Axiom of Choice and the Partition Principle

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Abstract

We introduce the Partition Principle (PP), an axiom introduced by Russell in the context of its similarities and differences with the Axiom of Choice (AC). We start by proving some properties of PP and AC, and show that AC entails PP. To address the problem of whether the converse holds, we develop the Zermelo-Fraenkel (ZF) set theory and examine its consistency and build a model in which AC fails. We follow this with a discussion of forcing, a technique introduced by Paul Cohen to build new models of set theory from existing ones, which have differing properties from the starting model. We conclude by examining candidate models called permutation models where AC fails, which may be useful as candidate models for forcing a model in which PP holds but AC does not. We conjecture that such a model exists, and that PP does not entail AC.

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⋮

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List of Symbols, Abbreviations, and Conventions

$\alpha, \beta, \gamma, \delta$	ordinals
κ, λ	cardinals
$0, \emptyset$	empty set
$\alpha^+, \alpha + 1$	successor of α
ON	class of all ordinals
AC	Axiom of Choice
PP	Partition Principle
SB	Schroeder-Bernstein theorem
dSB/SB*	Dual Schroeder-Bernstein theorem
ZF	Zermelo-Fraenkel axiomatization of sets
ZFC	ZF augmented with AC
ZF^-	ZF without the axiom of regularity/foundation
ZFA	ZF with atoms
ZF-P	ZF without P
p.o.	partial order
WF	well-founded
o.t.	order-type
OD	ordinal definable
HOD	hereditarily ordinal definable
TF	transfinite
tc	transitive closure

Chapter 1

The Axiom of Choice and the Partition Principle

We start by introducing our two main objects of study: The Axiom of Choice (AC) and the Partition Principle (PP). Both are axioms stipulating the existence of certain kinds of sets[1]. AC is the statement asserting that any onto function has an inverse-like injection. More formally:

Axiom 1.0.0.1 (AC)**.** *Let x, y be sets and $f : x \to y$ a surjection. Then, there exists a function $g : y \to x$ such that g is one-to-one, and $g(f(z)) = z$, for all $z \in x$. That is, g is a left-inverse of f. Alternatively, we say that if $\mathcal{C}$ is a non-empty collection of non-empty sets, there exists a set S such that for each $y \in \mathcal{C}$, there exists precisely one $x \in y$ in S, and a function $f : \mathcal{C} \to S$, so $f(C) = x$, so $x \in C$ for each $C \in \mathcal{C}$. Such a function h, assigning to each $C \in \mathcal{C}$ an element of itself is called a* ***choice*** *function.*

PP is similar, but asserts only that g is one-to-one. Formally:

Axiom 1.0.0.2 (PP)**.** *Let x, y, f be as in Axiom 1.0.0.1. Then, there exists $g' : y \to x$, such that g' is one-to-one.*

[1] For now, we will not worry about defining what a set is precisely, but adopt the naive idea that a set is a collection of objects.

It is immediately obvious that AC entails PP for we may choose $g' := g$ whose existence is guaranteed by AC. The converse, is not immediately obvious and remains an open problem in set theory since its introduction to the literature by Russell in 1905 (see [BRGHM14]).

A third axiom, the dual Schroeder-Bernstein theorem (dSB) asserts that if there are surjections from sets A to B and B to A, there is a bijection between A and B. That is:

Axiom 1.0.0.3 (dSB). *Let x, y be sets, and f, g surjections in either direction. Then there exists $h : x \to y$ a bijection.*

We can prove that PP entails dSB:

Theorem 1.0.0.4. $\mathsf{PP} \implies \mathsf{dSB}$.

Proof. Suppose PP holds and x, y, f, g are as in Axiom 1.0.0.3. Then there are f', g' injections $x \to y$ and $y \to x$ by PP. The result follows by Schroeder-Bernstein (SB). ■

Another natural question to ask becomes whether or not dSB entails AC. This problem is open as well. The goal of this thesis is to examine these relationships through the study of axiomatic set theory and Cohen's forcing. Both of these will enable us to prove independence results, where we say a certain sentence/property is **independent** of a collection of axioms if it can be neither proven nor disproven from those axioms. That there are such statements is a consequence of Gödel's incompleteness theorems, which we touch on briefly in the second chapter.

We start in the subsequent section by further examining some consequences/properties of the axioms introduced earlier, with a particular emphasis on PP. The second chapter focuses on the Zermelo-Fraenkel axioms (ZF), and other set-theoretic and model-theoretic constructions necessary to understand the chapters which come after it. In

Chapter 3, we exhibit a model of ZF by introducing the well-founded (WF) universe. We also show that we can produce a set model of ZF if we assume the existence of an inaccessible cardinal. The fourth chapter is dedicated to showing the independence of AC from the axioms of zf. We also show the independence of PP and dSB from ZF based on work to come later in this chapter. Chapter 5 discusses the technique of forcing and its use in independence results. We will prove that both the Continuum Hypothesis (CH) and AC are independent of ZF. Finally, Chapter 6 is dedicated to revisting the open problems stated earlier, and we conjecture some preliminary results.

1.1 Equivalents of PP

There are several well-known equivalents of AC; among them Zorn's Lemma, that every vector space has a basis, that every set is well-orderable, and that infinite Cartesian products of non-empty sets are non-empty. On the other hand, PP has few known equivalences; likely due to its lesser renown. However, we can demonstrate some equivalents to PP:

First define:

Definition 1.1.0.1. *Let x, y be sets. Then $x \leq y$ if there is an injective function $f : x \to y$. Moreover, we say $x \approx y$ if both $x \leq y$ and $y \leq x$. We say x and y are **equinumerous**. In particular, x, y are in bijection.*

Remark 1.1.0.2. *Equinumerous sets have the same cardinality. See Lemma 2.2.2.17.*

Theorem 1.1.0.3. *Let $\{A_i : i \in I\}$ be a family of sets. Let σ be the statement that there is an injection from $\bigcup_{i \in I} A_i$ to $\bigcup_{i \in I} \{i\} \times A_i$. That is, σ is the statement that a union always embeds into its disjoint union. Then,* PP *is equivalent to σ over* ZF.

Proof. Clearly there is a surjection via the projection map from the disjoint union to the union. Under PP, there is then an injection from the union to the disjoint union, so $\mathsf{PP} \implies \sigma$.

For the converse, let $f : A \to B$ be a surjection. Let $B_a = \{f(a)\}$ for each a. Then $B = \bigcup_{a \in A} B_a$. Moreover, $|A| = \bigcup_{a \in A} \{a\} \times B_a$, but this is the disjoint union, so there is an injection from $B \to A$. So $\sigma \implies \mathsf{PP}$.

■

The above is from Masai Higasikawa's paper, [MH95], in which he also demonstrates a number of cardinal properties which are consequences of PP.

Another reformulation (by Sierpinski in 1918) is as follows:

Theorem 1.1.0.4. *Let τ be the statement that if R is a relation that $|dom(R)| \leq |R|$. Then PP is equivalent to τ over ZF.*

Proof. There is a natural surjection from R to $\operatorname{dom}(R)$ by projection. Assuming PP there is an injection from $\operatorname{dom}(R)$ to R as required. So, $\mathsf{PP} \implies \tau$.

On the other hand, let $f : A \to B$ be a surjection. Let $R = f^{-1}$. Clearly R is a relation with $\operatorname{dom} R = B$. Then, invoking τ, there is an injection $B \to A$ so $\tau \implies \mathsf{PP}$.

■

One will note immediately that neither of these equivalents are particularly as nice as those for AC (i.e. the well-orderability of any set). However, they will be useful for showing some more consequences of PP in the subsequent section.

1.2 Unmeasurable sets

We start our discussion with some results from analysis[2]. It is a well-known fact that AC proves the existence of an unmeasurable set. We give a proof below:

Theorem 1.2.0.1 (with AC). *Let $\lambda : \mathcal{P}(\mathbb{R}) \to \mathbb{R}^+ \cup \{+\infty\}$, such that $\lambda((a,b]) = b-a$, $\lambda(x) = \lambda(x+\sigma)$ and if $\mathcal{C}$ is a collection of disjoint sets, then $\lambda(\bigcup \mathcal{C}) = \sum_{c\in\mathcal{C}} \lambda(c)$. Then, λ is not a total function. That is there exists a set x such that $\lambda(x)$ is undefined.*

Proof. Suppose for a contradiction that λ is total. We define an equivalence relation on $\mathbb{R}$ by asserting that $x \sim y$ if $y - x \in \mathbb{Q}$. Clearly this relation satisfies reflexivity, symmetry and transitivity[3]. Set $[x] = \{y : y - x \in \mathbb{Q}\}$ the equivalence classes of $\mathbb{R}$ under $\sim$, and set $\Lambda = \mathbb{R}/\sim$. Now invoking AC, for each equivalence class, we may choose a representative. In particular, we may choose this representative to lie in the interval $(0,1)$. Set this collection of representatives to be $\Omega \subseteq (0,1)$.

We notice that $\Omega+p, \Omega+q$ are disjoint or equal for $p,q \in \mathbb{Q}$. Suppose $z \in \Omega+p \cap \Omega+q$. Then there exists $x \in \Omega + p, y \in \Omega + q$ so that $z = x + p = y + q$. In particular, $y - x = p - q$, such that $x \sim y$. Since we have chosen precisely one representative from each class $x = y$, and so $p = q$, from which the proof follows. In particular, it follows that

$$\lambda\left(\bigcup_{-1<q<1} \Omega + q\right) = \sum_{-1<q<1} \lambda(\Omega + q) = \sum_{-1<q<1} \lambda(\Omega).$$

As well, the set in question lies within $(-1,2)$. At most then, it's measure is 3. That is, $\sum_{-1<q<1} \lambda(\Omega) \leq 3$. But this means $\lambda(\Omega) \not> 0$. However, we note that $\bigcup_{q\in\mathbb{Q}, -1<q<1} \Omega + q \supset (0,1)$ – contradiction.

■

Though it is conjectured that PP is weaker than AC (i.e. that it does not entail AC),

[2] The growth of the field of set theory was largely motivated by a need to make calculus rigorous.
[3] $(z - x) = (z - y) + (y - x)$.

it is at least strong enough to also prove the existence of an unmeasurable set. Indeed we have the following theorem:

Theorem 1.2.0.2 (with PP)**.** *Let λ be as in Theorem 1.2.0.1. Then λ is partial.*

We require a lemma first:

Definition 1.2.0.3. *For any set x, $[x]^\omega$ is the set of all countably infinite subsets of x.*

Lemma 1.2.0.4. PP *proves that $[\mathbb{R}]^\omega$ has the same cardinality as $\mathbb{R}$. That is, there is an injection $f : [\mathbb{R}]^\omega \to \mathbb{R}$.*

Proof. We will show there is an injection from $\mathbb{R}$ into $[\mathbb{R}]^\omega$ and a surjection as well.

To see there is an injection from $\mathbb{R}$ into $[\mathbb{R}]^\omega$, choose $x \in [\mathbb{R}]^\omega$. Then x is a countably infinite set of reals. Define $f : \mathbb{R} \to [\mathbb{R}]^\omega$ by

$$f(r) = \begin{cases} x \cup \{r\} & \text{if } r \notin x \\ x - \{r\} & \text{if } r \in x \end{cases}.$$

It is easy to verify f is an injection. To see there is a surjection from $\mathbb{R}$ onto $[\mathbb{R}]^\omega$ note by Fact 4.3 from [Hal19], that there is a bijection between $\mathbb{R}^\omega$ and $\mathbb{R}$. It suffices to construct a surjection from $\mathbb{R}^\omega$ onto $[\mathbb{R}]^\omega$. For $x \in \mathbb{R}^\omega$ map x to its range, or ω if its range is finite. One will verify that this is a surjection from $\mathbb{R}^\omega$ to $[\mathbb{R}]^\omega$. Now invoking PP, we have an injection from $[\mathbb{R}]^\omega$ into $\mathbb{R}$. The result follows by applying SB.

■

We also state without proof the Lebesgue density theorem:

Definition 1.2.0.5. *on $\mathbb{R}^n$, and let A be a Lebesgue measurable set. The* ***approximate density*** *of A in an ϵ-neighborhood about a point $x \in \mathbb{R}^n$ is defined as*

$$d_\epsilon = \frac{\mu(A \cap B_\epsilon(x))}{\mu(B_\epsilon(x))},$$

where $B_\epsilon(x)$ is the closed ϵ-ball about x, and μ is the Lebesgue measure. We define the ***density*** *about a point x to be $d(x) = \lim_{\epsilon \to 0} d_\epsilon(x)$.*

Theorem 1.2.0.6 (Lebesgue density theorem)**.** *Let d, A be as in Definition 1.2.0.5. Then, for almost every point $x \in A$, $d(x)$ exists and $d(x) = 1$.*

Proof of Theorem 1.2.0.2. We claim that if $[\mathbb{R}]^\omega \approx \mathbb{R}$, then there is an unmeasurable set. To see this, we let $f : [\mathbb{R}]^\omega \to \mathbb{R}$ be an injection. Define Q_x to be the set of all reals a rational distance from x. This set is clearly countable, and hence an element of $[\mathbb{R}]^\omega$. Now define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = f(Q_x)$. It is easy to see that $g(x) = g(y)$ only if $x - y \in \mathbb{Q}$. Since if $x - y \in \mathbb{Q}$, $Q_x = Q_y$. Now suppose for a contradiction that g is Lebesgue measurable [4]. Then, so too is $h(x) = g(x) - g(-x)$. Clearly h is non-zero only at irrational points since x and $-x$ are rational distance apart for rational x, and so $g(x) = g(-x)$, such that $h(x) = 0$ for rational x. Set $N = \{x : h(x) > 0\}$. Then, since h is Lebesgue measurable, we have N is Lebesgue measurable. For any rational, r, $h(r - x) = g(r - x) - g(x - r) = f(E_{r-x}) - f(E_{x-r}) = f(E_{-x}) - f(E_x) = g(-x) - g(x) = -h(x)$ [5]. But then $x \in N \iff r - x \notin N$. Now choose arbitrary $x \in N$. Since N is Lebesgue measurable, it is the case that

$$\lim_{r \to 0, r \in \mathbb{Q}} \frac{\mu(N \cap B_r(x))}{\mu(B_r(x))} = 1.$$

But $\mu(B_r(x)) = 0$ for r rational, while the numerator "blows up", which contradicts

[4] A function is Lebesgue measurable if it satisfies the properties in [Gar], Proposition 3.1.

[5] Notice that $E_{r-x} = E_{-x}$, since if $y \in E_{r-x}$, then, $r - x - y = p$, for p rational. But then $-x - y = p - r$. Similarly $E_{x-r} = E_x$, since if $y \in E_{x-r}$, then $x - r - y = p$, but then $x - y = p + r$.

the Lebesgue density theorem.

∎

Alternatively, see [Sie18] for the original proof.

1.3 How strong is PP?

Certainly, given its ability to prove the existence of a Lebesgue non-measurable set, PP is sufficiently strong since the existence of an unmeasurable set is independent of ZF. At the very least it appears strikingly similar to AC, so another question arises in that: Does PP give us any choice?

The answer is affirmative in a fairly strong sense, but first, we require a working definition of cardinals and $\aleph$-numbers:

Definition 1.3.0.1. *Given a binary relation R on a set x, we say $y \in x$ is R-**least** or R-**minimal** if $\forall z \in x(\neg zRy)$.*

Definition 1.3.0.2. *A set x is **well-ordered** if every non-empty subset y of x has an $\in$-least element, where $x \in y$ is read to mean "x is an element of y."*

Definition 1.3.0.3. *The **cardinality** of x, denoted $|x|$ is the collection $\hat{x}$ of all sets y so $x \approx y$.*

We will more formally introduce cardinality in a subsequent section. A cardinal is **infinite** if it contains an infinite set.

Definition 1.3.0.4. *An $\aleph$-**number** is an infinite cardinal which contains a well-ordered set.*

Theorem 1.3.0.5. PP *entails* $\mathsf{AC}_\aleph$ *for all $\aleph$-numbers κ (see Definition 2.2.2.20), where* AC_κ *is choice for families of cardinality κ.*

The proof is by Andrzej Pelc in 1978 (see [Pel78])

Proof. The proof is by transfinite induction. We will show that $\mathsf{PP}+\mathsf{AC}_{<\kappa}$ proves AC_κ for every $\aleph$-number κ [6]. The base case is easy. Indeed, let $\kappa = 2$. Then choice holds for families of size 1 since it suffices to choose an arbitrary element from the single non-empty element of the family. It suffices to show that $x_1 \times x_2$ is non-empty for non-empty sets x_1, x_2. But this does not even require choice. Indeed, fix $y \in x_2$ and $x \in x_1$. The result clearly follows. Indeed, the result follows easily for all $n \in \omega$. For the induction step then: $\langle x_\alpha : \alpha < \kappa \rangle$ be a family of non-empty sets. By the induction hypothesis, we have that $C_\gamma = \prod\limits_{\alpha<\gamma} x_\alpha$ is non-empty for all $\gamma < \kappa$. Define by TF-induction:

- $\lambda_\gamma = \aleph\left(\bigcup_{\delta<\gamma} D_\delta\right) + \sup_{\delta<\gamma} \lambda_\delta^+$
- $D_\gamma = C_\gamma \times \lambda_\gamma$

Let $\lambda = \sup \lambda_\gamma$ and $D = \bigcup D_\gamma$. There is clearly a natural surjection from D to λ via the projection map. So, by PP there is an injection from λ into D. To see such an injection is of the required type, see the proof in [Pel78].

■

[6] $\mathsf{AC}_{<\kappa}$ is AC_λ for all $\aleph$-numbers $\lambda < \kappa$.

Chapter 2

Preliminaries and ZF

Set theory started informally as the study of collections of objects and their properties. Most mathematicians are familiar with, and use set operations such as union, and intersection without much thought. However, deeper studies have revealed a need for considerable care when working with sets. When used properly, they provide a powerful framework in which to do mathematics. Improper use on the other hand can have catastrophic consequences and reveal deep-rooted flaws with the foundations of mathematics. As well, the axiomatization of the theory of sets enables mathematicians to understand the inherent limitations of using sets to study mathematics via independence results.

2.1 Preliminaries and some model theory

We assume the reader is familiar with basic first-order logic (FOL) and model theory (i.e. the notion of a theory, derivations, provability, etc.).

The language of set theory is the language of FOL with equality augmented with the binary element relation $\in$ (i.e. $\mathcal{L}_{\text{set}} = \{=, \in\}$). We will use $<, \in$ interchangeably. Our main objects of study, sets, will be denoted by lowercase Latin alphabets (i.e.

x, y, z are sets), and we will write $x \in y$ to mean "x is an element of y". Sometimes, we will denote a set by an uppercase Latin alphabet to specify that it is a superset of another set or that it is the set being quantified over. Moreover, our universe of discourse will consist only of sets, by which we mean that if y is a set and x is an element of y, then x is also a set. We say "y is 'hereditarily' a set". For more on this convention, the reader is directed to [Kun06].

Fix a theory, Γ in some language. We say that Γ **proves** a sentence σ if there is a derivation of σ from Γ and the underlying logic we have chosen, and write $\Gamma \vdash \sigma$. We say $\Gamma \models \sigma$ to mean that if for every model $\mathcal{M}$ of Γ we have $\mathcal{M} \models \sigma$. We will use $\Gamma \models \sigma$ and $\Gamma \vdash \sigma$ interchangeably in light of Gödel's completeness theorem which states (without proof):

Theorem 2.1.0.1 (Gödel's completeness theorem)**.** $\Gamma \vdash \sigma \iff \Gamma \models \sigma$*.*

We define the Peano theory of the natural numbers:

Axioms 2.1.0.2 (Peano arithmetic (PA))**.** *The language is FOL with a unary function S, where $S(x)$ is the successor function on x, and a constant symbol,* 0*. The axioms are:*

(PA1) $\forall n(S(n) \neq 0)$

(PA2) $\forall n \exists m(n \neq 0 \implies S(m) = n)$

(PA3) $\forall n, m(n = m \iff S(n) = S(m))$*.*

(PA4) *Let ϕ be a unary formula. Then if $\phi(0)$ and $\forall n(\phi(n) \implies \phi(S(n)))$, then $\phi(n)$ for every n (induction).*

We will later see that PA can be defined in ZF.

Finally, we assume Gödel's incompleteness theorems. In particular:

Theorem 2.1.0.3 (Gödel's Incompleteness Theorem)**.** *For any theory* Γ *strong enough to encode Peano Arithmetic* (PA)*, there exist sentences* σ*, such that* Γ *cannot prove* σ *or its negation.*

2.2 The need for axiomatization

As mentioned before, mathematicians have comfortably used sets as part of their work for several years before Zermelo and Fraenkel axiomatized their theory. In 1901 however, Russell demonstrated very succinctly, a need to exercise care when working with the objects so as to avoid paradoxical results.

In particular, he demonstrated a need to distinguish between sets and "very large collections of objects". His exact argument is postulated below:

Theorem 2.2.0.1 (Russell's Paradox)**.** *The set* $R = \{x : x \notin x\}$ *is not a set*[1]*.*

Proof. Suppose R is a set. We ask whether R is a member of itself. If $R \in R$, then $R \notin R$ by definition of R. However, if $R \notin R$, then $R \in R$, again by definition of R. In particular then, $R \in R \iff R \notin R$.

