergences when $\delta \to 0$, which just means that this infinitesimal – which was introduced for other reasons – is playing double duty; providing here also a near-hotpsot regularization for the singular integration (which could indeed have been regulated in other ways). To the extent that they contribute to non-redundant interactions these divergences can be renormalized into effective couplings, such as λ, thereby underlining that such self-couplings are generically always present in the effective theory.

What is *not* simply a small change to the effective description, even in the local limit, is the generation of an imaginary part of $\mathcal{A}$ seen in (4.24). This cannot be absorbed into λ without changing the reality properties of λ, and its appearance is a manifestation of the general probability loss away from the mean-field sector into its 'diffuse' complement. Since the relative size of $\tilde{g}^2/\beta$ to λ provides a measure of the relative importance of the real and imaginary parts of this coupling we should also only expect the mean-field description to be a good approximation when $\lambda \gg \tilde{g}^2/\beta$. The above discussion also suggests that if the ϕ self-coupling λ at the hotspot is ultimately induced by the microscopic ϕ–χ coupling, then its natural size is $\lambda \sim \tilde{g}^2/\xi$. If true, this would suggest the mean-field limit should appear to work best in the regime $\xi \ll \beta$.

A more explicit expression for the real part of $\overline{H}_{\rm int}$ is given by

$$
\begin{aligned}
\mathfrak{H}(t) &:= \frac{1}{2}\Big[\overline{H}_{\rm int}(t) + \overline{H}^\star_{\rm int}(t)\Big] \\
&\simeq \frac{\lambda}{2}\,\phi^2(t,\mathbf{0}) - \frac{i\tilde{g}^2}{2}\int_0^t {\rm d}s\;\Big(\mathscr{W}_\beta(s)\,\phi(t,\mathbf{0})\,\phi(t-s,\mathbf{0}) - \mathscr{W}^\ast_\beta(s)\,\phi(t-s,\mathbf{0})\,\phi(t,\mathbf{0})\Big) \\
&= \frac{\lambda}{2}\,\phi^2(t,\mathbf{0}) - \frac{i\tilde{g}^2}{2}\int_0^t {\rm d}s\;{\rm Re}[\mathscr{W}_\beta(s)]\;\Big[\phi(t,\mathbf{0})\,,\phi(t-s,\mathbf{0})\Big] \\
&\qquad\qquad + \frac{\tilde{g}^2}{2}\int_0^t {\rm d}s\;{\rm Im}[\mathscr{W}_\beta(s)]\;\Big\{\phi(t,\mathbf{0})\,,\phi(t-s,\mathbf{0})\Big\} \\
&= \frac{\lambda}{2}\,\phi^2(t,\mathbf{0}) + \frac{\tilde{g}^2}{4\pi}\mathcal{I}_+\int_0^t {\rm d}s\;{\rm Re}[\mathscr{W}_\beta(s)]\;\delta'(s) + \frac{\tilde{g}^2}{8\pi}\int_0^t {\rm d}s\;\delta'(s)\;\Big\{\phi(t,\mathbf{0})\,,\phi(t-s,\mathbf{0})\Big\}
\end{aligned}
\tag{4.25}
$$

where the last equality uses the free-field commutator, $[\phi(t,\mathbf{x})\,,\phi(t',\mathbf{x})] = i\delta'(t-t')\mathcal{I}_+/(2\pi)$, computed in (B.12), and that the imaginary part of the free thermal Wightman function is ${\rm Im}[\mathscr{W}_\beta(t)] = \delta'(t)/(4\pi)$, as computed in (B.27).

Integrating by parts, the above formula becomes (noting that the $\delta(t)$ factors vanish for $t > 0$)

$$
\begin{aligned}
\mathfrak{H}(t) &= \frac{\lambda}{2}\,\phi^2(t,\mathbf{0}) + \frac{\tilde{g}^2}{4\pi}\,\mathcal{I}_+\Big(- {\rm Re}[\mathscr{W}_\beta(0)]\;\delta'(0) - {\rm Re}[\mathscr{W}'_\beta(0)]\Big) \\
&\qquad - \frac{\tilde{g}^2}{8\pi}\Big(\delta(0)\cdot 2\phi^2(t,\mathbf{0}) + \phi(t,\mathbf{0})\partial_t\phi(t,\mathbf{0}) + \partial_t\phi(t,\mathbf{0})\phi(t,\mathbf{0})\Big)\,.
\end{aligned}
\tag{4.26}
$$

For the present purposes we may drop any terms that are proportional to $\mathcal{I}_+$, since these do not contribute to the dynamics because they drop out of commutators with fields. We can also (as always) omit redundant operators like $\phi\,\partial_t\phi$ — see the logic given below equation (4.24) — and after doing so we have

$$
\mathfrak{H}(t) \simeq \frac{1}{2}\left[\lambda - \frac{\tilde{g}^2}{2\pi}\,\delta(0)\right]\,\phi^2(t,\mathbf{0})\,,
\tag{4.27}
$$

showing once more how the real part of the mean-field Hamiltonian serves to renormalize[8] the self-interaction parameter λ, with

$$
\lambda \;\to\; \lambda_{\scriptscriptstyle R} := \lambda - \frac{\tilde{g}^2}{2\pi}\,\delta(0)\,.
\tag{4.28}
$$

[8]Note that the shift shown in (4.27) matches the shift in (4.24), when one interprets $\delta(0) = 1/(\pi\delta)$ (which follows from writing $\frac{1}{x-i\delta} = 1/x + i\pi\delta(x)$.

As usual we henceforth suppress the subscript 'R'.

The imaginary part of $H_{\rm int}(t)$ is similarly given by

$$\begin{aligned}\mathfrak{I}(t) &:= \frac{1}{2i}\Big[\overline{H}_{\rm int}(t) - \overline{H}^{\star}_{\rm int}(t)\Big] \\ &= -\frac{\tilde{g}^2}{2}\int_0^t {\rm d}s\;\Big(\mathscr{W}_\beta(t-s)\,\phi(t,\mathbf{0})\,\phi(s,\mathbf{0}) + \mathscr{W}^*_\beta(t-s)\,\phi(s,\mathbf{0})\,\phi(t,\mathbf{0})\Big)\,. \end{aligned} \tag{4.29}$$

Although this can also be written in terms of commutators and anti-commutators of ϕ, it turns out to be less useful to do so. After renormalizing the self-interaction parameter using (4.28) and combining terms, the complete mean-field Hamiltonian can therefore be written as

$$\begin{aligned}\overline{H}_{\rm int}(t) &= \mathfrak{H}(t) + i\,\mathfrak{I}(t) \\ &= \frac{\lambda}{2}\phi^2(t,\mathbf{0}) - \frac{i\tilde{g}^2}{2}\int_0^t {\rm d}s\;\Big(\mathscr{W}_\beta(t-s)\,\phi(t,\mathbf{0})\,\phi(s,\mathbf{0}) + \mathscr{W}^*_\beta(t-s)\,\phi(s,\mathbf{0})\,\phi(t,\mathbf{0})\Big)\,. \end{aligned} \tag{4.30}$$

4.3 Mean-field ϕ correlation function

The virtue of the hotspot model is that it can be solved exactly, making a comparison with mean-field predictions instructive about the latter's domain of validity. This comparison is most easily made using the $\langle\phi\,\phi\rangle$ correlation function, since this has a known form [23] — given explicitly at late times by (2.17) in the perturbative limit. To make this comparison we now evaluate the $\langle\phi\,\phi\rangle$ two-point function within the mean-field limit.

4.3.1 Mean-field contribution

Keeping in mind that the similarity transformation relating the Heisenberg and interaction pictures is $\phi_H(t,\mathbf{x}) = V^\star(t)\,\phi(t,\mathbf{x})\,V(t)$ where $V(t) = U_0^{-1}(t)\,U(t)$ (as before), the correlation function can be written

$$\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\,\phi_H(t',\mathbf{x}')\rho_H\Big] = \mathrm{Tr}\Big[V^\star(t)\,\phi(t,\mathbf{x})\,V(t)\,V^\star(t')\,\phi(t',\mathbf{x}')\,V(t')\rho(0)\Big]\,, \tag{4.31}$$

in which we also use that the two pictures agree at the initial time, so $\rho_H = \rho(0)$ given by (2.9). The mean-field result is obtained by using in this expression the approximate form

$$V(t) \simeq \overline{V}(t)\otimes\mathcal{I}_- \tag{4.32}$$

with $\overline{V}(t)$ given by (4.6).

With this replacement – and using the initial conditions (2.9) and (2.10) – the mean-field correlation function reduces to an in-in expectation in the observed ϕ sector, of the form

$$\begin{aligned}\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t',\mathbf{x}')\rho_H\Big]_{MF} &:= \mathrm{Tr}_+\Big[\overline{V}^\star(t)\phi(t,\mathbf{x})\overline{V}(t)\overline{V}^\star(t')\phi(t',\mathbf{x}')\overline{V}(t')\,\rho_+\Big] \\ &= \langle{\rm vac}|\,\overline{V}^\star(t)\phi(t,\mathbf{x})\overline{V}(t)\overline{V}^\star(t')\phi(t',\mathbf{x}')\overline{V}(t')\,|{\rm vac}\rangle \end{aligned} \tag{4.33}$$

where the interaction-picture state is evaluated at $t=0$. Evaluating $\overline{H}_{\rm int}$ using (4.20), we have

$$\overline{V}(t) \simeq \mathcal{I}_+ - i\int_0^t {\rm d}\tau\;\overline{H}_{\rm int}(\tau) \simeq \mathcal{I}_+ - i\int_0^t {\rm d}\tau\Big[\mathfrak{H}(\tau) + i\,\mathfrak{I}(\tau)\Big]\,, \tag{4.34}$$

where real and imaginary parts of $\overline{H}_{\rm int}(t)$ are given by (4.30). Inserting (4.34) into (4.33) and working to leading nontrivial order in $\overline{H}_{\rm int}$ then yields the quantity to be evaluated:

$$\begin{aligned} \mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t',\mathbf{x}')\rho_H\Big]_{MF} &\simeq \langle \mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle \\ &+i\int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\Big[\mathfrak{H}(\tau),\phi(t,\mathbf{x})\Big]\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle + i\int_0^{t'}\mathrm{d}\tau\ \langle\mathrm{vac}|\,\phi(t,\mathbf{x})\Big[\mathfrak{H}(\tau),\phi(t',\mathbf{x}')\Big]\,|\mathrm{vac}\rangle \\ &+\int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\Big\{\mathfrak{I}(\tau),\phi(t,\mathbf{x})\Big\}\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle + \int_0^{t'}\mathrm{d}\tau\ \langle\mathrm{vac}|\,\phi(t,\mathbf{x})\Big\{\mathfrak{I}(\tau),\phi(t',\mathbf{x}')\Big\}\,|\mathrm{vac}\rangle\,. \end{aligned} \tag{4.35}$$

4.3.2 Aside: $\overline{V}^\star$ vs $\overline{V}^{-1}$

We pause the main line of development here to settle a side issue that might bother the reader at this point. The issue is this: equation (4.35) is derived by substituting the mean-field expression (4.32) into the general correlator definition (4.31). For the full theory unitarity ensures $V^\star = V^{-1}$ but the same is *not* true for the mean-field limit, since we have seen $\overline{V}^\star \neq \overline{V}^{-1}$. So although nothing changes if we replace $V^\star \to V^{-1}$ in (4.31), making the replacement (4.32) in the result instead leads to

$$\mathcal{U}(t,\mathbf{x};t',\mathbf{x}') := \langle\mathrm{vac}|\,\overline{V}^{-1}(t)\,\phi(t,\mathbf{x})\,\overline{V}(t)\overline{V}^{-1}(t')\,\phi(t',\mathbf{x}')\,\overline{V}(t')\,|\mathrm{vac}\rangle \tag{4.36}$$

which differs from the right-hand side of (4.33). Using (4.36) would change (4.35) by replacing the anticommutators $\{\mathfrak{I},(\cdot)\}$ with commutators $[\mathfrak{I},(\cdot)]$ – an important difference in practice because the commutator is much easier to evaluate (as we do for completeness in Appendix B.1).

Which is right? This is partially a matter of definition, since it hinges on how the full result gets spit into mean-field and diffuse parts. The guiding principle in §4.1 is to make this split so that observables like (4.8) break into a sum of mean-field and diffuse pieces, with no interference terms. The same principle tells us to define the mean-field correlator using (4.33) rather than (4.36). Specialized to $t = t'$ both equations have the same form as (4.8), and it is only for (4.33) that mean-field and diffuse parts cleanly split, because $V^\star = \overline{V}^\star + \mathcal{V}^\star$ divides linearly while V^{-1} does not.

4.3.3 Equal-time Limits

With the mean-field/diffuse split in mind, we next specialize the correlation function to equal times ($t = t'$), so that (4.31) agrees with (4.8) with the choice $\mathcal{O}(t) = \phi(t,\mathbf{x})\phi(t,\mathbf{x}')$. As described above, this ensures the equal-time correlation function nicely splits into the sum of mean-field and diffuse parts

$$\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big] = \mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big]_{MF} + \mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big]_{\rm diff} \tag{4.37}$$

where we define the (equal-time) mean-field correlations as[9]

$$\begin{aligned} \mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big]_{MF} &:= \langle\mathrm{vac}|\,\overline{V}^\star(t)\phi(t,\mathbf{x})\phi(t,\mathbf{y})\overline{V}(t)\,|\mathrm{vac}\rangle \\ &= \langle\mathrm{vac}|\,\phi(t,\mathbf{x})\,\phi(t,\mathbf{x}')\,|\mathrm{vac}\rangle\ + i\int_0^t\mathrm{d}\tau\ \langle\mathrm{vac}|\Big[\mathfrak{H}(\tau),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big]\,|\mathrm{vac}\rangle \\ &\quad + \int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\Big\{\mathfrak{I}(\tau),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big\}\,|\mathrm{vac}\rangle \end{aligned} \tag{4.38}$$

c.f. (4.34), while the diffuse part of the correlations is

$$\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big]_{\rm diff} := \mathrm{Tr}\Big[\mathcal{V}^\star(t)\phi(t,\mathbf{x})\phi(t,\mathbf{y})\mathcal{V}(t)\rho_H\Big]\,. \tag{4.39}$$

[9]Note that this is *not* the same as taking $t = t'$ in the formula (4.34), since $\overline{V}(t)\overline{V}^\star(t) \neq \mathcal{I}_+$.

4.3.4 Equal-time mean-field correlation

This section explicitly evaluates the equal-time mean-field correlation (4.38), ending with a final form that reduces the mode sums to a single integration over explicit elementary functions.

