ON LINEAR POWER CONTROL POLICIES FOR ENERGY HARVESTING COMMUNICATIONS

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Abstract

In this thesis, we focus on linear online power control policies for the energy harvesting communications. In the first part of the thesis, greedy policy is investigated as a special case of linear policies. The tight upper and lower bounds on the greedy threshold (c^*) are provided in semi-universal settings where few parameters are known from the actual arrival distribution and clipped arrival distribution. Then the optimality region of the greedy policy is discussed. In the second part, various notions of optimal slope (s^*) linear policy are discussed. The numerical results show the existence of optimal linear policy with strictly better performance than the conventional fixed fraction policy in terms of the multiplicative ratio.

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Contents

List of Figures

Acronyms

AWGN Additive White Gaussian Noise. 3, 6, 7, 9

i.i.d Independent and identically distributed. 6, 32

MCR Mean to Capacity Ratio. 7, 27, 29

Chapter 1

Introduction

1.1 Background

Due to recent advances in wireless communication, wireless network services have become vital as they provide accessibility to distant locations or provide sensor measurements for different applications. Most of the wireless nodes rely on batteries for their operations instead of connecting to the power line. The lifetime of the wireless network is limited by the lifetime of batteries in the network. When a sufficient number of batteries are exhausted, the network will not achieve its designated goal and need to replace the batteries every few months which causing the increase of maintenance cost [1]. Thus, the ability to harvest energy from the environment highly increases the lifetime of the node and enhances its independence, self-reliance, and self-sustainability.

There are some energy harvesting technologies that have been developed successfully for the wireless network. In addition, there are two main types of energy harvesting system [2]. One is the *Harvest-Use-Store* architecture and the other is the *Harvest-Store-Use* architecture. In the first one, wireless nodes will use the energy from the energy harvesting system directly until there is no sufficient energy left. In the second one, the harvested energy is stored in a battery and the system consumes stored energy to power the wireless network. In this thesis, we focus on the *Harvest-Store-Use* architecture.

A particular problem in the energy harvesting communication systems is to find the optimal policy for energy consumption which has been studied intensely during recent years [3–21]. For the energy arrival, two different scenarios are considered while studying the energy consumption: offline and online. In the offline scenario, the energy-arrival time and the quantity of harvested energy are known in advance, so the energy arrival distribution does not have much significance when considering power control policy. In this case, the optimal policy is much easier to find and is mostly to keep the battery level at a fixed value to avoid overflow [4, 5, 12, 18].

In contrast to that, in the online scenario the wireless node does not have prior knowledge of the energy arrivals. Therefore, the distribution of energy arrival is critical, and the optimal policy highly depends on it. Although theoretically, the optimal policy can be found by solving the associated Bellman equation, it often does not have an explicit characterization. One exception is that under the condition of low battery capacity, the explicit solution of the Bellman equation is known, which is the greedy policy [20, 21].

As for the power control policy problem, the idea is to maximize the long-term average reward of the system, which is calculated based on the reward function depending on the type of the task assigned to node. By definition, the reward function r is a non-decreasing, Lipschitz and concave function from $[0, +\infty)$ to $[0, +\infty)$ and $r(0) = 0$ [21]. The concavity of the function is the main reason behind the need for power control policies, as if the reward function was linear, greedy policy would be always the optimal policy. The fact that reward's first-order derivative function $r'(.)$ saturates for large inputs demands that node splits the energy consumption between time slots. On the other hand, the limited battery capacity is the other bottleneck which demands that energy consumes as soon as possible to avoid battery overflow and loss of energy. Many of the existing works assume the AWGN channel capacity as their reward function which models a node transmitting information through a wireless channel [12–21].

Besides, the optimality of the greedy policy is discussed thoroughly in [20]. By using the Bellman equation in an indirect approach, they established necessary and sufficient conditions for the optimality of greedy policy, which translates to a threshold $c \leq c^*$ on the battery capacity as a function of energy arrival distribution. They also provide an easier way to evaluate upper and lower bounds for the value of c^*. Furthermore, they find a pair of semi-universal upper and lower bounds for threshold c^* when only $\underline{x}, \overline{x}$ and μ parameters are given from the arrival distribution.

The greedy policy and constant policy are two policies that become optimal in the limits when battery capacity goes to zero and to infinity respectively. However, neither of them can provide optimal performance in other non-asymptotic scenarios [18]. Hence, a simple fixed fraction policy is introduced in [18] that only uses p fraction of the battery level at each time slot where p is the (effective) mean of the arrival distribution. They further prove that for the AWGN capacity reward function, the Bernoulli distribution incurs the worst-case performance when using fixed fraction policy and also prove that its performance is within an additive and multiplicative gap from a universal upperbound. This proves that fixed fraction policy is universally near optimal for the given reward function. Later, [11] extends this result to the general case of concave and non-decreasing reward functions and provides the multiplicative and additive gaps.

In [21] authors discuss the general reward function and prove that for any stationary policy with given set of conditions (normal policies), the Bernoulli distribution is the worst-case scenario, i.e., all other distributions have better performance when using the same policy. Then for the general reward function they provide the optimal policy for the Bernoulli distribution, and prove that it is in fact a normal policy meeting the conditions. This concludes that the given policy is maximin optimal. They further prove that this policy is within a tighter multiplicative ratio than the aforementioned fixed fraction policy.

Although the maximin policy has strictly better performance than fixed fraction policy, it is a piece-wise linear policy with different parameters and thus has more complexity and computational cost than simply using a fixed fraction in every time slot.

1.2 Thesis Structure

Chapter 1 presents the introduction of the thesis.

Chapter 2 gives the problem statements and essential background.

In Chapter 3.1 we find the optimal semi-universal bounds $\underline{c}, \overline{c}$ for the case where parameters $\underline{x}, \overline{x}, \mu$ are known from the *arrival distribution*. The obtained bounds match the bounds in [Proposition 4,5 [20]] which proves that they are in fact the tightest possible semi-universal bounds with the given parameters. Moreover, in Chapter 3.2 we find another pair of optimal bounds $\underline{c}', \overline{c}'$ based on the parameters $\underline{x}, \overline{x}, \mu_c$ of the *clipped energy distribution* as well. In Chapter 3.3 we find the upperbound on the value of c^* when the parameters c, p are known from the clipped arrival distribution, and also show which distribution achieves the bounds provided. It turns out that the expression of bounds and the boundary distributions change in different intervals of p.

In Chapter 4 we discuss how to optimize the slope of a linear policy based on battery capacity(c) and effective mean to capacity ratio (p). We then provide numerical results to show that suitably optimized linear policies can have strictly better multiplicative ratio, compared to fixed fraction policy.

The conclusion is provided in Chapter 5.

Chapter 2

Problem Statement

The problem is about a discrete-time energy harvesting communication node with battery capacity c. In this system, the energy is first harvested at each time-slot and stored in the battery with the energy arrival $E^t = \{E_1, E_2, ..., E_t\}$. Then the energy is consumed by the node based on the power control policy to transmit data over AWGN wireless communication channel.

$$b_{t+1} = \min\{b_t + E_t,\ c\} - g_t \quad t = 1, 2, ... \tag{2.1}$$

where b_t and g_t are the battery level and the consumed energy at time-slot t respectively. The energy arrival E^t follows an i.i.d probability distribution Q. We also use these notations $\mu_c = \mathbb{E}\left[\min\{Q, c\}\right]$, $\mu = \mathbb{E}\left[Q\right]$, $\underline{x} \leq \inf_x\{Q(x) > 0\}$ and $\overline{x} \geq \sup_x\{Q(x) < 1\}$ for the clipped mean, actual mean and range lower and upper bounds respectively. Also, p is defined as the (effective) mean-to-capacity ratio

(MCR) of the clipped arrival distribution

$$p = \mathrm{MCR}_c(Q) = \frac{\mu_c}{c}. \tag{2.2}$$

The consumed energy g_t at each time-slot is the result of the power control policy which is a sequence of mappings:

$$g_t : E^t \longrightarrow \mathbb{R}_+, \quad t = 1, 2, ...$$

The reward function is defined as a mapping which indicates how much reward does the system gain in each time-slot as a function of the consumed energy at that time-slot. For the AWGN channel capacity, the reward function is defined as

$$r(x) = \frac{1}{2}\log(1 + \gamma x) \tag{2.3}$$

where γ is the fixed fading coefficient. With no loss of generality we assume $\gamma = 1$ as its effect can be absorbed in the other parameters of the problem. Therefore, the n-horizon expected throughput is defined as

$$\mathscr{T}_n(g^n) = \frac{1}{n}\mathbb{E}\left[\sum_{t=1}^{n} \frac{1}{2}\log(1 + g_t(E^t))\right] \tag{2.4}$$

and the long-term average throughput is obtained at the limit when $n \to \infty$. Therefore, the optimal policy problem is expressed in the following format:

$$\Theta = \sup_{g} \lim_{n\to\infty} \inf \mathscr{T}_n(g^n) \tag{2.5}$$

where $g = \{g_t\}_{t=1}^{\infty}$ is the policy. Because of b_t fully representing the state of the device, and E^t being an i.i.d process, and considering (2.1) and (2.3), the problem is a Markov Decision Process and its optimal policy is Markovian $g_t^*(E^t) = g_t^*(b_t)$ [18, 22]. Furthernote that E^t is a non-delayed regenerative process. Therefore, in [18] using [Lemma 1 [18]] and Fatou's lemma [23] they provided a more practical lowerbound for the infinite-horizon expected throughput

$$\lim_{n\to\infty} \inf \mathscr{T}_n(g^n) \geq \underline{\Gamma}(g) = \frac{1}{\mathbb{E}\left[L\right]}\mathbb{E}\left[\sum_{t=1}^{L}\frac{1}{2}\log(1+g_t(E^t))\right] \tag{2.6}$$

where L is the random variable representing the time it takes the system to reset. The summation is also on one duration of the epoch with mentioned properties. Moreover, the long-term expected throughput is further upperbounded by the following expression [Proposition 1 [18]]:

$$\lim_{n\to\infty} \inf \mathscr{T}_n(g^n) \leq \overline{\Gamma}(\mu_c) = \frac{1}{2}\log(1+\mu_c). \tag{2.7}$$

A subclass of time-invariant Markovian policies are called the stationary policies. As discussed in the previous section, we are further interested in a subclass of stationary policies called linear policies expressed as

$$g_t(b_t) = sb_t \tag{2.8}$$

where s is the slope of the line and the main parameter of the policy. Both greedy and fixed fraction policies are special cases of the above with $s = 1$ and $s = p$ respectively.

