Virtual Resolutions of Points in Sufficiently General Position in $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract

Minimal free resolutions are an important notion in algebraic geometry and commutative algebra. The minimal free resolution of a subvariety in projective spaces provides geometric properties of the subvariety. However, if the ambient space is the product of projective spaces, the minimal free resolution can be too long. On the other hand, virtual resolutions of a subvariety of products of projective spaces can be shorter and they still provide information about the subvariety. In this thesis, we investigate sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with generic Hilbert function and in particular, points in a sufficiently general positions. We find an explicit virtual resolution of ideals of a sufficiently general set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Our proof depends upon computing some values of the mutigraded Castelnuovo-Mumford regularity and using a result of Berkesch, Erman and Smith. We also generalize one of the Berkesch, Erman and Smith's result in a special case.

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CHAPTER 1

Introduction

Many invariants in algebraic geometry and commutative algebra may be defined in terms of free resolutions. A free resolution is an exact sequence of free modules. Let R be a Noetherian ring. For every R-module M, one can construct a free resolution of free R-modules F_i which fit into an exact sequence

$$\mathcal{F} : \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0,$$

as follows: define F_0 to be the free R-module whose basis elements are mapped to a set of generators of M. Then, we define F_1 to be the free R-module whose basis elements are mapped to generators of the kernel of the map $F_0 \to M$. We define each F_i, for $i > 1$ to be the free R-module whose basis elements are mapped to the generators of the kernel of the map $\varphi_{i-1} : F_{i-1} \to F_{i-2}$.

If M is a graded module over a graded ring, e.g. the polynomial ring $S = k[x_0, x_1, \ldots, x_n]$ over $n+1$ variables, then we can define a graded version of a free resolution. Hilbert proved that every finitely generated S-module has a finite graded free resolution of length at most $n + 1$. Among graded free resolutions, the minimal free resolutions are those for which the map $\varphi_l : F_l \to F_{l-1}$, takes the standard basis of F_l to a minimal generating set of ker $(\varphi_{\ell-1})$ for all ℓ, $\ell \geq 0$ The condition of minimality is important since without minimality, resolutions are not unique (up to isomorphism).

Minimal free resolutions give us some information of a subvariety in a projective space. As an example, we can compute the Hilbert function of a variety which is used for computing the dimension and the degree of the variety. However, when the ambient space is a product of projective spaces, minimal free resolutions over the coordinate ring can be too long. However, virtual resolutions, as first defined in [**BES20**] by Berkesch, Erman and Smith, can be much shorter and they still give us some of the geometric properties.

The definition of a virtual resolution is new and there is still much to learn about them. Here are some of the works on virtual resolutions. Berkesch, Erman, and Smith [**BES20**] constructed virtual resolutions. They proved that the set of virtual resolutions of a module determines its multigraded Castelnuovo–Mumford regularity. They also showed how to extract a virtual resolution from a minimal free resolution. Loper [**Lop19**] identified two algebraic conditions that characterize when a chain complex is virtual. Kennedy [**Ken20**] also gave an algebraic condition on a complex to guarantee it is a virtual resolution. In [**GLLM21**], Gao, Li, Loper and Mattoo investigated which sets of points have a virtual resolution that is a Koszul complex on a regular sequence. They provided conditions on sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$, some of which guarantee the points have this property, and some of which guarantee the points do not have this property. A Macaulay2 package was released by Almousa, Bruce, Loper, and Sayrafi in [**ABLS20**]. They introduced the *VirtualResolutions* package that has tools to construct, display, and study virtual resolutions for products of projective spaces. The package also has tools for generating curves in $\mathbb{P}^1 \times \mathbb{P}^2$, providing sources of interesting virtual resolutions. Recently, Berkesch, Klein, Loper, and Yang [**BKLY20**] continued the research program on the notion of a virtually Cohen–Macaulay property started by Berkesch, Erman, and Smith in [**BES20**] in two related ways. Firstly, when X is a product of projective spaces, they described a large new class of virtually Cohen–Macaulay Stanley–Reisner rings. Secondly, for an arbitrary smooth projective toric variety X, they developed homological tools for assessing the virtual Cohen–Macaulay property. They also used these tools to establish relationships among the arithmetically, geometrically, and virtually Cohen–Macaulay properties.

Let $\mathbb{P}^{\underline{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be the product of projective spaces, where the n_i's are positive integers. Let $S = k[x_{ij} : 1 \leq i \leq r,\ 0 \leq j \leq n_i]$ be the coordinate ring of $\mathbb{P}^{\underline{n}}$ and, $B = \bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$ be its irrelevant ideal. Berkesch, Erman, and Smith proved the following proposition:

Proposition 1.1. [**BES20**, Proposition 1.2.] *Every finitely-generated $\mathbb{Z}^r$-graded B-saturated S-module has a virtual resolution of length at most $|\underline{n}| := n_1 + n_2 + \cdots + n_r = \dim \mathbb{P}^{\underline{n}}$.*

Therefore, by Proposition 1.1, every finitely generated $\mathbb{Z}$-graded $(x_0, x_1, \cdots, x_n)$-saturated S-module where $S = k[x_0, x_1, \cdots, x_n]$ has a virtual resolution of length at most $n = \dim \mathbb{P}^n$. The Hilbert Syzygy Theorem also asserts the existence of a finite free resolution.

Theorem 1.2. *(Hilbert Syzygy Theorem) Let $S = k[x_0, x_1, \cdots, x_n]$. Then every finitely generated S-module has a finite free resolution of length at most $n + 1$.*

Hence, Proposition 1.1 generalizes this result.

Let X to be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$, and let I_X be its defining ideal in $S = k[x_0, x_1, y_0, y_1]$. Proposition 1.1, implies the existence of virtual resolutions of length at most two for I_X. In this thesis we find an explicit virtual resolution of length two for the ideal of finitely many points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$. Specifically, we prove the following theorem which is one of the main results of our thesis.

Theorem 1.3. *(Theorem 4.7) Let X be a set of sufficiently general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then I_X has a virtual resolution of length two. In particular, if s is even, then a virtual resolution is*

$$0 \to S(-s,-1)^2 \to \begin{array}{c} S(-s/2,-1)^2 \\ \oplus \\ S(-s,0) \end{array} \to S.$$

and, if s is odd,

$$0 \to S(-s,-1)^2 \to \begin{matrix} S\left(-\frac{s-1}{2},-1\right) \\ \oplus \\ S\left(-\frac{s+1}{2},-1\right) \\ \oplus \\ S(-s,0) \end{matrix} \to S.$$

is a virtual resolution of I_X.

The structure of the thesis is as follows.

We start Chapter 2 with the definitions of graded rings and graded modules. These results are needed to define minimal free resolutions in Section 2.2. Virtual resolutions are defined geometrically by Berkesch, Erman, and Smith in [**BES20**], but there is an algebraic reformulation of the geometric definition proved by Kennedy in [**Ken20**]. We will use this as our definition for virtual resolutions. In Section 2.3 we also introduce the concept of multigraded Castelnuovo-Mumford regularity defined in [**MS04**]. We need the notion of multigraded regularity for one of the main theorems in [**BES20**] that is the key result in the proof of Theorem 4.7. Lastly, we will introduce a few concepts from algebraic geometry. Most of the content of Chapter 2 can be found in [**CLO05**] and [**Eis95**].

We begin Chapter 3 by defining the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$. We continue by focusing on the properties of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Some of the main results of Chapter 3 are from [**GMR92**] and [**GMR96**]. At the end of Chapter 3, we explain what it means to have a set of points in sufficiently general position.

In Chapter 4, we start with an example to explain our strategy in proving our main theorem, Theorem 4.7. Then we provide a series of lemmas we need to prove our main theorem.

Finally, in Chapter 5, we state three conjectures, with supporting examples. The motivation behind these conjectures is that in [**BES20**, Theorem 4.1], Berkesch, Erman and Smith prove the existence of a virtual resolution for an ideal of a set of points. In these conjectures we try to find the virtual resolutions explicitly.

One of our conjectures is the following.

Conjecture 1.4. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ that has generic Hilbert function and let $I_X \subset S = k[x_0, x_1, y_0, y_1]$ be its defining ideal. Let $B^{(a,0)} = \langle x_0, x_1\rangle^a$. The smallest value of a where the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X has the following properties.*

(1) $a \leq s-1$.
(2) *If a yields such a virtual resolution of S/I_X, then $a+1$ does as well.*

Moreover, if a is the smallest value where the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X, then this virtual resolution will be of the form

$$0 \to S(-1,-s)^s \to \begin{matrix} S(-s,0) \\ \oplus \\ S(-s+1,-1)^s \end{matrix} \to S,$$

and for $i > 0$, the virtual resolution corresponding to $(a+i,0)$ is:

$$0 \to \begin{matrix} S(-s-i,0)^{i-1} \\ \oplus \\ S(-s-i,-1)^s \end{matrix} \to \begin{matrix} S(-s-i+1,0)^i \\ \oplus \\ S(-s-i+1,-1)^s \end{matrix} \to S.$$

The idea of the conjecture above is based on the following theorem by Berkesch, Erman and Smith [**BES20**, Theorem 4.1]. In this theorem they prove the existence of an $\underline{a} = (a,0)$ such that the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X. For the conjecture above we checked more than 20 different configurations of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and found the least a for each configuration.

THEOREM 1.5. [**BES20**, Theorem 4.1] *If $Z \subset \mathbb{P}^{\underline{n}}$ is a zero-dimensional scheme and I is the corresponding B-saturated S-ideal, then there exists an $\underline{a} \in \mathbb{N}^r$ with $a_r = 0$ such that the minimal free resolution of $S/(I \cap B^{\underline{a}})$ has length equal $|\underline{n}| = \dim \mathbb{P}^{\underline{n}}$. Moreover, any $\underline{a} \in \mathbb{N}^r$ with $a_r = 0$ and other entries sufficiently positive yields such a virtual resolution of S/I.*

In Chapter 5, we give a partial answer to the conjecture above. Ee show that for $a \geq s-1$ the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X. We prove the following proposition.

PROPOSITION 1.6. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ that has generic Hilbert function and let $I_X \subset S = k[x_0, x_1, y_0, y_1]$ be its corresponding B-saturated defining ideal. If $a = s-1$, then, the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution for S/I_X. Moreover, for every number $t \in \mathbb{N}$, where $t > s-1$, the minimal free resolution of $S/(I_X \cap B^{(t,0)})$ is also a virtual resolution.*

CHAPTER 2

Background and Preliminaries on Resolutions

Let $S = k[x_0, x_1, \ldots, x_n]$ be the polynomial ring in $n+1$ variables over an algebraically closed field k. In order to study the homogeneous ideals $I = I(V)$ of projective varieties V, we study their free resolutions. In this chapter we shall recall the background on the minimal free resolutions and virtual resolutions of I. An important fact is that these resolutions have an extra structure coming from grading on the ring S. Much of the content of this section can be found in [**CLO05**] and [**Eis95**].

1. Graded Modules

In this section we collect together all the results we will need about graded modules. We start with the definition of a graded ring.

Definition 2.1. A *graded ring* is a ring R together with a direct sum decomposition

$$R = \bigoplus_{i \geq 0} R_i,$$

as abelian groups, such that

$$R_i R_j \subseteq R_{i+j} \text{ for all } i, j \geq 0.$$

Thus R_0, is a subring of R, and each R_n is an R_0-module.

A *homogeneous element* of R is an element of one of the groups R_i, and a *homogeneous ideal* of R is an ideal generated by homogeneous elements. If $f \in R$, there is a unique expression for f of the form

$$f = \sum_i f_i \text{ with } f_i \in R_i.$$

The f_i are called the *homogeneous components* of f. One can enlarge these definitions to allow components of negative degrees. In that case we shall sometimes call the result a $\mathbb{Z}$-graded ring. More generally, one can construct a ring graded by any semigroup with identity. We will discuss such a case in Chapter 3.

Example 2.2. The polynomial ring $S = k[x_0, \ldots, x_n]$ is a graded ring, where S_i is the set of all homogeneous polynomials of degree i. Now, each S_i is a S_0-module, and since $S_0 = k$, each S_i is a k-vector space.

In the following definition we define graded modules over graded rings.

DEFINITION 2.3. If $R = \bigoplus_{i \geq 0} R_i$ is a graded ring, then a *graded module* over R is an R-module M with a decomposition

$$M = \bigoplus_{i \in \mathbb{Z}} M_i,$$

as abelian groups, such that $R_i M_j \subset M_{i+j}$ for all $i \geq 0$ and $j \in \mathbb{Z}$.

It is easy to see from the definition that each M_i is a module over the subring R_0.

EXAMPLE 2.4. Let $R^m = R \oplus R \oplus \cdots \oplus R$ (m times) for $m \geq 1$. Then R^m is a graded R-module. The modules R^m are called *free* R-modules. There is a standard basis of R^m given by the set of *coordinate vectors* $e_1 := (1, 0, \ldots, 0)$, $e_2 := (0, 1, 0, \ldots, 0)$, $\ldots$, $e_m := (0, \ldots, 0, 1)$.

DEFINITION 2.5. Given a graded R-module M, we define the *twisted module* $M(n)$, with $n \in \mathbb{Z}$, as the same R-module, but with the shifted grading

$$M(n)_k = M_{n+k}.$$

EXAMPLE 2.6. The R-module $R(d)$, by Definition 2.5 is a twisted module with grading $R(d)_k = R_{d+k}$ for all $k \in \mathbb{Z}$. The modules $(R^m)(d) = R(d)^m$ are called *shifted* or *twisted graded free modules over* R. The standard basis vectors e_i from Example 2.4 still form a module basis for $R(d)^m$, but they are now defined to be homogeneous elements of degree $-d$ in the grading, since $R(d)_{-d} = R_0$. More generally, we can consider graded free R-modules of the form

$$R(d_1) \oplus \cdots \oplus R(d_m)$$

for any integers $d_1, \ldots, d_m$, where the basis vector e_i is homogeneous of degree $-d_i$ for each i.

A graded module is said to be *finitely generated* if the underlying module is finitely generated. The generators may be taken to be homogeneous [**Bou**, page 367]. If M is a finitely generated graded S-module, for each $t \in \mathbb{Z}$, the degree t homogeneous part M_t is a finite dimensional vector space over k. This leads naturally to the definition of the Hilbert function [**CLO05**, page 280].

DEFINITION 2.7. If M is a finitely generated graded S-module, then the *Hilbert function* $H_M(t)$ is defined by

$$H_M(t) := \dim_k M_t.$$

EXAMPLE 2.8. The most basic example of a graded module is $S = k[x_0, x_1, \ldots, x_n]$ considered as a (free) module over itself. Since S_t is the vector space of homogeneous polynomials of deg t in $n+1$ variables, we have

$$H_S(t) = \dim_k S_t = \binom{t+n}{n}.$$

If we adopt the convention that $\binom{b}{a} = 0$ if $a > b$, then the above formula holds for all t. Similarly, the Hilbert function of the twisted module $S(d)$ is given by

$$H_{S(d)}(t) = \dim_k S(d)_t = \binom{t+d+n}{n}, \text{ for all } t \in \mathbb{Z}.$$

If M and N are two R-modules, then we can define an R-module homomorphism between them as follows.

Definition 2.9. An *R-module homomorphism* between two R-modules M and N is an R-linear map between M and N. That is, a map $\varphi : M \to N$ is an R-module homomorphism if for all $a \in R$ and all $f, g \in M$, we have

$$\varphi(af + g) = a\varphi(f) + \varphi(g).$$

Now, let M and N be two graded R-modules. We define a graded R-module homomorphism between them as follows.

Definition 2.10. Let M, N be graded R-modules. A homomorphism of R-modules $\varphi : M \to N$ is said to a *graded R-module homomorphism of degree d* if $\varphi(M_t) \subset N_{t+d}$ for all $t \in \mathbb{Z}$.

Example 2.11. Suppose that M is a graded R-module generated by homogeneous elements $f_1, \ldots, f_m$ of degrees $d_1, \ldots, d_m$. Then we can define a graded homomorphism

$$\varphi : R(-d_1) \oplus \cdots \oplus R(-d_m) \to M$$

by defining $\varphi(e_i) = f_i$ for all $1 \le i \le m$. Note that φ is onto because $f_1, f_2, \ldots, f_m$ generates M. Also, since e_i has degree d_i, it follows that φ is a graded R-module homomorphism of degree zero.

