THREE ESSAYS ON STOCHASTIC VOLATILITY WITH VOLATILITY MEASURES

Three Essays on Stochastic Volatility with Volatility Measures

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Abstract

This thesis studies realized volatility (RV), implied volatility (IV) and their applications in stochastic volatility models. The first essay uses both daytime and overnight high-frequency price data for equity index futures to estimate the RV of the S&P500 and NASDAQ 100 indexes. Empirical results reveal strong inter-correlation between the regular-trading-time and after-hour RVs, as well as a significant predictive power of overnight RV on daytime RV and vice versa. We propose a new day-night realized stochastic volatility (DN-SV-RV) model, where the daytime and overnight returns are jointly modeled with their RVs, and their latent volatilities are correlated. The newly proposed DN-SV-RV model has the best out-of-sample return distribution forecasts among the models considered. The second essay extends the realized stochastic volatility model by jointly estimating return, RV and IV. We examine how RV and IV enhance the estimation of the latent volatility process for both the S&P500 index and individual stocks. The third essay re-examines asymmetric stochastic volatility (ASV) models with different return-volatility correlation structures given RV and IV. We show by simulation that estimating the ASV models with return series alone may infer erroneous estimations of the correlation coefficients. The incorporation of volatility measures helps identify the true return-volatility correlation within the ASV framework. Empirical evidence on global equity market indices verifies that ASV models with additional volatility measures not only obtain significantly different estimations of the correlations compared to the benchmark ASV models, but also improve out-of-sample return forecasts

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Declaration of Authorship

I, Zehua Zhang, declare that this thesis, entitled, “Three Essays on Stochastic Volatility with Volatility Measures”, and the work presented in it, are my own. I confirm that the thesis comprises the following chapters:

- Is Overnight Volatility Overlooked?
- Estimating the Stochastic Volatility Model with Realized Volatility and Implied Volatility
- Improving Asymmetric Stochastic Volatility Models with Ex-post Volatility

This thesis is entirely my own original work unless otherwise indicated. Any use of the work of other authors is acknowledged at their point of use.

Introduction

The volatility of financial asset returns is a major topic in finance. As a measure of risk, volatility plays a central role in portfolio management, risk hedging and derivative pricing. Moreover, volatility itself is a tradable asset in modern financial practice, especially during a financial crisis. Modeling and forecasting volatility is of great importance for both researchers and practitioners.

Since the seminal paper by Engle (1982), numerous parametric volatility models have been proposed. The GARCH model (Generalized AutoRegressive Conditional Heteroskedasticity) of Bollerslev (1986) and the SV (Stochastic Volatility) model of Taylor (1986) are two popular and well-studied volatility models. On the other hand, realized volatility measures (realized volatility and bi-power variation), based on high frequency data, provide consistent and model-free estimations of the volatility (Andersen and Bollerslev 1998; Barndorff-Nielsen and Shephard 2001; Barndorff-Nielsen and Shephard 2004). Takahashi et al. (2009) combine the SV model with realized volatility (SV-RV) to jointly model return and realized volatility processes.

The first chapter of this thesis reveals the omitted overnight volatility by estimating the overnight realized volatility with the high frequency prices of index futures, which are traded almost 24 hours. The overnight realized volatility significantly improve the forecasting of the following daytime realized volatility, which is treated as the daily realized volatility by previous studies. This chapter further jointly estimates the daytime and overnight returns, which are jointly modeled with their realized volatilities. Both in-sample and out-of-sample empirical results indicate a strong correlation between the daytime and overnight volatility in the equity market. The volatility clustering is persistent during and after regular trading hours.

Another well-documented volatility measure is the implied volatility estimated from the option prices. As an ex-ante volatility measure, the predictive relation between implied volatility and realized volatility has been well-studied (Busch et al. 2011). The second chapter extends the stochastic volatility model with both realized volatility(RV) and implied volatility(IV), and examines whether RV and IV improve the stochastic volatility model in identifying the underlying volatility process and the out-of-sample forecast. The empirical results suggest that implied volatility, as an expectation of future volatility in the risk-neutral measure, is biased, and it may be insufficient to jointly model implied volatility and return with a single latent volatility factor.

The last chapter reexamines asymmetric stochastic volatility (ASV) models (Harvey and Shephard 1996; Jacquier et al. 2004) given realized volatility and implied volatility as volatility measures. We focus on the estimation of the return-volatility correlation coefficients according to different asymmetric stochastic models' specifications. Fitting the ASV models with return series alone could lead to misidentification of the correlation coefficients. The inclusion of volatility measures leads to return-volatility correlations that differ from the traditional ASV models and improves out-of-sample forecasting.

The thesis makes significant contributions to the existing financial econometrics literature by estimating the overnight volatility, examining the role of implied volatility, and clarifying the return-volatility correlation in the SV framework. The documented improvement in volatility forecasting has numerous further applications in derivative pricing and portfolio management.

Chapter 1

Is Overnight Volatility Overlooked?

1.1 Introduction

The stock market is open during the day, typically from 9:30 a.m. to 4:00 p.m. Eastern Standard Time. However, Cliff et al. (2008) report that the cumulative return from 1993 to 2006 by holding the SPY ETF is solely due to overnight returns. Sommer (2018) further shows that all gains by holding the S&P500 index (proxied by the SPY ETF) from 1993 to 2018 occur outside trading hours, and holding the equity-only portfolio during trading hours is an unprofitable investment strategy. Berkman et al. (2012) and Liu and Tse (2017) also examine this phenomenon for individual stocks, index ETFs and futures.

Information shocks are prone to arrive during the night. DeHaan et al. (2015) report that 21.5% of listed firms announced their earnings during trading hours in 2000, while only 2.2% firms did so in 2011. Important macroeconomic information announcements, like the release of the unemployment rate, are usually scheduled at 8:30 a.m. EST before the opening of major exchanges (e.g., NYSE). Besides, global economic information, especially from Asia-Pacific and European countries, affects the U.S. equity market during the night. Unfortunately, limited research, especially the work related to financial time series modeling and forecasting, focuses on the overnight return and volatility series.

There are two fundamental issues for modeling overnight returns, especially for major U.S. equity indexes. First, although the overnight return can be clearly defined as the price change during the non-trading hours, it would be misleading for major equity indexes like the S&P500 and Dow Jones Industrial Average (DJIA), since it depends how index provider sets the open price. The providers of the S&P500 and DJIA indexes have tended to set the open price according to the last close price for the last two decades. Ahoniemi and Lanne (2013) show that the overnight returns of the S&P 500 index are basically zero, especially before 2006, and they use the price of S&P 500 at 9:35 a.m. EST as the open price to calculate the overnight returns of the S&P500 index. Moreover, global market indices are also inconsistent in setting the open prices, as shown in Figure A1.1 in Appendix Section A1. The inconsistency of the open price explains the evidence regarding the role and mechanism of overnight returns. Previously work that documents the importance of overnight returns, like Gallo (2001), Tsiakas (2008), Tseng et al. (2012), Ahoniemi and Lanne (2013), and Dhaene and Wu (2020), focus mainly on international equity indexes or individual stocks, which are free from the open price issue. In this paper, we use the open and close prices of index exchange-traded funds (ETFs) to evaluate the daytime and overnight returns. The price of ETFs can fluctuate during the pre-opening price discovery process, so that the overnight information is naturally embodied in the open price. Moreover, the Redemption Mechanism of the ETF excludes significant deviations of the ETF open price from the underlying constituents' open price.

Second, and more importantly, it is hard to measure the overnight volatility under the current market microstructure. The realized volatility, estimated from high-frequency daytime prices (Barndorff-Nielsen and Shephard 2002), is a well-accepted consistent estimator of the volatility process. However, the absence of high-frequency overnight prices prevents researchers from directly measuring and studying the overnight volatility process. Many related papers use the squared overnight return (Fleming et al. 2003; Hansen and Lunde 2005; Ahoniemi and Lanne 2013; Todorova and Souček 2014; Fuertes et al. 2015; Maderitsch 2017) as a rough approximation to the overnight volatility. We show that the squared overnight returns are very noisy and hardly capture the overnight volatility. In this paper, we select the high-frequency price of overnight equity index futures (e.g. E-mini future contracts) to estimate the overnight realized volatility. Taylor (2007) estimates the overnight volatility from S&P500 E-mini futures and provides its risk management (value-at-risk) implications. The continuous trading time of the futures market facilitates price discovery overnight. With the rapid development of the futures market since the early 2000s, overnight futures trading became increasingly important and contains more pertinent information. For example, the average daily dollar volume of the S&P 500 E-mini futures is 192 billion, while the average dollar volume for the corresponding ETF is only 19.6 billion[1]. The

[1]https://www.cmegroup.com/education/courses/futures-vs-etfs/why-trade-futures-instead-of-etfs.html

high trading volume ensures adequate liquidity in the futures market and enhances price discovery for the underlying assets, which, in turn, improves the estimation of realized volatility measurement using futures data.

Our paper makes the following contributions to the existing literature. First, we use the future prices to directly establish the estimation of the overnight realized volatility. Thus, our empirical estimation avoids the rough approximation when estimating the overnight effect on daytime realized volatility. Modeling and forecasting realized volatility is one of the major foci and challenges for financial econometricians (Andersen et al. 2003; Andersen et al. 2005; Gonçalves and Meddahi 2009; Corsi 2009; Patton 2011). The realized volatility measure has substantial asset pricing implications for underlying stock returns (Bollerslev et al. 2009b; Barndorff-Nielsen et al. 2008) and estimating stochastic volatility models (Deo et al. 2006; Takahashi et al. 2009). We follow the well-documented heterogeneous autoregression (HAR) process to re-examine the predictive capability of overnight realized volatility on the following daytime open to close realized volatility, which is the well-studied daily realized volatility of previous studies. For the S&P 500 (NASDAQ 100) index, including the overnight realized volatility will reduce the out-of-sample forecasting mean squared error of the following daytime logarithmic realized volatility by 16.11% (13.88%) compared to the benchmark HAR-RV model. Along with overnight return and implied volatility, our best model has a 26.72% (23.17%) lower out-of-sample forecasting mean squared error compared to the benchmark HAR-RV model for the S&P 500 (NASDA-100) index. The improvement is substantial, which is a significant contribution to the existing literature on realized volatility forecasting.