Chapter 3

Greedy Policy

First, we study the special case of linear policy where $s = 1$. In [20] authors determine the region in which the greedy policy maximizes the long-term average throughput which depends on the arrival distribution. They prove that given the reward function $r(x)$ the greedy policy is optimal iff $c \leq c^*$, where

$$c^* = \sup \left\{ c \geq 0,\ r'(c) \geq \rho(c) \mathbb{E} \left[r'(x) | x \leq c \right] \right\} \tag{3.1}$$

with $\rho(x) = \mathbb{P}(Q < x)$. In the special case where the reward function is the AWGN channel capacity, $r(x) = \frac{1}{2} \log(1 + x)$ for $x \geq 0$, the threshold is given by:

$$c^* = \sup \left\{ c \geq 0,\ \frac{1}{1+c} \geq \int_{[0,c)} \frac{1}{1+x} dQ \right\}. \tag{3.2}$$

Because the inequality in (3.2) might not be easy to solve, a pair of lower and upper bounds was provided in [20] as a function of the cdf of energy arrival distribution $\rho(x)$.

3.1 Semi-Universal Bounds on c^* given $\underline{x}$, $\overline{x}$, μ of Actual Arrival Distribution

Our goal is to find the boundaries for the threshold of greedy policy's optimality (c^*) among all distributions with given $\underline{x}, \overline{x}$ and μ parameters. Therefore, in order to find the lower(upper) bounds of c^*, one must maximize(minimize) the RHS of the inequality condition of set in (3.2). Thus, define:

$$\overline{f}(\underline{x}, \overline{x}, \mu, c) = \max_{Q \in \mathcal{Q}_{\underline{x}, \overline{x}, \mu}} \int_{[0,c)} \frac{1}{1+x} dQ, \tag{3.3}$$

$$\underline{f}(\underline{x}, \overline{x}, \mu, c) = \min_{Q \in \mathcal{Q}_{\underline{x}, \overline{x}, \mu}} \int_{[0,c)} \frac{1}{1+x} dQ. \tag{3.4}$$

Then the expressions for the above optimization problems is given as follows:

Proposition 1. *Having the integral function* $\int_{[0,c)} \frac{1}{1+x} dQ$ *where* Q *is the set of actual arrival distributions with given parameters* $Q \in \mathcal{Q}_{\underline{x}, \overline{x}, \mu}$*, the integral function is bounded below by the following function:*

$$\underline{f}(\underline{x}, \overline{x}, \mu, c) = \begin{cases} \left.\begin{matrix} \frac{1}{1+\mu} & \mu < \frac{c-1}{2} \\ \frac{4(c-\mu)}{(c+1)^2} & \underline{x} < \frac{c-1}{2} \leq \mu \\ \frac{c-\mu}{(1+\underline{x})(c-\underline{x})} & \underline{x} \geq \frac{c-1}{2} \end{matrix}\right\} & \mu < c \\ 0 & \mu \geq c \end{cases} \tag{3.5}$$

Furthermore, the integral function is also bounded above by the following function:

$$\overline{f}(\underline{x},\overline{x},\mu,c) = \begin{cases} \left.\begin{array}{ll} \frac{\overline{x}-\mu}{(1+\underline{x})(\overline{x}-\underline{x})} & c \leq \overline{x}-\underline{x}-1 \\ \frac{1+\underline{x}+c-\mu}{(1+\underline{x})(1+c)} & c > \overline{x}-\underline{x}-1 \end{array}\right\} & \mu < c \\ \left.\begin{array}{ll} \frac{\overline{x}-\mu}{(1+\underline{x})(\overline{x}-\underline{x})} & c \leq \overline{x}-\underline{x}-1 \\ \frac{\overline{x}-\mu}{(1+c)(\overline{x}-c)} & c > \overline{x}-\underline{x}-1 \end{array}\right\} & \mu \geq c \end{cases} \tag{3.6}$$

Proof. We aim to simplify the infinite dimensional optimization problem into a simpler one. With no essential loss of generality we assume that Q is a probability mass function with support $x_0 \leq ... \leq x_N$. Further, assume $x_0, ..., x_M \in [\underline{x}, c)$ and $x_{M+1}, ..., x_N \in [c, \overline{x}]$:

$$\int_{[0,c)} \frac{1}{1+x} dQ = \sum_{i=0}^{M} \frac{1}{1+x_i} Q(x_i). \tag{3.7}$$

Therefore, the optimization problem is reformulated as follows:

$$\begin{aligned} \text{(max) or } \min_{Q(x_i), i=0,1,...,N} \; & \sum_{i=0}^{M} \frac{1}{1+x_i} Q(x_i) \\ \text{s.t. } & \sum_{i=0}^{N} x_i Q(x_i) = \mu \\ & \sum_{i=0}^{N} Q(x_i) = 1 \\ & Q(x_i) \geq 0, \; i = 0, 1, ..., N. \end{aligned} \tag{3.8}$$

This is a linear programming problem and its minimum(maximum) is attained at a certain vertex of the domain. Since the domain is given by the intersection of the probability simplex and the hyperplane $\sum_{i=0}^{N} x_i Q(x_i) = \mu$, its vertices must lie on the edges of the probability simplex. Therefore, it suffices to consider Q of the form:

$$Q(a) = \frac{b-\mu}{b-a}, \quad Q(b) = \frac{\mu - a}{b-a}, \quad \text{with } \underline{x} \le a \le \mu < b \le \overline{x}.$$

Therefore, the optimization problem reduces to the following simplified forms corresponding to the upper and lower bounds of c^* threshold:

$$\underline{f}(\underline{x}, \overline{x}, \mu, c) := \min_{a,b} \frac{b-\mu}{(1+a)(b-a)} 1_{a<c} + \frac{\mu - a}{(1+b)(b-a)} 1_{b<c} \tag{3.9}$$
$$\text{s.t. } \underline{x} \le a \le \mu < b \le \overline{x},$$
$$\overline{f}(\underline{x}, \overline{x}, \mu, c) := \max_{a,b} \frac{b-\mu}{(1+a)(b-a)} 1_{a<c} + \frac{\mu - a}{(1+b)(b-a)} 1_{b<c} \tag{3.10}$$
$$\text{s.t. } \underline{x} \le a \le \mu < b \le \overline{x},$$
$$\text{where} \quad 1_{x<c} = \begin{cases} 1, & x < c, \\ 0, & x \ge c. \end{cases}$$

Next we solve the above simplified optimization problems to find the upper and lower bounds for the integral function. Thus, define:

$$f(a,b) = \frac{b-\mu}{(1+a)(b-a)} 1_{a<c} + \frac{\mu - a}{(1+b)(b-a)} 1_{b<c}. \tag{3.11}$$

(I) If $a \leq \mu < b < c$, then $f(a,b) = \frac{1+a+b-\mu}{(1+a)(1+b)}$. We have

$$\begin{aligned}
\frac{\partial f}{\partial a} &= \frac{\mu - b}{(1+a)^2(1+b)} < 0, \\
\frac{\partial f}{\partial b} &= \frac{\mu - a}{(1+a)(1+b)^2} \geq 0, \\
\frac{\partial^2 f}{\partial a^2} &= \frac{-2(\mu - b)}{(1+a)^3(1+b)} > 0, \\
\frac{\partial^2 f}{\partial b^2} &= \frac{-2(\mu - a)}{(1+a)(1+b)^3} \leq 0.
\end{aligned}$$

Thus, $f(a,b)$ is a convex monotonically decreasing function of a and a concave monotonically non-decreasing function of b, throughout the given intervals. Therefore,

$$\begin{aligned}
\min_{a,b} f(a,b) &= \frac{1}{1+\mu} & &\text{attained when } a = \mu,\ b \downarrow \mu, \\
\max_{a,b} f(a,b) &= \frac{1+\underline{x}+c-\mu}{(1+\underline{x})(1+c)} & &\text{attained when } a = \underline{x},\ b \uparrow c.
\end{aligned}$$

(II) If $a \leq \mu < c \leq b$, then $f(a,b) = \frac{b-\mu}{(1+a)(b-a)}$, which implies

$$\begin{aligned}
\frac{\partial f}{\partial a} &= \frac{(\mu - b)(b - 2a - 1)}{(1+a)^2(b-a)^2}, \\
\frac{\partial f}{\partial b} &= \frac{\mu - a}{(1+a)(b-a)^2} \geq 0, \\
\frac{\partial^2 f}{\partial a^2} &= \frac{-2(\mu - b)[(1+a)(b-a) + (b-2a-1)^2]}{(1+a)^3(b-a)^3} > 0, \\
\frac{\partial^2 f}{\partial b^2} &= \frac{-2(\mu - a)}{(1+a)(b-a)^3} \leq 0.
\end{aligned}$$

Thus, $f(a,b)$ is convex function of a and monotonically non-decreasing function of b.

Minimum is attained by setting $b = c$. Then by using $\frac{\partial f}{\partial a}$ formula, the global minimum is obtained as follows:

$$\min_{a,b} f(a,b) = \begin{cases} \frac{1}{1+\mu} & \text{if } \mu < \frac{c-1}{2}, \quad \text{at } a = \mu, \\ \frac{c-\mu}{(1+\underline{x})(c-\underline{x})} & \text{if } \underline{x} \geq \frac{c-1}{2}, \quad \text{at } a = \underline{x}, \\ \frac{4(c-\mu)}{(c+1)^2} & \text{if } \underline{x} < \frac{c-1}{2} \leq \mu, \quad \text{at } a = \frac{c-1}{2}. \end{cases} \tag{3.12}$$

Likewise, the maximum is attained by setting $b = \overline{x}$. Then by using $\frac{\partial f}{\partial a}$ formula, the global maximum is obtained as follows:

$$\max_{a,b} f(a,b) = \begin{cases} \frac{\overline{x}-\mu}{(1+\underline{x})(\overline{x}-\underline{x})} & \text{if } \mu \leq \overline{x} - \underline{x} - 1, \quad \text{at } a = \underline{x}, \\ \frac{1}{1+\mu} & \text{if } \mu > \overline{x} - \underline{x} - 1, \quad \text{at } a = \mu. \end{cases} \tag{3.13}$$

(III) If $a < c \leq \mu < b$, then similarly to section II, $f(a,b)$ is a convex function of a and monotonically non-decreasing function of b.