Another example of a graded homomorphism is given by an $m \times p$ matrix A, all of whose nonzero entries are homogeneous polynomials of degree d in the ring R. Then A defines a graded homomorphism φ of degree d by matrix multiplication, i.e.,

$$\varphi : R^p \to R^m$$
$$f \mapsto Af.$$

We can also consider A as defining a graded homomorphism of degree zero from the shifted module $R(-d)^p$ to R^m. Similarly, if the entries of the jth column are all homogeneous polynomials of degree d_j, but the degree varies with the column, then A defines a graded homomorphism of degree zero

$$R(-d_1) \oplus \cdots \oplus R(-d_p) \to R^m$$

Still more generally, a graded homomorphism of degree zero

$$R(-d_1) \oplus \cdots \oplus R(-d_p) \to R(-c_1) \oplus \cdots \oplus R(-c_m)$$

is defined by an $m \times p$ matrix A where the i,jth entry $a_{ij} \in R$ is homogeneous of degree $d_j - c_i$ for all i, j. We will call a matrix A satisfying this condition for some collection d_j

of column degrees, and some collection c_i of row degrees, a *graded matrix* over R. Graded matrices appear in free resolutions of graded modules over R. We give an example after defining free resolutions (see Example 2.23).

We now give the definition of a regular sequence.

DEFINITION 2.12. If $I \subseteq S = k[x_0, x_1, y_0, y_1]$ is a bihomogeneous ideal, then a sequence $F_1, \ldots, F_r$ of elements is a regular sequence modulo I if and only if

1) $\langle I, F_1, F_2, \ldots, F_r\rangle \subset \langle x_0, x_1, y_0, y_1\rangle$
2) $\overline{F_1}$ is not a zero-divisor in S/I,
3) $\overline{F_i}$ is not a zerodividor in $S/\langle I, F_1, \ldots, F_{i-1}\rangle$.

In the following theorem, we see that the union of the associated primes of an R-module M consists of 0 and the set of zero-divisors on M (See [**Eis95**, Theorem 3.1]).

THEOREM 2.13. *Let R be a Noetherian ring and let M be a finitely generated nonzero R-module. The union of the associated primes of M consists of 0 and the set of zero-divisors on M.*

We can find the associated primes of a decomposable ideal from its minimal primary decomposition as follows (see [**AM69**, Proposition 4.7]).

PROPOSITION 2.14. *Let I be a decomposable ideal, let $I = \bigcap_{i=1}^{n} q_i$ be a minimal primary decomposition, and let $\sqrt{q_i} = p_i$. Then*

$$\bigcup_{i=1}^{n} p_i = \{x \in R : (I : x) \neq I\}$$

In particular, if the zero ideal is decomposable, the set D of zero-divisors of R is the union of the prime ideals belonging to 0.

2. Basic Algebraic Geometry Terminology

We start with the definition of projective spaces of dimension n over an algebraically closed field k.

DEFINITION 2.15. The *n-dimensional projective space* over the field k, denoted $\mathbb{P}^n$, is the set of equivalence classes of $\sim$ on $k^{n+1} \setminus \{(0, 0, \ldots, 0)\}$, where $\sim$ is defined on the nonzero points of k^{n+1} by setting $(x_0, x_1, \ldots, x_n) \sim (x'_0, x'_1, \ldots, x'_n)$ if there is a nonzero element $\lambda \in k$ such that $(x_0, x_1, \ldots, x_n) = \lambda(x'_0, x'_1, \ldots, x'_n)$. Thus, as a space we have

$$\mathbb{P}^n := (k^{n+1} \setminus \{(0, 0, \ldots, 0)\})/\sim$$

Each nonzero $(n+1)$-tuple $(x_0, x_1, \ldots, x_n) \in k^{n+1}$ defines a point P in $\mathbb{P}^n$, and we say that $[x_0 : x_1 : \cdots : x_n]$ are the homogeneous coordinates of P.

We define the projective algebraic set associated to a homogeneous ideal I.

DEFINITION 2.16. Given any homogeneous ideal I of $S = k[x_0, x_1, \cdots, x_n]$, we define the *projective algebraic set* $Z(I)$ associated to I to be

$$Z(I) = \{[a_0 : \cdots : a_n] \in \mathbb{P}^n \mid f(a_0, \dots, a_n) = 0 \text{ for all homogeneous } f \in I\}.$$

In the following definition we define projective varieties.

DEFINITION 2.17. A *projective variety* $V \subseteq \mathbb{P}^n$ is defined as

$$V = \mathbf{V}(f_1, f_2, \dots, f_s) = \{[a_0 : a_1 : \cdots : a_n] \in \mathbb{P}^n : f_i(a_0, a_1, \cdots, a_n) = 0 \text{ for all } 1 \leq i \leq s\},$$

where $f_i \in S = k[x_0, x_1, \dots, x_n]$. The *homogeneous coordinate ring* of V is defined to be the quotient ring

$$k[V] = S/I(V),$$

where $I(V) = \langle f_1, f_2, \dots, f_s \rangle$.

The next theorem gives the projective ideal-variety correspondence [**CLO15**, Theorem 10, page 384].

THEOREM 2.18. *Let* $B = \langle x_0, x_1, \dots, x_n \rangle \subseteq S = k[x_0, x_1, \dots, x_n]$ *be the irrelevant ideal. There is a bijective correspondence*

$$\{ \text{ non-empty subvarieties of } \mathbb{P}^n \} \iff \{ \text{ homogeneous radical ideals not equal to } B \}.$$

3. Minimal Free Resolutions

In this section our goal is to define a *minimal free resolution* of an R-module M. First, we define *free resolutions*. We then define minimal free resolutions by adding some conditions on free resolutions.

DEFINITION 2.19. Let M be an R-module. A *projective resolution* of M is a complex

$$\mathcal{F} : \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

of projective R-modules such that $\mathcal{F}$ has no homology, i.e., $\text{Im}\varphi_i = \ker\varphi_{i-1}$, except at F_0.

A free resolution is a projective resolution where all the projective modules are free modules.

DEFINITION 2.20. Let M be an R-module. A *free resolution* of M is an exact sequence of the form

$$\mathcal{F} : \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0$$

where for all i, $F_i \cong R^{r_i}$ is a free R-module for some positive integer r_i. If there is an l such that $F_{l+1} = F_{l+2} = \cdots = 0$, but $F_l \neq 0$, then we shall say that $\mathcal{F}$ is a *finite resolution of length* l.

We define the ith syzygy module of an R-module M as follows [**Pee11**, Page 38].

Definition 2.21. The kernel of the map $\varphi_{i-1} : F_{i-1} \to F_{i-2}$ is called the ith *syzygy module* of M and denoted by $\mathrm{Syz}_i^R(M)$. We can see that $\mathrm{Syz}_i^R(\mathrm{Syz}_j^R(M))$=$\mathrm{Syz}_{i+j}^R(M)$.

If R is a graded ring and M is a graded R-module we define graded free resolutions to be as follows.

Definition 2.22. A free resolution $\mathcal{F}$ is a *graded free resolution* if R is a graded ring, M is a graded R-module, the F_i are graded free R-modules, and the maps are homogeneous maps of degree 0.

In the following example, we give the graded free resolution of the ideal for the degree two Veronese surface.

Example 2.23. The degree two Veronese surface $V \subset \mathbb{P}^5$ is the image of the mapping given in homogeneous coordinates by

$$\varphi : \mathbb{P}^2 \to \mathbb{P}^5$$
$$[x_0 : x_1 : x_2] \mapsto [x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2].$$

The homogeneous ideal $I(V) \subset k[x_0, x_1, \ldots, x_5]$ is:

$$I(V) = \langle x_0x_3 - x_1^2, x_0x_4 - x_1x_2, x_0x_5 - x_2^2, x_1x_4 - x_2x_3, x_1x_5 - x_2x_4, x_3x_5 - x_4^2\rangle.$$

Using Macaulay2 [**GS**], we find that there exists a graded free resolution for $R/I(V)$ of the form

$$0 \to R(-4)^3 \xrightarrow{\varphi_3} R(-3)^8 \xrightarrow{\varphi_2} R(-2)^6 \xrightarrow{\varphi_1} R \to R/I(V) \to 0.$$

where

$$\varphi_1 = \begin{pmatrix} x_0x_3 - x_1^2 & x_0x_4 - x_1x_2 & x_0x_5 - x_2^2 & x_1x_4 - x_2x_3 & x_1x_5 - x_2x_4 & x_3x_5 - x_4^2 \end{pmatrix},$$

$$\varphi_2 = \begin{pmatrix} -x_2 & 0 & x_4 & 0 & x_5 & 0 & 0 & 0 \\ x_1 & -x_2 & -x_3 & x_4 & -x_4 & x_5 & 0 & 0 \\ 0 & x_1 & 0 & -x_3 & 0 & -x_4 & 0 & 0 \\ -x_0 & 0 & x_1 & x_2 & 0 & 0 & -x_4 & x_5 \\ 0 & -x_0 & 0 & 0 & x_1 & x_2 & x_3 & -x_4 \\ 0 & 0 & 0 & x_0 & -x_0 & 0 & -x_1 & x_2 \end{pmatrix},$$

and,

$$\varphi_3 = \begin{pmatrix} -x_4 & -x_5 & 0 \\ x_3 & x_4 & 0 \\ -x_2 & 0 & -x_5 \\ x_1 & 0 & x_4 \\ 0 & -x_2 & x_4 \\ 0 & x_1 & -x_3 \\ x_0 & 0 & x_2 \\ 0 & -x_0 & x_1 \end{pmatrix}.$$

The next theorem states that every finitely generated graded S-module has a graded resolution of finite length [**Eis95**, Theorem 1.13].

THEOREM 2.24. *(Hilbert Syzygy Theorem) Let $S = k[x_0, x_1, \ldots, x_n]$ be the polynomial ring in $n+1$ variables. Then every finitely generated graded S-module has a finite graded free resolution of length $\leq n+1$.*

We define a minimal free resolution as follows.

DEFINITION 2.25. Suppose that

$$\cdots \to F_l \xrightarrow{\varphi_l} F_{l-1} \to \cdots \to F_0 \to M \to 0$$

is a graded free resolution of M. Then the resolution is *minimal* if for every $l \geq 1$, the nonzero entries of the graded matrix of φ_l have positive degree.

EXAMPLE 2.26. By looking at the maps in Example 2.23 we can see that all the nonzero entries have positive degrees. Thus, the free resolution in Example 2.23 is minimal.

DEFINITION 2.27. Two graded resolutions $\cdots \to F_0 \xrightarrow{\varphi_0} M \to 0$ and $\cdots \to G_0 \xrightarrow{\psi_0} M \to 0$ are *isomorphic* if there are graded isomorphisms $\alpha_l : F_l \to G_l$ of degree zero for all $\ell \geq 0$ such that $\psi_0 \circ \alpha_0 = \varphi_0$ and, for every $l \geq 1$, the diagram

$$\begin{array}{ccc} F_l & \xrightarrow{\varphi_l} & F_{l-1} \\ {\scriptstyle \alpha_l}\downarrow & & \downarrow{\scriptstyle \alpha_{l-1}} \\ G_l & \xrightarrow{\psi_l} & G_{l-1} \end{array}$$

commutes, meaning $\alpha_{l-1} \circ \varphi_l = \psi_l \circ \alpha_l$.

The following theorem states that a finitely generated module M has a unique minimal resolution up to isomorphism (see [**CLO05**, Theorem 3.13]).

THEOREM 2.28. *Any two minimal resolutions of M are isomorphic in the sense of Definition 2.27.*

4. Virtual Resolutions

A virtual resolution was defined by Berkesch, Erman, and Smith in [**BES20**, Definition 1.1] as follows.

DEFINITION 2.29. A free complex $\mathcal{F} : \cdots \to F_2 \to F_1 \to F_0$ of $\mathrm{Pic}(X)$-graded S-modules is called a *virtual resolution* of a $\mathrm{Pic}(X)$-graded S-module M if the corresponding complex $\widetilde{F}$ of vector bundles on X is a locally-free resolution of the sheaf $\widetilde{M}$.

There is an equivalent algebraic condition for a complex $\mathcal{F}$ to be a virtual resolution proved by Kennedy in [**Ken20**, Theorem 4.9], and we will use this formulation instead of

Definition 2.29 above. Therefore, we will not give the precise definitions of all the terms used in Definition 2.29. But first, to state the algebraic conditions, we need to define the irrelevant ideal, saturation, homology modules and local cohomology modules.

Let $\mathbb{P}^{\underline{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ denote the product of projective spaces with dimension vector $\underline{n} := (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r_+$ over a field k. Let $S := k[x_{i,j} : 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the coordinate ring of $\mathbb{P}^{\underline{n}}$. If $e_1, e_2, \cdots, e_r$ is the standard basis of $\mathbb{Z}^r$, then the polynomial ring S has the $\mathbb{Z}^r$-grading induced by $\deg(x_{i,j}) := e_i$.

We define the irrelevant ideal of $\mathbb{P}^{\underline{n}}$ as follows.

Definition 2.30. Let $S := k[x_{i,j} : 1 \leq i \leq r, 0 \leq j \leq n_i]$ be a polynomial ring. The *irrelevant ideal* of S is defined by

$$B := \bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle.$$

We define the B-saturation of an ideal as follows:

Definition 2.31. Let B be the irrelevant ideal in $S := k[x_{i,j} : 1 \leq i \leq r, 0 \leq j \leq n_i]$. We define the *$B$-saturation* ideal of an ideal $I \subset S$ to be

$$(I : B^\infty) = \{f \in S \mid fB^n \subset I \text{ for some } n \in \mathbb{N}\}$$

If $I = (I : B^\infty)$, we say I is *B-saturated.*

Example 2.32. Let $S = k[x_0, x_1, y_0, y_1, y_2]$ be a polynomial ring associated to $\mathbb{P}^1 \times \mathbb{P}^2$ and let $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle$ be its irrelevant ideal. Let

$$I = \langle y_1 - 43y_2,\ y_0 - 28y_2,\ 18x_0y_2 - x_1y_2,\ 34776x_0^3 - 3516x_0^2x_1 + 106x_0x_1^2 - x_1^3 \rangle.$$

By using Macaulay2, we calculate the B-saturation of I to be

$$(I : B^\infty) = \langle y_1 - 43y_2, y_0 - 28y_2, 18x_0 - x_1 \rangle$$

Example 2.33. Let $S = k[x_0, x_1, y_0, y_1]$ be a polynomial ring associated to $\mathbb{P}^1 \times \mathbb{P}^1$ and let $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle$ be its irrelevant ideal. Let $I = \langle 2275y_0^2 - 100y_0y_1 + y_1^2, 2275x_1y_0 - 1944x_0y_1 - 11x_1y_1, 6825x_0y_0 - 267x_0y_1 + 2x_1y_1, 2916x_0^2 - 117x_0x_1 + x_1^2 \rangle$. By using Macaulay2, we can see that $I = (I : B^\infty)$, which means that, I is B-saturated.

We define the homology modules of a complex as follows.

Definition 2.34. A *complex of R-modules* $\mathcal{F}$

$$\mathcal{F} : \cdots \to F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots,$$

is a sequence of modules F_i and maps $F_i \xrightarrow{\varphi_i} F_{i-1}$ such that the compositions $F_{i+1} \xrightarrow{\varphi_{i+1}} F_i \xrightarrow{\varphi_i} F_{i-1}$ are all zero. The *homology* of this complex at F_i is the module

$$H_i(\mathcal{F}) = \ker\varphi_i / \mathrm{im}\varphi_{i+1}.$$

Now, we define the B-power torsion module of M as follows:

DEFINITION 2.35. For an S-module M, we define the *B-power torsion module of M* to be

$$\Gamma_B(M) := \{m \in M \mid B^t m = 0 \text{ for some } t \in \mathbb{N}\},$$

i.e., the set of all elements annihilated by some power of B.

It is easy to check that $\Gamma_B(M)$ is a submodule of M.

The following theorem gives the algebraic condition for a complex to be a virtual resolution [**Ken20**, Theorem 4.9].

THEOREM 2.36. *Let M be a finitely generated S-module and let*

$$\mathcal{F} := \cdots \to F_2 \to F_1 \to F_0$$

be a complex of free S-modules satisfying:

(1) *For each $i > 0$ there is some power t such that $B^t H_i(\mathcal{F}) = 0$, and*
(2) $H_0(\mathcal{F})/\Gamma_B(H_0(\mathcal{F})) \cong M/\Gamma_B(M)$.

Then $\mathcal{F}$ is a virtual resolution of M.