Moreover, we also model the overnight realized volatility with the HAR model and examine the predictive capability of daytime realized volatility for the following overnight close to open realized volatility. Similarly, we find significant improvement by incorporating daytime realized volatility. The estimation outcomes from the HAR models suggest strong correlation between daytime and overnight volatility.

Second, we propose a day-night realized stochastic volatility (DN-SV-RV) model to model jointly the daytime and overnight return and realized volatility. This innovative framework extends the SV model with realized volatility model (Takahashi et al. 2009). The model setup allows the inter-correlation between the daytime and overnight latent volatilities. Both the in-sample estimation and out-of-sample return distribution forecasts support a finding that the auto-correlation coefficient of the daytime (overnight) latent volatility will greatly decrease, which should be around 0.9 if modeled independently. We find that the daytime and overnight volatility processes are highly correlated. The results suggest that the volatility clustering effect is persistent throughout the day and night. When predicting the daytime (overnight) volatility at market open (close), the most recent information is the immediately preceding overnight (daytime) volatility. Moreover, the mean of overnight returns, after controlling the stochastic volatility, is not necessarily higher than that of the daytime returns. The stochastic volatility sheds light on explaining the observed high overnight returns (Berkman et al. 2012; Branch and Ma 2012; Liu and Tse 2017).

The rest of the paper is organized as follows. Section 1.2 motivates the importance of incorporating overnight information by comparing the statistics of daytime and overnight returns and by estimating the daytime and overnight volatility process with the SV model. Section 1.2.4 explains the approach that we used to estimated the overnight realized volatility from high-frequency data from the futures market. We model and forecast daytime and overnight realized volatility, and confirm their strong correlation in Section 1.3. Section 1.4 proposes the DN-SV-RV model and compares the model with benchmark SV and SV-RV models. Section 1.5 concludes the paper.

1.2 Motivations

1.2.1 Daily, Daytime and Overnight Returns

For the following, we use Eastern Standard Time and follow the trading calendar of the New York Stock Exchange (NYSE). Given daily close price S_t^C, $t = 1, 2, ...T$, the daily close to close return is defined as:

$$r_t \equiv \log S_t^C - \log S_{t-1}^C. \tag{1.1}$$

This is the return used by researchers and participants as the daily return. The daily return can be decomposed into two parts, the overnight close to open and the daytime open to close returns. Let S_t^O, $t = 1, 2, ...T$ denote the daily open price. Then the overnight return is defined as:

$$r_{N,t} \equiv \log S_t^O - \log S_{t-1}^C, \tag{1.2}$$

and the daytime return is defined as:

$$r_{D,t} \equiv \log S_t^C - \log S_t^O. \tag{1.3}$$

The daily return is the summation of the overnight return and daytime return ($r_t = r_{N,t} + r_{D,t}$). Now we will demonstrate the daily and overnight returns of major equity indices, index ETFs and individual stocks and discuss the issue of erroneous open prices of certain equity indices. For the rest of this paper, all returns will be scaled by 100 to represent percent returns.

1.2.2 Overnight Returns, Significant or Negligible?

The following Figure 1.1 shows the daily returns and the overnight returns of three major equity indices: the S&P 500, the NASDAQ 100 and the Dow Jones Industrial Average. For comparison, we also include the corresponding ETFs (SPY, QQQ and DIA respectively).

The left column of Figure 1.1 shows the daily returns and overnight returns of the indices from January 1st, 2001 to January 1st, 2020. The S&P 500 and DJIA overnight returns seem to be negligible, especially before 2016, while, the NASDAQ 100 overnight returns are significant. Considering the fact that these indices share many common stocks, the obvious difference in overnight returns is puzzling. On the other hand, overnight returns are significant for all three ETFs. SPY and DIA overnight returns differ significantly from those of the S&P 500 and DJIA indexes. In Appendix Section A1, Figure A1.2 further demonstrates the

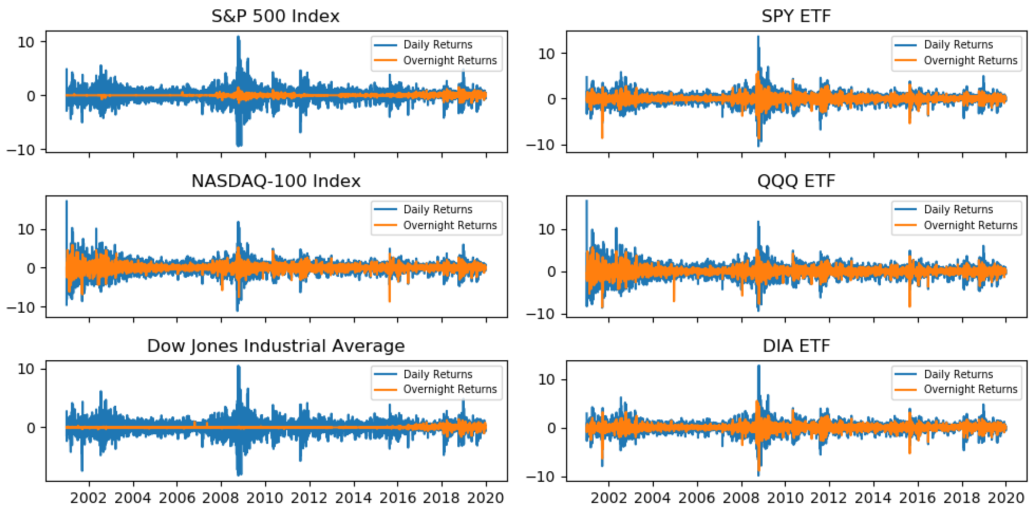

FIGURE 1.1: The daily and overnight returns of major indices and corresponding ETFs.

daily and overnight returns of sixteen representative companies included in the DJIA index. All component companies have significant overnight returns.

The pre-opening mechanism used by the stock exchanges only applies to tradable assets like individual stocks and ETFs (Cao et al. 2000; Madhavan and Panchapagesan 2000; Madhavan and Panchapagesan 2002; Angel and Wu 2001). The pre-open price discovery will reflect overnight information in the open price for individual stocks and ETFs. However, the index open prices is not determined through this pre-opening price discovery process. As a result, the open prices of equity indices do not necessarily represent the average opening prices of their component stocks.

For the rest of this paper, we use the 5-minute prices of the SPY and QQQ ETF, which are retrieved from the TAQ database, to set the open and close prices according to the NYSE calendar, and calculate the daytime, overnight and daily returns accordingly. Table 1.1 presents the summary statistics of daytime, overnight and daily returns.

	SPY			QQQ		
	r_t	$r_{N,t}$	$r_{D,t}$	r_t	$r_{N,t}$	$r_{D,t}$
Mean	0.0521	0.0317	0.0203	0.0681	0.0489	0.0192
Variance	0.8296	0.3268	0.4898	1.1639	0.4267	0.7223
Skewness	-0.4286	-0.9602	-0.5670	-0.3659	-1.5144	-0.4337
Kurtosis	4.1986	13.4697	5.0766	3.5565	23.9689	3.5079
Correlation Coefficients						
r_t	1.0000	-	-	1.0000	-	-
$r_{N,t}$	0.6402	1.0000	-	0.6161	1.0000	-
$r_{D,t}$	0.7786	0.0164	1.0000	0.7959	0.0134	1.0000
r_{t-1}	-0.0184	-0.0395	0.0084	-0.0109	-0.0349	0.0131
$r_{N,t-1}$	0.0146	-0.0434	0.0545	0.0369	-0.0477	0.0834
$r_{D,t-1}$	-0.0359	-0.0160	-0.0336	-0.0421	-0.0076	-0.0476

Sample period: 2009-10-29 to 2019-12-31.
Number of observations: 2560.

Table 1.1: Summary statistics and correlation of daily, daytime and overnight returns.

1.2.3 Overnight Return and Volatility

From Table 1.1, overnight returns have a higher mean and lower volatility, or higher Sharpe ratio, for the past ten years. The ratio of overnight volatility to daily volatility ($Var(r_{N,t})/Var(r_t)$) is 39% (0.37%) for the SPY (QQQ). Roughly speaking, nearly 40% of daily volatility is overnight volatility. This challenges previous work which focuses on modeling daytime volatility alone as it suggests that 40% of the infomation in the volatilty process is omitted.

Heteroscedasticity and volatility clustering are well-documented properties of financial return series. To examine the overnight volatility process, we fit the well-known stochastic volatility (SV) model, given by Eq. (1.4) and (1.5), with daytime, overnight and daily returns respectively. We use Bayesian MCMC to estimate the SV model and we report parameter estimation results in Table 1.2. The Bayesian method allows us to estimate the smoothed latent volatility process $h_t \approx \sum_{i=1}^{M} h_t^{(i)}$, where $h_t^{(i)}$ is the MCMC draw of the latent volatility. Figure 1.2 demonstrates the smoothed estimated latent volatility process of SPY.

$$y_t = \mu + exp(h_t/2)u_t, \qquad u_t \sim N(0,1), \tag{1.4}$$

$$h_t = \alpha + \delta h_{t-1} + \sigma_h v_t, \qquad v_t \sim N(0,1). \tag{1.5}$$

	SPY			**QQQ**		
	$r_{D,t}$	$r_{N,t}$	r_t	$r_{D,t}$	$r_{N,t}$	r_t
μ	0.0604	0.0443	0.0982	0.0781	0.0657	0.1376
	[0.009]	[0.007]	[0.012]	[0.012]	[0.009]	[0.015]
	(0.043, 0.078)	(0.030, 0.059)	(0.075, 0.121)	(0.054, 0.101)	(0.048, 0.083)	(0.108, 0.167)
α	-0.0795	-0.0822	-0.0446	-0.0565	-0.1039	-0.0225
	[0.017]	[0.019]	[0.011]	[0.013]	[0.024]	[0.009]
	(-0.115, -0.049)	(-0.122, -0.049)	(-0.068, -0.025)	(-0.084, -0.034)	(-0.154, -0.063)	(-0.041, -0.007)
δ	0.9386	0.9515	0.9419	0.9294	0.9252	0.9312
	[0.011]	[0.010]	[0.010]	[0.013]	[0.016]	[0.013]
	(0.914, 0.959)	(0.930, 0.970)	(0.920, 0.961)	(0.903, 0.952)	(0.892, 0.953)	(0.904, 0.954)
σ_h	0.3586	0.3042	0.3535	0.3560	0.3519	0.3519
	[0.031]	[0.030]	[0.030]	[0.032]	[0.038]	[0.032]
	(0.305, 0.423)	(0.248, 0.366)	(0.299, 0.416)	(0.297, 0.420)	(0.283, 0.430)	(0.292, 0.419)

The table reports the posterior mean, standard deviation in [] and 0.95 density interval in ().