■

Clearly in any "good" sense of the word, R is not an object that is particularly "nice," where again, "good" and "nice" are words we use to describe objects which are easy and intuitive to work with rigorously.

His now infamous paradox highlighted a glaring need for the formulation of a theory of sets. Over the years, several axiomatizations have been presented. The most commonly used one is that of Zermelo and Fraenkel, called ZF theory or just ZF. We present the axiomatization below:

[1] Here we use the term "set" to refer to something the average individual would recognize as being a collection of objects. Indeed, it is hard to envision (for someone not superhuman), how a set containing itself looks.

2.2.1 Axioms of set theory

Axioms 2.2.1.1. *The axioms of* ZF *are as follows:*

(ZF1) $\exists x(\forall y(y \notin x))$ *(empty set).*

(ZF2) $\forall x \forall y(x = y \iff (x \subseteq y \wedge y \subseteq x))$ *(extensionality*[2]*).*

(ZF3) *Given a formula* $\phi(x)$ *in* $\mathcal{L}_{set}$, $\forall x \exists y(y = \{z \in x : \phi(z)\})$ *(axiom schema of separation).*

(ZF4) $\forall x \forall y \exists z(z = \{x, y\})$ *(pairing).*

(ZF5) $\forall x \exists y(y = \bigcup x)$ *(union).*

(ZF6) $\forall x \exists y(y = \mathcal{P}(x))$ *(power-set).*

(ZF7) $\forall x(\exists z(z \in x) \implies \exists z((z \in x) \wedge \forall y(y \in x \implies \neg y \in z)))$ *(regularity/foundation).*

(ZF8) $\exists x(\emptyset \in x \wedge \forall y(y \in x \implies y \cup \{y\} \in x))$ *(infinity).*

(ZF9) *If* F *is a class function*[3]*, then the restriction of* F *to a set* x *is a set (axiom schema of replacement).*

Remark 2.2.1.2. *We notice that some axioms of the theory are redundant. Indeed,* (ZF1) *is a consequence of* (ZF3) *by setting* $\phi := x \neq x$*. For convenience however, we state certain logical consequences of the theory as axioms themselves.*

When augmented with AC, ZF becomes ZFC.

[2]Notice that here we have written a sentence not in the language of set theory. This is acceptable as long as we are able to express this symbol in the language. In particular, $x \subseteq y \iff \forall z(z \in x \implies x \in y)$. On many occasions, we shall use abbreviations such as this to refer to a formula of set theory for the sake of brevity. It is assumed that the reader will understand which formulas we are abbreviating in these situations.

[3]A function from the universe of all sets to itself.

The first of the nine axioms asserts the existence of an empty set; a set with no elements. The second asserts that equality behaves as it should; in particular that sets are defined by their members. The third, of these axioms states that for any set x, there exists another set y containing precisely those elements of x which satisfy a given property $\phi(x)$.

It is important to note that ZF3 is a schema, not an axiom, as it is in fact a blueprint for generating axioms for each of countably infinite number of well-formed formulas in $\mathcal{L}_{\text{set}}$. The same will be true of ZF9.

ZF4 asserts the existence of a set containing exactly two other named sets; ZF5, the existence of a set which is the union of a given set; and ZF6, the existence of a power-set for every set[4].

ZF7 states that every set has a minimal element with respect to membership, and ZF8 asserts the existence of an infinite set. ZF9 is straight-forward, but again really refers to an infinite collection of sentences for each class function.

2.2.2 Ordinals and cardinals

Using the axioms of union, power-set, etc, we can construct more complex sets. In particular, we will be able to construct ordinals and cardinals, both of which we will define later. For some preliminary definitions, the reader is directed to Section 1.3.

We start first with ordinals:

Definition 2.2.2.1. *A set x is **transitive** if for all $y \in x$, $y \subset x$.*

[4]Unions and power-sets are defined exactly as one would expect from the naïve theory of sets. As well, we may extend these notions to classes, and will do so informally throughout the body of this text.

Definition 2.2.2.2. *An **ordinal** is a transitive, well-ordered set under* $\in$*.*

Definition 2.2.2.3. $pred(x, y) = \{z \in x : z \in y\}$*.*

We notice by separation that $\text{pred}(x, y)$ is a set for any set x and $y \in x$. We state without proof the following which we require for some lemmas:

Theorem 2.2.2.4 ([Kun06], Theorem 6.3)**.** *Let* (A, R) *and* (B, S) *be two well-orderings. That is,* A *and* B *are sets with the binary relations* R, S *respectively such that* R, S *well-order* A, B*. Exactly one of the following holds:*

(i) There is an order-preserving bijection between A, B*. We say* A ***is order-isomorphic*** *to* B*, and write* $A \simeq B$ [5]*.*

(ii) A *is isomorphic to an **initial segment** of* B*. That is, there is* $y \in B$ *so* A *is isomorphic to* $pred(B, y)$*.*

(iii) B *is isomorphic to an initial segment of* A*.*

Lemma 2.2.2.5 (Properties of ordinals)**.** *Let* x *be an ordinal. Then:*

(i) If $y \in x$ *then* y *is an ordinal. Moreover,* $y = pred(x, y)$*.*

(ii) If y *is an ordinal and* $x \simeq y$*, then* $x = y$*.*

(iii) Exactly one of $x = y$*,* $x \in y$*, or* $y \in x$ *is true when* y *is an ordinal.*

(iv) If y, z *are ordinals and* $x \in y$ *and* $y \in z$*, then* $y \in z$*.*

(v) Every non-empty set of ordinals has a least element.

Proof. For (i), $y \in x$ implies $y \subset x$, so $\forall z \in y$, $z \in x$. Now if $w \in z$, since $z \in x$, $z \subset x$ and $w \in x$ as well. By well-ordering, $\{w, z, y\}$ has a least element. z, y cannot be least elements since $w \in z$ and $z \in y$. So, w is the least-element. Now, either

[5] We will drop order when the type of isomorphism is clear.

$w \in y$ or $w = y$. If $w = y$, then we have $y \in z \in y$, so $w \in y$, and y is transitive as required. To see that y is well-ordered, notice that any subset of y is a subset of x by transitivity. The result follows. That $y = \text{pred}\,(x, y)$ is clear from definition.

For (ii): We suppose $x \simeq y$ but $x \neq y$. Then, either $x \in y$ or $y \in x$. Let $f : y \to x$ be an isomorphism, and without loss of generality, assume $x \in y$. Then, $f(x) \in x$ and $f(x) \in y$ by transitivity. Then $f^2(x) \in f(x) \in x$, and so on. Continuing to argue along these lines, we construct a chain $... \in f^{n+1}(x) \in f^n(x) \in ...$ of elements in y with no least element. The result follows by contradiction.

(iii) Follows by Theorem 2.2.2.4, and (iv) by transitivity. Finally, for (v), let C be a non-empty set of ordinals. Choose $x \in C$, and take $x \cap C$. If $x \cap C$ is empty, then x is the minimal element for if not then there is $y \in x \cap C$, so $y \in C$ with $y \in x$. Otherwise, we have a non-empty set of elements of x and so there is a least element, y. Clearly y is minimal in C since if $z \in y$, with $z \in C$, then $z \in x$ by transitivity contradicting the minimality of y in $x \cap C$. The result follows.

■

We conclude by introducing the **class** of all ordinals **ON**.

Definition 2.2.2.6. *A **class** C is a collection of sets defined by a formula. That is $\phi(x) \iff x \in C$, for sets x, where ϕ is the formula defining the class.*

Remark 2.2.2.7. *Notice that any set y is a class. Indeed, consider the formula $\phi(x) := x \in y$. On the other hand, not every class is a set, for if we define $\psi(x) := x \notin x$, we obtain Russell's paradox.*

Theorem 2.2.2.8. $\neg \exists z \forall x (x \in z \iff x \text{ is an ordinal})$. *In particular, the collection of all ordinals is a class defined by the formula $\phi(x) := x$ is transitive and well-ordered.*

Proof. Suppose for a contradiction that there is such a set z. Then z is well-ordered and transitive by the previous lemma. Hence z is an ordinal. Hence, $z \in z$ – contradicting (ZF7). [6].

■

We denote the class of ordinals by **ON** henceforth.

We are able to define Peano arithmetic using ordinals. Indeed, set for all ordinals α, $S(\alpha) = \alpha \cup \{\alpha\}$.

Theorem 2.2.2.9. *Under S defined as above, the class of ordinals models* PA.

Proof. Clearly 0 is an ordinal, and the minimal element. Moreover, for each ordinal α, it is easy to see $\alpha \cup \alpha$ is an ordinal. It suffices to show $\alpha \cup \{\alpha\} = \beta \cup \{\beta\}$ iff $\alpha = \beta$. The reverse direction is obvious, so we only show the forward proof. If $\alpha \cup \{\alpha\} = \beta \cup \{\beta\}$, then $\alpha \in \beta \cup \{\beta\}$ and $\beta \in \alpha$. This contradicts the irreflexivity of $\in$. We conclude $\alpha \in \{\beta\}$ so $\alpha = \beta$. To see that induction holds, suppose for a contradiction that $\phi(0)$ holds, and for all n, $\phi(n) \implies \phi(n+1)$, but it is not the case for every n, that $\phi(n)$ holds. Then the set of $m \in \mathbf{ON}$ so $\phi(m)$ does not hold is a non-empty set of ordinals. By Lemma 2.2.2.5(v), this set has a least element, say n. But then $\phi(m)$ with $n = m + 1$ holds, and by assumption $\phi(n)$ holds - contradiction.

■

As a consequence of Gödel's incompleteness theorem then, there are statements σ in the language of set theory which cannot be proven nor disproven in ZF. We will discuss the consequences of this in greater detail in the next section.

From now on we will write $\alpha + 1$ in place of $S(\alpha)$ and Greek letters, $\alpha, \beta, \gamma, \delta$ are used for ordinals.

[6] Or alternatively, Russell's paradox, though this is only really disallowed by (ZF7)

Definition 2.2.2.10. *If β is such that there is α so $\alpha+1=\beta$, then β is a **successor ordinal**. Otherwise, if β is non-zero, β is called a **limit ordinal**.*

Lemma 2.2.2.11. *The collection of all natural numbers, ω is a limit ordinal.*

Proof. Suppose for a contradiction that ω was a successor ordinal. Then there exists α so $\alpha + 1 = \omega$. Then since $\alpha \in \omega$, we have $\alpha + 1 \in \omega$ – contradiction.

■

Ordinal arithmetic is defined via transfinite recursion (see Theorem B.0.0.1 for the details). We will not dwell on the specifics. As reference, the reader is directed to [Kun06] or [Jec03] for the details.

Definition 2.2.2.12 (Ordinal arithmetic)**.** *Ordinal addition is defined as:*

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
3. $\alpha + \beta = \lim\limits_{\delta<\beta} \alpha + \delta$*, for limit ordinals* β*, where* $\lim\limits_{\delta<\beta} \delta$ *is the smallest ordinal* γ *so* $\delta \in \gamma$ *for all* $\delta < \gamma$*. Notice that such a* γ *exists by Lemma 2.2.2.5(v).*

Similarly we can define ordinal multiplication and exponentiationTheorem B.0.0.1.

AC guarantees that every set is well-orderable. In particular then, a theorem (see [Kun06], Theorem 7.6) states that every set is order isomorphic to some unique ordinal α.

Definition 2.2.2.13. *For a set x, the unique α so $x \simeq \alpha$ is called the **order type (o.t.)** of x.*

With AC every set has an o.t.

Definition 2.2.2.14. *Call the universe of all sets V. Then V is a class defined by $\phi(x) := (x = x)$.*

In Chapter 3 we will see that we can define V using transfinite recursion on the ordinals. In particular $V = \bigcup_{\alpha} V_\alpha$ where V_α will be defined in Chapter 3 [7].

We give a more formal definition of cardinality than that given in Definition 1.3.0.3.

Definition 2.2.2.15 ([Hal19], p.54)**.** *We define the **cardinality** of x, $|x|$ to be the set $\{y \in V_\beta : x \approx y\}$. Here, β is the least ordinal such that there is a $y \in V_\beta$ so $y \approx x$.*

Remark 2.2.2.16. *Notice that such a β exists for every x since any set is in bijection with itself and since V is the universe of all sets, x appears in some V_α. In particular then, the set of all α for which there is $y \in V_\alpha$ so $y \approx x$ is a non-empty collection of ordinals and hence has a minimal element by Lemma 2.2.2.5. This corresponds to β.*

It is easy to see:

Lemma 2.2.2.17. *Cardinality is an equivalence class modulo equinumerosity.*

Moreover, we define the collection of **cardinal numbers** to be these equivalence classes.

Definition 2.2.2.18. *A cardinal number κ is **finite** if there is $n \in \kappa$ so $n \in \omega$ and **infinite** otherwise.*

Remark 2.2.2.19. *We may alternatively say that κ is finite if the elements of κ up to o.t. are elements of ω. That is, κ is finite if $\{o.t.(x) : x \in \kappa\} \subset \omega$.*

We restate the definition of an $\aleph$-number.

Definition 2.2.2.20. *An **$\aleph$-number** is an infinite cardinal number which contains a well-ordered set as an element.*

[7] Essentially the idea is to construct nested sets V_α so that the union of them all contains all "good" sets.

Remark 2.2.2.21. *With* AC *every infinite cardinal is an* $\aleph$*-number since every set is well-orderable.*

Lemma 2.2.2.22. *Every set in an* $\aleph$*-number is well-orderable. Moreover, the cardinality of all ordinals is an* $\aleph$*-number.*

Proof. Let κ be an $\aleph$-number. Then there is $x \in \kappa$ so x is well-orderable. But every other set $y \in \kappa$ is in bijection with x since κ is an equivalence class under equinumerosity. For $y \in \kappa$, choose $f : x \to y$ a bijection. Then define an order on y by $w <_y z$, if $f^{-1}(w) <_x f^{-1}(z)$ for $w, z \in y$, where $<_x$ is a well-order on x.

∎

Theorem 2.2.2.23 (Hartog's theorem)**.** *For every cardinal* κ*, there exists a smallest* $\aleph$*-number* $\aleph(\kappa)$ *such that* $\aleph(\kappa) \not\leq \kappa$*.*

Proof. Choose $A \in \kappa$, and let $\mathcal{R} \subset \mathcal{P}(A \times A)$ be the set of all well-orderings on subsets of A. Notice such a set exists by separation and power-set axioms. Now for any $R \in \mathcal{R}$ there is β so $\beta = \text{o.t.}\,(R)$. Moreover, for any $\alpha \in \text{o.t.}\,(R)$ there is $R' \in \mathcal{R}$ so $\text{o.t.}\,(R') = \alpha$ since α is a sub-ordering of β. In particular $\gamma = \{\text{o.t.}\,(R) : R \in \mathcal{R}\}$ is an ordinal (it is well-ordered trivially since it is a non-empty set of ordinals).

Now by definition there is for any $\beta \in \gamma$ a $R \in \mathcal{R}$ so $\text{o.t.}\,(R) = \beta$. Moreover, R is a well-ordering of some $S \subseteq A$. In particular then, we have $|\beta| \leq |A|$. However, if $|\gamma| \leq |A|$, we have $\gamma \in \gamma$ since if $|\gamma| \leq |A|$, then there is $S \subseteq A$ so $\text{o.t.}\,(S) = \gamma$, which is a contradiction. Let $\aleph(\kappa) = |\gamma|$. The result follows.

∎

Arithmetic can be defined analogously for cardinals as for ordinals (see [Jec03] or [Kun06] for details). We can also distinguish between limit cardinals and successor cardinals as with ordinals. For more on cardinal relations see [Hal19].

Finally, we are able to introduce the notion of **cofinality** which will be important in a subsequent chapter.

2.2.3 Cofinality

Fix x a limit ordinal.

Definition 2.2.3.1. *A sequence $(y_\epsilon)_{\epsilon\in\beta}$ for β a limit ordinal is said to be **cofinal** in x if $\lim_{\epsilon\to\beta} y_\epsilon = \alpha$.*

Equivalently, we may say $\sup(y_\epsilon)$ is α, where the sup of a collection of ordinals is the smallest set containing every element in the collection. For non-limit ordinals, we say the following:

Definition 2.2.3.2. *x is **cofinal** in y if for all $z \in y$, there is $w \in x$ such that $x \geq z$.*

Lemma 2.2.3.3. *$\{\alpha\}$ is cofinal in $\alpha + 1$ for every $\alpha \in$ **ON**.*

Proof. Certainly $\alpha \geq x$ for every $x \in \alpha + 1$. The result follows by definition.

■

Definition 2.2.3.4. *The **cofinality** of x is the least ordinal β so there is a β sequence converging to x.*

Lemma 2.2.3.5. *$cf(cf(\alpha)) = cf(\alpha)$.*

Proof. Let $\beta = \mathrm{cf}(\alpha)$ and $\gamma = \mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\beta)$. It is clearly $\gamma \leq \beta$. To see the reverse, we note that by assumption, there is a cofinal sequence $(y_\epsilon)_{\epsilon<\beta}$ in α, and a cofinal sequence $(z_\delta)_{\delta<\gamma}$ in β. In particular, $(y_{z_\delta})_{\delta<\gamma}$ is cofinal in α. The result follows.

■

Definition 2.2.3.6. *An infinite cardinal κ is said to be regular if for every x cofinal in κ, we have $|x| = \kappa$.*

Lemma 2.2.3.7. *Every successor cardinal is regular.*

Proof. Let κ be a successor cardinal. Then there is a cardinal λ so $\kappa = \lambda + 1$. If x is cofinal in κ, $\lambda \in x$. Suppose $|x| < \kappa$. Then $|x| \leq \lambda$. If $|x| = \lambda$, then $x \in \lambda$ and $\lambda \in x$ – contradiction. If $|x| < \lambda$, then there is a cardinal $\mu < \lambda$ so $x \in \mu$. But λ in x. Contradiction. The result follows.

■

Definition 2.2.3.8. *An uncountable cardinal κ is said to be **strongly inaccessible** if for every $\lambda < \kappa$, $2^\lambda < \kappa$, where 2^λ is |functions from λ onto 2| (*[Jec03]*, Equation 3.3).*

We prove the following in the next chapter:

Theorem 2.2.3.9. *The existence of a strongly inaccessible cardinal is independent of ZFC.*

Chapter 3

The well-founded universe

In this chapter we develop our first model of ZF. Because of Gödel's incompleteness theorem, we must assume the consistency of ZF in order to verify that our constructions are indeed models. We will also see that there is difficulty associated with building set models of ZF.

3.0.1 Model theory of sets

Definition 3.0.1.1. *Let ϕ be a formula of set theory. The* ***relativization*** *of ϕ to an $\mathcal{L}_{set}$-structure, $\mathcal{M}$, denoted $\phi^{\mathcal{M}}$, is formed by the following rules:*

(i) $(x \in y)^{\mathcal{M}} := (x \in y)$.

(ii) $(x = y)^{\mathcal{M}} := (x = y)$.

(iii) Relativization distributes over the the standard logical connectives.

(iv) Relativization over quantification restricts the domain of quantification to the underlying set of $\mathcal{M}$.

A model $\mathcal{M}$ of ZF is an $\mathcal{L}_{\text{set}}$-structure, satisfying all axioms of ZF relativized to $\mathcal{M}$. That is, $\mathcal{M} \models \phi^{\mathcal{M}}$, where ϕ is an axiom of set theory.

Example 3.0.1.2. *Let* $\phi := \exists y \forall x (x \notin y)$*. Then,* $\phi^{\mathcal{M}} := \exists y \in M \forall x \in M (x \notin y)$*. Note that* $\mathcal{M} = \{\{\emptyset\}, \in, =\}$ *is a model of* $\{\phi\}$*.*

Again, on account of Gödel's Second Incompleteness theorem, it will be impossible for us to prove the consistency of ZF. It is however possible for us to prove the relative consistency and independence of sentences assuming ZF, or fragments of ZF, are consistent.

Henceforth, it will be assumed that all sentences/formulas are sentences/formulas of $\mathcal{L}_{\text{set}}$ unless explicitly mentioned.

Lemma 3.0.1.3. *Let* S *be a set of sentences, and let* T *be a set of sentences such that from* T*, it is provable that* $\mathcal{M} \models S$ *where* $\mathcal{M}$ *is a non-empty structure. If* T *is consistent, then* S *is consistent. (We shall henceforth abbreviate this as* $Con(T) \implies Con(S)$*.)*

Proof. Suppose for a contradiction that S is inconsistent. Then there is a sentence χ such that $S \vdash (\chi \wedge \neg\chi)$. But then $T \models (\chi \wedge \neg\chi)^{\mathcal{M}}$. Applying relativization, $T \models \chi^M \wedge \neg\chi^M$. In particular, T is inconsistent – contradiction.

■

The reason for assuming the non-emptiness of $\mathcal{M}$ arises from the fact that if $M = \emptyset$, then $\text{Th}(\mathcal{M})$ is inconsistent, in the sense that we may find statements which when relativized to $\mathcal{M}$ (e.g. universal quantifications) are inconsistent regardless of whether T is consistent or not. This, we hopefully agree, is not a useful notion.