The following proposition [**BES20**, Proposition 1.2] shows the existence of a virtual resolution for a finitely generated $\mathbb{Z}^r$-graded B-saturated S-module.

PROPOSITION 2.37. *Every finitely generated $\mathbb{Z}^r$-graded B-saturated S-module has a virtual resolution of length at most $|\underline{n}| := n_1 + n_2 + \cdots + n_r = \dim \mathbb{P}^{\underline{n}}$.*

Theorem 2.45 below will give us a way to get a virtual resolution from the minimal free resolution of a module ([**BES20**, Theorem 1.3]). To state it, we first need to define the *multigraded Castelnuovo-Mumford regularity*, which is a generalization of Castelnuovo-Mumford regularity. This definition was first introduced by Maclagan-Smith in [**MS04**].

We start with the definition of Castelnuovo-Mumford regularity. The *Castelnuovo-Mumford regularity*, or simply the *regularity* of an ideal in S, is an important measure of how complicated the ideal is. Regularity is actually a property of a complex, defined as follows [**Eis05**].

DEFINITION 2.38. Let $S = k[x_0, x_1, \ldots, x_r]$ and let

$$\mathcal{F} : \cdots \to F_i \to F_{i-1} \to \cdots$$

be a graded complex of free S-modules, with $F_i = \sum_j S(-a_{i,j})$. The *Castelnuovo-Mumford regularity* of $\mathcal{F}$ is the supremum of the numbers $a_{i,j} - i$.

For a finitely generated graded S-module M,the regularity of M is defined to be the regularity of a minimal graded free resolution of M. We will write reg M for this number. If $X \subset \mathbb{P}^r$ is a projective variety and I_X is its associated ideal, then reg I_X is called the *regularity* of X, denoted reg X.

EXAMPLE 2.39. Let M be a free S-module. Then the regularity of M is the supremum of the degrees of a set of homogeneous minimal generators of M.

NOTATION 2.40. Let $\underline{p} = (p_1, \ldots, p_k)$. We denote by $\underline{p} + \mathbb{N}^k$ the set

$$\{(a_1, \ldots, a_k) \mid a_1 \geq p_1, \ldots, a_k \geq p_k\}$$

DEFINITION 2.41. Let $i \in \mathbb{Z}$ and set

$$\mathbb{N}^k[i] := \bigcup (\frac{i}{|i|}\underline{p} + \mathbb{N}^k) \subseteq \mathbb{Z}^k$$

where the union is over all $\underline{p} \in \mathbb{N}^k$ whose coordinates sum to $|i|$.

EXAMPLE 2.42. Let $i = -1$. Then $\mathbb{N}^k[i] = \mathbb{N}^k[-1] = \bigcup(-\underline{p} + \mathbb{N}^k) \subseteq \mathbb{Z}^k$, where the union is over all $\underline{p} \in \mathbb{N}^k$ whose coordinates sum to 1. Thus,

$$\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 0, 1)\}$$

are all the possible values for $\underline{p}$.

We defined $\Gamma_I(M)$ for an ideal I in S in Definition 2.35. We now define the ith local cohomology as follows. From this definition, $H^0_I(M) = \Gamma_I(M)$.

DEFINITION 2.43. Let I be an ideal in S. we define

$$H^i_I(M) \cong \lim_{n\to\infty} \operatorname{Ext}^i_S(S/I^n, M).$$

Since we will not use this definition directly, we will not give precise definition of the terms used in Definition 2.43.

For a finitely generated $\mathbb{N}^k$-graded S-module M, the multigraded regularity of M, which is a subset of $\mathbb{Z}^k$, is defined as follows.

DEFINITION 2.44. Let M be a finitely generated $\mathbb{N}^k$-graded S-module. If $\underline{m} \in \mathbb{Z}^k$, we say that M is a *$\underline{m}$-regular* if $H^i_B(M)_{\underline{p}} = 0$ for all $\underline{p} \in \underline{m} + \mathbb{N}^k[1-i]$ for all $i > 0$. The *multigraded regularity* of M, denoted $\operatorname{reg}_B(M)$, is the set of all $\underline{m}$ for which M is $\underline{m}$-regular.

The next theorem gives us a way to get a virtual resolution from the minimal free resolution of a module ([**BES20**, Theorem 1.3]).

THEOREM 2.45. *Let M be a finitely generated $\mathbb{Z}^r$-graded B-saturated S-module that is $\underline{d}$-regular. If G is the free subcomplex of a minimal free resolution of M consisting of all summands generated in degree at most $\underline{d} + \underline{n}$, then G is a virtual resolution of M.*

By this theorem, if I_X is the defining ideal of a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ which is $\underline{d}$-regular, and $\mathcal{F}$ is its minimal free resolution, then $S(-i, -j)$ appears in the resolution if $(i, j) \leq \underline{d} + (1, 1)$.

CHAPTER 3

Points in $\mathbb{P}^1 \times \mathbb{P}^1$

In this chapter we define the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$. Then, we explain algebraic properties of the defining ideal I_X of a set of points X in $\mathbb{P}^1 \times \mathbb{P}^1$. In Section 2 we provide some results we need about multigraded regularity for points in $\mathbb{P}^1 \times \mathbb{P}^1$. Much of the content of this section can be found in [**GVT15**].

1. Generic Points in $\mathbb{P}^1 \times \mathbb{P}^1$

We start by defining the biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$.

DEFINITION 3.1. The biprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ is defined as the set of equivalence classes of $(k^2 \setminus \{(0,0)\}) \times (k^2 \setminus \{(0,0)\})$ with respect to the relation $\sim$, where

$$(a_1, a_2) \times (b_1, b_2) \sim (a'_1, a'_2) \times (b'_1, b'_2)$$

if $(a_1, a_2) = (\lambda_1 a'_1, \lambda_1 a'_2)$ and $(b_1, b_2) = (\lambda_2 b'_1, \lambda_2 b'_2)$ for some nonzero $\lambda_1, \lambda_2 \in k$.

If $(a_1, a_2) \times (b_1, b_2) \in (k^2 \setminus \{(0,0)\}) \times (k^2 \setminus \{(0,0)\})$, then the equivalence class of $(a_1, a_2) \times (b_1, b_2)$ is called a *point* in $\mathbb{P}^1 \times \mathbb{P}^1$, denoted $[a_1 : a_2] \times [b_1 : b_2]$. It follows that $[a_0 : a_1]$, respectively $[b_0 : b_1]$, is a point of $\mathbb{P}^1$.

Let $S = k[x_0, x_1, y_0, y_1]$ be the coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ and let $B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1 \rangle$ be its irrelevant ideal. Then the polynomial ring S has the $\mathbb{N}^2$-grading induced by

$$\deg(x_0) = \deg(x_1) = (1, 0) \quad \text{and} \quad \deg(y_0) = \deg(y_1) = (0, 1). \tag{3.1}$$

REMARK 3.2. Let $S = k[x_0, x_1, y_0, y_1]$ and let $\mathbb{N} = \{0, 1, \dots\}$. Then S equipped with the grading in Equation 3.1 is an $\mathbb{N}^2$-graded (bigraded) ring, where $S = \bigoplus_{(i,j) \in \mathbb{N}^2} S_{i,j}$, and $S_{i,j}$ is the finite dimensional vector space over k that is spanned by all monomials of the form $x_0^{\alpha_0} x_1^{\alpha_1} y_0^{\beta_0} y_1^{\beta_1}$, where $\alpha_0 + \alpha_1 = i$ and $\beta_0 + \beta_1 = j$. Thus it can be seen that

$$\dim_k S_{i,j} = \binom{i+1}{i} \binom{j+1}{j} = (i+1)(j+1).$$

Compare this to the example for graded rings, given in Example 2.8.

We say that an element $F \in S$ is *bihomogeneous* if $F \in S_{i,j}$ for some $(i, j) \in \mathbb{N}^2$. If F is bihomogeneous, we say its degree is $\deg(F) = (i, j)$. Any polynomial $F \in S$ can be written uniquely as $F = F_1 + \cdots + F_t$ where each F_i is bihomogeneous. We call the F_i's the *bihomogeneous terms* of F. Suppose that $I = (F_1, \dots, F_r) \subseteq S$ is an ideal. If each F_i is bihomogeneous, then we say that I is a *bihomogeneous ideal*. Just as in the standard graded case, it can be shown that I is a bihomogeneous ideal if and only if for every $F \in I$, all of the bihomogeneous terms of F also belong to I.

We now define the bigraded modules over the bigraded ring S.

Definition 3.3. An S-module M is a *bigraded* S-module if it has a direct sum decomposition

$$M = \bigoplus_{(i,j)\in\mathbb{Z}^2} M_{i,j}$$

with the property that $S_{i,j}M_{k,l} \subseteq M_{i+k,j+l}$ for all (i,j), $(k,l) \in \mathbb{Z}^2$.

If I is a bihomogeneous ideal of S, then I and S/I are both examples of bigraded S-modules.

Notation 3.4. Let $\preceq$ denote the natural partial order on the elements of $\mathbb{Z}^2$ defined by $(a,b) \preceq (c,d)$ in $\mathbb{Z}^2$ if and only if $a \leq c$ and $b \leq d$.

Example 3.5. Another example of a bigraded module is the polynomial ring S but with a shifted grading. Specifically, let $(a,b) \in \mathbb{Z}^2$. Then $S(-a,-b)$ is the polynomial ring with a shifted bigrading: the (i,j)-th graded piece of $S(-a,-b)$ is defined to be

$$S(-a,-b)_{i,j} := S_{i-a,j-b}.$$

Note that $S_{i,j} = 0$ if $(0,0) \npreceq (i,j)$.

Since $S_{i,j} = 0$ if $(0,0) \npreceq (i,j)$, we can also consider S as an $\mathbb{N}^2$-graded ring.

The next lemma shows that if we have a nonzero ith syzygy of a degree $\underline{d}$, we must have at least two $(i-1)$th syzygies of degrees less than $\underline{d}$. In Chapter 4, we will use the following lemma and Theorem 2.45 to prove our main result.

Lemma 3.6. *Let I be a bigraded ideal in $S = k[x_0, x_1, y_0, y_1]$. If $S(-a,-b)$ appears in the ith step of the minimal free resolution of I, then there exist $S(-a_1,-b_1)$ and $S(-a_2,-b_2)$ in the $(i-1)$st step of the minimal free resolution, where, $(a_1,b_1) \prec (a,b)$ and $(a_2,b_2) \prec (a,b)$.*

Note that $a_1 = a$ or $b_1 = b$ is allowed, but not both.

Proof. Let

$$\mathcal{F} : \cdots \to F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} I \to 0$$

be the minimal free resolution for I. As we stated before, $\mathrm{Syz}(\mathrm{Syz}^S_{j-1}(I))=\mathrm{Syz}^S_j(I)$ for all $j > 1$. Let $\mathrm{Syz}^S_i(I) = \langle g_1, \ldots, g_t\rangle$ be a system of homogeneous generators. Let $f = (f_1, \ldots, f_t) \in \mathrm{Syz}^S_{i+1}(I)$ be an element in F_i. So, by the definition we have the relation $f_1g_1 + f_2g_2 + \cdots + f_tg_t = 0$. In particular, $t \geq 2$, i.e., there are at least two generators of $\mathrm{Syz}^S_i(I)$. Suppose that $\deg(f) = (a,b)$. We have $\varphi_i(f) \in \mathrm{Im}\varphi_i = \ker\varphi_{i-1}$, since we have an exact sequence. Hence, $\varphi_i(f) \in \mathrm{Syz}^S_i(I)$. Therefore, there exists $a_1, \ldots, a_t \in S$ such that $\varphi_i(f) = a_1g_1 + \ldots + a_tg_t$. Moreover, $g_i \in F_{i-1}$ and $f \in F_i$ and since $\mathcal{F}$ is a minimal free resolution, for every $j \geq 1$, the nonzero entries of the graded matrix of φ_j have positive degree. Hence, there should exist at least two nonzero generators g_k and g_l with degree less than the degree of f. □

As we stated earlier, a point $P \in \mathbb{P}^1 \times \mathbb{P}^1$ has the form $P = A \times B$ where $A, B \in \mathbb{P}^1$. Given a point $P = A \times B$, its associated bihomogeneous ideal is given by

$$I_P = \{F \in S \mid F(P) = 0\} \subset S = k[x_0, x_1, y_0, y_1].$$

The following theorem gives some properties about I_P. The proof of the theorem can be found in [**GVT15**, Theorem 3.1].

THEOREM 3.7. *Let I_P be the bihomogeneous ideal in the bigraded ring $S = k[x_0, x_1, y_0, y_1]$ associated with a point $P \in \mathbb{P}^1 \times \mathbb{P}^1$. Then*

(1) *I_P is a prime ideal of S.*
(2) *$I_P = \langle H, V\rangle$ where deg(H) $= (1, 0)$ and deg(V) $= (0, 1)$.*
(3) *Let $X = \{P_1, \ldots, P_s\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s distinct points and suppose that I_{P_i} is the ideal associated with the point P_i. Then $I_X = I_{P_1} \cap I_{P_2} \cap \cdots \cap I_{Ps}$.*

The following corollary is contained in the proof [**GVT15**, Theorem 3.1].

COROLLARY 3.8. *Let $P = A \times B \in \mathbb{P}^1 \times \mathbb{P}^1$. If $A = [a_0 : a_1] \in \mathbb{P}^1$ and $B = [b_0 : b_1] \in \mathbb{P}^1$, then $I_P = \langle a_1x_0 - a_0x_1, b_1y_0 - b_0y_1\rangle$.*

EXAMPLE 3.9. Let $X = \{[1 : 2] \times [3 : 4], [1 : 3] \times [1 : 4]\}$. Then

$$I_1 = I_{[1:2]\times[3:4]} = \langle x_1 - 2x_0, 4y_0 - 3y_1\rangle,$$

$$I_2 = I_{[1:3]\times[1:4]} = \langle x_1 - 3x_0, y_1 - 4y_0\rangle,$$

and, by the previous theorem $I_X = I_1 \cap I_2$. Therefore
$I_X = \langle 16y_0^2 - 16y_0y_1 + 3y_1^2, 4x_1y_0 - 12x_0y_1 + 3x_1y_1, 4x_0y_0 - 7x_0y_1 + 2x_1y_1, 6x_0^2 - 5x_0x_1 + x_1^2\rangle$.

We now introduce a way to present sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$. On $\mathbb{P}^1 \times \mathbb{P}^1$ there exist two families of lines $\{H_C\}$ and $\{V_C\}$, each parametrized by $C \in \mathbb{P}^1$, with the property that if $A \neq B \in \mathbb{P}^1$, then $H_A \cap H_B = \emptyset$ and $V_A \cap V_B = \emptyset$, and for all $A, B \in \mathbb{P}^1$, $H_A \cap V_B = A \times B$ is a point on $\mathbb{P}^1 \times \mathbb{P}^1$. We can thus view $\mathbb{P}^1 \times \mathbb{P}^1$ as a grid with horizontal and vertical rulings. A point $P = [a_0 : a_1] \times [b_0 : b_1] \in \mathbb{P}^1 \times \mathbb{P}^1$ can be viewed as the intersection of the horizontal ruling defined by the degree (1, 0) line $H = a_1x_0 - a_0x_1$ and the vertical ruling defined by the degree (0, 1) line $V = b_1y_0 - b_0y_1$ (see [**GVT15**, Page 22]).

Let $S = k[x_0, x_1, y_0, y_1]$ and let I be a bihomogeneous ideal of S. We define the Hilbert function for the bigraded module S/I as follows.

DEFINITION 3.10. Let I be a bihomogeneous ideal of $S = k[x_0, x_1, y_0, y_1]$. The *Hilbert function* of S/I is the numerical function $H_{S/I} : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$H_{S/I}(i, j) := \dim_k(S/I)_{i,j} = \dim_k S_{i,j} - \dim_k I_{i,j}.$$

NOTATION 3.11. When S/I is a bigraded ring, we write the output of the Hilbert function of S/I as an infinite matrix where the initial row and column are indexed with 0.

NOTATION 3.12. Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and let I_X be its ideal. We denote the Hilbert function of S/I_X by H_X.

The following definition (see [**GVT15**, Lemma 3.25]) distinguishes certain sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ by values of its Hilbert function.