TABLE 1.2: The SV model estimation results.

The prior distributions are: $p(\mu) \sim N(0,5)$, $p(\alpha) \sim N(0,5)$ and $p(\sigma_h^2) \sim IG(3/2, 0.5/2)$. $p(\delta) \sim N(0.9,5)I_{|\delta|<1}$, a normal distribution truncated to the stationary region (-1,1).

The SV model results confirm that the heteroscedasticity of the overnight return series and the overnight volatility process is highly correlated with the daytime volatility process. The estimated auto-correlation coefficient, δ, which is 0.9515 for

SPY (0.9252 for QQQ), supports the volatility clustering of the overnight return series. Figure (1.5) further demonstrates that the overnight return series shares a similar stochastic volatility pattern with the daytime and daily returns.

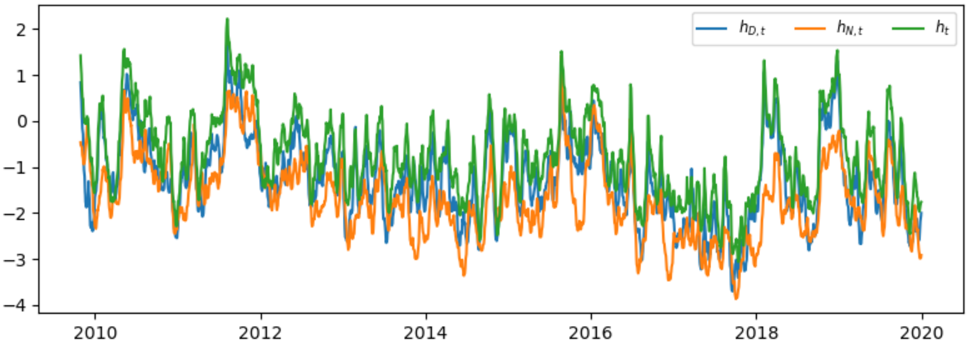

FIGURE 1.2: Estimated latent volatility process of the SV model.

Fitting the SV model with $r_{D,t}$, $r_{N,t}$ and r_t yields estimated smoothed latent volatility processes $h_{D,t}$, $h_{N,t}$ and h_t respectively.

1.2.4 Realized Volatility Measures and Jumps

1.2.5 Data

We collect 5-minute high-frequency S&P 500 E-mini and NASDAQ 100 E-mini future prices from Kibot. According to the CME Group, the trading hours of the E-mini contracts are Sunday 6:00 p.m. EST, to Friday 6:00 p.m. EST. After the close of trading on the NYSE, which is typically 4:00 p.m EST, there is a daily trading halt from 4:15 p.m. to 4:30 p.m. EST, Monday to Friday, and daily maintenance from 5:00 p.m. to 6:00 p.m. EST, Monday to Thursday. For every trading day, we can match a continuous overnight trading interval starting at 6:00 p.m. EST on the previous calendar day to the following market open.

Given the log asset price $\log(S_t)$ by continuous jump-diffusion process:

$$d\log(S_t) = \mu_t dt + \sigma_t dW_t + J_t dq_t, \tag{1.6}$$

where W_t is the standard Brownian motion and dq_t is the Poisson process with jump intensity λ_t. J_t follows a normal distribution with mean μ_J and variance σ_J, measuring the jump size of the underlying process. Define the intraday return over time interval Δ as:

$$r_{t,j} = \log(S_{t,j\cdot\Delta}) - \log(S_{t,(j-1)\cdot\Delta}), \tag{1.7}$$

where $\log(S_{t,j\cdot\Delta})$ is the log asset price at day t and time $j \cdot \Delta$. As shown in Barndorff-Nielsen and Shephard (2004), the realized variance and bipower variation of asset returns are

$$RV_t = \sum_{j=1}^{m} r_{t,j}^2 \xrightarrow{\Delta\downarrow 0} \int_{t-1}^{t} \sigma_s^2 ds + \int_{t-1}^{t} J_s^2 dq_s, \tag{1.8}$$

$$BV_t = \frac{\pi}{2}\frac{m}{m-1}\sum_{j=1}^{m} |r_{t,j}||r_{t,j-1}| \xrightarrow{\Delta\downarrow 0} \int_{t-1}^{t} \sigma_s^2 ds. \tag{1.9}$$

We also filter the cumulative squared jumps (Barndorff-Nielsen and Shephard 2004; Huang and Tauchen 2005; Andersen et al. 2007; Tauchen and Zhou 2011) as:

$$J_t^2 = (RV_t - BV_t) \times I_{(ZJ_t \geq \Phi_\alpha^{-1})}, \tag{1.10}$$

where

$$ZJ_t = \frac{RV_t - BV_t}{RV_t\sqrt{\left[\left(\frac{\pi}{2}\right)^2 + -5\right]\frac{1}{m}\max\left(1, \frac{TP_t}{BV_t^2}\right)}} \xrightarrow{d} N(0,1), \tag{1.11}$$

which converges to a standard normal distribution and

$$TP_t = m\mu_{4/3}^{-3}\frac{m}{m-2}\sum_{j=3}^{m}|r_{t,j-2}|^{4/3}|r_{t,j-1}|^{4/3}|r_{t,j}|^{4/3} \to \int_{t-1}^{t}\sigma_s^4 ds, \tag{1.12}$$

where $\mu_k = 2^{k/2}\Gamma((k+1)/2)/\Gamma(1/2)$ for $k > 0$. The squared cumulative jump is then filtered at significance level α of the z-test. For this paper, we choose $\alpha = 0.99$.

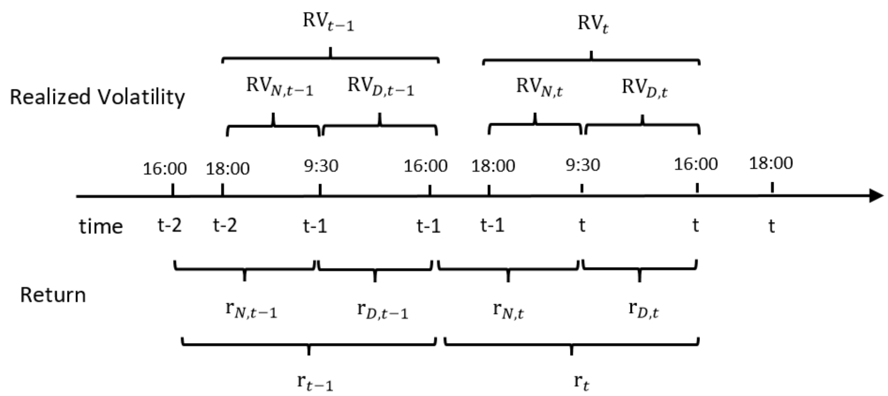

FIGURE 1.3: The time label of the daily, daytime and overnight RV and returns.

Figure 1.3 illustrates how we estimate the daytime, overnight and daily realized volatility and set the time label. We use the 5-minute prices to calculate the intraday returns. Using returns within the overnight time interval (from 6:00 p.m. of the previous calendar day to 9:30 a.m. of day t), we estimate the overnight realized volatility, denoted as $RV_{N,t}$. Similarly, daytime realized volatility $RV_{D,t}$ is estimated with intraday returns within trading hours. In addition, we can also estimate the daily realized volatility RV_t with intraday returns from 6:00 p.m. of the previous calendar day $t-1$ to 4:00 p.m. (1:00 p.m. if early closure) of day t.

Given our notation, $RV_{N,t}$, although labeled with day t, is observed before $RV_{D,t}$ and RV_t=$RV_{N,t} + RV_{D,t}$.

Notice that during the trading hours, both the ETF and E-Mini futures are traded. The daytime realized volatility measures calculated from the ETF and futures are almost identical. Considering the higher trading volume of the futures market and the consistency with the overnight realized volatility, we will use the daytime realized volatility calculated from futures prices in the following. In total, we collect 2,560 daily observations from 2009-08-29 to 2019-12-31.

Similarly, we can also estimate the overnight, daytime and daily bi-power variation ($BV_{N,t}$, $BV_{D,t}$ and BV_t) and squared overnight jumps ($J^2_{N,t}$, $J^2_{D,t}$ and J^2_t).

1.2.6 Overnight Realized Volatility and Jumps

We report the summary statistics of the realized volatility measures of the S&P 500 in Table 1.3. The summary statistics table for the NASDAQ 100 can be found in the Appendix, Table A1.1.

	RV_t	$RV_{N,t}$	$RV_{D,t}$	BV_t	$BV_{N,t}$	$BV_{D,t}$
Mean	0.8715	0.3611	0.5103	0.8132	0.3157	0.4875
Var	2.2285	0.4968	0.8454	2.1981	0.4059	0.8758
RV_t	1.0000					
$RV_{N,t}$	0.8933	1.0000				
$RV_{D,t}$	0.9388	0.6839	1.0000			
BV_t	0.9941	0.8891	0.9324	1.0000		
$BV_{N,t}$	0.8661	0.9825	0.6530	0.8713	1.0000	
$BV_{D,t}$	0.9441	0.7053	0.9922	0.9479	0.6733	1.0000

TABLE 1.3: Summary statistics and correlations of the S&P500 realized volatility measures.

The means of realized volatility and bi-power variation (daily, daytime and overnight respectively) match the variances of the corresponding returns in Table 1.1, which supports the accuracy of our realized volatility measures. The correlation coefficient between the overnight and daytime realized volatility, which is a lead-lag correlation since $RV_{N,t}$ is observed before $RV_{D,t}$, is 0.6839. This confirms the correlation between overnight and daytime volatility, but also challenges the method of estimating the whole day realized volatility by scaling the daytime realized volatility upward to match the daily close to close return volatility. Scaling actually assumes that the daytime and overnight realized volatility are perfectly correlated, which is not true given our results. The overnight volatility contains information that is independent of the daytime volatility and should not be omitted.