To attain the minimum, set $b = \lim_{\varepsilon \downarrow 0} \mu + \varepsilon$, then according to $\frac{\partial f}{\partial a}$, $f(a,b)$ the global minimum for this case is zero:

$$\min_{a,b} f(a,b) = \begin{cases} 0 & \text{if } c \leq \frac{\mu-1}{2}, \quad \text{as } a \uparrow c, \\ 0 & \text{if } \underline{x} > \frac{\mu-1}{2}, \quad \text{at } a = \underline{x}, \\ 0 & \text{if } \underline{x} \leq \frac{\mu-1}{2} < c, \quad \text{at } a = \frac{\mu-1}{2}. \end{cases} \tag{3.14}$$

Also, to obtain maximum, set $b = \overline{x}$, then due to $\frac{\partial f}{\partial a}$, when $a = \frac{\overline{x}-1}{2}$, $f(a,b)$ reaches the global maxima. It is clear that

$$\max_{a,b} f(a,b) = \begin{cases} \frac{\overline{x}-\mu}{(1+\underline{x})(\overline{x}-\underline{x})} & \text{if } c \leq \overline{x} - \underline{x} - 1, \quad \text{at } a = \underline{x}, \\ \frac{\overline{x}-\mu}{(1+c)(\overline{x}-c)} & \text{if } c > \overline{x} - \underline{x} - 1, \quad \text{as } a \uparrow c. \end{cases} \tag{3.15}$$

(IV) If $c \leq a \leq \mu < b$, then $f(a,b) = 0$.

Combining sections I, II, III and IV gives the results of $\underline{f}(\underline{x}, \overline{x}, \mu, c)$ and $\overline{f}(\underline{x}, \overline{x}, \mu, c)$ as shown in (3.5), (3.6).

□

Corollary 1. *The upperbound and lowerbound values for the c^* are found using the following expressions and are the tightest possible bounds covering all distributions with given $\underline{x}, \overline{x}, \mu$ parameters of the arrival distribution:*

$$\underline{c}(\underline{x}, \overline{x}, \mu) := \sup \left\{ c \in (\underline{x}, \overline{x}) : \frac{1}{1+c} \geq \overline{f}(\underline{x}, \overline{x}, \mu, c) \right\} \tag{3.16}$$

$$\overline{c}(\underline{x}, \overline{x}, \mu) := \sup \left\{ c \in (\underline{x}, \overline{x}) : \frac{1}{1+c} \geq \underline{f}(\underline{x}, \overline{x}, \mu, c) \right\} \tag{3.17}$$

Further, we find the expressions for these bounds:

Theorem 1. *The lowerbound for the threshold c^* among all arrival distributions with given $\underline{x}, \overline{x}, \mu$ parameters is:*

$$\underline{c}(\underline{x}, \overline{x}, \mu) = \begin{cases} \mu & \mu \geq \overline{x} - \underline{x} - 1, \\ \frac{(1+\underline{x})(\overline{x}-\underline{x})}{\overline{x}-\mu} - 1 & \mu < \overline{x} - \underline{x} - 1. \end{cases} \tag{3.18}$$

Proof. To prove this theorem, we combine (3.16) with the result of $\overline{f}(\underline{x}, \overline{x}, \mu, c)$ in Proposition 1. We have:

(I) if $c \leq \overline{x} - \underline{x} - 1$:

$$\frac{1}{1+c} \geq \frac{\overline{x} - \mu}{(1+\underline{x})(\overline{x} - \underline{x})} \implies c \leq \frac{(1+\underline{x})(\overline{x} - \underline{x})}{\overline{x} - \mu} - 1,$$

which together with the condition $c \leq \overline{x} - \underline{x} - 1$ gives:

$$c \leq \min \left\{ \overline{x} - \underline{x} - 1, \quad \frac{(1+\underline{x})(\overline{x} - \underline{x})}{\overline{x} - \mu} - 1 \right\}.$$

(II) if $c > \overline{x} - \underline{x} - 1$ and $\mu < c$:

$$\frac{1}{1+c} \geq \frac{1 + \underline{x} + c - \mu}{(1+c)(1+\underline{x})} \implies c \leq \mu.$$

The above obtained condition does not have any intersection with the case's condition, therefore, this set is empty.

(III) if $c > \overline{x} - \underline{x} - 1$ and $\mu \geq c$:

$$\frac{1}{1+c} \geq \frac{\overline{x} - \mu}{(1+c)(\overline{x} - c)} \implies c \leq \mu.$$

By finding the intersection of the above condition and the case's condition, we have:

$$\overline{x} - \underline{x} - 1 < c \leq \mu.$$

Thus, combining I, II and III cases, we find the interval for c as:

$$\left\{c \leq \min\left\{\overline{x} - \underline{x} - 1, \quad \frac{(1+\underline{x})(\overline{x} - \underline{x})}{\overline{x} - \mu} - 1\right\}\right\} \cup$$
$$\emptyset \cup$$
$$\{\overline{x} - \underline{x} - 1 < c \leq \mu\}.$$

Thus, under the following conditions, we have:

(i) if $\mu > \overline{x} - \underline{x} - 1$ the set reduces to:

$$\{c \leq \overline{x} - \underline{x} - 1\} \cup \emptyset \cup \{\overline{x} - \underline{x} - 1 < c \leq \mu\} = \{c \leq \mu\}$$

(ii) if $\mu \leq \overline{x} - \underline{x} - 1$ the set reduces to:

$$\left\{c \leq \frac{(1+\underline{x})(\overline{x} - \underline{x})}{\overline{x} - \mu} - 1\right\} \cup \emptyset \cup \emptyset = \left\{c \leq \frac{(1+\underline{x})(\overline{x} - \underline{x})}{(\overline{x} - \mu)} - 1\right\}$$

which proves the formula of the theorem.

$\square$

Theorem 2. *The upperbound for the threshold c^* among all arrival distributions with given $\underline{x}, \overline{x}, \mu$ parameters is:*

$$\overline{c}(\underline{x}, \overline{x}, \mu) = \begin{cases} \frac{\mu+\underline{x}+\sqrt{(\mu+\underline{x})^2-4(\underline{x}^2+\underline{x}-\mu)}}{2} & \mu < \frac{3\underline{x}+1}{2}, \\ \frac{4\mu+1}{3} & \mu \geq \frac{3\underline{x}+1}{2}. \end{cases} \tag{3.19}$$

Proof. To find the upper-limit, we combine (3.17) with the result of $\underline{f}(\underline{x}, \overline{x}, \mu, c)$ in Proposition 1. Therefore, we form the expressions for all cases:

(I) if $\mu < c$ and $\mu < \frac{c-1}{2}$:

$$\frac{1}{1+c} \geq \frac{1}{1+\mu} \implies \mu \geq c.$$

The derived condition does not match with the case's assumptions. Therefore, this case is an empty set.

(II) if $\mu < c$ and $\underline{x} < \frac{c-1}{2} \leq \mu$:

$$\frac{1}{1+c} \geq \frac{4(c-\mu)}{(c+1)^2} \implies c \leq \frac{4\mu+1}{3},$$

which together with the case's assumptions results into:

$$\max\{\mu, 2\underline{x}+1\} < c \leq \frac{4\mu+1}{3}.$$

(III) $\mu < c$ and $\underline{x} \geq \frac{c-1}{2}$:

$$\begin{aligned}&\frac{1}{1+c} \geq \frac{c-\mu}{(1+\underline{x})(c-\underline{x})} \implies \\ &\frac{\mu+\underline{x}-\sqrt{(\mu+\underline{x})^2-4(\underline{x}^2+\underline{x}-\mu)}}{2} \leq c \\ \leq &\frac{\mu+\underline{x}+\sqrt{(\mu+\underline{x})^2-4(\underline{x}^2+\underline{x}-\mu)}}{2}.\end{aligned}$$

Next we find the intersection of above condition and the case's assumptions. But because the smaller root of the above quadratic function is less than μ, then it simplifies to:

$$\mu < c \leq \min\left\{2\underline{x}+1, \quad \frac{\mu+\underline{x}+\sqrt{(\mu+\underline{x})^2-4(\underline{x}^2+\underline{x}-\mu)}}{2}\right\}.$$

Then we find the interval in which, each of the two terms are minimum:

- if $\underline{x} \geq \frac{2\mu-1}{3}$:

$$\mu < c \leq \frac{\mu+\underline{x}+\sqrt{(\mu+\underline{x})^2-4(\underline{x}^2+\underline{x}-\mu)}}{2}.$$

- if $\underline{x} < \frac{2\mu-1}{3}$:

$$\mu < c \leq 2\underline{x}+1.$$

(IV) if $c \leq \mu$: no constraint for this case.

