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Abstract

This thesis explores three important issues in credit risk modeling: the nonlinear credit risk stress testing models, the recovery term structure of point-in-time (PIT) loss given default (LGD), and the estimation of LGD by mixture beta regression model.

In the first essay of this thesis, we study the credit risk stress testing models. By incorporating the regime-switching and quantile regression techniques into credit risk stress testing models, we propose two new dynamic models that outperform the traditional linear regression model according to both the point estimate accuracy and the confidence interval breaches. This confirms the importance of nonlinear regression approaches in the estimation and the prediction of credit risk determinants. The proposed models perform especially well in capturing the extreme values on the tail of the estimated distribution of the credit risk measure. The proposed models could be used for both the International Financial Reporting Standard 9 (IFRS9) expected loss calculation and Basel's Advanced Internal Rating-Based (AIRB) regulatory capital calculation purposes.

In the second essay, we examine and model the time-series pattern of recovery throughout the bankruptcy and workout process of a retail credit portfolio; whereas other researchers are concerned with predicting the overall recovery rates of debt instruments, we model the amounts a creditor can recover at different points in time subsequent to the default event. This is of practical interest to commercial banks in managing the risk of their default loan portfolios. Like managing performing loan portfolios, banks must assign loss provision and

determine the capital requirement associated with non-performing (i.e., defaulted) loan portfolios. Given the fact that it usually takes two to three (up to five or more) years to complete the recovery process for a typical defaulted retail (corporate) loan, it is important to understand the time-varying risk characteristic of the defaulted portfolio as a function of its vintage in the recovery process. An accurate point-in-time (PIT) risk assessment enables financial institutions to manage their defaulted loan portfolios in a timely fashion.

In the third essay, we further extend our understanding of the distribution of LGD. For credit risk management purposes, the LGD of credit instruments is one of the key determinants in estimating capital requirements for financial institutions. To address the practical problems encountered in implementing LGD prediction model (e.g., in capturing the bimodal characteristic of the LGD distribution), we propose to develop a mixture beta regression LGD model. By using the maximum likelihood estimation and the method of moment approaches, the parameters of the mixture beta regression model can be estimated. Furthermore, we examine the impact of the systematic factors and model the time-series variation of the LGD distribution as a function of these systematic factors. Finally, through a number of empirical analyses, we demonstrate the superior performance of our proposed mixture beta models in comparison with the single-beta logit-linked model commonly considered in the literature.

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Chapter 1

Introduction

This thesis focuses on three important issues in credit risk modeling: the nonlinear credit risk stress testing models, the recovery term structure of point-in-time (PIT) loss given default (LGD), and the estimation of LGD by mixture beta regression model. In this chapter, we highlight the background and motivation of the research, as well as the main findings and contributions of the three essays.

Following the financial crisis in 2008, many new regulatory requirements have been introduced in the financial industry and specifically for banks. Most of them focus on stress testing and capital adequacy mainly due to the 2010 Dodd-Frank Act. Starting from 2011, new regulations in the United States require the submission of Comprehensive Capital Analysis and Review (CCAR) documentation for the financial industry. CCAR requires financial institutions to report on their internal procedures for managing capital, and financial institutions are required to include a discussion on the impact of various stress-tested scenarios in their final report. It involves examining how the banks' asset portfolios behave under historical and/or hypothetical stress conditions, usually by using a statistical model to establish the relationship between the key risk parameters and some macroeconomic variables that define the stress condition. For example, a historical stress scenario could be the bursting of the dot-com bubble (in 2001) or the global financial crisis (in 2007-2008). The economists in the banks would determine which macroeconomic variables are involved in each scenario and estimate the directions and magnitudes of the changes of these variables. Then, with the help of risk models, the changes in the key risk factors can be derived based on the changes in these macroeconomic variables. Finally, the expected loss and capital requirement, which are the major management concerns, are calculated with the estimated risk factors. To fulfill this regulatory objective, banks need to develop stress testing models that assist in making business decisions.

There are in general three key credit risk parameters: Probability of Default (PD), Loss-Given-Default (LGD), Exposure at Default (EAD). As mentioned in Basel's guidelines, financial institutions need to estimate these parameters so as to calculate the required risk measures such as the expected loss and the unexpected loss.

In the first essay, we focus on the stress testing models of the probability of default (PD) that are used in calculating regulatory capital. We contribute to the literature in several ways. First, previous PD stress testing models typically utilize linear regression techniques to establish the relation between PD and the selected macroeconomic variables based on historical data. The key risk parameters under the stress scenarios are then simulated contingent on the realizations of specific values of the macroeconomic variables that are consistent with the stress scenarios. In this study, we propose a couple of non-linear PD stress testing models that can better capture the dynamic behaviors of credit risk across different states (e.g., contraction versus expansion) of the economy. The two non-linear models are, respectively, the regime-switching model (Hamilton, 1989) and the quantile regression model (Koenker and Bassett, 1978). We conduct an empirical analysis to compare the performance of our proposed models with that of an OLS stress testing model, so as to have a better understanding of the advantages and disadvantages of each model. Finally, unlike the previous studies where the primary concern of the researchers is on the accuracy of the point estimate, we also examine the performance of the model in replicating tail events across the distribution of the predicted PD value. We find that the regime-switching model is the best among all as it outperforms other models (quatile regression models and OLS models) in producing the most accurate point estimation (based on the absolute average error) and breach counts (based on both the 80% and the 90% confidence intervals).

In the second essay, we examine and model the time-series pattern of recovery throughout the bankruptcy and workout process of a retail credit portfolio. This essay is a joint work with Dr. Donghui Chen (Scotiabank) and Dr. Peter Miu. As one of the key credit risk parameters, the recovery rate (or LGD) of a defaulted instrument attracts a lot of attention from both researchers and practitioners with the introduction of the Basel II Accord in 2006, under which the amount of regulatory capital required to be held by banks becomes a direct function of not only PD but also LGD and EAD of the loan portfolios.

To our understanding, all previous studies of recovery rate focus on the investigation of either the point-in-time (e.g., Krüger and Rösch, 2017) or the through-the-cycle (e.g., Jankowitsch, 2014) behavior of the *overall recovery rate* from the workout process rather than the *profile of recovery rate within* the workout process. In this study, we contribute to the aforementioned literature by examining and modeling the time-series pattern of recovery throughout the bankruptcy and workout process of a retail credit portfolio; whereas other researchers are concerned with predicting the overall recovery rates of debt instruments, we model the amounts a creditor can recover at different points in time after the default event. This is of practical interest to commercial banks in managing the risk of their default loan portfolios. Like managing performing loan portfolios, banks must assign loss provision and determine the capital requirement associated with non-performing (i.e., defaulted) loan portfolios. Given the fact that it usually takes two to three (up to five or more) years to complete the recovery process for a typical defaulted retail (corporate) loan, it is important to understand the time-varying risk characteristic of the defaulted portfolio as a function of its vintage in the recovery process. An accurate point-in-time (PIT) risk assessment enables financial institutions to manage their defaulted loan portfolios in a timely fashion.

The third essay investigates the distribution of LGD and proposes a new parameterization to extend the generalized beta regression model (GBR) by incorporating two beta distributions in modeling LGD. Huang and Oosterlee (2012) propose the GBR framework for estimation of the prediction of LGD. The idea is to utilize a monotonic, differentiable link function and a linear combination of predictors to model the mean and variance of the LGD distribution. Potential predictors can be macroeconomic variables or firm-level variables capturing the characteristics of the underlying assets. The link functions can be logit or probit functions.

In the third essay, we contribute to the literature on LGD modeling in a number of ways. First, we propose a new dual-beta regression LGD model that considers the probability weights of realizing the two underlying beta distributions as functions of macroeconomic variables. The general five-factor model and a simplified three-factor model are introduced. Second, with an extensive dataset on the recovery values of corporate defaults, we estimate the proposed models with a number of different macroeconomic variables that are expected to be associated with the recovery value. We conduct both in-sample and out-of-sample tests and demonstrate the superior performance of our proposed mixture beta distribution regression model when compared with the commonly used single-beta logit-link regression models. Third, we demonstrate how our proposed mixture distribution model can capture the time-varying behavior of the LGD distribution and how the probability weights assigned to the two underlying beta distributions vary with the business cycle. We find that our proposed models perform better in predicting the LGD distribution during recessionary periods[1].

[1] Most of the coding of these three chapters are finished by MATLAB, while some preliminary analysis and graphs are prepared by STATA and R.

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Chapter 2

Non-linear Credit Risk Stress Testing Modelling

2.1 Introduction

As a watershed event for regulators of financial institutions, the great depression in 1929 shows how the failure of banks impacted all of society. Among the lessons which were learned from that event, the most important one is the need to formulate an effective way to regulate banks. After some unsustainable attempts post-depression, such as rate-based competition, regulators came up with Basel I. By introducing a minimum capital requirement that depends on a bank's risk profile, it established a link between risk-taking behavior and capital requirements. Following several amendments to the capital accord in 1996 and 1998, the Basel II framework was introduced emphasizing the importance of capital adequacy.

Let us use an example to illustrate how capital requirements may affect a bank's risk management strategies. Assume there is a Bank A (a deposit-taking financial institution) with the capital structure depicted in Figure 2.1.1. As we know, assets are equal to the sum of liabilities and equities. For Bank A, the liabilities can be divided into two parts: deposits and debts, which we assume to be 60% and 30% of overall assets, respectively. So the equity amounts to 10% of assets. Although the rights of claim may vary between countries because of different legal traditions, the depositor usually ranks on the top in the hierarchy of claims. Bank A utilizes capital (i.e., shareholders' equity) raised to make investments as other financial institutions do. The returns of such investment opportunities are uncertain. If the returns are high, the equity holders take most of the profits, while the debtholders and depositors receive the interest as promised. If the returns turn out to be poor, the equity capital is the first to evaporate followed by debt and deposits. For Bank A, when the loss is higher than 10% of the asset value, it is insolvent and a default incident happens. In a more severe condition - suppose the loss is greater than 40% of assets - both the equity capital and debt together are not enough to absorb the loss. The depositors will then suffer from the poor investment decision made by the management of Bank A. As we see in this example, the liability holders face an asymmetric pattern of returns. The risk they take is just compensated by a very limited promised return (i.e., interest income), whereas the management and equity holders are incentivized to pick the risky investment opportunities to maximize their potential profits[2]. Regulators are therefore interested in finding the answers to the following two questions:

1. How can we protect the liability holders, especially the depositors?
2. How can we regulate the risk-taking behavior of equity holders?

A minimum regulatory capital requirement gives us a solution to both. Assuming we have a Bank B which has the same risk profile as that of Bank A. But unlike Bank A, which is

[2] The management's incentive to take risky behavior in making investment is profit driven, which is a rational choice (from the perspective of management) rather than personal preference. The personal attachment and business skills which may affect their strategies in making decisions are not in the scope of this chapter.

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not subject to any regulatory requirement, Bank B is required by the regulators to maintain a minimum capital ratio of 20% of the assets based on its risk profile. Generally speaking, the minimum required capital of a particular financial institution depends on the riskiness of its investments are. The riskier a bank's portfolio, the more equity capital is required to cushion the corresponding risk taken. Compared with Bank A, Bank B is more financially stable when faced with economic turbulence given its larger capital cushion. Bank B can absorb more financial losses before it becomes insolvent. Even with the extreme loss of 40% of asset value, Bank B's depositors can still recover all the money they deposited in the bank. For the equity holders of Bank B, given the regulatory capital requirement, more of their money is at stake and they will be less inclined to pursue a risky investment strategy. Since the minimum capital requirement is a direct function of the risk profile, it prevents the bank management from becoming too aggressive as they will need to raise more equity capital to provide a thicker cushion to protect liability holders. There is a trade-off between optimizing the capital requirement and the pursuit of an aggressive investment strategy.[3]

INSERT FIGURES 2.1.1 and 2.1.2 ABOUT HERE

The minimum capital requirement has proved to be an effective way to instill stability in the financial sector in the past decades. However, regulators start to notice that it is not wise to utilize a static regulator driven capital adequacy measure to deal with the risk profile and capital requirement of an active financial institution in a dynamic risk environment. The need for an internal and comprehensive assessment of the capital profile of a bank-led to the introduction of an Internal Capital Adequacy and Assessment Process

[3] Many banks maintain a capital level well above the minimum capital requirement as stipulated by the regulators. Such a strategy makes the banks more attractive from the perspective of depositors and debt holders, potentially lowering their costs of debt.

(ICAAP) whose objective is to allocate risk capital to all significant sources of risk and stress test the result so that the senior management and the board of directors are informed of any expected or projected capital shortfall.

Following the financial crisis in 2008, a number of new regulatory requirements were introduced in the financial industry and specifically for banks. Most of them focus on stress testing and capital adequacy mainly due to the 2010 Dodd-Frank Act. Starting in 2011, new regulations in the United States required the submission of Comprehensive Capital Analysis and Review (CCAR) documentation for the financial industry. CCAR requires financial institutions to report on their internal procedures for managing capital, and financial institutions are required to cover various stress-tested scenarios in their final report. This involves examining how the banks' asset portfolios behave under historical and/or hypothetical stress conditions, usually by using a statistical model to establish the relationship between key risk parameters and macro variables that define the stress condition. For example, a historical stress scenario could be the bursting of the dot-com bubble (in 2001) or the global financial crisis (in 2007-2008), while a hypothetical scenario might be nuclear warfare. Bank economists would determine which macro variables are involved in each scenario and estimate the directions and magnitudes of the change of each macro variable. Then, with the help of risk models, the changes in the key risk factors can be derived based on the changes in these macro variables. Finally, the expected loss and capital requirement, which are the major management concerns, are calculated with the estimated risk factors. To fulfill this regulatory objective, banks need to develop stress testing models that assist in making business decisions.

In addition to CCAR reporting, systemically important financial institutions (typically those with greater than $50 billion in assets) in the United States deemed too big to fail by the Financial Stability Board must include stress-tested reporting on planning for a bankruptcy scenario[4]. Currently, Basel III is in effect for banks as well. This is a universal reporting stress test that requires reporting documentation on banks' capital levels with specified requirements for stress testing of various designed crisis scenarios.

In Canada, all banks and investment firms that are regulated by the federal government are required to comply with the Basel Capital Adequacy Requirement (BCAR). BCAR focuses on the capital to risk-based asset ratio and an asset-to-capital multiple of the financial institutions and is required to be filed with the Office of the Superintendent of Financial institution (OSFI) on a quarterly basis. Two methodologies are available for calculating the capital requirements: the standardized and the internal rating-based (IRB) approaches. Prior approval from OSFI is required to use IRB. OSFI reviews stress testing programs of financial institutions as part of their supervisory process. Expecting to see evidence that the stress testing procedure is integrated into the financial institution's internal risk management process, OSFI may:[5]

1. Evaluate the consistency of the scenarios.
2. Assess the appropriateness of scenarios.

[4] In the US government's most recent reporting review of these financial institutions in 2016, there were eight too big to fail systemically important financial institutions.
[5] OSFI itself also conducts analysis on stress testing results for system-wide scenarios.

3. Assess the sufficiency of timing and frequency of stress testing.
4. Examine the capital adequacy under stress testing scenarios.

The present study focuses on the stress testing of credit risk exposure. Among all types of risks (e.g., market risk, operational risk, credit risk, etc.) defined by the Basel committee, credit risk is the most important one for commercial banks. Let us take the Bank of Montreal (BMO) as an example. In Table 2.1.1, we present the risk-weighted assets (RWA) of BMO calculated using the Advanced Internal Rating Based (AIRB) approach from 2012 to 2016. The amount of RWA in credit risk consistently represents more than 80% of the overall RWA of the bank. More importantly, the importance of credit risk is not isolated in a specific line of business of the bank. Based on the economic capital (EC) allocated to the different risk classes according to the 2016 annual report, BMO is heavily exposed to credit risk in all of its major banking businesses, namely personal banking, commercial banking, capital market, and corporate services (see Table 2.1.2).

INSERT TABLES 2.1.1 and 2.1.2 ABOUT HERE

In this study, we focus on one of the most important risk parameters that define the credit risk of an instrument – the probability of default (PD) of the obligor. The other two parameters are respectively loss-given-default (LGD) and exposure-at-default (EAD). To measure the magnitude of credit risk in daily operations (e.g., to calculate the expected loss of a loan), we need to estimate the PD, LGD, and EAD of the instruments.

The probability of default, which estimates the likelihood that a borrower will be unable to fulfill its promised obligation at the settlement date, concerns the quality of the obligor over a particular time horizon. As one of the most important topics in credit risk management, PD estimation has been well studied. There are many popular alternatives for estimating PD. For example, default probabilities can be estimated from historical data of default frequencies using regressions analysis. The PD of large publicly traded companies can be inferred from the market prices of their credit default swaps, options on stocks, and bonds. Banks also use models developed by external rating agencies, such as Moody's, to estimate the PD of corporations from historical default experiences. On the other hand, credit scoring models are commonly used to evaluate PD for individuals and small businesses.

PD by itself cannot fully define the credit risk of an instrument. Even if the PD of the borrower is very high, a bank may not be subject to any credit risk if the loan is sufficiently secured. This is because, in the event of the borrower defaulting on the loan, the bank may fully recover the money that it lends by liquidating the collateral underlying the secured loan. Besides PD, we also need to estimate the LGD of the instrument. LGD represents the share of an asset that is lost in the case of default (because of legal fees, transaction costs, degeneration of asset value, etc.). So LGD can also be expressed as one minus the recovery rate. LGD is influenced by key transaction characteristics such as the value of the collateral and the degree of subordination. The LGD calculation can be easily understood with the help of an example. Suppose a borrower defaults with an outstanding debt of $100 and the financial institution is able to liquidate the collateral for a net price of $50. Then the LGD is 50% (= ($100-$50) / $10~~5~~0). Typically expressed as a proportion of the outstanding amount, LGD is the total loss divided by the exposure-at-default (EAD), which is another crucial parameter in defining credit risk. EAD, also known as credit exposure, represents the exposure of the lender if the counterparty defaults on his debt. EAD is simply the current amount outstanding in the case of fixed-exposure instruments like loans. It becomes more complicated if it is a revolving exposure, such as a line of credit. EAD is then made up of two parts: the drawn amount and (part of) the undrawn commitment. Financial institutions need to estimate the amount of the facility that is likely to be drawn if a default occurs in the future. The product of PD, LGD, and EAD gives us the expected loss (EL) under an independent assumption of the risk parameters:

$$\text{Expected loss (EL)} = PD * E(LGD) * E(EAD), \qquad (1)$$

where PD = probability of default

LGD = Loss given default

EAD = Exposure at default

Let us illustrate the calculation involved with a numerical example (see Figure 2.1.3). Suppose we want to estimate EL for a line of credit issued to a borrower whose PD is 5%, LGD is 50% (70%) with a probability of 30% (70%), and EAD is either $40 or $60 with equal probability. As there are two potential outputs for each of the three parameters, we have eight possible outcomes in total. When there is no default (with a 95% probability), the loss will be zero for four of the eight possible outcomes. So, we focus on the remaining four outcomes. The tree in Figure 2.1.3 shows these four possible outcomes when we put PD, EAD, and LGD into consideration respectively and sequentially. The number on the left of each cell represents the probability of realizing a particular outcome of the corresponding parameter. The calculation of EL is quite straightforward if we assume PD, LGD, and EAD are independent of each other. Upon the default of the borrower (with a 5% probability), there is an equal chance of realizing an exposure of \$40 or \$60. If a \$40 EAD is realized, the credit loss will be either \$20 (=\$40 * 50%) if the realized LGD is 50% (with a 30% probability) or \$28 (=\$40 * 70%) if the realized LGD is 70% (with a 70% probability). If a \$60 EAD is realized, the credit loss will be either \$30 (=\$60 * 50%) if the realized LGD is 50% (with a 30% probability) or \$42 (=\$60 * 70%) if the realized LGD is 70% (with a 70% probability).[6] There are then two ways to calculate EL. One way is to evaluate the probability-weighted average loss across the four possible outcomes, i.e.,

$$EL=5\%*50\%*30\%*\$20+5\%*50\%*70\%*\$28+5\%*50\%*30\%*\$30 + 5\%*50\%*70\%*\$42 = \$1.6$$

Alternatively, we can simply use Eq. (1), i.e.,

$$E(LGD) = 50\%*30\%+70\%*70\% = 0.64$$

$$E(EAD) = \$40*50\%+\$60*50\% = \$50$$

Finally,

$$EL = 0.05*0.64*\$50 = \$1.6$$

INSERT FIGURE 2.1.3 ABOUT HERE

To calculate the capital requirement for both regulatory or internal risk management purposes, besides finding the EL, we also need to estimate the unexpected loss (UEL). UEL is the extreme amount of loss to be incurred under a small but plausible probability (e.g.,

[6] Without assuming PD, LGD and EAD are independent of each other, it is more complicated to calculate the probability of each outcome as we need to consider the correlations among the parameters.

0.05%). Banks need to ensure they have sufficient capital to survive even under such an extreme event. For example, an AA-rated financial institution would like to make sure it can survive with a 99.95% certainty. UEL, therefore, corresponds to an extreme tail event. The economic capital (EC) requirement can therefore be defined as the amount of UEL in excess of EL; whereas regulatory capital (RC) is the amount of capital a bank needs to hold as required by the financial regulator based on specific assumptions of EL and UEL assessments. The former is formulated for internal risk management and decision purposes, while the latter is the mandatory minimum capital required to maintain for regulatory purposes.

Our paper focuses on the stress testing models of PD that are used in calculating EL, UEL, and EC. We contribute to the literature in a number of ways. First, previous PD stress testing models typically utilize linear regression techniques to establish the relation between PD and selected macroeconomic variables based on historical data. The key risk parameters under the stress scenarios can then be simulated contingent on the realizations of specific values of the macroeconomic variables that are consistent with the stress scenarios. In this study, we propose a couple of non-linear PD stress testing models that can better capture the dynamic behaviors of credit risk across different states (e.g., contraction and expansion) of the economy. Besides, we conduct an empirical analysis to compare the performance of our proposed models with that of an OLS stress testing model, so as to have a better understanding of the advantages and disadvantages of each model and check the robustness of our proposed nonlinear against the OLS model. Finally, unlike the previous studies where the primary concern of the research is on the accuracy of the point estimate, we also examine the performance of the model in replicating tail events across the distribution of the predicted PD value. In this study, we focus on applying our non-linear model to PD. With some adjustments, the proposed methodology can be readily used in the modeling of LGD and EAD. Please note that the correlation impacts among assets in the portfolio is not in considered in this study as the focus is on PD estimation and prediction.

The rest of the paper is organized as below. In Section 2, we provide a literature review. We outline our proposed stress testing models in Section 3. In Section 4, we present the results of our empirical analysis with the proposed models. A validation exercise is then conducted and the results are summarized in Section 5. In Section 6, we conclude with a few remarks.

2.2 Literature Review

Credit risk is one of the major concerns among financial institutions that are active in the lending business. To assess credit risk, financial institutions need to estimate the probability of default (PD) of their retail and corporate customers. For a long time, The OLS model (and its extensions such as the logit model and probit model) is a common approach adopted by banks to estimate and predict PD for stress testing purposes. Bank stress tests can be categorized into two types based on their purposes: micro stress test and macro stress test. The former is a bank-level stress test to check the capital adequacy of a particular bank or investment firm. Miu and Ozdemir (2008) propose a stress testing model of the probability of default and migration rate concerning the Basel II requirement allowing for the robust use of external data, and further identify, examine, and quantify the impact of stress events. Yang and Du (2015) extend Miu and Ozdemir's model on stress testing by incorporating different asset correlations. The proposed models demonstrate the desired sensitivity to the risk factors as expected. A macro stress test is usually run by central banks to stimulate macroeconomic environment change to assess the resilience of the financial system as a whole rather than by individual institutions. Borio, Drehmann, and Tsatsaronis (2011) review the state of macro stress testing, assess its strengths and weaknesses, and discuss ways to improve its performance such as generating more realistic nonlinearities and feedback effects. Havrylchyk (2010) utilizes linear models to connect the explanatory variables with PD and demonstrates that macroeconomic shocks, e.g., changes in the interest rate and property prices, have an enormous influence on credit loss in South Africa.

This study is on micro-level credit risk stress testing. The purpose is to quantify the credit risk under an unfavorable economic environment or other scenarios so as to determine whether a financial institution has enough capital to withstand the impact. Stress testing provides risk managers with an idea of the possible impact of extreme (but plausible) shocks on their financial institutions. We would like to investigate the impact on credit portfolio (e.g., corporate loans, residential mortgages) of a financial institution. The stress testing exercise cannot answer "when will be the next crisis" but rather "what if there is a crisis?" For example:

- What happens if GDP falls by x% in a given year?
- What happens if the unemployment rate rises to x% in a given year?
- What happens if interest rates go up by x% in a given year?

Defining the stress scenarios is the first step of the stress testing procedure (see Figure 2.2.1). We then determine both the direction and magnitude of changes in the relevant macroeconomic variables based on the specific stress scenario. In the third step, we find out the relationships between the credit risk factors, like PD, LGD and EAD, and the macroeconomic variables with the help of statistical models usually referred to as stress testing models. Finally, with the simulated risk factors (PD, LGD, and EAD), EL and capital requirement can be calculated for internal risk management purposes and to satisfy regulatory requirements.

INSERT FIGURE 2.2.1 ABOUT HERE

Below we provide a review of the research on micro-level credit risk stress testing. For example, Misina, Tessier, and Dey (2006) build a stress testing model for the corporate loan portfolio of the Canadian banking sector following the procedure described in Figure 2.2.1 and conclude that PD is negatively correlated with GDP, while positively correlated with the interest rate. Specifically, a 200 basis point increase in the U.S. real interest rate from 0.18 percent to 2.18 percent will result in an expected loss of 4.6 percent and an unexpected loss (99% VaR) of 8.5 percent.

Although it is still a common practice to adopt a linear model (like those considered by Miu and Ozdemir (2008) in establishing the relation between macroeconomic factors and credit risk parameters, researchers and practitioners have started to understand that the relationship is unlikely to be identical in different phases of the economic cycle. This observation is mainly due to the asymmetric pattern of risk and returns for the creditor (i.e., the bank) in a loan contract. For the creditor, the upside gain is limited to the promised interest payment, while the downside risk in a default event could be substantial and depends on the financial situation of the debtor. Such an asymmetric pattern, which is due to the presence of capital structure, increases the sensitivity of credit risk during recessions. It makes the PD of banks' credit portfolios more sensitive to the business environment during downturns. This argument is supported by empirical evidence. For example, in BCBS (2005), it is shown that PD is more sensitive to changes in the business cycle under stressed conditions. Aguais, Forest, Wong, and Diaz-Ledezma (2004) point out the importance of developing separate ratings for short- and long-term exposures. Without considering this feature, we tend to underestimate the downside risk if we assume these variables to react in their usual ways by using one regression such as OLS to fit for all situations. In other words, the stress test would lose its original designed purposes.

Researchers started to look for alternative approaches to accommodate the asymmetric effect. Nickell, Perraudin, and Varotto, S. (2000) use a probit model to exam the dependence of rating transition probabilities on different stages of the business cycle, and confirm that the business cycle is the most important factor in explaining the variation of their transition probabilities using the Moody's data from 1970 to 1997. Bangi, Diebold, and Schuermann (2002) extend this idea to incorporate business cycles into the stress testing process and point out the potential usefulness of regime-switching models in credit risk stress testing. In the present study, we contribute to this line of research by utilizing nonlinear regression models that can capture the dynamic relationship between the PD and the explanatory variables in conducting stress testing, and we compare such frameworks with the traditional OLS model.

The first kind of nonlinear model we consider is the regime-switching model of Hamilton (1989). This nonlinear model characterizes the time series behavior in multiple regimes. By allowing for the switching among regimes, this model is able to capture complex dynamic patterns. Although not used extensively in credit risk modeling, the regime-switching model has been extensively used in empirical studies in finance. For example, Elliott, Siu, and Chan (2005) price volatility swap under Heston's stochastic volatility model with a regime-switching model. Ang and Timmermann (2012) study the financial market with regime-switching models and conclude that the regime-switching models are capable of capturing the stylized behavior of many financial series including fat tails, heteroskedasticity, skewness, and time-varying correlations. In credit risk modeling, Bruche and Gonzalez-Aguado (2006) propose and estimate a model with switching coefficients formalizing the idea that the default probability and recovery rates are negatively correlated and show that the proposed model fits the data well. Alexander and Kaeck (2008) demonstrate that credit default swap (CDS) spreads are extremely sensitive to stock return volatility during market turbulence. Nyberg (2017) estimates and forecasts

US interest rates and business cycles using a Markov switching vector autoregression (VAR) model. These studies make progress in incorporating the regime-switching model into financial analysis and show that the regime-switching model is a good candidate for recognizing state-contingent time series behavior. From our knowledge, we are the first to examine the use of regime-switching models in credit risk stress testing. In allowing for different behaviors of PD in different states of the economy, we can arrive at a more reliable prediction of credit risk with the estimated regime-switching coefficients than with the OLS model. We find that both the direction and the magnitude of the relation between PD and the explanatory variables may differ between recession and expansion states of the economy. The naïve use of a single OLS model will therefore lead to a biased estimation of PD.

The second kind of nonlinear model we consider is the quantile regression model. The quantile regression model accommodates different sensitivities between the dependent and the independent variables at different levels of the dependent variable. Unlike in the regime-switching model, where the regression equation switches between states in the time dimension, the quantile regression allows for the regression coefficients to vary cross-sectionally. Specifically, it models the whole distribution of the dependent variable by estimating the regression coefficients for different quantiles. The quantile regression model is first introduced by Koenker and Bassett (1978) and its use became widespread in the late 20^{th} century. The approach has been widely adopted in empirical research in finance. For example, Ma and Pohlman (2005) investigate stock market returns with quantile regressions. They show that quantile regression provides more accurate forecasts and more value-added portfolios than the traditional OLS method. Mensi et al. (2014) examine the asymmetric dependence structure among BRICS countries' stock markets using the quantile regression approach. Baur, Dimpfl, and Jung (2012) provide a comprehensive description of the dependence pattern of stock returns by investigating the return distribution with quantile regression and conclude that lower quantiles exhibit positive dependence while upper quantiles are negatively dependent. Lee and Zeng (2011) study the impact of changes in oil prices on stock returns of G7 countries. Their quantile regression estimates are quite different from those of the OLS model in many cases showing some crucial implications for the linkage between oil prices and the stock market.

We are not the first to use quantile regression in studying credit risks. Li and Miu (2010) establish a hybrid bankruptcy prediction model with dynamic loadings for both accounting ratio-based and market-based information. They find that the distance-to-default variable derived from the market-based model is statistically significant in explaining default for poor credit quality firms, while the z-score obtained from the accounting-based model is significant in forecasting default event of firms with good credit quality. Kruger and Rosch (2017) propose a quantile regression approach to get a comprehensive view of the loss-given-default distribution. They conclude that the middle quantiles and tail events are explained by observable covariates and unobservable random events, respectively. Due to its advantage in capturing the full distribution information and its robustness against outliers and fat tails relative to the OLS regression, the quantile regression models proposed in the above studies outperform the models constructed based on the classical conditional mean estimation method. These studies confirm the importance of nonlinear regression approaches in the estimation and prediction of risk determinants. In the present study, we extend the literature by examining the use of quantile regression in credit risk stress testing models. Specifically, we explore the different reactions of PD with respect to macroeconomic variables in different quantiles of the credit risk level. By capturing the different behavior of PD in high quantiles (high credit risk) vs. that of low quantiles (low credit risk), we are able to arrive at a more accurate estimation and prediction of PD over the stress periods.

2.3 Methodology and Data

2.3.1 Markov regime-switching Model

The Markov regime-switching model is a type of specification in which the main contribution is the flexibility in accommodating processes driven by heterogeneous states of the world. In this section, we give a brief introduction to the model. More details can be found in Hamilton (1994), and Kim and Nelson (1999).

Consider the following process of a random variable y_t given by:

$$y_t = \mu_{S_t} + \epsilon_t \qquad (1)$$

where $S_t = 1,2,\cdots,k$ denote the *k* regimes (or states). The residual ϵ_t follows a normal distribution with zero mean and variance given by $\sigma^2_{S_t}$. This is the simplest case of a dynamic model with switching regimes. Usually μ_{S_t} can be expressed as a linear combination of n independent variables ($i = 1 \; to \; n$) with coefficients β_{i,S_t}. Note that the intercept has *k* states, which means there are *k* values for both intercept and variance. If *k*

= 1, then the process becomes a simple linear regression model under the general condition. The number of parameters of the Markov regime-switching model will increase dramatically as the number of states and the number of independent variables increase.

In this study, we consider a two-state regime-switching model (i.e., k = 2). The two states can generally be interpreted as the contraction and expansion states of the economy. The limited monthly data we have essentially forbid us to consider a model with more than two states. By adopting a parsimonious model, we can ensure the robustness of our results. A two-state model is also intuitive, as the two states naturally represent the upturn and downturn of the economy over a business cycle.

Suppose there are n independent variables (e.g., macroeconomic variables). Then the two-state model can be represented by:

$$y_t = \beta_{0,1} + \sum_{i=1}^{n} \beta_{i,1} x_{i,t} + \epsilon_t \quad for\ state\ 1$$

$$y_t = \beta_{0,2} + \sum_{i=1}^{n} \beta_{i,2} x_{i,t} + \epsilon_t \quad for\ state\ 2,$$

where

$$\epsilon_t \sim (0, \sigma_1^2) \quad for\ state\ 1$$

$$\epsilon_t \sim (0, \sigma_2^2) \quad for\ state\ 2$$

For a credit risk model, y_t can represent a vector of default frequency (or its transformation). The different standard deviation in each state represents the uncertainty regarding the predictive power of the model in that state of the world. Going back to our setup, one would expect that PD during the contraction period is more volatile than that of the expansion period. This means that we can expect $\sigma^2_{contraction}$ to be higher than $\sigma^2_{expansion}$. This model also allows us to accommodate the possibility that the variation of PD with macroeconomic variables behaves differently across two states. Note that we do not identify the states, for example, contraction as State 1. Actually, here S_t is just an index of the states, while the interpretation depends on the values of the estimated parameters.

To complete the specification of the regime-switching model, we assume a constant transition probability matrix P that governs the switching between the two states:[7]

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},$$

where p_{12} and p_{21} are the probability of a switch from State 1 to State 2 and from State 2 to State 1 in the next period, respectively, while the probability of staying in State 1 and State 2 in the next period is given by p_{11} and p_{22}.

The filtered probability of realizing a certain state evolves over time and is updated according to the arrival of new information. Specifically, the updating of the probability of realizing State *j* (i.e., Pr ($S_t =$ j)) follows:

[7] The model can be generalized by specifying a time-varying transition probability matrix (see, e.g., Wang (2003)).

1. Setting the starting probability (at t = 0) of each state to be 0.5. That is, $\Pr(S_0 = \mathrm{j}) = 0.5$ for j = 1,2.

2. The probability of realizing State j at time t = 1 given the information set up to time t = 0 (ψ_0) is given by:

$$\Pr(S_1 = \mathrm{j}|\psi_0) = \sum_{i=1}^{2} p_{ij}(\Pr(S_0 = \mathrm{i}|\psi_0))$$

3. We then update the probability of each state with the new information available at time t=1. Suppose the likelihood function in each state is given by $f(y_1|S_1 = j, \psi_0)$. We use the following formula to update the probability of realizing State j:

$$\Pr(S_1 = \mathrm{j}|\psi_1) = \frac{f(y_1|S_1 = j, \psi_0)\Pr(S_1 = \mathrm{j}|\psi_0)}{\sum_{j=1}^{2} f(y_1|S_1 = j, \psi_0)\Pr(S_1 = \mathrm{j}|\psi_0)}$$

4. We then repeat the above steps over time until we arrive at the end of the sample period $t = T$. We therefore obtain a time series of filtered probabilities for each of the two states over our sample period.

The log-likelihood function of the model can be expressed with the set of parameters β, $\sigma_1, \sigma_2, p_{11} and\ p_{22}$:

$$lnL = \sum_{t=1}^{T} ln \sum_{j=1}^{2} f(y_t|S_t = j, \Theta)\Pr(S_t = \mathrm{j}|\psi_t) \quad (2)$$

The model can be calibrated by maximizing equation (2) to find the set of estimated parameters. In our empirical analysis, we will be comparing the performance of different specifications of the regime-switching model. We gauge their performance by calculating the Akaike information criterion (AIC) and Bayesian information criterion (BIC) of the models. With the built-in penalty for increasing the number of estimated parameters, the AIC and BIC measures allow us to arrive at an optimal model by incorporating the appropriate trade-off between the goodness of fit of the model and its complexity.

2.3.2 Quantile Regression Model

The quantile regression model is built on the method of minimum absolute deviation which was first proposed by Boscovich in 1757 and later developed by Laplace. With the growing interest in robust methods and extreme value modeling, this method becomes more popular in the late 20th century. Hao and Naiman (2007) and Wooldridge (2010) give a detailed introduction to the quantile regression method. In general, quantile regression aims at estimating specific quantiles of the response variable (i.e., the dependent variable). For stress testing purposes, one of the desired properties of the quantile regression model is that the quantile regression estimates are more robust and reliable against outliers in the response measurements.

Let us define τ as the quantile to be estimated; the median is τ = 0.5. For each observation i, let ε_i be the residual

$$\varepsilon_i = y_i - x_i'\widehat{\beta_\tau}$$

In order to estimate the parameters of the quantile regression model, we minimize the objective function:

$$c_\tau(\varepsilon_i) = (\tau 1\{\varepsilon_i \geq 0\} + (1-\tau)1\{\varepsilon_i < 0\})|\varepsilon_i| = (\tau - 1\{\varepsilon_i < 0\})\varepsilon_i, \qquad (3)$$

where $1\{\cdot\}$ is the indicator function. Please note that, in choosing the parameters $\widehat{\beta_\tau}$ that minimize $c_\tau(\varepsilon_i)$, we are in effect finding the $\widehat{\beta_\tau}$ that makes $x_i'\widehat{\beta_\tau}$ fit the quantiles of *y* given *x*.

As quantile regression is not estimated with the likelihood function, it is not appropriate to use AIC and BIC as the model selection criteria. To measure and compare the performance of quantile regression models, we first calculate the Pseudo-R^2 suggested by Koenker and Machado (1999). It measures the goodness of fit by comparing the sum of weighted deviation from the calibrated model with that based on the raw data:

$$R_\tau^2 = 1 - \frac{sum\ of\ weighted\ deviations\ about\ the\ estimated\ quantile}{sum\ of\ weighted\ deviations\ about\ the\ raw\ quantile}$$

$$= 1 - \frac{\sum_{y_i \geq \widehat{y_i}} \tau \cdot |y_i - \widehat{y_i}| + \sum_{y_i < \widehat{y_i}} (1-\tau) \cdot |y_i - \widehat{y_i}|}{\sum_{y_i \geq \bar{y}} \tau \cdot |y_i - \bar{y}| + \sum_{y_i < \bar{y}} (1-\tau) \cdot |y_i - \bar{y}|} \qquad (4)$$

The shortcoming with this statistic as a performance measure is similar to that of R^2 in the Ordinary Least Square (OLS) model, that is, Pseudo-R^2 will always be higher as the number of explanatory variables increases since there is no penalty for data mining. To avoid arriving at an unnecessarily complicated model, we adjust the Pseudo-R^2 in a similar fashion as we calculate adjusted R^2 in OLS regression. That is,

$$adjusted\ PseudoR^2 = 1 - \frac{(n-1)(1 - \text{Pseudo}R^2)}{n-p-1}$$

With adjusted Pseudo-R^2, we can more readily compare quantile regression models with different numbers of parameters.

2.3.3 Data

In this section, we describe the details regarding the preparation of variables for both quantile regression models and Markov regime-switching models.