The left column of Figure 1.4, from top to bottom, demonstrates the daytime, overnight and daily realized volatility. The fourth row shows the squared overnight returns. The right column shows the logarithmic RVs. We also include the estimated latent volatility process from the SV model (as in Figure 1.2) for comparison purposes. The logarithmic realized volatility matches the estimated latent volatility process. However, the squared overnight returns process is too noisy to use as an overnight volatility measure.

The last row of Figure 1.4 shows the filtered daytime (left) and overnight (right) squared jumps. They share the same vertical axis for comparison purposes. It is obvious that jumps are prone to occur outside the main trading hours. This is consistent with the observation in Table 1.1 that the kurtosis of SPY (QQQ) overnight returns is 13.45 (23.87), which is significantly higher than that of the daytime returns. Releasing important information outside the main trading hours, instead of avoiding shocks in the equity market, merely postpones the shocks. The estimated overnight squared jumps reflect the shocks from the overnight-traded futures market.

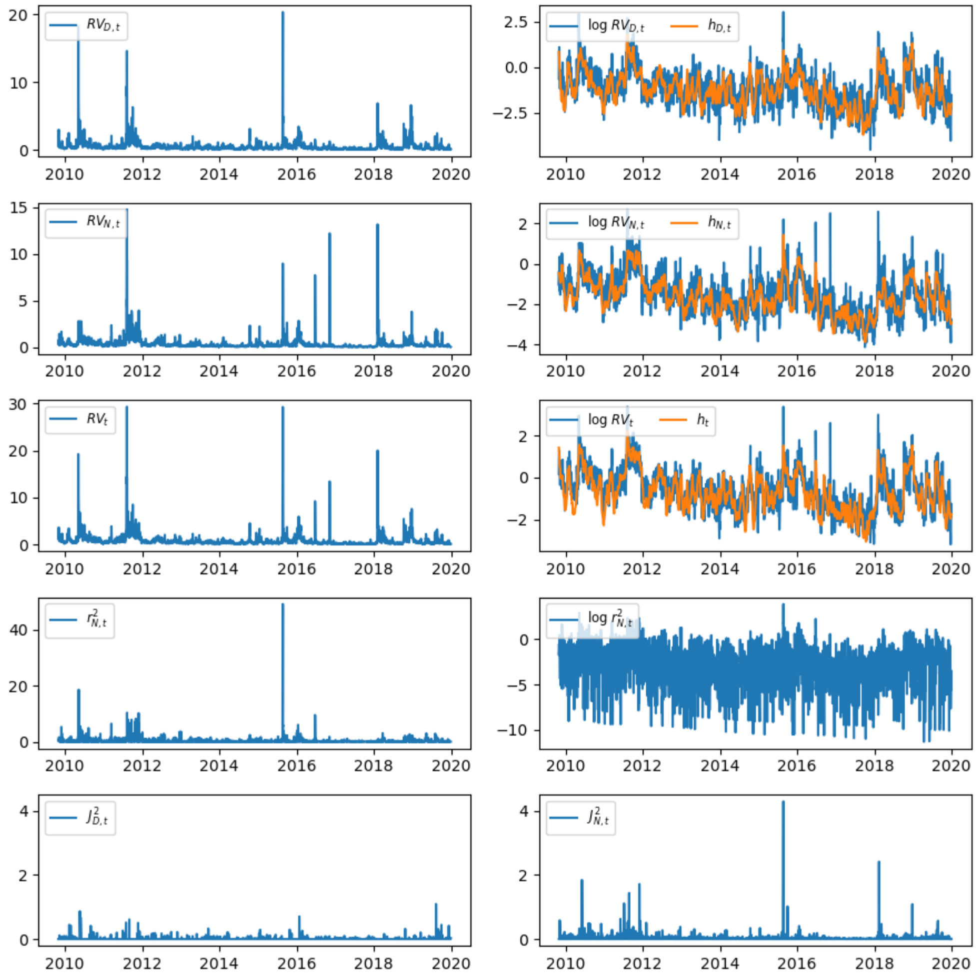

Figure 1.4: SPY realized volatility measures and squared jumps.

Given the high-frequency futures prices, we reveal the overnight volatility and jumps by estimating the realized volatility measures and squared jumps. The economic implications are numerous and not limited to this paper. We focus on the daytime and overnight volatility here. The next section models the daytime and overnight realized volatility process and investigates the correlation between daytime and overnight volatility.

1.3 Forecasting Daytime and Overnight RV

1.3.1 HAR-RV modeling of Realized Volatility

Predicting the future realized volatility, especially after assuming a constant return mean, is the key component in return distribution prediction (Bollerslev et al. 2009a; Maheu and McCurdy 2011). The heterogeneous autoregressive model for realized volatility (HAR-RV), developed by Corsi (2009), is popular for modeling the realized volatility process since it captures the long memory of the volatility process. Inasmuch as Andersen et al. (2003) have shown that the distribution of logarithmic realized volatility is close to normal, we will model the logarithmic realized volatility like Andersen et al. (2007) and Liu and Maheu (2009) and follow a parameterization similar to Patton and Sheppard (2015) for clear parameter interpretation.

First, we model the daytime realized volatility and focus on forecasting the following daytime realized volatility given all the information available at the market open. Following previous literature, all HAR-RV models in this section assume normal noises with zero mean and constant variance ($e_t \sim N(0, \sigma^2)$). The basic HAR-RV model is specified as follows:

$$\log RV_{D,t} = \mu + \rho_1 \log RV_{D,t-1} + \rho_5 \left(\frac{1}{4} \sum_{i=2}^{5} \log RV_{D,t-i} \right) + \rho_{22} \left(\frac{1}{17} \sum_{i=6}^{22} \log RV_{D,t-i} \right) + e_t. \tag{1.13}$$

Recent work (Corsi and Renò 2012; Wang et al. 2015) extend the HAR-RV by including the implied volatility, leverage effect, etc. Before adding more explanatory variables to capture more volatility-related effects or modifying the HAR-RV model with non-linear, non-Gaussian properties like Corsi et al. (2008), we need to ensure that the right hand side includes all the available information of the volatility process itself at the market open. Including the overnight realized volatility leads to Eq. (1.14):

$$\begin{aligned} \log RV_{D,t} =& \mu + \rho_1 \log RV_{D,t-1} + \rho_5 \left(\frac{1}{4} \sum_{i=2}^{5} \log RV_{D,t-i} \right) + \rho_{22} \left(\frac{1}{17} \sum_{i=6}^{22} \log RV_{D,t-i} \right) \\ &+ \theta_1 \log RV_{N,t} + \theta_5 \left(\frac{1}{4} \sum_{i=1}^{4} \log RV_{N,t-i} \right) + \theta_{22} \left(\frac{1}{17} \sum_{i=5}^{21} \log RV_{N,t-i} \right) + e_t. \end{aligned} \tag{1.14}$$

We include overnight return $r_{N,t}$, and lagged daytime return $r_{D,t-1}$ to control the asymmetry between return and volatility. We also include the lagged logarithmic implied volatility ($\log IV_{t-1}$, $\frac{1}{4}\sum_{i=2}^{5} \log IV_{t-i}$ and $\frac{1}{17}\sum_{i=6}^{22} \log IV_{t-i}$). For SPY (QQQ), the logarithmic implied volatility is calculatd from the VIX (VXN) index ($\log IV_t = 2(\log VIX_t - \log 21)$). Notice IV_{t-1} denotes the closing price of the volatility index so that $\log IV_{t-1}$ does not contain overnight information. We can classify all the right hand side variables into two sets: variables with and without overnight information. We expect that variables with overnight information (overnight realized volatility and overnight returns) should have strong predictive power.

To predict the overnight realized volatility at the market close, the benchmark HAR model is given by Eq. (1.15):

$$\log RV_{N,t} = \mu + \rho_1 \log RV_{N,t-1} + \rho_5 \left(\frac{1}{4}\sum_{i=2}^{5} \log RV_{N,t-i}\right) + \rho_{22}\left(\frac{1}{17}\sum_{i=6}^{22} \log RV_{N,t-i}\right) + e_t. \tag{1.15}$$

And we extend Eq. (1.15) with lagged daytime realized volatility as follows:

$$\begin{aligned}\log RV_{N,t} =& \mu + \rho_1 \log RV_{N,t-1} + \rho_5 \left(\frac{1}{4}\sum_{i=2}^{5} \log RV_{N,t-i}\right) + \rho_{22}\left(\frac{1}{17}\sum_{i=6}^{22} \log RV_{N,t-i}\right) \\ & \theta_1 \log RV_{D,t-1} + \theta_5 \left(\frac{1}{4}\sum_{i=2}^{5} \log RV_{D,t-i}\right) + \theta_{22}\left(\frac{1}{17}\sum_{i=6}^{22} \log RV_{D,t-i}\right) + e_t.\end{aligned} \tag{1.16}$$

Similarly, we also include lagged daytime return ($r_{D,t-1}$), lagged overnight return ($r_{N,t-1}$) and implied volatility ($\log IV^O_{t-1}$, $\frac{1}{4}\sum_{i=2}^{5} \log IV^O_{t-i}$ and $\frac{1}{17}\sum_{i=6}^{22} \log IV^O_{t-i}$). We choose the opening price of the volatility index so that $\log IV^O_{t-1}$ does not contain the lagged daytime information when predicting the overnight realized volatility at the market close. We can also classify the right hand variables for overnight realized volatility forecasting into two sets: variables with and without lagged daytime information.