Therefore, by combining the above cases I, II, III and IV, the region of c according to (3.17) is found:

$$\emptyset \cup$$
$$\left\{\max\{\mu, 2\underline{x}+1\} < c \leq \frac{4\mu+1}{3}\right\} \cup$$
$$\left\{\mu < c \leq \min\left\{2\underline{x}+1,\ \frac{\mu+\underline{x}+\sqrt{(\mu+\underline{x})^2 - 4(\underline{x}^2+\underline{x}-\mu)}}{2}\right\}\right\} \cup$$
$$\{c \leq \mu\}.$$

Finally, we find the union of the above sets under the following conditions:

(i) if $2\underline{x}+1 < \mu$:

$$\emptyset \cup \left\{\mu < c \leq \frac{4\mu+1}{3}\right\} \cup \emptyset \cup \{c \leq \mu\} = \left\{c \leq \frac{4\mu+1}{3}\right\}.$$

(ii) if $\mu \leq 2\underline{x}+1 < \frac{4\mu+1}{3}$:

This translates to $\frac{\mu-1}{2} \leq \underline{x} < \frac{2\mu-1}{3}$. Therefore, the union is:

$$\emptyset \cup \left\{2\underline{x}+1 < c \leq \frac{4\mu+1}{3}\right\} \cup \{\mu < c \leq 2\underline{x}+1\} \cup \{c \leq \mu\} = \left\{c \leq \frac{4\mu+1}{3}\right\}.$$

(iii) if $2\underline{x}+1 \geq \frac{4\mu+1}{3}$:

This translates to $\underline{x} \geq \frac{2\mu-1}{3}$, thus the union is:

$$\emptyset \cup \emptyset \cup \left\{ \mu < c \leq \frac{\mu + \underline{x} + \sqrt{(\mu+\underline{x})^2 - 4(\underline{x}^2 + \underline{x} - \mu)}}{2} \right\} \cup \{c \leq \mu\}$$
$$= \left\{ c \leq \frac{\mu + \underline{x} + \sqrt{(\mu+\underline{x})^2 - 4(\underline{x}^2 + \underline{x} - \mu)}}{2} \right\}.$$

With the above cases and the given c thresholds for each of them, (3.19) is resulted and the theorem is proved.

□

3.2 Semi-Universal Bounds on c^* given $\underline{x}$, μ_c of Clipped Arrival Distribution

In addition to bounds found on the greedy threshold (c^*) based on the parameters of actual arrival distribution in previous section, it is also possible to examine the bounds based on parameters of the clipped arrival distribution; i.e. the amount of energy stored in battery which cannot exceed the value c.

Therefore, we first define $\mathcal{Q}'_{\underline{x},\mu_c}$ as set of clipped distributions with given parameters, where μ_c is the clipped (effective) mean:

$$\mu_c = \mathbb{E}\left[Q'\right] = \mathbb{E}\left[\min(Q, c)\right].$$

Note that according to [Theorem 1 [20]], the greedy threshold c^* is bounded with $\underline{x} < c^* \leq \overline{x}$. Therefore, we don't need to consider parameter $\overline{x}$ in this case.

Proposition 2. *Having the integral function* $\int_{[0,c)} \frac{1}{1+x} dQ'$ *where* Q' *is the set of clipped arrival distributions with given parameters* $Q' \in \mathcal{Q}'_{\underline{x},\overline{x},\mu_c}$*, the integral function is bounded below by the following function:*

$$\underline{g}(\underline{x}, \mu_c, c) = \begin{cases} \frac{1}{1+\mu_c} & \mu_c < \frac{c-1}{2} \\ \frac{4(c-\mu_c)}{(c+1)^2} & \underline{x} < \frac{c-1}{2} \leq \mu_c \\ \frac{c-\mu_c}{(1+\underline{x})(c-\underline{x})} & \underline{x} \geq \frac{c-1}{2} \end{cases} \tag{3.20}$$

Furthermore, the integral function is bounded above by the following function:

$$\overline{g}(\underline{x}, \mu_c, c) = \frac{1+\underline{x}+c-\mu_c}{(1+\underline{x})(1+c)} \tag{3.21}$$

Proof. Using the similar procedure in the previous section, we assume that Q' is a probability mass function with support $\underline{x} \leq x_0 \leq ... \leq x_N \leq c$:

$$\int_{[0,c)} \frac{1}{1+x} dQ' = \sum_{i=0}^{N-1} \frac{1}{1+x_i} Q'(x_i). \tag{3.22}$$

Therefore, the optimization problem is reformulated as follows:

$$\begin{aligned} (\max) \text{ or } \min_{Q'(x_i), i=0,1,...,N} \; & \sum_{i=0}^{N-1} \frac{1}{1+x_i} Q'(x_i) \\ \text{s.t. } & \sum_{i=0}^{N} x_i Q'(x_i) = \mu_c \\ & \sum_{i=0}^{N} Q'(x_i) = 1 \\ & Q'(x_i) \geq 0, \; i = 0, 1, ..., N. \end{aligned} \tag{3.23}$$

Similarly, this is a linear programming problem and its (maximum) minimum is attained at a certain vertex of the domain. Since the domain is given by the intersection of the probability simplex and the hyperplane $\sum_{i=0}^{N} x_i Q'(x_i) = \mu_c$. Its vertices must lie on the edges of the probability simplex. Therefore, it suffices to consider Q' of the form (with different intervals from Q):

$$Q'(a) = \frac{b-\mu_c}{b-a}, \quad Q'(b) = \frac{\mu_c - a}{b-a}, \quad \text{with } \underline{x} \le a \le \mu_c < b \le c.$$

Then, the upper and lower bounds for the integral function can be obtained among the clipped distributions by solving the following optimization problems:

$$\begin{aligned} \underline{g}(\underline{x}, \mu_c, c) &:= \min_{a,b} \frac{b-\mu_c}{(1+a)(b-a)} + \frac{\mu_c - a}{(1+b)(b-a)} 1_{b<c} & (3.24)\\ &\text{s.t. } \ \underline{x} \le a \le \mu_c < b \le c, \\ \overline{g}(\underline{x}, \mu_c, c) &:= \max_{a,b} \frac{b-\mu_c}{(1+a)(b-a)} + \frac{\mu_c - a}{(1+b)(b-a)} 1_{b<c} & (3.25)\\ &\text{s.t. } \ \underline{x} \le a \le \mu_c < b \le c. \end{aligned}$$

(I) If $b < c$, then similar to section (I) in the proof of Proposition 1, the results are:

$$\min_{a,b} f(a,b) = \frac{1}{1+\mu_c} \qquad \text{attained when } a = \mu_c,\ b \downarrow \mu_c, \tag{3.26}$$

$$\max_{a,b} f(a,b) = \frac{1+\underline{x}+c-\mu_c}{(1+\underline{x})(1+c)} \qquad \text{attained when } a = \underline{x},\ b \uparrow c. \tag{3.27}$$

(II) If $b = c$, then $f(a,b)$ changes from (3.11) to

$$f(a) = \frac{c - \mu_c}{(1+a)(c-a)}, \tag{3.28}$$

which is similar to section (II) in the proof of Proposition 1, with constraint that $c = b$. Therefore, $f(a)$ is a convex function and:

$$\frac{df}{da} = \frac{(\mu_c - c)(c - 2a - 1)}{(1+a)^2(c-a)^2}.$$

For the minima,

$$\min_{a,b} f(a,b) = \begin{cases} \frac{1}{1+\mu_c} & \text{if } \mu_c < \frac{c-1}{2}, \quad \text{at } \ a = \mu_c, \\ \frac{c-\mu_c}{(1+\underline{x})(c-\underline{x})} & \text{if } \underline{x} \geq \frac{c-1}{2}, \quad \text{at } a = \underline{x}, \\ \frac{4(c-\mu_c)}{(c+1)^2} & \text{if } \underline{x} < \frac{c-1}{2} \leq \mu_c, \quad \text{at } a = \frac{c-1}{2}. \end{cases} \tag{3.29}$$

For the maxima,

$$\max_{a,b} f(a,b) = \begin{cases} \frac{c-\mu_c}{(1+\underline{x})(c-\underline{x})} & \text{if } \mu_c \leq c - \underline{x} - 1, \quad \text{at } a = \underline{x}, \\ \frac{1}{1+\mu_c} & \text{if } \mu_c > c - \underline{x} - 1, \quad \text{at } a = \mu_c. \end{cases} \tag{3.30}$$

Combining section I and II gives the proofs of (3.20) and (3.21). □

Corollary 2. *The upperbound and lowerbound values for the c^* are found using the following expressions and are the tightest possible bounds covering all distributions* *with given* $\underline{x}, \overline{x}, \mu_c$ *parameters of the clipped distributions:*

$$\underline{c}'(\underline{x}, \overline{x}, \mu_c) := \sup\left\{c \in (\mu_c, \overline{x}) : \frac{1}{1+c} \geq \overline{g}(\underline{x}, \mu_c, c)\right\}, \tag{3.31}$$

$$\overline{c}'(\underline{x}, \overline{x}, \mu_c) := \sup\left\{c \in (\mu_c, \overline{x}) : \frac{1}{1+c} \geq \underline{g}(\underline{x}, \mu_c, c)\right\}. \tag{3.32}$$

Further, we find the expressions for these bounds:

Theorem 3. *Assume having the set of all clipped distributions with given* $\underline{x}, \mu_c$ *parameters* ($Q_{\underline{x},\mu_c}$). *There exists a sequence of clipped distributions*

$$Q_n(x) = \begin{cases} \frac{\mu_c - \underline{x}}{b_n - \underline{x}} & x = b_n \\ \\ \frac{b_n - \mu_c}{b_n - \underline{x}} & x = \underline{x} \end{cases}$$

where $b_1, ..., b_n > \mu_c$ *is a sequence of numbers such that* $lim_{n\to\infty} b_n = \mu_c$*. Thus, a sequence of greedy thresholds* c_n^* *exists for these distributions which converges to the infimum bound of greedy thresholds among* $Q_{\underline{x},\mu_c}$ *as*

$$\underline{c}'(\underline{x}, \overline{x}, \mu_c) = \mu_c. \tag{3.33}$$

Proof. Given $c > \mu_c$, using supremum value in (3.27) from proof of proposition 2, we form the inequality of (3.31):

$$\frac{1}{1+c} \geq \lim_{\beta \uparrow 0} \frac{1 + \underline{x} + c + \beta - \mu_c}{(1+\underline{x})(1+c)} \implies c \leq \lambda(\underline{x}, \beta, \mu_c) \tag{3.34}$$

where

$$\lambda(\underline{x}, \beta, \mu_c) = \frac{\mu_c - \beta - 1 + \sqrt{(\mu_c - \beta - 1)^2 + 4(\mu_c + \beta \underline{x})}}{2}$$

is strictly greater than μ_c for $\beta < 0$ and converges to it when $\beta \to 0$, i.e.

$$(\forall \epsilon \in \mathbb{R}, \epsilon > 0; \exists \delta \in \mathbb{R}, \delta > 0 \text{ and } \exists N \in \mathbb{N}, N > 0; \forall n \geq N)$$
$$(0 < c - b_n \leq \delta \implies 0 < \lambda(\underline{x}, b_n - c, \mu_c) - \mu_c \leq \epsilon).$$