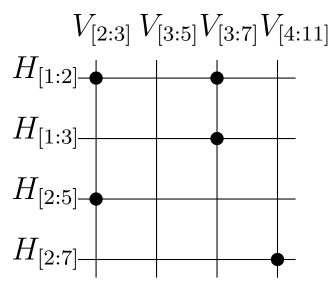
DEFINITION 3.13. Let X be a finite set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ with Hilbert function H_X. If

$$H_X(i,j) = \min\{(i+1)(j+1),\ s\} \text{ for all } (i,j) \in \mathbb{N}^2$$

then the Hilbert function is called *maximal*. A set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ is said to have *generic Hilbert function* if its Hilbert function is maximal.

We motivate this terminology in the following example.

EXAMPLE 3.14. Let X be the set of points given in the following diagram

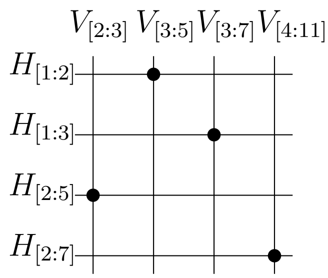


i.e., $X = \{[1:2]\times[2:3], [1:2]\times[3:7], [1:3]\times[3:7], [2:5]\times[2:3], [2:7]\times[4:11]\}$. By using Macaulay2 for computing the Hilbert function of I_X, we can see that this is an example of a set of points that does not have a generic Hilbert function.

$$H_X = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & \\ 2 & 4 & 5 & 5 & 5 & \\ 3 & 5 & 5 & 5 & 5 & \cdots \\ 4 & 5 & 5 & 5 & 5 & \\ 4 & 5 & 5 & 5 & 5 & \\ & & \vdots & & & \end{bmatrix}.$$

However in the following example, the points in the set Y have a generic Hilbert function.

Let Y be the set of points given in the following diagram



i.e., $Y = \{[1:2] \times [3:5], [1:3] \times [3:7], [2:5] \times [2:3], [2:7] \times [4:11]\}$. Notice that all points in Y lie on distinct horizontal and vertical lines.

Example 3.15. Let X be a set of four points that have a generic Hilbert function. Then its Hilbert matrix is

$$H_X = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & \\ 2 & 4 & 4 & 4 & 4 & \\ 3 & 4 & 4 & 4 & 4 & \cdots \\ 4 & 4 & 4 & 4 & 4 & \\ 4 & 4 & 4 & 4 & 4 & \\ & & \vdots & & & \end{bmatrix}.$$

If Y is a set of seven points in $\mathbb{P}^1 \times \mathbb{P}^1$ that have a generic Hilbert function. Then its Hilbert function is

$$H_Y = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 7 & \\ 2 & 4 & 6 & 7 & 7 & 7 & 7 & 7 & \\ 3 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & \\ 4 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & \cdots \\ 5 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & \\ 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & \\ 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & \\ & & & \vdots & & & & & \end{bmatrix}.$$

Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a set of points. Then the first difference and the second difference functions of H_X can be computed from the Hilbert function. As we will see, in some cases H_X will give us information about the resolution of X.

Definition 3.16. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a function. The *first difference* function of H, denoted ΔH, is the function $\Delta H : \mathbb{N}^2 \to \mathbb{N}$ defined by

$$\Delta H(i,j) := H(i,j) - H(i-1,j) - H(i,j-1) + H(i-1,j-1)$$

where $H(i,j) = 0$ if $(i,j) \not\succeq (0,0)$.

EXAMPLE 3.17. Continuing Example 3.15, the first difference matrix for H_X is

$$\Delta H_X = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & \\ 1 & 1 & -1 & -1 & 0 & \\ 1 & -1 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ & & \vdots & & & \end{bmatrix},$$

and the first difference matrix for H_Y is

$$\Delta H_Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & \\ 1 & 1 & 1 & 0 & -1 & -1 & -1 & 0 & \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \vdots & & & & & \end{bmatrix}.$$

DEFINITION 3.18. Let $H : \mathbb{N}^2 \to \mathbb{N}$ be a function. Let $\Delta H_X = (c_{i,j})$ be the first difference function. We define the *second difference* function to be $\Delta^2 H = \Delta H(i,j) - \Delta H(i-1,j) - \Delta H(i,j-1) + \Delta H(i-1,j-1)$.

EXAMPLE 3.19. Continuing Example 3.17, the second difference matrix for H_X is:

$$\Delta^2 H_X = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & \\ 0 & 0 & -2 & 0 & 2 & 0 & \\ 0 & -2 & 3 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ -1 & 2 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & & \vdots & & & & \end{bmatrix}.$$

and the second difference matrix for H_Y is

$$\Delta^2 H_Y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & 0 & \\ 0 & 0 & -2 & 1 & 2 & 0 & 0 & -1 & 0 & \\ 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & & \vdots & & & & & & \end{bmatrix}.$$

In order to derive some results about the resolution of I_X, we recall some results from homological algebra. The depth of a module S/I is an important invariant that is defined as follows.

Definition 3.20. The *depth* of S/I, denoted $\operatorname{depth}(S/I)$, is the length of the maximum regular sequence modulo I.

The projective dimension of an S-module M is defined as follows.

Definition 3.21. The *projective dimension* of an S-module M, denoted $\operatorname{proj-dim}(M)$, is the length of the minimal free resolution of M.

This means that if M admits a finite projective resolution, the minimal length among all finite projective resolutions of M is the projective dimension. If M does not admit a finite projective resolution, then by convention the projective dimension is said to be infinite. The projective dimension can be thought of as a measure of how far M is from being a free module, since finitely generated modules with projective dimension 0 are free. We note that over $S = k[x_0, x_1, \cdots, x_n]$ every finitely generated graded projective module is free. This explains why the length of a minimal free resolution is called the projective dimension [**MS13**, page 553].

Next, we will define the notion of a height of a prime ideal of S/I for a bihomegenous ideal I, and the Krull dimension of S/I.

Definition 3.22. If $I \subseteq S$ is a bihomogeneous ideal, then the *height* of a prime ideal P in S/I, denoted $\operatorname{ht}_{S/I}(P)$, is the largest integer t such that there exist prime ideals P_i of S/I for $0 \leq i \leq t$ such that $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_{t-1} \subsetneq P_t = P$. For any ideal I of S, the *Krull dimension* of S/I, denoted $\operatorname{K-dim}(S/I)$, is

$$\operatorname{K-dim}(S/I) := \sup\{\operatorname{ht}_{S/I}(P) \mid P \text{ a prime ideal of } S/I\}.$$

Example 3.23. The Krull dimension of S where $S = k[x_0, x_1, y_0, y_1]$, is the number of variables, which is four. To prove this, we know

$$(x_0, x_1, y_0, y_1) \supset (x_0, x_1, y_0) \supset (x_0, x_1) \supset (x_0) \supset (0).$$

is a sequence of prime ideals in S. By [**Eis95**, Theorem A], we can see there is no longer sequence of prime ideals for S. So, K-dim$(S) = 4$.

The following is a special case of **Auslander-Buchsbaum Formula** [**Eis95**, Theorem 19.9].

THEOREM 3.24. *Let I be a bihomogeneous ideal in the ring $S = k[x_0, x_1, y_0, y_1]$. Then*

$$\text{proj-dim}(S/I) + \text{depth}(S/I) = \text{K-dim}(S) = 4.$$

Given a finitely generated module M, we define the *minimal number of generators* of the module M, often denoted by $\mu(M)$, to be the smallest number of elements in any generating set of M. We call a sets of generators *unshortenable* if it has no proper subset that generates M. Unshortenable sets of generators are minimal, and any set of generators contains an unshortenable set [**CLO15**, Section 5.4].

Now, let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with associated ideal I_X. The bigraded minimal free resolution of I_X has either length two or three (see [**GMR92**, page 268]). We will see in Proposition 3.26 [**GMR92**, Proposition 3.3] that the bigraded minimal free resolution of I_X has the form

$$0 \to \bigoplus_{i=1}^{p} S(-a_{3i}, -a'_{3i}) \hookrightarrow \bigoplus_{i=1}^{n} S(-a_{2i}, -a'_{2i}) \to \bigoplus_{i=1}^{m} S(-a_{1i}, -a'_{1i}) \twoheadrightarrow I_X \to 0,$$

where the morphisms are of bidegree $(0, 0)$. With the notation of the resolution above, we set the following:

$$\alpha_{hk} := \#\{(a_{1i}, a'_{1i}) = (h, k)\}, \tag{3.2}$$

which gives us number of minimal generators of I of degree (h, k), and

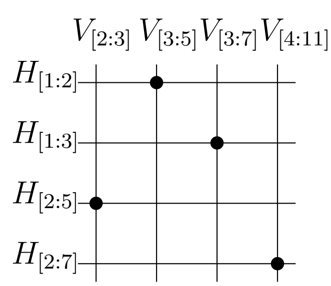
$$\beta_{hk} := \#\{(a_{2i}, a'_{2i}) = (h, k)\}, \tag{3.3}$$

that is the number of summands of the form $S(-h, -k)$ that appears in the first step of the minimal free resolution of I, and

$$\gamma_{hk} := \#\{(a_{3i}, a'_{3i}) = (h, k)\}, \tag{3.4}$$

which is the number of summands of the form $S(-h, -k)$ that appears in the second step of the minimal free resolution of I.

EXAMPLE 3.25. Let X be the following sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$



Thus, $I_X = I_{[1:2]\times[3:5]} \cap I_{[1:3]\times[3:7]} \cap I_{[2:5]\times[2:3]} \cap I_{[2:7]\times[4:1]}$. By using Macaulay2 we get the minimal free resolution

$$0 \to \begin{array}{c} S(-4,-2) \\ \oplus \\ S(-2,-4) \end{array} \to \begin{array}{c} S(-4,-1)^2 \\ \oplus \\ S(-2,-2)^3 \\ \oplus \\ S(-1,-4)^2 \end{array} \to \begin{array}{c} S(-4,0) \\ \oplus \\ S(-2,-1)^2 \\ \oplus \\ S(-1,-2)^2 \\ \oplus \\ S(0,-4) \end{array} \to I_X \to 0.$$

Thus, $\alpha_{21} = 2, \alpha_{12} = 2, \alpha_{04} = 1, \alpha_{40} = 1, \beta_{22} = 3, \beta_{14} = 2, \beta_{41} = 2, \gamma_{24} = 1$, and $\gamma_{42} = 1$. Also, $\alpha_{ij} = \beta_{ij} = \gamma_{ij} = 0$ for all other i, j.

Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with $H_X = (m_{ij})$, $\Delta H_X = (c_{ij})$ and $\Delta^2 H_X = (d_{ij})$. The following proposition gives us some information about the resolutions of points on $\mathbb{P}^1 \times \mathbb{P}^1$ [**GMR92**, Proposition 3.3].

PROPOSITION 3.26. *Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ and let*

$$0 \to \bigoplus_{i=1}^{p} S(-a_{3i}, -a'_{3i}) \hookrightarrow \bigoplus_{i=1}^{n} S(-a_{2i}, -a'_{2i}) \to \bigoplus_{i=1}^{m} S(-a_{1i}, -a'_{1i}) \twoheadrightarrow I_X \to 0$$

be the minimal free resolution of I_X. Then we have:

(i) $n + 1 = m + p$;

(ii) *the following relations between the given resolution of I_X and the functions $H_X = (m_{ij})$, $\Delta H_X = (c_{ij})$ and $\Delta^2 H_X = (d_{ij})$ hold:*
  a) $m_{rs} = (r+1)(s+1) - \sum_{h \le r\ k \le s} (r+1-h)(s+1-k)(\alpha_{hk} - \beta_{hk} + \gamma_{hk})$,
  b) $c_{rs} = \sum_{h \le r\ k \le s} (\alpha_{hk} - \beta_{hk} + \gamma_{hk})$,
  c) $d_{00} = 1$,
  d) *for every* $(r, s) \succ (0, 0)$ $d_{rs} = -\alpha_{rs} + \beta_{rs} - \gamma_{rs}$.

EXAMPLE 3.27. In Example 3.25 it can be seen that $n = 7, m = 6$, and, $p = 2$, and we have $n + 1 = 8 = m + p$. Also, from Examples 3.15, 3.17, and 3.19, we have the Hilbert function, the first difference function and the second difference function. By looking at the second difference function

$$\Delta^2 H_X = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & \\ 0 & 0 & -2 & 0 & 2 & \\ 0 & -2 & 3 & 0 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \\ -1 & 2 & -1 & 0 & 0 & \\ & & \vdots & & & \end{bmatrix}.$$

we see that for example, $3 = d_{22} = -\alpha_{22} + \beta_{22} - \gamma_{22} = 0 + 3 - 0 = 3$.

The following theorem gives us some information about the minimal generators of I_X. It is a special case of [**GMR96**, Theorem 4.3].

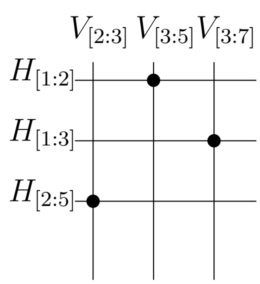
Theorem 3.28. *For each integer $s \geq 1$, there exists a dense open-subset U of $(\mathbb{P}^1 \times \mathbb{P}^1)^s$ such that for every $(P_1, \ldots, P_s) \in U$, the set of points $X = \{P_1, \ldots, P_s\}$ has the generic Hilbert function and the number α_{ij} of minimal generators of the homogeneous saturated ideal I_X of X can be read in the second difference function in the following way: for any degree (i, j) such that*

$$d_{ij} < 0 \text{ and } d_{is} > 0 \text{ for some } s > j \text{ or}$$
$$d_{ij} < 0 \text{ and } d_{rj} > 0 \text{ for some } r > i$$

we have $\alpha_{ij} = -d_{ij}$. Furthermore, these numbers give all the minimal generators of I_X.

Definition 3.29. A set of s points $X = \{P_1, \ldots, P_s\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is in *sufficiently general position* if $(P_1, \ldots, P_s)$ belongs to the open set of the above theorem.

Example 3.30. Let X be the following sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$



i.e., $X = \{[1:2] \times [3:5], [1:3] \times [3:7], [2:5] \times [2:3]\}$. Then its Hilbert function is

$$H_X = \begin{bmatrix} 1 & 2 & 3 & 3 & \\ 2 & 3 & 3 & 3 & \\ 3 & 3 & 3 & 3 & \cdots \\ 3 & 3 & 3 & 3 & \\ & & \vdots & & \end{bmatrix},$$

the first difference function for H_X is

$$\Delta H_X = \begin{bmatrix} 1 & 1 & 1 & 0 & \\ 1 & 0 & -1 & 0 & \\ 1 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \\ & & \vdots & & \end{bmatrix},$$

and its second difference function is

$$\Delta^2 H_X = \begin{bmatrix} 1 & 0 & 0 & -1 & \\ 0 & -1 & -1 & 2 & \\ 0 & -1 & 2 & -1 & \dots \\ -1 & 2 & -1 & 0 & \\ & & \vdots & & \end{bmatrix}.$$

For instance, we can see from function $\Delta^2 H_X = (d_{ij})$ that, $d_{03} = -1 < 0$ and $d_{13} = 2 > 0$. Hence by Theorem 3.28, $\alpha_{03} = -d_{03} = 1$. Also $\alpha_{12} = -d_{12} = 1$ and $\alpha_{11} = -d_{11} = 1$.Moreover, $\alpha_{21} = -d_{21} = 1$ and $\alpha_{03} = -d_{03} = 1$. We could also see this from the fact that $\Delta^2 H_X = (d_{ij})$ is symmetric. Therefore, the zeroth step of the minimal free resolution of X will be

$$S(-3,0) \oplus S(-2,-1) \oplus S(-1,-2) \oplus S(-1,-1) \oplus S(0,-3).$$

We saw that if X is a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ with generic Hilbert function, the structure of I_X, the defining ideal of X has interesting properties. One of them is given in the following proposition (See [**HVT04**, Proposition 2.3]).

PROPOSITION 3.31. *Let I_X be the defining ideal of s points $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ with generic Hilbert function. If $\underline{t} = (t_1, t_2) \in \mathbb{N}^2$ is such that $t_1 + t_2 \geq s$, and $t_2 > 0$, then $(I_X, y_0)_{\underline{t}} = S_{\underline{t}}$*

2. Multigraded Regularity for Points in $\mathbb{P}^1 \times \mathbb{P}^1$

We stated the definition of multigraded Castelnuovo-Mumford regularity in Definition 2.44. In this section we collect together all the results we need about this multigraded Castelnuovo-Mumford regularity of sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$ to prove our main theorem in Chapter 4.

