Types in Algebraically Closed Valued Fields: A Defining Schema for Definable 1-types

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June 2021

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Chapter 1

Introduction and Preliminaries

1.1 Introduction

Applied model theory concerns itself with algebraic structures through the lens of a model theorist, by using model theoretic tools to obtain algebraic results. Abraham Robinson was one of the first mathematician to work in this area, as his work focused on using methods in logic, especially model theory, to tackle problems in algebra and analysis. In 1956 Robinson proved a model completeness result in the theory of algebraically closed valued fields (ACVF) in his book *Complete theories*. His result can be used to derive that the theory of ACVF admits quantifier elimination. During the 1970's quantifier elimination results were proved for different theories of valued fields, different languages such as multi sorted languages. Much of this work was due to Macintyre, and Weispfenning.

The work of Françoise Delon [2] in the 1970s gives a classification of types in algebraically closed valued fields. Delon advanced the model-theoretic understanding of valued fields by giving precise descriptions of the one-types in an algebraically closed valued field of characteristic zero. In particular, she used stability-theorietic methods to describe which one-types are definable. However, this proof is non-constructive in a sense it does not provide a way to find the defining schema for those types which are definable. One of the goals in this thesis to describe how to find a defining scheme for the definable types.

We first talk about the structure of valued fields and the underlying topology we can define using the valuation. Given an element of our value group and from our valued field, we define open and closed K-definable balls, which are the simplest definable sets and form a basis of the valuation topology. Because our field is alegbraically closed and we have quantifier elimination we can conclude that the definable sets in one variable are finite Boolean combinations of balls. The work of Jan Holly [5], is used heavily to describe the definable sets, and that they are the finite union of Swiss cheeses—a ball with finitely many balls removed. It should be noted that Holly takes this a step further further to provide a schema of how to find the finite union of Swiss cheeses, even though we will not depend on it. To ensure that we can rely on defining formulas for the Swiss cheeses we prove a uniformity in parameters result which tells us that the formula will not change as the parameters vary. Understanding the definable sets of ACVF will lead to understanding the definable types of the theory, and help us supply the desired defining schema.

In the case where the type is definable, we show how to construct a defining schema. As motivation, we start off by looking at the theory of algebraically closed fields (ACF). The proof that the theory is stable, and hence all types are definable, is standard but we discuss explicitly how to go from the proof to describing a schema. We produce a similar result for definable types of ACVF. Lastly, from [8] we know that definable types are also invariant, but the converse does not necessarily hold. We know that not all the types are definable. We use Delon's classification of the 1-types to see which ones are definable, and prove the rest are invariant.

1.2 Preliminaries

In this section we go over some algebraic properties of valued fields and end with some discussion of the valuation topology.

1.2.1 Valued Fields

Definition 1.2.1. An *ordered abelian group* is an abelian group $(\Gamma, +)$ with a total ordering, $<$ such that

$$a < b \implies a + c < b + c$$

for all $a, b, c \in \Gamma$.

Definition 1.2.2. Let K be a field and Γ an ordered abelian group written multiplicatively. An *absolute value* is a function from K to Γ that satisfies

1. $|x| \geq 0$ (Non-negativity)
2. $|x| = 0$ if and only if $x = 0$ (Positive-definiteness)
3. $|xy| = |x||y|$ (Multiplicatively)
4. $|x + y| \leq |x| + |y|$ (Triangle Inequality)

for all $x, y \in K$. If we replace the triangle inequality with a slightly stronger property

4.′ $|x + y| \leq \max\{|x|, |y|\}$

then we obtain a *non-Archimedean* absolute value.

Definition 1.2.3. Let K be a field and Γ an ordered abelian group. A *valuation* v on K is a surjection $v : K \to \Gamma \cup \{\infty\}$ such that for $a, b \in K$ and $\gamma \in \Gamma$ the following hold:

1. $v(a) = \infty$ if and only if $a = 0$
2. $v(a + b) \geq \min\{v(a), v(b)\}$
3. $v(ab) = v(a) + v(b)$

4. $\gamma < \infty$ and $\gamma + \infty = \infty$

where Γ is the collection of values we obtain from v (which forms an ordered abelian group). We call Γ the *value group*. The second condition of the valuation is called *subadditivity*, which we will see later affects the topology.

Note. Some properties of valued fields that will be used implicitly throughout this paper include:

a. $v(1) = v(-1) = 0$ and $v(a/b) = v(a) - v(b)$.

b. If $v(a_1 + \cdots + a_k) > \min\{v(a_i) : 1 \leq i \leq k\}$ then there exists a $j \neq i$ such that $v(a_i) = v(a_j)$.

p-adic example:

Definition 1.2.4. Fix a prime p. The p-valuation is the function $v_p : \mathbb{Z}\backslash\{0\} \to \mathbb{R}$ such that for each nonzero $n \in \mathbb{Z}$, $v_p(n)$ is the unique positive integer such that

$$n = p^{v_p(n)}m$$

where $p \nmid m$. We extend v_p to the field of rational numbers by setting

$$v\left(\frac{a}{b}\right) = v(a) - v(b)$$

for $a/b \in \mathbb{Q}^{\times}$. It is convention $v_p(0) = \infty$.

Let $x \in \mathbb{Q} \setminus \{0\}$. Then for integers a, b not divisible by p and a unique integer r we can represent x as

$$x = \frac{ap^r}{b}.$$

$\mathbb{Q}_p$ **example:** The p-adic valuation is an example of a nontrivial valuation.
$\mathbb{Q}_p$ **example:** The value group corresponding to v_p is $\mathbb{Z}$.[1]

Definition 1.2.5. We define the *p-adic norm* to be

$$|x|_p = p^{-r}.$$

Furthermore, we define the *p-adic field* $\mathbb{Q}_p$ to be the completion of the rational numbers with respect to the p-adic norm.

[1] This valuation is discrete.

Definition 1.2.6. A *p-adic expansion* is a sum of the form

$$c_0 + c_1 p + c_2 p^2 + c_3 p^3 + \dots$$

where $0 \le c_i < p$ for each i.

Any element of $\mathbb{Q}_p$ is a limit of a p-adic expansion of the form

$$c_{-k} p^{-k} + c_{-k+1} p^{-k+1} + \dots c_0 + c_1 p + c_2 p^2 + c_3 p^3 \dots$$

The beginning $k+1$ terms

$$c_{-k} p^{-k} + c_{-k+1} p^{-k+1} + \dots + c_0$$

are the fractional part of the rational number.

Definition 1.2.7. Let $O_v = \{a \in K : v(a) \ge 0\} \subset K$. This is a ring since

- since $v(0) = \infty, 0 \in O_v$ and thus we have an additive identity,
- if $a \in O_v$ then $-a \in O_v$ because $v(-a) = v(a) + v(1) = v(a) + 0 = v(a)$,
- if $a, b \in O_v$ then $a + b \in O_v$ since $v(a+b) \ge \min\{v(a), v(b)\}$ and thus $v(a+b) \ge 0$,
- if $a, b. \in O_v$ then $a \cdot b \in O_v$ since $v(a \cdot b) = v(a) + v(b) \ge 0$.

We denote the maximal ideal of O_v [2] as $M_v = \{a \in K : v(a) > 0\}$

Definition 1.2.8. We define O_v/M_v to be the *residue field* of K with respect with the valuation v and denote the field as $\overline{K}$.

Definition 1.2.9. The *residual function*, res, is the function from $O_v \to \overline{K}$. We can extend it to K as a whole by taking $\text{res}(K \setminus O_v) = \infty$.

Note. If we have $v(a) \le v(b)$ then $b/a \in O_v$. Thus, if I is an ideal of O_v and $a \in I$ then for all b such that $v(b) \ge v(a)$, b is also an element of I

$\mathbb{Q}_p$ **example:** The ring O_v contains $\mathbb{Z}$ and $M_v \cap \mathbb{Z}$ is a prime ideal of the form $p\mathbb{Z}$.

[2] There is only one.

Henselian Fields

The following lemma we take from Engler and Prestel, is due to Hensel and will be very important.