2.3.3.1 Dependent Variable

We collect the information of default frequency (DF) in the United States from BankruptcyData. BankruptcyData has been collecting and storing data since the 1980s, giving us the country's largest collection of historical and current public company bankruptcy information. We note that companies with assets of less than one million are not included in their database. And we further exclude all firms that are private in their data sets. Here we use the default frequency (DF) as the definition of the probability of default. So DF and PD are interchangeable in this paper. For time period t:

$$DF_t = \frac{Number\ of\ firms\ that\ defaulted\ during\ 3\ months\ starting\ from\ t}{Total\ number\ of\ firms\ at\ t}$$

Our sample period is February 1987 to December 2015. The number of firms that defaulted during each month is from BankruptcyData and the total number of public-listed firms in the US in each month is based on data from the World Federation of Exchanges. In Table

2.3.1, we present the annual number of defaults observed for different industries over our sample period. We have about 3,000 default cases in total from 1987 to 2015. The number of default peaks in 2001, 2002, and 2009, which are the time windows for the bursting of the dot-com bubble and the global financial crisis. Among the different industries, the manufacturing sector suffers from the most default incidences compared to that of other industries.

INSERT TABLE 2.3.1 ABOUT HERE

To develop a PD stress testing model, we construct a time series of quarterly default frequency (DF) over our sample period. For example, the first DF is:

$$DF_{Feb1987} = \frac{Number\ of\ defaulted\ firms\ from\ Feb1987\ to\ Apr1987}{Number\ of\ firms\ at\ Feb1987}$$

, and the second data point is:

$$DF_{Mar1987} = \frac{Number\ of\ defaulted\ firms\ from\ Mar1987\ to\ May1987}{Number\ of\ firms\ at\ Mar1987}$$

In general, we have:

$$DF_t = \frac{Number\ of\ defaulted\ firms\ from\ t\ to\ t+2}{Number\ of\ firms\ at\ t}$$

and

$$DF_{t+1} = \frac{Number\ of\ defaulted\ firms\ from\ t+1\ to\ t+3}{Number\ of\ firms\ at\ t+1}$$

In Figure 2.3.1, we plot the calculated DF over the sample period from Feb 1987 to Dec 2015. DF peaks at the two recession periods: (1) the bursting of the dot-com bubble in 2001, and (2) the global financial crisis in 2008. There are significant variations of DF over the last three decades. The maximum and minimum DF are 1.6% and 0.3% respectively. The average DF is 0.46% and the standard deviation is 0.2%.

INSERT FIGURE 2.3.1 ABOUT HERE

The preliminary tests on the varaibles above indicates that the residuals of regressions are not normally distributed which may lead to biased results in the further analysis. So We conduct the Shapiro-Wilk test to check the normality of DF and we can reject the hypothesis that it follows a normal distribution (see Table 2.3.2). We then repeat the test on two commonly used transformations of DF: logit and probit transformations. We can again reject the normal distribution hypothesis for the logit transformation, but not for the probit transformation. In the rest of the study, in order to satisfy the normal distribution assumption of the models, we use the probit transformation of DF as our dependent variable.

INSERT TABLE 2.3.2 ABOUT HERE

2.3.3.2 Independent Variables

Six macroeconomic variables are considered as our potential candidates in developing our stress testing models: Gross domestic product (GDP), federal fund rate (FFR), unemployment rate (UE), 10-year Treasury bond yield (T10), 3-month treasury bill yield (T3), and corporate credit spread (CS). The choice of these variables is motivated by a number of previous studies,e.g., Havrylchyk (2010), and Misina, Tessier, and Dey (2006). Havrylchyk (2010) finds that credit risk is sensitive to GDP, UE, CS, and interest rates, confirming that higher interest rates and inflation increases have a positive effect on credit risk, while GDP and employment are negatively correlated with loan loss provisions in univariate and multivariate regressions. The empirical results of Misina, Tessier, and Dey (2006) suggest that a decrease in the US real GDP growth rate leads to an increase in the credit loss from loans. Below we describe how each of the macroeconomic variables is constructed.

- GDP: 3-month US GDP growth rate is calculated based on monthly GDP collected from Bloomberg.

$$GDP\ growth\ rate_t = \frac{GDP_{t+3} - GDP_t}{GDP_t}$$

- UE: 3-month change in the US unemployment rate is another popular indicator of economic condition, which is available on the International Labor Organization, ILOSTAT database.

$$UE\ change_t = Unemployment\ Rate_{t+3} - Unemployment\ Rate_t$$

- FFR, T10, and T3: These interest rates are collected from the Federal Reserve Bank of St. Louis on a monthly basis.
- CS: Credit Spread is calculated with the following formula:

$$CS = Moody's\ Baa\ \ Bond\ Yield(\%) - Moody's\ Aaa\ Bond\ Yield(\%)$$

where the yields are from the Federal Reserve Bank of St. Louis.

In Table 2.3.3, we report the summary statistics of the six macroeconomic variables over the sample period from February 1987 to December 2015. The pair-wise correlation coefficients of the macroeconomic variables together with the DF are presented in Table 2.3.4. PD is found to be positively correlated with UE and CS, while negatively correlated with GDP and all three interest rates. The intuition behind this observation is the logic that we expect PD, unemployment rate, and credit spread to be higher when the economy is in a downturn. On the other hand, interest rates and GDP growth rates tend to be lower during a recession indicating a negative correlation with PD.

INSERT TABLES 2.3.3 AND 2.3.4 ABOUT HERE

2.4 Results of Empirical Analysis

2.4.1 Preliminary Analysis

First of all, univariate analysis is performed for each explanatory variable using simple OLS regression:

$$\Phi^{-1}(DF_t) = intercept + \beta x_{t-s} + \varepsilon_t, \text{where s} = 1, 2, 3, 4 \quad (6)$$

The dependent variable is the probit transformation of DF. In order to capture the possibility of a lead-lag relation between DF and the macroeconomic variables, we consider different lagged versions of the univariate regression (lag s = 1, 2, 3, and 4). The results are reported in Table 2.4.1. All of the coefficients of the macroeconomic variables are statistically significant. Consistent with the pair-wise correlation results examined earlier, DF is significantly positively (negatively) associated with the unemployment rate and credit spread (GDP growth rate and interest rates). By examining the magnitude of the estimated coefficients across the different lags, we notice that, for FFR, T10, T3, and CS, the most recent data (i.e., lag 1) has the strongest influence on the current DF, while for GDP and UE, it takes a few periods (around lag 3) before their explanatory power reaches their peak. The possible explanation for this observation is that the response of DF is quicker to the change in interest rates and credit spread than to that of GDP and UE.

INSERT TABLE 2.4.1 ABOUT HERE

2.4.2 Model Selection

In this section, we develop PD stress testing models based on the regime-switching model and the quantile regression model, respectively. In doing so, we highlight the characteristics of these nonlinear approaches in capturing the dynamic effects of the macroeconomic factors that could be state-contingent and/or condition on the prevailing risk level of the economy.

We start by developing an optimal regime-switching model in explaining the observed time series of DF. We consider all the combinations of macroeconomic variables based on our data set. Using the probit transformation of DF, $\Phi^{-1}(DF_t)$ as our dependent variable, we start from regressing a single explanatory variable, two explanatory variables,…, until all six variables are exhausted. For the purpose of building a stress testing model, we are developing a prediction model where all the explanatory variables are lagged as we described in 4.1, that is, one month lag for interest rates and CS, and three months lag for UE and GDP. The regime-switching regression results are presented in Table 2.4.2. The univariate regime-switching regression results reported in Panel A suggest that, except for T3 and CS, there are significant regime-switching effects for the selected macroeconomic variables with respect to DF. The signs of the coefficients that are statistically significant are consistent with our expectation as stated in 3.3.2. Moreover, the magnitudes of these coefficients are greater in state 2 (bad state) compared to that of state1(good state). In Panel B, we report the best multivariate regime-switching models of combinations of two, three, four and five regressors in Columns (7) to (10), respectively, based on AIC and BIC The reason for not regressing all six variables in the same regression is that T3 and FFR are both short rates and are highly correlated. So we pick one each time to avoid multicollinearity issues in our regression.

INSERT TABLE 2.4.2 ABOUT HERE

Among all the regressions with different combinations of macroeconomic variables, the best performing regime-switching model (according to AIC and BIC) is the one with GDP, UE, T10, T3, and CS as the explanatory variables. The estimation results can be found in Column (10) of Panel B. All the estimated coefficients are statistically significant and of the expected sign in both States 1 and 2. Specifically, the GDP growth rate and interest rates are negatively associated with PD, while the credit spread and unemployment rates have a positive relationship with PD. This is consistent with our intuition. When PD is low, the economy is usually growing which tends to be accompanied by a high-interest rate, a low unemployment rate, a narrow credit spread, and vice versa. Based on the average DF and the standard deviation of the residuals, States 1 and 2 can be interpreted as the "good" (i.e., expansion) and the "bad" (i.e., contraction) states of the economy, respectively. The regime-switching model also provides us with the average time we spend in the two states. For our best regime-switching model, the expected durations are 240 and 79 months for States 1 and 2, respectively. It is therefore more likely we are in the "good" state than in the "bad" state.

In Figure 2.4.1, we plot some of the key time-series characteristics based on our best regime-switching model. The top subgraph is the time-series plot of DF. The middle subgraph shows the conditional standard deviation of the residuals. The one on the bottom exhibits the (smoothed) probability of realizing the two states as defined in Section 3.1. The bad (i.e., contraction) state is characterized by the larger conditional standard deviation. The two episodes of contraction periods predicted by the model are from Sep 1999 to Nov 2003 and from Oct 2008 to Jun 2010. They generally align with the timing of the bursting of the dot-com bubble in 2000 and the global financial crisis in 2008. Nevertheless, the predicted recession windows are slightly later than the timing of the two crises respectively.

This may be due to the fact that it usually takes more than a few months after a recession begins before we witness distressed companies filing for bankruptcy. We, therefore, expect a time lag between the predicted credit risk cycle and the general business cycle which is usually measured by GDP and other macroeconomic indicators. Based on the smooth probability plot (bottom subgraph of Figure 2.4.1), we notice that the regime-switching effect is very strong in the sense that the states are quite persistent over time (i.e., they do not switch from one to the other frequently). The estimated transition probabilities are 0.99 and 0.97 for States 1 and 2 respectively.

INSERT FIGURE 2.4.1 ABOUT HERE

In comparing the estimated coefficients between the two states of our best regime-switching model in Column 10 of Table 2.4.2 Panel B, we notice that the magnitudes are consistently larger for the bad state than for the good state. However, it does not necessarily mean that DF is more sensitive to the macroeconomic variables in the bad state because the regression is based on the probit transformation of DF, $\Phi^{-1}(DF_t)$, rather than DF itself. The estimated coefficients of our probit regime-switching models do not quantify the influence of the macroeconomic variables on DF. Rather, it represents a change in the Z-score of the normal distribution with respect to the changes of the right-hand side variables. To have a better understanding of the impact on DF, we calculate the marginal effect of the right-hand side variables. There are generally two methods of calculating marginal effects. One approach is to calculate the average marginal effect. It is the average change in the probability of default (i.e., DF) when a particular macroeconomic variable increases by one unit. The other one is called marginal effect at the mean, which is the marginal effect at the sample mean of the independent variable. Here we use the former approach as it is more appropriate to provide a realistic interpretation of the estimation results. Since probit regression is a non-linear model, the effect differs from point to point of the distribution. What the average marginal effect does is to compute the average effect across the different points.[8]

Table 2.4.3 presents the average marginal effect of the same regressions reported in Table 2.4.2, with the univariate and multivariate results in Panel A and B respectively. Unlike the results in Table 2.4.2, the difference in the magnitude of the marginal effects between the two states tells us the difference in the sensitivities of DF on the macroeconomic variables across the two states. From Table 2.4.3, it is clear that DF is less sensitive to changes in the macroeconomic variables in the good state (i.e., State 1) than in the bad state (i.e., State 2) for almost all of the macroeconomic variables under consideration.[9] For example, based on our best regime-switching model (see Column (10) of Table 2.4.3 Panel B), the same change in GDP growth rate would have approximately twice the amount of impact on PD during the bad state than in the good state. If we had developed a stress testing model with OLS, we would not be able the capture this state-contingent sensitivity, thus potentially leading to inaccurate predictions of capital requirements across the two states of the economy.

INSERT TABLE 2.4.3 ABOUT HERE

[8] More details regarding the similarities and differences between the two approaches of calculating the marginal effects can be found in Bartus (2005).

[9] We can interpret the difference between Table 2.4.2 and 2.4.3 as follows. As the good state is usually associated with small PD which is closer to the left tail of the PD distribution, the marginal effect on the tails is smaller than that of the points closer to the middle since the cumulative distribution function of a normal distribution has a steeper slope around the mean. This results in strong magnitudes of average marginal effects for bad states as shown in Table 2.4.3. By comparing the coefficients in Table 2.4.2 with the corresponding marginal effects in Table 2.4.3, it is clear that the magnitudes have been magnified in state two within each regression.

Next, we develop an optimal quantile regression model in explaining the observed time series of DF. We consider all the combinations of macroeconomic variables. As in the regime-switching model, we use the probit transformation of DF, $\Phi^{-1}(DF_t)$ as our dependent variable. We again use different lagged values of the macroeconomic variables to enhance the explanatory power. Specifically, we apply a one month lag for interest rates and CS and a three month lag for UE and GDP. We run the regressions at five different quantiles: 10%, 25%, 50%, 75%, and 90%. The lowest (highest) quantile corresponds to the lowest (highest) DF. Let us first examine the univariate quantile regression results as reported in Table 2.4.4. Most of the estimated coefficients (twenty-nine out of thirty) are statistically significant at the 5% level (see Panel A). A total of twenty-eight are significant at the 1% level. More importantly, the signs of coefficients are consistent with our expectations. Specifically, DF is negatively associated with the GDP growth rate and interest rates (T3, T10, and FFR), while it is positively related to the unemployment rate and credit spread. The corresponding average marginal effects of each of the macroeconomic variables at different quantiles of DF are reported in Panel C.[10] We notice a consistent trend of the magnitudes of the impact of macroeconomic variables on PD across the quantiles. Specifically, the effect is stronger, the higher the quantile (i.e., the larger is PD). In other words, PD becomes more sensitive to the macroeconomic variables as PD becomes higher. This is consistent with the findings in the regime-switching model where the sensitivity is found to be higher in the bad state of the economy. Lastly, the

[10] The same calculations that we use to transform the estimated coefficients to marginal effects in the regime-switching model are involved here for the quantile regression coefficients.

pseudo R-squared in panel B shows a similar increasing trend of explanatory power from low quantiles to high quantiles.

INSERT TABLE 2.4.4 ABOUT HERE

We exhausted all the combinations of macroeconomic variables in arriving at the best quantile regression model based on the adjusted pseudo R-squared defined in Section 3.2. The best model uses five macroeconomic variables: GDP, UE, FFR, T10, and CS. As in the univariate models, we run the multivariate regressions at five different quantiles: 10%, 25%, 50%, 75%, and 90%. The estimated coefficients (Panel A) and the corresponding average marginal effects (Panel B) of the best model are reported in Table 2.4.5. Most of the coefficients are statistically significant, especially for high quantile coefficients. Thus the influence of the macroeconomic variables is more statistically significant at higher levels of PD (i.e., during contraction periods). More importantly, GDP and interest rates are negatively related to PD, while UE and CS are positively associated with PD. This finding aligns with our intuition discussed in Section 3.3.2. Turning to the marginal effects reported in Panel B, as in the univariate quantile regression models, we observe a consistent pattern when comparing the effects across the five quantiles. The influence of the macroeconomic variables becomes stronger the higher the level of PD. For example, by comparing the average marginal effects between the 10% and 90% quantiles, we notice that the effect can be more than twice as much at high PD levels vs. low PD levels. Finally, based on both pseudo R-squared and adjusted pseudo R-squared, the prediction power at the 90% quantile is the highest among the quantile regressions.

In summary, the quantile regression results point to a non-uniform relation between PD and its key drivers, which is conditional on the level of credit risk. Ignoring such a structural pattern in developing PD stress testing models could potentially lead to a biased estimation of credit risk. Through a validation exercise, we attempt to provide more insight into this issue in the subsequent section.

INSERT TABLE 2.4.5 ABOUT HERE

2.5 Model Validation

2.5.1 Model Validation Process

In the previous sections, we demonstrate the use of a Markov regime-switching model and quantile regression model in developing a PD stress testing model. In doing so, we highlight how these nonlinear models can capture the dynamic relation of PD with the macroeconomic variables, which cannot be captured with an OLS model. Will these nonlinear models be able to outperform the traditional OLS model in terms of accuracy in predicting PD? How do the two nonlinear models compare with each other in terms of their performance? We would like to answer these questions in this section by conducting a number of out-of-sample model performance validation tests. We assess the performance based on the accuracy of the model in predicting both the central tendency and the tail events of the distribution. In this subsection, we outline the validation methodology involved. We will then discuss the validation results in the next subsection.

First of all, to gauge the performance of the proposed models, we need to define a benchmark model that is commonly used in practice. We consider the OLS model to be appropriate for this purpose as it is widely used both in the stress testing literature and in practice. Second, we need to select the same set of macroeconomic variables for all the models being tested in order to make sure that the prediction results are comparable. In this validation exercise, we include all the six macroeconomic variables (i.e., GDP, UE, FFR, T10, T3, and CS) in all the model estimations. Third, it is important to compare model performance in an out-of-sample setting. Since both the regime-switching model and the quantile regression model can be considered the generalized version of the OLS model involving more degrees of freedom, it is not surprising that they will outperform the OLS model in an in-sample comparison. We assess their out-of-sample performance on a rolling sample basis. Specifically, it involves the following steps:

1. The first calibration window includes the first 50% of the data points (a total of 172 observations) from Feb 1987 to Jun 2001. We calibrate all the candidate models with the realized observations of both PD and the macroeconomic variables to get the first set of estimated parameters. Assuming we were at the end of that calibration window (denoted as time t = 0). With the estimated model parameters, we predict the PD to be realized in the next month (i.e., t = 1) using the latest realized observations of the macroeconomic variables (MR0).
2. At t = 1, we observe the actual realized PD. We can then assess the accuracy by comparing the previously predicted PD (P1) with the realized PD (R1) at t = 1. The difference between P1 and R1 is our prediction error at t = 1 (TR1).

3. With one more observation from t = 1, we extend the calibration window by one month and recalibrate all the candidate models with 173 data points. We then repeat Step 1 to get our PD prediction for t = 2 (P2), followed by repeating Step 2 to calculate the second prediction error (TR2) by comparing the predicted value P2 with the actually realized value R2.

Repeating Step 1 to Step 3 while rolling our calibration window forward generates a series of prediction errors (TR1 to TR172) after we have exhausted all 344 observations in our full sample. The mean and the absolute mean of these prediction errors can be used as indicators measuring how good the models are. Besides conducting a one-month ahead prediction, we also consider the model performance in longer-term predictions (e.g., two-month, three-month, etc.). We use the same rolling out-of-sample approach. For example, again starting with the first calibration window from Feb 1987 to Jun 2001, we estimate all the candidate models, but now by regressing the PD two periods ahead (time t + 2) against the macroeconomic variables observed at time t. At the end of the calibration window, using the calibrated models, we then predict the PD to be realized in two months. Comparing this predicted value with the PD realized two months from now would give us the two-month ahead prediction error. As in the one-month prediction exercise, we then roll forward the calibration window to obtain a time-series of two-month ahead prediction errors. Longer-term prediction errors can be calculated in a similar fashion.

Besides judging the model performance by calculating the error in the point estimates, we also check the performance of each of the models for its accuracy in replicating the distribution of the prediction. Specifically, for each of the out-of-sample predictions for each of the models, we check if the realized PD actually lies within specific confidence intervals (e.g., 90% confidence interval) of the PD distribution as implied by the model. If there are too many breaches, we will conclude the model is incapable of accurately replicating the dispersion of the PD. This serves as our second measure of model performance in addition to assessing the accuracy in predicting the mean. For example, we will be doubtful about the performance of a model if we observe many more than seventeen breaches of the 90% confidence interval out of a total of 173 observations. We test the null hypothesis of a 10% breaches based on the binomial distribution. Suppose we observe forty breaches. According to the cumulative binomial probability, the corresponding p-value is 0.131, and thus we cannot reject the null hypothesis at the 10% confidence level.

There are a few details that are involved in assessing the out-of-sample performance of the regime-switching model and the quantile regression model worth mentioning. Unlike the OLS model, our two-state Markov regime-switching model gives us two sets of parameters, and thus two-point estimates of PD, at each point in time. To obtain an overall point estimate of PD, we calculate the weighted average of the two PDs with the estimated filtered probability of realizing each of the two states. Specifically, point estimation is calculated as follows:

$$\widehat{PD} = W_1\left(\widehat{PD}_1\right) + W_2\left(\widehat{PD}_2\right),$$

where W1 and W2 are the filtered probabilities of realizing State 1 and State 2.

A similar weighting scheme is used to arrive at the confidence intervals (CI) used in counting the number of breaches for out-of-sample prediction of the regime-switching model. Specifically, the CI corresponding to a particular z-score (Z_α) corresponding to α-percent of confidence level is calculated by using the weighted average of the two standard deviations (SD1 and SD2) of the residuals of the two states of the calibrated regime-switching model:

$$\Phi^{-1}(CI) = \Phi^{-1}(\widehat{PD}) \pm Z_\alpha(W_1SD_1 + W_2SD_2)$$

When the realized PD falls outside the CI of the predicted PD, we count it as a breach. In this validation exercise, we will be considering the 80% and 90% CI.

The out-of-sample prediction assessment for the quantile regression model is more complicated as there are no such filtered probabilities as in the regime-switching model for us to use as weights for the corresponding regressions from the quantile regression model. In order to determine which quantile regression to use in predicting the PD over the next period, we look at all the historical PD realized up to the current time period and determine how the last observed PD ranks among the historical PD observations. This rank ordering thus serves as our best guess of which quantile we are currently at based on the latest information. For example, if we observe a relatively high PD at time t, which falls in the largest 10% of all previously observed PDs, then we assume the PD at time t+1 will be in the 90%-100% range as that of time t. We therefore use the calibrated 95% quantile regression equation to predict the out-of-sample test. We will be switching among different quantile regression equations as we conduct the out-of-sample prediction over time, so as to select the most appropriate quantile regression equation to do the prediction. Similar to the regime-switching model, we construct the CI for the quantile regressions based on the distribution of the residuals obtained from the regressions. Unlike the regime-switching model, we do not need to calculate any probability-weighted average, as we use the standard deviation estimated for the selected quantile regression equation (e.g., the 95% quantile equation) in calculating the CI for that particular time period.

In general, the more quantiles we include in our quantile regression model, the more accurate we are able to depict the whole distribution of PD. The cost of having more quantiles is the complexity of the model. The number of quantiles we can consider is also restricted by the number of data points available. The prediction power may degenerate when we run out of degrees of freedom. Due to the limited number of observations in our data set, we consider three different levels of granularity of quantiles, by dividing the probability distribution into three, five or ten equal segments. In the least granular case of three segments, the quantiles are 16.7%, 50%, 83.3% respectively. In the five-segment case, the quantiles are 10%, 30%, 50%, 70%, and 90% respectively. In the most granular case of ten segments, they are 5%, 15%, 25%, 35%, 45%, 55%, 65%, 75%, 85%, and 95%. To serve as another benchmark, we also examine the performance of a (degenerated) quantile regression model at the median (i.e., the 50^{th} percentile). Altogether we, therefore, have six models to compare: (1) OLS model,

(2) 2-state Markov regime-switching model, (3) median quantile regression model, (4) 3-segment quantile regression model, (5) 5-segment quantile regression model, and (6) 10-segment quantile regression model.

2.5.2 Model Validation Result

In Table 2.5.1, we report the average prediction errors and the average absolute prediction errors calculated for each of the six models from the differences between the realized values and the predicted values of PD. We conduct the performance comparison for different prediction horizons, between the time of the information set we have and the time we try to make the prediction. “No gap” means that we have all the information until time t and we make a prediction for PD at t+1, while “4 months” means that we are predicting a PD value at time t+5. Not surprisingly, for all six models, the magnitude of the prediction errors increases with the length of the prediction horizon. Regardless of the prediction horizon, the Markov regime-switching model has the smallest absolute prediction error, meaning that it has the most accurate point estimation among all the six models. The 10-segment quantile regression actually slightly outperforms the regime-switching model for a couple of prediction horizons (“one month” and “two months”) if we assess based on the average prediction errors. Nevertheless, the performances of the other models are not significantly worse than these two “best” models. In particular, as the prediction horizon lengthens to more than three months, we do not see a clear advantage of using the more complicated nonlinear models in terms of their point estimates. Finally, among the four quantile regression models, we notice a general pattern of increasing accuracy from the least granular model (i.e., the median quantile regression) to the most granular one (i.e., the 10-segment quantile regression) overall prediction horizons. This observation is consistent with our intuition that the number of segments affects the prediction accuracy of the quantile regression model.

INSERT TABLE 2.5.1 ABOUT HERE

In Figure 2.5.1, we present the time-series plots of the PD point estimates of the six models over our sample period. Although all models show a similar trend, we see that the OLS model underestimates the credit risk during the crisis compared to those predicted by the regime-switching model and quantile regression models. In Figure 2.5.2, we plot the out-of-sample one-month ahead PD point estimates based on our regime-switching model (our best model) along with the actual realized PD. Based on the plot, the regime-switching model performs reasonably well in capturing the time-series variations of PD during both the expansion and the recession (e.g., the 2008-09 financial crisis) periods. Nevertheless, although the point estimates lie close to the realized PD values for most of the time, the predicted PD tends to be higher than the corresponding realized PD.

INSERT FIGURES 2.5.1 AND 2.5.2 HERE

To further examine the performance of the models, we turn our attention to the breaches of the out-of-sample predicted confidence intervals reported in Table 2.5.2. We count the breaches for the 80% and 90% confidence intervals respectively. Let's first focus on the full sample results in the first two columns of Table 2.5.2. According to the p-values, the regime-switching model is the only model for which we cannot reject the null hypothesis that it accurately replicates the PD distribution at both the 80% and 90% confidence intervals. Its performance is therefore considered to be superior to the other models under consideration. The 10-segment quantile regression model is likely to be our second best model. Although it has a significant p-value based on the usual 5% level for the 90% CI breaches, the null hypothesis cannot be rejected based on the 80% CI breaches.

Is our conclusion regarding model performance robust under different market conditions? We address this question by assessing the model performances during the contraction and the expansion subsample periods separately. We separate our sample into contraction and expansion periods following the definition of the National Bureau of Economic Research (NBER) and the Federal Reserve Bank of St.Louis. The two contraction periods are from Jul 2001 to Nov 2002 and then from Dec 2007 to Dec 2010; whereas the two expansion periods are from Dec 2002 to Nov 2007 and, then, from Jan 2011 to Dec 2015. The two contractions (expansion) periods together last for 53 (120) months. These subsample results are reported in Columns 3 to 6 of Table 2.5.2. Still, the regime-switching model and (to a lesser extent) the 10-segment quantile regression model outperforms the other models in the contraction periods. We cannot reject the null hypothesis that the regime-switching model correctly replicates the distribution PD for both the 80% and 90% CI. The difference in model performance is not that stark during the expansion periods. We cannot reject the null hypothesis for four out of the six models for the 80% CI during the expansion periods. Similar to what we have observed in the point estimation results, we notice that the granularity of the quantile regression models can help to improve the model performance, with the 10-segment (5-segment) quantile regression model outperforming the 5-segment (3-segment) model. Most importantly, the OLS model does a poor job of replicating the distribution of PD regardless of the market conditions.

INSERT TABLE 2.5.2 ABOUT HERE

In summary, based on the performance results, we conclude the regime-switching model is the best as it outperforms the other models in point estimation (based on absolute prediction error) and breaches count for both the 80% and the 90% CI. Although we notice an improvement in performance as we increase the granularity of the quantile regression segment, the 10-segment quantile regression model is still inferior to the two-state regime-switching model based on our validation results. Although the performance of the OLS model is not much worse than some of the nonlinear models in terms of its PD point estimates, its implied PD distribution in an out-of-sample setting is significantly different from what we observe in the actual data.

2.6 Conclusion

This chapter investigates the performances of two different types of dynamic models for credit risk stress testing: a regime-switching model and a quantile regression model. We utilize these statistical modeling approaches in a credit risk stress testing framework and compare their performances with that of the traditional OLS model. Putting all the test results into consideration, we conclude the regime-switching model is the best among all as it outperforms other models in producing the most accurate point estimation (based on the absolute average error) and breach counts (based on both the 80% and the 90% CI). Although we see an improvement in the performance as we increase the granularity of the segments of the quantile regression models, it is still generally inferior to the regime-switching model based on our validation tests. Introducing more segments in the quantile regression makes the model more complex, involving more model parameters. Future studies can focus on:

1. Optimizing the number of states of the regime-switching model and the number of segments for the quantile regression model.
2. Determining the macroeconomic factors that influence credit risk indicators.

The performance of the proposed models could be enhanced with a more detailed consideration of these two modeling aspects.

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Appendix

Table 2.1.1 Risk-weighted Assets (in millions) Pertaining to Different Risk Class

This table reports the Risk-Weighted Assets (RWA) of different risk classes from 2012 to 2016 at the Bank of Montreal (BMO). All the numbers are from BMO 2013-2016 annual reports.

	2016	2015	2014	2013	2012
Credit Risk	$222,499	$200,385	$185,387	$179,289	$171,955
Market Risk	$8,962	$10,262	$9,002	$9,154	$7,598
Operational Risk	$30,502	$28,538	$27,703	$26,651	$25,677
% of Credit Risk	84.94%	83.60%	83.47%	83.53%	83.79%
Total	$261,963	$239,689	$222,092	$215,094	$205,230

Table 2.1.2 Economic Capital Allocated to Different Lines of Business in BMO as of Dec 31, 2016

This table reports the Economic Capital (EC) allocated to different lines of business in 2016 at the Bank of Montreal (BMO). All the numbers are from BMO 2016 annual reports.

	Personal and Commercial Banking	BMO Capital Market	Corporate services
Credit Risk	78%	65%	64%
Market Risk	4%	14%	27%
Operational Risk	18%	21%	9%

Table 2.3.1 Summary Statistic of Defaults from BankruptcyData

The table reports the default events in history. There are two dimensions: year and industry. The industries are segmented as follows based on SIC codes:

Number of Defaults

Year	Agriculture	Construction	Manufacturing	Transportation & Communication	Trade	Finance	Services	Total
1987	1	3	9	1	4	0	4	22
1988	1	3	11	5	7	4	3	34
1989	0	3	20	5	16	14	7	65
1990	0	7	25	7	17	22	9	87
1991	1	2	38	13	24	17	14	109
1992	0	5	21	8	24	6	12	76
1993	0	8	30	6	11	6	9	70
1994	0	2	19	7	15	2	8	53
1995	0	4	12	9	24	7	11	67
1996	0	5	23	4	20	1	11	64
1997	0	4	20	15	14	6	7	66
1998	0	7	34	13	21	13	15	103
1999	0	13	53	20	24	8	25	143
2000	1	7	69	20	34	10	46	187
2001	1	6	85	48	45	13	68	266
2002	0	13	82	58	15	14	47	229
2003	0	8	66	27	21	10	44	176

2004	0	6	39	12	21	2	14	94
2005	0	3	44	13	10	5	15	90
2006	0	2	34	8	7	3	12	66
2007	0	5	36	7	12	6	14	80
2008	1	14	54	13	21	19	21	143
2009	1	29	97	14	15	32	25	213
2010	0	7	40	9	8	22	21	107
2011	0	10	30	11	13	8	15	87
2012	0	9	34	12	6	9	17	87
2013	1	5	26	10	8	7	14	71
2014	0	7	18	10	7	5	7	54
2015	0	38	20	2	6	4	9	79
Total	8	235	1089	387	470	275	524	2988

Table 2.3.2 Normality Test

The table presents the results of the Shapiro-Wilk normality test for original DF, logit transformation of DF and probit transformation of DF. Obs stands for the number of observations. W is the Shapiro-Wilk test statistic. Z statistics and p-values are reported in the last two columns. The same tests are conducted for residuals.

Variable	Obs	W	z	Prob>z
DF	347	0.9857	2.94	0.0016
Logit DF	347	0.9015	7.50	0.0000
Probit DF	347	0.99921	1.55	0.0607

Table 2.3.3 Summary of Macroeconomic Variables

This table reports the summary statistic of six macroeconomic variables. The sample period is from Feb 1987 to Dec 2015.

- GDP: GDP growth rate is calculated based on the monthly GDP collected from Bloomberg.
- UE: Monthly UE is another popular indicator of economic condition, which is available on the International Labor Organization, ILOSTAT database.
- FFR, T10 and T3: All these interest rates are collected from the Federal Reserve Bank of St. Louis on a monthly basis.
- CS: Credit Spread is calculated by the following formula:

$$CS = Moody's\ Baa\ \ Bond\ Yield(\%) - Moody's\ Aaa\ Bond\ Yield(\%),$$

where the yields are from the Federal Reserve Bank of St. Louis.

	GDP (%)	UE (%)	CS (%)	T10 (%)	T3 (%)	FFR (%)
No. Of Observations	347	347	347	347	347	347
Mean	1.02	-0.01	0.97	5.21	3.40	3.62
Median	1.08	-0.10	0.90	5.09	3.72	3.97
Std.dev	0.78	0.31	0.39	2.09	2.59	2.74
Max	3.00	1.50	3.38	9.52	9.14	9.85
Min	-2.65	-0.80	0.55	1.53	0.01	0.07

Table 2.3.4 Correlation Matrix of Variables

This table presents the correlation coefficients between all seven variables. As we can see, DF is positively correlated with GDP, UE and CS, while negatively correlated with the three interest rates.

	DF	*FFR*	*GDP*	*UE*	*TBill(3M)*	*TBill(10Y)*	*CS*
DF	1						
FFR	-0.33835	1					
GDP	-0.30848	0.2054	1				
UE	0.380797	0.010234	-0.23665	1			
TBill(3M)	-0.39003	0.987393	0.248645	-0.02359	1		
TBill(10Y)	-0.3544	0.617133	0.279243	-0.02828	0.651831	1	
CS	0.663234	-0.45539	-0.51905	0.423516	-0.48912	-0.46593	1

Table 2.4.1 Univariate OLS Analysis

This table presents the OLS regression results of DF with respect to each macroeconomic factor (lagged). *,**,*** correspond to statistical significance at the 10%, 5% and 1% level, respectively.

Panel A: lag 1						
	GDP	UE	CS	T10	T3	FFR
Intercept	0.0057***	0.0047***	0.0004	0.008***	0.0063***	0.0062***
Slope	-0.1066***	0.5061***	0.4395***	-0.0646***	-0.0496***	-0.0426***
R^2	0.08	0.2873	0.3395	0.211	0.19	0.1573

Panel A: lag 2						
	GDP	UE	CS	T10	T3	FFR
Intercept	0.0059***	0.0047***	0.0005	0.008***	0.0063***	0.0061***
Slope	-0.124***	0.5248***	0.4325***	-0.0627***	-0.0496***	-0.0398***
R^2	0.1086	0.31	0.3290	0.1985	0.17	0.137

Panel A: lag 3						
	GDP	UE	CS	T10	T3	FFR
Intercept	0.0061***	0.0047***	0.0007*	0.0078***	0.0062***	0.006***
Slope	-0.1404***	0.5283***	0.4145***	-0.0605***	-0.0441***	-0.0368***
R-sqaured	0.1392	0.316	0.3029	0.1844	0.1504	0.1174

Panel A: lag 4						
	GDP	UE	CS	T10	T3	FFR
Intercept	0.0061***	0.0047***	0.0009***	0.0077***	0.0061***	0.0059***
Slope	-0.1389***	0.5199***	0.3866***	-0.0582***	-0.0411***	-0.034***
R-sqaured	0.1368	0.3075	0.2644	0.1704	0.1308	0.1

Table 2.4.2 Univariate and Multivariate Regime-Switching Regression Models

Panel A presents the univariate regime-switching regression results and Panel B presents the selected multivariate regime-switching regression results. RS stands for regime-switching, which indicates whether regime-switching occurs in the corresponding regression. *,**,*** correspond to statistical significance at the 10%, 5% and 1% level, respectively.

Panel A												
	(1)		(2)		(3)		(4)		(5)		(6)	
	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2
GDP	-7.75***	-13.74***										
UE			27.13***	52.91***								
T10					-5.84***	-11.62***						
T3							N/A	N/A				
CS									N/A	N/A		
FFR											-0.5***	-0.8***
RS	Yes		Yes		Yes		No		No		Yes	
AIC	-3422.33		-3504.07		-3569.42		-3109.66		-3459.85		-3486.97	
BIC	-3391.60		-3473.35		-3538.65		-3078.89		-3429.08		-3456.19	

	Panel B							
	(7)		(8)		(9)		(10)	
	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2
GDP							-1.15***	-1.28***
UE					28.17***	35.71***	22.26***	27.79***
T10	-4.9***	6.51***	-6.94***	-7.03***	-3.67***	2.99***	-6.63***	-14.19***
T3					-0.92***	-2.23***	-0.86***	-0.95***
CS	11.72***	19.2***	9.09***	19.96***	5.43***	6.26***	4.27***	9.13***
FFR			-1.79***	-2.74***				
RS	Yes		Yes		Yes		Yes	
AIC	-3564.9		-3629.2		-3511.62		-3642.88	
BIC	-3449		-3583.1		-3450.85		-3588.43	

Table 2.4.3 Average Marginal Effect of Regime-Switching Regression Models

Panel A presents the univariate regime-switching regression marginal effects and Panel B presents the selected multivariate regime-switching regression marginal effects. RS stands for regime-switching, which indicates whether regime-switching occurs in the corresponding regression.

Panel A												
	(1)		(2)		(3)		(4)		(5)		(6)	
	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2
GDP	-0.06	-0.14										
UE			0.12	0.34								
T10					-0.05	-0.13						
T3							N/A	N/A				
CS									N/A	N/A		
FFR											-0.05	-0.09
RS	Yes		Yes		Yes		No		No		Yes	
AIC	-3422.33		-3504.07		-3569.42		-3109.66		-3459.85		-3486.97	
BIC	-3391.6		-3473.35		-3538.65		-3078.89		-3429.08		-3456.19	

Panel B								
	(7)		(8)		(9)		(10)	
	State 1	State 2	State 1	State 2	State 1	State 2	State 1	State 2
GDP							-0.02	-0.04
UE					0.10	0.22	0.08	0.28
T10	-0.04	0.12	-0.05	-0.07	-0.16	0.12	-0.03	-0.08
T3					-0.01	-0.03	-0.03	-0.05
CS	0.19	0.46	0.16	0.46	0.21	0.34	0.02	0.12
FFR			-0.21	-0.69				
RS	Yes		Yes		Yes		Yes	
AIC	-3564.9		-3629.2		-3511.88		-3642.62	
BIC	-3449		-3583.1		-3450.43		-3588.85	

Table 2.4.4 Univariate Quantile Regression Models

This table includes three panels that report the univariate quantile regression coefficients, the corresponding pseudo R^2 and average marginal effects respectively. The quantile regressions are based on five quantiles, which are 10%, 25%, 50%, 75% and 90%. *,**,*** correspond to statistical significance at the 10%, 5% and 1% level, respectively.

Panel A: Coefficients					
	0.1	0.25	0.5	0.75	0.9
GDP	-7.29***	-9.24**	-9.16***	-7.61***	-8.61***
UE	33.86***	37.32***	32.88***	30.04***	31.02***
T10	-5.82***	-5.92***	-5.95***	-5.94***	-7.21***
T3	-5.44***	-4.66***	-3.59***	-3.86***	-4.26***
CS	22.56	29.28***	32.27***	29.53***	26.31***
FFR	-4.98***	-4.55***	-3.16***	-3.44***	-4.08***

Panel B: Pseudo R^2					
	0.1	0.25	0.5	0.75	0.9
GDP	0.0208	0.0223	0.0309	0.0445	0.1103
UE	0.051	0.0525	0.0957	0.202	0.2878
T10	0.1596	0.1867	0.1605	0.1003	0.1505
T3	0.0976	0.1218	0.1385	0.1072	0.1332
CS	0.0035	0.1498	0.2232	0.2401	0.2233
FFR	0.0372	0.0404	0.0548	0.0863	0.1091

Panel C: Marginal Effect					
	0.1	0.25	0.5	0.75	0.9
GDP	-0.04***	-0.07**	-0.08***	-0.1***	-0.14***
UE	0.20***	0.34***	0.36***	0.36***	0.44***
T10	-0.04***	-0.05***	-0.06***	-0.07***	-0.14***
T3	-0.03***	-0.04***	-0.04***	-0.05***	-0.1***
CS	0.12	0.36***	0.47***	0.49***	0.5***
FFR	-0.26***	-0.33***	-0.39***	-0.42***	-0.55***

Table 2.4.5 Selected Multivariate Quantile Regression Models

Panel A presents the coefficients of the selected quantile regression models based on R^2 and adjusted R^2 and Panel B shows the corresponding average marginal effects. Variable definitions are provided in Table 3.3. The first row represents five different quantiles selected for the model. *,**,*** correspond to statistical significance at the 10%, 5% and 1% level, respectively.

