AN INTRODUCTION TO THE (META-) THEORY OF STRUCTURES

AN INTRODUCTION TO THE (META-) THEORY OF STRUCTURES

By

JOHN WILLIFORD DUSKIN, JR., B.Sc.

A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Master of Science

McMaster University

June 1963

MASTER OF SCIENCE (1963)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario.

TITLE: An Introduction to the (Meta-) Theory of Structures

AUTHOR: John Williford Duskin, Jr., B.Sc. (Georgia Institute of Technology)

SUPERVISOR: Professor G. Bruns

NUMBER OF PAGES: vi, 123

SCOPE AND CONTENTS: This thesis is intended as a self-contained, expository introduction to material found in Chapter IV of Bourbaki's Theorie des Ensembles. With a minimum of external reference, it presents all relevant logical and set-theoretic background material and then develops and extends the notions of "species of structure", "intrinsic terms", "canonical mappings", "processes of deduction", "morphism", etc. found in this work.

PREFACE

The material of this thesis is largely concerned with the formal explication of the naive notions of "mathematical structure", "isomorphism", "morphism", etc. which are fundamental in all of modern mathematics.

A first step toward such an explication was made by Birkhoff in 1935 with his notion of an "abstract algebra". In his paper (Birkhoff 35), he showed that by suitably abstracting the common properties of the purely algebraic systems such as groups, rings, fields, modules, etc. one could give a single definition which in particular specializations would give all these algebraic objects back again, and by the use of which a large number of theorems previously proved separately for each of these algebraic objects could be replaced by a single theorem for abstract algebras, which would give each of the previously proved theorems back as corollaries.

In spite of the power of this abstraction, its extension to cover other mathematical systems such as topologies never got beyond the employment of analogous notational conventions, e.g., in analogy to the definition of abstract algebra, a topological space was defined as a <u>pair</u> (X,V). In addition to this difficulty, there were a number of inelegancies of the original definition of abstract algebra which made their use cumbersome, e.g., in order to consider a module as an abstract algebra, one had to allow for the possibility of an infinite number of binary relations in addition to the finite number of ternary relations which sufficed in all other cases.

A meta-theory of mathematical structures of sufficient generality to cover algebraic, topological, and order structures was not forthcoming until 1957 when Bourbaki published Chapter IV of his <u>Theorie des Ensembles</u> (Bourbaki 57). In this chapter, Bourbaki presented a meta-theory which not only eliminated the inelegancies of Birkhoff's approach (which for algebraic structures it supercedes) but was presumably adequate for all presently known mathematical structure.

Unfortunately, in spite of the power and beauty of Bourbaki's approach, the apparent cumbersomeness of the notation to the "uninitiated" and the large amount of unfamiliar antecedent material necessary for its comprehension, have made this chapter one of the most neglected of all the volumes in Bourbaki's treatise. This thesis arises out of an attempt to obviate some of these difficulties.

To do this we have abstracted relevant material from Chapters I, II, and III of the <u>Theorie des Ensembles</u> (Bourbaki 54, 56) and have presented this material as parts I and II of this thesis. In general, proofs have been eliminated much in the manner of Bourbaki (58) which is unfortunately inadequate for our purposes.

Part III then presents in an amplified and extended fashion the material found in section 1 and part of section 2 of Bourbaki's Chapter IV, the remaining sections having already been presented by the author in a Departmental Seminar in the Fall of 1962.

It will be apparent to the reader familiar with the theory of "categories and functors" that much of the material considered in Chapter IV presents very close analogies to the subject matter of that theory and this thesis may also be viewed as a study preliminary to the rewriting of one of these "theories" in terms of the other.

ACKNOWLEDGMENTS

The author would like to express his sincere thanks to his advisor, Professor G. Bruns, and also to his Departmental Chairman, Professor B. Banaschewski, without both of whose sage advice and ready counsel this thesis would never have even reached its present stage.

Gratitude is also expressed to McMaster University for financial assistance during its writing and to Miss Barbara Harrison who undertook the unenviable task of typing an almost illegible manuscript.

TABLE OF CONTENTS

PART I

FORMAL MATHEMATICS

1. TERMS AND RELATIONS

A mathematical theory consists of signs, relations, terms, axioms, proofs and theorems. The meaning of each of these notions will become clear as we proceed.

The signs of a mathematical theory $\mathcal{T}$ fall into three distinct types.

1. Logical Signs: $\vee$, $\neg$, τ , $\square$
2. Letters: x, y, A, A', etc.
3. Specific Signs: e.g. in the theory of sets the specific signs are « $=$, $\in$, $\supset$ ».

Once the specific signs are specified for a particular theory $\mathcal{T}$, one may form the assemblages of $\mathcal{T}$, i.e. strings of signs of $\mathcal{T}$ in which each occurrence of the sign « $\square$ » may be joined by a horizontal line (called a bond) to the sign « τ » which ordinarily will occur to its left. For example « $\overline{\tau \in x \square}$ » and « $\tau \in x \square$ » are assemblages of the theory of sets.

In any assemblage of $\mathcal{T}$, we are permitted the operation of substitution, i.e. the replacement of one or more of the signs occurring in the assemblage by other signs or assemblages of $\mathcal{T}$. We shall use the following notation for such substitutions: If A and B designate assemblages of $\mathcal{T}$, and x designates a letter which may or may not figure in A, then (B|x)A will designate the simultaneous replacement of the letter x, in each of its occurrences in A, by the assemblage B. For example, ($\tau \in y \square$ | x)xy designates $\tau \in y \square$ y. Of course if x does not figure in A, then (B|x)A is just A. As an alternative to this notation, we shall occasionally use the following sort of notation: Suppose that we are given some assemblage R in which the letters x and y may or may not occur and we wish to call attention to the fact of the possibility of such an occurrence; under such circumstances, we shall write R $\{$x,y$\}$ to single out the possibility of the occurrence of x and/or y in R. If this has been done, then we shall use the notation R $\{$z,w$\}$ to designate the assemblage obtained by the simultaneous replacement of x by z and y by w in each of their respective possible occurrences in R. (This same notation will be used without limiting the number of letters which we may wish to call attention to in any particular assemblage of $\mathcal{T}$.)

It will become apparent that the exclusive use of assemblages would result in typographically - not to mention mentally - insurmontable difficulties; for this reason, we shall, at convenient spots, introduce abreviating symbols, notably words of ordinary language, to designate various assemblages. The introduction of these symbols is the object of the <u>definitions</u> of $\mathcal{T}$. For example the assemblage $\vee \neg$ will be represented by $\Rightarrow$.

Let A be an assemblage of $\mathcal{T}$; we designate by τ_x(A) the assemblage of $\mathcal{T}$ obtained in the following manner: One takes the assemblage A and in each occurrence of the letter x, one replaces it by the sign $\square$; this done, one writes to the left of the resulting assemblage the sign τ and joins each the occurrences of $\square$ by a bond to the τ .

For example, $\tau_x(\in xy)$ designates the assemblage $\overline{\tau \in \square}\, y$.

In developing some particular theory $\mathcal{T}$, we shall often concern ourselves with manipulations involving various substitutions in various assemblages. Because of the extreme length of such reasonings and the frequency of similar forms of such reasonings about substitutions, it is very convenient to group together the final result of a succession of certain manipulations over certain assemblages as *metamathematical substitution criteria*. Their justification of course does not belong to the formal mathematics itself but rather to the *metamathematics* of the theory. These criteria we shall designate by CS followed by a numeral. The first ones are the following:

CS1. *Let A and B be assemblages, x and x' letters. If x' does not figure in A, $(B \mid x)A$ is identical to $(B \mid x')(x' \mid x)A$.*

CS2. *Let A, B and C be assemblages, x and y distinct letters. If y does not figure in B, $(B \mid x)(C \mid y)A$ is identical to $(C' \mid y)(B \mid x)A$, where C' is the assemblage $(B \mid x)C$.*

CS3. *Let A be an assemblage, x and x' letters. If x' does not figure in A, $\tau_x(A)$ is identical to $\tau_{x'}(A')$, where A' is the assemblage $(x' \mid x)A$.*

CS4. *Let A and B be assemblages, x and y distinct letters. If x does not figure in B, $(B \mid y)\tau_x(A)$ is identical to $\tau_x(A')$, where A' is the assemblage $(B \mid y)A$.*

CS5. *Let A, B, and C be assemblages, x a letter. The assemblage*

$(C \mid x)(\neg A)$ is identical to $\neg A'$;

$(C \mid x)(\vee AB)$ " " " A'B';

<u>$(C \mid x)(\Rightarrow AB)$ is identical to $\Rightarrow A'B'$;</u>

<u>$(C \mid x)(SAB)$ " " " $SA'B'$</u>.

<u>where A' is $(C \mid x)A$, B' is $(C \mid x)B$, and S is a specific sign</u>.

A mathematical theory consists of certain rules which permit one to say which assemblages of the theory are <u>relations</u> or <u>terms</u> of the theory and other rules which permit one to say that certain assemblages are the <u>theorems</u> of the theory. The <u>description</u> of these rules which we will give here does not, of course, belong to the formal mathematics itself but rather to the metamathematics of the theory.

The specific signs of a mathematical theory fall into two distinct types, <u>relational</u> signs and <u>substantive</u> signs. Additionally, each specific sign is assigned one and only one whole number, called the <u>weight</u> of the specific sign. For example, in the theory of sets « $=$ » , and « $\in$ » are relational signs of weight 2, while « $\supset$ » is a substantive sign of weight 2.

We classify our assemblages into two species: A is of the <u>first species</u> if it commences by a τ , a substantive sign, or reduces to a letter, A is said to be of the <u>second species</u> in all other cases.

A <u>formative construction</u> of a theory $\mathcal{T}$ is a sequence of assemblages of $\mathcal{T}$ which possess the following property:

For each assemblage A of the sequence, one of the following conditionsis verified:

a) A is a letter.

b) There occurs in the sequence preceding A a second species assemblage B, such that A is $\neg B$.

c) There occurs in the sequence, preceding A, two (not necessarily distinct) assemblages B and C such that A is $\vee BC$.

d) There occurs in the sequence preceding A, a second species assemblage B and a letter x, such that A is $\tau_x(B)$.

e) There is a specific sign S of weight n of $\mathcal{T}$, and there occurs in the sequence preceding A, n first species assemblages $A_1,\ldots,A_n$, such that A is $SA_1A_2\ldots A_n$.

We call the terms of $\mathcal{T}$ the first species assemblages of $\mathcal{T}$, which figure in the formative constructions of $\mathcal{T}$. We call the relations of $\mathcal{T}$ the second species assemblages which so figure.

Example: In the theory of sets, where $\in$ is a relational sign of weight 2, the following sequence of assemblages is a formative construction:

(1) A

(2) A'

(3) A''

(4) $\in AA'$

(5) $\in AA''$

(6) $\neg \in AA'$

(7) $\vee \neg \in AA' \in AA''$

(8) $\tau \vee \neg \in \square A' \in \square A''$

Let us verify this fact. (1), (2), and (3) verify a) since they are all letters; (4) verifies e) since $\in$ is a relational sign of weight 2 and A and A' are first species assemblage which occur in the sequence preceding (4), similarly for (5); (6) verifies b) since $\in AA'$ is a second species assemblage occuring in the sequence preceding (6); (7) verifies c) since $\neg \in AA'$ and $\in AA''$ are both second species assemblages occurring in the sequences preceding (7); (8) verifies d) since (8) is simply $\tau_A(\vee \neg \in AA' \in AA'')$, the "argument" of which is (7) which is a second species assemblage. The final assemblage (8), since it commences with a τ, and is thus of the first species and is thus a term of the theory of sets, similarly (1), (2), and (3) are also terms, while (4), (5), (6) and (7) are all of the second species and hence are relations of the theory of sets.

We can now comment on the intuitive significance of our logical and specific signs in relation to the formally defined terms and relations of a theory. The terms of theory intuitively represent the objects, the description of which is the purpose of the theory, while the relations represent relations between the objects or the properties of the objects, or assertions about the objects of $\mathcal{T}$. With this in mind, we attach the interpretation of negation to $\neg$ so, that if A is an assertion, then $\neg A$ (not A) is an assertion; $\vee$ is to be interpreted as inclusive disjunction thus if A and B are assertions about objects, then $\vee AB$, (A or B) is an assertion of $\mathcal{T}$. Similarly if S is a specific sign and $A_1, \ldots, A_n$ are objects of $\mathcal{T}$, then $SA_1, \ldots, A_n$ represents an object of $\mathcal{T}$ (if S is a substantive sign) or a relation between objects of $\mathcal{T}$ (if S is a relational sign). Finally if R is a relation understood as an assertion about the object x, then $\tau_x(R)$ designates that object, which, if it exists, is privileged with possessing the property asserted by R.

It is clear from the specification of what constitutes a formative construction of $\mathcal{T}$, that the initial sign of a relation of $\mathcal{T}$ must be $\vee$, $\neg$, or a relational sign, while the initial sign of a term of $\mathcal{T}$ must be τ, a substantive sign, or else the term reduced to being simply a letter. In fact, once the specific signs of a theory $\mathcal{T}$ are specified, the terms and relations of $\mathcal{T}$ are effectively determined in the sense that given any assemblage of $\mathcal{T}$ one has at one's disposal an effective decision procedure which will enable one to determine whether the given assemblage is a term or a relation of $\mathcal{T}$ (cf. Bourbaki 1954, Appendix 1 to Chapter I).

In a more practical vain we present a collection of metamathematically justified *Formative Criteria* each of which summarizes chains of reasonings about the formative constructions of a theory . These criteria, when they appear here in the text are designated by CF and an appropriate numeral. The first eight of these are the following:

CF1. *If A and B are relations of a theory $\mathcal{T}$, $\vee AB$ is a relation of $\mathcal{T}$.*

CF2. *If A is a relation of $\mathcal{T}$, $\neg A$ is a relation of $\mathcal{T}$.*

CF3. *If A is a relation of $\mathcal{T}$, and x a letter $\tau_x(A)$ is a term of $\mathcal{T}$.*

CF4. *If $A_1, A_2, \ldots, A_n$ are terms of $\mathcal{T}$, and S is a relational (resp. substantive) sign of weight n of $\mathcal{T}$, $SA_1A_2, \ldots, A_n$ is a relation (resp. term) of $\mathcal{T}$.*

CF5. *If A and B are relations of $\mathcal{T}$, $\Rightarrow AB$ is a relation of $\mathcal{T}$.*

CF6. *Let $A_1, A_2, \ldots, A_n$ be a formative construction of $\mathcal{T}$, x and y letters. If y does not figure in any of the A_i, then $(y|\,x)A_1$, $(y|\,x)A_2, \ldots, (y|\,x)A_n$ is a formative construction of $\mathcal{T}$.*

CF7. *Let A be a relation (resp. term) of $\mathcal{T}$, x and y letters. Then $(y|\,x)A$ is a relation (resp. term) of $\mathcal{T}$.*

CF8. <u>Let A be a relation (resp. term) of $\mathcal{T}$, x a letter, and T a term of $\mathcal{T}$. Then (T | x)A is a relation (resp. term) of $\mathcal{T}$</u>.

We are now at the stage where we can describe the rules which enable us to determine which assemblages of $\mathcal{T}$ are the theorems of $\mathcal{T}$. Before we do this we shall make a few conventions which will greatly enhance the readability of the text. They are the following: we shall commonly write « not (A) » in place of « $\neg$A », « A $\Rightarrow$ B » in place of « $\Rightarrow$ AB », « A or B » in place of « $\vee$AB ». This, while enhancing the intuitive interpretation of the text, is not without its own difficulties. For example, our notation, heretofore was, in the manner of Lukasiewicz, « parenthesis free », but now to avoid interpretational ambiguities, we must make use of such auxiliary devices as parenthesis to render the meaning of our expressions clear, e.g. we write (A or B) or C for $\vee\vee$ABC to distinguish this from A or (B or C) which is the convention for $\vee$A $\vee$ BC.

2. THEOREMS AND PROOFS

The specification of the specific signs of $\mathcal{T}$ completely determines the terms and relations of $\mathcal{T}$. In order to construct the theorems of $\mathcal{T}$, we first write down a certain number of relations of which will be called the <u>explicit</u> <u>axioms</u> of $\mathcal{T}$; the letters which figure in the explicit axioms are called the <u>constants</u> of $\mathcal{T}$. <u>Intuitively</u> the constants represent the well determined objects, of the theory $\mathcal{T}$ and the explicit axioms represent the fundamental, or evident assertions that we wish to make about these well determined objects.

We next may write down one or more « rules » called the schemas of $\mathcal{T}$ which each must have the following properties: 1) The application of such a rule R must furnish a relation of $\mathcal{T}$; 2) if S is a relation furnished by such a rule, T a term of $\mathcal{T}$, and x a letter then the relation $(T \mid x)S$ must again be constrictible by means of an application of the rule R. Intuitivity. if x is a letter, then it represents a completely undetermined object so that if some assertion is made involving the letter x, which we wish to be true as an axiom, then this axiom must be of the sort that it be true for an arbitrary object T of theory $\mathcal{T}$. A relation furnished by the application of a schema of $\mathcal{T}$ will be called an implicit axiom of $\mathcal{T}$.

We are now in a position to make clear what we mean by a proof and a theorem of $\mathcal{T}$. We do this in the following manner.

We say that a demonstrative text of a theory $\mathcal{T}$ comprises:

1. An auxiliary formative construction of terms and relations of $\mathcal{T}$,
2. A demonstration (proof) of $\mathcal{T}$, i.e. a sequence of relations of $\mathcal{T}$ figuring in the auxiliary formative construction, such that, for each relation R of the sequence, at least one of the following conditions is verified:

   a_1) R is an explicit axiom of $\mathcal{T}$;

   a_2) R results from the application of a schema of $\mathcal{T}$ to the terms or relations figuring in the auxiliary formative construction;

   b) there are in the sequence two relations S, T preceding R, such that T is $S \Rightarrow R$.

We now say that a *theorem of* $\mathcal{T}$ is a *relation figuring in a proof of* $\mathcal{T}$. However, we should note that this notion is essentially relative to the state of development of the theory at a particular moment of writing: a relation of $\mathcal{T}$ *becomes* a theorem of $\mathcal{T}$ when one has successfully inserted it in a proof of $\mathcal{T}$. Thus to say that a relation of $\mathcal{T}$ is *not* a theorem of $\mathcal{T}$ may be without precise sense since it can only refer to the present stage of development of the theory. In lieu of « theorem of $\mathcal{T}$ » we will also say « *true* relation in $\mathcal{T}$ » or « proposition », « lemma » etc. If R is a relation of $\mathcal{T}$, x a letter and T a term of $\mathcal{T}$, and if $(T \mid x)R$ is a theorem of $\mathcal{T}$, we shall say that T verifies the relation R in $\mathcal{T}$ (or is a *solution* of R) when R is considered as a relation involving x.

A relation is said to be *false* in $\mathcal{T}$ if its negation is a theorem of $\mathcal{T}$. One can say that a theory $\mathcal{T}$ is *contradictory* if one has a relation at hand which is both true and false in $\mathcal{T}$. Here again, we should be on guard against saying that once we have a false relation R in $\mathcal{T}$ that « the relation R is *not true* in $\mathcal{T}$ » for this latter statement may not actually make good sense, since it essentially refers to the present stage of development of the theory.

We now shall present a number of metamathematically justified *deductive criteria* which permit us to abbreviate proofs in a theory $\mathcal{T}$. These will be designated by C followed by a numeral. The majority of these criteria will be presented without proof, but as the first five are immediate consequences of the notion of proof, we shall present them and their (meta-) proofs here.

C1. (<u>Modus ponens</u>) <u>Let A and B be relations of a theory $\mathcal{T}$. If A and A $\Rightarrow$ B are theorems of $\mathcal{T}$, then B is a theorem of $\mathcal{T}$.</u>

In effect let $R_1, \ldots, R_n$ be a demonstration of $\mathcal{T}$ where A figures, and $S_1, \ldots, S_p$ be a demonstration of $\mathcal{T}$ where A $\Rightarrow$ B figures. It is evident that $R_1, R_2, \ldots, R_n$, $S_1, \ldots, S_p$ is a demonstration of $\mathcal{T}$ in which A and A $\Rightarrow$ B figure. Thus

$$R_1, R_2, \ldots, R_n, \quad S_1, S_2, \ldots, S_p, \; B$$

is a demonstration of $\mathcal{T}$, so that B is a theorem of $\mathcal{T}$.

We present this meta-theorem and its meta-proof in full to demonstrate the general method of proof for all such criteria. This one criterion is particularly important as it is essentially the <u>only rule of inference available</u> in our construction of a mathematical theory. Thus our logic is <u>strictly</u> classical.

To illustrate how our formative criteria and substitution criteria are used in these meta-theorems, we present the following criterion and its meta-proof.

Let $\mathcal{T}$ be a theory, $A_1, \ldots, A_n$ its explicit axioms, x a letter, T a term of $\mathcal{T}$. Let $(T \mid x)\mathcal{T}$ be the theory whose signs and schemas are the same as those of $\mathcal{T}$, but whose explicit axioms are $(T \mid x)A_1$, $(T \mid x)A_2, \ldots, (T \mid x)A_n$.

C2. <u>Let A be a theorem of a theory $\mathcal{T}$, T a term of $\mathcal{T}$, x a letter. Then $(T \mid x)A$ is a theorem of $(T \mid x)\mathcal{T}$.</u>

In effect, let $R_1, R_2, \ldots, R_n$ be a demonstration of $\mathcal{T}$ where A figures. Consider the sequence $(T \mid x)R_1$, $(T \mid x)R_2, \ldots, (T \mid x)R_n$, which is a sequence of relations by CF8. One must see that this is a demonstration of $(T \mid x)\mathcal{T}$, which will establish the criterion. If R_k is an implicit axiom of $\mathcal{T}$, $(T \mid x)R_k$ is again an implicit axiom of $\mathcal{T}$ from the definition of schema of $\mathcal{T}$, and thus of $(T \mid x)\mathcal{T}$. If R_k is an explicit axiom of $\mathcal{T}$, then $(T \mid x)R_k$ is an explicit axiom of $(T \mid x)\mathcal{T}$. Finally, if R_k is preceded by the relations R_i and R_j, R_j being $R_i \Rightarrow R_k$, $(T \mid x)R_k$ is preceded by $(T \mid x)R_i$ and by $(T \mid x)R_j$, and this last relation is identical to $(T \mid x)R_i \Rightarrow (T \mid x)R_k$ by CS5.

C3. *Let A be a theorem of a theory $\mathcal{T}$, T a term of $\mathcal{T}$, and x a letter which is not a constant of $\mathcal{T}$. Then $(T \mid x)A$ is a theorem of $\mathcal{T}$* .

This is an immediate result of C2, since x, by hypothesis is not a constant of $\mathcal{T}$ and hence, by definition does not figure in the explicit axioms of $\mathcal{T}$.

In particular, if $\mathcal{T}$ has no explicit axioms, or if the explicit axioms of $\mathcal{T}$ contain no letters, C3 applies without restriction on the letter x.

A theory $\mathcal{T}'$ is said to be *stronger* than a theory $\mathcal{T}$ if all of the signs of $\mathcal{T}$ are signs of $\mathcal{T}'$, if all of the explicit axioms of $\mathcal{T}$ are theorems of $\mathcal{T}'$, and if the schemas of $\mathcal{T}$ are schemas of $\mathcal{T}'$.

The above notion has several consequences. One of these is that all of the terms and relations of $\mathcal{T}$ are again terms and relations of $\mathcal{T}'$ since all of the signs of $\mathcal{T}$ are signs of $\mathcal{T}'$ and hence any formative construction of $\mathcal{T}$ is *a forteori*, a formative construction of $\mathcal{T}'$. Another consequence is the following criterion.

C4. If a theory $\mathcal{T}'$ is stronger than a theory $\mathcal{T}$, all of the theorems of $\mathcal{T}$ are theorems of $\mathcal{T}'$.

Let $R_1, R_2, \ldots, R_n$ be a proof in $\mathcal{T}$. We shall show one after another, that each R_i is a theorem of $\mathcal{T}'$, which will establish the criterion. We suppose our assertion established for the relations preceding R_k and establish for R_k. If R_k is an axiom of $\mathcal{T}$, it is a theorem of $\mathcal{T}'$, by hypothesis. If R_k is preceded by the relations R_i and $R_i \Rightarrow R_k$, one has thus that R_i and $R_i \Rightarrow R_k$ are theorems of $\mathcal{T}'$, thus R_k is a theorem of $\mathcal{T}'$ by C1.

The preceding criterion was established by a strictly finitistic method which might best be called « experimental induction ». It is typical of the only additional method which we use in these meta-proofs.

If each of two theories $\mathcal{T}$ and $\mathcal{T}'$ is stronger than the other, one says that $\mathcal{T}$ and $\mathcal{T}'$ are equivalent. Then every theorem of $\mathcal{T}$ is a theorem of $\mathcal{T}'$, and vice versa. In particular every theory $\mathcal{T}$ is equivalent to itself.

C5. Let $\mathcal{T}$ be a theory, $A_1, \ldots, A_n$ its explicit axioms, $a_1, \ldots, a_n$ its constants, $T_1, \ldots, T_k$ terms of $\mathcal{T}$. Suppose that $(T_1 | a_1)(T_2 | a_2) \ldots (T_k | a_k)A_i$ (for $i = 1, 2, \ldots, n$) be theorems of a theory $\mathcal{T}'$, in which the signs of $\mathcal{T}$ are signs of $\mathcal{T}'$, and in which the schemas of $\mathcal{T}$ are schemas of $\mathcal{T}'$. Then, if A is a theorem of $\mathcal{T}$, $(T_1 | a_1) \ldots (T_k | a_k)A$ is a theorem of $\mathcal{T}'$.

In effect, $\mathcal{T}'$ is stronger than the theory $(T_1 | a_1)(T_2 | a_2) \ldots (T_k | a_k)\mathcal{T}$ and the criterion follows by application of C2 and C4.

When one deduces, by the preceding criterion, a theorem of $\mathcal{T}'$ from a theorem of $\mathcal{T}$, one says that one has applied in $\mathcal{T}'$, the results of $\mathcal{T}$. Intuitively, the axioms of $\mathcal{T}$ express properties of $a_1, \ldots, a_k$

and A express a property which is a consequence of these axioms. If the objects $T_1, \ldots, T_k$ possess in $\mathcal{T}'$ the properties expressed by the axioms of $\mathcal{T}$, they also possess the property A.

Note that under the hypothesis of C5, if the theory $\mathcal{T}$ involves a contradiction it is the same for $\mathcal{T}'$. For, in effect, if A and « not A » are theorems of $\mathcal{T}$, $(T_1 | a_1) \ldots (T_k | a_k)A$, and not $(T_1 | a_1)(T_2 | a_2) \ldots (T_k | a_k)A$ are theorems of $\mathcal{T}'$.

We have introduced the preceding five criteria because they are applicable to any theory $\mathcal{T}$ whatever. We have presented their meta-proofs also in full to illustrate the general methods whereby we establish all of such criteria. Hereafter, we shall limit our attention to particular theories, which will be supposed to contain certain particular schema. It will be made clear which particular theory we are referring to at any given moment. In general when we present certain criteria which are consequences of certain axioms or schema, we shall not give the appropriate meta-proofs, all of them being established by methods similar to those which justify C1 - C5.

3. LOGICAL THEORIES

We call a logical theory any theory $\mathcal{T}$ in which the schemas S1 to S4 together furnish implicit axioms.

S1. If A is a relation of $\mathcal{T}$, the relation (A or A) $\Rightarrow$ A is an axiom of $\mathcal{T}$.

S2. If A and B are relations of $\mathcal{T}$, the relation A $\Rightarrow$ (A or B) is an axiom of $\mathcal{T}$.

S3. <u>If A and B are relations of $\mathcal{T}$, the relation (A or B) $\Rightarrow$ (B or A) is an axiom of $\mathcal{T}$</u>.

S4. <u>If A, B and C are relations of $\mathcal{T}$, the relation (A $\Rightarrow$ B) $\Rightarrow$ ((C or A) $\Rightarrow$ (C or B)) is an axiom of $\mathcal{T}$</u>.

These four rules, which are in effect the Russell-Whitehead principals of tautology, addition, permutation and summation, respectively (cf. Russell-Whitehead 13, p. 96), merely serve to give a formal explication of the sense which we wish to attach to the words « or » and «implies » in ordinary mathematical usage. The theory $\mathcal{T}$ which has these four schema as its schema and no explicit axioms and only the two logical signs « $\vee$ » and « $\neg$ » is often called <u>the propositional calculus</u>.