Theorem 1.1. *(Hensel's Lemma) Let K be a field complete with respect to an absolute value v. Let $f \in O_v[X]$ be a polynomial , and let $a_0 \in O_v$ be such that $v(f(a_0)) > 2v(f'(a_0))$. Then there exists some $a \in O_v$ with $f(a) = 0$ and $v(a_0 - a) > v(f'(a_0))$*

$\mathbb{Q}_p$ **example:** Let $p = 3$ and consider the polynomial $f(X) = X^2 - 4$. Then $f(4) = 4^2 - 4 = 12$, but $f'(X) = 2X$ and thus $f'(4) = 8$. Furthermore, $v(f(4)) = v(12) = 1$ and $2v(f'(4)) = 0$ and satisfies the condition $v(f(4)) > 2v(f'(4))$. Thus Hensel's Lemma guarantees us that there is a unique 3-adic intger b such that $f(b) = 0$ and $v(4 - b) > v(f'(4))$. To find such a b notice $4 \equiv (1 + 3)^2 \text{mod} 9$. If we keep going we have that

$$4 \equiv (1 + 2 \cdot 3 + 2 \cdot 3^2)^2 \bmod 3^3$$

$$4 \equiv (1 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3)^2 \bmod 3^4$$
$$4 \equiv (1 + 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^3 + 2 \cdot 3^4)^2 \bmod 3^5$$
$$\vdots$$
$$4 \equiv (1 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-1})^2 \bmod 3^n$$
$$\vdots$$

And thus 4 is a perfect square. Admittedly this is not surprising, but what might be surprising is that it is the square of

$$1 + \sum_{n=1}^{\infty} 2 \cdot 3^n.$$

Definition 1.2.10. We say a valued field is *Henselian* if Hensel's Lemma is satisfied. In other words, if for all polynomials $f(T) \in O_v[T]$ and $a \in O_v$ such that $\text{res} f(a) = 0$ and $\text{res} \; f'(a) \neq 0$ there is a $b \in O_v$ such that $f(b) = 0$ and $\text{res}\ a = \text{res}\ b$.

$\mathbb{Q}_p$ **example:** The p-adics are Henselian.

We will often be presented with two fields, $K \subset L$. Let v be a valuation on K. It is not necessarily the case that the valuation of K extends uniquely to the valuation of L. This happens to be true if and only if (K, v) is Henselian.

Theorem 1.2. *Let (K, v) be a valued field. Then K is Henselian if and only if there is a unique extension of the valuation to an algebraic extension of K.*

Since the algebraic closure of an algebraically closed field is itself, and therefore doesn't have any nontrivial algebraic extensions, we have the following corollary.

Corollary. *Algebraically closed valued fields are Henselian.*

1.2.2 Topology of Valued Fields

We can discuss what an open ball looks like in valued fields, which will lead to how definable sets are classified.

Definition 1.2.11. An *open ball* is defined as

$$B^{op}(b_1, b_2) = \{x : v(x - b_1) > v(b_2)\}$$

for $b_1, b_2 \in K$. Similarly a *closed ball* is defined as

$$B^{cl}(b_1, b_2) = \{x : v(x - b_1) \geq v(b_2)\}$$

for $b_1, b_2 \in K$.

In this case we will refer to b_1 as the *center* of the ball, and $v(b_2)$ as the *radius* of the ball. If we choose not to specify whether the ball is open or closed we will simply write $B(b_1, b_2)$.

The topology of valued fields is generated by the open balls. The following results from [5] are important consequences of the subadditivity of the valuation.

Proposition 1.3. *(Note 3.4, [5])*

(i) Every point in a ball is a center of the ball.

(ii) *Given any two balls B and C, if their intersection is nontrivial then either $B \subseteq C$ or $C \subseteq B$*

Proof. (i) Let $B = B^{op}(b_1, b_2)$ and $b' \in B$, then $v(b' - b_1) > v(b_2)$. Assume towards a contradiction that there is $x \in B$ such that $x \notin B' = B^{op}(b', b_2)$ then

$v(x - b') \leq v(b_2)$. Thus we have

$$v(b_1 - b') = v(b_1 - x + x - b') =$$

$$\min\{v(b_1 - x), v(x - b')\} = v(x - b').$$

Hence, we have $B \subseteq B'$. The proof of the converse to show $B' \subseteq B$ is analogous, and thus $B' = B$ and therefore every point in B serves as the center of the ball.

(ii) Let $B = B^{op}(b_1, b_2)$ and $C = B^{op}(c_1, c_2)$ be open balls and $d \in B \cap C$. Let $b' \in B \setminus C$. By (i) we can take d to be the center of both balls, and thus

$$v(b_2) < v(d - b') < v(c_2)$$

and hence $v(b_2) < v(c_2)$.

Since C has the same center as B and $v(b_2) < v(c_2)$ then if $v(c_2) < v(c - y)$ we immediately have $v(b_2) < v(c - y)$, and thus $C \subseteq B$.

□

This brings us to a topological consequence of this proposition.

Corollary. *The ball $B = B^{op}(b_1, b_2)$ is closed, and the ball $B = B^{cl}(b_1, b_2)$ is open.*

Despite the corollary, we will still refer to the open balls as open (and similarly for the closed balls) for notational convenience.

All except (4) are in [5] but without proof, but since we will use the propositions heavily, we provide the arguments. Since the propositions only deal with the radius of the balls, we will use Greek letters to represent the radius as shorthand to make the argument a little cleaner.

Proposition 1.4. *(Note 3.4, [5]) Let B and C be balls with radii β and γ respectively, and $c \in C$. If*

1. *B and C are open balls, or*
2. *B and C are closed balls, or*
3. *B is closed and C is open*

and if $\beta \leq \gamma$ then either $C \subseteq B$ or $C \cap B = \emptyset$. Furthermore,

4. *B is open and C is closed,*

and if $\beta < \gamma$ then either $C \subset B$ or $C \cap B = \emptyset$.

Proof. Let B and C be balls as described above. If $B \cap C = \emptyset$ then we are done, so assume $c \in B \cap C$.

1. Suppose both B and C are open balls such that $\beta \leq \gamma$ Thus if $y \in C$ then $\gamma < v(c-y)$, but since $\beta \leq \gamma$ we have $\beta < v(c-y)$ and thus $y \in B$. Since y was an arbitrary element of C, this holds for every element of C and therefore $C \subseteq B$.

2. Now suppose both B and C are closed balls with $\beta \leq \gamma$. Then if $y \in C$ we have by definition $\gamma \leq v(c-y)$, but since $\beta \leq \gamma$ it follows $\beta \leq v(c-y)$, and thus $y \in B$. We therefore conclude $C \subseteq B$.

3. Let B be a closed ball and C be an open ball. Again let $y \in C$, then $\gamma < v(c-y)$, but since $\beta \leq \gamma$ we immediately have $\beta < v(c-y)$ and thus $y \in B$. Thus, $C \subseteq B$.

4. Let B be an open ball and C a closed ball. Then for any $y \in C$ we have $\gamma \leq v(c-y)$, but since $\beta < \gamma$, we have $\beta < v(c-y)$ and thus $y \in B$. Therefore $C \subseteq B$.

□

Chapter 2

Algebraically Closed Valued Fields

First we establish the language we are working in, along with some model theoretic results of it. We will then continue the chapter with some discussion on the definable sets of algebraically closed valued fields, and then end with a uniformity in parameters result.

2.1 Model Theory of Valued Fields

Now that we have an understanding of valued fields, we turn our attention to the algebraically closed valued fields. There are many choices of language for the theory of valued fields, and particularly the theory of algebraically closed valued fields, such as one sorted or multisorted languages.

2.1.1 Language

When discussing the model theory of valued fields we work in the three sorted language.

Definition 2.1.1. The *Denef-Pas Language* of valued fields $\mathcal{L}_{Pas}$ is the three sorted language with the following sorts and map:

1. The valued field K which has the language of rings $\mathcal{L}_{ring} = 0, 1, +, -, \cdot$
2. The value group Γ_K has the language of ordered abelian groups $\mathcal{L}_{OAG} = \{0, +, -, <, \infty\}$.

3. The residue field $\mathcal{K}$ which also has the language of rings

4. The valuation $v : K \to \Gamma_K$.

Furthermore the theory includes axioms for a (non-trivial) valuation, the axioms specifying the characteristics of the field and residual field, and the axioms stating the field is algebraically closed.

Theorem 2.1. *The theory of algebraically closed valued fields eliminates quantifiers in $\mathcal{L}_{Pas}$.*

2.1.2 Definable Sets

Definition 2.1.2. Let K be a valued field. A *Swiss cheese* is a nonempty set of the form

$$B \setminus (C_1 \cup \cdots \cup C_n)$$

where B is a K-definable ball, and $C_1, \ldots, C_n$ are K-definable subballs of B. We will often refer to B in the definition above as the *outer ball* and C_i as the *inner ball*.

Proposition 2.2. *(Proposition 3.6, [5]) Let S_1 and S_2 be two Swiss cheeses. Then $S_1 \cap S_2$ and $S_1 \cup S_2$ are both Swiss cheeses.*

Notice that since ACVF has quantifer elimination in the language of our choosing every formula can be written as a Boolean combination of formulas of the form $v(f(x)) = v(g(x))$ and $v(f(x)) < v(g(x))$, where $f(x), g(x) \in K[x]$. Since our field is algebraically closed, all polynomials can be written as a product of linear factors. In [5], we see that this will lead to a theorem that every K-definable set of ACVF is a finite Boolean combination of balls, which we will take as a fact. Furthermore, we will can actually say that every definable set of ACVF is a finite union of Swiss cheeses.

We introduce the following notation to write a general formula defining a Swiss cheese. Let B be the outer ball to the Swiss cheese with center a_0 and radius $v(a'_0)$ and let $C_1, \ldots, C_n$ be the inner balls with centers $a_1, \ldots, a_n$ and radii $v(a'_1), \ldots, v(a'_n)$ respectively.