			Panel A		
	0.1	0.25	0.5	0.75	0.9
GDP	-4.04**	-4.28**	-4.31	-4.89**	-4.81***
UE	24.14***	26.01***	22.01***	29.32***	26.41***
FFR	-2.02*	-2.03	-2.34	-2.26	-2.3***
T10	-5.48***	-5.82***	-5.41***	-5.55***	-6.44***
CS	4.86	4.13	5.52***	6.80***	7.08***
R^2	0.3356	0.3499	0.3553	0.3485	0.3856
adj-R^2	0.3258	0.3403	0.3458	0.3389	0.3765
			Panel B		
	0.1	0.25	0.5	0.75	0.9
GDP	-0.05	-0.06	-0.07	-0.09	-0.11
UE	0.21	0.25	0.24	0.38	0.50
FFR	-0.09	-0.093	-0.12	-0.1	-0.16
T10	-0.05	-0.05	-0.05	-0.06	-0.09
CS	0.14	0.18	0.23	0.27	0.39

Table 2.5.1 Point Estimation Comparison Result

This table presents the tracking errors of the point estimation results of six different models. The first column represents the lag between the date of an information set and the prediction point. For example, "0 Month" means predicting PD at time t+1 with information until time t, while "1 month" means predicting PD at t+2 with information until time t. The first row represents the comparison models. The minimum differences among all six models are highlighted. Please note that all numbers are in percentages.

	0 month		One month		Two months		Three months		Four months	
	average	abs_average	average	abs_average	average	abs_average	average	abs_average	average	abs_average
OLS	-0.30	0.40	-0.32	0.42	-0.32	0.44	-0.33	0.44	-0.34	0.46
MS	-0.28	0.37	-0.31	0.41	-0.32	0.42	-0.33	0.44	-0.33	0.45
Quantile (median)	-0.29	0.38	-0.31	0.43	-0.32	0.44	-0.33	0.44	-0.34	0.46
Quantile_3	-0.29	0.38	-0.31	0.42	-0.31	0.43	-0.34	0.44	-0.34	0.46
Quantile_5	-0.28	0.38	-0.32	0.41	-0.31	0.43	-0.33	0.44	-0.34	0.46
Quantile_10	-0.28	0.37	-0.30	0.41	-0.31	0.43	-0.33	0.44	-0.33	0.46

Table 2.5.2 Comparison of models based on CI Breach

Panel A and Panel B report the number of breaches (N) and corresponding probability (of getting N or more breaches) of all six different models for 80% CI and 90% CI, respectively. The second, fourth and sixth columns are the total number of breaches, breaches during contraction period and breaches during expansion periods respectively.

PANEL A: 80% CI

	Number of breaches	p	Contraction breaches	p	expansion breaches	p
OLS	60	0	22	0	38	0
MS	40	0.18	14	0.16	26	0.36
Quantile(median)	62	0	24	0	38	0
Quantile(3)	46	0.02	16	0.05	30	0.11
Quantile(5)	43	0.07	16	0.05	27	0.28
Quantile(10)	41	0.13	15	0.09	26	0.36

PANEL B: 90% CI

	Number of breaches	p	Contraction breaches	p	expansion breaches	p
OLS	35	0	13	0.0018	22	0
MS	23	0.0972	8	0.1558	15	0.22
Quantile(median)	38	0	16	0	22	0
Quantile(3)	34	0	12	0.0053	22	0
Quantile(5)	33	0.0002	12	0.0053	18	0.05
Quantile(10)	30	0.002	10	0.0355	18	0.05

Figure 2.1.1 Hypothetical Capital Structure of Bank A

This figure presents the hypothetical capital structure of bank A.

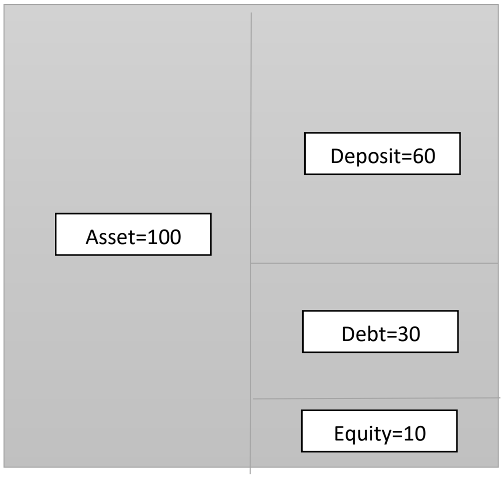


Figure 2.1.2 Hypothetical Capital Structure of Bank B

This figure presents the hypothetical capital structure of bank A.

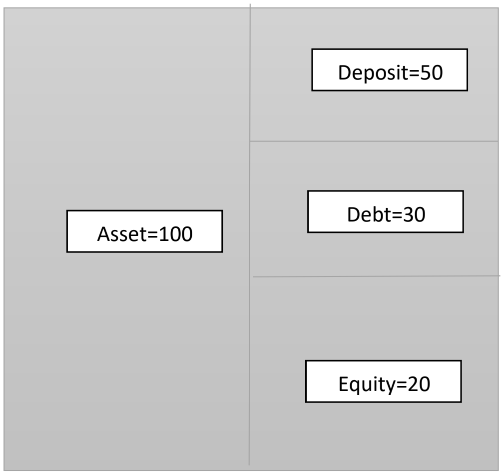


Figure 2.1.3: Binomial Tree of the Potential Loss

This figure shows all the possible paths leading to four outcomes with a potential loss. Three key factors are considered sequentially: PD, EAD and LGD, which are assumed to be independent of each other.

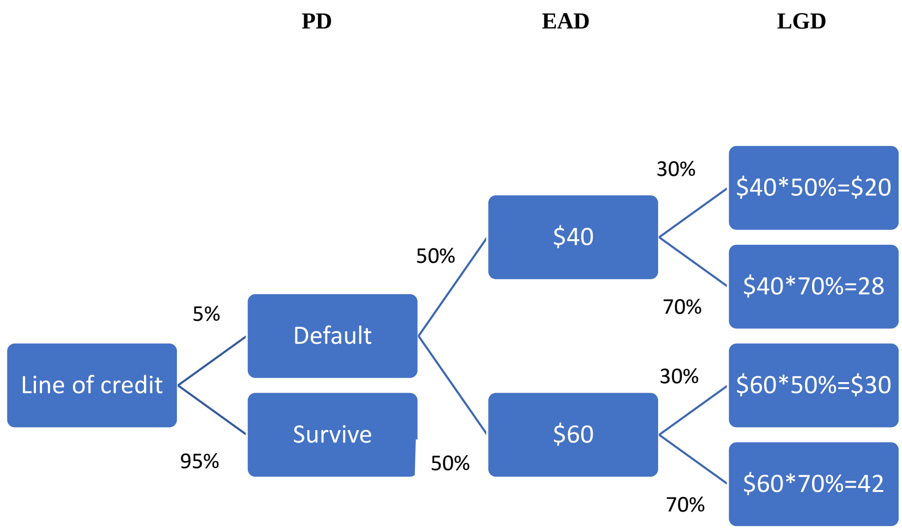


Figure 2.2.1 Stress Testing Procedure

The figure summarizes the procedure of stress testing. Usually, stress testing includes four parts as shown below starting from the left side.

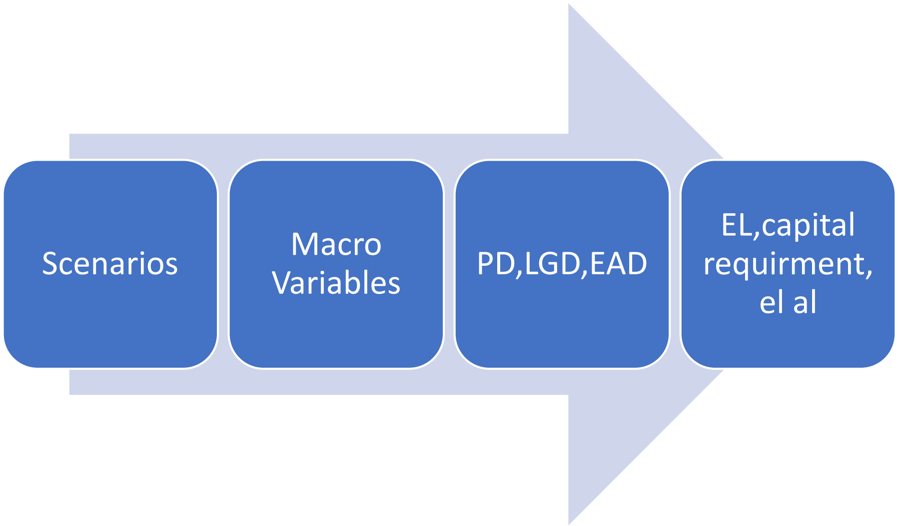


Figure 2.3.1 Summary of DF

This figure reports the DF calculated based on Equation number and the summary statistics are available in the table below. The sample period is from Feb 1987 to Dec 2015. From this graph we see two significant peaks, which in general, align with historical recessions: One is the dot-com bubble in 2001 and the other is the global financial crisis in 2008.

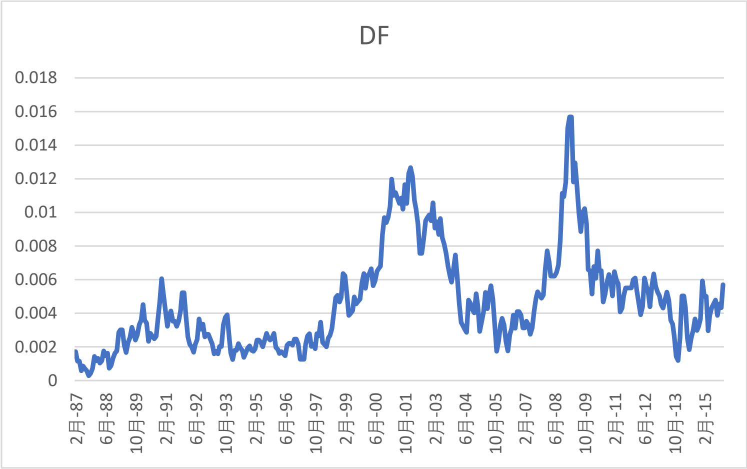


Figure 2.4.1 Regime-Switching Model Summary

The subgraph on the top is the visualization of time-series PD. The middle subgraph shows the conditional standard deviation of the equation. The one on the bottom exhibits the smoothed probability as defined in 3.1.

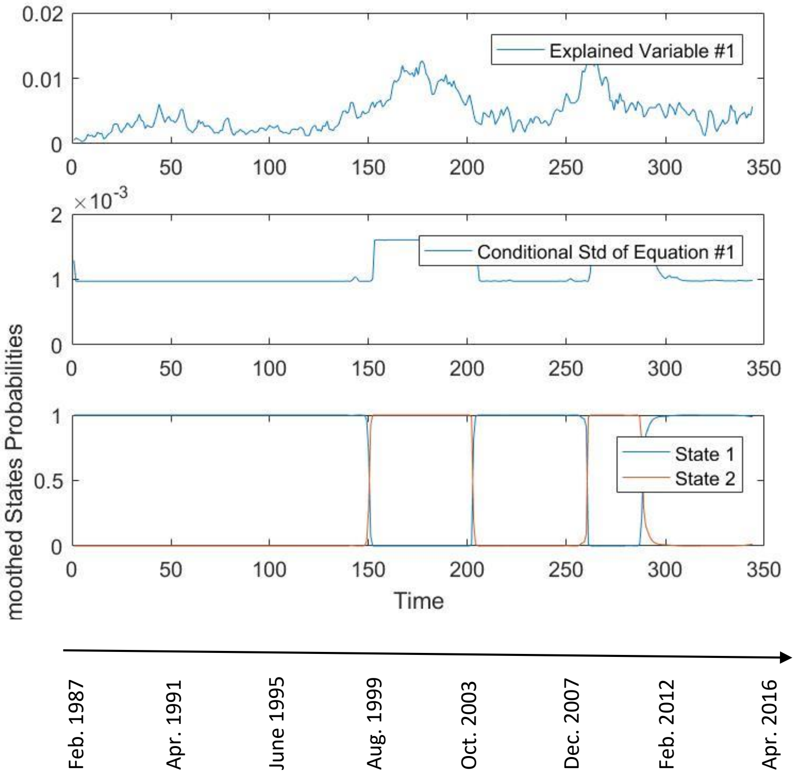


Figure 2.5.1 Point Estimation Result Based on Different Models

This graph presents the point estimation results based on six different models:

OLS model, Markov regime-switching model, Quantile regression model (median), Quantile regression model (three segments), Quantile regression model (five segments) and Quantile regression model (ten segments)

All point predictions are based on the most recently available information, which means we use the information up until time t-1 to predict PD at time t.

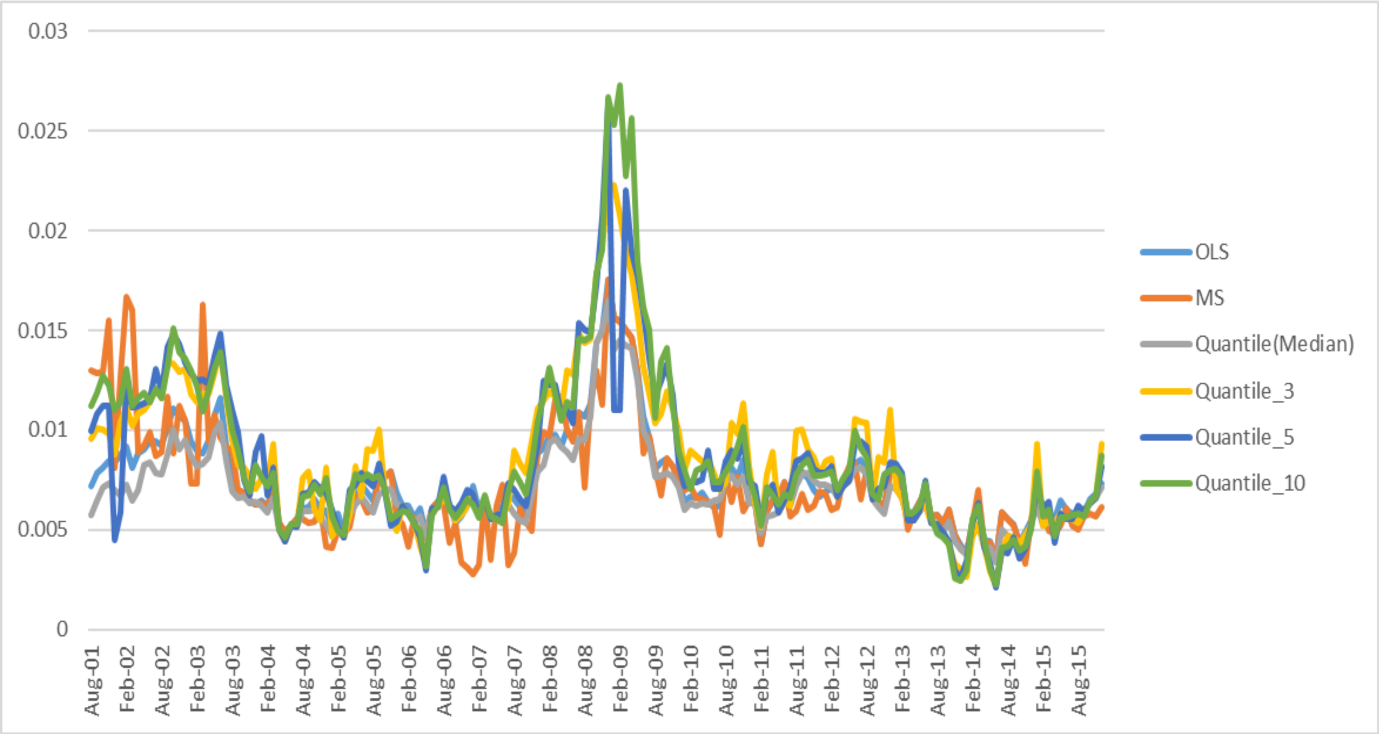
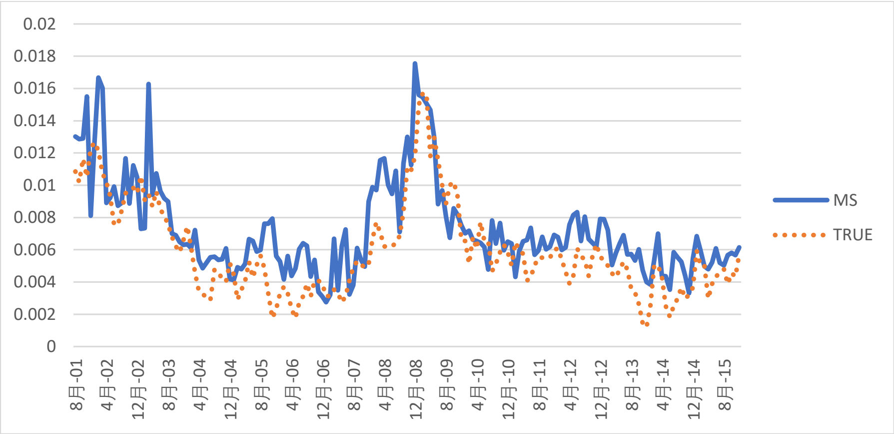
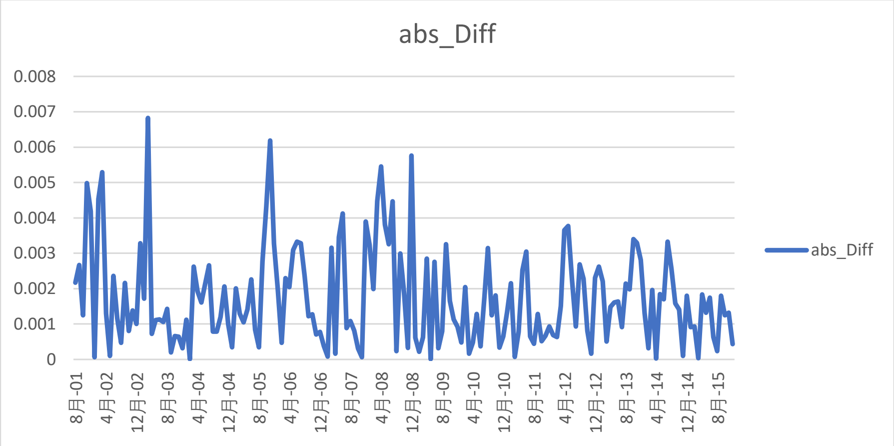

0.03
0.025
0.02
0.015
0.01
0.005
0
Aug-01
Feb-02
Aug-02
Feb-03
Aug-03
Feb-04
Aug-04
Feb-05
Aug-05
Feb-06
Aug-06
Feb-07
Aug-07
Feb-08
Aug-08
Feb-09
Aug-09
Feb-10
Aug-10
Feb-11
Aug-11
Feb-12
Aug-12
Feb-13
Aug-13
Feb-14
Aug-14
Feb-15
Aug-15
OLS
MS
Quantile(Median)
Quantile_3
Quantile_5
Quantile_10

Figure 2.5.2 Comparison between the True Value of PD and Predicted PD by MS Model

These graphs show the difference between the predicted PD by MS model and realized PD from Aug 2001 to Aug 2015. Please note the second graph presents the absolute value of differences, so all numbers are positive.





Chapter 3

Point-in-Time Model of Recovery Value of Defaulted Portfolio

(This essay is a joint work with Dr. Donghui Chen of Scotiabank and Dr. Peter Miu)

3.1 Introduction and Literature Review

The key determinants of the credit risk of a debt instrument (e.g., a bank loan) are the probability of default (PD) of the borrower or issuer, the expected loss-given-default (LGD) and the expected exposure-at-default (EAD) upon a default event. Among these three risk parameters, PD is the most studied; there has been substantial literature on the modeling and prediction of PD ever since Beaver (1966) and Altman (1968) published their findings on the use of accounting information to predict corporate bankruptcy. The present study focuses on LGD, which has, until recently, received much less attention than PD from both researchers and practitioners.[11] LGD is defined as the fraction of the outstanding loan exposure that cannot be recovered by creditors during the bankruptcy process (e.g., by liquidating the collateral). Thus, if the *recovery rate* is defined as the fraction of the recovered outstanding exposure, then LGD is simply one minus the recovery rate.

Recently, both researchers and practitioners have become more attentive to the modeling and prediction of recovery rate and LGD with the introduction of the Basel II Accord in

[11] There is a small but growing literature on EAD (e.g., Qi, 2009; Jacobs and Bag, 2010; Tong et al., 2016).

2006, under which the amount of (regulatory) capital required to be held by banks becomes a direct function of not only PD but also LGD and EAD of loan portfolios. Perhaps the first systematic study on recovery rate was conducted by Altman and Kishore (1996), who document the importance of industry effect and debt instrument seniority in estimating recovery rate and discover statistically significant differences between the recovery rate of the utility sector and those of other industry sectors. Besides the cross-sectional difference, a significant time-series variation of the recovery rate is also documented. For example, by examining the recovery rates on corporate bond defaults over the sample period of 1982 to 2002, Altman et al. (2005) reveal a significant time-series relation between the aggregated recovery rates and the supply and demand for the defaulted instruments that are dictated by the system-wide default rates.

In this study, we contribute to the aforementioned literature by examining and modeling the time-series pattern of recovery throughout the bankruptcy and workout process of a retail credit portfolio; whereas other researchers are concerned with predicting the *overall* recovery rates of debt instruments, we model the amounts a creditor can recover *at different points in time* subsequent to the default event. This is of practical interest to commercial banks in managing the risk of their default loan portfolios. Like managing *performing loan* portfolios, banks must assign loss provision and determine the capital requirement associated with *non-performing (i.e., defaulted) loan* portfolios. Given the fact that it usually takes two to three (up to five or more) years to complete the recovery process for a typical defaulted retail (corporate) loan, it is important to understand the time-varying risk characteristic of the defaulted portfolio as a function of its *vintage* in the recovery process.

An accurate point-in-time (PIT) risk assessment enables financial institutions to manage their defaulted loan portfolios in a timely fashion.

To manage their performing loan portfolios, Basel II Pillar 1's risk-weighted asset (RWA) calculation requires banks to develop a methodology to assess the LGD of their loans in a through-the-cycle (TTC) fashion. The TTC LGD can be interpreted as a cycle-neutral long-term average LGD measure. When combined with the necessary downturn LGD adjustment, this measure serves as input in calculating the required regulatory capital. As a result of the specific regulatory requirements, banks have been focusing their effort on trying to understand the TTC LGD of their debt instruments, typically estimated using historical defaulted loan data. In practice, the same TTC LGD parameters are also used directly for risk and portfolio management activities that are not directly governed by Basel II Pillar 1, such as calculating the economic capital of defaulted loan portfolios. More importantly, it is not uncommon that banks directly use the TTC LGD parameters, which are estimated for their *performing loan* portfolios, in assessing the risk of their *non-performing (defaulted) loan* portfolios. However, such an approach does not allow for an accurate measure of the time-varying portfolio risks of defaulted loan portfolios over the recovery process. Specifically, as recovery cash flows are realized throughout the recovery process, the uncertainty of the overall recovery value becomes smaller the longer the debt has defaulted. In addition, the amount of partially realized recovery cash flow, as an indicator of the prevailing state of the economy, should also be informative in updating our estimation of the residual recovery cash flow (and in turn the overall recovery value). Ignoring this PIT information could result in the underestimation (overestimation) of portfolio loss during the recessionary (expansionary) stage of the business cycle. Therefore, conducting PIT LGD assessments throughout the recovery process is of paramount importance for us to assess accurately the risk of the defaulted loans of different vintages and under different prevailing market conditions.[12] In this study, we propose adopting a PIT LGD model to capture these time-varying risk characteristics of the defaulted loan portfolios, while leveraging on the well-developed TTC LGD methodology that has been used in managing performing loan portfolios.

What do we know about the PIT nature of LGD and how LGD might be governed by firm-level and instrument-level characteristics? As pointed out by Altman et al. (2005), aggregated recovery rates are likely to be varying with the aggregated default rates in a positive fashion over time as a result of supply and demand shocks of defaulted assets. Some empirical studies also demonstrate that credit risk could be understated by using historical average TTC LGD. For example, Frye (2000) concludes that making simple recovery assumptions for collateralized loans can lead to errors because the same factors that result in an increase of default rates can also attribute to a decrease in the collateral value. Schuermann (2004) find out that the key determinants of LGD are the seniority of the debt instrument in the capital structure, the presence and quality of the collateral, the industry the debtor belongs to, and the business cycle; while Jankowitsch et al (2014) confirms the influences of firm fundamentals on the LGD of its debt instruments. Altman and Kalotay (2014) and Acharya et al (2007) explore the effect of macroeconomic variables such as industry and market-wide PD on LGD. The empirical relationship between the

[12] A PIT LGD is also instrumental for financial institutions in conducting stress testing and in calculating loss provision according to the new IFRS-9 requirement.

credit cycle and LGD has been confirmed by many studies, including research from Kansas City Federation (2014) and Frye and Jacobs (2012). Recently, a number of researchers propose the use of different PIT LGD models. For example, Chawla et al. (2016) proposed a framework to cover the estimation of PIT LGD and PIT EAD for wholesale exposures, and Krüger and Rösch (2017) propose a quantile regression approach to estimate the entire LGD distribution. They find that the middle quantiles are explained by observable covariates, while the tail events are driven by unobservable random events.[13]

All of the above studies focus on the investigation of the PIT behavior of the *overall* recovery rate from the workout process rather than the *profile* of recovery rate within the workout process. The understanding of the former will suffice in assessing the credit risk of a *performing* loan portfolio; however, the latter is required in order to accurately model the risk of a *defaulted* loan portfolio. To manage a defaulted loan portfolio, in addition to knowing the expected *overall* recovery rate, one must also understand the creditors' expected recovery amount *at different points in time* subsequent to the default event. In the present study, we propose and estimate a time-series model to capture the recovery rate profile of a retail portfolio of a bank. Contrary to a TTC LGD model, our proposed model allows for the dynamic updating of the expected future recovery rate based on the recovery information realized in the current time period. This dynamic feature of our model allows

[13] Our research is also related to the corporate finance literature on the bankruptcy and recovery process. For example, Bris et al. (2006) explore a comprehensive sample of corporate bankruptcies in US from 1995 to 2001. They found that bankruptcy costs are sensitive to the measurement method used that Chapter 11 seems to preserve assets better, allowing creditors to recover relatively more. The amount of recovery can also be affected by the presence of different stakeholders in the bankruptcy process. For example, Jiang et al. (2012) find that the presence of hedge funds in the Chapter 11 process can actually enhance the recovery values of junior creditors.

for more precise capturing of the time-varying nature of the portfolio risk across different phases of the business cycle; otherwise, we could be understating (overstating) risk and thus in turn capital requirement during recessionary (expansion) time periods.

We conduct an estimation of our proposed model using an internal data set of recovery information on a portfolio of defaulted retail instruments of a financial institution. The rich data set spans the time period from 2003 to 2011 and consists of recovery rate information defined along two-time dimensions: *the default date* (measured in absolute calendar time) and *the time since default occurred* (measured relative to the default date). Under the first time dimension, we track the recovery experience of thirty-four different cohorts defaulting respectively in thirty-four different quarters within our sample period, starting from 2003Q1 and ending in 2011Q2. Under the second time dimension, for each cohort, we track how much we recover in each of the next twenty quarters (i.e., over five years) subsequent to the respective default time of that cohort. By using the panel data regression method proposed by Baltagi and Li (1994, 1997), we estimate our time-series model to capture the evolution of the recovery rate profile of the defaulted portfolio over the sample period. Estimation results reveal a robust pattern of the long-term average recovery rate profile. More importantly, the findings indicate that the realized recovery rate does deviate from its long-term average and there is a significant time-series relation of the realized recovery rates. Our model utilizes this time-series relation to dynamically update our expectation of the overall recovery rate of a cohort conditional on the recovery rate realized up to the current time. By doing so, we enhance the accuracy of modeling the PIT portfolio risk over the business cycle. In comparing our proposed conditional model with the commonly-used unconditional approaches, our simulation analysis with the dataset confirms the ability of our proposed model to generate more realistic estimations of both the mean LGD and portfolio value-at-risk (VaR) measures.

The present study has a number of practical and business implications other than the pursuit of a more accurate prediction of LGD that reflects the dynamic nature of the recovery rate. First of all, the proposed model can be used to enhance the estimation of required economic capital under the Basel II's Advanced Internal Rating-Based (AIRB) regulatory requirement and the loss provision under IFRS 9 of the International Financial Reporting Standard for financial institutions. The model can also be used in the pricing of defaulted portfolios, e.g., when a financial institution is making an investment or divestment decisions regarding its defaulted portfolios. Finally, the proposed model can be used in the pricing of credit derivatives such as recovery swaps, which provide hedging against the uncertainty of recovery in default. Please note that this study is based on retail portfolio. We aware that the wholesale portfolio risk profile could be very different from the data we have, leading to potential different conclusions.

The remainder of the paper is organized as follows. We will describe the data sample being examined in Section 2, and outline the proposed model and the estimation methodology in Section 3. Then, in Section 4, we will estimate the proposed model using recovery rate sample information and examine how prevailing recovery rate information can be used in updating the conditional model. We will compare the proposed model with two unconditional models commonly adopted in practice to highlight the differences in the resulting value-at-risk (VaR) and economic capital measures in Section 5. Lastly, we will provide a conclusion in Section 6.

3.2 Data

The data set is derived from the internal database of a Bank and consists of detailed information on the Bank's defaulted retail credit portfolio, including the number of defaults, outstanding loan balances, and recovery rates (measured as fractions of the loan exposures) for each quarter from April 2003 to July 2011. In our sample, there are altogether 13,098 defaulted retail facilities with a combined default balance of about $268 million that are classified into four different segments based on the internal risk assessment conducted by the Bank. Specifically, Segment 1 represents the lowest risk (i.e., with the highest expected recovery rate), while Segment 4 represents the highest risk (i.e., with the lowest expected recovery rate).[14] In classifying them into the four segments, we assume the recovery behavior of the defaulted instruments to be sufficiently homogenous (different) within (among) the segment(s). The segment level analysis can be further extended with more data available.

Typically, financial institutions recover the money lent to defaulted retail customers through internal debt collection efforts; formal procedures are followed by debt collection call centers in determining the customers who should be called as well as the number of times follow-ups should be made. Occasionally, financial institutions may decide to

[14] The determination of the risk category is based on a rigorous segmentation model adopted by the Bank, which is a logistic regression model considering the loan-level characteristics, including security, loan-to-value ratio, bureau score, and behavior indicators (e.g., whether it has been delinquent in the prior twelve months). With the prediction values from the logistic model, the Bank then uses a decision tree to divide them into the four segments. The four segments are verified to be sufficiently distinct from each other by conducting various statistical tests.

involve third-party debt collectors in the recovery process. And for secured retail loans (e.g., residential mortgages), the recovery process also involves the liquidations of the collaterals (e.g., the underlying residential properties). It is a time-consuming process for debtors of defaulted loans to recover cash flow. In a retail setting, most of the recovery occurs within the first two years after the default date, but the process can extend up to five years or longer. The dataset we have consisted of quarterly recovery rate information for each defaulted instrument over the five-year period after its default date. It is worth mentioning that, though uncommon, the overall recovery rate measured as a fraction of the loan amount can be negative (in other words, LGD can be larger than unity). A negative recovery rate represents the situation where the cost of the collection effort is higher than the amount of money actually collected from the debtors. Upon screening the sample data, we note several incidences where the recorded recovery rates are lower than minus one (i.e., LGD higher than two); to minimize the likelihood of data errors, these extreme outliers are excluded in performing the analysis.

For each risk segment (i.e., Segments 1 to 4), we group the defaulted facilities into thirty-four cohorts based on their default dates. Specifically, facilities of the first cohort defaulted in the quarter ending April 2003 (i.e., in the months of February to April 2003). Facilities of the second cohort defaulted in the quarter ending July 2003, and so forth until the thirty-fourth cohort defaulted in the quarter ending July 2011. Then, for each cohort, we track how much is recovered (as fractions of the loan amounts) in each of the quarters from the default quarter up to five years after the default quarter (i.e., in total twenty quarters of recovery rate information).[15] Figure 3.2.1 depicts the timeline of the first three cohorts. In summary, there is a panel data of thirty-four cohorts times twenty quarters of recovery rate information for each risk segment.

The recovery rates realized from the first cohort (defaulted in the quarter ending April 2003) of Segment 4 over the twenty quarters are plotted in Figure 3.2.2. This particular cohort has altogether fifty-seven defaulted facilities with a total outstanding balance of about $588,000. For this cohort, most of the recovery cash flows are realized within the first three and a half years. It is noteworthy that negative recovery cash flows, though rarely occurring, transpired in the second quarter. This is likely because the money recovered from the debtors in that quarter cannot fully offset the costs incurred in the collection effort.

In Figure 3.2.3, we plot the mean recovery rates of each of the four risk segments over the twenty quarters since the respective default time. For each segment, the average recovery rate of each quarter is calculated across the thirty-four cohorts using their recovery rates realized in the same quarter since the respective default time. This provides a time profile of quarterly recovery rates measured in a relative fashion with respect to the default time rather than according to the absolute calendar time. It can therefore be interpreted as the long-term average profile of recovery rates. As expected, the recovery rates of Segment 1 (lowest recovery risk) are consistently higher than those of Segment 4 (highest recovery risk); the overall average LGD of the four segments are 0.76 (Segment 1), 0.82 (Segment

[15] We essentially assume the money that we can recover beyond the fifth year from the default date to be small enough that we can ignore

2), 0.87 (Segment 3), and 0.93 (Segment 4). Interestingly, all four segments have a similar hump-shape recovery rate profile that peaks at approximately one year after the default time and the majority of the recovery cash flows are realized within the first three years since the default. Conditional on the prevailing economic condition, the amount of recovery realized in any specific quarter since the respective default time is expected to deviate from the long-term profile depicted in Figure 3.2.3. Specifically, we expect to recover more (less) than the long-term average during an expansionary (a recessionary) time period. In the next section, we propose a time-series model to capture the time-varying deviation from the long-term profile, so as to allow us to arrive at a point-in-time LGD prediction.

INSERT FIGURES 3.2.1 to 3.2.3 ABOUT HERE

3.3 Proposed Model and Estimation Methodology

Our objective is to model the evolution of the mean recovery rates of a cohort as some of the quarterly recovery amounts have been realized over time. Given the persistence of the recovery rate as we go through the different phases of the economic cycle, we expect that the recovery rate realized in the current time period is informative in predicting the recovery rate of the subsequent period. Specifically, we expect the next period recovery rate will tend to be higher (lower) when a higher (lower) recovery rate is realized in the current period. We assume that the recovery rate can be decomposed into two components: the *expected* and *unexpected* components. The former captures both the long-term average profile of recovery rate and the information carried by the recovery rate realized in the last period while the latter captures any residual unexpected shocks. Let us illustrate by considering a T period model with the following timeline:

$$t = 0 \xrightarrow{Period\ 1} t = 1 \xrightarrow{Period\ 2} t = 2 \rightarrow \cdots \xrightarrow{Period\ T} t = T$$

For a specific cohort i, the recovery rate $\tilde{R}_{i,t}$ realized at quarter t (measured relative to the default time of the cohort) is modeled as the following time-series model with autoregressive residuals $\tilde{v}_{i,t}$.

$$\tilde{R}_{i,t} = c_t + \tilde{v}_{i,t} \qquad (1)$$

where

$$\tilde{v}_{i,t} = \rho \tilde{v}_{i,t-1} + \tilde{e}_{i,t} \qquad (2)$$

and $\tilde{e}_{i,t} \sim N\,(0, \sigma_e^2)$ for t = 1 to T.[16] The intercept c_t (t = 1 to T) captures the long-run average profile of recovery rates at different quarters since the default time. We expect that the intercepts are not only specific to the time t since the time of default, but also specific to the risk segment the defaulted facility belongs to. Specifically, we expect Segment 1 to have larger intercept values than Segment 4. However, we assume different cohorts of the same risk segment defaulting at different points in time share the same intercept profile c_t (t = 1 to T). Also note that, according to Equation (2), the unconditional mean and variance of $\tilde{v}_{i,t}$ are given by:

$$E(\tilde{v}_{i,t}) = 0 \qquad (3)$$

[16] We further assume that $\tilde{e}_{i,t}$ is independently and identically distributed across both cohort i and time t.

$$var(\tilde{v}_{i,t}) = \sigma_v^2 = \frac{\sigma_e^2}{1-\rho^2} \qquad (4)$$

Based on our model (Equations (1) and (2)), the *unexpected* component of the recovery rate is captured by the shocks $\tilde{e}_{i,t}$ that is independent of any of the previously realized recovery rates. On the other hand, the *expected* component is composed of the long-run average recovery rate c_t and the serially correlated component $\rho\tilde{v}_{i,t-1}$ of the residuals. The former captures the long-run average recovery profile; whereas the latter captures the persistence of recovery rate (governed by the autocorrelation coefficient ρ) as we progress through the economic cycle. Note that $\tilde{v}_{i,t}$ is a latent variable to be estimated by observing the recovery rate $\tilde{R}_{i,t}$. Different cohorts can start with different initial values $\tilde{v}_{i,0}$, which represent the prevailing economic conditions when the cohorts defaulted. If the economy is booming (weak), $\tilde{v}_{i,0}$ is expected to be positive (negative) as more (less) is expected to be recovered from the facilities.

It is important to emphasize the fact that both indices i and t are time indices. The former denotes the cohort under consideration, which is defined by its default quarter. For example, the facilities in Cohort 1 defaulted in the quarter ending April 2003, while those in Cohort 2 defaulted in the subsequent quarter ending July 2003, and so forth. Thus, Index i is a default date measure in terms of *absolute* calendar time. On the other hand, Index t is defined as the time passed since the default date of the cohort under consideration and is a *relative* time measure with respect to the default date, which varies from one cohort to the other. Therefore, our proposed model can accommodate the time-series variations of recovery rates along these two-time dimensions.

We estimate our proposed model with the panel data of retail recovery rates of the Bank as described in Section 2. We follow the approach adopted by Baltagi and Li (1994, 1997) in conducting a panel data regression with serially correlated disturbances. Baltagi and Li derive the spectral decomposition of the variance-covariance matrix of the composite error when the remainder error follows an AR(p) or an MA(q) process. With $T = 20$ (twenty quarters), Equation (1) can be expressed in the following matrix form:

$$R_i = I \times C + v_i, \quad \text{for } cohort\ i = 1, 2, \cdots, N \qquad (5)$$

where $R_i = [\, R_{i,1}\, R_{i,2}\, \cdots\, R_{i,20}]'$, I is the 20×20 identity matrix, $C = [c_1\ c_2\ \cdots\ c_{20}]'$, and $v_i = [\, v_{i,1}\, v_{i,2}\, \cdots\, v_{i,20}]'$ where the disturbances $v_{i,t}$ follows a stationary $AR(1)$ process of Equation (2). In the regression representation of Equation (5), the only regressors are the twenty dummy variables denoting the twenty different quarters $t = 1, 2, \ldots, 20$ since default date.[17] Each of these dummy variables has a coefficient c_t for $t = 1, 2, \ldots, 20$. With a total of thirty-four cohorts (i.e., $N = 34$), we have a panel regression model comprising thirty-four Equation (5) each for a different i. Our objective is to estimate the autocorrelation coefficient ρ, the parameter vector C, and the standard deviation of the shocks σ_e.