We should keep in mind the fact that if a logical theory $\mathcal{T}$ should prove contradictory, then every relation of $\mathcal{T}$ is a theorem of $\mathcal{T}$.

In all that follows, $\mathcal{T}$ will designate a logical theory.

C6. <u>Let A, B, C be relations of $\mathcal{T}$. If A $\Rightarrow$ B and B $\Rightarrow$ C are theorems of $\mathcal{T}$, A $\Rightarrow$ C is a theorem of $\mathcal{T}$</u>.

C7. <u>If A and B are relations of $\mathcal{T}$, B $\Rightarrow$ (A or B) is a theorem of $\mathcal{T}$</u>.

C8. <u>If A is a relation $\mathcal{T}$, A $\Rightarrow$ A is a theorem of $\mathcal{T}$</u>.

C9. <u>If A is a relation, and B a theorem of $\mathcal{T}$, A $\Rightarrow$ B is a theorem of $\mathcal{T}$</u>.

C10. <u>If A is a relation of $\mathcal{T}$, « A or (not A) » is a theorem of $\mathcal{T}$</u>.

C11. <u>If A is a relation of $\mathcal{T}$, « A $\Rightarrow$ (not not A) » is a theorem of $\mathcal{T}$.</u>

C12. <u>Let A and B be two relations of $\mathcal{T}$. The relation</u>

<u>(A $\Rightarrow$ B) $\Rightarrow$ ((not B) $\Rightarrow$ (not A))</u>

<u>is a theorem of $\mathcal{T}$.</u>

C13. <u>Let A, B, C be relations of $\mathcal{T}$. If A $\Rightarrow$ B is a theorem of $\mathcal{T}$, (B $\Rightarrow$ C) $\Rightarrow$ (A $\Rightarrow$ C) is a theorem of $\mathcal{T}$.</u>

C14. (Criterion of deduction). <u>Let A be a relation of $\mathcal{T}$, and $\mathcal{T}'$ be the theory obtained on adjoining A to the axioms of $\mathcal{T}$. If B is a theorem of $\mathcal{T}'$, then A $\Rightarrow$ B is a theorem of $\mathcal{T}$.</u>

<u>Remark</u>. In practice, one indicates that one is applying this criterion by a phrase of the following genere: « Suppose that A be true ». This phrase signifies that one is reasoning in the theory $\mathcal{T}'$. One remains in $\mathcal{T}'$ long enough to prove the relation B. This done, it is established that A $\Rightarrow$ B is a theorem of $\mathcal{T}$ and one then continues to reason in $\mathcal{T}$ without indicating the abandonment of $\mathcal{T}'$. The relation A that one has introduced as a new axiom is called the <u>auxiliary hypothesis</u> and the method of reasoning resting on C14 is called the <u>method of the auxiliary hypothesis</u>.

C15. <u>Let A be a relation of $\mathcal{T}$, and $\mathcal{T}'$ be the theory obtained on adjoining the axiom « not A » to the axioms of $\mathcal{T}$. If $\mathcal{T}'$ is contradictory, A is a theorem of $\mathcal{T}$.</u>

<u>Remark</u>. In practice, one indicates that one is employing this criterion by a phrase of the following genere: « Suppose that A be

false ». This phrase signifies that one is reasoning for the moment in $\mathcal{T}'$. One remains in $\mathcal{T}'$ long enough to establish two theorems of the form B and « not B ». This done, it is established that A is a theorem of $\mathcal{T}$, which one indicates in general by a phrase of the following genere: « But this (meaning Band « not B ») is absurd; thus A is true » . One then resumes reasoning in $\mathcal{T}$ as before. This general method of proof is called reductio ad absurdum.

C16. If A is a relation of $\mathcal{T}$, (not not A) $\Rightarrow$ A is a theorem of $\mathcal{T}$.

C17. If A and B are relations of $\mathcal{T}$,

$$((\text{not } B) \Rightarrow (\text{not } A)) \Rightarrow (A \Rightarrow B)$$

is a theorem of $\mathcal{T}$.

C18. Let A, B, C be relations of $\mathcal{T}$. If « A or B » , A $\Rightarrow$ C, and B $\Rightarrow$ C are theorems of $\mathcal{T}$, then C is a theorem of $\mathcal{T}$.

Remark. In order to prove C, it thus suffices when one has at one's disposal a theorem « A or B » , to prove C on adjoining B to the axioms of $\mathcal{T}$. The general method of proof which hangs on this criterion is called the method of case disjunction.

C19. Let x be a letter, A and B relations of $\mathcal{T}$ such that:

1. The letter x is not a constant of $\mathcal{T}$ and does not figure in B.
2. One has a term T of $\mathcal{T}$ such that $(T \mid x)A$ is a theorem of $\mathcal{T}$.

Let $\mathcal{T}'$ be the theory obtained on adjoining A to the axioms of $\mathcal{T}$. If B is a theorem of $\mathcal{T}'$, then B is a theorem of $\mathcal{T}$.

Intuitively, the method consists of the utilization, in order to prove B, of an arbitrary object x (called an auxiliary constant) which one supposes to be invested with certain properties which are expressed by A. It is evident that before one can make use of such and object, one must insure oneself of the existence of such objects. The theorem $(T \mid x)A$ guarantees this existence and is called the theorem of legitimation. In practice, one indicates the employment of this criterion by a phrase of the following genera: « let x be an object such that A ». The conclusion of the reasoning of course does not depend on x, as in the method of auxiliary hypothesis. The general method of proof which rests on C19 is called the method of the auxiliary constant.

Before we proceed further we make the following definitions of conjunction and equivalence. As with all such definitions, we have as an immediate result a formative criterion and a substitution criterion, which we shall present as usual without their immediate meta-proofs.

Definition 1. - Let A and B be assemblages. The assemblage

(not ((not A) or (not B))

will be designated by « A and B ».

CS6. Let A, B, and T be assemblages, x a letter. The assemblage $(T \mid x)$(A and B) is identical to « $(T \mid x)$A and $(T \mid x)$B ».

CF9. If A and B are relations of $\mathcal{T}$, « A and B » is a relation (called the conjunction of A and B).

C20. If A and B are theorems of $\mathcal{T}$, « A and B » is a theorem of $\mathcal{T}$.

C21. <u>If A, B are relations of $\mathcal{T}$, (A and B) $\Rightarrow$ A, (A and B) $\Rightarrow$ B are theorems of $\mathcal{T}$</u>.

<u>Definition 2</u>. - Let A and B be assemblages. The assemblage

(A $\Rightarrow$ B) and (B $\Rightarrow$ A)

will be designated by A $\Leftrightarrow$ B.

CS7. <u>Let A, B, and T be assemblages, x a letter. The assemblage (T| x)(A $\Leftrightarrow$ B) is identical to (T| x)A $\Leftrightarrow$ (T| x)B</u>.

CF10. <u>If A and B are relations of $\mathcal{T}$, A $\Leftrightarrow$ B is a relation of $\mathcal{T}$</u>.

If A and B are theorems of $\mathcal{T}$, one says that A and B are <u>equivalent</u> in $\mathcal{T}$ and if considered as relations in x, every term which verifies A also verifies B and vice versa.

C22. <u>Let A, B, and C be relations of $\mathcal{T}$. If A $\Leftrightarrow$ B is a theorem of $\mathcal{T}$, B $\Leftrightarrow$ A is a theorem of $\mathcal{T}$. If A $\Leftrightarrow$ B and B $\Leftrightarrow$ C are theorems of $\mathcal{T}$, A $\Leftrightarrow$ C is a theorem of $\mathcal{T}$</u>.

C23. <u>Let A and B be equivalent relations in $\mathcal{T}$, and C a relation of $\mathcal{T}$. Then, one has in $\mathcal{T}$ the following theorems</u>:

<u>(not A) $\Leftrightarrow$ (not B); (A $\Rightarrow$ C) $\Leftrightarrow$ (B $\Rightarrow$ C); (C $\Rightarrow$ A) $\Leftrightarrow$ (C $\Rightarrow$ B)</u>;

<u>(A and C) $\Leftrightarrow$ (Band C); (A or C) $\Leftrightarrow$ (B or C)</u>.

C24. <u>Let A, B, and C be relations of $\mathcal{T}$; one has in $\mathcal{T}$ the following theorems</u>:

(not not A) $\Leftrightarrow$ A; (A $\Rightarrow$ B) $\Leftrightarrow$ ((not B) $\Rightarrow$ (not A));

(A and A) $\Leftrightarrow$ A; (A and B) $\Leftrightarrow$ (B and A);

$(A \text{ and } (B \text{ and } C)) \Leftrightarrow ((A \text{ and } B) \text{ and } C))$;

$(A \text{ or } A) \Leftrightarrow A$; $(A \text{ or } B) \Leftrightarrow (B \text{ or } A)$;

$(A \text{ or } (B \text{ or } C)) \Leftrightarrow ((A \text{ or } B) \text{ or } C)$;

$(A \text{ and } (B \text{ or } C)) \Leftrightarrow ((A \text{ and } B) \text{ or } (A \text{ and } C))$;

$(A \text{ or } (B \text{ and } C)) \Leftrightarrow ((A \text{ or } B) \text{ and } (A \text{ or } C))$;

$(A \text{ and } (\text{not } B)) \Leftrightarrow \text{not } (A \Rightarrow B)$; $(A \text{ or } B) \Leftrightarrow ((\text{not } A) \Rightarrow B$.

C25. <u>If A is a theorem of $\mathcal{T}$ and B a relation of $\mathcal{T}$, $(A \text{ and } B) \Leftrightarrow B$ is a theorem of $\mathcal{T}$. If « not A » is a theorem of $\mathcal{T}$, $(A \text{ or } B) \Leftrightarrow B$ is a theorem of $\mathcal{T}$</u>.

4. QUANTIFIED THEORIES

So far we have made no use of the logical signs other than $\neg$ and $\vee$. We shall now develop the use of the only two remaining logical signs τ and $\square$.

<u>Definition 1</u>. - If R is an assemblage, and x a letter, the assemblage $(\tau_x(R) | x)R$ will be designated by « there exists an x such that R » or by $(\exists x)R$. The assemblage not $((\exists x)(\text{not } R))$ will be designated by « for all x, R » or by « whatever be x, R », or $(\forall x)R$. The abbreviated symbols $\exists$ and $\forall$ will be called the <u>existential</u> and <u>universal quantifiers</u>, respectively.

Since the letter x does not figure in the assemblage designated by $\tau_x(R)$; it thus <u>does not figure</u> in the assemblages designated by $(\exists x)R$ and $(\forall x)R$. It is thus that we see the usefulness of the rules governing the employment of τ and $\square$. This usage has the effect of binding free variables (letters) by effectively eliminating them from the corresponding assemblages.

CS8. Let R be an assemblage, x and x' letters. If x' does not figure in R, $(\exists x)R$ and $(\forall x)R$ are identical respectively to $(\exists x')R'$ and $(\forall x')R'$, where R' is $(x' | x)R$.

CS9. Let R and U be assemblages, x and y distinct letters. If x does not figure in U, $(U | y)(\exists x)R$ and $(U | y)(\forall x)R$ are identical respectively to $(\exists x)R'$ and $(\forall x)R'$, where R' is $(U | y)R$.

CF17. If R is a relation of a theory $\mathcal{T}$ and x a letter, $(\exists x)R$ and $(\forall x)R$ are relations of $\mathcal{T}$.

Intuitively, let us consider R as expressing a property of an object designated by x. By the intuitive signification of the term $\tau_x(R)$, to affirm $(\exists x)R$ amounts to saying that this is an object passing the property R. To affirm « not $(\exists x)(\text{not } R)$ » is to say that there are no objects with the property « not R », thus to say that every object possess the property R.

If in a logical theory $\mathcal{T}$, one has at one's disposal a theorem of the form $(\exists x)R$, where the letter x is not a constant of $\mathcal{T}$, this theorem may serve as the theorem of legitimation in the method of the auxiliary constant since it is identical to $(\tau_x(R) | x)R$ and thus $\tau_x(R)$ is the desired term T.

C26. Let $\mathcal{T}$ be a logical theory, R a relation of $\mathcal{T}$ and x a letter. The relations $(\forall x)R$ and $(\tau_x(\text{not } R) | x)R$ are equivalent in $\mathcal{T}$.

C27. If R is a theorem of a logical theory $\mathcal{T}$ in which the letter x is not a constant, $(\forall x)R$ is a theorem of $\mathcal{T}$.

C28. *Let τ be a logical theory, R a relation of τ and x a letter. The relations « not $(\forall x)R$ » and $(\exists x)(\text{not } R)$ are equivalent in τ.*

A theory   will be said to be *quantified* if the schemas S1 - S4 together with the schema S5 are among the schemas of τ. Often the theory τ which has the logical signs $\vee$, $\neg$, τ, and $\square$ together with just the schemas S1 through S5 is called the *first order functional calculus (without equality)*.

S5. *If R is a relation of τ, T a term of τ, and x a letter, the relation $(T \mid x)R \Rightarrow (\exists x)R$ is an axiom.*

Intuitively the above schema expresses that, if one has an object T for which the relation R, considered as expressing a property of x, is true, then R is true for the object $\tau_x(R)$, which is, of course, in accord with the intuitive signification of $\tau_x(R)$. It is clear also that $\tau_x(R)$ is just a version of Hilbert's « ε-operator » and that the above axiom-schema is just Hilbert's axiom for the ε-operator. Thus τ acts intuitively as a kind of single « selection operator » which may be used to represent a chosen object which satisfies the relation R (if such exists). It should be noted that its use gives no information about the particular object selected by the operator. For example, we know that $\tau_x(x = 1 \text{ or } x = 2 \text{ or } x = 3)$ must be 1, 2, or 3, but we have no means of determining which one of 1, 2, or 3, gets selected. It might also be noted in passing that many objections have been raised to the use of such an operator, most of which are similar to those which have been leveled against the « axiom of choice ». However, the use of the « τ-operator » as we have presented it here does not by itself make such an « axiom of choice » derivable in our system. The axiom of choice is derivable in our theory of sets, as we shall see, but this derivation is possible only through the use of the schema S8, which we present much later and not solely due to the presence of the « τ-operator » in our « underlying logic ». It's presence here does make our underlying logic of the « non-standard » variety however . (cf. Fraenkel 58, Section 77, p. 182 et seq. and Carnap 61 p.156 et. seq.)

From now on $\mathcal{T}$ will designate a quantified theory.

C29. Let R be a relation of $\mathcal{T}$, and x a letter. The relations « not $(\exists x)$R » and $(\forall x)$(not R) are equivalent in $\mathcal{T}$.

C28 and C29 permit us to derive the properties of one of the quantifiers from those of the other.

C30. Let R be a relation of $\mathcal{T}$, T a term of $\mathcal{T}$, x a letter. The relation $(\forall x)R \Rightarrow (T \mid x)R$ is a theorem of $\mathcal{T}$.

Let R be a relation of $\mathcal{T}$, by C26, C27, and C30, it amounts to the same (when x is not a constant of $\mathcal{T}$) to enunciate in $\mathcal{T}$ the theorem R, or the theorem $(\forall x)$R, or finally to give the metamathematical rule: if T is an arbitrary term of $\mathcal{T}$, $(T \mid x)R$ is a theorem of $\mathcal{T}$.

C31. Let R and S be relations of $\mathcal{T}$, and x a letter which is not a constant of $\mathcal{T}$. If $R \Rightarrow S$ (resp. $R \Leftrightarrow S$) is a theorem of $\mathcal{T}$, $(\forall x)R \Rightarrow (\forall x)S$ and $(\exists x)R \Rightarrow (\exists x)S$ (resp. $(\forall x)R \Leftrightarrow (\forall x)S$ and $(\exists x)R \Leftrightarrow (\exists x)S$) are theorems of $\mathcal{T}$.

C32. Let R and S be relations of $\mathcal{T}$, and x a letter. The relations

$$(\forall x)(R \text{ and } S) \Leftrightarrow ((\forall x)R \text{ and } (\forall x)S)$$

$$(\exists x)(R \text{ or } S) \Leftrightarrow ((\exists x)R \text{ or } (\exists x)S)$$

are theorems of $\mathcal{T}$.

C33. Let R and S be relations of $\mathcal{T}$, and x a letter which does not figure in R. The relations

$$(\forall x)(R \text{ or } S) \Leftrightarrow (R \text{ or } (\forall x)S)$$

$$(\exists x)(R \text{ and } S) \Leftrightarrow (R \text{ and } (\exists x)S)$$

are theorems of $\mathcal{T}$.

C34. Let R be a relation, x and y letters. The relations

$$(\forall x)(\forall y)R \Leftrightarrow (\forall y)(\forall x)R$$

$$(\exists x)(\exists y)R \Leftrightarrow (\exists y)(\exists x)R$$

$$(\exists x)(\forall y)R \Rightarrow (\forall y)(\exists x)R$$

By constrast, if $(\forall y)(\exists x)R$ is a theorem of $\mathcal{T}$, one may not conclude that $(\exists x)(\forall y)R$ is a theorem of $\mathcal{T}$. Intuitively to say that the relation $(\forall y)(\exists x)R$ is true signifies that being given an arbitrary object y, there is an object x such that R is a true relation between the objects x and y. But the object x in general will depend on the choice of the object y. To the contrary, to say that $(\exists x)(\forall y)R$ is true signifies that there is a fixed object x such that R is a true relation between this fixed object and every object y.

The definitions which follow are not strictly necessary but are highly useful because of the fact that most of the usual mathematical reasoning involving quantifiers is actually of the type which is embodied in the criteria which follow from these definitions.

Definition 2. - Let A and R be assemblages, and x a letter. We designate the assemblage $(\exists x)(A \text{ and } R)$ by $(\exists_A x)R$, and the assemblage « not $(\exists_A x)(\text{not } R)$ » by $(\forall_A x)R$. Read respectively « there exists an x of the type A such that R » and « for all x of the type A, R ». The abbreviated symbols $\exists_A$ and $\forall_A$ are called typical quantifiers. The letter x of course does not appear in either of these assemblages.

CS10. Let A and R be assemblages, x and x' letters. If x' figures neither in R nor in A, $(\exists_A x)R$ and $(\forall_A x)R$ are identical respectively to $(\exists_{A'} x')R'$ and $(\forall_{A'} x')R'$, where R' is $(x' \mid x)R$, and where A' is $(x' \mid x)A$.

CS11. Let A, R, and U be assemblages, x and y distinct letters. If x does not figure in U, the assemblages $(U \mid y)(\exists_A x)R$ and $(U \mid y)(\forall_A x)R$ are identical respectively to $(\exists_{A'} x)R'$ and $(\forall_{A'} x)R'$ where R' is $(U \mid y)R$ and where A' is $(U \mid y)A$.

CF12. Let A and R be relations of $\mathcal{T}$, and x a letter. Then $(\forall_A x)R$ and $(\exists_A x)R$ are relations of $\mathcal{T}$.

C35. Let A and R be relations of $\mathcal{T}$, x a letter. The relations $(\forall_A x)R$ and $(\forall x)(A \Rightarrow R)$ are equivalent in $\mathcal{T}$.

C36. Let A and R be relations of $\mathcal{T}$, and x a letter. Let $\mathcal{T}'$ be the theory obtained on adjoining A to the axioms of $\mathcal{T}$. If x is not a constant of $\mathcal{T}$, and if R is a theorem of $\mathcal{T}'$, $(\forall_A x)R$ is a theorem of $\mathcal{T}$.

In practice, one indicates the employment of this criterion by a phrase of the following genere: « Let x be an arbitrary object such that A » . In the theory $\mathcal{T}'$ thus constituted, one seeks to prove R. One may not naturally affirm that the relation R is itself a theorem of $\mathcal{T}$, of course.

C37. Let A and R be relations of $\mathcal{T}$, x a letter. Let $\mathcal{T}'$ be the theory obtained on adjoining to the axioms of $\mathcal{T}$ the relations A and «not R». If x is not a constant of $\mathcal{T}$, and if $\mathcal{T}'$ is contradictory

$(\forall_A x)R$ is a theorem of $\mathcal{T}$.

In practice one says: « Suppose that there exists an object x verifying A, for which R be false. » One then seeks to establish a contradiction.

The usefulness of typical quantification comes from the fact that the properties of typical quantifiers are analogous to those of quantifiers.

C38. Let A and R be relations of $\mathcal{T}$, x a letter. The relations $\text{not } (\forall_A x)R \Leftrightarrow (\exists_A x)(\text{not } R)$, $\text{not } (\exists_A x)R \Leftrightarrow (\forall_A x)(\text{not } R)$ are theorems of $\mathcal{T}$.

C39. Let A, R, and S be relations of $\mathcal{T}$, and x a letter which is not a constant of $\mathcal{T}$. If the relation $A \Rightarrow (R \Rightarrow S)$ (resp. $A \Rightarrow (R \Leftrightarrow S)$) is a theorem of $\mathcal{T}$, the relations

$$(\exists_A x)R \Rightarrow (\exists_A x)S, \quad (\forall_A x)R \Rightarrow (\forall_A x)S$$

$$(\text{resp. } (\exists_A x)R \Leftrightarrow (\exists_A x)S, \quad (\forall_A x)R \Leftrightarrow (\forall_A x)S)$$

are theorems of $\mathcal{T}$.

C40. Let A, R, and S be relations of $\mathcal{T}$, and x a letter. The relations

$$(\forall_A x)(R \text{ and } S) \Leftrightarrow ((\forall_A x)R \text{ and } (\forall_A x)S)$$

$$(\exists_A x)(R \text{ or } S) \Leftrightarrow ((\exists_A x)R \text{ or } (\exists_A x)S)$$

are theorems of $\mathcal{T}$.

C41. Let A, R, and S be relations of $\mathcal{T}$, and x a letter which does not figure in R. The relations

$$(\forall_A x)(R \text{ or } S) \Leftrightarrow (R \text{ or } (\forall_A x)S)$$

$$(\exists_A x)(R \text{ and } S) \Leftrightarrow (R \text{ and } (\exists_A x)S)$$

are theorems of $\mathcal{T}$.

C42. Let A, B, R be relations of $\mathcal{T}$, x and y letters. If x does not figure in B, and if y does not figure in A, the relations

$$(\forall_A x)(\forall_B y)R \Leftrightarrow (\forall_B y)(\forall_A x)R$$

$$(\exists_A x)(\exists_B y)R \Leftrightarrow (\exists_B y)(\exists_A x)R$$

$$(\exists_A x)(\forall_B y)R \Rightarrow (\forall_B y)(\exists_A x)R$$

are theorems of $\mathcal{T}$.

5. EQUALITY THEORIES

We call an equality theory a theory $\mathcal{T}$ in which figures a relational sign of weight 2 denoted = (which we read « equals »), and in which the schemas S1 through S5 together with the schemas S6 and S7 furnish implicit axioms; if T and U are terms of $\mathcal{T}$, the assemblage $=TU$ is a relation of $\mathcal{T}$ (called the relation of equality) by CF4; we designate it in practice by $T = U$ or $(T) = (U)$. The theory which has solely the relational sign = (in addition to the logical signs) and has only the schemas S1 - S7 and no explicit axioms is often called the first order functional calculus with equality.

S6. Let x be a letter, T and U terms of $\mathcal{T}$, and $R\{x\}$ a relation of $\mathcal{T}$; the relation $(T = U) \Rightarrow (R\{T\} \Leftrightarrow R\{U\})$ is an axiom of $\mathcal{T}$.

S7. If R and S are relations of $\mathcal{T}$ and x is a letter, the relation $((\forall x)(R \Leftrightarrow S)) \Rightarrow (\tau_x(R) = \tau_x(S))$ is an axiom.

Intuitively, the schema S6 signifies that if two objects are equal, then they have the same properties. The schema S7 is an extension of our usual intuition. It signifies that, when two properties of an object x are equivalent, then the selected objects $\tau_x(R)$ and

$\tau_x(S)$ (selected from the objects which verify R and those which verify S, if such exist) are thesame. The schema is often Ackermann's axiom for the ε-operator, rephrased for our operator τ. The presence of the quantifier $(\forall x)$ is essential here, otherwise we can obtain the theorem $(\forall x)(x = y)$ which is certainly not to be desired as for example in the theory of sets we will have the theorem $(\exists x)(\exists y)(x \neq y)$.

C43. <u>Let x be a letter, T and U terms of $\mathcal{T}$, and $R\{x\}$ a relation of $\mathcal{T}$; the relations (T = U and $R\{T\}$) and (T = U and $R\{U\}$) are equivalent</u>.

The following theorems hold in any theory $\mathcal{T}_0$ which has the same signs as an equality theory but only the schemas S1 - S7.

<u>Theorem 1</u>. - $x = x$.

<u>Theorem 2</u>. - $(x = y) \Leftrightarrow (y = x)$.

<u>Theorem 3</u>. - $((x = y) \text{ and } (y = z)) \Rightarrow (x = z)$.

C44. <u>Let x be a letter, T, U, $V\{x\}$ be terms of $\mathcal{T}_0$. The relation $(T = U) \Rightarrow (V\{T\} = V\{U\})$ is a theorem of $\mathcal{T}_0$</u>.

One says that a relation of the form T = U, where T and U are terms of $\mathcal{T}$, is an <u>equation</u>; a <u>solution</u> (in $\mathcal{T}$) of the <u>relation</u> T = U considered as an equation in a letter x, is thus a term V of $\mathcal{T}$ such that $T\{V\} = U\{V\}$ is a theorem of $\mathcal{T}$ as is consistent with the previous definition of solution of a relation.

Let T and U be two terms of $\mathcal{T}$ and let $x_1, x_2, \ldots, x_n$ be the letters figuring in T and not in U. If the relation $(\exists x_1)(\exists x_2)\ldots(\exists x_n)(T = U)$ is a theorem of $\mathcal{T}$, one says that U <u>may be put in the</u>

form T (in $\mathcal{T}$). Let R be a relation of $\mathcal{T}$, y a letter. Let V be a solution (in $\mathcal{T}$) of R, considered as a relation in y. If every solution (in $\mathcal{T}$) of R, considered as a relation in y, may be put in the form V, one says that V is the *complete* (or *general*) solution of R (in $\mathcal{T}$).

Let R be an assemblage, x a letter, Let y and z be letters distinct from themselves, distinct from x and not figuring in R. Let y' and z' be two other letters with the same properties. By CS8, CS9, CS2, CS5, and CS6, the assemblages

$$(\forall y)(\forall z)(((y \mid x)R \text{ and } (z \mid x)R) \Rightarrow (y = z))$$

and

$$(\forall y')(\forall z')(((y' \mid x)R \text{ and } (z' \mid x)R) \Rightarrow (y' = z'))$$

are identical. If R is a relation of $\mathcal{T}$, the assemblage thus defined is a relation of $\mathcal{T}$, which will be designated by « there exists at most one x such that R ». The letter x does not figure in this assemblage. When this relation is a theorem of $\mathcal{T}$, one says that R is *unique* in x in $\mathcal{T}$.

C45. *Let R be a relation of $\mathcal{T}$, and x a letter which is not a constant of $\mathcal{T}$. If R is unique in x in $\mathcal{T}$, R $\Rightarrow (x = \tau_x(R))$ is a theorem of $\mathcal{T}$. Conversely, if, for a term T of $\mathcal{T}$ not containing x, R $\Rightarrow (x = T)$ is a theorem of $\mathcal{T}$, R is unique in x in $\mathcal{T}$.*

Let R be a relation of $\mathcal{T}$. The relation

« $(\exists x)$R and there exists at most one x such that R »

will be designated by « there exists one and only one x such that R ». If this relation is a theorem of $\mathcal{T}$, one says that R is a *functional relation in x in $\mathcal{T}$*.

C46. <u>Let R be a relation of $\mathcal{T}$, and x a letter which is not a constant of $\mathcal{T}$. If R is functional in x in $\mathcal{T}$, R $\Leftrightarrow$ (x = τ_x(R)) is a theorem of $\mathcal{T}$. Conversely, if, for a term T of $\mathcal{T}$ not containing x, R $\Leftrightarrow$ (x = T) is a theorem of $\mathcal{T}$, R is functional in x in $\mathcal{T}$</u> .

When a relation R is functional in x in $\mathcal{T}$, R is thus equivalent to the relation, often more manageable, x = τ_x(R). Thus one generally introduces an abbreviated symbol Σ to represent the term τ_x(R). Such a symbol is called a <u>functional symbol</u> in $\mathcal{T}$. Intuitively Σ will represent the unique object which possess the property defined by R. *For example in a theory where « y is a real number $\geqslant$ o » is a theorem, the relation « x is a real number $\geqslant$ o and y = x^2 » is functional in x, we take as corresponding functional symbol $\sqrt{y}$ or $y^{\frac{1}{2}}$.*

C47. <u>Let x be a letter which is not a constant of $\mathcal{T}$, and let R $\{x\}$ and S $\{x\}$ be two relations of $\mathcal{T}$. If R $\{x\}$ is functional in x in $\mathcal{T}$, the relation S $\{\tau_x(R)\}$ is equivalent to ($\exists$ x)(R $\{x\}$ and S $\{x\}$)</u>.