We define $\beta^{op}(b, b')$ to be the formula $v(x - b) > v(b')$, which defines an open ball. Likewise, we let $\beta^{cl}(b, b')$ be the formula $v(x-b) \geq v(b')$ for a closed ball.

Then to describe a Swiss cheese we need to know how many inner balls are removed, and whether or not the outer ball and inner balls are open or closed. Let $\Sigma = \{op, cl\}$ be an alphabet and Σ^* be the set of all finite strings generated from Σ.

Let $\sigma = \alpha_0\alpha_1 \dots \alpha_n \in \Sigma^*$ be a finite string of length $n+1$ indicating whether or not each ball is open or closed. To express a Swiss cheese with n many inner balls removed:

$$B^{\alpha_0}(a_0, a_0') \setminus (B^{\alpha_1}(a_1, a_1') \cup B^{\alpha_2}(a_2, a_2') \cup \dots \cup B^{\alpha_n}(a_n, a_n'))$$

we write

$$S_n^{\sigma}(x; a_0, a_0', a_1, a_1', \dots, a_n, a_n')$$

which is the formula

$$\beta^{\alpha_0}(x; a_0, a_0') \wedge \neg(\beta^{\alpha_1}(x; a_1, a_1') \vee \dots \vee \beta^{\alpha_n}(x; a_n, a_n')).$$

Example 2.1.1. For example,

$$S_2^{op,cl,op}(x; a_0, a_0', a_1, a_1', a_2, a_2')$$

is the formula

$$\beta^{op}(x; a_0, a_0') \wedge \neg(\beta^{cl}(x; a_1, a_1') \vee \beta^{op}(x; a_2, a_2'))$$

which is defining the Swiss cheese

$$B^{op}(a_0, a_0') \setminus (B^{cl}(a_1, a_1') \cup B^{op}(a_2, a_2'))$$

For convenience, we let $\overline{a}$ be a tuple of length $2(n+1)$ and denote the formula describing a Swiss cheese as $S_n^{\sigma}(x; \overline{a})$. Furthermore, we let

$$\mathcal{S}(x; \overline{a}, \sigma, n, m) = \bigvee_{i=1}^{m} S_{n_i}^{\sigma_i}(x; \overline{a}_i)$$

be a formula defining the union of m many Swiss cheeses, where σ_i is the information telling us which balls are open and closed in each of the m many Swiss cheeses, n_i denotes how many inner balls are removed in defining an ith Swiss cheese, and for each i, $\overline{a_i}$ is a tuple of length $2(n_i+1)$ However, σ, n and m are not parameters of the formula.

The following is an important result we will heavily use from [5], which we state without proof.

Theorem 2.3. *Every K-definable set of ACVF is a finite union of Swiss cheeses. In other words, for every definable set X there exists $\sigma, n, m, \overline{a}$ such that X is defined by $\mathcal{S}(x; \overline{a}, \sigma, n, m)$.*

We now end this chapter with a uniformity in parameters result.

Theorem 2.4. *Let $\varphi(x, y)$ be a formula in ACVF describing a definable set. Then there are a σ, n, m such that for all b there are parameters $\overline{a}$ such that*

$$\varphi(x; b) \leftrightarrow \mathcal{S}(x; \overline{a}, \sigma, n, m).$$

Proof. Let $\varphi(x; y)$ be a formula in ACVF and $\mathcal{S}(x; z, \sigma, n, m)$ be formulas describing a finite union of Swiss cheeses. Assume towards a contradiction uniformity in parameters is not true, i.e. for all σ, n, m there is a y such that for all z

$$\varphi(x; y) \not\equiv \mathcal{S}(x, z, \sigma, n, m).$$

Now consider the theory T which consists of all formulas of the form

$$\exists y\ \forall z\ [\varphi(x; y)\ \wedge\ \neg\mathcal{S}(x, z, \sigma, n, m)]\ \vee\ [\neg\varphi(x; y)\ \wedge\ \mathcal{S}(x, z, \sigma, n, m)]$$

for all σ, n, m along with the theory of ACVF. We claim U has a model for every finite subset $U \subset T$. Let $\mathcal{M} \vDash$ ACVF and let φ_i be formulas of the form above. By assumption, each φ_i is realized in a model $\mathcal{M}_i$ of ACVF. Thus, there is a corresponding $y_i \in M_i$ that realizes the formula. Because ACVF is model complete, $y_i \in M$ where M is the underlying set of $\mathcal{M}$. However, this is true of all $i \in \mathbb{N}$ and thus, any finite collection of the formulas φ_i is realized, since they are individually realized in $\mathcal{M}$. Therefore, there is a model for each finite subset $U \subset T$.

However, since we have a model for every finite $U \subset T$ then by the Compactness Theorem, there is a model $\mathfrak{A}$ that satisfies the whole theory. Therefore in $\mathfrak{A}$ there is a choice of parameter for which the set defined by $\varphi(x; b)$ is not a finite union of Swiss cheeses, a contradiction to the fact that the set defined by $\varphi(x; b)$ is definable and therefore by Theorem 2.3 is a finite union of Swiss cheeses. ∎

Chapter 3

Definable Types in ACVF

In this chapter we will explicitly describe a defining schema, that will later be used for the definable types. We will first obtain some model theoretic results about types and then move on to discuss the definable types and defining schema in ACF, and then end with definability in ACVF.

3.1 Types

In this section we first look at the necessary background to familiarize ourselves with the model theoretic notion of a *type*.

3.1.1 Preliminaries

Definition 3.1.1.

1. Let $\mathcal{M}$ be an $\mathcal{L}$ structure. A *partial type*, p over a set $A \subset M$, in a variable x is a set of $\mathcal{L}(A)$- formulas in x with parameters from M.

2. If x is an n-tuple then we call p a *partial n-type.*

3. We say a partial type is *consistent* if for every finite subset U of p there is an $m \in M$ such that

$$\mathcal{M} \vDash \varphi(m) \text{ for all } \varphi(x) \in U.$$

4. A *complete type* over M is a maximal consistent partial type over M.

5. Let $A \subseteq M$. We write $S_n(A)$ to denote the *space of complete n-types over A*.

Unless otherwise specified, we will be working with complete types.

The single turnstile symbol $\vdash$ is a binary relation used to represent syntactic consequence in the study of formal languages. Having one proposition on the left and one on the right we read $P \vdash Q$ to mean that Q is derivable or provable from P in the given axiomatic system. In the context of types we use the definition of $\vdash$ from [8]: a type p concentrates on a definable set S if p contains a formula defining S. In fact, when we write $p \vdash \varphi$ we mean that the *type* p contains the formula φ.

The double turnstile symbol $\vDash$, on the other hand, is a binary relation often used to show semantic consequence with a collection of sentences on the left and a singular sentence on the right. Hence $\Sigma \vDash \varphi$ is understood to mean if every sentence in the set Σ holds, then the sentence φ also holds.

In model theory the double turnstile has a slightly different use, and is meant to show *satisfaction* in a model on the lefthand side and a collection of sentences on the right. For example when we write $\mathcal{M} \vDash \Sigma$ to mean that $\mathcal{M}$ is a model for Σ. To bring back the idea of how this denotes the semantic consequence, we can interpret the model theoretic notion of $\Sigma \vDash \varphi$ as "if $M \vDash \Sigma$ then $M \vDash \varphi$".

Definition 3.1.2. Let $\mathcal{M}$ and $\mathcal{L}$ be as above, $A \subseteq M$ and $b \in M^n$. We define

$$tp(b/A) := \{\varphi(x) : \varphi \in \mathcal{L}(A), \mathcal{M} \vDash \varphi(b)\}$$

to be the *type of b over A*.

Definition 3.1.3. Let p be a partial n-type over A. We say $b \in M^n$ *realizes* p if $p \subset tp(b/A)$. We say p is *finitely realized in $\mathcal{M}$* if every finite subset of p is realized in $\mathcal{M}$. We say a type p is *finitely satisfiable* in a set A if for every formula $\varphi(x; b) \in p$ there is an $a \in A$ such that $\varphi(a; b)$ holds.

Note. Admittedly, it is a bit bizarre that a type is finitely satisfiable just because we can find a witness for every one formula. An equivalent way to say p is *finitely satisfiable* in A is if any finite subset of p is realized in A. Since the type is assumed to be complete, it is maximally consistent. Thus if there are $\varphi_1(x; b), \ldots, \varphi_k(x; b)$ such that there are $a_1, \ldots, a_k$ that make

$\varphi_1(a_1;b),\ldots,\varphi_k(a_k;b)$ hold, then since p is maximally consistent, the type contains the formula

$$\bigwedge_{i=1}^{k}\varphi_i(x;b)$$

and thus there is an $a^* \in A$ such that

$$\bigwedge_{i=1}^{k}\varphi_i(a^*;b)$$

holds. And thus we have that any finite subset of p is realized in A, since the conjunction of any collection of formulas in p is also in p.