The terms involving the real part $\mathfrak{H}$ of the mean-field Hamiltonian involve commutators are simplest and so all integrals can be evaluated explicitly. The quantity to be evaluated is

$$i\int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\Big[\mathfrak{H}(\tau),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big]|\mathrm{vac}\rangle = \frac{i\lambda}{2}\int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\Big[\phi^2(\tau,\mathbf{0}),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big]|\mathrm{vac}\rangle \tag{4.40}$$

(which is why we could ignore terms in $\mathfrak{H}(t)$ proportional to the $\mathcal{I}_+$ in (4.27)). Using the single-field commutator

$$\Big[\phi(\tau,\mathbf{0}),\phi(t,\mathbf{x})\Big] = \frac{i}{4\pi|\mathbf{x}|}\Big(\delta\big[\tau-(t-|\mathbf{x}|)\big]-\delta\big[\tau-(t+|\mathbf{x}|)\big]\Big)\mathcal{I}_+\ , \tag{4.41}$$

evaluated in Appendix A.3 allows this term to be written

$$\Big[\phi^2(\tau,\mathbf{0}),\phi(t,\mathbf{x})\Big] = \frac{i}{2\pi|\mathbf{x}|}\Big(\delta\big[\tau-(t-|\mathbf{x}|)\big]-\delta\big[\tau-(t+|\mathbf{x}|)\big]\Big)\phi(\tau,\mathbf{0})\ , \tag{4.42}$$

which in turn implies

$$\begin{aligned}\Big[\phi^2(\tau,\mathbf{0}),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big] &= \Big[\phi^2(\tau,\mathbf{0}),\phi(t,\mathbf{x})\Big]\phi(t,\mathbf{x}')+\phi(t,\mathbf{x})\Big[\phi^2(\tau,\mathbf{0}),\phi(t,\mathbf{x}')\Big]\\ &= \frac{i}{2\pi|\mathbf{x}|}\Big(\delta\big[\tau-(t-|\mathbf{x}|)\big]-\delta\big[\tau-(t+|\mathbf{x}|)\big]\Big)\phi(\tau,\mathbf{0})\phi(t,\mathbf{x}')\\ &\quad+\frac{i}{2\pi|\mathbf{x}'|}\Big(\delta\big[\tau-(t-|\mathbf{x}'|)\big]-\delta\big[\tau-(t+|\mathbf{x}'|)\big]\Big)\phi(t,\mathbf{x})\phi(\tau,\mathbf{0})\ .\end{aligned} \tag{4.43}$$

Only δ-functions with singularities at the retarded times $t-|\mathbf{x}|$ and $t-|\mathbf{x}'|$ contribute in the regime of interest, so

$$\begin{aligned}&i\int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\Big[\mathfrak{H}(\tau),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big]|\mathrm{vac}\rangle\\ &= -\frac{\lambda}{4\pi}\int_0^t\mathrm{d}\tau\left(\frac{\delta\big[\tau-(t-|\mathbf{x}|)\big]}{|\mathbf{x}|}\langle\mathrm{vac}|\,\phi(\tau,\mathbf{0})\phi(t,\mathbf{x}')\,|\mathrm{vac}\rangle+\frac{\delta\big[\tau-(t-|\mathbf{x}'|)\big]}{|\mathbf{x}'|}\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(\tau,\mathbf{0})\,|\mathrm{vac}\rangle\right)\\ &= -\frac{\lambda}{4\pi|\mathbf{x}|}\,\Theta(t-|\mathbf{x}|)\,\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|,\mathbf{0})\phi(t,\mathbf{x}')\,|\mathrm{vac}\rangle-\frac{\lambda}{4\pi|\mathbf{x}'|}\,\Theta(t-|\mathbf{x}'|)\,\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t-|\mathbf{x}'|,\mathbf{0})\,|\mathrm{vac}\rangle\\ &= -\frac{\lambda\Theta(t-|\mathbf{x}|)}{16\pi^3|\mathbf{x}|\big(-(-|\mathbf{x}|-i\delta)^2+|\mathbf{x}'|^2\big)}-\frac{\lambda\Theta(t-|\mathbf{x}'|)}{16\pi^3|\mathbf{x}'|\big(-(|\mathbf{x}'|-i\delta)^2+|\mathbf{x}|^2\big)}\end{aligned} \tag{4.44}$$

In the limit that the transients have passed – *i.e.* once $t-|\mathbf{x}|>0$ and $t-|\mathbf{x}'|>0$ – this simplifies to

$$i\int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\Big[\mathfrak{H}(\tau),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big]|\mathrm{vac}\rangle = -\frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|\big)} \tag{4.45}$$

which agrees with the λ-dependent part of equal-time correlation functions computed in (2.19) and (3.74), (evaluated using the renormalized coupling $\lambda=\lambda_R$).

The more complicated contribution involves the imaginary part of $\overline{H}_{\rm int}$, in which we evalaute $\mathfrak{I}$ using (4.29) to get

$$\begin{aligned}
&\int_0^t \mathrm{d}\tau\ \langle \mathrm{vac}| \Big\{ \mathfrak{I}(\tau), \phi(t,\mathbf{x})\phi(t,\mathbf{x}') \Big\} |\mathrm{vac}\rangle \\
&= -\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\ \langle \mathrm{vac}| \left\{ \frac{\tilde{g}^2 \Big(\mathscr{W}_\beta(\tau-\tau')\phi(\tau,\mathbf{0})\phi(\tau',\mathbf{0}) + \mathscr{W}^*_\beta(\tau-\tau')\phi(\tau',\mathbf{0})\phi(\tau,\mathbf{0}) \Big)}{2}, \phi(t,\mathbf{x})\phi(t,\mathbf{x}') \right\} |\mathrm{vac}\rangle \\
&= -\frac{\tilde{g}^2}{2} \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\ \mathscr{W}_\beta(\tau-\tau')\, \langle \mathrm{vac}| \Big\{ \phi(\tau,\mathbf{0})\phi(\tau',\mathbf{0}), \phi(t,\mathbf{x})\phi(t,\mathbf{x}') \Big\} |\mathrm{vac}\rangle\ ,
\end{aligned} \tag{4.46}$$

where the last equality changes integration variables $\tau \leftrightarrow \tau'$ in one of the two terms and uses the property $\mathscr{W}^*_\beta(t) = \mathscr{W}_\beta(-t)$ of the Wightman function. The formula (4.38) therefore takes the final form

$$\begin{aligned}
\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H \Big]_{MF} \simeq\ & \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|\big)} \\
& - \frac{\tilde{g}^2}{2} \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\ \mathscr{W}_\beta(\tau-\tau')\, \langle \mathrm{vac}| \Big\{ \phi(\tau,\mathbf{0})\phi(\tau',\mathbf{0}), \phi(t,\mathbf{x})\phi(t,\mathbf{x}') \Big\} |\mathrm{vac}\rangle
\end{aligned} \tag{4.47}$$

in the regime $t - |\mathbf{x}| > 0$ and $t - |\mathbf{x}'| > 0$ (after transients have passed from the switch-on of couplings at $t = |\mathbf{x}| = 0$).

The matrix element in this last expression is evaluated in Appendix B.2 with the result

$$\begin{aligned}
\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H \Big]_{MF} \simeq\ & \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} \left[1 - \frac{\tilde{g}^2}{4\pi^2} \left(\frac{\zeta(3)t}{\pi\beta^3} - \int_0^\infty \mathrm{d}p\, p\, \frac{\mathrm{d}\mathcal{D}_\beta(p,\delta)}{\mathrm{d}p} \right) \right] \\
& - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|\big)} - \frac{\tilde{g}^2}{16\pi^4|\mathbf{x}||\mathbf{x}'|} \int_0^\infty \mathrm{d}p\ \sin(k|\mathbf{x}|) \int_0^\infty \mathrm{d}k\ \sin(k|\mathbf{x}'|) \\
& \times \left\{ \frac{\mathcal{C}_\beta(p)+\mathcal{C}_\beta(k)}{p+k} \sin\big[(p+k)t\big] - \frac{\mathcal{D}_\beta(p,\delta)+\mathcal{D}_\beta(k,\delta)}{p+k} \Big[1 + \cos\big[(p+k)t\big] \Big] \right\}
\end{aligned} \tag{4.48}$$

where the Riemann-Zeta function evaluates to $\zeta(3) \simeq 1.202$ and the functions $\mathcal{C}_\beta$ and $\mathcal{D}_\beta$ are given by

$$\mathcal{C}_\beta = \frac{p}{4\pi} \coth\left(\frac{\beta p}{2} \right) \quad \text{and} \quad \mathcal{D}_\beta = \frac{p}{2\pi^2} \log\left(\frac{2\pi e^\gamma \delta}{\beta} \right) + \frac{p}{2\pi^2} \mathrm{Re}\left[\psi^{(0)}\left(-i\, \frac{\beta p}{2\pi} \right) \right]\ , \tag{4.49}$$

where $\psi^{(0)}$ is the digamma function defined by $\psi^{(0)}(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log\Gamma(z)$, and where δ is (as usual) to be taken to zero at the end (after renormalization).

Although the $\tilde{g}^2$-independent terms in (4.48) agree with the perturbative and Markovian results (2.19) and (3.74), those that include $\tilde{g}^2$ do not. This difference is due to the contributions of the diffuse evolution first entering at this order, as we now show.

4.3.5 Including diffuse correlations

The above calculation omits the diffuse correlations (4.39), and to the order we work it suffices to use the lowest-order expression (4.13) for $\mathcal{V}$:

$$\mathcal{V}(t) \simeq -i \int_0^t \mathrm{d}\tau\ \delta H_{\rm int}(\tau) = -i \int_0^t \mathrm{d}\tau\ g_a\, \phi(\tau,\mathbf{0}) \otimes \chi^a(\tau,\mathbf{0})\,. \tag{4.50}$$

Using this in (4.39) then gives

$$\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big]_{\mathrm{diff}} \simeq \tilde{g}^2 \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\; \mathscr{W}_\beta(\tau-\tau')\;\langle\mathrm{vac}|\,\phi(\tau,\mathbf{0})\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\phi(\tau',\mathbf{0})\,|\mathrm{vac}\rangle\;. \tag{4.51}$$

We now show that adding (4.51) to the mean-field result (4.47) reproduces the perturbative expression for the full correlator given in (2.19). Summing these mean-field and diffuse contributions gives the result

$$\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big] \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|\big)} + \mathcal{I}_\beta(t,\mathbf{x},\mathbf{x}') \tag{4.52}$$

where the last term is given by the following combination of matrix elements

$$\begin{aligned}\mathcal{I}_\beta(t,\mathbf{x},\mathbf{x}') := &-\frac{\tilde{g}^2}{2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\; \mathscr{W}_\beta(\tau-\tau')\,\langle\mathrm{vac}|\,\Big\{\phi(\tau,\mathbf{0})\phi(\tau',\mathbf{0}),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\Big\}\,|\mathrm{vac}\rangle \\ &+\tilde{g}^2\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\; \mathscr{W}_\beta(\tau-\tau')\;\langle\mathrm{vac}|\,\phi(\tau,\mathbf{0})\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\phi(\tau',\mathbf{0})\,|\mathrm{vac}\rangle\;.\end{aligned} \tag{4.53}$$

These can be usefully (but tediously) re-written as a double commutator plus a remainder,

$$\mathcal{I}_\beta(t,\mathbf{x},\mathbf{x}') \;=\; \mathcal{P}_\beta(t,\mathbf{x},\mathbf{x}') + \mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') \tag{4.54}$$

where the double commutator is

$$\mathcal{P}_\beta(t,\mathbf{x},\mathbf{x}') := -\tilde{g}^2\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\; \mathscr{W}_\beta(\tau-\tau')\,\langle\mathrm{vac}|\,\Big[\phi(\tau,\mathbf{0}),\phi(t,\mathbf{x})\Big]\Big[\phi(\tau',\mathbf{0}),\phi(t,\mathbf{x}')\Big]\,|\mathrm{vac}\rangle \tag{4.55}$$

while the remainder becomes

$$\begin{aligned}\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') := \frac{\tilde{g}^2}{2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\; \mathscr{W}_\beta(\tau-\tau')\,\langle\mathrm{vac}|\,\bigg(&\Big[\phi(\tau,\mathbf{0}),\phi(t,\mathbf{x})\Big]\phi(\tau',\mathbf{0})\phi(t,\mathbf{x}') \\ -\phi(\tau,\mathbf{0})\Big[\phi(\tau',\mathbf{0}),\phi(t,\mathbf{x})\Big]\phi(t,\mathbf{x}') &+ \phi(t,\mathbf{x})\Big[\phi(\tau,\mathbf{0}),\phi(t,\mathbf{x}')\Big]\phi(\tau',\mathbf{0}) \\ -\phi(t,\mathbf{x})\phi(\tau,\mathbf{0})&\Big[\phi(\tau',\mathbf{0}),\phi(t,\mathbf{x}')\Big]\bigg)\,|\mathrm{vac}\rangle\;.\end{aligned} \tag{4.56}$$

These integrals evaluate (see Appendix B.3) in the regime $t-|\mathbf{x}|>0$ and $t-|\mathbf{x}'|>0$ to

$$\mathcal{P}_\beta(t,\mathbf{x},\mathbf{x}') = -\frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\Big[\frac{\pi}{\beta}(-|\mathbf{x}|+|\mathbf{x}'|-i\delta)\Big]}\;, \tag{4.57}$$

and

$$\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') = \frac{\tilde{g}^2}{64\pi^4|\mathbf{x}||\mathbf{x}'|}\bigg[\frac{1}{(|\mathbf{x}|-|\mathbf{x}'|+i\delta)^2} - \frac{1}{(|\mathbf{x}|+|\mathbf{x}'|)^2}\bigg]\;. \tag{4.58}$$

When we use these in (4.52) the overall correlation function is therefore

$$\begin{aligned}\mathrm{Tr}\Big[\phi_H(t,\mathbf{x})\phi_H(t,\mathbf{x}')\rho_H\Big] \;\simeq\; &\frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} - \frac{\lambda}{16\pi^3|\mathbf{x}||\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|\big)} \\ &-\frac{\tilde{g}^2}{64\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\Big[\frac{\pi}{\beta}(-|\mathbf{x}|+|\mathbf{x}'|-i\delta)\Big]} + \frac{\tilde{g}^2}{16\pi^4}\,\frac{1}{(|\mathbf{x}|^2-|\mathbf{x}'|^2)^2}\;.\end{aligned} \tag{4.59}$$

This last expressions agrees perfectly with the perturbative Heisenberg-picture result computed in [23], once this is evaluated in the equal-time limit – see eq. (2.19) – inside the future light-cone of the event at $t=|\mathbf{x}|=0$ where the coupling switches on.

4.3.6 Domain of validity of mean-field methods

We see from these calculations that the mean-field correlator does *not* in general agree with the Heisenberg-picture result, even if this comparison is only made at leading order in λ and $\tilde{g}^2$. The comparison of the previous section shows that the difference between the mean-field and Heisenberg-picture answers is precisely given by the diffuse contribution that must be small for mean-field methods to apply.

The difference between mean-field result (4.48) and the corresponding Heisenberg-picture answer (4.59) lies completely in their $\tilde{g}^2$ dependence; the term involving the self-coupling λ is identical in both cases. Since $\tilde{g}$ and λ both have dimensions of length, another scale must appearin the comparison of λ and $\tilde{g}^2$, and the explicit evaluation – *e.g.* (4.59) – shows this scale to be either β or a combination of $|\mathbf{x}|$ and $|\mathbf{x}'|$.

Because our interest is typically where $|\mathbf{x}|$ and $|\mathbf{x}'|$ are much larger than β the relative size of the λ-dependent term and the largest of $\tilde{g}^2$ corrections is set by the relative size of λ and $\tilde{g}^2/\beta$, suggesting that mean-field methods provide a reliable approximation in the regime $\lambda \gg \tilde{g}^2/\beta$.

In summary, we see that mean-field methods can apply to the hotspot problem, but only in some parts of parameters space such as when $\lambda \gg \tilde{g}^2/\beta$. Where it does apply the resulting effective Hamiltonian can be nonlocal, both in the angular directions of $\mathcal{S}_\xi$ and in time, due to the nonlocality of the χ correlations with which the external ϕ field interacts.

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A Useful intermediate steps

This appendix gathers together many intermediate steps not given in the main text, including the evaluation of several of the integrals encountered there. Our goal is to be as explicit as possible.

A.1 Kernel equations for the gaussian ansatz

This section evaluations the implications of the master equation (3.37) for the gaussian kernels in the ansatz (3.39).