The last inequality implies that under given assumptions,

$$\mu_c < \lambda(\underline{x},\, b_n - c,\, \mu_c) \leq \mu_c + \epsilon. \tag{3.35}$$

Also, from (3.34), we have

$$c_n \leq \lambda(\underline{x}, b_n - c, \mu_c)$$

where c_n is the variable representing the battery capacity value corresponding to the case where arrival distribution is formed by b_n. Combining them together and finding the supremum according to (3.31) suggests that

$$c_n^* = \mu_c + \epsilon$$

which proves the desired result. □

Theorem 4. *The upperbound for the threshold c^* among all clipped energy distributions with given $\underline{x}, \overline{x}, \mu_c$ parameters is:*

$$\overline{c}'(\underline{x}, \overline{x}, \mu_c) = \begin{cases} \frac{\mu_c+\underline{x}+\sqrt{(\mu_c+\underline{x})^2-4(\underline{x}^2+\underline{x}-\mu_c)}}{2} & \mu_c < \frac{3\underline{x}+1}{2} \\ \frac{4\mu_c+1}{3} & \mu_c \geq \frac{3\underline{x}+1}{2} \end{cases} \tag{3.36}$$

Proof. The mathematical derivation is the same as $\overline{c}(\underline{x}, \overline{x}, \mu)$ except for the $c \leq \mu$ parts. □

3.3 Upperbound on c^* Greedy Threshold

Assuming a family of clipped arrival distributions $\mathcal{Q}'_{c,p}$ where c is the battery capacity (or the clipping point of stored energy distribution) and p is the mean to capacity ratio as:

$$\text{MCR}_c(Q') = \frac{\mu_c}{c} = p.$$

It is tempting to investigate if the greedy policy threshold c^* is different among members of the $\mathcal{Q}'_{c,p}$ family, and finding its boundary values and corresponding distributions. For this purpose, we first find the minimum value of the RHS integral in (3.2).

Proposition 3.

$$let\ \nu(c,p) = \min_{Q' \in \mathcal{Q}'_{c,p}} \int_{[0,c)} \frac{1}{1+x} dQ', then:$$

- *For* $p \in (0, \frac{1}{2})$*:*

$$\nu(c,p) = \begin{cases} 1-p & c \leq 1, \\ \frac{4c(1-p)}{(c+1)^2} & 1 < c \leq \frac{1}{1-2p}, \\ \frac{1}{1+cp} & c > \frac{1}{1-2p}. \end{cases} \tag{3.37}$$

- *For* $p \in [\frac{1}{2}, 1)$*:*

$$\nu(c,p) = \begin{cases} 1-p & c \leq 1, \\ \frac{4c(1-p)}{(c+1)^2} & c > 1. \end{cases} \tag{3.38}$$

Proof. The first part of the proof is similar to the one in proof of Proposition 2. We assume that the support of Q' is given as $0 = x_0 \leq ... \leq x_N = c$. Therefore, with the hyperplane $\sum_{i=0}^{N} x_i Q'(x_i) = pc$, consider Q' of the following form:

$$Q'(a) = \frac{b-pc}{b-a}, \quad Q'(b) = \frac{pc-a}{b-a}, \quad \text{with } \ 0 \leq a \leq pc < b \leq c.$$

This reduces the optimization problem to the following simplified form:

$$\begin{aligned} \nu(c,p) := \min_{a,b} \ & \frac{b-pc}{(1+a)(b-a)} + \frac{pc-a}{(1+b)(b-a)} 1_{b<c} \\ \text{s.t. } \ & 0 \leq a \leq pc < b \leq c, \end{aligned} \tag{3.39}$$

$$\text{where} \quad 1_{x<c} = \begin{cases} 1, & x < c, \\ 0, & x \geq c. \end{cases}$$

Furthermore, the result of this optimization problem is also similar to $\underline{g}(\underline{x}, \mu_c, c)$ (3.20) in Proposition 2, which is:

$$\nu(c,p) = \begin{cases} \frac{1}{1+cp} & \frac{c-1}{2} > cp, \text{ attained when } a = cp, b = c \text{ or } b \downarrow cp, \\ \frac{4c(1-p)}{(c+1)^2} & 0 < \frac{c-1}{2} \leq cp, \text{ attained when } a = \frac{c-1}{2}, b = c, \\ 1-p & \frac{c-1}{2} \leq 0, \text{ attained when } a = 0, b = c. \end{cases}$$

Then, split the result with respect to the value of p. When $p \in (0, \frac{1}{2})$, $\nu(c,p)$ reformulates into 3.37 and when $p \in [\frac{1}{2}, 1)$, $\nu(c,p)$ reformulates into 3.38. □

Thus, using the above proposition, the boundaries can be found:

Theorem 5. *The highest possible threshold for the battery capacity in which the greedy policy is optimal (c^*) among the set of ${\mathcal{Q}'}_{c,p}$ clipped arrival distributions depends on the range of p (MCR) as follows:*

- $p \in (0, \frac{1}{2})$*: The highest possible threshold is* $\frac{p}{1-p}$ *achieved by* $\tilde{B}_p$*. No non-constant clipped distribution* Q' *has a greedy threshold above that.*

- $p \in [\frac{1}{2}, \frac{3}{4})$*: The highest possible threshold is* $\frac{1}{3-4p}$ *achieved by the following distribution. No non-constant clipped distribution has a greedy threshold above that value.*

$$Q'(x) = \begin{cases} \frac{2c(1-p)}{c+1} & x = \frac{c-1}{2}, \\ \frac{2pc-c+1}{c+1} & x = c. \end{cases}$$

- $p \in [\frac{3}{4}, 1)$*: There is no uniform upperbound on the greedy policy threshold among the set of all non-constant distributions.*

Proof. To obtain the greedy threshold, we form the inequality in (3.2), and replace the RHS with the values from Proposition 3.

- For $p \in (0, \frac{1}{2})$:

$$\frac{1}{1+c} \geq \nu(c,p) = \begin{cases} 1-p & c \leq 1, \\ \frac{4c(1-p)}{(c+1)^2} & 1 < c \leq \frac{1}{1-2p}, \\ \frac{1}{1+cp} & c > \frac{1}{1-2p}, \end{cases}$$

$$\implies \begin{cases} c \leq \frac{p}{1-p} & c \leq 1 \implies p \leq \frac{1}{2} \quad \text{(valid)}, \\ c \leq \frac{1}{3-4p} & 1 < c \leq \frac{1}{1-2p} \implies \frac{3}{4} < p \leq 1 \quad \text{(invalid)}, \\ p \geq 1 & c > \frac{1}{1-2p} \quad \text{(invalid)}, \end{cases}$$

$$\therefore c \leq \frac{p}{1-p}.$$

Also, to find the boundary distribution, recall the values of a, b under different condition interals in proof of Proposition 3:

$$\begin{cases} Q'(a) = 1-p & a = 0, \\ Q'(b) = p & b = c, \end{cases}$$

which is the same as $\tilde{B}_p$ Bernoulli distribution:

$$\tilde{B}_p(x) = \begin{cases} 1-p & x = 0, \\ p & x = c. \end{cases}$$

Combining the equations 3.2 and 3.7, the threshold condition for $\tilde{B}_p$ reduces to $c \leq \frac{p}{1-p}$.

- For $p \in [\frac{1}{2}, 1)$:

$$\frac{1}{1+c} \geq \nu(c,p) = \begin{cases} 1-p & c \leq 1, \\ \frac{4c(1-p)}{(c+1)^2} & c > 1, \end{cases}$$

$$\implies \begin{cases} c \leq \frac{p}{1-p} & c \leq 1, \\ (3-4p)c \leq 1 & c > 1, \end{cases}$$

$$\implies \begin{cases} p \leq \frac{1}{2} & c \leq 1 \quad \text{(invalid)}, \\ c \leq \frac{1}{3-4p} & p < \frac{3}{4}; \quad c > 1 \implies p > \frac{1}{2} \quad \text{(valid)}, \\ c \geq 0 & p \geq \frac{3}{4}; \quad c > 1 \quad \text{(valid)}, \end{cases}$$

$$\therefore \begin{cases} c \leq \frac{1}{3-4p} & \frac{1}{2} \leq p < \frac{3}{4}, \\ \text{no upperbound} & \frac{3}{4} \leq p \leq 1. \end{cases}$$

Again, to find the boundary distribution, recall the values of a, b corresponding to the conditions from proof of Proposition 3:

$$\begin{cases} Q'(a) = \frac{2c(1-p)}{c+1} & a = \frac{c-1}{2}, \\ Q'(b) = \frac{2pc-c+1}{c+1} & b = c, \end{cases}$$

which gives the proof of Theorem 5.

□

Chapter 4

Optimal Linear Policy

We will study the optimal slope s among the family of the linear policies as described in (2.8). As proved in [Proposition 5 [18]], the n-horizon expected throughput of any i.i.d distribution is higher than the throughput of i.i.d Bernoulli case with the same clipped mean for any power control policy. Therefore, finding the linear policy with highest possible throughput for the Bernoulli arrivals, will establish the optimal linear policy, as it is guaranteed that the performance would be at least as good as the Bernoulli case.

Further, under the condition of Bernoulli(p) energy arrivals, the epoch length in (2.6) is distributed as $L \sim \text{Geometric}(p)$. Also, the linear policy (2.8) can be written in the following format:

$$g(E^t) = cs(1-s)^{n-j_t} \tag{4.1}$$

where j_t is the most recent time in which system reset to $b_{j_t} = c$. With these settings, for the Bernoulli(p) energy arrivals and when using the linear policies, the lowerbound in 2.6 reduces to:

$$\begin{aligned}\underline{\Gamma}(c,p,s) &= p\sum_{l=1}^{\infty} p(1-p)^{l-1}\sum_{i=1}^{l}\frac{1}{2}\log(1+cs(1-s)^{i-1}) \\ &= \frac{p}{2}\sum_{n=0}^{\infty}(1-p)^n\log(1+cs(1-s)^n) \qquad (4.2)\\ &= \frac{1}{2}\mathbb{E}_N[\log(1+cs(1-s)^N)] \qquad (4.3)\end{aligned}$$

where expectation is over $N \sim$ Geometric Failure(p). The universal upperbound in (2.7) can be written as

$$\overline{\Gamma}(c,p) = \frac{1}{2}\log(1+cp). \tag{4.4}$$

To find the optimal linear policy, one can maximize (4.2) by finding the optimal s^* as a function of $c > 0$ and $p \in (0,1]$:

$$s^*(c,p) = \arg\max_{s\in(0,1]} \underline{\Gamma}(c,p,s). \tag{4.5}$$

Although it is not easy to find the analytical expression for $s^*(c,p)$, it is possible to find it numerically as this is a one-time calculation as a part of the initial process. Thus, under the following conditions the optimal s^* is obtained:

- if $c \leq \frac{p}{1-p}$: The optimal value $s^*(c,p) = 1$ which means greedy policy is optimal. This was discussed in Chapter 3.