Example 3.1.1. Let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ then the collection

$$\{f(T) \neq 0 : f(T) \in \mathbb{Q}[T],\ f(T) \not\equiv 0\}$$

is a partial type and is finitely satisfiable. It is important to note that this type is not realized in $\overline{Q}$. Furthermore we take all consistent Boolean combinations of these formulas we will have an example of a complete type.

The next few definitions will be important in our discussion of invariant types.

Definition 3.1.4. Let $\mathcal{M}$ be a model and p a type. Let $p \in S(\mathcal{M})$. We say p is *definable* over a set B if for all $\varphi(x;y)$ there exists $d_\varphi(y) \in \mathcal{L}(B)$ such that for all $b \in \mathcal{M}$, $p \vdash \varphi(x;b)$ if and only if $b \vDash d_\varphi(y)$.

Definition 3.1.5. If all types over all models of a theory T are definable, we say the theory is *stable*.

Example 3.1.2. The theory of algebraically closed fields is stable. The theory of valued fields, however, is not.

Definition 3.1.6. Let $\mathcal{M}$ be a model and p a type. Let $p \in S_x(\mathcal{M})$. We say p is A-invariant if $\sigma p = p$ for all $\sigma \in Aut(\mathcal{M}/A)$. In other words, if σ applied to every formula of p is still a formula in p. And thus, p is *A-invariant* if for every formula $\varphi(x;y)$ and tuples b, b' from the model, if $b \equiv_A b'$ then

$$p \vdash \varphi(x;b) \iff p \vdash \varphi(x;b').$$

Additionally, p is *invariant* if it's A-invariant for some $A \subset \mathcal{M}$.

Proposition 3.1. *Definable types are invariant.*

Proof. If p is definable it is definable over some A such that $|A| \leq |T|$, where T is the theory. If $b \equiv_A b'$ then there is some $\sigma \in Aut(\mathcal{M}/A)$ such that $\sigma(b) = b'$. Then we have

$$\begin{aligned} \varphi(x; b) \in p &\iff b \vDash d_\varphi(y) \\ &\iff \sigma(b) \vDash \sigma(d_\varphi(y)) \\ &\iff b' \vDash d_\varphi(y) \quad \text{(since } \sigma \text{ fixes } A \text{ and the parameters are from } A) \\ &\iff \varphi(x, b') \in p \end{aligned}$$

and thus, p is invariant. □

Proposition 3.2. *If a type is finitely satisfiable in A it is A-invariant.*

Proof. Let p be finitely satisfiable in A and let $p \vdash \varphi(x; b)$. Then if $b \equiv_A b'$ there is a $\sigma \in Aut(\mathcal{M}/A)$ such that $\sigma(b) = b'$. Thus there is an $a \in A$ such that

$$\begin{aligned} p \vdash \varphi(x; b) &\implies a \vDash \varphi(x; b) \text{ (since) } p \text{ is finitely satisfiable.)} \\ &\iff \sigma(a) \vDash \sigma(\varphi(x; b)) \\ &\iff a \vDash \varphi(x; b') \text{ (since } \sigma \text{ fixes } A \text{ and } \sigma(b) = b') \end{aligned}$$

thus $a \vDash \varphi(x; b) \wedge \varphi(x; b')$. Since p is a complete type it must contain either $\varphi(x; b')$ or $\neg\varphi(x; b')$. However, since $a \vDash \varphi(x; b) \wedge \varphi(x; b')$ and p is consistent, $\varphi(x; b) \wedge \varphi(x; b') \in p$ and thus $\varphi(x; b') \in p$.

□

3.1.2 Definable Types in ACF

In this section we illustrate one of the goals of this thesis by high lighting those goals in the example of algebraically closed fields in the pure field language. Algebraically closed fields were looked at by Alfred Tarski who proved that the theory admitted quantifier elimination

The following well known theorem will be necessary to talk about the definable sets of algebraically closed fields.

Theorem 3.3. *The theory of algebraically closed fields admits quantifier elimination.*

Let $K \subset F$ where K is an algebraically closed field and let $Spec(K[x])$ be the set of all prime ideals of $K[x]$. We prove that every type in ACF is definable by proving there is a bijection between the types $p \in S_n(K)$ and the prime ideals

$$I_p := \{f \in K[x] : \text{“}f(x) = 0\text{”} \in p\}$$

for x and n-tuple.

First we verify that I_p is a prime ideal. First to show I_p is an ideal take f and g in I_p, then “$f(x) = 0''$ $\in p$ and “$g(x) = 0''$ $\in p$. Because p is complete (and therefore consistent) there is a model that realizes the formulas, so there is an a such that $f(a) = 0$ and $g(a) = 0$. And thus $f(a) + g(a) = 0$. And therefore “$f(x) + g(x) = 0$” is in p which implies $f + g \in I_p$. If $f \in I_p$ and $g \in K[x]$ then their product is in I_p. Indeed because $f \in I_p$ we have “$f(x) = 0$” $\in p$ and thus since p is complete, there is a model that realizes the formula and thus, there is an a from the model such that $f(a) = 0$. But then $f(a) \cdot g(a) = 0$ for any $g \in K[x]$ and thus for similar reasons as above “$f(x) \cdot g(x) = 0$” $\in p$ and therefore $f \cdot g \in I_p$. Lastly, if $f \cdot g \in I_p$ then “$f(x) \cdot g(x) = 0$” $\in p$ and since p is complete there is a model that realizes that formula. Thus there is a witness a such that $f(a) \cdot g(a) = 0$, so at least one of $f(a) = 0$ or $g(a) = 0$ and so at least one of “$f(x) = 0$” or “$g(x) = 0$” is in p. Without loss of generality, let's say “$f(x) = 0''$ $\in p$. Then there is a model and a realization such that, $f(a) = 0$. Therefore, $f \in I_p$, making I_p a prime ideal.

Theorem 3.4. *The map $\alpha : S_n(K) \to Spec(K[x])$ where $\alpha(p) = I_p$ is a bijection.*

Proof. Let p, q be two distinct types. To prove it is injective notice that by quantifier elimination, there is is a quantifier free formula φ such that $\varphi \in p$ and $\varphi \notin q$, in other words $\neg\varphi \in q$.

Claim. *We can assume without loss of generality that φ is an atomic formula.*

Proof. (of Claim) This is a consequence of the fact that the theory admits quantifier elimination and all quantifier free formulas can be written as a finite

Boolean combination of atomic formulas. To see why the claim is true let's take two atoms "$f(x) = 0$" and "$g(x) = 0$" and consider the formula

$$\varphi := f(x) = 0 \wedge g(x) = 0$$

and assume $\varphi \in p$ and $\neg\varphi \in q$. Then since p is maximally consistent, in order $\varphi \in p$ we must have "$f(x) = 0$" $\in p$ and "$g(x) = 0$" $\in p$. However,the negation of φ is

$$f(x) \neq 0 \vee g(x) \neq 0$$

which is assumed to be in q. Since q is maximally consistent then one of the following three cases must happen:

1. "$f(x) \neq 0$" $\in q$ and "$g(x) \neq 0$" $\in q$
2. "$f(x) \neq 0$" $\in q$ and "$g(x) = 0$" $\in q$
3. "$f(x) = 0$" $\in q$ and "$g(x) \neq 0$" $\in q$

However, none of these cases have q contain "$f(x) = 0$" and "$g(x) = 0$", which are both in p. Thus in all cases, $p \neq q$ is witnessed by an atomic formula.

A very similar argument will show that if the formula

$$\psi := f(x) = 0 \vee g(x) = 0$$

disrupts the injectivity of α then in fact there is an atomic formula that disrupts the injectivity as well. □

This makes φ the formula "$f(x) = 0''$ for $f \in K[x]$ [1] But since $\neg\varphi \in q$ we have $f \notin I_q$ and thus $I_p \neq I_q$.

To show α is surjective, let $I \in Spec(K[x])$ and let $a_i = x_i/I \in K[x]/I$ for $i = 1, \ldots, n$ and $a = (a_1, \ldots, a_n)$. Then for all $f \in K[x]$ we have that $f(a) = 0$ if and only if $f \in I$. Indeed assume $f(a) = 0$. Then

$$f((a_1, \ldots, a_n)) = f(x_1/I, \ldots, x_n/I)) = 0$$

and therefore $f \in I$. If $f \in I$ then $f(a) = 0$ because $a_i \in K[x]/I$. Thus $I_{tp(a/K)} = I$, making the map surjective. □

[1]Or possibly we picked the negation "$f(x) \neq 0$".

Now that we have a bijection between the types and the ideals we can conclude that the types are definable (and invariant by Proposition 3.1) because the ideals are finitely generated. That is, in order to determine whether or not a formula $\varphi(x; b)$ is in the type, we check whether or not the polynomials which occur in the atoms of the formula ("$f(x) = 0$" or "$f(x) \neq 0$") are in the ideal determined by the type. Since the ideals are finitely generated, for each ideal I_p we let $\{g_1, \ldots, g_{k_p}\}$ be the generators for each ideal. Then to know whether or not the polynomial f (with coefficients determined by the parameters b) is in the ideal, is to know if it is a polynomial combination of $\{g_1, \ldots, g_{k_p}\}$

3.2 Construction of Parameter Dependent Formulas

Now we work towards how to construct a defining formula for definable types in ACVF. Since we know every definable set in ACVF can be written as a finite union of Swiss cheeses, we go about figuring out how to write out the defining formula for each Swiss cheese, that is only dependent on the *parameters*.

