Lyapunov-based Control of Nonlinear Processes Systems: Handling Input Constraints and Stochastic Uncertainty

Maaz Mahmood, B.Eng., B.Sc., M.Sc.

August 2020

Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

to the

Department of Chemical Engineering
Faculty of Engineering

McMaster University Hamilton, ON, Canada

Title:	Lyapunov-based Control of Nonlinear Processes Systems: Handling Input Constraints and Stochastic Uncertainty
Author:	Maaz Mahmood, B.Eng., B.Sc., M.Sc.
	McMaster University, Hamilton, ON, Canada
Degree (year):	Doctor of Philosophy (2020)
Department:	Chemical Engineering
Supervisor:	Dr. Prashant Mhaskar
Number of pages:	105

Abstract

This thesis develops Lyapunov-based control techniques for nonlinear process systems subject to input constraints and stochastic uncertainty. The problems considered include those which focus on the null-controllable region (NCR) for unstable systems. The NCR is the set of states in the state-space from where controllability to desired equilibrium point is possible. For unstable systems, the presence of input constraints induces bounds on the NCR and thereby limits the ability of any controller to steer the system at will. Common approaches for applying control to such systems utilize Control Lyapunov Functions (CLFs) . Such functions can be used for both designing controllers and also preforming closed–loop stability analysis. Existing CLF-based controllers result in closed–loop stability regions that are subsets of the NCR and do not guarantee closed–loop stability from the entire NCR. In effort to mitigate this shortcoming, we introduce a special type of CLF known as a Constrained Control Lyapunov Function (CCLF) which accounts for the presence of input constraints in its definition. CCLFs result in closed–loop stability regions which correspond to the NCR. We demonstrate how CCLFs can be constructed using a function defined by the NCR boundary trajectories for varying values of the available control capacity. We first consider linear systems and utilize available explicit characterization of the NCR to construct CCLFs. We then develop a Model Predictive Control (MPC) design which utilizes this CCLF to achieve stability from the entire NCR for linear anti-stable systems. We then consider the problem of nonlinear systems where explicit characterizations of the NCR boundary are not available. To do so, the problem of boundary construction is considered and an algorithm which is computationally tractable is developed and results in the construction of the boundary trajectories. This algorithm utilizes properties of the boundary pertaining to control equilibrium points to initialize the controllability minimum principle. We then turn to the problem of closed–loop stabilization from the entire NCR for nonlinear systems. Following a similar development as the CCLF construction for linear systems, we establish the validity of the use of the NCR as a CCLF for nonlinear systems. This development involves relaxing the conditions which define a classical CLF and results in CCLF-based control achieving stability to an to an equilibrium manifold. To achieve stabilization from the entire NCR, the CCLF-based control design is coupled with a classical CLF-based controller in a hybrid control framework. In the final part of this thesis, we consider nonlinear systems subject to stochastic uncertainty. Here we design a Lyapunov-based model predictive controller (LMPC) which provides an explicitly characterized region from where stability can be probabilistically obtained. The design exploits the constraint-handling ability of model predictive controllers in order to inherent the stabilization in probability characterization of a Lyapunov-based feedback controller. All the proposed control designs along with the NCR boundary computation are illustrated using simulation results.

Acknowledgements

I wish to express my sincere gratitude to my supervisor and mentor, Dr.Prashant Mhaskar, for the motivation, guidance and support he has provided me with over the years. Dr.Mhaskar has been incredibly patient and is responsible for igniting my interest in the topic of control. It has been a great privilege and honor to work with him over the years.

I would also like to thank Dr.Shahin Sirouspour and Dr.Thomas Adams who served on my committee and were the source of insightful comments, much encouragement and challenging questions.

I am grateful for my parents and my brothers for their unconditional support. Last but not least, I would like to thank my beloved wife and daughter without whom none of this would be possible.

Contents

Chapter 1

Introduction

The demand for more effective process control systems has increased over last decade. This is mainly driven by the requirement for industrial processes to become more economically efficient, safer and more reliable. This has motivated considerable research effort which has focused on developing both the theoretical foundations and the practical applications of control systems. The developments in this body of work address a plethora of challenges and problems which arise in real world process control systems. Among these problems, there does exist those which are application specific and others which are generic and apply to almost process control systems. Such generic issues include the need for the control system to provide adequate performance in the presence of complex process dynamics characterized by nonlinearity, constraints and uncertainty. The failure to appropriately account for these process characteristics in the control design can lead to poor control performance and even closed-loop instability.

The presence of nonlinear dynamics such has oscillations, steady state multiplicity is common in chemical process systems. The use of linear models can be applied in a vicinity of an operating point under certain assumptions, however the aforementioned challenges can only be adequately addressed using nonlinear models. This realization has motivated the development of nonlinear process control techniques. Such techniques build on the well established study of nonlinear dynamical systems. A common phenomena of nonlinear systems is the occurrence of multiple equilibria. Often, some of the equilibrium points are unstable and also are the desired operating points due to economic considerations. In general, the control problem is more difficult in the presence of open–loop instability [1].

In addition to process nonlinearites, the control and operation of chemical processes contains physical limitations due to the inherent physical limitation of all control actuators. Such constraints are hard in the sense they are always enforced and are in additional layer of complexity that control designs need to account for. The application of control schemes which ignore the presence of input constraints can result in input saturation which can have undesirable effects on both performance and stability of the closed–loop system. For open-loop unstable systems, saturation can cause the unstable modes to push the state away from the region from where any controller is able to steer the state back to the desired equilibrium point. This region is known as the null-controllable region (NCR) and is independent of what controller is used.

These considerations have motivated the development of several control designs strategies which account for nonlinearites and input constraints while also providing a region from where closed–loop stability can be achieved.

The resulting closed–loop stability regions are dependent on the control design and thus would be subsets of the NCR. Such control design predominately use Lyapunov-based techniques as this approach provides a common framework to perform both control design and stability analysis.

The classical direct method of Lyapunov uses the idea of a scalar energy function. Specifically, if this energy function along paths governed by a dynamical system is decreasing, then the system must be approaching equilibrium point. With this energy function one is able to analyze the stability of the system without solving the differential equations which define the dynamical system. This is incredibly helpful as analytical solutions to the dynamical system are seldom available. The challenge with Lyapunov-based techniques lies in being able to construct such an energy function which is satisfies condition on the time derivative.

In the context of control, the energy function is called a Control Lyapunov Function (CLF) and can be used to perform both control design and stability analysis. The classical definition of a CLF does not explicitly consider the presence of input constraints and only requires the time derivative be negative locally around an equilibrium point. This lack of consideration for input constraints can degrade CLF based control designs. This is especially important for unstable systems as the NCR is a finite subset of the state–space. In this context, the CLF would not capture to energy of the system close to the boundary of the NCR where the control action is critical to maintain stability. This gives rise to the problem of designing a controller where the closed–loop domain of attraction is equal or as close as possible to the NCR can be considered. A key contribution of this work is extend this energy function to limits of the controllability boundary to be able achieve stabilization from all states possible. To this end, we extend the definition of a CLF to consider input constraints and introduce the notion of a Constrained Control Lyapunov Function (CCLF). These functions explicitly account for the presence of input constraints by maximizing the estimate of the NCR over the set of all possible CLFs. The problem of CCLF construction is considered. The key idea in the proposed CCLF construction is to utilize the boundary of the NCR for varying values of the available control capacity to define an energy function. We show how this construction results in a meaningful CCLF and thus enables stabilization from the entire NCR. Moreover we provide explicit control designs which can be implemented to achieve stabilization from the entire NCR. This approach necessitates the availability of the characterization of the NCR. We first consider linear systems where explicit characterization of the NCR are available and utilized directly within the CCLF construction. We then utilize this construction Lyapunov-based model predictive controller coupled with an auxiliary control design to achieve stabilization from all initial conditions in the NCR. In general for nonlinear systems, explicit characterization of the NCR for nonlinear systems are not available. In this work we consider the problem of devising a computationally tractable procedure to generate the NCR for unstable nonlinear systems. Our approach extends the well-known Controllability Minimum Principle to generate the trajectories which form the boundary of the NCR. We then show use such boundary trajectories can be used to define a CCLF for general nonlinear systems. This development involves relaxing the conditions which define a classical CLF and results in CCLF-based control achieving stability to an to an equilibrium manifold. Stabilization from the entire NCR is then shown to be achieved using a hybrid control scheme which couples a classical CLF-based control design with a CLF based control design. Following this, we then utilize the NCR boundary characterizations to design a controller to enable stabilization from the entire NCR.

In addition to handling complex process dynamics characterized by nonlinearity and input constraints, its important to consider model uncertainty within the control design. The model uncertainty can be the result of unknown process parameters and exogenous disturbances and can cause poor control performance and even closed–loop instability. Therefore the design of robust controllers which account for such uncertainty has been the topic of considerable research effort ([2], [3]). The use of Lyapunov-based MPC (LMPC) designs [4], [5] has been a popular choice to address this problem as the design handles the presence of uncertainty, constraints and optimality considerations. Moreover, LMPC provides explicit characterizations of states from where stability can be achieved. Existing approaching on LMPC handle the uncertainty under the assumption of bounded disturbances resulting in conservative control action. In the context of unbounded stochastic uncertainty, the concept of stability must be understand in a probabilistic sense. That is, stability can be obtained with an associated probability. Lyapunov techniques for stochastic systems are well developed and can be used to derive regions in the state–space from where stability with an associated probability can be attained. In this work, we utilize such probability measures in a Lyapunov-based stochastic MPC design to obtain stabilization (in probability) of nonlinear stochastic systems with unbounded disturbances and allow for the characterization from where stability in probability can be obtained.

1.1 Thesis Outline

Motivated by the discussion above, in this thesis, we are considering the problem of designing Lyapunov-based control designs for nonlinear systems in the presence of input constraints and model stochastic uncertainity. The rest of this thesis is organized as follows:

Chapter 1: The problem of control of linear systems with input constraints is considered. The notion of control Lyapunov functions (CCLFs) is first generalized and relaxed to define a constrained control Lyapunov function (CCLF) and a constructive procedure for CCLF is subsequently presented.

Chapter 2: The construction of constrained control Lyapunov functions (CCLF) is utilized within a Lyapunov-based model predictive controller coupled with an auxiliary control design to achieve stabilization from all initial conditions in the null-controllable region. Illustrative simulation results as well as an application to a nonlinear chemical process example is presented to demonstrate the efficacy of the results.

Chapter 3: Here we focus on the problem of designing a constructive procedure for constructing the NCR of general nonlinear systems. To this end, a controllability minimum principle based computationally tractable approach for constructing the null controllable region is presented. Simulation results are used to illustrate the computation of the NCR for several examples.

Chapter 4: Using the NCR boundary constructive procedure for nonlinear systems in Chapter 3, we address the problem of designing a controller to enable stabilization from the entire NCR. First the validity of the use of the NCR as a constrained control Lyapunov function is established. The analysis reveals the ability of a CCLF based control design to drive the system to an equilibrium manifold. The CCLF based control design is then utilized within a hybrid control framework that guarantees the ability to stabilize to the origin. Simulation results are used to illustrate the implementation of the control design

Chapter 5: We design a Lyapunov-based model predictive controller (LMPC) for nonlinear systems subject to stochastic uncertainty. The LMPC design provides an explicitly characterized region from where stability can be probabilistically obtained. The key idea is to use stochastic *Lyapunov-based feedback* controllers, with well characterized stabilization in probability to design constraints in the LMPC that allows the inheritance of the stability properties by the LMPC. The application of the proposed LMPC method is illustrated using a nonlinear chemical process system example.

Chapter 6: The contributions of the research are summarized and directions of future work are presented. .

1.2 References

[1] G. Stein, "Respect the unstable", *IEEE Control Systems Magazine*, vol. 23, no. 4, pp. 12–25, 2003.

[2] L. Corless and G. Leitmann, "Controller design for uncertain systems via lyapunov functions", in *1988 American Control Conference*, 1988, pp. 2019–2025.

[3] C. Kravaris and S. Palanki, "Robust nonlinear state feedback under structured uncertainty", *AIChE Journal*, vol. 34, no. 7, pp. 1119–1127, 1988.

[4] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Stabilization of nonlinear systems with state and control constraints using Lyapunov-based predictive control", *Syst. & Contr. Lett.*, vol. 55, pp. 650–659, 2006.

[5] M. Mahmood and P. Mhaskar, "Enhanced stability regions for model predictive control of nonlinear process systems", *AIChE J.*, vol. 54, pp. 1487–1498, 2008.

Chapter 2

Constrained Control Lyapunov Functions for Linear Systems

The results in this chapter have been published in:

Journal Articles

[1] M. Mahmood and P. Mhaskar, "On constructing constrained control lyapunov functions for linear systems", *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1136–1140, 2011.

2.1 Introduction

Input constraints are ubiquitous in control and operation of all control systems. These constraints usually arise due to the physical limitation of control actuators such as pumps or valves. It is well established that neglecting these constraints while designing controllers can lead to significant performance deterioration and even closed–loop instability. This has motivated considerable research effort towards the problem of designing controllers in the presence of input constraints (see e.g. [2], [3] and references therein). Traditionally, Lyapunov theory has served as a powerful tool for stability analysis and control system design. The idea of a Lyapunov function was extended [4], [5] in the context of control design to yield control Lyapunov functions (CLF). For continuous-time linear time-invariant systems, there exist a well known method to construct CLFs, which essentially involves finding a positive definite solution of a Riccati equation. More recently, a universal construction procedure which involves solving a linear Lyapunov equation was derived [6]. However, both procedures are derived under the assumption of unconstrained control action.

When considering linear open-loop unstable systems, one measure of the suitability of a given CLF is how well stability regions estimate for a given CLF compares with the set of initial conditions from where the system can be stabilized in the presence of constraints (the so-called null controllability region). Currently, there exists no systematic framework to choose parameters when designing the control

Lyapunov functions to explicitly account for the presence of constraints to maximize the closed–loop stability region estimate. The topic of global [7]–[10] and semi-global [11] stabilization of LTI systems with bounded controls has been extensively studied under the assumption that the open-loop system is asymptotically null controllable with bounded controls (ANCBC). i.e., the system has to be stabilizable in the usual linear systems sense. It is established that for open–loop unstable systems, global and semi-global stability is generally not possible with constrained controls. That is, the presence of constraints limits the set of initial conditions from where a process can be stabilized at a desired equilibrium point irrespective of the type of input manipulation used. Thus, feedback controllers must be designed with the goal of achieving a closed–loop domain of attraction which is equal or as close as possible to the null-controllable region.

There exist several results regarding the design of linear feedback controllers which provide stability estimates using quadratic Lyapunov functions (e.g [12]–[14]). In the direction of constrained stabilization for unstable LTI systems, the concept of invariant sets has played a significant role (see e.g. [15]). Such sets are used to estimate the domain of attraction under linear feedback control. Recently [16] the problem of estimating the domain of attraction using invariant ellipsoids has been considered where a sufficient condition is derived in terms of an auxiliary feedback matrix for determining if a given ellipsoid is contractively invariant under saturated linear state feedback. This condition is used to formulate an optimization problem to find a maximal invariant ellipsoid set and also to simultaneously design the corresponding linear feedback gain. The quadratic functions which yield the invariant sets are simple to use, however, the ellipsoid estimates they provide are inherently conservative. This conservativeness can be partially alleviated by constructing composite Lyapunov functions based on a set of quadratic functions to provide better estimates of the closed–loop domain of attraction [17]. In addition, polyhedral sets have been employed to construct invariant sets [15], [18], [19]. Such sets are inherently more flexible as they can form any convex shape with the complex representation being the tradeoff.

In the direction of characterization of the null-controllable region for unstable constrained LTI systems, recent results [20] have provided a closed form expression for generating the null-controllable region. The problem of stabilization from the entire null-controllable region (for planar-unstable systems) has also been considered [21] where a saturated linear state feedback is designed that results in a closed-loop system having a domain of attraction that is arbitrarily close to the null controllable region. An important contribution of the characterization of the null controllable region [20] is that of providing a natural objective in the design of CCLF's- that of designing a CCLF that can be used to construct a control law to stabilize from all states in the null controllable region. In summary, a review of the existing literature yields several results on (essentially unconstrained) CLF construction that enable control designs with stability regions that approximate the null controllable region, there exist no results on the definition and construction of a constrained CLF for linear systems, which naturally yields the null-controllable region as the set of states from where stabilization is achievable.

Motivated by the above considerations, this work considers the problem of defining and developing a constructive procedure for constrained CLFs for unstable LTI systems. To this end, first in Section 3.2, we outline the class of systems and the required definitions needed. In Section 2.3 we then show how the boundary of the null-controllable region can be used to construct a constrained CLF.

2.2 Preliminaries

We consider continuous-time LTI systems with input constraints, described by:

$$\dot{x} = Ax(t) + Bu(t),\ u \in \mathcal{U} \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}^m$ denotes the vector of manipulated input taking values in a nonempty convex subset $\mathcal{U}$ of $\mathbb{R}^m$, where $\mathcal{U} = \{u \in \mathbb{R}^m : \|u\|_{\text{¥}} \leq u_{max}\}$, and $u_{max} \in \mathbb{R}^+$ denotes the upper bound on the magnitude of each manipulated input u_i. Without loss of generality, we assume that the input constraints for each manipulated input is identical. If this were not the case, the B matrix can be 'adjusted' to absorb the true (symmetric) input constraint. For a vector $x \in \mathbb{R}^n$ we denote by $\|x\|_{\text{¥}} = \max\limits_i \|x_i\|$ the infinity norm. The matrix norm induced by the Euclidean vector norm of a matrix $P \in \mathrm{R}^{n\times m}$ is given by $\|P\| = \sqrt{s_{max}}$, where s_{max} is the largest eigenvalue of the matrix P^TP (also known as the largest singular-value of P). We denote by $l_{min}(P)$ and $s_{min}(P)$ as the minimum eigenvalue and minimum singular-value of a matrix P respectively. We denote the trace of a matrix P by $\mathrm{tr}(P)$. The notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x^TQx$ for all vectors $x \in \mathbb{R}^n$, where Q is a positive definite symmetric matrix and x^T denotes the transpose of the vector x. We consider systems where A is anti-stable (all eigenvalues of A are in open right-half plane), and that (A, B) is a controllable pair. A state x_0 is said to be null controllable if there exists a $T \in [0, \text{¥})$ and an admissible control $u(t) \in \mathcal{U}$ such that the state trajectory $x(t)$ of the system of Eq.3.1 satisfies $x(0) = x_0$ and $x(T) = 0$, and the set of all null controllable states is called the null controllable region. We denote the null controllable region of the system of Eq.3.1 with input constraint u_{max} by $\mathcal{C}^{u_{max}}$. A state $x_e^{u_e}$ is said to be an equilibrium point with input u_e if $Ax_e^{u_e} + Bu_e = 0$. The set of all equilibrium points which are contained in $\mathcal{C}^{u_{max}}$ are denoted by $\mathcal{E}$. It follows that $\mathcal{E} = \{A^{-1}Bu_e : u_e \in \mathrm{R}^m\}$. Similarly, we denote the set of all equilibrium points with input values contained in the set $\mathcal{U}$, by $\mathcal{E}_{\mathcal{U}}$. It also follows that $\mathcal{E}_{\mathcal{U}} = \{A^{-1}Bu_e : u_e \in \mathcal{U}\}$. A set T^* is said to be positively controlled invariant if for all $x(0) \in T^*$, there exists an input trajectory $u(t)$, such that the state trajectory $x(t)$ of the system of Eq.3.1 satisfies $x(t) \in T^*$, for all $t \geq 0$. A supporting hyperplane of a convex set C is a plane such that C lies entirely on one side of the plane, and C contains at least one point on the hyperplane. The Minkowski sum of two convex sets C and D is defined as $C \oplus D = \{c + d : c \in C,\ d \in D\}$ (the resultant set is known to be convex). The boundary points of $C \oplus D$ can be computed from points on the boundaries of C and D where the outward unit normal vectors are equal. For convex sets with non-smooth boundaries, the notion of normal vectors must be generalized using supporting hyperplanes. Specifically, a vector is called a normal vector at a point x if it is normal to a hyperplane at x. We denote the interior, closure and boundary of a set X by $\mathring{X}$, $\overline{X}$ and $\mathrm{bd}(X)$ respectively. The notation $X \backslash Y$, where X and Y are sets, refers to the relative complement, defined by $X \backslash Y = \{x \in X : x \notin Y\}$. A compact and convex set $S \subset \mathbb{R}^n$ with the origin in the interior of the set is called a C-set. For any $x \in S$ the Minkowski functional or gauge functional is given by

$$j_S(x) = \inf\{ l > 0 : x \in lS\} \tag{2.2}$$

The level sets of j_s are essentially the set S linearly scaled. This function satisfies the following properties [22]:

Proposition 2.1. *[22] The Minkowski gauge function has the following properties:*

1. *Positive definiteness:* $0 \leq j_S(x) \leq$ ¥ *and* $j_S(x) > 0$ *for* $x \neq 0$
2. *Positive homogeneous:* $j_S(lx) = l\, j_S(x)$ *for* $l \geq 1$
3. *Sub-additivity:* $j_S(x_1 + x_2) \leq j_S(x_1) + j_S(x_2)$
4. *Lipschitz continuity*
5. *Convexity*

To accommodate non-smooth Lyapunov functions we recall the following generalized derivative, and subgradient:

Definition 2.1. *[23] For a locally Lipschitz function* $V : \mathrm{R}^n \to \mathrm{R}$, *the upper-right Dini directional derivative of* V *with respect to Eq.3.1 at* x *is*

$$D^+_{Ax+Bu}V(x) = \limsup_{h \to 0^+} \frac{V(x + h(Ax + Bu)) - V(x)}{h} \tag{2.3}$$

and we denote $D^+_{Ax+Bu}V(x) = D^+V(x)$.

Definition 2.2. *[23] For a locally Lipschitz function* $V : \mathrm{R}^n \to \mathrm{R}$, *the vector* $z \in \mathrm{R}^n$ *is a subgradient of* V *at* x *if*

$$V(y) - V(x) \geq z^T(y - x), \ \forall y \in \mathrm{R}^n \tag{2.4}$$

Furthermore, the subdifferential ¶$V(x)$ *is the set of all the subgradients at* x.

If V is differentiable at x, then $D^+V(x)$ reduces to the usual directional derivative $\nabla V(x)^T(Ax + Bu)$. Moreover, if V is convex (and possibly nondifferentiable), then $D^+V(x)$ can be computed as [23]:

$$D^+V(x) \quad = \quad \sup_{z \in \P V(x)} z^T(Ax + Bu) \tag{2.5}$$

The following condition is proposed for use in subsequent definitions (to enable the use of a relaxed version of LaSelle's invariance principle):

Condition 2.1. *Given a continuous, positive definite, and radially unbounded function* $V : \mathsf{R}^n \rightarrow \mathsf{R}$*, let E be the set of all points where* $D^+_{Ax}V(x) = \inf_{u \in \mathsf{R}^m} D^+_{Bu}V(x) = \inf_{u \in \mathsf{R}^m} D^+_{Ax+Bu}V(x) = 0$*. For all* $x_0 \in E \backslash 0$*, there exists an input trajectory* $u(t)$ *under which the state trajectory* $x(t)$ *escapes* $E \backslash 0$ *for all* $t \geq 0$ *and satisfies* $D^+V(x)|_{u(t)} \leq 0$*.*

We now state a generalized version of the classical definition of a control Lyapunov function (CLF) for the system in Eq.3.1.

Definition 2.3. *A continuous, convex, positive definite, and radially unbounded function* $V : \mathsf{R}^n \rightarrow \mathsf{R}$ *such that*

$$\inf_{u \in \mathbb{R}^m} D^+V(x) \leq 0 \tag{2.6}$$

for all $x \in U \subseteq \mathsf{R}^n$*, where U is compact, and Condition 1 is satisfied is a CLF for the system in Eq.3.1.*

Note that the infimum in Eq.3.3 is not taken over a bounded constrained input variable set $\mathcal{U}$ but rather over all values in $\mathbb{R}^m$. That is, we present a generalized definition of a CLF (where the input constraints are not accounted for) in preparation to our definition of a constrained CLF. Finally, using Condition 1 together with Lasalles invariance principle, the requirement of strict negative definiteness of time derivative of a CLF is relaxed. Condition 1 ensures that for all points within the set where the time derivative of the function V can at best be made zero, there exists an input trajectory which can make the states escape this set while maintaining the inequality of Eq.3.3.

Consider W defined as the set induced by the level sets of V,

$$\mathsf{W}(V, c) \quad = \quad \{x \in \mathsf{R}^n : V(x) \leq c\} \tag{2.7}$$

and P as the region of the state space where the time derivative can be made negative semi-definite,

$$\mathsf{P}(V) \quad = \quad \{x \in \mathsf{R}^n : \inf_{u \in \mathcal{U}} D^+V(x) \leq 0\} \tag{2.8}$$

It follows that $\mathsf{W}(V, c)$ is an estimate of the stabilizable region of the origin (using the CLF V) if $\mathsf{W}(V, c) \subseteq \mathsf{P}(V)$. Moreover, for a given Lyapunov function V, the maximal estimate of the stabilizable region can be determined with the largest level set of V which is completely contained within $\mathsf{P}(V)$. We will denote this maximum level set by $c_{max}(V)$:

$$c_{max}(V) = \sup\{c \in \mathbb{R} : \mathsf{W}(V,c) \subseteq \mathsf{P}(V)\} \tag{2.9}$$

Let $g(V,c)$ denote the volume function of $\mathsf{W}(V,c)$ given by:

$$g(V,c) = \int \cdots \int_{\mathsf{W}(V,c)} dx_1 dx_2 \ldots dx_n \tag{2.10}$$

For $c \leq c_{max}(V)$, $g(V,c)$ is the volume of the estimate of the stabilizable region. We are now ready to postulate the definition of a constrained control Lyapunov function (CCLF).

Definition 2.4. *Let $\mathcal{V}$ denote the set of all locally Lipschitz CLFs. A locally Lipschitz CLF $V_c : \mathsf{R}^n \to \mathsf{R}$ such that*

$$g(V_c, c_{max}(V_c)) = \max_{V \in \mathcal{V}} g(V, c_{max}(V)), \tag{2.11}$$

is a CCLF for the system in Eq.3.1.

Remark 2.1. *The generalized definition of a CLF presented above is adapted from an earlier definition [24]. The definition in the present work differs from the traditional definition of a CLF in that the differentiability requirement is relaxed with the use of the Dini derivative [23] and the strict negative definiteness is also relaxed by imposing conditions analogous to the LaSalle Invariance Principle (in a control design, this would necessitate utilizing an auxiliary controller to handles states in this set where strict negative definiteness of the CCLF is not achievable in conjunction with a controller designed to cause a decay in the CCLF). Since a CCLF is by definition a CLF which maximizes the volume of the estimate of the stability region, the generalized definition of a CLF widens the search space for this maximization. As will be shown in Section 2.3, these relaxed requirements are key for the construction of CCLFs that result in the stabilization from the entire null-controllable region. Note also that while there exists construction procedures for CLFs, there is a lack of results on the construction of CCLFs, and simply 'saturating' the control action in a control design that uses an 'unconstrained' CLF does not yield the largest possible stability region.*

2.3 Using the null-controllable region to construct CCLFs

In this section, we present a construction procedure for CCLFs (based on null-controllable region characterizations) that can be used to design controllers that possess a stability region equal to the null controllable region. The key idea is to define a gauge function using the null-controllable set. The time derivative of this function is shown to achieve negative semi-definiteness over the entire null controllable region, as well as coincide with the 'level sets' of the function. As a result, the estimate of the controllable region generated by this function coincides with the null-controllable region and hence is maximal, making this function a CCLF.

We consider the system of Eq.3.1 with input constraint $u_{max} = 1$, and for ease of notation we let $\mathcal{C}^1 = \mathcal{C}$. The set $\mathcal{C}$ is characterized as (see [20]):

$$\mathcal{C} = \bigcup_{T\in[0,\infty)} \{x = -\int_0^T e^{-At}Bu(t)dt : u(t) \in \mathcal{U}\} \tag{2.12}$$

If A is anti-stable, it can be shown that this set is bounded, strictly convex, and open with the origin in the interior of the set. Furthermore, it can be shown [20] that the null controllable region of the multi-input system of Eq.3.1 is the Minkowski sum of the null controllable regions of the single input subsystems

$$\dot{x}(t) = Ax(t) + b_i u_i(t), \ |u_i(t)| \leq 1 \tag{2.13}$$

where $B = [b_1 \ b_2 \ \ldots b_m]$ and u_i denotes the ith component of the vector u. Specifically, let $\mathcal{C}_i$ denote the null controllable region of the subsystem of Eq.2.13 then

$$\begin{aligned} \mathcal{C} &= \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \cdots \oplus \mathcal{C}_m \\ &= \{x_1 + x_2 + \cdots + x_m : x_i \in \mathcal{C}_i, i = 1, \ldots, m\} \end{aligned} \tag{2.14}$$

Hence the convexity of the null-controllable region for multi-input systems is preserved from the null-controllable region of the single input subsystems.

Using the null-controllable region $\mathcal{C}$ in conjunction with gauge functionals, we define the following candidate CCLF:

$$V_{\mathcal{C}}(x) = \phi_{\mathcal{C}}(x) = \inf\{\lambda > 0 : x \in \lambda\overline{\mathcal{C}}\} \tag{2.15}$$

The continuity, positiveness definiteness, and radially unboundedness of $V_{\mathcal{C}}$ follow from Proposition 2.1. It is established [20] that the boundary of the null-controllable region for single input systems is covered by extremal trajectories of the respective time reversed system. The magnitude of the input variable for such extremal trajectories is shown to be equal to the magnitude of the input constraint. Hence, differentiability of the function $V_{\mathcal{C}}$ should be expected. However, this is not the case as the boundary of $\mathcal{C}$ can contain corner points, as shown by the following argument. Note (as shown in [20], Theorem 3.1) that the boundary of the set $\mathcal{C}_i$ can be determined by a function $\Phi : S^n \to \mathbb{R}^n$ which maps the surface of a unit ball S^n to the boundary of $\mathcal{C}_i$. This mapping is given by

$$\Phi(h) = \int_{-\infty}^{0} e^{At} b_i sign(he^{At}b_i)dt \tag{2.16}$$

The function Φ maps S^n continuously but not one-to-one (in general) onto the boundary of $\mathcal{C}_i$. It can be shown that each $\bar{x} = \Phi(\bar{h})$ which is on the boundary of $\mathcal{C}_i$ has an outward unit normal vector equal to $\hbar$. Since the mapping may not be one-to-one, the boundary of $\mathcal{C}_i$ can contain points which have a non-unique normal. Since the Minkowski sum will not "smooth-out" such points, the function $V_{\mathcal{C}}(x)$ is in general non-differentiable.

It follows that the level set $V_{\mathcal{C}}(x) = a$ defines the boundary of the null-controllable region with input constraint $u_{max} = a$, which is also the boundary of $\mathcal{C}^a$.

$$\mathsf{W}(V_{\mathcal{C}}, c) \quad = \quad \{x \in \mathsf{R}^n : V_{\mathcal{C}}(x) \leq c\} = \overline{\mathcal{C}^c} \tag{2.17}$$

Theorem 2.1 below states that the set $\mathsf{P}(V_{\mathcal{C}})$ for the function $V_{\mathcal{C}}$ contains completely the set $\mathsf{W}(V_{\mathcal{C}}, u_{max})$, and that Condition 1 is satisfied for the function $V_{\mathcal{C}}$.

Theorem 2.1. *For the system of Eq.3.1 with input constraint u_{max}, for every x in $\mathring{\mathsf{W}}(V_{\mathcal{C}}, u_{max})$, there exists a $u \in \mathcal{U}$ for which the time derivative of $V_{\mathcal{C}}$ achieves negative semi-definiteness. That is,*

$$\mathring{\mathsf{W}}(V_{\mathcal{C}}, u_{max}) \subseteq \mathsf{P}(V_{\mathcal{C}}) \tag{2.18}$$

Furthermore, the function $V_{\mathcal{C}}$ satisfies Condition 1.