[17] It will be a straightforward extension to incorporate other time-varying regressors as our independent variables. For example, we can test the effects of selected macroeconomic variables (e.g., GDP growth rate) on the recovery rate profile by adding them as regressors.

Baltagi and Li (1997) suggest the following estimator for ρ,

$$\hat{\rho} = \frac{\tilde{Q}_1 - \tilde{Q}_2}{\tilde{Q}_0 - \tilde{Q}_1}, \tag{6}$$

where $\tilde{Q}_s = \sum_{i=1}^{N} \sum_{t=s+1}^{T} \hat{v}_{i,t} \hat{v}_{i,t-s} / N(T-s)$ and $\hat{v}_{it}$ denotes the OLS residuals from Equation (1). With the estimated $\hat{\rho}$, we follow the two-step process of Baltagi and Li (1994) to estimate parameter vector C. The objective of the two-step transformation is to allow us to use OLS to obtain the coefficients while correcting for the bias introduced by the serial correlation under the panel data setting. In brief, let M be a $T \times T$ matrix that removes the serial correlation in the disturbances, in the sense that $Mv_i \sim N\ (0,\ \sigma^2 I)$. In Step 1, we apply the M transformation on the vector of observed recovery rates R_i to remove the serial correlation and obtained the transformed vector R_i^*.

$$R_i^* = MR_i \tag{7}$$

Then, in Step 2, we perform a further transformation on the transformed observations R_i^*, from which we subtract a fraction of a weighted average value of the observations R_i^*. After correcting for the bias introduced by the serial correlation with these two-step transformations, we can then conduct a OLS regression with R_i^* to estimate vector $C = [c_1\ \ c_2\ \cdots\ c_{20}]'$ and σ_e.[18] In the next section, we report the results of our model estimations and discuss the implementation of the calibrated model.

[18] In conducting our panel data regression, we ignore any cross-sectional (i.e., across cohort) random effect.

3.4 Model Estimation and Implementation

We estimate our proposed model for each risk segment of the Bank's recovery data set based on the estimation method described in Section 3. The estimated values of the autocorrelation coefficient ρ, the parameter vector C, and the standard deviation of the shocks σ_e of each segment are reported in Table 3.4.1. Consistent with our expectation of in general positively autocorrelated recovery rates, the estimated AR (1) coefficient ρ is found to be positive for three of the four segments. Based on the magnitudes of the estimated σ_e, the higher the average recovery rate of the segment, the higher the variability of the shocks. In other words, it is more difficult to predict the recovery rate profile of Segment 1 than that of Segment 4. All of the estimated values of $c_1, c_2, \cdots,$ and c_{20}, which together represent the long-run average recovery rate profile, are at least weakly statistically significant. Consistent with our expectation, Segment 1 (lowest recovery risk) has in general the highest long-run average recovery rates, while Segment 4 (highest recovery risk) has the lowest. Tracking the estimated value of c_t over the elapsed time t since the default date, the hump shape pattern of the long-run average recovery rate profile is quite salient and robust. The recoveries in the first year following the default time are relatively low. For all four segments, the highest expected recoveries are realized in the second year after default, from the fourth to the seventh quarter. The expected recovery rates then gradually diminish following the peak for the rest of the five-year recovery period.

INSERT TABLE 3.4.1 ABOUT HERE

Our model allows us to utilize the time-series relation of recovery rates to dynamically update our expectation of the overall recovery rate of a cohort conditional on the recovery rate realized up to the current time. Intuitively, as we observe a recovery rate that is higher (lower) than the long-run average in the current quarter, it suggests the economic condition is better (worse) than average, and thus we would expect the recovery rate in the subsequent quarter will also be higher (lower) as the prevailing economic condition persists. By exploiting this time-series relation, we can arrive at a more accurate assessment of the PIT portfolio risk than if such timely information is ignored.

Let us first examine the dynamic updating rule based on our proposed model. Then, with the model parameters estimated above, we demonstrate how the updating rule is applied in calculating the conditional LGD of the Bank's retail defaulted portfolio as new information is being incorporated over time. In the illustration below, we focus on a particular cohort i. Without loss of generality, we can therefore suppress subscript i in the following derivation and focus on the description of how the recovery rate evolves over t. Let us start with the relatively simple case of a two-period problem. That is, we assume recovery cash flows will only be realized over two quarters after the default date. We will generalize to the multi-period case later. The objective here is to predict the overall recovery rate $\tilde{R}$, which is the sum of the recovery rates to be realized in the first and the second quarters. It is a random variable at $t = 0$ (i.e., the default date), which can be expressed as (according to Equations (1) and (2)):

$$\tilde{R} = \tilde{R}_1 + \tilde{R}_2$$

$$= c_1 + \tilde{v}_1 + c_2 + \rho\tilde{v}_1 + \tilde{e}_2 \qquad (8)$$

Therefore, based on Equation (3), the expected value of the overall recovery rate is:

$$E_0(\tilde{R}) = c_1 + c_2 \qquad (9)$$

and, according to Equation (4), the variance of the overall recovery rate is:

$$var_0(\tilde{R}) = (1+\rho)^2 var(\tilde{v}_1) + var(\tilde{e}_1)$$

$$= (1+\rho)^2 \frac{\sigma_e^2}{1-\rho^2} + \sigma_e^2$$

$$= (\frac{1+\rho}{1-\rho} + 1)\sigma_e^2$$

$$= \frac{2\sigma_e^2}{1-\rho}$$

(10)

At $t = 1$ (i.e., at the end of the first quarter after default), *conditional* on observing R_1, we have:

$$\tilde{R} = R_1 + \tilde{R}_2 = R_1 + c_2 + \rho v_1 + \tilde{e}_2$$

$$= R_1 + c_2 + \rho(R_1 - c_1) + \tilde{e}_2 \qquad (11)$$

Now, the only random variable is the recovery rate $\tilde{R}_2$ (or, in other words, the shock $\tilde{e}_2$) to be realized in the subsequent quarter. Therefore, the *conditional* (i.e., updated) expected values and variances of the second quarter and the overall recovery rate become:

$$E_1(\tilde{R}_2) = c_2 + \rho(R_1 - c_1) \quad (12)$$

$$E_1(\tilde{R}) = R_1 + c_2 + \rho(R_1 - c_1) \quad (13)$$

$$var_1(\tilde{R}) = var_1(\tilde{R}_2) = \sigma_e^2 \quad (14)$$

The updating effect can also be illustrated by examining the changes in the conditional mean and the conditional variance of the overall recovery rate from $t = 0$ to $t = 1$. Comparing Equations (9) and (13), the change in the expected value is:

$$E_1(\tilde{R}) - E_0(\tilde{R}) = R_1 + c_2 + \rho(R_1 - c_1) - c_1 - c_2$$

$$= (1 + \rho)(R_1 - c_1), \quad (15)$$

whereas, by comparing Equations (10) and (14), the change in variance is:

$$var_1(\tilde{R}) - var_0(\tilde{R}) = \sigma_e^2 - \frac{2\sigma_e^2}{1 - \rho}$$

$$= -\frac{1+\rho}{1-\rho}\sigma_e^2 \quad (16)$$

From Equation (15), we notice that the direction and magnitude of the update to our expectation of the overall recovery rate is governed by how the realized recovery rate in the first quarter R_1 compared with the long-run average recovery rate c_1 of the same quarter. If an unexpectedly high recovery rate is realized in the first quarter (i.e., $(R_1 - c_1) > 0$), we will upwardly adjust our expectation, and vice versa, when a lower than expected recovery rate is realized in the first quarter (i.e., $(R_1 - c_1) < 0$). A larger deviation from the long-run average will result in a larger adjustment. On the other hand, according to Equation (16), the conditional variance of the overall recovery rate is always decreasing (regardless of the realized recovery rate in the first quarter) as part of the repayment is realized.

Let us now generalize the problem and examine the multi-period case where recovery cash flows are realized over T periods (i.e., T quarters) after the default time. At the time of default (i.e., at $t = 0$), the overall recovery rate is the sum of the quarterly recovery rates from $t = 1 \; to \; T$. That is,

$$\tilde{R} = \tilde{R}_1 + \tilde{R}_2 + \cdots + \tilde{R}_T$$

$$= c_1 + c_2 + \cdots + c_{T-1} + c_T + \tilde{v}_1 + \tilde{v}_2 + \cdots + \tilde{v}_T = \sum_{i=1}^{T}(c_i + \tilde{v}_i) \qquad (17)$$

Therefore, the expected overall recovery rate is:

$$E_0(\tilde{R}) = E_0(\tilde{R}_1 + \tilde{R}_2 + \cdots + \tilde{R}_T)$$

$$= c_1 + c_2 + \cdots + c_{T-1} + c_T = \sum_{i=1}^{T} c_i \qquad (18)$$

whereas the variance of the overall recovery rate as at $t = 0$ (refer to the Appendix for the detailed derivations) is:

$$var_0\big(\tilde{R}\big) = var_0(\tilde{R}_1 + \tilde{R}_2 + \cdots + \tilde{R}_T)$$

$$= \left(\sum_{i=0}^{T-1} \rho^i\right)^2 \frac{\sigma_e^2}{1-\rho^2} + \left(\sum_{i=0}^{T-2} \rho^i\right)^2 \sigma_e^2 + \left(\sum_{i=0}^{T-3} \rho^i\right)^2 \sigma_e^2 + \cdots + \left(\sum_{i=0}^{1} \rho^i\right)^2 \sigma_e^2 + \sigma_e^2 \qquad (19)$$

What are the conditional expectation and variance of the overall recovery rate when we are within the recovery process and some quarterly recovery rates have already been realized?

Suppose we are at the end of the period $t^* - 1$ $(T \geq t^* > 1)$ and we observe the historical quarterly recovery rate of $R_1, R_2, \cdots, R_{t^*-1}$ realized in time periods $1, 2, \ldots, t^* - 1$ respectively. Conditional on this information, our updated expectation of the overall recovery rate is given by (refer to the Appendix for the detailed derivations):

$$E_{t^*-1}(\tilde{R}) = E(R_1 + R_2 + \cdots + R_{t^*-1} + \tilde{R}_{t^*} + \cdots + \tilde{R}_T)$$

$$= \sum_{i=1}^{t^*-1} R_i + \sum_{i=t^*}^{T} c_i + (R_{t^*-1} - c_{t^*-1}) \sum_{i=1}^{T-t^*+1} \rho^i \qquad (20)$$

Comparing the *conditional* expectation of the overall recovery rate at $t^* - 1$ (Equation (20)) with its initial (*unconditional*) expectation (Equation (18)), we have:

$$E_{t^*-1}(\tilde{R}) - E_0(\tilde{R}) = (R_{t^*-1} - c_{t^*-1}) \sum_{i=0}^{T-t^*+1} \rho^i + \sum_{i=1}^{t^*-2} (R_i - c_i) \qquad (21)$$

From Equation (21), we notice that the direction and magnitude of the update of our expectation is dictated by how the realized quarterly recovery rates $R_1, R_2, \cdots, R_{t^*-1}$ compared with their respective long-run average recovery rates $c_1, c_2, \cdots, c_{t^*-1}$. Any unexpectedly high (low) realized recovery rates will result in an upward (a downward) adjustment of our expectations. The larger the deviations from the long-run averages, the larger will be the adjustment. With autocorrelation coefficient ρ lying between 0 and 1, an unexpectedly high recovery rate realized in the last quarter (i.e., $(R_{t^*-1} - c_{t^*-1}) > 0$) will have a positive but diminishing impact on the expected values of the recovery rates to be realized in the subsequent quarters. On the other hand, an unexpectedly low recovery rate in the last quarter (i.e., $(R_{t^*-1} - c_{t^*-1}) < 0$) will have a negative but diminishing impact on the expected values of the recovery rates to be realized in the upcoming quarters.

At the end of period $t^* - 1$, the conditional variance of the overall recovery rate can be expressed as (refer to the Appendix for the detailed derivations):

$$var_{t^*-1}(\tilde{R}) = var_{t^*-1}(R_1 + R_2 + \cdots + R_{t^*-1} + \tilde{R}_{t^*} + \cdots + \tilde{R}_T)$$

$$= (\sum_{i=0}^{T-t^*} \rho^{\,i})^2 \frac{\sigma_e^2}{1-\rho^2} + (\sum_{i=0}^{T-t^*-1} \rho^i)^2 \sigma_e^2 + (\sum_{i=0}^{T-t^*-2} \rho^i)^2 \sigma_e^2 + \cdots + (\sum_{i=0}^{1} \rho^i)^2 \sigma_e^2 + \sigma_e^2 \quad (22)$$

Unsurprisingly, when ρ is positive, the conditional variance of the overall recovery rate is monotonically decreasing as time passes and when more quarterly recovery cash flows are realized. Thus, the longer the time since the default date, the lower the risk of the defaulted cohort. It is also important to note that the conditional variance is always smaller than the initial (*unconditional*) variance (Equation (19)).

Below we demonstrate the updating of the expected overall recovery rate for a cohort of Segment 1 that defaulted in 2006Q3. The resultant PIT expected recovery rates of the cohort over the sample period from 2006Q3 to 2011Q3 are plotted in Figure 4, together with the levels of the TTC (i.e., unconditional) expected value and the actual realized overall recovery rate indicated. Note that with the facilities in the cohort just defaulted, we have yet to observe any recovery rate information in 2006Q3. We, therefore, start with the TTC expected value given by the sum of the long-run average recovery rates c_i. Specifically, according to Equation (18), our expected overall recovery rate as perceived in 2006Q3 is (according to the estimated values of c_i of Segment 1 presented in Table 1):

$$E_0(\tilde{R}) = c_1 + c_2 + \cdots + c_{20} = 0.2436$$

In the subsequent quarter 2006Q4, a recovery rate of -0.0133 (i.e., $R_1 = -0.0133$) is realized. This is the first realized quarterly recovery rate for this cohort. Given this information, we therefore update our assessment of the expected overall recovery rate by following the updating Equation (20). That is (according to the estimated values of c_i and ρ of Segment 1 presented in Table 1),

$$E_1(\tilde{R}) = R_1 + \sum_{i=2}^{20} c_i + (R_1 - c_1) \sum_{i=1}^{19} \rho^i$$

$$= -0.0133 + 0.2299 + (-0.0133 - 0.0137) \times \sum_{i=1}^{19} 0.0245^i = 0.2160$$

In other words, with the negative recovery rate realized in the first quarter, we downwardly adjust our expectation from the initial (unconditional) value of 0.2436 to 0.2160. Our PIT assessment of the expected overall recovery rate is updated in each of the subsequent quarters in a similar fashion using Equation (20) as more recovery information is realized as time passes. As depicted in Figure 3.4.1, the dynamic updating allows our conditional model to more accurately approximate the actual overall recovery rate of 0.1882 for this cohort as eventually fully realized in 2011Q3. Without updating, the TTC expected value of 0.2436 grossly overstates the actual recovery rate of 0.1882, thus leading to an underestimation of the recovery risks.

INSERT FIGURE 3.4.1 ABOUT HERE

3.5 Model Applications in Portfolio Risk Management

In this section, we illustrate how our proposed model can be used to enhance the risk management of a defaulted facilities portfolio. There are two key risk measures to estimate: value-at-risk (VaR) and the required economic capital. The VaR of a portfolio can be defined as the critical loss of the portfolio such that it will only be exceeded on very rare occasions, e.g., with only a 0.10% or even 0.05% probability of occurrence (i.e., with corresponding confidence levels of 99.90% or 99.95% respectively). The choice of the appropriate probability of occurrence is governed by the risk appetite of the financial institution. On the other hand, the required economic capital that needs to be set aside by the financial institution to cushion the unexpected loss from the portfolio can be defined by the difference between VaR and the expected loss of the portfolio. By subtracting the *expected* loss (which has already been booked as loss provision) from the extreme VaR loss amount, we determine the *unexpected* component of the loss, which will have to be absorbed by the shareholders' capital of the financial institution in protecting its debtholders and deposit holders.

In the previous sections, we propose and estimate a time-series model of recovery rate profile based on our sample portfolio of defaulted retail facilities. The proposed model allows us to utilize realized recovery rates to update our expectation of the overall recovery rate of a cohort. We are essentially forming an expectation of the aggregated recovery rate of the portfolio of facilities making up the cohort. In order to measure the VaR and in turn the economic capital requirement, in addition to knowing the mean recovery rate of the portfolio or cohort, we also need to find out the cross-sectional variability of the recovery rate of the individual defaulted facilities. For the same aggregated mean recovery rate, the higher the cross-sectional variability of recovery rates, the higher is the VaR, since VaR is driven not by the mean of the distribution but the extreme tail events. Thus, prior to using the conditional model to generate VaR and economic capital, we need an assessment of the cross-sectional variability of recovery rate within the cohort.

We hypothesize that cross-sectional variability in facility-level recovery rate is an increasing function of the aggregated mean recovery rate. In other words, the distribution becomes more dispersed as the mean of the distribution increases. Once we establish this relationship empirically, we can then estimate the cross-sectional variability of the recovery rate of a cohort by knowing its mean recovery rate as predicted by our conditional model estimated in the previous section. With this objective, we conduct a regression analysis with the mean recovery rate of each cohort and the standard deviation of the realized recovery rates of the facilities within that cohort. For each segment, we have thirty-four data points from the thirty-four cohorts defaulted at different quarters over our sample period from April 2003 to July 2011. As an example, in Table 3.5.1, we present the means and the standard deviations of the realized recovery rates of the facilities that defaulted in the 2003Q1 (Cohort 1), 2003Q2 (Cohort 2), and 2003Q3 (Cohort3) of Segment 1. There are fifteen defaulted facilities in Cohort 1, and the mean and standard deviation of the overall recovery rates of these fifteen defaulted facilities are 0.3105 and 0.3665, respectively, in five years after default. In the second quarter of 2003, eighteen facilities defaulted with the mean and the standard deviation of recovery rates equal to 0.2998 and

0.3504. After calculating the means and standard deviations for all thirty-four cohorts up to the last one defaulting in the quarter ending July 2011, we regress the standard deviations of the thirty-four cohorts against an intercept term, the means, and the squared-means. The regressions are conducted at both the segment level and for the overall portfolio. The results, reported in Table 3.5.2, indicate that the linear models are statistically significant for all the segments and the overall portfolio. Consistent with our hypothesis, all the coefficients are positive and strongly statistically significant. In other words, the larger the mean recovery rate, the larger is the standard deviation. On the other hand, the quadratic term is only statistically significant for the overall portfolio and Segment 1. The estimated coefficients of both of these regressions are negative, suggesting a diminishing effect of the increase in the mean on the standard deviation as the mean increases. The fittings of the linear and quadratic functions of the overall portfolio are graphically presented in Figure 3.5.1. Given the above regression results, we adopt the quadratic (linear) function for Segment 1 (Segments 2, 3, and 4) in the subsequent simulations of VaR and economic capital.

INSERT FIGURE 3.5.1, TABLES 3.5.1 AND 3.5.2 ABOUT HERE

Below we compare the performance of our proposed conditional recovery model (*Conditional Model*) with those of two models commonly used in practice: (i) *Benchmark Model* - an unconditional recovery model without adjusting the variance according to the stage of recovery; and (ii) *Unconditional Model* - an unconditional recovery model with an ad-hoc adjustment of the residual variance. We compare the predicted portfolio losses, the portfolio VaR, and the corresponding economic capital requirements from the three models by conducting a simulation exercise with real-life retail defaulted portfolio.

Firstly, let us highlight the key differences among the three models under consideration before revealing the simulation results. We will be implementing the models on the same defaulted portfolio consisting of twenty cohorts of different default dates, which represents a snapshot of the aggregated defaulted portfolio at a certain point in time. Suppose Cohort 1 consists of facilities defaulted in the last quarter. Cohort 2 consists of those defaulted two quarters ago and thus we have only witnessed recovery information for one quarter. Cohort 3 consists of those defaulted three quarters ago with only two quarters of recovery rates observed, and so forth, until Cohort 20, in which the defaulted facilities approach the end of the 5-year recovery process and there remains one more quarter of recovery rate information to be realized. The *Benchmark Model* assumes the mean recovery rate for each of the twenty cohorts, regardless of their historical default dates, is identical and equal to the unconditional average given by Equation (18) for the specific segment to which they belong. The model-implied portfolio risk will therefore be static over time, representing a TTC measure of the long-run average recovery rate profile of that segment. Thus, we ignore the information conveyed in observing the amount of money that has already been (partially) recovered up to the current point in time. The cross-sectional standard deviations of the facilities' recovery rates of each of the cohorts are also assumed to be identical to each other and static over time. The standard deviation is calculated using the linear (quadratic) function estimated earlier for Segments 2, 3, and 4 (Segment 1) and with the unconditional mean given by Equation (18). The *Unconditional Model* adopts the same assumptions of the *Benchmark Model*, except that the cross-sectional standard deviations of different cohorts are adjusted to reflect the fact that the variability of recovery rates becomes smaller as the remaining time till the end of the 5-year recovery process becomes shorter. Specifically, the variance is assumed to decrease in a quadratic fashion with the remaining time. That is, for Cohort j ($j = 1, 2, \ldots, 20$), the variance is:

$$variance_j = (\frac{20-j+1}{20})^2 \times variance_{unadjusted} \tag{23}$$

Unlike the *Benchmark* and *Unconditional Models*, in which the model-implied portfolio risk is static over time, reflecting a TTC condition, our proposed *Conditional Model* dynamically updates the expected recovery rate based on the realized recovery rate information, thus giving us a PIT measure of portfolio risks that varies over the business cycle. The updating is conducted based on the methodology outlined in Section 3.

In Table 3.5.3, we present the means and variances of the recovery rates of the *Benchmark* and *Unconditional Model*. The mean recovery rates are calculated based on Equation (18) and the values of $c_1, c_2, \cdots, \text{and } c_{20}$, which together represent the long-run average recovery rate profile, as estimated in Section 4. The corresponding variances of the *Benchmark Model* are calculated using the linear (quadratic) function estimated earlier for Segments 2, 3, and 4 (Segment 1). The same variance is applied to all cohorts. For example, for Segment 1, all cohorts have the same variance of 0.1245. For the *Unconditional Model*, we adjust the variances of different cohorts using Equation (23). For example, the variances of Cohorts 1 and 2 of Segment 1 are calculated as:

$$variance_1 = (\frac{20-1+1}{20})^2 \times variance_{unadjusted} = 1 \times 0.1245 = 0.1245$$

$$variance_2 = (\frac{20-2+1}{20})^2 \times variance_{unadjusted} = 0.95^2 * 0.1245 = 0.1124$$

With the estimated means and variances of the recovery rates, we conduct a number of Monte Carlo simulations to calculate the VaR and the economic capital requirement as described below by assuming a beta distribution for the recovery rates.

INSERT TABLE 3.5.3 ABOUT HERE

To ensure our simulation results are not affected by the changing portfolio composition, we focus on a specific *snapshot* of our defaulted facilities portfolio and examine how the model-implied portfolio risks vary over time as the risk parameters are undated under our proposed *Conditional Model*. The snapshot represents the portfolio composition as observed in 2011Q4. The default balances and corresponding numbers of defaulted facilities of each of the cohorts of the snapshot portfolio are reported in Table 3.5.4. The portfolio composition information is presented in two dimensions: time after default (TIME) and risk segment (Segment1 to Segment 4). The TIME of 0.25Y denotes the cohort consisting of facilities that defaulted in the past three months (i.e., Cohort 1). On the other hand, the TIME of 5Y denotes the cohort consisting of facilities that defaulted from 4.75 years (57 months) to 5 years (60 months) ago (i.e., Cohort 20). As we can see from Table 3.5.4, since Segment 1 is the largest portfolio in our dataset, most observations are from the segment that defaulted during 2009 and 2010 as a result of the financial crisis.

INSERT TABLE 3.5.4 ABOUT HERE

Since portfolio composition is fixed throughout the simulation process, the means and variances of recovery rates are identical overtime under the *Benchmark Model* and the *Unconditional Model* as reported in Table 3.5.3. The corresponding model-implied VaR and economic capital requirement are therefore also static over time. For the proposed *Conditional Model*, the risk parameters (i.e., mean and variance of recovery rates) for each cohort are updated over time based on the latest recovery information as described in Section 4. Therefore, we expect to obtain a more accurate PIT estimation of the portfolio risk by using the *Condition Model*. Let us examine in detail the calculations involved in the updating process. Suppose the Bank is holding the snapshot portfolio (as described above) in 2008Q1 and thus observe the realized recovery rates of all twenty cohorts up to 2008Q1. The realized recovery rates of Segment 1 are presented in Table 3.5.5. Under Columns 1 to 19, we have the quarterly realized recovery rates for the twenty cohorts of Segment 1 that defaulted respectively from 2003Q2 to 2008Q1. Specifically, the recovery rates realized in the first quarter after the respective default date are presented under Column 1, whereas those realized in the 19th quarter after the default date are presented under Column 19. As we can see from Table 3.5.5, the cohort that defaulted in the most recent quarter 2008Q1 (i.e., Cohort 1) has no recovery information available yet. Cohort 2, which defaulted one quarter prior to Cohort 1, has only one quarter's recovery rate information available. The Bank has so far recovered 0.0051 from that cohort (see row 2 in Table 3.5.5). For Cohort 3, there are two quarters of recovery information available as of 2008Q1: 0.0103 recovered in the first quarter after default and 0.0132 recovered in the second quarter after default (see row 3 in Table 3.5.5). Finally, for Cohort 20, a total of nineteen quarters of recovery data are observed (see the last row of Table 3.5.5).

INSERT TABLE 3.5.5 ABOUT HERE

The means and variances of the recovery rates for each cohort of Segment 1 are reported in the last two columns of Table 3.5.5. Let us illustrate the calculations involved by using

Cohort 2 as an example. First of all, the mean overall recovery rate is updated according to Equation (20). That is,

$$E_{t^*-1}(\tilde{R}) = \sum_{i=1}^{t^*-1} R_i + \sum_{i=t^*}^{T} c_i + (R_{t^*-1} - c_{t^*-1}) \sum_{i=1}^{T-t^*+1} \rho^i$$

Since there is the only one realized recovery rate of 0.0051 for Cohort 2, the first term is $R_1 = 0.0051$. To calculate the second term, we sum up all the long-run recovery rates c_i from $i = 2$ to 20, based on the estimated values presented in Table 1, $\sum_{i=2}^{20} c_i = 0.2299$. The last term is $(R_1 - c_1)\sum_{i=1}^{19} \rho^i = (0.0051 - 0.0137)\sum_{i=1}^{19} 0.0245^i = -0.0002$. The updated mean overall recovery rate, therefore, equals the sum of these three terms. That is $0.0051 + 0.2299 - 0.0002 = 0.2348$. Based on this conditional mean value, we then use the quadratic function estimated earlier in this section for Segment 1 (reported in Table 3.5.2) to estimate the corresponding cross-section standard deviation of the recovery rates for Cohort 2. Numerically,

$$\begin{aligned} std.dev. = {} & intercept + linear\ coefficient * mean + quadratic\ coefficient \\ & * mean^2 = 0.054 + 1.9729 * 0.2348 + (-2.9395) * 0.2348^2 \\ & = 0.3552 \end{aligned}$$

These mean and standard deviation of Cohort 2 are reported in the last two columns of Table 3.5.5 (see the second row). The means and standard deviations of the other cohorts are calculated in a similar fashion using Equation (20) and the quadratic function that relates the standard deviation to the corresponding mean. Using the means and standard deviations of recovery rates of all the twenty cohorts of Segment 1, we can therefore assess the portfolio risks of Segment 1 of our snapshot portfolio as perceived in 2008Q1 based on the *Conditional Model* (as described below). By repeating the above calculations based on the same snapshot portfolio but using recovery rate information observed up to 2008Q2, we will then be able to assess the portfolio risks as of 2008Q2. In the subsequent simulation exercise, we track the portfolio risks of our snapshot portfolio on a quarterly basis following the above procedure up to 2011Q3, always using the most up-to-date recovery information in updating our risk parameters. We contrast this conditional portfolio risk assessment with those from the *Benchmark* and *Unconditional Models*. The key differences among the *Benchmark, Unconditional,* and *Conditional Models* are further summarized in Table 3.5.6.

INSERT TABLE 3.5.6 ABOUT HERE

With the estimated mean and the variance of the recovery rates for each cohort under the *Benchmark* and *Unconditional Models* as reported in Table 3.5.3, we simulate the LGD (i.e., one minus the recovery rate) of each of the 8,670 defaulted facilities in the snapshot portfolio by assuming the random LGD value follows a beta distribution with parameters α and β calculated based on the mean and the variance of the cohort to which the default facilities belong.[19]

$$\alpha = \frac{mean}{variance} * (-mean^2 + variance) \tag{24}$$

$$\beta = (mean - mean^2 + variance) * \frac{1-mean}{variance} \tag{25}$$

[19] The traditional beta distribution is a bounded distribution between zero and one. In practice, we do observe LGD values exceeding one. In other words, the Bank loses more than it lent. It is likely due to the different direct or indirect costs incurred by the Bank during the recovery process (e.g., collection costs, legal fees). We therefore conduct an adjustment on the beta distribution to extend the range of the beta distribution to beyond one.

With the LGD value generated for each defaulted facility, we can then calculate the portfolio-level LGD by aggregating the facility-level LGD weighted by their respective default balances within the snapshot portfolio as presented in Table 3.5.4. Repeating the above random sampling procedures 200,000 times gives us an empirical portfolio loss distribution, from which the VaR at the 99.95% confidence level under the *Benchmark* and *Unconditional Models* are calculated to be 0.954 and 0.915 respectively. Subtracting the portfolio-level mean loss rates from these VaR values gives us the required economic capitals of $0.185 and $0.147 per dollar of exposure under the *Benchmark* and *Unconditional Models* respectively. Unsurprisingly, the *Benchmark Model* gives us higher portfolio risk measures than the *Unconditional Model*; nevertheless, there is no guarantee that the former will always result in a conservative risk measure given the changing economic condition over time.

Unlike the *Benchmark* and *Unconditional Models* where the risk parameters are time-invariant, the portfolio risk measures implied by the *Conditional Model* vary over time as new recovery rate information is realized. To calculate the time-varying VaR and economic capital requirement, we, therefore, need to repeat the simulation procedures for each quarter over our sample period from 2008Q1 to 2011Q3. The procedure is described in Figure 3.5.2. For each quarter, the simulation process starts with the latest recovery information (e.g., the cohort recovery rate information realized up to 2008Q1 as reported in Table 3.3.6). Together with the long-term recovery parameters presented in Table 3.4.1, the updated expected overall recovery rate for each cohort is then calculated (just like the calculation for Cohort 2 we demonstrate earlier in this section). Next, using the estimated linear or quadratic function governing the relation between the cross-sectional standard deviation of recovery rates and their mean as reported in Table 3.5.2, the corresponding cohort-specific standard deviations are estimated. With the estimated mean and the standard deviation, we then determine the α and β parameters of the LGD beta distribution based on Equations (24) and (25). The LGD value of each defaulted facility in the snapshot portfolio is then generated with the fitted beta distribution for each cohort and each segment. The facility-level LGD are then aggregated to arrive at the portfolio loss using the default balance per facility. This process is repeated 200,000 times resulting in the portfolio loss distribution for us to determine the VaR and in turn the economic capital requirement for that quarter. The above procedure is then repeated in the next quarter (2008Q2) as new recovery rate information is realized to calculate the PIT portfolio risk measures (i.e., VaR and economic capital) in the next quarter, and so forth for each of the subsequent quarters until we arrive at the last quarter (2011Q3) of our sample period.

INSERT FIGURE 3.5.2 ABOUT HERE

The simulation results for the overall snapshot portfolio are presented in Figure 3.5.3. In Panel A, we plot the expected loss rates from different models together with the actual realized loss rate. Unlike the *Benchmark* and *Unconditional Models*, which only capture the long-run mean, the proposed *Conditional Model* can track the variation of the actual realized loss rate over time quite well. Turning to the VaR results in Panel B, we notice the *Benchmark* and *Unconditional Models* significantly understate the PIT VaR generated by the *Conditional Model* in the aftermath of the 2009 financial crisis.[20] Finally, the economic

[20] We expect there is a certain time lag between an economic downturn and the peak of the resulting credit loss from a retail portfolio. It can take a few quarters before we witness defaults to occur and facilities entering the recovery process.

capital assessments based on the three models are presented in Panel C. We do not observe the significant underestimation of the PIT economic capital by the *Benchmark Model*. Note that economic capital is simply the difference between VaR and the expected loss rate. It appears that there is a similar amount of underestimation in both the mean and the tail (as measured by the VaR) of the distribution based on the *Benchmark Model*, thus being able to preserve an appropriate economic capital assessment. On the other hand, the *Unconditional Model*, which is commonly used in practice, can significantly underestimate both the VaR and the economic capital requirement.

INSERT FIGURE 3.5.3 ABOUT HERE

3.6 Conclusion

We propose a conditional model to capture the time-series variations of the recovery rate profile that can incorporate up-to-date recovery information in predicting the ultimate recovery rate and we estimate the proposed model using a sample of defaulted facilities of a retail credit portfolio. Based on the simulation results covering a sample period including the recent financial crisis, we demonstrate that the proposed model can generate more realistic PIT portfolio risk measures over time in comparison to commonly used models. The proposed model captures both the dynamic evolution of the mean and the variance of recovery rate over time and is simple to implement for both facility-level and portfolio-level risk analysis. With the time-series panel regression setup, the model can be readily extended to incorporate other macroeconomic variables that may also drive the variations of recovery rate over time. The flexibility and effectiveness of the proposed model make it a viable candidate to replace the models currently used in practice.

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Appendix

Derivations of Equations (19), (20), and (22)

Derivation of Equation (19):

$$var_0(\tilde{R}) = var_0(\tilde{R}_1 + \tilde{R}_2 + \cdots + \tilde{R}_T)$$

$$= var_0(\tilde{v}_1 + \tilde{v}_2 + \cdots + \tilde{v}_T)$$

$$= (1 + \rho + \cdots + \rho^{T-1})^2 var_0(\tilde{v}_1) + (1 + \rho + \cdots + \rho^{T-2})^2 \sigma_e^2 + (1 + \rho + \cdots + \rho^{T-3})^2 \sigma_e^2 + \cdots + (1 + \rho)^2 \sigma_e^2 + \sigma_e^2$$

$$= (1 + \rho + \cdots + \rho^{T-1})^2 \frac{\sigma_e^2}{1 - \rho^2} + (1 + \rho + \cdots + \rho^{T-2})^2 \sigma_e^2 + (1 + \rho + \cdots + \rho^{T-3})^2 \sigma_e^2 + \cdots + (1 + \rho)^2 \sigma_e^2 + \sigma_e^2$$

$$= (\sum_{i=0}^{T-1} \rho^i)^2 \frac{\sigma_e^2}{1 - \rho^2} + (\sum_{i=0}^{T-2} \rho^i)^2 \sigma_e^2 + (\sum_{i=0}^{T-3} \rho^i)^2 \sigma_e^2 + \cdots + (\sum_{i=0}^{1} \rho^i)^2 \sigma_e^2 + \sigma_e^2$$

Derivation of Equation (20):

$$E_{t^*-1}(\tilde{R}) = E(R_1 + R_2 + \cdots + R_{t^*-1} + \tilde{R}_{t^*} + \cdots + \tilde{R}_T)$$

$$= R_1 + R_2 + \cdots + R_{t^*-1} + c_{t^*} + c_{t^*+1} + \cdots + c_{T-1} + c_T + (\rho + \rho^2 + \cdots + \rho^{T-t^*} + \rho^{T-t^*+1}) v_{t^*-1}$$

$$= R_1 + R_2 + \cdots + R_{t^*-1} + c_{t^*} + c_{t^*+1} + \cdots + c_{T-1} + c_T + (\rho + \rho^2 + \cdots + \rho^{T-t^*} + \rho^{T-t^*+1})(R_{t^*-1} - c_{t^*-1})$$

$$= \sum_{i=1}^{t^*-1} R_i + \sum_{i=t^*}^{T} c_i + (R_{t^*-1} - c_{t^*-1}) \sum_{i=1}^{T-t^*+1} \rho^i$$

Derivation of Equation (22):

$$var_{t^*-1}(\tilde{R}) = var_{t^*-1}(R_1 + R_2 + \cdots + R_{t^*-1} + \tilde{R}_{t^*} + \cdots + \tilde{R}_T)$$

$$= (1 + \rho + \rho^2 + \cdots + \rho^{T-t^*})^2 var(v_{t^*}) + \left(1 + \rho + \cdots + \rho^{T-t^*-1}\right)^2 \sigma_e^2 + \cdots$$
$$+ (1+\rho)^2 \sigma_e^2 + \sigma_e^2$$

$$= (1 + \rho + \rho^2 + \cdots + \rho^{T-t^*})^2 \frac{\sigma_e^2}{1-\rho^2} + \left(1 + \rho + \cdots + \rho^{T-t^*-1}\right)^2 \sigma_e^2 + \cdots + (1+\rho)^2 \sigma_e^2$$
$$+ \sigma_e^2$$

$$= (\sum_{i=0}^{T-t^*} \rho^i)^2 \frac{\sigma_e^2}{1-\rho^2} + (\sum_{i=0}^{T-t^*-1} \rho^i)^2 \sigma_e^2 + (\sum_{i=0}^{T-t^*-2} \rho^i)^2 \sigma_e^2 + \cdots + (\sum_{i=0}^{1} \rho^i)^2 \sigma_e^2 + \sigma_e^2$$

Table 3.4.1 Estimated Coefficients for Different Segments

This table presents the estimated coefficients (c, ρ and σ²) for all four segments. *,**,*** correspond to statistical significance at the 10%, 5% and 1% level, respectively. Please note that there is no significant information for ρ.

	Seg1	Seg2	Seg3	Seg4
ρ	0.0245	-0.0234	0.0985	0.0316
σ²	3.67E-04***	9.48E-05***	6.47E-05***	1.60E-05***
c1	0.0137***	0.00172**	-0.0034***	-0.0053***
c2	0.0069***	-0.0002**	0.0001**	-0.0039***
c3	0.0157***	0.0052***	0.0035***	0.0005**
c4	0.0300***	0.0230***	0.0173***	0.0088***
c5	0.0300***	0.0230***	0.0137***	0.0091***
c6	0.0308***	0.0183***	0.0155***	0.0067***
c7	0.0213***	0.0170***	0.0124***	0.0064***
c8	0.0191***	0.0145***	0.0109***	0.0069***
c9	0.0133***	0.0127***	0.0105***	0.0071***
c10	0.0105***	0.0090***	0.0097***	0.0066***
c11	0.0106***	0.0096***	0.0076***	0.0043***
c12	0.0116***	0.0078***	0.0063***	0.0044***
c13	0.0059***	0.0064***	0.0069***	0.0039***
c14	0.0064**	0.0064***	0.0050***	0.0039***
c15	0.0072**	0.0063***	0.0046***	0.0028***
c16	0.0018***	0.0042***	0.0049***	0.0025***
c17	0.0030*	0.0032**	0.0030***	0.0023***
c18	0.0032*	0.0029**	0.0018*	0.0018***
c19	0.0023*	0.0039***	0.0027**	0.0016***
c20	0.0003*	0.0013*	0.0005*	0.0010*

Table 3.5.1 An Example of Data for Mean and Standard Deviation Calibration

This table presents the means and standard deviations of three consecutive cohorts (from Q1,2003 to Q3,2003) as an example of data collected leading to the result in Table 3.

	Default during Q1-2003	Default during Q2-2003	Default during Q3-2003
No. of facilities	15	18	10
Mean	0.3105	0.2998	0.2364
Standard Deviation	0.3665	0.3504	0.3625

Table 3.5.2 Mean and Standard Deviation Calibration Results

This table shows the mean and standard deviation calibration results for both the linear model and the quadratic model. The corresponding p-values are in parentheses. The R-squared are also reported in the last column. The first table includes all four segments, while the others only include data points from each segment.