PART 2

ELEMENTARY SET THEORY

1. THE THEORY OF SETS

The <u>theory of sets</u> is a theory in which figure the relational signs $=$, $\in$, and the substantive sign $\supset$ (all of which are to be of weight 2). It contains the schemas S1 - S8 and the explicit axioms A_1 - A_5. These explicit axioms, as will be seen, contain no letters, thus the theory of sets has <u>no constants</u>. Thus the theory of sets is an equality theory and all of our previous results are applicable in it.

From now on, unless we expressly mention the contrary, all of our reasoning will be assumed to take place in a theory stronger than the theory of sets and may thus be assumed to be the theory of sets itself. It will be apparent, from the sequential development which follows, which particular theory weaker than the theory of sets in which the reasoning necessarily takes place.

If T and U are terms, the assemblage $\in$ TU is a relation (called the <u>relation of membership</u>) which we shall in practice denote in one of the following manners: $T \in U$, $(T) \in (U)$, « T belongs to U » , « T is a member of U » ,

<u>etc</u>. The negation will be denoted by $T \notin U$.

From the naive point of view, much of mathematics may be considered as collections or « sets » of objects. We shall not formalize this notion, and in the formalist interpretation which follows the word « set » may be considered as strictly synonymous with « term of the theory of sets » ; in particular, such phrases as « let x be a set » are in principal, totally superflous; since every letter is a term. Such phrases will be introduced solely to facilitate the intuitive interpretation of the text.

Definition 1. - The relation designated by $(\forall z)((z \in x) \Rightarrow (z \in y))$, in which only the letters x and y figure, will be denoted by $x \subseteq y$, $y \supseteq x$, « x is contained in y » , « x is a subset of y » , etc.

CS12. Let T, U, and V be assemblages, and x a letter. The assemblage $(V | x)(T \subseteq U)$ is identical to $(V | x)T \subseteq (V | x)U$.

CF13. If T and U are terms, $T \subseteq U$ is a relation (called the relation of inclusion).

From now on we will not explicitly state the substitution, and formative criteria which result from the definitions.

Proposition 1 - $x \subseteq x$

Proposition 2 - $(x \subseteq y \text{ and } y \subseteq z) \Rightarrow (x \subseteq z)$.

The following axiom is called the axiom of extensionality:

A1. $(\forall x)(\forall y)((x \subseteq y \text{ and } y \subseteq x) \Rightarrow (x = y))$.

Intuitively, this axiom expresses that two sets with the same elements are equal.

C48. Let R be a relation, x a letter, y a letter distinct from x and not figuring in R. The relation $(\forall x)((x \in y) \Leftrightarrow R)$ is unique in y.

Let R be a relation, x a letter. If y and y' designate letters distinct from x and <u>not figuring</u> in R, the relations $(\exists y)(\forall x)((x \in y) \Leftrightarrow R)$ and $(\exists y')(\forall x)((x \in y') \Leftrightarrow R)$ are identical by CS8. The relation thus defined will be designated by $\mathrm{Coll}_x(R)$.

When $\mathrm{Coll}_x(R)$ is a theorem of a theory $\mathcal{T}$, one says that R is <u>collective in</u> x in $\mathcal{T}$. If this is the case, one may introduce an auxiliary constant a, distinct from x, from the constants of $\mathcal{T}$, and not figuring in R, with the axiom of introduction $(\forall x)((x \in a) \Leftrightarrow R)$, or, which amounts to the same if x is not a constant of $\mathcal{T}$, $(x \in a) \Leftrightarrow R$. Intuitively, to say that R is collective in x is to say that there exists a set a such that the objects x possessing the property R are precisely the elements of a.

Example 1. - The relation $x \in y$ is evidently collective in x.

Example 2. - The relation $x \notin x$ is not collective in x; i.e., (not $\mathrm{Coll}_x(x \notin x)$) is a theorem. Reasoning by <u>reductio ad absurdum</u>: assume that x   x is collective. Let a be an auxiliary constant, distinct from x and from the constants of the theory, with the axiom of introduction $(\forall x)((x \notin x) \Leftrightarrow (x \in a))$. Then the relation $(a \notin a) \Leftrightarrow (a \in a)$ is true by C30. The method of case disjunction proves at first that $a \notin a$ is true, since the relation $a \in a$ is true, which is absurd. It is by this simple technique that <u>Russell's paradox</u> is eliminated in this set theory.

C49. <u>Let R be a relation and x a letter. If R is collective in x, the relation $(\forall x)((x \in y) \Leftrightarrow R)$, where y is a letter distinct from x and not figuring in R, is functional in y.</u>

Very frequently, in what follows, we dispose of a theorem of the form $\mathrm{Coll}_x(R)$. We then introduce to represent the term $y(\forall x)((x \in y) \Leftrightarrow R)$, which does not depend on the choice of the letter y (distinct from x and not figuring in R) a functional symbol; in what follows, we utilise the symbol $\mathcal{E}_x(R)$ or $\{x \mid R\}$; the corresponding term does not contain the letter x. It is this term that we mean when we speak of « the set of all x such that R » . Then by definition the relation $(\forall x)((x \in \mathcal{E}_x(R)) \Leftrightarrow R)$ is identical to $\mathrm{Coll}_x(R)$; consequently the relation R is thus equivalent to $x \in \mathcal{E}_x(R)$.

C50. Let R and S be two relations and x a letter. If R and S are collective in x, the relation $(\forall x)(R \Rightarrow S)$ is equivalent to $\mathcal{E}_x(R) \subset \mathcal{E}_x(S)$; the relation $(\forall x)(R \Leftrightarrow S)$ is equivalent to $\mathcal{E}_x(R) = \mathcal{E}_x(S)$.

The following axiom is called the axiom of pairing:

A2. $(\forall x)(\forall y)\mathrm{Coll}_z(z = x \text{ or } z = y)$.

This axiom expresses that, if x and y are objects, there exists a set whose only elements are x and y.

Definition 2. - The set $\mathcal{E}_z(z = x \text{ or } z = y)$, whose only elements are x and y will be denoted by $\{x,y\}$.

The set $\{x,x\}$ will be designated simply by $\{x\}$, and will be called the set whose only element is x.

The following schema is called the schema of selection and union:

S8. Let R be a relation, x and y distinct letters, X and Y distinct letters distinct from x and y and not figuring in R. The relation

$$(\forall y)(\exists X)(\forall x)(R \Rightarrow (x \in X)) \Rightarrow (\forall Y)\mathrm{Coll}_x((\exists y)((y \in Y) \text{ and } R))$$

is an axiom.

Intuitively, the relation $(\forall y)(\exists X)(\forall x)(R \Rightarrow (x \in X))$ signifies that, for every object y, there exists a set X((which may depend on y), such that the objects x which are in the relation R with the given object y are the elements of X (without necessarily constituting all of the set X). The schema affirms that, if this is the case, and if Y is an arbitrary set, there exists a set whose elements are exactly all of the objects x which find themselves in the relation R with an object y out of the set Y.

C51. Let P be a relation, A a set, and x a letter not figuring in A. The relation « P and $x \in A$ » is collective in x.

The set $\mathcal{E}_x(P \text{ and } x \in A)$ is called the set of $x \in A$ such that P.

C52. Let R be a relation, A a set, x a letter not figuring in A. If the relation $R \Rightarrow (x \in A)$ is a theorem, then R is collective in x.

C53. Let T be a term A a set, x and y distinct letters. Suppose that x does not figure in A and that y figures neither in T nor in A. The relation $(\exists x)(y = T \text{ and } x \in A)$ is collective in y.

The relation $(\exists x)(y = T \text{ and } x \in A)$ will be read as « y may be put in the form T for an x belonging to A ». The set $\mathcal{E}_y((\exists x)(y = T \text{ and } x \in A))$ is generally called the set of objects of the form T for $x \in A$.

By C51, the relation ($x \notin A$ and $x \in X$) is collective in x.

Definition 3. - Let A be a subset of a set X. The set $\mathcal{E}_x(x \notin A \text{ and } x \in X)$ is called the complement of A with respect to X and is designated by $C_x A$ or $X - A$ or CA.

Theorem 1. - The relation $(\forall x)(x \notin X)$ is functional in X.

The term $\tau_X((\forall x)(x \notin X)$ corresponding to this functional relation will be represented by the functional symbol $\emptyset$, and will be called the void or empty set. (The term designated by $\emptyset$ is thus $\tau \neg\neg \in \tau \neg\neg \in \square\square\square$) The relation $(\forall x)(x \notin X)$, is then equivalent to $X = \emptyset$, which is read « the set X is empty ». We have as theorems $x \notin \emptyset$, $\emptyset \subseteq X$, $C_x X = \emptyset$, $C_x \emptyset = X$. Also if $R\{x\}$ is a relation, the relation $(\forall x)((x \in \emptyset) \Rightarrow R\{x\})$ is true. Furthermore $\emptyset \neq \{x\}$ is a theorem and hence $(\exists x)(\exists y)(x \neq y)$ is also.

There does not exist a set all of whose objects are elements; i.e., « not $(\exists X)(\forall x)(x \in X)$ » is a theorem. For, in effect, if there existed such a set, every relation would be collective by C52. But, as we have seen the relation $x \notin x$ is not collective.

It is interesting to note that $(x = y) \Leftrightarrow (\forall X)((x \in X) \Leftrightarrow (y \in X))$ is a theorem.

As we have noted, the sign $\supset$ is in this theory a substantive sign of weight 2. If T, U are terms, $\supset TU$ is thus a term, which we will in practice designate by (T,U).

The axiom of ordered pairs (or of couples) is the following axiom:

A3. $(\forall x)(\forall x')(\forall y)(\forall y')(((x,y) = (x',y')) \Rightarrow (x = x' \text{ and } y = y'))$.

By C44, the relation $(x,y) = (x',y')$ is <u>equivalent</u> to « $x = x'$ and $y = y'$ ».

The relation $(\exists x)(\exists y)(z = (x,y))$ will be designated by « <u>z is an ordered pair</u> » or « <u>z is a couple</u> ». If z is an ordered pair, the relations $(\exists y)(z = (x,y))$ and $(\exists x)(z = (x,y))$ are functional in x and y respectively by A3. The terms $\tau_x((\exists y)(z = (x,y)))$ and $\tau_y((\exists x)(z = (x,y)))$ will be designated by pr_1z and pr_2z respectively, which will be called the <u>first coordinate</u> (or <u>first projection</u>) and <u>second coordinate</u> (or <u>second projection</u>) of z.

Let $R\{x,y\}$ be a relation, the letters x and y being distinct and figuring in R. Let z be a letter distinct from x and y and not figuring in R. Designate by $S\{z\}$ the relation $(\exists x)(\exists y)(z = (x,y)$ and $R\{x,y\})$; it is thus a relation which contains a letter not figuring in R, and which is <u>equivalent</u> to « z is an ordered pair and $R\{pr_1z, pr_2z\}$ ». $R\{x,y\}$ is equivalent to $S\{(x,y)\}$, and to $(\exists z)(z = (x,y)$ and $S\{z\})$. This means that a relation between the objects x and y may be interpreted as a property of the ordered pair formed by these objects.

<u>Theorem 2</u>. - The relation

$$(\forall X)(\forall Y)(\exists Z)(\forall z)((z \in Z) \Leftrightarrow (\exists x)(\exists y)(z = (x,y) \text{ and } x \in X \text{ and } y \in Y))$$

is true, i.e., whatever be X and Y, the relation « z is an ordered pair and $pr_1z \in X$ and $pr_2z \in Y$ » is collective in z.

<u>Definition 3</u>. - Being given two sets X and Y, the set $\mathcal{E}_z((\exists x)(\exists y)(z = (x,y)$ and $x \in X$ and $y \in Y))$ is called the <u>product of X and Y</u> and is designated by $X \times Y$.

The relation $z \in X \times Y$ is thus equivalent to « z is an ordered pair and $pr_1z \in X$ and $pr_2z \in Y$ ».

Proposition 3. - If A' and B' are two non-empty sets, the relation $A' \times B' \subseteq A \times B$ is equivalent to « $A' \subseteq A$ and $B' \subseteq B$ ».

Proposition 4. - Let A and B be two sets. The relation $A \times B = \emptyset$ is equivalent to « $A = \emptyset$ or $B = \emptyset$ ».

If A, B, and C are sets, one lets $(A \times B) \times C = A \times B \times C$. An element $((x,y),z)$ of $A \times B \times C$ (which is written also as (a,b,c)) is called a triplet. Similarly, one may define a multiplet $(x_1, x_2, \ldots, x_n)$.

The relation $\{\{x\}, \{x,y\}\} = \{\{x'\}, \{x',y'\}\}$ is equivalent to « $x = x'$ and $y = y'$ ». This is known as the Kuratowski definition of the ordered pair (x,y), i.e., $(x,y) = \{\{x\}, \{x,y\}\}$. If $\mathcal{T}_0$ is the theory of sets and $\mathcal{T}_1$ the theory with the same schemas and explicit axioms as $\mathcal{T}_0$, with the exception of the axiom A3, it can be shown, utilizing the Kuratowski definition of the ordered pair, that if $\mathcal{T}_1$ is not contradictory, then neither is $\mathcal{T}_0$. This gives a relative consistency proof for A3.

Definition 4. - G is said to be a graph iff every element of G is an ordered pair, i.e., if the relation $(\forall z)(z \in G \Rightarrow z$ is an ordered pair) is true.

If G is a graph, the relation $(x,y) \in G$ is expressed often by « y is corresponded to x by G ».

Let G be a letter distinct from x and y, x and y being distinct letters, and let $R\{x,y\}$ be a relation in which G does not figure. If the relation $(\exists G)(G$ is a graph and $(\forall x)(\forall y)(((x,y) \in G) \Leftrightarrow R)))$ is true one says that R admits a graph (with respect to the letters x and y). The graph G is unique b[y] the axiom of extensionality, and is called the graph of R with respect to x and y.

Proposition 5. - Let G be a graph. There exists a unique set A and a unique set B which possess the following properties:

1) the relation $(\exists y)((x,y) \in G)$ is equivalent to $x \in A$;

2) the relation $(\exists x)((x,y) \in G)$ is equivalent to $y \in B$.

The sets $A = \mathcal{E}_x((\exists y)((x,y) \in G))$ and $B = \mathcal{E}_y((\exists x)((x,y) \in G))$ are called the respective first and second projections of the graph G, or the set of definition and the set of values of G, and are designated by $pr_1 \langle G \rangle$ and $pr_2 \langle G \rangle$, respectively.

Remark. The relation $x = y$ does not admit a graph since if it did exist, its first projection would be the set of all objects, which we have noted does not exist.

Definition 5. - A triplet $\Gamma = (G,A,B)$, where A and B are sets and G is a graph such that $pr_1 \langle G \rangle \subseteq A$ and $pr_2 \langle G \rangle \subseteq B$ is said to be a correspondence between A and B. G is called the graph of Γ, A the set of departure and B the set of arrival) of Γ.

If $(x,y) \in G$, one says again that « y is corresponded to x by the correspondence Γ ». If $x \in pr_1\langle G \rangle$, one says that the correspondence Γ is defined for the object x, and $pr_1 \langle G \rangle$ is called the domain (or set) of definition of Γ; for $y \in pr_2 \langle G \rangle$, one says that y is a value taken by Γ and $pr_2\langle G \rangle$ is called the range (or set) of values of Γ.

If $R\{x,y\}$ is a relation admitting a graph G (wrt. x and y), and if A and B are two sets such that $pr_1 \langle G \rangle \subseteq A$ and $pr_2 \langle G \rangle \subseteq B$, one says that R is a relation between an element of A and an element of B (relative to x and y). One says that the correspondence $\Gamma = (G,A,B)$ is the correspondence between A and B defined by the relation R (wrt. x and y).

subset Y of B, the image $\overset{-1}{\Gamma}\langle Y\rangle$ of Y by $\overset{-1}{\Gamma}$ is again called the inverse image of Y by Γ .

Let G and G' be two graphs. Designate by A the set pr_1G and by C the set pr_2G'. The relation $(\exists y)((x,y) \in G$ and $(y,z) \in G')$ entails that $(x,z) \in A\times C$; it thus admits a graph w.r.t. x and z.

Definition 9. - Let G and G' be graphs. We call the graph w.r.t. x and z of the relation $(\exists y)((x,y) \in G$ and $(y,z) \in G')$ the composition of G' and G. It will be designated by $G'\circ G$.

Proposition 6. - Let G and G' be two graphs. The inverse graph of $G'\circ G$ is $\overset{-1}{G}\circ\overset{-1}{G'}$.

Proposition 7. - Let G_1, G_2, G_3 be graphs. One then has $(G_3\circ G_2)\circ G_1 = G_3\circ(G_2\circ G_1)$.

Proposition 8. - Let G and G' be graphs and A a set. Then one has $(G'\circ G)\langle A\rangle = G'\langle G\langle A\rangle\rangle$.

Definition 10. - Let $\Gamma = (G,A,B)$ and $\Gamma' = (G',B,C)$ be two correspondences such that the set of arrival of Γ is identical to the set of departure of Γ'. We call the composition of Γ' and Γ the correspondence $(G'\circ G,A,C)$. It is denoted by $\Gamma'\circ\Gamma$.

Definition 11. - If A is a set, the set Δ_A of objects of the form (x,x), for $x \in A$, is called the diagonal of A×A. The correspondence $I_A = (\Delta_A,A,A)$ is called the identity correspondence of A.

Definition 12. - One says that a graph F is a functional graph if, for every x, there exists at most one object corresponded to x by F. One says that a correspondence $f = (F,A,B)$ is a function if its graph F is a functional graph, and if its set of departure A is equal to its domain of definition $pr_1 F$. In other words, a correspondence $f = (A\ ,A,B)$ is a function if, for every $x \in A$, the relation $(x,y) \in F$ is functional in y; the unique object corresponded to x by f is called the value of f for the element x in A, and is designated by $f(x)$ or f_x (or $F(x)$, or F_x).

If f is a function, F its graph and x an element of the domain of definition of f, the relation $y = f(x)$ is thus equivalent to $(x,y) \in F$.

Let A and B be sets; one calls a mapping (or application) of A into B a function f whose set of departure (which is thus equal to its set of definition since f is a function) is equal to A and whose set of arrival is equal to B; one also says that such a function is defined in A and takes its values in B. This is abbreviated by $f\colon A \longrightarrow B$.

In certain cases, a functional graph is also called a family; the domain is then called the set of indices, and the set of values is called (by abuse of language) the set of elements of the family. When the set of indices is the product of two sets, one speaks of a double family. Similarly, a function whose set of arrival is E is often called a family of elements of E. When every element of E is a subset of a set F, one speaks of a family of subsets of F.

We will often use the word « function » in place of « functional graph » in that which follows.

Example - $(\emptyset,\emptyset,\emptyset)$ is called the void function and the identity correspondence, being a function, is called the identity mapping.

One says that two functions f and g coincide in a set E if E is contained in the sets of definition of f and of g, and if $f(x) = g(x)$

for every $x \in E$. To say that $f = g$ amounts to saying that f and g have the same domain of definition A, the same set of arrival B, and coincide in A.

Let $f = (F,A,B)$ and $g = (G,C,D)$ be two functions. To say that $F \subseteq G$ amounts to saying that the domain of f, A, is contained in the domain C of g. If in addition $B \subseteq D$, one says that g is an extension of f to C.

C54. Let T and A be two terms, x and y distinct letters. Suppose that x does not figure in A, and that y figures neither in T nor in A. Let R be the relation « $x \in A$ and $y = T$ ». The relation R admits a graph F with respect to the letters x and y. This graph is functional; its first projection is A; its second projection is the set of objects of the form T for $x \in A$. For every $x \in A$, one has $F(x) = T$.

If C is a set containing the set B of objects of the form T for $x \in A$ (y not figuring in C), the function (F,A,C) is also designated by the notation $x \to T$ ($x \in A$, $T \in C$). The assemblage corresponding to this in the formal mathematics contains neither x nor y and does not depend on the choice of the letter y verifying the preceding conditions. When the context is sufficiently explicit, one may be content with the notations $x \to T$ ($x \in A$), $(T)_{x \in A}$, or $x \to T$ and even simply T or (T). *For example, on may speak of « the function x^3 » or « $x \to 2x$ » in some specific contexts involving the real numbers *.

Proposition 9. - If f is a mapping of A into B, and g a mapping of B into C, $g \circ f$ is a mapping of A into C.

The function gof is written also $x \rightarrow g(f(x))$, or simply gf if no confusion is likely.

Definition 13. - Let f be a mapping of A into B, One says that f is an injection (or 1-1 mapping), or is an injective mapping, if two distinct elements of A have distinct images under f ($x \neq y \Rightarrow f(x) \neq f(y)$). One says that f is a surjection, or that f is a surjective mapping (or is an onto mapping), if $f(A) = B$. One says that f is a bijection or bijective mapping (or 1-1, onto mapping) if f is at once injective and surjective.

In lieu of injection, one may say that f is a biunique. In lieu of surjection, one may say that f is a mapping of A onto B, or a parametric representation of B by means of A (here, A is called the set of parameters of the representation). If f is bijective one may also say that f places A in a 1-1 correspondence with B. A bijection of A onto A is also called a permutation.

Example - If $A \subseteq B$, the mapping of A into B whose graph is the diagonal of A is injective and is called the canonical injection of A into B.

Proposition 10. - Let f be a mapping of A into B. In order that $\overset{-1}{f}$ be a function, it is necessary and sufficient that f be bijective.

Where f is bijective, $\overset{-1}{f}$ is called the inverse mapping of f; $\overset{-1}{f}$ is bijective, $\overset{-1}{f} \circ f$ is the identity mapping of A and $f \circ \overset{-1}{f}$ is the identity mapping of B.

Let $f: A \rightarrow B$; for every subset X of A one has that $X \subseteq \overset{-1}{f}\langle f\langle X\rangle\rangle$ and for every subset Y of B, one has $f\langle \overset{-1}{f}\langle Y\rangle\rangle \subseteq Y$. If f is a surjection $f\langle \overset{-1}{f}\langle Y\rangle\rangle = Y$ for every $Y \subseteq B$. If f is an injection, for every $X \subseteq A$, $\overset{-1}{f}\langle f\langle X\rangle\rangle = X$.

Proposition 11. - Let f be a mapping of A into B. If there exists a mapping r (resp. s) of B into A such that $r \circ f$ (resp. $f \circ s$) is the identity mapping of A (resp. B), f is injective (resp. surjective). Conversely, if f is surjective, there exists a mapping s of B into A, such that $f \circ s$ is the identity mapping of B. If f is injective and if $A \neq \emptyset$, there exists a mapping r of B into A such that $r \circ f$ is the identity mapping of A.

Corollary. Let A and B be sets, f a mapping of A into B, g a mapping of B into A. If $g \circ f$ is the identity mapping A and $f \circ g$ the identity mapping of B, f and g are both bijective and $g = \overset{-1}{f}$.

Definition 14. - Let f be an injective mapping (resp. surjective mapping) of A into B. Every mapping r (resp. s) of B into A such that $r \circ f$ (resp. $f \circ s$) is the identity mapping of A (resp. B) is called a retraction or left inverse (resp. section or right inverse) associated with f.

A function of two arguments is a function whose domain of definition is a set of ordered pairs.

Definition 15. - Let u be a mapping of A into C and v a mapping of B into D. The mapping $z \rightarrow (u(pr_1 z), v(pr_2 z))$ of $A \times B$ into $C \times D$ is called the canonical extension of u and v to the product set $A \times B$, or simply the product of u and v when no confusion is likely and is designated by $u \times v$ or (u,v).

Its set of values is $u(A) \times v(B)$. If u and v are injective (resp. surjective), then $u \times v$ is injective (resp. surjective) and if u and v are bijective, ther $u \times v$ is bijective and its inverse mapping is $u \times v$. If u' is a mapping of C into E and v' a mapping of D into F, one has that

$$(u' \times v') \circ (u \times v) = (u' \circ u) \times (v' \circ v).$$

Let X be a family, I its set of indices. In order to facilitate the intuitive interprelation of what follows, we shall say that X is a <u>family of sets</u>.

If $(X, I, \mathfrak{G})$ is a <u>family of subsets of a set E</u> (i.e., a family of elements whose sets of arrival ($\mathfrak{G}$ is such that the relation $Y \in \mathfrak{G}$ entails $Y \subseteq E$), we shall use the notation $(X_i)_{i \in I}$ $(X_i \in \mathfrak{G})$, or simply $(X_i)_{i \in I}$; by abuse of notation, we shall use the notation $(X_i)_{i \in I}$ for an arbitrary family of sets, with I for the set of indices.

As the relation $(\forall x)((i \in I \text{ and } x \in X_i) \Rightarrow (x \in X_i))$ is true, S5 allows us to conclude that the relation

$$(\forall i)(\exists Z)(\forall x)((i \in I \text{ and } x \in X_i) \Rightarrow (x \in Z)$$

is true. In virute of S8, the relation $(\exists i)(i \in I \text{ and } x \in X_i)$ is thus <u>collective</u> in x.

<u>Definition 16</u>. - Let $(X_i)_{i \in I}$ be a family of sets (resp. a family of subsets of a set E). The <u>union</u> of this family designated by $\bigcup_{i \in I} X_i$, is the set

$$\{ x \mid (\exists i)(i \in I \text{ and } x \in X_i) \} ,$$

i.e., the set of those x which belong to at least one set out of the family $(X_i)_{i \in I}$.

It is immediate if $I = \emptyset$, one has $\bigcup_{i \in I} X_i = \emptyset$ as the relation $(\exists i)(i \in I \text{ and } x \in X_i)$ is then false.

Suppose that $I \neq \emptyset$. If $\alpha \in I$, the relation $(\forall i)(i \in I) \Rightarrow (x \in X_i))$ entails $x \in X_\alpha$, thus, in virtue of C52, this relation is <u>collective</u> in x.

<u>Definition 17</u>. - Let $(X_i)_{i \in I}$ be a family of sets whose set of indices I is not void. The <u>intersection</u> of this family, designated by $\bigcap_{i \in I} X_i$, is the set $\{ x \mid (\forall i)((i \in I) \Rightarrow (x \in X_i)) \}$, i.e., the set of those x which belong to all of the sets in the family $(X_i)_{i \in I}$.

N.B. If $I = \emptyset$, the relation $(\forall i)((i \in I) \Rightarrow (x \in X_i))$ is <u>not collective</u> in x, for if it were the resulting set would be the « set of all objects » which does not exist.

If $(X_i)_{i \in I}$ is a family of subsets of a set E, and if $I \neq \emptyset$, the relation « $x \in E$ and $(\forall i)((i \in I) \Rightarrow (x \in X_i))$ » is equivalent to $(\forall i)((i \in I) \Rightarrow (x \in X_i))$; consequently, it is collective in x and the set of x verifying this relation is equal to $\bigcap_{i \in I} X_i$. When $I = \emptyset$, the relation « $x \in E$ and $(\forall i)(i \in I) \Rightarrow (x \in X_i))$ » is equivalent to $x \in E$, it is thus again collective in x, and the set of all x verifying this relation is E.

<u>Definition</u> 18. - Let $(X_i)_{i \in I}$ be a family of subsets of a set E. The <u>intersection</u> of this family, designated by $\bigcap_{i \in I} X_i$, is the set $\{ x \mid x \in E \text{ and } (\forall i)((i \in I) \Rightarrow (x \in X_i)) \}$, i.e., the set of all x which belong to E and all of the sets of the family $(X_i)_{i \in I}$.

<u>Definition 19</u>. - Let $\mathfrak{F}$ be a family of sets, and let Φ be the family of sets defined by the identity mapping of $\mathfrak{F}$. The union of the

sets of Φ, and (if $\mathfrak{F}$ is non void) the intersection of the sets of Φ are called respectively the <u>union</u> and <u>intersection</u> of the sets of $\mathfrak{F}$, and are designated by $\bigcup_{X \in \mathfrak{F}} X$ and $\bigcap_{X \in \mathfrak{F}} X$.

If A and B are sets, one lets

$$A \cup B = \bigcup_{X \in \{A,B\}} X \quad \text{and} \quad A \cap B = \bigcap_{X \in \{A,B\}} X .$$

The intersection $X \cap A$ is called the <u>trace of X over A</u>. If $\mathfrak{F}$ is a family of sets, one also calls the <u>trace of $\mathfrak{F}$ over A</u>, the set of traces over A of the sets belonging to $\mathfrak{F}$.