Example 3.0.1.4. *Let* $\mathcal{M} = \{\emptyset, \in, =\}$*. Then* $\mathcal{M} \models (\forall x \in M (x \notin x))$*. As well,* $\mathcal{M} \models \forall x \in M (x \in x)$ *since* M *is empty and every universal quantification holds vacuously. Here* ϕ *and* ψ *(the two statements above) are fundamentally incompatible. Given any set of sentences* T *then, it is clearly the case that a contradiction regarding* $\mathcal{M}$ *will be provable from* T *regardless of its consistency.*

This in particular will be our approach to consistency proofs for sentences in/not in ZF, relative to ZF.

Example 3.0.1.5. *Let $\mathcal{M} = \{0\}$. Then, $\mathcal{M}$ is a model of extensionality, and $\forall y(y = 0)$ assuming only extensionality.*

We will drop the use of the calligraphic font-face when referring to models at times when it will be understood that we are talking about models.

3.1 Absoluteness results

We now set out on our task at hand: to find a model of ZF. It is not a requirement yet, that we introduce absolute notions, but we will do so to illustrate how to use absoluteness results to verify properties of models. At the end of the subsequent section, we will see how to prove the same theorem proven in the section without absoluteness. In some cases however, this will not always be possible. We begin with some definitions:

Definition 3.1.0.1. *Let M be a class defined by Φ, then M is* ***transitive*** *if for every $x \in M$, we have $x \subset M$ [1]. $\mathcal{M}$ as an $\mathcal{L}_{set}$-structure is a transitive model if its underlying domain is transitive.*

Definition 3.1.0.2. *Let $\mathcal{M}$ be a transitive model, and let ϕ be a formula with free variables $x_1, ..., x_n$. Then ϕ is* ***absolute*** *if for all $x_1, ..., x_n \in M$ $\phi(x_1, ..., x_n) \iff \phi(\vec{x})^{\mathcal{M}}$ [2].*

Definition 3.1.0.3. *A formula, ϕ is Δ_0 if it satisfies one of:*

(i) ϕ is atomic.

[1] Note that we are using $x \in M$ as an abbreviation for $\Phi(x)$, where $x \in M \iff \Phi(x)$. Similarly, $x \subset M$ is an abbreviation of $\forall y(y \in x \implies \Phi(y))$.

[2] $\vec{x} = x_1, ..., x_n$.

(ii) ϕ *is a formula involving standard logical connectives and* Δ_0 *formulas.*

(iii) ϕ *is of the form* $\exists x \in y(\psi(x))$ *or* $\forall x \in y(\psi(x))$*, where* ψ *is* Δ_0*.*

Lemma 3.1.0.4. *Let* $\mathcal{M}$ *be a transitive model. Then every* Δ_0 *formula is absolute over* $\mathcal{M}$*.*

Proof. (i) follows from definition of relativization of formulas involving only $\in$ or $=$. (ii) is by induction. As an example, suppose $\chi := \phi \vee \psi$, where ϕ, ψ are Δ_0. Then, $\chi^{\mathcal{M}} \iff (\phi \wedge \psi)^{\mathcal{M}} \iff \phi^{\mathcal{M}} \wedge \phi^{\mathcal{M}} \iff \phi \wedge \psi \iff \chi$. Suppose ψ is of the form $\exists x \in y(\phi(x))$ and assume by induction that ϕ is absolute. Then ψ is equivalent to $\exists x(x \in y \wedge \phi(x))$. Relativized to $\mathcal{M}$, this becomes $\exists x \in M(x \in y \wedge \phi(x)^{\mathcal{M}})$. Using the fact that ϕ is absolute, we have $\psi \iff \exists x \in M(x \in y \wedge \phi(x))$. Since $y \in M$, $x \in M$, so $\psi \iff \exists x(x \in y \wedge \phi(x))$, as required.

■

Clearly, this notion of Δ_0 formulae is powerful, as it allows us to prove that certain statements will hold in transitive models without having to assume the consistency of the entirety of ZF.

Lemma 3.1.0.5. *The following sentences/notions are* Δ_0*:*

(i) $z = \{x, y\}$.

(ii) $x = (x, y)$.

(iii) $x = \emptyset$.

(iv) $x \subset y$.

(v) x *is transitive.*

(vi) x *is an ordinal.*

(vii) x *is a limit ordinal.*

(viii) x *is a natural number.*

(ix) $x = \omega$.

(x) $z = x \times y$.

(xi) $z = x - y$ [3].

(xii) $z = x \cap y$.

(xiii) $z = \bigcup x$.

(xiv) $z = dom(X)$.

(xv) $z = ran(X)$.

(xvi) x *is a relation.*

(xvii) f *is a function.*

(xviii) $y = f(x)$.

(xix) $g = f_{|x}$

(xx) x *is an inductive set*

Proof. For (i), $z = \{x, y\}$ when $x \in z \wedge y \in z \wedge (\forall w \in z)(w = x \vee w = y)$. This is clearly a conglomerate of Δ_0 formulas and is hence itself Δ_0. (ii), (iii) are similar.

For (iv), $x \subset y$ when $(\forall z \in x)(z \in y)$

For (v), x is transitive when $(\forall w \in x)(w \subset x)$. Again, since (iv) is Δ_0 as we have just shown, the entire formula is Δ_0.

[3] $x - y = \{z \in x : z \notin y\}$. Such an operation is well-defined by separation.

Before we proceed to (vi), we show that the notion "w has an ϵ-minimal element" is absolute. We have that this holds when $(\exists y \in x)((\forall w \in x)(w \notin y))$. This is also clearly absolute as it is made up of absolute notions.

For (vi), x is an ordinal when

$$(x \text{ is transitive}) \wedge ((\forall w \in x)((w \text{ has an } \epsilon\text{-minimal element}) \vee (w = \emptyset))).$$

We have already shown all of these notions are absolute, and so since (vi) is a conjunction of absolute notions, it is itself absolute.

Skipping ahead a little bit, for (xii), $z = x - y$ when $(\forall w \in z)(w \in x \wedge w \notin y)$.

For (xiv), we have that $z \in \text{dom}(X)$ when $(\exists x \in X)(\exists y \in X)(\exists w \in y)(x = (z, w))$. In particular then $z = \text{dom}(X)$ when $(\forall y \in z)(y \in \text{dom}(X)) \wedge (\forall x \in \text{dom}(X))(x \in z)$.

We will stop here for the purpose of brevity, but it is quite easily verifiable that all the above are indeed Δ_0. The interested reader is directed to [Kun06] or [Jec03] for the complete proofs.

■

3.2 The well-founded universe and an easy consistency result

We will now put this idea of absoluteness to good use, to prove the following theorem:

Definition 3.2.0.1. ZF^- *is* ZF *without regularity/foundation.*

Theorem 3.2.0.2. *Working in* ZF^-, ZF *is consistent.*

Unsurprisingly, we must demonstrate the existence of a model. That is, we must show there is some set/class (in our case it will be a class), such that the sentences of ZF relativized to our class hold within the class using only axioms from ZF that are not regularity/foundation.

Before we begin the proof, we will define a class and demonstrate some of its properties.

Definition 3.2.0.3. *Define by transfinite recursion the following sets:*

- $R(0) = 0$.
- $R(\alpha + 1) = \mathcal{P}(R(\alpha))$.
- $R(\alpha) = \bigcup_{\epsilon<\alpha} R(\epsilon)$, *for* α *a limit ordinal.*

The class of well-founded sets V *is defined as* $\bigcup_{\alpha\in\mathbf{ON}} R(\alpha)$.

Lemma 3.2.0.4. *For each* α, $R(\alpha)$ *is transitive.*

Proof. The proof is by transfinite induction on α. For $\alpha = 0$, the result is obvious. Suppose the result holds for α. Then, $R(\alpha + 1) = \mathcal{P}(R(\alpha))$. We choose an element $x \in R(\alpha+1)$. This element is a subset of $R(\alpha)$. We choose $y \in x$. Then $y \in R(\alpha)$. By the transitivity of $R(\alpha)$, $y \subset R(\alpha)$, and so it is an element of $R(\alpha+1)$. In particular, $x \subset R(\alpha + 1)$ as desired. For the case of α a limit ordinal, we assume the hypothesis for all ordinals $\beta < \alpha$. Then choosing $x \in R(\alpha)$ implies that there exists $\beta < \alpha$ for which $x \in R(\beta)$. Since the hypothesis is true for β the result follows.

■

Lemma 3.2.0.5 (Nesting property)**.** $R(\beta) \subset R(\alpha)$ *for* $\beta < \alpha$.

Proof. The proof is again by transfinite induction. The result follows easily for $\alpha = 0$. Suppose the result holds for α. Choose $\beta < \alpha+1$. If $\beta = \alpha$, then by definition $R(\alpha+1)$,

$R(\alpha) \in R(\alpha) + 1$. The result follows by Lemma 3.2.0.4. If $\beta < \alpha$, then $R(\beta) \subset R(\alpha)$ by the induction hypothesis. Since $R(\alpha) \subset R(\alpha + 1)$, the result follows. The result is obvious for the limit ordinal case.

■

Remark 3.2.0.6. *We note that V is hence a transitive class, as it is a union of nested transitive sets.*

Until this point, we have proven some properties about V which make it a potential candidate as a model of ZF. Namely, we have demonstrated that it is a transitive class. However, we would also like to demonstrate that V is a "good" model, in the sense that we can do all the math we care to do in V without too much fuss. That is, we want to show that math is doable in ZF[4]. To this end we have a few more lemmas.

Definition 3.2.0.7. *Let $x \in V$. We define rank(x) to be the least α such that $x \in R(\alpha + 1)$.*

Lemma 3.2.0.8. *For any α, $R(\alpha) = \{x : \text{rank}(x) < \alpha\}$.*

Proof. Pick $x \in R(\alpha)$. Then, $\text{rank}(x) < \alpha$ by definition. Now pick x from the RHS. Then, by the nesting property (Lemma 3.2.0.5) the result follows.

■

Lemma 3.2.0.9. *If $y \in V$, then $\forall x \in y(x \in V \wedge \text{rank}(x) < \text{rank}(y))$.*

Proof. Let $x \in y$. Let $\text{rank}(y) = \alpha$. Then $y \in R(\alpha + 1)$, from which it follows that $y \subset R(\alpha)$ and $x \in R(\alpha)$. So, $x \in V$ and $\text{rank}(x) < \text{rank}(y)$.

■

Lemma 3.2.0.10. *$\text{rank}(y) = \sup\{\text{rank}(x) + 1 : x \in y\}$.*

Proof. Let $\alpha =$ RHS. Then $\alpha \leq \text{rank}(y)$. On the other hand, by definition of α, it is the case that $\text{rank}(x) < \alpha$ for all $x \in y$ as otherwise $\alpha + 1 \in \alpha$. So $y \subset R(\alpha)$, and is hence an element of $R(\alpha + 1)$. So $\text{rank}(y) \leq \alpha$. The result follows.

[4]Otherwise, there would not be much of a point to any of this.

■

Lemma 3.2.0.11. *If $\alpha \in \mathbf{ON}$, then $\alpha \in V$, and rank$(\alpha) = \alpha$.*

Proof. The proof is by transfinite induction on α. If $\alpha = 0$, then $\alpha \subset R(0)$, and so $\alpha \in R(1)$. In particular rank$(0) = 0$. Suppose $\alpha = \beta + 1$ and rank$(beta) = \beta$ rank$(\alpha) = \alpha$, so $\beta \in R(\beta + 1)$. Then $\beta + 1 \subset R(\beta + 1)$. Finally, we take the case where α is a limit ordinal. We have rank$(\alpha) = \sup\{\text{rank}(\beta) + 1 : \beta \in \alpha\}$. This must be α for otherwise α would be a successor ordinal.

■

Lemma 3.2.0.12. *If $x \in V$, then the following are also elements of V:*

i) $\bigcup x$

ii) $\mathcal{P}(x)$

iii) $\{x\}$

Moreover, the rank of each of these is less than rank$(x) + \omega$.

Proof. Let rank$(x) = \alpha$. For (i), $x \subset R(\alpha)$. Pick $z \in \bigcup x$ then $\exists y \in x$ so $z \in y$. Now, $y \in R(\alpha)$. So, by transitivity, we conclude $z \in R(\alpha)$. In particular, rank$(z) < \alpha$ for all $z \in \bigcup x$. By Lemma 3.2.0.10, that rank$(\bigcup x) = \alpha$ follows.

For (ii), pick $y \in \mathcal{P}(x)$. Then $y \subset x$. So $y \subset R(\alpha)$, and thus rank$(y) \leq R(\alpha + 1)$. Then $\mathcal{P}(x) \in R(\alpha + 2)$. The case for (iii) is similar.

■

Lemma 3.2.0.13. *If $x \in V$, then the following are also elements of V:*

i) $x \times y$

ii) $x \cup y$

iii) $x \cap y$

iv) $\{x, y\}$

v) $\langle x, y \rangle$

vi) ${}^y x$

All have rank less than $\max(rank(x), rank(y)) + \omega$*.*

Sketch of proof. For (i), we notice that $x \times y$ is a subset of $\mathcal{P}(\mathcal{P}(\mathcal{P}(x \cup y)))$ and we may proceed as in the proof of the previous lemma. (ii), (iii) are obvious. The other cases are similar.

■

Lemma 3.2.0.14. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ *are all elements of* $R(\omega + \omega)$*.*

Sketch of proof. We note that $\mathbb{Z}$ can be defined in ZF as a set of equivalence classes of ω. In particular, $\mathbb{Z}$ is a subset of the set containing sets of pairs of ω. Any arbitrary element of $\omega \times \omega$ has at most $\text{rank}\,(=) \max(\text{rank}\,(x), \text{rank}\,(y)) + 2$. A set of these elements then has rank at most $\text{rank}\,(=) \max(\text{rank}\,(x), \text{rank}\,(y)) + 3$. A set of these sets, has at most $\text{rank}\,(=) \max(\text{rank}\,(x), \text{rank}\,(y)) + 4$. The result follows.

The other cases are similar.

■

While it is nice to know that the particular class we have constructed has these recognizable properties necessary to do math, we move onto the proof of the theorem from a little earlier and demonstrate that V is indeed a model of ZF.

We need one more lemma about V.

Lemma 3.2.0.15. $x \in V \iff x \subset V$*.*

Proof. If $x \in V$, then by the transitivity of V, it is obvious that $x \subset V$. Now, suppose $x \subset V$. Let $\alpha = \sup\{\text{rank}\,(y) + 1 : y \in x\}$. Then $x \subset R(\alpha)$ such that $x \in R(\alpha + 1) \subset V$.

■

And now we may prove the theorem:

Proof of Theorem 3.2.0.2. We show each of the axioms of ZF hold:

(extensionality) Extension is equivalent to

$$((\forall u \in x)(u \in y) \wedge (\forall v \in y)(v \in x)) \iff X = Y.$$

This is easily shown to be Δ_0, so it is true in V.

(separation) Let ϕ be a formula in $\mathcal{L}_{\text{set}}$. We must show that for any $x \in V$, there exists $y \in V$ such that $y = \{z \in x : \phi(z)\}$. Since $x \in V$, we have that $x \subset V$ by transitivity, so y as defined is a subset of V. Applying Lemma 2.3.13, we have the $y \in M$.

(pairing) Let $x, y \in V$. Define $z = \{x, y\}$. Then $z \subset V$, so $z \in V$.

(union) This follows by absoluteness of union. That is we must show for $x \in V$, that $\exists(y \in V)(y = \bigcup x)$. This was shown to be absolute, so assuming union, we are done.

(power-set) Follows since separation holds. That is, we may define the power-set of $x \in V$ by a formula $\phi \in \mathcal{L}_{\text{set}}$.

(infinity) We have $\omega \in V$ by Lemma 2.3.10.

(regularity) Let $x \in V$, and let $x \neq 0$ (which is absolute). Consider the set $\{\text{rank}\,(y) : y \in x\}$. Choose y with least rank. Then y is minimal in x. Clearly, $y \in V$.

(replacement) The result follows again by transitivity. That is, let F be a class function. Let $I = \{(x, y) \in V : F(x, y, p)^V\}$. Then $I \subset V$, so $I \in V$. The result follows. Notice that we have implicitly assumed I to be a set by assuming replacement under ZF.

■

Note we have omitted $\mathsf{ZF1}$ from our proof as it follows from $\mathsf{ZF4}$.

The idea again here is to assume the axioms of ZF^- to then show that they all hold in V. For example, forgetting about absoluteness for the time, to show extensionality holds in V, we show that the axiom of extensionality holds in V. In particular, we suppose the familiar axiom of extensionality for all sets holds. That is, given $y, z \in V$, we suppose $x \in y$ and $x \in z$. Then, since extensionality holds, the result holds. Of course, this is a bit harder for more complicated formulas, since certain formulas "look" different inside smaller models.

Similarly, we consider the axiom of separation. Relativized to V, it says that for any $\mathcal{L}_{\text{set}}$ formula and any set in V, it is the case that there is a subset of that set containing precisely those elements satisfying the given formula. Assuming the axiom of separation as it is stated in ZF, there does indeed exist a set x' of any given set x satisfying a given $\mathcal{L}_{\text{set}}$ formula. But this is a subset of $x \in V$, and so $x' \subset V$, and so $x' \in V$.

The notion of absoluteness serves to generalize these types of reasoning. As another, we consider the power-set axiom.

Example 3.2.0.16. *We wish to show the existence of a power-set of $x \in V$ which is also an element of V. We notice that every subset of x is an element of V by transitivity. So the power-set as defined in ZF is an element of V. Again, it is important to make this remark as in these types of proofs it is important to be aware* *that we are assuming* ZF*. In particular, here we assume the existence of a power-set in its familiar form.*

We conclude the section with yet another example.

Example 3.2.0.17. *We will show that if α is a limit ordinal, then $V(\alpha)$ satisfies extensionality, pairing, separation, union, power-set, and regularity.*

Extensionality is expressed as $\forall z \in V_\alpha((z \in x \iff z \in y) \implies x = y)$. But this follows since we assumed extensionality in ZF by the transitivity of V_α. Now, we choose $x, y \in V_\alpha$. Let $z = \{x, y\}$. It suffices to show $z \in V_\alpha$ since $z = \{x, y\}$ is absolute for transitive models. Now since $V_\alpha = \bigcup_{\beta<\alpha} V(\beta)$, we have that there exists a maximal β such that $x, y \in V_\beta$. Then $z \in V_{\beta+1} \subset V_\alpha$ since α is a limit ordinal. For separation, choose $x \in V_\alpha$. Then there exists $\beta < \alpha$ so $x \in V_\beta$. Then $x \subset V_\beta$. In particular then $\{z \in x : \phi^{V_\alpha}(\bar{u}, z)\} \subset V_\beta$. The result follows by the fact that α is a limit ordinal (i.e. $\beta+1 < \alpha$ for $\beta < \alpha$). Union follows almost identically to separation and absoluteness. Power set also follows by transitivity arguments and absoluteness. Finally, to see regularity, we note that its relativization to V_α is the following formula, $(\forall x \in V_\alpha)(\exists y \in x(y \cap x = 0))$. Or $(\exists y \in V_\alpha)(y \in x \wedge y \cap x = \emptyset)$ But certainly this follows by transitivity and absoluteness and the assumption that regularity holds in ZF*.*

Notice that it was not necessary to actually use relativization to prove V to be a model of ZF.

Moreover, one may show that the axiom of regularity is equivalent to the statement that V is the universe of all sets.

Proof. It suffices to show that V contains all sets. Indeed, let x be a set, and suppose for a contradiction that $x \notin V$. If every element of x is an element of V, then $x \in V$ by the reverse transitivity. So there is at least one $y \in x$ so $y \notin V$. If every element of y is in V then $y \in V$ again, so there is $z \in y$ so $z \notin V$. Continuing to argue in this manner, we get an infinite descending chain of sets contradicting regularity/foundation.

■

We make a very important distinction here between this proof and the earlier proof. In particular, the latter fails to show the consistency of the regularity axiom with the other axioms.

And henceforth, when working in ZF we will indeed take V as our universe of all sets. A natural question to ask having constructed V, is does this model admit choice? We have the following lemma:

Lemma 3.2.0.18. *If $x \in V$, then x is well-orderable iff $(x$ (is well orderable)$)^V$*

Proof. For the forward case, suppose x is well-orderable and R well-orders x. Then $R \subset x \times x$, but $x \times x \in V$ so $R \subset V$, and so $R \in V$ by reverse transitivity.

For the reverse: suppose there is $R \in V$ so R well-orders x. But every subset of x is in V. It follows R well-orders x.

■

In particular, we have just shown that AC is consistent with ZF^- assuming ZF^- and AC are consistent.