All Segments	Intercept	Mean	$Mean^2$	R^2
Linear model	0.1568*** (< 0.0001)	0.7961*** (< 0.0001)	-	0.8486
Quadratic model	0.1163*** (< 0.0001)	1.3928*** (< 0.0001)	-1.7357*** (< 0.0001)	0.8817

Segment 1	Intercept	Mean	$Mean^2$	R^2
Linear model	0.2159*** (< 0.0001)	0.5507*** (< 0.0001)	-	0.5991
Quadratic model	0.0540 (0.3385)	1.9729*** (0.0002)	-2.9395*** (0.0044)	0.6927

Segment 2	Intercept	Mean	$Mean^2$	R^2
Linear model	0.1961*** (< 0.0001)	0.6211*** (< 0.0001)	-	0.5680
Quadratic model	0.1125* (0.0776)	1.6076** (0.0298)	-2.7997 (0.1683)	0.5940

Segment 3	Intercept	Mean	$Mean^2$	R^2
Linear model	0.1739***	0.7092***	-	0.5578

	(< 0.0001)	(< 0.0001)		
Quadratic model	0.09941** (0.0428)	1.7708*** (0.0100)	-3.391 (0.1052)	0.5943

Segment 4	Intercept	Mean	$Mean^2$	R^2
Linear model	0.1360*** (< 0.0001)	0.9006*** (< 0.0001)	-	0.7192
Quadratic model	0.1518*** (< 0.0001)	0.4494 (0.2387)	2.6669 (0.2204)	0.7327

Table 3.5.3 Mean and Variance of Recovery Rates of Benchmark and Unconditional Model

This table presents the mean and variance of the recovery rates of the benchmark and unconditional models used in portfolio risk simulations. The mean and variance for the benchmark model are constant across cohorts and time, whereas, for the unconditional model, the variances are decreasing from the most recently defaulted cohort (Cohort 1) to the earliest default cohort (Cohort 20) according to Equation (23).

		Benchmark Model			
	Cohort	Segment 1	Segment 2	Segment 3	Segment 4
Mean recovery rates	All	0.2441	0.1758	0.1306	0.0706
Variance of recovery rates	All	0.1245	0.0940	0.0727	0.0410
		Unconditional Model			
		Segment 1	Segment 2	Segment 3	Segment 4
Mean recovery rates	All	0.2441	0.1758	0.1306	0.0706
Variance of recovery rates	1	0.1245	0.0940	0.0727	0.0410
	2	0.1124	0.0848	0.0656	0.0370
	3	0.0910	0.0687	0.0531	0.0300
	4	0.0658	0.0496	0.0384	0.0217
	5	0.0421	0.0318	0.0246	0.0139
	6	0.0237	0.0179	0.0138	0.0078
	7	0.0116	0.0088	0.0068	0.0038
	8	0.0049	0.0037	0.0029	0.0016
	9	0.0018	0.0013	0.0010	0.0006
	10	0.0005	0.0004	0.0003	0.0002
	11	0.0001	0.0001	0.0001	0.0000
	12	0.0000	0.0000	0.0000	0.0000
	13	0.0000	0.0000	0.0000	0.0000
	14	0.0000	0.0000	0.0000	0.0000
	15	0.0000	0.0000	0.0000	0.0000
	16	0.0000	0.0000	0.0000	0.0000
	17	0.0000	0.0000	0.0000	0.0000
	18	0.0000	0.0000	0.0000	0.0000
	19	0.0000	0.0000	0.0000	0.0000
	20	0.0000	0.0000	0.0000	0.0000

Table 3.5.4 Snapshot Portfolio Used in Simulation

This table presents the default balances and the corresponding number of facilities that defaulted as of 2011Q4 in Panel A and Panel B, respectively. The portfolio composition information is presented in two dimensions: time after default (TIME) and risk segment (Segment1 to Segment 4). The TIME of 0.25Y denotes the cohort consisting of facilities that defaulted in the past three months (i.e., Cohort 1). On the other hand, the TIME of 5Y denotes the cohort consisting of facilities that defaulted from 4.75 years (57 months) to 5 years (60 months) ago (i.e., Cohort 20).

PANEL A: DEFAULT BALANCE (2011Q4 SNAPSHOT)				
TIME	Segment 1	Segment 2	Segment 3	Segment 4
0.25Y	$ 6,341,210	$ 5,505,888	$ 4,077,823	$ 2,597,039
0.5Y	$ 7,240,639	$ 3,623,269	$ 1,639,160	$ 147,552
0.75Y	$ 9,720,200	$ 2,024,387	$ 315,953	$ -
1Y	$ 12,258,295	$ 497,528	$ -	$ 4,430
1.25Y	$ 12,283,221	$ 146,911	$ -	$ -
1.5Y	$ 12,076,539	$ 316,195	$ -	$ -
1.75Y	$ 10,740,089	$ 133,186	$ 25,596	$ 25,819
2Y	$ 12,263,599	$ 92,334	$ 34,438	$ -
2.25Y	$ 11,341,960	$ 69,032	$ 4,912	$ 7,046
2.5Y	$ 8,051,168	$ 137,756	$ -	$ 143
2.75Y	$ 7,880,610	$ 82,063	$ 12,634	$ -
3Y	$ 6,127,697	$ 83,167	$ 9,259	$ -
3.25Y	$ 4,866,032	$ 45,887	$ -	$ -
3.5Y	$ 4,407,207	$ 36,404	$ -	$ -
3.75Y	$ 3,176,088	$ 31,020	$ -	$ -
4Y	$ 2,490,440	$ 27,901	$ 3,315	$ -
4.25Y	$ 1,927,589	$ 62,825	$ 3,435	$ -
4.5Y	$ 2,037,338	$ 15,301	$ -	$ -
4.75Y	$ 1,407,993	$ 68,491	$ -	$ -
5Y	$ 6,711,157	$ 145,890	$ 25,056	$ -

PANEL B: NO. OF ACCOUNTS (2011Q4 SNAPSHOT)				
TIME	Segment 1	Segment 2	Segment 3	Segment 4
0.25Y	490	269	191	129
0.5Y	416	159	84	8
0.75Y	521	111	21	0
1Y	672	36	0	1
1.25Y	585	10	0	0
1.5Y	624	23	0	0
1.75Y	579	13	2	1
2Y	608	12	4	0
2.25Y	424	8	1	1
2.5Y	403	10	0	1
2.75Y	329	6	1	0
3Y	303	6	1	0
3.25Y	245	4	0	0
3.5Y	232	4	0	0
3.75Y	166	2	0	0
4Y	149	2	1	0
4.25Y	128	3	1	0
4.5Y	110	3	0	0
4.75Y	90	8	0	0
5Y	433	22	4	0

Table 3.5.5 Realized Cohort Recovery Inputs for Conditional Model Simulation

This table presents the recovery rates of Segment 1 as of 2008Q1. Under Columns 1 to 19, we have the quarterly realized recovery rates for the 20 cohorts of Segment 1 that defaulted respectively from 2003Q2 to 2008Q1. Specifically, the recovery rates realized in the first quarter after the respective default date are presented under Column 1, whereas those realized in the 19th quarter after the default date are presented under Column 19.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	mean	Std
cohort1 (default 2008Q1)	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2436	0.3602
cohort2	0.0051	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2348	0.3552
cohort3	0.0103	0.0132	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2467	0.3618
cohort4	0.0026	-0.0129	0.0036	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2003	0.3312
cohort5	0.0373	-0.0155	-0.0022	-0.0052	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1907	0.3234
cohort6	-0.0133	0.0054	-0.0151	0.0598	0.0506	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2352	0.3554
cohort7	0.0003	-0.0080	0.0652	0.0400	0.0004	0.0065	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2202	0.3459
cohort8	0.0032	-0.0172	-0.0068	0.0723	0.0451	-0.0017	0.0482	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2390	0.3576
cohort9	0.0175	-0.0085	-0.0090	0.0861	0.0395	0.0463	-0.0014	0.0485	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2957	0.3804
cohort10	-0.0073	-0.0039	-0.0010	0.0368	-0.0054	0.0384	0.0431	0.0013	0.0010	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.1654	0.2999
cohort11	0.0203	0.0010	-0.0095	0.1284	0.0044	0.0000	0.0399	0.0431	0.0027	0.0057	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2881	0.3784
cohort12	0.0086	-0.0096	-0.0083	0.0443	0.0706	0.0178	0.0144	0.0371	0.0072	0.0398	0.0077	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2712	0.3729
cohort13	-0.0084	-0.0076	0.1308	0.0493	0.0000	0.0879	0.0081	0.0284	0.0000	0.0017	0.0013	0.0036	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.3249	0.3847
cohort14	0.0046	-0.0065	-0.0076	0.0508	0.0566	0.0569	0.0028	0.0535	0.0007	0.0025	0.0574	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.2959	0.3804
cohort15	0.0619	0.0352	0.0861	0.0125	0.0408	0.0052	0.0092	0.0150	-0.0018	0.0409	0.0004	0.0000	0.0382	0.0075	0.0000	0.0000	0.0000	0.0000	0.0000	0.3688	0.3818
cohort16	0.0695	0.0042	0.0084	0.0625	0.1217	0.0021	-0.0005	0.0040	0.0011	0.0051	0.0050	0.0073	0.0024	0.0104	0.0129	0.0000	0.0000	0.0000	0.0000	0.3269	0.3848
cohort17	0.0330	0.0869	-0.0021	0.0050	0.0000	0.0412	0.0031	0.0013	0.0351	0.0010	0.0015	0.0009	0.0004	0.0029	0.0020	0.0004	0.0000	0.0000	0.0000	0.2214	0.3467
cohort18	0.0121	-0.0051	-0.0104	0.0952	-0.0027	0.0865	0.0010	0.0010	0.0000	0.0000	0.0044	0.0119	0.0074	0.0042	0.0061	0.0040	0.0039	0.0000	0.0000	0.2251	0.3492
cohort19	0.0629	-0.0069	0.0508	0.0003	0.0050	0.0299	0.0036	0.0092	0.0060	0.0075	0.0093	0.0765	0.0342	-0.0002	0.0028	0.0019	0.0033	0.0028	0.0000	0.3014	0.3816
cohort20 (default 2003Q2)	-0.0020	-0.0056	0.0230	-0.0017	-0.0030	0.0655	0.0713	0.0364	0.0000	0.0000	0.0549	0.0644	0.0000	0.0073	0.0000	0.0000	0.0000	0.0000	0.0000	0.3107	0.3832

Table 3.5.6 Simulation Setup for Three Different Models

This table presents the key differences between the means and variances of each cohort for the three models under consideration.

	Mean	Variance
Benchmark Model	the universal mean of mean recovery (constant across cohort and overtime)	the universal variance of mean recovery (constant across cohort and overtime)
Unconditional Model	the universal mean of mean recovery (constant across cohort and overtime)	the universal variance of mean recovery (decrease in quadratic fashion constant over time)
Conditional Model	dynamic	dynamic

Figure 3.2.1 Cohort Formation Process

This figure illustrates the timeline of the first three cohorts of defaulted facilities in our sample data. Twenty quarter-end recovery amounts are collected for each cohort. Cohort 1 contains the facilities that defaulted in 2003Q1 (i.e., the quarter ending April 2003) and their recovery information is collected until 2008Q1 (i.e., the quarter ending April 2008). Cohorts 2 and 3 consists of those facilities that defaulted respectively in 2003Q2 and 2003Q3. Their quarterly recovery values are tracked from their default quarters up to the end of 2008Q2 and 2008Q3, respectively.

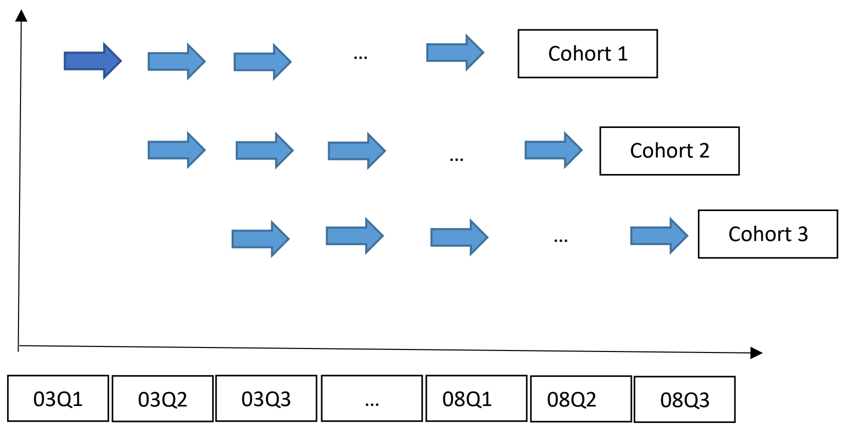


Figure 3.2.2 Quarterly Recovery Rates of Cohort 1 in Segment 4

This figure presents the quarterly average recovery rates of Cohort 1 in Segment 4 of our sample of defaulted retail facilities. The recovery rate is expressed as a fraction of the loan balance. Cohort 1 consists of those facilities defaulted in the quarter ending April 2003 (i.e., 2003Q1). Average recovery rates of the facilities are observed on a quarterly basis from the first quarter after default (i.e., rec_1) to the 20th quarter (i.e., rec_20) after default. We therefore assume a 5-year recovery process.

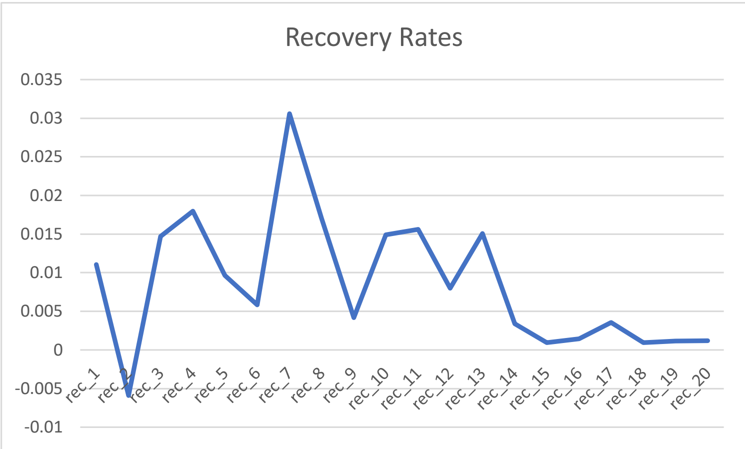


Figure 3.2.3 Segment-Level Recovery Rates

This figure presents the average recovery information in two dimensions: time and segment. This figure includes the average recovery amounts for all four segments and over twenty quarters after default.

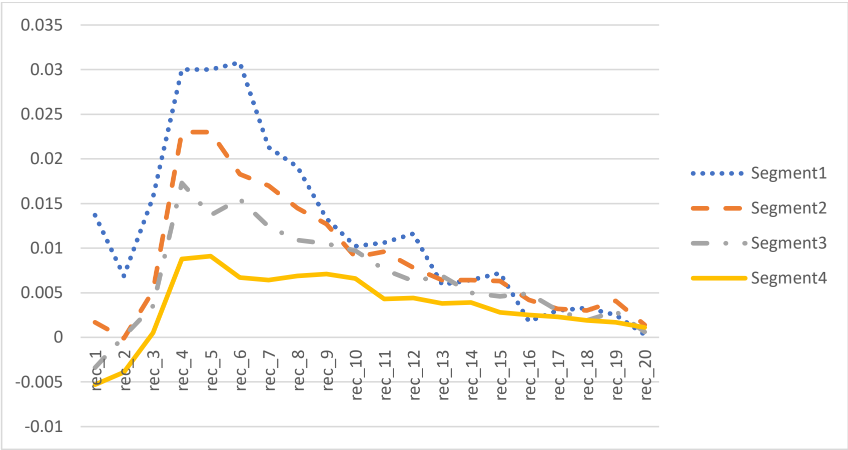


Figure 3.4.1 PIT Expected Overall Recovery Rates of a Cohort

This figure presents the updating of the expected overall recovery rate for a cohort of Segment 1 that defaulted in 2006Q3 based on the proposed conditional model. The resultant PIT expected recovery rates of the cohort over the sample period from 2006Q3 to 2011Q3 are plotted, together with the levels of the TTC (i.e., unconditional) expected value and the actual realized overall recovery rate indicated.

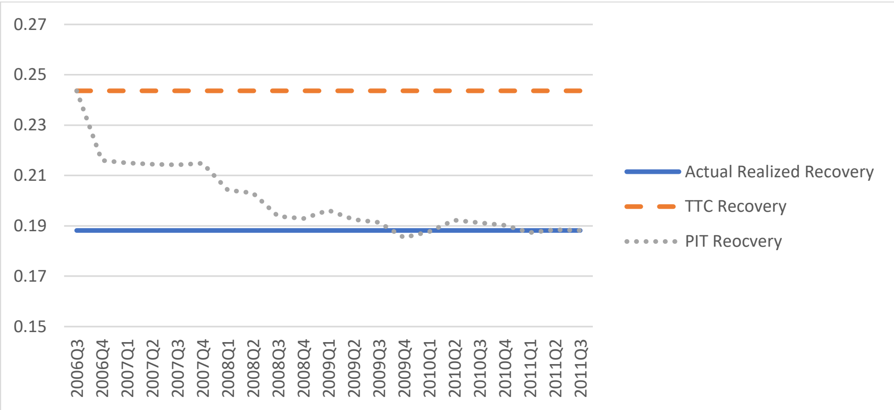


Figure 3.5.1 Mean and Variance Calibration

This figure shows the linear and quadratic regression results for the overall portfolio (i.e., all segments). The standard deviations are regressed against the corresponding means as shown in Table 3.

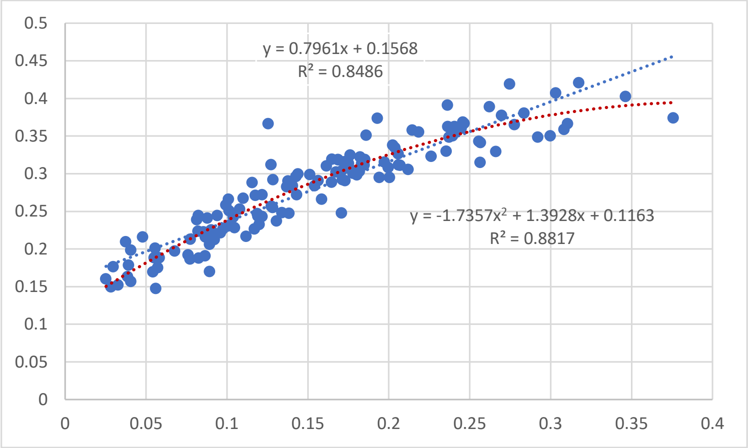


Figure 3.5.2 Simulation Procedure

This figure describes the simulation process for each cohort. Each cohort has a unique LGD beta distribution based on its recovery mean and recovery variance. Each facility is assigned a random draw within the corresponding LGD beta distribution to generate loss per facility together with default balance and the number of facilities. The facility-level loss amounts are aggregated to arrive at the portfolio-level loss rate. By repeating 200,000 times, we obtain the empirical distribution of the portfolio loss rate to calculate VaR and economic capital.

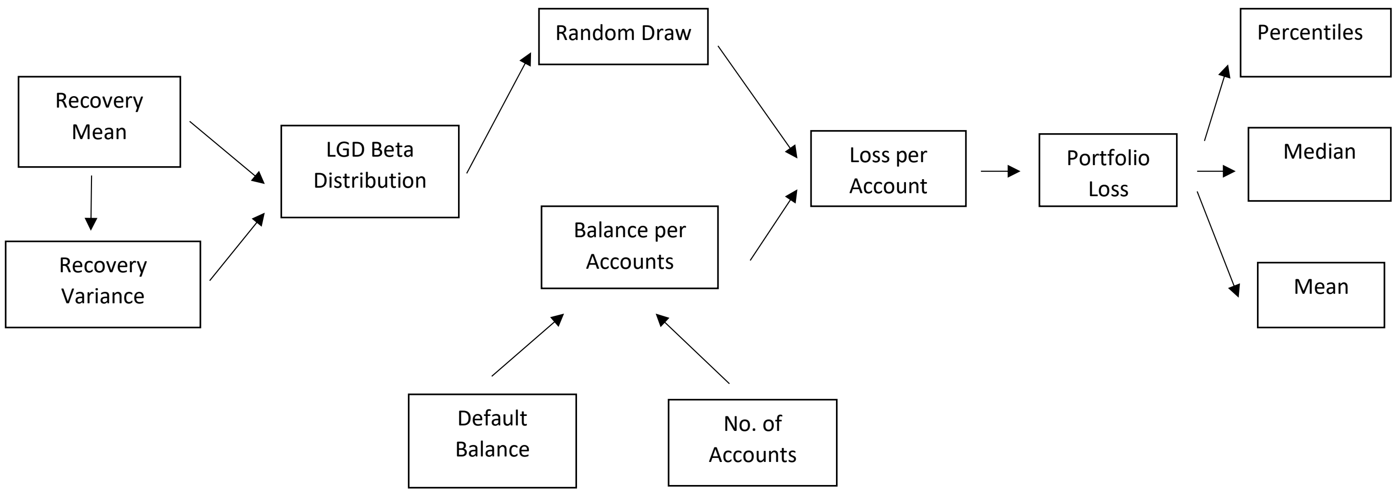
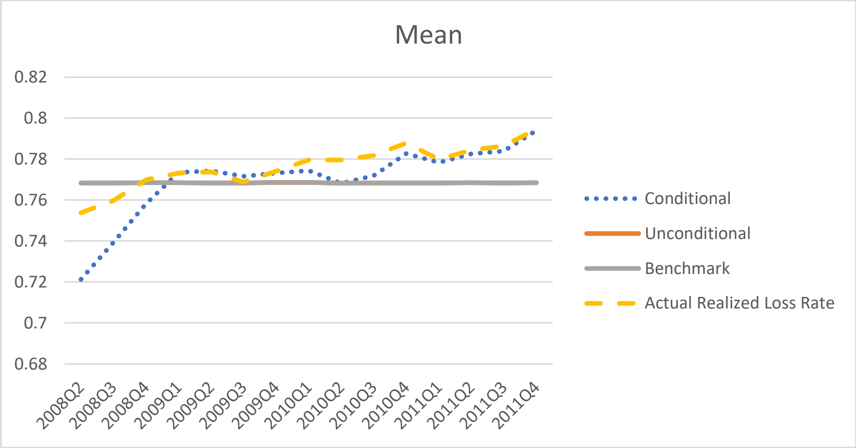


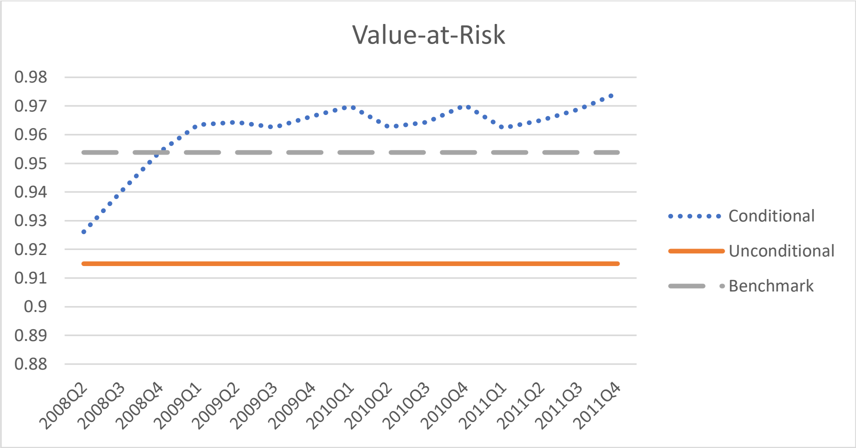
Figure 3.5.3 Simulation Results

This figure shows the expected portfolio loss rate, the value-at-risk (VaR), and the economic capital estimated by the three models (*Benchmark*, *Unconditional*, *Conditional Models*) based on the simulation results over the sample period from 2008Q1 to 2011Q3.

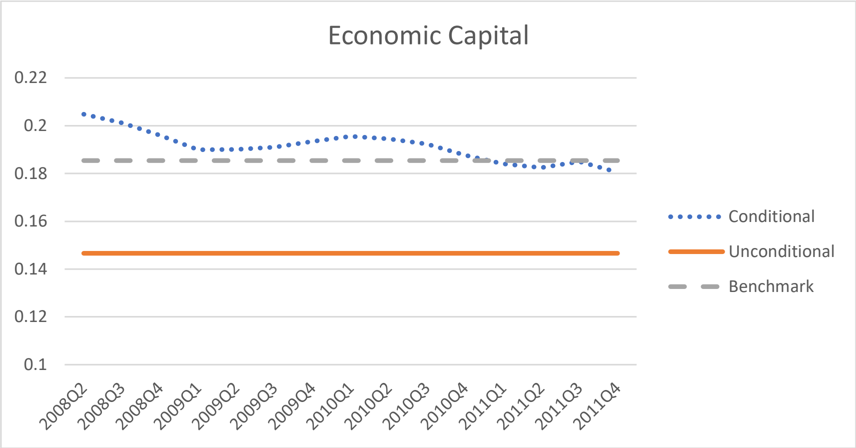
Panel A: Expected loss



Panel B: Value-at-risk



Panel C: Economic capital



Chapter 4

Estimating Loss-Given-Default by Mixture Beta Regression Model

4.1 Introduction and Literature Review

Credit risk is one of the most important risks financial institutions are facing in the modern financial world. Probability of Default (PD), Loss Given Default (LGD) and Exposure at Default (EAD) are the three key determinants in defining the credit risk of corporate bonds and loans. PD is used to describe the likelihood that a borrower will be unable to meet its debt obligation over a particular time horizon and EAD is the total value a lender is exposed to when a default happens. The focus of this paper is LGD. By definition, LGD is the share of an asset that is lost when a borrower defaults, measuring the severity of the credit loss given the default event. Under Basel II for financial institutions, it is also an important parameter in credit risk modeling and in assessing capital requirements. Recovery Rate (RR) measures the amount recovered by the creditor from default as a fraction of the outstanding loan balance. It can therefore be expressed as one minus LGD. Summing up to one, RR and LGD carry exactly the opposite economic meaning as discussed in this paper. In practice, RR is the discounted value of the net cash flow received by the creditor minus any loss or costs incurred during the collection process expressed as a fraction of the exposure at default. For tradeable instruments (e.g., corporate bonds), the discounted market value at resolution or the price of traded debt at default can also be used as a proxy of RR.

Different from the Probability of Default (PD), which has been extensively studied in both the academic and practitioner literature, LGD attracts relatively less attention from researchers. The present study contributes to our understanding of the time-series behavior of the distribution of LGD. Traditionally, in credit risk modeling, the focus has been primarily on the modeling of the probability of default (PD), while the recovery rate (or LGD) is assumed to be a constant without necessarily recognizing its potential variation (both cross-sectionally and over time). As the understanding and modeling of LGD are still in their infancy and far from satisfactory, its research is still on the way to recovery. Previous studies on LGD mainly focus on analyzing the factors influencing LGD, the relationship between LGD and PD, and the probability density function of LGD. The modeling of LGD is intriguing and can also be challenging given that the distribution of LGD is quite different from a normal distribution that is commonly used for statistical models. First, the LGD distribution is supposed to be bounded between 0 and 1. Second, it is not uncommon to encounter bimodal LGD distribution that exhibits fat tails.

Previous studies on credit risk can be broadly classified into two categories: theoretical papers dealing with credit risk modeling and empirical studies analyzing past defaults. The two main credit risk modeling approaches are the structural model starting from Black and Scholes (1973) and Merton (1974) and the reduced-form model pioneered by Jarrow and Turnbull (1995) and Duffie and Singleton (1999). The former assumes that default is driven by the process generating the value of the borrower's asset. When the asset value is lower than the firm's debt level at maturity, default occurs and the residual asset value defines the rate of recovery. In other words, since the payment of the debt at maturity is the smaller of the two: the residual value of the firm or the face value of its debt, so the payoff of debt is equivalent to the face value of the debt minus a put option whose strike price is equal to the face value of the debt and maturity is equal to the maturity of its debt. Based on this framework, Merton estimates the PD and LGD of debts. Researchers have since extended the structural model by relaxing some of the unrealistic assumptions made by Merton. For example, the Black-Cox model (Black and Cox, 1976) extends the Merton framework by allowing intermediate default and the Geske model (Geske, 1979) applies the framework to interest-paying corporate bonds. Under the structural model, LGD is an endogenous variable and is positively related to the probability of default (PD). Despite the fact that the structural model provides an intuitive way to model both PD and LGD within a consistent framework, it still suffers from several drawbacks. First, as one of the most important parameters in the model, the firm's value is not directly observable and its estimation is far from trivial. Second, Merton's model cannot handle a complex capital structure. Last, the assumption of continuous firm value is at odds with reality.

Contrary to the structural model, the reduced form credit risk model (e.g., that of Jarrow and Turnbull, 1995, and Madan and Unal, 1998) does not explicitly condition default on the firm's capital structure, rather, it allows separate assumptions regarding PD and LGD making recovery rate an exogenous variable. The reduced form approach models the default event as a statistical process using the hazard rate to represent PD in a continuous-time framework and recovery is usually treated as another input of the model. For example, in Jarrow and Turnbull (1995), default is modeled as a Poisson process stopping at the first jump. There are some empirical studies of credit derivatives basing on the reduced-form model. For example, Chen, Cheng, Fabozzi and Liu (2008) provide an explicit solution to the valuation of credit default swap based on the reduced-form model. The recovery can be assumed to be zero, a constant, or to follow a stochastic process. There are in general two methods to parameterize the recovery rate in the reduced-form approach: as a fraction of the market value (Duffie and Singleton, 1999) or as a fraction of the face value. The former assumes that, in case of the default, creditors are compensated based on the rest of the market value of the risky bonds, while the recovery value is based on the face value of the bonds in the latter. These are the extensions of the reduced-form model following Jarrow and Turnbull coming up with different approaches in the modeling of recovery rates. Since PD and LGD in the reduced form framework are independent of each other and not necessarily related to the firm value as that in the structural model, it overcomes some of the disadvantages of the structural model mentioned above, e.g. the unobserved asset value. However, since there is a lack of an underlying economic model (like Merton's firm value and shareholder model), we cannot intuitively explain the behavior of the observed credit risk measures (e.g., the term structure of credit spread and the correlation of PD and LGD) by using the reduced form model.

There is an active and growing literature on the modeling of LGD. Through empirical works investigating past default events, researchers have a better understanding of the stochastic nature of LGD in their endeavor to improve the modeling of LGD. Altman et al.

(2005), Grunert and Weber (2008) investigate the factors that may affect LGD, confirming that PD, the size of the company, the intensity of the client relationship, and the creditworthiness of the borrower all play a pivotal role in dictating LGD. Altman et al. (2005), Hu and Perraudin (2002), and Rosch and Scheule (2005) document a positive relationship between PD and LGD. Creditors, therefore, expect to recover less the higher the default probability of the debtors. If this relationship is ignored in credit risk modeling, a financial institution might underestimate the expected loss of its credit portfolios. In turn, its capital requirement will also be underestimated. Besides, some studies focus on examining the statistical properties of LGD, for example, Gert (2014) investigates different techniques of backtesting LGD. More recently, researchers have been adopting unconventional approaches and parameters in modeling LGD, e.g., Heng (2018) considers time-to–recovery as an explanatory factor of LGD.

Another strand of LGD research is on its distribution. Earlier researchers adopt different variations/transformations of the normal distribution. For example, the logit-normal distribution model by Dullmann and Trapp (2004), and by Rosch and Scheule (2005), the normal distribution model by Frye (2000), log-normal distribution model Pykhtin (2003) and the probit-normal distribution model by Anderson and Sidenius (2003) are introduced. Dullmann and Trapp (2004) compare the performance of the normal model, log-normal model, and the logit-normal model using S&P's LGD data from 1982 to 1992. Judging from the *p*-values of Sharpiro-test and Jarque-Bera-test, they find that the log-normal model is inferior to the other two models in fitting the LGD distribution. The normal distribution model by Frye assumes that recovery is a linear function of the normal risk factor associated with the Vasicek distribution. This idea is further expanded in log-normal and probit-normal models to accommodate the LGD specifications. These assumptions are used in modeling recovery rates to avoid downward-biased estimation of economic capital.

By far, the beta distribution is the most popular class of distribution used in the modeling of LGD. Beta distribution has been widely used in the industry in estimating LGD (e.g., in Moody's *Losscalc*, J.P. Morgan's *Credit Metrics*, and KMV's *Efficient Frontier*). The main advantage is that it has the support of [0, 1]. Besides, with its two shape parameters α and β, it is quite flexible in fitting the commonly observed LGD distribution shapes as depicted in Figure 4.1.1 (e.g., bell shape, U-shaped, J-shaped, left-skewed, and right-skewed shapes).

INSERT FIGURE 4.1.1 ABOUT HERE

Adopting the beta distribution within the regression model framework, Huang and Oosterlee (2012) propose the Generalized Beta Regression model (GBR) of LGD. The idea is to utilize a monotonic, differentiable link function and a linear combination of predictors to model the mean and variance of the LGD distribution. Potential predictors can be macroeconomic variables or firm-level variables capturing the characteristics of the underlying assets. The link functions can be logit or probit functions. Either the least-squares method or the maximum likelihood estimation (MLE) method can be used to estimate the model parameters. Jacob and Ahmet (2011) also develop a simultaneous equation model based on the beta-link generalized linear model (BLGLM), which can be considered as an extension of the GBR framework by using a mixture of cumulative beta distributions as the link function. The use of a mixture of beta distributions has proven to be a worthwhile extension in modeling LGD since it has been shown that the single-beta distribution is not flexible enough to accommodate the LGD distributions encountered in practice. As pointed out by Schuermann (2004), single-beta distribution cannot model the bimodal and the fat tail of the LGD distribution sometimes observed in practice. The research by Renault and Scaillet (2004), using the S&P's default data observed between 1981 and 1999, suggests that LGD distribution is far from a (single) beta distribution. No matter how we change the two shape parameters of the beta distribution, we cannot replicate the observed bimodal shape. We admit that it is easier for researchers to fit LGD distribution by using just two parameters with beta distribution, but such convenience is at a cost of losing the accuracy in fitting the observed LGD data. The resulting errors, especially in the tails of the LGD distribution, could be too large to be ignored by financial institutions in assessing their capital requirements.

The use of a mixture of distributions provides the modeler the flexibility to capture different features of a data sample that consists of multiple components with different types of distribution. It thus facilitates the detailed description of complex data systems. Mixture distributions have been used in answering research questions in diverse areas, such as ecology, bioinformatics, astronomy, computer science, economics, engineering, robotics, and biostatistics. For instance, in genetics, the location of quantitative traits on a chromosome and interpretation of microarrays both relate to mixture distributions. While in computer science, spam filters and web context analysis start from a mixture assumption to distinguish spams from regular emails and to group pages by topics. Specifically, Ji et al. (2005) propose a beta-mixture model approach in solving a variety of problems in bioinformatics related to a large number of correlation coefficients. In their experiments, the subsamples of a single variable behave differently. The finite mixture model is typically used to analyze data of this type. Recently, researchers have applied different kinds of mixture models in addressing research questions in economics and finance. For example, Andreas and Ulrich (2009) investigate the viability of finite mixtures of Gaussians to model marginal distributions of the stock market in some sub-periods. Roman (2012) also proposes a mixture model to explain the behavior of daily price changes and trading volume on the financial market.

Given the restrictions imposed by the single-beta distribution in modeling LGD and the unrealistic implications of such a model as mentioned earlier, we propose a mixture beta regression LGD model that accommodates the dynamic changes in the LGD distribution over the business cycle as defined by various macroeconomic variables. In formulating our mixture beta model, we are motivated by the time-varying bimodal distribution of recovery rate as observed in practice. These different shapes of bimodal distributions can be seen as a combined distribution of LGD with time-varying weights over the business cycle. To capture this facet of the LGD distribution, unlike the commonly used beta-linked regression models where the shape parameters of the beta distribution vary with the underlying variables (e.g., Simas and Rocha, 2010), we model the probability weights of realizing different beta distributions as functions of the underlying variables. Through extensive empirical analysis, we demonstrate that our proposed model can outperform the commonly used models in both in-sample and out-of-sample settings, resulting in a more accurate fitting of the observed time-varying LGD distribution.[21]

In this study, we contribute to the literature on LGD modeling in a number of ways. First, we propose a new dual-beta regression LGD model that considers the probability weights of realizing the two distributions as functions of macroeconomic variables. The general five-factor model and a simplified three-factor model are introduced. Second, with an extensive dataset on the recovery values of corporate defaults, we estimate the proposed models with a number of different macroeconomic variables that are expected to be associated with the recovery value. We conduct both in-sample and out-of-sample tests and demonstrate the superior performance of our proposed mixture beta distribution regression model when compared with the commonly-used single-beta logit-link regression models. Third, we demonstrate how our proposed mixture distribution model can capture the time-varying behavior of the LGD distribution and how the probability weights assigned to the two underlying beta distributions vary with the business cycle. We find that our proposed models perform better in predicting the LGD distribution during recessionary periods.

The rest of the paper is organized as follows. Section 4.2 elaborates on the mixture beta distribution framework, including the parameter estimation methodology and the regression setup. In section 4.3 we present the details of data that contains the recovery

[21] We are not the first to apply a mixture distribution model in examining recovery rates. Altman and Kalotay (2010) introduce a Bayesian approach to model the distribution of the discounted ultimate recoveries on defaulted debts using mixtures of normal distributions. Specifically, they model recovery rates using a weighted mixture of several normal distributions and show that the technique is flexible enough to accommodate important idiosyncratic features of recovery distributions. The problem with this approach is that the support of the mixed normal distribution is $[-\infty, \infty]$, which is not realistic for LGD.

rates and macroeconomic variables that are involved in this paper. Section 4.4 shows the empirical results based on the mixture beta distribution framework and then compare the performance of the proposed models with that of the commonly used single-beta regression model. The last section of Chapter 4 summarizes the important conclusions of this essay.

4.2 Methodology

4.2.1 Beta Distribution of LGD

As mentioned in the introduction and literature review section, the beta distribution has been widely used in the modeling of LGD. It is by far the most popular distribution assumption adopted by researchers in studying recovery rate and LGD in the literature. It is also commonly used in practice in the management of default risk by financial institutions. A static beta distribution assumption is utilized in popular credit risk models, like J.P. Morgan's Credit Metrics and KMV's Portfolio Manager, in the modeling of LGD.

The beta distribution is a two-parameter distribution and its probability density function $f(x)$ can be expressed as:

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad (1)$$

where $B(\cdot)$ denotes the beta function and $\Gamma(\cdot)$ denotes the gamma function. It has support [0, 1]. The two parameters, α and β, are referred to as the *shape* parameters of the beta distribution. With both α and β greater than 0, we can ensure a proper probability density function. The mean and variance of variable x are given by:

$$\mu = E[x] = \frac{\alpha}{\alpha+\beta} \tag{2}$$

$$\sigma^2 = var[x] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\mu(1-\mu)}{\alpha+\beta+1} \tag{3}$$

We can also define a *dispersion parameter*, φ, as the sum of α and β as it related to the variance in Equation (2). The two shape parameters can then be expressed as functions of μ and φ.

$$\alpha = \mu\varphi \tag{4}$$

$$\beta = (1-\mu)\varphi \tag{5}$$

By varying the two shape parameters, we can arrive at different distribution shapes, e.g., bell shape, U-shaped, J-shaped (as depicted in Figure 4.1.1), in modeling LGD.

Using the single-beta distribution, Giese (2006) and Bruche (2010) study LGD by modeling the shape parameters, α and β, as functions of different explanatory variables. For example, Bruche (2008) first specifies the arrival of defaults by assuming a discrete hazard rate function in the form of:

$$\lambda_t = [1 + \exp(\gamma_0 + \gamma_1 X_t)]^{-1} \tag{6}$$

where X_t are credit variables or macroeconomic variables. And then the random recovery value is drawn from a beta distribution, of which the shape parameters depend on the same set of variables that are governing the hazard rate. Finally, to ensure the positivity of α and β, an exponential specification is imposed on them. With this joint modeling of hazard rate and recovery value, Bruche confirms the dependency of PD and LGD and proposes an econometric model allowing for the time variation in PD and LGD.