We estimated the HAR-RV model using the Bayesian MCMC method. For model comparison, we calculate the forecasting mean squared error (FMSE) to measure the accuracy of mean forecasting and the logarithmic predictive likelihood (LPL) to measure the accuracy of distributional forecasting (Geweke and Amisano 2010; Gelman et al. 2014). FMSE and LPL are defined as follows:

$$FMSE_M = \frac{1}{T-s} \sum_{t=s}^{T-1} \left((y_{t+1} - \hat{y}_{M,t+1})^2 | \mathcal{F}_t \right), \tag{1.17}$$

$$LPL_M = \sum_{t=s}^{T-1} \log p_M(y_{t+1} | \mathcal{F}_t). \tag{1.18}$$

Here y_{t+1} is the observation to predict and $\hat{y}_{M,t+1}$ is the prediction of y_{t+1} with available information at time t ($\mathcal{F}_t$ denotes the available information set) according to model M. Since the HAR model is essentially a linear model with normally distributed noise, $p_M(y_{t+1}|\mathcal{F}_t)$ is a normal likelihood. Let $x_{1:t}$ denote the set of explanatory variable vectors $x_1, x_2, ...x_t$ and $y_{1:t}$ denote the dependent variable vector $y_1, y_2, ...y_t$. By fitting the HAR model M with $y_{1:t}$ and $x_{1:t}$, we have the MCMC draws of correlation coefficients $\beta_{M,t}^{(i)}$ and variance $\sigma_{M,t}^{(i)}$ ($i = 1, 2...N$, N denotes the number of MCMC iterations). Then $\hat{y}_{M,t+1}$ and $p_M(y_{t+1}|\mathcal{F}_t)$ of model M can be estimated as:

$$\hat{y}_{M,t+1} \approx \frac{1}{N} \sum_{i=1}^{N} {x_{t+1}}' \beta_{M,t}^{(i)}, \tag{1.19}$$

$$p_M(y_{t+1}|\mathcal{F}_t) \approx \frac{1}{N} \sum_{i=1}^{N} f_N(y_{t+1} | {x_{t+1}}' \beta_{M,t}^{(i)}, \sigma_{M,t}^{(i)}). \tag{1.20}$$

$f_N(x|\mu, \sigma^2)$ denotes the likelihood function of a normal distribution with mean μ and variance σ^2. The total sample size is 2560 ($T = 2560$). We set the initial sample size to be 60 (s=60) and the number of MCMC iteration to be 10,000 ($N = 10,000$). We also set 1,000 burn-in iterations for parameter convergence.

Clearly, a lower $FMSE_M$ and a higher LPL_M indicate better predictive performance of model M. To compare two models M_1 and M_2, we focus on the logarithmic predictive Bayes factor (LBF), which is defined as: $LPL_{M_1} - LPL_{M_2}$. A logarithmic predictive Bayes factor greater than 5 lends strong support for model M_1 over model M_2. In addition, we also compare two models by the percentage change of $FMSE$ ($FMSE\ P.C. = (FMSE_{M_1} - FMSE_{M_2})/FMSE_{M_2}$).

Tables 1.4 and 1.5 summarize the results of forecasting SPY daytime realized volatility without and with overnight information. The benchmark model is column 1 in Table 1.4. The last two rows of Tables 1.4 and 1.5 report the FMSE percentage change ($FMSE\ P.C.$) and logarithmic Bayes factor (LBF) of each model compared to the benchmark HAR-RV model. From Table 1.4, including the implied volatility (column 3 in Table 1.4) significantly improves the daytime realized volatility forecasting with a logarithmic Bayes factor of 152.07, which is strong evidence of forecasting improvement. Moreover, further including the lagged daytime return $r_{D,t-1}$ in addition to the implied volatility yields little improvement as column 4 of Table 1.4. Implied volatility, as a forward looking volatility measure, has already incorporated the leverage effect, so we observe little marginal improvement of $r_{D,t-1}$. Without overnight information, we have about 11% lower $FMSE$ compared to the benchmark model.

However, including the overnight realized volatility (overnight return) leads to a logarithmic Bayes factor of 221.08 (135.28), which is strongly significant. Including both overnight realized volatility and returns produces a logarithmic Bayes factor as high as 296.59. According to the $FMSE$, incorporating overnight realized volatility will decrease the $FMSE$ by 16%, which is greater than the best model without overnight information. Including both overnight realized volatility and return will will decrease the $FMSE$ by 21%. The results strongly support the value of overnight information in forecasting the daytime realized volatility. Conditional on the overnight realized volatility and return, implied volatility still has incremental information and our best model decreases the $FMSE$ by 26% compared to the benchmark HAR-RV model.

Similarly, Tables 1.6 and 1.7 summarize the results of forecasting SPY overnight realized volatility without and with daytime information. The benchmark model is column 1 in Table 1.6. As expected, the lagged daytime realized volatility yields the greatest marginal improvement (column 2 in Table 1.7 with a logarithmic Bayes factor of 310.56 and 22% decrease in $FMSE$), followed by the lagged daytime return (column 1 in Table 1.7 with a logarithmic Bayes factor of 155.89 and 12% percent decrease in $FMSE$). Again, we observe significant predictive power of implied volatility (open price of the implied volatility) on the overnight realized volatility, which also support the informativeness of implied volatility.

Regarding the parameters, overnight realized volatility $\log RV_{N,t}$ has a positive effect on the following daytime realized volatility $\log RV_{D,t}$ (coefficient is 0.4851), and lagged daytime realized volatility $\log RV_{D,t-1}$ has a positive effect on the following overnight realized volatility $\log RV_{N,t}$ (coefficient is 0.4660). The correlations indicate that the volatility process is continuous and higher overnight volatility is likely to be followed by higher daytime volatility and vice versa. Consistent with the well-documented leverage effect, overnight return (lagged daytime return) is negatively correlated with the following daytime (overnight) realized volatility.

All findings for the realized volatility of the S&P500 index in this section are consistent with those for the NASDAQ 100 index. Tables A1.2, A1.3, A1.4 and A1.5 in the Appendix Section A2 summarize the results of the HAR-RV models for the NASDAQ 100 index. Consistent results support the robustness of our conclusion. The strong correlation between the daytime and overnight realized volatility calls for jointly modeling the daytime and overnight return and realized volatility, which captures the correlation between the daytime and overnight volatility processes. We will propose a day-night stochastic volatility model with realized volatility in the next section.

	(1)	(2)	(3)	(4)
μ	-0.1014	-0.1010	-0.1780	-0.1799
	[0.022]	[0.021]	[0.026]	[0.027]
	(-0.144, -0.059)	(-0.143, -0.059)	(-0.230, -0.126)	(-0.231, -0.128)
$\log RV_{D,t-1}$	0.5694	0.5184	0.3073	0.3078
	[0.018]	[0.019]	[0.022]	[0.022]
	(0.533, 0.605)	(0.481, 0.556)	(0.264, 0.350)	(0.264, 0.351)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{D,t-i}$	0.2513	0.2917	0.2669	0.2670
	[0.024]	[0.024]	[0.034]	[0.034]
	(0.204, 0.297)	(0.245, 0.339)	(0.201, 0.332)	(0.200, 0.334)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{D,t-i}$	0.0984	0.1073	0.1641	0.1604
	[0.021]	[0.021]	[0.041]	[0.041]
	(0.057, 0.139)	(0.067, 0.148)	(0.084, 0.243)	(0.080, 0.240)
$r_{D,t-1}$	-	-0.1468	-	-0.0361
		[0.017]		[0.017]
		(-0.179, -0.114)		(-0.070, -0.003)
$\log IV_{t-1}$	-	-	1.2322	1.1648
			[0.068]	[0.077]
			(1.098, 1.364)	(1.012, 1.313)
$\frac{1}{4}\sum_{i=2}^{5}\log IV_{t-i}$	-	-	-0.7356	-0.6697
			[0.088]	[0.095]
			(-0.906, -0.565)	(-0.855, -0.481)
$\frac{1}{17}\sum_{i=6}^{22}\log IV_{t-i}$	-	-	-0.2225	-0.2182
			[0.071]	[0.071]
			(-0.363, -0.079)	(-0.357, -0.079)
FMSE	0.3278	0.3185	0.2908	0.2910
LPL	-2157.47	-2120.76	-2005.40	-2005.30
FMSE P.C.	-	-2.84%	-11.29%	-11.23%
LBF	-	36.71	152.07	152.17

The table reports the posterior mean, standard deviation in [] and 0.95 density interval in ().

Table 1.4: Forecasting S&P500 daytime realized volatility without overnight information.

	(1)	(2)	(3)	(4)	(5)
μ	-0.0846	-0.0304	-0.0249	-0.1467	-0.1473
	[0.020]	[0.026]	[0.025]	[0.040]	[0.040]
	(-0.125, -0.044)	(-0.081, 0.021)	(-0.074, 0.025)	(-0.225, -0.069)	(-0.226, -0.069)
$\log RV_{D,t-1}$	0.5791	0.3234	0.3644	0.2274	0.2285
	[0.018]	[0.020]	[0.020]	[0.022]	[0.021]
	(0.545, 0.613)	(0.285, 0.363)	(0.325, 0.403)	(0.185, 0.270)	(0.186, 0.271)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{D,t-i}$	0.2565	0.2179	0.2023	0.2474	0.2449
	[0.023]	[0.035]	[0.033]	[0.035]	[0.035]
	(0.212, 0.301)	(0.150, 0.285)	(0.137, 0.268)	(0.178, 0.317)	(0.178, 0.315)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{D,t-i}$	0.0890	0.2147	0.2026	0.2637	0.2638
	[0.020]	[0.050]	[0.048]	[0.046]	[0.047]
	(0.050, 0.129)	(0.115, 0.312)	(0.108, 0.297)	(0.173, 0.356)	(0.172, 0.357)
$\log RV_{N,t}$	-	0.4851	0.4109	0.2840	0.2821
		[0.022]	[0.022]	[0.023]	[0.023]
		(0.443, 0.527)	(0.369, 0.453)	(0.239, 0.328)	(0.236, 0.327)
$\frac{1}{4}\sum_{i=1}^{4}\log RV_{N,t-i}$	-	-0.1361	-0.0809	-0.0862	-0.0812
		[0.039]	[0.038]	[0.039]	[0.039]
		(-0.214, -0.059)	(-0.154, -0.007)	(-0.162, -0.012)	(-0.158, -0.003)
$\frac{1}{17}\sum_{i=5}^{21}\log RV_{N,t-i}$	-	-0.1786	-0.1710	-0.1764	-0.1797
		[0.050]	[0.048]	[0.055]	[0.055]
		(-0.275, -0.079)	(-0.265, -0.077)	(-0.285, -0.070)	(-0.288, -0.072)
$r_{N,t}$	-0.3195	-	-0.2300	-0.2675	-0.2678
	[0.019]		[0.018]	[0.018]	[0.018]
	(-0.356, -0.283)		(-0.266, -0.194)	(-0.302, -0.234)	(-0.302, -0.233)
$r_{D,t-1}$	-	-	-	-	-0.0229
					[0.016]
					(-0.054, 0.008)
$\log IV_{t-1}$	-	-	-	0.9672	0.9249
				[0.068]	[0.073]
				(0.833, 1.103)	(0.780, 1.069)
$\frac{1}{4}\sum_{i=2}^{5}\log IV_{t-i}$	-	-	-	-0.5603	-0.5218
				[0.084]	[0.088]
				(-0.726, -0.396)	(-0.693, -0.347)
$\frac{1}{17}\sum_{i=6}^{22}\log IV_{t-i}$	-	-	-	-0.1546	-0.1496
				[0.073]	[0.073]
				(-0.296, -0.012)	(-0.292, -0.006)
FMSE	0.2943	0.2750	0.2589	0.2402	0.2405
LPL	-2022.19	-1936.39	-1860.88	-1766.49	-1767.42
FMSE P.C.	-10.22%	-16.11%	-21.02%	-26.72%	-26.63%
LBF	135.28	221.08	296.59	390.98	390.05

The table reports the posterior mean, standard deviation in [] and 0.95 density interval in ().