3.2.1 Defining Formulas for Containment

Let B and C be K-definable balls. We find a formula that only depends on parameters that will hold if and only if $C \subseteq B$. Notice that since B and C could either be open or closed, we have four cases to consider.

Lemma 3.5. *1. Suppose $B = B^{op}(b_1, b_2)$ and $C = B^{op}(c_1, c_2)$ are open. We show the formula*

$$v(b_2) < v(b_1 - c_1) \wedge v(b_2) \leq v(c_2) \qquad (\Phi.1(b_1, b_2, c_1, c_2))$$

holds if and only if $C \subseteq B$.

2. Let $B = B^{op}(b_1, b_2)$ be an open ball and $C = B^{cl}(c_1, c_2)$ be a closed ball. Then

$$v(b_2) < v(b_1 - c_1) \wedge v(b_2) < v(c_2) \qquad (\Phi.2(b_1, b_2, c_1, c_2))$$

holds if and only if $C \subseteq B$.

3. If $B = B^{cl}(b_1, b_2)$ is a closed ball and $C = B^{op}(c_1, c_2)$ is open.

$$v(b_2) \leq v(b_1 - c_1) \wedge v(b_2) \leq v(c_2) \qquad (\Phi.3(b_1, b_2, c_1, c_2))$$

holds if and only if $C \subseteq B$.

4. Finally, suppose both $B = \{x : v(x - b_1) \geq v(b_2)\}$ and $C = \{x : v(x - c_1) \geq v(c_2)\}$ are closed balls. Then $C \subseteq B$ if and only if the same formula as above

$$v(b_2) \leq v(b_1 - c_1) \wedge v(b_2) \leq v(c_2) \qquad (\Phi.3(b_1, b_2, c_1, c_2))$$

holds.

Proof. 1. Suppose $v(b_2) < v(b_1 - c_1)$. That is to say $c_1 \in B$. Thus, we know that $B \cap C \neq \emptyset$ and so we either have $C \subseteq B$ or $B \subseteq C$.

Now assume $v(b_2) \leq v(c_2)$, then by Proposition 1.4 we have that $C \subseteq B$.

To show that $C \subseteq B$ implies the formula above, notice that if $C \subseteq B$, then in particular $c_1 \in B$. Thus $v(b_2) < v(c_1 - b_1)$ by definition of B. To show that $C \subseteq B$ implies $v(b_2) \leq v(c_2)$ prove the contrapositive:

$$v(c_2) < v(b_2) \implies C \nsubseteq B.$$

But, $C \nsubseteq B$ is equivalent to $C \cap B = \emptyset$ or $B \subseteq C$. If $C \cap B$ is empty then there we are done. Otherwise, let $d \in C \cap B$ serve as the center of each of the K-definable balls. Let $y \in B$, then $v(b_2) < v(y - d)$, but since $v(c_2) < v(b_2)$, this implies $v(c_2) < v(y - d)$ and so $y \in C$. Thus $B \subseteq C$. Therefore, we have proved the contrapositive of our original claim, and conclude if $C \subseteq B$ then $v(b_2) < v(b_1 - c_1)$ and $v(b_2) \leq v(c_2)$.

2. Similar to the argument above, assume $\Phi.2(b_1, b_2, c_1, c_2)$ holds, then if $v(b_2) < v(b_1 - c_1)$ then in particular, $c_1 \in B$. Thus one of B or C contains the other.

   Now assume $v(b_2) < v(c_2)$, then by Proposition 1.4 we have $C \subseteq B$.

   For the converse, assume $C \subseteq B$. Again, since $C \subseteq B$ we have $c_1 \in B$, so $v(b_2) < v(b_1 - c_1)$ holds. Now to show $C \subseteq B$ implies $v(b_2) < v(c_2)$ we again prove the contrapositive

$$v(c_2) \leq v(b_2) \implies C \cap B = \emptyset \ \vee \ B \subseteq C.$$

Again, if $C \cap B$ is empty we are done. So assume $d \in C \cap B$. Then we use d as the center of both balls and conclude $B \subseteq C$, as in the above argument, and thus showing $C \subseteq B$ implies $\Phi.2$ holds.

3. Again, if $v(b_2) \leq v(b_1 - c_1)$ then $c_1 \in B$ by definition. Thus, since B and C have nontrivial intersection, one ball contains the other. Now assume $v(b_2) \leq v(c_2)$, then by Proposition 6 we have $C \subseteq B$.

   To show that $C \subseteq B$ implies the formula above, we run the argument in the same fashion of the previous two cases. First, notice that if $C \subseteq B$, then in particular $c_1 \in B$. Thus $v(b_2) \leq v(c_1 - b_1)$ by definition of B. To show that $C \subseteq B$ implies $v(b_2) \leq v(c_2)$ we prove the contrapositive again:
$$v(c_2) < v(b_2) \implies C \not\subseteq B.$$
   There is a subtle change in inequalities, so we repeat the proof to address this. Recall if $C \not\subseteq B$ then either $C \cap B = \emptyset$ or $B \subseteq C$. If $B \cap C = \emptyset$ then we are done, so let $d \in B \cap C$. Let $y \in B$, then we have $v(b_2) \leq v(y - d)$. But if $v(c_2) < v(b_2)$, then $v(c_2) < v(y - d)$ and so $y \in C$. Thus if $v(c_2) < v(b_2)$ then $B \subseteq C$. Therefore, we have proved the contrapositive of our original claim, and conclude if $C \subseteq B$ then $v(b_2) \leq v(b_1 - c_1)$ and $v(b_2) \leq v(c_2)$.

4. Since $v(b_2) \leq v(b_1 - c_1)$ we have B and C have nontrivial intersection and thus either $B \subseteq C$ or $C \subseteq B$. Then we let c_1 serve as the center of both C and B and observe that if $v(c_2) \leq v(c_1 - y)$ then $v(b_2) \leq v(c_1 - y)$ since $v(b_2) \leq v(c_2)$. The converse of this argument is exactly the argument of the converse in the above case.

□

3.2.2 Defining Formulas for Empty Intersection

Now we work towards finding a formula that will hold if and only if $B \cap C = \emptyset$.

Lemma 3.6. *1. Let $B = B^{op}(b_1, b_2)$ and $C = B^{op}(c_1, c_2)$ be open. We show the formula*
$$[v(c_1 - b_1) \leq v(b_2)] \ \wedge \ [v(c_1 - b_1) \leq v(c_2)] \qquad (\Psi.1(b_1, b_2, c_1, c_2))$$
holds if and only if $B \cap C = \emptyset$.

2. If $B = B^{op}(b_1, b_2)$ is open and $C = B^{op}(c_1, c_2)$ is close then

$$[v(c_1 - b_1) \leq v(b_2)] \ \wedge \ [v(c_1 - b_1) < v(c_2)] \qquad (\Psi.2(b_1, b_2, c_1, c_2))$$

holds if and only if $B \cap C = \emptyset$.

3. Let $B = B^{cl}(b_1, b_2)$ and $C = B^{cl}(c_1, c_2)$ be closed, then

$$[v(c_1 - b_1) < v(b_2)] \ \wedge \ [v(c_1 - b_1) < v(c_2)] \qquad (\Psi.3(b_1, b_2, c_1, c_2))$$

holds if and only if $B \cap C = \emptyset$.

Proof. 1. Notice that if $v(c_1 - b_1) \leq v(b_2)$ then $c_1 \notin B$, so since $c_1 \in C \setminus B$ we have $B \not\subseteq C$. Similarly, if $v(c_1 - b_1) \leq v(c_2)$ then that tells us $b_1 \notin C$. And so $b_1 \in B \setminus C$ and thus $B \not\subseteq C$. Thus since we have $C \not\subseteq B$ and $B \not\subseteq C$ we conclude that $B \cap C = \emptyset$.

To see why $B \cap C = \emptyset$ implies formula $\Psi.1(b_1, b_2, c_1, c_2)$ notice that $B \cap C = \emptyset$ implies $C \not\subseteq B$ and $B \not\subseteq C$. Let $c \in C \setminus B$, then $v(c - b_1) \leq v(b_2)$.

Claim.

$$v(c_1 - b_1) \leq v(b_2).$$

Proof. (of claim) Suppose $v(b_2) < v(c_2)$. From above we have $v(c - b_1) \leq v(b_2)$. This holds if and only if

$$v(c - c_1 + c_1 - b_1) \leq v(b_2).$$

Notice that $v(c - c_1 + c_1 - b_1) = \min\{v(c - c_1), v(c_1 - b_1)\}$ since $v(c - c_1) \neq v(c_1 - b_1)$ or else $b_1 \in C$. However, if $v(c - c_1) \leq v(c_1 - b_1)$ then since c and c_1 are both in C we have $v(c_2) < v(c - c_1)$ but this implies that $v(c_2) < v(c_1 - b_1)$ making $b_1 \in C$. This is a contradiction of the fact that B and C are disjoint, so $v(c_1 - b_1) < v(c - c_1)$ and therefore $v(c_1 - b_1) \leq v(b_2)$.