<u>Definition 20</u>. - We say that a family of sets $(X_i)_{i \in I}$ is a <u>cover of a set</u> E if $E \subseteq \bigcup_{i \in I} X_i$.

<u>Definition 21</u>. - We say that two sets A and B are <u>disjoint</u> (or <u>without common element</u>) if $A \cap B = \emptyset$. If this is not so, we say that A and B <u>meet each other</u>. Let $(X_i)_{i \in I}$ be a family of sets; we say that the sets of this family are <u>mutually disjoint</u> (or <u>two by two disjoint</u>) if the conditions $i \in I$, $x \in I$, $i \neq x$ entail $X_i \cap X_x = \emptyset$.

<u>Definition 22</u>. - We call a <u>partition of a set</u> E a family of <u>non void</u> and mutually disjoint subsets of E, which is a cover of E.

<u>Definition 23</u>. - Let $(X_i)_{i \in I}$ be a family of sets. We call the <u>sum of this family of sets</u>, the union of the family of sets $X_i \times \{i\}$ $(i \in I)$.

<u>Proposition 12</u>. - Let $(X_i)_{i \in I}$ be a family of mutually disjoint sets. Let A be its union and S its sum. Then there exists a biunique mapping of A onto S.

All of the usual properties of unions and intersections follow from the above definitions and will not be presented here. To outline our development of set theory further, we give another axiom called the *axiom of the power set*.

A4. $(\forall X)\mathrm{Coll}_Y(Y \subseteq X)$.

This axiom signifies that, for every set X, there exists a set whose elements are all of the subsets of X, viz. the set $\{Y \mid Y \subseteq X\}$. We will designate this set by $\mathfrak{P}(X)$, and will call it the *power set of X* or the *set of subsets of X*. Clearly, if $X \subseteq X'$, then $\mathfrak{P}(X) \subseteq \mathfrak{P}(X')$.

Definition 24. - Let A and B be two sets, Γ a correspondence between A and B. The *function* $X \longrightarrow \Gamma\langle X\rangle$ $(X \in \mathfrak{P}(X),\ \Gamma\langle X\rangle \in \mathfrak{P}(B)$ is called the *canonical extension of* Γ *to the power set* (or *set of subsets*) of A, and will be denoted by $\hat{\Gamma}$. It is a mapping of $\mathfrak{P}(A)$ into $\mathfrak{P}(B)$.

If Γ' is a correspondence between B and a set C, the formula $(\Gamma' \circ \Gamma)\langle X\rangle = \Gamma'\langle \Gamma\langle X\rangle\rangle$ shows that the canonical extension of $\Gamma' \circ \Gamma$ to the set of subsets is the mapping $\hat{\Gamma}' \circ \hat{\Gamma}$.

Proposition 13. - 1. If f is a surjection of a set E over a set F, the canonical extension $\hat{f}$ is a surjection of $\mathfrak{P}(E)$ onto $\mathfrak{P}(F)$.

2. If f is an injection of E into F, the canonical extension $\hat{f}$ is an injection of $\mathfrak{P}(E)$ into $\mathfrak{P}(F)$.

3. If f is a bijection of E into F, the canonical extension $\hat{f}$ is a bijection of $\mathfrak{P}(E)$ onto $\mathfrak{P}(F)$.

Let E and F be sets. The graph of a mapping of E into F is a subset of ExF. The set of elements of $\mathfrak{P}(E\mathrm{x}F)$ which possess the property of being graphs of mappings of E into F is thus a subset of $\mathfrak{P}(E\times F)$ which we designate by F^E. The set of triplets $f = (G,E,F)$, for $G \in F^E$ is thus the <u>set of mappings</u> of E into F, which we designate by $\mathfrak{F}(E,F)$. It is clear that $G \longrightarrow (G,E,F)$ is a bijection called the <u>canonical bijection of F^E onto</u> $\mathfrak{F}(E,F)$. The existence of this bijection permits the immediate translation of every proposition relative to the set F^E into a proposition relative to $\mathfrak{F}(E,F)$ and vice-versa.

Let $(X_i)_{i \in I}$ be a family of sets, F a functional graph with I for domain of definition, and such that, for every $i \in I$, one has $F(i) \in X_i$, then for every $i \in I$, one has $F(i) \in A = \bigcup_{i \in I} X_i$, and consequently F is an element of $\mathfrak{P}(I\times A)$. The functional graphs with the preceding property thus forms a subset of $\mathfrak{P}(I\times A)$.

<u>Definition 25</u>. - Let $(X_i)_{i \in I}$ be a family of sets. The set of functional graphs F, with I for a set of definition, and such that $F(i) \in X_i$ for every $i \in I$, is called the <u>product of the family of sets</u> $(X_i)_{i \in I}$ and is designated by $\prod_{i \in I} X_i$. The mapping $F \longrightarrow F(i)$ ($F \in \prod_{i \in I} X_i$, $F(i) \in X_i$) is called the <u>coordinate function</u> (or <u>projection</u>) of index i, and is denoted pr_i.

We often use the notation $(x_i)_{i \in I}$ to designate the elements of $\prod_{i \in I} X_i$

Let A and B be sets and let α and β be two distinct objects (e.g., $\emptyset$ and $\{\emptyset\}$). Consider the graph (obviously functional) $\{(\alpha,A), (\beta,B)\}$ which is nothing other than the family $(X_i)_{i \in \{\alpha,\beta\}}$ such that $X_\alpha = A$ and $X_\beta = B$. For every pair $(x,y) \in A\times B$, let $f_{x,y}$ be the functional graph $\{(\alpha,x),(\beta,y)\}$. It is immediate that the function

$(x,y) \to f_{x,y}$ is a bijection of $A \times B$ onto $\prod_{i \in \{\alpha,\beta\}} X_i$, whose inverse mapping is $g \to (g(\alpha), g(\beta))$; these two mappings are called <u>canonical</u>. This correspondence is used to prove properties of the product of two sets by means of the properties of the product of a family of sets.

<u>Proposition 14</u>. - Let $(X_i)_{i \in I}$ be a family of sets such that $X_i \neq \emptyset$ for every $i \in I$. Being given a mapping g of $J \subseteq I$ into $A = \bigcup_{i \in I} X_i$, such that $g(i) \in X_i$ for every $i \in J$, there exists an extension f of g to I such that $f(i) \in X_i$ for every $i \in I$.

Proof. In effect, for every $i \in I - J$, designate by T_i the term $\tau_y(y \in X_i)$. As $X_i \neq \emptyset$ by hypothesis, one has that $T_i \in X_i$ for every $i \in I - J$. If G is the graph of g, the graph $G \cup (\bigcup_{i \in I - J} \{(i, T_i)\})$ is the graph of the desired function f.

<u>Corollary 1</u>. - Let $(X_i)_{i \in I}$ be a family of sets such that for every $i \in I$, one has $X_i \neq \emptyset$. Then, for every $\alpha \in I$, the projection pr_α is a mapping of $\prod_{i \in I} X_i$ onto X_α.

<u>Corollary 2</u>. - Let $(X_i)_{i \in I}$ be a family of sets. For $\prod_{i \in I} X_i = \emptyset$ it is necessary and sufficient that there exist an $i \in I$ such that $X_i = \emptyset$.

We have seen that, if one has a family $(X_i)_{i \in I}$ of non void sets, one may introduce (by means of an auxiliary constant) a function f with I for its domain of definition, which is such that $f(i) \in X_i$ for every $i \in I$. One says in practice: Take in each set X_i an element x_i. Intuitively, one has thus «chosen» an element x_i in each of the X_i; the introduction of the logical sign τ and the criteria which govern its employment have allowed us to dispense with an appeal to the «axiom of choice» to legitimate this operation. In fact, Proposition 14 with g the void function is often called the «axiom of choice» [cf. Bourbaki 58, Section 4, No. 10] and Corollary 2, which is equivalent to it is usually called the «multiplicative axiom» [cf Russell 19, p. 117 et seq.] It is with this simple four line proof that the *axiom of choice becomes derivable in our system*.

Let $R\{x,y\}$ be a relation, x and y being distinct letters. One says that the relation R is *symmetric* (with respect to the letters x and y) if one has that $R\{x,y\} \Rightarrow R\{y,x\}$. From this definition, it is immediate that $R\{x,y\}$ is equivalent to $R\{y,x\}$.

Let z be a letter which does not figure in R. One says that $R\{x,y\}$ is *transitive* (with respect to the letters x and y) if one has that $(R\{x,y\}$ and $R\{y,z\}) \Rightarrow R\{x,z\}$.

If $R\{x,y\}$ is at once symmetric and transitive, one says that $R\{x,y\}$ is an *equivalence relation* (with respect to the letters x and y), and use the notation $x = y$ (mod. R) in lieu of $R\{x,y\}$. If R is an equivalence relation one has that $R\{x,y\} \Rightarrow (R\{x,x\}$ and $R\{y,y\})$ in virtue of the definition.

Let $R\{x,y\}$ be a relation. One says that the relation R is *reflexive in* E (wrt. x and y) if the relation $R\{x,x\}$ is equivalent to $x \in E$.

One calls an *equivalence relation in* E an equivalence relation which is reflexive in E. If this is so then R admits a graph. One calls an *equivalence in a set* E a correspondence which has E as its set of departure and arrival, whose graph F is such that the relation $(x,y) \in F$ is an equivalence relation in E.

Let f be a function, E its set of definition, F its graph. The relation « $x \in E$ and $y \in E$ and $f(x) = f(y)$ » is an equivalence relation in E, called the equivalence relation <u>associated</u> with f. The criterion which follows will show at every equivalence relation R on E is of this type. Let G be the graph of R. For every $x \in E$, the (non void) set $G(x) \subseteq E$ is called the <u>equivalence class of x with respect to R</u>. An element of such a class is called a <u>representative</u> of this class. The set of equivalence classes with respect to R (i.e., the set of objects of the form G(x) for $x \in E$) is called the <u>quotient set</u> of E by R and is designated by E/R; the mapping $x \rightarrow G(x)(x \in E)$ whose domain is E and whose set of arrival is E/R is called the <u>canonical mapping</u> (<u>surjection</u>) of E onto E/R.

C55. <u>Let R be an equivalence relation in a set E and ν the canonical mapping of E onto E/R. One has that</u>

$$\underline{R\{x,y\} \Leftrightarrow (\nu(x) = \nu(y))} .$$

Let R be an equivalence relation in a set E. The quotient set E/R is a subset of $\mathfrak{P}(E)$, and the identity mapping of E/R is a <u>partition</u> of E. Conversely every partition of E, $(X_i)_{i \in I}$ defines an equivalence relation on E, viz., $(\exists\, i)(i \in I$ and $x \in X_i$ and $y \in X_i)$. Every subset S of E such that for each $i \in I$, the set $S \cap X_i$ is reduced to a single element is called a <u>system of representatives</u> of the equivalence classes with respect to R.

Let $R\{x,x'\}$ be an equivalence relation, and $P\{x\}$ a relation. One says that $P\{x\}$ is <u>compatible with the equivalence relation</u> $R\{x,x'\}$ (with respect to x), if, given that y designates a letter which figures neither in P nor in R, one has

$$(P\{x\} \text{ and } R\{x,y\}) \Rightarrow P\{y\}.$$

C56. Let $R\{x,x'\}$ be an equivalence relation in a set E, $P\{x\}$ a relation wherein the letter x' does not figure, compatible (with respect to x) with the equivalence relation $R\{x,x'\}$; then if t does not figure in $P\{x\}$, the relation « $t \in E/R$ and $(\exists x)(x \in t$ and $P\{x\})$ » is equivalent to the relation « $t \in E/R$ and $(\forall x)((x \in t) \rightarrow (P\{x\}))$ ».

The relation « $t \in E/R$ and $(\exists x)(x \in t$ and $P\{x\})$ » is called the relation deduced from $P\{x\}$ by passage to quotients.

Let R be an equivalence relation in a set E, and f a function whose domain is E. One says that f is compatible with the relation R if the relation $y = f(x)$ is compatible (with respect to x) with the relation $R\{x,x'\}$.

C57. Let R be an equivalence relation in a set E, and let g be the canonical mapping of E onto E/R. In order thata mapping f of E into F be compatible with R, it is necessary and sufficient that f may be put in the form $h \circ g$, h being a mapping of E/R into F. The mapping h is uniquely determined by f; if s is a section associated with g, one has that $h = f \circ s$.

The mapping h is said to be the mapping deduced from f by passage to quotients with respect to R.

Let f be a mapping of a set E into a set F, and let R be the equivalence relation associated with f. Then f is compatible with R and the mapping h deduced from f by passage to quotients is an injection of E/R into F. Let k be the mapping of E/R onto $f\langle E\rangle$ which has the same graph as h; k is thus a bijection. If j is the canonical injection of $f\langle E\rangle$ into F and ν the canonical mapping of E onto E/R, one may write $f = j \circ k \circ \nu$; this relation is called the canonical decomposition of f.

Let f be a mapping of a set E into a set F, R an equivalence relation in E, S an equivalence relation in F. Let u be the canonical mapping of E onto E/R and v the canonical mapping of F onto F/S. One says that f is compatible with the equivalence relations R and S if $v \circ f$ is compatible with R. The mapping h of E/R into F/S deduced from $v \circ f$ by passage to quotients with respect to R is then called the mapping deduced from f by passage to quotients with respect to R and S; it is characterical by the relation $v \circ f = h \circ u$.

Let $R\{x,y\}$ be an equivalence relation not necessarily possessing a graph. It is immediate that if x, x', and y are three distinct letters the relation $R\{x,x'\}$ entails $R\{x,y\} \Leftrightarrow R\{x',y\}$, thus also the relation $(\forall y)(R\{x,y\} \Leftrightarrow R\{x',y\})$. By means of S7 we see that if one lets $\Theta\{x\} = \tau y(R\{x,y\})$, the relation $R\{x,x'\}$ implies that $\Theta\{x\} = \Theta\{x'\}$. For the other part note that, by definition, $R\{x,\Theta\{x\}\}$ is nothing other then the relation $(\exists y)R\{x,y\}$, which is equivalent to $R\{x,x\}$. We conclude that the relation ($R\{x,x\}$ and $R\{x',x'\}$ and $\Theta\{x\} = \Theta\{x'\}$) is equivalent to $R\{x,x'\}$. The term $\Theta\{x\}$ is called the class of objects equivalent to x (for the relation R).

Suppose that T be a term such that the relation

$$(1) \quad (\forall y)(R\{y,y\} \Rightarrow (\exists x)(x \in T \text{ and } R\{x,y\}))$$

is true. Then the relation $(\exists x)(R\{x,x\}$ and $z = \Theta\{x\})$ is collective in z. Let $\textcircled{R}$ be the set of objects of the form $\theta\{x\}$ for $x \in T$. We call $\textcircled{R}$ the <u>set of classes of equivalent objects</u> with respect to R.

Let $R\{x,y\}$ be a relation, x and y being distinct letters. One says that R is an <u>order relation</u> (or <u>partial order relation</u>) <u>with respect to the letters x and y</u> (or <u>between x and y</u>) if the relations

$$(R\{x,y\} \text{ and } R\{y,z\}) \Rightarrow R\{x,z\}$$

$$(R\{x,y\} \text{ and } R\{y,x\}) \Rightarrow (x = y)$$

$$R\{x,y\} \Rightarrow (R\{x,x\} \text{ and } R\{y,y\})$$

are true.

One calls an <u>order relation in a set</u> E an order relation $R\{x,y\}$ with respect to two distinct letters x and y such that the relation $R\{x,x\}$ is equivalent to $x \in E$.

One calls an <u>order over a set</u> E a correspondence $\Gamma = (G,E,E)$ with E as its set of departure and arrival such that the relation $(x,y) \in G$ is an order relation in E.

If $R\{x,y\}$ is an order relation, we shall often use the notation $x \leqslant y$ in lieu of $R\{x,y\}$ and speak of $\leqslant$ in place of R.

We write $x < y$ for the relation « $x \leqslant y$ and $x \neq y$ ».

C58. <u>Let $\leqslant$ be an order relation, x and y being two distinct letters. The relation $x \leqslant y$ is equivalent to « $x < y$ and $x = y$ ». Each of the relations « $x \leqslant y$ and $y < z$ », « $x < y$ and $y \leqslant z$ » entail $x < z$.</u>

We often write $x \leqslant y \leqslant z$ for « $x \leqslant y$ and $y \leqslant z$ », etc.

Definition 26. - Let E be an ordered set. One says that an element a E is the *least element* (resp. *greatest element*) of E, if for every $x \in E$ one has $a \leqslant x$ (resp. $x \leqslant a$).

Definition 27. - One says that two elements x,y of an ordered set E are *comparable* if the relation « $x \leqslant y$ or $y \leqslant x$ » is true. A set E is said to be *totally ordered* if it is ordered and if any two elements of E are comparable. One then says that the order over E is a *total order* and the corresponding order relation is a *total order relation*.

Let E be an ordered set, a and b two elements of E such that a b then we make the following definition

1) $[a,b] = \{x \mid x \in E \text{ and } a \leqslant x \leqslant b\}$
2) $[a,b[\; = \{x \mid x \in E \text{ and } a \leqslant x < b\}$
3) $]a,b] = \{x \mid x \in E \text{ and } a < x \leqslant b\}$
4) $]\leftarrow,x] = \{x \mid x \in E \text{ and } x \leqslant a\}$
5) $]\leftarrow,\rightarrow[\; = \{x \mid x \in E\}$.

These are called respectively the *closed interval* a,b, the *right half open interval* a,b, the *left half open interval* a,b, etc. following in the usual terminology.

One says that a relation $R\{x,y\}$ is a *well ordering relation between x and y* if R is an order relation between x and y and if for every non empty subset of E over which $R\{x,y\}$ induces an order relation (i.e., $x \in E \Rightarrow R\{x,x\}$), E ordered by this relation admits a least element.

Definition 28. - One says that E is *well ordered* if it is ordered and if every non empty subset of E admits a least element.

Definition 29. - In an ordered set E, one calls a segment of E a subset S of E such that the relations $x \in S$, $y \in E$ and $y \leqslant x$ entail $y \in S$.

Proposition 15. - In a well ordered set E, every segment of E distinct from E is an interval $]\leftarrow, a[$, where $a \in E$.

For every element of a well ordered set E, we use the notation S_x for the segment $]\leftarrow, x[$ which we call segment with extremity x.

Let us now consider ourselves in a theory $\mathcal{T}$ where E is a set well ordered by a relation denoted $x \leqslant y$. We now can enunciate the following criterion called the principle of transfinite induction (or recurrence):

C59. Let $R\{x\}$ be a relation of $\mathcal{T}$ (x not being a constant of $\mathcal{T}$), such that the relation

$$(x \in E \text{ and } (\forall y)((y \in E \text{ and } y < x) \Rightarrow R\{y\})) \Rightarrow R\{x\}$$

is a theorem of $\mathcal{T}$. Under these conditions, the relation $(x \in E) \Rightarrow R\{x\}$ is a theorem of $\mathcal{T}$.

In the application of C59, the relation $x \in E$ and $(\forall y)((y \in E$ and $y < x) \Rightarrow R\{y\})$ is usually called the inductive hypothesis.

For every mapping g of a segment S of E into a set F, and for every $x \in S$, we shall designate by $g^{(x)}$ the mapping of the segment $S_x = \ ]\leftarrow, x[$ of E onto $g(S_x)$, which coincides with g in S_x. With this notation, we have the following criterion called the definition of a mapping by transfinite induction:

C60. Let u be a letter, $T\{u\}$ a term of the theory $\mathcal{T}$. There exists a set U and a mapping f of E onto U such that, for every $x \in E$, one has $f(x) = T\{f^{(x)}\}$. In addition, the set U and the mapping f are determined in a unique manner by these conditions.

Most often, one applies the preceding criterion in a case where there exists a set F such that, for every mapping h of a segment of E onto a subset of F, one has that $T\{h\} \in F$. Then the set U obtained by application of C60 is a subset of F.

Definition 30. - One says that a set X is equipotent to a set Y if there exists a bijection of X onto Y. We denote Eq(X,Y) the relation « X is equipotent to Y ».

The relation Eq(X,Y) is clearly an equivalence relation, which is reflexive in every set. It does not, however, possess a graph.

Definition 31. - The set $\tau_{Z}(\mathrm{Eq}(X,Z))$ is called the cardinal of X (or the power of X) and is denoted by Card (X).

We note that Card (X) is nothing other than the class of objects equivalent to X for the relation of equipotence. (cf. p55).

As Eq(X,X) is true, Card (X) is equipotent to X by S5 and we have the following proposition:

Proposition 16. - In order that two sets X and Y be equipotent, it is necessary and sufficient that their cardinals be equal.

N.B. To say that $\mathfrak{n}$ is a cardinal means that there exists a set X such that $\mathfrak{n} = \mathrm{Card}(X)$.

Example. We use the notation 0 for the Card ($\emptyset$). The only set equipotent to $\emptyset$ being $\emptyset$, one has that $0 = \text{Card}(\emptyset) = \emptyset$.

Example. All one element sets are equipotent since $\{(a,b)\}$ is the graph of a bijection of $\{a\}$ onto $\{b\}$, in particular, they are equipotent to $\{\emptyset\}$. We denote by 1 the cardinal

$$\text{Card}(\{\emptyset\}) = \tau_Z(\text{Eq}(\{\emptyset\},Z)).$$

Here it is important not to confuse the mathematical <u>term</u> designated by the symbol « 1 » and the word « one » of ordinary language. The term designated by « 1 » is equal, by definition, to the term designated by the symbol

$$\begin{aligned}
&\tau_Z((\exists u)(\exists U)(u = (U,\{\emptyset\},Z) \text{ and } U \subseteq \{\emptyset\} \times Z \text{ and} \\
&\quad (\forall x)((x \in \{\emptyset\}) \Rightarrow (\exists y)((x,y) \in U)) \text{ and} \\
&\quad (\forall x)(\forall y)(\forall y')(((x,y) \in U \text{ and } (x,y') \in U) \Rightarrow (y = y')) \text{ and} \\
&\quad\quad (\forall y)((y \in Z) \Rightarrow (\exists x)((x,y) \in U)))).
\end{aligned}$$

The actual assemblage designated by this symbol consists of course of hundreds of signs, each one of which is one of the signs

$$\tau, \square, \vee, \neg, =, \in, \text{ and } \supset .$$

Example. We denote by 2, the cardinal Card $(\{\emptyset,\{\emptyset\}\})$, etc.

<u>Proposition 17</u>. - The relation $R\{\mathfrak{a},\mathfrak{b}\}$:

« $\mathfrak{a}$ and $\mathfrak{b}$ are cardinals and $\mathfrak{a}$ is equipotent to a subset of $\mathfrak{b}$»

is a well ordering relation.

We shall denote the relation $R\{\mathfrak{a}, \mathfrak{b}\}$ by $\mathfrak{a} \leq \mathfrak{b}$.

Definition 32. - Let $(\mathfrak{a}_i)_{i \in I}$ be a family of cardinals. The cardinal of the product set (resp. sum) of the sets $\mathfrak{a}_i$ is called the cardinal product (resp. cardinal sum) of the $\mathfrak{a}_i$ and is denoted by $\prod_{i \in I} \mathfrak{a}_i$ (resp. $\sum_{i \in I} \mathfrak{a}_i$).

Proposition 18. - Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be cardinals, then

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}, \qquad \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a},$$

$$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}, \quad \mathfrak{a}(\mathfrak{b}\mathfrak{c}) = (\mathfrak{a}\mathfrak{b})\mathfrak{c}, \text{ and}$$

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}.$$

Definition 33. - Let $\mathfrak{a}$ and $\mathfrak{b}$ be cardinals; the cardinal of the set of mappings of $\mathfrak{b}$ into $\mathfrak{a}$ (Card($\mathcal{F}(\mathfrak{b}, \mathfrak{a})$)) is denoted by $\mathfrak{a}^{\mathfrak{b}}$, by abuse of notation.

Proposition 19. - Let X be a set and $\mathfrak{a}$ its cardinal; the cardinal of the set $\mathfrak{P}(X)$ is $2^{\mathfrak{a}}$.

Proposition 20. - For every cardinal $\mathfrak{a}$, one has that $2^{\mathfrak{a}} > \mathfrak{a}$.

This is the celebrated theorem of Cantor.

Corollary. - There does not exist a set of which every cardinal is an element.

Definition 34. - One says that a cardinal $\mathfrak{a}$ is finite if $\mathfrak{a} \neq \mathfrak{a} + 1$; a finite cardinal is also called a natural number. One says that a set E is finite if Card (E) is a finite cardinal; one also says that Card (E) is the number of elements of E.

The following criterion is called the <u>principle of induction</u>:

C61. <u>Let $R\{n\}$ be a relation in a theory $\mathcal{T}$ (n not being a constant of $\mathcal{T}$). Suppose that the relation</u>

<u>$R\{0\}$ and $(\forall n)((n$ is a natural number and $R\{n\}) \Rightarrow R\{n+1\}$</u>

<u>is a theorem of $\mathcal{T}$. Under these conditions, the relation</u>

<u>$(\forall n)((n$ is a natural number$) \Rightarrow R\{n\})$</u>

<u>is a theorem of $\mathcal{T}$</u>.

In applications of the above criterion, the relation

« n is a natural number and $R\{n\}$ » or simply $R\{n\}$

is called the <u>inductive hypothesis</u>.

The following criteria, which are consequences of the above are also known as <u>induction principals</u>:

1) Let $S\{n\}$ be the relation

$(\forall p)((n$ is a natural number and p is a natural number and

$p < n) \Rightarrow R\{p\})$,

and suppose that $S\{n\} \Rightarrow R\{n\}$. Then the relation

$(\forall n)((n$ is a natural number$) \Rightarrow R\{n\})$

is true.

2) « <u>induction after</u> k »: Let k be a natural number, $R\{n\}$ be a relation such that the relation

$R\{k\}$ and $(\forall n)((n$ is a natural number $\geq k$ and $R\{n\}) \Rightarrow R\{n+1\})$

is true. Then the relation

$$(\forall n)((n \text{ is a natural number } \geq k) \Rightarrow R\{n\}$$

is true.

3) « <u>induction limited to an interval</u> »: Let a and b be two natural numbers such that $a \leq b$, and let $R\{n\}$ be a relation such that one has

$$R\{a\} \text{ and } (\forall n)((n \text{ is a natural number and } a \leq n < b \text{ and } R\{n\}) \Rightarrow R\{n+1\}).$$

Then the relation

$$(\forall n)((n \text{ is a natural number and } a \leq n \leq b) \Rightarrow R\{n\})$$

is true.

4) « <u>descending induction</u> » : Let a and b be two natural numbers such that $a \leq b$, and let $R\{n\}$ be a relation such that one has

$$R\{b\} \text{ and } (\forall n)((n \text{ is a natural number and } a \leq n \leq b \text{ and } R\{n+1\}) \Rightarrow R\{n\}).$$

Then the relation

$$(\forall n)((n \text{ is a natural number and } a \leq n \leq b) \Rightarrow R\{n\})$$

is true.

<u>Definition 35</u>. - One says that a set is <u>infinite</u> if it is not finite.

In particular, a cardinal is infinite if it is not a natural number.

We introduce the following axiom called the <u>axiom of infinity</u>:

A5. <u>There exists an infinite set</u>.

It is not known whether or not the above axiom is independent of the foregoing axioms. This problem is still an open question. By placing it here, we presume it to be independent.

<u>Proposition 21</u>. - The relation «x is a natural number» is collective in x.

We designate by $\mathbb{N}$ the set of natural numbers. The cardinal of $\mathbb{N}$ is denoted by $\aleph_0$.

<u>Definition 36</u>. - One says that a set is <u>denumberable</u> (or countable) if it is equipotent to a subset of natural numbers $\mathbb{N}$.

For every infinite cardinal $\mathfrak{m}$ one has that Card $(\mathbb{N}) \leqslant \mathfrak{m}$.

The set $\mathbb{N}$ is indeed well ordered and one may apply C60, which we rewrite here using the same notation as before as

C62. <u>Let u be a letter, $T\{u\}$ a term. There exists a set U and a mapping f of $\mathbb{N}$ onto U such that for every natural number n, one has that $f(n) = T\{f^{(n)}\}$, where $f^{(n)}$ is the mapping of $[o,n[$ onto $f([o,n[)$ which coincides with f in $[o,n[$. The set U and the mapping f are then uniquely determined by this condition</u>.

From C61 follows the following criterion called the <u>definition of a mapping by induction</u>:

C63. <u>Let $S\{v\}$ and a be two terms. There exists a set V and a mapping f of $\mathbb{N}$ onto V such that $f(0) = a$ and for every natural number $n \geqslant 1$, $f(n) = S\{f(n-1)\}$. In addition, the set V and the mapping f are uniquely determined by these conditions</u>.