If X is a set of points with generic Hilbert function in $\mathbb{P}^1 \times \mathbb{P}^1$, we can compute $\text{reg}_B(X)$ from $H_X(\underline{i})$ by the following theorem. This theorem is a special case of [**MS04**, Proposition 6.7].

THEOREM 3.32. *Let X be a set of points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ with generic Hilbert function. Then $\underline{i} \in \text{reg}_B(X)$ if and only if $H_X(\underline{i}) = |X|$.*

Specially, if X is a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ that has the generic Hilbert function, then we have:

COROLLARY 3.33. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ that has the generic Hilbert function. Then $(s-1, 0) \in \text{reg}_B(X)$.*

PROOF. When X is a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ that have the generic Hilbert function, by Definition 3.13 its Hilbert function is maximal. Thus by Theorem 3.32

$$\text{reg}_B(X) = \{(i,j) | \dim_k S_{i,j} \geq s\}.$$

Since

$$\dim_k S_{s-1,0} = \binom{s-1+1}{1}\binom{1+0}{1} = s \geq s,$$

by Remark 3.2, we conclude that $(s-1,0) \in \mathrm{reg}_B(X)$. □

COROLLARY 3.34. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ with the generic Hilbert function and let I_X be its ideal. Then for any $\underline{d} = (i,j) \succeq (s-1,0)$, I_X is $\underline{d}$-regular.*

PROOF. From Corollary 3.33, $(s-1,0) \in \mathrm{reg}_B(X)$. Notice that if $H(i,j) = |X|$, then for any $\underline{d} \succeq (i,j)$, $H(\underline{d}) = |X|$ (see [**GVT15**, Theorem 3.27]. Hence by Theorem 3.32 for any $\underline{d} = (i,j) \succeq (s-1,0)$, I_X is $\underline{d}$-regular. □

CHAPTER 4

Virtual Resolutions of Points in $\mathbb{P}^1 \times \mathbb{P}^1$

The main result of this chapter, Theorem 4.7, finds an explicit virtual resolution of length two for a set of s points in sufficiently general position in $\mathbb{P}^1 \times \mathbb{P}^1$. In [**BES20**], Berkesch, Erman and Smith only proved the existence of a virtual resolution of length n for a set of points in a multi-projective space, where n is the dimension of the space. However, in this chapter we find such virtual resolutions explicitly, for certain sets of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

We start with an example of four points with the generic Hilbert function in $\mathbb{P}^1 \times \mathbb{P}^1$, and find a virtual resolution for it. This example also illustrates how we prove the main theorem.

Example 4.1. In Examples 3.15, 3.17 and 3.19, we found the Hilbert function, first difference function, and second difference function for a set of four points with generic Hilbert function. By Proposition 3.26, the resolution must be of the form:

$$0 \to \bigoplus_{i=1}^{p} S(-a_{3i}, -a'_{3i}) \to \bigoplus_{i=1}^{n} S(-a_{2i}, -a'_{2i}) \to \bigoplus_{i=1}^{m} S(-a_{1i}, -a'_{1i}) \to I_X \to 0$$

Our strategy is to compute some of the constants a_{ij} and a'_{ij} in the resolution and then use Theorem 2.45 to find a virtual resolution of length two. We follow the same notation as introduced in Equations 3.2, 3.3 and 3.4 in Section 3.1. In particular, α_{ij} denotes the number of minimal generators of I_X of degree (i, j). By looking at the second difference function for X, $\Delta^2 H_X = (d_{ij})$, in Example 3.19, we see that $d_{40} = d_{04} = -1$, and, $d_{41} = d_{14} = 2$. So, by Theorem 3.28, $\alpha_{40} = \alpha_{04} = 1$. By the same argument, $\alpha_{12} = \alpha_{21} = 2$, and since there are no other entries d_{ij} of $\Delta^2 H_X = (d_{ij})$ that satisfies the conditions of Theorem 3.28, the generators are only of degrees $(4, 0)$, $(0, 4)$, $(1, 2)$, and, $(2, 1)$. So the resolution will be of the form

$$0 \to \bigoplus_{i=1}^{p} S(-a_{3i}, -a'_{3i}) \to \bigoplus_{i=1}^{n} S(-a_{2i}, -a'_{2i}) \to \begin{array}{c} S(-2,-1)^2 \\ \oplus \\ S(-1,-2)^2 \\ \oplus \\ S(-4,0) \\ \oplus \\ S(0,-4) \end{array} \to I_X \to 0.$$

Let β_{ij} and γ_{ij} be the number of syzygies of degree (i, j) and the number of second syzygies of degree (i, j), respectively. In Lemma 3.6, we proved that if we have an sth syzygy of degree (i, j), then there exist at least two $(s-1)$th syzygies with degrees strictly less than (i, j). So, $\beta_{ij} = 0$ for (i, j) less than or equal to the degrees of the generators. Moreover, $\alpha_{40} = \alpha_{04} = 1$ and $\alpha_{i0} = \alpha_{0i} = 0$ for $i \neq 4$, so, β_{i0} is zero by Lemma 3.6. Since $\alpha_{12} = \alpha_{21} = 2$, there may exist syzygies of degrees $(3, 1)$, $(1, 3)$, $(4, 1)$, and $(1, 4)$. From the second difference matrix, we have $0 = d_{13}$, and as we stated earlier, $\alpha_{13} = 0$. So, by Proposition 3.26(ii) part (d), we have $0 = d_{13} = 0 + \beta_{13} - \gamma_{13}$. So, $\beta_{13} = \gamma_{13}$. However,there are no syzygies of degrees less than $(1, 3)$, therefore, by Lemma 3.6, $\gamma_{13} = 0$. Hence by Lemma 3.26(ii) part (d), $\beta_{13} = 0$. Thus, since there are no $\beta_{ij} \neq 0$, for $(i, j) \prec (1, 4)$, by Lemma 3.6, $\gamma_{14} = 0$.

Since all the functions H_X, ΔH_X and $\Delta^2 H_X$ are symmetric, γ_{41} is also 0. This proves that there are no second syzygies of degrees $(i, 1)$ or $(1, j)$ for $i, j \leq 4$. Moreover, since $d_{14} = 2 = -\alpha_{14} + \beta_{14} - \gamma_{14}$, and $\alpha_{14} = \gamma_{14} = 0$ by Theorem 3.28, we conclude $\beta_{14} = 2$, and by symmetry, $\beta_{41} = 2$.

Since X is a set of 4 points that has generic Hilbert function, by Corollary 3.33, I_X is $\underline{d}$-regular for $\underline{d} = (3, 0)$. Since $\underline{n} = (1, 1)$, then by Theorem 2.45, the free subcomplex of a minimal free resolution of I_X consisting of all the summands generated in degree at most $(3, 0) + (1, 1) = (4, 1)$ is a virtual resolution of I_X. However, since we proved $\gamma_{ij} = 0$ for all $(i, j) \preceq (4, 1)$, the virtual resolution is of length two.

In particular,

$$0 \to S(-4,-1)^2 \to \begin{matrix} S(-2,-1)^2 \\ \oplus \\ S(-4,0) \end{matrix} \to S,$$

is a virtual resolution.

The example above shows how to use the Theorem 2.45 and Corollary 3.33 to find a virtual resolution of length two for a sufficiently general set of points.

Now we consider a more general case. Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points that is in sufficiently general position. Then X has the generic Hilbert function, i.e., $H_X(i, j) = \min\{(i + 1)(j + 1), s\}$. Let $H_X = (m_{ij})$, $\Delta H_X = (c_{ij})$, and $\Delta^2 H_X = (d_{ij})$ be the Hilbert function, the first difference function, and the second difference function, respectively. As we saw in Theorem 3.28 and Proposition 3.26, the d_{ij}'s give us information about α_{ij}, β_{ij}, and, γ_{ij}.

The next lemma finds α_{i0}, β_{i0}, and, γ_{i0} by computing the d_{i0}. We need this lemma in order to find virtual resolutions for a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$.

LEMMA 4.2. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points in sufficiently general position, and let I_X be its defining ideal. Then for all $0 \leq i \leq s$ we have*

$$\alpha_{i0} = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta_{i0} = \gamma_{i0} = 0$$

PROOF. By the definition of the first and the second difference functions we get the following relations:

$$c_{ij} = m_{ij} + m_{(i-1)(j-1)} - m_{(i-1)j} - m_{i(j-1)}, \text{ and } d_{ij} = c_{ij} + c_{(i-1)(j-1)} - c_{(i-1)j} - c_{i(j-1)}.$$

From these relations we get

$$d_{i0} = c_{i0} + c_{(i-1)(0-1)} - c_{(i-1)0} - c_{i(0-1)} = c_{i0} - c_{(i-1)0},$$

since $c_{(i-1)(-1)}$ and $c_{i(-1)}$ are zero. However,

$$c_{i0} = m_{i0} + m_{(i-1)(0-1)} - m_{(i-1)0} - m_{i(0-1)} = m_{i0} - m_{(i-1)0},$$

since $m_{(i-1)(-1)}$ and $m_{i(-1)}$ are zero. Therefore, we have

$$d_{i0} = m_{i0} - 2m_{(i-1)0} + m_{(i-2)0}.$$

The same procedure will give a relation for d_{i1}. We have the following expressions

$$d_{i0} = m_{i0} - 2m_{(i-1)0} + m_{(i-2)0} \quad \text{and} \quad d_{i1} = m_{i1} - 2m_{(i-1)1} + m_{(i-2)1} - 2d_{i0}. \tag{4.1}$$

Moreover, since X has a generic Hilbert function, $m_{ij} = H_X(i,j) = \min\{(i+1)(j+1), s\}$. From Equation 4.1 and the fact that $m_{ij} = H_X(i,j) = \min\{(i+1)(j+1), s\}$ we get

$$d_{i0} = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = s \\ 0 & \text{otherwise.} \end{cases}$$

We see that $d_{s0} = -1$, and $d_{s1} = 2$, so by Theorem 3.28, $d_{s0} = -\alpha_{s0} = -1$, and $\alpha_{i0} = 0$ for all $i < s$. Also, by Proposition 3.26 (ii), $d_{i0} = -\alpha_{i0} + \beta_{i0} - \gamma_{i0}$, so $\beta_{i0} = \gamma_{i0}$ for all i. Since $\alpha_{i0} = 0$ for $i < s$, by Lemma 3.6, $\beta_{i0} = 0$ for all i, so $\gamma_{i0} = 0$ for all $0 \leq i \leq s$.

Therefore, for all $0 \leq i \leq s$,

$$\alpha_{i0} = \begin{cases} 1 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \beta_{i0} = \gamma_{i0} = 0$$

□

In order to find the nonzero values of d_{i1} for all i, we will check two cases. First, we assume that s is even.

LEMMA 4.3. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points in sufficiently general position, where s is even, and let I_X be its defining ideal. Then for all $0 \leq i \leq s$ we have*

$$\alpha_{i1} = \begin{cases} 2 & \text{if } i = \frac{s}{2} \\ 0 & \text{otherwise} \end{cases}, \quad \beta_{i1} = \begin{cases} 2 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \gamma_{i1} = 0.$$

PROOF. Because $H_X(i,j) = \min\{(i+1)(j+1), s\}$ we have

$$H_X(i,1) = m_{i1} = \begin{cases} 2(i+1) & \text{if } i < \frac{s}{2} - 1 \\ s & \text{if } i \geq \frac{s}{2} - 1. \end{cases}$$

So, by 4.1 we get,

$$d_{i1} = \begin{cases} -2 & \text{if } i = \frac{s}{2} \\ 2 & \text{if } i = s \\ 0 & \text{otherwise.} \end{cases}$$

We see that $d_{\frac{s}{2}1} = -2 < 0$ and $d_{s1} = 2 > 0$. So by Theorem 3.28, $d_{\frac{s}{2}1} = -2 = -\alpha_{\frac{s}{2}1}$, and for $i \neq \frac{s}{2}$, $\alpha_{i1} = 0$. So, by Lemma 3.6, we can conclude that $\beta_{i1} = 0$ for $i \leq \frac{s}{2}$. (Notice that by Lemma 4.2, $\alpha_{i0} = 0$ for $i \leq \frac{s}{2}$).

By Proposition 3.26 (ii), we have $d_{i1} = -\alpha_{i1} + \beta_{i1} - \gamma_{i1}$. For $i < \frac{s}{2}$, $d_{i1} = \alpha_{i1} = \beta_{i1} = 0$, so $\gamma_{i1} = 0$. For $i = \frac{s}{2}$, $d_{i1} = -\alpha_{i1} = -2$ and $\beta_{i1} = 0$ so $\gamma_{i1} = 0$. For $i = \frac{s}{2}+1$, $d_{i1} = \alpha_{i1} = 0$. Therefore, by Proposition 3.26 (ii), we have $\beta_{i1} = \gamma_{i1}$. However, $\beta_{j1} = \beta_{j0} = 0$ for all $j \leq i$, i.e., there are no first syzygies of degree less than $(i,1)$. Hence, $\gamma_{i1} = 0$. If we continue this process, we see that for $i < s$, $\beta_{i1} = \gamma_{i1} = 0$. For $i = s$, $\alpha_{i1} = 0$, so by Proposition 3.26 (ii), $2 = d_{i1} = \beta_{i1} - \gamma_{i1}$. However, $\beta_{j1} = \beta_{j0} = 0$ for $j < i$, so there are no syzygies of degree less than $(i,1)$. Hence $\gamma_{i1} = 0$ and $\beta_{i1} = 2$. Therefore,

$$\alpha_{i1} = \begin{cases} 2, & \text{if } i = \frac{s}{2} \\ 0, & \text{otherwise} \end{cases}, \quad \beta_{i1} = \begin{cases} 2, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad \gamma_{i1} = 0 \quad \text{for all } 0 \leq i \leq s.$$

□

In the next lemma, we prove a similar result for the case s is odd.

LEMMA 4.4. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points in sufficiently general position, where s is odd, and let I_X be its defining ideal. Then for all $0 \leq i \leq s$ we have*

$$\alpha_{i1} = \begin{cases} 1 & \text{if } i = \frac{s-1}{2} \text{ or } i = \frac{s+1}{2} \\ 0 & \text{otherwise} \end{cases}, \quad \beta_{i1} = \begin{cases} 2 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \gamma_{i1} = 0.$$

PROOF. In this case

$$m_{i1} = \begin{cases} 2(i+1), & \text{if } i < \frac{(s-1)}{2} \\ s, & \text{if } i \geq \frac{(s-1)}{2}. \end{cases} \tag{4.2}$$

By 4.1 we get,

$$d_{i1} = \begin{cases} -1, & \text{if } i = \frac{s-1}{2} \text{ or } i = \frac{s+1}{2} \\ 2, & \text{if } i = s \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

We see that $d_{i1} = -1 < 0$ for $i = \frac{s-1}{2}$ and $i = \frac{s+1}{2}$, and $d_{s1} = 2 > 0$ so by Theorem 3.28, $d_{\frac{s-1}{2}1} = -1 = -\alpha_{\frac{s-1}{2}1}$, $d_{\frac{s+1}{2}1} = -1 = -\alpha_{\frac{s+1}{2}1}$, and $\alpha_{i1} = 0$ for other values of i. By Lemma 3.6, there are no syzygies of degrees $(i, 1)$, for $i \leq \frac{s-1}{2}$. So, $\beta_{i1} = 0$ for $i \leq \frac{s-1}{2}$. Moreover, by Lemma 3.6, $\gamma_{i1} = 0$ for $i \leq \frac{s+1}{2}$. By Proposition 3.26 (ii) we have, $d_{\frac{s+1}{2}1} = -\alpha_{\frac{s+1}{2}1} + \beta_{\frac{s+1}{2}1} - \gamma_{\frac{s+1}{2}1}$. Also, we know $d_{\frac{s+1}{2}1} = \alpha_{\frac{s+1}{2}1} = -1$ and $\gamma_{\frac{s+1}{2}1} = 0$. So, $\beta_{\frac{s+1}{2}1} = 0$. So far, we know $\beta_{i1} = \gamma_{i1} = 0$ for $i \leq \frac{s+1}{2}$. By Lemma 3.6, $\gamma_{\frac{s+3}{2}1} = 0$. By Proposition 3.26 (ii), $d_{i1} = -\alpha_{i1} + \beta_{i1} - \gamma_{i1}$, and by the fact that $d_{i1} = \alpha_{i1} = \gamma_{i1} = 0$, we get $\beta_{i1} = 0$. If we continue this process, we see that for $i < s$, $\beta_{i1} = \gamma_{i1} = 0$. For $i = s$, $\alpha_{i1} = 0$, so by Proposition 3.26 (ii), $d_{i1} = \beta_{i1} - \gamma_{i1}$. However, $\beta_{j1} = \beta_{j0} = 0$ for $j < i$, so by Lemma 3.6, there are no first syzygies of degree less than $(i, 1)$. Hence $\gamma_{i1} = 0$ and $\beta_{i1} = 2$. Therefore, for all $0 \leq i \leq s$,

$$\alpha_{i1} = \begin{cases} 1, & \text{if } i = \frac{s-1}{2} \text{ or } i = \frac{s+1}{2} \\ 0, & \text{otherwise} \end{cases}, \quad \beta_{i1} = \begin{cases} 2, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad \gamma_{i1} = 0$$

□

By the results from Lemma 4.3 and 4.4 we have the following corollary.