Table 1.5: Forecasting S&P500 daytime realized volatility with overnight information.

	(1)	(2)	(3)	(4)
μ	-0.1101	-0.1056	-0.3213	-0.3193
	[0.024]	[0.024]	[0.042]	[0.043]
	(-0.157, -0.062)	(-0.152, -0.059)	(-0.404, -0.238)	(-0.402, -0.235)
$\log RV_{N,t-1}$	0.5063	0.4698	0.2688	0.2617
	[0.019]	[0.019]	[0.022]	[0.022]
	(0.469, 0.544)	(0.432, 0.508)	(0.226, 0.312)	(0.218, 0.305)
$\frac{1}{4}\sum_{i=2}^{5} \log RV_{N,t-i}$	0.3411	0.3824	0.2952	0.3040
	[0.025]	[0.025]	[0.035]	[0.035]
	(0.292, 0.389)	(0.333, 0.431)	(0.226, 0.363)	(0.235, 0.373)
$\frac{1}{17}\sum_{i=6}^{22} \log RV_{N,t-i}$	0.0843	0.0792	0.1001	0.0981
	[0.021]	[0.021]	[0.046]	[0.045]
	(0.043, 0.125)	(0.039, 0.120)	(0.011, 0.187)	(0.009, 0.186)
$r_{N,t-1}$	-	-0.1620	-	-0.1061
		[0.019]		[0.018]
		(-0.198, -0.125)		(-0.142, -0.071)
$\log IV_{t-1}^{O}$	-	-	1.0504	0.9611
			[0.064]	[0.066]
			(0.924, 1.175)	(0.830, 1.091)
$\frac{1}{4}\sum_{i=2}^{5} \log IV_{t-i}^{O}$	-	-	-0.4829	-0.3838
			[0.085]	[0.088]
			(-0.648, -0.318)	(-0.553, -0.209)
$\frac{1}{17}\sum_{i=6}^{22} \log IV_{t-i}^{O}$	-	-	-0.1545	-0.1665
			[0.075]	[0.075]
			(-0.302, -0.004)	(-0.312, -0.020)
FMSE	0.2895	0.2814	0.2602	0.2571
LPL	-2000.41	-1965.42	-1866.76	-1852.06
FMSE P.C.	-	-2.80%	-10.12%	-11.19%
LBF	-	34.99	133.65	148.35

The table reports the posterior mean, standard deviation in [] and 0.95 density interval in ().

TABLE 1.6: Forecasting S&P500 overnight realized volatility without daytime information.

	(1)	(2)	(3)	(4)	(5)
μ	-0.0951	-0.1140	-0.1075	-0.2613	-0.2597
	[0.023]	[0.024]	[0.023]	[0.038]	[0.038]
	(-0.139, -0.051)	(-0.160, -0.068)	(-0.153, -0.062)	(-0.336, -0.187)	(-0.334, -0.186)
$\log RV_{N,t-1}$	0.5064	0.1973	0.2401	0.1543	0.1529
	[0.018]	[0.020]	[0.020]	[0.022]	[0.022]
	(0.471, 0.541)	(0.158, 0.237)	(0.200, 0.280)	(0.111, 0.198)	(0.110, 0.195)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{N,t-i}$	0.3426	0.2778	0.2661	0.2785	0.2753
	[0.023]	[0.033]	[0.032]	[0.035]	[0.035]
	(0.297, 0.389)	(0.214, 0.341)	(0.204, 0.328)	(0.209, 0.349)	(0.208, 0.345)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{N,t-i}$	0.0893	0.2658	0.2534	0.1753	0.1814
	[0.020]	[0.042]	[0.041]	[0.051]	[0.051]
	(0.050, 0.128)	(0.183, 0.348)	(0.174, 0.333)	(0.075, 0.275)	(0.080, 0.281)
$\log RV_{D,t-1}$	-	0.4569	0.3847	0.3142	0.3023
		[0.017]	[0.018]	[0.019]	[0.019]
		(0.422, 0.491)	(0.349, 0.420)	(0.277, 0.351)	(0.264, 0.340)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{D,t-i}$	-	-0.0167	0.0147	-0.0309	-0.0128
		[0.032]	[0.031]	[0.032]	[0.032]
		(-0.081, 0.045)	(-0.045, 0.075)	(-0.094, 0.029)	(-0.076, 0.051)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{D,t-i}$	-	-0.1971	-0.1775	-0.1198	-0.1280
		[0.043]	[0.042]	[0.042]	[0.042]
		(-0.281, -0.112)	(-0.258, -0.095)	(-0.202, -0.040)	(-0.208, -0.045)
$r_{D,t-1}$	-0.2606	-	-0.1572	-0.1707	-0.1733
	[0.014]		[0.014]	[0.014]	[0.014]
	(-0.288, -0.233)		(-0.185, -0.130)	(-0.197, -0.145)	(-0.200, -0.146)
$r_{N,t-1}$	-	-	-	-	-0.0717
					[0.017]
					(-0.105, -0.039)
$\log IV^{O}_{t-1}$	-	-	-	0.6273	0.5746
				[0.063]	[0.063]
				(0.503, 0.754)	(0.452, 0.701)
$\frac{1}{4}\sum_{i=2}^{5}\log IV^{O}_{t-i}$	-	-	-	-0.2729	-0.2176
				[0.077]	[0.079]
				(-0.426, -0.122)	(-0.371, -0.060)
$\frac{1}{17}\sum_{i=6}^{22}\log IV^{O}_{t-i}$	-	-	-	-0.0490	-0.0525
				[0.068]	[0.068]
				(-0.182, 0.084)	(-0.184, 0.082)
FMSE	0.2556	0.2257	0.2150	0.2066	0.2053
LPL	-1844.52	-1689.85	-1629.67	-1580.43	-1573.28
FMSE P.C.	-11.71%	-22.04%	-25.73%	-28.64%	-29.08%
LBF	155.89	310.56	370.74	419.98	427.13

The table reports the posterior mean, standard deviation in [] and 0.95 density interval in ().

Table 1.7: Forecasting S&P500 overnight realized volatility with daytime information.

1.4 A New Model Estimating the Daytime and Overnight Return and RV Jointly

1.4.1 Model Specification

The correlation coefficients between daytime and overnight returns, from Table 1.1, are close to zero for both SPY and QQQ, which suggests little linear correlation. Accordingly, the benchmark joint model for daytime and overnight return series treats them independently and fits two SV models, as follows:

$$r_{N,t} = \mu_N + \exp(h_{N,t}/2)u_{N,t}, \qquad u_{N,t} \sim N(0,1), \tag{1.21}$$

$$h_{N,t} = \alpha_N + \delta_N h_{N,t} + \sigma_{hN} v_{N,t}, \qquad v_{N,t} \sim N(0,1), \tag{1.22}$$

$$r_{D,t} = \mu_D + \exp(h_{D,t}/2)u_{D,t}, \qquad u_{D,t} \sim N(0,1), \tag{1.23}$$

$$h_{D,t} = \alpha_D + \delta_D h_{D,t-1} + \sigma_{hD}, v_{D,t} \qquad v_{D,t} \sim N(0,1). \tag{1.24}$$

Inasmuch as we have daytime and overnight realized volatility, we can extend the stochastic volatility model by jointly estimating the return and realized volatility as following SV-RV models (see Takahashi et al. (2009)):

$$r_{N,t} = \mu_N + \exp(h_{N,t}/2)u_{N,t}, \qquad u_{N,t} \sim N(0,1), \tag{1.25}$$

$$\log RV_{N,t} = \xi_N + h_{N,t} + \sigma_{RVN} e_{N,t}, \qquad e_{N,t} \sim N(0,1), \tag{1.26}$$

$$h_{N,t} = \alpha_N + \delta_N h_{N,t} + \sigma_{hN} v_{N,t}, \qquad v_{N,t} \sim N(0,1), \tag{1.27}$$

$$r_{D,t} = \mu_D + \exp(h_{D,t}/2)u_{D,t}, \qquad u_{D,t} \sim N(0,1), \tag{1.28}$$

$$\log RV_{D,t} = \xi_D + h_{D,t} + \sigma_{RVD} e_{D,t}, \qquad e_{D,t} \sim N(0,1), \tag{1.29}$$

$$h_{D,t} = \alpha_D + \delta_D h_{D,t-1} + \sigma_{hD} v_{D,t}, \qquad v_{D,t} \sim N(0,1). \tag{1.30}$$

The constant term ξ (in Eq. (1.26) and (1.29)) scales the realized volatility to match the return volatility. A negative ξ (or $\exp(\xi) < 1$) indicates that the realized volatility underestimates the return volatility. Takahashi et al. (2009) fit the SV-RV model with daily close to close returns for the Tokyo stock price index (TOPIX) and its corresponding daytime realized volatility (to be specific, trading hour realized volatility since Tokyo Stock Exchange has a lunch break). The posterior mean of ξ is -1.0707, which implies $RV_t = 0.34 \times \exp(h_t)$. Takahashi et al. (2009) points out that the overnight (or non-trading hour) effect is much more significant than the microstructure noise in estimating realized volatility. Takahashi et al. (2009) also uses the scaling method by Hansen and Lunde (2005) to scale the daytime realized volatility to match the daily return volatility. However, either scaling the daytime realized volatility before fitting the models or adding constant term ξ in the model assumes a perfect correlation between the daytime realized volatility and overnight volatility. As we have shown that the daytime and overnight realized volatility are not perfectly correlated, scaling the daytime realized volatility could be inferior to estimating the overnight realized volatility directly.