This complete our summary of the theory of sets.

Finally we summarize here the signs and axioms and schemas of the theory of sets.

Signs: Logical: $\vee$, $\neg$, τ, $\square$

Letters: x, y, A, B, etc.

Specific signs: relational (of weight 2): $=$, $\in$

substantive (of weight 2): $\supset$

Axioms and Schemas of the Theory of Sets

Principal of Tautology

S1. If A is a relation of $\mathcal{T}$, the relation (A or A) $\Rightarrow$ A is an axiom of $\mathcal{T}$.

Principal of Addition

S2. If A and B are relations of $\mathcal{T}$, the relation A $\Rightarrow$ (A or B) is an axiom of $\mathcal{T}$.

Principle of Permutation

S3. If A and B are relations of $\mathcal{T}$, the relation (A or B) $\Rightarrow$ (B or A) is an axiom of $\mathcal{T}$.

Principle of Summation

S4. If A, B, and C are relations of $\mathcal{T}$, the relation (A $\Rightarrow$ B) $\Rightarrow$ ((C or A) $\Rightarrow$ (C or B)) is an axiom of $\mathcal{T}$.

Hilbert's ε-formula

S5. If R is a relation of $\mathcal{T}$, T a term of $\mathcal{T}$, and x a letter, the relation $(T \mid x)R \Rightarrow (\exists x)R$ is an axiom of $\mathcal{T}$.

S6. Let x be a letter, T and U terms of $\mathcal{T}$, and $R\{x\}$ a relation of $\mathcal{T}$; the relation $(T = U) \Rightarrow (R\{T\} \Leftrightarrow R\{U\})$ is an axiom of $\mathcal{T}$.

Ackermann's Axiom (as a schema)

S7. If R and S are relations of $\mathcal{T}$ and x a letter, the relation $((\forall x)(R \Leftrightarrow S)) \Rightarrow (\tau_x(R) = \tau_x(S))$ is an axiom of $\mathcal{T}$.

Schema de selection et reunion

S8. Let R be a relation, x and y distinct letters, X and Y distinct letters distinct from x and y and not figuring in R. The relation

$$(\forall y)(\exists X)(\forall x)(R \Rightarrow (x \in X)) \Rightarrow (\forall Y)\mathrm{Coll}_x((\exists y)((y \in Y) \text{ and } R))$$

is an axiom.

Extensionality Axiom

A1. $(\forall x)(\forall y)((x \subseteq y \text{ and } y \subseteq x) \Rightarrow (x = y))$.

Pairing Axiom

A2. $(\forall x)(\forall y)\mathrm{Coll}_z(z = x \text{ or } z = y)$.

Ordered Pairs Axiom

A3. $(\forall x)(\forall x')(\forall y)(\forall y'(((x,y) = (x',y')) \Rightarrow (x = x' \text{ and } y = y'))$.

Power Set Axiom

A4. $(\forall X)\mathrm{Coll}_Y(Y \subseteq X)$.

The Axiom of Infinity

A5. There exists an infinite set.

PART III

THE THEORY OF STRUCTURES

It has been our purpose in the preceding two sections to describe and then present a formal language sufficient for the purposes of modern mathematics. Since most of modern mathematics investigates what might be called « structured sets » , it is one of the primal purposes of the theory of structures to explicate the more or less vague notion of mathematical structure within the framework of our formal language.

Let us think for amoment of what we usually mean when we speak of a mathematical structure. For example, when we speak of a partially ordered set E, we are usually thinking that we are given a set E, certain elements of which are related two by two in some particular fashion. That is for some x and y in E we have that $x \leq y$, i.e., the ordered pair of elements (x,y) satisfy the order relation $R\{x,y\}$. Now as we have noted before such a binary relation between elements of a set is equivalent to defining a particular subset of the product set ExE and thus a particular element of the power set $\mathfrak{P}(E\times E)$. Conversely if we are given a particular element S of the power set $\mathfrak{P}(E\times E)$, about which we assert certain relations, i.e., $S \circ S = S$ and $S \cap S^{-1} = \Delta_E$ we say that such an element which satisfies the particular relations i.e., the axioms (or by <u>conjunction</u> the axiom) of a partial order, defines over E (or <u>supplies E with</u>) <u>the structure of a partially ordered set</u>.

As another example, what do we mean when we speak of the topological space E? We usually are then thinking that we have a set E together with a certain distinguished collection of subsets of E, i.e., a subset of $\mathfrak{P}(E)$ or equivalently, a <u>single element</u> S of $\mathfrak{P}(\mathfrak{P}(E))$, called the <u>system of open sets</u> of E, which satisfies certain relations, called the <u>axioms</u> of a topological space. We may then say that the giving of such an element S of $\mathfrak{P}(\mathfrak{P}(E))$ which satisfies the particular axioms of a topological space defines over E (or supplies E with) the <u>structure of a topological space</u>.

As a final example, let us consider what we mean when we speak of a group with operators. Ordinarily, we would say that we have a set E and a set A, which may be presumed to already have a structure of its own (as in the case of, say, A-modules) together with two laws of composition, one of which is said to be internal and the other involving A and E which is called external. Now the internal law of composition (e.g., addition) is nothing other than a function from ExE into E, i.e., a subset S_1 of (ExE)xE or equivalently an <u>element</u> of $\mathfrak{P}((E\times E)\times E)$, the external law of composition is nothing other than a correspondence from AxE into E, i.e., a subset S_2 of (AxE)xE or <u>equivalently</u> an element of $\mathfrak{P}((A\times E)\times E)$ which satisfies certain relations with respect to the internal law (viz., it is distributive). Thus to say that E is a group with a set of operators A, is equivalent to asserting the existence of a pair $S = (S_1, S_2) \in \mathfrak{P}((E\times E)\times E) \times \mathfrak{P}((A\times E)\times E)$

which satisfies the axioms of a group with operators. The pair (S_1,S_2) thus may be said to supply E with the structure of a group with operators. In this case the set E usually is considered to play the principal role while the term A is said to play an auxiliary role.

Several observations might be made from the consideration of examples such as the foregoing ones.

We generally speak of one (or more) sets as having a structure when we have defined certain relations between members or subsets or between subsets and members or between sets of subsets and members and so forth. In all such cases, these relations define a single member of a set obtained from the basic set (or sets) by the formation of power sets and cartesian products. Conversely, to define such relations on the basic sets (or their subsets, etc.) is equivalent to the specification of a certain member of a particular set (obtained from the basic sets by means of the formation of cartesion products and power sets) which satisfies certain properties.

If we were to consider all such possible formationsobtained from the basic sets by means of cartesian products and power sets taken in any possible order as defining a sort of « ladder of sets with the basic sets as its base » , then the consideration of a particular « rung » of this ladder will be equivalent to the consideration of a particular « type » of relation defined over the basic sets of the ladder. Any particular such rung will itself be characterized by its scheme of formation, i.e., some method which tells one the order in which one is to take the cartesion products and power sets of sets obtained from performing such operations on the basic sets, e.g., how the rung $\mathfrak{P}(E\times E)$ is obtained from the base set E.

By means of such observations as these, we can arrive at some tentative views as to the noti n of what a «species of structure», may consist of and some general requirements that such a notion must satisfy. First we have noted that the consideration of any particular variety or «type» of relation («type of structure») that may be defined over a given collection of sets is equivalent to the consideration of one single element of one particular set which is itself a «rung» of the «ladder of sets» which has the given collection of sets as its «base» . Furthermore, it is apparent that some of these «base sets» will play a «principal» role while others will only play an «auxiliary» role, and these roles will have to be noted as such.

Being given such a collection of sets and noting which ones are to play a principal role and which are to play an auxiliary role we then may specify the ty e of relation or «type of structure» that we wish to consider over tl ese «base sets» by means of some particular «rung» of the «ladder of sets» with the given sets as base. We may then take a particular member of such a rung and say that it is a «structure» over the base sets providing it satisfies certain relations relative to it and the base sets.

We would all agree that for any given collection of sets, such a device will define what we would all call a «structure» over the given sets. It is apparent that if such a process is to be adequate in all cases that we would like to have all structures of the exact same «variety» to be given the same name. Thus we must arrive at some notion of a «species of structure» which is independent of the particular choice of base sets over which we define our structures in the sense that any other «structure» satisfying the «same» relations would be given the same name..

Moreover, any relations which are to be taken as <u>axioms</u> for such a structure must be independent of the particular sets which appear in their formulation in the sense that if S is a structure over the base set, which is thus presumed to satisfy some relation $R\{x,S\}$ and if we have a bijection of this base set x onto another set y, the corresponding relation $R\{y,S'\}$ must be equivalent to $R\{x,S\}$.

I.e., the relations which are to be taken as <u>axioms</u> for a certain species of structure must be in some sense «transportable» relative to the particular «typification» of the structure S for bijections of base sets.

All of the preceding analysis is necessarily vague and is intended to only be of a heuristic nature, to aid the intuitive understanding of that which follows. It is hoped that by keeping the first few examples in mind together with the preceding «analysis» what follows will be more intelligible and at least plausible.

We noted that «types» of relations over given sets could be specified by means of a particular «rung» of the «ladder of sets» with given sets as base» and that such rungs could be characterized by giving their particular «scheme of construction» . To first make this notion clear, we will employ the natural numbers in their meta-mathematical usage, i.e., to specify «ranges of a certain order». Their use here has nothing to do with the <u>mathematical theory</u> of the natural numbers which we outlined in Part II. Their usage here may be considered here as analogous to their usage as abbreviated expressions for «first one writes down this and second one writes down that», etc.

<u>Definition 1</u>. - By a <u>construction schema S for a rung</u> we mean a finite sequence of pairs of natural numbers $c_1, c_2, \dots, c_n$ $(c_i = (a_i, b_i))$ satisfying the following conditions:

(a) If $b_i = 0$, then $1 \leqslant a_i \leqslant i - 1$.

(b) If $a_i \neq 0$ and $b_i \neq 0$, then $1 \leqslant a_i \leqslant i - 1$ and $1 \leqslant b_i \leqslant i - 1$.

These two conditions imply that $c_1 = (0, b_1)$ with $b_1 > 0$ for if not then either $a_1 \neq 0$ and $b_1 \neq 0$ or $a_1 \neq 0$ and $b_1 = 0$, and we have that by (b) in the first case $1 \leqslant a_1 \leqslant 0$ which is impossible and in the second case by (a), that $1 \leqslant a_1 \leqslant 0$ which is also impossible.

Thus if $n = \max \{ b_i \mid (0, b_i) \in S \}$ then we say that $S = (c_1, c_2, \dots, c_m)$ is a construction schema <u>over n terms</u>.

<u>Definition 2</u>. - Let $S = (c_1, \dots, c_m)$ be a construction schema over n terms, and let $E_1, \dots, E_n$ be n terms of a theory $\mathcal{T}$ which is stronger than the theory of sets. Then by the <u>construction</u>, <u>of schema</u> S (or S-<u>construction</u>), over $E_1, \dots, E_n$, we mean a sequence $A_1, A_2, \dots, A_m$ of m terms of $\mathcal{T}$ defined recusively by the following conditions:

(a) If $c_i = (0, b_i)$, then A_i is the term E_{b_i}.

(b) If $c_i = (a_i, 0)$, then A_i is the term $\mathfrak{P}(A_{a_i})$

(c) If $c_i = (a_i, b_i)$ with $a_i \neq 0$ and $b_i \neq 0$, then A_i is the term $A_{a_i} x A_{b_i}$.

<u>Definition 3</u>. - The final term A_m of the S-construction over $E_1, \dots, E_n$ is called the <u>rung</u>, <u>of schema</u> S (or S-<u>rung</u>) <u>over the base sets</u> $E_1, \dots, E_n$ and is denoted by $S(E_1, \dots, E_n)$.

Example 1. - $S_1 = ((0,2), (0,1), (1,0), (2,0), (4,0), (5,3))$ is a rung construction schema over $n = 2$ terms as may be seen immediately from Definition 1. The S_1-construction over the base sets E_1, E_2 is the following sequence E_2, E_1, $\mathfrak{P}(E_2)$, $\mathfrak{P}(E_1)$, $\mathfrak{P}(\mathfrak{P}(E_1))$, $\mathfrak{P}(\mathfrak{P}(E_1) \times \mathfrak{P}(E_2))$ the term A_6 is $\mathfrak{P}(\mathfrak{P}(E_1)) \times \mathfrak{P}(E_2)$ and is thus the S_1-rung over E_1, E_2, i.e., $S_1(E_1,E_2) = \mathfrak{P}(\mathfrak{P}(E_1) \times \mathfrak{P}(E_2))$.

More than one schema can give rise to the same rung as the following example will show. (We shall give it in its full detail):

Example 2. - Let $S_2 = ((0,1), (0,2), (1,0), (3,0), (2,0), (4,5))$, then the S_2-construction over E_1,E_2 is

$c_1 = (0,1)$	implies	$A_1 = E_1$	by condition (a)
$c_2 = (0,2)$	"	$A_2 = E_2$	" (a)
$c_3 = (1,0)$	"	$A_3 = \mathfrak{P}(E_1)$	" (b)
$c_4 = (3,0)$	"	$A_4 = \mathfrak{P}(\mathfrak{P}(E_1))$	" (b)
$c_5 = (2,0)$	"	$A_5 = \mathfrak{P}(E_2)$	" (b)
$c_6 = (4,5)$	"	$A_6 = \mathfrak{P}(\mathfrak{P}(E_1)) \times \mathfrak{P}(E_2)$	" (c)

Thus $S_1(E_1,E_2) = S_2(E_1,E_2)$ while $S_1 \neq S_2$. This fact, however, causes no particular difficulties as we shall see.

We now turn our attention to some other possible schemas which may be constructed out of given ones.

Let $S = (c_1,\ldots,c_r)$ and $S' = (c_1',\ldots,c_s')$ be two rung construction schemas over n terms. We can define a rung construction schema over n terms denoted by SxS' such that $SxS'(E_1,\ldots,E_n) = S(E_1,\ldots,E_n) x S'(E_1,\ldots,E_n)$.

This is accomplished by first defining c_{r+1} for $1 \leq i \leq s$ by

$$c_{r+1} = \begin{cases} c_i' \text{ if } c_i' = (0,b_i') \\ (a_i' + r,0) \text{ if } c_i' = (a_i',0) \\ (a_i' + r,\ b_i' + s) \text{ if } c_i' = (a_i',b_i') \text{ and} \\ \quad a_i' \neq 0 \text{ and } b_i' \neq 0. \end{cases}$$

Then the sequence $(c_1,\ldots,c_r,\ c_{r+1},\ldots,c_{r+s})$ is a rung construction schema S" over n terms, and one has

$$S''(E_1,\ldots,E_2) = S'(E_1,\ldots,E_n),$$

so that if finally we let $c_{r+s+1} = (r,r+s)$, the sequence $(c_1,\ldots,c_{r+s+1})$ is the desired schema SxS'.

We can define in a similar fashion (only more simply) a schema denoted by $\mathfrak{P}(S)$, comprising r+1 pairs of integers which has the property that $\mathfrak{P}(S)(E_1,\ldots,E_n) = \mathfrak{P}(S(E_1,\ldots,E_n))$.

We now shall show that to every schema we can associate a mapping which has several interesting properties. Our previous analysis has given us no motivation for this notion, but its importance will readily become apparent when we formulate our notion of «transportable relations» and isomorphisms of structures.

Let $S = (c_1,\ldots,c_m)$ be rung construction schema over n terms. Let $E_1,\ldots,E_n$, $E_1',\ldots,E_n'$ be sets (terms of $\mathcal{T}$) and let $f_1,\ldots,f_n$ be terms of $\mathcal{T}$ such that the relations « $f_i: E_i \longrightarrow E_i'$ » are theorems of $\mathcal{T}$ for $1 \leq i \leq n$. Let $A_1,\ldots,A_m$ (resp. $A_1,\ldots,A_m$) be the S-construction over $E_1,\ldots,E_n$ (resp. $E_1',\ldots,E_n'$). We now define recusively a sequence of m terms $g_1,\ldots,g_m$ such that for each i $(1 \leq i \leq m)$ $g_i: A_i \longrightarrow A_i'$ subject to the following conditions:

(a) If $c_i = (0,b_i)$ so that $A_i = E_{b_i}$ and $A_i' = E'_{b_i}$, then g_i is the mapping f_{b_i}.

(b) If $c_i = (a_i,0)$ so that $A_i = \mathfrak{P}(A_{a_i})$ and $A_i' = \mathfrak{P}(A'_{a_i})$ then g_i the canonical extension $\hat{g}_{a_i}$ of g_{a_i} to the power set (Part II, Def. 24).

(c) If $c_i = (a_i,b_i)$ with $a_i,b_i \neq 0$ so that $A_i = A_{a_i} x A_{b_i}$ and $A_i' = A'_{a_i} x A'_{b_i}$, then g_i is the canonical extension $g_{a_i} x g_{b_i}$ of g_{a_i} and g_{b_2} to the product set $A_{a_i} x A_{b_i}$. (Part II, Def. 15)

Definition 4. - The final so defined term g_m of this sequence is called the canonical extension, of schema S (or canonical S-extension) of the mappings $f_1,\ldots,f_n$ and is designated by $\langle f_1,\ldots,f_n\rangle^S$

As a consequence of this definition, we have that

$$\langle f_1,\ldots,f_n\rangle^S: \quad S(E_1,\ldots,E_n) \longrightarrow S(E_1',\ldots,E_n').$$

Example. - As in the preceding example 2, let $S = ((0,1), (0,2), (1,0), (3,0), (2,0), (4,5))$ which is schema over two terms. Let E_1, E_2, E_1', E_2' be terms and $f_1: E_1 \longrightarrow E_1'$ and $f_2: E_2 \longrightarrow E_2'$, then we have one after another

$A_1 = E_1 \Rightarrow g_1 = f_1: \quad E_1 \longrightarrow E_1'$ by (a)

$A_2 = E_2 \Rightarrow g_2 = f_2: \quad E_2 \longrightarrow E_2'$ by (a)

$A_3 = \mathfrak{P}(E_1) \Rightarrow g_3 = \hat{f}_1: \mathfrak{P}(E_1) \longrightarrow \mathfrak{P}(E_1')$ by (b)

$A_4 = \mathfrak{P}(\mathfrak{P}(E_1)) \Rightarrow g_4 = \hat{\hat{f}}_1: \mathfrak{P}(\mathfrak{P}(E_1)) \longrightarrow \mathfrak{P}(\mathfrak{P}(E_1'))$ by ( )

$A_5 = \mathfrak{P}(E_2) \Rightarrow g_5 = \hat{f}_2: \mathfrak{P}(E_2) \longrightarrow \mathfrak{P}(E_2')$ by (b)

$A_6 \quad \mathfrak{P}(\mathfrak{P}(E_1)) x \mathfrak{P}(E_2) \Rightarrow g_6 = \hat{\hat{f}}_1 x \hat{f}_2: \mathfrak{P}(\mathfrak{P}(E_1)) x \mathfrak{P}(E_2) \longrightarrow$

$\mathfrak{P}(\mathfrak{P}(E_1') x \mathfrak{P}(E_2')$ by (c)

thus $\langle f_1,f_2\rangle^S = \hat{\hat{f}}_1 x \hat{f}_2$

From the elementary properties of the two canonical extensions used in the above definition which we outlined in Part II we obtain the following criteria:

CST1. If $f_i : E_i \longrightarrow E_i'$ and $f_i' : E_i' \longrightarrow E_i''$ for $1 \leq i \leq n$, then for every rung construction schema S for a rung over n terms,

$$\langle f_1' \circ f_1, f_2' \circ f_2, \ldots, f_n' \circ f_n \rangle^S = \langle f_1', f_2', \ldots, f_n' \rangle^S \circ \langle f_1, f_2, \ldots, f_n \rangle^S.$$

CST2. <u>If f_i is injective (resp. surjective) for $1 \leq i \leq n$, then $\langle f_1, \ldots, f_n \rangle^S$ is injective (resp. surjective).</u>

CST3. If $f_i : E_i \longrightarrow E_i'$ is a bijection and $\overset{-1}{f_i}$ its inverse bijection for $1 \leq i \leq n$, then $\langle f_1, \ldots, f_n \rangle^S$ is a bijection and $\langle \overset{-1}{f_1}, \ldots, \overset{-1}{f_n} \rangle^S$ its inverse bijection, i.e., $(\langle f_1, \ldots, f_n \rangle^S)^{-1} = \langle \overset{-1}{f_1}, \ldots, \overset{-1}{f_n} \rangle^S$.

With the notion of canonical extensions of mappings at hand we can make precise our vague notion of « transportability » which we noted that all relations which may be taken as axioms for a « species of structure » must satisfy. We shall go into this notion in some detail and shall develop a collection of criteria which will enable us to decide just how restrictive this notion is.

<u>Definition 5</u>. - Let $\mathcal{C}$ be a theory stronger than the theory of sets, $x_1, \ldots, x_n, s_1, \ldots, s_p$ distinct letters (distinct from themselves and from the constants of $\mathcal{C}$), $A_1, \ldots, A_m$ terms of $\mathcal{C}$ in which none of the letters x_i ($1 \leq i \leq n$) and s_j ($1 \leq j \leq p$) figure, and finally let $S_1, \ldots, S_p$ be rung construction schemas over $n + m$ terms. Under these conditions we will say that the relation $T\{x_1, \ldots, x_n, s_1, \ldots, s_p, A_1, \ldots, A_m\}$:

« $s_1 \in S_1(x_1,\ldots,x_n, A_1,\ldots,A_m)$ and $s_2 \in S_2(x_1,\ldots,x_n, A_1,\ldots,A_m)$ and

... and $s_p \in S_p(x_1,\ldots,x_n, A_1,\ldots,A_m)$ »

is a typification of the letters $s_1,\ldots,s_p$.

Definition 6. - Let $R\{x_1,\ldots,x_n, s_1,\ldots,s_p\}$ be a relation of $\mathcal{T}$, possibly containing certain of the letters x_i, s_j (and possibly other letters). Then to say that R is transportable (in $\mathcal{T}$) for the typification T, with the $x_i (1 \leq i \leq n)$ considered as principal base sets, and the $A_k (1 \leq k \leq m)$ considered as auxiliary base sets is to say that the following condition is satisfied:

Let $y_1,\ldots,y_n, f_1,\ldots,f_n$ be letters distinct from themselves and from the $x_i (1 \leq i \leq n)$, the $s_j (1 \leq j \leq p)$, and the constants of $\mathcal{T}$, and also from the letters which figure in R or in the $A_k (1 \leq k \leq m)$. Let $I_k (1 \leq k \leq m)$ be the identity mapping of A_k onto itself. Then the relation

(1) « $T\{x_1,\ldots,x_n, s_1,\ldots,s_p\}$ and ($f_1 : x_1 \longrightarrow y_1$ is a bijection) and ... and ($f_n : x_n \longrightarrow y_n$ is a bijection) »

implies, in $\mathcal{T}$, the relation

(2) $R\{x_1,\ldots,x_n, s_1,\ldots,s_p\} \Leftrightarrow R\{y_1,\ldots,y_n, s_1',\ldots,s_p'\}$

where

(3) $s_j' = \langle f_1,\ldots,f_n, I_1,\ldots,I_m \rangle^{S_j}(s_j)$ for $1 \leq j \leq p$.

(We may formulate a simpler definition in case the auxiliary base sets do not appear.)

The relation (1) above is called the transport relation for the typification T.

The relation (2) means (in words) that the relation R, possibly involving the letters $x_1,\ldots,x_n$, $s_1,\ldots,s_n$, is equivalent to the relation R with each occurrence of an x_i replaced by a y_i and each occurrence of an s_j replaced by its « image » under the canonical extension of the f_i, I_k by the schema S_j.

To give a trivial example, suppose that n = p = 2 and that T is « $s_1 \in x_1$ and $s_2 \in x_1$ », then the relation « $s_1 = s_2$ » is transportable (since the relation of transport for this T implies that

$$s_1 = s_2 \Leftrightarrow f_1(s_1) = f_1(s_2))$$

while the relation $x_1 = x_2$ is <u>not</u> (since $x_1 = x_2 \not\Leftrightarrow y_1 = y_2$).

We shall develop a number of criteria which will greatly facilitate the determination of whether or not a given relation is transportable.

For brevity, the terms x_i, s_j, and A_k will be referred to as the <u>initial</u> letters and terms of the criterion. We shall use the notation S(x,A) for the rung $S(x_1,\ldots,x_n, A_1,\ldots,A_m)$, where S is a rung construction schema over n+m letters. We shall also use the notation $T\{x,s,A\}$ (or $T\{x,s\}$, or simply T) to designate a particular typification « $s_1 \in S_1(x,A)$ and ... and $s_p \in S_p(x,A)$ » where $s_1,\ldots,s_p$ are p rung construction schemas over n+m letters, the x_i, s_j, A_k being the initial letters and terms of the criterion. In each of the criteria considered, there being the further question of relations of $\mathcal{T}$, denoted in general by U, U', U",..., these relations and terms will be considered as possibly involving the initial letters of the criterion. We shall also designate by $\mathcal{T}_c(x,s,A,y,f)$ (or simply $\mathcal{T}_c$) the theory obtained upon adjoining the relation of transport (1), to the axioms of $\mathcal{T}$. Thus if S is a rung construction schema over n+m terms, and if we designate by f^S the term of $\mathcal{T}_c$ denoted by $\langle f_1,\dots,f_n, I_1,\dots,I_n\rangle^S$, the relation

$$« f^S : S(x,A) \longrightarrow S(y,A) \text{ is a bijection} »$$

is (by CST3) a <u>theorem</u> of $\mathcal{T}_c$. Also with s_j' defined as in (3), for every assemblage $W\{x,s\}$, we designate by $W\{y,s'\}$ the assemblage obtained on replacing each of the x_i by y_i and each of the s_j by s_j' in W.

With these notations, to say that the <u>relation R is transportable</u> (<u>in</u> $\mathcal{T}$) <u>for the typification</u> T is the same as saying that the relation $« R\{x,s\} \Leftrightarrow R\{y,s\} »$ is a <u>theorem of</u> $\mathcal{T}_c$.

<u>Definition 7</u>. - With these same notations, we say that a <u>term</u> U is <u>of type</u> (s,x,A) <u>for the typification</u> T (or by abuse of language, <u>of type</u> (S(x,A) or <u>of type</u> S) if the relation

$$T \Rightarrow (U \in S(x,A))$$

is a theorem of $\mathcal{T}$.

<u>Definition 8</u>. - We say that U is a <u>transportable term of type</u> (s,x,A) (or <u>of type</u> S(x,A) or <u>of type</u> S) <u>for the typification</u> T if the the following conditions are satisfied:

1) U is of type S(x,A) (for T);
2) the relation $U\{y,s'\} = f^S(U\{x,s\})$ is a theorem of $\mathcal{T}_c$.

Remember that if $\mathcal{T}'$ is a theory stronger than $\mathcal{T}$, every relation (resp. term) of $\mathcal{T}$ which is transportable for a typification T is again transportable for the same typification when considered as a relation (resp. term) of $\mathcal{T}'$. Note also that the preceding definitions (in a simpler form) extend to the case where there are no letters s_j occurring and similarly for all of the criteria (it will suffice to replace T by any true relation of $\mathcal{T}$).

As an immediate example we may note that the term Card (x) is not transportable since there is no rung of which Card (x) is a member, but the relation

« Card (x) $\leqslant$ Card (y) »

is transportable since it is equivalent to « x is equipotent to a subset of y » which is transportable as we shall soon see.

For brevity we shall say "transportable" in lieu of "transportable for the typification T" where no confusion will arise. In the same criterion "transportable" will always mean for the same typification unless expressly noted otherwise.

CT1. If none of the letters $x_1,\ldots,x_n$, $s_1,\ldots,s_p$ figure in a relation R, then R is transportable. The term $\emptyset$ is transportable of type $\mathfrak{P}(S)$ (whatever be the schema S).

CT2. For the typification T$\{x,s,A\}$, x_i is a transportable term of type $\mathfrak{P}(x_i)$, s_j is a transportable term of type $S_j(x,A)$ and A_k is a transportable term of $\mathfrak{P}(A_k)$.

These criteria are an immediate result of the definitions.

CT3. If R and R' are transportable relations, then so are the relations « not R », « R or R' », « R and R' », « R $\Rightarrow$ R' », « R $\Leftrightarrow$ R' ».