Proof. *The proof of this Theorem is divided in two parts. In the first part we show that $\mathring{\mathsf{W}}(V_{\mathcal{C}}, u_{max}) \subseteq \mathsf{P}(V_{\mathcal{C}})$. In the second part we show that the function $V_{\mathcal{C}}$ satisfies Condition 1.*

Part 1: Let $x \in \mathring{\mathsf{W}}(V_{\mathcal{C}}, u_{max})$. Since the set $\mathring{\mathsf{W}}(V_{\mathcal{C}}, u_{max}) = \mathcal{C}^{u_{max}}$, it follows that $x \in \mathcal{C}^{u_{max}}$. We must show that $\inf_{u \in \mathcal{U}} D^+ V_{\mathcal{C}}(x) \leq 0$. Since x is in the interior of the set $\mathcal{C}^{u_{max}}$, it follows that there exists a $u^*_{max} < u_{max}$, such that $x \in \mathrm{bd}(\mathcal{C}^{u^*_{max}})$. Since $\mathcal{C}^{u^*_{max}}$ is the Minkowski sum of the sets $\mathcal{C}_i^{u^*_{max}}$ for $i = 1, \ldots, m$, we can decompose x as the sum of m points, each of which lies on the boundary of $\mathcal{C}_i^{u^*_{max}}$: $x = x_1 + \cdots + x_m$, where $x_i \in \mathrm{bd}(\mathcal{C}_i^{u^*_{max}})$, $i = 1, \ldots, m$. Recall that the boundary of the Minkowski sum of the sets $\mathcal{C}_i^{u^*_{max}}$ is computed from points on the boundaries of $\mathcal{C}_i^{u^*_{max}}$ where the outward unit normal vectors are equal. Hence, x along with each x_i have outward normal vectors which are parallel. Here the notion of a normal vector at a point is the generalized normal to the hyperplane at a point. Since the boundary of each $\mathcal{C}_i^{u^*_{max}}$ is covered by a extremal trajectory, and is convex, it follows that $\sup\limits_{z \in \P V_{\mathcal{C}}(x)} z^T(Ax_i + b_i u^*_{max_i}) = 0$, for some $u^*_{max_i}$ such that $|u^*_{max_i}| = u^*_{max}$. Let $u_i =$

$u^*_{max_i} + u'_i$, then $u'_i \in [-u_{max} - u^*_{max_i}, u_{max} - u^*_{max_i}] = \mathbf{U}' \ni 0$. Using Eq.2.5, we obtain

$$
\begin{aligned}
D^+V_{\mathcal{C}}(x) &= \sup_{z\in\partial V_{\mathcal{C}}(x)} z^T(Ax(t) + Bu(t)) \\
&= \sup_{z\in\partial V_{\mathcal{C}}(x)} z^T \left(\sum_{i=1}^{m} (Ax_i + b_i u_i(t))\right) \\
&= \sup_{z\in\partial V_{\mathcal{C}}(x)} z^T \left(\sum_{i=1}^{m} Ax_i + b_i(u^*_{max_i} + u'_i)\right) \\
&= \sup_{z\in\partial V_{\mathcal{C}}(x)} z^T \left(\sum_{i=1}^{m} Ax_i + b_i u^*_{max_i} + \sum_{i=1}^{m} b_i u'_i\right) \\
&= \sup_{z\in\partial V_{\mathcal{C}}(x)} z^T \sum_{i=1}^{m} b_i u'_i
\end{aligned}
\tag{2.19}
$$

We have established an upper-bound for the directional Dini derivative. We must show that the infimum over the constrained control set is negative semi-definite (essentially achieved by setting $u' = 0$ in the expression for the Dini-derivative):

$$
\begin{aligned}
\inf_{u\in\mathcal{U}} D^+V_{\mathcal{C}}(x) &\leq \inf_{u'\in\mathcal{U}'} \sup_{z\in\partial V_{\mathcal{C}}(x)} z^T \sum_{i=1}^{m} b_i u'_i \\
&\leq 0
\end{aligned}
\tag{2.20}
$$

Thus, $x \in P(V_{\mathcal{C}})$, and hence $\Omega(V_{\mathcal{C}}, u_{max}) \subseteq P(V_{\mathcal{C}})$.
Part 2: Let E be the set of all points where $D^+_{Ax}V_{\mathcal{C}}(x) = \inf_{u\in\mathbb{R}^m} D^+_{Bu}V_{\mathcal{C}}(x) = \inf_{u\in\mathbb{R}^m} D^+_{Ax+Bu}V_{\mathcal{C}}(x) = 0$. For every $x_0 \in E\backslash 0$, we must show there exists an input trajectory $u(t)$, such that the closed–loop trajectory $x(t)$ fails to remain within $E\backslash 0$ for all $t \geq 0$ while satisfying $D^+V_{\mathcal{C}}(x)|_{u(t)} \leq 0$. By definition, since x_0 is in the null-controllable region, there exists at least one admissible input trajectory which drives the system to the origin. Hence the trajectory under the input cannot remain in E for all times, as the Lyapunov function value must eventually decay. It remains to show that a stabilizing input trajectory can always be found while maintaining $D^+V_{\mathcal{C}}(x)|_{u(t)} \leq 0$ for all times. We proceed to show this in general true for all states in $\mathcal{C}^{u_{max}}$ by contradiction, i.e., we assume that for a given $x_0 \in \mathcal{C}^{u^*_{max}}$, all stabilizing input trajectories $u(t)$ result in $D^+V_{\mathcal{C}}(x(t))|_{u(t)} > 0$ and we denote the earliest time that this happens as T with x_T being the state (i.e., $D^+V_{\mathcal{C}}(x(t))|_{u(t)} > 0$). Since the input is stabilizing, we know that x_T is in the null-controllable region, and hence there exists a $u^*_{max} < u_{max}$, such that x_T lies on the boundary of the null-controllable region with input constraint u^*_{max}. That is, $x_T \in \mathrm{bd}(\mathcal{C}^{u^*_{max}})$. Let u_T denote the set of all admissible input trajectory which stabilize x_T. Out of all possible trajectories in u_T let

$$
u^*_1 = \min_{|u(t)|_\infty \leq u_{max}} \max_t V_{\mathcal{C}}(x_u(t)) \tag{2.21}
$$

where $x_u(t)$ denotes the state profile corresponding to an input profile of $u(t)$. Thus u^*_1 represents the minimum (over all possible stabilizing trajectories) of the maximum (over time) that the function $V_{\mathcal{C}}(\cdot)$ takes. It follows that the closed–loop trajectory must stay *within the interior of the set* $\mathcal{C}^{u_{max}}$*. Hence,*

$$u_1^* < u_{max} \tag{2.22}$$

Let $u_1^* = u_{max}^* + g < u_{max}$ *with* $g > 0$*. Since* $x_T \in \text{bd}(\mathcal{C}^{u_{max}^*})$*, it follows that* $x_T \in \mathcal{C}^{u_{max}^*+g/2}$*. Denoting*

$$u_2^* = \min_{|u(t)|_\infty \leq u_{max}^*+g/2\ t,\ x(0)=x_0} \max V_{\mathcal{C}}(x_{u_2}(t)) \tag{2.23}$$

and similar to Eq.2.22, it follows that $u_2^* < u_{max}^* + g/2$*. Furthermore, noting that the minimizations of Eq.2.21 and Eq.2.23 are exactly the same, albeit with a larger constraint in Eq.2.21 compared to Eq.2.23, we get that* $u_1^* = u_{max}^* + g \leq u_2^* < u_{max}^* + g/2$*, which is a contradiction, implying* g *cannot be a positive real number. Thus we have that for all states within* $\mathcal{C}^{u_{max}}$*, an input trajectory exists which drives the state to the origin while maintaining* $D^+V_{\mathcal{C}}(x)|_{u(t)} \leq 0$ *for all times. This completes the proof of Theorem 2.1.*

A consequence of Theorem 2.1, is that a control law that uses $V_{\mathcal{C}}$ (and ensures negative semi-definiteness of the CCLF derivative) could possess a stability region which is equal to the null-controllable region and thus is maximal. This is formalized in Corollary 3.1 below.

Corollary 2.1. *For the system of Eq.3.1 with input constraints* u_{max}*, the function* $V_{\mathcal{C}}$ *is a CCLF.*

Proof. *The maximal level set of* $V_{\mathcal{C}}$ *is* $c_{max}(V_{\mathcal{C}}) = u_{max}$ *(Theorem 2.1). Since* $V_{\mathcal{C}} = u_{max}$ *defines the closure of the null-controllable region, this set indeed has the maximum volume for all possible control Lyapunov functions.*

Remark 2.2. *The boundary of the null-controllable region as the level sets of a Lyapunov function has been used [25] to show that a saturated linear feedback law cannot stabilize from the entire null-controllable region. Note that the existing results [25] do not use the null-controllable region to construct control Lyapunov functions, or to develop a stabilizing control law, but only as an analysis tool within the proof of the main result. The results [25], however, further motivate the need of defining and constructing a CCLF that can eventually be used to stabilize from the entire null-controllable region.*

Remark 2.3. *The use of polyhedral functions as control Lyapunov functions has recently received more attention [15], [18], [19]. Specifically, polyhedral Lyapunov functions [18], [19] have been generated which approximate with arbitrary precision the largest closed–loop contractively invariant region. However, comparisons of this region with the null-controllable region are not made. It follows from the analysis in this section that the null-controllable set, albeit invariant, is not contractively invariant. Therefore, the closed–loop contractively invariant region given by the polyhedral Lyapunov functions [15], [18], [19] are only subsets of the null-controllable region.*

2.4 Conclusions

This works considered linear systems with input constraints and defined and presented a constructive procedure for constrained control Lyapunov functions (CCLFs).

2.5 References

[1] M. Mahmood and P. Mhaskar, “On constructing constrained control lyapunov functions for linear systems”, *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1136–1140, 2011.

[2] D. S. Bernstein and A. N. Michel, “A chronological bibliography on saturating actuators”, *Int. J. Rob. & Nonl. Contr.*, vol. 5, pp. 375–380, 1995.

[3] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, “Constrained model predictive control: Stability and optimality”, *Automatica*, vol. 36, pp. 789–814, 2000.

[4] E. D. Sontag, “A Lyapunov-like characterization of asymptotic controllability”, *SIAM J. Control Optimization*, vol. 21, pp. 462–471, 1983.

[5] Z. Artstein, “Stabilization with relaxed controls”, *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 11, pp. 1163–1173, 1983.

[6] X. S. Cai and Z. Z. Han, “Universal construction of control Lyapunov functions for linear systems”, *Lat. Am. Appl. Res.*, vol. 36, pp. 15–22, 2006.

[7] A. R. Teel, “Global stabilization and restricted tracking for multiple integrators with bounded controls”, *Syst. & Contr. Lett.*, vol. 18, pp. 165–171, 1992.

[8] H. J. Sussmann, E. D. Sontag, and Y. Yang, “A general result on the stabilization of linear systems using bounded controls”, *IEEE Trans. Aut. Contr.*, vol. 39, pp. 2411–2425, 1994.

[9] E. D. Sontag and H. J. Sussmann, “Nonlinear output feedback design for linear systems with saturating controls”, in *Proc. 29th IEEE Conf. Decision and Control*, IEEE Publications, 1990, pp. 3414–3416.

[10] B. Zhou and G. Duan, “Global stabilization of linear systems via bounded controls”, *Syst. & Contr. Lett.*, vol. 58, pp. 54–61, 2009.

[11] Z. Lin and A. Saberi, “Semi-global exponential stabilization of linear systems subject to "input saturation" via linear feedbacks”, *Syst. & Contr. Lett.*, vol. 21, pp. 225–239, 1993.

[12] P. Gutman and P. Hagander, “A new design of constrained controllers for linear systems”, *IEEE Trans. Aut. Contr.*, vol. 30, pp. 22–33, 1985.

[13] T. Lauvdal and T. Fossen, “Stabilization of linear unstable systems with control constraints”, in *Proceedings of 36th IEEE Conference on Decision and Control*, San Diego, CA, 1997, pp. 4504–4509.

[14] R. M. Dolphus and W. E. Schmitendorf, “Stability analysis for a class of linear controllers under control constraints”, *Dynamics and Control*, vol. 5, pp. 191–203, 1995.

[15] F. Blanchini, “Set invariance in control”, *Automatica*, vol. 35, pp. 1747–1767, 1999.

[16] T. Hu, Z. Lin, and B. M. Chen, “An analysis and design method for linear systems subject to actuator saturation and disturbance”, *Automatica*, vol. 38, no. 2, pp. 351–359, 2002.

[17] T. Hu and Z. Lin, “Composite quadratic Lyapunov functions for constrained control systems”, *IEEE Trans. Aut. Contr.*, vol. 48, pp. 440–450, 2003.

[18] F. Blanchini and S. Miani, “Constrained stabilization of continuous-time linear systems”, *Syst. & Contr. Lett.*, vol. 28, 95–102(8), 1996.

[19] ——, “Constrained stabilization via smooth Lyapunov functions”, *Syst. & Contr. Lett.*, vol. 35, pp. 155–163, 1998.

[20] T. Hu, Z. Lin, and L. Qiu, “An explicit description of null controllable regions of linear systems with saturating actuators”, *Sys. & Contr. Lett.*, vol. 47, pp. 65–78, 2002.

[21] ——, “Stabilization of exponentially unstable linear systems with saturating actuators”, *IEEE Trans. Aut. Contr.*, vol. 46, pp. 973–979, 2001.

[22] D. G. Luenberger, *Optimization by Vector Space Methods*. New York, NY, USA: John Wiley & Sons, Inc., 1997.

[23] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Boston, USA: Birkhäuser, 2008.

[24] E. D. Sontag and H. J. Sussmann, “Nonsmooth control-lyapunov functions”, in *Proc. IEEE Conf. Decision and Control*, IEEE Publications, 1995, pp. 2799–2805.

[25] T. Hu and Z. Lin, “On semiglobal stabilizability of antistable systems by saturated linear feedback”, *IEEE Trans. Aut. Contr.*, vol. 47, pp. 1193–1198, 2002.

Chapter 3

Constrained Control Lyapunov Function Based Model Predictive Control Design

The results in this chapter have been published in:

Journal Articles

[1] M. Mahmood and P. Mhaskar, "Constrained control lyapunov function based model predictive control design", *International Journal of Robust and Nonlinear Control*, vol. 24, no. 2, pp. 374–388, 2014.

3.1 Introduction

The presence of input constraints is ubiquitous in all applications of control systems. These constraints often represent the physical limitations of control actuators (e.g., pumps, valves). Failure to account for such constraints within the controller design can lead to significant performance deterioration and even closed–loop instability. While possible for a system of integrators [2], [3], it is well established that global and semi-global stability is generally not possible with constrained controls for continuous linear time-invariant systems which are open–loop unstable. That is, the presence of constraints limits the set of initial conditions from where a process can be stabilized at a desired equilibrium point irrespective of the type of input manipulation used. This set is known as the *null-controllable region*. The desire to make most use of the available control effort has motivated considerable research effort towards the problem of designing controllers with the goal of achieving a closed–loop domain of attraction which is equal or as close as possible to the null-controllable region.

In particular, the problem of stabilization from the entire null-controllable region has been considered [4] where a saturated linear state feedback controller is designed that results in a closed-loop system having a domain of attraction that is arbitrarily close to the null controllable region. The result [4], however, only considers planar-unstable systems. There exist several results regarding the design of linear feedback controllers which provide stability estimates using quadratic Lyapunov functions (e.g., [5]–[7]). One direction of work make use of the concept of invariant sets (see e.g., [8]) which are used to estimate the domain of attraction under linear feedback control. The most common form of invariant sets used are ellipsoidal since they result from the level sets of quadratic Lyapunov functions.

Recently [9] the problem of estimating the domain of attraction using ellipsoid sets has been considered. In [9] a sufficient condition is derived in terms of an auxiliary feedback matrix for determining if a given ellipsoid is contractively invariant under saturated linear state feedback. This condition is used to formulate an optimization problem to find a maximal invariant ellipsoid set and also to simultaneously design the corresponding linear feedback gain. The quadratic functions which yield the invariant sets are simple to use, however, the ellipsoid estimates they provide are inherently conservative. This has motivated the use of different forms of Lyapunov functions. In particular, the work in [10] constructs composite Lyapunov functions based on a set of quadratic functions. This composite Lyapunov function is shown to provide better estimates of the closed–loop domain of attraction. In addition, polyhedral sets have been employed to construct invariant sets [8], [11], [12], providing inherently more flexibility as they can approximate any convex shape at the cost of increasing the complexity of the representation.

Another control design which has been used for this problem is that of model predictive control schemes with the use of Lyapunov-based stability constraints [13], [14]. Such designs allow explicit characterization of the stability region, via mimicking the stability properties of Lyapunov-based bounded controllers, without assuming initial feasibility of the optimization problem. More recently, a model predictive controller was designed [15] which better utilizes the constraint handling capabilities of model predictive controllers and thereby enhances the set of initial conditions from where stability is achieved. Recently [16], the idea of a CLF was extended to account for input constraints; resulting in the concept of constrained CLFs (CCLFs), which explicitly accounts for presence of input constraints by maximizing the estimate of the null-controllable region over the set of all possible CLFs. The results [16], however, do not demonstrate the use of the CCLF within a control design, and the need to couple it with an auxiliary controller to establish closed–loop stability from all initial conditions in the null controllable region.

Motivated by the above considerations, this work considers unstable LTI systems and presents a control design utilizing CCLFs to enable stabilizing the system from all initial conditions within the null-controllable region. The rest of the manuscript is organized as follows: First, in Section 3.2, we outline the class of systems and the required definitions needed. In addition, in this section we review the recent results from [16] where the boundary of the null-controllable region is used to construct a constrained CLF. In Section 3.3, a control design which uses the CCLF to stabilize the entire null-controllable region is presented. Following this, in Section 3.3.2, an auxiliary control design is presented and subsequently illustrated through simulation examples in Section 3.3.3. In Section 3.4, application to a nonlinear CSTR example is presented and in Section 3.5, the results are summarized.

3.2 Preliminaries

We consider continuous-time LTI systems with input constraints, described by:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ u &\in \mathcal{U} \end{aligned} \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x \in \mathbb{R}^n$ denotes the vector of state variables, $u \in \mathbb{R}^m$ denotes the vector of manipulated input taking values in a nonempty convex subset $\mathcal{U}$ of $\mathbb{R}^m$, where $\mathcal{U} = \{u \in \mathbb{R}^m : \|u\|_{\text{¥}} \leq u_{max}\}$, and $u_{max} \in \mathbb{R}^+$ denotes the upper bound on the magnitude of each manipulated input u_i. Without loss of generality, we assume that the input constraints for each manipulated input is identical. If this were not the case, the B matrix can be 'adjusted' to absorb the true (symmetric) input constraint. For a vector $x \in \mathbb{R}^n$ we will denote by $\|x\|$ the Euclidean vector norm, and by $\|x\|_{\text{¥}} = \max_i \|x_i\|$ the infinity norm. The matrix norm induced by the Euclidean vector norm of a matrix $P \in \mathsf{R}^{n\times m}$ is given by $\|P\| = \sqrt{\mathsf{s}_{max}}$, where s_{max} is the largest eigenvalue of the matrix P^TP (also known as the largest singular-value of P). The notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x^TQx$ for all vectors $x \in \mathbb{R}^n$, where Q is a positive definite symmetric matrix and x^T denotes the transpose of the vector x. We denote the closure of a set X by $\overline{\mathrm{X}}$. The notation $X\backslash Y$, where X and Y are sets, refers to the relative complement, defined by $X\backslash Y = \{x \in X : x \notin Y\}$. We consider systems where A is anti-stable (all eigenvalues of A are in open right-half plane), and that (A, B) is a controllable pair. A state x_0 is said to be null controllable if there exists a $T \in [0, \text{¥})$ and an admissible control $u(t) \in \mathcal{U}$ such that the state trajectory $x(t)$ of the system of Eq.3.1 satisfies $x(0) = x_0$ and $x(T) = 0$, and the set of all null controllable states is called the null controllable region. We will denote the null controllable region of the system of Eq.3.1 with input constraint u_{max} by $\mathcal{C}^{u_{max}}$. A state $x_e^{u_e}$ is said to be an equilibrium point with input u_e if $Ax_e^{u_e} + Bu_e = 0$. The set of all equilibrium points which are contained in $\mathcal{C}^{u_{max}}$ and have corresponding input values contained in the set $\mathcal{U}$ are denoted by $\mathcal{E}_{\mathcal{U}}$. It follows that $\mathcal{E}_{\mathcal{U}} = \{x_e : u_e \in \mathcal{U}, x_e \in \mathcal{C}^{u_{max}}\}$. Within the simulation examples, numerical integration is performed using the `MATLAB` solver `ODE15s` and the optimization problems are solved using the `MATLAB` subroutine `FMINCON`.

To accommodate non-smooth Lyapunov functions we recall the following generalized derivative:

Definition 3.1. *[17] For a locally Lipschitz function $V : \mathsf{R}^n \to \mathsf{R}$, the upper-right Dini directional derivative of V with respect to Eq.3.1 at x is*

$$D^+_{Ax+Bu}V(x) \quad = \quad \limsup_{h \to 0^+} \frac{V(x + h(Ax + Bu)) - V(x)}{h} \tag{3.2}$$

and we denote $D^+_{Ax+Bu}V(x) = D^+V(x)$.

The following condition enables the use of a relaxed version of LaSalle's invariance principle:

Condition 3.1. *[16] Given a continuous, positive definite, and radially unbounded function $V : \mathsf{R}^n \to \mathsf{R}$, let E be the set of all points where $D^+_{Ax}V(x) = \inf_{u \in \mathsf{R}^m} D^+_{Bu}V(x) = \inf_{u \in \mathsf{R}^m} D^+V(x) = 0$. For all $x_0 \in E \backslash 0$, there exists an input trajectory $u(t)$, such that the closed–loop trajectory is not invariant in the set $E \backslash 0$ and satisfies $D^+V(x)|_{u(t)} \leq 0$.*

We now state a generalized version of the classical definition of a control Lyapunov function (CLF) for the system in Eq.3.1.

Definition 3.2. *[16] A continuous, convex, positive definite, and radially unbounded function $V : \mathsf{R}^n \to \mathsf{R}$ such that*

$$\inf_{u \in \mathbb{R}^m} D^+V(x) \leq 0 \tag{3.3}$$

for all $x \in \mathsf{R}^n$ and Condition 1 is satisfied is a CLF for the system in Eq.3.1.

Consider W defined as the set induced by the level sets of V,

$$\mathsf{W}(V, c) \quad = \quad \{x \in \mathsf{R}^n : V(x) \leq c\} \tag{3.4}$$

and P as the region of the state space where the time derivative can be made negative semi-definite,

$$\mathsf{P}(V) \quad = \quad \{x \in \mathsf{R}^n : \inf_{u \in \mathcal{U}} D^+V(x) \leq 0\} \tag{3.5}$$

It follows that $\mathsf{W}(V, c)$ is an estimate of the stabilizable region of the origin (using the CLF V) if $\mathsf{W}(V, c) \subseteq \mathsf{P}(V)$. Moreover, for a given Lyapunov function V, the maximal estimate of the stabilizable region can be determined with the largest level set of V which is completely contained within $\mathsf{P}(V)$. We will denote this maximum level set by $c_{max}(V)$:

$$c_{max}(V) \quad = \quad \sup\{c \in \mathbb{R} : \mathsf{W}(V, c) \subseteq \mathsf{P}(V)\} \tag{3.6}$$

Let $g(V, c)$ denote the volume function of $\mathsf{W}(V, c)$ given by:

$$g(V, c) \quad = \quad \int \cdots \int_{\mathsf{W}(V,c)} dx_1 dx_2 \ldots dx_n \tag{3.7}$$

For $c \leq c_{max}(V)$, $g(V, c)$ is the volume of the estimate of the stabilizable region. We are now ready to postulate the definition of a constrained control Lyapunov function (CCLF).

Definition 3.3. *[16] Let $\mathcal{V}$ denote the set of all locally Lipschitz CLFs. A locally Lipschitz CLF $V_c : \mathbb{R}^n \to \mathbb{R}$ such that*

$$g(V_c, c_{max}(V_c)) = \max_{V \in \mathcal{V}} g(V, c_{max}(V)), \tag{3.8}$$

is a CCLF for the system in Eq.3.1.

Consider the system of Eq.3.1 with input constraint $u_{max} = 1$, and for ease of notation let $\mathcal{C}^1 = \mathcal{C}$. The set $\mathcal{C}$ is characterized as (see [18]):

$$\mathcal{C} = \bigcup_{T \in [0, \infty)} \{x = -\int_0^T e^{-At} Bu(t)dt : u(t) \in \mathcal{U}\} \tag{3.9}$$

Using the null-controllable region $\mathcal{C}$ in conjunction with gauge functionals, we define the following candidate CCLF:

$$V_{\mathcal{C}}(x) = \varphi_{\mathcal{C}}(x) = \inf\{\lambda > 0 : x \in \lambda\overline{\mathcal{C}}\} \tag{3.10}$$

The continuity, positiveness definiteness, and radial unboundedness of $V_{\mathcal{C}}$ follow from Proposition 3.12 in [17].

It follows that the level set $V_{\mathcal{C}}(x) = \alpha$ defines the boundary of the null-controllable region with input constraint $u_{max} = \alpha$, which is also the boundary of $\mathcal{C}^{\alpha}$. Note it also follows that $\mathcal{C}^{\alpha} = \alpha\mathcal{C}$.

$$\Omega(V_{\mathcal{C}}, c) = \{x \in \mathbb{R}^n : V_{\mathcal{C}}(x) \leq c\} = \overline{\mathcal{C}^c} \tag{3.11}$$

Theorem 3.1. *[16] For the system of Eq.3.1 with input constraints u_{max}, the function $V_{\mathcal{C}}$ is a CCLF.*

3.3 CCLF-based control design

After the result of [19], there has been an abundance of results on the design of stabilizing CLF-based feedback schemes. However, none have been able to achieve stabilization from the entire null-controllable region (due to the designs being based on CLF's that inherently do not take constraints into account). The CCLF defined in Eq.3.10 was shown to achieve negative semi-definiteness of the time derivative over the entire null-controllable region, and hence can be used within a CLF-based feedback scheme to achieve stabilization. In this section we present a predictive control design (a discrete version of which was presented in [15], shown to achieve practical stability) which is able to achieve stability from the entire null-controllable region and also incorporate optimality considerations.

3.3.1 Model Predictive Control Formulation

The predictive controller that guarantees stabilization from all initial conditions in $\mathcal{C}^{u_{max}}$ takes the form:

$$u_{MPC} = argmin\{J(x,t,u(\cdot))|u(\cdot) \in \mathcal{U}\} \tag{3.12}$$

$$s.t.\ \dot{x} = Ax + Bu \tag{3.13}$$

$$D^{+}V_C(x(t)) \leq 0,\ \ \forall\ t \in [t, t+T) \tag{3.14}$$

$$x(t) \neq x(t)\ \ \forall\ t \in (t, t+T] \tag{3.15}$$

Eq.3.13 is the linear model describing the time evolution of the state x. The performance index is given by

$$J(x,t,u(\cdot)) = \int_t^{t+T} \|x^u(s;x,t)\|_Q^2 + \|u(s)\|_R^2\ ds + rV_C(x(T)) \tag{3.16}$$

where $r > 0$, Q is a positive semi-definite symmetric matrix and R is a strictly positive definite symmetric matrix. $x^u(s;x,t)$ denotes the solution of Eq.3.1, due to control u, with initial state x at time t. The computed minimizing control trajectory $u^0(\cdot)$ over a specified time horizon T is applied to the plant at time t and the procedure is repeated indefinitely.

The result in Theorem 3.2 below states how under the continuous implementation of the above predictive controller, stabilization of the system in Eq.3.1 and feasibility of the optimization problem can be achieved for all initial conditions in the null-controllable region.

Theorem 3.2. *Consider the system of Eq.3.1 with input constraint u_{max} under the MPC law of Eqs.3.12–3.16. Then, given any $x_0 \in \mathcal{C}^{u_{max}}$, the optimization problem of Eqs.3.12–3.16 is feasible for all times, and $\lim_{t\to\infty} x(t) = 0$.*

Proof. *We divide the proof into two parts: In part 1 we show feasibility of the optimization problem, and in part 2, we show the implementation of the optimal solution results in closed–loop stability.*

Part 1: *Since $x_0 \in \mathcal{C}^{u_{max}}$, it follows from Theorem 3.1 that there exists some input trajectory such that the constraints in Eqs.3.14–3.15 are satisfied. Note in particular that the satisfaction of the constraint in Eq.3.15 follows via Condition 1. Hence the optimization problem of Eqs.3.12–3.16 is feasible for all times.*

Part 2: *Having established the feasibility of the optimization problem in Part 1 above, we proceed to show closed–loop stability. The satisfaction of the constraints in Eqs.3.14–3.15 ensures that the value of the Lyapunov function is non-increasing along the closed–loop trajectory. By condition 1, the closed–loop trajectory will not be invariant for all states in the set $E\backslash 0$. Therefore, using an extension of the Lasalles invariance principle implies that the closed–loop system is asymptotically stable. That is, $\lim_{t\to\infty} x(t) = 0$. This completes the proof of Theorem 3.2.*

Remark 3.1. *Note that the use of the CCLF renders the 'contractive' constraint different from standard MPC based control designs. In particular, the constraint of Eq.3.14 requires the generalized derivative, instead of the standard derivative of the Lyapunov function to be negative semi-definite (instead of negative definite). Furthermore, as opposed to typical 'contractive' MPC designs, feasibility of the optimization problem is guaranteed (not assumed) from all initial conditions in a well characterized set, and not just a subset of the null-controllable region, but from all initial conditions in the null controllable region. Note also that while the results of Theorem 3.2 are derived under the assumption of continuous implementation of the control action, in practice the results can be implemented with an 'implement and hold' approach. This, and other practical issues are addressed in Section 3.3.2.*

Remark 3.2. *Using the idea that the value of the CCLF at a given state $\bar{x}$ represents the value of the input constraint u^*_{max} which renders that state $\bar{x}$ on the boundary of the null-controllable region $\mathcal{C}^{u^*_{max}}$, an alternate interpretation of the control design can be developed. In particular, the value of the CCLF at a given state represents the minimum control action required to achieve stabilization. Thus, the predictive control design computes a control action which drives the process in a direction where the minimum control action required to achieve stabilization decreases.*

Remark 3.3. *While extensive results exist on the stabilization of linear systems, for anti-stable systems most control laws only provide stability guarantees for subsets of the null-controllable region. In particular, the work in [20] provides stability guarantees for subsets (which can get arbitrarily close to the null controllable region) of the null controllable region, and the control design becomes practically impossible to implement as larger stability regions are sought. Moreover, in model predictive control approaches, the idea used is to estimate the time that it would take for all initial conditions in the 'desired' stability region to reach the origin and to incorporate this information via large or variable horizon, leading to computationally expensive optimization problems. In all of these approaches, the idea remains the same: require the state to go to the origin (or some neighborhood of the origin) by some time (the horizon) and pick a large enough horizon to ensure feasibility of the optimization problem. When the horizon is variable, the optimization problem is in general difficult to solve since the number of decision variables in the optimization problem itself keep changing. When the horizon is fixed, the number of decision variables that have to be retained grows as larger and larger subsets of the null controllable region are desired as the stability region. Note that in our result, feasibility from the null controllable region is achieved via the appropriate choice of the control Lyapunov function. In contrast, existing predictive controllers, which assume initial feasibility of the optimization problem, are not guaranteed to be feasible from all initial conditions in the null controllable region. The existing predictive controllers, however, can very well be used in conjunction with the proposed controller within a hybrid predictive control framework (along the lines of [21], [22]) to enable stabilization from the null-controllable region.*

Remark 3.4. *The use of the boundary of the null-controllable region as the level sets of a Lyapunov function was also used within [23] where it was shown that a saturated linear feedback law cannot in general stabilize from the entire null-controllable region. The result in [23] considered three-dimensional LTI systems with three unstable modes to illustrate the point. Note that [23] does not use the null-controllable region based Lyapunov function to construct control Lyapunov functions, or to develop a stabilizing control law, but only as an analysis tool within the proof of the main result. The results of [23], however, further motivate the need of defining and constructing a CCLF that can eventually be used to stabilize from the entire null-controllable region (we illustrate stabilization from the entire null controlable region for the example of [23] in Section 3.3.3).*

Remark 3.5. *The constraint in Eq.3.15 ensures that if the state at a given time is an equilibrium point with an admissible input value u_e, that is, it is contained within the set of equilibrium points $\mathcal{E}_{\mathcal{U}}$, then the computed control action which satisfies the constraint in Eq.3.14 must be different than u_e (to prevent the closed–loop system getting 'stuck' at this value of the state). This is achieved by ensuring that the state during the next time step does not become equal to the value of the state at the time of the control calculation (Eq.3.15). Again, in contrast to existing MPC designs, Eq.3.15 is an additional constraint that is required to be enforced and is instrumental in yielding the null controllable region as the closed–loop stability region (i.e., it prevents the state getting stuck at some internal point). The feasibility of this constraint follows from the fact that V_C satisfies Condition 1. We note that for single-input systems the set $\mathcal{E}_{\mathcal{U}}$ must be contained within the set E as defined in Condition 1. This follows from the fact that the vectors Ax and B are parallel on the set $\mathcal{E}_{\mathcal{U}}$, and hence the system can only transverse in a single direction regardless of the magnitude of the control action applied. This restriction to a single direction under the continuous implementation of the predictive controller can necessitate instantaneous switching of the control action, and hence make practical implementation a problem. When such a situation exists, an auxiliary continuous control law presented in section 3.3.2 can be utilized to stabilize from all initial conditions within the set $\mathcal{E}_{\mathcal{U}}$.*

3.3.2 Auxiliary control design

Although the predictive control design in Eqs.3.12–3.16 is able to stabilize from all initial conditions in the null-controllable region, implementing the control action in a discrete fashion could result in chattering [24] of the control action. In this section we present an auxiliary continuous control design that augments the predictive controller in Eqs.3.12–3.16 to stabilize all states in the set $\mathcal{E}_{\mathcal{U}}$ without requiring instantaneous switching of the control action by switching (one time) the control law instead. In particular, the auxiliary controller is implemented when the state lands within the set $\mathcal{E}_{\mathcal{U}}$. The key idea in the design of the auxiliary controller is to track a moving equilibrium value. Since all states within $\mathcal{E}_{\mathcal{U}}$ are equilibrium points with admissible control values, we construct a linear feedback control law with the goal of stabilization at a moving equilibrium point. We show that an admissible feedback can always be found which makes the closed-loop system stay arbitrarily close to the moving equilibrium. By making the equilibrium decay to the origin, closed–loop stability of the origin follows.

We begin with some preparatory definitions. Let $x_s \in \mathcal{E}_{\mathcal{U}}$ denote an equilibrium point for the system in Eq.3.1 and $u_s \in \mathcal{U}$ the unique corresponding admissible equilibrium input value. It follows that $x_s = -A^{-1}Bu_s$. We design the control law as follows: Let x_s, and hence also u_s, be a continuously differentiable function of time, i.e., $x_s(t) = -A^{-1}Bu_s(t)$. $u_s(t)$ is designed to decay exponentially as

$$\dot{u}_s(t) = -ku_s(t) \tag{3.17}$$

where $k > 0$ is a design parameter. It follows that,

$$\begin{aligned} \dot{x}_s(t) &= -A^{-1}B\dot{u}_s(t) \\ &= A^{-1}Bku_s(t) \\ &= A^{-1}Bke^{-kt}u_s(0) \end{aligned} \tag{3.18}$$

We denote the deviation of the state from the (instantaneous) equilibrium state $x_s(t)$ as $\tilde{x}(t) = x(t) - x_s(t)$. The theorem presented below states that for all initial conditions within the set $\mathcal{E}_{\mathcal{U}}$, a decay rate k and a gain matrix K (defined below) can be found such that a linear state feedback control law generates admissible control values and the closed–loop state $x(t)$ remains arbitrarily close to the moving equilibrium $x_s(t)$. Furthermore, as the (desired) equilibrium point decays to the origin, the state $x(t)$ follows, resulting in closed–loop stability.