A similar argument will hold if $v(c_2) < v(b_2)$.

$\square$

Similarly, for $B \not\subseteq C$ to hold there must be a $b \in B$ such that $b \in B$ but $b \notin C$, i.e. $v(b - c_1) \leq v(c_2)$. And thus we get $\Psi.1(b_1, b_2, c_1, c_2)$ to hold.

2. If $v(c_1 - b_1) \leq v(b_2)$ then c_1 will not be in B and thus $B \not\subseteq C$. Likewise if $v(c_1 - b_1) < v(c_2)$ then $b_1 \notin C$ and thus, $C \not\subseteq B$ and we again conclude $B \cap C = \emptyset$.

   Now to show that $B \cap C = \emptyset$ implies formula $\Psi.2$, again notice that $B \cap C = \emptyset$ implies $C \not\subseteq B$ and $B \not\subseteq C$. Thus, there is a $c \in C$ such that $c \in C \setminus B$, thus $v(c - b_1) \leq v(b_2)$ by definition of B. Likewise, there is also a $b \in B \setminus C$ and thus $v(c_1 - b) < v(c_2)$. Therefore, with the help of the claim, we have $B \cap C = \emptyset$ implies formula $\Psi.2(b_1, b_2, c_1, c_2)$.

   Note. If we let $B = B^{cl}(b_1, b_2)$ be a closed ball and $C = B^{op}(b_1, b_2)$ open. If you reverse the roles of B and C play in the above case, then we're in the situation of one open ball and one closed ball. Thus $\Psi.2(b_1, b_2, c_1, c_2)$ will hold if and only if $B \cap C = \emptyset$ for the same reasons as above.

3. Again, if $v(c_1 - b_1) < v(b_2)$ then by how we defined of B, $c_1 \notin B$ so $B \not\subseteq C$. Likewise, if $v(c_1 - b_1) < v(c_2)$ then $b_1 \notin C$ by definition, so $B \not\subseteq C$ and thus $B \cap C = \emptyset$.

   And as we did in the previous two cases, to show $B \cap C = \emptyset$ implies formula $\Psi.3(b_1, b_2, c_1, c_2)$, we have $B \cap C = \emptyset$ implies $C \not\subseteq B$ and $B \not\subseteq C$. Thus we can find a $c \in C \setminus B$ which implies $v(c - b_1) < v(b_2)$ or a $b \in B \setminus C$ which implies $v(b - c_1) < v(c_2)$.

□

3.2.3 Defining Formula for Containment in a Definable Set

Let $\varphi(x; \bar{b})$ be a formula defining a finite union of Swiss cheese.We develop an explicit way to see whether or not a ball D is contained in the Swiss cheeses:

$$\bigcup_{i=1}^{k} (B_i \setminus C_1^i \cup C_2^i \cup \cdots \cup C_{n_i}^i)$$

and then we utilize the above lemmas for the construction.

In order for D to be a subset of the Swiss cheeses it must be the case that $D \subseteq B_i$ for a fixed i and $D \cap C_m^i = \emptyset$ for all m. We have the following cases to consider.

1. The ball $D = B^{op}(d_1, d_2)$ is open. First we look at the formula needed to ensure $D \subseteq B_i$ for some i. Let $\Phi.1(d_1, d_2, b_{1,l}, b_{2,l})$ be the formula $\Phi.1$ applied to K-definable balls D and open balls B_l where $1 \leq l \leq j$ [2]. Likewise, let $\Phi.3(d_1, d_2, b_{1,l}, b_{2,l})$ be the formula $\Phi.3$ applied to K-definable balls D and closed balls B_l for $j+1 \leq l \leq k$. Thus, if there j many open balls and $k-j$ many closed balls then

$$D \subseteq \left(\bigcup_{l=1}^{j} B_l \ \cup \bigcup_{l=j+1}^{n} B_l \right)$$

if and only if

$$\bigvee_{l=1}^{j} \Phi.1(d_1, d_2, b_{1,l}, b_{2,l}) \ \vee \bigvee_{l=j+1}^{k} \Phi.3(d_1, d_2, b_{1,l}, b_{2,l})$$

holds.

Now to make sure D does not intersect with any C^i_j we make use of $\Psi.1$ and $\Psi.3$. Let $\Psi.1(d_1, d_2, c_{1,l'}, c_{2,l'})$ be the formula applied to the K-definable balls D and open balls $C^i_{l'}$, for $1 \leq l' \leq n'$ where n' is the number of open balls removed in the finite union of Swiss cheeses. Likewise let $\Psi.3(d_1, d_2, c_{1,l'}, c_{2,l'})$ be the formula applied to D and closed balls $C^i_{l'}$ for $n'+1 \leq l' \leq n''$ where n'' is the number of closed balls removed in the finite union of Swiss cheeses. Thus, if there are $n'+n'' = n$ many inner balls removed then

$$D \cap \left(\bigcup_{l'=1}^{n'} C^i_{l'} \ \cup \bigcup_{l'=n'+1}^{n''} C^i_{l'} \right) = \emptyset$$

holds if and only if

$$\bigwedge_{l'=1}^{n'} \Psi.1(d_1, d_2, c_{1,l'}, c_{2,l'}) \ \wedge \bigwedge_{l'=n'+1}^{n''} \Psi.3(d_1, d_2, c_{1,l'}, c_{2,l'})$$

holds.

[2]Since we do not know which outer ball D will be contained in, we must range over all possible outer balls.

Therefore, D is a subset of the union of Swiss cheeses if and only if

$$\begin{aligned}\Big(\bigvee_{l=1}^{j}\Phi.1(d_1,d_2,b_{1,l},b_{2,l})&\\ \wedge\bigvee_{l=j+1}^{k}\Phi.3(d_1,d_2,b_{1,l},b_{2,l})\Big)&\\ \wedge\Big(\bigwedge_{l'=1}^{n'}\Psi.1(d_1,d_2,c_{1,l'},c_{2,l'})&\\ \wedge\bigwedge_{l'=n'+1}^{n''}\Psi.3(d_1,d_2,c_{1,l'},c_{2,l'})\Big)&\end{aligned} \tag{3.1}$$

2. The ball $D = B^{cl}(d_1, d_2)$ is closed.

   Similar to the case above, we can construct a formula that holds if and only if D is a subset of the Swiss cheeses, if we make use of formulas $\Phi.2, \Phi.3, \Psi.2$ and $\Psi.3$ and obtain

$$\begin{aligned}\Big(\bigvee_{l=1}^{j}\Phi.2(d_1,d_2,b_{1,l},b_{2,l})&\\ \wedge\bigvee_{l=j+1}^{k}\Phi.3(d_1,d_2,b_{1,l},b_{2,l})\Big)&\\ \wedge\Big(\bigwedge_{l'=1}^{n'}\Psi.2(d_1,d_2,c_{1,l'},c_{2,l'})&\\ \wedge\bigwedge_{l'=1}^{n''}\Psi.3(d_1,d_2,c_{1,l'},c_{2,l'})\Big)&\end{aligned} \tag{3.2}$$

3.3 Defining Schema

Now we have established the necessary background to come up with an explicit schema for the definable types. Recall that all definable sets in ACVF can be written as a finite union of Swiss cheeses. To see if the formula is in the type we ask whether or not the smallest ball is contained in the union of outer balls of the Swiss cheeses, and if it is not contained in the inner balls removed.

Lemma 3.7. *Suppose $K \vDash ACVF$, $m \in M \succeq K$. If $tp(m/K)$ is definable then the intersection of all K-definable balls (open or closed) containing m is a non-empty definable subset of K.*

Proof. Let $tp(m/K)$ be definable and let $\phi(x; y, z) = v(x - y) > v(z)$ and $\psi(x; y, z) = v(x - y) \geq v(z)$. Let d_ϕ and d_ψ be the defining formulas of ϕ and ψ. Then the intersection of all K-definable balls containing m is defined by

$$\Phi(x) = \forall y \forall z((d_\phi(y, z) \rightarrow \phi(x, y, z))) \wedge (d_\psi(y, z) \rightarrow \psi(x, y, z)).$$

Since $m \in M, M \vDash \exists x \Phi(x)$. And since $M \succeq K, K \vDash \exists x \Phi(x)$ making $\Phi(K)$ nonempty. $\square$

It follows from the lemma that if $tp(m/K)$ is definable then there is a smallest K-definable ball containing it.

We end this chapter with the following theorem to develop a schema.