CT4. <u>If the terms U and U' are transportable of types S and S', respectively, then (U,U') is transportable of type SxS'. If U and U' are transportable of type $\mathfrak{P}$(S) and $\mathfrak{P}$(S') respectively, then UxU' is transportable of type $\mathfrak{P}$(SxS') and $\mathfrak{P}$(U) is transportable of type $\mathfrak{P}$($\mathfrak{P}$(S)).</u>

CT5. <u>If U and U' are transportable terms of the same type S, the relation U = U' is transportable. If U is transportable of type S and U' is transportable of type $\mathfrak{P}$(S), then the relation U $\in$ U' is transportable. If U and U' are transportable of type $\mathfrak{P}$(S), then the relation U $\subseteq$ U' is transportable.</u>

These criteria are the result of the definition and the properties of canonical extensions.

CT6. <u>For every rung construction schema S over n+m terms, S(x,A) is a transportable term of type $\mathfrak{P}$(S(x,A)) for the typification T$\{$ x,s,A$\}$.</u>

This is a result of CT2 and CT4 applied one after another over the S-construction.

CT7. <u>If R is a relation such that T $\Rightarrow$ R is valid in $\mathcal{T}$, then R is transportable for T. If U and U' are two terms such that T $\Rightarrow$ (U = U') is valid in $\mathcal{T}$, and if U is transportable of type S for T, then so is U'.</u>

The second part of the criterion is a result of the definition of a transportable term and of schema S6 applied in the theory $\mathcal{T}_c$. - For the other part, the relation T$\{$ x,s,A$\}$ is transportable (for the typification T$\{$ x,s,A$\}$ in virtue of CT3, CT5 and CT6; the relation

$T\{x,s,A\} \Leftrightarrow T\{y,s',A\}$ is thus a theorem of $\mathcal{T}_c$, and hence similarly so is $T\{y,s',A\}$. The hypothesis on R entails that $R\{x,s\}$ is a theorem of $\mathcal{T}_c$; thus $R\{x,s'\}$ is a theorem of $\mathcal{T}_c$ and one has in concusion that the relation $R\{x,s\} \Leftrightarrow R\{y,s'\}$ is also a theorem of $\mathcal{T}_c$, hence the first part of the criterion.

CT8. Let z be a letter distinct from both the constants of and the letters figuring in the typification $T\{x,s,A\}$. Let S be a rung construction schema over n+m letters, and let T' be the typification

$$\text{« } T\{x,s,A\} \text{ and } z \in S(x,A) \text{ .»}$$

Finally, let R be a relation containing no z. Under these conditions, if R is transportable (in $\mathcal{T}$) for the typification T', R is transportable for the typification T in the theory $\mathcal{T}'$ obtained by adjoining to the axioms of $\mathcal{T}$ the relation $S(x,A) \neq \emptyset$.

This result is obtained easily by the method of the auxiliary constant.

The preceding criterion is applied notably in the following two cases:

a) the rung $S(x,A)$ is of the form $\mathfrak{P}(x)$;

b) the schema S is identical to one of the schemas S_j $(1 \leqslant j \leqslant p)$ involved in the typification T.

In these two cases one concludes from CT8 that R is transportable in the theory $\mathcal{T}$ for the typification T in case $S(x,A) \neq \emptyset$ is a theorem of $\mathcal{T}_c$.

CT9. Let R be a transportable relation for the typification T and let R' be a relation such that $T \Rightarrow (R \Leftrightarrow R')$ is a theorem of $\mathcal{T}$. Then the relation R' is transportable for T.

In effect, the same reasoning as that in the criterion CT8 shows that the relations $R\{x,s\} \Leftrightarrow R'\{x,s\}$ and $R\{y,s'\} \Leftrightarrow R'\{y,s'\}$ are theorems of $\mathcal{T}_c$, since by hypothesis, the relation $R\{x,s\} \Leftrightarrow R\{y,s'\}$ is valid in $\mathcal{T}_c$, it is the same for $R'\{x,s\} \Leftrightarrow R'\{y,s'\}$.

CT10. For the typification $T\{x,s,A\}$, let U be a term of type $\mathfrak{P}(S_j)$ in which the letter s_j does not figure. For U to be transportable for T, it is necessary and sufficient that the relation $s_j \in U$ be transportable for T.

The condition is necessary in virtue of CT5. Conversely, if it is satisfied, the relation

$$(s_j \in U\{x_1,\dots,x_n, s_1,\dots,s_{j-1}, s_{j+1},\dots,s_p\}) \Leftrightarrow (f^{S_j}(s_j) \in U\{y_1,\dots,y_n, s_1',\dots,s_{j-1}',\dots,s_p'\})$$

is true in $\mathcal{T}_c$. As, in the theory $\mathcal{T}_c$, f^{S_j} is bijective, it is a result that the relation $U\{y,s'\} = f^{S_j}(U\{x,s\})$ is a theorem of $\mathcal{T}_c$ which establishes the criterion.

CT11. For the typification $T\{x,s,A\}$, let U be a term of type S_j in which the letter s_j does not figure. For U to be transportable for T, it is necessary and sufficient that the relation $s_j = U$ be transportable for T.

Proof is similar to that of CT10.

CT12. Let z be a letter distinct from the constants of $\mathcal{C}$ and from the letters figuring in the typification $T\{x,s,A\}$, and let U be a term of type S (resp. $\mathfrak{P}(S)$) for T in which the letter z does not figure. Then the following three conditions are equivalent:

a) U is transportable of type S (resp. $\mathfrak{P}(S)$) for T;

b) U is transportable of type S (resp. $\mathfrak{P}(S)$) for the typification « $T\{x,s,A\}$ and $z \in S(x,A)$ »;

c) the relation $z = U$ (resp. $z \in U$) is transportable for the typification « $T\{x,s,A\}$ and $z \in S(x,A)$ ».

The equivalence of b) and c) results from CT10 and CT11 and a) evidently entails b). For the remainder, the method of the auxiliary constant shows that b) entails that U is transportable ffor T in the theory obtained on adjoining to $\mathcal{C}$ the axiom $S(x,A) \neq \emptyset$. But if U is of type S, the hypothesis (in $\mathcal{C}$) entails the relation $U \in S(x,A)$, and consequently the relation $S(x,A) \neq \emptyset$; this last is thus a theorem of $\mathcal{C}$, which proves that in this case, U is transportable for T in the theory $\mathcal{C}$. If U is of type $\mathfrak{P}(S)$, the relation « $T\{x,s,A\}$ and $S(x,A) \neq \emptyset$ » entails $U = \emptyset$ in $\mathcal{C}$, and then U is transportable for T in the theory obtained on adjoining to $\mathcal{C}$ the axiom $S(x,A) = \emptyset$, in virtue of CT1; the conclusion then results by the method of the case disjunction.

CT13. Let R be a relation transportable for the typification $T\{x,s,A\}$. Then for every index j ($1 \leq j \leq p$), the term

« the set of the $s_j \in S_j(x,A)$ such that R »

is transportable of type $\mathfrak{P}(S_j)$ (for T).

In effect, if one designates this term by U, it is clear that U is of type $\mathfrak{P}(S_j)$ and that s_j does not figure in it. Now in $\mathcal{C}$, T entails the relation $(s_j \in U) \Leftrightarrow (s_j \in S_j(x,A)$ and R), and the relation « $s_j \in S_j(x,A)$ and R » is transportable for T (Criteria CT5, CT6, and CT3). One thus has the conclusion desired with the aid of CT9 and CT10.

CT14. <u>For the typification T{x,s,A}, let R be a transportable relation, and let U be a term, transportable of type $\mathfrak{P}(S_j)$. Then the relations</u>

$$\underline{(\exists s_j)(s_j \in U \text{ and } R)}$$

$$\underline{(\forall s_j)(s_j \in U) \Rightarrow R)}$$

<u>are transportable for T</u>.

In effect, let U' be the term « the set of $s_j \in S_j(x,A)$ such that R ». In $\mathcal{C}$, the relation T entails the relation $(U \subseteq U') \Leftrightarrow ((\forall s_j)((s_j \in U) \Rightarrow R)$. As U' is transportable of type $\mathfrak{P}(S_j)$ for T by means of CT13, the second assertion of the criterion results from CT5 and CT3; the first is then deduced with the aid of CT3 and CT9.

CT15. <u>For the typification T{x,s,A}, let U be a transportable term of type S, U' a transportable term of type $\mathfrak{P}(S_j)$, such that s_j does not figure in U. Then the term</u>

« <u>the set of objects of the form U for $s_j \in U'$</u> »

<u>is transportable of type $\mathfrak{P}(S)$ for T</u>.

In effect, let z be a letter distinct from the letters introduced in the preceding. The term considered is the set V of the $z \in S(x,A)$

such that one has $(\exists s_j)(s_j \in U'$ and $z = U)$. Applying successively CT5, CT3, and CT13, one observes that V is transportable of type $\mathfrak{P}(S)$ for the typification « $T\{x,s,A\}$ and $z \in S(x,A)$ ». The conclusion is then obtained with the aid of CT12.

CT16. Let R be a transportable relation for the typification T. If, in $\mathcal{C}$, the relation « T and R » is functional in s_j, the term τ_{s_j}(T and R) is transportable of type S_j.

Let V be this term, which is evidently of type S_j. In $\mathcal{C}$, the relation T entails $(s_j = V) \Leftrightarrow$ (T and R) and s_j does not figure in V, one concludes the criterion with the aid of CT9 and CT11.

By contrast, if one does not suppose that « T and R » be functional in s_j, the conclusion of criterion CT16 is inexact. Suppose for example that $\mathcal{C}$ be the theory of sets, that $n = p = 1$, $m = 0$, and that T and R be both identical to the relation $s_1 \quad x_1$. If $\tau_{s_1}(R)$ be transportable for T, the relation of transport entails the equality

$$f_1(\tau s_1(s_1 \in x_1)) = \tau_t(t \in f_1(x_1)).$$

This consequently entails that for every set E, the image of $\tau_x(x \in E)$ for every bijection of E onto a set F is the element $\tau_x(x \in F)$, which is absurd, for example, for every set with two elements.

CT17. Let R be a transportable relation, U a transportable term of type S_j, U' a transportable term of type S'. Then the relation $(U \mid s_j)R$ is transportable, and the term $(U \mid s_j)U'$ is transportable of type S'.

In effect, be V the set of $s_j \in S_j(x,A)$ such that R, V is a transportable term (CT13), and the relation T entails (in $\mathcal{C}$) the relation

$$((U \mid s_j)R) \Longleftrightarrow (U \in V).$$

Consequently $(U \mid s_j)R$ is transportable (CT9). Let z be a letter distinct from those already introduced; the relation $z = (U \mid s_j)U'$ is identical to $(U \mid s_j)(z = U')$, and $z = U'$ is transportable for this typification. The conclusion results from CT12 when we show that the term $(U \mid s_j)U'$ is of type S' for the typification T. Now in $\mathcal{T}$, the relation T entails thus the relation $(U \mid s_j)T$, and since s_j does not figure in the term $U' \in S'(x,A)$, T entails finally the relation $(U \mid s_j)U' \in S'(x,A)$ (criterion C2).

CT18. <u>Let U be a transportable term for T, of type $\mathfrak{P}(\mathfrak{P}(S))$. Then the term $\bigcup_{x \in U} X$ is transportable of type $\mathfrak{P}(S)$, and so is the term $\bigcap_{x \in U} X$ when T entails $U \neq 0$.</u>

CT19. <u>If U and U' are transportable terms of type $\mathfrak{P}(S)$, then so are the terms $U \cup U'$, $U \cap U'$ and $S(x,A) - U$.</u>

CT20. <u>If U is transportable of type $S \times S'$, then $pr_1 U$ and $pr_2 U$ are transportable of types S and S' respectively. If U' is transportable of type $\mathfrak{P}(S \times S')$, then $pr_1 \langle U' \rangle$ and $pr_2 \langle U' \rangle$ are transportable of types $\mathfrak{P}(S)$ and $\mathfrak{P}(S')$ respectively.</u>

We give the demonstration for example, of the first part of CT18: Let z and t be two letters distinct from themselves and from the letters already introduced; the relation T entails the relation

$$(z \in U \text{ and } t \in z) \Rightarrow (t \in S(x,A) \text{ and } z \in \mathfrak{P}(S(x,A))).$$

It thus suffices to show that the set of $t \in S(x,A)$ such that $(\exists z)(z \in U \text{ and } t \in z)$ is transportable of type $\mathfrak{P}(S)$, for the typification T. Now this term is of type $\mathfrak{P}(S)$ for T, and is transportable of type $\mathfrak{P}(S)$ for the typification « T $\{x,s,A\}$ and $z \in \mathfrak{P}(S(x,A))$ and $t \in S(x,A)$ »; as it contains neither z nor t, one has the desired conclusion by CT12. The demonstrations of the other criteria are analogous.

N.B. In that which follows, we will make no distinction between a correspondence and its graph.

CT21. If U is transportable of type $\mathfrak{P}(S \times S')$, and if U' is transportable of type $\mathfrak{P}(S' \times S'')$, then $U' \circ U$ is transportable of type $\mathfrak{P}(S \times S'')$ and U^{-1} is transportable of type $\mathfrak{P}(S' \times S)$.

CT22. If U is transportable of type $\mathfrak{P}(S \times S')$ and V transportable of type $\mathfrak{P}(S)$, then the term $U\langle V \rangle$ is transportable of type $\mathfrak{P}(S')$.

CT23. If U is transportable of type $\mathfrak{P}(S)$, then the identity mapping I_U of U onto itself is transportable of type $\mathfrak{P}(S \times S)$.

CT24. Suppose that U be transportable of type $\mathfrak{P}(S)$, U' transportable of type $\mathfrak{P}(S')$, and V transportable of type $\mathfrak{P}(S \times S')$. Then the relations

« V is a mapping of U into U' »

« V is an injection of U into U' »

« V is a surjection of U onto U' »

« V is a bijection of U onto U' »

are transportable.

We give the demonstration for the first relation which we designate by R. It is immediate that, in $\mathcal{C}$, the typification T entails the relation

$$R \Leftrightarrow ((V\langle U \rangle \subseteq U') \text{ and } V \circ V^{-1} = I_{V\langle U \rangle}).$$

The conclusion thus results from CT9 and the criteria CT21, CT22, CT23, CT5 and CT3.

CT25. Let U, U', U", and V be transportable terms of types respectively S, $\mathfrak{P}(S)$, $\mathfrak{P}(S')$, and $\mathfrak{P}(S\times S')$ for a typification T. Suppose that the relation T entails the relations « V: U' ⟶ U" » and $U \in U'$. The term V(U) is then transportable of type S". If moreover W' is a term transportable of type $\mathfrak{P}(S)$ and if the relation T entails the relation $W' \subseteq U'$, then the term « the restriction of V to W' » is transportable of type $\mathfrak{P}(S\times S')$.

CT26. If R is a transportable relation, then the graph w.r.t. s_j and s_k of the relation

$$\text{« } s_j \in S_j(x,A) \text{ and } s_k \in S_k(x,A) \text{ and } R \text{ »}$$

is a transportable term of type $\mathfrak{P}(S_j \times S_k)$.

CT27. Suppose that for two distinct indices j and k, the schemas S_j and S_k are the same, and for a typification T, let U be a transportable term of type $\mathfrak{P}(S_j)$ and let R be a transportable relation. Suppose in addition that the relation T entails the relation

$$\text{« } R \text{ is an equivalence relation in } U \text{ between } s_j \text{ and } s_k \text{ ».}$$

Then the term U/R is transportable of type $\mathfrak{P}(\mathfrak{P}(S_j))$ and the canonical mapping of U onto U/R is a transportable term of type $\mathfrak{P}(S_j \times \mathfrak{P}(S_j))$.

CT28. For a typification T, let V be a transportable term of type $\mathfrak{P}(S\times S')$, then the canonical extension of V to $\mathfrak{P}(S(x,A))$ and $\mathfrak{P}(S'(x,A))$ is a transportable term of type $\mathfrak{P}(\mathfrak{P}(S)\times \mathfrak{P}(S'))$. Let U, U_1, U_1' and U_1' be transportable terms of types respectively $\mathfrak{P}(S)$, $\mathfrak{P}(S")$, $\mathfrak{P}(S')$, and $\mathfrak{P}(S"')$; let V be a transportable term of type $\mathfrak{P}(S"\times S"')$, and suppose that the relation T entails the relations

«V is a mapping of U into U'» and « V_1 is a mapping of U_1 into U_1' ». Then the canonical extension of V and V_1 to $U \times U_1$ is a transportable term of type $\mathfrak{P}((S \times S'') \times (S' \times S'''))$.

CT29. Let U, U', and U" be three transportable terms of types respectively $\mathfrak{P}(S)$, $\mathfrak{P}(S')$, and $\mathfrak{P}(S'')$. Then the canonical bijection of $(U \times U') \times U''$ onto $U \times (U' \times U'')$ and the canonical bijection of $U \times U'$ onto $U' \times U$ are transportable terms of types respectively

$$\mathfrak{P}(((S \times S') \times S'') \times (S \times (S' \times S''))) \text{ and } \mathfrak{P}((S \times S') \times (S' \times S)).$$

CT30. Let U and U' be two transportable terms of types respectively $\mathfrak{P}(S)$ and $\mathfrak{P}(S')$. Then the set of mappings of U into U' is a transportable term of type $\mathfrak{P}(\mathfrak{P}(S \times S'))$.

CT31. For a typification T, let U be a transportable term of type $\mathfrak{P}(S_j)$, V a transportable term of type $\mathfrak{P}(S_j \times \mathfrak{P}(S))$; suppose that the relation T entails the relation « V is a mapping of U into $\mathfrak{P}(S(x,A))$ » and that s_j figures in neither U nor V. Then the terms $\prod_{s_j \in U} V(s_j)$ and $\bigcup_{s_j \in U} V(s_j)$ are transportable of types $\mathfrak{P}(\mathfrak{P}(S_j \times S))$ and $\mathfrak{P}(S)$ respectively. If T entails the relation $U \neq 0$, then the term $\bigcap_{s_j \in U} V(s_j)$ is transportable of type $\mathfrak{P}(S)$.

We are now finally ready to explicate the notion of « species of structure » .

Definition 9. - Let $\mathcal{T}$ be a theory stronger than the theory of sets (which of course may be the theory of sets itself). A species of structure in $\mathcal{T}$ is a text (specification) Σ formed of the following assemblages:

1. A certain number of letters $x_1, \ldots, x_n$, s distinct from themselves and from the constants of $\mathcal{T}$. (The letters x_i are called the <u>principal base sets of</u> Σ ; the letter s is called the <u>generic structure of</u> Σ .)

2. A certain number of terms $A_1, \ldots, A_m$ of $\mathcal{T}$ (called the <u>auxiliary base sets of</u> Σ) in which none of the x_i, s figure.

3. A typification $T\{x_1, x_2, \ldots, x_n, s\}$: $s \in S(x_1, \ldots, x_n, A_1, \ldots, A_m)$ where S is a rung construction schema over n+m terms (called the <u>typical characterization of</u> Σ). (S may be the product of rung construction schemas $S_1 x S_1 x \ldots x S_p$, then S will be a « multiplet » $(s_1, \ldots, s_p)$.)

4. A relation $R\{x_1, \ldots, x_n, s\}$ which is transportable (in $\mathcal{T}$) for the typification T, with the x_i as principal base sets, and the A_k as auxiliary base sets. (R is called the <u>axiom of</u> Σ .) (R may of course be the <u>conjunction</u> of one or more transportable relations which will then be called the axioms of Σ .)

<u>Definition 10</u>. - The <u>theory of the species of structure</u> Σ is that theory $\mathcal{T}_\Sigma$ which has the same axioms schemas as $\mathcal{T}$, the same explicit axioms as $\mathcal{T}$, and the axiom « T and R » ; the constants of $\mathcal{T}_\Sigma$ are then the constants of $\mathcal{T}$ and the letters which figure in T or in R.

Definition 11. - Let $\mathcal{T}'$ be a theory stronger than $\mathcal{T}$ and let $E_1,\dots,E_n$, U be terms of $\mathcal{T}'$. We say that (in the theory $\mathcal{T}'$) U is a structure of species Σ (or Σ-structure) over the principal base sets $E_1,\dots,E_n$, with $A_1,\dots,A_m$ for auxiliary base sets if the relation

$$\text{«}\ T\{E_1,\dots,E_n,U\} \text{ and } R\{E_1,\dots,E_n,U\}\ \text{»}$$

is a theorem of $\mathcal{T}'$.

It is then the case that for every theorem $B\{x_1,\dots,x_n,s\}$ of the theory $\mathcal{T}_\Sigma$, the relation $B\{E_1,\dots,E_n,U\}$ is a theorem of $\mathcal{T}'$.

Definition 12. - We say that (in $\mathcal{T}'$) the principal base sets $E_1,\dots,E_n$ are supplied (or furnished) with the structure U. For brevity we often will under such conditions say that $E_1,\dots,E_n$ is a Σ-set.

It is clear then that U is an element of the set $S(E_1,\dots,E_n, A_1,\dots,A_m)$. The set of those elements V of $S(E_1,\dots,E_n, A_1,\dots,A_m)$ which satisfy the relation $R\{E_1,\dots,E_n,V\}$ is thus the set of Σ-structures over $E_1,\dots,E_n$. It may be empty, for example, if the axioms of Σ are contradictory!

Definition 13. - By abuse of language, in the theory of sets, the specification of n distinct letters without typical characterization or axiom is considered as the species of structure Σ_0 called the species of structure of a set over the n principal base sets $x_1,\dots,x_n$.

Example 1. Let $\mathcal{T}$ be the theory of sets and consider the species of structure, without auxiliary base set, consisting of the principal base E, the typical characterization $s \in \mathfrak{P}(E\times E)$ and the axiom «$s \circ s = S$ and $s \cap s^{-1} = \Delta_A$» (where Δ_A is the diagonal of $A\times A$), which is indeed a transportable relation for the typification $s \in \mathfrak{P}(E\times E)$ as is shown by application of the definition or by CT2, CT21, CT5, CT19, CT25, and CT3. This species of structure is of course the <u>species of structure of a (partially) ordered set</u>. The theory of this species of structure is nothing other than the <u>theory of (partially) ordered sets</u> which has two constants, the letters E and S. (For the sake of completeness we mention that $\mathfrak{P}(E\times E) = S(E)$ where $S = ((0,1), (1,1), (2,0))$ although the importance of the schemas lies more in their <u>existence</u> than in any particular example of their use.)

Example 2. Again let $\mathcal{T}$ be the theory of sets and consider the species of structure of a <u>topological space</u> which has one principal base set E, no auxiliary base set, typical characterization $V \in \mathfrak{P}(\mathfrak{P}(E))$ and axiom

$$\text{«}\ (\forall V')(V' \subseteq V) \Rightarrow \Big(\bigcup_{X\in V'} X \in V\Big) \text{ and } (\forall X)(\forall Y)(X \in V \text{ and } Y \in V) \Rightarrow ((X \cap Y) \in V) \text{ and } E \in V\ \text{»}.$$

That this axiom is indeed a transportable relation for the typification $V \in \mathfrak{P}(\mathfrak{P}(E))$ may be seen from the definition or by consulting CT18, CT14, CT19, CT5, CT3, and CT2, etc. A structure of this species is of course a <u>topology</u> and the relation «$X \in V$» is expressed by «X is <u>open</u> for the topology V». (Again for expository completeness, one may take $S = ((0,1), (1,0), (2,0))$.) The <u>theory of topological spaces</u> has two constants E and V.

We may within this context say what one means by an <u>algebraic structure</u>.

A <u>species of algebraic structure</u> Σ (in a theory stronger than the theory of sets) defined over the principal base sets $x_1,\ldots,x_n$ and auxiliary base sets $A_1,\ldots,A_m$ has a generic structure of the form $(s_1,\ldots,s_p)$ and a typical characterization of the form

$$\langle\langle\ s_1 \in T_1 \text{ and } s_2 \in T_2 \text{ and } \ldots \text{ and } s_p \in T_p \rangle\rangle$$

where each T_j is obtained by replacing in the term $\mathfrak{P}((u \times v)\times v)$ each of the letters u and v by one of the terms x_i or A_k. In addition the <u>axiom</u> of Σ is written in the form « P and Q » , where P is the relation

$$\langle\langle\ s_1 \text{ is a functional graph and } \ldots \text{ and } s_p \text{ is a functional graph} \rangle\rangle ,$$

(which thus expresses that the s_i are the graphs of the laws of composition, (<u>external</u> if $s_i \in \mathfrak{P}((A_k \times x_i)\times x_2))$ and <u>internal</u> if $s_i \in \mathfrak{P}((x_i \times x_i)\times x_i)$). The relation Q, which expresses the supplementary conditions which the laws of composition satisfy, is generally called (by abuse of language) the <u>axiom</u> of Σ (or if a conjunction of several relations, the <u>axioms</u> of Σ). The axiom is as always required to be a transportable relation for the typification $(s_1,\ldots,s_p) \in T_1 \times\ldots\times T_p$. A structure of such a species will be called an <u>algebraic structure</u>.

We shall now give two examples of algebraic structure species.

Example 3. Let $\mathcal{T}$ be the theory of sets; in $\mathcal{T}$, the <u>species of (algebraic) structure of a group</u> has one principal base set x_1, no auxiliary base sets and a typical characterization $s_1 \in \mathfrak{P}((x_1 \times x_1)\times x_1)$ with axiom « s_1 is a law of composition of a group over x_1» . This axiom is indeed transportable for the typification T: $s_1 \in \mathfrak{P}((x_1 \times x_1) \times x_1)$ since it is equivalent to the conjunction of the following relations:

R_1: « s_1 is a law of composition everywhere defined over x_1 » which is transportable by CT24.

R_2: (associativity) $s_1 \circ (s_1 \times I_{x_1}) = S_1 \circ (I_{x_1} \times S_1) \circ J$, where J denotes the canonical mapping of $(x_1 \times x_1) \times x_1$ onto $x_1 \times (x_1 \times x_1)$; R_2 is transportable by means of CT21, CT23, and CT28.

R_3: (unit element) « $(\exists z)(z \in x_1$ and $(\forall z')((z' \in x_1) \Rightarrow (s_1(z,z') = z'$ and $s_1(z',z)) = z')))$ » which is certainly transportab/ility for the typification « T and $z \in x_1$ and $z' \in x_1$ »; transportable for T then results from CT8 and case disjunction where one observes that upon adjoining the relation $x_1 = \emptyset$ to $\mathcal{C}$, R_3 is false and hence transportable by CT7 and CT3.

R_4: (inverses) « $(\forall z)(\forall z')((z \in x_1$ and $z' \in x_1) \Rightarrow ((\exists z'')(z'' \in x_1$ and $s_1(z,z'') = z')$ and $(\exists z''')(z''' \in x_1$ and $s_1(z''',z) = z'))))$ » which is transportable for T by similar reasoning as for R_3.

The theory of groups $\mathcal{C}_\Sigma$ thus has two constants, the set x_1 and the law of composition s_1. In the theory of sets $\mathcal{C}$ we have two terms « the set of real numbers » and «the addition of real numbers ». If we substitute these terms for x_1 and s_1 respectively in the explicit axioms of $\mathcal{C}_\Sigma$, we obtain theorems of $\mathcal{C}$. Thus by C5 we may « apply the results of the theory of groups to the addition of real numbers » . One says that one has constructed a model for the theory of groups within the theory of sets. Also since the theory of groups is stronger than the theory of sets, we may apply the results of the theory of sets to the theory of groups, but if the theory of groups should prove contradictory, then the theory of sets is also.

Example 4. Take for $\mathcal{C}$, the theory of the species of structure of a field, which has (among others) the constant K as its unique principal base sets. In $\mathcal{C}$, the <u>species of structure of a (left) vector space over</u> K has E for principal base sets, K for auxiliary base set and for typical characterization $V \in \mathcal{P}((E \times E) \times E) \times \mathcal{P}((K \times E) \times E)$. $pr_1 V$ is of course the addition and $pr_2 V$ is the scalar multiplication. Its axioms are the familiar axioms for a vector space over K which are all transportable relations as may be seen by the transportability criteria already developed.

We shall now proceed to define the important notions of <u>isomorphism</u> and <u>transport of structures</u>.