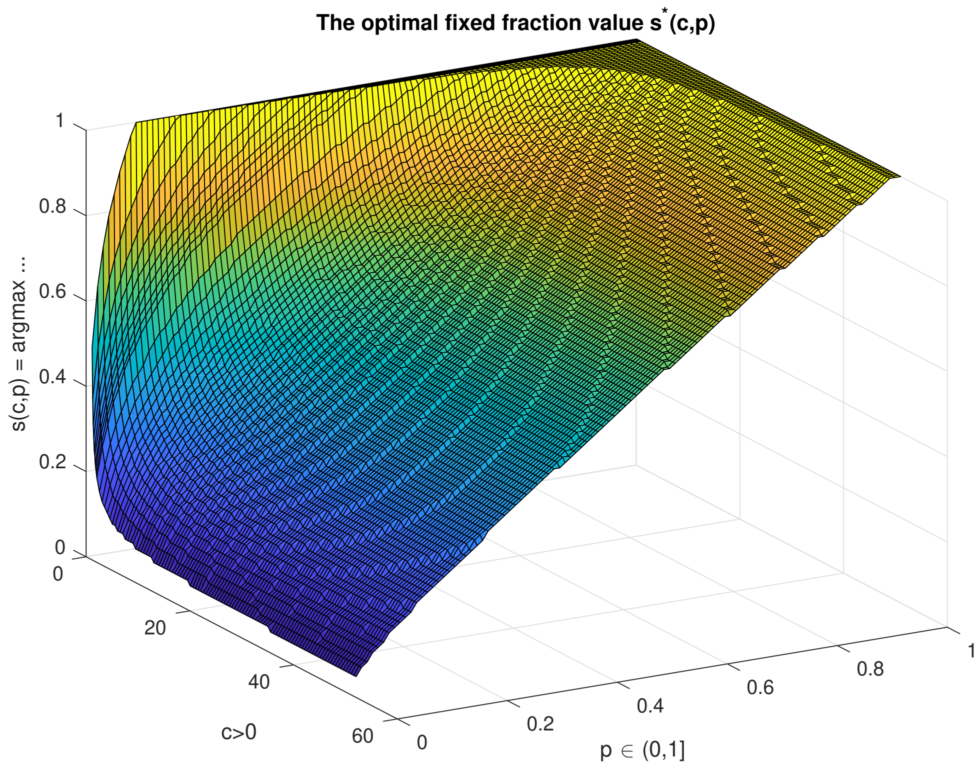


FIGURE 4.1: Optimal value of s^*. The upper corner shows the region in which the greedy policy is optimal. The plot shows that $s^*(c,p) \geq p$ and converges to p when $c \to \infty$.

- if $c > \frac{p}{1-p}$: Optimal value can be obtained by numerically solving the following transcendental equality:

$$\frac{\partial \underline{\Gamma}(c,p,s)}{\partial s}\bigg|_{s=s^*} = \frac{1}{2}\mathbb{E}_N\left[\frac{c(1-(N+1)s^*)(1-s^*)^{N-1}}{1+cs^*(1-s^*)^N}\right] = 0$$

$$\Longrightarrow \mathbb{E}_N\left[\frac{s^*(N+1)-1}{1+cs^*(1-s^*)^N}\right] = \frac{s^*}{p} - 1. \tag{4.6}$$

A 3-d plot of the surface of $s^*(c,p)$ is shown in figure 4.1.

Note 1. *By examining the numerical results and plot, we note that the optimal $s^*(c,p)$ is a monotonically non-decreasing function of p for any given c, and is a monotonically non-increasing function of c for any given p.*

Proposition 4. *The asymptotic regimes of $s^*(c,p)$ among capacity c is as follows:*

$$\lim_{c\to\infty} s^*(c,p) = p \tag{4.7}$$

$$\lim_{c\to 0} s^*(c,p) = 1. \tag{4.8}$$

Proof.

- when $c \downarrow 0$:
  - if $c \le \frac{p}{1-p}$: then $s(c,p) = 1$ according to Chapter 3.
  - if $c \ge \frac{p}{1-p}$: this forces $p \to 0$ as well.

$$\begin{aligned}
&\frac{\partial \underline{\Gamma}(c,p,s)}{\partial s} = \frac{p}{2}\sum_{n=0}^{\infty}(1-p)^n \frac{c(1-s)^{n-1}(1-(n+1)s)}{1+cs(1-s)^n} \Rightarrow \\
&\lim_{c\to 0}\frac{\partial \underline{\Gamma}(c,p,s)}{\partial s} = \frac{cp}{2}\sum_{n=0}^{\infty}(1-p)^n\left[(1-s)^n - ns(1-s)^{n-1}\right] \\
&= \frac{cp}{2}\left[\frac{1}{p+s-ps} - \frac{s(1-p)(1-s)}{(1-s)(p+s-ps)^2}\right] > 0 \\
&\implies \underline{\Gamma}(c,p,s) \text{ is a increasing function,}
\end{aligned}$$

$$\therefore \lim_{c\to 0} s^*(c,p) = 1.$$

- when $c \uparrow \infty$:

$$\lim_{c\to\infty} \frac{p}{2}\sum_{n=0}^{\infty}(1-p)^n \log(1+cs(1-s)^n) = \lim_{c\to\infty} \log(cs) + \frac{1-p}{p}\log(1-s)$$
$$\frac{\partial}{\partial s} = \frac{c}{cs} + \frac{1-p}{p}\frac{-1}{1-s} = 0$$
$$\therefore \lim_{c\to\infty} s^*(c,p) = p.$$

$\square$

Corollary 3. *Combining the monotonicity of s^* and the above proposition, it follows that $s^*(c,p) \geq p$ for any $c > 0$.*

Proposition 5. *In the asymptotic scenario when $c \to +\infty$ and $p \to 0$ simultaneously, assume $s := ap$, where a is a fixed real number in $a \in [1,+\infty)$. Also, let $b := cp$ be a fixed non-zero number $b \in (0,+\infty)$. The lowerbound $\underline{\Gamma}(c,p,s)$ converges to:*

$$\underline{\Gamma}_{asym}(a,b) = \frac{1}{2}\int_0^{\infty} e^{-x}\log(1+abe^{-ax})dx \tag{4.9}$$
$$= \frac{1}{2}\mathbb{E}_X\left[\log(1+abe^{-aX})\right] \tag{4.10}$$

where X is a continous random variable with $X \sim exp(1)$.

Proof.

$$
\begin{aligned}
\lim_{\substack{c\to\infty\\ p\to 0\\ b=cp;s=ap}} \underline{\Gamma}(c,p,s) &= \lim \frac{1}{2}\sum_{n=0}^{\infty} p(1-p)^n \log(1+ab(1-ap)^n) \\
&= \lim \frac{1}{2}\sum_{n=0}^{\infty} p(1-p)^{\frac{1}{p}.np} \log(1+ab(1-ap)^{\frac{1}{p}.np}) \\
&= \lim \frac{1}{2}\sum_{n=0}^{\infty} pe^{-np} \log(1+abe^{-a.np}) \\
&= \int_0^\infty e^{-x}\log(1+abe^{-ax})dx.
\end{aligned}
$$

□

4.1 Multiplicative Ratio Optimal Linear policy

Next we form the multiplicative ratio by using the optimal s^*, and express the lowerbound on it as follows:

$$
\underline{F}^* = \inf_{c>0, p\in(0,1]} \frac{\underline{\Gamma}(c,p,s^*(c,p))}{\overline{\Gamma}(c,p)}. \tag{4.11}
$$

A 3-d plot of the multiplicative ratio when using s^* is shown in figure 4.2 as a function of $c > 0$ and $p \in (0, 1]$.

The numerical results show that the infimum of $F(s^*)$ happens at $c \to \infty, p \to 0$, which using Proposition 5 simplifies to:

$$
\underline{F}^* = \inf_{b\in(0,+\infty)} \frac{\int_0^\infty e^{-x}\log(1+a^*be^{-a^*x})dx}{\log(1+b)}. \tag{4.12}
$$

By numerically solving optimization problem (4.12), the multiplicative ratio is

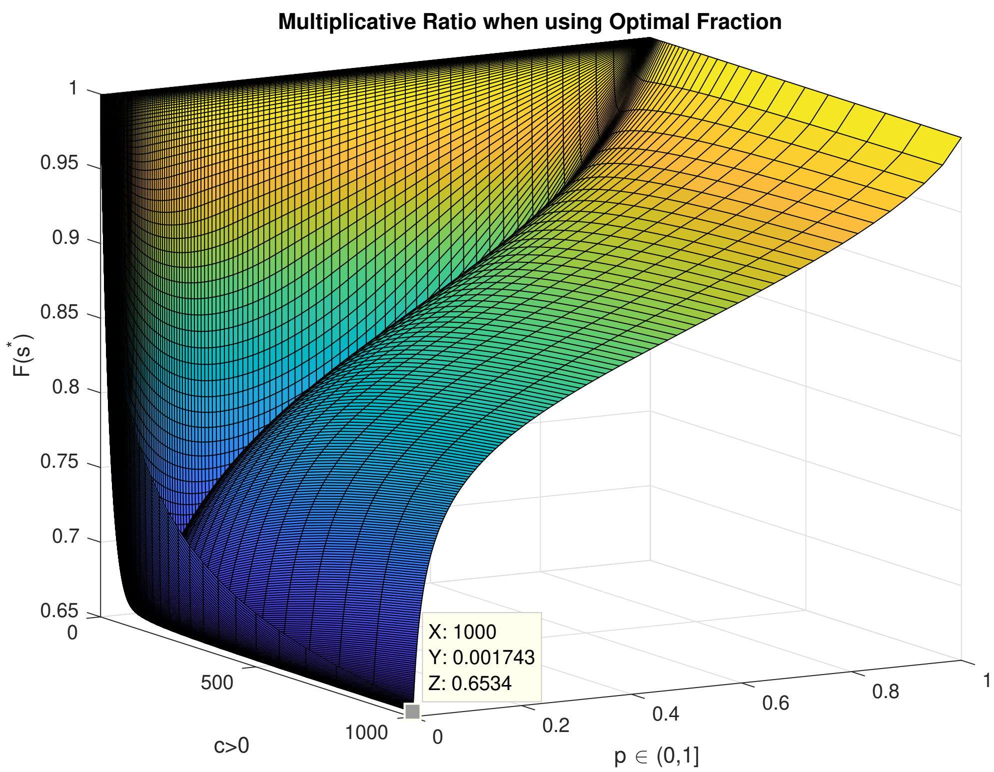


FIGURE 4.2: The multiplicative ratio when using optimal s^*.