The Generalized Beta Regression (GBR) model extends the idea and provides a different parameterization of the beta distribution. To be specific, rather than modeling α and β directly, GBR models the mean and standard deviation of beta distribution with credit variables and macroeconomic variables. This approach can provide a more intuitive interpretation of the parameters. Here we provide a detailed description of the GBR framework as proposed by Huang and Oosterlee (2008). The model proposed by them is, hereafter, referred to as the *logit-link beta regression model*. The shape parameters of the beta distribution are formulated as follow,

$$\alpha = \mu\varphi$$

$$\beta = (1-\mu)\varphi$$

Then a monotonic, differentiable link function, e.g. a logit function is used to connect the linear predictors η and the mean,

$$\mu = \frac{e^{\eta}}{1+e^{\eta}} \tag{7}$$

which guarantees a mapping to [0,1] for LGD. Here, the mean can be modeled by linear regression functions that are driven by selected macroeconomic factors and the dispersion parameter is considered to be a nuisance parameter. Jacob (2011) discusses some more complicated linked functions and recommends using beta distribution as opposed to Gaussian distribution. The motivation of using beta distribution here is to restrict the recovery rate within the range of (0,1). Another contribution of Jacob's paper is that it examines a wide range of firm-level and instrument level variables that potentially affect LGD.

As mentioned in the introduction section, some of the previous studies mentioned the limitations of current models when fitting the bimodal LGD. This motivates us to use a mixture beta distribution as a potential alternative to accommodate such difficulties.

4.2.2 Mixture Beta Regression Model

4.2.2.1 Mixture of Beta Distributions

To overcome the shortcomings of the single-beta distribution in accommodating different kinds of multi-modal LGD distribution observed in practice, we propose the use of a mixture of distributions. A mixture distribution can be expressed as a weighted linear combination of a number of (say n) probability density functions $p_1(x)$, $p_2(x)$, $\cdots$, $p_n(x)$. Specifically, the probability density function $f(x)$ of the mixture distribution can be expressed as:

$$f(x) = \sum_{i=1}^{n} \omega_i p_i\,(x)$$

where ω_i $(i = 1\ to\ n)$ denotes the weight assigned to the underlying probability distribution i. The weight ω_i can be constant or vary over time with some co-variates. Although the underlying $p_i(x)$ are individually probability density functions, a general linear combination of the n probability density functions does not necessarily give us a proper probability density function, since it is possible to be negative or it may integrate to some values other than 1. To ensure we will arrive at a proper density function for the mixture distribution, we will focus on a convex combination of probability density functions, where $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$.

Unlike in some of the previous studies that consider a mixture of normal distributions, we examine a mixture of beta distributions with bounded support from 0 to 1, which is most appropriate to represent the fractional amount of default loss out of, for example, a fixed par value of a defaulted bond. Although in our subsequent analysis, we focus on a mixture of two beta distributions (i.e., *n* =2), the framework can be readily extended to incorporate more than two underlying beta distributions. As we will demonstrate in our empirical analysis (see Section 4), a dual-beta distribution is flexible enough to capture the dynamic changes in LGD distribution over the business cycle that cannot be accommodated by a single-beta approach. To ensure we have a proper mixture distribution density, we restrict that the two weights, ω_1 and ω_2, add up to one by introducing a single weight parameter ρ ($0 \leq \rho \leq 1$), where $\omega_1 = \rho$ and $\omega_2 = 1 - \rho$. The probability density function of our mixture beta distribution can therefore be expressed as:

$$f(x) = \rho \frac{1}{B(\alpha_1, \beta_1)} x^{\alpha_1 - 1}(1-x)^{\beta_1 - 1} + (1-\rho)\frac{1}{B(\alpha_2, \beta_2)} x^{\alpha_2 - 1}(1-x)^{\beta_2 - 1} \qquad (8)$$

where $0 \leq x \leq 1$, $B(\alpha_1, \beta_1)$ and $B(\alpha_2, \beta_2)$ are two beta functions. Here, $f(x)$ is the mixed probability density function based on two different beta distributions and the parameter ρ governs how much weight we assign to the two distributions. If ρ is larger (smaller) than 0.5, we put more weight on the first (second) beta distribution with shape parameters α_1 and β_1 (α_2 and β_2). Obviously, $f(x)$ meets all the requirements of being a probability density function with five parameters, α_1, $\beta_1, \alpha_2, \beta_2,$ and ρ, that need to be estimated. When ρ is 0 or 1, $f(x)$ becomes the probability density function of a (single) beta distribution. Thus, the beta distribution is a special case of the mixture beta distribution model. With three more parameters, the mixture beta distribution is a generalized version of single beta distribution with only two shape parameters. Hereafter, we refer to this generalized mixture beta distribution as our *five-factor* mixture beta distribution model, given that it is governed by a total of five parameters. In Figure 4.2.1, we plot the probability distribution density of our mixture beta distribution with $\alpha_1 = 4$, $\beta_1 = 10, \alpha_2 = 8, \beta_2 = 3,$ and $\rho = 0.33$. In Figure 4.2.2, we plot the probability density separately for the two underlying beta distributions. The first (Beta distribution 1) with shape parameters $\alpha_1 = 4$ and $\beta_1 = 10$. The second (Beta distribution 2) with shape parameters $\alpha_2 = 8$ and $\beta_2 = 3$. In Table 4.2.1, we present the model-implied mean LGD, standard deviations of LGD and the corresponding dispersion factors of the two individual beta distributions and the mixture beta distribution. With a lower mean LGD of 0.286, Beta distribution 1 can be interpreted as a “Good” LGD distribution, representing the distribution under a good state of the economy. On the other hand, with a higher mean LGD of 0.727, Beta distribution 2 can be interpreted as a “Bad” LGD distribution, representing the distribution under a bad state of the economy. By varying the weight ρ, we can construct mixture beta distributions that represent different degrees of resemblance to the “Good” versus the “Bad” distribution. For example, when the economy is recovering from a downturn, we naturally expect a gradually larger (smaller) weight to be assigned to the “Good” (“Bad”) distribution. This enhances the flexibility in the modeling of time-varying LGD distribution and allows us to generate bimodal LGD distributions (see Figure 4.2.1) that we observe in practice, which cannot be replicated by using single beta distribution.

INSERT FIGURES 4.2.1 and 4.2.2, AND TABLES 4.2.1 ABOUT HERE

A special case of the generalized mixture beta distribution model introduced above can be obtained by restricting $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_1$. These restrictions bring us the following mixture beta distribution.

$$f(x) = \rho \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} + (1-\rho)\frac{1}{B(\alpha,\beta)} x^{\beta-1}(1-x)^{\alpha-1}, 0 \leq x \leq 1 \quad (9)$$

This is a simplified version of the above *five-factor* model. Hereafter, we refer to it as our *three-factor* mixture beta distribution model, since it is governed by three parameters, α, β, and ρ. In simplifying the mixture model, we are giving up some flexibility in fitting the observed LGD distribution comparing to the five-factor model. Nevertheless, the three-factor model is still able to meet the bi-modal LGD distribution requirement. To demonstrate that, in Figure 4.2.3, we plot the distribution density function of a three-factor model with $\alpha = 4.0$, $\beta = 1.8$, and $\rho = 0.35$. By setting $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_1$, the two underlying beta distributions, as depicted in Figure 4.2.4, are in fact mirror images of each other. The statistics of the two underlying beta distributions (Beta distribution 1 and Beta distribution 2) and the three-factor mixture beta distribution are presented in Table 4.2.2. With a lower (higher) mean LGD, Beta distribution 1 (Beta distribution 2) can therefore be interpreted as the “Good” (“Bad”) distribution representing the good (bad) state of the economy. It is important to point out that the restrictions imposed in obtaining the three-factor model result in the mixture beta distribution having the same standard deviation and dispersion factor as the two underlying beta distributions.

INSERT FIGURES 4.2.3 AND 4.2.4, AND TABLE 4.2.2 ABOUT HERE

4.2.2.2 Time-Series Model of LGD

Previous empirical studies tell us that LGD distribution does not remain static over time. How much a creditor can recover from a defaulted debtor is expected to be highly procyclical. The creditor tends to recover less in, say, liquidating the collateral when the workout process coincides with a recessionary period of the business cycle; whereas the recovery value is likely to be higher under a booming economy. We, therefore, expect LGD to vary over time in a counter-cyclical fashion. In this section, we would like to develop a model that can systematically explain/predict the time-series variations of our mixture beta distributions of LGD, as introduced in the previous section. It is important to note that any time-series variation does not only involve the first (i.e., the mean) and the second (i.e., the standard deviation) moments of the LGD distribution, but also the *shape* of the overall distribution. In some credit risk management applications, we can be more concerned about the *tails* of the LGD distribution, which are heavily related to the shape of the distribution, than the central tendency of the distribution. For example, the value-at-risk of a credit portfolio is governed by extreme tail events. It is therefore important to adopt a modeling approach that will allow us to capture how the shape of the distribution might vary over time.

In the literature, researchers incorporate macroeconomic variables in their models by using different link functions e.g., the logit function, beta distribution, and the Gaussian distribution. For example, in the GBR approach, the models' LGD mean and dispersion is logit or probit functions of the explanatory variables; whereas Bruche (2010) directly models the shape parameters of the beta distribution instead. In our mixture beta distribution setup, there are multiple ways that we can incorporate time-series explanatory variables (e.g., GDP growth rate, stock market return, etc.) in driving the dynamic changes in the LGD distribution. Similar to previous studies, we can model the shape parameters α and β as functions of the explanatory variables. Alternatively, we can directly model how the mean and standard deviation of the distribution change over time. In addition, our mixture distribution model offers us a unique way to model the effects of the covariates via the weights ρ and $1-\rho$ assigned to the two underlying beta distributions. In this third modeling approach, we essentially assume that the chance of realizing the first vs. the second beta distribution varies with the explanatory variables. As explained earlier, the two underlying beta distributions can be interpreted as representing, respectively, the "Good" vs. "Bad" economic conditions, with the former (latter) LGD distribution characterized by a lower (higher) means and a heavier left (right) tail. As the weights on the two underlying beta distributions vary with the predictor variables over time, the overall shape of the mixture distribution also changes over time. It is this third approach that we are taking in developing our time series model. Specifically, we model our weight parameter ρ as a logit function of a linear combination of explanatory variables Y_t specified as:

$$\rho_t = 0.5 + \frac{0.5}{1+e^{Y_t\gamma+\varepsilon_t}} \quad (10)$$

where γ is a vector of coefficients of the explanatory variables to be estimated, and ε_t is the residual. This specification ensures that $0.5 \leq \rho \leq 1.0$, and thus $0 \leq 1 - \rho \leq 0.5$. In other words, the weight on the first (second) underlying beta distribution is between 0.5 and 1.0 (0 and 0.5). This specification does not impose any restriction on our mixture beta distribution since we do not restrict the relative values of the shape parameters α_1, β_1, α_2, and β_2 of the two beta distribution. When $\rho = 1.0$, the mixture beta distribution degenerates into a single beta distribution. When $\rho = 0.5$, we assign equal weights to the two beta distributions. We contribute to the literature by examining the channel through which the explanatory variables drive the LGD distribution via the weight parameter of the mixture distribution. This approach facilitates the interpretation of how the systematic factors govern the relative importance of the two underlying beta distributions, with which we can model the dynamics of the recovery rates over the business cycles. Specifically, we expect the underlying beta distribution that represents the "Bad" economic condition to be weighted heavier during a downturn than during an expansionary phase of the cycle, vice versa for the other underlying beta distribution representing the "Good" economic condition.

Finally, it is important to note that the proposed approach of incorporating the time-series variables via the weight parameter is applicable to both the three- and five-factor mixture distribution models outlined in Section 2.2.1. In Section 4, based on the above time-series model, we will test the explanatory power of a number of commonly considered independent variables, e.g., GDP growth rate and unemployment rate, in explaining the systematic variation of LGD. We will also compare the performance of our proposed model with other models considered in the literature. Before we discuss the empirical results, in the next section, we first outline the model estimation methodologies involved.

4.2.3 Parameter Estimation

After setting up the mixture beta distribution model, the estimation of parameters is described in this section. There are two different approaches to estimate the models – *method of moments* and *maximum likelihood estimation*.

4.2.3.1 Method of Moments Approach

In the method of moments approach, we estimate the model parameters by matching the moments implied by the model and the empirical moments based on the observations. For our general five-factor mixture beta distribution, there are altogether five parameters (α_1, $\beta_1, \alpha_2, \beta_2,$ and ρ) to be estimated by matching the first five moments of the LGD distribution. The five model-implied moments can be expressed as:

$$E\left(x^j\right) = \int_0^1 x^j f(x)dx, j = 1,2,3,4,5 \qquad (11)$$

where $f(x) = \rho \frac{1}{B(\alpha_1,\beta_1)} x^{\alpha_1 - 1}(1-x)^{\beta_1 - 1} + (1-\rho)\frac{1}{B(\alpha_2,\beta_2)} x^{\alpha_2 - 1}(1-x)^{\beta_2 - 1}$

Let $x_1, x_2, \cdots, x_n$ be n observations of LGD (or recovery rate) in our sample. The five sample moments $\hat{E}\left(x^j\right)$ can be estimated by:

$$\hat{E}\left(x^j\right) = \frac{1}{n}\sum_{i=1}^{n} x_i^j , j = 1,2,3,4,5 \qquad (12)$$

By solving the five equations in matching the five model-implied moments ($E(x^j)$) with their sample counterparts ($\hat{E}(x^j)$), it is straightforward to obtain the estimated values of α_1, $\beta_1, \alpha_2, \beta_2,$ and ρ. But the results of the generalized method of moments approach can be biased (Bowman and Shenton, 1998). Thus, in our empirical analysis, we do not use the method of moments approach in conducting the main model estimations. It is however used in the estimation of the *initial starting values* of the parameters for the iterative model estimation process involved in the maximum likelihood estimation (MLE), as outlined in the subsequent section.

4.2.3.2 Maximum Likelihood Approach

The probability density function $f(x)$ of recovery rate x of our general five-factor mixture beta distribution can be expressed as:[22]

$$f(x) = \rho \frac{1}{B(\alpha_1,\beta_1)} x^{\alpha_1-1}(1-x)^{\beta_1-1} + (1-\rho)\frac{1}{B(\alpha_2,\beta_2)} x^{\alpha_2-1}(1-x)^{\beta_2-1} \quad (13)$$

We further specify the weight parameter ρ as a function of a vector of explanatory variables Y.

$$\rho_t = 0.5 + \frac{0.5}{1+e^{Y_t\gamma+\varepsilon_t}} \quad (14)$$

We conduct the estimation with panel data of historical recovery rates. For each year of our sample period, we observe the recovery rates of facilities defaulted within that year.

[22] We essentially follow the approach of Ferrari (2004), Huang (2011), and Cribari-Neto and Zeileis (2010) in formulating our maximum likelihood estimation methodology. Their models are based on single-beta regression, while ours is based on mixed beta regression instead.

Let $x_{n,t}$ represents the recovery rate of the nth defaulted asset in the tth year ($n = 1\ to\ N_t,\ t = 1\ to\ T$). Based on the density function of the five-factor model, the log-likelihood $\mathcal{L}(\blacksquare)$ of realizing the panel data of observations is:

$$\mathcal{L}(x_{1,1}, x_{2,1}, \cdots, x_{N_T,T}, \mathrm{Y}_1, \mathrm{Y}_2, \ldots, \mathrm{Y}_T\ ;\ \alpha_1, \beta_1, \alpha_2, \beta_2, \gamma)$$

$$= \sum_{t=1}^{T} \sum_{n=1}^{N_t} \ln \left[\rho \frac{1}{B(\alpha_1,\beta_1)} x_{n,t}^{\alpha_1-1} \left(1 - x_{n,t}\right)^{\beta_1-1} + (1-\rho) \frac{1}{B(\alpha_2,\beta_2)} x_{n,t}^{\alpha_2-1} (1 - x_{n,t})^{\beta_2-1}\right] \quad (15)$$

The values of parameters that jointly maximize this log-likelihood function are the maximum likelihood estimators. The model parameters $\theta_j \in \left\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma\right\}$ can be estimated by solving the following set of equations obtaining by setting the partial derivative of the log-likelihood function with respect to each estimator θ_j to zero.

$$\frac{\partial \mathcal{L}(\blacksquare)}{\partial \theta_j} = 0$$

Closed-form representations of the partial derivatives are however not readily available. We therefore resort to the numerical method to obtain an approximate solution. Starting from a set of initial values of the estimate parameter θ, an iterative process is conducted to numerically estimate the likelihood function and its derivatives, until we converge to a set of estimates that maximize the objective function in satisfying a number of pre-specified tolerance criteria. After obtaining the point estimates, the first order and second order partial derivatives of the log-likelihood function with respect to the parameters θ are utilized to estimate and derive the Fisher information matrix, which allows us to estimate the asymptotic standard errors of the maximum likelihood estimates of the model parameters. With the estimated asymptotic standard errors, we can then model the variation of the recovery rate and check the statistical significance of estimated parameters.

The three-factor model is a special case of the generalized five-factor model stated above, with its log-likelihood function as follows,

$$\mathcal{L}(x_{1,1}, x_{2,1}, \cdots, x_{N_T,T}, \mathrm{Y}, \mathrm{Y}_2, \dots, \mathrm{Y}_T\ ;\ \alpha, \beta, \gamma)$$

$$= \sum_{t=1}^{T} \sum_{n=1}^{N_t} \ln \left[\rho \frac{1}{B(\alpha,\beta)} x_{n,t}^{\alpha-1} \left(1 - x_{n,t}\right)^{\beta-1} + (1-\rho) \frac{1}{B(\beta,\alpha)} x_{n,t}^{\beta-1} \left(1 - x_{n,t}\right)^{\alpha-1}\right] \quad (16)$$

4.2.3.3 Goodness-of-fit tests

A number of goodness-of-fit tests are used in measuring the performance of the different models examined in our empirical analysis. We consider three standard approaches that are widely used in practice: Log-Likelihood function (LL), Akaike information criterion (AIC) and, Bayesian information criterion (BIC).

The LL is commonly used to measure the goodness-of-fit of statistical models by measuring the probability of realizing the observed data sample. As the logarithm transformation is strictly increased, we judge the model performance by comparing the log-likelihood function. With the assumption of independence of each observation, the overall log-likelihood is the sum of the log-likelihood of each individual observation of the sample data. The higher the LL, the better the goodness-of-fit of a model. This is consistent with the log-likelihood functions utilized in the MLE method outlined in the previous section.

The more complicated and the more degrees of freedom of a model, the higher is its LL. It does not necessarily mean that the model performs better than others in a parsimonious fashion. We might actually be *overfitting* the model. To remedy this deficiency of judging solely based on LL, we use the AIC and BIC measures to help us to strike the appropriate trade-off between the goodness-of-fit of the model and the simplicity of the model. Both measures are founded on information theory and likelihood function. AIC and BIC estimate the relative amount of information lost by a given model. In other words, there is a *penalty* if more parameters are introduced in a model as the degree of freedom decreases. The two performance measures are formally defined as:

$$AIC = 2k - 2\ln(L)$$

$$BIC = kln(n) - 2\ln(L)$$

where L is the maximized value of the likelihood function, k is the number of parameters in the model, and n is the number of observations. The lower the value of AIC (or BIC), the better is the performance of a model. As we can see, AIC and BIC are similar in their formulation while with a different penalty for the number of parameters. The penalty term under AIC is $2k$, while it is $k \cdot ln(n)$ for BIC. A detailed discussion on the comparison of AIC and BIC measures can be found in Burnham and Anderson (2002, 2004). In practice, BIC is a more popular measure for model selection purposes than AIC.

4.3 Data

4.3.1 Recovery Data

4.3.1.1 Data description

Facility-level recovery value data of defaulted US companies are obtained from the Standard & Poor's (S&P's) CreditPro database. The recovery rate in the S&P's database is calculated by discounting the ultimate recovery value back to the time of the default event and expressed as a dollar amount per notional value ($1,000) of the defaulted asset. The ultimate recovery value of a defaulted debt is computed by one of the following methods:

a. *Emergency pricing* - the debt trading price at the point of emergence from the default;
b. *Settlement pricing* - the debt trading price at the emergence of those instruments received in the workout process in exchange; or
c. *Liquidity event pricing* - values of the debts received in the settlement at their respective liquidity events.

The dataset includes both public and private US companies that have bank loans or bonds that are greater than fifty million dollars. And the companies must also have all recovery information available and fully completed the restructuring process to be qualified for inclusion. The recovery information of defaulted instruments from 1987 to 2012 is examined. Out of 4,347 defaulted instruments, there are 386 senior secured bonds, 1,228 senior unsecured bonds, 564 senior subordinated bonds, 332 subordinated bonds, 54 junior subordinated bonds, 818 term loans, 848 revolving credit, and 17 others. The defaulted companies belonged to thirty-eight different industries and came from all 51 different states and districts of the United States. Please note that we adjust all recovery rates that are lower than zero to zero and those higher than one to one to ensure that all recovery rate observations fall within the range of (0,1).

4.3.1.2 Summary Statistics

In Table 4.3.1, we present the number of recovery rate observations and the mean recovery rate by industry over the sample period from 1987 to 2012. The recovery rate is expressed as a proportion of its notional value (i.e., recovery amount per $1). Most of the industries are well represented in our historical data sample, with the most defaulted cases (516) from Telecommunication, while Insurance has the least number of defaulted cases (4). The mean recovery rate varies across the industries. The Consumer Nondurable and the Utility industries have mean recovery rates of over 80%, which are the highest among all the industries; whereas the lowest average recovery rate is from Government (18%) and the Insurance industry (27%). Out of the 38 industries, 33 have average recovery rates fall within the range of 40% to 70%. In Table 4.3.2, for each of the years from 1987 to 2012, we present the number of observations, mean, standard deviation, and the 25^{th} and 75^{th} percentiles of recovery rate. It is shown that the recovery rate varies over time. The lowest average recovery rate appeared in 1998 (37%), while the highest in 2007 (78%). Not surprisingly, the average recovery rate tends to be lower during the recessionary period than during the expansionary period. Out of the 26 years, six (1987, 1989, 1998, 2000, 2001, and 2008) have average recovery rates lower than 50%. Most of them are recession years.

INSERT TABLES 4.3.1 AND 4.3.2 ABOUT HERE

In Table 4.3.3, we present more detailed summary statistics of the discounted recovery rates of the full sample and two subsamples – *Recession* and *Expansion* – based on the bankruptcy date of the debt contract. The US Federal Reserves defines 1990-1991, 2001-2002, and 2008-2009 as global recessions. We follow them in defining the timing of the recession and expansion periods but extending the recession period windows to also include those defaulted incidents that occurred within one year ahead of the defined periods. Since the recovery workout process tends to be longer than a year, this treatment is necessary to ensure that we are correctly classifying a recovery observation, of which the default date is within a year prior to a recession period, as an observation of the recession subsample.

The mean recovery rate of the full sample is 0.56, suggesting that creditors on average recover slightly more than half of the money owed. With a standard deviation of 0.378, the variation in the recovery rate is substantial. The skewness and kurtosis statistics suggest that the recovery rate distribution is quite different from a normal distribution. Although the minimum and maximum values are zero and one, respectively as expected, it is worth noting that 25th and 75th percentiles are zero and one as well, which means both tails are heavily weighted in the distribution of recovery rate. Turning to the subsample results. Not surprisingly, the recovery rate in the expansion period is higher than that in the recession period. The median recovery rate during an expansion (0.65) is 0.15 higher than the recovery rate during the recession (0.50). This is consistent with our intuition that creditors tend to recover more in disposing/liquidating the default company assets when the economy is doing good. Based on the large standard deviations, the variation in the recovery rate within each of the two subsamples remains significant. Comparing the other statistics of the two subsamples indicates that the distribution of recovery rates observed during the recession and the expansion periods behaves quite differently. The null hypothesis of equality of distribution is rejected by a Kolmogorov-Smirnov test at the 90% confidence level.

INSERT TABLE 4.3.3 ABOUT HERE

4.3.2 Macroeconomic Variables

4.3.2.1 Data Description

To explain the time-series variation of the LGD distribution, we consider four macroeconomic variables that are examined in the literature as potential systematic factors that impact the recovery rate: the Real Gross Domestic Product (*GDP*), Unemployment Rate (*UE*), Standard & Poor's 500 Index (*SP500*) and the Probability of Default (*PD*). Havrylchyk (2010) finds that credit risk is sensitive to GDP and UE, confirming that GDP and employment are negatively correlated with loan loss provisions in both univariate and multivariate regressions. On the other hand, the empirical results of Misina (2006) suggest that a decrease in the US real GDP growth rate leads to an increase in the credit loss of loan. The S&P 500 index growth rate is commonly used as an indicator of economic growth in many recovery rate studies (e.g., Jacob and Karagozoglu (2011)). Finally, PD is well documented to be correlated with LGD based on both theoretical models (e.g., Merton's model) and empirical studies (e.g., Altman (2005); Havrylchyk (2010); Misina (2006)).

The definitions of these variables are as follow:

- ***GDP***: Annual US GDP growth rate is calculated based on the annual US GDP level collected from Bloomberg.

$$GDP\ growth\ rate_t = \frac{GDP_{t+1} - GDP_t}{GDP_t} \tag{17}$$

- ***UE***: The unemployment rate is another popular indicator of economic condition. We obtain the US unemployment rate from the Bureau of Labor Statistics. According to the definition of the US Department of Labor, a person is defined as unemployed if they do not have a job and have actively looked for work for at least four weeks and still available for work. Temporarily laid off persons are also included as unemployed. *UE* is formally defined as:

$$UE_{\ t} = \frac{Number\ of\ Unemployed_{\ t}}{Labor\ Force_t} \tag{18}$$

- ***SP500***: *SP500* is the annual return on the S&P 500 index (i.e., the annual growth rate of the S&P 500 index) calculated based on the S&P 500 index level obtained from Bloomberg.

$$S\&P\ 500\ growth\ rate_t = \frac{S\&P\ 500\ index_{t+1} - S\&P\ 500\ index_t}{S\&P\ 500\ index_t} \qquad (19)$$

- ***PD***: Here, we use the annual default frequency of US corporations as our PD measure. Specifically, for each year *t*, we calculate PD by dividing the number of US firms defaulted in that year (obtained from *Bankruptcydata.com*) by the total number of US firms in that year (obtained from the *World Bank database*).

$$PD_t = \frac{Number\ of\ firms\ that\ defaulted\ during\ year\ t}{Total\ number\ of\ firms\ at\ t} \qquad (20)$$

4.3.2.2 Summary Statistics of Macroeconomic Variables

In Table 4.3.4, we present the summary statistics of the selected macroeconomic variables over the sample period from 1987 to 2012. The means of GDP, UE, PD and SP500 are 2.7%, 5.9%, 1.6%, and 8.7%, respectively. Based on the relative value of the standard deviation and the mean, SP500 (PD) exhibits the highest (lowest) time-series variation. The time-series variations of the macroeconomic variables from 1987 to 2012 are presented in Figure 4.3.1. We can see that GDP and SP500 show similar time-series variations that are negatively correlated with those of UE and PD. This observation is consistent with our intuition as we expect GDP and SP500 to be high during economic expansion and low during the recession, and vice versa for UE and PD. This opposite behavior is most noticeable during the 2008-2009 financial crisis. During that time, both GDP and SP500 attain their lowest values of -2.5% and -38%, respectively; whereas both UE and PD reach their highest values of 10% and 4.2%, respectively. Finally, it is important to point out that our sample period of 1987 to 2012 covers more than a couple of business cycles, including a number of market downturns (e.g., the early 1990s recession, the bursting of the dot-com bubble, the 2008-09 financial crisis) and economic expansion episodes (e.g., during the late 1990s and the mid-2000s).

INSERT TABLE 4.3.4 AND FIGURE 4.3.1 ABOUT HERE

In our empirical analysis, we consider different combinations of these macroeconomic variables in explaining the time-series variation of the observed LGD distribution. Besides, the contemporaneous values, we also consider the explanatory power of the lagged values of these macroeconomic variables. To avoid any statistical issues as a result of multicollinearity, we calculate the pair-wise correlations of these macroeconomic variables (and their lags) and make sure we exclude any combinations of independent variables that are highly correlated with each other in our regression framework. The pair-wise correlations are presented in Table 4.3.5. As expected, GDP is positively correlated with SP500, but negatively correlated with both UE and PD. Although the magnitude of most of the correlations is not considered to be high, there are a few highly (positively or negatively) correlated pairs. We set a threshold of +0.6/-0.6 for the pair-wise correlation to be too large for including both variables as the independent variables in our regression framework. For example, GDP and UE (lag 1) would not appear in the same regression as their correlation coefficient of -0.8 breaches the limit of -0.6.

INSERT TABLE 4.3.5 ABOUT HERE

4.4 Empirical Results

4.4.1 Three-factor Model vs. Five-factor Model

In this section, we compare the three-factor and five-factor models by assessing their ability to explain the time-series variation of the weight parameter ρ of the mixed beta distribution over time. For both models, we adopt the same logit function for ρ as proposed in Section 2.2.2 (see Equation (10)). We consider the univariate version of the logit function using the lagged-one values of macroeconomic variables of GDP, SP500, PD, and UE separately in explaining the time-series variations of ρ. We are therefore making a one-year prediction on the weight parameter based on the macroeconomic variables. The estimation results are presented in Table 4.4.1 based on the full sample period from 1987 to 2012. The dependent variable is the recovery rate of the defaulted facilities. In each of these four univariate models and for both the three-factor and the five-factor versions of the model, the first (second) beta distribution is considered as the "Good" ("Bad") distribution, given its relatively higher (lower) mean value of recovery rate. All the coefficients on the macroeconomic variables as presented in Table 4.4.1 are significant at least at the 10% confidence level, and the signs of the coefficients are consistent with our expectation. Specifically, according to the estimated coefficients, the lower (higher) the values of GDP and SP500 (UE and PD), the lower the weight we assigned to the "Good" beta distribution. This is consistent with our expectation that, in an economic downturn (e.g., in 2009), the lower (higher) GDP growth rate and S&P 500 return (unemployment rate and default frequency) are associated with a lower chance of realizing a high recovery rate, given the lower weight assigned to the "Good" recovery distribution. This, therefore, confirms our expectation that GDP and SP500 are positively correlated with recovery rate, while PD and UE are negatively correlated with the recovery rate. For example, during our sample period, the GDP growth rate ranges from the lowest value of -2.5% (in 2009) to the highest value of 4.8% (in 1999). Based on the estimated coefficient, the weight of the "Good" beta distribution (ρ) ranges from approximately 0.615 (in 2009) to 0.852 (in 1999). The goodness-of-fit results – likelihood value (LL), AIC, and BIC – indicate that the five-factor models are consistently superior to the three-factor models by having a higher LL and lower AIC and BIC. Among the four explanatory variables, using GDP gives us both the best three-factor model and the best five-factor model in terms of goodness-of-fit.

INSERT TABLE 4.4.1 ABOUT HERE

To further judge the performance of the three-factor and five-factor models, we also compare the accuracy of their one-year predictions of the weight parameter ρ in year t based on the macroeconomic variables observed in year $t-1$. Specifically, the one-year predictions are obtained by using the eight univariate models (i.e., four three-factor models and four five-factor models) presented in Table 4.4.1, together with corresponding macroeconomic variables; whereas the accuracy is measured against the *realized* weight parameter ρ obtained by fitting the respective mixture beta model (i.e., either the three-factor or the five-factor model) with the actual observed recovery rates in year t. The time-series plots of the one-year predicted ρ of each of the eight models in comparison with the realized ρ are presented in Figure 4.4.1.

We notice that the predicted ρ, in general, follows the same time-series pattern as the realized ρ. In particular, both the predicted and realized ρ tend to be lower during the three episodes of the market downturn, namely the US recession in the early 1990s, in the aftermath of the bursting of the dot-com bubble in 2001, and the 2008-09 financial crisis. Since ρ is the weight assigned to the first beta distribution, both the three-factor and five-factor models are assigning a lower weight to the first beta distribution during market downturns compared to that during the market expansion. This is an intuitive finding given the fact that the first (second) beta distribution is the "Good" ("Bad") distribution with a higher (lower) average recovery rate. The models appropriately capture the time-series variations in recovery rate by correctly producing a lower recovery rate during market downturns through adjusting the weights assigned to the two underlying beta distributions. A visual inspection of the plots reveals that the five-factor models in general give us a better prediction of the realized weight over time than the three-factor models. Nevertheless, the performance varies among the four five-factor (and also among the four three-factor models). Specifically, it seems that the five-factor model with GDP as the explanatory variable provides us with a tighter fit of the time-series change in the realized ρ than the other three five-factor models.

INSERT FIGURE 4.4.1 ABOUT HERE

To further quantify the performance of the models, we calculate the mean differences and the root mean squared differences (RMSD) of the one-year predicted ρ from the realized ρ for our four three-factor models and four five-factor models over the sample period from 1988 to 2012.

$$\text{Mean difference} = \frac{\sum_{t=1988}^{2012}(\text{Realized } \rho_t - Predicted\ \rho_t)}{25} \quad (21)$$

$$RMSD = \sqrt{\frac{\sum_{t=1988}^{2012}(\text{Realized } \rho_t - Predicted\ \rho_t)^2}{25}} \quad (22)$$

The results reported in Table 4.4.2 suggest that the five-factor models are consistently superior to the three-factor models resulting in a smaller error based on both measures, regardless of the explanatory variable used. In the subsequent empirical analysis, we, therefore, focus our attention on using the five-factor version of the mixture beta distribution model.

INSERT TABLE 4.4.2 ABOUT HERE

4.4.2 Selection of Explanatory Variables

In this section, we consider different combinations of explanatory variables (GDP, UE, PD, and SP500) in formulating our five-factor mixture beta distribution. We would like to identify the models that will give us the best goodness-of-fit in explaining the observed distribution of recovery rate over our same period from 1987 to 2012. We start by examining the univariate versions of the five-factor model each time using only a single macroeconomic variable. In the previous section, we have examined the univariate results by using the lagged-one values of the variables. Here we also consider the explanatory power of the lagged-zero (i.e., contemporaneous), lagged-one, and lagged-two values of the macroeconomic variables. The univariate estimation results are presented in Table 4.4.3. Besides the estimated intercepts and coefficients, we also present the respective likelihood function, AIC, and BIC. The sign of all the estimated coefficients is consistent with our expectation of the directional relation between recovery rate and the respective explanatory variable. Based on these univariate results, GDP (lag 1) has the best explanatory power in terms of the statistical significance of its coefficient, the likelihood value, AIC, and BIC. Five coefficients out of the total 12 univariate models are statistically significant. They are from the univariate models with GDP (lag 1), UE (lag 1), UE (lag 2), PD (lag 1), and SP500 (lag 0), respectively, as the explanatory variables. They are Model no. 4, 5, 6, 7, and 10 as presented in Table 4.4.3.

INSERT TABLE 4.4.3 ABOUT HERE

In Table 4.4.4, we present the estimation results for the five-factor models with different combinations of more than one explanatory variable. We exhaust all combinations of two, three, and four explanatory variables selected out of the five significant univariate variables – GDP (lag 1), UE (lag 1), UE (lag 2), PD (lag 1), and SP500 (lag 0) – as identified in the previous univariate analysis.[23] In doing so, however, we exclude those combinations involving variables with pair-wise correlation coefficients of magnitude larger than 0.6 (see Table 4.3.5), so as to avoid any issue of multicollinearity. Since GDP (lag 1) is highly correlated with all other significant univariate variables, it is not included in any combination.

INSERT TABLE 4.4.4 ABOUT HERE

All the estimated coefficients of the seven multi-variable models (i.e., Model no. 13 to 19) presented in Table 4.4.4 are of the expected sign. Specifically, without any exception, the

[23] We do not consider models with both UE (lag 1) and UE (lag 2) as explanatory variables.

coefficients of UE and PD (SP500) are positive (negative). A majority of the estimated coefficients are statistically significant. Nevertheless, the performance of these models does vary in terms of their goodness-of-fit as measured by their AIC and BIC. We rank all the univariate models (i.e., Model no. 1 to 12 in Table 4.3) together with all the multi-variable models (i.e., Model no. 13 to 19 in Table 4.4) based on their AIC and BIC, and arrive at the following three five-factor models as our best model candidates:

- *The best model* (with the lowest AIC/BIC among all the models): With GDP (lag 1) as the single explanatory variable [Model no. 5 in Table 4.4.3]
- *The second best model* (with the second-lowest AIC/BIC): With SP500 (lag 0), PD (lag 1), and UE (lag 2) as the explanatory variables [Model no. 18 in Table 4.4.4]
- *The third best model* (with the third-lowest AIC/BIC): With SP500 (lag 0) and UE (lag 2) as the explanatory variables [Model no. 15 in Table 4.4.4]

In the subsequent section, we will demonstrate the superior performance of these three five-factor models in explaining the time-series variation of the recovery rate distribution in comparison with that of the commonly-used logit-link regression models.

4.4.3 Comparison with Other Recovery Rate Model

To highlight the importance of using a mixed distribution, in this section, we compare the performance of our proposed five-factor mixture beta distribution models with single-beta distribution time-series recovery rate models considered by other researchers. In the literature, the logit-link regression model has been commonly used to explain the time-series variation of the observed recovery rates. We consider a specific version of that as examined by Huang and Oosterlee (2012). In particular, the mean μ of the (single) beta distribution is formulated as a logit function of the explanatory variables, while the dispersion parameter φ is constant over time. The model can be expressed as:

$$\alpha = \mu\varphi \tag{23}$$

$$\beta = (1 - \mu)\varphi \tag{24}$$

$$\mu = \frac{e^{\eta}}{1+e^{\eta}} \tag{25}$$

where η is a linear function of the explanatory variables that change over time. First of all, we compare the goodness-of-fit of our proposed mixture beta distribution models with that of the single-beta logit-link models based on the full sample period from 1987 to 2012. To ensure we have a *fair* comparison, we use the same set of macroeconomic variables in driving both models. We consider three different versions of our mixture beta distribution (i.e., the *best, second-best,* and *third-best* models as identified in Section 4.2) and compare their goodness-of-fit with three different versions of the single-beta logit-link model each with the same combinations of macroeconomic variables as in our *best*, *second-best*, and *third-best* models, respectively. The estimation results together with the model performance are reported in Table 4.4.5.

INSERT TABLE 4.4.5 ABOUT HERE

Firstly, let us take a look at the estimation results of the three logit-linked models. The signs of coefficients of each macroeconomic variable are consistent with our expectations.

Specifically, the higher (lower) the GDP and SP500 (PD and UE), the higher is the mean of the recovery rate distribution. Four out of the six estimated coefficients are also statistically significant (at least at the 10% level). Turning to the estimated shape parameters of the five-factor mixture beta distribution models. Comparing the shape parameters α_1, β_1, α_2, and β_2 across the three mixture beta distribution model, we notice that the two underlying beta distributions of our best model are not the same as those of our second best and third best models. Nevertheless, for all three models, the first beta distribution (with shape parameters α_1 and β_1) can always be interpreted as the "Good" distribution in the sense that it gives us a higher mean recovery rate than the second beta distribution (with shape parameters α_2 and β_2), which can be labeled as the "Bad" distribution. Taking the best model as an example, with $\alpha_1 = 1.237$ and $\beta_1 = 0.829$, the mean recovery rate of the first beta distribution is 0.599; whereas, with $\alpha_2 = 4.343$ and $\beta_2 = 6.867$, the mean recovery rate of the second beta distribution is 0.387. Thus, during the expansion period, we expect the first ("Good") beta distribution to be weighted heavier than the second ("Bad") beta distribution, and vice versa during the recession period. This expectation is confirmed by the negative coefficient of -31.281 estimated for the explanatory variable of the lagged-one GDP. During the expansion period, as the GDP growth rate is higher, with the negative coefficient, the logit function of Equation (10) gives us a higher weight parameter ρ that is closer to 1. Since ρ is the weight on the first beta distribution, the model indeed assigns a heavier weight to the "Good" beta distribution during a good time. On the other hand, during a recession when the GDP growth rate is low (or even negative), the model correctly assigns a lower weight to the first beta distribution. Besides having different means, the shape of the first and second beta distributions is also quite different (see Figure 4.4.2). We can see that the “Good” beta distribution is an “inverse L” shape, while the “Bad” one is a bell shape.