Let Σ be a species of structure in a theory $\mathcal{C}$, over n principal base sets $x_1, \ldots, x_n$, with m auxiliary base sets $A_1, \ldots, A_m$. Let S be the rung construction schema over n+m letters which figures in the typical characterization of Σ, and let R be the axiom of Σ. In a theory $\mathcal{C}'$ stronger than $\mathcal{C}$, let U be a Σ-structure over $E_1, \ldots, E_n$ and U' also be a Σ-structure over $E_1', \ldots, E_n'$. Finally in $\mathcal{C}'$ let $f_i : E_i \rightarrow E_i'$ be a bijection for $1 \leq i \leq n$. Under these conditions we make the following definition:

<u>Definition 14</u>. - The multiplet of mappings $(f_1, \ldots, f_n)$ is called an <u>isomorphism of the sets</u> $E_1, \ldots, E_n$ <u>supplied with the structure U onto the sets</u> $E_1', \ldots, E_n'$ <u>supplied with the structure</u> U' if (in $\mathcal{C}'$),

$$(4) \quad \langle f_1, \ldots, f_n, I_1, \ldots, I_m \rangle^S (U) = U',$$

where $I_k : A_k \longrightarrow A_k$ is the identity mapping.

Let $\overset{-1}{f_i}$ be the inverse bijection of f_i for $1 \leqslant i \leqslant n$. Then it is an immediate result of CST3 that $\langle \overset{-1}{f_1}, \ldots, \overset{-1}{f_n}, I_1, \ldots, I_m \rangle^S(U') = U$ and hence that $(f_1, \ldots, f_n)$ is an isomorphism of $E_1', \ldots, E_n'$ supplied ~~with U;~~ onto $E_1, \ldots, E_n$ supplied with U. We say that these isomorphisms are inverses of each other.

Definition 15. - We say that $E_1', \ldots, E_n'$ supplied with U' is isomorphic to $E_1, \ldots, E_n$ supplied with U if there exists an isomorphism of $E_1, \ldots, E_n$ onto $E_1', \ldots, E_n'$, furthermore we then say that the structures U and U' are isomorphic.

CST1 and the preceding definitions immediately give the following criterion:

CST4. Let U, U' and U" be three Σ-structures over $E_1, \ldots, E_n$, $E_1', \ldots, E_n'$ and $E_1'', \ldots, E_n''$ respectively. Let $f_i : E_i \longrightarrow E_i'$ and $g_i : E_i' \longrightarrow E_i''$ be bijections for $1 \leqslant i \leqslant n$. Then if $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ are isomorphisms, $(g_1 \circ f_1, g_2 \circ f_2, \ldots, g_n \circ f_n)$ is an isomorphism.

One usually calls an isomorphism of $E_1, \ldots, E_n$ onto $E_1, \ldots, E_n$ (for the same structure) an automorphism of $E_1, \ldots, E_n$. It is then a result of CST4 and the definitions that the automorphisms of $E_1, \ldots, E_n$ form a group.

The following criterion gives another reason for the requirement that the axiom of a species of structure be a transportable relation.

CST5. In a theory $\mathcal{T}'$ stronger than $\mathcal{T}$, let U be a Σ-structure over $E_1, \ldots, E_n$ and f_i be a bijection E_i onto a set E_i' for $1 \leqslant i \leqslant n$. Then there exists over $E_1', \ldots, E_n'$ a Σ-structure (which is unique)

<u>such that $(f_1,\dots,f_n)$ is an isomorphism of $E_1,\dots,E_n$ onto $E_1',\dots,E_n'$</u>.

In effect the desired structure is nothing other than the term U' defined by the relation (4). For what remains it suffices to verify that this term is a Σ-structure, i.e., that the relation $R\{E_1',\dots,E_n',U'\}$ is true in $\mathcal{C}'$. But this is an immediate result of $R\{x_1,\dots,x_n,s\}$ being <u>transportable</u>, for then $R\{E_1',\dots,E_n',U\}$ is <u>equivalent</u> in $\mathcal{C}'$ to the relation $R\{E_1,\dots,E_n,U\}$ which is true in $\mathcal{C}'$ by hypothesis.

<u>Definition 16</u>. - We say that the structure U' is obtained by <u>transport of the structure U to the sets</u> $E_1',\dots,E_n'$ <u>by means of the bijections</u> $f_1,\dots,f_n$.

It thus amounts to say that two Σ-structures are isomorphic if one may be deduced from the other by <u>structure transport</u>.

<u>Definition 17</u>. - If two <u>arbitrary</u> structures of the same species are <u>necessarily isomorphic</u>, one says that the species of structure is <u>univalent</u>.

This is indeed the case for classical Eucliden geometry and also for the following species of structure:

1. The species of an <u>infinite monogenic group</u> ($\cong \mathbf{Z}$)
2. The species of a <u>prime field of characteristic</u> o ($\cong \mathbf{Q}$)
3. The species of a <u>complete, archemidian ordered field</u> ($\cong \mathbf{R}$)
4. The species of an <u>algebraically closed, connected, locally compact commutative field</u> ($\cong \mathbf{C}$)
5. The species of a <u>connected, locally compact, non-commutative field</u> ($\cong \mathbf{K}$).

(In fact for **Q** and **R** there are <u>no</u> automorphisms other than the identity mapping, but this is not always the case as $(x \longrightarrow -x)$: $\mathbf{Z} \rightarrow \mathbf{Z}$).

It is interesting to observe that the preceding structures are those which lie at the base of <u>classical mathematics</u>. By consrast the species of group, partially ordered set, topological space etc. (part of modern mathematics) are not univalent!

We shall now consider the notion of « relative transportability » . (We chall use the notations already developed for the transportability criteria.)

Let be a species of structure in $\mathcal{T}$, with $x_1,\dots,x_n$ for principal base sets, $A_1,\dots,A_m$ for auxiliary base sets and s_o for its generic structure; let $s_o \in S_o(x_1,\dots,x_n\ A_1,\dots,A_m)$ be the typical characterization which we will designate by T_o, and let P be the axiom of Σ ; P is thus transportable for T_o by definition.

<u>Definition 18</u>. - We shall say that a relation R is <u>transportable</u> (in $\mathcal{T}$) <u>relative to</u> Σ , <u>for the typification</u> « T_o and T » , when the relation $P \Rightarrow R$ is transportable (in $\mathcal{T}$) for « T_o and T » and the following conditions are satisfied:

1. the initial letters of T are $x_1,\dots,x_n,s_o$ (and possibly additional letters $x_1',\dots,x_r'$, $s_1,\dots,s_p$); the initial terms are $A_1,\dots,A_m$ (and possibly additional terms $A_1',\dots,A_s'$ of $\mathcal{T}$ not containing any of the initial letters of T);

2. T is of the form

   « $s_1 \in S_1(x,x',A,A')$ and ... and $s_p \in S_p(x,x',A,A')$ » ,

   where the S_j $(1 \leqslant j \leqslant p)$ are rung construction schemas over n+r+m+s letters.

We shall show that this definition is equivalent to the following assertion concerning R:

Definition 18'. - The relation $R\{x,x',s_o,s\} \Leftrightarrow R\{y,y',s_o',s'\}$ is a theorem of the theory $(\mathcal{T}_c)_\Sigma$, obtained by adjoining to the axioms of $\mathcal{T}$ the transport relation for the typification « T_o and T » and the axiom $P\{x,s\}$.

(N.B. - This condition does not signify that R is transportable in $\mathcal{T}_\Sigma$ for « T_o and T » since the x_i and s_o are constants of $\mathcal{T}_\Sigma$.)

Suppose in effect that R is transportable (in $\mathcal{T}$) relative to Σ for « T_o and T »; then the relation

(1) $(P\{x,s_o\} \Rightarrow R\{x,x',s_o,s\}) \Leftrightarrow (P\{y,s_o'\} \Rightarrow R\{y,y',s_o',s'\})$

is a theorem of $\mathcal{T}$. Also $P\{x,s_o\} \Leftrightarrow P\{y,s_o'\}$ is a theorem of $\mathcal{T}$ since P is transportable for T_o (in $\mathcal{T}$). In $\mathcal{T}_c$, the relation (1) is thus equivalent to

(2) $(P\{x,s_o\} \Rightarrow R\{x,x',s_o,s\}) \Leftrightarrow (P\{x,s_o\} \Rightarrow R\{y,y',s_o',s'\})$.

But in $(\mathcal{T}_c)_\Sigma$, $R\{x,x',s_o,s\}$ and $(P\{x,s_o\} \Rightarrow R\{x,x',s_o,s\})$ are equivalent relations; similarly, $R\{y,y',s_o',s\}$ and $(P\{x,s_o\} \Rightarrow R\{y,y',s_o',s\}$ are equivalent in $(\mathcal{T}_c)_\Sigma$. Therefore one concludes that $R\{x,x',s_o,s\} \Leftrightarrow R\{y,y',s_o',s\}$ is a theorem of $(\mathcal{T}_c)_\Sigma$.

Conversely, suppose that Definition 18' holds, then in $\mathcal{T}_c$, the relation

(3) $P\{x,s_o\} \Rightarrow (R\{x,x',s_o,s\} \Leftrightarrow R\{y,y',s_o',s\})$ is a theorem; now it is well known that the relations $B \Rightarrow (C \Leftrightarrow D)$ and $(B \Rightarrow C) \Leftrightarrow (B \Rightarrow D)$ are equivalent in every logical theory; but (2) is a theorem of $\mathcal{T}_c$ and consequently also (1), which thus proves our assertion.

Definition 19. - We will say that a term U of $\mathcal{T}$ is transportable of type S (in $\mathcal{T}$) relative to Σ, for the typification «T_o and T» if in $(\mathcal{T}_c)_\Sigma$, the relations $U \in S(x,x',A,A')$ and $U\{y,y',s_o',s'\} = f^S(U\{x,x',s_o,s\})$ are theorems.

It is possible to verify that the criteria (CT) still hold when one replaces «transportable» by «transportable relative to Σ» and (in CT7, CT9, CT16), the theory $\mathcal{T}$ by the theory $\mathcal{T}_\Sigma$. The majority of the relations and terms that one considers in the theory of a species of structure Σ are transportable relative to Σ for some suitable typification, e.g., in the theory of groups, the «neutral element», the «subgroup generaled by w, where w is a subset of the group», etc. are relatively transportable.

Suppose that R is a transportable relation relative to Σ, for a typification «T_o and T», where r = 0. In a theory $\mathcal{T}'$ stronger than $\mathcal{T}$, let $\mathcal{S}$ (resp. $\mathcal{S}'$) be a Σ-structure over $E_1,\dots,E_n$ (resp. $E_1',\dots,E_n'$) and $(g_1,\dots,g_n)$ be an isomorphism of $E_1,\dots,E_n$ supplied with $\mathcal{S}$ onto $E_1',\dots,E_n'$ supplied with $\mathcal{S}'$. Furthermore let $C_1,\dots,C_p$ be terms of $\mathcal{T}'$ such that the relations

$$C_j \in S_j(E_1,\dots,E_n, A_1,\dots,A_m, A_1',\dots,A_s')$$

are theorems of $\mathcal{T}'$ for $1 \leqslant j \leqslant p$. Let g^S be the canonical extension of $g_1,\dots,g_n$ and the identity mappings of A_k and A_h' ($1 \leqslant k \leqslant m$, $1 \leqslant h \leqslant s$) to a rung of type S over $E_1,\dots,E_n, A_1,\dots,A_m, A_1',\dots,A_s'$; one has in particular that $g^{S_o}(\mathcal{S}) = \mathcal{S}'$. Under these conditions the relation

$$R\{E_1,\dots,E_n,\mathcal{S}, C_1,\dots,C_p\} \Leftrightarrow R\{E_1',\dots,E_n',\mathcal{S}', g^{S_1}(C_1),\dots,g^{S_p}(C_p)\}$$

is a theorem of $\mathcal{T}'$.

In effect, if, in the term $f^{S_j}(s_j)$ we substitute g_i for f_i, E_i for x_i, E_i' for y_i, $\mathfrak{S}$ for s_o, and C_l for s_l $(1 \leqslant i \leqslant n,\ 1 \leqslant l \leqslant p)$ we obtained the term $g^{S_j}(C_j)$ $(1 \leqslant j \leqslant p)$. Since the same substitution effected in P, T_o, T, and in the transport relation for « T_o and T » give theorems of $\mathcal{T}'$, our assertion is an immediate result of the definition of a transportable relation relative to Σ.

Similarly from the definition, we may observe that if U is a transportable term of type S relative to Σ, for the typification « T_o and T » (with r = o), <u>the relation</u>

$$g^S(U\{E_1,\ldots,E_n, \mathfrak{S}, C_1,\ldots,C_p\}) = U\{E_1',\ldots,E_n', \mathfrak{S}', g^{S_1}(C_1),\ldots,g^{S_p}(C_p)$$

<u>is a theorem of</u> $\mathcal{T}'$.

<u>Definition 20</u>. - We say that a term $V\{x_1,\ldots,x_n,s_o\}$ of is <u>intrinsic for</u> s_o, <u>of type</u> T, provided it contains no letters other than the constants of $\mathcal{T}_\Sigma$, and is transportable relative to Σ for the typification T_o.

Because of the importance of this notion we shall restate this definition in full:

<u>Definition 20'</u>. - Let Σ be a species of structure in a theory $\mathcal{T}$, over n principal base sets, $x_1,\ldots,x_n$, with m auxiliary base sets $A_1,\ldots,A_m$; with $s_o \in T_o(x_1,\ldots,x_n, A_1,\ldots,A_m)$ as typical characterization for Σ. Let T be a rung construction schema over n+m terms. One says that a term $V\{x_1,\ldots,x_n,s_o\}$ which contains no letters other than the constants of $\mathcal{T}_\Sigma$ is <u>intrinsic for</u> s_o, <u>of type</u> $T(x_1,\ldots,x_n, A_1,\ldots,A_m)$ if it satisfies the following conditions:

1. The relation $V\{x_1,\ldots,x_n,s_o\} \in T\{x_1,\ldots,x_n, A_1,\ldots,A_m\}$ is a theorem of $\mathcal{T}_\Sigma$.

2. Let $(\mathcal{T}_c)_\Sigma$ be the theory obtained by adjoining to the axioms of $\mathcal{T}_\Sigma$ the axioms « $f_i : x_i \longrightarrow y_i$ is a bijection» for $1 \leqslant i \leqslant n$ the letters f_i, y_i being distinct from themselves and from the constants of $\mathcal{T}_\Sigma$. Let s_o' be the structure obtained on transporting s_o by $(f_1,\ldots,f_n)$, i.e., $s_o' = \langle f_1,\ldots,f_n, I_1,\ldots,I_n\rangle^{T_o}(s_o)$. Then

$$V\{y_1,\ldots,y_n,s_o'\} = \langle f_1,\ldots,f_n, I_1,\ldots,I_m\rangle^{T}(V\{x_1,\ldots,x_n,s_o\})$$

is a theorem of $(\mathcal{T}_c)_\Sigma$.

It can be shown that in the theory of groups, says, the neutral element, the group of commutators, the center, and the groups of automorphisms, etc. are intrinsic.

Let $V\{x_1,\ldots,x_n,s_o\}$ be an intrinsic term for s_o, of type T. It is immediate that the relation « $(f_1,\ldots,f_n)$ is an <u>automorphism</u> of $x_1,\ldots,x_n$ supplied with s_o» entails in $\mathcal{T}_\Sigma$, the relation $f^T(V) = V$; we shall under such conditions say that V is <u>invariant</u> for all of the automorphisms of $x_1,\ldots,x_n$ supplied with s_o. This latter condition, it should be emphasized, is not sufficient to guarantee <u>intrinsicity</u>, however.

In view of the conventions introduced concerning « the species of structure of a set » , to say that a relation (resp. term) is <u>transportable relative to the species of structure of a set</u> simply means that the relation (resp. term) is transportable in the unrelativised meaning of the term.

<u>**Definition 21.**</u> - When a term V, intrinsic for s_o, is such that in addition the relation « V is a correspondence between X and Y » (resp. « V is a mapping of V_1 into V_2 ») is a theorem of $\mathcal{T}_\Sigma$ (V_1 and V_2 being two terms also intrinsic for s_o), we say that V is a <u>canonical correspondence</u> (resp. <u>mapping</u>) for s_o. The terminology of « <u>canonical mapping</u> » introduced <u>in the theory of sets</u> is thus in accord with the conventions already introduced.

We shall now give an equivalent characterization of intrinsic mappings in the most common special case.

Let U_1 and U_2 be two terms of $\mathcal{T}$ which are intrinsic for s_o of types $\mathfrak{P}(S_1)$ and $\mathfrak{P}(S_2)$ respectively. Then a mapping $V: U_1 \to U_2$ is canonical for s_o if and only if in $(\mathcal{T}_c)_\Sigma$ $\langle f_1,\dots,f_n; I_1,\dots,I_m\rangle^{S_2} \circ$

$$V\{x_1,\dots,x_n,s_o\} = V\{y_1,..y_n,s_o'\} \circ \langle f_1,\dots,f_n; I_1,\dots,I_m\rangle^{S_1}$$

i.e., with our usual abbreviated notation, in $(\mathcal{T}_c)_\Sigma$, the following diagram is commutative:

$$\begin{tikzcd}
U_1\{x,s_o\} \arrow[r, "f^{S_1}"] \arrow[d, "V\{x,s_o\}"'] & U_1\{y,s_o'\} \arrow[d, "V\{y,s_o'\}"] \\
U_2\{x,s_o\} \arrow[r, "f^{S_2}"] & U_2\{y,s_o'\}
\end{tikzcd}$$

The above assertion is an immediate consequence of the definitions for intrinsicity when we recall that V is intrinsic, i.e., canonical under the hypothesis of the theorem iff $f^{S_1 \times S_2}\langle V\{x,s_o\}\rangle = V\{y,s_o'\}$, and that we always have $f^{S_1 \times S_2} = f^{S_1} \times f^{S_2}$.

We now shall consider the important notion of a « process of deduction » .

Definition 22. - Let Θ be a second species of structure in the theory $\mathcal{T}$, over r principal base sets $u_1,\dots,u_r$, with p auxiliary base sets $B_1,\dots,B_p$; let $t \in T(u_1,\dots,u_r, B_1,\dots,B_p)$ be the typical characterization of Θ . We call a process of deduction of a structure of species Θ from a structure of species Σ any sequence of r+1 terms $\mathcal{P}, U_1,\dots,U_r$ each intrinsic for s_o, and such that $\mathcal{P}$ is a Θ-structure over $U_1,\dots,U_r$ in the theory $\mathcal{T}_\Sigma$. (By abuse of language we will occasionally refer to the single term $\mathcal{P}$ as the process of deduction.)

Definition 23. - Let $\mathcal{T}'$ be a theory stronger than $\mathcal{T}$. If, in $\mathcal{T}'$, $\mathcal{S}$ is a Σ-structure over $E_1,\dots,E_n$, then $\mathcal{P}\{E_1,\dots,E_n,\mathcal{S}\}$ is a Θ-structure over the r sets $F_j = U_j\{E_1,\dots,E_n,\mathcal{S}\}$ $(1 \le j \le r)$, said to have been deduced from $\mathcal{S}$ by the process $\mathcal{P}$, or to have been subordinated to $\mathcal{S}$.

The hypothesis that the terms $\mathcal{P}$, $U_1,\dots,U_r$ are intrinsic for s_o entails the following criterion:

CST6. Let $(g_1,\dots,g_n)$ be an isomorphism of $E_1,\dots,E_n$, supplied with a Σ-structure $\mathcal{S}$ onto $E_1',\dots,E_n'$, supplied a Σ-structure $\mathcal{S}'$. If U_j is of type $\mathfrak{P}(T_j)$, let $h_j = \langle g_1,\dots,g_n, I_1,\dots,I_m\rangle^{T_j}$ $(1 \le j \le r)$ and let $F_j' = U_j\{E_1',\dots,E_n',\mathcal{S}'\}$ $(1 \le j \le r)$, then $(h_1,\dots,h_r)$ is an isomorphism of $F_1,\dots,F_r$ onto $F_1',\dots,F_r'$ when supplied respectively with the Θ-structures deduced from $\mathcal{S}$ and $\mathcal{S}'$ by the process $\mathcal{P}, U_1,\dots,U_r$.

Definition 24. - The mappings $(h_1,\dots,h_r)$ are said to be the isomorphism deduced from $(g_1,\dots,g_n)$ by the process $\mathcal{P}, U_1,\dots,U_r$.

Suppose that $\mathcal{P}, U_1, \ldots, U_r$ and $\mathcal{P}', U_1', \ldots, U_r'$ are both processes of deduction of a Θ-structure from a Σ-structure. Let $(V_1, \ldots, V_r)$ be a sequence of <u>canonical mappings</u> such that $V_j : U_j \longrightarrow U_j'$ is a bijection for $1 \leqslant j \leqslant r$. If, furthermore $(V_1, \ldots, V_r)$ is an isomorphism of $U_1, \ldots, U_r$ supplied with $\mathcal{P}$ onto $U_1', \ldots, U_r'$ supplied with $\mathcal{P}'$, we say that $(V_1, \ldots, V_r)$ defines a <u>canonical equivalence of the process of deduction</u> P and P'.

Let us suppose that the hypothesis of CST6 are satisfied and let us use the following notational conventions:

Let $D_j(x_1, \ldots, x_n) = U_j \{ x_1, \ldots, x_n, s_o \}$, $D_j(g_1, \ldots, g_n) = \langle g_1, \ldots, g_n, I_1, \ldots, I_m \rangle^{T_j}$ for $1 \leqslant j \leqslant r$ and $D_j'(x_1, \ldots, x_n) = U_j' \{ x_1, \ldots, x_n, s_o \}$ and $D_j'(g_1, \ldots, g_n) = \langle g_1, \ldots, g_n, I_1, \ldots, I_m \rangle^{T_j}$ and finally $F_j(x_1, \ldots, x_n) = V_j \{ x_1, \ldots, x_n, s_o \}$ for $1 \leqslant j \leqslant r$, then under the hypothesis of CST6, the following r diagrams are commutative

$$\begin{tikzcd}
D_j(E_1, \ldots, E_n) \arrow[rr, "D_j(g_1, \ldots, g_n)"] \arrow[d, "F_j(E_1, \ldots, E_n)"'] & & D_j(E_1', \ldots, E_n') \arrow[d, "F_j(E_1', \ldots, E_n')"] \\
D_j'(E_1, \ldots, E_n) \arrow[rr, "D_j'(g_1, \ldots, g_n)"] & & D_j'(E_1', \ldots, E_n')
\end{tikzcd} \quad (1 \; j \; r)$$

CST6 implies that $D_j(g_1, \ldots, g_n)$ $(1 \leqslant j \leqslant r)$ are isomorphisms and also that the $D_j'(g_1, \ldots, g_n)$ are isomorphisms. If $(F_1, \ldots, F_r)$ is a canonical equivalence, then it is also an isomorphism.

It is clear that the terms $x_1, \ldots, x_n$ are intrinsic for s_o. In many cases the terms $U_1, \ldots, U_r$ are certain of the letters $x_1, \ldots, x_n$, in such cases we speak of the Θ-structure deduced from s_o by the process $\mathcal{P}$ as <u>underlying</u> s_o. (cf. Example 1)

Suppose that Θ has the <u>same</u> base sets (both principal and auxiliary) as Σ, and also the same typical characterization. If furthermore, the axiom of Σ <u>implies</u> (in $\mathcal{T}$) the axiom of Θ, it is clear that the term s_0 is a process of deduction of a Θ-structure from a Σ-structure. We then say that Θ is <u>less rich</u> than Σ or that Σ is <u>more rich</u> than Θ. Every Σ-structure in a theory $\mathcal{T}'$ stronger than $\mathcal{T}$ is then also a Θ-structure. (cf. Example 3).

In the case that φ is a multiplet $(\varphi_1,\ldots,\varphi_q)$, one also says that the terms $\varphi_1,\ldots,\varphi_q$ constitute a process of deduction of a Θ-structure from a Σ-structure.

Example 1. The species of structure of a <u>topological group</u> has a single principal base set E, no auxiliary base sets, and a generic structure which is a pair (s_1, s_2) (s_1 being the internal law of composition over E and s_2 being the system of open sets of the topology of E). Each of the terms s_1 and s_2 is a <u>process of deduction</u> furnishing respectively the <u>underlying structure of a group</u> and of the <u>underlying structure of topology</u>. Similarly, from the structure of a module we can deduce the underlying structure of an abelian group. From the structure of a ring we can deduce the underlying structure of an abelian group and also a multiplicative semigroup, <u>etc</u>.

Example 2. If Σ and Θ the species of structure of a group (resp. ring). We may define a process of deduction associating to each group structure (resp. ring structure) the structure of a group (resp. ring) over its <u>centre</u>. If Σ is the structure of a module over a commutative ring with a unit K and Θ is the species of structure of an algebra over K we can define a process of deduction which assigns to each module over K its <u>tensor algebra</u> and its <u>exterior algebra, etc</u>.

Example 3. The species of structure of a <u>totally ordered set</u> (obtained by the adjunction of the axiom « $S \cup S^{-1} = E \times E$ » to the axioms of the structure of an ordered set is <u>richer</u> than the species of structure of an order. Similarly the species of an abelian group is richer than the species of a group and the species of a compact topology space is richer than the species of a topology, etc.

It is well known that there is « more than one way of defining a topology » (e.g., by means of open sets and closure operators) and that an abelian group and a unitary $\mathbf{Z}$-module are the « same thing » . We now show that such naive notions of « equivalence » of various species of structure can be given a satisfactory formal meaning by means of « process of deduction » .

<u>Definition 25</u>. - In the same theory $\mathcal{T}$, let Σ and Θ be two species of structure with the <u>same</u> principal base sets $x_1, \ldots, x_n$. Let s and t be the generic structures, respectively of Σ and Θ and suppose that the following conditions are satisfied:

1. One has a process of deduction $\varphi\{x_1, \ldots, x_n, s\}$ for a Θ-structure over $x_1, \ldots, x_n$ from a Σ-structure over $x_1, \ldots, x_n$.
2. One has a process of deduction $\vartheta\{x_1, \ldots, x_n, t\}$ of a Σ-structure over $x_1, \ldots, x_n$ from a Θ-structure over $x_1, \ldots, x_n$.
3. The relation $\vartheta\{x_1, \ldots, x_n, \varphi\{x_1, \ldots, x_n, s\}\} = s$ is a theorem of $\mathcal{T}_\Sigma$ and the relation $\varphi\{x_1, \ldots, x_n, \vartheta\{x_1, \ldots, x_n, t\}\} = t$ is a theorem of $\mathcal{T}_\Theta$.

Under these conditions we say that the species of structure Σ and Θ are <u>equivalent by intermediation of the process of deduction</u> φ and ϑ .

In this case for each theorem $B\{x_1,\ldots,x_n,s\}$ of $\mathcal{T}_\Sigma$, the relation $B\{x_1,\ldots,x_n,\vartheta\}$ is a theorem of $\mathcal{T}_\Theta$, and conversely, for each theorem $C\{x_1,\ldots,x_n,t\}$ of $\mathcal{T}_\Theta$, the relation $C\{x_1,\ldots,x_n,\varphi\}$ is a theorem of $\mathcal{T}_\Sigma$.

<u>Definition 26</u>. - If U is a Σ -structure, one says that the structure deduced from U by the process φ is <u>equivalent</u> to U.

Our criterion CST6 has as an immediate consequence the following criterion:

CST7. <u>Let $\mathcal{S}$ and $\mathcal{S}'$ be two Σ -structures over $(E_1,\ldots,E_n)$ and $(E_1',\ldots,E_n')$ respectively. Let $\mathcal{S}_0$ and $\mathcal{S}'_0$ be Θ-structures equivalent respectively to $\mathcal{S}$ and $\mathcal{S}'$. Then in order that $(g_1,\ldots,g_n)$ be an isomorphism of the structures $\mathcal{S}_0$ and $\mathcal{S}'_0$, it is necessary and sufficient that $(g_1,\ldots,g_n)$ be an isomorphism of the structures $\mathcal{S}$ and $\mathcal{S}'$</u> .

Example. Let Σ be the species of structure of a topology with E as its base set and V its generic structure. Consider the relation « $x \in E$ and $X \subseteq E$ and $(\forall U)((U \in V \text{ and } x \in U) \Rightarrow (X \cap U \neq \emptyset))$ » ; it admits a graph Φ with respect to the pair (X,x) and $\Phi \subseteq \mathfrak{P}(E)\times E$. $\Phi\{E,V\}$ is then a term of $\mathcal{T}$ (called the « set of pairs (X,x) such that x is in the <u>closure</u> of X for the topology V ») and we can prove that the following relations are theorems of $\mathcal{T}_\Sigma$:

(1) $\Phi(\emptyset) = \emptyset$,

(2) $(\forall Y)((Y \subseteq E) \Rightarrow (Y \subseteq \Phi(Y))$,

(3) $(\forall Y)(\forall Z)((Y \subseteq E \text{ and } Z \subseteq E) \Rightarrow ((\Phi(Y \cup Z) = \Phi(Y) \cup \Phi(Z)))$,

(4) $(\forall Y)((Y \subseteq E) \Rightarrow (\Phi(\Phi(Y)) = \Phi(Y)))$.