Our SV-RV specification (Eq. (1.25) to (1.30)) treats daytime and overnight return separately, and there is no mismatch between return and corresponding realized volatility. As we will show in the following empirical results, ξ_N and ξ_D are close to zero, which supports the accuracy of our daytime and overnight realized volatility estimation.

On the other hand, both SV-RV and SV specifications treat the daytime and overnight volatility processes independently. Although the overnight and daytime returns have little linear correlation, the results in the previous section suggest a strong correlation between the daytime and overnight volatility ($h_{D,t}$ and $h_{N,t}$). To address this correlation, we propose the DN-SV-RV (day-night SV-RV) specification in Eq. (1.31) to (1.36).

$$r_{N,t} = \mu_N + \exp(h_{N,t}/2)u_{N,t}, \qquad u_{N,t} \sim N(0,1), \tag{1.31}$$

$$\log RV_{N,t} = \xi_N + h_{N,t} + \sigma_{RVN}e_{N,t}, \qquad e_{N,t} \sim N(0,1), \tag{1.32}$$

$$h_{N,t} = \alpha_N + \beta_N h_{D,t-1} + \delta_N h_{N,t-1} + \sigma_{hN}v_{N,t}, \qquad v_{N,t} \sim N(0,1), \tag{1.33}$$

$$r_{D,t} = \mu_D + \exp(h_{D,t}/2)u_{D,t}, \qquad u_{D,t} \sim N(0,1), \tag{1.34}$$

$$\log RV_{D,t} = \xi_D + h_{D,t} + \sigma_{RVD}e_{D,t}, \qquad e_{D,t} \sim N(0,1), \tag{1.35}$$

$$h_{D,t} = \alpha_D + \beta_D h_{N,t} + \delta_D h_{D,t-1} + \sigma_{hD}v_{D,t}, \qquad v_{D,t} \sim N(0,1). \tag{1.36}$$

The major difference between this specification and the traditional SV or SV-RV approach is that the daytime latent volatility $h_{D,t}$ not only depends on its own lagged term $h_{D,t-1}$, but also on the overnight latent volatility $h_{N,t}$, and this correlation is captured by the parameter β_D in Eq. (1.36). Similarly, β_N in Eq. (1.33) captures the correlation between the lagged daytime latent volatility $h_{D,t-1}$ and the following overnight latent volatility $h_{N,t}$. Moreover, the simple SV or SV-RV model will be a special case of the DV-SV-RV model since the DN-SV-RV model will degenerate into an SV-RV model if $\beta_D = \beta_N = 0$. If there is no correlation between the daytime and overnight volatility, jointly modelling the daytime and overnight returns (and realized volatility) would be equivalent to modelling them independently.

To fit the DN-SV-RV model, we need to consider the daytime and overnight return and realized volatility jointly. It is not necessary that the daytime return/realized volatility series has the same length as that of the overnight return/realized volatility series. If both daytime and overnight series have t observations, we fit the model at the market close of day t, and we use all the past daytime and overnight information to predict the following overnight return and volatility. If the daytime series has length t while the overnight series has length $t + 1$, we fit the model at the market open of day $t + 1$ to predict the following daytime return and volatility. This is important for decomposing the likelihood of daytime and overnight observations and for out-of-sample prediction.

The likelihood of return and realized volatility $\{r_{D,t}, r_{N,t}, \log RV_{D,t}, \log RV_{N,t}\}_{t=1}^{T}$ of model M, conditional on parameter set Θ, is:

$$\begin{aligned}
&p_M\left(\{r_{D,t}, r_{N,t}, \log RV_{D,t}, \log RV_{N,t}\}_{t=1}^{T}|\Theta\right) \\
=&\prod_{t=1}^{T} p_M\left(r_{D,t}, r_{N,t}, \log RV_{D,t}, \log RV_{N,t} \mid r_{D,:t-1}, r_{N,:t-1}, \log RV_{D,:t-1}, \log RV_{N,:t-1}, \Theta\right).
\end{aligned} \tag{1.37}$$

Here $y_{:t}$ denotes $\{y_1, y_2, ...y_t\}$. For SV-RV, since the daytime and overnight volatility are modeled independently, the likelihood will be:

$$\begin{aligned}
&p_M\left(\{r_{D,t}, r_{N,t}, \log RV_{D,t}, \log RV_{N,t}\}_{t=1}^{T}|\Theta\right) \\
=&\prod_{t=1}^{T} p_M\left(r_{N,t}, \log RV_{N,t}|r_{N,:t-1}, \log RV_{N,:t-1}, \Theta\right) \\
&\quad\prod_{t=1}^{T} p_M\left(r_{D,t}, \log RV_{D,t}|r_{D,:t-1}, \log RV_{D,:t-1}, \Theta\right).
\end{aligned} \tag{1.38}$$

However, for the DN-SV-RV model, the likelihood will be:

$$\begin{aligned}
&p_M\left(\{r_{D,t}, r_{N,t}, \log RV_{D,t}, \log RV_{N,t}\}_{t=1}^{T}|\Theta\right) \\
=&\prod_{t=1}^{T} p_M\left(r_{N,t}, \log RV_{N,t}|r_{N,:t-1}, r_{D,:t-1}, \log RV_{N,:t-1}, \log RV_{D,:t-1}, \Theta\right) \\
&\quad\prod_{t=1}^{T} p_M\left(r_{D,t}, \log RV_{D,t}|r_{D,:t-1}, r_{N,:t}, \log RV_{D,:t-1}, \log RV_{N,:t}, \Theta\right).
\end{aligned} \tag{1.39}$$

We can see the difference between DN-SV-RV and other models in information updating. For example, at every market open, new overnight return and realized volatility observations will be available. The SV and SV-RV only update the overnight part of the model (Eq. (1.21) to (1.22) or (1.25) to (1.27)) and leave the daytime part unchanged. This represents modeling daytime series alone without overnight information, and the prediction of the daytime return and volatility is not affected even though the new overnight return and realized volatility observations are available. The DN-SV-RV model updates the joint model at every market open so that the prediction of the following daytime return and volatility will be affected by the newly available overnight information. To update information at the market close, when the new daytime return and realized volatility observations are available, the DN-SV-RV model update the joint model similarly.

We use Bayesian MCMC to estimate the DN-SV-RV model (details about the posterior distributions of latent volatility $h_{D,t}$ and $h_{N,t}$ are covered in Appendix Section A3). We show the MCMC iteration steps and parameter prior distributions for the DN-SV-RV model here. The SV-RV/SV model can be treated as a nested version of the DN-SV-RV model, and the parameters have the same prior distributions as the DN-SV-RV model:

- $\mu_N, \mu_D \mid h_N, r_N, h_D, r_D$: Conjugate normal prior $N(0, 5)$.
- $h_D \mid h_N, r_D, \log RV_D$: Metropolis-Hasting with tailored proposal distribution.
- $h_N \mid h_D, r_N, \log RV_N$: Metropolis-Hasting with tailored proposal distribution.
- $\alpha_N, \beta_N, \delta_N, \alpha_D, \beta_D, \delta_D \mid h_N, h_D$: Conjugate normal prior $N(0, 5)$ for α_N and α_D and truncated normal prior $N(0.9, 5)I_{|\delta|<1}$ for β_N, δ_N, β_D and δ_D.
- $\xi_N, \xi_D, \mid h_N, h_D, \log RV_D, \log RV_N$: Conjugate normal prior $N(0, 5)$.

- $\sigma_h N, \sigma_h d, \sigma_R VN \mid h_N, h_D, \log RV_D, \log RV_N$: Conjugate inverse gamma prior $IG(\frac{3}{2}, \frac{0.5}{2})$.

We set flat prior distributions for all parameters to avoid subjective bias. Also, we set the total number of MCMC iterations to be 100,000, and we keep 1 of every 10 draws. We also set 5,000 burn-in iterations for parameter convergence.

To compare the out-of-sample performance, we focus on forecasting the daytime and overnight return distributions. We use the following logarithmic predictive likelihood of model M:

$$LPL_M = \sum_{t=s}^{T-1} (\log p_M(r_{N,t+1}, r_{D,t+1} | r_{D,:t}, r_{N,:t}, \log RV_{D,:t}, \log RV_{N,:t})) \tag{1.40}$$

Given our sample size $T = 2560$, we use the initial $s = 560$ observation to initially fit the models. Moreover, we can further decompose the predictive likelihood in Eq. (1.40) into daytime and overnight parts, as follows:

$$\begin{aligned} & p_M(r_{N,t+1}, r_{D,t+1} | r_{D,:t}, r_{N,:t}, \log RV_{D,:t}, \log RV_{N,:t}) \\ = & \, p_M(r_{N,t+1} | r_{D,:t}, r_{N,:t}, \log RV_{D,:t}, \log RV_{N,:t}) \\ & \quad p_M(r_{D,t+1} | r_{D,:t}, r_{N,:t+1}, \log RV_{D,:t}, \log RV_{N,:t+1}) \end{aligned} \tag{1.41}$$

Then the logarithmic predictive likelihood in Eq. (1.40) can be decomposed into overnight and daytime parts, as follows:

$$LPL_{N,M} = \sum_{t=s}^{T-1} \left(\log p_M(r_{N,t+1}|r_{D,:t}, r_{N,:t}, \log RV_{D,:t}, \log RV_{N,:t})\right) \tag{1.42}$$

$$LPL_{D,M} = \sum_{t=s}^{T-1} \left(\log p_M(r_{D,t+1}|r_{D,:t}, r_{N,:t+1}, \log RV_{D,:t}, \log RV_{N,:t+1})\right) \tag{1.43}$$

We have $LPL_M = LPL_{N,M} + LPL_{D,M}$ given conditional independence.