First we compute the time-derivative of the above ansatz

$$\frac{\partial \sigma_S(t,\varphi_1,\varphi_2)}{\partial t} = \bigg[\frac{\partial_t \mathcal{N}(t)}{\mathcal{N}(t)} - \int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{y}\ \bigg\{\frac{1}{2}\partial_t \mathcal{A}_1(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{x})\varphi_1(\mathbf{y}) + \frac{1}{2}\partial_t \mathcal{A}_2(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{x})\varphi_2(\mathbf{y}) + \partial_t \mathcal{B}(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{x})\varphi_2(\mathbf{y})\bigg\}\bigg]\sigma_S(t,\varphi_1,\varphi_2)\ . \tag{A.1}$$

and we note the RHS of equation (3.37)

$$\text{RHS of (3.37)} = -\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\Big[-\frac{\delta^2}{\delta\varphi_1(\mathbf{x})^2} + \big|\boldsymbol{\nabla}\varphi_1(\mathbf{x})\big|^2 + \frac{\delta^2}{\delta\varphi_2(\mathbf{x})^2} - \big|\boldsymbol{\nabla}\varphi_2(\mathbf{x})\big|^2\Big]\sigma_S(t,\varphi_1,\varphi_2) \\ -\frac{\tilde{g}^2}{4\pi\beta}\big(\varphi_1(\mathbf{0})-\varphi_2(\mathbf{0})\big)^2\sigma_S(t,\varphi_1,\varphi_2) - \frac{i\lambda}{2}\big(\varphi_1(\mathbf{0})^2-\varphi_2(\mathbf{0})^2\big)\sigma_S(t,\varphi_1,\varphi_2)\ . \tag{A.2}$$

We first need to compute the functional derivative

$$\frac{\delta\sigma_S[t,\varphi_1,\varphi_2]}{\delta\varphi_1(\mathbf{x})} = \bigg(-\int d^3\mathbf{y}\ \big[\mathcal{A}_1(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{y}) + \mathcal{B}(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{y})\big]\bigg)\,\sigma_S[t,\varphi_1,\varphi_2] \tag{A.3}$$

which assumes the symmetry $\mathcal{A}_j(\mathbf{x},\mathbf{y};t) = \mathcal{A}_j(\mathbf{y},\mathbf{x};t)$. From there we have

$$\begin{aligned}\frac{\delta^2\sigma_S[t,\varphi_1,\varphi_2]}{\delta\varphi_1(\mathbf{x})^2} &= \bigg(-\mathcal{A}_1(\mathbf{x},\mathbf{x};t) + \Big\{\int d^3\mathbf{y}\ \big[\mathcal{A}_1(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{y}) + \mathcal{B}(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{y})\big]\Big\}^2\bigg)\,\sigma_S[t,\varphi_1,\varphi_2] \\ &= \bigg(-\mathcal{A}_1(\mathbf{x},\mathbf{x};t) + \int d^3\mathbf{y}\int d^3\mathbf{z}\ \big[\mathcal{A}_1(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{y}) + \mathcal{B}(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{y})\big] \\ &\qquad \times\big[\mathcal{A}_1(\mathbf{x},\mathbf{z};t)\varphi_1(\mathbf{z}) + \mathcal{B}(\mathbf{x},\mathbf{z};t)\varphi_2(\mathbf{z})\big]\bigg)\sigma_S[t,\varphi_1,\varphi_2]\ .\end{aligned} \tag{A.4}$$

Since $\mathcal{B}$ is *not* symmetric, the other φ_2-derivative differs slightly from (A.4) (note the variable being integrated in $\mathcal{B}$ here) where

$$\begin{aligned}\frac{\delta^2\sigma_S[t,\varphi_1,\varphi_2]}{\delta\varphi_2(\mathbf{x})^2} &= \bigg(-\mathcal{A}_2(\mathbf{x},\mathbf{x};t) + \int d^3\mathbf{y}\int d^3\mathbf{z}\ \big[\mathcal{A}_2(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{y}) + \mathcal{B}(\mathbf{y},\mathbf{x};t)\varphi_1(\mathbf{y})\big] \\ &\qquad \times\big[\mathcal{A}_2(\mathbf{x},\mathbf{z};t)\varphi_2(\mathbf{z}) + \mathcal{B}(\mathbf{z},\mathbf{x};t)\varphi_1(\mathbf{z})\big]\bigg)\sigma_S[t,\varphi_1,\varphi_2]\ ,\end{aligned} \tag{A.5}$$

which implies that

$$\begin{aligned}\frac{\text{RHS of (3.37)}}{\sigma_S(t,\varphi_1,\varphi_2)} &= -\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\Big[\mathcal{A}_1(\mathbf{x},\mathbf{x};t) - \mathcal{A}_2(\mathbf{x},\mathbf{x};t) + |\boldsymbol{\nabla}\varphi_1(\mathbf{x})|^2 - |\boldsymbol{\nabla}\varphi_2(\mathbf{x})|^2\Big] \\ &\quad + \int \mathrm{d}^3\mathbf{x}\int d^3\mathbf{y}\int d^3\mathbf{z}\ \bigg(\frac{i\,\big[\mathcal{A}_1(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{y})+\mathcal{B}(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{y})\big]\big[\mathcal{A}_1(\mathbf{x},\mathbf{z};t)\varphi_1(\mathbf{z})+\mathcal{B}(\mathbf{x},\mathbf{z};t)\varphi_2(\mathbf{z})\big]}{2} \\ &\qquad - \frac{i\big[\mathcal{A}_2(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{y})+\mathcal{B}(\mathbf{y},\mathbf{x};t)\varphi_1(\mathbf{y})\big]\big[\mathcal{A}_2(\mathbf{x},\mathbf{z};t)\varphi_2(\mathbf{z})+\mathcal{B}(\mathbf{z},\mathbf{x};t)\varphi_1(\mathbf{z})\big]}{2}\bigg) \\ &\quad + \bigg(-\frac{\tilde{g}^2}{4\pi\beta} - \frac{i\lambda}{2}\bigg)\varphi_1(\mathbf{0})^2 + \frac{\tilde{g}^2}{2\pi\beta}\varphi_1(\mathbf{0})\varphi_2(\mathbf{0}) + \bigg(-\frac{\tilde{g}^2}{4\pi\beta} + \frac{i\lambda}{2}\bigg)\varphi_2(\mathbf{0})^2\ .\end{aligned} \tag{A.6}$$

We next need to collect the terms that are proportional to the various possible powers of $\varphi_1(\mathbf{x})$ and $\varphi_2(\mathbf{x})$ and so on (note that we re-label some integration variables here):

$$\begin{aligned}\frac{\text{RHS of (3.37)}}{\sigma_S(t,\varphi_1,\varphi_2)} &= -\bigg(\frac{\tilde{g}^2}{4\pi\beta} + \frac{i\lambda}{2}\bigg)\varphi_1(\mathbf{0})^2 + \frac{\tilde{g}^2}{2\pi\beta}\varphi_1(\mathbf{0})\varphi_2(\mathbf{0}) - \bigg(\frac{\tilde{g}^2}{4\pi\beta} - \frac{i\lambda}{2}\bigg)\varphi_2(\mathbf{0})^2 \\ &\quad -\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\Big[\mathcal{A}_1(\mathbf{x},\mathbf{x};t) - \mathcal{A}_2(\mathbf{x},\mathbf{x};t)\Big] - \frac{i}{2}\int \mathrm{d}^3\mathbf{x}\ \Big(|\boldsymbol{\nabla}\varphi_1(\mathbf{x})|^2 - |\boldsymbol{\nabla}\varphi_2(\mathbf{x})|^2\Big) \\ &\quad +\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\int \mathrm{d}^3\mathbf{z}\ \big[\mathcal{A}_1(\mathbf{z},\mathbf{x};t)\mathcal{A}_1(\mathbf{z},\mathbf{y};t) - \mathcal{B}(\mathbf{x},\mathbf{z};t)\mathcal{B}(\mathbf{y},\mathbf{z};t)\big]\varphi_1(\mathbf{x})\varphi_1(\mathbf{y}) \\ &\quad +i\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\int \mathrm{d}^3\mathbf{z}\ \big[\mathcal{A}_1(\mathbf{z},\mathbf{x};t)\mathcal{B}(\mathbf{z},\mathbf{y};t) - \mathcal{B}(\mathbf{x},\mathbf{z};t)\mathcal{A}_2(\mathbf{z},\mathbf{y};t)\big]\varphi_1(\mathbf{x})\varphi_2(\mathbf{y}) \\ &\quad +\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\int \mathrm{d}^3\mathbf{z}\ \big[-\mathcal{A}_2(\mathbf{x},\mathbf{z};t)\mathcal{A}_2(\mathbf{x},\mathbf{z};t) + \mathcal{B}(\mathbf{z},\mathbf{x};t)\mathcal{B}(\mathbf{z},\mathbf{y};t)\big]\varphi_2(\mathbf{x})\varphi_2(\mathbf{y})\end{aligned} \tag{A.7}$$

Note that the above is equal to the quantity (with the time-derivative we computed above)

$$\frac{\text{LHS of (3.37)}}{\sigma_s(t,\varphi_1,\varphi_2)} = \frac{\partial_t \mathcal{N}(t)}{\mathcal{N}(t)} + \int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{y} \,\bigg\{ -\frac{1}{2}\partial_t \mathcal{A}_1(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{x})\varphi_1(\mathbf{y}) \\ -\frac{1}{2}\partial_t \mathcal{A}_2(\mathbf{x},\mathbf{y};t)\varphi_2(\mathbf{x})\varphi_2(\mathbf{y}) - \partial_t \mathcal{B}(\mathbf{x},\mathbf{y};t)\varphi_1(\mathbf{x})\varphi_2(\mathbf{y})\bigg\} \tag{A.8}$$

and so we need to get the RHS into this form. We then use integration by parts (twice) to write

$$\begin{aligned} \int \mathrm{d}^3\mathbf{x}\, |\boldsymbol{\nabla}\varphi_1(\mathbf{x})|^2 &= -\int \mathrm{d}^3\mathbf{x}\; \varphi_1(\mathbf{x})\nabla^2_{\mathbf{x}}\varphi_1(\mathbf{x}) \\ &= -\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\; \delta^3(\mathbf{x}-\mathbf{y})\varphi_1(\mathbf{y})\nabla^2_{\mathbf{x}}\varphi_1(\mathbf{x}) \\ &= -\int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\; \varphi_1(\mathbf{x})\varphi_1(\mathbf{y})\; \nabla^2_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}) \end{aligned} \tag{A.9}$$

which gives

$$\begin{aligned} \frac{\text{RHS of (3.37)}}{\sigma_s(t,\varphi_1,\varphi_2)} &= -\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\Big[\mathcal{A}_1(\mathbf{x},\mathbf{x};t) - \mathcal{A}_2(\mathbf{x},\mathbf{x};t)\Big] \\ &+ \int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\;\varphi_1(\mathbf{x})\varphi_1(\mathbf{y})\bigg\{\frac{i}{2}\nabla^2_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}) - \bigg(\frac{\tilde{g}^2}{4\pi\beta} + \frac{i\lambda}{2}\bigg)\delta^3(\mathbf{x})\delta^3(\mathbf{y}) \\ &\qquad + \frac{i}{2}\int \mathrm{d}^3\mathbf{z}\;\big[\mathcal{A}_1(\mathbf{z},\mathbf{x};t)\mathcal{A}_1(\mathbf{z},\mathbf{y};t) - \mathcal{B}(\mathbf{x},\mathbf{z};t)\mathcal{B}(\mathbf{y},\mathbf{z};t)\big]\bigg\} \\ &+ \int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\;\varphi_2(\mathbf{x})\varphi_2(\mathbf{y})\bigg\{-\frac{i}{2}\nabla^2_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}) - \bigg(\frac{\tilde{g}^2}{4\pi\beta} - \frac{i\lambda}{2}\bigg)\delta^3(\mathbf{x})\delta^3(\mathbf{y}) \\ &\qquad + \frac{i}{2}\int \mathrm{d}^3\mathbf{z}\;\big[-\mathcal{A}_2(\mathbf{x},\mathbf{z};t)\mathcal{A}_2(\mathbf{y},\mathbf{z};t) + \mathcal{B}(\mathbf{z},\mathbf{x};t)\mathcal{B}(\mathbf{z},\mathbf{y};t)\big]\bigg\} \\ &+ \int \mathrm{d}^3\mathbf{x}\int \mathrm{d}^3\mathbf{y}\;\varphi_1(\mathbf{x})\varphi_2(\mathbf{y})\bigg[\frac{\tilde{g}^2}{2\pi\beta}\delta^3(\mathbf{x})\delta^3(\mathbf{y}) + i\int \mathrm{d}^3\mathbf{z}\;\big[\mathcal{A}_1(\mathbf{z},\mathbf{x};t)\mathcal{B}(\mathbf{z},\mathbf{y};t) - \mathcal{B}(\mathbf{x},\mathbf{z};t)\mathcal{A}_2(\mathbf{z},\mathbf{y};t)\big]\bigg]\,. \end{aligned} \tag{A.10}$$

Setting LHS = RHS gives four equations. The constant piece gives

$$\frac{1}{\mathcal{N}(t)}\frac{\partial \mathcal{N}}{\partial t} = -\frac{i}{2}\int \mathrm{d}^3\mathbf{x}\Big[\mathcal{A}_1(\mathbf{x},\mathbf{x};t) - \mathcal{A}_2(\mathbf{x},\mathbf{x};t)\Big]\,, \tag{A.11}$$

while coefficient of $\varphi_1(\mathbf{x})\varphi_1(\mathbf{y})$ gives

$$\begin{aligned} \frac{\partial \mathcal{A}_1(\mathbf{x},\mathbf{y};t)}{\partial t} &= -i\nabla^2_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}) + \bigg(\frac{\tilde{g}^2}{2\pi\beta} + i\lambda\bigg)\,\delta^3(\mathbf{x})\delta^3(\mathbf{y}) \\ &\quad + \int \mathrm{d}^3\mathbf{z}\;\big[-i\mathcal{A}_1(\mathbf{z},\mathbf{x};t)\mathcal{A}_1(\mathbf{z},\mathbf{y};t) + i\mathcal{B}(\mathbf{x},\mathbf{z};t)\mathcal{B}(\mathbf{y},\mathbf{z};t)\big]\,, \end{aligned} \tag{A.12}$$

the coefficient of $\varphi_2(\mathbf{x})\varphi_2(\mathbf{y})$ gives (the first term in the integral has used the symmetry of $\mathcal{A}_2$)

$$\begin{aligned} \frac{\partial \mathcal{A}_2(\mathbf{x},\mathbf{y};t)}{\partial t} &= i\nabla^2_{\mathbf{x}}\delta^3(\mathbf{x}-\mathbf{y}) + \bigg(\frac{\tilde{g}^2}{2\pi\beta} - i\lambda\bigg)\,\delta^3(\mathbf{x})\delta^3(\mathbf{y}) \\ &\quad + \int \mathrm{d}^3\mathbf{z}\;\big[i\mathcal{A}_2(\mathbf{x},\mathbf{z};t)\mathcal{A}_2(\mathbf{y},\mathbf{z};t) - i\mathcal{B}(\mathbf{z},\mathbf{x};t)\mathcal{B}(\mathbf{z},\mathbf{y};t)\big]\,, \end{aligned} \tag{A.13}$$

and the coefficient of $\varphi_1(\mathbf{x})\varphi_2(\mathbf{y})$ gives

$$\frac{\partial \mathcal{B}(\mathbf{x},\mathbf{y};t)}{\partial t} = -\frac{\tilde{g}^2}{2\pi\beta}\delta^3(\mathbf{x})\delta^3(\mathbf{y}) + \int \mathrm{d}^3\mathbf{z}\;\big[-i\mathcal{A}_1(\mathbf{z},\mathbf{x};t)\mathcal{B}(\mathbf{z},\mathbf{y};t) + i\mathcal{B}(\mathbf{x},\mathbf{z};t)\mathcal{A}_2(\mathbf{z},\mathbf{y};t)\big]\,. \tag{A.14}$$

A.2 The Calculation of $\mathcal{M}^{-1}(\mathbf{x},\mathbf{x}';t)$

We here compute the matrix $\mathcal{M}^{-1}(\mathbf{x},\mathbf{x}';t)$ appearing in the correlator (3.72). We do so by going to momentum space and perturbing in the interactions. Defining the momentum-space version of $\mathcal{M}$ using the expression

$$\mathcal{M}(\mathbf{x},\mathbf{x}';t) := \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\; e^{+i\mathbf{k}\cdot\mathbf{x}} M(\mathbf{k},\mathbf{q};t) e^{-i\mathbf{q}\cdot\mathbf{x}'} \tag{A.15}$$

for which the inverse-matrix condition

$$\int \mathrm{d}^3\mathbf{z}\; \mathcal{M}^{-1}(\mathbf{x},\mathbf{z};t)\mathcal{M}(\mathbf{z},\mathbf{x}';t) = \delta^3(\mathbf{x}-\mathbf{x}') \tag{A.16}$$

takes the form

$$\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\; M^{-1}(\mathbf{k},\mathbf{p};t) M(\mathbf{p},\mathbf{q};t) = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{q})\;, \tag{A.17}$$

where $M^{-1}(\mathbf{k},\mathbf{q};t)$ denotes the momentum-space components of $\mathcal{M}^{-1}(\mathbf{x},\mathbf{x}';t)$.