Theorem 3.8. 1. *Let p be a definable type and $\varphi(x; y)$ a formula defining finite union of Swiss cheeses that describes $X_{\varphi(x;\bar{b})}$. Also let D be the smallest ball containing the type. Then $p \vdash \varphi(x; \bar{b})$ if and only if $D \subset \bigcup B_j$ and $(C_i^j \subset D \vee C_i^j \cap D = \emptyset)$ where the B_js are the outer balls C_i^js are the inner balls with respect to the outer ball B^j.*

2. *Furthermore, the defining schema is the right hand side of the biconditional.*

Proof. 1. Assume $p \vdash \varphi(x; \bar{b})$ and let B_j an outer ball of the finite union of Swiss cheeses. Then since B_j contains realizations of the type, D and B_j are not disjoint, and therefore are nested. Then Since D is the smallest ball containing the type, we cannot have $B_j \subset D$, so the only alternative is $D \subset B_j$. Furthermore, since D contains a realization of the type, $D \not\subseteq C_i^j$, so either $C_i^j \subset D$ or $C_i^j \cap D = \emptyset$.

For the converse, assume $D \subset B_j$ and $C_i^j \cap D = \emptyset$. Then a realization a of the type is in $D \setminus C_i^j$ and thus $\varphi(x; b) \in p$. Now assume $D \subset B_j$ and $C_i^j \subset D$, and assume towards a contradiction $a \in C_i^j$. If $a \in C_i^j$ then the formula $x \in C_i^j$ is in the type, and thus C_i^j would be a smaller ball that realizes the type, a contradiction to our selection of D. Thus, if $C_i^j \subset D$ then $a \in D \setminus C_i^j$ and we are done.

2. Now for the defining schema. Given a formula $\varphi(x;\bar{b})$ for a definable set, we write it as $\mathcal{S}(x;\bar{a},\sigma,n,m)$, which is uniform in parameters, which we know exists by 2.4. Thus, by part (1.) we know whether or not the formula is in the type just by checking the right hand side of the biconditional, which is only dependent on the parameters.

   (a) D is a subset of the union of the outer balls of the Swiss cheeses that are describing $X_{\varphi(x;\bar{b})}$, and

   (b) D is disjoint from all the inner balls removed from the Swiss cheeses.

   Since the definable sets are a finite union of Swiss cheeses, we have schema.

□

Chapter 4

Classification of Types in ACVF

Suppose K is a valued field. We give the classification of 1-types over K where the valued field and the residue field are of any characteristic. In other words, if L is an elementary extension of K and a, b are elements of the extension, we give the necessary and sufficient conditions for $tp(a/K) = tp(b/K)$.

4.1 Residual, Valuational, and Immediate Types

In the case of algebraically closed fields, we had a way of classifying types with ideals. Despite the fact that we won't be working with ideals, we define a new object to serve a similar purpose as the ideals served when classifying the definable types of algebraically closed fields.

Definition 4.1.1. Let $K \subseteq M$ be two valued fields $m \in M$ then we define the set

$$J_K(m) := \{v(m-k) : k \in K\}$$

Definition 4.1.2. Let $K \subseteq M$ be two valued fields, $m \in M$. To discuss the type of m over K we introduce the set

$$I_K(m) = \{g \in \Gamma_K : \exists k \in K \text{ such that } M \models v(m-k) \geq g\}$$

Proposition 4.1. $I_K(m) = J_K(m) \cap \Gamma_K$.

Proof. First we show $I_K(m) \subseteq J_K(m) \cap \Gamma_K$. Notice that $I_K(m)$ is an initial segment of Γ_K because if $g \in I_K(m)$ then there is a k such that $v(m-k) \geq g$, but for any element $g' \in \Gamma_K$ such that $g' \leq g$ we have $v(m-k) \geq g'$ and thus $g' \in I_K(m)$. To see why $I_K(m) \subset J_K(m)$ assume towards a contradiction there was a $g \in I_K(m)$ such that $g \notin J_K(m)$, i.e. $v(m-k) > g$ for all $k \in K$. Let $b \in K$ be such that $g = v(b)$. Then for all $k \in K$

$$\begin{aligned} v(b) < v(m-k) &\implies v(b-m+m) < v(m-k) \\ &\implies \min\{v(b-m), v(m)\} < v(m-k). \end{aligned}$$

However, $\min\{v(b-m), v(m)\} \neq v(m)$ because then $v(m) < v(m-k)$ wouldn't hold for $k = 0$. On the other hand, $\min\{v(b-m), v(m)\} \neq v(m-b)$ because since $b \in K$, we can take $k = b$ and thus $v(m) < v(m-k)$ still would not hold. Thus, there is no $g \in I_K(m)$ such that $g \notin J_K(m)$.

To see why the converse containment holds, let $g = v(m-k) \in J_K(m) \cap \Gamma_K$. Then $v(m-k) \geq g$, and thus we have that $g \in I_K(m)$. □

4.1.1 Major Results from Delon

The following definitions and results are credited to [5] and will serve as fruitful later on.

Definition 4.1.3. Let $K \subseteq M$ be two valued fields, $a \in M$; we have the three following algebraic definitions:

1. We say m is **residual over** K if $J_K(m) \subset \Gamma_K$ and $J_K(m)$ has a largest element.
2. We say m is **valuational over** K if $J_K(m) \nsubseteq \Gamma_K$ and equal to $I_K(m) \cup \{v(m-k_0)\}$ where $k_0 \in K$ such that $v(m-k_0) > I_K(m)$
3. We say m is **immediate over** K if $J_K(m) \subset \Gamma_K$ and $J_K(m)$ does not have a greatest element.

Lemma 4.2. *The cases in the above definition are in fact all the possible cases.*

Proof. It is clear to see that the residual case and the immediate case define two different instances. Now, to see why the valuational case is necessary and sufficient to complete the classification of the types, suppose for a contradiction that there is another case where there is a k_1 such that $v(m - k_1) \notin I_K(m)$ and $v(m - k_1) < v(m - k_0)$. Notice then that

$$v(m - k_1) = v(m - k_0 + k_0 - k_1) = v((m - k_0) + (k_0 - k_1))$$

$$= \min\{v(m - k_0), v(k_0 - k_1)\}$$

since $v(m - k_0) \neq v(k_0 - k_1)$ by assumption of $v(m - k_0)$. However, since $v(m - k_1) < v(m - k_0)$ we must have $v(m - k_1) = v(k_0 - k_1)$, a contradiction since $v(k_0 - k_1) \in \Gamma_K$. Therefore, there are not other cases to consider. $\square$

One of Delon's major results is the following theorem that characterizes the necessary information needed to classify the three types, which we will use in the next section. Here when we say *characterized* we mean that $tp(a/K)$ is characterized by A if there is $\sigma \in Aut(M/A)$ such that $\sigma(a) = a'$, then there exists $\tau \in Aut(M/K)$ such that $\tau(a) = a'$.

Theorem 4.3. 1. *The residual type is characterized by the tuple $(a, b) \in K^* \times K$ such that the residue of $am + b$ is not in $\overline{K}$ and by $tp(res(am + b)/K)$.*

2. *The valuational type is characterized by a field element $a \in K$ such that $v(m - a) \notin \Gamma_K$ and by $tp(v(m - a)/\Gamma_K)$.*

3. *And the immediate type is characterized by a sequence $(g_\alpha; b_\alpha)_{\alpha<\alpha_0}$ that satisfies $v(m - b_\alpha) = g_\alpha$ and is cofinal in $I_K(m)$.*

4.2 Definable Types

We are now ready to classify the definable types of ACVF.

Definition 4.2.1. Let G be a divisible ordered abelian group. Then a 1-type p over G is a *cut* if there exists $H_1, H_2 \subset G$ such that

1. H_1 does not have a least element, and H_2 does not have a greatest element.

2. $H_1 \cap H_2 = \emptyset$ and $G = H_1 \cup H_2$

3. For all $h \in H_1$ and $g \in H_2$, $g < h$.

4. For all $h \in H_1$ and $g \in H_2$, $x < h_1 \in p$ and $h_2 < x \in p$.

Theorem 4.4. *Let M be a field extension of K and $m \in M$. Suppose the type $tp(m/K)$ is definable. Then the there is a smallest ball containing it by Lemma 3.7 and we have two cases to consider:*

1. If the smallest ball is closed, then $tp(m/K)$ is a residual type.

2. If the the smallest ball is open, then $tp(m/K)$ is a valuational type.

In other words, the definable types in ACVF are residual types and valuational types.

Proof. 1. The K-definable ball is closed.

The smallest ball containing $tp(m/K)$ is a closed ball $D = B^{cl}(d_1, d_2)$. So for any $v(b_2) > v(d_2)$ and for any $b_1 \in D$ we have $m \notin B^{cl}(b_1, b_2)$, i.e. $v(m - b_1) < b_2$. Thus we have

$$v(d_1) \leq v(m - b_1) < v(b_2)$$

which means $\{v(m - k) : k \in M\}$ has a largest element, which according to Definition 4.1.3, is the property to determine the type is residual.