$\underline{F}^* \approx 0.65$, which is 30% improvement over the fixed fraction policy $\underline{F}_{\text{ffp}} = \frac{1}{2}$ in [18]. The infimum happens when $b \approx 1.8$ as $c \to +\infty$ and $p \to 0$. The plot of cp at infimum points versus c is shown in figure 4.3a. Furthermore, at the same infimum curve of multiplicative ratio, where $c \to +\infty, p \to 0$, the optimal $s^*(c, p)$ converges to $s^*(c, p) \approx 2.29p$ (i.e. $a^* = 2.29$) which differs from optimal value in (4.7). The plot of convergence of a^* over the infimum curve versus c is shown in figure 4.3b.

4.2 c-Universal Mult-Ratio Optimal Linear Policy

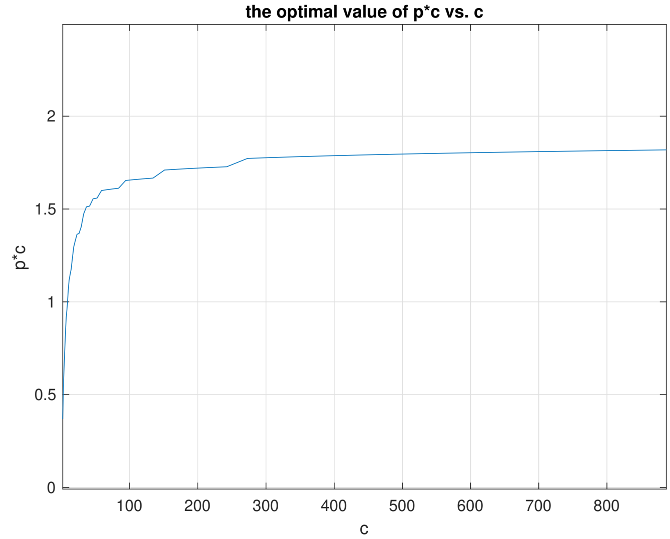
We are further interested in obtaining the optimal slope s that maximizes the multiplicative ratio, when the value of c is the worst-case possible. This provides a c-universal max-min optimal $s^\times(p)$ that would be available regardless of the value of c to ensure a minimum attainable performance for any battery capacity.

$$s^\times(p) = \arg \max_{s \in (0,1]} \inf_{c>0} \frac{\underline{\Gamma}(c, p, s)}{\overline{\Gamma}(c, p)}. \tag{4.13}$$

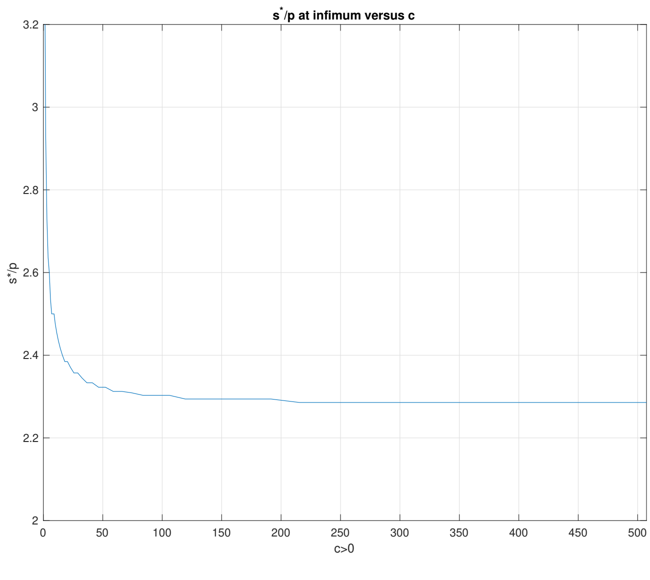
The plot in figure 4.4 shows $s^\times$ as a function of p. Also, the infimum for the multiplicative ratio of worst-case c when using $s^\times$ is given by the following minimization.

$$\underline{F}^\times = \inf_{c>0, p \in (0,1]} \frac{\underline{\Gamma}(c, p, s^\times(p))}{\overline{\Gamma}(c, p)}. \tag{4.14}$$

Proposition 6. *When using the sub-optimal $s^\times(p)$ linear fraction, the multiplicative ratio will still have the same infimum as when using the optimal linear fraction*



(A) Value of cp at the infimum points versus c. It appears in the plot that cp converges to a non-zero value as c goes to infinity. It specifically converges to $cp \to 1.80$



(B) Value of s^*/p at the infimum points versus c. Although the optimal s^* converges to p when $c \to \infty$ for larger values of p, but when $p \to 0$ as well, and specifically at the infimum points, the optimal fraction converges to $s^*(c, p) \approx 2.29p$

Figure 4.3: asymptotic regimes of cp and s^*/p at infimum points of multiplicative ratio $\underline{F}(s^*)$ versus c.

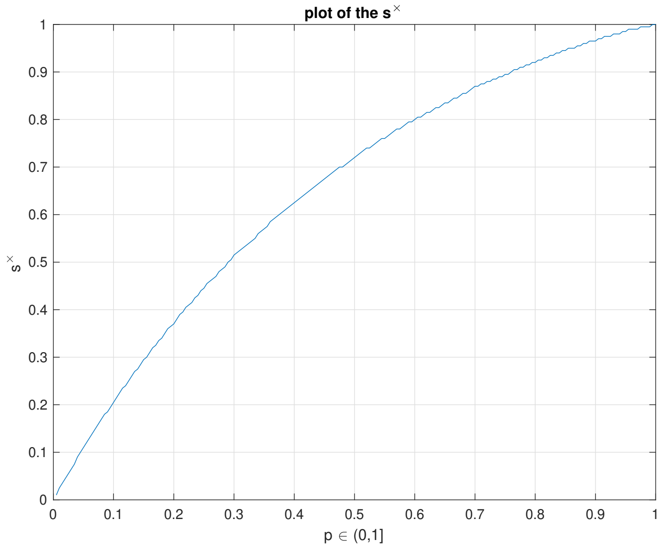


FIGURE 4.4: $s^{\times}(p)$ is the slop of c-Universal Mult-Ratio Optimal Linear Policy which maximizes the throughput multiplicative ratio for the worst-case c. It is the best possible fraction in case of unknown or worst-case capacity c which is a function of p.

policy. i.e.

$$\underline{F}^* = \underline{F}^{\times}.$$

Proof. By using the formula (4.13), we can rewrite the definition for $s^{\times}$ as

$$s^{\times}(p) = \arg\max_{s} \frac{\underline{\Gamma}(c^{\times}, p, s)}{\overline{\Gamma}(c^{\times}, p)}, \tag{4.15}$$

where

$$c^{\times}(s, p) = \arg\min_{c} \frac{\underline{\Gamma}(c, p, s)}{\overline{\Gamma}(c, p)}. \tag{4.16}$$

Next we have that for any given fixed c, p values that maximizes the multiplicative ratio is the same value that maximizes the throughput ($s^*(c, p)$). Because the denominator does not depend on variable s:

$$\begin{aligned} \arg\max_{s} \frac{\underline{\Gamma}(c,p,s)}{\overline{\Gamma}(c,p)} &= \arg\max_{s} \underline{\Gamma}(c,p,s) \\ &= s^*(c,p). \end{aligned} \tag{4.17}$$

Therefore, looking back at (4.15), although $c^\times(s,p)$ is a function of s,p, but when we fix it as an unknown value, the above property (4.17) still holds, which means

$$s^\times(p) = s^*(c^\times, p). \tag{4.18}$$

Next, we form the two multiplicative ratio gaps. For the suboptimal $s^\times$ we have by definition

$$\begin{aligned} \underline{F}^\times &= \inf_{c,p} \frac{\underline{\Gamma}(c,p,s^\times(p))}{\overline{\Gamma}(c,p)} \\ &= \inf_{c,p} \frac{\underline{\Gamma}(c,p,s^*(c^\times,p))}{\overline{\Gamma}(c,p)} \\ &= \inf_{p} \frac{\underline{\Gamma}(c^\times,p,s^*(c^\times,p))}{\overline{\Gamma}(c^\times,p)}. \end{aligned}$$

In addition, for the optimal multiplicative ratio, we have by definition

$$\begin{aligned} \underline{F}^* &= \inf_{c,p} \frac{\underline{\Gamma}(c,p,s^*(c,p))}{\overline{\Gamma}(c,p)} \\ &= \inf_{p} \inf_{c} \frac{\underline{\Gamma}(c,p,s^*(c,p))}{\overline{\Gamma}(c,p)} \\ &= \inf_{p} \frac{\underline{\Gamma}(c^\times,p,s^*(c^\times,p))}{\overline{\Gamma}(c^\times,p)}, \end{aligned}$$

where the third equality holds from the definition of $c^\times$ in (4.16). Thus, comparing above results $\underline{F}^\times = \underline{F}^*$. □

The numerical results of $\underline{F}^\times$ also indicate that the infimum happens approximately at 0.65 which is the same as $\underline{F}^*$ and confirm the above proposition.