INSERT FIGURE 4.4.2 ABOUT HERE

Based on the LL, AIC, and BIC presented in Table 4.4.5, the goodness-of-fit of the proposed five-factor mixture beta distribution models is consistently better than that of the single-beta logit-link regression models. Specifically, the former has a lower AIC and BIC and a higher LL than the latter, indicating its superior performance in modeling the observed recovery rates.

There are two follow-up issues we want to address: (a) How different is the relative performance of the two kinds of models in different phases of the economic cycle? Does the superior performance of the mixture beta distribution model exist only during specific economic conditions? (b) The above model comparison is conducted in an in-sample setting. How do the two kinds of models perform in an out-of-sample setting?

To address these two issues. We conduct the Kolmogorov–Smirnov test (KS test) to gauge the accuracy in modeling the recovery rate distribution observed each year from 1990 to 2012. The KS test is a nonparametric test of which the test statistic quantifies the *distance* between the sample distribution and the reference distribution. So it can be used to test whether two distributions differ. The null hypothesis here is that the model-implied distribution is identical to the observed distribution. We first conduct the test in an in-sample setting. Using all the sample data from 1987 to 2012, we estimate the parameters of our three five-factor mixture beta distribution models and the three single-beta logit-link models. Using the estimated parameters together with the observed macroeconomic variables, we calculate the model-implied recovery rate distribution for each year from 1990 to 2012. The reason why we choose the estimation window starting from 1990 rather than 1987 is there are very few observations before 1990 leading to potentially biased estimation results for the earlier time period from 1987 to 1989. The respective KS statistics are then calculated to check whether the null hypothesis can be rejected or not for each model and each of these years. In addition to the six models, we also calculate the KS statistics based on a constant single-beta distribution obtained by pooling all the recovery rate observations overall years. The *p*-values of the KS statistics are presented in Table 4.4.6.

INSERT TABLE 4.4.6 ABOUT HERE

Except for in 2008, the model-implied recovery rate distributions of all the three single-beta logit-link models are (at least weakly) significantly different from the observed distributions for each year from 1990 to 2012 according to the KS statistics. This finding, therefore, casts doubt on the accuracy of the commonly-used logit-link models in explaining the variations of the recovery rate distribution over time. Not surprisingly, with highly significant *p*-values throughout the years, the constant single-beta distribution also fails horribly in fitting the observed recovery rates. The performance of the proposed mixture beta distribution models is much more promising than both the single-beta logit-link models and the constant single-beta distribution. For most of the years over our sample period, the model-implied distributions are not significantly different from the observed distributions according to the KS statistics. As expected, the best and second-best mixture beta distribution models tend to perform better than the third-best model. Nevertheless, in 1992, 1996, 1997, 2007, and 2012, there are significant differences (at least weakly) between the model-implied and observed distributions regardless of which of the three mixture-beta models we use. It is important to point out that the US economy was in general booming during these five years.[24] Based on the *p*-values, it seems that the proposed mixture-beta distribution models perform better during the recessionary periods than during the expansionary periods. Most notably, in 2001-2002 and 2008-2010, the high *p*-values reflect the close resemblance between the model-implied and observed recovery rate distributions.

To confirm the robustness of our conclusions and to address the concern of *overfitting* the sample data, we repeat the above analysis but in an out-of-sample setting. Specifically, in predicting the recovery rate distribution in a specific *target year*, we estimate the parameters of all the models under consideration using the subsample data observed within the calibration window from 1987 up to and including the year prior to the target year. To ensure we have sufficient data in our estimation process, the first target year is 1995. In predicting the recovery rate distribution in 1995, we, therefore, use all the data points from 1987 to 1994 to calibrate the parameters of each model. After obtaining the model-implied distributions for 1995, we then calculate the KS statistics so as to compare the performance of the models for that target year. Then, for the target year of 1996, the calibration window is extended for a year to include the observations in 1995, i.e., covering the period of 1987-

[24] All the five years are outside the recessionary time periods of 1990-1991, 2001-2002, and 2009-2009 as defined by the US Federal Reserves.

1995. This calibration window extension process is repeated for each of the subsequent years until the last target year of 2012, which uses the data observed between 1987 and 2011. The *p*-values of the resultant KS statistics of all the models are presented in Table 4.4.7.

INSERT TABLE 4.4.7 ABOUT HERE

As expected, all the models have a more difficult time replicating the observed distribution in this out-of-sample setting than in the previous in-sample setting. There are more instances where the model-implied distributions are significantly different from the observed distributions.[25] Without any exception, the single-beta logit-link models fail to produce accurate enough recovery rate distributions in each year from 1995 to 2012. The null hypothesis of identical distribution is consistently rejected according to the KS test (at least at the 5% confidence level). The performance of the mixture-beta distributions models is consistently better. In particular, their ability to accurately replicate the observed recovery rate distributions during the recessionary periods of 2001-2002 and 2008-2010 can still be confirmed by the KS statistics in this out-of-sample test.

Finally, to visualize the fitting of the observed recovery rate distributions, we present the implied distributions of all the models under consideration in comparison to the kernel density of the observed distribution for each year from 1995 to 2012 in Figure 4.4.3 from

[25] Nevertheless, we do observe a trend of an improvement of the out-of-sample test results over time. As the calibration window lengthens over time, more data are included in the parameter estimation, and thus the better is the performance of the models.

the out of sample results. The graphs give us a clear view of the relative performance of the different models under consideration. It is obvious that the mixture beta five-factor model plots are consistently closer to that of the benchmark nonparametric distribution compared to those of the single-beta logit-link models. The superior performance of the mixture beta models is particularly stark in the recession years such as 2000 and 2008 as the mixture beta models can replicate the details of the recovery distribution better by adjusting the weights of two individual beta distributions. For example, the weight on the "Good" beta distribution has decreased from 0.73 to 0.59 from 2007 to 2008, thus altering the shape of the mixture distribution from an "inverse L" shape to a bell shape and shifting the overall mean of recovery rate to the left (i.e., lower end) as shown in the R2008 subplot of Figure 4.4.3. This kind of time-series variation in the distribution shape cannot be accommodated by the single beta logit-link models.

INSERT FIGURE 4.4.3 ABOUT HERE

4.5 Conclusion

In this paper, we propose a new LGD model – a dual-beta mixture regression model – which is different from all other previous models. We show that this approach is flexible enough to accommodate the important features of the LGD distribution. Rather than just treating it as a static LGD distribution model, we also examine how we can allow the mixture beta LGD distribution to be driven by the systematic risk factors through linking the weights of the underlying beta distributions to the macroeconomic variables. To our knowledge, this is the first study that uses this approach to model LGD. The results enable some interesting insights that complement and extend the findings in the literature. For example, based on our corporate default data sample, the positive correlation between PD and LGD is confirmed, which means, when the economy is good, both PD and LGD are relatively low and vice versa. Besides, the macroeconomic variables such as GDP, UE, and SP500 are also associated with the realized recovery rates in an intuitive fashion based on our empirical tests. By comparing the proposed mixture beta regression model with other commonly used models in practice, we find that the estimation accuracy is much improved, especially for modeling the peaks and tails of the LGD distributions.

Future studies can be extended to incorporate three or more beta distributions in the mixture beta model and to test the significance of other macroeconomic variables. The former could improve the accuracy of the model, but at the price of becoming more difficult to be estimated as more parameters are involved. Fortunately, statistical techniques like AIC, BIC, root mean squared deviation, coefficient of determination, KS test statistic and average deviation on cumulative distribution function can be utilized to facilitate the estimation process. The difficulty in the latter is in the determination of the potential macroeconomic variables (and the lags of them) that are both statistically significant and economically meaningful in the model.

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Table 4.2.1 Five-factor Model Example

The mean, standard deviation, and dispersion factor of each individual beta distribution in a five-factor model example is shown in the table below,

	Mean	Standard Deviation	Dispersion factor
Beta distribution 1 (Good)	0.286	0.069	14
Beta distribution 2 (Bad)	0.727	0.087	11
Mixture beta distribution	0.582	0.081	11.99

Table 4.2.2 Three-factor Model Example

The mean, standard deviation, and dispersion factor of each individual beta distribution of a three-factor model example is shown in the table below,

	Mean	Standard Deviation	Dispersion factor
Beta distribution 1 (Good)	0.310	0.159	5.8
Beta distribution 2 (Bad)	0.690	0.159	5.8
Mixture beta distribution	0.443	0.159	5.8

Table 4.3.1 Recovery Rate by Industry

This table presents the number of recovery rate observations and the mean recovery rate by industries.

Industry	No. of observations	Mean recovery rate
AEROSPACE/DEFENSE	24	0.44
AIRLINES	89	0.42
AUTOMOTIVE	192	0.60
BUILDING MATERIALS	115	0.65
CHEMICALS	130	0.60
COMPUTERS & ELECTRONICS	257	0.56
CONSTRUCTION	52	0.49
CONSUMER NONDURABLES	11	0.88
ENTERTAINMENT AND LEISURE	92	0.56
ENVIRONMENTAL SERVICES	10	0.59
FINANCIAL SERVICES	96	0.57
FOOD AND BEVERAGE	119	0.62
FOREST PRODUCTS	96	0.63
GAMING AND HOTELS	104	0.49
GOVERNMENT	15	0.18
HEALTHCARE	175	0.55
HOME FURNISHINGS	49	0.63
INSURANCE	4	0.27
LEASING	71	0.65
MACHINERY	85	0.66
MANUFACTURING	20	0.59
MEDIA	262	0.62
METALS & MINING	90	0.66
OIL & GAS	214	0.54
PERSONAL SERVICES	17	0.54
PRINTING & PUBLISHING	98	0.58
PROFESSIONAL&BUSINESS SERVICES	58	0.44
REAL ESTATE	120	0.49
RETAIL FOOD & DRUG	138	0.53
RETAILING	414	0.52
SECURITIES & TRUSTS	38	0.38
SHIPPING & SHIP BUILDING	32	0.65
STEEL	87	0.58
TELECOMMUNICATIONS	516	0.45
TEXTILE & APPAREL MFG.	149	0.61
TRANSPORTATION	70	0.61
UTILITIES	230	0.80
WHOLESALE TRADE	8	0.66
Grand Total	**4,347**	0.56

Table 4.3.2 Recovery Rate by Year

This table presents the number of observations, mean recovery rate, the standard deviation of recovery rate, and the 25th and 75th percentile of recovery rate per year from 1987 to 2012.

Year	No. of observations	Mean recovery rate	Standard deviation of recovery rate	25th percentile of recovery rate	75th percentile of recovery rate
1987	32	0.47	0.36	0.18	0.83
1988	78	0.57	0.38	0.18	1.00
1989	109	0.40	0.35	0.08	0.58
1990	173	0.52	0.39	0.10	1.00
1991	241	0.54	0.38	0.16	0.97
1992	203	0.58	0.36	0.28	1.00
1993	160	0.59	0.37	0.26	1.00
1994	67	0.65	0.37	0.35	1.00
1995	83	0.61	0.37	0.28	1.00
1996	80	0.63	0.35	0.36	1.00
1997	69	0.61	0.37	0.25	1.00
1998	71	0.37	0.36	0.06	0.55
1999	179	0.55	0.37	0.18	1.00
2000	286	0.47	0.39	0.08	0.86
2001	527	0.46	0.38	0.09	0.82
2002	632	0.50	0.37	0.14	0.92
2003	347	0.69	0.34	0.41	1.00
2004	150	0.70	0.30	0.54	0.97
2005	129	0.73	0.27	0.61	0.95
2006	65	0.68	0.37	0.42	1.00
2007	33	0.78	0.27	0.79	0.99
2008	122	0.48	0.30	0.22	0.66
2009	401	0.64	0.37	0.30	1.00
2010	52	0.59	0.36	0.31	0.94
2011	26	0.62	0.39	0.44	0.99
2012	32	0.72	0.33	0.49	1.00
Grand Total	**4,347**	**0.56**	**0.37**	**0.28**	**0.92**

Table 4.3.3 Summary Statistics of Recovery Rate

This table presents the detailed summary statistics of the discounted recovery rates of the full sample and two subsamples – *Recession* and *Expansion* – based on the bankruptcy date of the debt contract.

	Full sample	Recession subsample	Expansion subsample
Mean	0.56	0.42	0.65
Median	0.59	0.50	0.65
Standard Deviation	0.38	0.45	0.37
Kurtosis	-1.44	-1.56	-1.28
Skewness	-0.12	0.00	-0.22
Minimum	0.00	0.00	0.00
25^{th} percentile	0.00	0.00	0.06
75^{th} percentile	1.00	0.88	1.00
Maximum	1.00	1.00	1.00
No. of observation	4,347	2,096	2,251

Table 4.3.4 Summary Statistics of Macroeconomic Variables

This table presents the summary statistics of the selected macroeconomic variables: *GDP*, *UE*, *PD*, and *SP500*. The data sources of the variables are as follow:

- *GDP*: Bloomberg
- *UE*: Bureau of Labor Statistics
- *PD*: BankruptcyData and World Bank database
- *SP500*: Bloomberg

	GDP	UE	PD (%)	SP500 (%)
Mean	0.027	0.059	1.65	8.73
Standard Deviation	0.0036	0.020	0.23	18.06
Min	-0.024	0.039	0.37	-38.49
Max	0.048	0.099	4.19	34.11
25th percentile	0.019	0.049	0.63	-1.54
75^{th} percentile	0.040	0.065	2.48	26.31
skewness	-1.47	1.30	0.90	-0.84
Kurtosis	2..32	1.77	-0.26	0.61
median	0.032	0.056	1.33	10.70

Table 4.3.5 Correlative Matrix of Macroeconomic Variables

This table presents the correlation matrix of the macroeconomic variables and their lags. We set a threshold of +0.6/-0.6 for the pair-wise correlation to be too large for including both variables as the independent variables in our regression framework. For example, GDP and UE (lag 1) would not appear in the same regression as their correlation coefficient of -0.8 breaches the limit of -0.6.

	GDP	*UE*	*PD*	*SP500*	*GDP lag 1*	*UE lag 1*	*PD lag 1*	*SP lag 1*	*GDP lag 2*	*UE lag 2*	*PD lag 2*	*SP lag 2*
GDP	1.00											
UE	-0.72	1.00										
PD	-0.64	0.39	1.00									
SP500	0.24	-0.06	-0.22	1.00								
GDP lag 1	0.45	-0.23	-0.19	0.67	1.00							
UE lag 1	-0.80	0.80	0.42	-0.35	-0.72	1.00						
PD lag 1	-0.19	-0.13	0.51	-0.55	-0.60	0.31	1.00					
SP lag 1	0.08	0.11	-0.10	0.02	0.26	-0.09	-0.22	1.00				
GDP lag 2	0.14	0.13	0.05	0.04	0.44	-0.21	-0.17	0.70	1.00			
UE lag 2	-0.59	0.46	0.20	-0.33	-0.80	0.82	0.34	-0.38	-0.68	1.00		
PD lag 2	0.32	-0.42	-0.07	0.05	-0.19	-0.10	0.49	-0.56	-0.60	0.32	1.00	
SP lag 2	-0.11	0.22	-0.16	0.01	0.07	0.12	-0.10	0.02	0.25	-0.07	-0.21	1.00

Table 4.4.1 Three-factor Model vs. Five-factor Model

In this table, we present the estimated parameters and coefficients of the three-factor models (Panel A) and the five-factor models (Panel B). For both types of models, we adopt the same logit function for the weight parameter ρ as proposed in Section 2.2.2 (see Equation (10)). We consider the univariate version of the logit function using the lagged-one values of macroeconomic variables of GDP, SP500, PD, and UE separately in explaining the time-series variations of ρ. The estimations are conducted over the full sample period from 1987 to 2012. The dependent variable is the recovery rate of the defaulted facilities.

Panel A：Three-factor model				
	GDP	UE	PD	SP500
Shape parameter				
α	2.46***	4.05	3.78*	2.16*
β	1.95**	2.74*	2.19**	1.75**
Coefficients of explanatory variable				
γ	-12.28***	1.09*	0.39**	-0.03**
Goodness-of-fit				
LL	167.1	112.8	134.0	145.7
AIC	15.89	17.67	19.89	18.12
BIC	17.67	19.43	20.98	20.31

Panel B：Five-factor model				
	GDP	UE	PD	SP500
Shape parameter				
α_1	1.24	1.34	2.87	4.13
β_1	0.82	0.67	1.92	3.70
α_2	4.34	5.87	3.12	4.65
β_2	6.87	9.87	6.12	8.05
Coefficients of explanatory variable				
γ	-31.28***	19.32***	0.33***	-0.04**
Goodness-of-fit				
LL	288.1	195.4	211.4	229.87
AIC	9.42	13.67	12.19	11.19
BIC	10.95	15.78	14.32	13.86

Table 4.4.2 Performance of Three-factor Model vs. Five-factor Model

This table presents the mean differences and root mean squared differences (RMSD) of the one-year predicted weights for time t (from both the five-factor model and the three-factor model) from the realized weights based on the realized recovery rates at time t. The mean difference and RMSD are calculated from the equations below.

$$\text{Mean difference} = \frac{\sum_{t=1988}^{2012}(\text{Realized } \rho_t - Predicted\ \rho_t)}{25}$$

$$RMSD = \sqrt{\frac{\sum_{t=1988}^{2012}(\text{Realized } \rho_t - Predicted\ \rho_t)^2}{25}}$$

Three-factor model				
	GDP	UE	PD (%)	SP500 (%)
Average Difference	-0.0235	-0.0253	-0.0201	0.0636
RMSD	0.0890	0.0921	0.0774	0.1142
Five-factor model				
	GDP	UE	PD (%)	SP500 (%)
Average Difference	-0.0042	-0.0050	-0.0063	0.0139
RMSD	0.0340	0.0584	0.0512	0.0494

Table 4.4.3 Selection of Explanatory Variables – Univariate Results

This table presents the coefficients and significances of all univariate models. The Likelihood, AIC and BIC are also reported for comparison purposes.

		MODELS											
	LAG	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
GDP	0	-8.85											
	1					-31.28***							
	2									-9.75			
UE	0		1.42										
	1						19.32**						
	2										20.93***		
PD	0			0.06									
	1							0.33***					
	2											0.09	
SP500	0				-0.04***								
	1								-0.04				
	2												0.00
Intercept		-0.17	-0.50	-0.51*	-0.22	0.42***	-1.57***	-0.95***	-0.36*	-0.16	-1.69***	-0.56*	-0.42***
Likelihood		135.76	106.85	91.28	229.87	288.18	195.47	211.42	149.87	110.93	200.14	137.97	120.63
AIC		46.17	58.32	74.31	11.19	9.42	13.67	12.19	42.19	55.80	11.03	45.23	52.91
BIC		49.79	60.53	76.85	13.86	10.95	15.78	14.32	45.86	59.02	12.76	47.44	57.38

Table 4.4.4 Selection of Explanatory Variables – Combinations of Variables

This table presents the coefficients and significances of selected combinations of macroeconomic variables. Please note that the pair of independent variables with correlation coefficients of out of the range from -0.6 to 0.6 are excluded to avoid multicollinearity. The correlation coefficients are reported in section 3.2.4.

	MODELS						
	(13)	(14)	(15)	(16)	(17)	(18)	(19)
GDP (lag 1)							
UE (lag 1)	11.79			0.27**			12.50*
UE (lag 2)			14.45**		0.25**	0.17	
PD (lag 1)		0.19*		13.21*	14.82**	10.14	0.15
SP500 (lag 0)	-0.02***	-0.02**	-0.02***			-0.01*	-0.01**
Intercept	-0.96**	-0.59**	-1.14**	-1.65***	-1.74***	-1.18**	-1.31***
Likelihood	202.18	215.82	235.61	212.34	221.30	246.07	233.22
AIC	13.51	12.10	10.93	12.15	11.59	10.87	11.11
BIC	15.07	13.98	12.43	14.27	13.51	11.96	12.62

Table 4.4.5 Model Performance – Mixture Beta Distribution Model vs. Single-Beta Logit-Link Regression Model

This table presents the coefficients, significances AIC, BIC and LL results of six different models: Three best five-factor models and three best Jacob's models. As expected, five-factor models are superior to their corresponding versions of Jacob's models based on AIC, BIC and LL.

	Five-factor mixture beta distribution model			Single-beta logit-link regression model		
	Best	Second best	Third best	Best	Second best	Third best
Shape parameters						
α_1	1.237***	1.348**	1.389**			
β_1	0.829***	0.677**	0.610**			
α_2	4.343**	4.646**	4.911***			
β_2	6.867**	8.057**	8.131***			
φ	4.352**	4.694**	4.760**	6.291**	8.137*	8.787*
Coefficients of explanatory variables						
GDP (lag 1)	-31.281***			12.600***		
SP500 (lag 0)		-0.018***	-0.013**		0.014**	0.008*
PD (lag 1)			0.151			-3.121
UE (lag 2)		14.450**	12.501*		-0.897**	-8.420
Intercept	0.42***	-1.14	-1.31	0.01**	0.97*	1.17
Goodness-of-fit						
LL	288.1	251.2	250.8	163.1	154.2	127.7
AIC	9.42	12.27	13.09	24.14	26.52	29.01
BIC	10.95	12.97	14.54	26.55	27.31	31.18

Table 4.4.6 In-Sample Tests Results

This table presents the in-sample results from 1990 to 2012 of the models in comparison. The KS statistics are calculated for every year of each model against the realized recovery rates in that year. The corresponding *p* values are reported below. For example, for a model with a *p*-value larger than 0.10, we cannot reject the null hypothesis that the model-implied recovery rates and the realized recovery rates are from the same distribution at a confidence level of α = 10%.

In Sample								
	No. of observations	Mixed beta 5-factor best model	Mixed beta 5-factor second-best model	Mixed beta 5-factor third best model	Logit-link best model	Logit-link second-best model	Logit-link third best model	Constant single Beta model
1990	173	0.273	0.261	0.248	0.061	0.008	0.011	0.009
1991	241	0.181	0.173	0.179	0.043	0.006	0.005	0.011
1992	203	0.089	0.079	0.072	0.098	0.009	0.008	0.013
1993	160	0.248	0.254	0.243	0.032	0.011	0.012	0.011
1994	67	0.122	0.109	0.102	0.006	0.009	0.011	0.008
1995	83	0.101	0.103	0.082	0.011	0.014	0.008	0.009
1996	80	0.075	0.052	0.061	0.013	0.011	0.009	0.012
1997	69	0.071	0.041	0.037	0.097	0.012	0.007	0.011
1998	71	0.233	0.187	0.191	0.031	0.009	0.013	0.005
1999	179	0.284	0.292	0.294	0.044	0.024	0.011	0.014
2000	286	0.078	0.131	0.147	0.009	0.031	0.014	0.080
2001	527	0.263	0.209	0.211	0.021	0.019	0.011	0.013
2002	632	0.292	0.304	0.263	0.033	0.033	0.012	0.006
2003	347	0.203	0.182	0.182	0.011	0.009	0.009	0.012
2004	150	0.174	0.098	0.081	0.009	0.008	0.008	0.014
2005	129	0.193	0.164	0.154	0.006	0.009	0.012	0.005
2006	65	0.121	0.112	0.092	0.012	0.012	0.011	0.009
2007	33	0.087	0.051	0.044	0.011	0.011	0.011	0.008
2008	122	0.373	0.334	0.327	0.108	0.103	0.008	0.013
2009	401	0.298	0.278	0.282	0.083	0.052	0.022	0.007
2010	52	0.276	0.251	0.241	0.007	0.018	0.011	0.011
2011	26	0.202	0.193	0.173	0.022	0.022	0.009	0.014
2012	32	0.099	0.042	0.068	0.011	0.011	0.008	0.005

Table 4.4.7 Out-of-Sample Tests Results

This table presents the out-of-sample results from 1995 to 2012 of the models in comparison. The KS statistics are calculated for every year of each model against the realized recovery rates in that year. The corresponding *p* values are reported below. For example, for a model with a *p*-value larger than 0.10, we cannot reject the null hypothesis that the model-implied recovery rates and the realized recovery rates are from the same distribution at a confidence level of α = 10%.

Out of Sample								
	No. of observations	Mixed beta 5-factor best model	Mixed beta 5-factor second-best model	Mixed beta 5-factor third best model	Logit-link best model	Logit-link Second best model	Logit-link third best model	Constant single Beta model
1995	83	0.014	0.007	0.012	0.011	0.008	0.005	0.014
1996	80	0.011	0.005	0.009	0.005	0.012	0.009	0.011
1997	69	0.005	0.009	0.005	0.009	0.011	0.009	0.012
1998	71	0.022	0.014	0.013	0.013	0.008	0.013	0.013
1999	179	0.153	0.136	0.138	0.012	0.011	0.011	0.007
2000	286	0.065	0.101	0.062	0.014	0.007	0.012	0.009
2001	527	0.162	0.122	0.131	0.007	0.005	0.011	0.009
2002	632	0.134	0.118	0.103	0.009	0.012	0.008	0.006
2003	347	0.071	0.085	0.098	0.005	0.013	0.007	0.013
2004	150	0.062	0.061	0.063	0.012	0.008	0.005	0.011
2005	129	0.136	0.125	0.134	0.006	0.014	0.013	0.014
2006	65	0.059	0.063	0.047	0.011	0.007	0.012	0.009
2007	33	0.063	0.051	0.033	0.014	0.009	0.014	0.008
2008	122	0.287	0.268	0.261	0.027	0.031	0.007	0.005
2009	401	0.257	0.243	0.242	0.022	0.018	0.009	0.013
2010	52	0.235	0.211	0.188	0.015	0.023	0.011	0.014
2011	26	0.202	0.182	0.171	0.023	0.005	0.005	0.011
2012	32	0.095	0.044	0.068	0.021	0.024	0.008	0.008

Figure 4.1.1 Beta Distributions

This figure shows the different shapes of beta distributions result from different pairs of shape parameters. The shape parameters of beta distributions are as follow:

- Solid line - $\alpha = 0.5, \beta = 0.5$
- Long dash line - $\alpha = 2, \beta = 5$
- Dotted line - $\alpha = 1, \beta = 3$
- Dashed line - $\alpha = 5, \beta = 1$
- Dot dash line - $\alpha = 2, \beta = 2$

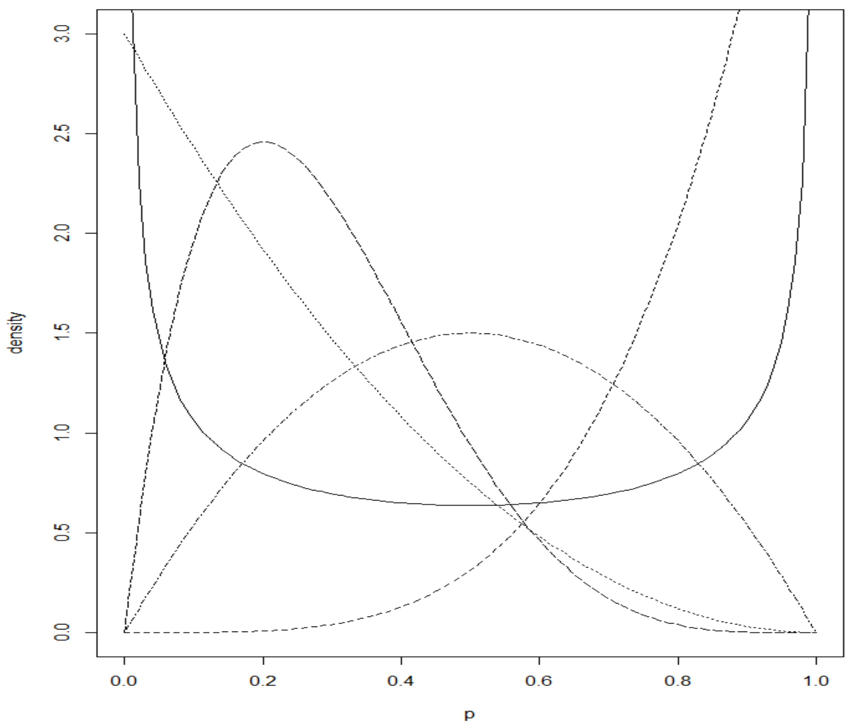


Figure 4.2.1 Five-factor Mixture Beta Distribution

This figure plots an example of five-factor mixture beta distribution.

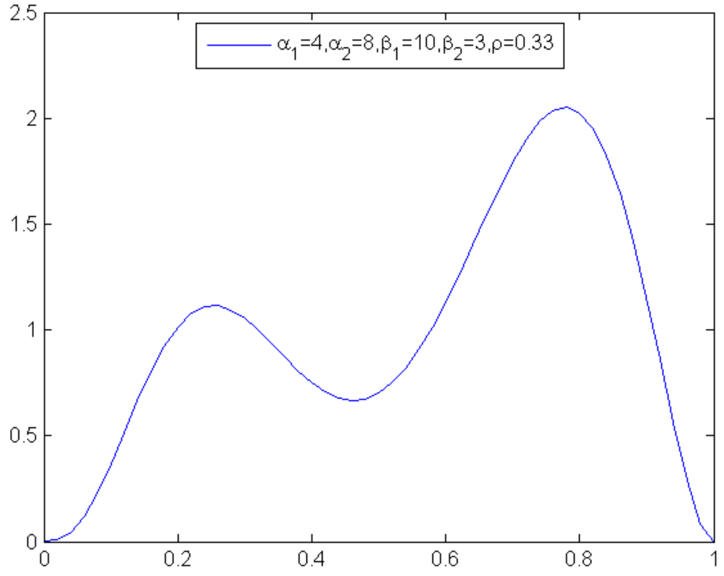


Figure 4.2.2 Individual Beta Distributions in Five-factor Model

This figure shows two individual beta distributions in a five-factor model.

- Dash line - $\alpha = 4, \beta = 10$
- Solid line - $\alpha = 8, \beta = 3$

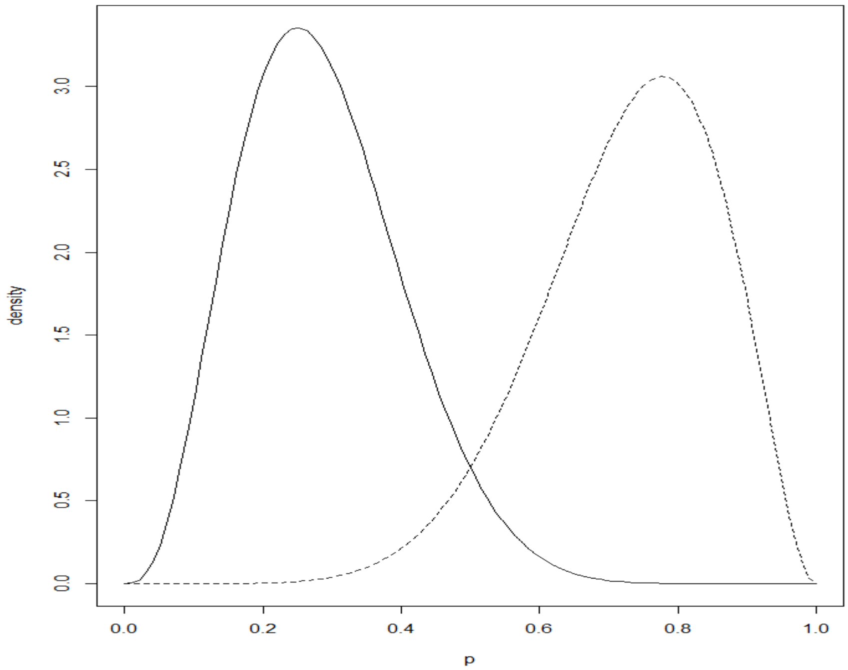


Figure 4.2.3 Three-factor Mixture Beta Distribution

This figure plots an example of three-factor mixture beta distribution.

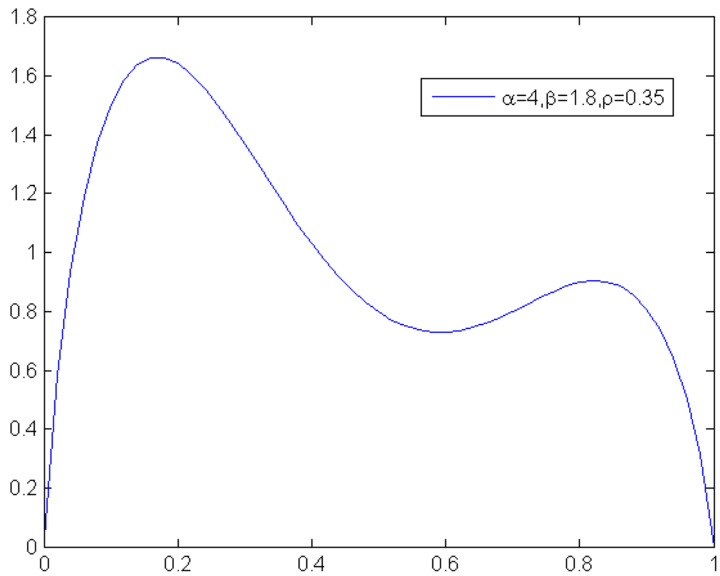


Figure 4.2.4 Individual Beta Distributions in Three-factor Model

This figure shows two individual beta distributions in a three-factor model.

- Dash line - $\alpha = 4, \beta = 1.8$
- Solid line - $\alpha = 1.8, \beta = 4$

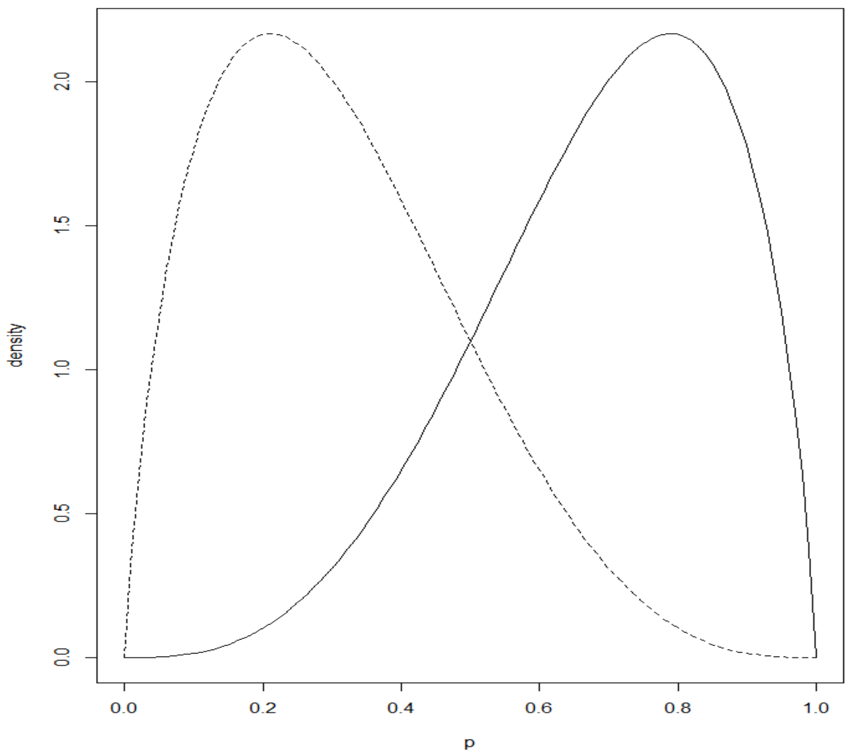
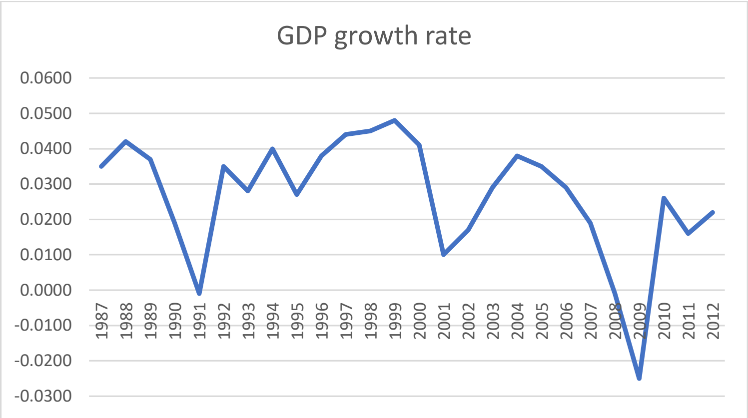
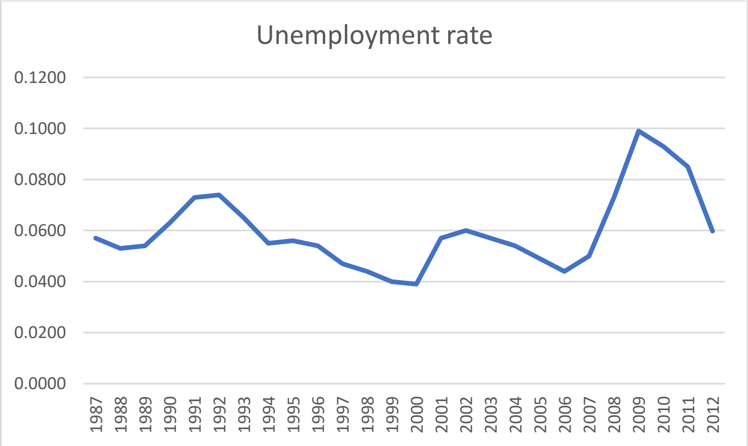
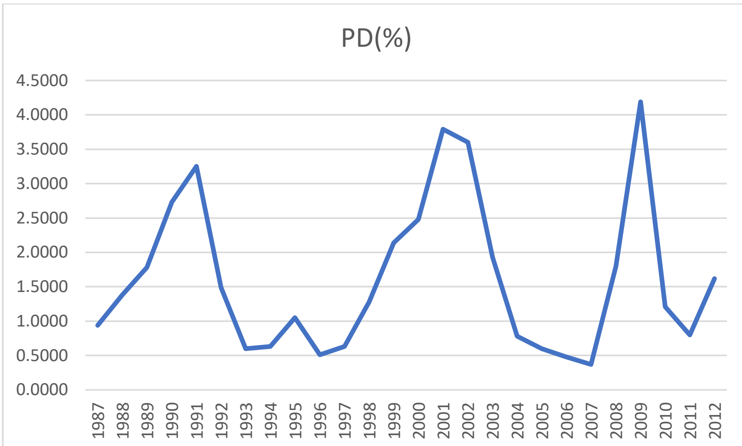


Figure 4.3.1 Time-Series Variations of Macroeconomic Variables

These graphs present the time-series variation of the four macroeconomic variables – *GDP*, *UE*, *PD*, and *SP500* – from 1987 to 2012. We can see that GDP and SP500 show similar time-series variations that are negatively correlated with those of UE and PD. This observation is consistent with our intuition as we expect GDP and SP500 to be high during economic expansion and low during the recession, and vice versa for UE and PD.