Theorem 3.3. *Given any $\epsilon > 0$, and $x(0) \in \mathcal{E}_{\mathcal{U}}$, there exists a constant k, and gain matrix K, such that the feedback control law $u(t) = K\tilde{x}(t) + u_s(t)$ remains admissible, that is $u(t) \in \mathcal{U}$ and $\|\tilde{x}(t)\| < \epsilon$, for all times $t \geq 0$. Furthermore, as $x_s(t) \to 0$, we have $x(t) \to 0$.*

Proof. *For an $x(0) \in \mathcal{E}_{\mathcal{U}}$, we denote the unique corresponding admissible equilibrium input value $u_s(0)$. Pick K so that $A + BK$ has eigenvalues within the open left complex-plane and determine an ϵ^* such that $\|K\| \leq \frac{u_{max} - \|u_s(0)\|_\infty}{\epsilon^*}$. Given any $\epsilon > 0$, denote $\epsilon^{**} = \min\{\epsilon, \epsilon^*\}$. We first show that for such a choice of K, under the feedback $u(t) = K\tilde{x}(t) + u_s(t)$, the norm of the deviation variable $\tilde{x}(t)$ remains less than ϵ for all times.*

$$\begin{aligned} \tilde{x} &= x - x_s \\ \dot{\tilde{x}} &= \dot{x} - \dot{x}_s \\ &= Ax + Bu - A^{-1}Bke^{-kt}u_s(0) \\ &= A\tilde{x} + BK\tilde{x} + Ax_s + Bu_s - A^{-1}Bke^{-kt}u_s(0) \\ &= (A + BK)\tilde{x} - A^{-1}Bke^{-kt}u_s(0) \end{aligned} \tag{3.19}$$

Let $A_C = A + BK$. It follows that the solution of the non-autonomous linear system is given by,

$$\begin{aligned} \tilde{x} &= e^{A_C t}\tilde{x}(0) - \int_0^t e^{A_C(t-t)} A^{-1} B k e^{-kt} u_s(0) dt \\ &= 0 - \int_0^t e^{A_C(t-t)} A^{-1} B k e^{-kt} u_s(0) dt \ (\textit{since } \tilde{x}(0) = 0) \end{aligned} \tag{3.20}$$

We take the norm of the above expression.

$$\begin{aligned} \|\tilde{x}\| &= \| \int_0^t e^{A_C(t-t)} A^{-1} B k e^{-kt} u_s(0) dt \| \\ &\leq \int_0^t \|e^{A_C(t-t)} A^{-1} B k e^{-kt} u_s(0)\| dt \\ &\leq \int_0^t \|A^{-1} B k e^{-kt} u_s(0)\| \|e^{A_C(t-t)}\| dt \\ &\leq k\|A^{-1} B u_s(0)\| \int_0^t \|e^{A_C(t-t)}\| dt \end{aligned} \tag{3.21}$$

Since A_c is stable, we know that there exists $\alpha > 0$, and $\beta > 0$ such that $\|e^{A_c t}\| \leq \alpha e^{-\beta t}$. Hence

$$\begin{aligned} \|\tilde{x}\| &\leq k\|A^{-1} B u_s(0)\| \int_0^t \|e^{\beta(t-t)}\| dt \\ &= \tfrac{\alpha}{\beta}(1 - e^{-\beta t}) k \|A^{-1} B u_s(0)\| \\ &\leq \tfrac{\alpha\|A^{-1} B u_s(0)\|}{\beta} k \end{aligned} \tag{3.22}$$

We choose the constant k, such that $k < \frac{\beta\epsilon^{**}}{\alpha\|A^{-1}Bu_s(0)\|}$ so that we get

$$\|\tilde{x}\| \leq \epsilon \tag{3.23}$$

This holds for any K which places the eigenvalues of A_C in the open left-plane. It remains to show that the feedback law $u(t) = K\tilde{x}(t) + u_s(t)$ remains admissible.

$$\begin{aligned} \|u(t)\|_\infty &= \|K\tilde{x}(t) + u_s(t)\|_\infty \\ &\leq \|K\tilde{x}(t)\|_\infty + \|e^{-kt} u_s(0)\|_\infty \\ &\leq \|K\|\|\tilde{x}(t)\| + \|e^{-kt}\|\|u_s(0)\|_\infty \\ &< \|K\|\epsilon^{**} + \|u_s(0)\| \\ &\leq \tfrac{u_{max} - \|u_s(0)\|_\infty}{\epsilon^*}\epsilon^{**} + \|u_s(0)\|_\infty \\ &\leq u_{max} \end{aligned} \tag{3.24}$$

Therefore the control law remain admissible. Since $\|\tilde{x}\| < \epsilon$ for all times, and $\tilde{x}(t) \to 0$, as well as $x_s(t) \to 0$ as $t \to \infty$, $x(t) \to 0$. This completes the proof of Theorem 3.3.

Remark 3.6. *Note that the intent and function of the auxiliary controller is inherently different from the 'terminal' set (see, e.g., [25]) and 'terminal' controller used in existing MPC designs, as well as from the idea of sliding mode controllers. In particular, the set upon reaching which the 'auxiliary' controller is activated is not a set close to the origin (as in the MPC designs with a terminal set); it could very well be a hyper plane cutting across the entire null-controllable region. The second key difference is that the MPC is not intended to drive the system states towards this set (as is done in existing MPC designs, or the idea behind sliding mode controllers). The auxiliary control design is only in place so that if the* *system state happens to enter this set, instead of possibly being driven in and out of this set by the MPC (resulting in chattering), the system is smoothly driven to the origin. We finally note also that since the MPC presented is a continuous time formulation and uses the null-controllability region as the CCLF, it requires suitable modifications (as with most other MPC designs) for the purpose of online implementation (see the simulation example for details).*

3.3.3 Simulation examples

To ease implementation of the control design, we make use of the polyhedral approximations of the null controllable regions derived in [18]. In particular, using the polyhedral set notation used in [17], the polyhedral approximation of the null-controllable region $\mathcal{C}$ is given by

$$\hat{\mathcal{C}} \quad = \quad \{x : \|Fx\|_{¥} \leq 1\} \tag{3.25}$$

where F is a proper $\mathsf{R}^{r \times n}$ full column rank matrix. The rows of the matrix F represent faces or sectors of the polytope $\hat{\mathcal{C}}$ which approximates $\mathcal{C}$. The rows can be formed by using the analytic characterization given in [18] to generate points which are on the boundary of the null-controllable region. By using such points as vertices and taking the convex hull, linear sectors can be generated which join together to approximate the boundary of the null-controllable region. The the polytope has a total of r sectors, with each sector containing n vertices.

Similar to Eq.3.10, we define the gauge functional of the set $\hat{\mathcal{C}}$ as the approximate CCLF $V_{\hat{\mathcal{C}}}$, which is given by

$$V_{\hat{\mathcal{C}}}(x) \quad = \quad \|Fx\|_{¥} \tag{3.26}$$

We note that the quality of the approximation of the set $\hat{\mathcal{C}}$ is directly related to the number of rows of the matrix F, which in turns grows significantly with the dimension n. Therefore there exists a tradeoff between the quality of approximation and the computational burden of dealing with a large number of sectors for the polytope. Nevertheless, the polyhedral representation renders the CCLF approximation in a functional form which can easily be implemented within the control design presented in Section 3.3. Furthermore, the constraint of Eq.3.14 is implemented as $V_{\mathcal{C}}(x(t + \mathsf{D})) \leq V_{\mathcal{C}}(x(t))$ with $\mathsf{D} = 0.05$.

To construct the matrix F (for both the examples in this section), points on the null-controllable regions of each single-input subsystem are first generated using the expressions given in [18]. The Minkowski sum of all the boundaries of the null-controllable regions for each of the single-input subsystems is then computed. By taking the convex hull of this Minkowski sum, the approximate boundary of the null-controllable region is computed. In addition, this convex hull yields the faces of the polytope which define the rows of the matrix F in Eq.3.26. The first example illustrates the use of the Minkowski sum to generate and use the CCLF, while the second example illustrates the need to use the auxiliary controller (and demonstrate stabilization of the example in [23]).

Example 3.1. *Consider a linear system of the form of Eq.3.1 with* $A = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.8 & -2.0 \\ 0 & 2 & 0.8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$ *and* $u_{max} = 1$. *The matrix* A *is exponentially unstable with the eigenvalues:* $0.8, 0.8 \pm 2i$. *We demonstrate the ability of the predictive controller to stabilize by using a couple of initial conditions. In designing the predictive controller, the weights are chosen as* $Q = 0.001I$, $R = 0.1I$, *and* $r = 10$. *We first pick an initial condition* $x_0 = [3.1737,\ -0.3450,\ 0.5322]$, *where* $V_C(x_0) = 0.97$ *and try to stabilize it using the proposed predictive controller. As can be seen from the solid line in Fig.3.1, closed–loop stability is achieved. Similarly, we also consider the initial condition* $x_0 = [-1.284,\ 1.7559,\ -1.1127]$, *where* $V_C(x_0) = 0.97$. *To demonstrate the effect of the choice of the objective function, for this initial condition we use:* $Q = 0$, $R = 0$, *and* $r = 1$, *which essentially requires the optimization problem to compute a control action that causes as fast a decay of the CCLF as possible. As expected (shown by the dashed line in Fig.3.1), the trajectory first travels along the path of sharpest decay (and results in a bang-bang control action) up-until the point where the extreme values of the control actions do not result in the sharpest decay of the CCLF. Closed–loop stability is again achieved. Fig.3.1 also shows the evolution of the Lyapunov function for these two scenarios (solid and dashed lines respectively).*

Example 3.2. *Next consider a linear system of the form of Eq.3.1 with* $A = \begin{bmatrix} 0.2 & 1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $u_{max} = 1$. *The system* A *is exponentially unstable with the eigenvalues of* A *being:* 0.2, 0.2, *and* 0.4. *In this example we use:* $Q = 0$, $R = 0$, *and* $r = 1$ *(that is we again minimize the value of the CCLF at every time step), and demonstrate the need for the use of the auxiliary controller. To demonstrate the stabilization properties of the proposed predictive controller, we pick two initial conditions* $x_0 = [2.4570,\ -2.6762,\ -2.0325]$, *where* $V_C(x_0) = 0.98$ *and* $x_0 = [3.6679,\ -3.6736,\ -2.3106]$, *where* $V_C(x_0) = 0.95$. *As can be seen from the solid and bold solid lines in Fig.3.2, closed–loop stability is achieved in both cases. Fig.3.2 also shows the evolution of the Lyapunov function for these two scenarios (solid and dashed lines respectively). Note that at time* $t = 257$ *for the first initial condition, and* $t = 235$ *for the second initial condition, the state reaches the set* $\mathcal{E}_U$ *and hence the control law is switched from the predictive controller to the auxiliary control design from Section 3.3.2. The gain matrix K is chosen as* $K = [-0.7200\ \ -7.3800\ 10.0000]$, *which places the closed–loop eigenvalues at* -0.1, -0.4, *and* -0.6. *The constant* k *is chosen to ensure the deviation variable* $\tilde{x}(t)$ *remains less than* $\epsilon = 0.01$. *As can be seen from the state–space* *evolution in Fig.3.2, the state is able to track the moving steady–state and closed–loop stability is achieved. Note that under the predictive controller, the CCLF value does not strictly decrease. In fact, just prior to the switch to the auxiliary controller, the Lyapunov function value remains constant, that is the state transverses on a level set of the Lyapunov function, underscoring the need for the design and use of the auxiliary controller.*

Remark 3.7. *One recognizes that it is not possible to practically implement the continuous-time MPC law given by Eqs.3.12–3.16, thus a discretized version of the constraint in Eq.3.14 is used and the MPC formulation is implemented with an 'implement and hold' approach. Using a sufficiently small time step for the control implementation, in conjunction with the switched auxiliary control law of section 3.3.2, results in a implementable control scheme that would yield practical stability.*

Remark 3.8. *The use of polyhedral functions as control Lyapunov functions has recently received more attention [8], [11], [12]. Specifically, the work in [11], [12] generates polyhedral Lyapunov functions which approximates with arbitrary precision the largest closed–loop contractively invariant region. However, comparisons of this region with the null-controllable region are not made. It follows from the analysis in this section that the null-controllable set, albeit invariant, is not contractively invariant. Therefore, the closed–loop contractively invariant region given by the polyhedral Lyapunov functions in [8], [11], [12] are only subsets of the null-controllable region.*

3.4 Application to nonlinear CSTR example

In the previous section, we showed the construction of CCLF's based on the knowledge of the null controllable region for linear systems. In this section we demonstrate an application of the idea to nonlinear systems and to this end, present an MPC design with the CCLF restricted to a quadratic form, albeit determining the 'best' quadratic form of the CCLF.

To this end, consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. The mathematical model for the process takes the form:

$$\begin{aligned} \dot{C_A} &= \frac{F}{V}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT_R}} C_A \\ \dot{T_R} &= \frac{F}{V}(T_{A0} - T_R) + \frac{(-\Delta H)}{\rho c_p} k_0 e^{\frac{-E}{RT_R}} C_A + \frac{Q}{\rho c_p V} \end{aligned} \tag{3.27}$$

where C_A denotes the concentration of the species A, T_R denotes the temperature of the reactor, Q is the heat added to the reactor, V is the volume of the reactor, k_0, E, ΔH are the pre-exponential constant, the activation energy, and the enthalpy of the reaction and c_p and ρ are the heat capacity and fluid density in the reactor. The values of all process parameters can be found in Table 3.1. The control objective is to stabilize the reactor at the unstable equilibrium point $(C_A^s,\ T_R^s) = (9.83\ Kmol/m^3, 344.23\ K)$

using the rate of heat input, Q as the manipulated input with the following constraint: $|Q| \leq 32\ KJ/s$.

We consider quadratic CCLFs of the form $V(x) = x^T Px$, where P is a symmetric positive definite matrix. The choice of the Lyapunov function is therefore equivalent to choosing the matrix P. This construction procedure for CCLFs can be interpreted as consisting of two components. The first being the satisfaction of a local CLF, and the second being the maximization of the stability region estimate. The condition of a candidate P matrix being a local CLF is checked using the linearized system matrices of Eq.5.38. The second condition of maximization of the level set is implemented using the nonlinear equations. In particular, the boundaries of the set P are mapped as nonlinear equations. Then the maximization of a given matrix P to find the largest level set within the P region is computed using a nonlinear program. This results in the following dual layer optimization problem.

$$\begin{aligned} J(P) &= \max_{P} \max_{c} \sqrt{\frac{c^2}{\det(P)}} \\ s.t.\ p_1 &> 0 \\ p_{12} &> 0 \\ p_1 - p_{12}^2 &> 0 \\ \mathsf{W}(P,c) &\subseteq \mathsf{P}(P) \end{aligned} \tag{3.28}$$

where $P = \begin{bmatrix} p_1 & p_{12} \\ p_{12} & 1 \end{bmatrix}$, $\mathsf{W}(P,c)$ is the set induced by the level-set $x^T Px = c$, and $\mathsf{P}(P)$ is the region of the state space where the time derivative can be made negative semi-definite. Note that the boundaries of the set $\mathsf{P}(P)$ can be found by setting the time derivative of V to zero and taking extremal control values. The objective function in the above optimization problem is the area of the estimate of the closed–loop stability region given by the quadratic CCLF $x^T Px$. The first three constraints ensure the matrix P is a local CLF (i.e. it is a CLF for the linearized system). These conditions are necessary and sufficient conditions for quadratic CLFs [26]. Finally, the last constraint forces the level-set is contained within the region P. The optimization searches over the space of quadratic CCLFs which are maximal.

For the purpose of comparison, we first construct a quadratic Lyapunov function by solving the Riccati inequality with the linearized system matrices A_{lin},B_{lin} and $Q_{riccati} = \begin{bmatrix} 1 & 0 \\ 0 & 0.0001 \end{bmatrix}$. This results in the following matrix:

$$P = \begin{pmatrix} 1 & -0.0032 \\ -0.0032 & 0.0013 \end{pmatrix}$$

where $A_{lin}^T P + PA_{lin} - PB_{lin}B_{lin}^T P + Q_{riccati} = 0$. The estimate of the region of controllability can be seen in Fig.3.3 and is denoted by $\mathsf{W}_{riccati}$. The enclosed area is 546.2.

The solution to the optimization problem results in the matrix:

$$P = \begin{pmatrix} 1 & 0.3779 \\ 0.3779 & 0.1428 \end{pmatrix}$$

The estimate of the region of controllability can also be seen in Fig.3.3 and is denoted by W_{CCLF}. The enclosed area is 1.3853×10^6, depicting a stability region clearly larger than the one obtained by solving the Riccati inequality.

To illustrate the enhancement in the set of initial conditions from where closed–loop stability can be achieved using the constructed CCLF, we pick an initial condition $C_A(0) = 22\ kmol/m^3$, $T_R(0) = 386.23\ K$ outside $W_{riccati}$ but inside W_{CCLF}. A Lyapunov-based predictive controller that requires the value of the Lyapunov function to decrease is implemented. A standard objective function comprising of a penalties on the state and input trajectories is used. The parameters in the objective function are chosen as $Q = qI$, with $q = 0.1$, and $R = 0.1$. A control and prediction horizon of 1 min is used, along with a sampling time of 0.01 min. Using the quadratic Lyapunov function generated using the Riccati inequality, we see that the Lyapunov function cannot be made to decay from this initial condition, and the state trajectory is unable to reach the desired steady–state. This can be seen as the dashed line in Fig.3.3. In contrast, using the quadratic CCLF, the CCLF value can be made to decay, and the system eventually reaches the desired steady–state (as shown by the solid line in Fig.3.3).

3.5 Conclusions

This works considered linear systems with input constraints with the objective of designing a controller that guarantees stability from all initial conditions in the null controllable region (the set of initial conditions from where the system can be stabilized). To this end, a recently developed procedure for construction of constrained control Lyapunov functions (CCLF) was utilized within a Lyapunov-based model predictive controller coupled with an auxiliary control design to achieve stabilization from all initial conditions in the null-controllable region. Illustrative simulation examples were presented and implementation to nonlinear systems was demonstrated via a chemical reactor example.

TABLE 3.1: Chemical reactor parameters and steady–state values.

V	=	0.1	m^3
R	=	8.314	$KJ/Kmol \cdot K$
C_{A0_s}	=	10.0	$Kmol/m^3$
T_{A0_s}	=	310.0	K
Q_s	=	0.0	KJ/min
DH	=	-4.78×10^4	$KJ/Kmol$
k_0	=	72×10^9	min^{-1}
E	=	8.314×10^4	$KJ/Kmol$
c_p	=	0.239	$KJ/kg \cdot K$
r	=	1000.0	kg/m^3
F	=	100×10^{-3}	m^3/min
T_{Rs}	=	395.33	K
C_{As}	=	0.57	$Kmol/m^3$

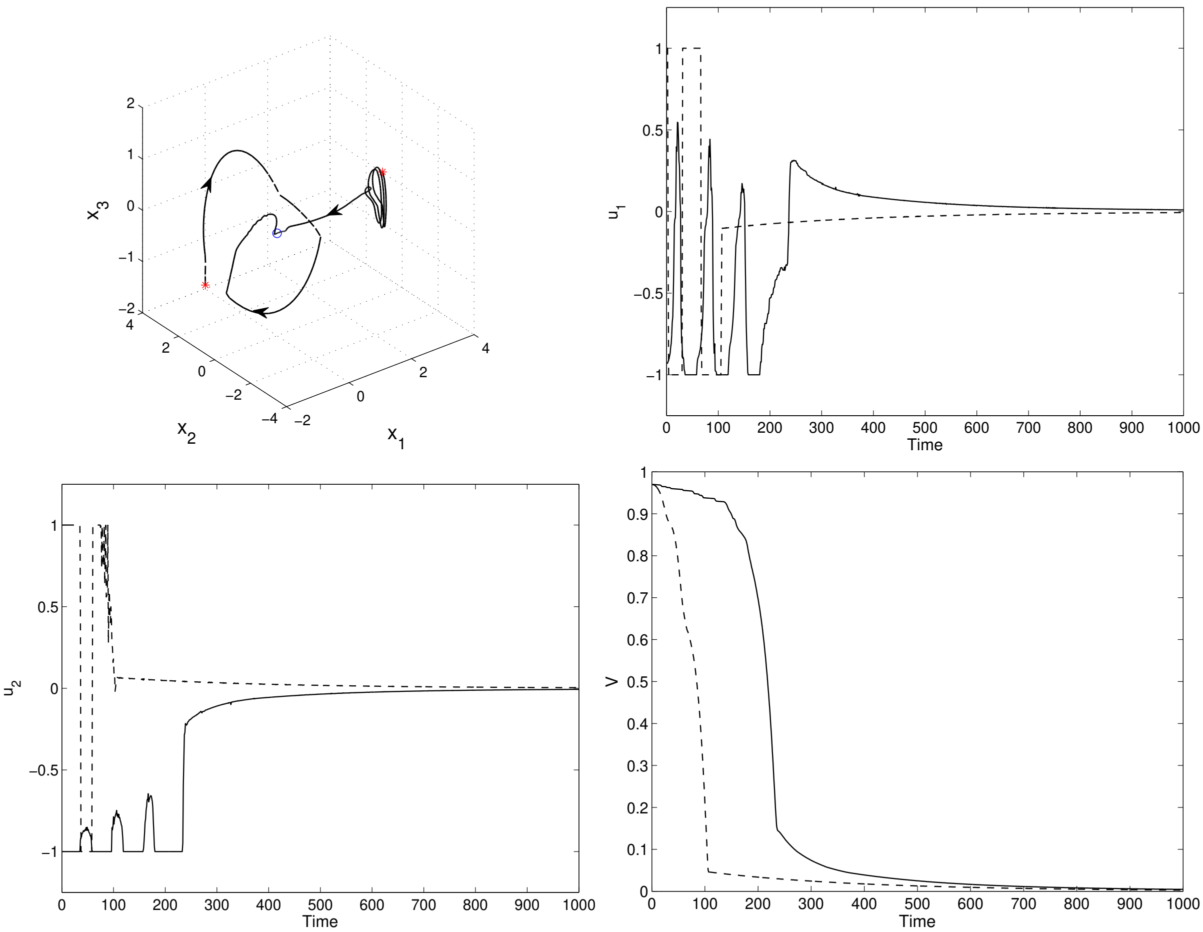

FIGURE 3.1: State–space trajectories and profiles of the inputs u_1, u_2 and the Lyapunov function for Example 1. The figure demonstrates stabilization from two initial conditions: $x_0 = [3.1737, \ -0.3450, \ 0.5322]$ (solid line), and $x_0 = [-1.284, \ 1.7559, \ -1.1127]$ (dashed line).

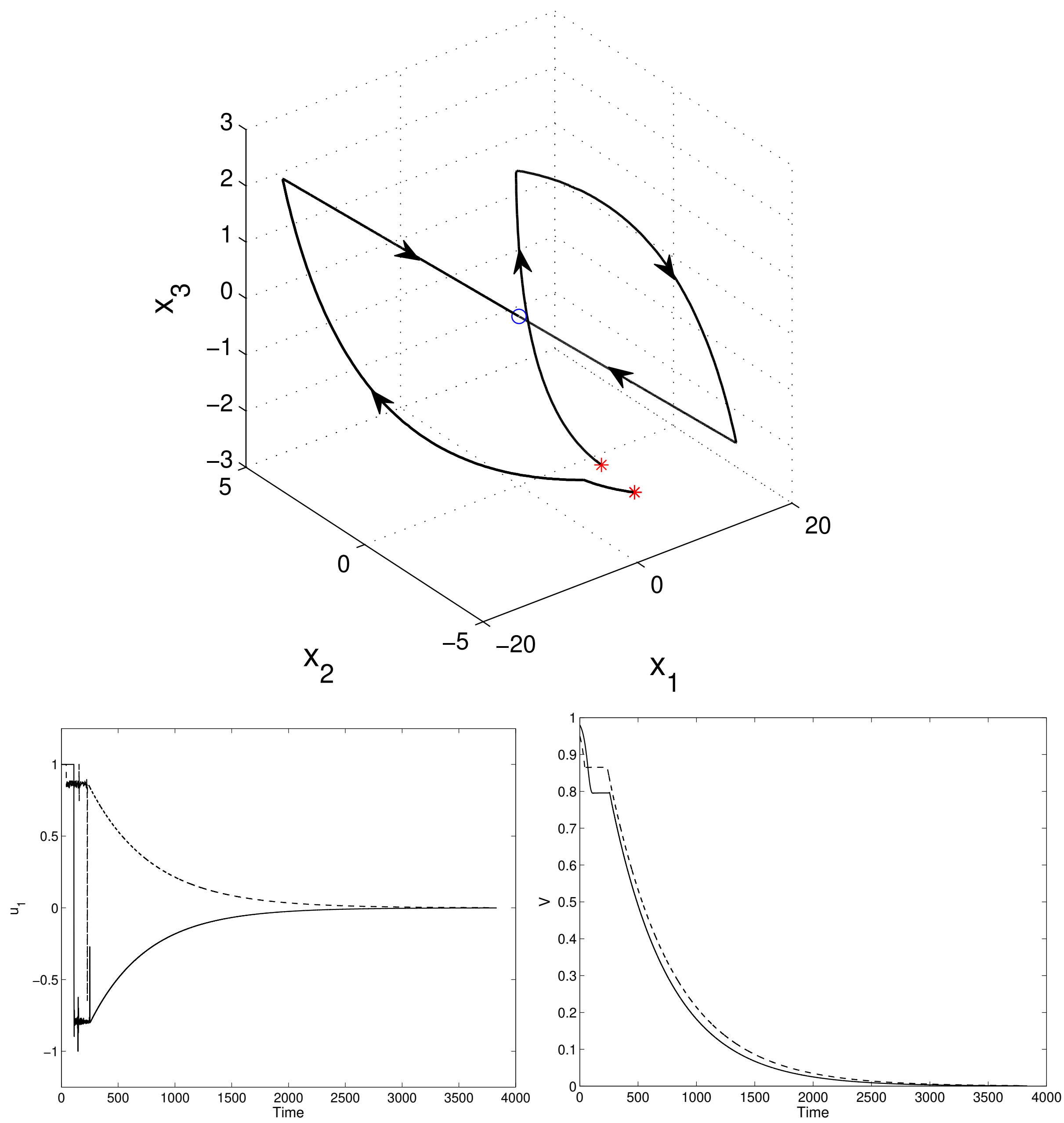

FIGURE 3.2: State–space trajectories, input profile and the Lyapunov function evolution for Example 2. Initial conditions: $x_0 = [2.4570, -2.6762, -2.0325]$ (solid lines), and $x_0 = [3.6679, -3.6736, -2.3106]$ (bold solid line depicting the state trajectory and dashed line depicting the input profile and Lyapunov function). The simulation example demonstrates the need for the auxiliary controller.

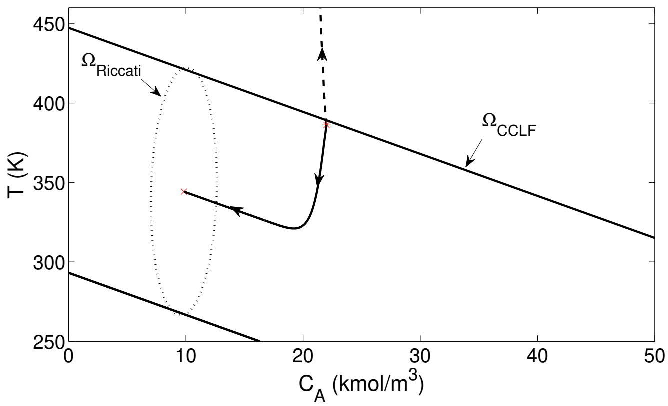

FIGURE 3.3: CCLF constructed for nonlinear CSTR. Estimates of controllable region given by proposed optimization (W_{opt}), and by solving Riccati inequality ($W_{riccati}$). The dotted lines show the implementation of an existing contractive MPC design resulting in instability while the solid lines show the implementation of the proposed CCLF based MPC design from the same initial condition leading to stabilization.)

3.6 References

[1] M. Mahmood and P. Mhaskar, "Constrained control lyapunov function based model predictive control design", *International Journal of Robust and Nonlinear Control*, vol. 24, no. 2, pp. 374–388, 2014.

[2] A. R. Teel, "Global stabilization and restricted tracking for multiple integrators with bounded controls", *Syst. & Contr. Lett.*, vol. 18, pp. 165–171, 1992.

[3] A. Saberi, Z. L. Lin, and A. R. Teel, "Control of linear systems with saturating actuators", *IEEE Trans. Aut. Contr.*, vol. 41, pp. 368–378, 1996.

[4] T. Hu, Z. Lin, and L. Qiu, "Stabilization of exponentially unstable linear systems with saturating actuators", *IEEE Trans. Aut. Contr.*, vol. 46, pp. 973–979, 2001.

[5] P. Gutman and P. Hagander, "A new design of constrained controllers for linear systems", *IEEE Trans. Aut. Contr.*, vol. 30, pp. 22–33, 1985.

[6] T. Lauvdal and T. Fossen, "Stabilization of linear unstable systems with control constraints", in *Proceedings of 36th IEEE Conference on Decision and Control*, San Diego, CA, 1997, pp. 4504–4509.

[7] R. M. Dolphus and W. E. Schmitendorf, "Stability analysis for a class of linear controllers under control constraints", *Dynamics and Control*, vol. 5, pp. 191–203, 1995.

[8] F. Blanchini, "Set invariance in control", *Automatica*, vol. 35, pp. 1747–1767, 1999.

[9] T. Hu, Z. Lin, and B. M. Chen, "An analysis and design method for linear systems subject to actuator saturation and disturbance", *Automatica*, vol. 38, no. 2, pp. 351–359, 2002.

[10] T. Hu and Z. Lin, "Composite quadratic Lyapunov functions for constrained control systems", *IEEE Trans. Aut. Contr.*, vol. 48, pp. 440–450, 2003.

[11] F. Blanchini and S. Miani, "Constrained stabilization of continuous-time linear systems", *Syst. & Contr. Lett.*, vol. 28, 95–102(8), 1996.

[12] ——, "Constrained stabilization via smooth Lyapunov functions", *Syst. & Contr. Lett.*, vol. 35, pp. 155–163, 1998.

[13] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Predictive control of switched nonlinear systems with scheduled mode transitions", *IEEE Trans. Automat. Contr.*, vol. 50, pp. 1670–1680, 2005.

[14] ——, "Stabilization of nonlinear systems with state and control constraints using Lyapunov-based predictive control", *Syst.* & *Contr. Lett.*, vol. 55, pp. 650–659, 2006.

[15] M. Mahmood and P. Mhaskar, "Enhanced stability regions for model predictive control of nonlinear process systems", *AIChE J.*, vol. 54, pp. 1487–1498, 2008.

[16] ——, "On constructing constrained control Lyapunov functions for linear systems", *IEEE Trans. Automat. Contr.*, vol. 56, pp. 1136–1140, 2011.

[17] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Boston, USA: Birkhäuser, 2008.

[18] T. Hu, Z. Lin, and L. Qiu, "An explicit description of null controllable regions of linear systems with saturating actuators", *Sys. & Contr. Lett.*, vol. 47, pp. 65–78, 2002.

[19] E. D. Sontag, "A "universal" construction of artstein's theorem on nonlinear stabilization", *Syst. & Contr. Lett.*, vol. 13, pp. 117–123, 1989.

[20] T. Hu, Z. Lin, and Y. Shamesh, "Semi-global stabilization with guaranteed regional performance of linear systems subject to actuator saturation", *Syst. & Contr. Lett.*, vol. 43, 203–210(8), 2001.

[21] N. H. El-Farra, P. Mhaskar, and P. D. Christofides, "Uniting bounded control and MPC for stabilization of constrained linear systems", *Automatica*, vol. 40, pp. 101–110, 2004.

[22] P. Mhaskar, N. H. El-Farra, and P. D. Christofides, "Hybrid predictive control of process systems", *AIChE J.*, vol. 50, pp. 1242–1259, 2004.

[23] T. Hu and Z. Lin, "On semiglobal stabilizability of antistable systems by saturated linear feedback", *IEEE Trans. Aut. Contr.*, vol. 47, pp. 1193–1198, 2002.

[24] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell time", in *Proceedings of 38th IEEE Conference on Decision and Control*, Phoenix, AZ, 1999, pp. 2655–2660.

[25] D. Limon, T. Alamo, and E. F. Camacho, "Enlarging the domain of attraction of MPC controllers", *Automatica*, vol. 41, pp. 629–635, 2005.

[26] X. S. Cai and Z. Z. Han, "Universal construction of control Lyapunov functions for linear systems", *Lat. Am. Appl. Res.*, vol. 36, pp. 15–22, 2006.

Chapter 4

Controllability Minimum Principle Based Construction of the Null Controllable Region for Nonlinear Systems

The results in this chapter have been submitted for publication to the following:

Journal Articles

[1] M. Mahmood, T. Homer, and P. Mhaskar, "Controllability minimum principle based construction of the null controllable region for nonlinear systems", *Submitted.*, 2020.

4.1 Introduction

In order to achieve improved control performance and closed–loop stability, the complexity of the system dynamics must be considered. This complexity can manifest itself in many forms, one such form being nonlinear dynamics. Such behavior often arises in chemical processes due to radiative heat transfer phenomena, complex reaction mechanisms, and Arrhenius temperature dependence of reaction rates. Nonlinear systems often exhibit multiple equilibria, of which few can be unstable. Such unstable equilibria can be desired operating points due to economic considerations. Thus the stabilization of the system to unstable equilibrium points is an important control problem. Moreover, the presence of input constraints makes this task more demanding. Accounting for input constraints has considerable practical implications as all actuators have physical limitations. Coupling the presence of input constraints with unstable equilibrium can give rise to the scenario where the input constraints limit the set of states which can be stabilized to the origin. This set has been termed the *null controllable region* (NCR) [2].