Now consider the species of structure Θ, with principal base set E, generic structure W and, typical characterization $W \in \mathfrak{P}(\mathfrak{P}(E)\times E)$, and axiom $W(\emptyset) = \emptyset$ and $(\forall Y)(Y \subseteq E \Rightarrow Y \subseteq W(Y))$ and $(\forall Y)(\forall Z)((Y \subseteq E \text{ and } Z \subseteq E) \Rightarrow (W(Y \cup Z) = W(Y) \cup W(Z)))$ and $(\forall Y)((Y \subseteq E) \Rightarrow (W(W(Y)) = WY))$.

Now consider the relation « $U \in E$ and $(\forall x)(x \in U \Rightarrow x \notin W(E - U))$ ». The set of all $U \in \mathfrak{P}(E)$ which satisfy this relation is a subset $\vartheta\{E,W\}$ of $\mathfrak{P}(E)$ and we can show that the following relations are theorems of $\mathcal{T}_\Theta$:

(1) $E \in \vartheta$,

(2) $(\forall M)(M \subseteq \vartheta \Rightarrow \{\bigcup X \mid X \in M\} \in \vartheta)$,

(3) $(\forall X)(\forall X)((X \in \vartheta \text{ and } Y \in \vartheta) \Rightarrow (X \cap Y) \in \vartheta))$.

Thus the terms $\Phi\{E,V\}$ and $\vartheta\{E,W\}$ verify conditions 1 and 2 and also 3 of Definition 25 and hence the species Σ and Θ are equivalent and we can consider a Θ-structure as a topology by means of the <u>process of deduction</u> $\vartheta\{E,W\}$.

We shall now show that the notion of intrinsicity can be extended so that we can define the notion of a « process of deduction from two species of structure furnishing a structure of a third species» .

In a theory $\mathcal{T}$ stronger than the theory of sets, let Σ be a species of structure over n principal base sets $x_1,\ldots,x_n$, m auxiliary base sets $A_1,\ldots,A_m$, with $s \in S(x_1,\ldots,x_n, A_1,\ldots,A_m)$ as typical characterization and $R_\Sigma\{x_1,\ldots,x_n,s\}$ as axiom. Also in $\mathcal{T}$, let Φ be a species of structure with o principal base sets $v_1,\ldots,v_o$, q auxiliary base sets $C_1,\ldots,C_q$, with $w \in W(v_1,\ldots,v_o, C_1,\ldots,C_q)$ as typical characterization and R_Φ for axiom. In addition let $\mathcal{C}_{\Sigma,\Phi}$ denote the theory obtained by adjoining to the axioms of $\mathcal{C}$, the axiom «R_Σ and R_Φ» , so that the constants of $\mathcal{C}_{\Sigma,\Phi}$ are the constants of $\mathcal{C}$ together with the letters which figure in R_Σ or in R_Φ .

<u>Definition 27</u>. - A term U of $\mathcal{C}$ will be said to be <u>bi-intrinsic</u> for (s,w), of type $V(x_1,\ldots,x_n, v_1,\ldots,v_o, A_1,\ldots,A_m, C_1,\ldots,C_q)$ provided U contains no letters other than the constants of $\mathcal{C}_{\Sigma,\Phi}$ and satisfies the following conditions:

1. the relation $U\{x, v, s, w\} \in V(x,v,A,C)$ is a theorem of $\mathcal{C}_{\Sigma,\Phi}$ where V is a rung construction schema over n+o+m+q letters.
2. let $(\mathcal{C}_c)_{\Sigma,\Phi}$ be the theory obtained by adjoining to the axioms of $\mathcal{C}_{\Sigma,\Phi}$, the axioms « $f_i : x_i \to y_i$ is a bijection » $(1 \leq i \leq n)$ and « $g_j : v_j \longrightarrow z_j$ is a bijection » $(1 \leq j \leq o)$ (the letters y_i, f_i, g_j, z_j being distinct from themselves and from the constants of $\mathcal{C}_{\Sigma,\Phi}$); let I_i be the identity mapping of A_i for $1 \leq i \leq m$ and let I_j' be the identity mapping of the C_j for $1 \leq j \leq q$. Then if s' is the structure obtained on transport of s by $(f_1,\ldots,f_n)$ and w' is the structure obtained on transport of w by $(g_1,\ldots,g_o)$, then

   $U\{y_1,\ldots,y_n, z_1,\ldots,z_n, s',w'\} =$

   $\langle f_1,\ldots,f_n, g_1,\ldots,g_n, I_1,\ldots,I_m, I_1',\ldots,I_q'\rangle^V (U')$

(where $U' = U\{x_1,\ldots,x_n, v_1,\ldots,v_o,s,w\}$)

is a theorem of $(\mathcal{C}_c)_{\Sigma,\Phi}$.

The above definition of bi-intrinsicity is thus equivalent to the requirement that U contains no letters other than the constants of $\mathcal{T}_{\Sigma,\Phi}$ and be relatively transportable both for Σ and Φ.

Example. For any species of structure Σ and Φ having only one principal base set, say x and y respectively, the term $\mathcal{F}(x,y)$ (« the set of all mappings of x into y ») is bi-intrinsic for (s,w).

Definition 28. - We shall call a process of deduction of a Θ-structure from a Σ-structure and a Φ-structure any sequence of r+1 terms of $\mathcal{T}$, $\mathcal{P}, U_1, \ldots, U_r$, each bi-intrinsic for (s,w), such that $\mathcal{P}$ is a Θ-structure over $U_1, \ldots, U_r$ in $\mathcal{T}_{\Sigma,\Phi}$, i.e.,

« $\mathcal{P}\{x,v,s,w\} \in T(U_1\{x,v,s,w\}, \ldots, U_r\{x,v,s,w\}, B_1, \ldots, B_p)$ and

$R_\Theta\{U_1\{x,v,s,w\}, \ldots, U_r\{x,v,s,w\}, \mathcal{P}\{x,v,s,w\}\}$ »

are theorems of $\mathcal{T}_{\Sigma,\Phi}$.

As an immediate consequence of this definition we have that if $\mathcal{T}'$ is a theory stronger than $\mathcal{T}$ in which $\mathcal{S}$ is a Σ-structure over $E_1, \ldots, E_n$ and $\mathcal{W}$ a Φ-structure over $F_1, \ldots, F_o$, then $\mathcal{P}\{E_1, \ldots, E_n, F_1, \ldots, F_o, \mathcal{S}, \mathcal{W}\}$ is a Θ-structure over $U_1\{E_1, \ldots, E_n, F_1, \ldots, F_o, \mathcal{S}, \mathcal{W}\}, \ldots, U_r\{E_1, \ldots, E_n, F_1, \ldots, F_o, \mathcal{S}, \mathcal{W}\}$.

Definition 29. - The Θ-structure $\mathcal{P}\{E_1, \ldots, E_n, F_1, \ldots, F_o, \mathcal{S}, \mathcal{W}\}$ is said to be the Θ-structure deduced from the pair (of Σ and Φ-structures) $(\mathcal{S}, \mathcal{W})$ by the process of deduction $\mathcal{P}, U_1, \ldots, U_r$.

In virute of the definition of bi-intrinsic terms, we have the following criterion for such a process of deduction $\mathcal{P}, U_1, \ldots, U_r$.

CST6'. Let $(f_1,\ldots,f_n)$ be an isomorphism of $E_1,\ldots,E_n$ supplied with $\mathcal{S}$ onto $E_1',\ldots,E_n'$ supplied with $\mathcal{S}'$ ($\mathcal{S}$ and $\mathcal{S}'$ both being Σ -structures) and let $(g_1,\ldots,g_o)$ be an isomorphism of $F_1,\ldots,F_o$ supplied with $\mathcal{W}$ onto $F_1',\ldots,F_o'$ supplied with $\mathcal{W}'$ ($\mathcal{W}$ and $\mathcal{W}'$ both being Φ -structures), then if U_j is of type $\mathfrak{P}(V_j)$ and we let $h_j = \langle f_1,\ldots,f_n, g_1,\ldots,g_o, I_1,\ldots,I_m, I_1',\ldots,I_q' \rangle^{V_j}$ for $1 \leqslant j \leqslant r$, we have that $(h_1,\ldots,h_r)$ is an isomorphism of the r sets $U_j\{E_1,\ldots,E_n, F_1,\ldots,F_o, \mathcal{S}, \mathcal{W}\}$ $(1 \leqslant j \leqslant r)$ onto the r sets $U_j\{E_1',\ldots,E_n', F_1',\ldots,F_o', \mathcal{S}', \mathcal{W}'\}$ $(1 \leqslant j \leqslant r)$ supplied respectively with the structures $\mathcal{P}\{E_1,\ldots,E_n, F_1,\ldots,F_o, \mathcal{S}, \mathcal{W}\}$ and $\mathcal{P}\{E_1',\ldots,E_n', F_1',\ldots,F_o', \mathcal{S}', \mathcal{W}'\}$ deduced from $(\mathcal{S}, \mathcal{W})$ and $(\mathcal{S}', \mathcal{W}')$ by the process of deduction $\mathcal{P}, U_1,\ldots,U_r$.

In effect,

$$h_j = \langle f,g,I,I' \rangle^{V_j}: \quad U_j\{E,F,\mathcal{S},\mathcal{W}\} \rightarrow U_j\{E_1',F_1',\mathcal{S}',\mathcal{W}'\}$$

is a bijection for $1 \leqslant j \leqslant r$ since U_j is bi-intrinsic for (s,w) and $(f_1,\ldots,f_n)$ and $(g_1,\ldots,g_o)$ are both isomorphisms (so that the respective structures obtained on transport of $\mathcal{S}$ and $\mathcal{W}$ are indeed $\mathcal{S}'$ and $\mathcal{W}'$).

Similarly

$$\langle h_1,\ldots,h_r, I_1'',\ldots,I_p'' \rangle^T = \langle f_1,\ldots,f_n, g_1,\ldots,g_n, I_1,\ldots,I_m, I_1',\ldots,I_q' \rangle^V$$

(where I_i'' is the identity mapping of B_k for $1 \leqslant k \leqslant p$) since $\mathcal{P}$ being of type T over $U_1,\ldots,U_r, B_1,\ldots,B_p$ implies that it must be of type V over $x_1,\ldots,x_n, v_1,\ldots,v_o, A_1,\ldots,A_m, C_1,\ldots,C_q$. Thus the required bi-intrinsicity and the fact that the (f) and (g) are isomorphisms implies that $\langle h_1,\ldots,h_r, I_1'',\ldots,I_p'' \rangle^T (\mathcal{P}\{E,F,\mathcal{S},\mathcal{W}\}) = \mathcal{P}\{E',F',\mathcal{S}',\mathcal{W}'\}$ and hence that $(h_1,\ldots,h_r)$ is an isomorphism.

<u>Definition 30</u>. - The mappings $(h_1,\dots,h_r)$ are said to be the <u>mappings deduced from</u> $((f_1,\dots,f_n), (g_1,\dots,g_o))$ <u>by the process of deduction</u> $\mathcal{P},U_1,\dots,U_r$.

N.B. The immediately preceding notions can be generalized without difficulty to <u>tri-intrinsic</u>, indeed n-<u>intrinsic terms</u> and the consequent definitions of processes of deduction from three or indeed n species can be then immediately formulated. The analog of CST6' will then follow just as easily as it has here.

We now come to the important notion of « morphisms » . For simplification, we for the moment assume the species of structure under consideration here have only a single (necessarily principal) base set.

Let Σ be a species of structure in a theory $\mathcal{T}$ stronger than the theory of sets and let x,y,s,t be four distinct letters, distinct from themselves and from the constants of $\mathcal{T}_\Sigma$. We shall use the notation $\mathfrak{F}(x,y)$ to designate the set of mappings of x into y.

Suppose that we are given a term $\sigma\{x,y,s,t\}$ of $\mathcal{T}$ which verifies the following conditions:

(MO_I) The relation « s is a Σ-structure over x and t is a Σ-structure over y » implies, in $\mathcal{T}$, the relation $\sigma\{x,y,s,t\} \subseteq \mathfrak{F}(x,y)$.

(MO_{II}) If, in a theory $\mathcal{T}'$ stronger than $\mathcal{T}$, we let E,E', and E" be three sets supplied with Σ-structures $\mathcal{S}$, $\mathcal{S}'$, and $\mathcal{S}''$, then the relation « $f \in \sigma\{E,E',\mathcal{S},\mathcal{S}'\}$ and $g \in \sigma\{E',E'',\mathcal{S}',\mathcal{S}''\}$ » implies the relation $g \circ f \in \sigma\{E,E'',\mathcal{S},\mathcal{S}''\}$.

(MO_{III}) Given, in a theory $\mathcal{T}'$ stronger than $\mathcal{T}$, two sets E and E' supplied with the Σ-structures $\mathcal{S}$ and $\mathcal{S}'$ respectively, then for a bijection $f: E \rightarrow E'$ to be an isomorphism, it is necessary and sufficient that $f \in \sigma\{E,E',\mathcal{S},\mathcal{S}'\}$ and $f^{-1} \in \sigma\{E',E,\mathcal{S}',\mathcal{S}\}$.

Definition 31. - If Σ and σ are given, we express the relation $f \in \sigma\{x,y,s,t\}$ by saying that f is a morphism (or σ-morphism) of x, furnished with s, into y, furnished with t. If (in a theory $\mathcal{C}'$ stronger than $\mathcal{C}$) E and E' are two sets furnished with Σ-structures $\mathcal{S}$ and $\mathcal{S}'$, the term $\sigma\{E,E',\mathcal{S},\mathcal{S}'\}$ is called the set of σ-morphisms of E into E' and if the context is clear simply by Hom(E,E') or Mor(E,E').

(MO_{III}) and the properties of bijections give the following criterion:

CST8. Let E and E' be two sets, each furnished with a Σ-structure. Let $f: E \rightarrow E'$ be a σ-morphism and $g: E' \rightarrow E$ also be a σ-morphism. If $g \circ f: E \rightarrow E$ is the identity mapping and $f \circ g: E' \rightarrow E'$ the identity mapping, then f is an isomorphism of E onto E' and g is its inverse isomorphism.

In case the species Σ consists of more than one principal base set, say $x_1,\dots,x_n$ and one or more auxiliary base sets $A_1,\dots,A_m$ then a σ-morphism is a system $(f_1,\dots,f_n)$, where $f_i: x_i \rightarrow y_i$ $(1 \leq i \leq n)$ such that the system verifies the analogous statements of (MO_I), (MO_{II}), and (MO_{III}).

N.B. It may be possible to define more than one term σ which satisfies (MO_I) - (MO_{III}) so that the notion of morphism in contrast to that of isomorphism is not uniquely determined by the specification of Σ.

We shall now outline the construction of a theory in which most of our previous results may be subsumed and the metamathematical device of rung construction schemas may be eliminated. This theory may tentatively be called the <u>theory of structures</u>.

Let A be an assemblage (of a theory $\mathcal{T}$) in which <u>only</u> letters and substantive signs figure. Let us call the <u>length</u> of A the total number of signs which figure in A and the <u>weight</u> of A the sum of the weights of the signs which figure in A. If A has the form A'BA" where A', B and A" are also assemblages, we shall say that the assemblage B is a <u>segment</u> of A((<u>proper segment</u> if B $\neq$ A). If A' is void we shall say that B is an <u>initial segment</u> of A. We shall say that such an assemblage A is <u>balanced</u> if its length is one greater than its weight and if for every proper initial segment B of A, we have that the length of A is less than the weight of B. If A is a balanced assemblage and begins with a substantive sign then A may be put in the form $fB_1,\ldots,B_p$, where f is a substantive sign of weight p (≥ 1) and all of the B_i are balanced. We call the assemblages B_i the assemblages <u>antecedent</u> to A.

Let $\mathcal{T}$ be a theory stronger than the theory of sets in which P is a substantive sign of weight 1, X a substantive sign of weight 2. Let $x_1,\ldots,x_n$ be distinct letters, each of which has weight 0. Let T be a balanced assemblage of the foregoing signs, i.e., of P , X , $x_1,\ldots,x_n$; such an assemblage will be called a <u>rung type over</u> $x_1,\ldots,x_n$.

From now on let $E_1,\ldots,E_n$ be n terms of a theory stronger than the theory of sets. For every rung type T over $x_1,\ldots,x_n$, we define a term $T(E_1,\ldots,E_n)$ in the following manner:

1. if T is a letter x_i, $T(E_1,\ldots,E_n)$ is the set E_i;
2. if T is of the form $\mathbf{P}$U, where U is the assemblage antecedent to T, $T(E_1,\ldots,E_n)$ is the assemblage $\mathfrak{P}(U(E_1,\ldots,E_n))$;
3. if T is of the form $\mathbf{X}$UV, where U and V are the assemblages antecedent to T, $T(E_1,\ldots,E_n)$ is the set $U(E_1,\ldots,E_n) \times V(E_1,\ldots,E_n)$.

It may be easily shown that, for each rung type T over $x_1,\ldots,x_n$, $T(E_1,\ldots,E_n)$ is a rung over the terms $E_1,\ldots,E_n$, and conversely (reasoning by induction over the length of the rung type or over the construction schema for the rung). Moreover every rung over n distinct terms may be written in one and only one manner in the form $T(x_1,\ldots,x_n)$, where T is a rung type.

The term $T(E_1,\ldots,E_n)$ will be called the realization of the rung type T over the terms $E_1,\ldots,E_n$.

In a fashion similar to the above definition but in analogy to Definition 4 we can show that one may associate to a rung type T over n letters, and to n mappings $f_1,\ldots,f_n$, a canonical extension of these mappings and we may then deduce that if two rung construction schemas S and S' over n terms are such that $S(x_1,\ldots,x_n) = S'(x_1,\ldots,x_n)$, the x_i being distinct letters, that one has $\langle f_1,\ldots,f_n\rangle^{S} = \langle f_1,\ldots,f_n\rangle^{S'}$.

Now let $\mathcal{T}$ be a theory stronger than the theory of sets, in which $\mathbf{P}$ and $\mathbf{P}^-$ are substantive signs of weight 1, $\mathbf{X}$ and $\mathbf{X}^-$ are substantive signs of weight 2.

For every assemblage A of these signs and n distinct letters $x_1,\ldots,x_n$, we define the variance of A in the following manner.

First we define the variance of the letters x_i and also the signs $\mathbf{P}$ and $\mathbf{X}$ as 0; we say that $\mathbf{P}^-$ and $\mathbf{X}^-$ have variance 1. Finally we call the variance of A the binary sum of the variances of the individual signs which figure in A, i.e., A is of 0 variance if there an <u>even</u> number of signs of variance 1, and 1 otherwise.

We now call a <u>signed rung type</u> a balanced assemblage A of the preceding signs satisfying the following two conditions:

1. The assemblages antecedent to A are signed rung types;
2. If A begins with the sign $\mathbf{X}$, the two antecedent assemblages must have 0 variance; if A begins with the sign $\mathbf{X}^-$, the two antecedent assemblages must have variance 1.

A signed rung type will be said to be <u>covariant</u> if it has variance 0, <u>contravariant</u> if it has variance 1.

If in a signed rung type A we replace $\mathbf{P}^-$ by $\mathbf{P}$ and $\mathbf{X}^-$ by $\mathbf{X}$, we obtain a rung type A*; every realization of the rung type A* over n terms $E_1,\ldots,E_n$ will be said to be a <u>realization of the signed rung type</u> A over $E_1,\ldots,E_n$ and will be denoted by $A(E_1,\ldots,E_n)$.

Let $E_1,\ldots,E_n$, $E_1',\ldots,E_n'$ be sets, and $f_i\colon E_i \longrightarrow E_i'$ be mappings for $1 \leqslant i \leqslant n$. We can easily show that to each signed rung type S over $x_1,\ldots,x_n$, we may associate a mapping $\{f_1,\ldots,f_n\}^S$ which has the following definitive properties:

1. if S is covariant (resp. contravariant), then
   $\{f_1,\ldots,f_n\}^S\colon S(E_1,\ldots,E_n) \longrightarrow S(E_1',\ldots,E_n')$
   (resp. $\{f_1,\ldots,f_n\}^S\colon S(E_1',\ldots,E_n') \longrightarrow S(E_1,\ldots,E_n)$);

2. if S is a letter x_i, $\{f_1,\ldots,f_n\}^S$ is f_i;

3. if S is $\mathbf{P}$ T (resp. $\mathbf{P}^-$ T), and if $g = \{f_1,\ldots,f_n\}^T$: $F \longrightarrow F'$, then $\{f_1,\ldots,f_n\}^S = \hat{g}$ (resp. $\overset{-1}{g}$);

4. if S is $\mathbf{X}$ TU or $\mathbf{X}^-$TU, where T and U are the antecedent assemblages, and if $\{f_1,\ldots,f_n\}^T = g: F \longrightarrow F'$ and $\{f_1,\ldots,f_n\}^U = h: G \longrightarrow G'$, then $\{f_1,\ldots,f_n\}^S = g \times h : F \times G \longrightarrow F' \times G'$.

The mapping $\{f_1,\ldots,f_n\}^S$ will be called the <u>signed canonical extension of the mappings</u> $f_1,\ldots,f_n$ <u>with respect to the signed rung type</u> S.

Of course if S is a rung type (i.e., when $\mathbf{P}^-$ and $\mathbf{X}^-$ do not figure in S) the signed canonical extension $\{f_1,\ldots,f_n\}^S = \langle f_1,\ldots,f_n\rangle^S$.

It may also be shown that if f_i: $E_i \longrightarrow E_i'$ and f_i': $E_i' \longrightarrow E_i''$ $(1 \leqslant i \leqslant n)$, one has for a covariant signed rung type S that

$$\{f_1' \circ f_1,\ldots,f_n' \circ f_n\}^S = \{f_1',\ldots,f_n'\}^S \circ \{f_1',\ldots,f_n'\}^S,$$

while for a contravariant signed rung type S

$$\{f_1' \circ f_1,\ldots,f_n' \circ f_n\}^S = \{f_1,\ldots,f_n\}^S \circ \{f_1',\ldots,f_n'\}^S.$$

Also, we have that if f_i: $E_i \longrightarrow E_i'$ is a bijection and $\overset{-1}{f_i}$ the inverse bijection for $1 \leqslant i \leqslant n$, then $\{f_1,\ldots,f_n\}^S$ is a bijection and $\{\overset{-1}{f_1},\ldots,\overset{-1}{f_n}\}^S$ its inverse bijection. Moreover in this case if S* is the (unsigned) rung type corresponding to the signed rung type S, $\{f_1,\ldots,f_n\}^S$ is equal to $\langle f_1,\ldots,f_n\rangle^{S^*}$ or to $\langle \overset{-1}{f_1},\ldots,\overset{-1}{f_n}\rangle^{S^*}$ depending on whether S is covariant or contravariant.

Let us call a signed rung type T *proper* if it has the form $\mathbf{P}$ U where U is the assemblage antecedent to T.

We define a *category type* C over $x_1,\ldots,x_n$ to be a balanced assemblage of proper signed rung types and the sign $\mathbf{X}$ all the antecedent assemblages of which are category types.

If C is a category type, then every realization of the rung type C* will be said to be a *realization of the category type* C and will be denoted by $C(E_1,\ldots,E_n)$.

Let $E_1,\ldots,E_n$, $E_1',\ldots,E_n'$ be sets and let $f_i : E_i \longrightarrow E_i'$ for $1 \leq i \leq n$. To each category type C over $x_1,\ldots,x_n$ we may associate a term $[f_1,\ldots,f_n]^C$ with the following properties:

1. if C is a signed rung type, then $[f_1,\ldots,f_n]^C = \{f_1,\ldots,f_n\}^C$;
2. if C is of the form $\mathbf{X}$ TU where T and U are assemblages not concordant (i.e., not having the same variance), then $[f_1,\ldots,f_n]^C$ is $([f_1,\ldots,f_n]^T, [f_1,\ldots,f_n]^U)$.

The term $[f_1,\ldots,f_n]^C$ will be called the *canonical extension of the mappings* $f_1,\ldots,f_n$ w.r.t. *to the category type* C.

If C is a category type over $x_1,\ldots,x_n$, then if $\mathbf{P}S_1,\ldots,\mathbf{P}S_p$ are the p proper signed rung types which figure in C, $[f_1,\ldots,f_n]^C$ may be written as $(\{f_1,\ldots,f_n\}^{\mathbf{P}S_1},\ldots,\{f_1,\ldots,f_n\}^{\mathbf{P}S_p})$.

Now let C be a category type over n+m letters. Let Σ be a species of structure with $x_1,\ldots,x_n$ for principal base sets, $A_1,\ldots,A_m$ for auxiliary base sets, whose typical characterization is of the form $s \in C(x_1,\ldots,x_n, A_1,\ldots,A_m)$. We shall show that one may define a notion of σ-morphism for this species of structure in the following manner:

Being given n sets $E_1,\ldots,E_n$ supplied with a Σ-structure $U = (U_{i_1},\ldots,U_{i_p})$, and a mapping $f_i: E_i \longrightarrow E_i'$ for $1 \leq i \leq n$, we say that $(f_1,\ldots,f_n)$ is a σ-morphism if and only if the mappings f_i verify the following conditions:

for each signed rung type $\mathbf{P}\, S_{i_j}$ figuring in C

1. if S_{i_j} is a covariant signed rung type

$$\{f_1,\ldots,f_n, I_1,\ldots,I_m\}^{S_{i_j}} \langle U_{i_j}\rangle \subseteq U'_{i_j}$$

2. if S_{i_j} is a contravariant rung type

$$\{f_1,\ldots,f_n, I_1,\ldots,I_m\}^{S_{i_j}} \langle U'_{i_j}\rangle \subseteq U_{i_j} .$$

That the mappings $(f_1,\ldots,f_n)$ which satisfy these conditions satisfy (MO_I) (MO_{II}) and (MO_{III}) follows immediately from the definitions and the properties of the canonical extension of the mapping to signed rung types which we have already outlined.

Example 1. Let Σ be the species of structure of an ordered set with

$$S_0 \in (\mathbf{P}(x \times x)\ (E) = \mathfrak{P}(E \times E)$$

as typical characterization then the above definition of σ-morphism gives the set of mappings $f: E \longrightarrow E'$ such that $f x f\langle \mathcal{S}\rangle \subseteq \mathcal{S}'$ i.e., such that $(u,v) \in \mathcal{S} \Rightarrow f x f(u,v) \in \mathcal{S}'$, but $f x f(u,v) = (f(u), f(v))$ so that in the usual notation the relation $(u,v) \in \mathcal{S} \Rightarrow (f(u),f(v)) \in \mathcal{S}'$ becomes $u \leqslant v \Rightarrow f(u) \leqslant' f(v)$ which is usually expressed by saying that f is an <u>increasing mapping</u>. If we use the contravariant category type

$\mathbf{P}(x \times^{-} x)$ to define the structure, the corresponding notion of σ-morphism gives these mappings $f: E \longrightarrow E'$ such that $u \leqslant v \Rightarrow f(v) \leqslant' f(u)$, i.e., it gives the <u>decreasing mappings</u> of E into E'. Both of these notions of morphism are the usual definitions of morphism for order sets.

Example 2. Let Σ be a species of algebraic structure having a single internal law of composition which is determined by the category type $\mathbf{P}((x \times x) \times x)$ then the above defined notion of σ-morphism gives those mappings $f: E \longrightarrow E'$ such that $\mathfrak{I}'(f(x), f(y)) = f(\mathfrak{I}(x,y))$ for $x, y \in E$ which are indeed the <u>homomorphisms</u> of E into E'. Using $\times^{-}$ we would get the <u>anti-homomorphisms</u> of E into E'. If we have more than one internal law of composition and/or an external law of composition, we again get the usual notion of homomorphism for such algebraic structures.

Example 3. Let Σ be the species of a topology with its typical characterization given by the category type $\mathbf{P}(\mathbf{P}(x))$. The above notion of σ-morphism gives those mappings $f: E \rightarrow E'$ such that $X \in V \Rightarrow f\langle X\rangle \in V'$ where V and V' are the topologies on E and E' respectively, i.e., it gives the <u>open mappings</u> of E into E'. Using the category type $\mathbf{P}(\mathbf{P}^{-}(x))$ we get those mappings f such that $X' \in V' \Rightarrow f^{-1}(X') \in V$ i.e., we get the <u>continuous mappings</u> of E into E'.

Example 4. Let Σ be the species of a topological group with the typical characterization given by the category type $\mathbf{P}((x \times x) \times x) \times \mathbf{P}(\mathbf{P}^{-}(x))$ then the above notion of σ-morphism gives the <u>continuous homomorphisms</u> of E into E'.