A major caveat however, to all we have done in this section is that we have failed to construct a "true" model of ZF. While V satisfies every sentence of ZF relativized to it, V fails to be a proper model as it is a class rather than a set. This is expected since if we could exhibit a set model of ZF, we would have that ZF is consistent. Consequently, we require assuming the existence of an inaccessible cardinal to exhibit a set model of ZF. We have the following:

Theorem 3.2.0.19. *If κ is an inaccessible cardinal, then V_κ is a model of $\mathsf{ZF}-\mathsf{P}$.*

Proof. By V_κ here, we mean V_α so α is the minimal ordinal in κ. We will prove some auxiliary facts first:

Choose $x \in V_\kappa$. Then $|x| < \kappa$. Indeed, if $x \in V_\kappa$, there is β so $x \in V_\beta$ with $\beta < \kappa$ since κ is a limit. In particular, $x \subset V_\beta$ by transitivity. So the cardinality of x is at most the cardinality of V_β. We can prove by TF-induction that for all $\beta < \kappa$, $|V_\beta| < \kappa$. The result holds trivially for the base case V_0. For successor ordinals the result holds since κ is strongly inaccessible so $\lambda < \kappa$ implies $2^\lambda < \kappa$. For limit ordinals, the result follows since the union of β-many sets each less than cardinality κ has cardinality less than κ. The result follows.

Now, extensionality, foundation, separation, and infinity all hold easily by the definition V_α for ordinals α. It suffices to show the axiom of union holds. Indeed, choose $\mathcal{C}$ a collection of sets in V_κ. Then every set $x \in \mathcal{C}$ is such that $|x| < \kappa$. But $\mathcal{C}$ also has less than κ sets by nesting and the earlier induction. It follows that union holds.
Ultimately then, we see V_κ is a model of $\mathsf{ZF}-$P, and moreover, V_κ is a set for any κ, strongly inaccessible.

■

To see power-set holds, see [Jec03], Theorem 12.13.

Chapter 4

The constructible universe and AC

Having shown that ZF is a "nice" theory, we will now construct a new universe in which we can talk about AC more meaningfully, which will hence be the main object of this paper along with its equivalent formulations and consequences.

Definition 4.0.0.1. *Let M be a class. $Def(M) = \{x \subset M : x \text{ is definable with parameters from } M\}$.*

Definition 4.0.0.2. *Define by transfinite induction the following:*

- $L(0) = 0$
- $L(\alpha + 1) = Def(L(\alpha))$
- $L(\alpha) = \bigcup_{\epsilon<\alpha} L(\epsilon)$.

  The class of all constructible sets $L = \bigcup_{\alpha \in \mathbf{ON}} L(\alpha)$.

Again, we begin with some properties of this constructible universe.

Lemma 4.0.0.3. *For every α, we have $L(\alpha)$ is transitive.*

Proof. The proof is by transfinite induction. The result is obvious for $\alpha = 0$. Suppose the hypothesis for smaller ordinals. Pick $x \in L(\alpha+1)$, and $y \in x$. Then $y \in x \subset L(\alpha)$ so $y \in L(\alpha)$. In particular then, $y \subset L(\alpha)$ by transitivity, and then y is definable as $y = \{z \in L(\alpha) : z \in y\}$. Since y is definable over $L(\alpha)$, it is an element of $\mathrm{Def}(L(\alpha+1))$ as desired. The limit ordinal case is trivial and exactly as in Lemma 2.3.3.

■

Lemma 4.0.0.4. *$L(\beta) \subset L(\alpha)$ for $\beta < \alpha$.*

Proof. By transfinite induction. The result is obvious for $\alpha = 0$. Suppose the hypothesis for smaller ordinals. Choose an element x from $L(\alpha)$. Then, x is definable with parameters in $L(\alpha - 1)$ (assuming α is a successor ordinal; we treat the limit ordinal case shortly). Since the hypothesis holds for smaller ordinals, we have that these parameters are elements of $L(\alpha)$. So, x is definable with parameters in $L(\alpha)$, and hence an element of $L(\alpha + 1)$ as desired. The limit ordinal case is obvious.

■

Remark 4.0.0.5. *Again we note that being the union of transitive sets that L is itself a transitive class.*

Lemma 4.0.0.6. *For all α, $\alpha \in L$. In particular $\alpha \in L(\alpha + 1)$ for every α.*

Proof. The proof is by induction. For $\alpha = 0$, we note that $0 \subset L(0)$, is definable with parameters in $L(0)$. Hence, $0 \in L(1)$. Now, suppose the hypothesis for smaller ordinals. Take $\alpha+1$. Then $\alpha \in L(\alpha+1)$. But $\alpha+1 = \{x \in L(\alpha+1) : x \text{ is an ordinal}\}$. This is a definable notion with parameters in $L(\alpha + 1)$ and the result follows. That is, $\alpha + 1 \in L(\alpha + 2)$. For limit ordinals α, the result follows as well.

■

Finally, we prove a model theoretic theorem along the lines of the Lowenheim-Skolem theorem, which will guarantee absoluteness of certain formulas in models without the property that $x \in M \iff x \subset M$.

We actually state a weaker version of the theorem. A stronger version can be proven if AC is assumed, but we will not need this for now.

Theorem 4.0.0.7 (Reflection Principle)**.** *Let $\phi(x_1, ..., x_n) \in \mathcal{L}_{set}$. For each M_0, there exists $M \supset M_0$, such that*

$$\phi(x_1, ..., x_n)^M \iff \phi(x_1, ..., x_n),$$

for $x_1, ..., x_n \in M$. Moreover, there exists such a transitive M.

In particular, if M satisfies the above, then we say M reflects ϕ.

While the Lowenheim-Skolem theorem guarantees "smaller" models, the Reflection Principle guarantees "larger" models[1]. Before we prove the Reflection Principle, we require a lemma.

Lemma 4.0.0.8. *The following hold:*

(i) Let $\phi(p_1, ..., p_n, x) \in \mathcal{L}_{set}$. For each set M_0, there exists $M \supset M_0$, such that

$$\textit{if } \exists x \phi(p_1, ..., p_n, x) \textit{ then } (\exists x \in M)(\phi(p_1, ..., p_n, x)),$$

for every $p_1, ..., p_n \in M$.

(ii) If $\phi_1, ..., \phi_k \in \mathcal{L}_{set}$, then for each M_0 there exists $M \supset M_0$ such that

$$\textit{if } \exists x \phi(p_1, ...p_n, x) \textit{ then } (\exists x \in M)(\phi(p_1, ..., p_n, x)).$$

Proof. For (i): Define $C = \{x : \phi(u_1, ..., u_n, x)\}$, and define $\dot{C} = \{x \in C : (\forall z \in C)(\text{rank}\,(x) \leq \text{rank}\,(z))\}$. For every $u_1, ..., u_n$ we define $H(u_1, ..., u_n) = \dot{C}$. Clearly $H(u_1, ..., u_1 = n)$ has the property that $\exists x \phi(u_1, ..., u_n, x) \implies \exists x \in H(u_1, ..., u_n)\phi(u_1, ..., u_n, x)$.

Construct M as follows: Let $M_{i+1} = M_i \cup \bigcup\{H(u_1, ..., u_n) : u_1, ..., u_n \in M_i\}$ for $i \in \omega$. Then set $M = \bigcup\limits_{i \in \omega} M_i$. Now, suppose there exists x such that $\phi(p_1, ..., p_n, x)$ for

[1]Forgetting for the time begin, that AC is necessary to talk about sizes of sets in a nice way.

$p_i \in M$. Then, there exists k so $p_i \in M_k$ for all i (simply take $k = \max\{i : p_i \in M_i\}$). Then, we have that there is $x \in M_{k+1}$ so $\phi(p_1, ..., p_n, x)$ holds by construction.

For the (ii), when constructing M, we let

$$M_{i+1} = M_i \cup \{H(u_1, ..., u_n) : \phi_i(u_1, ..., u_n)), i \in k\}.$$

The proof is then identical to that of (ii).

■

Moreover, it is actually possible to choose $M \supset M_0$ transitive in each of the above proofs. We will not work through the details, but simply say that

Proof of Theorem 4.0.0.7. Take $\phi(x_1, ..., x_n)$ a formula in $\mathcal{L}_{\text{set}}$. We must show that there exists an $M \supset M_0$ satisfying the absoluteness property. Let $\phi_1, ...\phi_k$ be the sub-formulas of ϕ. The proof is by induction on the complexity of ϕ and its sub-formulas.

We begin with the simplest case. Let ϕ be of the form $x \in y$. Then clearly the desired property holds. Now we suppose that for formulas ϕ, ψ the property holds. It follows by the definition of absoluteness that ϕ, ψ with any combination of logical connectives also satisfies the desired property. The case where ϕ is bound by an existential quantifier remains. We must show that

$$\exists x \phi(x_1, ..., x_n, x) \implies ((\exists x \in M)(\phi(x_1, ..., x_n, x)))^M,$$

for transitive $M \supset M_0$. However, this formula relativized to M is $(\exists x \in M)(\phi^M(x_1, ..., x_n,))$. By the induction hypothesis, we may express this as $(\exists x \in M)(\phi(x_1, ..., x_n, x))$, since ϕ is of lesser complexity, and hence reflected. But such a transitive $M \supset M_0$ exists by the earlier lemma, and so we are done.

■

We are now ready to prove that L is a model of ZF.

Theorem 4.0.0.9. *L is a model of ZF.*

Proof. We show σ^L holds for each $\sigma \in$ ZF as we did for V.

(extensionality) Let $y, z \in L$. Let $x \in V$. Then assuming extensionality, it is obvious that $x \in y$ and $x \in z$ implies $y = z$.

(pairing) Let $x, y \in L$. We must show there exists $z \in L$ such that $z = \{x, y\}$ is true in L. Let α be such that $x, y \in L_\alpha$. Then we may define a set $z = \{u \in L_\alpha : u = x \vee u = y\}$, Clearly, this is a formula with parameters in L_α and is hence an element of $L_{\alpha+1}$. Since $z = \{x, y\}$ is absolute, the result follows.

(separation) Let $\phi \in \mathcal{L}_{\text{set}}$. Let x be a named set. Then set $z = \{y \in x : \phi(u_1, ..., u_n, y)\}$. Take α to be such that $u_1, ..., u_n, x \in L_\alpha$. Then, by the reflection theorem, there exists transitive M which contains containing all of z. The result follows.

(union) Let $x \in L$. Define $u = \{z \in L : \exists y(y \in x \wedge z \in y)\}$. Then $u \in L_{\alpha+1}$. As well $u = \bigcup x$ is absolute.

(power-set) Define $z = \mathcal{P}(x) \cap L$. Then it is the case that $(\forall y \in L)(y \subset x \iff y \in z)$. Moreover, z is definable by the formula $\{y \in L_\alpha : y \subset x\}$, where α is the rank of x. So $z \in L_{\alpha+2}$. As well, the formula from earlier is absolute, and the result follows.

(infinity) We note $\omega \in L$. The result follows by absoluteness.

(regularity) Let $x \in L$ be non-empty. Then by regularity, there exists $y \in x$ so $y \cap x = \emptyset$. But this $y \in L$ by transitivity. The result follows by absoluteness.

(replacement) Let $x \in L$, and let $\phi \in \mathcal{L}_{\text{set}}$ be a class function. Then, we

■

Having defined several concepts, we come to the climax of this section. We will demonstrate that L is a model of AC. In particular, we will show that every set in L looks like it can be well-ordered in L. Of course, doing this is not so easy. Expressing AC in the language of sets is not particularly easy. Instead, we will do it in a more roundabout way by proving the following:

- First, that $(V = L)^L$.
- And second, that $V = L \implies AC$.

While the claims are perhaps not themselves intuitive, it is clear that if we show that these two statements hold that L will indeed be a model for AC. This will be the topic of our next subsection.

4.0.1 Consistency of AC

As before, we begin with several definitions and lemmas.

Theorem 4.0.1.1 (Gödel's Normal Form Theorem)**. ??**
There exist operations $G_1, ..., G_{10}$ such that if $\phi(u_1, ..., u_n)$ is a Δ_0 formula, then there is a composition G of $G_1, ..., G_{10}$ such that for all $x_1, ..., x_n$, it is the case that $G(\bar{x}) = \{\bar{u} : \bar{u} \in \bar{x} \wedge \phi(u_1, ..., u_n)\}$.

Here compositions, G, if $G_1, ..., G_{10}$ are referred to as Gödel operations, and $G_1, ..., G_{10}$ are as follows:

$$G_1(x, y) = \{x, y\}$$
$$G_2(x, y) = x \times y$$
$$G_3(x, y) = \{(u, v) : u \in x \wedge v \in y \wedge u \in v\}$$
$$G_4(x, y) = x - y$$
$$G_5(x, y) = x \cap y$$
$$G_6(x) = \bigcup x$$
$$G_7 = \operatorname{dom}(x)$$
$$G_8(x) : \{(u, v) : (v, u) \in x)\}$$
$$G_9(x) = \{(u, v, w) : (u, w, v) \in x\}$$

$G_{10} = \{(u, v, w) : (w, v, u) \in x\}$

Proof sketch. The proof is on the complexity of ϕ. We first make a few easy simplifications. Namely we restrict our logical connectives to $\neg$ and $\wedge$ since any other formula may be written using only these symbols. As well, we only permit quantification to be existential, and omit equality from our language (since it is expressible using only the $\in$ symbol). Moreover, any existential quantification is of the form

$$(\exists u_{m+1} \in u_i)(\phi(u_1, ..., u_{m+1})).$$

We begin with the case that $\phi(u_1, ..., u_n)$ is atomic. In this case it is of the form $u_i \in u_j$, where $i \neq j$. We suppose $n = 2$. Then, we have the case $u \in v$. But this is just $G_3(x, y)$. Or, assuming the permutation, just $G_8 \circ G_3$.

We move to the case where $n > 2$. Suppose the induction holds for $k < n$. There are several cases to consider the case where ϕ remains atomic. Suppose ϕ is of the form $u_i \in u_j$ for $i, j \neq n$. Then we have that there exists a formula G such that $G(x_1, ..., x_{n-1}) = \{(u_1, ..., u_{n-1}) : u_1 \in X_1, ..., u_{n-1} \in x_{n-1} \wedge u_i \in u_j\}$. But this is just $G_3(G, X_n)$.

Now we suppose $i, j \neq n-1$. By the earlier case there is a G such that $G(x_1, ..., x_n) = \{(u_1, ..., u_n, u_{n-1}) : u_1 \in x_1, ..., u_n \in x_n, u_{n-1} \in x_{n-1} \wedge x_i \in x_j\}$. But then the desired set is just $G_9 \circ G$. There are two other sub-cases to consider, but the proofs are similar.

We move onto the case where $\phi = \neg\psi$. There is G defining ψ by the induction hypothesis. Then $x_1 \times ... \times x_n - G$ is the desired set, and this is expressible using G_2. For conjunctions we can clearly take the intersection using G_5. Finally, we show that if ϕ is of the form $(\exists u_{n+1} \in u_n)\phi(u_1, ..., u_{n+1})$. Certainly there is

■

We state the following related lemma without proof. This should be obvious enough since the Gödel operations are all Δ_0 notions.

Lemma 4.0.1.2. *If G is a Gödel operation then the property $Z = G(x_1, ..., x_n)$, can be written as a Δ_0 formula.*

From these two propositions follow two very important facts which we will make use of in the proof of our central result for this subsection.

Remark 4.0.1.3. *We note here that these two lemmas may be generalized to all ϕ and we will use this result without proof, for the proofs are very similar and by induction on the complexity of formulas we are handling.*

Corollary 4.0.1.4. *If M is a transitive class closed under Gödel operations, then M satisfies Δ_0-separation.*

Proof. Let $\phi(u, p_1, ..., p_n)$ be a Δ_0-formula. By Gödel's normal form theorem, there exists $G(x, \{p_1\}, ..., \{p_n\})$ such that $G = \{(u, p_1, ..., p_n) : u \in x \wedge \phi(u, p_1, ..., p_n)\}$. But then $y = \text{dom}...\text{dom}\,(G)$ where we iterate dom n times is the desired set, and exists in M by closure under Gödel operations.

∎

Corollary 4.0.1.5. *For every transitive set M:*

$$def(M) = cl(M \cup \{M\}) \cap \mathcal{P}(M),$$

where cl is taken to mean closure under the Gödel operations.

Proof. Let $x \in \text{def}\,(M)$. Then there exists a formula so $x = \{u \in M : \phi(u, p_1, ..., p_n)\}$, where $p_1, ..., p_n$ are parameters. But this formula can be generated by some Gödel operation on M and its elements by the earlier remark, so the result follows. In particular, we see that this is a generalization of Δ_0-separation.

∎

Until now, we have restricted ourselves to the set of Δ_0-formulas. However, it is desirable to be able to talk about a wider range of formulas. For this purpose we introduce the following notions:

Definition 4.0.1.6. *A formula ϕ is said to be Σ_1 if it satisfies one of the following:*

- *ϕ is Δ_0.*
- *It is a conjunction or disjunction of Σ_1 formulas.*
- *ϕ is of the form, $\exists x\psi$ where ψ is Σ_1.*
- *ϕ is of the form $\exists x(x \in y \wedge \psi)$ or of the form $\forall x(x \in y \implies \psi)$ and ψ is Π_1.*

Definition 4.0.1.7. *A formula ϕ is said to be Π_1 if it satisfies one of the following:*

- *ϕ is Δ_0*
- *ϕ is a conjunction or disjunction of Π_1 formulas.*
- *ϕ is of the form $\forall x\psi$ where ψ is Π_1.*
- *ϕ is of the form $\exists x(x \in y \wedge \psi)$, or of the form $\forall x(x \in y \implies \psi)$, where ψ is Π_1.*

These two types of formulas will enable us to discuss the constructions introduced at the beginning of this subsection with greater ease. First, we demonstrate some of their properties:

Definition 4.0.1.8. *A formula ϕ is said to be upwards absolute if $\phi^M \implies \phi$ for M a transitive class.*

We define absolute downwards similarly.

Lemma 4.0.1.9. *All Σ_1 formulas are absolute upwards and all Π_1 formulas are absolute downwards.*

Proof. For the first part, if ϕ is Δ_0, the result is obvious. If it is of the form $\exists x\phi$, this is again obvious. Conjunctions and disjunctions are also easy to handle and so too is the case where ϕ is of the form $\exists x(x \in y \wedge \psi)$. On the other hand, if ϕ is of the form $\forall x(x \in y \implies \psi)$, it is not immediately obvious. However, taking the relativization, we have that this becomes $(\forall x \in M)(x \in y \implies \psi)^M = (\forall x \in M)(x \in y \implies \phi^M)$. But by the transitivity of M if this holds, then so too does the desired result.

The proof for Π_1 formulas is almost identical.

■

We have one more definition and then we will be ready to prove the main result.

Definition 4.0.1.10. *A formula is Δ_1 if it is both Σ_1 and Π_1.*

We proceed to the main result after an example:

Example 4.0.1.11. *"E is well-founded relation on P" is a Δ_1 property. To see this, we note that the following is Π_1:*

E is a relation on p and $\forall x(0 \neq x \wedge x \subset p \implies \exists a(a \in x \wedge a$ is E-minimal$))$.

This is because each of the formulas in the conjunction are Π_1, so the conjunction is itself Π_1. Moreover, the binding of the conjunction by the universal quantifier is Π_1, and so too is the statement in the outer conjunction, E is a relation on p.

Now, we note that E is well-founded on a set p if and only if there is a function $f : p \to ON$ such that $xEy \iff f(x) \in f(y)$. This is clearly Σ_1 by similar observations. Hence, the result follows.

Now (the main results):

Lemma 4.0.1.12. *Let G be a Σ_1 class function and let F be defined on ON by $F(\alpha) = G(F|\alpha)$. Then F is a Σ_1 function on the ordinals.*

Proof. We have that ON is a Σ_1 class. That is, it is defined by the formula, x is a transitive set of transitive sets (i.e. x is transitive $\wedge \forall y(y \in x \implies y$ is transitive), where x is transitive is known to be Δ_0). Then it suffices to show that $y = F(\alpha)$ if and only if $\exists f(f$ is a function $\wedge$ $\text{dom}(f) = \alpha \wedge (\forall \beta < \alpha)(f(\beta) = G(f|\beta) \wedge y = G(f)))$. In particular, to show F is a Σ_1 function, we must show that $F(x) = y$ if and only if $P(x, y)$ holds is absolute. Since ordinals are a Σ_1 class defined by the given formula, it is the case that, every instance of α, β, ordinals, may be replaced with a formula that is Σ_1. Since G is also Σ_1 and the the other notions are either Σ_1 (in the case of $\forall \beta < \alpha(f(\beta) = G(f|\beta)))$ or Δ_0 (in the case of f is a function), the entire formula characterizing F the function, is Σ_1, and the result follows.

∎

Lemma 4.0.1.13. *(x is constructible)^L $\iff$ (x is constructible). In particular, constructibility is absolute for inner models, where we define an inner model to be one that contains every ordinal and that is closed under the Gödel operations.*

Proof. We may encode the notion that x is constructible by saying,

$$(x \text{ is constructible})^L \iff \exists \alpha \in L(x \in L_\alpha^L).$$

In particular, we will show that the formula on the right-hand side is Δ_1. To establish this, we establish that $x \in L_\alpha^L$ is both Σ_1 and Π_1.