4.3 Additive Gap Optimal Linear policy

In addition to above ratio, one can also consider the additive gap for the above policies and find the point in which the maximum gap occurs.

$$\overline{G}^* = \sup_{c>0, p\in(0,1]} \overline{\Gamma}(c,p) - \underline{\Gamma}(c,p,s^*(c,p)). \tag{4.19}$$

It is obvious that the gap should be upperbounded by $\overline{G}^* \leq \frac{1}{2}$ which is obtained using p fixed fraction policy as in [Proposition 3 [18]] (Note the difference is that we use natural logarithm instead). But because supremum happens at $cp \to +\infty$ and the fact that $\lim_{cp\to+\infty} s^*(c,p) = p$ (i.e. it converges to fixed fraction policy) suggests that

$$\overline{G}^* = \frac{1}{2}.$$

Besides, consider that both the fixed fraction policy and linear fraction policy are suboptimal whereas the maximin optimal policy in [21] is optimal for Bernoulli but still it achieves the upperbound $\frac{1}{2}$. This again proves that the additive gap will not be improved, and it achieves $\frac{1}{2}$ for both fixed fraction policy and linear fraction policy.

Note 2. *Although the extremum points of both additive gap G^* and multiplicative ratio F^* happen at $c \to +\infty$ and $p \to 0$, they occur at different asymptotic regimes. As the supremum of G^* happens when $c \to +\infty$ for any given p, and then $p \to 0$ afterwards, thus $cp \to \infty$. But the infimum of F^* happens when $c \to +\infty, p \to 0$ at the same time while their relation converges to $cp \to 2.29$ as shown in 4.3b.*

4.4 c-Universal Additive-Gap Optimal Linear Policy

Similar to section 4.2, we also introduce the c-universal policy which attempts to minimize the additive gap of performance at worst-case value of c.

$$s^+(p) = \arg \min_{s \in (0,1]} \sup_{c>0} \overline{\Gamma}(c,p) - \underline{\Gamma}(c,p,s) \tag{4.20}$$

It turns out that the policy is similar to fixed fraction policy, which is $s^+(p) = p$. The plot of this policy is shown in figure 4.5.

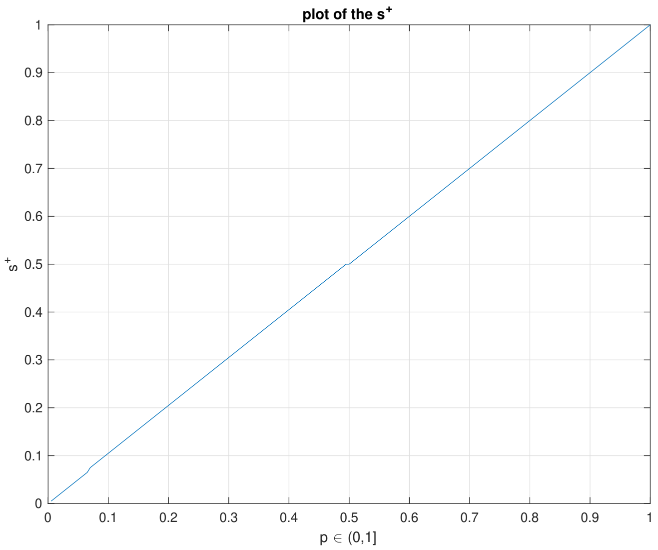


FIGURE 4.5: $s^+(p)$ is the slope of c-Universal Additive-Gap Optimal Linear Policy which minimizes the throughput additive gap for the worst-case c. The plot shows that this policy coincides with the fixed fraction policy

Chapter 5

Conclusion

In this thesis, we investigated optimality and near-optimality of linear policies for energy harvesting communications. In Chapter 3, we found the optimality region of the greedy policy and obtained tight semi-universal bounds on the greedy threshold c^* based on given parameters of arrival distributions. A pair of semi-universal optimal bounds on c^* were obtained, when parameters $\underline{x}, \overline{x}, \mu$ are known from the *arrival distribution* ($Q_{\underline{x},\overline{x},\mu}$). The semi-universal bounds matches the bounds in [proposition 4,5 [20]] which proved them to be optimal. Moreover, similar semi-universal bounds were obtained for the case where same parameters are available from the *clipped energy distribution*. Furthermore, the bounds on $\mathcal{Q}_{c,p}$ showed that the previously known greedy threshold $c \leq \frac{p}{1-p}$ and its corresponding arrival distribution are not always the largest possible interval of optimality of greedy policy, but only true when $p \leq \frac{1}{2}$. The importance of these bounds on the greedy policy is that the lowerbound ensures that for any arrival distribution with given parameters, if battery capacity is less than the lowerbound, then the greedy policy is optimal regardless of the distribution. Also, in the upperbounds' case, it ensures that given the known parameters, if battery capacity is larger than the upperbound, then the greedy policy is definitely not optimal regardless of the distribution.

Then, in Chapter 4 we considered the problem of optimizing linear policies based on the (effective) mean of the energy arrivals and the battery capacity of the node. The numerically solvable transcendental equation for finding the optimal slope was given. The asymptotic regimes showed that the conventional fixed fraction policy is only optimal when the value of $c \to +\infty$ while p is fixed. We also showed that the multiplicative ratio could be improved over the fixed fraction policy.

Bibliography

[1] Victor Shnayder, Mark Hempstead, Bor-rong Chen, Geoff Werner Allen, and Matt Welsh. Simulating the power consumption of large-scale sensor network applications. In *Proceedings of the 2nd International Conference on Embedded Networked Sensor Systems*, SenSys '04, page 188–200, New York, NY, USA, 2004. Association for Computing Machinery.

[2] Xiaofan Jiang, Joseph Polastre, and David Culler. Perpetual environmentally powered sensor networks. In *Proceedings of the 4th International Symposium on Information Processing in Sensor Networks*, IPSN '05, page 65–es. IEEE Press, 2005.

[3] Vinod Sharma, Utpal Mukherji, Vinay Joseph, and Shrey Gupta. Optimal energy management policies for energy harvesting sensor nodes. *IEEE Transactions on Wireless Communications*, 9(4):1326–1336, 2010.

[4] Omur Ozel, Kaya Tutuncuoglu, Jing Yang, Sennur Ulukus, and Aylin Yener. Transmission with energy harvesting nodes in fading wireless channels: Optimal policies. *IEEE Journal on Selected Areas in Communications*, 29(8):1732–1743, 2011.

[5] Jing Yang and Sennur Ulukus. Optimal packet scheduling in an energy harvesting communication system. *IEEE Transactions on Communications*, 60(1):220–230, 2012.

[6] Chin Keong Ho and Rui Zhang. Optimal energy allocation for wireless communications with energy harvesting constraints. *IEEE Transactions on Signal Processing*, 60(9):4808–4818, 2012.

[7] Pol Blasco, Deniz Gunduz, and Mischa Dohler. A learning theoretic approach to energy harvesting communication system optimization. *IEEE Transactions on Wireless Communications*, 12(4):1872–1882, 2013.

[8] Qingsi Wang and Mingyan Liu. When simplicity meets optimality: Efficient transmission power control with stochastic energy harvesting. In *2013 Proceedings IEEE INFOCOM*, pages 580–584, 2013.

[9] Masoud Badiei Khuzani and Patrick Mitran. On online energy harvesting in multiple access communication systems. *IEEE Transactions on Information Theory*, 60(3):1883–1898, 2014.

[10] Sennur Ulukus, Aylin Yener, Elza Erkip, Osvaldo Simeone, Michele Zorzi, Pulkit Grover, and Kaibin Huang. Energy harvesting wireless communications: A review of recent advances. *IEEE Journal on Selected Areas in Communications*, 33(3):360–381, 2015.

[11] Ahmed Arafa, Abdulrahman Baknina, and Sennur Ulukus. Online fixed fraction policies in energy harvesting communication systems. *IEEE Transactions on Wireless Communications*, 17(5):2975–2986, 2018.

[12] Kaya Tutuncuoglu and Aylin Yener. Optimum transmission policies for battery limited energy harvesting nodes. *IEEE Transactions on Wireless Communications*, 11(3):1180–1189, 2012.

[13] Omur Ozel and Sennur Ulukus. Achieving awgn capacity under stochastic energy harvesting. *IEEE Transactions on Information Theory*, 58(10):6471–6483, 2012.

[14] Rahul Srivastava and Can Emre Koksal. Basic performance limits and tradeoffs in energy-harvesting sensor nodes with finite data and energy storage. *IEEE/ACM Transactions on Networking*, 21(4):1049–1062, 2013.

[15] Jie Xu and Rui Zhang. Throughput optimal policies for energy harvesting wireless transmitters with non-ideal circuit power. *IEEE Journal on Selected Areas in Communications*, 32(2):322–332, 2014.

[16] Ramachandran Rajesh, Vinod Sharma, and Pramod Viswanath. Capacity of gaussian channels with energy harvesting and processing cost. *IEEE Transactions on Information Theory*, 60(5):2563–2575, 2014.

[17] Yishun Dong, Farzan Farnia, and Ayfer Özgür. Near optimal energy control and approximate capacity of energy harvesting communication. *IEEE Journal on Selected Areas in Communications*, 33(3):540–557, 2015.

[18] Dor Shaviv and Ayfer Özgür. Universally near optimal online power control for energy harvesting nodes. *IEEE Journal on Selected Areas in Communications*, 34(12):3620–3631, 2016.

[19] Ali Zibaeenejad and Jun Chen. The optimal power control policy for an energy harvesting system with look-ahead: Bernoulli energy arrivals. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pages 116–120, 2019.

[20] Ye Wang, Ali Zibaeenejad, Yaohui Jing, and Jun Chen. On the optimality of the greedy policy for battery limited energy harvesting communications. *CoRR*, abs/1909.07895, 2019.

[21] Shengtian Yang and Jun Chen. A maximin optimal online power control policy for energy harvesting communications. *IEEE Transactions on Wireless Communications*, 19(10):6708–6720, 2020.

[22] Martin L Puterman. *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons, 2014.

[23] R Durrett. Probability: Theory and examples cambridge university press, 2010.