To solve for $M^{-1}(\mathbf{k},\mathbf{q};t)$ we perturb about the free-vacuum solution, writing

$$\begin{aligned} M(\mathbf{p},\mathbf{q};t) &= 2(2\pi)^3|\mathbf{p}|\delta^3(\mathbf{p}-\mathbf{q}) + \mathfrak{m}(\mathbf{p},\mathbf{q};t) \\ M^{-1}(\mathbf{k},\mathbf{q};t) &= \frac{(2\pi)^3}{2|\mathbf{k}|}\delta^3(\mathbf{k}-\mathbf{q}) + \mathfrak{i}(\mathbf{k},\mathbf{q};t)\,, \end{aligned} \tag{A.18}$$

where in both lines the first term is just the free-field result — see Appendix A.3 — and the second term is the perturbation that is to be solved to linear order in $\tilde{g}^2$ and λ. Inserting these into the relation (A.17) gives at linear order

$$\frac{1}{2|\mathbf{k}|}\cdot \mathfrak{m}(\mathbf{k},\mathbf{q};t) + \mathfrak{i}(\mathbf{k},\mathbf{q};t)\cdot 2|\mathbf{q}| \;\simeq\; 0 \tag{A.19}$$

and so $\mathfrak{i}(\mathbf{k},\mathbf{q};t) \simeq -\mathfrak{m}(\mathbf{k},\mathbf{q};t)/(4\,|\mathbf{k}|\,|\mathbf{q}|)$. Using expression (3.71) giving $\mathcal{M}$ in terms of $\mathcal{A}_j$ and $\mathcal{B}$, together with the solutions (3.59) through (3.61), then implies

$$\mathfrak{i}(\mathbf{k},\mathbf{q};t) = -\frac{1}{2|\mathbf{k}|\;|\mathbf{q}|}\mathrm{Re}\left[\left(\lambda - \frac{i\tilde{g}^2}{2\pi\beta}\right)\frac{1-e^{-i\left(|\mathbf{k}|+|\mathbf{q}|\right)t}}{|\mathbf{k}|+|\mathbf{q}|}\right] - \frac{1}{2|\mathbf{k}|\;|\mathbf{q}|}\mathrm{Re}\left[\frac{i\tilde{g}^2}{2\pi\beta}\cdot\frac{1-e^{-i\left(|\mathbf{k}|-|\mathbf{q}|\right)t}}{|\mathbf{k}|-|\mathbf{q}|}\right]\,. \tag{A.20}$$

The desired position-space inverse is now found by Fourier transforming:

$$\begin{aligned} \mathcal{M}^{-1}(\mathbf{x},\mathbf{x}';t) &= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\; e^{+i\mathbf{k}\cdot\mathbf{x}} M^{-1}(\mathbf{k},\mathbf{q};t)\, e^{-i\mathbf{q}\cdot\mathbf{x}'} \\ &= \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2} + \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\; e^{+i\mathbf{k}\cdot\mathbf{x}}\, \mathfrak{i}(\mathbf{k},\mathbf{q};t)\, e^{-i\mathbf{q}\cdot\mathbf{x}'} \end{aligned} \tag{A.21}$$

where the first term is the free result computed in §A.3. The angular integrals are simple because $\mathfrak{i}(\mathbf{k},\mathbf{q};t)$ depends only on $|\mathbf{k}|$ and $|\mathbf{q}|$, and so

$$\begin{aligned} &\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\; e^{+i\mathbf{k}\cdot\mathbf{x}}\, \mathfrak{i}(\mathbf{k},\mathbf{q};t)\, e^{-i\mathbf{q}\cdot\mathbf{x}'} \\ &\quad= \frac{1}{4\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}q\, \Bigg(-\frac{1}{2kq}\mathrm{Re}\left[\left(\lambda - \frac{i\tilde{g}^2}{2\pi\beta}\right)\frac{1-e^{-i(k+q)t}}{k+q}\right] \\ &\qquad\qquad -\frac{1}{2kq}\mathrm{Re}\left[\frac{i\tilde{g}^2}{2\pi\beta}\cdot\frac{1-e^{-i(k-q)t}}{k-q}\right]\Bigg) k\sin(k|\mathbf{x}|)q\sin(q|\mathbf{x}'|) \\ &\quad= -\frac{\lambda}{8\pi^4|\mathbf{x}||\mathbf{x}'|}\;\mathrm{Re}\big[I_1(\mathbf{x},\mathbf{x}',t)\big] + \frac{\tilde{g}^2}{16\pi^5\beta|\mathbf{x}||\mathbf{x}'|}\;I_2(\mathbf{x},\mathbf{x}',t) \end{aligned} \tag{A.22}$$

where we define the integrals

$$I_1(\mathbf{x},\mathbf{x}',t) := \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}q \; \frac{1-e^{-i(k+q)t}}{k+q} \; \sin(k|\mathbf{x}|)\sin(q|\mathbf{x}'|) \tag{A.23}$$

and

$$I_2(\mathbf{x},\mathbf{x}',t) := \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}q \; \bigg(- \frac{\sin\big[(k+q)t\big]}{k+q} + \frac{\sin\big[(k-q)t\big]}{k-q} \bigg) \sin(k|\mathbf{x}|)\sin(q|\mathbf{x}'|) \; . \tag{A.24}$$

To compute I_1 we use the Schwinger parametrization trick, which uses the identity

$$\frac{1}{p} = \int_0^\infty \mathrm{d}\zeta \; e^{-p\zeta} \,, \tag{A.25}$$

for any parameter $p > 0$ to rewrite the factor of $(k+q)^{-1}$ in the integrand. This gives

$$\begin{aligned} I_1(\mathbf{x},\mathbf{x}',t) &= \int_0^\infty \mathrm{d}\zeta \int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}q \; (1-e^{-i(k+q)t})\sin(k|\mathbf{x}|)\sin(q|\mathbf{x}'|)e^{-(k+q)\zeta} \\ &= \int_0^\infty \mathrm{d}\zeta \; \bigg\{ \int_0^\infty \mathrm{d}k \; e^{-k\zeta} \sin(k|\mathbf{x}|) \bigg\} \bigg\{ \int_0^\infty \mathrm{d}q \; e^{-q\zeta} \sin(q|\mathbf{x}'|) \bigg\} \\ &\qquad - \int_0^\infty \mathrm{d}\zeta \; \bigg\{ \int_0^\infty \mathrm{d}k \; e^{-k(\zeta+it)} \sin(|\mathbf{x}|k) \bigg\} \bigg\{ \int_0^\infty \mathrm{d}q \; e^{-q(\zeta+it)} \sin(|\mathbf{x}'|q) \bigg\} \\ &= \int_0^\infty \mathrm{d}\zeta \; \bigg[\bigg\{ \frac{|\mathbf{x}|}{\zeta^2+|\mathbf{x}|^2} \bigg\} \bigg\{ \frac{|\mathbf{x}'|}{\zeta^2+|\mathbf{x}'|^2} \bigg\} - \bigg\{ \frac{|\mathbf{x}|}{(\zeta+it)^2+|\mathbf{x}|^2} \bigg\} \bigg\{ \frac{|\mathbf{x}'|}{(\zeta+it)^2+|\mathbf{x}'|^2} \bigg\} \bigg] \end{aligned} \tag{A.26}$$

leaving an elementary integral over ζ. Performing this integral we find that I_1 evaluates to

$$I_1(\mathbf{x},\mathbf{x}',t) = \frac{|\mathbf{x}||\mathbf{x}'|}{|\mathbf{x}|^2-|\mathbf{x}'|^2} \bigg(\frac{i}{|\mathbf{x}'|} \log\bigg|\frac{1+t/|\mathbf{x}'|}{1-t/|\mathbf{x}'|}\bigg| + \frac{\pi\Theta(t-|\mathbf{x}'|)}{2|\mathbf{x}'|} - \frac{i}{|\mathbf{x}|} \log\bigg|\frac{1+t/|\mathbf{x}|}{1-t/|\mathbf{x}|}\bigg| - \frac{\pi\Theta(t-|\mathbf{x}|)}{2|\mathbf{x}|} \bigg) \,. \tag{A.27}$$

Only the real part of this expression

$$\mathrm{Re}\big[I_1(\mathbf{x},\mathbf{x}',t)\big] = \frac{|\mathbf{x}||\mathbf{x}'|}{|\mathbf{x}|^2-|\mathbf{x}'|^2} \bigg[\frac{\pi}{2|\mathbf{x}'|}\,\Theta(t-|\mathbf{x}'|) - \frac{\pi}{2|\mathbf{x}|}\,\Theta(t-|\mathbf{x}|) \bigg] \tag{A.28}$$

appears in (A.22).

To compute I_2 it proves easier to first differentiate with respect to t, leading to

$$\begin{aligned} \frac{\partial I_2(\mathbf{x},\mathbf{x}',t)}{\partial t} &= 2\int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}q \; \sin(tk)\sin(tq)\sin(|\mathbf{x}|k)\sin(|\mathbf{x}'|q) \\ &= \frac{1}{2} \int_0^\infty \mathrm{d}k \; \Big(\cos\big[(t-|\mathbf{x}|)k\big] - \cos\big[(t+|\mathbf{x}|)k\big] \Big) \\ &\qquad \times \int_0^\infty \mathrm{d}q \; \Big(\cos\big[(t-|\mathbf{x}'|)q\big] - \cos\big[(t+|\mathbf{x}'|)q\big] \Big) \\ &= \frac{\pi^2}{2} \Big(\delta(t-|\mathbf{x}|) - \delta(t+|\mathbf{x}|) \Big) \Big(\delta(t-|\mathbf{x}'|) - \delta(t+|\mathbf{x}'|) \Big) \end{aligned} \tag{A.29}$$

where the last line uses the real part of the Fourier transform of a Heaviside step function. Since $\delta(t-|\mathbf{x}|) = \delta(t-|\mathbf{x}'|) = 0$ for $t > 0$, $|\mathbf{x}| > 0$ and $|\mathbf{x}'| > 0$ this simplifies to

$$\frac{\partial I_2(\mathbf{x},\mathbf{x}',t)}{\partial t} = \frac{\pi^2}{2}\delta(t-|\mathbf{x}|)\delta(t-|\mathbf{x}'|) \;=\; \frac{\pi^2}{2}\delta(t-|\mathbf{x}|)\delta(|\mathbf{x}|-|\mathbf{x}'|)\,. \tag{A.30}$$

Integrating with respect to t with the initial condition $I_2(0, \mathbf{x}, \mathbf{x}') = 0$ (from the definition (A.24)) then gives

$$I_2(\mathbf{x}, \mathbf{x}', t) = \frac{\pi^2}{2}\Theta(t - |\mathbf{x}|)\delta(|\mathbf{x}| - |\mathbf{x}'|) \ . \tag{A.31}$$

Putting everything together gives

$$\begin{aligned} \mathcal{M}^{-1}(\mathbf{x}, \mathbf{x}'; t) &= \frac{1}{4\pi^2|\mathbf{x} - \mathbf{x}'|^2} - \frac{\lambda}{8\pi^4|\mathbf{x}||\mathbf{x}'|} \,\mathrm{Re}\big[I_1(\mathbf{x}, \mathbf{x}', t)\big] + \frac{\tilde{g}^2}{16\pi^5\beta|\mathbf{x}||\mathbf{x}'|} \, I_2(\mathbf{x}, \mathbf{x}', t) \\ &= \frac{1}{4\pi^2|\mathbf{x} - \mathbf{x}'|^2} - \frac{\lambda}{16\pi^3(|\mathbf{x}|^2 - |\mathbf{x}'|^2)}\left[\frac{1}{|\mathbf{x}'|}\,\Theta(t - |\mathbf{x}'|) - \frac{1}{|\mathbf{x}|}\,\Theta(t - |\mathbf{x}|)\right] \\ &\qquad + \frac{\tilde{g}^2}{32\pi^3\beta|\mathbf{x}||\mathbf{x}'|}\,\Theta(t - |\mathbf{x}|)\delta(|\mathbf{x}| - |\mathbf{x}'|) \,. \end{aligned} \tag{A.32}$$

This is the expression quoted in (3.73) of the main text.

A.3 Schrödinger-picture equal-time free-field correlator

In this Appendix we compute the $\langle \phi \, \phi \rangle$ correlator for free fields at equal times $t = t'$, as a check on Schrödinger picture methods. Using the field basis and the vacuum wave-functional the equal-time Wightman function is given by the functional integral

$$\begin{aligned} \langle \mathrm{vac} |\, \phi_S(\mathbf{x})\phi_S(\mathbf{x}') \,| \mathrm{vac} \rangle &= \int \mathcal{D}[\varphi(\cdot)] \;\; \langle \varphi(\cdot) |\, \phi_S(\mathbf{x})\phi_S(\mathbf{x}') \,| \mathrm{vac} \rangle \, \langle \mathrm{vac} | \varphi(\cdot) \rangle \\ &= \int \mathcal{D}[\varphi(\cdot)] \;\, \varphi(\mathbf{x})\varphi(\mathbf{x}')\langle \varphi(\cdot) | \mathrm{vac} \rangle \langle \mathrm{vac} | \varphi \rangle \\ &= \mathcal{N}_0 \int \mathcal{D}\varphi \;\, \varphi(\mathbf{x})\varphi(\mathbf{x}') \; e^{-\frac{1}{2}\int \mathrm{d}^3\mathbf{x} \int \mathrm{d}^3\mathbf{y} \; 2\mathcal{E}(\mathbf{x} - \mathbf{y})\varphi(\mathbf{x})\varphi(\mathbf{y})} \\ &= [2\mathcal{E}]^{-1}(\mathbf{x} - \mathbf{x}') \end{aligned} \tag{A.33}$$

where we use the free Gaussian solution in the second-last line, where $\mathcal{E}(\mathbf{x} - \mathbf{x}') = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} |\mathbf{k}| e^{i(\mathbf{x} - \mathbf{x}')\cdot\mathbf{k}}$ given in (3.33). In order to perform the Gaussian integral we use the standard gaussian results

$$\int_{-\infty}^{\infty} \prod \mathrm{d}\xi_r \; \xi_{r_1}\xi_{r_2} e^{-\frac{1}{2}\sum_{r,s} K_{rs}\xi_r\xi_s} \; = \; \det\left(\frac{K}{2\pi}\right)^{-1/2} (K^{-1})_{r_1 r_2} \tag{A.34}$$

$$\int_{-\infty}^{\infty} \prod \mathrm{d}\xi_r \; e^{-\frac{1}{2}\sum_{r,s} K_{rs}\xi_r\xi_s} \; = \; \det\left(\frac{K}{2\pi}\right)^{-1/2} \tag{A.35}$$

The latter formula determines $\mathcal{N}_0 = \sqrt{\det(\frac{2\mathcal{E}}{2\pi})}$. Now we need to invert the "matrix" $2\mathcal{E}$ here, where the matrix $[2\mathcal{E}]^{-1}$ is defined by

$$\int \mathrm{d}^3\mathbf{z} \; [2\mathcal{E}]^{-1}(\mathbf{x} - \mathbf{z}) 2\mathcal{E}(\mathbf{z} - \mathbf{x}') \;\; = \;\; \delta^3(\mathbf{x} - \mathbf{x}') \ , \tag{A.36}$$

We solve the above in Fourier space, by writing $[2\mathcal{E}]^{-1}(\mathbf{x} - \mathbf{x}') = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \mathfrak{I}_{\mathbf{k}} e^{i(\mathbf{x} - \mathbf{x}')\cdot\mathbf{k}}$ for some function $\mathfrak{I}_{\mathbf{k}}$ in momentum space which we solve for here. The above equation then implies

$$2|\mathbf{k}|\mathfrak{I}_{\mathbf{k}} \;\; = \;\; 1 \ , \tag{A.37}$$

which means that $\mathfrak{I}_{\mathbf{k}} = (2|\mathbf{k}|)^{-1}$. We Fourier transform this to position space to find that

$$[2\mathcal{E}]^{-1}(\mathbf{x}-\mathbf{x}') = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2|\mathbf{k}|}\right]e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}} = \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty \mathrm{d}k\ \sin(k|\mathbf{x}-\mathbf{x}'|) = -\frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\cdot \mathrm{Im}\left[\int_{-\infty}^{\infty}\mathrm{d}k\ \Theta(k)e^{-i|\mathbf{x}-\mathbf{x}'|k}\right] \tag{A.38}$$

From here we use Weinberg's formula (6.2.15) in [78] for the Fourier transform of a Heaviside step function, where (in the limit $\delta \to 0^+$)

$$\Theta(x) = -i\int_{-\infty}^{\infty}\frac{\mathrm{d}y}{2\pi}\cdot\frac{e^{+iyx}}{y-i\delta} \qquad \Longleftrightarrow \qquad \int_{-\infty}^{\infty}\mathrm{d}x\ \Theta(x)e^{-iyx} = \frac{-i}{y-i\delta} \tag{A.39}$$

which gives us

$$[2\mathcal{E}]^{-1}(\mathbf{x}-\mathbf{x}') = -\frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\left(-\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right)\ , \tag{A.40}$$

which tells us that the free correlation function is

$$\langle \mathrm{vac}|\,\phi_S(\mathbf{x})\phi_S(\mathbf{x}')\,|\mathrm{vac}\rangle\ =\ +\frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2}\ , \tag{A.41}$$

which is the correct answer for the Wightman function (for equal times $t = t'$).