The realization of the existence of an (finite) NCR naturally gives rise to the problem of designing a controller where the closed–loop domain of attraction is equal or as close as possible to the NCR. Solution approaches to this problem have been dominated by Lyapunov-based techniques. The use of Lyapunov functions, although rooted in the topic of stability analysis of dynamical systems, has since extended to the domain of control analysis and design through the idea of control Lyapunov function (CLF) ([3], [4]). CLFs are generalized Lyapunov functions which are used to construct stabilizing control laws while also providing estimates of the controllability region. Although construction procedures for CLFs do exists, such procedures focus on unconstrained CLF construction. That is, the construction of the CLF does not recognize the presence of input constraints, but the analysis of the controllability region accounts for the presence of input constraints after the fact.

The results in [5] extended the idea of a CLF to account for input constraints, resulting in the concept of constrained control Lyapunov functions (CCLF). These functions explicitly account for the presence of input constraints by maximizing the estimate of the NCR over the set of all possible CLFs. Such functions also enable control designs with optimal stability regions which are equal to the NCR. Most existing results, however, have focused on linear systems [6]. Some recent results have considered nonlinear systems and provided some practical control designs which utilize CCLFs within a predictive control framework [7], [8]. However, the formal development of stabilization using CCLFs for nonlinear systems along with explicit control design is still lacking. More recently, the work in [9] presented some computational techniques for NCR construction. This work utilizes a simulation-based approach resulting in high computational cost both for NCR construction and utilization in the control design.

In summary, a general, computationally tractable procedure to generate the NCR for nonlinear systems remains an open problem. In this work we consider this problem for unstable nonlinear systems where constraints on the control action induce a NCR boundary. Our approach provides a boundary condition for the well-known Controllability Minimum Principle resulting in computationally tractable procedure to generate the trajectories which form the boundary of the NCR. The rest of the manuscript is organized as follows: first, in Section 5.2, we outline the class of systems and the required notion, definitions, and assumptions. In addition, we formulate the problem statement and review results on characterizing the NCR. In Section 4.3, a construction procedure to generate the NCR bounded trajectories is presented. In Section 5.4, several examples are presented to demonstrate the NCR construction procedure. Finally, in Section 6.6 we summarize our results.

4.2 Preliminaries

4.2.1 Notation

If X is a set X°,$\overline{X}$ and ¶X denotes the interior of X, the closure of X, and the boundary of X respectively. The Euclidian norm on R^n is denoted by $\|\cdot\|$.

4.2.2 Problem Formulation

Consider single-input nonlinear systems that are affine in the control

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{4.1}$$

where $x(t) \in M \subset \mathrm{R}^n$, M being an open connected set, denotes the state vector and $u(t) \in \mathcal{U}(A)$ denotes the scalar control input where $\mathcal{U}(A) = \{u : \mathrm{R} \to A \subset \mathrm{R}, \text{ locally integrable}\}$ are the admissible controls. We assume $f, g : \mathrm{R}^n \to \mathrm{R}^n$ are smooth analytic vector fields defined over a smooth domain M which contains the origin in its interior with $f(0) = 0$. We assume the input constraint range is symmetric $U_m = [-m, m]$, $m > 0$ and we refer to the system in Eq.5.1 with control constraint set $\mathcal{U}(U_m)$ as S. We also consider the *unforced system* $\dot{x}(t) = f(x(t)) + g(x(t))u_0$ obtained from Eq.5.1 using a constant input value of $u_0 \in \mathcal{U}(U_m)$ and the *nominal system* obtained using the nominal input value of $u = 0$. A solution of the system in Eq.5.1 from initial condition x_0 and an admissible control u at time t is called the *controlled solution* and is denoted by $j(t, x_0, u(t))$. A point x_{eq} is an equilibrium point of the system if there is a constant control action $u(t) = u_0$, such that $f(x_{eq}) + g(x_{eq})u_0 = 0$. Without loss of generality, it is assumed that the origin is an isolated equilibrium point of the nominal system (i.e. $f(0) = 0$). An equilibrium point x_{eq} is said to be hyperbolic if none of the eigenvalues of the Jacobian for the linearized system (linearized at x_{eq}) have zero real parts. For a given equilibrium point, if all the eigenvalues of Jacobian have nonzero imaginary parts then we call this equilibrium point *purely periodic*. Also, a hyperbolic equilibrium point is said to be *unstable* if at least one eigenvalue has a positive real part and *stable* otherwise. For a hyperbolic equilibrium point x_{eq}, we can decompose the tangent space $T_{x_{eq}}(M)$ as the direct sum of stable $E^s(x_{eq})$ and unstable $E^u(x_{eq})$ subspaces. The subspaces are denoted as $E^s(x_{eq}) = \{v^1, v^2, \cdots, v^{n_s}\}$, $E^u(x_{eq}) = \{w^1, w^2, \cdots, w^{n_s}\}$. The vectors $v^1, v^2, \cdots, v^{n_s}$ and $w^1, w^2, \cdots, w^{n_s}$ are generalized eigenvectors whose eigenvalues are stable (negative) and unstable (positive). Note that $n_s + n_u = n$. A dynamical system is said to be *structurally stable* if perturbations to the system preserve the qualitative characteristics of the dynamics. That is, there exists a homeomorphism which maps orbits of the original system to orbits of the perturbed system while also preserving the direction of time. We say that y is an *w-limit point* of x associated with the controlled solution $j(t, x_0, u(t))$ if there exists a sequence $\{t_i\}$ with $\lim_{i\to¥} t_i = ¥$ such that $\lim_{i\to¥} j(t_i, x, u(t_i)) = y$. The set of all w-limit points of x associated with $j(t, x_0, u(t))$ is called the *w-limit set* of x, denoted by $w(x, u)$. Let the multiplicity of the equilibrium points (for a given input value) be k (note that due to the assumption of the structural stability below, the multiplicity is independent of the value of u). The equilibrium set $\hat{\mathcal{E}}_i(A) \subset M \times \mathrm{R}$ of the system S and its projection into M, $\mathcal{E}_i(A)$ are defined by

$$\hat{\mathcal{E}}_i(A) = \{(x_i, u_0) \in M \times \mathrm{R} | f(x_i) + g(x_i)u_0 = 0, u_0 \in A\} \tag{4.2}$$

where i indexes the multiplicity of the equilibrium points. Further,

$$\mathcal{E}_i(A) = \{x_i \in M \,|(x_i, u_0) \in \hat{\mathcal{E}}_i(A)\} \tag{4.3}$$

Finally let $\hat{\mathcal{E}}(A) = \bigcup_{i=0}^{k-1} \hat{\mathcal{E}}_i(A)$, and $\mathcal{E}(A) = \bigcup_{i=0}^{k-1} \mathcal{E}_i(A)$. We use the subscript 0 to denote the subset which contains the origin (i.e $0 \in \mathcal{E}_0(A)$). As stated in the assumptions below the origin is assumed to be an unstable equilibrium point.

The system Σ is called *controllable* from x_1 to x_2 in finite time T if there exists an admissible control trajectory $u(t) \in \mathcal{U}(U_m)$, $t \leq T$ such that the solution trajectory satisfies $\varphi(T, x_1, u) = x_2$. The set of all points x where y is controllable from in time T for the system Σ using admissible inputs from $\mathcal{U}(U_m)$ is the *time T controllable set* and is denoted by $\mathcal{C}_{U_m}(y, T)$. We also write $\mathcal{C}_{U_m}(y) = \bigcup_{0 \leq t} \mathcal{C}_{U_m}(y, t)$ and refer to this as the *controllable set*. The system Σ is called *small-time locally controllable* (STLC) at an equilibrium point x_{eq} if $\mathcal{C}_{U_m}(x_{eq}, T)$ contains a neighborhood of x_{eq} for all $T > 0$. We also say the system Σ is STLC *with small controls* at an equilibrium point x_{eq} if $\mathcal{C}_{U_m}(x_{eq}, T)$ contains a neighborhood of x_{eq} for all $T > 0$. The system Σ is called *large-time locally controllable* (LTLC) at an equilibrium point x_{eq} if there exists a time $T > 0$ such that $\mathcal{C}_{U_m}(x_{eq}, T)$ contains a neighborhood of x_{eq}. In this paper the focus is on the *null-controllable region* (NCR) $\mathcal{C}_{U_m}(0)$ which will be abbreviated by $\mathcal{C}_m$. We recall that the boundary of the NCR $\partial\mathcal{C}_m$ is semi-permeable (if $x(t)$ starts in the exterior of $\mathcal{C}_m$, it can never reach $\partial\mathcal{C}_m$) and comprised of solution trajectories.

Consider the system Σ under the following assumptions:

1. The nominal system is structurally stable for all $u \in \mathbb{R}$.

2. The origin of the system is unstable (i.e. the set $\mathcal{E}(U_m)$ is nonempty and contains the origin). Moreover, the linearized system around each $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}(U_m)$ is controllable, not purely periodic and hyperbolic.

3. The system Σ around each $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}(U_m)$ is LTLC with U_m only if its STLC with U_m.

4. The set $\mathcal{C}_m$ is open, connected and diffomorphic to $\mathbb{R}^n$.

5. The sets $\mathcal{E}_i(U_m)$, $i \neq 0$ are outside of $\overline{\mathcal{C}_m}$.

6. There exists a smooth function $u_{eq} : \mathcal{E}_0(U_m) \to \mathbb{R}$ such that for each $x_{eq} \in \mathcal{E}_0(U_m)$, $f(x_{eq}) + g(x_{eq})u_{eq}(x_{eq}) = 0$ and $\mathcal{E}_0(U_m) \subset M$.

7. Let $\varphi(t, x^\star, u^\star)$ denote a controlled solution which forms part of the boundary $\partial\mathcal{C}_m$. The limit set $\omega(x^\star, u^\star)$ is nonempty and contains only equilibrium points.

This setup leads to the following two problems:

1. Characterizing the boundary trajectories which define the set $\mathcal{C}_m$.

2. Designing a stabilizing control action for the system Σ from all initial conditions in $\mathcal{C}_m$. This problem is address in Chapter 4 of this Thesis.

Remark 4.1. *In general, the equilibrium points induced by different input values can bifurcate and also change properties such as controllability [10]. The structural stability of the system ensures that equilibrium points do not bifurcate and simplifies the development which follows. Our analysis can be extended to handle the situation where the equilibrium points bifurcate by carefully considering which branch of the bifurcation needs to be analyzed further. In the interest of focusing on the key result in this paper, we leave this analysis to future work.*

Remark 4.2. *The set $\mathcal{E}_0(U_m)$ defines a continuum of equilibrium points which contains the origin and is obtained by varying the input in the unforced system smoothly over U. The function $u_{eq} : \mathcal{E}_0(U_m) \to \mathsf{R}$ which maps each control equilibrium value to the corresponding equilibrium point is assumed to be smooth. A sufficient condition for this assumption to hold is given in [11] and uses the dimension of function g and the boundedness of the function u_{eq}. Moreover, a geometric condition on f and g was derived in [12] to verify the set $\mathcal{E}_0(U_m)$ lies in the interior of $\mathcal{C}_m$. Under Assumption 5, we consider the problem where there is only a single equilibrium manifold $\mathcal{E}_0$ within $\overline{\mathcal{C}_m}$ and other branches arises due to multiplicity of the equilibria lie outside of the NCR.*

Remark 4.3. *Assumption 2 refers to the classic Kalman-Rank-Condition and is a sufficient condition for the system S to be STLC* with small controls *at x_{eq}. Note that the unstable nature of the unforced system along with the constraints on the control input render the system only locally controllable. This ensures the boundary trajectories exists and makes the problem at hand well defined.*

Remark 4.4. *Using results from linear control theory [13], one can show that the equilibrium manifold $\mathcal{E}_i$ corresponding to a set of purely periodic equilibrium points will be contained within the interior of the NCR. In this work, we focus on systems with non-purely periodic equilibria and show that the equilibrium points corresponding to extremal values of the control input lie on the boundary of the NCR. This fact will then be used to generate the boundary trajectories of the NCR. Note that the periodic nature of the equilibrium can be easily verified by analyzing the linearized system.*

Remark 4.5. *Under Assumption 3 we preclude systems which are LTLC with strictly positive (or negative) controls. In Lemma 4.3 we show that STLC of the linearized system with strictly positive (or negative) controls is a sufficient condition for STLC of the nonlinear system with strictly positive (or negative) controls. This result is key in being able to show that control equilibrium points with extremal values of the control are on the boundary of the NCR.*

Remark 4.6. *In general, the possible dynamical behavior of nonlinear system of the form in Eq.5.1 is very rich. These system's can exhibit highly complex behavior such as bounded orbits which are non-periodic. Assumption 7 restricts the class of systems to a simplified subclass where the long-term behavior as $t \to ¥$ of the NCR boundary trajectories approaches an equilibrium state. Empirical evidence suggests that many unstable systems satisfy this criteria, and there also exists a sufficient condition to verify this assumption [14].*

Remark 4.7. *Under Assumptions 1-5 we consider a particular case of small-time local controllability where presence of input constraints causes the size of the controllability region to be limited. This corresponds to the situation where the available input control set is uniformly related to the size of the NCR: $\mathcal{C}_{m_1} \subset \mathcal{C}_{m_2}$ for any two control subsets U_{m_1}, U_{m_2} of the input constraint set U_m where $0 \in U^{\circ}_{m_1}$ and $U_{m_1} \subset U_{m_2} \subseteq U_m$. That is, the presence of input constraints induces a limit on the controllability. Another way to express this is that with larger control action magnitude, it should be possible to stabilize a (strictly) larger set of initial conditions. Situations where this would not happen are when larger control action magnitude expands the NCR to the point where the region would absorb other nominally stable equilibrium points. This situation is demonstrated in Example 4.1. Note that in this situation since these newly included equilibrium points are stable, increasing the capacity of the control input may not change the boundary of the NCR. From a control standpoint, the problem at hand is more challenging for systems which exhibit this property.*

Example 4.1. *Consider the simple scalar example $\dot{x} = -0.5x^3 + 1.5x + u$ where, $x \in \mathsf{R}$, $|u| \leq u_{\max}$. If $u_{\max} \leq 1$, then $\mathcal{E}_0 = [-u_{\max}, u_{\max}]$ and the NCR is simply given by the set $x \in (-u_{\max}, u_{\max})$. However, if $u_{\max} > 1$, then the NCR is R as it contains the stable equilibrium manifold branches $\mathcal{E}_1$ and $\mathcal{E}_2$ which does not satisfy Assumption 5.*

4.2.3 Characterizing the Null-Controllable Region

The problem of characterizing the NCR experiences a sharp jump in complexity when transitioning from scalar to multi-dimensional systems. To understand this better, consider the following example:

Example 4.2. *Consider the simple scalar example $\dot{x} = f(x) + g(x)u$ where, $x \in \mathsf{R}$ $|u| \leq 1$, satisfying Assumptions 1-7. In this case the NCR is simply given by the set $x \in (x_{min}, x_{max})$, where $0 = f(x_{max}) + g(x_{max})$ and $0 = f(x_{min}) - g(x_{min})$, where for the sake of simplicity, we assume $x_{min} < x_{max}$ and there exist a single unique solution to the above equation. That is, the NCR is simply the (open) set of the equilibrium points as the input is varied over the allowable range. The simplicity of the analysis completely breaks down as the dimension of the state space increases from one.*

The problem of characterizing the NCR has been tackled in modern control theory and can be broadly classified into two approaches: 1) Direct approaches 2) Simulation-based approaches. In general, explicit characterizations of the boundary of the NCR are not available. However, there does exist a direct approach for generating the boundaries of the NCR using optimal control techniques. This approach is known as the *controllability minimum principle* and resembles the well-known Pontryagin minimum principle [15]. This result is often referred to as the abnormal form of the Pontryagin minimum principle since the adjoint variable associated to the cost function is zero.

Theorem 4.1 (Controllability Minimum Principle, [15])**.** *Let $u^*(t)$ be an admissible control trajectory which generates a boundary trajectory $x^*(t) \in \P\mathcal{C}_m$ for all $t \in [0, t_1]$. Then* *there exists a non-zero continuous solution* $l(t)$ *to the adjoint equations* $\dot{l}^T = -\frac{\P H(x,l,u)}{\P x}$, $t \in [0, t_1]$, *where* $H(x^*, l, u^*) = \min_{|u| \leq m} H(x^*, l, u) = 0$, *and* $H(x, l, u) = l^T(f(x) + g(x)u)$

Specifically, this theorem provides a necessary condition for trajectories which move along the boundary $\P\mathcal{C}_m$, where the adjoint variable can be understood as orthogonal to the NCR. Note that H is simply a scalar product of the tangent vector to the solution trajectory and the costate vector l. The solution trajectories that satisfy the conditions of Theorem 5.3 result in $H = 0$, thus ensuring that l is orthogonal to the solution trajectories.

Remark 4.8. *The conditions laid out in Theorem 5.3 are in general for any nonlinear system. In the present manuscript, they are written for a nonlinear control affine system, and for the system* S, *it can be further shown that the admissible control trajectory must be of the form of a time–optimal control law (bang–bang)* $u(t) = -\mathrm{sgn}(l^T(t)g(x(t)))$ *(we utilize this fact later in the proposed algorithm).*

Remark 4.9. *Note that Theorem 5.3 is not prescriptive, in the sense that it is not directly suited to compute the control law. This is because no boundary conditions are defined for* x *and* l *except* $H = 0$. *This lack of appropriate boundary conditions makes the application of Theorem 5.3 to determine a control law very difficult. In particular, we require the knowledge of a point on* $\P\mathcal{C}_m$ *in order to make the solution well defined. In the present manuscript, we show that the boundary* $\P\mathcal{C}_m$ *will contain equilibrium points which can be used to initialize the above problem. While the idea of using equilibrium points to initialize the problem in Theorem 5.3 to determine the control law has been mentioned in [16], one of the key contributions in the present work is to show that these equilibrium points also reside on the boundary of the NCR, and thus we utilize this fact to compute the NCR.*

Remark 4.10. *In general, explicit characterizations of the NCR are not available owning to the fact that general nonlinear dynamical systems do not always have explicit solutions. For linear systems, there does exist explicit characterizations of the NCR [2] and Theorem 5.3 for such systems becomes a necessary and sufficient condition, and this has been utilized [2] to provide explicit expressions which can be used draw out the boundary of the NCR for various linear systems of interest. The work in [17] does explore the problem of characterization of the NCR for nonlinear systems in terms of basins of attraction of the equilibrium points but does not utilize the controllability minimum principle. In the next section, we show how Theorem 5.3 can be applied in a computationally tractable procedure to obtain a non-explicit characterization of the NCR.*

4.3 Generating the Boundary Trajectories of the NCR

In this section we focus on the problem of generating the boundary trajectories of the NCR. The general approach is to use control-equilibrium points corresponding to extremal values of the input control range as boundary conditions in the controllability minimum principle. Towards this end, we first establish some properties of the

NCR in relation to the equilibrium manifold $\mathcal{E}_0$. Subsequently, we use these properties to derive boundary conditions in the controllability minimum principle. Finally, we outline two algorithms which apply the controllability minimum principle with the new boundary conditions to generate the NCR boundary trajectories.

4.3.1 Control Equilibrium Points and the Null-Controllable Region

In this subsection, we will derive properties of the NCR relating to control equilibrium set $\mathcal{E}_0$. We first show controllability region for each equilibrium point in the open set $\mathcal{E}_0(U_{\mathsf{m}}^{\circ})$ will be same and that the equilibrium manifold $\mathcal{E}_0(U_{\mathsf{m}})$ is contained within the closure of the NCR.

Lemma 4.1. *Let x_{eq}, y_{eq} be two equilibrium points in $\mathcal{E}_0(U_{\mathsf{m}}^{\circ})$. Then $\mathcal{C}_{U_{\mathsf{m}}}(x_{eq}) = \mathcal{C}_{U_{\mathsf{m}}}(y_{eq})$ and $\mathcal{E}_0(U_{\mathsf{m}}) \subseteq \overline{\mathcal{C}_{\mathsf{m}}}$.*

Proof. *Consider the set of equilibrium points $\mathcal{E}_0(U_{\mathsf{m}})$. Since the system is structurally stable, this set is a connected $(n-1)$-dimensional manifold which contains 0. It follows from Assumption 2 that each point in the subset $\mathcal{E}_0(U_{\mathsf{m}}^{\circ})$ is STLC using an admissible control action in U_{m}. That is, the set $\mathcal{C}_{U_{\mathsf{m}}}(x_{eq})$ contains a neighborhood of x_{eq} for every $x_{eq} \in \mathcal{E}_0(U_{\mathsf{m}}^{\circ})$. Since this manifold is connected it follows that the system can transverse from one point x_{eq} in the manifold to another y_{eq} (or vice versa). That is, $\mathcal{C}_{U_{\mathsf{m}}}(x_{eq}) = \mathcal{C}_{U_{\mathsf{m}}}(y_{eq})$. This also implies $\mathcal{E}_0(U_{\mathsf{m}}^{\circ}) \subseteq \mathcal{C}_{\mathsf{m}}$. It follows that $\mathcal{E}_0(U_{\mathsf{m}}) \subseteq \overline{\mathcal{C}_{\mathsf{m}}}$.*

We now consider linear systems and derive an important property regarding equilibrium points corresponding to extremal values of the scalar input range and the trajectories which form the boundaries around these equilibrium points. Consider linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4.4}$$

where $A \in \mathsf{R}^{n\times n}$, $B \in \mathsf{R}^{n}$.

Lemma 4.2. *Consider the linearized system of S in the form of Eq.4.4 with non-negative inputs $u(t) \in \mathcal{U}([0,\mathsf{m}])$. Then there exists a trajectory $x^*(t)$ on $\P\mathcal{C}_{[0,\mathsf{m}]}(0)$ such that $x^*(0) \neq 0$ and $x^*(t_1) = 0$, $u^*(t_1) = \mathsf{m}$ for some $t_1 > 0$.*

Proof. *Recall by Assumption 2, the linearized system around $x_{eq} \in \mathcal{E}_0$ is not purely periodic and controllable. Using non-negative control input $u(t) \in \mathcal{U}([0,\mathsf{m}])$, one can show using similar arguments as Lemma 3.1 in [13] that with such a control input range and non-purely periodic equilibrium points, the origin will be on the boundary of the NCR $\mathcal{C}_{[0,\mathsf{m}]}(0)$ for the linear system. Let $l_1, \ldots, l_n$ denote the eigenvalues of A. These would also be the eigenvalues of A^T. For each l_i there is a corresponding eigenvector $v_i \in \mathsf{R}^n$, $v_i \neq 0$ such that $A^T v_i = l_i v_1$. Let $B_\perp$ be a vector with is orthogonal to B. Since the system is controllable the vector $B_\perp$ can be expressed as a linear combination of eigenvectors $B_\perp = c_1 v_1 + \ldots + c_n v_n$, $c_i \in \mathsf{R}$. We may assume all the scalars c_i are one ($c_i = 1$) and replace each v_i by $c_i v_i$ if necessary. $B_\perp = v_1 + \ldots + v_n$. Let $x(t)$ be the solution to the reverse-time* *system in Eq.5.36 for* $t \leq 0$ *with* $x(0) = 0$ *and* $u(t) \in \mathcal{U}([0, m])$*. Let* $h(t) = B_{\perp}^T x(t)$*, and* $r_i(t) = v_i^T x(t)$ *then* $h(t) = (v_1 + \ldots + v_n)^T x(t) = r_1(t) + \ldots + r_n(t)$*. As shown in Lemma 3.1 from [13] the functions* r_i *will satisfy the dynamic equation*

$$\dot{r}(t) = l_i r(t) + (v_i^T B) u(t) \tag{4.5}$$

It follows that $h(t) \geq 0$*. That is the* $x(t)$ *must reside in the half-space* $\{x \in \mathrm{R}^n : B_{\perp}^T x(t) \geq 0\}$ *for* $t \leq 0$*. Therefore the NCR* $\mathcal{C}_{[0,m]}(0)$ *must be bounded by the plane with normal* $B_{\perp}^T$ *and which defines this half space. Since* $\dot{x} = Bm$ *with* $u = m$*, it follows that starting from* $x(0) = 0$ *the reverse time system will transverse tagentially on the hyperplane* $B_{\perp}^T x(t) = 0$ *and thus the trajectory starting at* $x(0) = 0$ *using* $u(0) = m$ *must be in the boundary of the NCR. Therefore there exists a trajectory* $x^*(t)$ *on* $\P\mathcal{C}_{[0,m]}(0)$ *such that* $x(t_1) = 0$*,* $u(t_1) = m$ *for some* $t_1 > 0$*.*

Lemma 4.2 provides a property of a trajectory on the NCR boundary for linear systems. Specifically, the Lemma states that the boundary of the controllability region $\P\mathcal{C}_{[0,m]}(0)$ will contain the origin and always have a trajectory which approaches the origin. For the nonlinear system S, we are able to use this result from the linearized system to show that the equilibrium points corresponding to extremal values of the scalar input range also reside on the boundary of the NCR.

Lemma 4.3. *Consider* $x_{eq} \in \mathcal{E}_0(\P U_m)$ *then* $x_{eq} \in \P\mathcal{C}_m$*.*

Proof. *Note that the effective input constraint range for the equilibrium point* $x_{eq} \in \mathcal{E}_0(\P U)$ *will have zero as an end point. Using Lemma 4.2, we know that with strictly positive (or negative) control input range and non-purely periodic equilibrium points, the equilibrium* x_{eq} *corresponding to extremal values of the input range will be on the boundary of the NCR for the linearized system. The controllability of the linearized system around* x_{eq} *implies the existence of a locally diffomorphic map* $y \to x(T, y)$ *where* $y = [y_1, y_2, \ldots, y_n] \in B_e$*,* y *near 0. Suppose* x_{eq} *is STLC using the control range* U_m*. This implies there exists an input function defined on* $[0, T]$ *which can steer the system from* x_{eq} *to points which form a span of* R^n*. Since the map* y *is difformorphic, a neighbourhood of the nonlinear system must be mapped to a neighbourhood of the linear system. This implies the linear system is controllable. It follows from Lemma 4.2 that this is a contradiction. Therefore* x_{eq} *is not STLC. Under Assumption 3, this implies* x_{eq} *is also not LTLC. Therefore there is no control trajectory that will drive* x_{eq} *to the origin. Using Lemma 4.1 along with the fact that* $\mathcal{E}_0(U_m)$ *is connected, it follows that every neighborhood of* x_{eq} *will contain points in* $\mathcal{C}_m$*. Therefore* x_{eq} *must be on the boundary of the NCR,* $x_{eq} \in \P\mathcal{C}_m$*.*

In addition to showing the extremal equilibrium points reside on the boundary of the NCR, in the next Lemma we show that the boundary of the controllable region $\P\mathcal{C}_{U_m}(x_{eq})$ for any extremal equilibrium $x_{eq} \in \mathcal{E}_0(\P U_m)$ contains a trajectory which reaches this equilibrium point in finite time.

Lemma 4.4. *Consider the system of* S*. Then for each* $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}_0(\P U_{\mathsf{m}})$ *there exists a trajectory* $x^*(t)$ *on* $\P\mathcal{C}_{U_{\mathsf{m}}}(x_{eq})$ *such that* $x^*(0) \neq x_{eq}$ *and* $x^*(t_1) = x_{eq}$*,* $u^*(t_1) = -u_{eq}$ *for some* $t_1 > 0$*.*

Proof. *Let* $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}_0(\P U_{\mathsf{m}})$*. Using Lemma 4.2 and a similar argument as Lemma 4.3, it follows that each* $x_{eq} \in \P\mathcal{C}_{U_{\mathsf{m}}}(x_{eq})$*. Using the difformorphic map* y*, all control trajectories of the linearized system in a neighbourhood of* x_{eq} *with the input range* $[-\mathsf{m}, \mathsf{m}]$ *can be mapped to corresponding control trajectories of the nonlinear system using the same input range* $[-\mathsf{m}, \mathsf{m}]$*. That is, this map will preserve the boundary trajectories which form the set* $\P\mathcal{C}_{U_{\mathsf{m}}}(x_{eq})$*. Since the boundary of the linearized system will contain a trajectory* $x^*(t)$ *on* $\P\mathcal{C}_{U_{\mathsf{m}}}(x_{eq})$ *such that* $x^*(0) \neq x_{eq}$ *and* $x^*(t_1) = x_{eq}$*,* $u^*(t_1) = -u_{eq}$ *for some* $t_1 > 0$*, it follows that the nonlinear system will also contain such a trajectory.*

Remark 4.11. *Lemma 4.1 states that the controllability region for any two equilibrium points in the interior of the set* $\mathcal{E}_0(U_{\mathsf{m}})$ *will be equal and that the entire equilibrium manifold* $\mathcal{E}_0(U_{\mathsf{m}})$ *is contained within the closure of the NCR. The main idea in the proof of this Lemma is that since each equilibrium* $x_{eq} \in \mathcal{E}_0(U_{\mathsf{m}}^{\circ})$ *is STLC, there exists an admissible input trajectory which can drive the state to glide along the equilibrium manifold* $\mathcal{E}_0$*. Therefore every point on the manifold* $\mathcal{E}_0(U_{\mathsf{m}}^{\circ})$ *is controllable to any other point in* $\mathcal{E}_0(U_{\mathsf{m}}^{\circ})$*. Lemma 4.1 is used in Lemma 4.2 where we derive an important property regarding the boundary trajectory for linear systems with only positive control inputs. This property is used in Lemma 4.3 to show that for the nonlinear system* S *the extremal points of the set* $\mathcal{E}_0(\P U_{\mathsf{m}})$ *are on the boundary of the NCR. It is also used in Lemma 4.4 to show that the boundary of the controllability region for* x_{eq}*,* $\P\mathcal{C}_{U_{\mathsf{m}}}(x_{eq})$ *will contain trajectories which reach the extremal equilibrium point in finite time. The key idea in the proofs for Lemmas 4.3 and 4.4 is to use the qualitative equivalence of the local behavior around the equilibrium between the nonlinear system and the linearized system. Both these properties are used in Section 4.3.2 to devise an algorithm to generate the boundary trajectories of the NCR* $\P\mathcal{C}_{\mathsf{m}}$*.*

4.3.2 Generating the Boundary Trajectories Using Controllability Minimum Principle

In this section we focus on using the derived properties to initialize the Controllability Minimum Principle in order to enable the application of the minimum principle to construct the boundary trajectories of the NCR. We divide the problem based on the nature of the instability of the origin. We first focus on anti-stable systems where the NCR is bounded. In such systems the boundaries are shown to always emanate from the extremal control equilibrium points and thus the equilibrium point can be used directly as an initial condition with the minimum principle. Following this, we look at semi-stable systems where the NCR is smooth. In such systems, the boundary of the NCR is shown to be comprised of trajectories which approach the stable modes of the equilibrium points. In this case, the Controllability Minimum Principle is initialized using states within the stable manifold of the control equilibrium point.

Anti-Stable Systems

By exploiting the derived properties of the NCR, we are able to devise a procedure to solve for the time-optimal trajectories which generate the boundary of the NCR $\P\mathcal{C}_m$. We first focus on systems where the origin is anti-stable (i.e at all eigenvalues are positive) and where the NCR is bounded.

In Lemma 4.4 we showed that the boundary of the controllability region for extremal equilibrium points will contain trajectories which reach the extremal equilibrium point in finite time. Here we show that the boundary of the NCR for a general nonlinear system S which is anti-stable has a similar property. Specifically, the boundary will always have a trajectory which reaches the extremal control equilibrium point in finite time. This property is used to design an algorithm to generate the boundary trajectories.

Lemma 4.5. *Consider the system* S *where* $x = 0$ *is an anti-stable equilibrium point of the nominal system and the NCR* $\mathcal{C}_m$ *is bounded. Let* $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}_0(\P\mathcal{U}_m)$ *be an equilibrium pair. There exists a trajectory* $x^*(t)$ *on the boundary* $\P\mathcal{C}_m$ *such that* $x^*(0) \neq x_{eq}$, $x^*(t_1) = x_{eq}$*, and* $u^*(t_1) = -u_{eq}$ *for some* $t_1 > 0$.

Proof. *By Lemma 4.4, the boundary of the set* $\P\mathcal{C}_{\mathcal{U}_m}(x_{eq})$ *will contain a trajectory* $x^*(t)$ *such that* $x^*(0) \neq x_{eq}$, $x^*(t_1) = x_{eq}$*, and* $u^*(t_1) = -u_{eq}$ *for some* $t_1 > 0$*. We now show that for the system* S *where* $x = 0$ *is anti-stable the set* $\bar{\mathcal{C}}_{\mathcal{U}_m}(x_{eq})$ *is invariant for any equilibrium point in the manifold* $\mathcal{E}_0(\mathcal{U}_m)$*. It has already been shown in Lemma 4.1 that if* x_{eq}*,* y_{eq} *are two equilibrium points in* $\mathcal{E}_0(\mathcal{U}_m^\circ)$*. Then* $\mathcal{C}_{\mathcal{U}_m}(x_{eq}) = \mathcal{C}_{\mathcal{U}_m}(y_{eq})$*. Let* $z_{eq} \in \mathcal{E}_0(\P\mathcal{U}_m)$ *and* $x_{z_{eq}} \in \mathcal{C}_{\mathcal{U}_m}(z_{eq})$*. It remains to show that* $\mathcal{C}_{\mathcal{U}_m}(z_{eq}) = \bar{\mathcal{C}}_{\mathcal{U}_m}(x_{eq})$*. Since* z_{eq} *is on the boundary of the NCR* $\P\mathcal{C}_{\mathcal{U}_m}(x_{eq})$ *and this boundary is semi-permeable, the state* $x_{z_{eq}}$ *can only reach* z_{eq} *starting from within the NCR* $\mathcal{C}_{\mathcal{U}_m}(x_{eq})$ *or starting from the boundary* $\P\mathcal{C}_{\mathcal{U}_m}(x_{eq})$*. That is* $x_{z_{eq}} \in \bar{\mathcal{C}}_{\mathcal{U}_m}(x_{eq})$*. Let* $x_{x_{eq}} \in \bar{\mathcal{C}}_{\mathcal{U}_m}(x_{eq})$*. Since the system is anti-stable, starting at* z_{eq} *we can reach any equilibrium point in reverse time. That is each equilibrium point is reachable from* z_{eq}*, therefore each equilibrium point is controllable to* z_{eq}*. Since all other points* $\bar{\mathcal{C}}_{\mathcal{U}_m}(x_{eq})$ *are controllable to every equilibrium point, it follows that* $x_{x_{eq}} \in \mathcal{C}_{\mathcal{U}_m}(z_{eq})$*. Thus,* $\mathcal{C}_{\mathcal{U}_m}(z_{eq}) = \bar{\mathcal{C}}_{\mathcal{U}_m}(x_{eq})$ *and using Lemma 4.4 we get that there exists a trajectory* $x^*(t)$ *on the boundary* $\P\mathcal{C}_m$ *such that* $x^*(0) \neq x_{eq}$, $x^*(t_1) = x_{eq}$*, and* $u^*(t_1) = -u_{eq}$ *for some* $t_1 > 0$.