Now, we note that $x \in L_\alpha^L$ iff the following holds:

$$\exists \beta \in \alpha(x \in \text{cl}\,(L_\beta^L)).$$

If we can show that $\text{cl}\,(L_\beta^L)$ is Δ_1 then we will be done. In particular, we will first show that $L_\beta^L = L_\beta$ for every β. This proof is by induction. We have that $L_0^L = L_\beta$ since both are 0. The result also follows easily for the case where β is a limit ordinal assuming that the result holds for $\gamma < \beta$. In the successor ordinal case, we have that $L(\beta+1)^L = \text{Def}(L_\beta)^L$. But we have that x is definable if there exists a Gödel operation G such that $x = G(x_1, ..., x_n)$. In particular, we may write the closure under the Gödel operations, y as,

$$\exists w(w \text{ is a function } \wedge \text{ dom}(w) = \omega \wedge y = \text{ ran}(\omega) \wedge w(0) = L_\beta$$
$$(\forall n \in \text{dom}(w))(w(n+1) = w(n) \cup \{G_i(x,y) : x \in W(n), y \in W(n), i = 1, ..., 10\})))$$

The idea here is that we take the closure of L_β since this is how we build up our definable set. Moreover, we are allowed to do this since the class function mapping ordinals to the corresponding to L_α is absolute and hence the method of construction is the same. This was demonstrated in Lemma 4.0.1.12. In particular then, since this formula is absolute (as one may verify[2] quite easily), we have the following chain of consequence:

$$(x \text{ is constructible})^L \iff \exists \alpha \in L(x \in L^L_\alpha) \iff \exists \alpha(x \in L^L_\alpha) \iff \exists \alpha(L_\alpha) \iff (x \text{ is constructible})$$

Here, the first iff follows by definition of relativization, the second by the fact that L is an inner model, the third by what we have just demonstrated, and the fourth again by definition.

■

There are several different methods to arrive at this result. In particular, Jech makes use of Gödel operations to represent the closure of L_α. For a method where definability of a formula is utilized, the interested reader is referred to Kunen's text (specifically Chapters 4, 5, and 6). Regardless, we arrive at the following very important consequence:

Theorem 4.0.1.14. $(V = L)^L$.

[2]Several of these notions have been shown to be absolute in earlier Lemmas.

Proof. By the earlier result, we have that x is constructible in L if and only if x is constructible. Now clearly, by how we have have constructed the constructible universe, every set is constructible, so in particular, every set looks constructible to L. Hence $(V = L)^L$.

■

As well, (perhaps a fact which will become more important as we proceed through this paper), we have the following result:

Theorem 4.0.1.15. *L is the smallest inner model of ZF.*

Proof. Let M be any other inner model of ZF. Then, M is closed under the Gödel operations. In particular $L^M = L$ and the result follows[3].

■

At last, we conclude the following:

Theorem 4.0.1.16. *The Axiom of Choice is consistent with ZF.*

Proof. We have shown $(V = L)^L$. It remains to show that this implies that $L \models \text{AC}$. We do this by constructing a well-ordering of the class L as follows:

We essentially construct what are referred to as end extensions for each α. In particular, for $\alpha < \beta$, we have that $x <_\alpha y \implies x <_\beta y$ and $x \in L_\alpha$ and $y \in L_\beta - L_\alpha$ implies $x <_\beta y$.

Using rank arguments, one easily sees that $x \in y \in L_\alpha$ implies $x <_\alpha y$.

Now if α is a limit ordinal, we define $<_\alpha = \bigcup\limits_{\beta<\alpha} <_\beta$. Note that this is well-defined by the notion of end extensions.

[3]For a detailed proof, the reader is directed to Jech.

For the case where $\alpha +!$ is a successor ordinal and $<_\alpha$ is defined, we define $<_{\alpha+1}$ as follows:

We note $L_{\alpha+1} = P(L_\alpha) \cap \mathrm{cl}\,(L_\alpha \cup \{L_\alpha\}) = P(L_\alpha) \cap \bigcup\limits_{n=0}^{\infty} w_n^\alpha$, where we define $w_0^\alpha = L_\alpha \cup \{L_\alpha\}$ and $w_{i+1}^\alpha = w_i^\alpha \cup \{G_i(x,y) : x, y \in w_i^\alpha \wedge i = 1, ...10\}$. Note that we are essentially closing our ordering under the Gödelian closure we have previously defined and exploited several times now. Now set the following:

- Set $<_{\alpha+1}^0$ to be the ordering that extends L_α to $L_\alpha \cup \{L_\alpha\}$ by setting L_α to be the last element.
- $<_{\alpha+1}^{n+1}$ given that $<_{\alpha+1}^n$ is defined is defined as follows: $x <_{\alpha+1}^{n+1}$ if and only if $x <_{\alpha+1}^n$ or $x \in W_n^\alpha$ and y is not, or neither are elements of the previous construction and: (i) the least i such that there exists, u, v so $x = G_i(u,v)$ is less than the least j such that $y = G_j(s,t)$ again for arbitrary s, t [4], or (ii) the aforementioned $i = j$ and the w_n^α-least u such that there exists v so $x = G_i(u,v)$ is $<_n^\alpha$the $<_n^\alpha$-least element s so there exists t so $y = G_{i=j}(s,t)$, or (iii) where the conditions of (ii) holds but $u = s$; in which case we extend the idea of (ii) to the second element in $G_i(u,v)$.

We notice that this is all well-defined, for if none of these conditions holds, $x = y$. We then set

$$<_{\alpha+1} = \bigcup <_{\alpha+1}^n .$$

Again, it is easy to check this is compatible since in each iteration we do not modify the existing ordering of the elements in our construction. Finally, we set $<_L$ to be the ordering such that $x <_L y$ if and only if there exists α such that $x <_\alpha y$. Again this is well-defined, since if x, y exists, then certainly they have a defined rank and

[4] We note this is well-defined since $i = 1, ..., 10$ which is a finite set. Trivially we may impose the canonical ordering on these elements as ordinals.

hence they exist somewhere along the ordinal hierarchy of L and are hence ordered in some construction given above.

■

We note that it is easy to apply this methodology to obtain consistency proofs for other results such as the Continuum Hypothesis (CH), etc. The following section is dedicated to developing the theory of inner models in greater detail as this will become of use when we begin our discussion of forcing.

Chapter 5

Forcing

In this chapter, we will demonstrate the independence of the continuum hypothesis (CH) and AC by building models which satisfy the statements and models which satisfy their negations. To do this, we will make use of the technique of forcing developed by Paul Cohen.

Forcing as an idea is straightforward enough. Starting with a set model of ZF (which exists if ZF is consistent), M, which is referred to as the ground model, the goal is to add elements to M necessary to satisfy some other sentence(s), Γ.

Example 5.0.0.1. *Consider the theory of fields of characteristic* 0 *of which the rationals,* $\mathbb{Q}$ *are a model. Consider also the sentence* $\sigma := \exists x(x \cdot x = 2)$*. Then* $\mathbb{Q} \models ZF + \neg\sigma$ *since* $\sqrt{2} \notin \mathbb{Q}$*. In order to "force"* $\mathbb{Q}$ *to model* σ*, it is necessary to add* $\sqrt{2}$ *to the domain of discourse. However, doing this alone will not enable us to construct a model, say* $\mathbb{Q}'$ *of fields* $+\sigma$*. In particular, the theory of fields demands that every non-zero element has a multiplicative inverse, while* $\sqrt{2} \in \mathbb{Q}'$ *does not as* $\frac{1}{\sqrt{2}} \notin \mathbb{Q}'$*. The easy fix to this in algebra, is to take the field extension* $\mathbb{Q}(\sqrt{2})$*, which will satisfy the required conditions.*

Forcing can be thought of as an analogous method for sets by which elements are added to a ground model (defined in Definition 5.0.0.2) to obtain a new model of the original theory along with some new sentences. Such elements are sometimes called **witnesses** or **witnesses to a property**, where the property is specified beforehand.

Definition 5.0.0.2. *A* ***ground model*** M *is a countable transitive model of ZF or ZFC.*

Remark 5.0.0.3. *Notice that for a ground model* M *to exist we must assume* $Con(ZF)$*. Since every model of ZF contains an inductive set by Section 2.2.1(8) (i.e. (ZF8)), every model of ZF is at least infinite. Then, invoking Lowenheim-Skolem, there exists a countable model. As well, since the* $\in$ *relation on every model of ZF is extensional, set-like, and well-founded by (ZF2, ZF7, and the fact that the domain of discourse consists of only sets), it follows by a theorem of Mostowski, known as the Mostowski collapse lemma*[1]*, that this countable model is isomorphic structurally to a transitive model of ZF (which is also countable hence). We set* M *to be such a model henceforth.*

Before beginning our discussion of forcing however, it is necessary to introduce several new definitions and notations.

5.1 The machinery

The ideas of partial orders and boolean algebras are intrinsic to forcing. We begin with their definitions and some preliminary remarks.

Definition 5.1.0.1. *A* ***partial order (p.o.)*** *is a set* P *along with a relation* $R \subseteq P \times P$*, satisfying the following:*

- $\forall x \in P(xRx)$.
- $\forall x, y \in P(xRy \wedge yRx \implies x = y)$.
- $\forall x, y, z \in P(xRy \wedge yRz \implies xRz)$.

[1]See Appendix A for a statement of the theorem and a proof.

In particular, the relation R is reflexive, symmetric, and transitive. We often write $\leq$ in place of R.

Definition 5.1.0.2. *A **boolean algebra** is a set B containing distinguished elements 0 and 1 with operations $\neg, \vee, \wedge$ satisfying the following:*

- $\forall x(1 \vee x = 1)$ *and* $\forall x(1 \wedge x = x)$.
- $\forall x(0 \vee x = x)$ *and* $\forall x(0 \wedge x = 0)$.
- $\forall x(\neg x \vee x = 1)$ *and* $\forall x(\neg x \wedge x = 0)$.
- $\forall x(x \vee x = x)$ *and* $\forall x(x \wedge x = x)$.
- $\wedge, \vee$ *are commutative, and associative.*
- $\vee, \wedge$ *distribute over one another.*
- $\neg$ *satisfies De Morgan's laws.*

*A boolean algebra B is said to be **complete** if for any two elements $x, y \in B$, $x \wedge y, x \vee y \in B$.*

We write B^+ to mean $B - \{0\}$ for a complete boolean algebra, B.

Remark 5.1.0.3. *One notices that we may induce a partial ordering on a complete boolean algebra B by asserting that $x \leq y$ whenever $x \vee y = y$. Certainly then, $x \leq x$. To see that symmetry holds, suppose $x \leq y$ and $y \leq x$. Then $x \vee y = y$ and $y \vee x = x$. But $x \vee y = y \vee x$ by commutivity, such that $x = y$. For transitivity, suppose $x \leq y$ and $y \leq z$. Then $x \vee y = y$ and $y \vee z = z$. Then $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$ as required. Under this ordering, it is easy to see that 1 and 0 are maximal/minimal elements respectively.*

We may induce the same partial ordering on B by stipulating that $x \wedge y = x \iff x \leq y$.

In a similar manner, one can construct a complete boolean algebra from a partial order under certain conditions, and this interplay between partial orders, and boolean algebras is critical to the technique of forcing. For the proof we require some more definitions:

Definition 5.1.0.4. *A* ***forcing notion*** *is a partial order $P \subset M$ where M is a ground model. We interpret $p < q$ for $p, q \in P$ to mean "p is* ***stronger*** *than q". We say p and q are* ***compatible*** *if $\exists r \leq p, q$. If there does not exist an $r \leq p, q$, we say p and q are* ***incompatible****, and write $p \perp q$. Finally, we say $D \subset P$ is* ***dense*** *in P if for every $p \in P$ there is $q \in D$ so $q \leq p$* [2]*.*

Definition 5.1.0.5. *A non-empty set $F \subset P$ is a* ***filter*** *if F is upwards closed, and any two elements are compatible.*

Definition 5.1.0.6. *A filter $F \subset P$ is said to be P****-generic*** *or just* ***generic*** *if it non-trivially intersects every dense subset of P.*

Finally:

Definition 5.1.0.7. *A p.o. P is said to be* ***separative*** *if for any $p, q \in P$ such that $p \not\leq q$, there is an $r \leq p$ so r and q are incompatible.*

Theorem 5.1.0.8. *Every separative p.o. P can be embedded in a complete boolean algebra which respects the ordering on P under the induced ordering defined in Remark 5.1.0.3.*

Proof. Let P be a partial order. We will call a set $U \subset P$ a **cut** if $p \leq q$ and $q \in U$ implies $p \in U$ for any $p, q \in P$ (i.e. if U is downwards closed). For every $p \in P$, set $U_p = \{q \in P : q \leq p\}$. It is clear U_p is a cut for every $p \in P$. We say a cut U is **regular** if whenever $p \notin U$, there exists $q \leq p$ and $U_q \cap U = \emptyset$. Every U_p is regular, since given $q \notin U_p$, we have that $q \not\leq p$, such that since P is separative, there exists

[2] One will notice that this is a sort of "anti-cofinality."

$r \leq q$ so $r \perp p$. In particular, there is no $s \leq r, p$. And so $U_r \cap U_p = \emptyset$. We also notice that every cut contains some U_p. As well, we claim the intersection of two regular cuts is regular. Take, u, v regular cuts, and set $w = u \cap v$. Suppose $p \notin w$. Then $p \notin u$ or $p \notin w$. But if $p \notin u$, then there exists $q \leq p$ so $U_q \cap u = \emptyset$, such that $U_q \cap w = \emptyset$. The result is identical when $p \notin v$.

Now, define for a regular cut u, $\bar{u} = \{p : (\forall q \leq p) u \cap U_q \neq \emptyset\}$. This is a regular cut by definition. In fact, it is the least regular cut containing u. To see that it contains u, choose $p \in u - \bar{u}$. Then, $\exists q \leq p$, such that $U_q \cap u = \emptyset$. Since $p \in u$, we have that $q \in u$ so $U_q \subset u$, but then $u = \emptyset$, which contradicts the regularity of u. It cannot be that $U_q = \emptyset$, since $q \in U_q$ by definition. Now, take $v \supset u$, a regular cut. Choose $p \in \bar{u} - v$. Then $p \notin v$, so by regularity, $\exists q \leq p$ such that $U_q \cap v = \emptyset$. Now $p \in u$, so $q \in u$, so $U_q \subset u$, so $u \cap U_q \neq \emptyset$. But then $v \cap U_q$ cannot be empty – contradiction.

We let B be the set of all regular cuts of P, and define $u \wedge v = u \cap v$, and $u \vee v = \overline{u \cup v}$. Finally, we define $\neg u = \{p : U_p \cap u = \emptyset\}$. That $\neg u$ is a cut follows from the inclusion of U_q in U_p for $q \leq p$. It is easy to verify this is a regular cut, and that the required properties for complete boolean algebras hold under these operations. We naturally set $\emptyset = 0$ and $P = 1$. Then $p \in P$ corresponds to $U_p \in B$, and for no $p \in P$ does p correspond to 0. We will call this embedding e henceforth.

■

Henceforth, we will use $\cdot$ in place of $\wedge$, $+$ in place of $\vee$, and $-$ in place $\neg$, for the purpose of conforming to the standard literature (e.g. [Jec03], [Hal19]).

Corollary 5.1.0.9. *$\{U_p : p \in P\}$ are dense in B as defined in the proof of Theorem 5.1.0.8.*

Proof. Pick $u \in B$. Then u is a regular cut. Since every cut contains some U_p, there is p so $U_p \subseteq u$. That is, $p \leq u$, where $\leq$ is the ordering induced by $\subset$ on the cuts. The result follows.

■

Moreover, every partial order P can be reduced to a separative partial order Q.

Proposition 5.1.0.10. *Let P be a partial order. Then there exists a separative partial order Q, and an embedding $h : P \to Q$ such that $x \leq y \implies h(x) \leq h(y)$, and such that x and y are compatible in P iff $h(x), h(y)$ are compatible in Q.*

Proof. See [Jec03], Lemma 14.1.

■

Corollary 5.1.0.11. *Given a partial order P it is always possible to embed it in a complete boolean algebra B such that P is dense in B.*

Finally, we state a lemma which will be used much later in this chapter.

Lemma 5.1.0.12. *If P is a partial order and $\mathcal{D}$ a countable collection of dense subsets of P, then there exists a $\mathcal{D}$-generic filter on P.*

Proof. Since $\mathcal{D}$ is countable, we can enumerate the dense subsets of P in $\mathcal{D}$ as $D_0, ..., D_n, ...$. Now, choose p_0 in D_0, and $p_1 \in D_1$ such that $p_1 \leq p_0$. Proceeding in this manner, choose for each D_i, p_i so $p_i \leq p_{i-1}$. This is possible since each D_i is dense in P. That is, since $p_0 \in P$, and D_1 is dense in P, there is $p_1 \in D_1$ so $p_1 \leq p_0$, and so on. Finally, set

$$G := \{q \in P : \exists n(q \geq p_n)\}.$$

We claim G is a $\mathcal{D}$-generic filter. Certainly, G intersects each element in $\mathcal{D}$ non-trivially. As well, G is non-empty and upwards closed. It remains to show that for $p, q \in G$, we can find $r \leq p, q$. That is, we must show that p, q are compatible. Since

$p, q \in G$, there exist n, m such $p \geq p_n$ and $q \geq p_m$. Take $k = \min(p_m, p_n)$, then $p, q \geq k$, and $k \in G$ so the result follows.

■

5.1.1 Density, compatibility, and genericity

The utility of the notions of density, compatibility, and genericity are not immediately obvious. Recalling that the goal of forcing is to add new elements to our ground model, we can illustrate how these concepts come together to make this possible.

We begin by introducing a forcing notion, P:

Definition 5.1.1.1 (Cohen forcing notion.). *Define a p.o. P to be the set of all finite functions from $\kappa \times \omega$ to the set $\{0, 1\} = 2$, with ordering by function extension (i.e. $g \leq f$, if g extends f), for κ a fixed an uncountable cardinal. That is,*

$$P = \{f : \kappa \times \omega \to 2 : f \text{ is finite}\}.$$

P is called the Cohen forcing notion, and was used originally to prove the independence of CH.

Suppose we wanted to "build" an infinite function using this p.o. The idea would naturally be to "glue" together infinitely many finite functions in P to obtain a new function. However, we must be careful in doing this.

In particular, if p, q differ on outputs for a particular input, then there is no way to glue p, q together without obtaining a non-function. As well, in order to obtain an infinite function, f, for each $g \in P$, it is a requirement that there is a h extending g.

These requirements are encapsulated in the notions of compatibility and density respectively, and combined in the notion of genericity. In this sense, the p.o. controls what kind of elements we can add to our ground model (i.e. which functions we can construct in the case of our example), while the generic filter G, actually builds them. We will see more on this interplay later.

5.2 Boolean-valued models and the forcing language

The motivation for using boolean algebras lies in a need to express statements about $M[G]$ (the extended model), from inside of M. The idea of forcing again, is to start with a ground model and construct an extended model $M[G]$ using a forcing notion P and a P-generic filter G not necessarily in M.

Consequently, it is necessary to be able to talk about truth in $M[G]$ from within M, which "knows" nothing about the elements of $M[G]$. For this purpose, we introduce the boolean-valued model and the forcing language. The boolean-valued model makes use of complete boolean algebras to discuss the truth of certain statements, and the forcing language is designed to make use of this notion of truth to establish properties of $M[G]$.

5.2.1 Boolean-valued models and truth

Fix B a complete boolean algebra.

Definition 5.2.1.1. *A* ***boolean-valued structure*** *in the language of set theory,* $\mathfrak{A}$ *is a boolean universe* A *and functions of two variables with values in* B*,*

$$||x = y||, \quad ||x \in y||,$$

satisfying the following:

(i) $||x = x|| = 1$

(ii) $||x = y|| = ||y = x||$

(iii) $||x = y|| \cdot ||y = z|| \leq ||x = z||$

(iv) $||x \in y|| \cdot ||v = x|| \cdot ||w = y|| \leq ||v \in w||$

If $B = \{0, 1\}$ and $A = V$, it becomes apparent that we are assigning to each sentence of set theory, a truth value.

Definition 5.2.1.2. *We define* $||\phi(a_1, ..., a_n)|| \in B$ *for* $a_1, ..., a_n \in A$ *by induction:*

(a) If ϕ *is atomic, then* $||\phi||$ *is as in Definition 5.2.1.1.*

(b) Otherwise, define by induction on the complexity of the formula (i.e. $||\neg\phi|| = -||\phi||$, $||\phi \vee \psi|| = ||\phi|| + ||\psi||$, $||\phi \wedge \psi|| = ||\phi|| \cdot ||\psi||$.*)*

(c) If ψ *is of the form* $\forall x \phi(x)$*, then* $||\psi(x)|| = \prod_{a \in A} ||\phi(a)||$*. Take the sum for existential quantification.*

Again, if $B = 2$, then we are really defining a notion of "truth" for formulas. We say a formula is **valid** if it maps to 1 under the truth assignment $|| * ||$. Here are some additional properties:

Lemma 5.2.1.3 (Properties of the boolean-valued truth map)**.** *The following are true:*

(i) $||\phi \implies \psi||$ *is valid if* $||\phi|| \leq ||\psi||$.