B Mean-field details

This appendix collects various intermediate steps encountered in the mean-field calculations of §4.

B.1 Correlators using $\overline{V}^{-1}$

We first compute the mean-field correlator of eq. (4.36) that would have been obtained if the transition to mean field methods had been done using V^{-1} rather than $V^\star$ in (4.31).

Starting with (4.36) leads to the following expression at leading nontrivial order in $\overline{H}_{\mathrm{int}}$:

$$\mathscr{U}(t,\mathbf{x};t',\mathbf{x}') \simeq \langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle + i\int_0^t \mathrm{d}\tau\ \langle\mathrm{vac}|\left[\overline{H}_{\mathrm{int}}(\tau),\phi(t,\mathbf{x})\right]\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle + i\int_0^{t'}\mathrm{d}\tau\ \langle\mathrm{vac}|\,\phi(t,\mathbf{x})\left[\overline{H}_{\mathrm{int}}(\tau),\phi(t',\mathbf{x}')\right]|\mathrm{vac}\rangle \tag{B.1}$$

where only terms linear in $\overline{H}_{\mathrm{int}}$ are kept. The first term in (B.1) is simply the free Wightman function

$$\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle = \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]}\ , \tag{B.2}$$

while the commutator in the subleading terms is evaluated in Appendix §B.1.2, giving

$$\begin{aligned}\left[\overline{H}_{\mathrm{int}}(\tau),\phi(t,\mathbf{x})\right] \ =\ & \frac{i\lambda}{4\pi|\mathbf{x}|}\Big[\delta\big(\tau-(t-|\mathbf{x}|)\big)-\delta\big(\tau-(t+|\mathbf{x}|)\big)\Big]\phi(\tau,\mathbf{0})\\ &+\frac{\tilde{g}^2}{4\pi|\mathbf{x}|}\Big[\Theta(t-|\mathbf{x}|)\Theta\big(\tau-[t-|\mathbf{x}|]\big)\mathscr{W}_\beta\big(\tau-[t-|\mathbf{x}|]\big)\\ &\qquad -\Theta\big(\tau-[t+\mathbf{x}]\big)\mathscr{W}_\beta\big(\tau-[t+\mathbf{x}]\big)\Big]\phi(\tau,\mathbf{0})\\ &+\frac{\tilde{g}^2}{4\pi|\mathbf{x}|}\Big[\delta\big(\tau-(t-|\mathbf{x}|)\big)-\delta\big(\tau-(t+|\mathbf{x}|)\big)\Big]\int_0^\tau\mathrm{d}\tau'\ \mathscr{W}_\beta(\tau')\phi(\tau-\tau',\mathbf{0})\,.\end{aligned} \tag{B.3}$$

Using this in (B.1) we get, after some manipulations,

$$
\begin{aligned}
i \int_0^t \mathrm{d}\tau \ \langle\mathrm{vac}| \Big[\overline{H}_{\rm int}(\tau), \phi(t,\mathbf{x})\Big]\phi(s,\mathbf{x}')\,|\mathrm{vac}\rangle &= -\frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\,\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|,\mathbf{0})\phi(s,\mathbf{x}')\,|\mathrm{vac}\rangle \\
&+\frac{i\tilde{g}^2\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\int_0^{|\mathbf{x}|}\mathrm{d}\tau'\ \mathscr{W}_\beta\big(\tau'\big)\,\langle\mathrm{vac}|\,\phi(\tau'+t-|\mathbf{x}|,\mathbf{0})\phi(s,\mathbf{x}')\,|\mathrm{vac}\rangle \\
&+\frac{i\tilde{g}^2\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\int_0^{t-|\mathbf{x}|}\mathrm{d}\tau'\ \mathscr{W}_\beta(\tau')\,\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|-\tau',\mathbf{0})\phi(s,\mathbf{x}')\,|\mathrm{vac}\rangle
\end{aligned}
\tag{B.4}
$$

and so the mean-field correlation function becomes

$$
\begin{aligned}
\mathcal{U}(t,\mathbf{x};t',\mathbf{x}') \simeq\ & \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]} \\
&-\frac{\lambda\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\,\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle - \frac{\lambda\Theta(t'-|\mathbf{x}'|)}{4\pi|\mathbf{x}'|}\,\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|,\mathbf{0})\,|\mathrm{vac}\rangle \\
&+\frac{i\tilde{g}^2\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\int_0^{|\mathbf{x}|}\mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\,\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|+\tau,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle \\
&+\frac{i\tilde{g}^2\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\int_0^{t-|\mathbf{x}|}\mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\,\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|-\tau,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle \\
&+\frac{i\tilde{g}^2\Theta(t'-|\mathbf{x}'|)}{4\pi|\mathbf{x}'|}\int_0^{|\mathbf{x}'|}\mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\,\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|+\tau,\mathbf{0})\,|\mathrm{vac}\rangle \\
&+\frac{i\tilde{g}^2\Theta(t'-|\mathbf{x}'|)}{4\pi|\mathbf{x}'|}\int_0^{t'-|\mathbf{x}'|}\mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\,\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|-\tau,\mathbf{0})\,|\mathrm{vac}\rangle\ .
\end{aligned}
\tag{B.5}
$$

This expression simplifies further in the regime where all of $t-|\mathbf{x}|$, $|\mathbf{x}|$, $t'-|\mathbf{x}'|$ and $|\mathbf{x}'|$ are much greater than β, because in this case the narrowness of the Wightman function — $\mathscr{W}_\beta(\tau) \propto e^{-2\pi\tau/\beta}$ for $\tau \gg \beta$ — makes it a good approximation to approximate the upper integration limits by ∞ (with only exponentially small error). Under these assumptions we also know $\Theta(t-|\mathbf{x}|) = \Theta(t'-|\mathbf{x}'|) = 1$ and so get

$$
\begin{aligned}
\mathcal{U}(t,\mathbf{x};t',\mathbf{x}')\ \simeq\ & \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]} \\
&-\frac{\lambda}{4\pi}\left[\frac{\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle}{|\mathbf{x}|}+\frac{\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|,\mathbf{0})\,|\mathrm{vac}\rangle}{|\mathbf{x}'|}\right] \\
&+\frac{i\tilde{g}^2}{4\pi|\mathbf{x}|}\int_0^\infty\mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\bigg[\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|+\tau,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle+\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|-\tau,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle\bigg] \\
&+\frac{i\tilde{g}^2}{4\pi|\mathbf{x}'|}\int_0^\infty\mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\bigg[\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|+\tau,\mathbf{0})\,|\mathrm{vac}\rangle+\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|-\tau,\mathbf{0})\,|\mathrm{vac}\rangle\bigg]\ .
\end{aligned}
\tag{B.6}
$$

The integrals involving $\mathscr{W}_\beta(\tau)$ above are computed in Appendix B.1.3. These contain a divergence from the $\tau \to 0$ limit, but this has the same structure as does the second line of (B.6) so the divergence can be absorbed into λ. Once this is done, and using the explicit form (B.2) for the free Wightman function, we find

$$\begin{aligned}\mathcal{U}(t,\mathbf{x};t',\mathbf{x}') \simeq\ & \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]} \\ &+\frac{\lambda}{32\pi^3|\mathbf{x}||\mathbf{x}'|}\left[-\frac{1}{t-t'+|\mathbf{x}|+|\mathbf{x}'|-i\delta}+\frac{1}{t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta}\right] \\ &-\frac{\tilde{g}^2}{128\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(t-t'+|\mathbf{x}|+|\mathbf{x}'|-i\delta)\right]}-\frac{i\tilde{g}^2}{64\pi^4\beta^2|\mathbf{x}||\mathbf{x}'|}\,\mathrm{Im}\;\psi^{(1)}\left[1+\frac{i(t-t'+|\mathbf{x}|+|\mathbf{x}'|)}{\beta}\right] \\ &+\frac{\tilde{g}^2}{128\pi^2\beta^2|\mathbf{x}||\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta)\right]}+\frac{i\tilde{g}^2}{64\pi^4\beta^2|\mathbf{x}||\mathbf{x}'|}\,\mathrm{Im}\;\psi^{(1)}\left[1+\frac{i(t-t'-|\mathbf{x}|-|\mathbf{x}'|)}{\beta}\right]\ ,\end{aligned} \tag{B.7}$$

where $\psi^{(1)}(z) := (\mathrm{d}/\mathrm{d}z)^2\ln\Gamma(z)$ is the polygamma function of order 1. Recall that the derivation of (B.7) assumes $|\mathbf{x}|\ , t-|\mathbf{x}|\ , |\mathbf{x}'|\ , t'-|\mathbf{x}'| \gg \beta$.

The equal-time limit of eq. (B.7) is given by

$$\mathcal{U}(t,\mathbf{x};t,\mathbf{x}') \simeq \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|^2}-\left(\lambda-\frac{i\tilde{g}^2}{2\pi\beta}\right)\frac{1}{16\pi^3|\mathbf{x}||\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|\big)} \tag{B.8}$$

where the assumption $|\mathbf{x}|+|\mathbf{x}'| \gg \beta$ used in the above derivation allows use of the large-z approximation $\mathrm{Im}[\psi^{(1)}(1+iz)] \simeq -1/z$. Notice that the $\tilde{g}^2$ term in this expression does not satisfy the hermiticity condition $\mathcal{U}^*(t,\mathbf{x};t,\mathbf{x}') = \mathcal{U}(t,\mathbf{x}';t,\mathbf{x})$ because the density matrix is evaluated at $t=0$ and the $\tilde{g}^2$ contributions ensure the effective mean-field hamiltonian that evolves to general t is also not hermitian.

We next evaluate the commutators required in the above, starting with the unequal-time commutator of the field itself.

B.1.1 Field commutators at unequal times

For later use in Appendix B.1.2, we here compute the commutator $[\phi(t,\mathbf{x}),\phi(t',\mathbf{y})]$ of interaction-picture fields at unequal times. This can be done using the standard field expansion in terms of creation and annihilation operators, but it is simpler to obtain it directly from the Wightman function given in (B.2). This can be done because the commutator of two free fields is a c-number, and so is equal to its expectation value in the vacuum, giving

$$\begin{aligned}\Big[\phi(t,\mathbf{x}),\phi(t',\mathbf{x}')\Big] &= \Big(\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle-\langle\mathrm{vac}|\,\phi(t',\mathbf{x}')\phi(t,\mathbf{x})\,|\mathrm{vac}\rangle\Big) \\ &= \frac{1}{4\pi^2}\left[\frac{1}{-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2}-\frac{1}{-(t-t'+i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2}\right] \\ &= \frac{1}{8\pi^2|\mathbf{x}-\mathbf{x}'|}\left[\frac{1}{(t-t')+|\mathbf{x}-\mathbf{x}'|-i\delta}-\frac{1}{(t-t')-|\mathbf{x}-\mathbf{x}'|-i\delta}\right. \\ &\qquad\left.-\frac{1}{(t-t')+|\mathbf{x}-\mathbf{x}'|+i\delta}+\frac{1}{(t-t')-|\mathbf{x}-\mathbf{x}'|+i\delta}\right]\end{aligned} \tag{B.9}$$

where a factor of the unit operator, $\mathcal{I}_+$, is implicit everywhere on the right-hand side.

Using the Sochocki-Plemelj identity $(z-i0^+)^{-1}-(z+i0^+)^{-1} = 2i\pi\delta(z)$ for infinitesimal and positive 0^+, the above becomes

$$\Big[\phi(t,\mathbf{x}),\phi(t',\mathbf{x}')\Big] = \frac{i}{4\pi|\mathbf{x}-\mathbf{x}'|}\Big[\delta\big(t-t'+|\mathbf{x}-\mathbf{x}'|\big)-\delta\big(t-t'-|\mathbf{x}-\mathbf{x}'|\big)\Big]\mathcal{I}_+\,, \tag{B.10}$$

which reduces when $t = t'$ to the standard equal-time commutator when $\mathbf{x} \neq \mathbf{x}'$:

$$\Big[\phi(t,\mathbf{x}), \phi(t,\mathbf{x}')\Big] = 0 \ . \tag{B.11}$$

Specializing (B.9) to the commutator for two fields at different times but with $\mathbf{x} = \mathbf{x}'$, instead gives

$$\Big[\phi(t,\mathbf{x}), \phi(t',\mathbf{x})\Big] = \frac{1}{4\pi^2}\left[-\frac{1}{(t-t'-i\delta)^2} + \frac{1}{(t-t'+i\delta)^2}\right]\mathcal{I}_+ = \frac{i}{2\pi}\,\delta'(t-t')\,\mathcal{I}_+ \tag{B.12}$$

where the limit $\delta \to 0$ is taken in the last equality and $\delta'(x)$ denotes the derivative of the Dirac delta function with respect to its argument.

B.1.2 The Commutator $[\overline{H}_{\rm int}(\tau), \phi(t,\mathbf{x})]$

The next intermediate step required is the commutator with ϕ of the local mean-field Hamiltonian $\overline{H}_{\rm int}(t)$ defined in (4.20):

$$\Big[\overline{H}_{\rm int}(\tau), \phi(t,\mathbf{x})\Big] = \frac{1}{2}\lambda\left[\phi^2(\tau,\mathbf{0}), \phi(t,\mathbf{x})\right] - i\tilde{g}^2\int_0^\tau {\rm d}s\, \mathscr{W}_\beta(\tau')\Big[\phi(\tau,\mathbf{0})\phi(\tau-s,\mathbf{0}), \phi(t,\mathbf{x})\Big] \ . \tag{B.13}$$

Using

$$\Big[\phi^2(\tau,\mathbf{0}), \phi(t,\mathbf{x})\Big] = -\phi(\tau,\mathbf{0})\Big[\phi(t,\mathbf{x}), \phi(\tau,\mathbf{0})\Big] - \Big[\phi(t,\mathbf{x}), \phi(\tau,\mathbf{0})\Big]\phi(\tau,\mathbf{0}) \tag{B.14}$$

with the result (B.10) we have

$$\Big[\phi^2(\tau,\mathbf{0}), \phi(t,\mathbf{x})\Big] = \frac{i}{2\pi|\mathbf{x}|}\Big[\delta\big(\tau-(t-|\mathbf{x}|)\big) - \delta\big(\tau-(t+|\mathbf{x}|)\big)\Big]\phi(\tau,\mathbf{0}) \ . \tag{B.15}$$

Similarly

$$\begin{aligned}\Big[\phi(\tau,\mathbf{0})\phi(\tau-s,\mathbf{0}), \phi(t,\mathbf{x})\Big] \;=\; & \frac{i}{4\pi|\mathbf{x}|}\Big[\delta\big(s-[\tau-(t-|\mathbf{x}|)]\big) - \delta\big(s-[\tau-(t+|\mathbf{x}|)]\big)\Big]\phi(\tau,\mathbf{0}) \\ & + \frac{i}{4\pi|\mathbf{x}|}\Big[\delta\big(\tau-(t-|\mathbf{x}|)\big) - \delta\big(\tau-(t+|\mathbf{x}|)\big)\Big]\phi(\tau-s,\mathbf{0})\end{aligned} \tag{B.16}$$

and so

$$\begin{aligned}[\overline{H}_{\rm int}(\tau), \phi(t,\mathbf{x})] \;=\; & \frac{i\lambda}{4\pi|\mathbf{x}|}\Big[\delta\big(\tau-(t-|\mathbf{x}|)\big) - \delta\big(\tau-(t+|\mathbf{x}|)\big)\Big]\phi(\tau,\mathbf{0}) \\ & + \frac{\tilde{g}^2}{4\pi|\mathbf{x}|}\,\phi(\tau,\mathbf{0})\int_0^\tau {\rm d}s\, \mathscr{W}_\beta(s)\Big[\delta\big(s-[\tau-(t-|\mathbf{x}|)]\big) - \delta\big(s-[\tau-(t+|\mathbf{x}|)]\big)\Big] \\ & + \frac{\tilde{g}^2}{4\pi|\mathbf{x}|}\Big[\delta\big(\tau-(t-|\mathbf{x}|)\big) - \delta\big(\tau-(t+|\mathbf{x}|)\big)\Big]\int_0^\tau {\rm d}s\, \mathscr{W}_\beta(s)\,\phi(\tau-s,\mathbf{0}) \ .\end{aligned} \tag{B.17}$$

Performing the s-integrals using the delta functions gives

$$\int_0^\tau {\rm d}s\, \mathscr{W}_\beta(s)\,\delta\big(s-[\tau-(t-|\mathbf{x}|)]\big) = \Theta(t-|\mathbf{x}|)\Theta\big(\tau-[t-|\mathbf{x}|]\big)\mathscr{W}_\beta\big(\tau-[t-|\mathbf{x}|]\big) \ , \tag{B.18}$$

where the step functions express the conditions under which the delta function has support within the integration range: $0 < \tau-(t-|\mathbf{x}|) < \tau$, which in turn implies $\tau > t-|\mathbf{x}| > 0$. Similarly

$$\int_0^\tau {\rm d}s\, \mathscr{W}_\beta(s)\delta\big(s-[\tau-(t+|\mathbf{x}|)]\big) = \Theta\big(\tau-[t+\mathbf{x}]\big)\mathscr{W}_\beta\big(\tau-[t+\mathbf{x}]\big) \, . \tag{B.19}$$

Putting the above terms together gives the result (B.3).