In the following Theorem, we use the property in Lemma 4.5 to be able to use the control equilibrium points as boundary conditions within the Controllability Minimum Principle.

Theorem 4.2. *Consider the system* S *where* $x = 0$ *is an anti-stable equilibrium point of the nominal system and the NCR* $\mathcal{C}_m$ *is bounded. Let* $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}_0(\P\mathcal{U}_m)$ *be an equilibrium pair. Then there exists a trajectory* $x^*(t)$ *on the boundary of the NCR* $\P\mathcal{C}(\mathcal{U}_m)$ *such that for all* $t \in [-t_1, 0]$ *there exists a non-zero continuous solution* $l(t)$ *to the adjoint equations* $\dot{l}^T = -\frac{\P H(x,l,u)}{\P x}$, $t \in [-t_1, 0]$*, where* $H(x^*, l, u^*) = \min_{|u| \leq m} H(x^*, l, u) = 0$, $H(x, l, u) = l^T(f(x) + g(x)u)$*, and* $x^*(0) = x_{eq}$, $u^*(0) = -\operatorname{sgn}(u_{eq})m$.

Proof. *It follows from Lemma 4.5 that the boundary of the NCR will have a trajectory which starts at* $x^*(0) = x_{eq}$ *with* $u^*(0) = -\operatorname{sgn}(u_{eq})m$ *and transverses in reverse time. Since the Controllability Minimum Principle is a necessary condition, this trajectory must also satisfy* $\dot{l}^T = -\frac{¶H(x,l,u)}{¶x}$, $t \in [-t_1, 0]$, *where* $H(x^*, l, u^*) = \min_{|u|\leq m} H(x^*, l, u) = 0$, $H(x, l, u) = l^T(f(x) + g(x)u)$.

We now apply this theorem to derive the following algorithm to generate the boundary of the NCR for anti-stable systems.

1. Find all the equilibrium points in the set $\mathcal{E}_0(¶U_m)$
2. For each equilibrium point $x^* \in \mathcal{E}_0(¶U_m)$ with corresponding equilibrium control u^*, and choose an initial costate vector $l(0)$ which is orthogonal to $f(x^*) - g(x^*)u^*$ and satisfies $\operatorname{sgn}(g(x^*)l(0)) = -u^*$
3. Using the equations from Theorem 4.2 with initial state being x^*, $u^*(0) = -u^*$, $l(0)$ simulate the system in reverse-time for time $-t_1$, where $t_! > 0$
4. If $H(x(-t_1))$ is not zero, then restart with a different costate vector $l(0)$
5. If $H(x(-t_1))$ is zero and $x(-t_1)$ is in e-ball of any equilibrium point $\bar{x}^* \in \mathcal{E}_0(¶U_m)$ then the resulting trajectory will be on the boundary of the NCR
6. If $x(-t_1)$ is still not close any equilibrium point $\bar{x}^* \in \mathcal{E}_0(¶U_m)$ then go back to step 5 with $t_2 = 2t_1$

Algorithm 1: To determine the boundary trajectories of $¶\mathcal{C}_m$ for anti-stable systems

Remark 4.12. *The key idea in proof for Lemma 4.5 is to show the equivalence of any two controllability regions. This uses the fact that the region is bounded and the equilibrium points are anti-stable. Note that for semi-stable systems this property is in general not true. This can be easily demonstrated using a linear system.*

Remark 4.13. *Algorithm 1 provides a systematic procedure to generate boundary trajectories of the NCR for anti-stable systems. The procedure applies the controllability minimum principle with the initial state being a control equilibrium point and the initial control value being the negative of the corresponding equilibrium control. The optimal control problem is solved from this initial condition in reverse time until the state reaches close to an equilibrium point. Since the region is assumed to be bounded the criteria of reaching close to an equilibrium point can be used. Note that the boundary trajectories include ones which emanate from an equilibrium point and converge back to the same starting equilibrium point. Any example of such behavior can be seen in three dimensional systems.*

Semi-Stable Systems

In this section we continue to study properties of the boundary of the NCR by considering systems which are semi-stable (i.e at least one eigenvalue is positive and at least one is negative) and where the boundary of the NCR is smooth. By assuming the existence of a feedback controller which can stabilize all initial conditions in the NCR the problem of NCR characterization is cast into the problem of characterizing the closed–loop stability region. The problem of characterizing and computing stability regions for nonlinear dynamical system has received considerable attention. One key result is that of [14] which shows how the boundary of the stability region consists of the stable manifolds of all the equilibrium points (and/or closed orbits) on the stability boundary. In the previous section we showed how if the origin is not purely periodic then the extremal control equilibrium points will be on the boundary of the NCR. In this section, we draw a connection with the stability region results by showing the stable manifolds which form the boundary in this setting correspond to the extremal control equilibrium points. This implies that the boundary trajectories will be tangential to the stable manifold at the extremal equilibrium points. This property is used to design an algorithm to construct the boundary of the NCR for this subclass of nonlinear systems.

Suppose there exists a state–feedback stabilizing law $k(x) : \mathsf{R}^n \to \mathcal{U}(U_m)$ for the system S such that the right-hand side of closed–loop system

$$\dot{x}(t) = f(x(t)) + g(x(t))k(x(t)) \tag{4.6}$$

is C^1 for $x(t) \in M \subset \mathsf{R}^n$ and the origin is asymptotically stable for all $x \in \mathcal{C}_m$. We also assume that $k(x_{eq}) = u_{eq}$, where $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}_0(¶U_m)$. The closed–loop stability region under the control action $k(x)$ is denoted by $\mathcal{S}_k$. Since the control law $k(x)$ stabilizes all initial conditions in the NCR, it follows that $\mathcal{S}_k = \mathcal{C}_m$. For an equilibrium point y the set $\mathcal{W}^s(y, u(t))$ denotes the stable manifold defined by

$$\mathcal{W}^s(y, u(t)) = \{x \in M : j(t, x, u(t)) \to y \text{ as } t \to ¥\} \tag{4.7}$$

This main results from [14] are adapted to our setting of control systems and are stated below.

Theorem 4.3 ([14])**.** *Consider the system* S *with* $u(t) = k(x(t))$*. Suppose that* $x = 0$ *is an asymptotically stable equilibrium point in the closed–loop and that the following conditions hold*

1. *All equilibrium points on the stability boundary* $¶\mathcal{S}_k$ *are hyperbolic*
2. *The stable and unstable manifolds of the equilibrium points on the stability boundary* $¶\mathcal{S}_k$ *satisfy the transversality condition*
3. *Every trajectory on the stability boundary* $¶\mathcal{S}_k$ *approaches one of the equilibrium points as* $t \to ¥$

Let x_i, $i = 1, 2, \ldots$ be the equilibrium points on the stability boundary $\P\mathcal{S}_k$. Then, the boundary can be characterized as:

$$\P\mathcal{S}_k = \mathop{[}\limits_i \mathcal{W}^s(x_i, k(x)) \tag{4.8}$$

Under the assumptions of Theorem 4.3 with the control law $u = k(x)$ the closed–loop stability region can be characterized using the stable manifolds of the equilibrium points on the boundary. In the previous section we showed how the boundary will contain the extremal control equilibrium points. Since the controller is assumed to have the property $k(x_{eq}) = u_{eq}$, such extremal control equilibrium points will be on the boundary of the closed–loop stability region. Since the explicit description of $k(x)$ is not available we cannot directly apply Theorem 4.3 to generate the boundary trajectories. However, since the trajectories form the stable manifold of these equilibrium points, some information about the direction of the tangent vector at the equilibrium point is known. Specifically, the tangent vector will be within stable eigenspace of the extremal control equilibrium point. This property is formalized in Lemma 4.6.

Lemma 4.6. *Consider the system* S *where $x = 0$ is a semi-stable equilibrium point of the nominal system, under the smooth stabilizing feedback law $u(t) = k(x(t))$. Suppose the conditions in Theorem 4.3 hold. Let $x_{eq} \in \mathcal{E}_0(\P U_m)$ be an extremal control equilibrium point. Then the tangent space of boundary curve $\P\mathcal{C}_m$ at x_{eq} is contained within $E^s(x_{eq})$.*

Proof. *Consider the closed–loop system* S *with $u(t) = k(x(t))$. By assumption, the controller is able to stabilize all initial conditions within the NCR thus the closed–loop stability region coincides with the NCR. It follows from Theorem 4.3 that the boundary of the NCR contains the stable manifolds of the equilibirum points on the boundary. From Lemma 4.3 it follows that the boundary of the NCR will contain extremal control equilibrium points $x_{eq} \in \mathcal{E}_0(\P U_m)$. By assumption $k(x_{eq}) = u_{eq}$, thus the stable manifold of the extremal control equilibrium point will be within the boundary of the NCR. It follows that the tangent space of the boundary curve $\P\mathcal{C}_m$ at x_{eq} is contained within $E^s(x_{eq})$.*

In the following Theorem, we use the property in Lemma 4.6 to be able to use the control equilibrium points and the direction of the tangent space of the boundary trajectories as boundary conditions within the Controllability Minimum Principle.

Theorem 4.4. *Consider the system* S *where $x = 0$ is an semi-stable equilibrium point of the nominal system. Let $(x_{eq}, u_{eq}) \in \hat{\mathcal{E}}_0(\P U_m)$ be an equilibrium pair. Then there exists a trajectory $x^*(t)$ on the boundary of the NCR $\P\mathcal{C}_m$ such that for all $t \geq 0$ there exists a non-zero continuous solution $l(t)$ to the adjoint equations $\dot{l}^T = -\frac{\P H(x,l,u)}{\P x}$, $t \geq 0$, where $H(x^*, l, u^*) = \min_{|u| \leq m} H(x^*, l, u) = 0$, $H(x, l, u) = l^T(f(x) + g(x)u)$, and $x^*(t) \to x_{eq} \in \mathcal{E}_0(\P U_m)$, $u^*(t) \to u_{eq} \in \P U_m$*

Proof. *It follows from Lemma 4.6 that the boundary of the NCR will be composed of the stable manifold of extremal equilibrium points. That is, $x^*(t) \to x_{eq} \in \mathcal{E}_0(\P U_m)$, $u^*(t) \to$*

$u_{eq} \in \P U_m$. Since the Controllability Minimum Principle is a necessary condition, this trajectory must also satisfy $\dot{l}^T = -\frac{\P H(x,l,u)}{\P x}$, *$t \geq 0$, where* $H(x^*, l, u^*) = \min_{|u| \leq m} H(x^*, l, u) = 0$, $H(x, l, u) = l^T(f(x) + g(x)u)$.

The property presented in Theorem 4.4 is used as a boundary condition to construct the trajectories in the NCR.

1. Find all the equilibrium points in the set $\mathcal{E}_0(\P U_m)$
2. For each equilibrium point $x^* \in \mathcal{E}_0(\P U_m)$ with corresponding equilibrium control u^* find a normalized stable eigenvector y of the Jacobian
3. Find the point of intersection x^{**} of this stable eigenvector with the boundary of an e-ball of the equilibrium point x^* and choose an initial costate vector $l(0)$ which is orthogonal to $f(x^{**}) + g(x^{**})u^*$ and satisfies $\mathrm{sgn}(g(x^{**})l(0)) = u^*$
4. Using the equations from Theorem 4.4 with initial state being the intersection point x^{**}, $u^*(0) = u^*$, and $l(0)$ solve the optimal control problem in reverse time for some time $-t_1$
5. If $H(x(-t_1))$ is not zero, then restart with a different costate vector $l(0)$
6. If $H(x(-t_1))$ is zero and $x(-t_1)$ is not contained in the NCR (determined by checking the feasibility of the boundary value problem) then the resulting trajectory will be on the boundary of the NCR
7. If $x(-t_1)$ is not contained in the NCR then repeat the simulation for time $t_2 = \frac{1}{2}t_1$

Algorithm 2: To determine the boundary trajectories of $\P\mathcal{C}_m$ for semi-stable systems

Remark 4.14. *The development in this section assumes the existence of the stabilizing feedback control law $u = k(x)$ and does not require the explicit availability of this controller. Determining condition for which such stabilizing feedback control law exist is the topic of large body of work (e.g. [18]–[20]). In general nonlinear systems which are controllable require discontinuous feedback. However, in the current work the existence of a stabilizing feedback follows from Assumption 2 [21]. Note also that Lemma 4.6 states that the tangent vectors of the boundary trajectories of the NCR for semi-stable systems at the extremal control equilibrium points will be contained within the stable subpace of the linearized system at the extremal control equilibrium point. This property is used to construct the boundary of the NCR.*

Remark 4.15. *The property presented in Lemma 4.6 is under the conditions given in Theorem 4.3. Condition 1 holds under Assumption 2. In [14], it was shown that Condition 3 holds for many dynamical systems arising in physical system models. Moreover they provide a sufficient condition for this assumption.*

Remark 4.16. *Algorithm 2 provides a systematic procedure to generate boundary trajectories of the NCR for semi-stable systems. The procedure applies the controllability minimum principle with the initial state being close to a control equilibrium point in the direction of the stable manifold and the initial control value corresponding to the extremal control equilibrium point. The optimal control problem is solved from this initial condition in reverse time.*

Remark 4.17. *In general, explicit descriptions of the boundary of the NCR do not exist for nonlinear systems, mostly due to the fact that nonlinear systems do not necessaritly have closed–form solution as opposed to linear systems For classes of systems where closed–form solutions do exist, explicit descriptions of the NCR can be derived. The simulation example presented in Section 4.4.1 demonstrates how the proposed procedure recovers the same explicit description available for linear systems. For all other classes of systems where closed–form solutions do not exist, one must resort to numerical procedures to construct the boundary of the NCR, in which case the distinguishing feature of a computing scheme is the computational complexity involved. The proposed algorithms are more computationally tractable than possible brute-force solutions. In particular, a possible brute- force solution would discretize the state-space and attempt to determine the feasibility of an (practically) infinite-horizon optimal control problem at each point in the state-space. For example, if each dimension is discretized into n_d nodes then the procedure would require n_d^n optimization problems to be solved and thus suffer from the curve of dimensionality. In our proposed procedure only the equilibrium manifold $\mathcal{E}_0$ needs to be discretized and thus only n_d optimization problems are required to be solved.*

4.4 Simulation Examples

In this section we demonstrate the proposed procedure to construct the boundary of the NCR to stabilize all initial conditions in the NCR using several simulation examples. The simulations also show the implementation of existing control design and one based on utilizing the NCR construction that leads to stabilization. Details on the control design are outside the scope of the present manuscript.

4.4.1 Linear Systems

We start with linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad |u(t)| \leq 1 \tag{4.9}$$

where $A \in \mathsf{R}^{n\times n}$, $B \in \mathsf{R}^n$.

The work in [2] derives explicit descriptions of the boundary for linear systems which are anti-stable using extremal trajectories of the time reversed system. Here we show how the proposed approach recovers the results from [2]. Recall that for linear systems the time-optimal control condition give in the Controllability Minimum Principle is both necessary and sufficient. Hence the algorithms presented in the last section simplify to allow the time-optimal controls which are used to form the boundary trajectories to be explicitly determined. Specifically, the time-optimal controls given by the Controllability Minimum Principle have the following form (*bang–bang*) $u(t) = -\text{sgn}(I^T(t)B) = \text{sgn}(c'e^{-At}B)$, $c \neq 0$. If A has only real eigenvalues, it can be shown that the term $c'e^{-At}B$ has at most $n-1$ zeros. For anti-stable systems with real eigenvalues, since the extremal equilibrium points on the boundary trajectories will be reached in finite time (as shown in Lemma 4.4), the optimal control trajectory which solves the Controllability Minimum Principle will have a switch point upon reaching the extremal equilibrium point. Therefore prior to reaching this extremal equilibrium point the control can have at most $n-2$ switches. Therefore the trajectories which form the boundary of the NCR consist of those which start at the extremal equilibrium points $\pm A^{-1}B$ and use the set of bang-bang controls with $n-2$ or less switches. This results in a simplification of Algorithm 1: Simply start at the equilibrium points $\pm A^{-1}B$ and simulate the time-reverse system for all possible bang-bang controls with $n-2$ or less switches until the trajectory reaches another equilibrium point or returns back to the starting equilibrium point. For nonlinear systems which are topologically equivalent to linear systems, this procedure can also be utilized.

Example 4.3. *Consider a second-order anti-stable linear system from [2] where* $A = \begin{bmatrix} 0 & -0.5 \\ 1 & 1.5 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. *The eigenvalues of the system are 0.5, 1. To generate the boundary of the NCR, we use Algorithm 1 by starting at the equilibrium point* $x_0 = x_{eq} = -A^{-1}b = [1,\ 0]^T$ *and apply the time-optimal control* $u = -1$ *in reverse time to generate the boundary trajectory corresponding to this equilibrium point and satisfying the Controllability Minimum Principle. Similarly we start at* $x_0 = x_{eq} = A^{-1}B = [-1,\ 0]^T$ *and apply* $u = 1$ *in reverse time to generate the other trajectory which forms the boundary.*

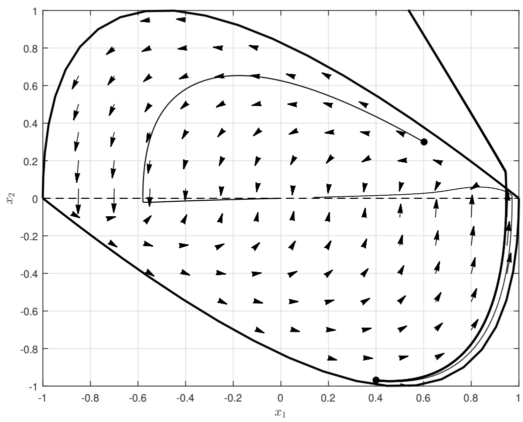

FIGURE 4.1: Phase plane of the anti-stable planar linear system given in Example 4.3.

4.4.2 Nonlinear Systems

Example 4.4. *Consider the 3d nonlinear system*

$$\begin{aligned}\dot{x_1} &= 0.5x_1 + x_2 + x_2^3 \\ \dot{x_2} &= x_2 + x_3 \\ \dot{x_3} &= 2x_3 + x_3^3 + u\end{aligned}$$

where $|u(t)| \leq 1$. The nominal system has a single equilibrium point at the origin and the linearized system around the origin is anti-stable with eigenvalues 0.5, 1, 2. To generate the boundary of the NCR, we again use Algorithm 1. The algorithm starts with the initial conditions being the extremal equilibrium points which are $x_0 = x_{eq} = \pm[1.0932, -0.4534, 0.4534]^T$ and solves for the time-optimal control trajectories that satisfy the Controllability Minimum Principle. Since this system is topologically equivalent to the linearized system, the family of trajectories which form the boundary are formed using set of bang-bang controls with $n-2$ or less switches.

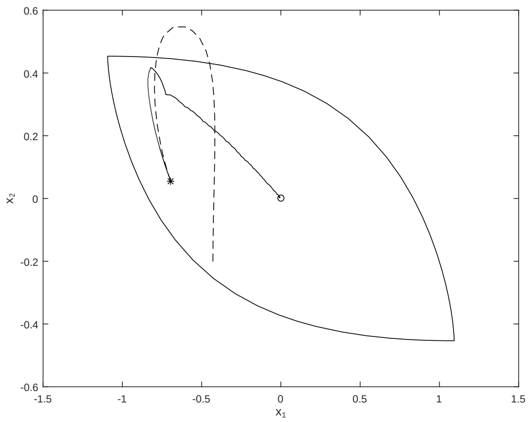

FIGURE 4.2: Phase plane (x_1, x_2) of the 3d nonlinear system given in Example 5.2.

Example 4.5. *Consider a continuous stirred tank reactor where an irreversible, first-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. We use simplified mathematical model for the process which was presented in [22] and takes the form:*

$$\dot{x_1} = -fx_1k(x_2) + q(x_{1f} - x_1) \tag{4.10}$$

$$\dot{x_2} = bfx_1k(x_2) - (q + d)x_2 + qx_{2f} + du \tag{4.11}$$

FIGURE 4.3: Phase plane (x_1, x_3) of the 3d nonlinear system given in Example 5.2.

FIGURE 4.4: Phase plane (x_2, x_3) of the 3d nonlinear system given in Example 5.2.

where $x = [x_1\ x_2]^T$, x_1 is the dimensionless concentration, and x_2 is the dimensionless temperature. The dimensionless cooling jacket temperature u is the control input. The system has three equilibrium points: 1) $x_{s1} = [0.8560, 0.8859]^T$, 2) $x_{s2} = [0.5528, 2.7517]^T$, 3) $x_{s3} = [0.2354, 4.7050]^T$. The equilibrium points x_{s1}, and x_{s3} are stable, whereas x_{s2} is unstable with one stable and one unstable mode. The control objective is to stabilize the reactor at the unstable equilibrium point x_{s2}. We let $\hat{x} = x - x_{s2}$ denote the deviation variable.

Similar to the previous examples, the results in Figure 5.5 show the phase-plane of the closed–loop system. Also depicted in Figure 5.5 is boundary of the NCR (solid line) constructed using developed procedure in algorithm 2 and set $\mathcal{E}_0$ (the dashed line).

4.5 Conclusions

In this work, we considered the problem of developing a construction procedure for CCLFs for general unstable nonlinear systems where constraints on the controller

FIGURE 4.5: Phase plane of the planar nonlinear system given in Example 5.3.

induces a NCR boundary. We first presented a procedure to allow the computation of the boundary trajectories by using a boundary condition for the well-known Controllability Minimum Principle. Following this, we show how CCLFs can be constructed using this boundary characterization.

4.6 References

[1] M. Mahmood, T. Homer, and P. Mhaskar, "Controllability minimum principle based construction of the null controllable region for nonlinear systems", *Submitted.*, 2020.

[2] T. Hu, Z. Lin, and L. Qiu, "An explicit description of null controllable regions of linear systems with saturating actuators", *Systems & Control Letters*, vol. 47, no. 1, pp. 65–78, 2002.

[3] E. D. Sontag, "A Lyapunov-like characterization of asymptotic controllability", *SIAM J. Control Optimization*, vol. 21, pp. 462–471, 1983.

[4] Z. Artstein, "Stabilization with relaxed controls", *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 11, pp. 1163–1173, 1983.

[5] M. Mahmood and P. Mhaskar, "On constructing constrained control lyapunov functions for linear systems", *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1136–1140, May 2011.

[6] M. Mahmood and P. Mhaskar, "Constrained control lyapunov function based model predictive control design", *International Journal of Robust and Nonlinear Control*, vol. 24, no. 2, pp. 374–388, 2014.

[7] T. Homer and P. Mhaskar, "Constrained control lyapunov function-based control of nonlinear systems", *Systems & Control Letters*, vol. 110, pp. 55–61, 2017.

[8] ——, “Utilizing null controllable regions to stabilize input-constrained nonlinear systems”, *Computers & Chemical Engineering*, vol. 108, pp. 24–30, 2018.

[9] T. Homer, M. Mahmood, and P. Mhaskar, “A trajectory-based method for constructing null controllable regions”, *International Journal of Robust and Nonlinear Control*, vol. 30, no. 2, pp. 776–786, 2020.

[10] W. Kang, “Bifurcation and normal form of nonlinear control systems, part i”, *SIAM Journal on Control and Optimization*, vol. 36, no. 1, pp. 193–212, 1998.

[11] M. H. Shin and M. J. Chung, “Parameterization and stabilization of the equilibrium points of affine non-linear control systems”, *International Journal of Systems Science*, vol. 33, no. 4, pp. 301–311, 2002.

[12] F. Colonius and W. Kliemann, “On control sets and feedback for nonlinear systems”, *IFAC Proceedings Volumes*, vol. 25, no. 13, pp. 275–282, 1992, 2nd IFAC Symposium on Nonlinear Control Systems Design 1992, Bordeaux, France, 24-26 June.

[13] S. H. Saperstone and J. A. Yorke, “Controllability of linear oscillatory systems using positive controls”, *SIAM Journal on Control*, vol. 9, no. 2, pp. 253–262, 1971.

[14] H. D. Chiang, M. W. Hirsch, and F. F. Wu, “Stability regions of nonlinear autonomous dynamical systems”, *Automatic Control, IEEE Transactions on*, vol. 33, no. 1, pp. 16–27, Jan. 1988.

[15] W. J. Grantham and T. L. Vincent, “A controllability minimum principle”, English, *Journal of Optimization Theory and Applications*, vol. 17, no. 1-2, pp. 93–114, 1975.

[16] W. J. Grantham, “Estimating controllability boundaries for uncertain systems”, in *Renewable Resource Management: Proceedings of a Workshop on Control Theory Applied to Renewable Resource Management and Ecology Held in Christchurch, New Zealand January 7 – 11, 1980*, T. L. Vincent and J. M. Skowronski, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 1981, pp. 151–162.

[17] N. Kapoor and P. Daoutidis, “On the dynamics of nonlinear systems with input constraints”, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 9, no. 1, pp. 88–94, 1999.

[18] E. D. Sontag, “A ‘universal’ construction of artstein’s theorem on nonlinear stabilization”, *Systems & Control Letters*, vol. 13, no. 2, pp. 117–123, 1989.

[19] E. Sontag and H. Sussmann, “Remarks on continuous feedback”, in *1980 19th IEEE Conference on Decision and Control including the Symposium on Adaptive Processes*, Dec. 1980, pp. 916–921.

[20] M. Kawski, “Stabilization of nonlinear systems in the plane”, *Systems & Control Letters*, vol. 12, no. 2, pp. 169–175, 1989.

[21] R. Sepulchre, G. Campion, and V. Wertz, “Some remarks about periodic feedback stabilization”, *IFAC Proceedings Volumes*, vol. 25, no. 13, pp. 403–408, 1992, 2nd IFAC Symposium on Nonlinear Control Systems Design 1992, Bordeaux, France, 24-26 June.

[22] P. B. Sistu and B. W. Bequette, “Nonlinear predictive control of uncertain processes: Application to a cstr”, *AIChE Journal*, vol. 37, no. 11, pp. 1711–1723, 1991.

Chapter 5

Constrained Control Lyapunov Function Based Bounded Control Design

The results in this chapter have been submitted for publication to the following:

Journal Articles

[1] M. Mahmood, T. Homer, and P. Mhaskar, “Constrained control Lyapunov function based bounded control design”, Submitted, 2020.

5.1 Introduction

The control and operation of process systems needs to grapple with several challenges, including nonlinearity and constraints. One of the limitations imposed by the presence of input constraints is to limit the set of initial conditions from where the system can be stabilized. This set has been termed thenull controllable region (NCR) [2]. The recognition of the NCR also provides a natural benchmark for control design- in terms of whether on not they are able to stabilize from the entire null controllable region. Control designs tackling this objective have often taken a Lyaupunov- based approach. In particular, the notion of Lyapunov functions has been generalized to the problem of control analysis and design in the form of control Lyapunov function (CLF) ([3], [4]), enabling estimating the controllability/stability region. While constructive procedures for CLF's exist, most of the procedures inherently do not recognize the presence of input constraints.

Some efforts that exploit the system structure include quadratic functions for feedback linearizable systems and back-stepping techniques for systems in strict feedback form [5]–[7]. The work in [8] provides a CLF-based controller which respects input saturations while in other approaches CLFs are utilized for the control design and stability regions are computed [9]–[11]. The resulting estimates of the NCR from all such CLF-based control designs do not necessarily capture the entire NCR.

The results in [12] extended the idea of a CLF to account for input constraints, resulting in the concept of constrained control Lyapunov functions (CCLF) for linear systems. These functions explicitly account for the presence of input constraints by maximizing the estimate of the NCR over the set of all possible CLFs. Such functions also enable control designs with optimal stability regions which are equal to the NCR. Most existing results, however, have focused on linear systems [13]. Some recent results have considered nonlinear systems and provided some practical control designs which utilize CCLFs within a predictive control framework [14], [15]. However, the formal development of stabilization using CCLFs for nonlinear systems along with explicit control design is still lacking. More recently, the work in [16] presented some computational techniques for NCR construction. This work utilizes a simulation-based approach resulting in high computational cost both for NCR construction and utilization in the control design.

In this work, we consider the problem of developing a CCLF construction procedure for general nonlinear systems. We assume the ability to construct the boundary of the NCR using a computationally tractable approach and focus on the problem of constructing CCLFs using the available boundary characterization. We show that a functional defined such that the levels sets correspond to the boundaries of the NCR for different input constraints does not satisfy the classical definition of a CLF. To alleviate this, we present a new general definition of a CLF albeit with the trade-off that the CLF only enables stabilization to some (potentially non-zero) equilibrium point. Under this general definition, the boundary of the NCR is able to be used as a CCLF. An explicit CCLF-based control design is then presented which results in stabilization to the equilibrium manifold. To achieve stabilization to the origin, a bounded controller which maneuvers the state along the equilibrium manifold to drive it to the origin is presented. This controller is coupled with CCLF-based controller to form a hybrid control scheme which achieves stabilization for all initial conditions in the NCR. The rest of the manuscript is organized as follows: first, in Section 5.2, we outline the class of systems and the required notion, definitions, and assumptions. In addition, we formulate the problem statement and review results on characterizing the NCR. In Section 5.3.2 we show how to construct CCLFs and design a hybrid control scheme which is able to stabilize all initial conditions within the NCR. In Section 5.4, several examples are presented to demonstrate the CCLF-based control design. Finally, in Section 6.6 we summarize our results.

5.2 Preliminaries

5.2.1 Notation

If X and Y are sets X°, $\overline{X}$, ∂X, and $Y \setminus X$ denotes the interior of X, the closure of X, the boundary of X, and the relative complement of X with respect to a set Y respectively. The Euclidian norm on $\mathbb{R}^n$ is denoted by $\|\cdot\|$.

To accommodate non-differentiable Lyapunov functions and discontinuous controllers we need to introduce the notion of generalized derivatives and gradients. In this work we utilize Clarke generalized derivatives and gradients [17]. For a locally Lipschitz scalar function $V : \mathbb{R}^n \to \mathbb{R}$ the generalized gradient of V at x is given by

$$\partial V(x) = \mathrm{co}\{x \in \mathbb{R}^n : \exists x_i \in D_V, x_i \to x, \nabla V(x_i) \to x\}$$

where co denotes the convex hull of a set where D_V denotes the set of points at which the gradient ∇V exists. The Lie derivative of V with respect to a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ at x is defined by

$$L_f V(x) = \max_{x \in \partial V(x)} x \cdot f(x)$$

A function $k : [0, a) \to [0, \infty)$ is a class $\mathcal{K}$ function if it is continuous, strictly increasing and $k(0) = 0$. A scalar function $W : \mathbb{R}^n \to \mathbb{R}$ is called *proper* if it is radially unbounded, i.e. $\lim_{\|x\| \to \infty} W(x) = +\infty$.