2. The K-definable ball is open.

The smallest ball containing $tp(m/K)$ is an open ball K -definable $D = B^{op}(d_1, d_2)$. So for any $v(b_2) > v(d_2)$ and for any $b_1 \in D$ we have $m \notin B^{op}(b_1, b_2)$, i.e. $v(m - b_1) < v(b_2)$. Thus we have

$$v(d_2) < v(m - b_1) < v(b_2)$$

making $v(m - b_1) \notin \Gamma_K$ a cut in Γ_K since the collection of values below $v(m - b_1)$ and the collection of values above $v(m - b_1)$ act as H_1 and H_2 in Definition 4.2.1, Thus the type is valuational.

□

Lastly, we describe the cases where the intersection of all K-definable balls containing $tp(m/K)$ is not a K-definable ball.

Theorem 4.5. *Let $tp(m/K)$ be a type. If there is no smallest K-definable ball containing $tp(m/K)$ then by Lemma 3.7 it is not definable. There are again two cases to consider:*

1. *If the radii of the balls containing the type are bounded above, then the type is valuational.*

2. *If the radii of the balls containing the type are unbounded, then the type is immediate.*

Thus the non-definable types in ACVF are valuational types and immediate types.

Proof. The contrapositive of Lemma 3.7 tells us that these types are not definable. Let b_i be some realization of the type, and let Let $G = \{v(b) :$ there is a K definable ball of radius $v(b)$ containing $m\} \subset \Gamma_K$.

1. The set G is bounded above.

   Suppose G is bounded above and let $\sup G = v(d)$. Let D be a ball containing the realizations of the type, with radius $v(d)$ and containing m. We know $v(d) \notin \Gamma_K$ because there is no smallest K-definable ball containing $tp(m/K)$, making D not a K-definable ball, but an M-definable ball. Since G is a subset of an o-minimal group, it defines a cut. Thus, for all $c \in D$ and for all $v(b) \in G$ we have

   $$v(d) < v(m - c) < v(b)$$

   which defines our cut which is not definable in Γ_K. Therefore by Definition 4.1.3, it is a valuational type.

2. The set G is unbounded.

   If $tp(m/K)$ is unbounded, then for all $v(b) \in G$ there is $v(b') \in G$ such that $v(b') > v(b)$ and thus by Definition 4.1.3 since there is no largest valuation, we have that $tp(m/K)$ is an immediate type.

□

4.3 Invariant Types

The definable types discussed above are all invariant by Proposition 3.1. However, we have two cases which are not definable: the valuational case when the cut is leads to the type being bounded above but not contained in a smallest

K-definable ball, and the immediate types which are unbounded . We show they are invariant nevertheless.

Recall that to show a type p is invariant we must show $\sigma p = p$ for all $\sigma \in Aut(\mathcal{M}/A)$ for some $A \subset \mathcal{M}$. To show the the types that were not definable are invariant, we first need to describe the set A.

Theorem 4.6. *Suppose $tp(m/K)$ is a valuational type and let $A = \{b\} \cup \{v(m-b)\}$ as in where $v(m-b) \notin \Gamma_K$, as in the classification in Theorem 4.3. Then the type is invariant*

Proof. Let $p = tp(m/K)$. We show that the type is invariant by showing $p \vdash \varphi(x; c)$ if and only if $p \vdash \varphi(x; c')$ for $c \equiv_A c'$. Since all definable sets can be written as a finite union of Swiss cheeses, we take $\varphi(x; c) \in \mathcal{S}(x, \bar{c}, \tau, n, m)$. Since Swiss cheeses are a Boolean combination of balls, we first observe applying an automorphism to a ball.

Let $m \in B(d_1, d_2)$. Then since $v(m - b)$ is not in Γ_K, we know $v(m - b)$ defines a cut by the proof of Theorem 4.4 . Thus $v(m-b)$ is either greater than $v(m - d_1)$ or less than $v(m - d_1)$. Assume first $v(d_2) < v(m - d_1) < v(m - b)$. Then and thus $B(b, m - b) \subseteq B(d_1, d_2)$ by Proposition 1.4, and therefore

$$K \vDash \forall x(v(x - b) > v(m - b) \to v(x - d_1) > v(d_2)).$$

Let $\sigma \in Aut(K/A)$. Then

$$K \vDash \forall x(v(x - b) \geq v(m - b) \to v(x - \sigma(d_1)) > \sigma(v(d_2))$$

holds as well, and thus $m \in B(\sigma(d_1), \sigma(d_2))$.

For the converse suppose $m \in B(\sigma(d_1), \sigma(d_2))$.Then as before, since $\sigma^{-1} \in Aut(K/A)$ we have

$$B(b, m - b) \subseteq B(\sigma^{-1}(\sigma(d_1)), \sigma^{-1}(\sigma(d_2))) = B(d_1, d_2)$$

and thus $m \in B(d_1, d_2)$.

Now suppose $v(m - b) < v(a - d_1)$, then $B(d_1, d_2) \subseteq B(b, m - b)$ and thus we have

$$K \vDash \forall x[v(x - d_1) > v(d_2) \to v(x - b) > v(m - b)].$$

Taking $\sigma \in Aut(K/A)$ we have

$$K \vDash \forall x[v(x - \sigma(d_1)) > \sigma(v(d_2)) \to v(x - b) > v(m - b)]$$

but this implies $B(\sigma(d_1)\sigma(d_2)) \subseteq B(b, m-b)$ and thus $B(\sigma(d_1), \sigma(b_2))$ only shifts its position in $B(b, b-m)$ and still contains $m568$.

Similar to the converse in the above case, we use the fact that $\sigma^{-1} \in Aut(K/A)$ to obtain if $m \in B(\sigma(d_1), \sigma(d_2))$ and then $m \in B(d_1, d_2)$.

Now we see what happens when σ acts on a Swiss cheese. Let $m \in B(d_1, d_2)$ and let $S_n^\tau(x, \bar{c}) \in tp(m/K)$ define a Swiss cheese

$$S = B \setminus (C_1 \cup \cdots \cup C_n)$$

where $B = B(b_1, b_2)$ and $C_i = B(c_i, c_i')$ for $1 \le i \le n$. Then if $m \in S$ then $m \in \sigma(B)$ and $m \notin \sigma(C_i)$. Therefore $m \in \sigma(S)$ and since any definable set can be written as a finite union of Swiss cheeses, we have $tp(m/K)$ is A-invariant. □

Theorem 4.7. *Suppose $tp(m/K)$ is an immediate type. Then as in the classification in Theorem 4.3, let $A = \{b_\alpha\} \cup \{v(m-b_\alpha)\}$ where $v(m-b_\alpha)$ is cofinal in $J_m(K) = \{v(m-k) : k \in K\}$. Then the type is invariant.*

Proof. As before, we prove the type is invariant by showing $m \in \mathcal{S}(x, \bar{c}, \tau, n, m)$ if and only if $m \in \sigma(\mathcal{S}(x, \bar{c}, \tau, n, m))$ where $\mathcal{S}(x, \bar{c}, \tau, n, m)$ is a finite union of Swiss cheeses.

Let $p = tp(m/K)$. For the forward direction, let $m \in B(d_1, d_2)$. Since $v(m-b_\alpha)$ is cofinal, there is an α such that $v(d_2) < v(m-d_1) \le v(m-b_\alpha)$. Then $v(d_2) < v(m-b_\alpha)$ and thus $B(b_\alpha, m-b_\alpha) \subseteq B(d_1, d_2)$ by an earlier proposition. Therefore

$$K \vDash \forall x(v(x-b_\alpha) \ge v(m-b_\alpha) \to v(x-d_1) > v(d_2)).$$

Let $\sigma \in Aut(K/A)$, then

$$K \vDash \forall x(v(x-b_\alpha) \ge v(m-b_\alpha) \to v(x-\sigma(d_1)) > \sigma(v(d_2))$$

also holds. Hence

$$B(b_\alpha, m-b_\alpha) \subseteq B(\sigma(d_1), \sigma(d_2))$$

and thus $m \in B(\sigma(d_1), \sigma(d_2))$.

Now suppose $m \in B(\sigma(d_2), \sigma(d_2))$. Then since the inverse of an automorphism is again an automorphism, $\sigma^{-1} \in Aut(K/A)$, so we can repeat the above argument to obtain

$$B(b_\alpha, m - b_\alpha) \subseteq B(\sigma^{-1}(\sigma(d_1)), \sigma^{-1}(\sigma(d_2))) = B(d_1, d_2)$$

and thus $m \in B(d_1, d_2)$.

Now that we know how invariant types behave with a singular K-definable ball we can consider Swiss cheeses. Let $m \in B(d_1, d_2)$ and let $S_n^\tau(x, \overline{c}) \in tp(m/K)$ that defines a Swiss cheese

$$S = B \setminus (C_1 \cup \cdots \cup C_n)$$

where $B = B(b_1, b_2)$ and $C_i = B(c_i, c_i')$ for $1 \leq i \leq n$.

Then if $m \in S$ then $m \in \sigma(B(b_1, b_2))$ and $m \notin C_i$ for $1 \leq i \leq n$. Thus $m \in \sigma(S)$ and therefore $tp(m/K)$ is A-invariant since every definable set of ACVF can be written as a finite union of Swiss cheeses. $\square$

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