PD(%)
4.5000
4.0000
3.5000
3.0000
2.5000
2.0000
1.5000
1.0000
0.5000
0.0000
1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012

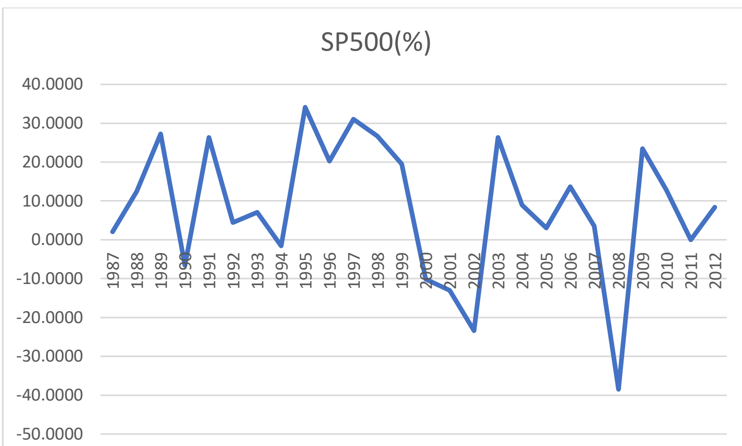
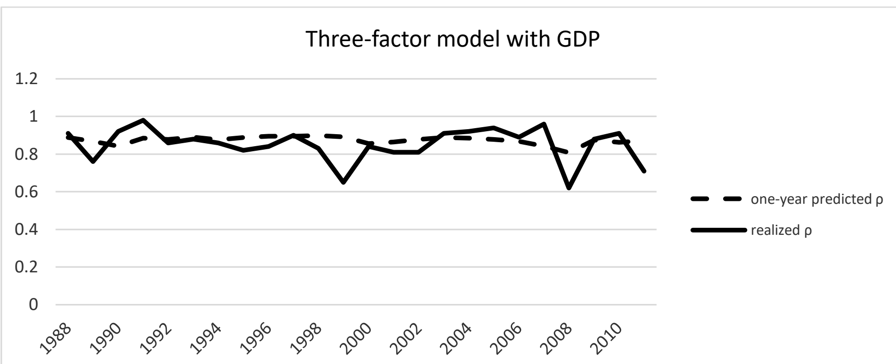
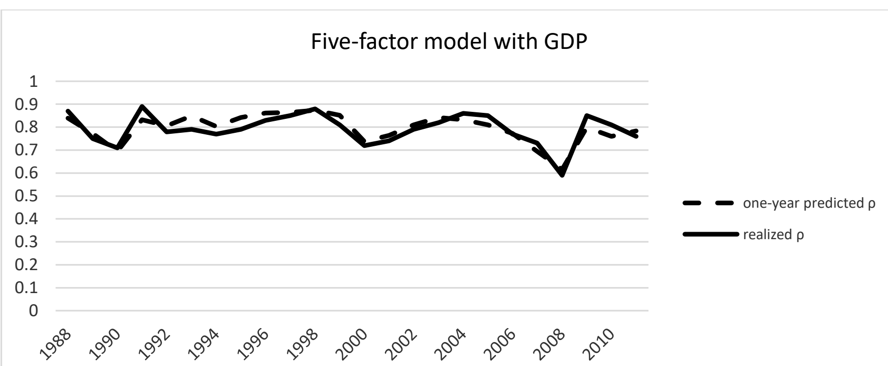
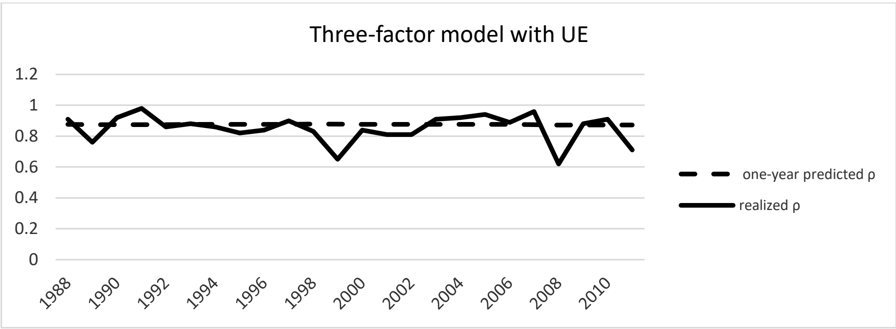

SP500(%)
40.0000
30.0000
20.0000
10.0000
0.0000
-10.0000
-20.0000
-30.0000
-40.0000
-50.0000
1987 1988 1989 1990 1991 1992 1993 1994 1995 1996 1997 1998 1999 2000 2001 2002 2003 2004 2005 2006 2007 2008 2009 2010 2011 2012

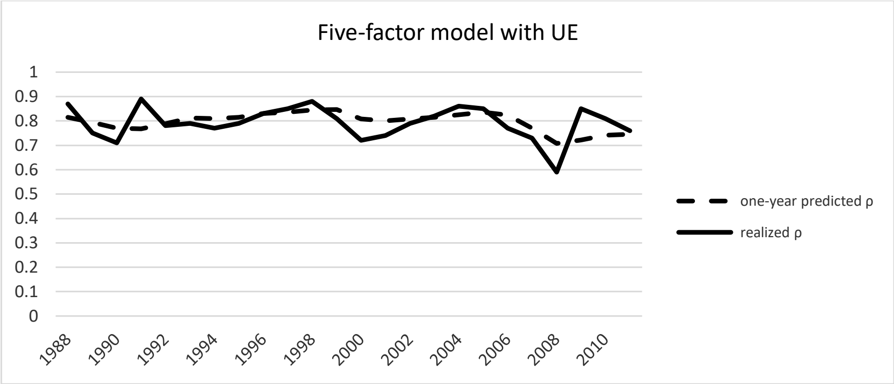
Figure 4.4.1 Predicted weights vs. Realized weights

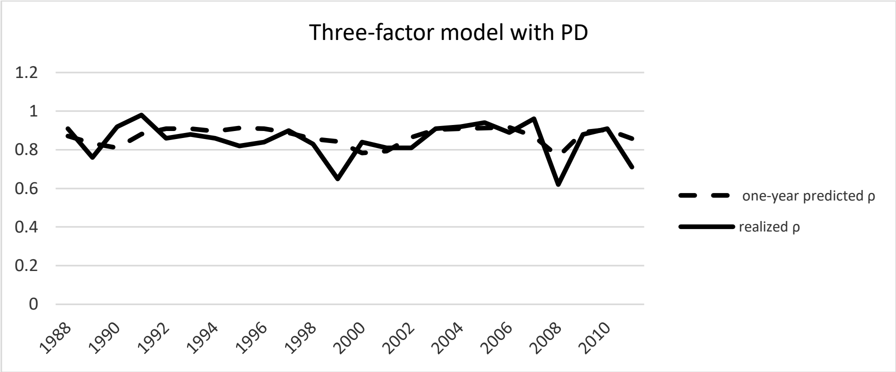
We present the time-series plots of the one-year prediction of the weight parameter ρ of each of the eight univariate mixture beta models, in comparison with the realized weight parameter. There are four three-factor models and four five-factor models based on the four macroeconomic variables: GDP, UE, PD, and SP500.

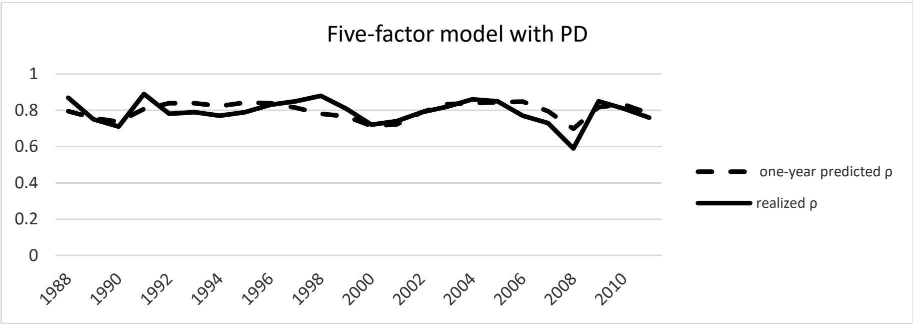


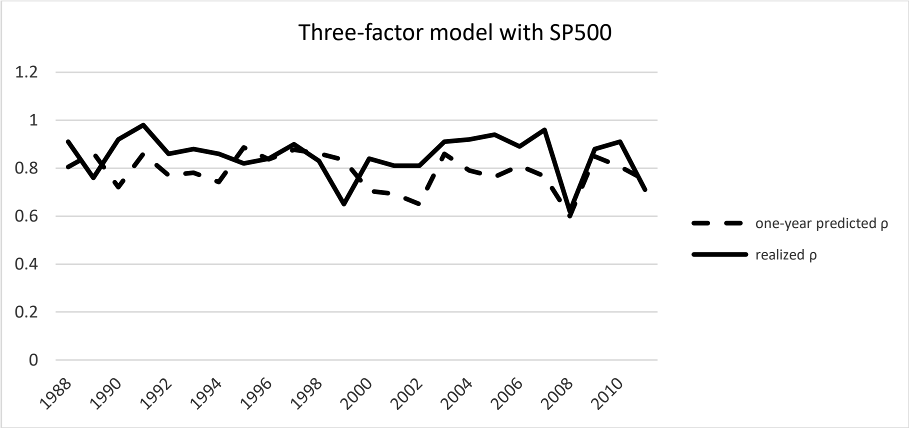



Three-factor model with UE
1.2
1
0.8
0.6
0.4
0.2
0
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
one-year predicted ρ
realized ρ


Five-factor model with UE
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
one-year predicted ρ
realized ρ


Three-factor model with PD
1.2
1
0.8
0.6
0.4
0.2
0
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
one-year predicted ρ
realized ρ


Five-factor model with PD
1
0.8
0.6
0.4
0.2
0
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
one-year predicted ρ
realized ρ


Three-factor model with SP500
1.2
1
0.8
0.6
0.4
0.2
0
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
one-year predicted ρ
realized ρ

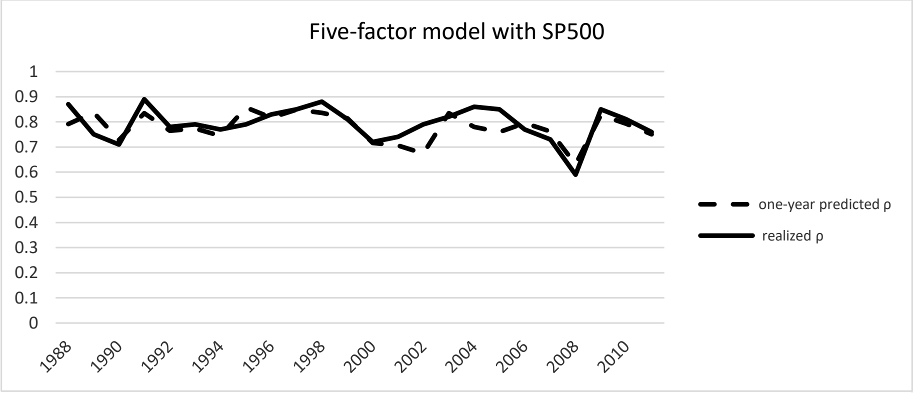

Five-factor model with SP500
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1988
1990
1992
1994
1996
1998
2000
2002
2004
2006
2008
2010
one-year predicted ρ
realized ρ

Figure 4.4.2 "Good" Beta Distribution vs. "Bad" Beta Distribution

This figure shows that the shapes of the "Good" beta distribution (solid line) and "Bad" beta distribution (dotted line) are quite different in the five-factor models

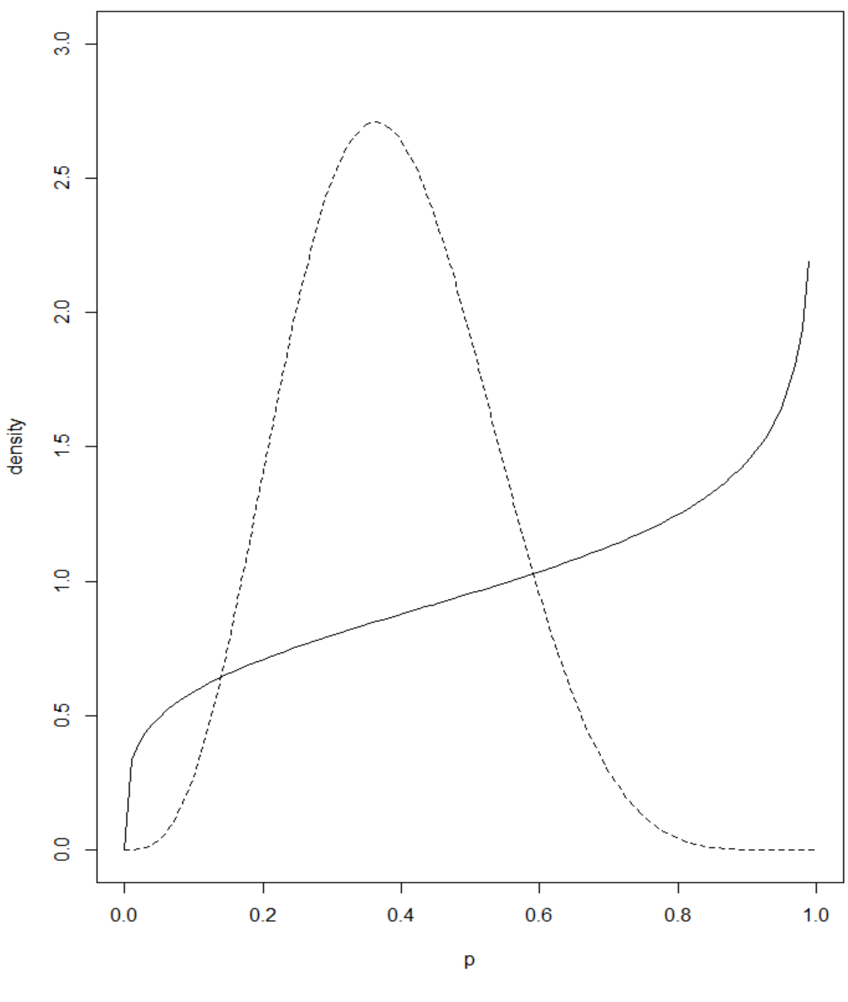
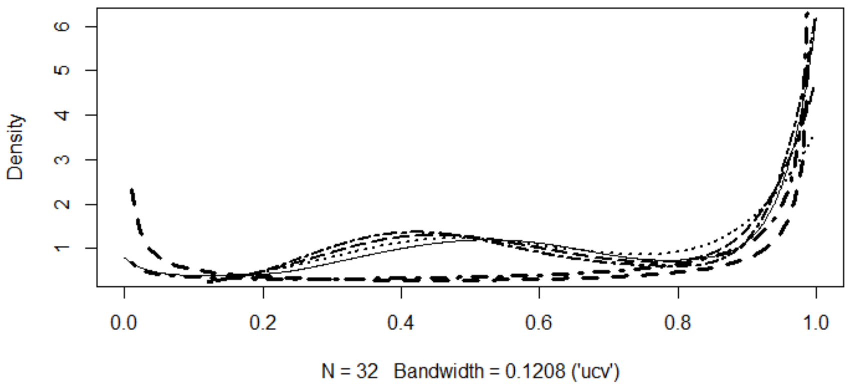


Figure 4.4.3 Out-of-Sample Models Comparison

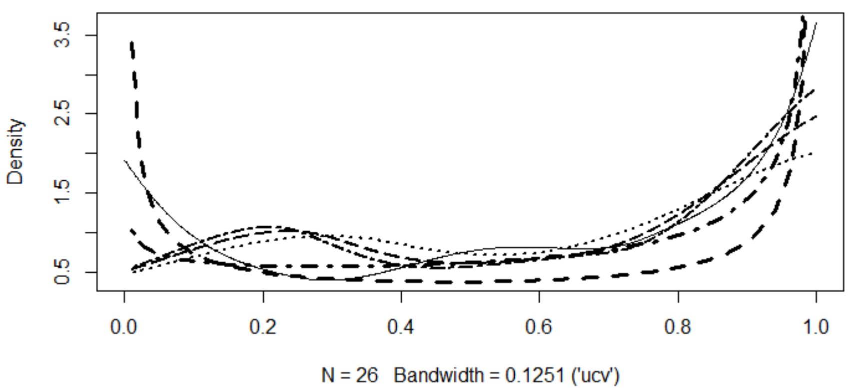
This figure shows the predicted distributions of all models against nonparametric recovery rates distribution (beta kernel) from 1995 to 2012. The legends are as follow:

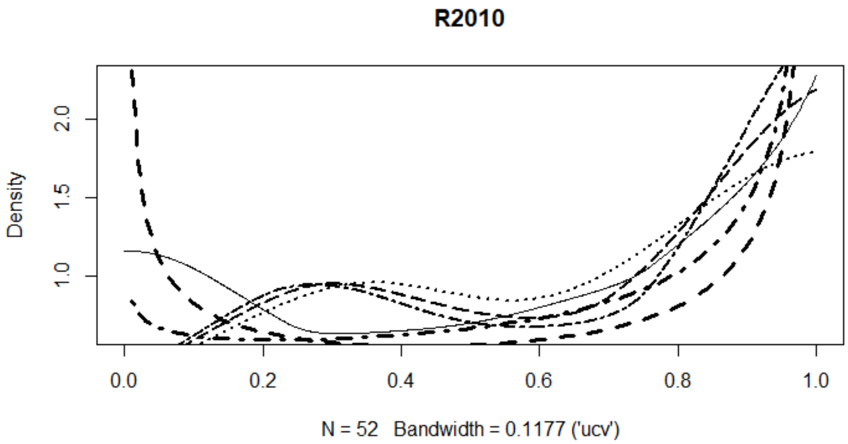
- Solid line - Nonparametric distribution
- Two dash line – Mixture beta five-factor model (Best)
- Long dash line – Mixture beta five-factor model (Second best)
- Dotted line – Mixture beta five-factor model (Third best)
- Dashed line – Single-beta logit-link model (Best)
- Dot-dash line – Single-beta logit-link model (Second best)

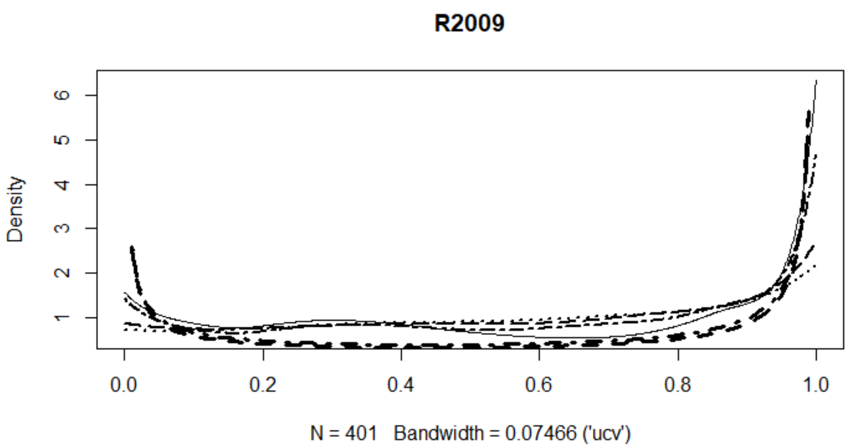
R2012

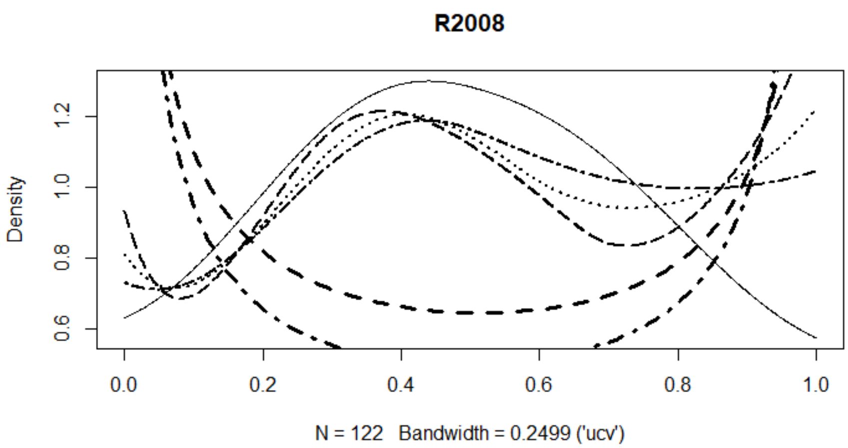

Density
0.0
0.2
0.4
0.6
0.8
1.0
N = 32 Bandwidth = 0.1208 ('ucv')

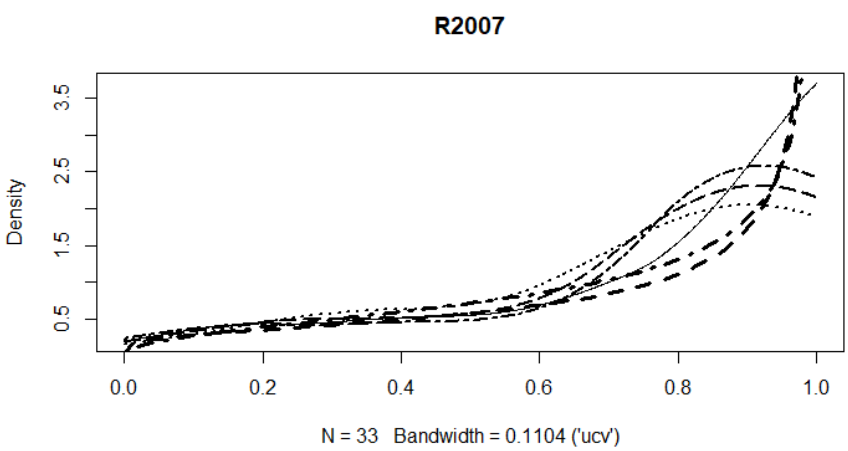
R2011

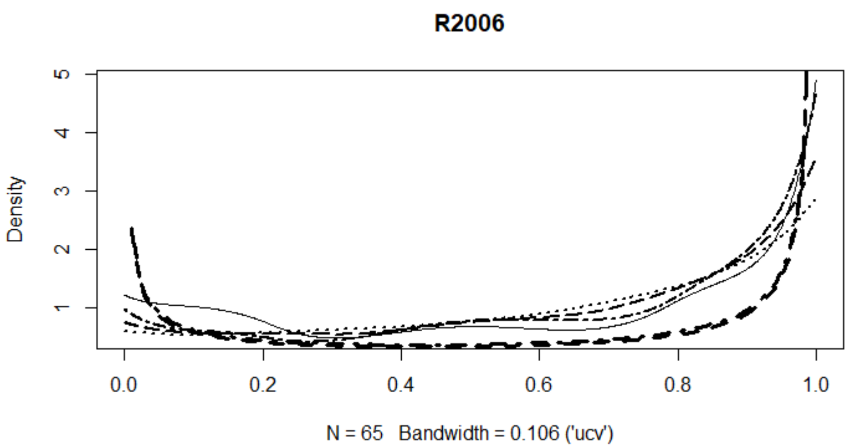

Density
0.5
1.5
2.5
3.5
0.0
0.2
0.4
0.6
0.8
1.0
N = 26 Bandwidth = 0.1251 ('ucv')

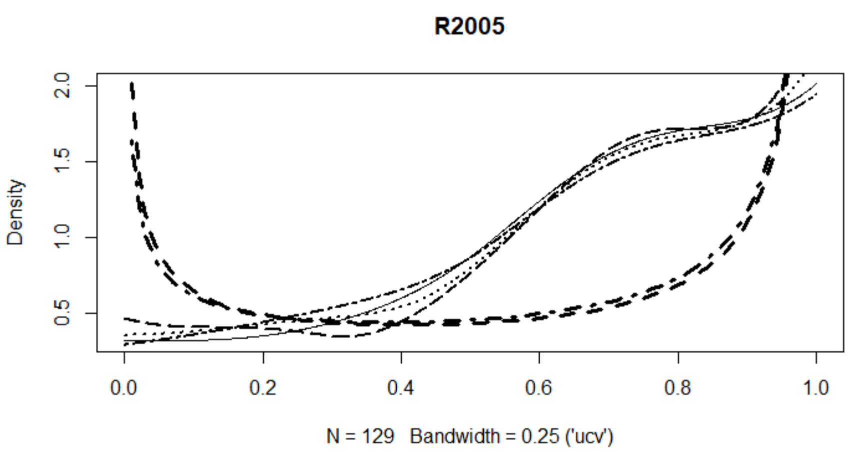

R2010
Density
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0
N = 52 Bandwidth = 0.1177 ('ucv')

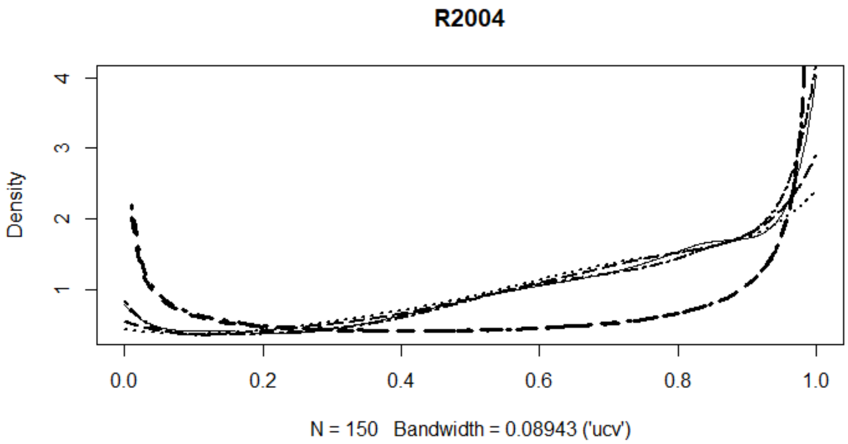

R2009
Density
6
5
4
3
2
1
0.0
0.2
0.4
0.6
0.8
1.0
N = 401 Bandwidth = 0.07466 ('ucv')

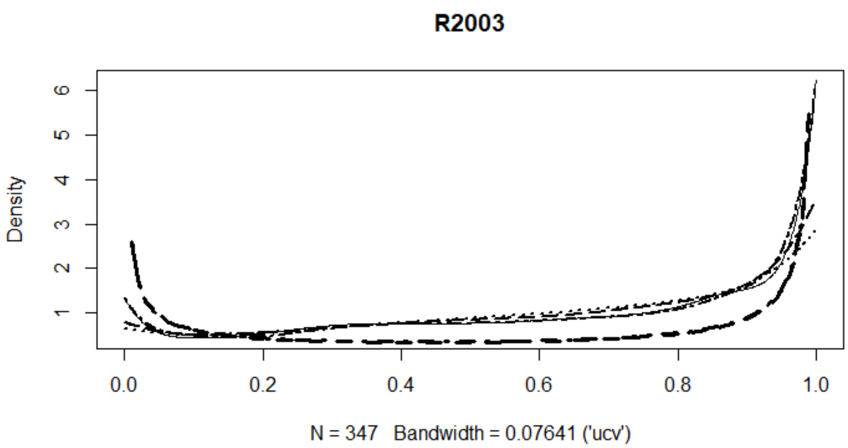

R2008
Density
0.6
0.8
1.0
1.2
0.0
0.2
0.4
0.6
0.8
1.0
N = 122 Bandwidth = 0.2499 ('ucv')

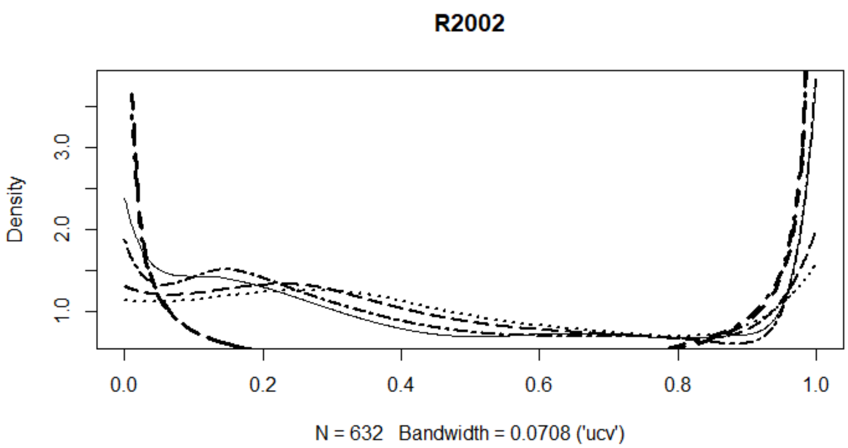

R2007
Density
0.5
1.5
2.5
3.5
0.0
0.2
0.4
0.6
0.8
1.0
N = 33 Bandwidth = 0.1104 ('ucv')

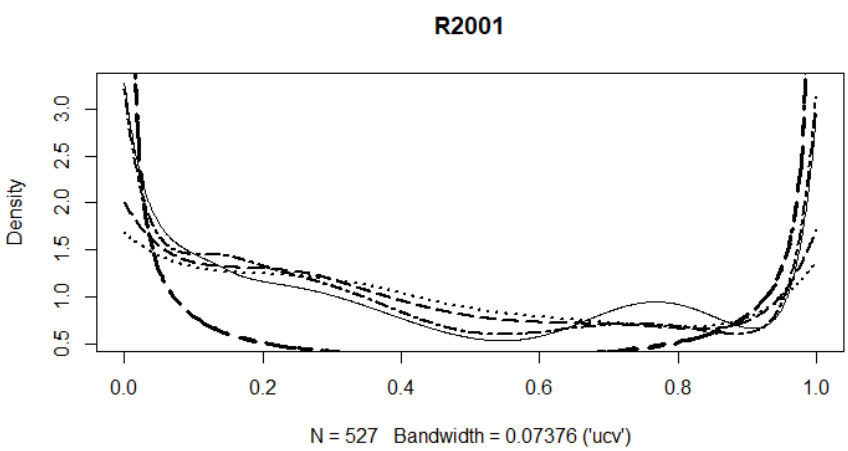

R2006
Density
N = 65 Bandwidth = 0.106 ('ucv')

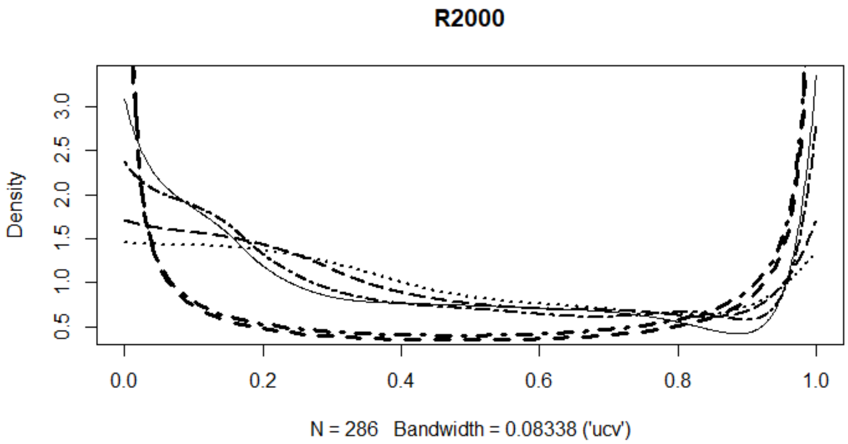

R2005
Density
N = 129 Bandwidth = 0.25 ('ucv')

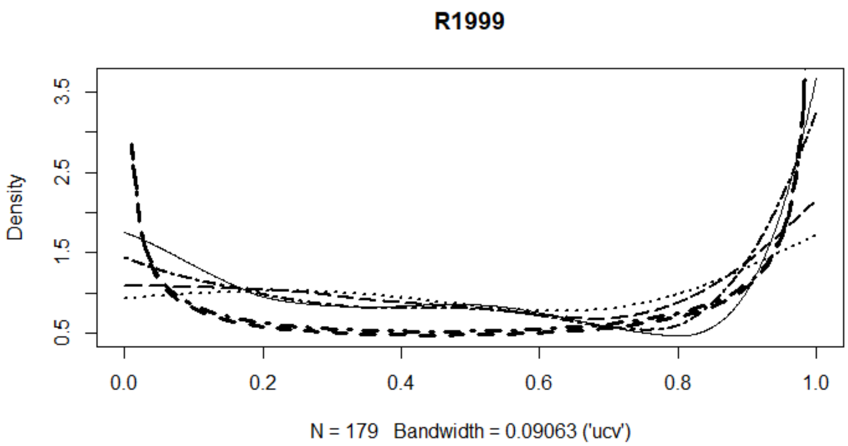

R2004
Density
4
3
2
1
0.0
0.2
0.4
0.6
0.8
1.0
N = 150   Bandwidth = 0.08943 ('ucv')

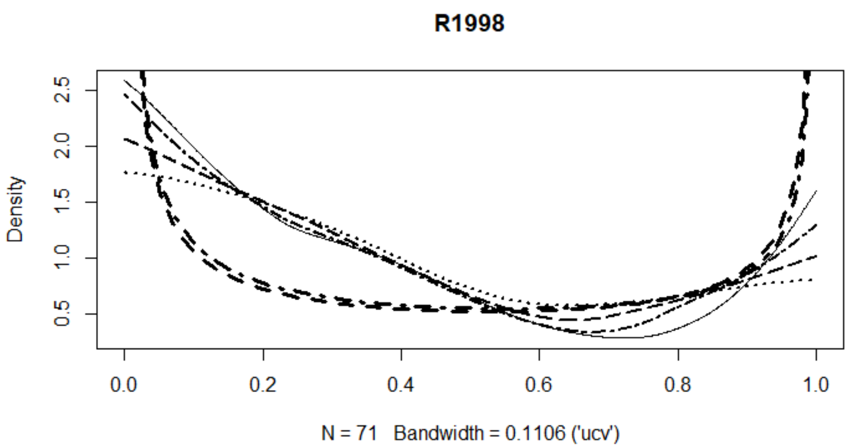

R2003
Density
6
5
4
3
2
1
0.0
0.2
0.4
0.6
0.8
1.0
N = 347   Bandwidth = 0.07641 ('ucv')

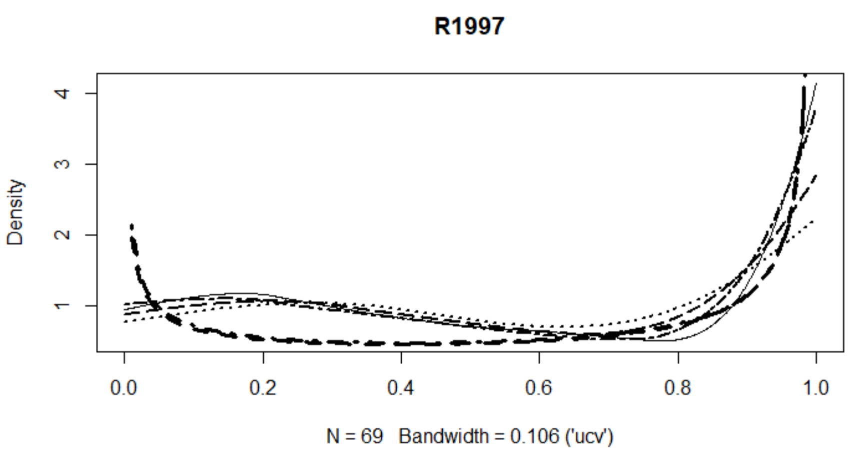

R2002
Density
3.0
2.0
1.0
0.0
0.2
0.4
0.6
0.8
1.0
N = 632 Bandwidth = 0.0708 ('ucv')


R2001
Density
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0
N = 527 Bandwidth = 0.07376 ('ucv')

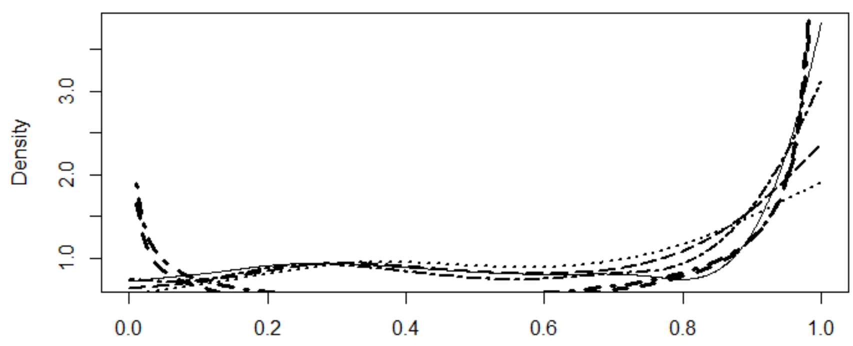

R2000
Density
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0
N = 286 Bandwidth = 0.08338 ('ucv')


R1999
Density
3.5
2.5
1.5
0.5
0.0
0.2
0.4
0.6
0.8
1.0
N = 179 Bandwidth = 0.09063 ('ucv')


R1998
Density
2.5
2.0
1.5
1.0
0.5
0.0
0.2
0.4
0.6
0.8
1.0
N = 71 Bandwidth = 0.1106 ('ucv')

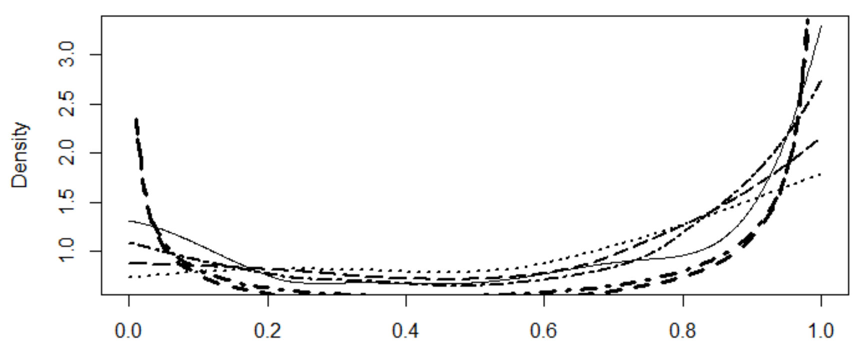

R1997
Density
4
3
2
1
0.0
0.2
0.4
0.6
0.8
1.0
N = 69 Bandwidth = 0.106 ('ucv')

R1996



N = 80 Bandwidth = 0.1027 ('ucv')

R1995



N = 83 Bandwidth = 0.1044 ('ucv')

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Chapter 5

Conclusions

This thesis focuses on three important issues in credit risk modeling: the nonlinear credit risk stress testing models, the recovery term structure of point-in-time loss given default (LGD), and the estimation of LGD by mixture beta regression model. In this chapter, we summarize the research conducted and the main findings of the three essays in Chapters 2, 3, and 4 of this thesis.

In chapter two, we investigate the performances of the regime-switching model and the quantile regression model in credit risk stress testing. We utilize these statistical modeling approaches under a credit risk stress testing framework and compare their performances with that of the traditional OLS model. We demonstrate that the regime-switching model is the best among all as it outperforms other models (in comparison) in producing the most accurate point estimation. Although we see an improvement in the model performance as we increase the granularity of the segments of the quantile regression models, it is still generally inferior to the regime-switching model.

In chapter three, we propose a conditional model to capture the time-series variations of the recovery rate profile that can incorporate up-to-date recovery information in predicting the ultimate recovery rate and we estimate the proposed model using a sample of defaulted facilities of a retail credit portfolio. Based on the simulation results covering a sample period including the recent financial crisis, we demonstrate that the proposed model can generate more realistic PIT portfolio risk measures over time in comparison to commonly used models. The proposed model captures both the dynamic evolution of the mean and the variance of recovery rate over time and is simple to implement for both facility-level and portfolio-level risk analysis. With the time-series panel regression setup, the model can be readily extended to incorporate other macroeconomic variables that may also drive the variations of recovery rate over time. The flexibility and effectiveness of the proposed model make it a viable candidate to replace the models currently used in practice.

In chapter four, we propose a new approach – a dual-beta mixture regression model – in the modeling of LGD distribution which is different from all other previous models. We show that this approach is flexible enough to accommodate the important features of the LGD distribution. Rather than just treating it as a static LGD distribution model, we also examine how we can allow the mixture beta distribution to be driven by the systematic risk factors through linking the weights of the underlying beta distributions to the time-varying macroeconomic variables. The results enable some interesting insights that complement and extend the findings in the literature. For example, the intuitive impacts of the macroeconomic variables such as GDP, PD, UE, and SP500 on the realized recovery rate are confirmed under our model setup.

In summary, this thesis studies three important aspects of credit risk modeling. We propose several new models in estimating, predicting, and stress testing PD and LGD. For PD stress testing purposes, our model extends previous studies by utilizing the regime-switching and the quantile regression techniques. For LGD, we conduct, to our knowledge, the first study to investigate the recovery term structure during the recovery process. Finally, in the third essay, we shift our focus to LGD estimation by incorporating mixture beta distribution and linking the weights of the underlying distributions with the macroeconomic variables to reflect the credit cycle dynamically. All models proposed are statistically proven to be either superior to models that are best practice in use (PD stress testing models and LGD mixture beta regression model) or be useful in addressing risk management issues currently encountered by the financial institutions (PIT recovery term structure model).

An interesting observation of these studies is that we find that the credit cycle, business cycle and the recovery cycle are not contemporaneous. Although this topic is not in the scope of our studies, it can improve our understanding of the mechanics of insolvency and recovery processes by investigating the interactions among these three different cycles and further assisting financial institutions to predict next cycles and prevent them from suffering potential unexpected losses.