B.1.3 The $\mathscr{W}_\beta$ Integrals

This section computes the integrals

$$\mathcal{J}_1(t,\mathbf{x},t',\mathbf{x}') := \int_0^\infty \mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\bigg[\langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|+\tau,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle + \langle\mathrm{vac}|\,\phi(t-|\mathbf{x}|-\tau,\mathbf{0})\phi(t,\mathbf{x}')\,|\mathrm{vac}\rangle \bigg]\,, \tag{B.20}$$

and

$$\mathcal{J}_2(t,\mathbf{x},t',\mathbf{x}') := \int_0^\infty \mathrm{d}\tau\ \mathscr{W}_\beta(\tau)\bigg[\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|+\tau,\mathbf{0})\,|\mathrm{vac}\rangle + \langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|-\tau,\mathbf{0})\,|\mathrm{vac}\rangle \bigg]\,, \tag{B.21}$$

which appear in eq. (B.6) above. Writing the free Wightman function (B.2) as a mode sum yields

$$\begin{aligned}
\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle &= \frac{1}{4\pi^2\big[-(t-t'-i\delta)^2+|\mathbf{x}-\mathbf{x}'|^2\big]} \\
&= \frac{1}{8\pi^2|\mathbf{x}-\mathbf{x}'|}\bigg[\frac{1}{t-t'+|\mathbf{x}-\mathbf{x}'|-i\delta} - \frac{1}{t-t'-|\mathbf{x}-\mathbf{x}'|-i\delta}\bigg] \\
&= \frac{i}{8\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty \mathrm{d}p\ \bigg[e^{-ip(t-t'+|\mathbf{x}-\mathbf{x}'|-i\delta)} - \int_0^\infty \mathrm{d}p\ e^{-ip(t-t'-|\mathbf{x}-\mathbf{x}'|-i\delta)}\bigg] \\
&= \frac{1}{4\pi^2|\mathbf{x}-\mathbf{x}'|}\int_0^\infty \mathrm{d}p\ e^{-ip(t-t'-i\delta)}\sin\big(p|\mathbf{x}-\mathbf{x}'|\big)\,,
\end{aligned} \tag{B.22}$$

which allows (B.20) to be written as

$$\begin{aligned}
\mathcal{J}_1(t,\mathbf{x},t,\mathbf{x}') &= \int_0^\infty \mathrm{d}\tau\ \frac{\mathscr{W}_\beta(\tau)}{4\pi^2|\mathbf{x}'|}\int_0^\infty \mathrm{d}p\ \bigg[e^{-ip(t-|\mathbf{x}|+\tau-t'-i\delta)}\sin(p|\mathbf{x}'|) + e^{-ip(t-|\mathbf{x}|-\tau-t'-i\delta)}\sin(p|\mathbf{x}'|)\bigg] \\
&= \frac{1}{4\pi^2|\mathbf{x}'|}\int_0^\infty \mathrm{d}p\ e^{-ip(t-|\mathbf{x}|-t'-i\delta)}\sin(p|\mathbf{x}'|)\Big[\mathcal{C}_\beta(p) + i\mathcal{K}_\beta\Big]
\end{aligned} \tag{B.23}$$

with the definitions

$$\mathcal{C}_\beta(p) := 2\int_0^\infty \mathrm{d}\tau\ \mathrm{Re}\,[\mathscr{W}_\beta(\tau)]\cos(p\tau) \qquad \text{and} \qquad \mathcal{K}_\beta := 2\int_0^\infty \mathrm{d}\tau\ \mathrm{Im}\,[\mathscr{W}_\beta(\tau)]\cos(p\tau)\,. \tag{B.24}$$

The first integral $\mathcal{C}_\beta$ was computed in [27, 29] and gives

$$\mathcal{C}_\beta(p) = \frac{p}{4\pi}\coth\left(\frac{\beta p}{2}\right)\,. \tag{B.25}$$

Meanwhile to compute $\mathcal{K}_\beta$ (which turns out to be divergent as well as p-independent), note that the imaginary part of $\mathscr{W}_\beta$ actually vanishes if τ is fixed but nonzero as $\delta \to 0$, since it can be written as

$$\mathrm{Im}\,[\mathscr{W}_\beta(\tau)] = \frac{i}{8\beta^2}\left\{\frac{1}{\sinh^2\left[\frac{\pi}{\beta}(\tau-i\delta)\right]} - \frac{1}{\sinh^2\left[\frac{\pi}{\beta}(\tau+i\delta)\right]}\right\} \tag{B.26}$$

and so the complete contribution comes only from the regime near $\tau \to 0$, for which

$$\mathrm{Im}\,[\mathscr{W}_\beta(\tau)] \simeq \frac{i}{8\pi^2}\left[\frac{1}{(\tau-i\delta)^2} - \frac{1}{(\tau+i\delta)^2}\right] \to \frac{1}{4\pi}\,\delta'(\tau)\,, \tag{B.27}$$

as $\delta \to 0$, which follows from the the Sochocki-Plemelj identity. The required integral then is

$$\mathcal{K}_\beta = 2\int_0^\infty \mathrm{d}\tau\ \frac{\delta'(\tau)}{4\pi}\cos(p\tau) = -\frac{1}{2\pi}\,\delta(0) \tag{B.28}$$

which displays a divergence that ultimately gets absorbed into the coupling parameter λ.

Combining the above into (B.23) yields

$$\mathscr{J}_1(t,\mathbf{x},t',\mathbf{x}') = \frac{1}{4\pi^2|\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; e^{-ip(t-|\mathbf{x}|-t'-i\delta)}\sin(p|\mathbf{x}'|)\left[\frac{p}{4\pi}\coth\left(\frac{\beta p}{2}\right) - \frac{i}{2\pi}\,\delta(0)\right] \tag{B.29}$$

The divergent term simplifies using (B.22), leading to

$$\begin{aligned}\mathscr{J}_1(t,\mathbf{x},t',\mathbf{x}') \;=\; & -\frac{i}{2\pi}\,\delta(0)\,\langle \mathrm{vac}|\,\phi(t-|\mathbf{x}|,\mathbf{0})\phi(t,\mathbf{x}')\,|\mathrm{vac}\rangle \\ & -\frac{i}{32\pi^3|\mathbf{x}'|}\int_0^\infty \mathrm{d}p\; p\coth\left(\frac{\beta p}{2}\right)\left[e^{-ip(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)} - e^{-ip(t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta)}\right]\end{aligned} \tag{B.30}$$

which can be integrated using

$$\int_0^\infty \mathrm{d}p\; p\coth\left(\frac{\beta p}{2}\right)e^{-ip\tau-i\delta} = -\frac{\pi^2}{\beta^2\sinh^2\left[\frac{\pi}{\beta}(\tau-i\delta)\right]} - \frac{2i}{\beta^2}\,\mathrm{Im}\left[\psi^{(1)}\left(1+\frac{i\tau}{\beta}\right)\right], \tag{B.31}$$

where $\psi^{(1)}(z) := \frac{\mathrm{d}}{\mathrm{d}z}\log\big(\Gamma(z)\big)$ is the Polygamma function of order 1 (for a derivation of this integral see Appendix B.1.4). The final result found by inserting (B.31) into (B.30) is then

$$\begin{aligned}\mathscr{J}_1(t,\mathbf{x},t',\mathbf{x}') = & -\frac{i}{2\pi}\,\delta(0)\,\langle \mathrm{vac}|\,\phi(t-|\mathbf{x}|,\mathbf{0})\phi(t',\mathbf{x}')\,|\mathrm{vac}\rangle \\ & +\frac{i\pi^2}{32\pi^3\beta^2|\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)\right]} - \frac{1}{16\pi^3\beta^2|\mathbf{x}'|}\,\mathrm{Im}\left[\psi^{(1)}\left(1+\frac{i(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}{\beta}\right)\right] \\ & -\frac{i\pi^2}{32\pi^3\beta^2|\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(t-t'-|\mathbf{x}|-|\mathbf{x}'|-i\delta)\right]} + \frac{1}{16\pi^3\beta^2|\mathbf{x}'|}\,\mathrm{Im}\left[\psi^{(1)}\left(1+\frac{i(t-t'-|\mathbf{x}|-|\mathbf{x}'|)}{\beta}\right)\right]\end{aligned} \tag{B.32}$$

and in an almost identical calculation the integral (B.21) evaluates to

$$\begin{aligned}\mathscr{J}_2(t,\mathbf{x},t',\mathbf{x}') = & -\frac{i}{2\pi}\,\delta(0)\,\langle \mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t'-|\mathbf{x}'|,\mathbf{0})\,|\mathrm{vac}\rangle \\ & +\frac{i\pi^2}{32\pi^3\beta^2|\mathbf{x}|\sinh^2\left[\frac{\pi}{\beta}(t-t'+|\mathbf{x}|+|\mathbf{x}'|-i\delta)\right]} - \frac{1}{16\pi^3\beta^2|\mathbf{x}|}\,\mathrm{Im}\left[\psi^{(1)}\left(1+\frac{i(t-t'+|\mathbf{x}|+|\mathbf{x}'|)}{\beta}\right)\right] \\ & -\frac{i\pi^2}{32\pi^3\beta^2|\mathbf{x}|\sinh^2\left[\frac{\pi}{\beta}(t-t'-|\mathbf{x}|+|\mathbf{x}'|-i\delta)\right]} + \frac{1}{16\pi^3\beta^2|\mathbf{x}|}\,\mathrm{Im}\left[\psi^{(1)}\left(1+\frac{i(t-t'-|\mathbf{x}|+|\mathbf{x}'|)}{\beta}\right)\right].\end{aligned} \tag{B.33}$$

Using these in (B.6) then gives the result (B.7).

B.1.4 One-Sided Fourier Transform of $p\coth(\beta p/2)$

Here we derive the integral (B.31). To this end we use the identity $\coth\left(\frac{\beta p}{2}\right) = 1 + \frac{2}{e^{\beta p}-1}$ to write the LHS of (B.31) as

$$\int_0^\infty \mathrm{d}p\; p\coth\left(\tfrac{\beta p}{2}\right)e^{-ip(\tau-i\delta)} \;=\; \int_0^\infty \mathrm{d}p\; pe^{-ip(\tau-i\delta)} + 2\int_0^\infty \mathrm{d}p\;\frac{p\cos(\tau p)}{e^{\beta p}-1} + 2i\int_0^\infty \mathrm{d}p\;\frac{p\sin(\tau p)}{e^{\beta p}-1}\;. \tag{B.34}$$

where the limit $\delta \to 0$ can be safely taken in the latter two integrals (since they are both convergent at $\tau = 0$). The first integral evaluates to

$$\int_0^\infty \mathrm{d}p\; pe^{-ip(\tau-i\delta)} \;=\; -\frac{1}{(\tau-i\delta)^2}\;, \tag{B.35}$$

and the second integral is given in equation (3.951.5) of [82] (which converges for any $\mathrm{Re}[\beta] > 0$)

$$\int_0^\infty \mathrm{d}p\; \frac{p\cos(\tau p)}{e^{\beta p}-1} \;=\; \frac{1}{2\tau^2} - \frac{\pi^2}{2\beta^2\sinh^2\left(\frac{\pi\tau}{\beta}\right)}\,, \tag{B.36}$$

and the third integral can be exactly evaluated as[10]

$$\int_0^\infty \mathrm{d}p\; \frac{p\sin(\tau p)}{e^{\beta p}-1} \;=\; -\frac{1}{\beta^2}\mathrm{Im}\left[\psi^{(1)}\left(1+\frac{i\tau}{\beta}\right)\right] \tag{B.37}$$

where $\zeta(3) \simeq 1.202$. Putting the above altogether (in the limit $\delta \to 0$) gives formula (B.31).

B.2 Integrals appearing in the Equal-Time Correlator

Here we simplify the integral

$$\mathcal{M}_\beta(t,\mathbf{x},\mathbf{x}') := -\frac{\tilde{g}^2}{2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\; \mathscr{W}_\beta(\tau-\tau')\,\langle \mathrm{vac}|\,\{\phi(\tau,\mathbf{0})\phi(\tau',\mathbf{0}),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\}\,|\mathrm{vac}\rangle \tag{B.38}$$

appearing in the equal-time mean-field correlator (4.47).

B.2.1 Four-Point Wightman Functions

First we note the functional form of the four-point Wightman functions appearing in the above, where

$$\begin{aligned}
&\langle \mathrm{vac}|\,\phi(t_1,\mathbf{x}_1)\phi(t_2,\mathbf{x}_2)\phi(t_3,\mathbf{x}_3)\phi(t_4,\mathbf{x}_4)\,|\mathrm{vac}\rangle \\
&\quad= \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E_k}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p}\Big[e^{-iE_k(t_1-t_4)+i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_4)}e^{-iE_p(t_2-t_3)+i\mathbf{p}\cdot(\mathbf{x}_2-\mathbf{x}_3)} \\
&\quad+e^{-iE_k(t_1-t_3)+i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_3)}e^{-iE_p(t_2-t_4)+i\mathbf{p}\cdot(\mathbf{x}_2-\mathbf{x}_4)} + e^{-iE_k(t_1-t_2)+i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}e^{-iE_p(t_3-t_4)+i\mathbf{p}\cdot(\mathbf{x}_3-\mathbf{x}_4)}\Big]
\end{aligned} \tag{B.39}$$

where the commutation relations $[\mathfrak{a}_\mathbf{k},\mathfrak{a}_\mathbf{p}] = [\mathfrak{a}^*_\mathbf{k},\mathfrak{a}^*_\mathbf{p}] = 0$ and $[\mathfrak{a}_\mathbf{k},\mathfrak{a}_\mathbf{p}] = \delta^3(\mathbf{k}-\mathbf{p})$ have been used, as well as the expectation values

$$\begin{aligned}
\langle \mathrm{vac}|\,\hat{\mathfrak{a}}_\mathbf{k}\hat{\mathfrak{a}}_\mathbf{l}\hat{\mathfrak{a}}^*_\mathbf{p}\hat{\mathfrak{a}}^*_\mathbf{q}\,|\mathrm{vac}\rangle &= \delta^3(\mathbf{k}-\mathbf{q})\delta^3(\mathbf{p}-\mathbf{l}) + \delta^3(\mathbf{k}-\mathbf{p})\delta^3(\mathbf{q}-\mathbf{l}) \\
\langle \mathrm{vac}|\,\hat{\mathfrak{a}}_\mathbf{k}\hat{\mathfrak{a}}^*_\mathbf{l}\hat{\mathfrak{a}}_\mathbf{p}\hat{\mathfrak{a}}^*_\mathbf{q}\,|\mathrm{vac}\rangle &= \delta^3(\mathbf{k}-\mathbf{l})\delta^3(\mathbf{p}-\mathbf{q})
\end{aligned} \tag{B.40}$$

In terms of free (two-point) Wightman functions, the above has the simple form

$$\begin{aligned}
\langle \mathrm{vac}|\,\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\,|\mathrm{vac}\rangle &= \langle \mathrm{vac}|\,\phi(x_1)\phi(x_4)\,|\mathrm{vac}\rangle\langle \mathrm{vac}|\,\phi(x_2)\phi(x_3)\,|\mathrm{vac}\rangle \\
&\quad+\langle \mathrm{vac}|\,\phi(x_1)\phi(x_3)\,|\mathrm{vac}\rangle\langle \mathrm{vac}|\,\phi(x_2)\phi(x_4)\,|\mathrm{vac}\rangle \\
&\quad+\langle \mathrm{vac}|\,\phi(x_1)\phi(x_2)\,|\mathrm{vac}\rangle\langle \mathrm{vac}|\,\phi(x_3)\phi(x_4)\,|\mathrm{vac}\rangle
\end{aligned} \tag{B.41}$$

using the shorthand $x_j = (t_j,\mathbf{x}_j)$.