5.2.2 Problem Formulation

Consider single-input nonlinear systems that are affine in the control

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \tag{5.1}$$

where $x(t) \in M \subseteq \mathbb{R}^n$, M being an open connected set, denotes the state vector and $u(t) \in U(A)$ denotes the scalar control input where $U(A) = \{u : \mathbb{R} \to A \subseteq \mathbb{R},$ locally integrable$\}$ are the admissible controls. We assume $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are smooth analytic vector fields defined over a smooth domain M which contains the origin in its interior with $f(0) = 0$. We assume the input constraint range is symmetric $U_m = [-m, m]$, $m > 0$ and we refer to the system in Eq. 5.1 with control constraint set $U(U_m)$ as S. We also consider the unforced system $\dot{x}(t) = f(x(t)) + g(x(t))u_0$ obtained from Eq. 5.1 using a constant input value of $u_0 \in U(U_m)$ and the nominal system obtained using the nominal input value of $u = 0$. A solution of the system in Eq. 5.1 from initial condition x_0 and an admissible control u at time $t \geq 0$ is called the *controlled solution* and is denoted by $\varphi(t, x_0, u(t))$. This solution φ is piecewise differentiable and $u(t)$ is piecewise continuous for all $t \in \mathbb{R}^+$. The region $\mathcal{R}(U(U_m)) \subseteq M$ is positively invariant if for all $x_0 \in \mathcal{R}$, there exists a control law $u(t)$ such that $\varphi(t, x_0, u(t)) \in \mathcal{R}$, and $u(t) \in U(U_m)$ for all $t \geq 0$. A point x_{eq} is an equilibrium point of the system if there is a constant control action $u(t) = u_0$, such that $f(x_{eq}) + g(x_{eq})u_0 = 0$. Without loss of generality, it is assumed that the origin is an isolated equilibrium point of the nominal system (i.e. $f(0) = 0$). An equilibrium point x_{eq} is said to be hyperbolic if none of the eigenvalues of the Jacobian for the linearized system (linearized at x_{eq}) have zero real parts. For a given equilibrium point, if all the eigenvalues of Jacobian have nonzero imaginary parts then we call this equilibrium point purely periodic. Also, a hyperbolic equilibrium point is said to be *unstable* if at least one eigenvalue has a positive real part and *stable* otherwise. For a hyperbolic equilibrium point x_{eq}, we can decompose the tangent space $T_{x_{eq}}(M)$ as the direct sum of stable $E^s(x_{eq})$ and unstable $E^u(x_{eq})$ subspaces. The subspaces are denoted as $E^s(x_{eq}) = \{v^1, v^2, \dots, v^{n_s}\}$, $E^u(x_{eq}) = \{w^1, w^2, \dots, w^{n_s}\}$. The vectors $v^1, v^2, \dots, v^{n_s}$ and $w^1, w^2, \dots, w^{n_s}$ are generalized eigenvectors whose eigenvalues are stable (negative) and unstable (positive). Note that $n_s + n_u = n$. A dynamical system is said to be *structurally stable* if perturbations to the system preserve the qualitative characteristics of the dynamics. That is, there exists a homeomorphism which maps orbits of the original system to orbits of the perturbed system while also preserving the direction of time. We say that y is an ω-limit point of x associated with the controlled solution $\varphi(t, x_0, u(t))$ if there exists a sequence $\{t_i\}$ with $\lim_{i\to\infty} t_i = \infty$ such that $\lim_{i\to\infty} \varphi(t_i, x, u(t_i)) = y$. The set of all ω-limit points of x associated with $\varphi(t, x_0, u(t))$ is called the ω-limit set of x, denoted by $\omega(x, u)$. Let the multiplicity of the equilibrium points (for a given input value) be k (note that due to the assumption of the structural stability below, the multiplicity is independent of the value of u). The equilibrium set $\hat{E}_i(A) \subset M \times R$ of the system S and its projection into M, $E_i(A)$ are defined by

$$\hat{E}_i(A) = \{(x_i, u_0) \in M \times R \mid f(x_i) + g(x_i)u_0 = 0, u_0 \in A\} \tag{5.2}$$

where i indexes the multiplicity of the equilibrium points. Further,

$$E_i(A) = \{x_i \in M \mid (x_i, u_0) \in \hat{E}_i(A)\} \tag{5.3}$$

Finally let $\hat{E}(A) = \bigcup_{i=0}^{k-1} \hat{E}_i(A)$, and $E(A) = \bigcup_{i=0}^{k-1} E_i(A)$. We use the subscript 0 to denote the subset which contains the origin (i.e $0 \in E_0(A)$). As stated in the assumptions below the origin is assumed to be an unstable equilibrium point.

The system S is called *controllable* from x_1 to x_2 in finite time T if there exists an admissible control trajectory $u(t) \in U(U_m)$, $t \leq T$ such that the solution trajectory satisfies $\varphi(T, x_1, u) = x_2$. The set of all points x where y is controllable from in time T for the system S using admissible inputs from $U(U_m)$ is the time T controllable set and is denoted by $C_{U_m}(y, T)$. We also write $C_{U_m}(y) = \bigcup_{0 \leq t} C_{U_m}(y, t)$ and refer to this as the *controllable set*. The system S is called *small-time locally controllable* (STLC) at an equilibrium point x_{eq} if $C_{U_m}(x_{eq}, T)$ contains a neighborhood of x_{eq} for all $T > 0$. We also say the system S is *STLC with small controls* at an equilibrium point x_{eq} if $C_{U_m}(x_{eq}, T)$ contains a neighborhood of x_{eq} for all $T > 0$. The system S is called *large-time locally controllable* (LTLC) at an equilibrium point x_{eq} if there exists a time $T > 0$ such that $C_{U_m}(x_{eq}, T)$ contains a neighborhood of x_{eq}. In this paper the focus is on the *null-controllable region* (NCR) $C_{U_m}(0)$ which will be abbreviated by C_m. We recall that the boundary of the NCR ∂C_m is semi-permeable (if $x(t)$ starts in the exterior of C_m, it can never reach ∂C_m) and comprised of solution trajectories.

Consider the system S under the following assumptions:

1. The nominal system is structurally stable for all $u \in \mathbb{R}$.

2. The origin of the system is unstable (i.e. the set $E(U_m)$ is nonempty and contains the origin). Moreover, the linearized system around each $(x_{eq}, u_{eq}) \in \hat{E}(U_m)$ is controllable, not purely periodic and hyperbolic.

3. The system S around each $(x_{eq}, u_{eq}) \in \hat{E}(U_m)$ is LTLC with U_m only if its STLC with U_m.

4. The set C_m is open, connected and diffomorphic to $\mathbb{R}^n$.

5. The sets $E_i(U_m)$, $i \neq 0$ are outside of $\overline{C}_m$.

6. There exists a smooth function $u_{eq} : E_0(U_m) \to \mathbb{R}$ such that for each $x_{eq} \in E_0(U_m)$, $f(x_{eq}) + g(x_{eq})u_{eq}(x_{eq}) = 0$ and $E_0(U_m) \subset M$.

7. Let $\phi(t, x^\star, u^\star)$ denote a controlled solution which forms part of the boundary ∂C_m. The limit set $\omega(x^\star, u^\star)$ is nonempty and contains only equilibrium points.

This setup leads to the following two problems:

1. Characterizing the boundary trajectories which define the set C_m. This problem was addressed in Chapter 3. The present chapter assumes the existence of a CCLF to address the next objective.

2. Designing a stabilizing control action for the system S from all initial conditions in C_m.

Remark 5.1. Assumption 6 ensures the set $E_0(U_m)$ defines a continuum of equilibrium points which contains the origin and is obtained by varying the input in the unforced system smoothly over U. This along with Assumption 2 preclude systems with equilibrium points x_{eq} such that $g(x_{eq}) = 0$. Moreover, under Assumption 5, we consider the problem where there is only a single equilibrium manifold E_0 within $\overline{C}_m$ and other branches arises due to multiplicity of the equilibria lie outside of the NCR.

5.3 Stabilization of the NCR

In this section we focus on the problem of stabilization of the entire NCR. The objective is to synthesize a Lyapunov-based state feedback controller that enforces closed–loop stability in the presence of input constraints. Moreover, the synthesized controller must guarantee closed–loop stability of the origin for all initial states in the NCR.

In the current context, the equilibrium manifold E_0 is a connected set which contains the origin. It has been shown that one can steer the system starting from any equilibrium point in E_0 to any other equilibrium point in E_0 using an admissible control action [18]. Thus the problem achieving stabilization to the origin can be reduced to that of first driving the state to any equilibrium point in E_0.

Thus, in our approach this problem is split into two independent stabilization sub-problems: One is that of driving any initial condition with the NCR to the equilibrium manifold and the other is the problem of sliding along the equilibrium manifold to reach the origin.

To tackle the problem of stabilizing all initial conditions within the NCR to the equilibrium manifold we introduce a new type of Control Lyapunov Function (CLF) [4] which generalizes the classical notion of a CLF. Specifically this new notion relaxes the condition of definiteness on the Lyapunov function and derivative of the Lyapunov function. This relaxation is critical to the use of the boundary of the NCR appropriately as a CLF, and in turn, enable stabilization from the entire NCR.

Recall that the classical definition of a *control Lyapunov function* (CLF) is a smooth positive-definite proper function $V : M \to R$ such that

$$\inf_{u \in U_m} L_{f+gu}V < 0, \; x \in D \tag{5.4}$$

Where D is a neighbourhood of the origin. Consider W defined as the set induced by the level sets of V,

$$W(V,c) = \{x \in R^n : V(x) \leq c\} \tag{5.5}$$

It follows that $W(V,c)$ is an estimate of the NCR C_m (using the CLF V) if $W(V,c) \subseteq D$. We denote the largest value of level set contained within D as c_{max}.

$$c_{max} = \arg\max_{c} W(V,c) \subseteq D \tag{5.6}$$

5.3.1 Generalized Control Lyapunov Functions

We now present a new type of CLF known as an generalized control Lyapunov function (g-CLF) which is a generalization of the classical CLF. In particular, the g-CLF does not require the definiteness of the Lyapunov function and derivative of the Lyapunov function.

Definition 5.1. Let D be a neighbourhood containing the origin. For the system S a *generalized control Lyapunov function* (g-CLF) is a continuous function $V : M \to R$ such that

1. $V(0) = 0, V(x) \geq 0$.
2. $\inf_{u \in U_m} L_{f+gu}V \leq 0, \; x \in D$.
3. $w(x, u) \in E_0$ for all $x \in G_V$

where

$$G_V = \{x \in D \,|\, \inf_{u \in U_m} L_{f+gu}V = 0\} \tag{5.7}$$

and G_V be the largest invariant set in G_V and u is the control input corresponding to each controlled invariant solution $\phi(t, x, u(t)) \in G_V$.

We now formulate the next result which is a generalization of the well-known LaSalle's Invariance Theorem [19] in the setting of control systems. Specifically, the results show that given a g-CLF, one can design a feedback controller which results in the decay of g-CLF and thus resulting in stabilization to some point in the equilibrium manifold E_0.

Theorem 5.1. If there exists a proper g-CLF V as defined in Definition 5.1 then there exists a control law such that the closed–loop system for all $x \in W(V, c_{max})$ will approach an equilibrium point in E_0. That is, $x(t) \to x_l \in E_0$ as $t \to \infty$.

Proof. By Definition 5.1, for every $x \in W(V, c_{max})$ there is a control action u_0 which results in the generalized directional derivative $L_{f+gu_0}V$ to be non-positive. It follows from generalized Artstien theorem ([20], [21]) that there is a feedback law $u(x)$ (in a sample-and-hold sense) which results in the closed–loop trajectory to evolve in the direction where the function $V(x)$ is non-increasing over time. Since $V(x)$ is proper, it follows that W is compact and thus using similar arguments as the classical Lasalle's invariance Theorem, one can show that $x(t)$ will approach the largest invariant set G_V as $t \to \infty$. Since all $x \in G_V$ will approach the limit set $w(x, u) \in E_0$ it follows that the closed–loop system for all $x \in W(V, c_{max})$ will approach E_0. That is, $x(t) \to x_l \in E_0$.

We now consider functions which are not proper and show that stabilization to the set where $V = 0$ can be achieved provided the Lyapunov function derivative is strongly bounded away from zero. Let G_V^0 be the set of states in the state space where $V = 0$.

$$G_V^0 = \{x \in D \mid V(x) = 0\} \tag{5.8}$$

Theorem 5.2. Suppose there exists a g-CLF V as defined in Definition 5.1 such that given $c_d < c_{max}$, $\inf_{u \in U_m} L_{f+gu} V(x) \leq -cV(x)^a$ for some $a \in (0, 1]$, $c > 0$ for all $x \in W(V, c_d)$. Then for all $c_d < c_{max}$, there exists a control law such that the closed–loop system for all $x \in W(V, c_d)$ will approach an equilibrium point in E_0. That is, $x(t) \to x_l \in E_0$ as $t \to \infty$.

Proof. If the set $W(V, c_{max})$ is compact then the result follows from Theorem 5.1. If the set $W(V, c_{max})$ is not compact then by assumption for some $0 < c_d < c_{max}$ $\inf_{u \in U_m} L_{f+gu} V \leq -cV(x)^a$ for some $a \in (0, 1]$ and $c > 0$ for all $x \in W(V, c_d)$. It follows from the comparison lemma that the closed–loop trajectory will reach the set $G_V^0 = \{x \in D \mid V(x) = 0\}$ in finite time. Since $G_V^0 \subseteq G_V$, we can use a similar argument as the proof of Theorem 5.1 leads to $x(t) \to x_l \in E_0$.

Remark 5.2. The above definition generalizes the classical definition of a CLF in three directions: 1) The definition allows the function $V(x)$ to be positive semi-definite and the derivative to be negative semi-derivative. This is similar to the definition of a weak CLF provided in [21] and is used in Theorem 5.1 to state a version of Lasalle's invariance Theorem in the context of control. 2) As the function $V(x)$ decays, the state approaches a point on the equilibrium manifold E_0 (not necessarily the origin). 3) The function $V(x)$ is permitted to be radially bounded (i.e. the set $W(V, c_{max})$ need not be compact) provided the decay of the function is strongly bounded away from zero. By relaxing the conditions on the Lyapunov function, the class of functions which can be used to analyze the system controllability and design control laws is enlarged. This is critical for being able to define the notion of CCLF. An example of function which is a g-CLFs but does not meet the criteria of a classical CLFs is provided in Example 5.1.

Remark 5.3. Theorem 5.1 shows that with a g-CLF V, there is a controller which can be used to drive the closed–loop system to an equilibrium point in the equilibrium manifold E_0. Since g-CLFs are a generalization of the classic CLFs, the existence of g-CLFs follows from Lyapunov converse theorems. Note that the results in this section are presented generically for any control law which satisfies the conditions laid out in Definition 5.1. Using a g-CLF, one can easily design a control law which satisfies these conditions. Perhaps the easiest control designs to adopt for g-CLF based controllers is one which uses optimal control techniques. In such control designs the conditions in Definition 5.1 can be specified as constraints in the optimization problem. In subsection 5.3.3 we present an explicit feedback law which is able to stabilize all initial conditions within the NCR.

5.3.2 Constrained Control Lyapunov Functions

In working towards the goal of stabilizing all initial conditions within the NCR we devise a special type of g-CLF where the guaranteed stability region to the equilibrium manifold $W(V, c_{max})$ coincides with the boundary of the NCR. This g-CLF will be known as a constrained control Lyapunov function (CCLF) and is defined as follows:

Definition 5.2. For the system S a constrained control Lyapunov function (CCLF) is a g-CLF $V(x)$ such that

$$V(x) \to c_{max} \text{ as } x \to \partial C_m \tag{5.9}$$

Let u_V be a CCLF-based controller. It follows that this controller will result in the stability region of closed–loop system being the same as the NCR. We now show how to construct CCLFs for the system S using the boundary trajectories of the NCR. In preparation for this construction we introduce the concept of an input capacity map.

Definition 5.3. The input capacity map is a function $g : \mathbb{R}^n \to \mathbb{R}$ that provides the input constraint u_{max} required to make a state vector x be on the boundary of the NCR C_m.

$$g(x) = \{q \in \mathbb{R} : x \in \partial C_q\} \tag{5.10}$$

The level sets of the function g provide the boundary of the NCR for different input constraints.

In preparation for the remainder of the results in this section we recall the controllability minimum principle. The theorem provides a necessary condition for trajectories which move along the boundary ∂C_m, where the adjoint variable can be understood as orthogonal to the NCR and thus a normal vector.

Theorem 5.3 (Controllability Minimum Principle, [22]). Let $u^*(t)$ be an admissible control trajectory which generates a boundary trajectory $x^*(t) \in \partial C_m$ for all $t \in [0, t_1]$. Then there exists a non-zero continuous solution $\lambda(t)$ to the adjoint equations $\dot{\lambda}^T = -\frac{\partial H(x,\lambda,u)}{\partial x}$, $t \in [0, t_1]$, where $H(x^*, \lambda^*, u^*) = \min_{|u| \leq m} H(x^*, \lambda^*, u) = 0$, and $H(x, \lambda, u) = \lambda^T(f(x) + g(x)u)$

In effort to utilize the function g to construct a CCLF, we now show some properties of the function g using the above principle. Specifically, we first show that the time derivative of the function g along the trajectories of the system is non-increasing for some admissible control input value. Following this we show that for the set of state from where g is not differentiable is a subset of the set of states from where decrease of the function g is not possible. Finally we show that the limit set $w(x, u)$ is contained in the equilibrium manifold E_0 for $x \in G_g$.

Lemma 5.1. $\inf_{u \in U_m} L_{f+gu} g(x) \leq 0$ for all $x \in C_m$

Proof. Let $x \in C_m$ and $g(x) = u_{max}$. Since the boundary of the NCR is comprised of solution trajectories there will always be a control action which can keep the trajectory on the boundary. That is, there is a u such that $|u| = u_{max}$ and $L_{f+gu} g = 0$. We now show that if the boundary of the NCR at x is differentiable then the evolution of $g(x)$ over time can be made to be non-increasing. From the controllability minimum principle there is a non-zero continuous solution $\lambda(t)$ to the adjoint equations $\dot{\lambda}^T = -\frac{\partial H(x,\lambda,u)}{\partial x}$, $t \in [0, t_1]$, where $H(x^*, \lambda^*, u^*) = \min_{|u| \leq |u^*|} H(x^*, \lambda^*, u) = 0$, and $H(x, \lambda, u) = \lambda^T(f(x) + g(x)u)$. The vector $\lambda(t)$ provides a normal to the boundary of this NCR ∂C_{u^*}. If the boundary is differentiable at x then the direction of this normal vector will be unique. Since $H(x^*, \lambda^*, u^*) = \min_{|u| \leq u^*} H(x^*, \lambda^*, u) = \lambda^T(f(x^*) + g(x^*)u^*) = 0$, one is able to find an admissible control action u such that $|u| \leq |u^*|$ which makes $H(x^*, \lambda^*, u) \leq 0$. This trajectory would never point to the exterior of the boundary of the NCR and thus the value of $g(f(s, x_0, u(s))$ would be non-increasing in time. That is $\inf_{u \in U_m} L_{f+gu} g \leq L_{f+gu} g \leq 0$.

We now derive a property relating to the points on $g(x)$ which are not differentiable. This result follows from Theorem 5.3.

Lemma 5.2. Let $x \in C_m \setminus E_0$ and suppose $g(x)$ is not differentiable. Then $\inf_{u \in U_m} L_{f+gu} g(x) = 0$

Proof. Since we know by Lemma 5.1 that $\inf_{u \in U_m} L_{f+gu} g(x) \leq 0$ we suppose $\inf_{u \in U_m} L_{f+gu} g(x) < 0$. Since $g(x)$ is not differentiable, there is at least two normal vectors x_1, x_2 within $\partial g(x)$ such that $x_1 \neq x_2$ and $\inf_{u \in U_m} x_2(f(x) + g(x)u) < 0$ and $\inf_{u \in U_m} x_2(f(x) + g(x)u) < 0$.

Since f and g are continuous functions of x, it follows that there is some u_{x_1} and u_{x_2} within U_m such that $x_1(f(x)+g(x)u_{x_1}) = 0$ and $x_2(f(x)+g(x)u_{x_2}) = 0$. Since $x \in E_0$, $f(x)+g(x)u_{x_1} \neq 0$ and $f(x)+g(x)u_{x_2} \neq 0$. Also since both x_1, x_2 are within $\partial g(x)$, and the function $g(x)$ is defined using the boundary of the NCR for different input constraint values the trajectories defined by $f(x)+g(x)u_{x_1}$ and $f(x)+g(x)u_{x_2}$ must both transverse on the boundary. Therefore both these trajectories must satisfy the controllability minimum principle which implies the continuity of the normal vector, $x_1 = x_2$ resulting in a contradiction. Therefore the $\inf_{u \in U_m} L_{f+gu} g(x) = 0$.

We now show that the limit set $w(x, u)$ of all states $x \in G_g$ must be within the equilibrium manifold E_0.

Lemma 5.3. $w(x, u) \in E_0$ for all $x \in G_g$.

Proof. All trajectories in the set $x \in G_g$ must transverse the boundary of the NCR, C_{u_0} for some u_0. Under Assumption 7 the limit set $w(x, u)$ of all trajectories on the boundary of the NCR must approach an equilibrium point. Therefore $w(x, u) \in E_0$.

We now show that the input capacity function g satisfies the criteria in Definition 5.2 and thus is a CCLF.

Theorem 5.4. Let k be a class K function. Then the function $k(g(x))$ is a CCLF for the system S.

Proof. Clearly, $k(g(x)) \geq 0$ for all x and $k(g(0)) = 0$. It follows from Lemma 5.1 that $\inf_{u \in U_m} L_{f+gu} k(g(x)) \leq 0$ for all $x \in C_m$. Therefore the function $k(g(x))$ is a g-CLF as per Definition 5.1. By definition of $g(x)$, the level sets of $k(g(x))$ will correspond to the boundary of the NCR. That is, $V(x) = k(g(x)) \to c_{max}$ as $x \to \partial C_m$.

Remark 5.4. Note that this formulation does not require the availability of a closed-form expression for the function g. In general, such closed-form expressions are not available as the level sets of the function g are solution trajectories which define the boundary of the NCR and explicit characterizations of these boundary trajectories are not available in general. Any method which can provide a discrete numerical representation can be utilized. Other possible solutions include those which rely on brute-force computation to obtain the NCR by discretizing the state-space and solving an optimal control to determine if each point can be stabilized by some admissible control action. These methods are computationally expensive and suffer from the curse of dimensionality. Another approach is that which was presented in the companion manuscript and provides a computationally tractable numerical procedure by using available properties of the NCR and can be scaled to higher dimensions. The use of such a representation would involve sampling the boundary data and employing numerical methods such has finite difference schemes and interpolation to obtain the value and the first derivatives of the CCLF which are required for the proposed control designs.

Remark 5.5. The notion of a CCLF for nonlinear constrained systems was presented in [23] using a predefined subset of the state–space. This differs from the above definition which defines the Lyapunov function such that the maximal level set from where the function can be made to decrease corresponds to the NCR.

Remark 5.6. Using the input capacity map as a CLF is an intuitive idea however, as shown in the examples in section 5.4 this function does not meet the criteria of a classical CLF. In particular, for anti-stable systems this function can not be made to be decreasing for points in E_0. Also for semi-stable systems this function is radially bounded. However, as shown in Theorem 5.4 this function does satisfy the criteria of a CCLF and can be used to stabilize all initial conditions in the NCR to the equilibrium manifold E_0.

Remark 5.7. The set C_m need not be compact and thus the function $g(x)$ may not be proper. For such cases, the resulting CCLF must additionally satisfy the criteria where the time-derivative is strongly bounded in the neighborhood of the set C_g^0 to ensure the state reaches this set in finite time. Since this set is a subset of C_g it follows from Definition 5.1 that the trajectory will approach an equilibrium point on the equilibrium manifold E_0. This can be observed for semi-stable systems such as the one presented in Example 5.1. Here C_g^0 the set are the stable manifolds of the origin.

Remark 5.8. For the system of Eq. 5.1, the traditional use of CLFs are in a sense only applicable locally (for a sufficiently small neighborhood of the origin), and do not account for the fundamental limitations arising due to input constraints. This property diminishes the value of CLF-based approaches as local stabilization can often be achieved using linear state-feedback designed based on the linearizing the nonlinear system around an equilibrium point. The definition of a CCLF, however, incorporates input constraints by ensuring the resulting region of closed–loop stability coincides with that of the boundary of the NCR. Note also that while there exists construction procedures for CLFs, there is a lack of results on the construction of g-CLFs and CCLFs. By using a traditional CLF-based control design and simply saturating the control action would result in a stability region which is a subset of the NCR and thus is sub-optimal.

5.3.3 CCLF-based Controller

In this subsection we present an explicit CCLF-based control law which is able to stabilize all initial conditions within the NCR to the equilibrium manifold. This control law will drive any initial condition within the NCR to lower level sets of the CCLF until the state reaches an equilibrium point on the manifold E_0 thereby achieving stability to an equilibrium point which may not be the origin. Let $V(x)$ be a CCLF constructed using the input capacity function given in Section 5.3.2, $V(x) = g(x)$. In the case where $V(x)$ is not proper, we assume that for every subset $W(V, c_d) \quad W(V, c_{max})$ the time derivative of V is strongly bounded away from zero for all $x \in W(V, c_d)$, i.e. $\inf_u L_{f+gu} V \quad cV(x)^a$. The proposed control design is based on the bounded control law in [9] and is adapted for generalized CLFs.

Consider the static state feedback law $u_{BC}(x)$:

$$= \begin{cases} \frac{L_fV + \sqrt{(L_fV)^2 + (mL_gV)^4}}{(L_gV)^2 \, 1 + \sqrt{1 + (mL_gV)^2}} L_gV, & \inf_u L_{f+gu}V < 0 \\ u, & \inf_u L_{f+gu}V = 0 \end{cases} \tag{5.11}$$

where u is the optimal control input which causes $x(t)$ to transverse on the boundary $\partial C_{g(x)}$.

Theorem 5.5. Consider the constrained nonlinear system S under the CCLF-based feedback control law given in Eq.5.11. Then the closed–loop system is asymptotically stable to an equilibrium point $x_f \in E_0$ for all $x \in C_m$.

Proof. We first show that the controller satisfies the input constraints within the NCR. Then using a Lyapunov argument we show that the state-feedback controller is stabilizing to an equilibrium point for all initial conditions in the NCR. If $\min_u L_{f+gu}V = 0$, then $u_{BC} = u$ which will always be less than u_m for all $x \in C_m$. If $\min_u L_{f+gu}V < 0$, the expression in Eq.5.11 can be shown to be less than u_m using similar arguments as made in [10]. Therefore the control law given in Eq.5.11 will always satisfy the input constraint $u_{BC}(x) \le m$. Since $V(x)$ is a CCLF it will satisfy $\inf_u L_{f+gu}V \le 0$. Consider x such that $\inf_u L_{f+gu} = 0$, the control action is simply $u = u$ and the state will stay on the boundary of the NCR. Now consider x such that $\inf_u L_{f+gu}V < 0$. It follows from Lemma 5.2 that $V(x)$ is differentiable at x and control law reduces to the bounded control law given in [10] and can be shown to result in $L_{f+gu_{BC}}V = L_fV + L_gVu_{BC} \le 0$. Therefore the nonlinear system S under the CCLF-based feedback control law given in Eq.5.11 results in the closed–loop system satisfying $L_{f+gu_{BC}}V \le 0$ for all $x \in C_m$. Using Theorem 5.1 and Lemma 5.3 it follows that the closed–loop system is asymptotically stable to an equilibrium point $x_f \in E_0$ for all $x \in C_m$.

Remark 5.9. Recall that when $V(\cdot)$ is differentiable, the minimum value of $L_fV + L_gVu$ is given by $u = \mathrm{sgn}(L_gV)$. In the present scenario, where the function $g(x) \le V(x)$ may not be differentiable everywhere the minimum value of $L_{f+gu}V$ may not be attained when $u = \mathrm{sgn}(L_gV)$, and therefore needs to be computed using the generalized derivative.

Remark 5.10. Theorem 5.5 shows that the explicit controller formula in Eq.5.11 is able to stabilize all initial conditions in the NCR to the equilibrium manifold E_0. This control law differs from the bounded control law in [9] in that it used the generalized CLF which can be non-differentiable and not strictly decreasing along the state trajectory. For the states where the NCR is not differentiable, Lemma 5.2 shows that the decrease in the Lyapunov function is not possible thus the control law in Eq.5.11 simply applies the control action corresponding the NCR with input constraint given by the input capacity function $g(x)$. This will result in the trajectory remaining on the boundary of the NCR. The existence of manifolds where the decrease in the Lyapunov function is not possible can be easily observed by looking at linear anti-stable systems with dimension greater than two. Such manifolds often coincide with switching surfaces and highlight the importance of handling such a scenario in the control law.

5.3.4 Local Stabilization Around the Equilibrium Manifold

In the previous section we developed a CCLF-based control scheme which achieved stabilization to some equilibrium point on the manifold E_0. In this section we develop a nonlinear Lyapunov-based controller which drives all initial conditions in a neighbourhood of the equilibrium manifold E_0 to origin. This control design uses a classical CLF parameterized over a set of moving equilibrium points. Speci cally, by setting the dynamics of the target equilibrium points we are able to design a control law which maneuvers the closed–loop state along the equilibrium manifold resulting in stability of nonlinear system to the origin while also providing an explicit region of guaranteed stability. This control design allows for the stabilization locally around the equilibrium manifold and will be referred to as the localized bounded controller. In the subsequent section we combine this localized bounded controller with the CCLF-based controller under a hybrid control scheme to be able to stabilize all initial conditions within the NCR.

The set of equilibrium points can be paramterized using the corresponding equilibrium control action. That is, there is a mapping $l : R ! R^n$ such that

$$f(x_l(l)) + g(x_l(l))l = 0, \text{ for all } l \ 2 \ U \tag{5.12}$$

This mapping takes the equilibrium control action $u = l$ to the corresponding equilibrium point $x_l \ 2 \ E_0$.

Suppose the target equilibrium point x_l evolves according to the following dynamic:

$$x_l = k_x(x_l) \tag{5.13}$$

This dynamic implies a corresponding dynamic for the control input l :

$$l = k_l(l) \tag{5.14}$$

These dynamics are designed to be suf ciently smooth and also globally exponentially stable to the unique zero solution ($x_l = 0, l = 0$).

We de ne z and n as the deviation of the state x from the target equilibrium point x_l and control u from the corresponding equilibrium control u_l :

$$z(t) = x(t) \quad x_l(t) \tag{5.15}$$

$$n(t) = u(t) \quad u_l(t) \tag{5.16}$$

The dynamics of this deviation variable can be derived as follows

$$z(t) = x(t) - x_l(t) \tag{5.17}$$
$$= f(x(t)) + g(x(t))u(t) - k(x_l(t)) \tag{5.18}$$
$$= f(z(t) + x_l(t)) + g(z(t) + x_l(t))(n(t) + u_l(t)) - k(x_l(t)) \tag{5.19}$$

We define the augmented state as $x = [z \; x_l]^T \in R^{2n}$. We combine the dynamics of the deviation variable z and the target equilibrium point x_l to give the dynamics of the augmented system

$$x(t) = f_x(x) + g_x(x)n(t) \tag{5.20}$$

where

$$f_x = [\, f(z(t) + x_l(t)) + g(z(t) + x_l(t))u_l(t) - k(x_l(t)) \;\; k(x_l(t))\,]^T \tag{5.21}$$
$$g_x = [\, g(z(t) + x_l(t)) \;\; 0]^T \tag{5.22}$$

Consider the classical CLF for the z-subsystem $V_z(z)$. Given an initial state x we set the initialization of the target equilibrium x_l state to be one which minimizes the distance to the set E_0:

$$x_l^0(x) = \arg\min_{x_0} \|x_0 - x\|, \; x_0 \in E_0 \tag{5.23}$$

Let l^0 denote the corresponding equilibrium control input to x_l^0. Note that in general, there could be more than one equilibrium point that results in the same minimum distance. We simply pick one of those equilibrium points. Let $n_{max}^{l^0} = m - |l^0|$ and $F_x^{l^0}$ be the set defined as $F_x^{l^0} =$

$$\left\{ x \in R^{2n} : L_{f_x}V_z(z) + r_z V_z(z) - n_{max}^{l^0}|L_{g_x}V_z(z)|, r_z > 0 \right\} \tag{5.24}$$

Assume that the set $F_x^{l^0}$ contains the origin and a neighborhood of the origin. Let W_x be defined as a level set of V_x completely contained in $F_x^{l^0}$ for some $c_x > 0$.

$$W_x(c_x) = \{x \in R^{2n} : V_x(x) \le c_x\} \tag{5.25}$$

We construct the following localized bounded control law:

$$n(x) = \begin{cases} -k(x)L_{g_x}V_z(z) & , L_{g_x}V_z(z) \ne 0 \\ 0 & , L_{g_x}V_z(z) = 0 \end{cases} \tag{5.26}$$

where

$$k(x) = \frac{L_{f_x} V_z + \sqrt{(L_{f_x} V_z)^2 + (n^{l\,0}_{max} L_{g_x} V_z)^4}}{(L_{g_x} V_z)^2 \; 1 + \sqrt{1 + (n^{l\,0}_{max} L_{g_x} V_z)^2}} \tag{5.27}$$

Eq.5.27 is an explicit state-feedback controller based on the bounded controller in [9]. Theorem 5.6 that follows shows that stability of the origin is guaranteed for all initial conditions in $W_x(c_x)$. The key idea in the proposed controller design is that the state x converges to the target equilibrium point x_l which evolves over time to reach the origin.

Theorem 5.6. Consider the constrained nonlinear system S with initial condition $x_0 \in C_m$. Suppose the initialization of the target equilibrium state l is determined using Eq.5.23 and the corresponding augmented state initial conditon satisfies $x_0 \in W_x(c_x)$. Then the origin of closed–loop system under the state feedback control law given in Eq.5.26 is asymptotically stable.

Proof. Consider the constrained nonlinear system S under the state feedback control law given in Eq.5.26. Using similar arguments as the proof of Theorem 1 from [10] we can show that

$$|n| \quad n^{l\,0}_{max} \tag{5.28}$$

Since $n^{l\,0}_{max} = m \; | \; l\,^0|$, it follows that $|n| \quad m$ Now consider the representation of the constrained nonlinear system using the augmented state x given in Eq.5.20 under the state feedback control law given in Eq.5.26. Clearly if the augmented state converges to the origin so does the state x, i.e. $x \to 0$ implies $x \to 0$. Thus we proceed to establish the asymptotic stability of the origin for the augmented system. To this end, consider the control Lyapunov function V_z for the z-subsystem in Eq.5.20. Using similar arguments as the proof of Theorem 1 from [10] results in the time-derivative of V_z along the closed–loop trajectories of x satisfying

$$V_z \qquad r_z V_z \tag{5.29}$$

for all $x_0 \in W_x(c_x)$. Since the l-subsystem is globally exponentially stable it follows by converse Lyapunov theorems that there is a continuously differentiable Lyapunov function $V_l(x_l)$ such that

$$V_l \qquad r_l V_l \tag{5.30}$$

for some $r_l > 0$. Consider the composite CLF $V_x(x) = V_x(z, x_l) = V_z(z) + V_l(x_l)$. Therefore the time-derivative of V_x along the closed–loop tajectories of x satisfy

$$V_x \qquad r_z V_z \quad r_l V_l \tag{5.31}$$

$$= r_x V_x \tag{5.32}$$

for some $r_x > 0$. Hence the origin of the nonlinear system S is asymptotically stable.