(ii) Two provably equivalent formulas have the same boolean truth value.

Proof. For (i) we have that $||\phi \implies \psi|| = ||\neg\phi \vee \psi|| = -||\phi|| + ||\psi||$. Now, if $||\phi|| \leq ||\psi||$, we have $||\phi|| + ||\psi|| = ||\psi||$. In particular

$$-||\phi|| + ||\psi|| = -||\phi|| + ||\phi|| + ||\psi|| = 1,$$

as required. For (ii), we can use the fact that if two sentences are provably equivalent, $\phi \implies \psi$ and $\psi \implies \phi$. Then we have $1 = -||\phi|| + ||\psi||$, so $||\phi|| \leq ||\psi||$ and vice versa. The result follows by definition of the fact that B is a partial order. ■

Similarly, we have

$$||x = y|| \cdot ||\psi(x)|| \leq ||\psi(y)||.$$

The idea is that predicate calculus works in this scheme. In particular, we have the following theorem:

Theorem 5.2.1.4. *All the axioms of first-order logic hold in a boolean-valued model. That is, all axioms of first-order logic are valid. In particular, we have:*

1. $||\phi \implies (\psi \implies \phi)|| = 1$.
2. $||\phi \wedge \psi \implies \phi|| = 1$.
3. $||(\phi \implies \psi) \implies ((\phi \implies \neg\psi) \implies \neg\phi)| = 1$.
4. $||\neg\phi \implies (\phi \implies \psi)|| = 1$.
5. *If* $||\phi|| = 1$ *and* $||\phi \implies \psi|| = 1$*, then* $||\psi|| = 1$ [3].

Proof. We have the following:

1.
$$\begin{aligned} ||\phi \implies (\psi \implies \phi)|| &= -||\phi|| + (-||\psi|| + ||\phi||) \\ &= -||\phi|| + 1 \\ &= 1 \end{aligned}$$

2.
$$\begin{aligned} ||\phi \wedge \psi \implies \phi|| &= -(||\phi|| \cdot ||\psi||) + ||\phi|| \\ &= -||\phi|| - ||\psi|| + ||\phi|| \\ &= 1 + (-||\psi||) \\ &= 1 \end{aligned}$$

[3] This is **modus ponens**.

3. Argue similarly to the first two (it is just manipulations of boolean operations).

4. Same as the first three.

■

We can now define a particular boolean class model:

Definition 5.2.1.5. *We define V^B as a generalization of V. That is, instead of defining sets of elements at each ordinal stage, we define sets which act as functions on their elements sending them to B.*

- $V_0^B = \emptyset$
- $V_{\alpha+1}^B =$ *the set of all functions x with dom$(x) \subset V_\alpha^B$ and values in B*
- $V_\alpha^B = \bigcup_{\beta<\alpha} V_\beta^B$*, if α is a limit ordinal.*

Then set V^B to be the union over all ordinals of V_α^B such that V^B is a class.

Example 5.2.1.6. *For instance, V_1^B is the set of functions mapping the empty set to every possible element in B.*

Now define:

Definition 5.2.1.7. *For $x \in V^B$, $\rho(x) =$ the least α such that $x \in V_{\alpha+1}^\beta$.*

This is analogous to our definition of rank in V.

Finally define the boolean-valued function on V^B as follows:

- $||x \in y|| = \sum_{t \in \text{dom}\,(y)} ||x = t|| \cdot y(t)$
- $||x \subset y|| = \prod_{t \in \text{dom}\,(x)} (-x(t) + ||t \in y||)$
- $||x = y|| = ||x \subset y|| \cdot ||y \subset x||$

Under these, we will show that V^B is a boolean-valued model of ZF.

We require a lemma:

Lemma 5.2.1.8. *Let $x \in V^B$. Then for $t \in dom\,(x)$, $x(t) \leq ||t \in x||$ if for all $t \in dom\,(x)$, we have $||t = t|| = 1$.*

Proof. We have

$$\begin{aligned} ||t \in x|| &= \sum_{s \in \mathrm{dom}\,(x)} ||s = t|| \cdot x(s) \\ &= ||t = t|| \cdot x(t) + \sum_{s \in \mathrm{dom}\,(x), s \neq t} ||s = t|| \cdot x(s) \\ &\geq x(t) \end{aligned}$$

whereby the first and second equalities are by definition, and the inequality holds since if $|| \cdot ||$ is a boolean-valued map, $||t = t|| = 1$.

∎

First we show V^B is a boolean-valued structure:

V^B is a boolean-valued structure. We must verify the conditions in Definition 5.2.1.1. Definition 5.2.1.1(ii) is obvious by symmetry. For (i), the proof is by induction on $\rho(x)$ (note that it suffices to show that $||x \subset x|| = 1$): For the base case, we set $\rho(x) = 0$. But then x is the zero function so the product evaluates to 1 (as convention for the empty product). Now, suppose the result holds for $\rho(y) < k$. Suppose $\rho(x) = k$. Now for $t \in \mathrm{dom}\,(x)$, we have that $t \in V_k^B$, so $||t = t|| = 1$ by the induction hypothesis. By definition, we have

$$||t \in x|| = \sum_{s \in \mathrm{dom}\,(x)} ||t = s|| \cdot x(t).$$

Since $t \in \mathrm{dom}\,(x)$, we have $||t = t|| \cdot x(t) \leq ||t \in x||$ by Lemma 5.2.1.8. In particular, $x(t) \leq ||t \in x||$. By Lemma 5.2.1.3(i), we have $x(t) \implies ||t \in x||$. But this holds for every $t \in \mathrm{dom}\,(x)$ and the product evaluates to 1 and so $||x \subset x|| = 1$.

For (iii), the proof is by induction on triples $(\rho(x), \rho(y), \rho(z))$. For the base case, all will be 0. In particular then $x = y = z = 0$, where here 0 is the 0 function. Then clearly by (i) $||x = y|| = ||y = z|| = ||x = z|| = 1$ so the inequality holds. For the induction step, it will be enough to prove that

$$||x \subset y|| \cdot ||y = z|| \leq ||x \subset z||, \tag{$*$}$$

by symmetry. This amounts to showing that for $t \in \mathrm{dom}\,(x)$ that $||y = z|| \cdot (x(t) \implies ||t \in y||) \leq x(t) \implies ||t \in z||$, for any t. Take as the induction hypothesis,

$$\forall y, z \in V^B \; ||u \subset y|| \cdot ||y = z|| \leq ||u \subset z||,$$

for all $u \in \mathrm{dom}\,(x)$. It follows by this hypothesis that $||t \in y|| \cdot ||y = z|| \leq ||t \in z||$. Rewriting $(*)$ gives

$$||y = z|| \cdot (-x(t) + ||t \in y||) = ||y = x|| \cdot (-x(t)) + ||t \in y|| \cdot ||y = z||.$$

But this is less than

$$-x(t) \cdot ||y = z|| + ||t \in z|| \leq -x(t) + ||t \in z||,$$

and the result follows[4].

(iv) may be verified similarly[5].

■

Finally, we will verify that V^B is a model of ZFC.

[4]The final inequality is obtained by recalling that $\cdot$ refers the the meet/glb of two elements.
[5]See [Jec03], Lemma 14.16 for the details.

Theorem 5.2.1.9. *V^B is a class model of* ZF.

Proof. We must verify the axioms of ZF are valid in V^B. That is, for each σ, an axiom of ZF, we must show $||\sigma|| = 1$.

For **extensionality**, we must verify

$$\forall z \in V^B \; ||x \in y \iff x \in z|| \implies ||y = z||.$$

We have

$$||x = y|| = \prod_{z \in \mathrm{dom}\,(x)} (x(z) \implies ||z \in y||) \cdot \prod_{z \in \mathrm{dom}\,(y)} (y(z) \implies ||z \in x||),$$

using the definition of the boolean map on the subset relation. But using Definition 5.2.1.2(b) and Definition 5.2.1.2(a), we have that this is

$$||\forall x (x \in y \implies x \in z)|| \cdot ||\forall x (x \in z \implies x \in y)||.$$

The result follows since this is equivalent to what we want to show.

For separation, we want to show that given $x \in V^B$, there exists $y \in V^B$ such that

$$\prod_{z \in V^B} ||z \in y \iff (z \in x \cdot \phi(z))|| = 1.$$

Define y by $\mathrm{dom}\,(y) = \mathrm{dom}\,(x)$, and $y(z) = x(z) \cdot ||\phi(z)||$ for all $z \in V^B$. Such a y exists since it is a function into B with $\mathrm{dom}\,(y) \subset V^B_\alpha$, where α corresponds to x. We must verify that $||\forall z \in y (z \in x \cdot \phi(z))|| = ||\forall z \in x (\phi(z) \implies z \in y)|| = 1$. For the first, we must show that

$$\prod_{z \in \mathrm{dom}\,(y)} (y(z) \implies ||z \in x|| \cdot ||\phi(z)||) = 1.$$

If $z \in \mathrm{dom}\,(y)$, then $y(z) = x(z) \cdot ||\phi(z)||$. But $x(z) \leq ||z \in x||$, so

$$y(z) = x(z) \cdot ||\phi(z)|| \leq ||z \in x \wedge \phi(z)||.$$

The result follows by Lemma 5.2.1.3(i). As separation holds, so too does pairing.

The union, power-set, regularity/foundation, and replacement axioms are similar to the above. The reader is directed to [Jec03] or [Bos15] for the proofs. For infinity, we need to introduce the concept of names. For the proof, see Lemma 5.2.1.12 below.

■

Definition 5.2.1.10 (Canonical names)**.** *Every set in V has a **canonical name** in V^B. By induction on $\rho(x)$:*

- $\check{0} = 0$
- $\check{x} = \{(\check{y}, 1) : y \in x\}$

The canonical name for a generic ultrafilter G, is $\dot{G}$, with dom$(\dot{G}) = \{\check{u} : u \in B\}$, and $\dot{G}(\check{u}) = u$.

Lemma 5.2.1.11. *If $\phi(x_1, ..., x_n)$ is a Δ_0 formula, then*

$$\phi(x_1, ..., x_n) \iff ||\phi(\check{x}_1, ..., \check{x}_n)|| = 1.$$

Proof. By complexity of ϕ. For the base case, ϕ is either $x \in y$ or $y \in x$. Indeed by definition $x \in y$ iff $\check{y}(\check{x}) = 1$. As well, if $x = y$, then $\check{x} = \check{y}$, by induction on rank, and the result follows for equality. The results for conjunction, disjunction and negation follow by Definition 5.2.1.2. It remains to verify the case where ϕ is of the form $(\exists x \in y)(\phi(x))$. We have that

$$||(\exists \check{x} \in \check{y})(\phi(\check{x}))|| = \sum_{\check{t} \in \mathrm{dom}\,(\check{x})} (\check{y}(\check{t}) \cdot ||\phi(\check{t})||).$$

The result easily follows, since if there were such an $\check{x}$, then the sum would evaluate to 1 by the inductive hypothesis. Otherwise the sum would be 0.

■

That infinity holds in V^B follows as a direct consequence, since "ω is inductive" is Δ_0 (see Lemma 3.1.0.5).

Lemma 5.2.1.12. $V^B \models$ *Infinity.*

Proof. ω is an inductive set, so $||\check{\omega}$ is an inductive set$|| = 1$, since the notion is Δ_0.

■

Remark 5.2.1.13. *Again, we see that the boolean algebra acts as a means of expressing the truth of statements in our universe within the language of set theory.*

5.2.2 The forcing relation

We are finally ready to introduce the forcing relation and prove the main theorems of forcing, Theorem 5.2.2.1, Theorem 5.2.2.4, and Theorem 5.2.2.5, which we state just below[6]:

Theorem 5.2.2.1 (Generic Model Theorem)**.** *Let M be a transitive set model of ZFC and let P be a notion of forcing in M. If $G \subset P$ is generic over P, then there exists a transitive structure $M[G]$ such that:*

(i) *$M[G]$ is a model of ZFC,*

(ii) *$M \subset M[G]$ and $G \in M[G]$,*

(iii) $\mathbf{ON}^{M[G]} = \mathbf{ON}^M$,

(iv) *$M[G]$ is the smallest such model satisfying these properties (i.e. if N is a transitive model of ZF such that $M \subset N$ and $G \in N$, then $M[G] \subset N$).*

[6]We also define the forcing relation $\Vdash$, just below in Definition 5.2.2.3

Again, let M be our ground model and P a forcing notion. Then there is a complete boolean algebra $B = B(P)$ so P is embedded in B via an order-preserving map (call the embedding map e), and so P is dense in B (see Theorem 5.1.0.8, , Proposition 5.1.0.10, and the following corollary). Define $M^P := M^B$ to be the B-valued model defined similarly to Definition 5.2.1.5:

Definition 5.2.2.2. *By transfinite induction:*

- $M_0^B = \emptyset$
- $M_{\alpha+1}^B =$ *the set of all functions x with $dom(x) \subset M_\alpha^B$ and values in B*
- $M_\alpha^B = \bigcup\limits_{\beta<\alpha} M_\beta^B$, *if α is a limit ordinal.*

Set $M^B = \bigcup\limits_{\alpha} M_\alpha^B$.

It is easy to verify that M^B with the same boolean-valued map as earlier is a model of ZF.

Definition 5.2.2.3. *Elements of $M^P = M^B$ are called P-**names**[7] We denote names with dotted letters. We set $\dot{0} = 0$. The **forcing language** is the language of set theory augmented with names. We define the **forcing relation**:*

$$p \Vdash \phi(\dot{a}_1, ..., \dot{a}_n) \iff e(p) \leq ||\phi(\dot{a}_1, ..., \dot{a}_n)||,$$

where e is the map from P to $B(P)$ as defined in Theorem 5.1.0.8.

In particular, every set in M has a canonical name in M^B, denoted again with a check. We define this by induction on rank:

- $\check{0} = 0$
- $\check{x} = \{\check{y} : y \in x, y \in M\}$

[7]Sometimes we drop the P if the partial order is clear from context.

Sometimes, we will write x in place of $\check{x}$ for $x \in M$ [8].

We can now state the remaining two theorems:

Theorem 5.2.2.4 (Main Theorem of Forcing)**.** *Let P be a forcing notion in a ground set model M. Then for every $G \subset P$ generic over M and sentence σ of $\mathcal{L}_{set}$,*

$$M[G] \models \sigma \iff (\exists p \in G) p \Vdash \sigma.$$

Theorem 5.2.2.5 (Properties of forcing)**.** *Let P be a forcing notion on a ground model M, and let M^P be the class of all names in M [9]. Then, the following are true:*

(i) If $p \Vdash \phi$ and $q \leq p$, then $q \Vdash \phi$.

(ii) There does not exist p such that $p \Vdash \phi$ and $p \Vdash \neg\phi$.

(iii) For every p there is a $q \leq p$ such that $q \Vdash \phi$ or $q \Vdash \neg\phi$ [10].

(iv) $p \Vdash \phi$ if and only if no $q \leq p$ forces $\neg\phi$.

(v) $p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi, \psi$.

(vi) $p \Vdash \forall x \phi$ iff $p \Vdash \phi(\dot{a})$ for every $\dot{a} \in M^P$.

(vii) $p \Vdash \phi \vee \psi$ iff $\forall q \leq p \exists r \leq q (r \Vdash \phi \vee r \Vdash \psi)$.

(viii) $p \Vdash \exists x \phi$ iff $\forall q \leq p \exists r \leq q \exists \dot{a} \in M^P r \Vdash \phi(\dot{a})$, where $\dot{a}$ is a name.

Proof of Theorem 5.2.2.5. (i): If $q \leq p$ $e(q) \leq e(p)$. (ii): If so, then $e(p) \leq ||\phi||$ and $e(p) \leq ||\neg\phi||$. So, $e(p) \leq 0$ since $e(p) \cdot e(p) \leq ||\phi|| \cdot ||\neg\phi|| = 0$ (contradiction, since e is only a map into B^+). (iii) Suppose $p \cdot ||\phi|| \neq 0$ (this is equivalent to stating $||\phi|| \neq 0$). Then, by density, there is a $q \leq p$, so $e(q) \leq ||\phi||$. Similar for the case with $\neg\phi$. (iv): For the forward direction, suppose $p \Vdash \phi$. Then $e(p) \leq ||\phi||$. Now suppose

[8] Notice by definition that $\dot{x} = \check{x}$ for $x \in M$.

[9] We will define this rigorously in Definition 5.2.2.2.

[10] We say q **decides** ϕ.

there is some $q \leq p$ so q forces $\neg\phi$. Since $q \leq p$, q also forces ϕ contradicting (ii). (v): If $p \Vdash \phi \wedge \psi$, then $e(p) \leq ||\phi|| \cdot ||\psi||$. (vi), (vii), (viii) follow from the previous observations; for proofs, the reader is directed to [Jec03] or [Bos15].

■

Corollary 5.2.2.6. *We can also introduce a name for M. In particular since $a \in \check{M}$ if and only if $\exists x \in M(a = x)$, we set the name for M to be:*

$$p \Vdash \dot{a} \in \check{M} \iff \forall q \leq p \exists r \leq q \exists x (r \Vdash \dot{a} = \check{x}).$$

This defines M in M^B.

In particular, we have p forces an a in M only if there is an $x = a$, where we identify x with $\check{x} \in M$ by Theorem 5.2.2.5. Indeed, for $x \in M$, we have

$$p \Vdash \dot{x} \in \check{M} \iff \forall q \leq p \exists r \leq q \exists y (r \Vdash \dot{x} = \check{y}),$$

since we may choose $y = x$, and $||\check{x} = \dot{x}|| = 1$ since $|| \cdot ||$ is a B-valued map.

Finally, we set the name for G to be as follows:

Remark 5.2.2.7.

$$p \Vdash q \in \dot{G} \iff e(p) \leq e(q).$$

To see that this defines $G \in M^B$, notice that $q \in G$ if there is $p \leq q$ so $p \in G$, since filters are upwards closed. But $e(p), e(q)$ are identified with p, q. The result follows.

Remark 5.2.2.8. *Essentially, we verify that p is compatible with q and less than q as generic ultrafilters are upwards closed. Alternatively we write*

$$p \Vdash q \in \dot{G} \iff \forall r \leq p \exists s \leq r (s \leq q).$$

This is the condition for compatibility and implication discussed earlier in the chapter.

We notice every p forces every axiom of ZFC and every sentence provable from ZFC since they are all valid (and 1 is the maximum element of the complete boolean algebra). Finally, we can define the extension $M[G]$ and prove Theorem 5.2.2.1 and Theorem 5.2.2.4. Again, let M be a set model of ZFC so M^P is a boolean-valued model of ZFC, where B is the boolean completion of a forcing notion P and G is a generic ultrafilter on B.

Definition 5.2.2.9. *For every $x \in M^P$, define x^G by induction on $\rho(x)$:*

- $0^G = 0$
- $x^G = \{y^G : x(y) \in G\}$, *where $x \in M^P$.*

We call x^G the ***interpretation*** *of x by G. Set $M[G] = \{x^G : x \in M^P\}$.*

We have the following:

Lemma 5.2.2.10. *The following are true for all names $x, y \in M^P$:*

(i) $x^G \in y^G \iff ||x \in y|| \in G$.

(ii) $x^G = y^G \iff ||x = y|| \in G$.

Proof. By induction on $\rho(x), \rho(y)$:

For (ii):

$$\begin{aligned} ||x \subset y|| &\iff \prod_{t \in \mathrm{dom}\,(x)} (x(t) \implies ||t \in y||) \in G \\ &\iff \forall t \in \mathrm{dom}\,(x)(x(t) \in G \implies ||t \in y|| \in G). \\ &\iff (\forall t(x(t) \in G \implies t^G \in y^G)) \quad \text{by (i)} \\ &\iff \{t^G : x(t) \in G\} \subset y^G \end{aligned}$$

Here, the second equivalence follows by the fact that if $(x(t) \implies ||t \in y||) \in G$, iff $x(t) - ||t \in y|| \in G$ iff $x(t) + ||t \notin y|| \in G$. But this is the same as saying

$x(t) \in G \implies ||t \in y|| \in G.$

The result follows by symmetry. For the proof of (i), see [Jec03].

■

Theorem 5.2.2.11. *$M[G]$ satisfies*

$$M[G] \models \phi(x_1^G, ..., x_n^G) \iff ||\phi(x_1, ..., x_n)|| \in G.$$

Proof. By induction on the complexity of formulas. We have already proven equality and element inclusion in the above lemma.

For example, for disjunction. Suppose $M[G] \models \phi \wedge \psi$. Then $M[G] \models \phi, \psi$. But by the induction hypothesis, $||\phi||, ||\psi|| \in G$. But then the result follows by the fact that G is a filter and hence closed under $\cdot$. $\neg, \vee$ are similar.

For existential statements, suppose $M[G] \models \exists x(\phi)$. Then there is $x \in M[G]$ so $M[G] \models \phi(x)$. But then $\exists x \in M^P$ so $M[G] \models \phi(x^G)$, but this holds only if $\exists x \in M^P$ so $||\phi(x)|| \in G$. By upwards closure, this is only if $\sum_{x \in M^P} ||\phi(x)|| \in G$. The result follows.