COROLLARY 4.5. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points in sufficiently general position, and let I_X be its defining ideal. Let γ_{ij} be the number of summands of the form $S(-i, -j)$ that appears in the second step of the minimal free resolution of I_X. Then, for $(i, j) \preceq (s, 1)$, we have $\gamma_{ij} = 0$.*

The following lemma proves the existence of a virtual resolution of length two for a set of points in $\mathbb{P}^1 \times \mathbb{P}^1$ with generic Hilbert function.

LEMMA 4.6. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points with generic Hilbert function. Then I_X a virtual resolution of length two.*

PROOF. Notice that by Corollary 3.33, if $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a set of s points with generic Hilbert function, then $(s-1, 0) \in \text{reg}_B(X)$. Also, by Corollary 3.33, as $(s-1, 0) \in \text{reg}_B(X)$, then for any $\underline{d} = (i, j) \succeq (s-1, 0)$, I_X is $\underline{d}$-regular. Since I_X is $\underline{d}$-regular for $\underline{d} = (s-1, 0)$

and $\underline{n} = (1, 1)$, then by Theorem 2.45 the free subcomplex of a minimal free resolution of I_X consisting of all summands generated in degree at most $(s-1, 0) + \underline{n} = (s, 1)$ is a virtual resolution of I_X. However, since we proved $\gamma_{ij} = 0$ for all $(i, j) \preceq (s, 1)$, the virtual resolution obtained by keeping all summands generated in degrees at most $(s, 1)$ and removing the rest, has length two. $\square$

Now we have all the materials to find virtual resolutions of a set of points.

THEOREM 4.7. *Let X be a set of sufficiently general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then I_X has a virtual resolution of length two. In particular, if s is even, then a virtual resolution is*

$$0 \to S(-s, -1)^2 \to \begin{matrix} S(-s/2, -1)^2 \\ \oplus \\ S(-s, 0) \end{matrix} \to S.$$

and, if s is odd,

$$0 \to S(-s, -1)^2 \to \begin{matrix} S\left(-\frac{s-1}{2}, -1\right) \\ \oplus \\ S\left(-\frac{s+1}{2}, -1\right) \\ \oplus \\ S(-s, 0) \end{matrix} \to S.$$

is a virtual resolution of I_X.

PROOF. We check two cases.

(i) s is odd:
In this case by Lemma 4.4 we have

$$\alpha_{i1} = \begin{cases} 1, & \text{if } i = \frac{s-1}{2} \text{ or } i = \frac{s+1}{2} \\ 0, & \text{otherwise} \end{cases}, \quad \beta_{i1} = \begin{cases} 2, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases}, \quad \text{and} \quad \gamma_{i1} = 0 \quad \text{for } 0 \le i \le s.$$

By Lemma 4.6, if we trim the minimal free resolution of I_X to get the free subcomplex consisting of all summands generated in degree at most $(s, 1)$ we get a virtual resolution of I_X of length two.

So, the resolution will be

$$0 \to S(-s, -1)^2 \to \begin{matrix} S\left(-\frac{s-1}{2}, -1\right) \\ \oplus \\ S\left(-\frac{s+1}{2}, -1\right) \\ \oplus \\ S(-s, 0) \end{matrix} \to S.$$

(ii) s is even:
In this case, by Lemma 4.3 we have

$$\alpha_{i1} = \begin{cases} 2 & \text{if } i = \frac{s}{2} \\ 0 & \text{otherwise} \end{cases}, \quad \beta_{i1} = \begin{cases} 2 & \text{if } i = s \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad \gamma_{i1} = 0.$$

Again, by Lemma 4.6 the free subcomplex consisting of all summands generated in degree at most $(s,1)$ is a virtual resolution of I_X of length two. So the resolution will be

$$0 \to S(-s,-1)^2 \to \begin{matrix} S(-s/2,-1)^2 \\ \oplus \\ S(-s,0) \end{matrix} \to S.$$

□

In the next theorem we will find virtual resolutions of a set of points by finding the positive components of the second difference matrix. We identify other vectors (i,j) such that when we trim the resolution by keeping all terms with $(a,b) \leq (i,j)+(1,1)$ and removing the rest, we get a virtual resolution.

THEOREM 4.8. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points, and let $\Delta^2 H_X = (d_{ij})$ be the second difference function for Hilbert function of X. If $d_{ij} > 0$ and $(i,j) \neq (0,0)$, then I_X is (i,j)-regular.*

PROOF. For every $i > 0$, $\deg X = i \cdot q_i + r_i$ with $0 \leq r_i < i$, then

$$d_{i-1j} = \begin{cases} r_i - i & \text{for } j = q_i \\ -r_i & \text{for } j = q_i + 1 \\ 2(i-1-r_{i-1}) & \text{for } j = q_{i-1} \\ 2r_{i-1} & \text{for } j = q_{i-1}+1 \\ r_{i-2} - (i-2) & \text{for } j = q_{i-2} \\ -r_{i-2} & \text{for } j = q_{i-2}+1 \\ 0 & \text{otherwise} \end{cases}.$$

(See [**GMR94**, Page 201]). So, the only positive entries happen when $d_{i-1q_{i-1}} = 2(i-1-r_{i-1})$ or $d_{i-1(q_{i-1}+1)} = 2r_{i-1}$.

However,

$$((i-1)+1)(q_{i-1}+1) = (i-1)q_{i-1}+(i-1)+q_{i-1}+1 > (i-1)q_{i-1}+r_{i-1}+q_{i-1}+1 = s+q_{i-1}+1 > s,$$

and

$$((i-1)+1)((q_{i-1}+1)+1) = (i-1)q_{i-1} + 2(i-1) + q_{i-1} + 2$$

where

$$(i-1)q_{i-1}+2(i-1)+q_{i-1}+2 > (i-1)q_{i-1}+r_{i-1}+(i-1)+q_{i-1}+2 = s+(i-1)+q_{i-1}+2 > s \ (*)$$

As we stated before, if $S = k[x_0,x_1,y_0,y_1]$, and $S = \bigoplus_{(i,j)\in\mathbb{N}^2} S_{i,j}$, then,

$$\dim_k S_{i,j} = \binom{i+1}{1}\binom{j+1}{1}.$$

Moreover, by the definition of multigraded regularity, if X is a set of s points with generic Hilbert function,

$$\operatorname{reg}_B(X) = \{(i,j) \mid \dim_k S_{i,j} \geq s\}.$$

By $(*)$, $\underline{d_1} = (i-1, q_{i-1})$, and $\underline{d_2} = (i-1, q_{i-1}+1)$ are in $\text{reg}_B(X)$. □

COROLLARY 4.9. *Let $X \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a set of s points, and let $\Delta^2 H_X = (d_{ij})$ be the second difference function for Hilbert function of X. Let G be the free subcomplex of a minimal free resolution of I_X consisting of all summands generated in degree at most $(i, q_{i-1}+1)$, where $s = i \cdot q_i + r_i$ and $d_{i-1q_{i-1}} = 2(i-1-r_{i-1})$. Then G is a virtual resolution of I_X.*

PROOF. As we proved in Theorem 4.8, $\underline{d_1} = (i-1, q_{i-1})$ is in $\text{reg}_B(X)$. By Theorem 2.45, the subcomplex of minimal free resolution of I_X consisting of all the summands generated in degree at most $d_1 + (1,1)$ is a virtual resolution of I_X. □

CHAPTER 5

Future Directions

In this chapter we discuss three conjectures. All of the conjectures are related to the following theorem of Berkesch, Erman and Smith [**BES20**, Theorem 4.1]. Let $\mathbb{P}^{\underline{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_r}$ be the product of projective spaces with dimension vector $\underline{n} := (n_1, n_2, \ldots, n_r) \in \mathbb{N}^r$ over a field k. Let $S := k[x_{i,j} : 1 \leq i \leq r, 0 \leq j \leq n_i]$ be the coordinate ring of $\mathbb{P}^{\underline{n}}$ and let $B := \bigcap_{i=1}^r \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$ be its irrelevant ideal. For $\underline{a} \in \mathbb{N}^r$, we define $B^{\underline{a}}$ to be

$$B^{\underline{a}} := \bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle^{a_i}.$$

THEOREM 5.1. [**BES20**, Theorem 4.1] *If $Z \subset \mathbb{P}^{\underline{n}}$ is a zero-dimensional scheme and I is the corresponding B-saturated S-ideal, then there exists an $\underline{a} \in \mathbb{N}^r$ with $a_r = 0$ such that the minimal free resolution of $S/(I \cap B^{\underline{a}})$ has length equal to $|n| = \dim \mathbb{P}^{\underline{n}}$. Moreover, any $\underline{a} \in \mathbb{N}^r$ with $a_r = 0$ and other entries sufficiently positive yields such a virtual resolution of S/I.*

The theorem above only proves the existence of $\underline{a}$. However, in the following conjecture we try to find $\underline{a}$ explicitly and this will give us an infinite number of virtual resolutions for a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ that has the generic Hilbert function.

CONJECTURE 5.2. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ that has generic Hilbert function and let $I_X \subset S = k[x_0, x_1, y_0, y_1]$ be its corresponding B-saturated defining ideal. (1) The smallest value of $a \in \mathbb{N}$ where the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X is $a = s - 1$. (2) For every number $t \in \mathbb{N}$, where $t > s - 1$, the minimal free resolution of $S/(I_X \cap B^{(t,0)})$ is also a virtual resolution of S/I_X.*

Moreover, if $a \in \mathbb{N}$ is the smallest value where the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X, then this virtual resolution will be of the form

$$0 \to S(-s,-1)^s \to \begin{matrix} S(-s,0) \\ \oplus \\ S(-s+1,-1)^s \end{matrix} \to S,$$

and for $i > 0$, the virtual resolution corresponding to $(a+i, 0)$ is:

$$0 \to \begin{matrix} S(-s-i,0)^{i-1} \\ \oplus \\ S(-s-i,-1)^s \end{matrix} \to \begin{matrix} S(-s-i+1,0)^i \\ \oplus \\ S(-s-i+1,-1)^s \end{matrix} \to S.$$

Below, we show that for $a \geq s-1$ the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X. This gives a partial answer to Conjecture 5.2.

PROPOSITION 5.3. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1$ that has generic Hilbert function and let $I_X \subset S = k[x_0, x_1, y_0, y_1]$ be its corresponding B-saturated defining ideal. If $a = s-1$, then, the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution for S/I_X. Moreover, for every number $t \in \mathbb{N}$, where $t > s-1$, the minimal free resolution of $S/(I_X \cap B^{(t,0)})$ is also a virtual resolution.*

PROOF. Let $X = \{P_1, \ldots, P_s\}$ be a set of s points. Without loss of generality, we can assume that each $P_i = [1 : A_i] \times [1 : B_i]$, and therefore $I_{P_i} = \langle A_i x_0 - x_1, B_i y_0 - y_1 \rangle$.

First, we prove that the depth of $S/I_X \cap B^{(a,0)}$ is 2 for $a \geq s-1$. Then by the Auslander-Buchsbaum Formula, Theorem 3.24, we can see that proj-dim$(S/I_X \cap B^{(a,0)}) = 2$.

Claim 1: The depth of $S/I_X \cap B^{(\underline{a},0)}$ for $a = s-1$, is 2.

Proof: To see that the depth is 2, we need to show that the maximal length of a regular sequence is 2. We begin by showing that there exists a regular sequence of length 2 . More precisely, we claim, $\{y_0, x_0 + y_1\}$ is a regular sequence for $I_X \cap B^{(s-1,0)}$.

By Definition 2.12, to prove that $\{y_0, x_0 + y_1\}$ is a regular sequence for $I_X \cap B^{(s-1,0)}$, we need to show the following:

(1) $\langle I_X \cap B^{(s-1,0)}, y_0, x_0 + y_1 \rangle \subset \langle x_0, x_1, y_0, y_1 \rangle$,
(2) y_0 is a non-zero-divisor in $S/I_X \cap B^{(s-1,0)}$,
(3) $x_0 + y_1$ is a non-zero-divisor in $S/\langle I_X \cap B^{(s-1,0)}, y_0 \rangle$.

We can see that (1) is true since $I_X \cap B^{(s-1,0)}$ is a bihomogeneous ideal with generators of degrees at least $(s-1, 0)$.

In order to show that y_0 is a non-zero-divisor in $S/\langle I_X \cap B^{(s-1,0)} \rangle$, we use Theorem 2.13 and Proposition 2.14. From these results it follows that we only need to show that y_0 is not in the Ass$(I_X \cap B^{(s-1,0)})$.

We first compute Ass$(I_X \cap B^{(s-1,0)})$. Let $S = k[x_0, x_1, y_0, y_1]$, and let $I_X \cap B^{(s-1,0)}$ be as above. The primary decomposition of $I_X \cap B^{(s-1,0)}$ is

$$I_X \cap B^{(s-1,0)} = (\bigcap_{i=1}^{s} I_{P_i}) \cap B^{(s-1,0)},$$

since each I_{P_i} is a prime ideal and therefore a primary ideal. Moreover, $B^{(s-1,0)}$ is also a primary ideal. To see this, we need to prove that for every $f, g \in S$, where $fg \in B^{(s-1,0)}$, either $f \in B^{(s-1,0)}$ or $g^m \in B^{(s-1,0)}$ for some integer $m > 0$. Since g is a polynomial, we can write g as a sum of monomials, $g = g_1 + g_2 + \cdots + g_r$. We have two cases, i) $\deg(g_i) \succeq (1,0)$ for all $1 \leq i \leq r$. ii) There exists some j, where $\deg(g_j) = (0, b)$ for some integer b. If case (i) happens, then $\deg(g)^{s-1} \succeq (s-1, 0)$. Therefore, $g^{s-1} \in B^{(s-1,0)}$. If case (ii) happens, then $\deg(f)$ must be at least $(s-1, 0)$, since $\deg(fg) \succeq (s-1, 0)$. Therefore, $f \in B^{(s-1,0)}$. Hence $B^{(s-1,0)}$ is a primary ideal.

Therefore, we have

$$\text{Ass } (I_X \cap B^{(s-1,0)}) = \{I_{P_1}, \ldots, I_{P_s}, \langle x_0, x_1\rangle\},$$

since $B^{(s-1,0)} = \langle x_0, x_1\rangle^{s-1}$, therefore $\sqrt{B^{(s-1,0)}} = \langle x_0, x_1\rangle$.

From Theorem 2.13, we can see that y_0 is not in the union of the associated primes of $I_X \cap B^{(s-1,0)}$, since we took each P_i to be in the form $[1 : A_i] \times [1 : B_i]$ and hence each I_{P_i} is $\langle A_i x_0 - x_1, B_i y_0 - y_1\rangle$. Therefore, y_0 is a non-zero-divisor in $S/I_X \cap B^{(s-1,0)}$. This proves (2).

We now prove (3). To do this, we again find associated primes, this time of $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$. We assumed that $P_i = [1 : A_i] \times [1 : B_i]$ and $I_{P_i} = \langle A_i x_0 - x_1, B_i y_0 - y_1\rangle$ for each i. We claim that

Claim 2:

$$\langle I_X \cap B^{(s-1,0)}, y_0\rangle = (\bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle) \cap \langle B^{(s-1,0)}, y_0\rangle. \tag{5.1}$$

is a primary decomposition for $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$.