Remark 5.11. The implementation of this state-feedback controller requires the initial target equilibrium point x_l^0 to be initialized using Eq.5.23. This initial target equilibrium point is used to determine the corresponding target equilibrium control u_l^0 which in turns determines the input constraint $u_{max}^{l\,0}$ used directly in the control design. This initial target equilibrium point affects the size of input constraint and ultimately the size of the guaranteed stability region. Practically this controller is to be used when the initial state x_0 is close to the equilibrium manifold E_0. Following the initialization, the target equilibrium point x_l evolves according to the dynamics given in Eq.5.13. Since the dynamics are globally asymptotically stable, the target equilibrium point converges to the origin. The state $x(t)$ under the feedback controller 5.27 evolves to track this moving equilibrium point. The set $W_x(c_x)$ defines the region of the state-space where the state $x(t)$ is guaranteed to converge to the moving equilibrium point. Note that the dynamics of x_l are a design variable which affect the size of the stability region $W_x(c_x)$ and can be chosen such that $W_x(c_x)$ contains the entire equilibrium manifold E_0.

Remark 5.12. Theorem 5.6 provides an explicit state-feedback control law which is able to stabilize all initial conditions within an explicitly characterized region of the state-space of the augmented state x. The projection of this region from the augmented state x to the state x is a region around the equilibrium manifold E_0. That is, this controller stabilizes initial conditions near the equilibrium manifold to the origin.

5.3.5 Uniting CCLF-based Control with the Localized Bounded Controller

The CCLF-based controller is able to drive all initial conditions within the NCR to the equilibrium manifold and the localized bounded controller is able to drive all initial conditions near the equilibrium manifold in a manner which slides along the equilibrium manifold and eventually reaches the origin.

In this subsection we present a hybrid control design which unites the CCLF-based controller with the localized bounded controller to achieve stabilization to the origin from all initial conditions with the NCR.

Consider the nonlinear system S for which the CCLF-based controller and a localized bounded controller of Eq.5.26 have been designed. We now formulate a set of switching laws to orchestrate the transition between the CCLF-based controller and a localized bounded controller to result in closed–loop stability of the origin from all initial conditions within the NCR. Consider the system S cast as a switched system of the form:

$$\dot{x}(t) = f(x(t)) + g(x(t))u_{i(t)}(t) \tag{5.33}$$

$$i(t) \in \{1, 2\} \tag{5.34}$$

where $i : [0,\infty) \to \{1, 2\}$ is a switching signal that indexes the control input $u(\cdot)$ which is assumed to be CCLF-based controller if $i = 1$ and the localized bounded controller if $i = 2$. This signal is assumed to be a piece-wise continuous (from the right) function of time. This signal implies that on a finite interval of time a finite number of switches between the CCLF-based controller and the localized bounded controller. We consider the problem of designing a switching law that provides a switching time to ensure the transition between the controller will result in closed–loop stability of the origin. Below we present a switching scheme which addresses the above problem.

Theorem 5.7. Consider the constrained nonlinear system of Eq.5.33 with any initial condition within the NCR, $x_0 \in C_m$. Let T_{switch} be the earliest time for which the augmented state x with the initialization of the target equilibrium state using Eq.5.23 is within the set $\Omega_x(c_x)$ where $\Omega_x(c_x)$ was defined in Eq.5.25. Then, the switching rule given by

$$i(t) = \begin{cases} 1, & 0 \le t < T_{switch} \\ 2, & t \ge T_{switch} \end{cases} \tag{5.35}$$

results in the origin of the switched closed–loop system being asymptotically stable.

Proof. The proof of this Theorem uses the fact that if the state of the augmented system x resides in $\Omega_x(c_x)$ when the controller is switched, then the localized bounded controller will drive the augmented system x (and thus also the state $x(t)$) to the origin. We consider two cases:

- Let $x_0 \in C_m$ be such that the initialization of the target equilibrium state using Eq.5.23 results in the augmented state satisfying $x_0 \in \Omega_x(c_x)$. Under the switching rule in Eq.5.35, the controller will immediately switch to $i = 2$ and use the localized bounded controller of Eq.5.26. Using Theorem 5.6 the closed–loop system will asymptotically approach the origin.

- Let $x_0 \in C_m$ such that the initialization of the target equilibrium state using Eq.5.23 results in $x_0 \notin \Omega_x(c_x)$ thus under the switching rule in Eq.5.35, $i = 1$ the CCLF-based controller remains active. Using Theorem 5.5 the closed–loop system is asymptotically stable to an equilibrium point $\bar{x} \in E_0$. Therefore there exists a time $T_{switch} > 0$ such that the augmented state x corresponding to the closed–loop state $x(T_{switch})$ under the initialization of the target equilibrium state using Eq.5.23 resides in $\Omega_x(c_x)$. Under the switching rule in Eq.5.35, the controller will switch to $i = 2$ at $t = T_{switch}$ and use the localized bounded controller of Eq.5.26. Using similar arguments as above, the closed–loop system will asymptotically approach the origin.

Remark 5.13. Theorem 5.7 describes the switching scheme to achieve closed–loop stability of nonlinear systems with input constraints for all initial conditions within the NCR. The strategy is comprised of CCLF-based controller, the localized bounded controller and a high-level supervisor that orchestrates the switching between the controllers. The implementation procedure of this hybrid control strategy is as follows:

Given the nonlinear system S and the corresponding NCR characterization, design a CCLF-based controller.

Design a localized bounded controller by defining the dynamics of Eq. 5.13 and calculate an estimate of the stability region $W_x(c_x)$ for the augmented state.

Given any $x_0 \in C_m$, check if corresponding augmented state is within $W_x(c_x)$, and switch to the localized bounded controller to achieve asymptotically stability of the origin.

Otherwise, proceed with the CCLF-based controller until the corresponding augmented state is within $W_x(c_x)$ then switch to the localized bounded controller to achieve asymptotically stability of the origin.

Remark 5.14. There exists a lack of results on the stabilization from the entire NCR. Even for linear systems, the results in [24] shows how saturated linear feedback law cannot stabilize from the entire NCR for systems of dimension greater than two. The proposed CCLF-based control design is able to stabilze from the entire NCR for general nonlinear systems. Stabilization of a three dimensional system is demonstrated in Example 5.2 below.

5.4 Simulation Examples

In this section we demonstrate the control design to stabilize all initial conditions in the NCR using several simulation examples.

5.4.1 Linear Systems

We start with linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad |u(t)| \le 1 \tag{5.36}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$.

Example 5.1. Consider a second-order semi-stable linear system with $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The eigenvalues of the system are -1, 2.

The stable manifold of the system is $W_s = \{x \mid x_1 + x_2 = 0\}$. Let $V(x) = 4(x_1 + x_2)^2$. Note that this function does not meet the criteria of a classical CLF since $V(x) = 0$ for all $x \in W_s$. However, this function does meet both criteria of a g-CLF: (1) $V(0) = 0$, and $V(x) \ge 0$, and 2) $L_f V = 4V(x)$, $L_g V = \mathrm{sgn}(x_1 + x_2)\sqrt{16V(x)}$. For all $x \in W = \{x : V(x) < \infty\}$, there is some $c > 0$ and $a \in [0, 1)$ such that $\min_u L_f V + L_g V u = 4V(x) - \sqrt{16V(x)} \le -cV^a(x)$. To demonstrate the ability of proposed controller to stabilize all initial conditions in the NCR we use a phase-plane of the closed–loop system. This is depicted in Figure 5.1 where the outer solid line represents the boundary of the NCR and the dashed line connecting the two extremal equilibrium points represents the set E_0. The arrows show the direction of the closed–loop system trajectories under the CCLF-based control law in Eq.5.11 and demonstrate that all initial conditions in the NCR are driven to the origin. Several closed–loop simulation runs were performed to demonstrate the stabilization of the proposed controller, but are not shown here for the sake of brevity.

FIGURE 5.1: Phase plane of the semi-stable planar linear system given in Example 5.1.

5.4.2 Nonlinear Systems

Example 5.2. Consider the 3d nonlinear system

$$
\begin{aligned}
x_1 &= 0.5x_1 + x_2 + x_2^3 \\
x_2 &= x_2 + x_3 \\
x_3 &= 2x_3 + x_3^3 + u
\end{aligned}
$$

where $|u(t)|$ 1. The nominal system has a single equilibrium point at the origin and the linearized system around the origin is anti-stable with eigenvalues 0.5, 1, 2.

To demonstrate the ability of proposed controller to achieve stability in the NCR we use 2d phase-planes projections of the closed–loop system. Figure 5.2 is the x_1, x_2 projection, Figure 5.3 is the x_1, x_3 projection, and Figure 5.4 is the x_2, x_3 projection. The solid line in each figure is the outer projection of the boundary of the NCR. We picked the initial condition $x_0 = [\ 0.6936, 0.0539, 0.3326]^T$ to show the stabilization to the origin using the hybrid controller given by the switching rule in Eq.5.35. Under this switching rule, the controller starts with the CCLF-based controller until it reaches the set E_0, upon which it switches to the localized bounded in Eq. 5.26 and drives the state to the origin. This is compared with a nonlinear Lyapunov-based bounded controller using the Lyapunov function $V = x^T Px$,

$P = \begin{bmatrix} 15.05 & 17.70 & 3.89 \\ 17.70 & 39.36 & 10.68 \\ 3.89 & 10.68 & 7.04 \end{bmatrix}$. As can be seen from the dashed lines in phase-plot gures, closed–loop stability from this initial condition is not achieved using the Lyapunov-based bounded controller.

FIGURE 5.2: Phase plane (x_1, x_2) of the 3d nonlinear system given in Example 5.2.

FIGURE 5.3: Phase plane (x_1, x_3) of the 3d nonlinear system given in Example 5.2.

Example 5.3. Consider a continuous stirred tank reactor where an irreversible, rst-order exothermic reaction of the form $A \xrightarrow{k} B$ takes place. We use simpli ed mathematical model for the process which was presented in [25] and takes the form:

$$x_1 = \quad f\, x_1 k(x_2) + q(x_{1f} \quad x_1) \tag{5.37}$$

$$x_2 = bf\, x_1 k(x_2) \quad (q + d)x_2 + q x_{2f} + du \tag{5.38}$$

FIGURE 5.4: Phase plane (x_2, x_3) of the 3d nonlinear system given in Example 5.2.

where $x = [x_1\ x_2]^T$, x_1 is the dimensionless concentration, and x_2 is the dimensionless temperature. The dimensionless cooling jacket temperature u is the control input. The system has three equilibrium points: 1) $x_{s1} = [0.8560, 0.8859]^T$, 2) $x_{s2} = [0.5528, 2.7517]^T$, 3) $x_{s3} = [0.2354, 4.7050]^T$. The equilibrium points x_{s1}, and x_{s3} are stable, whereas x_{s2} is unstable with one stable and one unstable mode. The control objective is to stabilize the reactor at the unstable equilibrium point x_{s2}. We let $\hat{x} = x - x_{s2}$ denote the deviation variable. Similar to the previous examples, the results in Figure 5.5 show the phase-plane of the closed–loop system under the CCLF-based control law in Eq.5.11 and demonstrates the ability to stabilize all initial conditions in the NCR. Also depicted in Figure 5.5 is boundary of the NCR (solid line). Also the figure shows the stabilization from the initial condition $x_0 = [0.09222, 0]^T$ using a CCLF-based control law in Eq.5.11. This is compared with a nonlinear Lyapunov-based bounded controller using the Lyapunov function $V = x^T P x$, $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. As can be seen from the dotted-dashed lines in Fig.5.5, closed–loop stability from this initial condition is not achieved using the Lyapunov-based bounded controller.

FIGURE 5.5: Phase plane of the planar nonlinear system given in Example 5.3.

5.5 Conclusions

In this work, we considered the problem developing a construction procedure for CCLFs for general unstable nonlinear systems where constraints on the controller induces a NCR boundary. Two control designs are presented: 1) A CCLF-based controller which results in stabilization to the equilibrium manifold and 2) a bounded controller which maneuvers the state along the equilibrium manifold to drive it to the origin. These two controllers are coupled using a hybrid control scheme to achieve stabilization for all initial conditions in the NCR. The results are demonstrated using multiple simulation examples.

5.6 References

[1] M. Mahmood, T. Homer, and P. Mhaskar, "Constrained control Lyapunov function based bounded control design", *Submitted*, 2020.

[2] T. Hu, Z. Lin, and L. Qiu, "An explicit description of null controllable regions of linear systems with saturating actuators", *Systems & Control Letters*, vol. 47, no. 1, pp. 65–78, 2002.

[3] E. D. Sontag, "A Lyapunov-like characterization of asymptotic controllability", *SIAM J. Control Optimization*, vol. 21, pp. 462–471, 1983.

[4] Z. Artstein, "Stabilization with relaxed controls", *Nonlinear Analysis: Theory, Methods & Applications*, vol. 7, no. 11, pp. 1163–1173, 1983.

[5] P. Kokotović and M. Arcak, "Constructive nonlinear control: A historical perspective", *Automatica*, vol. 37, no. 5, pp. 637–662, 2001.

[6] M. Krstic, P. V. Kokotovic, and I. Kanellakopoulos, *Nonlinear and Adaptive Control Design*, 1st. USA: John Wiley & Sons, Inc., 1995.

[7] R. A. Freeman and P. V. Kototovic, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. USA: Birkhauser Boston Inc., 1996.

[8] K. Galloway, K. Sreenath, A. D. Ames, and J. W. Grizzle, "Torque saturation in bipedal robotic walking through control lyapunov function-based quadratic programs", *IEEE Access*, vol. 3, pp. 323–332, 2015.

[9] Y. Lin and E. D. Sontag, "A universal formula for stabilization with bounded controls", *Systems & Control Letters*, vol. 16, no. 6, pp. 393–397, 1991.

[10] N. H. El-Farra and P. D. Christo des, "Bounded robust control of constrained multivariable nonlinear processes", *Chemical Engineering Science*, vol. 58, no. 13, pp. 3025–3047, 2003.

[11] P. D. Christo des and N. H. El-Farra, *Control of Nonlinear and Hybrid Process Systems: Designs for Uncertainty, Constraints and Time-Delays*. Berlin, Germany: Springer-Verlag, 2005.

[12] M. Mahmood and P. Mhaskar, "On constructing constrained control lyapunov functions for linear systems", *IEEE Transactions on Automatic Control*, vol. 56, no. 5, pp. 1136–1140, May 2011.

[13] M. Mahmood and P. Mhaskar, "Constrained control lyapunov function based model predictive control design", *International Journal of Robust and Nonlinear Control*, vol. 24, no. 2, pp. 374–388, 2014.

[14] T. Homer and P. Mhaskar, "Constrained control lyapunov function-based control of nonlinear systems", *Systems & Control Letters*, vol. 110, pp. 55–61, 2017.

[15] ——, "Utilizing null controllable regions to stabilize input-constrained nonlinear systems", *Computers & Chemical Engineering*, vol. 108, pp. 24–30, 2018.

[16] T. Homer, M. Mahmood, and P. Mhaskar, "A trajectory-based method for constructing null controllable regions", *International Journal of Robust and Nonlinear Control*, vol. 30, no. 2, pp. 776–786, 2020.

[17] F. H. Clarke, *Optimization and Nonsmooth Analysis*. Wiley New York, 1983.

[18] J.-M. Coron, *Control and Nonlinearity*. USA: American Mathematical Society, 2007.

[19] J. LaSalle, "Stability theory for ordinary differential equations", *Journal of Differential Equations*, vol. 4, no. 1, pp. 57–65, 1968.

[20] F. Clarke, "Lyapunov functions and feedback in nonlinear control", in *Optimal Control, Stabilization and Nonsmooth Analysis*, M. S. de Queiroz, M. Malisoff, and P. Wolenski, Eds., Berlin, Heidelberg: Springer Berlin Heidelberg, 2004, pp. 267–282.

[21] E. D. Sontag and H. J. Sussmann, "Nonsmooth control-lyapunov functions", in *Proceedings of 1995 34th IEEE Conference on Decision and Control*, vol. 3, 1995, 2799–2805 vol.3.

[22] W. J. Grantham and T. L. Vincent, "A controllability minimum principle", English, *Journal of Optimization Theory and Applications*, vol. 17, no. 1-2, pp. 93–114, 1975.

[23] M. Sznaier, R. Suárez, and J. Cloutier, "Suboptimal control of constrained nonlinear systems via receding horizon constrained control lyapunov functions", *International Journal of Robust and Nonlinear Control*, vol. 13, no. 3-4, pp. 247–259, 2003.

[24] T. Hu and Z. Lin, "On semiglobal stabilizability of antistable systems by saturated linear feedback", *IEEE Trans. Aut. Contr*, vol. 47, pp. 1193–1198, 2002.

[25] P. B. Sistu and B. W. Bequette, "Nonlinear predictive control of uncertain processes: Application to a cstr", *AIChE Journal*, vol. 37, no. 11, pp. 1711–1723, 1991.

Chapter 6

Lyapunov-based Model Predictive Control of Stochastic Nonlinear Systems

The results in this chapter have been published in:

Journal Articles

[1] M. Mahmood and P. Mhaskar, "Lyapunov-based model predictive control of stochastic nonlinear systems", Automatica, vol. 48, no. 9, pp. 2271–2276, 2012.

6.1 Introduction

Accounting for the system complexity in the control design is central to achieving improved performance and closed–loop stability. Such characteristics can include highly nonlinear behavior, uncertainty (typically in the form of additive disturbances and/or uncertain model parameters), and input constraints. Neglecting these characteristics at the control design stage can lead to performance degradation or even closed-loop instability. Owing to the constraint handling ability of nonlinear model predictive control (NMPC), along with the ability to incorporate an explicit system model, the NMPC framework has been widely utilized to design robust, constrained optimization based controllers. In most NMPC approaches, the manipulated input trajectory is computed at each sampling time via solving a dynamic optimization problem, where a cost function is minimized subject to a nonlinear dynamic system and input/state constraints. Several research studies dealing with NMPC have focused on issues such as feasibility, stability, constraint satisfaction, and uncertainty [2]–[4] including Lyapunov-based NMPC (LMPC) designs [5]–[7] that provide a priori (i,e, before controller implementation or testing for feasibility), an explicit characterization of initial conditions from where stability and feasibility of the closed–loop system is guaranteed in the presence of constraints and bounded uncertainty.

All the aforementioned work on NMPC is however, dominated by use of deterministic system models with very few results using the stochastic nature of the process in the control design. In the existing results, stochastic disturbance is handled via inherently `worst-case' robust NMPC schemes (e.g. [8]–[10]) where the uncertainty term is assumed to be bounded, however, such formulations are typically very numerically expensive, and as a result can impede on-line implementation. Moreover, the assumption of bounded disturbance, without the use of statistical information about the disturbances, can lead to conservative control laws and thus degrade system performance. A natural alternative is to use stochastic unbounded system disturbance in the controller design.

This alternative approach has recently been pursued under the framework of stochastic MPC where the disturbances are modeled as random variables and the expected value of a cost function is minimized. In this direction, one line of work has focused on probabilistic input constraints and not hard input constraints. In particular, the works [11]–[14] consider this problem for linear systems. For nonlinear systems, this direction has only been pursued for state constraints and not input constraints [15], [16]. In [17], the stochastic programming problem is recast as a deterministic problem with bounded disturbance and then solved over a nite horizon. The performance of the controller on a plant with unbounded disturbance is then subsequently estimated. Also recently, advancements for the problem of stochastic MPC for linear systems have been developed [18]–[20]. Speci cally, these results have addressed the problem of developing a tractable receding horizon controller in the presence of input constraints and stochastic unbounded disturbance for linear systems. In addition, the results provide conditions to ensure mean-square stability.

Although stochastic MPC circumvents the challenge of determining an a priori bound on the disturbance and also the conservatism originating from the use of a worst-case framework, it gives rise to several other challenging issues. Namely, the optimization problem is generally a stochastic program which induces significant computational burden. For example, the cost function requires the explicit calculation of a conditional expectation and/or probability associated with multidimensional random variables. For general nonlinear systems, this is a non-trivial task, and often requires resorting to probability density approximation techniques [21].

Another conceptual challenge in the extension of MPC to stochastic systems is the concept of stability. Developments in probabilistic robust control have shown that instead of stability guarantees under worst-case realizations of the uncertainty, control system performance with stochastic uncertainty can be improved by introducing a well-de ned risk of instability. That is, the closed–loop trajectory will be only be able to reach a desired target region with an associated probability. Yet, such developments have not been applied within the MPC framework. While Lyapunov techniques for stability analysis and control design for stochastic nonlinear systems do exist, the results are not as nely polished as their deterministic counterparts.

The key hurdle in the use of Lyapunov techniques for stochastic systems is the presence of an additional Hessian term within the stochastic derivative. Nevertheless, there do exist stabilizing (in a suitable stochastic sense) control laws using stochastic Lyapunov techniques that provide explicitly-de ned regions of attraction (in a probabilistic sense) for the closed–loop system [22]–[24]. In fact, such results shadow the deterministic counterpart, and have recently been used to derive regions of attraction with well de ned risk measures [25].

Deterministic Lyapunov-based control designs have been recently united with predictive control schemes to provide an explicit characterization of the states from where closed–loop stability is guaranteed in the presence of constraints and bounded uncertainty via a hybrid structure comprising a Lyapunov-based controller and MPC controllers [26] as well through the development of Lyapunov-based model predictive controllers [5]–[7]. Such results, however, do not exist for the stochastic counterpart of the problem. Hence, the MPC approach doesstand to gain from the theoretical development in the area of stochastic Lyapunov-based bounded controllers. The incorporation of explicit optimality considerations in the control design in the MPC framework, together with explicit characterizations of states with well de ned risk of instability measures which are derived using stochastic Lyapunov techniques, therefore becomes a meaningful goal.

Motivated by the above considerations, in this work we propose a stochastic Lyapunov-based NMPC for nonlinear systems with unbounded disturbances (with Ito noise and subject to input constraints). In particular, we utilize the information on the distribution of the uncertain variables (instead of traditionally used worst-case bounds) to develop model predictive controllers that yield less conservative (albeit probabilistic, with well characterized probabilities) stability region estimates while handling uncertainty. The rest of the paper is organized as follows: In Section 6.2, we outline the notation, describe the class of nonlinear stochastic systems studied, review some preliminary background and introduce a general class of Lyapunov-based feedbackcontrollers. In Section 6.3, we derive properties of the stochastic Lyapunov-based feedbackcontrollers subject to sample and hold control action. In Section 6.4, the proposed SLMPC design is presented and the stability properties inherited from the Lyapunov-based feedbackcontroller are established. In Section 6.5, the theoretical results are demonstrated on a continuous stirred tank chemical reactor (CSTR) example. Finally in Section 6.6, we summarize our results.

6.2 Preliminaries

In this section, we present the notation, provide a system description, and also review pertinent assumptions and existing results.

6.2.1 Notation

Throughout the paper, $\mathbb{R}^n$ denotes the real n-dimensional space and $(\Omega, \mathcal{F}, P)$ a general probability space. The function $X : \Omega \to \mathbb{R}$ is a random variable if for every Borel subset B of $\mathbb{R}$, $\{X \in B\} = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$. We recall that an event happens almost surely if it happens with probability one. We use the notation $E(\cdot)$ and $P(\cdot|\cdot)$ $(E(\cdot|\cdot))$ to denote the expectation and conditional probability (expectation) respectively. For a given vector or matrix v, v^T denotes its transpose, and $\mathrm{Tr}\{v\}$ denotes its trace when v is square. The notation $\|\cdot\|$ is used to denote the Euclidean norm of a vector, and the notation $\|\cdot\|_Q$ refers to the weighted norm, defined by $\|x\|_Q^2 = x^T Q x$ for all $x \in \mathbb{R}^n$, where Q is a positive definite symmetric matrix. The notation B_δ and B_δ^Q are used to denote the open balls around the origin defined by $B_\delta = \{x \in \mathbb{R}^n : \|x\| < \delta\}$ and $B_\delta = \{x \in \mathbb{R}^n : \|x\|_Q < \delta\}$ respectively. The notation $L_f c$ denotes the standard Lie derivative of a scalar function $c(\cdot)$ with respect to the vector function $f(\cdot)$. The notation $X \setminus Y$, where X and Y are sets, refers to the relative complement, defined by $X \setminus Y = \{x \in X : x \notin Y\}$. We denote the closure, interior, and boundary of the set X, by $\overline{X}$, X°, and ∂X respectively.

6.2.2 System Description

We consider stochastic nonlinear systems with input constraints, characterized by the following stochastic differential equation (SDE):

$$
\begin{gathered}
dx(t) = f(x(t))dt + g(x(t))u(t)dt + h(x(t))dW(t) \\
x(t_0) = x_0, \quad u \in U
\end{gathered}
\tag{6.1}
$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of stochastic state variables with initial state x_0, $u \in \mathbb{R}^m$ denotes the vector of manipulated inputs and $W(t)$ denotes a standard q-dimensional independent Wiener process defined on the probability space $(\Omega, \mathcal{F}, P)$. We recall that the system state $x(t)$ is actually a function of two variables $x : [0,\infty) \times \Omega \to \mathbb{R}$, where $x(t,\cdot)$ is a random variable for each $t \in [0,\infty)$. For each $\omega \in \Omega$ we call $x(\cdot,\omega) : [0,\infty) \to \mathbb{R}$ a realization, a sample path or a trajectory of the stochastic process and abbreviate $x(t,\omega)$ with $x_\omega(t)$. The input vector $u(t)$ takes on values in a nonempty convex subset U of $\mathbb{R}^m$, where $U = \{u \in \mathbb{R}^m : u_{min} \leq u \leq u_{max}\}$, $u_{min} \in \mathbb{R}^m$ and $u_{max} \in \mathbb{R}^m$ denote the lower and upper bounds on the manipulated input. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are vector fields and the term $f(x(t)) + g(x(t))u(t)$ characterizes the deterministic drift. The function $h : \mathbb{R}^n \to \mathbb{R}^{n \times q}$ is the diffusion matrix. We assume the processes $f(x(t))$, $g(x(t))$, and $h(x(t))$ are non-anticipating, so that the corresponding Ito integrals are well defined and also sufficiently smooth on their domains of definition. To ensure existence and uniqueness of solutions to Eq. 6.1, we assume that for all $t \in [0,\infty)$, the functions f, g, and h are locally Lipschitz continuous. Without loss of generality, it is assumed that the origin is the equilibrium point of the unforced and undisturbed

system (i.e. $f(0) = 0$), that should be stabilized. The first hitting time τ_X of a compact set X containing x_0 is defined as the first time the state trajectory $x(t)$ reaches the boundary of the set X. Using this we define $t_X(t) = \min\{t, \tau_X\}$.
In preparation of our results, we introduce the following definitions and propositions.

Definition 6.1. Given a C^2 function $V : \mathbb{R}^n \to \mathbb{R}$, the operator $\mathcal{L}$, known as the infinitesimal generator associated with the system in Eq.6.1 is defined as follows:

$$\mathcal{L}V(x) = L_fV(x) + L_gV(x)u(t) + \frac{1}{2}\mathrm{Tr}\{h(x)^T \frac{\partial^2 V}{\partial x^2} h(x)\} \tag{6.2}$$

where $L_gV = [L_{g_1}V, \ldots, L_{g_m}V]$.

Throughout the manuscript, we assume the terms L_fV, L_gV and $h(x)^T \frac{\partial^2 V}{\partial x^2} h(x)$ are locally Lipschitz. We recall the following definition from stochastic calculus.

Proposition 6.1. (Ito) [27] Given a C^2 function $V : \mathbb{R}^n \to \mathbb{R}$ and the solution $x(t)$ of the system in Eq.6.1, then

$$dV(x(t)) = \mathcal{L}V(x(t))\,dt + L_hV(x(t))\,dW(t) \tag{6.3}$$

We now recall Dynkin's formula.

Proposition 6.2. (Dynkin) [27] The solution $x(t)$ of the system in Eq.6.1 satisfies on $t \in [0, T]$ the following equation,

$$E(V(x(T))) - V(x_0) = E\int_0^T \mathcal{L}V(x(s))\,ds \tag{6.4}$$

Finally, we recall a key property of Brownian Motion concerning hitting times.

Proposition 6.3. [28] The distribution of the first hitting time of the set B_d by a q-dimensional Brownian process W is given by

$$P(\tau_{B_d} > T) = \sum_{r=1}^{\infty} \xi_{q,r} \exp\left(-\frac{k_{q,r}^2}{2d^2}T\right) \tag{6.5}$$

where $k_{q,r}$ are positive roots of the Bessel function $J_n(z)$ with $n = q/2 - 1$ and

$$\xi_{q,r} = \frac{1}{2^{n-1}\Gamma(n+1)} \frac{k_{q,r}^{n-1}}{J_{n+1}(k_{q,r})} \tag{6.6}$$

Remark 6.1. In contrast to the deterministic time derivative of the Lyapunov function, the infinitesimal generator $\mathcal{L}V$ contains an additional second-order derivative term. This is the main difference in stochastic Lyapunov analysis, and gives rise to several challenges in the stability analysis.

6.2.3 Stochastic Lyapunov-Based Controller

We assume that there exists a twice differentiable stochastic control Lyapunov function (SCLF) $V : \mathbb{R}^n \to \mathbb{R}$. We first characterize the set P for which negative definiteness of the infinitesimal generator $\mathcal{L}V$ can be achieved while satisfying manipulated input constraints.

$$P = \{x \in \mathbb{R}^n : \inf_{u \in U} \mathcal{L}V(x) + rV(x) \le 0\} \tag{6.7}$$

The parameter r is to be defined later. The $\inf_{u \in U}$ can be easily computed by determining the sign of the elements within the L_hV and L_gV vectors. The set f_c is defined as the set induced by level set $V = c$. Without loss of generality, we assume the largest level set of V which is contained within the set P is $V = 1$. That is, $f_1 = \sup_{c \in \mathbb{R}} \{x \in \mathbb{R}^n : x \in P, V(x) \le c\}$. To this end, we omit the subscript for the set f_1, and denote this maximal level set as f.

We note that the assumption of the existence of a SCLF is equivalent to the existence of a feedback control law $f : \mathbb{R}^n \to \mathbb{R}^m$ which result in the closed–loop system being locally stable in probability [23]. We also assume that this feedback control law $f(x)$ will result in the closed–loop system achieving negative definiteness of $\mathcal{L}V$ over f. That is

$$\mathcal{L}V(x(t))|_{u(t)=f(x(t))} + rV(x(t)) \le 0, \quad \forall x \in f \tag{6.8}$$

In the remainder, we will refer to $f(x)$ as the *Lyapunov-based controller*. The feedback law $f(x)$ will be used in the design of the SLMPC controller.

Remark 6.2. The *Lyapunov-based controllers* define a general class of feedback control laws which results in the closed–loop system achieving negative definiteness of the drift of the Lyapunov function derivative over the set f. Note that the set f is defined independently of any control law and depends only on the input constraints, Lyapunov function, and system dynamics. Explicit stabilizing (in probability) control laws that provide explicit characterization of the closed–loop region of attraction have been developed using stochastic Lyapunov techniques [23]. However such characterizations will always be subsets of the region f. Hence the class of aforementioned *Lyapunov-based controllers* is in some sense a super set of all feedback controller which are able to achieve negative definiteness of the Lyapunov function derivative over a compact region. For the purpose of this work, we assume henceforth the existence of a Lyapunov-based controller that we will first analyze further and then use to design a stochastic Lyapunov-based MPC in Section 6.4.

6.3 Properties of the Lyapunov-Based Controller

In this section, we derive the properties of the Lyapunov-based controller which will be subsequently used in Section 6.4 to design a Lyapunov-based MPC scheme. To this end, first we establish risk margins for the Lyapunov-based controller when implemented in a discrete (sample and hold) fashion with a suf ciently small hold time (D). Using this result along with the recent work in [25], we then derive risk margins for the Lyapunov-based controller to achieve stability from an explicitly de ned set (f).

6.3.1 Sample-and-Hold Implementation

We rst investigate the properties of the Lyapunov-based controller u = f (x) when applied in a sample-and-hold fashion. These properties will be subsequently used in the design of the proposed MPC scheme. In particular, with discrete-time implementation of the control action on a continuous-time dynamical system with stochastic unbounded uncertainty, one must consider the impact of the uncertainty on the closed–loop system intra-sample time. We will derive risk margins for states within the set f to remain invariant under the discrete (sample and hold) implementation of the Lyapunov-based controller with a suf ciently small hold time (D). This is formalized in Lemma 6.1 below.