■

Corollary 5.2.2.12. *$M[G]$ is a model of* ZFC*.*

Proof. Follows by the above theorem, and the fact that every sentence of ZFCis valid and G is a filter (i.e. $1 \in G$).

■

Corollary 5.2.2.13. *$M[G]$ is transitive.*

Proof. By definition of $M[G]$.

■

Corollary 5.2.2.14. *Theorem 5.2.2.4 holds.*

Proof. Suppose $M[G] \models \sigma(x_1^G, ..., x_n^G)$, then $||\sigma|| \in G$ by Theorem 5.2.2.11. But then it must be that there is p which forces $||\sigma|| \in \dot{G}$, as desired.

■

Finally:

Lemma 5.2.2.15. *For every $x \in V^B$,*

$$||x \textit{ is an ordinal }|| = \sum_{\alpha \in \mathbf{ON}} (||x = \check{\alpha}||).$$

Proof. Obvious.

■

Proof of Theorem 5.2.2.1(iii). Clearly $\mathbf{ON}^M \subset \mathbf{ON}^{M[G]}$. To see the reverse, suppose for a contradiction that there is an ordinal in $M[G]$ not in M. Then since the ordinals are well-ordered in $M[G]$, there is a least ordinal γ so $\gamma \in M[G]$, but $\gamma \notin M$. Let $x \in M^P$ be such that $\gamma := x^G$. Then $\{y : x(y) \in G\}$ is a set in M but this is γ – contradiction.

■

As a very brief summary:

The **forcing notion** P determines which elements are members of the **generic ultrafilter** G. G in turn describes which relations hold inside our new universe. Proving that our new universe is also a model of ZFC gives the desired results depending on the choice of P.

5.3 Independence of CH

Having outlined the technique of forcing, we are finally ready to do our first independence proof. We start by proving the independence of the Continuum Hypothesis (CH), which asserts the following:

Axiom 5.3.0.1 (CH). *$|\mathfrak{c}| = \aleph_1$, where $\mathfrak{c}$ is the continuum.*

Using the ideas developed above, we will be able to show that assuming the self-consistency of ZFC, that we may prove that ZFC+$|\mathfrak{c}| = \aleph_\alpha$, for any $\alpha \geq 0$ is consistent. In particular, CH is independent of ZFC.

Let M be a ground model, P be the set of finite partial functions from $\kappa \times \omega$ to $\{0, 1\}$. (To see that this is in fact a forcing notion, see Definition 5.1.1.1.).

Define dense subsets as follows:

Definition 5.3.0.2. $D_{x,n} = \{p \in P : (x, n) \in dom(p)\}$.

Remark 5.3.0.3. *Notice $D_{x,n}$ are dense for we can always extend our partial functions by a single element.*

Let G be generic over the $D_{x,n}$. Define $f = \cup G$.

Lemma 5.3.0.4. *f is a function and $dom(f) = \kappa \times \omega$.*

Proof. Follows since the $D_{x,n}$ are dense and G is a filter. That is G intersects each $D_{x,n}$ non-trivially, so no two functions in the intersections can be incompatible since otherwise $0 \in G$.

■

Next, define for each $\lambda < \kappa$, $f_\lambda = f(\lambda, \cdot)$.

Lemma 5.3.0.5. *If $\lambda \neq \mu$, then $f_\lambda \neq f_\mu$.*

Proof. Notice $D = \{p \in P : \exists n(p(\lambda, n) \neq p(\mu, n))\}$ is dense since we are dealing with partial functions and can always extend the function to contain some n it already does not. Then $G \cap D \neq 0$. The result follows.

■

In particular, for every $\lambda \in \kappa$, there is a $f_\lambda \in M[G]$ from $\kappa \to \{0,1\}^\omega$ since $G \in M[G]$ and $M[G]$ is a model of ZF by the forcing theorems such that it contains $\cup G$. But then we have constructed an injection from κ to $\mathfrak{c}^M$ for arbitrarily infinite (but fixed) κ. It only remains to show that cardinals in $M, M[G]$ are the same.

We know $\aleph_\alpha^M \leq \aleph_\alpha^{M[G]}$ since $M[G]$ extends M. For a contradiction: suppose the reverse inequality does not hold. In particular then, there must be a β so there is a bijection from $\aleph_\beta$ to $\aleph_{\beta+1}$ in $M[G]$. Let κ, κ^+ be sets of these cardinalities and call the bijection $\dot{f}$. In particular

$$M[G] \models \dot{f} \text{ is a bijection from } \check{\kappa} \text{ to } \check{\kappa}^+.$$

Then there is a p which forces this by Theorem 5.2.2.4.

We can show using forcing that this is impossible:

Let $F_p \in M$ be a function from κ to subsets of κ^+ defined as follows: For $y \in \kappa^+$, $y \in F_p(x)$ iff there is $p_{x,y} < p$ so

$$p_{x,y} \Vdash \dot{f}(\check{x}) = \check{y}.$$

Notice $p_{x,y}$ is incompatible with $p_{x,y'}$ for $y \neq y'$ since they force $\dot{f}$ to take different values by Theorem 5.2.2.5. In particular, $p_{x,y}$ will force the sentence $\dot{f}(\check{x}) = \check{y}$ while $p_{x,y'}$ forces the negation. Then, supposing $\exists q \in P$ so $q \leq p_{x,y}, p_{x,y'}$, by Theorem 5.2.2.5(i), q will force a statement and its negation contradicting Theorem 5.2.2.5(ii).

The above argument also guarantees that F_p is a well-defined function.

By a theorem from Jech then, we have that any incompatible collection of conditions in M is at most countable[11]. But then each F_p is at most a countable function, for any $p \in P$. In particular the union of the F_p cannot be all of κ^+.

However, since $\dot{f} \in M[G]$ is a bijection $\kappa \to \kappa^+$, we have for any $y \in \kappa^+$ that there is $x \in \kappa$ and $p_{x,y} < p$ so $p_{x,y} \Vdash \dot{f}(\check{x}) = \check{y}$ so that every element of κ^+ is in some $F_p(x)$. Contradiction and the result follows.

5.4 Independence of AC

We start with a complete boolean algebra B, and set $\pi : B \to B$ an automorphism.

Proposition 5.4.0.1. *π induces an automorphism on V^B defined as follows:*

- $\pi(0) = 0$.
- $dom(\pi(x)) = \pi(dom(x))$ *and* $\pi(x)(\pi(y)) = \pi(x(y))$ *for* $\pi(y) \in dom(\pi(x))$.

Proof. To see injectivity, suppose $\mathrm{dom}(\pi(x)) = \mathrm{dom}(\pi(x'))$. Then, by definition, $\pi(\mathrm{dom}(x)) = \pi(\mathrm{dom}(x'))$. By induction, we conclude $\mathrm{dom}(x) = \mathrm{dom}(x')$. As well, we have $\pi(x)(\pi(y)) = \pi(x')(\pi(y))$. But then, $\pi(x(y)) = \pi(x'(y))$. Since $x(y), x'(y) \in B$, and π is an automorphism on B, we have $x(y) = x'(y)$. Hence $x = x'$.

Surjectivity is similar.

∎

[11] See Theorem 9.18 in [Jec03].

Lemma 5.4.0.2. *Let ϕ be a formula. Then,*

$$||\phi(\pi x_1, ..., \pi x_n)|| = \pi||\phi(x_1, ..., x_n)||.$$

Proof. By induction on the complexity of formulas. See [Jec03] for details.

■

Now, start with a ground model M, such that $(V = L)^M$. In particular then, M is a model of AC. Let P, our collection of forcing notions be the set of finite functions from $\omega \times \omega \to \{0, 1\}$. If p extends q (i.e. $p \supset q$), then $p < q$. Moreover, fix $G \subset P$, P-generic over M and define

$$a_i = \{n \in \omega : (\exists p \in G) p(i, n) = 1\}.$$

Let $A = \{a_i\}$. Let $\dot{A}, \dot{a}_i$ be the canonical names for A, a_i. As well:

- $\mathrm{dom}\,(\dot{a}_i) = \{\check{n} : n \in \omega\}$ and $\dot{a}_i(\check{n}) = \sum\{p \in P : p(i, n) = 1\}$.
- $\mathrm{dom}\,(\dot{A}) = \{\dot{a}_i : i \in \omega\}$, and $\dot{A}(\dot{a}_i) = 1$.

Lemma 5.4.0.3. *If $i \neq j$, then every p forces $\dot{a}_i \neq \dot{a}_j$.*

Proof. For any given p, there is a $q < p$ so for some $n \in \omega$ $q(i, n) = 1$ and $q(j, n) = 0$. Just extend p by a single element. The result follows by Theorem 5.2.2.1.

■

HOD is the collection of all sets which are ordinal definable and have ordinal definable transitive closure. Now let N be the class of sets ordinal definable over A in $M[G]$. Clearly $A \in N$. Moreover N is a model of ZF (see **??**). We will show that A cannot be well-ordered in N. In particular, we will show that there does not exist $f : A \to \mathbf{ON}$ in N which is injective.

Proof. For a contradiction, suppose $f : A \to \mathbf{ON}$ is such an injection in N. Then there is a sequence $s = \langle x_0, ..., x_m \rangle$ of elements in A so f is ordinal definable from s, A. Obviously, every $a \in A$ is ordinal definable over A, s.

In particular, $M[G] \models a$ is the unique set $\phi(a, \alpha_1, ..., \alpha_n, s, A)$, for ordinals $\alpha_1, ..., \alpha_n$. Using forcing, we will be able to show this is impossible.

Set $\dot{a}$ the name for a, $\dot{x}_i$ the names for the x_i in s, and $\dot{s}$ the name for the sequence, s.

Let p be such that

$$p \Vdash \phi(\dot{a}, \check{\alpha}_1, ...\check{\alpha_n}, \dot{s}, \dot{A}).$$

We note that since the $\dot{a}_i$ we defined earlier are just sets of integers, they are clearly ordinal definable. Then there exists $p' \leq p$, and $i, i_0, ..., i_k$ so

$$p' \Vdash \dot{a} = \dot{a}_i, \dot{x_0} = \dot{a}_{i_0}, ...\dot{x}_k = \dot{a}_{i_k}.$$

Fix $j \in \omega$ so $j \neq i$ and for all m, $(j, m) \notin \mathrm{dom}\,(p')$. We can find such a j since p' is a finite function. Indeed there is some j on which p' is undefined on every m. Let π be the permutation on ω interchanging i, j and leaving every other element unchanged.

This induces an automorphism on P by $\mathrm{dom}\,(\pi(p)) = \{(\pi x, m) : (x, m) \in \mathrm{dom}\,(p)\}$ and $\pi p(\pi x, m) = p(x, m)$. In turn, this induces an automorphism on $B(P)$ by $\pi(u) = \sum\{\pi p : p \leq u\}$. In particular, this enables us to rewrite the result of Lemma 5.4.0.2 as

$$p \Vdash \phi(\dot{x}_1, ..., \dot{x}_n) \iff \pi p \Vdash \phi(\pi\dot{x}_1, ..., \pi\dot{x}_n),$$

by Theorem 5.2.2.4.

We verify $\pi(\dot{a}_i) = \dot{a}_j$ using the definitions from earlier. In particular $\operatorname{dom}(\pi\dot{a}_i) = \{\pi\check{n} : n \in \omega\} = \{\check{n} : n \in \omega\} = \operatorname{dom}(\dot{a}_j)$ since π is an automorphism on ω. Moreover, $\pi\dot{a}_i(\check{n}) = \sum\{\pi p : \pi p(i, n) = 1\}$. Since π interchanges only i, j, the result follows. The proof is identical for the case with $\dot{a}_j$. Clearly $\pi(\dot{A}) = A$ by definition, and $\pi(\dot{s}) = \dot{s}$.

Notice finally that $p', \pi p'$ are compatible since $i \neq j$ and j was chosen so $(j, m) \notin \operatorname{dom} p$ for any m. In particular, there is $q \leq p', \pi p'$. Set $q = p' \cap \pi p'$. We have the following:

$$p' \Vdash \phi(\dot{a}_i, \alpha_1, ..., \alpha_n, \dot{s}, \dot{A}).$$

$$\pi p' \Vdash \phi(\dot{a}_j, \alpha_1, ..., \dot{s}, \dot{A}).$$

As a result,

$$q \Vdash \phi(\dot{a}_i, ...), \phi(\dot{a}_j, ...).$$

By Lemma 5.4.0.3, $q \Vdash \dot{a}_i \neq \dot{a}_j$. Contradiction.

■

Chapter 6

The axioms revisted

The question of whether PP is at least independent of ZF remains, since this is a minimal requirement for it to potentially entail AC. This can be resolved in several ways. On one hand, it is known that dSB is independent of ZF, so PP must be too for otherwise ZF would model dSB as a consequence of PP [citation required]. Another method would be to use forcing to build a model of ZF $+\neg$ PP. Note that any model of ZFC is automatically a model of PP, so there is no need to construct one for the independence result.

A particular difficulty associated with this is that it is hard to pin down the precise difference between PP and AC. We start with a preliminary treatment here:

Remark 6.0.0.1. *If PP implies AC over ZF, there must be constructions in ZF which allow us to generate choice-like sets given PP-like sets.*

In particular, we have the following proposition:

Proposition 6.0.0.2. *PP $\implies$ AC over ZF if and only if it is possible to construct a choice-like injection $g : B \to A$ given that one exists.*

That is to say, PP guarantees given a surjection $f : A \to B$, there exists an injection $g : B \to A$. If it is the case that from this single injection it is possible to construct enough injections using only constructions in ZF, such that one constructed injection is choice-like, then it is the case that PP implies AC. In particular, we note that this is a notion intrinsically tied to cardinality it would seem since the number of injections one may need to construct is a pigeon-hole like problem and hence deeply connected to combinatorial set theory.

Proof of Proposition 6.0.3. The reverse case is easy. Suppose $g : B \to A$ an injection guarantees the existence of a choice-like injection $h : B \to A$, and suppose the partition principle is true. Then given $f : A \to B$ a surjection, we are guaranteed there exists an injection $g : B \to A$. Moreover, such an injection guarantees a choice like injection, so in particular AC holds.

The forward case likely holds since if $\mathsf{PP} \implies \mathsf{AC}$, then PP must force the existence of a choice function. However, the exact mechanism is not known.

■

In particular, to construct a choice-like injection from an injection, it seems necessary to be able to permute elements in the range of a set. For this reason, it is necessary to introduce the idea of ZF with atoms, known as ZFA, and permutation models.

Indeed in [SBdFB20], [SBdF19], and [SBdFB21], the authors purport to have built a model of PP without AC using an immersion of ZFA in their new meta-language, Flow. Whether or not their construction is valid remains to be seen however.

6.1 ZFA and permutation models

ZFA is ZF augmented with new objects known as atoms, which are not sets. That is, there is some collection A of objects which are not sets (and contain no sets either). We hence require new axioms and a new language to describe the behaviour of these objects. Fix $\mathcal{L}_{\mathsf{ZFA}} = \{\in, A\}$ along with the other logical connectives/quantifiers, and equality, and modify the axioms of empty set and extensionality to the following:

$$\exists x(\notin A \wedge \forall z(z \notin x))$$

$$\forall x \forall y((x, y \notin A) \implies \text{extensionality})$$

Moreover, to describe atoms, we add the following axiom, which we will refer to as the axiom of atoms:

$$\forall x(x \in A \iff (x \neq \emptyset \wedge \forall z(z \notin x)))$$

We can also construct a universe of ZFA using TF-induction, similar to V. Indeed, set:

- $M_0 = A$
- M_α is defined analogously to V_α for all ordinals $\alpha \neq 0$.

Clearly $V \subset M$. In particular, we call V the **kernel** of M. Moreover, it is easy to see that M is transitive by similar arguments.

The utility of M in defining models where PP might hold but AC to fail lies in the fact that ZFA cannot distinguish between atoms. That is, $\forall x, y \in A$, there is no way to show $x \neq y$ or that $x = y$ from inside ZFA. In this manner, we are able to induce permutations on the universe via an automorphism on A that doesn't greatly alter its structure.

Indeed, using ZFA, it is possible to construct a model where not all vector spaces have a basis (a statement known to be consistent with AC). At the very least, it is a good starting point for searching for models of ZF with PP but without AC.

We will demonstrate an example of a special type of model known as a permutation model, in which AC fails to hold, and discuss the challenges associated with constructing a proof to show that PP holds in a such a type of model. We start with some more terminology and notation. Fix M a model of ZFA, and a collection of atoms A.

Definition 6.1.0.1. *Let π be an automorphism on A. We can extend π to an automorphism on the universe M by defining*

$$\pi(x) = \begin{cases} \emptyset & \text{if } x = \emptyset \\ \pi x & \text{if } x \in A \\ \{\pi y : y \in x\} & \text{otherwise} \end{cases} .$$

It is easy to see the following is true:

Lemma 6.1.0.2. *$\pi(x) = x$ for all $x \in V$.*

The proof is by TF-induction.

We define in addition the following:

Definition 6.1.0.3. *Let $\mathcal{G}$ be a group of permutations (automorphisms) on A. Then define for $a \in A$, the **symmetry group** of a to be $sym_{\mathcal{G}}(a) = \{\pi \in \mathcal{G} : \pi a = a\}$.*

One will verify immediately that $\mathrm{sym}_{\mathcal{G}}(a)$ is a subgroup of $\mathcal{G}$ for any $a \in A$.

Definition 6.1.0.4. *Let $\mathcal{G}$ be a group of permutations (automorphisms) on A. Then a set of subgroups of $\mathcal{G}$, $\mathcal{F}$, is said to be a **normal filter** if for all subgroups H, K of $\mathcal{G}$, the following hold:*

- *$\mathcal{G} \in \mathcal{F}$.*

- *$H \in \mathcal{F}$ and $H \subseteq K$ implies $K \in \mathcal{F}$.*

- *$\mathcal{F}$ is closed under intersections.*

- *If $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$.*

- *For each $a \in A$, $sym_{\mathcal{G}}(a) \in \mathcal{F}$.*

Definition 6.1.0.5. *We say a set $x \in M$ is **symmetric** if $sym_{\mathcal{G}}(x) \in \mathcal{F}$. A set is **hereditarily symmetric** if in addition every element of its tc is symmetric.*

We call $\hat{\mathcal{M}}$, the set of all hereditarily symmetric sets of $\mathcal{M}$ a **permutation model**. Note that $\hat{\mathcal{M}}$ is not a model of ZF since it is provable in ZF that the empty set is unique, but every atom $a \in A$ is an element of $\hat{\mathcal{M}}$ but contains no other elements. However, by the Jech-Sochor embedding theorem (see [Hal19], Theorem 17.2), arbitrarily large fragments of permutation models can be embedded into models of ZF.

The **Fraenkel permutation models** are of the following type: Let $I = \text{fin}\,(A)$, the set of all finite subsets of A.

Lemma 6.1.0.6. *For every set $S \in I$, set $fix_{\mathcal{G}} = \{\pi \in \mathcal{G} : \pi a = a$ for every $a \in S\}$. Then the set $\mathcal{F} := \{H : H$ is a subgroup of $\mathcal{G}$ and $\exists S \in I(H \supseteq S)\}$ is a normal filter.*

Proof. Note $\{a\}$ is a finite subset of A for arbitrary $a \in A$. Moreover, $\text{fix}_{\mathcal{G}}(\{a\}) = \text{sym}_{\mathcal{G}}(a)$, and the latter is a subgroup of $\mathcal{G}$ and the former an element of I, so $\mathcal{G} \in \mathcal{F}$. If $H \in \mathcal{F}$, then there is $S \in I$ so $H \supseteq \text{fix}\,(S)$ [1]. If $K \supseteq H$ then clearly $K \supseteq \text{fix}\,(S)$, so $K \in \mathcal{F}$. Now, let $H, K \in \mathcal{F}$. Then, there exist $S, S' \in I$ so $H \supseteq \text{fix}\,(S), K \supseteq \text{fix}\,(S')$. We claim $H \cap K \supseteq \text{fix}\,(S \cup S')$. Indeed, choose $\pi \in \text{fix}\,(S \cup S')$, then $\pi \in \text{fix}\,(S)$ and $\pi \in \text{fix}\,(S')$. In particular, $\pi \in H$ and $\pi \in K$, so $\pi \in H \cap K$. Since $S, S' \in I$, both S, S' are finite such that their union is also finite. The result follows. The remaining two properties are verified similarly.

■

[1] We will drop $\mathcal{G}$ from now on.

Lemma 6.1.0.7. *Moreover, x is symmetric iff there is a set of atoms $E_x \in I$ so $\mathit{fix}_{\mathcal{G}}(E_x) \subseteq \mathit{sym}_{\mathcal{G}}(x)$. We call E_x the **support** of x.*

Proof. Let x be symmetric. Then $\mathrm{sym}\,(x) \in \mathcal{F}$. But then by definition, there exists $S \in I$ so $\mathrm{sym}\,(x) \supseteq \mathrm{fix}\,(S)$ so the result follows. For the converse, suppose there is such an $E_x \in I$, then by definition of $\mathcal{F}$, $\mathrm{sym}\,(x) \in \mathcal{F}$.