Proof: We will show that

$$(\bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle) \cap \langle B^{(s-1,0)}, y_0\rangle = \langle y_0, y_1, \prod_{i=1}^{s}(A_i x_0 - x_1)\rangle \cap \langle B^{(s-1,0)}, y_0\rangle, \tag{5.2}$$

$$\langle y_0, y_1, \prod_{i=1}^{s}(A_i x_0 - x_1)\rangle \cap \langle B^{(s-1,0)}, y_0\rangle = \langle y_0, y_1 x_0^{s-1}, y_1 x_0^{s-2} x_1, \ldots, y_1 x_0 x_1^{s-2}, y_1 x_1^{s-1}, \prod_{i=1}^{s}(A_i x_0 - x_1)\rangle, \tag{5.3}$$

and,

$$\langle I_X \cap B^{(s-1,0)}, y_0\rangle = \langle y_0, y_1 x_0^{s-1}, y_1 x_0^{s-2} x_1, \ldots, y_1 x_0 x_1^{s-2}, y_1 x_1^{s-1}, \prod_{i=1}^{s}(A_i x_0 - x_1)\rangle. \tag{5.4}$$

If we prove Equation 5.2, 5.3, and, 5.4, then we have shown that Equation 5.1 is indeed the primary decomposition of $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$, as we proved each ideal is primary.

First we prove that

$$\bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle = \langle y_0, y_1, \prod_{i=1}^{s}(A_i x_0 - x_1)\rangle. \tag{5.5}$$

To prove RHS $\subseteq$ LHS, we can see that y_0 and y_1 are in $\bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle$. To show that $\prod_{i=1}^{s}(A_i x_0 - x_1) \in \bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle$, notice that for each i, $\prod_{i=1}^{s}(A_i x_0 - x_1) = (A_i x_0 - x_1)\prod_{j\neq i}(A_j x_0 - x_1)$. Therefore, for each i, $\prod_{i=1}^{s}(A_i x_0 - x_1) \in \langle y_0, y_1, A_i x_0 - x_1\rangle$. Hence RHS $\subseteq$ LHS as desired.

For the other inclusion, let $f \in \bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle$. Therefore, for each i, we can write $f = y_0 r_{1i} + y_1 r_{2i} + (A_i x_0 - x_1) r_{3i}$, where r_{ri} is a polynomial in x_0 and x_1. Notice that $f \in \langle y_0, y_1, A_i x_0 - x_1\rangle$ for all i. Let j and k be fixed integers between 1 and s. $f = y_0r_{1j} + y_1r_{2j} + (A_ix_0 - x_1)r_{3j} \in \langle y_0, y_1, A_kx_0 - x_1\rangle$. Therefore $A_kx_0 - x_1$ divides r_{3j}. If we do the same process for all i between 1 and s, we see that f can be written as $f = y_0r_1 + y_1r_2 + (A_ix_0 - x_1)r_3$ where for each i, $A_ix_0 - x_1$ divides r_3. Hence $f \in \langle y_0, y_1, \prod_{i=1}^{s}(A_ix_0 - x_1)\rangle$. This completes the proof of Equation 5.2.

We now prove Equation 5.3. To simplify our notation, we define

$$J_1 := \langle y_0, y_1, \prod_{i=1}^{s} A_ix_0 - x_1\rangle \cap \langle B^{(s-1,0)}, y_0\rangle,$$

and,

$$J_2 := \langle y_0, y_1x_0^{s-1}, y_1x_0^{s-2}x_1, \ldots, y_1x_0x_1^{s-2}, y_1x_1^{s-1}, \prod_{i=1}^{s}(A_ix_0 - x_1)\rangle.$$

We first show $J_1 \subseteq J_2$. Let $f \in J_1$. Therefore, we have $f \in \langle y_0, y_1, \prod_{i=1}^{s}(A_ix_0 - x_1)\rangle$ and $f \in \langle B^{(s-1,0)}, y_0\rangle$. From $f \in \langle y_0, y_1, \prod_{i=1}^{s} A_ix_0 - x_1\rangle$, we have $f = r_1y_0 + r_2y_1 + r_3\prod_{i=1}^{s}(A_ix_0 - x_1)$. Since $f \in \langle B^{(s-1,0)}, y_0\rangle$, we know $r_1y_0 + r_2y_1 + r_3\prod_{i=1}^{s}(A_ix_0 - x_1) \in \langle B^{(s-1,0)}, y_0\rangle$. Notice that $\deg\prod_{i=1}^{s}(A_ix_0 - x_1) = (s, 0)$, so $\prod_{i=1}^{s}(A_ix_0 - x_1) = (s, 0) \in B^{(s-1,0)}$ and hence $r_3\prod_{i=1}^{s}(A_ix_0 - x_1) \in B^{(s-1,0)}$. Hence $r_2y_1 \in \langle B^{(s-1,0)}, y_0\rangle$. We can write $r_2 = t_1y_0 + t_2$, where t_2 is a polynomial in x_0, x_1, and y_1. Since, $r_2y_1 = t_1y_0y_1 + t_2y_1 \in B^{(s-1,0)}$, and t_2 is a polynomial in x_0, x_1, and y_1, we have $\deg(t_2) \succeq (s-1, 0)$. Hence, $r_2 \in \langle B^{(s-1,0)}, y_0\rangle$. Therefore, $f \in J_2$, and $J_1 \subseteq J_2$ as desired.

Now, let $f \in J_2$. So, we can write

$$f = t_1y_0 + t_2y_1x_0^{s-1} + t_3y_1x_0^{s-2}x_1 + \cdots + t_sy_1x_0x_1^{s-2} + t_{s+1}y_1x_1^{s-1} + t_{s+2}\prod_{i=1}^{s}(A_ix_0 - x_1),$$

for $t_i \in S$, $1 \leq i \leq s+2$. We can see that $f \in \langle y_0, y_1, \prod_{i=1}^{s}(A_ix_0 - x_1)\rangle$ and $f \in \langle B^{(s-1,0)}, y_0\rangle$ and hence $f \in J_1$. Thus, $J_1 = J_2$.

Now we prove Equation 5.4.

First, we show that $\langle I_X \cap B^{(s-1,0)}, y_0\rangle \subseteq J_2$. To see this, let $f \in \langle I_X \cap B^{(s-1,0)}, y_0\rangle$. Therefore, $f = r_1y_0 + r_2g$ where $g \in I_X \cap B^{(s-1,0)}$. If $g \in I_X \cap B^{(s-1,0)}$, we have two cases, (i) $\deg g = (a, 0)$, where $a \geq s$ (ii) $\deg g \succeq (s-1, 1)$. If $\deg g = (a, 0)$ for $a \geq s$, we can write $g = r_3\prod_{i=1}^{s}(A_ix_0 - x_1)$ where $\deg r_3 = (a-s, 0)$.Therefore, $f = r_1y_0 + r_2r_3\prod_{i=1}^{s}(A_ix_0 - x_1)$ which is clearly in J_2. If $\deg g \succeq (s-1, 1)$, we can write g as finite sum $\sum_j c_jx_0^{a_0}x_1^{a_1}y_0^{b_0}y_1^{b_1}$, where $(a_0 + a_1, b_0 + b_1) = \deg g$ and $c_j \in k$. Therefore, $a_0 + a_1 \geq s - 1$ and $b_0 + b_1 \geq 1$ which concludes that $f \in J_2$.

We now prove that $J_2 \subseteq \langle I_X \cap B^{(s-1,0)}, y_0\rangle$.

We can see that y_0 and $\prod_{i=1}^{s}(A_ix_0 - x_1)$ are in $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$. Therefore it suffices to prove that each monomial $y_1x_0^ix_1^{s-1-i}$ for $0 \leq i \leq s-1$ is in $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$. We can prove this by Proposition 3.31 as follows. If we let $\underline{t} = (s-1, 1)$, then from the Proposition 3.31, we can see that $(I_X, y_0)_{\underline{t}} = S_{\underline{t}}$. Notice that $(I_X \cap B^{(s-1,0)}, y_0)_{\underline{t}} = (I_X, y_0)_{\underline{t}}$. Therefore, all the monomials $y_1x_0^ix_1^{s-1-i}$ are in $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$. This proves Equation 5.4.

To prove that this is indeed a primary decomposition, we first prove that each $\langle y_0, y_1, A_i x_0 - x_1\rangle$ is a prime ideal.

To see this, let $f, g \in S$. We can write $f = f_1(A_i x_0 - x_1) + f_2 y_0 + f_3 y_1 + f_4$ and $g = g_1(A_i x_0 - x_1) + g_2 y_0 + g_3 y_1 + g_4$, where f_4 and g_4 are polynomials in x_0 and x_1. We now prove that if $fg \in \langle y_0, y_1, A_i x_0 - x_1\rangle$, then either $f \in \langle y_0, y_1, A_i x_0 - x_1\rangle$ or $g \in \langle y_0, y_1, A_i x_0 - x_1\rangle$. If $fg \in \langle y_0, y_1, A_i x_0 - x_1\rangle$, since f_4 and g_4 are polynomials purely in x_0 and x_1, it must follow that $f_4 g_4 \in \langle y_0, y_1, A_i x_0 - x_1\rangle$. So, $A_i x_0 - x_1$ divides $f_4 g_4$. We can conclude that either $A_i x_0 - x_1 \mid f_4$ or $A_i x_0 - x_1 \mid g_4$, and therefore $f \in \langle y_0, y_1, A_i x_0 - x_1\rangle$ or $g \in \langle y_0, y_1, A_i x_0 - x_1\rangle$. Since every prime ideal is primary, each $\langle y_0, y_1, A_i x_0 - x_1\rangle$ is primary.

We now prove that $\langle B^{(s-1,0)}, y_0\rangle$ is a primary ideal. Let $f, g \in S$ where $fg \in \langle B^{(s-1,0)}, y_0\rangle$. We can write $f = f_1 y_0 + f_2$ and $g = g_1 y_0 + g_2$ where f_2 and g_2 are polynomials in x_0, x_1, and y_1. Since $fg \in \langle B^{(s-1,0)}, y_0\rangle$ and $f_1 g_1 y_0^2 + f_1 g_2 y_0 + f_2 g_1 y_0 \in \langle B^{(s-1,0)}, y_0\rangle$, it must follow $f_2 g_2 \in B^{(s-1,0)}$, and since $B^{(s-1,0)}$ is a primary ideal, we have either $f_2 \in B^{(s-1,0)}$ or $g_2^m \in B^{(s-1,0)}$ for some m. Therefore, either $f \in \langle B^{(s-1,0)}, y_0\rangle$, or $g^m \in \langle B^{(s-1,0)}, y_0\rangle$. This proves Equation 5.1 is a primary decomposition.

Next, we need to find the associated primes of $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$. To do this, by Theroem 2.14, we only need to find the radical of the ideals in the primary decomposition of $\langle I_X \cap B^{(s-1,0)}, y_0\rangle$. Since $\langle y_0, y_1, A_i x_0 - x_1\rangle$ is a prime ideal for all i, we only need to find $\sqrt{\langle B^{(s-1,0)}, y_0\rangle}$.

We claim that $\sqrt{\langle B^{(s-1,0)}, y_0\rangle} = \langle y_0, x_0, x_1\rangle$. Notice that $\langle B^{(s-1,0)}, y_0\rangle = B^{(s-1,0)} + \langle y_0\rangle$.

We also have $\sqrt{\langle B^{(s-1,0)}, y_0\rangle} = \sqrt{\sqrt{B^{(s-1,0)}} + \sqrt{\langle y_0\rangle}}$. To see this, notice that $\langle B^{(s-1,0)}, y_0\rangle = B^{(s-1,0)} + \langle y_0\rangle$. Moreover, we have $B^{(s-1,0)} + \langle y_0\rangle \subseteq \sqrt{\sqrt{B^{(s-1,0)}} + \sqrt{\langle y_0\rangle}}$. Therefore $\sqrt{B^{(s-1,0)} + \langle y_0\rangle} \subseteq \sqrt{\sqrt{B^{(s-1,0)}} + \sqrt{\langle y_0\rangle}}$. For the other inclusion, let $f \in \sqrt{\sqrt{B^{(s-1,0)}} + \sqrt{\langle y_0\rangle}}$. Then $f^m \in \sqrt{B^{(s-1,0)}} + \sqrt{\langle y_0\rangle}$ for some integer $m > 0$. This means that $f^m = g + h$ where $g^l \in B^{(s-1,0)}$ and $h^n \in \langle y_0\rangle$ for some integers $l, n > 0$. Then $f^{m(l+n)} = (f^m)^{l+n} \in B^{(s-1,0)} + \langle y_0\rangle$, therefore $f \in \sqrt{B^{(s-1,0)} + \langle y_0\rangle}$. This proves that $\sqrt{\langle B^{(s-1,0)}, y_0\rangle} = \sqrt{\sqrt{B^{(s-1,0)}} + \sqrt{\langle y_0\rangle}}$.

However, $\sqrt{B^{(s-1,0)}} = \langle x_0, x_1\rangle$ and $\sqrt{\langle y_0\rangle} = \langle y_0\rangle$. Therefore $\sqrt{\sqrt{B^{(s-1,0)}} + \sqrt{\langle y_0\rangle}} = \sqrt{\langle x_0, x_1\rangle + \langle y_0\rangle} = \sqrt{\langle x_0, x_1, y_0\rangle}$. Since $\langle x_0, x_1, y_0\rangle$ is a prime ideal, $\sqrt{\langle x_0, x_1, y_0\rangle} = \langle x_0, x_1, y_0\rangle$. So, $\sqrt{\langle B^{(s-1,0)}, y_0\rangle} = \langle x_0, x_1, y_0\rangle$, as desired. This proves Claim 2.

By the above discussion we can conclude that

$$\text{Ass }(I_X \cap B^{(s-1,0)}, y_0\rangle) = \{\langle y_0, y_1, A_1 x_0 - x_1\rangle, \ldots, \langle y_0, y_1, A_s x_0 - x_1\rangle, \langle y_0, x_0, x_1\rangle\}$$

and we can also see that $x_0 + y_1$ is not contained in Ass $(I_X \cap B^{(s-1,0)}, y_0\rangle)$. Therefore, by Theorem 2.13, it is a non-zero-divisor in $I_X \cap \langle B^{(s-1,0)}, y_0\rangle$.

We have proved that $S/I_X \cap B^{(s-1,0)}$ has a regular sequence of length 2. This proves that the depth of $I_X \cap B^{(s-1,0)}$ is at least 2. However, the depth of $I_X \cap B^{(s-1,0)}$ cannot be more than 2. This follows from the fact that depth $S/I_X \cap B^{(s-1,0)} \leq \text{K-}\dim S/I_X \cap B^{(s-1,0)}$. Moreover, $\text{K-}\dim S/I_X \cap B^{(s-1,0)} \leq \text{K-}\dim S/I_X$. Since $\text{K-}\dim S/I_X = 2$ (see [**GVT15**, Lemma 4.2] for the proof), we conclude that depth $S/I_X \cap B^{(s-1,0)} = 2$. This completes the proof of Claim 1.

In fact, we can also prove that $S/I_X \cap B^{(a,0)}$ has a regular sequence of length 2 for $a \geq s$. In order to prove this, we follow the same strategy. We show that $y_0, x_0 + y_1$ is a regular sequence for $S/I_X \cap B^{(a,0)}$ for $a \geq s$. Again, we need to show the following for $a \geq s$:

1) $\langle I_X \cap B^{(a,0)}, y_0, x_0 + y_1\rangle \subset \langle x_0, x_1, y_0, y_1\rangle$,
2) y_0 is a non-zero-divisor in $S/I_X \cap B^{(a,0)}$, and
3) $x_0 + y_1$ is a non-zero-divisor in $S/\langle I_X \cap B^{(a,0)}, y_0\rangle$.

We can see that (1) is true. In order to show that y_0 is a non-zero-divisor in $S/\langle I_X \cap B^{(a,0)}\rangle$, we show that y_0 is not in the union of the associated primes of $I_X \cap B^{(a,0)}$. The primary decomposition of $I_X \cap B^{(a,0)}$ is

$$I_X \cap B^{(a,0)} = (\bigcap_{i=1}^{s} I_{P_i}) \cap B^{(a,0)}.$$

Therefore, we have

$$\text{Ass } (I_X \cap B^{(a,0)}) = \{I_{P_1}, \ldots, I_{P_s}, \langle x_0, x_1\rangle\}.$$

We can see that y_0 is not in the union of the associated primes of $I_X \cap B^{(a,0)}$. Therefore, y_0 is a non-zero-divisor in $S/I_X \cap B^{(a,0)}$. This proves (2).

To prove (3), again, we find Ass $(I_X \cap B^{(a,0)}, y_0\rangle)$.