Lemma 6.1. Consider the system in Eq.6.1, under the Lyapunov-based controller u = f (x) designed using the SCLF V, > 0, and the accompanying set f. Let u(t) = u(0) for all t D. Then, given any probability l 2 [0, 1), there exists positive real numbers D := D (l), and d < d^0 < 1, such that if D 2 (0, D], then

$$P\left(\sup_{t \in [0, t_{f \setminus f_d}(D)]} LV(x(t)) < 0\right) \geq 1 - l, \quad x_0 \in f \setminus f_d \tag{6.9}$$

$$P\left(\sup_{t \in [0, t_{f \setminus W_d}(D)]} V(x(t)) \leq d^0\right) \geq 1 - l, \quad x_0 \in f_d \tag{6.10}$$

Proof. We will show that for any given probability l, if D is chosen small enough, discrete implementation of the control action given by the Lyapunov-based feedback controller f (x) will preserve satisfaction of the requirement Eq.6.7 with probability of at least l for initial conditions in f nf d. In addition, for initial conditions within the set f d, if D is chosen small enough, the state trajectory remains within the set f with probability of at least l. Let the set A_B be the set of realizations of the random disturbance W_t over the time interval [0, D] which are bounded by B.

$$A_B := \left\{ w : \sup_{t \in [0, D]} \|W_t\| \leq B \right\} \tag{6.11}$$

It follows from Proposition 6.3, that given any probability l one can choose B small enough so that P(A_B) = 1 l. Using the Holder continuity of each sample path x_w(t), that there exists a K^1 := K^1(l) for all w 2 A_B such that $\sup_{t \in [0, D]} \|x_w(t) - x_0\| \leq K^1 (D)^g$ for all

$x_0 \in P$, where $g < 1/2$. Hence, it follows that

$$P\left(\sup_{t\in[0,D]} \|x(t) \quad x_0\| \quad K^1(D)^g\right) \quad 1 \quad l \tag{6.12}$$

To this end, we let $d^0 = \inf_{y\in R^n\setminus B^Q_d} V(y)$, where Q is a symmetric positive definite matrix. Note that since $V(\,)$ is a continuous function of the state x, $V(x) \quad d^0$ implies $\|x\|_Q \quad d$. Now consider a "ring" close to the boundary of f, described by $M := \{x \in R^n : d \quad V(x) \quad 1\} = f \setminus f_d$, for a $0 < d \quad 1$, with d to be determined later. The definition of the set P in Eq.6.7 implies that for all $x(0) \in f$,

$$\begin{aligned} LV(x(t)) \quad &= \quad L_fV + L_gVu(t) + \tfrac{1}{2}\mathrm{Tr}\{h(x)^T\tfrac{\partial^2 V}{\partial x^2}h(x)\} \\ & \quad\quad rV(x) \end{aligned} \tag{6.13}$$

Furthermore, if the control action is held constant until a time D, where D is a positive real number ($u(t) = u(x_0) := u_0 \;\forall t \in [0,D]$) then, $\forall t \in [0, t_{f\setminus f_d}(D)]$,

$$\begin{aligned} LV(x(t)) \quad &= \quad L_fV(x(t)) + L_gV(x(t))u_0 + \tfrac{1}{2}\mathrm{Tr}\{h(x(t)^T\tfrac{\partial^2 V(x(t))}{\partial x(t)^2}h(x(t))\} \\ &= \quad L_fV(x_0) + L_gV(x_0)u_0 + \tfrac{1}{2}\mathrm{Tr}\{h(x_0)^T\tfrac{\partial^2 V(x_0)}{\partial x^2}h(x_0)\} \\ &+ \quad (L_fV(x(t)) \quad L_fV(x_0)) + (L_gV(x(t))u_0 \quad L_gV(x_0)u_0) \\ &+ \quad \tfrac{1}{2}\mathrm{Tr}\{h(x(t))^T\tfrac{\partial^2 V(x(t))}{\partial x(t)^2}h(x(t))\} \quad \tfrac{1}{2}\mathrm{Tr}\{h(x_0)^T\tfrac{\partial^2 V(x_0)}{\partial x^2}h(x_0)\} \end{aligned} \tag{6.14}$$

Since $x_0 \in M \quad f$, $L_fV(x_0) + L_gV(x_0)u_0 + \frac{1}{2}\mathrm{Tr}\{h(x_0)^T\frac{\partial^2 V(x_0)}{\partial x^2}h(x_0)\} \quad rV(x_0)$. By definition, for all $x_0 \in M$, $V(x_0) \quad d$, therefore

$$L_fV(x_0) + L_gV(x_0)u_0 + \frac{1}{2}\mathrm{Tr}\{h(x_0)^T\frac{\partial^2 V(x_0)}{\partial x^2}h(x_0)\} \quad rd$$

Since the functions $L_fV(\,)$, $L_gV(\,)$, $h(x(t)^T\frac{\partial^2 V(x(t))}{\partial x(t)^2}h(x(t))$, and $V(\,)$ are locally Lipschitz in the state $x(t)$, we have that one can find positive real numbers K^2, K^3, K^4 and K^5 such that $\|L_fV(x(t)) \quad L_fV(x_0)\| \quad K^3\|x(t) \quad x_0\|$, $\|L_gV(x(t))u_0 \quad L_GV(x_0)u_0\| \quad K^2\|x(t) \quad x_0\|$, $\|\frac{1}{2}\mathrm{Tr}\{h(x(t))^T\frac{\partial^2 V(x(t))}{\partial x^2}h(x(t))\} \quad \frac{1}{2}\mathrm{Tr}\{h(x_0)^T\frac{\partial^2 V(x_0)}{\partial x^2}h(x_0)\}\| \quad K^4\|x(t) \quad x_0\|$, and $\|V(x(t)) \quad V(x_0)\| \quad K^5\|x(t) \quad x_0\|$ almost surely. We let e be a positive real number such that

$$e \;<\; rd \tag{6.15}$$

It follows that for all w such that $\sup_{t\in(0,D]}\|x_w(t) \quad x_0\| \quad K^1(D)^g$, and a choice of D such that, $D \;<\; \left(\frac{rd \quad e}{(K^1K^2 + K^1K^3 + K^1K^4)}\right)^{1/g}$ that we get that $LV(x_w(t)) \quad e < 0$, for all $t \quad t_{f\setminus f_d}(D)$. Hence it follows that

$$P\left(\sup_{t\in[0,t_{f\setminus f_d}(D)]} LV(x(t)) < 0\right) \quad 1 \quad l \tag{6.16}$$

This implies that, given d^0, if we pick d such that

$$d < d^0 \tag{6.17}$$

and nd a corresponding value of D then if the control action is computed for any $x \in M$, and the `hold' time is less than D, we get that $L V$ remains negative with a probability greater than l during the interval $[0, t_{f\ nf_d}(D)]$.
Using a similar line of reasoning, for initial conditions within the set f_d, and for all w such that $\sup_{t \in [0, D]} \|x_w(t) - x_0\| \leq K^1(D)^g$, and a choice of D such that $D < \left(\frac{d^0 - d}{K^5}\right)^{1/g}$ we get that $V(x_w(t)) \leq d^0$, for all $t \leq t_{f_{d^0}}(D^0)$. Hence it follows that

$$P\left(\sup_{t \in [0, D]} V(x(t)) \leq d^0\right) \geq 1 - l \tag{6.18}$$

Let $D = \min\left\{\left(\frac{rd\ e}{(K^1K^2 + K^1K^3 + K^1K^4)}\right)^{1/g}, \left(\frac{d^0 - d}{K^5}\right)^{1/g}\right\}$, then for $D \in (0, D]$, we get Eqs. 6.9–6.10.

Remark 6.3. For continuous-time systems under continuous implementation of the control action, suf cient Lyapunov conditions for stochastic stability (analogous to the deterministic conditions) can be derived by ensuring negative de niteness of the in nitesimal generator L on the outer boundary of a set. For systems with continuous-time dynamics and discrete implementation of the control action, one cannot use this condition as the in nitesimal generator may become positive during the sampling period or escape a desired target region before a new sample is obtained. The risk of such events is quanti ed in Lemma 6.1. In particular, Lemma 6.1 states that for any desired probability l and initial conditions within $f\ nf_d$, the control law $f(x)$, when implemented in a sample-and-hold fashion will result in the in nitesimal generator L maintaining negative de niteness over the sampling period with a probability of at least $1 - l$ provided that the sampling time is suf ciently small. Likewise, for initial conditions within f_d, the probability that the state trajectory will remain within the set f_{d^0} before a new sample is obtained is also at least $1 - l$.

6.3.2 Characterizing Stability in Probability Regions

Having established the risk margins associated with implementing the Lyapunov-based controller in a sample and hold fashion, we now establish the closed–loop stability (in probability) regions. That is, we show that for all initial conditions within the set f stability can be achieved with an associated well-de ned probability. The notion of stability used here is from the recent work [25] and is formalized in Theorem 6.1 below.

Theorem 6.1. Consider the system in Eq. 6.1, under the Lyapunov-based controller $u = f(x)$ designed using the SCLF V, $r > 0$, and the accompanying set f. Let $u(t) = u(jD)$ for all $jD \leq t < (j+1)D$, and $u(jD) = f(x(jD))$, $j = 0, \ldots, \infty$. Then, given any positive real number d, and probability $\lambda \in [0,1)$, there exists positive real numbers $D^* := D^*(\lambda)$, $d < a$ and probabilities $a, b \in [0,1)$ such that if $D \in (0, D^*]$, then the following will hold for the closed–loop system:

(i) $P\left(\sup_{t\geq 0} \|x(t)\|_Q \leq d\right) \geq (1-b)(1-\lambda),\ x_0 \in f_d$

(ii) $P\left(\sup_{t\geq 0} V(x(t)) < 1,\ \sup_{t\geq 0} \|x(t + t_{\mathbb{R}^n \setminus f_d})\|_Q < d,\ t_{\mathbb{R}^n \setminus f_d} < \infty\right) \geq (1-a)(1-b)(1-\lambda)^2$, $x_0 \in f_a \setminus f_d$

Proof. Using the result from Lemma 6.1 we have that for any probability λ there exists a hold time small enough such that under the discrete implementation of the control action $\mathcal{L}V$ remains negative definite on $f \setminus f_d$ and will remain within the set $f_{d'}$ for $x_0 \in f_d$ with probability of at least $1-\lambda$. We now proceed to show that there exists probabilities a and b, such that conditions i) and ii) stated above hold. In the remainder of the proof, we abbreviate all probabilities and expectations conditional on the event in Eq.6.12 with the superscript. This part follows very similar lines as the proof of Theorem 1 in [25].

We begin with part i). Note that it suffices to show the complementary event with $x_0 \in \partial f_d$. Using Lemma 6.1, it follows from Proposition 6.2 and the discrete implementation of the $f(x)$, that

$$E\left(V(x(t_{f \setminus f_{d'}}))\right) < V(x_0) \tag{6.19}$$

for all $x_0 \in f \setminus f_d$. One can prove using similar arguments as in [25], that

$$P\left(\|x(t)\|_Q > d \text{ for some } t > 0\right) \leq \frac{V(x_0)}{\inf_{y \in \mathbb{R}^n \setminus \bar{B}^Q_d} V(y)} \tag{6.20}$$

for all $x_0 \in \partial f_{d'}$. Recall from the proof of Lemma 6.1 that

$$d' = \inf_{y \in \mathbb{R}^n \setminus \bar{B}^Q_d} V(y) \tag{6.21}$$

Using Eq.6.17, there exists $b < 1$ such that $\frac{d}{d'} \leq b$, and hence,

$$\sup_{x_0 \in \partial f_d} P\left(\|x(t)\|_Q > d \text{ for some } t > 0\right) \leq \sup_{x_0 \in \partial f_{d'}} P\left(\|x(t)\|_Q > d \text{ for some } t > 0\right) \leq b \tag{6.22}$$

By passing to the complementary events, we get part i). To prove part ii) we assume $x_0 \in f_a \setminus f_d$, for some positive $a < 1$. Again following a similar argument as Theorem 1 in [25], it can be shown that

$$P\left(t_{f \setminus f_d} < \infty\right) = 1 \tag{6.23}$$

We now show,

$$\inf_{x_0 \in f_a \setminus f_d} P\left(t_{\mathbb{R}^n \setminus f_d} < t_f\right) \geq 1 - a \tag{6.24}$$

Using the fact that $\left\{t_{\mathbb{R}^n \setminus f_d} > t_f\right\} \subseteq \left\{V(x(t_{f \setminus f_d})) \geq 1\right\}$, along with the Chebyshev's inequality, it follows that

$$
\begin{aligned}
P \quad t_{R^n nf_d} > t_f \qquad & P \quad V(x(t_{f\ nf_d})) \quad 1 \\
& E \quad V(x(t_{f\ nf_d})) \\
& < V(x_0) \\
\sup_{x_0 2 f_a nf_d} P \quad t_{R^n nf_d} > t_f \qquad & < \sup_{x_0 2 f_a nf_d} V(x_0) \\
& < a
\end{aligned}
\tag{6.25}
$$

By the continuity of the $x(t)$ and since f_d f, it follows that

$$
P \quad t_{R^n nf_d} = t_f \quad = \quad 0 \tag{6.26}
$$

Using Eqs.6.25, 6.26 with the complementary events implies Eq.6.24. The remainder of proof again follows the same lines as the proof of Theorem 1 in [25], where the complementary events in conjunction with Bayes formula is used to show part ii).

Remark 6.4. Theorem 6.1 establishes that the practical stochastic stabilization of the Lyapunov-based feedback controllers can be achieved from the region f_a with the upper bound on the probability being a function of the initial state, the desired target region, and the sampling time. Each component represents a risk factor which multiplies to give the overall risk of resulting in instability. Since the region is control law independent, it follows that the derived risk margins are also independent of any control law and only depend on the Lyapunov function, system dynamics, constraints, and sample and hold time.

The results presented in this section establish the stability risk margins associated with the Lyapunov-based controller when implemented in a sample-and-hold fashion. Lemma 6.1 derived lower bounds on the probability of the in nitesimal generator maintaining negative de niteness over a sampling period and the state trajectory escaping a target region before a new sample is obtained. Conditional on these events, Theorem 6.1 further derived the risks of the escaping the region f before reaching a subset of the target region. These properties will be inherited by the proposed LMPC design, which simultaneously incorporates optimality considerations which improve closed-loop performance. This result is presented in the upcoming section.

6.4 Lyapunov-Based MPC Design

Existing LMPC designs only consider uncertainty with nite support and are unable to handle stochastic unbounded disturbance. These controller provide explicit characterizations of regions from where the stability of the closed–loop system and feasibility of the optimization problem is guaranteed. In the presence of stochastic unbounded disturbance, such results no longer hold. In this section, we propose a new robust model predictive control design which provides a systematic way of handling stochastic disturbance which is unbounded. This design takes into account the probabilistic information of the disturbance and provides an explicitly characterized region from where stabilization in probability can be achieved. With the use of appropriate constraints within the optimization problem, we show that the proposed receding horizon controller is an implicit form of a Lyapunov-based feedback controller. Hence, this predictive control scheme inherits all the stability and robustness properties of the Lyapunov-based feedback controller when it is applied in a sample-and-hold fashion, while also incorporating optimality considerations. Consider now the receding horizon implementation of the control action computed by solving an optimization problem of the form:

$$
\begin{aligned}
u_{MPC}(\hat{x}) &:= \arg\min\{J(\hat{x},t,u(\cdot)) \mid u(\cdot) \in S\} && (6.27)\\
\text{s.t.}\quad \frac{d\hat{x}}{d\tilde{t}} &= f(\hat{x}(\tilde{t})) + g(\hat{x}(\tilde{t}))u(\tilde{t}) && (6.28)\\
\hat{x}(t) &= x(t) && (6.29)\\
LV(\hat{x}(t)) + rV(\hat{x}(t)) &\quad 0 && (6.30)
\end{aligned}
$$

where $S = S(t,t+T)$ is the family of piecewise continuous functions (functions continuous from the right), with period D, mapping $[t,t+T]$ into U. Note in this formulation the model in Eq.6.28 is defined in continuous time and the control input is a piecewise constant function, which results in the closed-loop system being a nonlinear sample-data system with sampling time D. $\hat{x}(\cdot)$ denotes the predicted trajectory of the nominal stochastic system for the input trajectory computed by the SLMPC. Note that the model in Eq.6.28 is a deterministic approximate nonlinear model describing the time evolution of the state $\hat{x}(\cdot)$ without any disturbance. Hence the predicted values need not and in general will not be the same as the actual system values. Note also that the system model used to predict the future dynamics of the system is initialized by the actual state of the system. A control $u(\cdot)$ in S is characterized by the sequence $\{u[j]\}$ where $u[j] := u(jD)$ and satisfies $u(t) = u[j]$ for all $t \in [jD,(j+1)D)$. The performance index is given by

$$
J(\hat{x},t,u(\cdot)) = \int_t^{t+T} \|\hat{x}^u(\tilde{t};\hat{x},t)\|^2_{Q_w} + \|u(\tilde{t})\|^2_{R_w}\, d\tilde{t} \qquad (6.31)
$$

where Q_w and R_w are positive semi-definite, and strictly positive definite, symmetric matrices, respectively, and $\hat{x}^u(\tilde{t};\hat{x},t)$ denotes the solution of Eq.6.28, due to control u, with initial state $\hat{x}$ at time t and T is the specified horizon. The minimizing control $u^0_{MPC}(\cdot) \in S$ is then applied to the plant over the interval $[t,t+D)$ and the procedure is repeated indefinitely. Feasibility of the optimization problem and stability properties of the closed–loop system under the predictive controller are formalized in Theorem 6.2 below.

Theorem 6.2. Consider the constrained system of Eq.6.1 under the MPC law of Eqs.6.27–6.30. Then, given any positive real number d and probability $l \in [0,1)$, there exists probabilities a, $b \in [0,1)$, and positive real number $D := D(l)$, and $d < a$, such that if

$\mathsf{D} \in (0, \mathsf{D}^*]$ *and* $x_0 \in \mathtt{f}_{\mathsf{a}}$, *then the optimization problem of Eqs.6.27-6.30 will be initially feasible and the following will hold for the closed–loop system:*

(i) $\mathsf{P}\left(\sup_{t\geq 0} \|x(t)\|_Q \leq d \right) \geq (1-b)(1-l), \; x_0 \in \mathtt{f}_d$

(ii) $\mathsf{P}\left(\sup_{t\geq 0} V(x(t)) < 1, \sup_{t\geq 0} \|x(t + t_{\mathsf{R}^n \backslash \mathtt{f}_d^\circ})\|_Q < d, \; t_{\mathsf{R}^n \backslash \mathtt{f}_d^\circ} < ¥ \right) \geq (1-\mathsf{a})(1-b)(1-l)^2,$
$x_0 \in \mathtt{f}_{\mathsf{a}} \backslash \mathtt{f}_d^\circ$

Proof. *The proof of the theorem comprises of two parts. In part 1, we show that for all* $x \in \mathtt{f} \supset \mathtt{f}_{\mathsf{a}}$, *where* $\mathsf{a} < 1$, *the optimization problem of Eqs.6.27–6.30 is guaranteed to be initially feasible. In part 2, we prove the closed–loop system under the receding horizon discrete implementation of the MPC law of Eqs.6.27–6.30 will result in 1) convergence of the state trajectory to a desired neighborhood of the origin, and 2) remaining within this neighborhood therein after.*

Part 1: *Consider some* $x_0 \in \mathtt{f} \supset \mathtt{f}_s$, *where* $s < 1$, *under receding horizon implementation of the predictive controller of Eqs.6.27–6.30, with a prediction horizon* $T = N\mathsf{D}$, *where* D *is the hold time and* $1 \leq N < ¥$ *is the number of the prediction steps. We first analyze the constraint of Eq.6.30 for feasibility. Since* $\mathtt{f} \in \mathsf{P}$ *and* $x_0 \in \mathtt{f}$, *this implies that there exists a* $u^* \in S$ *such that* $\mathcal{L}V(x(t)) + rV(x(t)) \leq 0$. *Therefore, for all* $x_0 \in \mathtt{f}$, *the solution comprising of* u^* *as the first element followed by* $N-1$ *zeros is a feasible solution to constraint of Eq.6.30.*

Part 2: *It follows from the initial feasibility of the optimization problem for all states within* $\mathtt{f}$ *in Part 1, that the proposed MPC controller is an implicit representation of a* Lyapunov-based controller. *Therefore the implementation of the MPC controller inherits the stability properties of the* Lyapunov-based controller *established in Section 6.3.*

Remark 6.5. *Note that the use of the nominal system without the stochastic term for the dynamics of the prediction state* $\hat{x}$ *is just one possible choice. A natural alternative would be to use the* average *dynamics of the state* $x(t)$. *For the case of linear systems, the average dynamics actually reduces to the aforementioned dynamics without the stochastic term (by the martingale property of the Ito integral). However, for the general nonlinear case, computation of the* average *dynamics requires the state transition density which is usually not available in closed–form and hence requires approximation. The recent work in [21] develops a framework where the prediction state uses axis-aligned Gaussian mixtures to approximate transition densities. Nevertheless, regardless of the choice of the model dynamics, the established stability results still hold. It is only the optimality of the control law which is influenced by the choice of model dynamics.*

Remark 6.6. *Theorem 6.2 above establishes that for all* $x_0 \in \mathtt{f}_{\mathsf{a}}$ *initial feasibility of the optimization problem in Eqs.6.27–6.30, along with closed–loop stability in a probabilistic sense. Similar notions of stability can be found in [25] and [29], where the state trajectory is required to reach a target subset of the region* $\mathtt{f}$ *in finite time and then remain within a target set* B_d^Q *therein after. The difference between the definition in the present work and that from [25] is that this definition uses compact sets which are not indexed (or remain* *static), as opposed to using sequences of compact sets. As described in [25], the numbers* a *and* b *are risk margins which quantify the risk of escaping the set* f *and remaining close to the target respectively. Owing to the stochastic non-vanishing disturbance along with the discrete-time implementation of the controller, this notion of stability is a more practical representation of achievable closed–loop dynamics. In the case of the limit as* D *and* s *go to zero, the* Lyapunov-based feedback *controller enforces asymptotic stability.*

Remark 6.7. *In many practical applications, distributional information about the disturbance term can be quantified. Most existing robust MPC approaches ignore such distributional information (by invoking worst-case bounds within the control design over realized disturbances), which can lead to conservative estimates of the stability region and also unnecessarily aggressive control action. In contrast, in the proposed robust MPC control design, the probabilistic nature of the disturbance is explicitly used within the predictive control formulation and also used to characterize regions of stability.*

Remark 6.8. *The proposed Lyapunov-based MPC design is novel in that it is the first to unify Lyapunov-based stochastic control results within a predictive control framework to be able to account for stochastic unbounded disturbance. This direction has been pursued in the deterministic setting, but has yet to be explored in the stochastic setting. Moreover, this design provides a systematic way to assess the risk of ending up with instability.*

Remark 6.9. *The proposed design differs from existing Lyapunov-based robust designs [30], [31] in being able to explicitly account for unbounded disturbances. In particular, the constraint in Eq.6.14 includes an additional Hessian term which arises with the use of Ito calculus. In contrast, the aforementioned robust Lyapunov-based designs assume the disturbance has finite support. Such designs use a constraint within the MPC formulation so that the worst case effect of the disturbances on the Lyapunov function derivative is countered to ensure closed–loop stability. In the presence of disturbances with infinite support, the results on guaranteed stability and recursive feasibility of the MPC scheme collapse. While results with absolute guarantees are not possible in the stochastic setting, the proposed SLMPC scheme provides measures (which are not achievable with the previous schemes) of being able to achieve stability. Moreover, enforcing the infinitesimal generator $\mathcal{L}$ to be negative definite renders the closed–loop system process $x(t)$ a super-martingale. That is* $\mathsf{E}\left(V(x(t))\right) \leq V(x(0))$ *for all $t \geq 0$ and $x(0) \in$ f. If one were to implement a Lyapunov-based robust MPC design which can counter the disturbance under the assumption that it will take only values from a predefined bounded set, then the super-martingale property will in general not hold. Note that the robust Lyapunov-based MPC schemes [30], [31] use an additional term in the Lyapunov function derivative of the form*

$$L_{s} V w^{max} = \frac{\P V}{\P x} s w^{max}$$

where w^{max} denotes the assumed bound on the disturbance. Depending on the choice of V and system parameter s*, the use of this term over the Hessian term in the Lyapunov constraint may not imply negative definiteness of the infinitesimal generator $\mathcal{L}$. Additionally, under the assumption of convexity of the Lyapunov function, the additional Hessian* *term will be positive semi-definite. Hence, in addition to being a key link in establishing the probabilistic stability properties of the closed–loop stochastic system, this term provides a robustness property to the controller by countering the effect of the uncertainty on the Lyapunov-function derivative.*

6.5 Simulation Results

Consider a continuous stirred tank reactor where a reaction of the form $A \xrightarrow{k} B$ takes place, described by:

$$\begin{aligned} dC_A &= \left(\frac{F}{V_R}(C_{A0} - C_A) - k_0 e^{\frac{-E}{RT_R}} C_A \right) dt \\ &+ \sigma_{C_A} (C_A - C_A^s)\, dw^{C_A}(t) \\ dT_R &= \left(\frac{F}{V_R}(T_{A0} - T_R) + \frac{(-\Delta H)}{\rho_d c_p} k_0 e^{\frac{-E}{RT_R}} C_A \right. \\ &+ \left. \frac{Q_R}{\rho_d c_p V_R} \right) dt + \sigma_{T_R} (T_R - T_R^s)\, dw^{T_R}(t) \end{aligned} \tag{6.32}$$

where C_{A0}, T_{R0} and C_A, T_R denote the concentration of species A, and temperature in the inlet stream and reactor respectively. The variable Q_R is the heat added to the reactor. The model considers stochastic uncertainty due to errors in the process parameters and process noise in the inlet flow rate F and inlet temperature T_{R0}. The terms $w^{C_A}(t)$ and $w^{T_R}(t)$ are independent standard Brownian motions with $\sigma_{C_A} = 0.1$ and $\sigma_{T_R} = 0.2$. The values of the process parameters can be found in Table 6.1 (for more details, see [6]). The SDE in Eq.6.32 is simulated using the `MATLAB` SDE toolbox* with an integration step-size of 0.0001. The control objective is to stabilize the reactor at the unstable equilibrium point $(C_A^s,\ T_R^s)$ subject to constraints: $|Q_R| \leq 90$ kJ/s and $C_{A0} \leq 2$ kmol/m^3. We consider a quadratic control Lyapunov function of the form $V(x) = x^T P x$, where $x = (C_A - C_A^s, T_R - T_R^s)$ with $P = \begin{pmatrix} 0.3333 & 0.0215 \\ 0.0215 & 0.0024 \end{pmatrix}$. We demonstrate the theoretically derived probability bounds in Theorem 6.2 using probabilities obtained/observed via monte-carlo simulations. To this end, we focus on the complementary sub-events $\mathcal{A}_a = \{\tau_{\mathbb{R}^n \setminus \Pi_d^\circ} > \tau_\Pi, x_0 \in \Pi_a \setminus \Pi_d^\circ\}$ and $\mathcal{A}_d = \{\exists t,\ \|x(t)\| > d, x_0 \in \partial\Pi_d\}$. Note these sub-events imply the events given in Theorem 6.2 using Bayes Theorem (as in the proof of Theorem 6.1). That is, we will estimate the *risk* of 1) starting outside the target region and reaching the target region before hitting the boundary of stability region Π (a) and 2) starting within the target region and remaining there (b). This is done by first discretizing the set

*http://sdetoolbox.sourceforge.net

of points on the level curves $\mathtt{f}_a$ and $\mathtt{f}_d$ and then performing 1000 closed–loop simulations from each one of these points under the implementation of proposed predictive controller with a prediction horizon of $T = 2\mathrm{D}$ over a time interval $[0, T_f]$. The probability estimates are then computed from each of these points, which in turn are used to determine an estimate of the upper bound on the probability over the entire set. A discretization time of $\mathrm{D} = 0.02$ min is used with $Q_w = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$, $R_w = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.01 \end{pmatrix}$ and $r = 0.001$. The optimization problem is solved using `MATLABs` subroutine `fmincon`. All simulations are preformed with $T_f = 2$ min, and each boundary of the set $\mathtt{f}_a$ is discretized into 25 points. The estimated probabilities are always less than the respective values of a, and b (summarized in Tables ??) supporting the theoretically derived upper bounds on the probabilities in Theorem 6.2.

TABLE 6.1: Chemical reactor parameters and steady–state values.

Parameter			Units
V_R	=	0.1	m^3
R	=	8.314	$KJ/Kmol \cdot K$
C_{A0}	=	1.0	$Kmol/m^3$
T_{A0}	=	350.0	K
Q_R^s	=	0.0	KJ/min
$\mathrm{D}H$	=	-4.78×10^4	$KJ/Kmol$
k_0	=	72×10^9	min^{-1}
E	=	8.314×10^4	$KJ/Kmol$
c_p	=	0.239	$KJ/kg \cdot K$
r	=	1000.0	kg/m^3
F	=	100×10^{-3}	m^3/min
T_R^s	=	388.48	K
C_A^s	=	0.8076	$Kmol/m^3$
s_{C_A}	=	0.1	
s_{T_R}	=	0.2	

TABLE 6.2: Probability estimates for different values of b and a.

b	$\sup_{x_0 \in \P \mathtt{f}_d} \mathrm{P}(\mathcal{A}_d)$	a	$\sup_{x_0 \in \mathtt{f}_a \setminus \mathtt{f}_d^\circ} \mathrm{P}(\mathcal{A}_a)$
0.5	0.388	0.5	0.005
0.6	0.510	0.6	0.103
0.7	0.620	0.7	0.222
0.8	0.750	0.8	0.429
0.9	0.870	0.9	0.825

6.6 Conclusions

In this work, a predictive control design was proposed for the constrained stabilization (in probability) of nonlinear stochastic systems with unbounded disturbance. First, a general class of *Lyapunov-based feedback controllers* was studied. Using stochastic Lyapunov-based techniques, key properties regarding the discrete implementation and closed–loop stability (in probability) region for this class of controllers was derived. The results were then united with a predictive control framework to derive the proposed Lyapunov-based stochastic MPC. The key idea in the proposed control design was to use stochastic Lyapunov techniques to derive constraints which are enforced within the optimization problem of the receding horizon controller. Upon feasibility of the optimization problem, the MPC scheme inherits the stability properties of the Lyapunov-based controllers. Moreover, the Lyapunov techniques were used to establish risk margins for achieving stability from a well characterized set of initial conditions. In particular, it was shown that the value of the Lyapunov function provides an upper bound on the probability of the state trajectory becoming unstable. The theoretically derived bounds on this probability were empirically demonstrated via simulation on an unstable CSTR example.

6.7 References

]

Chapter 7

Conclusions and Future Work

In this chapter, we summarize the main contributions of this work and discuss future research directions.

7.1 Conclusions

In this work, we addressed the problem of control of nonlinear systems in the presence of input constraints and stochastic uncertainty. Specifically, the work focused on achieving stabilization via control from all initial conditions from where control is possible (i.e. the NCR). The problem scope focuses on unstable systems where the presence of input constraints results in the NCR being a subset of the state space. This makes the use of control more important as instability can result from poor control design.

In Chapters 2 and 3 we first consider linear systems and present a constructive procedure for constrained control Lyapunov functions (CCLFs). In addition we consider the objective of designing a controller that guarantees stability from all initial conditions in the NCR. The aforementioned procedure for construction of CCLFs was utilized within a Lyapunov-based model predictive controller coupled with an auxiliary control design to achieve stabilization from all initial conditions in the NCR. Illustrative simulation examples were presented and implementation to nonlinear systems was demonstrated via a chemical reactor example.

In Chapter 4 we considered the problem developing a construction procedure for CCLFs for general unstable nonlinear systems. A procedure was developed which allows for the computation of the boundary trajectories by using a boundary condition for the well-known Controllability Minimum Principle. Following this, we show how CCLFs can be constructed using this boundary characterization. In Chapter 5, we turned to the problem of stabilization of the entire NCR for general nonlinear systems. Two control designs are presented: 1) A CCLF-based controller which results in stabilization to the equilibrium manifold and 2) a bounded controller which maneuvers the state along the equilibrium manifold to drive it to the origin. These two controllers are coupled using a hybrid control scheme to achieve stabilization for all initial conditions in the NCR. The results are demonstrated using multiple simulation examples.

In Chapter 6, we consider the problem of control for nonlinear stochastic systems with unbounded disturbance. A predictive control design was proposed to achieve constrained stabilization (in probability). The results build on a general class of *Lyapunov-based feedback controllers* and utilize stochastic Lyapunov-based techniques. Specifically, several key properties regarding the discrete implementation and closed–loop stability (in probability) region for this class of controllers were first derived. Following this, the results were united with a predictive control framework to derive the proposed Lyapunov-based stochastic MPC. The main idea in the proposed control design was to derive appropriate constraints which are enforced within the optimization problem of the receding horizon controller and which use established stochastic Lyapunov properties. The MPC scheme inherits the stability properties of the Lyapunov-based controllers, upon feasibility of the optimization problem. Finally, the Lyapunov techniques were used to derive risk margins for achieving stability from a well characterized set of initial conditions. In particular, it was shown that the value of the Lyapunov function provides an upper bound on the probability of the state trajectory becoming unstable. The theoretically derived bounds on this probability were empirically demonstrated via simulation on an unstable CSTR example.

7.2 Future Work

We suggest the following topics for future research. The general theme for these suggestions is to generalize the results to wider class of systems.

- The development of CCLF construction procedures along with control designs for both linear (Chapter 2 and 3) and nonlinear (Chapter 4 and 5) are done in the context of process systems which does not account for the presence of parametric uncertainty and disturbances. Extending the notion of the CCLF to achieve stabilization in the presence of input constraints along with robustness against parametric uncertainty and disturbances is a natural direction. By assuming the uncertainty and disturbances are another input into the plant and remains bounded, the results for characterizing the NCR could be extended to define the set of states within the state–space from where stabilization is possible under such an uncertainty input. This would then result in a robust CCLF which can be used to design control laws resulting guarantees stabilization from the entire NCR. This direction can be further extended to address the unavailability of some of the states for measurement (output-feedback problem). Specifically, the robust CCLF control design could be re-purposed to handle estimation errors in the state feedback controller and combining the control design with a nonlinear observer.

- The work in Chapters 4 and 5 focus on nonlinear systems with a single control input. A topic of future research is to extend the NCR construction and the

CCLF-based control design for nonlinear systems with multiple inputs. A key challenge in the construction of the CCLF in this context is the NCR characterization. For linear systems, the NCR for multi-input systems can be decomposed as the super-set of the NCR's for each single-input subsystem. This decomposition does not hold for general nonlinear systems. However, one could explore if there is a class of nonlinear systems where such a decomposition does hold resulting in a natural generalization of the presented CCLF-based control designs. Extending the nonlinear single-input construction procedure to multi-input systems would involve using a manifold of equilibrium points in place of single control equilibrium points as a boundary condition which would increase the computational complexity of the procedure.

- The work in Chapters 4 and 5 also focus on nonlinear systems under the assumption that the target equilibrium point is structurally stable. A topic of future research is to extend the problem to relax this assumption. For such systems, the equilibrium point can bifurcate for varying values of the control action and thus would impact the construction procedure of the NCR. The proposed procedure would need to be modified to account for new equilibrium branches.