[10] Note the integral representation $\psi^{(1)}(z) = \int_0^\infty \mathrm{d}q\; q\, e^{-zq}/(1-e^{-q})$ which follows from formula (5.9.12) of [83]. This implies $\psi^{(1)}(1+iy) = \int_0^\infty \mathrm{d}q\; q\, e^{-iyq}/(e^q-1)$, and then taking the imaginary part of this gives (B.37).

B.2.2 One-Sided Fourier Transform of $\mathscr{W}_\beta(t)$

Here we compute the integrals

$$\int_0^\infty \mathrm{d}\sigma\; \mathscr{W}_\beta(\sigma)e^{\pm ip\sigma} = \frac{1}{2}\bigg(\big[\mathcal{C}_\beta + i\mathcal{K}_\beta\big] \pm \big[-\mathcal{S}_\beta + i\mathcal{D}_\beta\big]\bigg) \tag{B.42}$$

where the function $\mathcal{C}_\beta(p) = \frac{p}{4\pi}\coth\left(\frac{\beta p}{2}\right)$ is given in (B.25) and the divergent constant $\mathcal{K}_\beta = -\delta(0)/(2\pi)$ is given in (B.28), and we furthermore define

$$\begin{aligned} \mathcal{S}_\beta(p) &:= 2\int_0^\infty \mathrm{d}\sigma\; \mathrm{Im}\left[\mathscr{W}_\beta(\sigma)\right]\sin(p\sigma) \\ \mathcal{D}_\beta(p,\delta) &:= 2\int_0^\infty \mathrm{d}\sigma\; \mathrm{Re}\left[\mathscr{W}_\beta(\sigma)\right]\sin(p\sigma) \end{aligned}$$

The functions $\mathcal{S}_\beta$ and $\mathcal{D}_\beta$ have also been computed in [27, 29] (where β is replaced by either the Unruh or Hawking temperatures). These functions take the form

$$\begin{aligned} \mathcal{S}_\beta(p) &= -\frac{p}{4\pi}\,, \\ \mathcal{D}_\beta(p,\delta) &= \frac{p}{2\pi^2}\log\left(\frac{2\pi e^{\gamma}\delta}{\beta}\right) + \frac{p}{2\pi^2}\mathrm{Re}\left[\psi^{(0)}\left(-i\,\frac{\beta p}{2\pi}\right)\right]\,, \end{aligned} \tag{B.43}$$

where γ is the Euler-Mascheroni constant and $\psi^{(0)}(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. Note that the function $\mathcal{D}_\beta$ has a δ-divergence, where $\delta > 0$ is the regulator appearing the correlation function

$$\mathscr{W}_\beta(\sigma) = -\frac{1}{4\beta^2\sinh^2\left(\frac{\pi(\sigma - i\delta)}{\beta}\right)}\,. \tag{B.44}$$

Putting this all together in (B.42) we find that

$$\begin{aligned} \int_0^\infty \mathrm{d}\sigma\; W_\beta(\sigma)e^{\pm ip\sigma} = &\left[\frac{p}{8\pi}\coth\left(\frac{\beta p}{2}\right) - \frac{i}{4\pi^2\delta}\right] \\ &\pm\left[\frac{p}{8\pi} + i\left(\frac{p}{4\pi^2}\log\left(\frac{2\pi e^{\gamma}\delta}{\beta}\right) + \frac{p}{4\pi^2}\mathrm{Re}\left[\psi^{(0)}\left(-i\,\frac{\beta p}{2\pi}\right)\right]\right)\right]\,. \end{aligned} \tag{B.45}$$

B.2.3 The Integral $\mathcal{M}_\beta$

It turns out that it is easiest to express the integral $\mathcal{M}_\beta$ in the nested-integral form of (4.46), where

$$\begin{aligned} \mathcal{M}_\beta(t,\mathbf{x},\mathbf{x}') = &-\frac{\tilde{g}^2}{2}\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}\tau'\; \mathscr{W}_\beta(\tau-\tau')\,\langle\mathrm{vac}|\,\{\phi(\tau,\mathbf{0})\phi(\tau',\mathbf{0}),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\}\,|\mathrm{vac}\rangle \\ &-\frac{\tilde{g}^2}{2}\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}\tau'\; \mathscr{W}^*_\beta(\tau-\tau')\,\langle\mathrm{vac}|\,\{\phi(\tau',\mathbf{0})\phi(\tau,\mathbf{0}),\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\}\,|\mathrm{vac}\rangle\,. \end{aligned} \tag{B.46}$$

By expanding the anti-commutators above and also using $[\phi(t,\mathbf{x}),\phi(t,\mathbf{x}')] = 0$, the above can be manipulated into the form

$$\begin{aligned} \mathcal{M}_\beta(t,\mathbf{x},\mathbf{x}') = -\tilde{g}^2\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}\sigma\;\mathrm{Re}\bigg[\mathscr{W}_\beta(\sigma)\bigg(&\langle\mathrm{vac}|\,\phi(\tau,\mathbf{0})\phi(\tau-\sigma,\mathbf{0})\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\,|\mathrm{vac}\rangle \\ &+\langle\mathrm{vac}|\,\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\phi(\tau,\mathbf{0})\phi(\tau-\sigma,\mathbf{0})\,|\mathrm{vac}\rangle\bigg)\bigg] \end{aligned} \tag{B.47}$$

where the change of variable $\tau' = \tau - \sigma$ has also been made. Using the formula (B.39) the four-point correlators can be written in momentum space as

$$\begin{aligned}&\langle\text{vac}|\,\phi(\tau,\mathbf{0})\phi(\tau-\sigma,\mathbf{0})\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\,|\text{vac}\rangle + \langle\text{vac}|\,\phi(t,\mathbf{x})\phi(t,\mathbf{x}')\phi(\tau,\mathbf{0})\phi(\tau-\sigma,\mathbf{0})\,|\text{vac}\rangle \\ &= \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E_k}\bigg(2e^{-iE_p\sigma}e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \\ &\qquad +2\mathrm{Re}\left[e^{-iE_pt+i\mathbf{p}\cdot\mathbf{x}}e^{-iE_kt+i\mathbf{k}\cdot\mathbf{x}'}e^{+i(E_p+E_k)\tau}\big(e^{-iE_p\sigma}+e^{-iE_k\sigma}\big)\right]\bigg)\,.\end{aligned}\tag{B.48}$$

With this, the integral $\mathcal{M}_\beta$ splits into two pieces

$$\mathcal{M}_\beta(t,\mathbf{x},\mathbf{x}') = \mathcal{M}^{(1)}_\beta(t,\mathbf{x},\mathbf{x}') + \mathcal{M}^{(2)}_\beta(t,\mathbf{x},\mathbf{x}')\tag{B.49}$$

where

$$\mathcal{M}^{(1)}_\beta(t,\mathbf{x},\mathbf{x}') := -2\tilde{g}^2\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E_k}\int_0^t\mathrm{d}\tau\int_0^\tau\mathrm{d}\sigma\;\mathrm{Re}\Big[\mathscr{W}_\beta(\sigma)e^{-iE_p\sigma}e^{+i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\Big]\tag{B.50}$$

and

$$\begin{aligned}\mathcal{M}^{(2)}_\beta(t,\mathbf{x},\mathbf{x}') \;:=\; &-2\tilde{g}^2\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E_p}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3 2E_k}\int_0^t\mathrm{d}\tau\int_0^\tau\mathrm{d}\sigma\;\mathrm{Re}[\mathscr{W}_\beta(\sigma)] \\ &\times\mathrm{Re}\left[e^{-iE_pt+i\mathbf{p}\cdot\mathbf{x}}e^{-iE_kt+i\mathbf{k}\cdot\mathbf{x}'}e^{+i(E_p+E_k)\tau}\big(e^{-iE_p\sigma}+e^{-iE_k\sigma}\big)\right]\,.\end{aligned}\tag{B.51}$$

First we focus on simplifying $\mathcal{M}^{(1)}$ above. The $\mathbf{k}$-integration is easily done, and then integrating the $\mathbf{p}$-angles away yields

$$\mathcal{M}^{(1)}_\beta(t,\mathbf{x},\mathbf{x}') = -\frac{\tilde{g}^2}{8\pi^4|\mathbf{x}-\mathbf{x}'|^2}\int_0^\infty\mathrm{d}p\;p\int_0^t\mathrm{d}\tau\int_0^\tau\mathrm{d}\sigma\;\mathrm{Re}\Big[\mathscr{W}_\beta(\sigma)e^{-ip\sigma}\Big]\,.\tag{B.52}$$

By switching the order of integration in the (τ,σ)-plane the above integral can be written as

$$\begin{aligned}\mathcal{M}^{(1)}_\beta(t,\mathbf{x},\mathbf{x}') &= -\frac{\tilde{g}^2}{8\pi^4|\mathbf{x}-\mathbf{x}'|^2}\int_0^\infty\mathrm{d}p\;p\int_0^t\mathrm{d}\sigma\int_\sigma^t\mathrm{d}\tau\;\mathrm{Re}\Big[\mathscr{W}_\beta(\sigma)e^{-ip\sigma}\Big] \\ &= -\frac{\tilde{g}^2}{8\pi^4|\mathbf{x}-\mathbf{x}'|^2}\int_0^\infty\mathrm{d}p\;p\int_0^t\mathrm{d}\sigma\;(t-\sigma)\mathrm{Re}\Big[\mathscr{W}_\beta(\sigma)e^{-ip\sigma}\Big] \\ &= -\frac{\tilde{g}^2}{8\pi^4|\mathbf{x}-\mathbf{x}'|^2}\int_0^\infty\mathrm{d}p\;p\;\bigg[t\int_0^t\mathrm{d}\sigma\;\Big(\mathrm{Re}[\mathscr{W}_\beta(\sigma)]\cos(p\sigma)+\mathrm{Im}[\mathscr{W}_\beta(\sigma)]\sin(p\sigma)\Big) \\ &\qquad -\frac{\mathrm{d}}{\mathrm{d}p}\int_0^t\mathrm{d}\sigma\;\Big(\mathrm{Re}[\mathscr{W}_\beta(\sigma)]\sin(p\sigma)-\mathrm{Im}[\mathscr{W}_\beta(\sigma)]\sin(p\sigma)\Big)\bigg]\end{aligned}\tag{B.53}$$

To simplify the integrals, we next assume that we probe times

$$t \gg \beta\,,\tag{B.54}$$

so that the upper limit on the σ-integrals can be taken to be $\simeq\infty$ (since $\mathscr{W}_\beta(\sigma)\propto e^{-2\pi\sigma/\beta}$). Upon doing so the σ-integrals in the above may be expressed in terms of the functions given in (B.42), where

$$\mathcal{M}^{(1)}_\beta(t,\mathbf{x},\mathbf{x}') \simeq -\frac{\tilde{g}^2}{16\pi^4|\mathbf{x}-\mathbf{x}'|^2}\int_0^\infty\mathrm{d}p\;p\;\left[t\big(\mathcal{C}_\beta(p)+\mathcal{S}_\beta(p)\big)-\frac{\mathrm{d}\mathcal{D}_\beta(p,\delta)}{\mathrm{d}p}+\frac{\mathrm{d}\mathcal{K}_\beta(p,\delta)}{\mathrm{d}p}\right]\,.\tag{B.55}$$

Note that $\partial_p \mathcal{K}_\beta(p,\delta) = 0$ from the functional form (B.28). Furthermore, using the functional forms of $\mathcal{C}_\beta$ and $\mathcal{S}_\beta$ we note the value of the integral

$$\int_0^\infty \mathrm{d}p\; p\,\big(\mathcal{C}_\beta(p) + \mathcal{S}_\beta(p)\big) \;=\; \int_0^\infty \frac{\mathrm{d}p\; p^2}{2\pi(e^{\beta p}-1)} \;=\; \frac{\zeta(3)}{\pi\beta^3} \tag{B.56}$$

where $\zeta(3) \simeq 1.202$ (with ζ the Riemann-Zeta function). With this we get

$$\mathcal{M}^{(1)}_\beta(t,\mathbf{x},\mathbf{x}') \simeq -\frac{\tilde{g}^2\left(\frac{\zeta(3)t}{\pi\beta^3} - \int_0^\infty \mathrm{d}p\; p\,\frac{\mathrm{d}\mathcal{D}_\beta(p,\delta)}{\mathrm{d}p}\right)}{16\pi^4|\mathbf{x}-\mathbf{x}'|^2} \tag{B.57}$$

where we note that the remaining momentum-integral appears to be ultraviolet divergent in the momentum p.

Moving on to the second integral $\mathcal{M}^{(2)}$, we first integrate away the angles in the momentum integrals and simplify to get

$$\begin{aligned}\mathcal{M}^{(2)}_\beta(t,\mathbf{x},\mathbf{x}') \;=\; &-\frac{\tilde{g}^2}{16\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}k \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\sigma\;\sin(|\mathbf{x}|p)\sin(|\mathbf{x}'|k) \\ &\times \mathrm{Re}\Big[e^{-i(p+k)t}e^{+i(p+k)\tau}(e^{-ip\sigma}+e^{-ik\sigma})\mathrm{Re}[\mathscr{W}_\beta(\sigma)]\Big]\,.\end{aligned} \tag{B.58}$$

We again can switch the order of integration in the (τ,σ)-plane giving us

$$\begin{aligned}\mathcal{M}^{(2)}_\beta(t,\mathbf{x},\mathbf{x}') \;=\; &-\frac{\tilde{g}^2}{16\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p \int_0^\infty \mathrm{d}k \int_0^t \mathrm{d}\sigma \int_\sigma^t \mathrm{d}\tau\;\sin(|\mathbf{x}|p)\sin(|\mathbf{x}'|k) \\ &\times \mathrm{Re}\Big[e^{-i(p+k)t}e^{+i(p+k)\tau}(e^{-ip\sigma}+e^{-ik\sigma})\mathrm{Re}[\mathscr{W}_\beta(\sigma)]\Big]\,.\end{aligned} \tag{B.59}$$

The τ-integration can now be easily performed such that

$$\begin{aligned}\mathcal{M}^{(2)}_\beta(t,\mathbf{x},\mathbf{x}') \;=\; &-\frac{\tilde{g}^2}{16\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\sin(|\mathbf{x}|p)\int_0^\infty \mathrm{d}k\;\sin(|\mathbf{x}'|k)\int_0^t \mathrm{d}\sigma \\ &\times \mathrm{Re}\bigg[\frac{ie^{-i(p+k)t}}{k+p}(e^{+ip\sigma}+e^{+ik\sigma})\mathrm{Re}[\mathscr{W}_\beta(\sigma)] - \frac{i}{k+p}(e^{-ip\sigma}+e^{-ik\sigma})\mathrm{Re}[\mathscr{W}_\beta(\sigma)]\bigg]\,.\end{aligned} \tag{B.60}$$

As noted above, we assume $t \gg \beta$ and so the σ-integrals can be expressed in terms of the functions $\mathcal{C}_\beta$ and $\mathcal{D}_\beta$ where

$$\begin{aligned}\mathcal{M}^{(2)}_\beta(t,\mathbf{x},\mathbf{x}') \;\simeq\; &-\frac{\tilde{g}^2}{16\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\sin(|\mathbf{x}|p)\int_0^\infty \mathrm{d}k\;\sin(|\mathbf{x}'|k) \\ &\times \mathrm{Re}\bigg[\frac{ie^{-i(p+k)t}}{k+p}\big(\mathcal{C}_\beta(p)+i\mathcal{D}_\beta(p,\delta)+\mathcal{C}_\beta(k)+i\mathcal{D}_\beta(k,\delta)\big) \\ &\qquad - \frac{i}{k+p}\big(\mathcal{C}_\beta(p)-i\mathcal{D}_\beta(p,\delta)+\mathcal{C}_\beta(k)-i\mathcal{D}_\beta(k,\delta)\big)\bigg] \\ =\; &-\frac{\tilde{g}^2}{16\pi^4|\mathbf{x}||\mathbf{x}'|}\int_0^\infty \mathrm{d}p\;\sin(|\mathbf{x}|p)\int_0^\infty \mathrm{d}k\;\sin(|\mathbf{x}'|k) \\ &\times\bigg(\frac{\mathcal{C}_\beta(p)+\mathcal{C}_\beta(k)}{p+k}\sin\big((p+k)t\big) - \frac{\mathcal{D}_\beta(p,\delta)+\mathcal{D}_\beta(k,\delta)}{p+k}\big[1+\cos\big((p+k)t\big)\big]\bigg)\,.\end{aligned} \tag{B.61}$$

B.3 Quantities entering with the diffuse correlator

This Appendix computes quantities that arise in §4.3.5 where the diffuse contributions to the Wightman function are computed.