■

We hence introduce the second Fraenkel model:

Set $A = \bigcup_{i\in\omega} P_i$, where $P_i = \{a_i, b_i\}$ and $P_i \cap P_j = \emptyset$ for $i \neq j$, and set I the set of finite subsets of A and $\mathcal{F}$ the collection of filters generated by fixtures of I. Let $\mathcal{G}$ be the group of automorphisms leaving disjoint pairs of atoms P_n fixed[2]. The second Fraenkel model $\mathcal{V}_{F_2}$, is the class of all hereditarily symmetric subsets relative to $\mathcal{F}$. We have the following theorem:

Theorem 6.1.0.8 ([Hal19], 7.6)**.** *The following are true:*

(i) For each $n \in \omega$, P_n the set $P_n \in \mathcal{V}_{F_2}$.

(ii) The sequence $\langle P_n : n \in \omega\rangle$ is in $\mathcal{V}_{F_2}$.

(iii) There is no choice function on $\{P_n : n \in \omega\}$ in $\mathcal{V}_{F_2}$.

Proof. For (i), it suffices to show P_n is symmetric since every atom is symmetric by definition of normal filter, $\mathcal{F}$. But this is obviously true since $\pi P_n = \pi\{a_n, b_n\} = \{\pi a_n, \pi b_n\} = P_n$ by choice of $\mathcal{G}$. (ii) is similar. We have $\pi\langle P_n : n \in \omega\rangle = \langle \pi P_n : n \in \omega\langle = \langle P_n : n \in \omega\rangle$ by Lemma 6.1.0.2 and (i). In particular, we have demonstrated that $\{P_1, ..., P_n, ...\}$ is countable.

For (iii), suppose for a contradiction that f is a choice function on $\{P_n : n \in \omega\}$. Then f is symmetric, and so by Lemma 6.1.0.7, f has a support E_f. For a contradiction, let

[2]It is easy to verify this is a group.

$s = \{a_0, b_0, ..., a_k, b_k\}$ be a support of f. And let $\pi \in \text{fix}_{\mathcal{G}}(s)$ be such that $\pi a_{k+1} = b_{k+1}$. Notice such a π exists in the fixture. Then, $\pi f(k+1) \neq f(k+1)$. But this means π is not in the fixture since it is not in the symmetry group of f. But then s is not a support. As well, this is true for any $k \in \omega$. So f has no support. Hence, f is not symmetric, and so there is no choice function on $\{P_n : n \in \omega\}$ as desired.

■

We conclude that $V_{F_2} \models \neg\mathsf{AC}_{\omega,2}$ (i.e. AC for families of size ω, with elements of cardinality 2).

Notice that we can take $s = \{a_0, b_0, ..., a_k, b_k\}$ a support of f in the above proof since if $s' = \{a_{i_0}, ..., b_{i_k}\}$ is an actual support, then so too is any extension because fixtures decrease in extensions.

The idea to exhibit a model of $\mathsf{ZF}-\mathsf{PP}$ would be to modify this construction and invoke the Jech-Sochor theorem.

The failing of AC in the second Fraenkel model is due to the choice of I and $\mathcal{G}$. Indeed, fixing I, and modifying $\mathcal{G}$ to be the group of automorphisms leaving $Q_i = \{a_{i1}, a_{i2}, ..., a_{ik}\}$ disjoint for $i \neq j$ fixed, results in a model of $\neg\mathsf{AC}_{\omega,k}$. We can similarly let I be count(A) (i.e. the countable subsets of A, with A instead uncountable). Let $\mathcal{P}$ be a collection of uncountably many disjoint pairs. Then, by the same techniques, as in the last lemma, we can obtain a model of $\neg\mathsf{AC}_{\mathfrak{c},k}$. In [Bru16], the author demonstrates a model negating several forms of weak choice setting I to be the finite partitions of a countable set of atoms A.

The challenge is associated with finding an appropriate $\mathcal{G}, I$.

Appendices

Appendix A

Mostowski collapse

Recall that we invoked the Mostowski's collapsing lemma to prove the existence of a countable transitive set model of ZF given its consistency in Chapter 5. We give a treatment of the theorem and a proof here.

We begin with some terminology:

Definition A.0.0.1. *We say a relation R on a class A is* ***set-like*** *if for every $x \in A$, $\{y : yRx\}$ is a set.*

Definition A.0.0.2. *We say a relation R on a class A is* ***well-founded*** *if every non-empty subset of A has an R-least element.*

Definition A.0.0.3. *Let R be a relation on a class A. For $x \in A$, we define the* ***extension of*** *x to be the class $Ext_R(x) := \{y : yRx\}$.*

Definition A.0.0.4. *We say a relation R on a class A is* ***extensional*** *if $Ext_R(x) \neq Ext_R(y)$ for distinct $x, y \in A$.*

We are ready to state the theorem:

Theorem A.0.0.5 (Mostowski's collapsing lemma (ZF-P))**.** *If a relation R on a class A is set-like, well-founded, and extensional, then there exists a unique transitive class*

M with a relation $\in$ such that (A, R) is structurally isomorphic to $(M, \in)$, and the isomorphism, π is unique.

We require a lemma first:

Lemma A.0.0.6. *If R is well-founded and set-like on A, then every non-empty subclass of A has an R-least element.*

Proof. First, define the following:

- $\mathrm{pred}\,(A, x, R) = \{y \in A : yRx\}$.
- $\mathrm{pred}^0(A, x, R) = \mathrm{pred}\,(A, x, R)$.
- $\mathrm{pred}^{n+1}(A, x, R) = \bigcup\{\mathrm{pred}\,(A, y, R) : y \in \mathrm{pred}^n(A, x, R)\}$.
- $\mathrm{cl}\,(A, x, R) = \bigcup\{\mathrm{pred}^n\,(A, x, R) : n \in \omega\}$.

Notice that since R is set-like, all of the above are sets.

We claim further, that if R is set-like on A, and $x \in A$, then for all $y \in \mathrm{cl}\,(A, x, R)$, $\mathrm{pred}\,(A, y, R) \subset \mathrm{cl}\,(A, x, R)$. Indeed, let $y \in \mathrm{cl}\,(A, x, R)$. Then $y \in \mathrm{pred}^n\,(A, x, R)$, for some $n \in \omega$. But, $\mathrm{pred}^{n+1}(A, x, R)$ contains $\mathrm{pred}\,(A, y, R)$ as a subset by definition.

Now, set X a subclass of A, and fix $x \in X$. If x is R-minimal in X we are done. If not, then $\mathrm{cl}\,(A, x, R) \cap X$ is a set since R is set-like, and has an R-minimal element y. Such y is also R-minimal in X. Suppose not. Then there is zRy, so $z \in X$. As well, $z \in \mathrm{cl}\,(A, x, R)$ by the earlier observation. Contradiction, so there can be no such z.

■

Proof of Theorem A.0.0.5. We define the isomorphism π by well-founded induction. That is: $\pi(x) = \{\pi(y) : y \in A \wedge yRx\}$. Define M to be the range of π. We show the following are true:

(i) $xRy \implies \pi(x) \in \pi(y)$.

(ii) M is transitive.

(i) is obvious by the definition of π. (ii) is also by definition.

To see that π is one-to-one, consider the set $S := \{x \in A : \exists y \in A(x \neq y) \wedge (\pi(x) = \pi(y))\}$, and choose x R-minimal from S, and a corresponding y. That is, choose $x \in S$, R-minimal in S (such an x exists since R is well-founded and set-like, by the earlier lemma), and $y \neq x$ so $\pi(x) = \pi(y)$. Since R is extensional, it must be that there exists zRx so $\neg zRy$ or vice versa. Without loss of generality, assume there is z so zRx and $\neg zRy$. Since $\pi(x) = \pi(y)$, and $\pi(z) \in \pi(x)$ by (i), it must be that there is $w \neq z$ such that $\pi(z) = \pi(w)$. But this contradicts the minimality of x. Hence, the set must be empty, and it follows that π is one-to one. We conclude π is an isomorphism as required.

We will not need to make use of the fact that π, M are unique, but the interested reader is directed to §3.5 in [Kun06].

■

Appendix B

Transfinite recursion

Throughout the body of text, we make a number of constructions by transfinite recursion without justification. That we are able to do this is a consequence of the Transfinite Recursion which we state without proof below[1]

Theorem B.0.0.1 (Transfinite Recursion)**.** *Let G be a class function. Then there exists unique F, a function on ordinals such that $F(\alpha) = G(F_{||\alpha})$.*

In particular, Theorem B.0.0.1 states that if G is a class function, then there is a unique function F defined on the ordinals that is constructed iteratively by "feeding" into G a segment of the function, where we define a segment to mean the function F restricted onto some ordinal.

For clarity, we give some examples:

Example B.0.0.2 (Ordinal Addition)**.** *We may define ordinal addition by transfinite recursion. In particular, for an ordinal α we set:*

- $\alpha + 0 := \alpha$
- $\alpha + (\beta + 1) := (\alpha + \beta) + 1$, *for every* β

[1] The reader interested in a proof is directed to [Jec03], [Kun06], or any other standard set theory text.

- $\alpha + \beta := \lim_{\epsilon < \beta} (\alpha + \epsilon)$, *for* β *a limit ordinal*

In particular we define a class function G_α *such that*

$$G_\alpha(x) = \begin{cases} \textit{undefined} & \textit{if } x \textit{ is not a function} \\ \alpha & \textit{if } x = \emptyset \\ x(\beta) \cup \{x(\beta)\} & \textit{if } dom(x) = \beta + 1 \\ \bigcup_{\epsilon < \beta} x(\beta) & \textit{if } \beta \textit{ is a limit ordinal} \end{cases}.$$

One verifies that $F(\alpha) = G(F_{|\alpha})$ *is standard ordinal addition on* α. *In particular* $F(0) = G(F_{|0}) = G(\emptyset) = \alpha$. *Then* F *after one iteration of recursion is the set* $\{(0, \alpha)\}$. *Then* $F(1) = G(F_{|1}) = G(\{(0, \alpha)\})$. *Now, the domain of this is* 1. *So in particular,* $G(\{(0, \alpha)\})$ *is* $x(0) \cup \{x(0)\} = \alpha \cup \{\alpha\} = \alpha + 1$, *as desired. A similar result holds for the limit ordinal case as the reader may verify.*

Example B.0.0.3 (Ordinal Multiplication)**.** *We may define ordinal multiplication by transfinite recursion. For an ordinal* α *we set:*

- $\alpha \cdot 0 = 0$

- $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$ *for every* β

- $\alpha \cdot \beta = \lim_{\epsilon < \beta} (\alpha \cdot \epsilon)$

Here, we may take our class function G *to be as follows:*

$$G_\alpha(x) = \begin{cases} \textit{undefined} & \textit{if } x \textit{ is not a function} \\ 0 & \textit{if } x = 0 \\ +_\alpha(x(\beta)) & \textit{if } dom(x) = \beta + 1 \\ \bigcup_{\epsilon < \beta} x(\beta) & \textit{if } \beta \textit{ is a limit ordinal} \end{cases}$$

Now, $F(0) = 0$*. Hence, after one iteration of the recursion* $F = \{(0,0)\}$*. Then* $F(1) = G(F_{|1}) = G(\{(0,0)\})$ *whose domain is* 1*. Then* $G(\{(0,0)\}) = +_\alpha(x(0)) = +_\alpha(0) = \alpha$ *as desired. Now, after two iterations, our domain is* $\{0,1\}$*. So* $F(2) = G(\{(0,0),(1,\alpha)\})$*, whose domain is* $\{0,1\} = 2$*. Hence* $G(\{(0,0),(1,\alpha)\}) = +_\alpha(F(1) = \alpha+\alpha)$*, as desired. We can show that the class function gives the desired output for the case of limit ordinals.*

Similarly, it is possible to define ordinal exponentiation by transfinite recursion (where we utilize the previously defined multiplication operation). As well, we also define several models of ZF using transfinite recursion:

Example B.0.0.4. *Consider the well-founded universe* V *which we previously defined in the main body of text (Definition 3.2.0.3). We check this is well-defined by demonstrating a class function which gives its construction.*

We set:

$$G = \begin{cases} \text{undefined} & \text{if } x \text{ is not a function} \\ 0 & \text{if } x = 0 \\ \mathcal{P}(\S(\beta)) & \text{if } dom(x) = \beta + 1 \\ \bigcup_{\epsilon<\beta} x(\epsilon) & \text{if } \beta \text{ is a limit ordinal} \end{cases}$$

In particular then, $F(0) = 0$*,* $F(1) = G(\{(0,0)\}) = \mathcal{P}(x(0)) = \{0\}$*,* $F(2) = G(\{(0,0),(1,1)\}) = \mathcal{P}(1) = \{0,\{0\}\} = 2$*,* $F(3) = \mathcal{P}(2)$*, and so on. The interested reader again easily verifies that the formula holds for the limit ordinal case.*

In particular, these constructions are useful since they enable us to ensure that at every stage of construction that our image is a set.

Appendix C

Transitive closures

There is another way of proving the consistency of regularity/foundation with the other axioms of ZF. We define:

Definition C.0.0.1. *Let x be a set. Then we define the* ***transitive closure*** *of x by*

$$tr\ cl(x) = \bigcup\{\cup^n x : n \in \omega\},$$

where $\cup^0 = x$, and $\cup^{n+1}x = \bigcup(\cup^n x)$.

Note that for any set x, $\mathrm{tr\ cl}(x)$ is a set.

Lemma C.0.0.2. *The following are true in $ZF^{-}-P$:*

(a) $A \subset tr\ cl(A)$.

(b) $tr\ cl(A)$ *is transitive.*

(c) If $A \subset T$ *and* T *is transitive,* $tr\ cl(A) \subset T$.

(d) If A *is transitive then* $tr\ cl(A) = A$.

(e) If $x \in A$, *then* $tr\ cl(x) \subset tr\ cl(A)$.

(f) $tr\ cl(A) = A \cup \{tr\ cl(x) : x \in A\}$.

Proof. (a) obvious. For (b) pick $x \in \text{tr cl}(A)$. Then $x \in \cup^n A$. But then for $y \in x$, $y \in \cup^{n+1} A$. So $x \subset \text{tr cl}(A)$ as desired. For (c), notice that $A \subset \cup^0 T$ and in general $\cup^n A \subset \cup^n T$. The result follows. (d) is obvious. For (e), notice $x \in A$ so $x \in \text{tr cl}(A)$ since $A \subset \text{tr cl}(A)$ by (a). But then by (b), $x \subset \text{tr cl}(A)$. Applying (c) to x gives the result. For (f), the reverse inclusion is obvious. To see the forward inclusion notice that A is a subset of the RHS, and that the RHS is transitive. The result follows by (c).

■

Lemma C.0.0.3. *The following are equivalent in* ZF^-*:*

(a) $A \in \boldsymbol{V}$.

(b) $\textit{tr cl}(A) \in V$

(c) $\in$ *is well-founded on* $\textit{tr cl}(A)$.

Proof. To see (a) implies (b), notice that $\cup^n A \in V$ for every $n \in \omega$ and that V is closed under unions. For (b) $\Longrightarrow$ (c) take $B \subset \text{tr cl}(A)$. Take y with $\text{rank}(y) = \alpha$, where $\alpha = \min(\text{rank}(y) : y \in B)$. Then y is $\in$-minimal in B. (c) $\Longrightarrow$ (a) follows from Kunen 3.3 and that $A \subset \text{tr cl}(A)$.

■

From the above, we have the very important theorem:

Theorem C.0.0.4. *TFAE:*

(a) Axiom of regularity.

(b) $\forall x(\in$ *is well-founded on* $x)$.

(c) the universe of all sets $= V$.

Appendix D

Some more forcing properties

We show a few more properties of generic filters and the forcing relation. These results are not invoked anywhere in the main body of text, but they are used in practice when doing forcing.

Fix M a ground model, $P \in M$ a p.o., and G a generic filter.

Lemma D.0.0.1. *If* $\{q : q \Vdash \sigma\}$ *is dense below* p*, then* $p \Vdash \sigma$*.*

Proof. It suffices in view of Theorem 5.2.2.5(iv) to show that there is no $q \leq p$ so $q \Vdash \neg\sigma$. Suppose for a contradiction such a q exists. Then by density of $\{q : q \Vdash \sigma\}$ below p there is $q_0 \leq q$ so $q_0 \Vdash \sigma$. But then, $q_0 \Vdash \sigma$ and by Theorem 5.2.2.5(i), $q_0 \Vdash \neg\sigma$. Hence, there can be no such q, and in view of Theorem 5.2.2.5(iv) $p \Vdash \sigma$ as required.

■

Lemma D.0.0.2. $||(\exists y \in x)\phi(y)|| = \sum_{y \in dom(x)} x(y) \cdot ||\phi(y)||$.

Proof. We have the following:

$$\begin{aligned}
||(\exists y \in x)\phi(y)|| &= ||\exists y(y \in x \wedge \phi(y))|| \\
&= \sum ||y \in x \wedge \phi(y)|| \\
&= \sum ||y \in x|| \cdot ||\phi(y)|| \\
&\geq \sum_{y \in \mathrm{dom}\,(x)} ||y \in x|| \cdot ||\phi(y)|| \\
&\geq \sum_{y \in \mathrm{dom}\,(x)} x(y) \cdot ||\phi(y)||
\end{aligned}$$

Here, the first inequality follows by the fact that we are restricting our domain, and the second follows from Theorem 5.2.1.4(vi). As well, we have:

$$\begin{aligned}
||(\exists y \in x)\phi(y)|| &= ||\exists y(y \in x \wedge \phi(y))|| \\
&= \sum ||y \in x \wedge \phi(y)|| \\
&= \sum ||y \in x|| \cdot ||\phi(y)|| \\
&= \sum \sum_{t \in \mathrm{dom}\,(x)} ||y = t|| \cdot x(t) \cdot ||\phi(y)|| \\
&\leq \sum_{y \in \mathrm{dom}\,(x)} x(y) \cdot ||\phi(y)||
\end{aligned}$$

The inequality uses the fact that the sums may be reordered and that

$$||x = y|| \cdot ||\phi(y)|| \leq ||\phi(x)||.$$

The result follows.

■

Similarly:

Lemma D.0.0.3. $||(\forall y \in x)\phi(y)|| = \prod_{y \in dom\,(x)} (x(y) \implies ||\phi(y)||)$.

Proof. We have the following:

$$\begin{aligned}
||(\forall y \in x)\phi(y)|| &= ||\forall y(y \in x \implies \phi(y))|| \\
&= \prod -||y \in x|| + ||\phi(y)|| \\
&\leq \prod_{y \in \mathrm{dom}\,(x)} -x(y) + ||\phi(y)|| \\
&= \prod_{y \in \mathrm{dom}\,(x)} x(y) \implies ||\phi(y)||
\end{aligned}$$

The inequalities follow for the same reasons as before. The reverse inequality follows similarly to before.

■

Another result of interest is that the generic set G is not an element of M to generate a non-trivial extension.

Lemma D.0.0.4. *$G \notin M$ if P is a p.o. such that $\forall p \in P$, there are $q, r \in P$ so $q, r \leq p$ but $q \perp r$.*

Proof. Let $D = P - G$. Pick $p \in P - G$. Then there exists $q, r \in P$ so $q, r \leq p$ but $q \perp r$. Now, since G is a generic filter, at most one of these elements is in G since elements of filters are compatible. Without loss of generality, take $q \in G$. Then $r \in D = P - G$. Clearly D is dense. Now suppose $G \in M$. Then by transitivity, $G \subset M$, so $D \in M$. But $G \cap D = \emptyset$, which contradicts the genericity of G.

■

Remark D.0.0.5. *Notice that the forcing notion Definition 5.1.1.1 satisfies the condition that for all $p \in P$ there exist q, r so $q \perp r$. Hence $M[G]$ as constructed is different from M since $G \notin M$ but $G \in M[G]$.*

Again, we reiterate that it is the p.o. that endows the generic filter G with certain properties that make the extensions $M[G]$ interesting.

Appendix E

Ordinal definability

We define the classes, **OD** and **HOD** and describe them.

Definition E.0.0.1. *A set x is **ordinal definable** if there exists a formula ϕ, so*

$$x = \{y : \phi(y, \alpha_1, ..., \alpha_n)\}.$$

It is not easy to see the property of ordinal definability is definable in $\mathcal{L}_{\text{set}}$.

We define the class of ordinal definable sets as:

$$\mathbf{OD} = \bigcup_{\alpha} \text{cl}\{V_\beta : \beta < \alpha\},$$

where cl is closure under the Gödel operations defined in **??**.
We have:

Lemma E.0.0.2. *There is a definable, injective map from* **ON** *to* **OD***.*

For a proof, see Chapter 13 in [Jec03]. Then define:

Definition E.0.0.3. **HOD** *is the class of **hereditarily ordinal definable** sets, defined as:*

$$\mathbf{HOD} = \{x : tc\,(x) \subset \mathbf{OD}\},$$

where tc is the transitive closure of a set (see Appendix C).

The following is true:

Theorem E.0.0.4. HOD *is a transitive model of* ZFC.

Proof. See Chapter 13 in [Jec03].

■

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