Claim 3: The primary decomposition of $\langle I_X \cap B^{(a,0)}, y_0\rangle$ is

$$\langle I_X \cap B^{(a,0)}, y_0\rangle = (\bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle) \cap \langle B^{(a,0)}, y_0\rangle. \tag{5.6}$$

Proof: First, we show that

$$\langle I_X \cap B^{(a,0)}, y_0\rangle = (\bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle) \cap \langle B^{(a,0)}, y_0\rangle.$$

To prove this, we prove that

$$(\bigcap_{i=1}^{s} \langle y_0, y_1, A_i x_0 - x_1\rangle) \cap \langle B^{(a,0)}, y_0\rangle = \langle y_0, y_1, \prod_{i=1}^{s} (A_i x_0 - x_1)\rangle \cap \langle B^{(a,0)}, y_0\rangle, \tag{5.7}$$

and

(5.8)

$$\langle y_0, y_1, \prod_{i=1}^{s}(A_ix_0-x_1)\rangle\cap\langle B^{(a,0)}, y_0\rangle = \langle y_0, y_1x_0^a, y_1x_0^{a-1}x_1, \ldots, y_1x_0x_1^{a-1}, y_1x_1^a, \prod_{i=1}^{s}(A_ix_0-x_1)\rangle.$$

We proved Equation 5.7 in the proof when $a = s-1$. We now prove Equation 5.8.

Let $F = \prod_{i=1}^{s}(A_ix_0 - x_1)$. We show that

$$\langle y_0, y_1, F\rangle \cap \langle B^{(a,0)}, y_0\rangle =$$

$$\langle y_0, y_1x_0^a, y_1x_0^{a-1}x_1, \ldots, y_1x_0x_1^{a-1}, y_1x_1^a, Fx_0^{a-s}, Fx_0^{a-s-1}x_1, \ldots, Fx_0x_1^{a-s-1}, Fx_1^{a-s}\rangle.$$

For simplicity, we let

$$J_3 = \langle y_0, y_1x_0^a, y_1x_0^{a-1}x_1, \ldots, y_1x_0x_1^{a-1}, y_1x_1^a, Fx_0^{a-s}, Fx_0^{a-s-1}x_1, \ldots, Fx_0x_1^{a-s-1}, Fx_1^{a-s}\rangle.$$

To prove this, first let $f \in \langle y_0, y_1, F\rangle \cap \langle B^{(a,0)}, y_0\rangle$. Therefore, we have $f \in \langle y_0, y_1, F\rangle$ and $f \in \langle B^{(a,0)}, y_0\rangle$. From $f \in \langle y_0, y_1, F\rangle$, we have $f = r_1y_0 + r_2y_1 + r_3F$ where r_2 and r_3 are polynomials in $x_0, x_1$and y_1. We also have $f \in \langle B^{(a,0)}, y_0\rangle$, and since $B^{(a,0)}$ is a monomial ideal, it concludes $r_2, r_3F \in B^{(a,0)}$, which means that r_2 can be written as a finite sum, $\sum t_jx_0^{a-m_j}x_1^{m_j}$, where $t_j \in S$. Also, since $r_3F \in B^{(a,0)}$ and $\deg F = (s, 0)$, $\deg r_3 \succeq (a-s, 0)$. So, $f \in J_3$.

Now, let $f \in J_3$. So, we can write

$f = t_1y_0+t_2y_1x_0^a+t_3y_1x_0^{a-1}x_1+\cdots+t_{a+1}y_1x_0x_1^{a-1}+t_{a+2}y_1x_1^a+t_{a+3}Fx_0^{a-s}+t_{a+4}Fx_0^{a-s-1}x_1+\cdots+t_{2a-s+2}Fx_0x_1^{a-s-1}+t_{2a-s+3}Fx_1^{a-s}$.

It is easy to see $f \in \langle y_0, y_1, F\rangle$ and $f \in \langle B^{(s-1,0)}, y_0\rangle$ and hence $f \in \langle y_0, y_1, F\rangle \cap \langle B^{(s-1,0)}, y_0\rangle$.

Now we prove

$$\langle I_X \cap B^{(a,0)}, y_0\rangle = J_3.$$

First, we show that

$$\langle I_X \cap B^{(a,0)}, y_0\rangle \subseteq J_3$$

To see this, let $f \in \langle I_X \cap B^{(a,0)}, y_0\rangle$. Therefore, $f = r_1y_0 + r_2g$ where $g \in I_X \cap B^{(a,0)}$. If $g \in I_X \cap B^{(a,0)}$, we have two cases, (i) $\deg g = (p, 0)$, where $p \geq a$ (ii) $\deg g \succeq (a, 1)$. If case (i) happens, since $g \in I_X$, then $g = rF$ for some $r \in S$ and hence, $f \in J_3$. If case (ii) happens, then $g = r_1y_0 + r_2y_1$ where $\deg r_i \succeq (a, 0)$, and therefore, $f \in J_3$.

We now prove that

$$J_3 \subseteq \langle I_X \cap B^{(a,0)}, y_0\rangle$$

To see this, it is easy to see y_0 and $Fx_0^{a-s-i}x_1^i$ for $0 \leq i \leq a-s$ are in $\langle I_X \cap B^{(a,0)}, y_0\rangle$. We now prove that each monomial $y_1x_0^ix_1^{a-i}$ is in $\langle I_X \cap B^{(a,0)}, y_0\rangle$. By Proposition 3.31, if we let $\underline{t} = (a, 1)$, then we can see that $(I_X, y_0)_{\underline{t}} = S_{\underline{t}}$. Notice that $(I_X \cap B^{(a,0)}, y_0)_{\underline{t}} = (I_X, y_0)_{\underline{t}}$. Therefore, all the monomials $y_1x_0^ix_1^{a-i}$ are in $\langle I_X \cap B^{(a,0)}, y_0\rangle$. This proves Equation 5.6. We have seen in the proof of the case $a = s-1$ that the ideals in RHS of Equation

5.6 are primary ideals. Therefore, Equation 5.6 is indeed the primary decomposition of $\langle I_X \cap B^{(a,0)}, y_0\rangle$. Hence,

$$\text{Ass } (I_X \cap B^{(a,0)}, y_0\rangle) = \{\langle y_0, y_1, A_1x_0 - x_1\rangle, \ldots, \langle y_0, y_1, A_sx_0 - x_1\rangle, \langle y_0, x_0, x_1\rangle\}.$$

We can see that $x_0 + y_1$ is not in

$$\text{Ass } (I_X \cap B^{(a,0)}, y_0\rangle) = \{\langle y_0, y_1, A_1x_0 - x_1\rangle, \ldots, \langle y_0, y_1, A_sx_0 - x_1\rangle, \langle y_0, x_0, x_1\rangle\}.$$

Therefore, it is a non-zero-divisor in $\langle I_X \cap B^{(a,0)}, y_0\rangle$.

We proved that $S/I_X \cap B^{(a,0)}$ has a regular sequence of length at least 2 and since depth $S/I_X \cap B^{(a,0)} \leq$ K- dim $S/I_X \cap B^{(a,0)} \leq 2$, depth $S/I_X \cap B^{(a,0)} = 2$.

□

In the following example we find the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ for different values of a by using Macaulay2.

Example 5.4. Let

$$X = \{[1:0]\times[1:2], [2:1]\times[2:3], [3:2]\times[3:4], [4:3]\times[4:5], [5:4]\times[5:6]\}$$

be a set of 5 points in $\mathbb{P}^1 \times \mathbb{P}^1$ with the generic Hilbert function. If $a = 0$, and $\underline{a} = (0,0)$,

$$0 \to \begin{array}{c} S(-3,-3) \\ \oplus \\ S(-5,-2) \\ \oplus \\ S(-2,-5) \end{array} \to \begin{array}{c} S(-3,-2)^2 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-5,-1)^2 \\ \oplus \\ S(-1,-5)^2 \end{array} \to \begin{array}{c} S(-2,-1) \\ \oplus \\ S(-1,-2) \\ \oplus \\ S(-3,-1) \\ \oplus \\ S(-1,-3) \\ \oplus \\ S(-5,0) \\ \oplus \\ S(0,-5) \end{array} \to S \to S/I_X \to 0.$$

If $a = 1$, and $\underline{a} = (1,0)$,

$$0 \to \begin{array}{c} S(-3,-3) \\ \oplus \\ S(-5,-2) \end{array} \to \begin{array}{c} S(-3,-2)^2 \\ \oplus \\ S(-2,-3)^2 \\ \oplus \\ S(-5,-1)^2 \end{array} \to \begin{array}{c} S(-2,-1) \\ \oplus \\ S(-1,-2) \\ \oplus \\ S(-3,-1) \\ \oplus \\ S(-1,-3) \\ \oplus \\ S(-5,0) \end{array} \to S \to S/(I_X \cap B^{(1,0)}) \to 0.$$

If $a = 2$, and $\underline{a} = (2, 0)$,

$$0 \to S(-5,-2) \to \begin{array}{c} S(-3,-2)^3 \\ \oplus \\ S(-5,-1)^2 \end{array} \to \begin{array}{c} S(-2,-1) \\ \oplus \\ S(-3,-1) \\ \oplus \\ S(-2,-2)^2 \\ \oplus \\ S(-5,0) \end{array} \to S \to S/(I_X \cap B^{(2,0)}) \to 0.$$

If $a = 3$, and $\underline{a} = (3, 0)$,

$$0 \to S(-5,-2) \to \begin{array}{c} S(-4,-1) \\ \oplus \\ S(-5,-1)^2 \\ \oplus \\ S(-4,-2)^2 \end{array} \to \begin{array}{c} S(-3,-1)^3 \\ \oplus \\ S(-3,-2) \\ \oplus \\ S(-5,0) \end{array} \to S \to S/(I_X \cap B^{(3,0)}) \to 0.$$

If $a = 4$, and $\underline{a} = (4, 0)$,

$$0 \to S(-5,-1)^5 \to \begin{array}{c} S(-5,0) \\ \oplus \\ S(-4,-1)^5 \end{array} \to S \to S/(I_X \cap B^{(4,0)}) \to 0.$$

If $a = 5$, and $\underline{a} = (5, 0)$,

$$0 \to S(-6,-1)^5 \to \begin{array}{c} S(-5,0) \\ \oplus \\ S(-5,-1)^5 \end{array} \to S \to S/(I_X \cap B^{(5,0)}) \to 0.$$

If $a = 6$, and $\underline{a} = (6, 0)$,

$$0 \to \begin{array}{c} S(-7,-1)^5 \\ \oplus \\ S(-7,0) \end{array} \to \begin{array}{c} S(-6,0)^2 \\ \oplus \\ S(-6,-1)^5 \end{array} \to S \to S/(I_X \cap B^{(6,0)}) \to 0.$$

If $a = 7$, and $\underline{a} = (7, 0)$,

$$0 \to \begin{array}{c} S(-8,-1)^5 \\ \oplus \\ S(-8,0)^2 \end{array} \to \begin{array}{c} S(-7,0)^3 \\ \oplus \\ S(-7,-1)^5 \end{array} \to S \to S/(I_X \cap B^{(7,0)}) \to 0.$$

As we can see in the example above, the least a that yields a virtual resolution is $a = 4 = 5 - 1$.

Our next conjecture is about virtual resolutions of a finite set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the natural projection morphism onto the ith coordinate. Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We denote $|\pi_i(X)|$ to be the number of distinct ith coordinates that appear in X.

Conjecture 5.5. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ where $|\pi_1(X)| = |\pi_2(X)| = s$. Let $I_X \subset S = k[x_0, x_1, y_0, y_1, z_0, z_1]$ be its defining ideal. If the minimal free resolution of $S/(I_X \cap B^{\underline{a}})$ where $\underline{a} = (a_1, a_2, 0)$ is a virtual resolution of S/I_X of length 3, then $\boldsymbol{a}' = (a_1 + 1, a_2, 0)$, and, $\boldsymbol{a}'' = (a_1, a_2 + 1, 0)$ is also a virtual resolution of S/I_X of length 3.*

Example 5.6. Let $X = \{[1 : 45] \times [1 : 7] \times [1 : 9], [1 : 21] \times [1 : 25] \times [1 : 32], [1 : 48] \times [1 : 20] \times [1 : 31], [1 : 2] \times [1 : 13] \times [1 : 32], [1 : 44] \times [1 : 1] \times [1 : 12]\}$ and let I_{P_i} be the defining ideal of P_i, for $i = 1, 2, \ldots, 5$

$$I_{P_1} = \langle 45x_0 - x_1, 7y_0 - y_1, 9z_0 - z_1\rangle$$

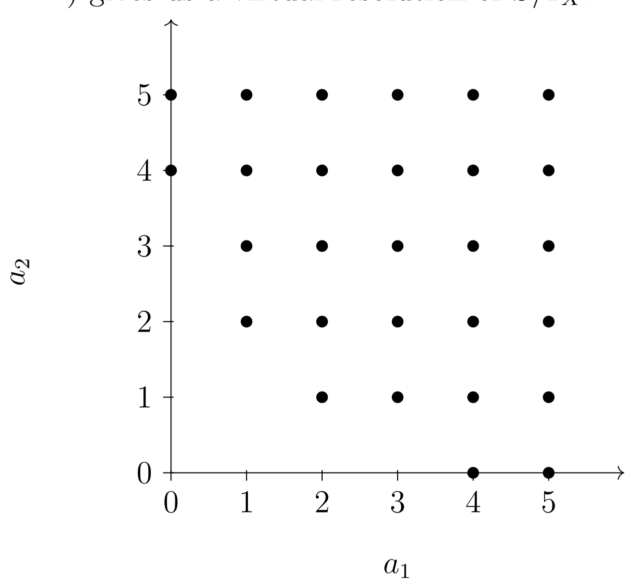
$$I_{P_2} = \langle 21x_0 - x_1, 25y_0 - y_1, 32z_0 - z_1\rangle$$

$$I_{P_3} = \langle 48x_0 - x_1, 20y_0 - y_1, 31z_0 - z_1\rangle$$

$$I_{P_4} = \langle 2x_0 - x_1, 13y_0 - y_1, 32z_0 - z_1\rangle$$

$$I_{P_5} = \langle 44x_0 - x_1, y_0 - y_1, 12z_0 - z_1\rangle$$

The following diagram shows all $(a_1, a_2) \in \mathbb{N}^2$ for $(a_1, a_2) \preceq (5, 4)$, such that $S/(I_X \cap B^{(a_1,a_2,0)})$ gives us a virtual resolution of S/I_X.



Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ be the natural projection morphism onto the first coordinate. Let X be a set of points in $\mathbb{P}^1 \times \mathbb{P}^2$. We denote $|\pi_1(X)|$ to be the number of distinct first coordinates that appear in X.

CONJECTURE 5.7. *Let X be a set of s points in $\mathbb{P}^1 \times \mathbb{P}^2$ where $|\pi_1(X)| = s$. Let $I_X \subset S = k[x_0, x_1, y_0, y_1, y_2]$ be its defining ideal. Then smallest value of a where the minimal free resolution of $S/(I_X \cap B^{(a,0)})$ is a virtual resolution of S/I_X has the following properties:*

(1) *The virtual resolution is of the form*

$$0 \to S^s \to S^m \to S^n \to S$$

(2) $a = s - 1$
(3) $m = 3s$

EXAMPLE 5.8. Let $X = \{[43 : 4 : 40] \times [1 : 1], [30 : 5 : 24] \times [1 : 38], [22 : 14 : 49] \times [1 : 7], [1 : 4 : 13] \times [1 : 14], [23 : 10 : 15] \times [1 : 26]\}$ and I_{P_i} be the defining ideal of P_i, for $i = 1, 2, \ldots, 5$ where

$$I_{P_1} = \langle -4y_0 + 43y_1, -40y_0 + 43y_2, x_0 - x_1 \rangle$$
$$I_{P_2} = \langle -5y_0 + 30y_1, -24y_0 + 30y_2, 38x_0 - x_1 \rangle$$
$$I_{P_3} = \langle -16y_0 + 22y_1, -49y_0 + 22y_2, 7x_0 - x_1 \rangle$$
$$I_{P_4} = \langle -4y_0 + y_1, -13y_0 + y_2, 14x_0 - x_1 \rangle$$
$$I_{P_5} = \langle -10y_0 + 23y_1, -15y_0 + 23y_2, 26x_0 - x_1 \rangle$$

By using Macaulay2 we get the following virtual resolution of length 3, where $a = 4$:

$$0 \to S^5 \to S^{15} \to S^5 \to S$$

In order to get the conjectures above, we checked more than 20 different configurations of sets of points for each case, until we found the right condition to have the properties explained in the conjectures.

Lastly, we hope that the ideas presented in this thesis will help to find the answers of these conjectures.

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