B.3.1 The integral $\mathcal{P}_\beta$

First we compute the integral $\mathcal{P}_\beta$ defined in (4.55). Using the commutator (4.41) we easily find that (4.55) simplifies to

$$\begin{aligned}\mathcal{P}_\beta(t,\mathbf{x},\mathbf{x}') &= -\tilde{g}^2 \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\, \mathscr{W}_\beta(\tau-\tau') \frac{i\delta\big(\tau-[t-|\mathbf{x}|]\big)}{4\pi|\mathbf{x}|} \frac{i\delta\big(\tau'-[t-|\mathbf{x}'|]\big)}{4\pi|\mathbf{x}'|} \langle \mathrm{vac}|\, \mathcal{I}_+^2 \,|\mathrm{vac}\rangle \\ &= +\frac{\tilde{g}^2}{16\pi^2|\mathbf{x}||\mathbf{x}'|}\Theta(t-|\mathbf{x}|)\Theta(t-|\mathbf{x}'|)\mathscr{W}_\beta(-|\mathbf{x}|+|\mathbf{x}'|) \\ &= -\left(\frac{\tilde{g}^2}{64\pi^2\beta^2}\right) \frac{\Theta(t-|\mathbf{x}|)\Theta(t-|\mathbf{x}'|)}{|\mathbf{x}||\mathbf{x}'|\sinh^2\left[\frac{\pi}{\beta}(-|\mathbf{x}|+|\mathbf{x}'|-i\delta)\right]} \end{aligned} \tag{B.62}$$

This is the result quoted as (4.57) in the main text, and it agrees precisely with the $\mathcal{O}(\tilde{g}^2)$ part of the correlator given in (2.19).

B.3.2 The integral $\mathcal{Q}_\beta$

Next we compute the integral $\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{y})$ defined in (4.56). Using the commutator (4.41) allows this integral to be rewritten as

$$\begin{aligned}\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') = \frac{\tilde{g}^2}{2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\, \mathscr{W}_\beta(\tau-\tau')\, \langle \mathrm{vac}|\, \tfrac{i\delta[\tau-(t-|\mathbf{x}|)]\phi(\tau',\mathbf{0})\phi(t,\mathbf{x}') - i\delta[\tau'-(t-|\mathbf{x}|)]\phi(\tau,\mathbf{0})\phi(t,\mathbf{x}')}{4\pi|\mathbf{x}|} \,|\mathrm{vac}\rangle \\ + \frac{\tilde{g}^2}{2}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\, \mathscr{W}_\beta(\tau-\tau')\, \langle \mathrm{vac}|\, \tfrac{i\delta[\tau-(t-|\mathbf{x}'|)]\phi(t,\mathbf{x})\phi(\tau',\mathbf{0}) - i\delta[\tau'-(t-|\mathbf{x}|)]\phi(t,\mathbf{x})\phi(\tau,\mathbf{0})}{4\pi|\mathbf{x}'|} \,|\mathrm{vac}\rangle\ .\end{aligned} \tag{B.63}$$

Relabelling τ and τ' in the first term of each of the lines above allows this to be rewritten as

$$\begin{aligned}\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') &= \frac{\tilde{g}^2}{4\pi|\mathbf{x}|}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\, \mathrm{Im}[\mathscr{W}_\beta(\tau-\tau')]\delta\big[\tau'-(t-|\mathbf{x}|)\big]\, \langle \mathrm{vac}|\, \phi(\tau,\mathbf{0})\phi(t,\mathbf{x}') \,|\mathrm{vac}\rangle \\ &\quad + \frac{\tilde{g}^2}{4\pi|\mathbf{x}'|}\int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\, \mathrm{Im}[\mathscr{W}_\beta(\tau-\tau')]\delta\big[\tau'-(t-|\mathbf{x}'|)\big]\, \langle \mathrm{vac}|\, \phi(t,\mathbf{x})\phi(\tau,\mathbf{0}) \,|\mathrm{vac}\rangle\ .\end{aligned} \tag{B.64}$$

Performing the integrations over τ' now gives

$$\begin{aligned}\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') &= \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{4\pi|\mathbf{x}|}\int_0^t \mathrm{d}\tau\, \mathrm{Im}\big[\mathscr{W}_\beta\big(\tau-[t-|\mathbf{x}|]\big)\big]\, \langle \mathrm{vac}|\, \phi(\tau,\mathbf{0})\phi(t,\mathbf{x}') \,|\mathrm{vac}\rangle \\ &\quad + \frac{\tilde{g}^2\Theta(t-|\mathbf{x}'|)}{4\pi|\mathbf{x}'|}\int_0^t \mathrm{d}\tau\, \mathrm{Im}\big[\mathscr{W}_\beta\big(\tau-[t-|\mathbf{x}'|]\big)\big]\, \langle \mathrm{vac}|\, \phi(t,\mathbf{x})\phi(\tau,\mathbf{0}) \,|\mathrm{vac}\rangle\ ,\end{aligned} \tag{B.65}$$

and using the explicit form for the free Wightman functions as well as $\mathrm{Im}[\mathscr{W}_\beta(t)] = \delta'(t)/(4\pi)$ makes the integrand explicit:

$$\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') = \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{64\pi^4|\mathbf{x}|}\int_0^t \mathrm{d}\tau\, \frac{\delta'\big(\tau-[t-|\mathbf{x}|]\big)}{-(\tau-t-i\delta)^2+|\mathbf{x}'|^2} + \frac{\tilde{g}^2\Theta(t-|\mathbf{x}'|)}{64\pi^4|\mathbf{x}'|}\int_0^t \mathrm{d}\tau\, \frac{\delta'\big(\tau-[t-|\mathbf{x}'|]\big)}{-(t-\tau-i\delta)^2+|\mathbf{x}|^2}\ . \tag{B.66}$$

Integrating this by parts yields

$$\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') = \frac{\tilde{g}^2\Theta(t-|\mathbf{x}|)}{64\pi^4|\mathbf{x}|}\left[\frac{\delta(|\mathbf{x}|)}{|\mathbf{x}'|^2} - \frac{\delta(t-|\mathbf{x}|)}{-(|\mathbf{x}|+i\delta)^2+|\mathbf{x}'|^2} - \int_0^t \mathrm{d}\tau\;\delta\big(\tau-[t-|\mathbf{x}|]\big)\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{1}{-(\tau-t-i\delta)^2+|\mathbf{x}'|^2}\right\}\right]$$
$$+\frac{\tilde{g}^2\Theta(t-|\mathbf{x}'|)}{64\pi^4|\mathbf{x}'|}\left[\frac{\delta(|\mathbf{x}'|)}{|\mathbf{x}|^2} - \frac{\delta(t-|\mathbf{x}'|)}{-(|\mathbf{x}'|+i\delta)^2+|\mathbf{x}|^2} - \int_0^t \mathrm{d}\tau\;\delta\big(\tau-[t-|\mathbf{x}'|]\big)\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\frac{1}{-(t-\tau-i\delta)^2+|\mathbf{x}|^2}\right\}\right] \tag{B.67}$$

where boundary terms with $\delta(|\mathbf{x}|)$ factors never contribute (since $|\mathbf{x}|>0$). In the regime of interest, $t-|\mathbf{x}|>0$ and $t-|\mathbf{x}'|>0$, the other boundary terms also do not contribute (since the Heaviside functions all turn on).

All that is left is to perform the δ-function integrations which gives the final result

$$\begin{aligned}\mathcal{Q}_\beta(t,\mathbf{x},\mathbf{x}') &= \frac{\tilde{g}^2}{64\pi^4|\mathbf{x}|}\left[\frac{1}{2|\mathbf{x}'|\big(|\mathbf{x}|-|\mathbf{x}'|+i\delta\big)^2} - \frac{1}{2|\mathbf{x}'|\big(|\mathbf{x}|+|\mathbf{x}'|+i\delta\big)^2}\right] \\ &\qquad + \frac{\tilde{g}^2}{64\pi^4|\mathbf{x}'|}\left[\frac{1}{2|\mathbf{x}|\big(|\mathbf{x}|-|\mathbf{x}'|+i\delta\big)^2} - \frac{1}{2|\mathbf{x}|\big(|\mathbf{x}|+|\mathbf{x}'|-i\delta\big)^2}\right] \\ &= \frac{\tilde{g}^2}{64\pi^4|\mathbf{x}||\mathbf{x}'|}\left[\frac{1}{(|\mathbf{x}|-|\mathbf{x}'|+i\delta)^2} - \frac{1}{(|\mathbf{x}|+|\mathbf{x}'|)^2}\right],\end{aligned} \tag{B.68}$$

where the last line safely takes $\delta\to 0$. This is the result quoted in (4.58).

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Chapter 5

Conclusions and Outlook

5.1 Summary

The aim of thesis is to motivate the use of Open EFT techniques for gravitational applications, specifically to backgrounds which contain an event horizon (which the OQS framework naturally applies to due to the way horizons separate quantum degrees of freedom). This was done by studying two Open EFT toy models (both involving black holes) in the hopes that this helps pave the way for more realistic calculations involving EFTs of gravity. As argued here (and explicitly shown in the toy models studied), quantum systems interacting in gravitational backgrounds are susceptible to secular growth breakdowns of perturbation theory, and the main focus in this thesis has been in resumming these breakdowns in order to make reliable late time predications (without resorting to solving the theory exactly).

The late time resummations performed have been a result of taking a Markovian limit which is naturally a hierarchy of scales argument. We have paid especially close attention to when the Markovian approximation is valid, and one of the main results of this thesis is that the Markovian regime is surprisingly rigid, usually applying in a small region of available parameter space. In the toy models considered here we find furthermore that there is no danger of straying into positivity-violating regions of parameter space so long as one is careful to remain in the domain of validity of approximations taken.

In §2, the late-time thermalization of a qubit is tracked as it hovers just outside the event horizon of a Schwarzschild black hole. One of the main themes of this chapter is *control over approximations* even though the mathematics in the Schwarzschild background are in general very complicated — this is demonstrated by the way in which the qubit self-correlations can be simplified near the horizon, as well as in the manner in which the Markovian approximation is constrained (under the assumption that the qubit reduced density matrix varies slowly compared to the correlation time $\sim r_s$ of the qubit self-correlations). The rigidness of the Markovian approximation was demonstrated here since this requires the qubit gap to be small compared to the Hawking temperature, which moreover gave rise to state evolution which is positivity-preserving for all times (not needing any extra approximations like the rotating-wave approximation often used in the literature).

In §3 and §4 the hotspot model is studied, a solvable toy model of a black hole meant to emphasize the OQS features thought to be present in more authentic black hole backgrounds (like Schwarzschild or Kerr).

In §3, PPEFT techniques are used to set up the hotspot model as an effective theory meant to describe observables over macroscopically large distance scales compared to the size of the black hole. Since the field theory is here Gaussian, the main focus in this chapter is in calculating the correlation functions associated with the open system (*aka.* the quantum field corresponding to the exterior of the toy black hole) and so §3 can be largely regarded as a detour from Open EFT approximation techniques. Solving for the correlation functions requires some care since the calculation involves small-distance "Colomb-like" singularities due to the point-particle type interaction of the hotspot. The PPEFT framework is used to argue that any divergences encountered do not end up contributing to physical observables (which is done by relating the divergences to renormalization-group invariant quantities).

In the final paper of §4, Open EFT approximation techniques are applied to the open system (*ie.* the exterior field) of the hotspot model. The mean-field approximation (describing evolution using a non-local Hamiltonain) is shown to apply only when the hotspot interaction decouples, and so happens to not be useful for this particular model. In contrast, the state of the exterior field is shown to have a Markovian limit which agrees with the exact correlators of §3 only for field modes with energies much smaller than the temperature set by the hotspot interior. Decoherence of the open system is also observed, but in a way which leaves much to be desired since the timescales for decoherence end up being (UV) cutoff dependent.

5.2 Future Work

The Open EFT approximations used in §4 give rise to decoherence timescales which are UV cutoff dependent — this result heuristically suggests that the hotspot exterior decoheres, but lacks precision in that the nothing physical can depend on the cutoff of a theory. One line of future work on the hotspot model is to find a way to take the Markovian approximation which gives an unambiguous answer for the decoherence timescale. One possible way forward is to use the spherical symmetry of the hotspot interaction and attempt a spherical harmonic decomposition of reduced density matrix for ϕ, such that

$$\hat{\varrho}_A \;=\; \bigotimes_{\omega,\ell,m} \hat{\varrho}_A^{(\omega,\ell,m)} \tag{5.1}$$

and derive a simpler equation of motion for each individual component $\hat{\varrho}_A^{(\omega,\ell,m)}$. This is similar to the strategy employed in [92], in which the reduced density matrix of the open system is split apart according to momenta $\boldsymbol{k}$ (resulting from the spatial translation invariance of the interaction studied there). It is likely that having a family of Nakajima-Zwanzig-type equations for each component $\hat{\varrho}_A^{(\omega,\ell,m)}$ will contain far more information than the equation (3.22) examined in §4 (which studied of all the spherical harmonics at once) — furthermore, since the environment correlations in the model are the same as those in [93] it is likely that the same methods might apply and even non-Markovian effects in the evolution of $\hat{\varrho}_A^{(\omega,\ell,m)}$ can be studied so as to better pin down a physical decoherence timescale.

Another project currently underway has to do with the theory of inflation described in equation (1.2). As mentioned, the theory of inflation reproduces the observed density fluctuations in the Cosmic Microwave Background (CMB) to a high degree of precision. However, one puzzle associated with the measured spectrum of the CMB is to explain how the observed *classical* density fluctuations could possibly have a quantum origin — in current work [94] this is resolved though an Open EFT picture in which the observable modes are decohered through their interaction with the unobserved modes (behind the de Sitter horizon) in this particular EFT of gravity (described in [16]). The main result of our calculation is that the observable modes in the CMB have decohered long ago (explaining why we see no quantum correlations in the sky) and that any quantum corrections to the power spectrum are extremely small (*ie.* undetectable with current technology). There have been many simliar calculations done along these lines: some for the simpler problem of a spectator field living in de Sitter space and others which examine the full EFT of inflation [96, 97, 98, 99, 100, 101, 102], however of particular interest in our work is the derivation and assessment of the domain of validity of the derived Lindblad equations (and so the late-time behaviour).

5.3 Conclusions

As discussed in the introduction, standard EFTs are based on a hierarchy of energy scales, but at each level of approximation energy is still conserved. Open EFTs are different in this regard in that the division between the measured sector (the open system) and the unmeasured sector (the environment) is not defined in terms of conserved quantities like energy, which is what results in the emergence of unusual phenomena such as decoherence, dissipation and entanglement effects normally not encountered in Wilsonian EFTs. To a large extent, this thesis is a step forward towards a better understanding of these phenomena in EFTs of gravity, as well as reliably making late time inferences.

In conclusion, the frameworks of Effective Field Theories and Open Quantum Systems are two powerful tools which share the common ground of being able to describe EFTs of gravity with horizons. Although a great deal of progress has been made in understanding these subjects individually, this thesis aims to show that there is still much to be learned and utilized in their merger in Open EFTs.

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