For the SV-RV model, the predictive likelihood is estimated as:

$$\begin{aligned} &p_M(r_{N,t+1}|r_{D,:t}, r_{N,:t}, \log RV_{D,:t}, \log RV_{N,:t}) \\ &= p_M(r_{N,t+1}|r_{N,:t}, \log RV_{N,:t}) \\ &\approx \frac{1}{N}\sum_{i=1}^{N} f_N\left(r_{N,t+1}|\mu_N^{(i)},\ \exp(\alpha_N^{(i)} + \delta_N^{(i)} h_{N,t}^{(i)} + \sigma_{hN}^{(i)} v_{N,t}^{(i)})\right) \end{aligned} \tag{1.44}$$

and

$$\begin{aligned} &p_M(r_{D,t+1}|r_{D,:t}, r_{N,:t+1}, \log RV_{D,:t}, \log RV_{N,:t+1}) \\ &= p_M(r_{D,t+1}|r_{D,:t}, \log RV_{D,:t}) \\ &\approx \frac{1}{N}\sum_{i=1}^{N} f_N\left(r_{D,t+1}|\mu_D^{(i)},\ \exp(\alpha_D^{(i)} + \delta_D^{(i)} h_{D,t}^{(i)} + \sigma_{hD}^{(i)} v_{D,t}^{(i)}\right) \end{aligned} \tag{1.45}$$

Here $\mu_N^{(i)}$, $\alpha_N^{(i)}$, $\delta_N^{(i)}$, $h_{N,t}^{(i)}$ and $\sigma_{hN}^{(i)}$ are MCMC draws by fitting the overnight part of the SV-RV model with overnight returns $\{r_{N,i}\}_{i=1}^t$ and overnight realized volatility $\{\log RV_{N,i}\}_{i=1}^t$ jointly. Similarly, $\mu_N^{(i)}$, $\alpha_D^{(i)}$, $\delta_D^{(i)}$, $h_{D,t}^{(i)}$ and $\sigma_{hD}^{(i)}$ are MCMC draws by fitting the daytime part of the SV-RV model with daytime return $\{r_{D,i}\}_{i=1}^t$ and daytime realized volatility $\{\log RV_{D,i}\}_{i=1}^t$. $v_{N,t}^{(i)}$ and $v_{D,t}^{(i)}$ are iid draws from a standard normal distribution. For the SV model, it is the same as Eq. 1.44 and 1.45 to evaluate the logarithmic predictive likelihood. However, we fit the SV model without realized volatility.

For the DN-SV-RV model, the predictive likelihood is esimated as:

$$
\begin{aligned}
& p_M(r_{N,t+1}|r_{D,:t}, r_{N,:t}, \log RV_{D,:t}, \log RV_{N,:t}) \\
\approx\ & \frac{1}{N}\sum_{i=1}^{N} f_N\left(r_{N,t+1}|\mu_N^{(i)},\ \exp(\alpha_N^{(i)} + \beta_N^{(i)} h_{D,t}^{(i)} + \delta_N^{(i)} h_{N,t}^{(i)} + \sigma_{hN}^{(i)} v_N^{(i)})\right) \qquad (1.46) \\
& p_M(r_{D,t+1}|r_{D,:t}, r_{N,:t+1}, \log RV_{D,:t}, \log RV_{N,:t+1}) \\
\approx\ & \frac{1}{N}\sum_{i=1}^{N} f_N\left(r_{D,t+1}|\mu_D^{(i)},\ \exp(\alpha_D^{(i)} + \beta_D^{(i)} h_{N,t+1}^{(i)} + \delta_D^{(i)} h_{D,t}^{(i)} + \sigma_{hD}^{(i)} v_D^{(i)})\right) \qquad (1.47)
\end{aligned}
$$

$\mu_N^{(i)}$, $\alpha_N^{(i)}$, $\beta_N^{(i)}$, $\delta_N^{(i)}$, $h_{D,t}^{(i)}$, $h_{N,t}^{(i)}$ and $\sigma_{hN}^{(i)}$ are MCMC draws from fitting the DN-SV-RV with $\{r_{N,i}\}_{i=1}^t$, $\{r_{D,i}\}_{i=1}^t$, $\{\log RV_{N,i}\}_{i=1}^t$ and $\{\log RV_{D,i}\}_{i=1}^t$. This step is to predict $r_{N,t+1}$ given the information available at market close on day t. On the next day, when the market is open, $r_{N,t+1}$ and $\log RV_{N,t+1}$ are available. Then we fit the DN-SV-RV model with $\{r_{N,i}\}_{i=1}^{t+1}$, $\{r_{D,i}\}_{i=1}^t$, $\{\log RV_{N,i}\}_{i=1}^{t+1}$ and $\{\log RV_{D,i}\}_{i=1}^t$ and draw $\mu_D^{(i)}$, $\alpha_D^{(i)}$, $\beta_D^{(i)}$, $\delta_D^{(i)}$, $h_{D,t}^{(i)}$, $h_{N,t+1}^{(i)}$ and $\sigma_{hD}^{(i)}$ accordingly.

Table 1.8 summarizes the estimation results for SV (Eq (1.21) to (1.24)), SV-RV (Eq (1.25) to (1.30)) and DN-SV-RV (Eq (1.31) to (1.36)) for SPY and QQQ. As we have stated before, the key difference between DN-SV-RV and the traditional SV-RV or SV is β_D and β_N. The estimation results support our expectation that the correlation between daytime and overnight volatility is strong. Taking the SPY result as an example, $\beta_D = 0.5885$, which indicates a strong correlation between overnight volatility and following daytime volatility. Similarly,

$\beta_N = 0.3901$ indicates a strong correlation between daytime volatility and following overnight volatility. Moreover, if the daytime or overnight return series is modelling with SV or SV-RV independently, we find robust volatility clustering as δ_D and δ_N are around 0.9. However, as we have noted, δ_D only captures the correlation between today's daytime volatility and yesterday's daytime volatility, and there is an overnight gap between them. Incorporating the overnight volatility into the daytime volatility equation (incorporating the daytime volatility into the overnight volatility equation) will significantly decrease the auto-correlation coefficient $\delta_D = 0.3244$ ($\delta_N = 0.5439$).

After controlling the overnight (daytime) volatility, the auto-correlation of the daytime (overnight) volatility is much weaker. However, there is no contradiction to the well-documented volatility clustering. The daytime and overnight volatility, like the daytime and overnight returns, are the converse of each other. Although the overnight volatility has a different long-run mean and variance from that of the daytime volatility (overnight volatility is smaller), they are driven by the same risk factor, and the volatility clustering should be persistent throughout the day and night. When predicting the daytime volatility, the most recent information is not the lagged daytime volatility, but the overnight volatility and vice versa. Previous studies that focus on daytime volatility alone use only a subset of the volatility information. The overnight realized volatility and the DV-SV-RV complete those overnight gaps.

	SPY			QQQ		
	SV	SV-RV	DN-SV-RV	SV	SV-RV	DN-SV-RV
μ_D	0.0604	0.0775	0.0882	0.0781	0.1086	0.1236
	[0.009]	[0.009]	[0.008]	[0.012]	[0.012]	[0.011]
	(0.043, 0.078)	(0.061, 0.095)	(0.071, 0.105)	(0.054, 0.101)	(0.086, 0.131)	(0.102, 0.145)
α_D	-0.0795	-0.1504	0.1098	-0.0565	-0.1155	0.1661
	[0.017]	[0.019]	[0.036]	[0.013]	[0.015]	[0.033]
	(-0.115, -0.049)	(-0.188, -0.115)	(0.040, 0.182)	(-0.084, -0.034)	(-0.146, -0.086)	(0.102, 0.233)
β_D	-	-	0.5885	-	-	0.4832
			[0.035]			[0.033]
			(0.520, 0.659)			(0.417, 0.547)
δ_D	0.9386	0.8889	0.3244	0.9294	0.8642	0.3779
	[0.011]	[0.012]	[0.033]	[0.013]	[0.014]	[0.033]
	(0.914, 0.959)	(0.865, 0.912)	(0.260, 0.390)	(0.903, 0.952)	(0.836, 0.891)	(0.314, 0.445)
σ_D	0.3586	0.4265	0.4833	0.3560	0.4259	0.4815
	[0.031]	[0.017]	[0.010]	[0.032]	[0.018]	[0.012]
	(0.305, 0.423)	(0.394, 0.461)	(0.463, 0.503)	(0.297, 0.420)	(0.391, 0.461)	(0.457, 0.503)
μ_N	0.0443	0.0497	0.0524	0.0657	0.0720	0.0762
	[0.007]	[0.007]	[0.007]	[0.009]	[0.008]	[0.008]
	(0.030, 0.059)	(0.036, 0.064)	(0.039, 0.066)	(0.048, 0.083)	(0.055, 0.088)	(0.059, 0.092)
α_N	-0.0822	-0.1162	-0.2673	-0.1039	-0.1206	-0.3300
	[0.019]	[0.017]	[0.029]	[0.024]	[0.017]	[0.032]
	(-0.122, -0.049)	(-0.150, -0.085)	(-0.325, -0.211)	(-0.154, -0.063)	(-0.155, -0.089)	(-0.396, -0.270)
β_N	-	-	0.3901	-	-	0.4172
			[0.029]			[0.033]
			(0.336, 0.449)			(0.356, 0.486)
δ_N	0.9515	0.9332	0.5439	0.9252	0.9156	0.5250
	[0.010]	[0.009]	[0.032]	[0.016]	[0.010]	[0.034]
	(0.930, 0.970)	(0.916, 0.950)	(0.481, 0.604)	(0.892, 0.953)	(0.894, 0.935)	(0.456, 0.589)
σ_N	0.3042	0.3261	0.3803	0.3519	0.3501	0.4133
	[0.030]	[0.014]	[0.016]	[0.038]	[0.015]	[0.017]
	(0.248, 0.366)	(0.300, 0.354)	(0.350, 0.412)	(0.283, 0.430)	(0.321, 0.381)	(0.380, 0.446)
ξ_D	-	0.1276	0.1372	-	0.0466	0.0536
		[0.030]	[0.028]		[0.030]	[0.027]
		(0.069, 0.187)	(0.081, 0.192)		(-0.012, 0.106)	(-0.000, 0.108)
σ_{RVD}	-	0.3101	0.1682	-	0.3061	0.1829
		[0.015]	[0.018]		[0.016]	[0.021]
		(0.280, 0.338)	(0.136, 0.205)		(0.275, 0.336)	(0.144, 0.227)
ξ_N	-	0.1497	0.1672	-	-0.0102	0.0120
		[0.030]	[0.028]		[0.030]	[0.029]
		(0.090, 0.208)	(0.112, 0.223)		(-0.069, 0.047)	(-0.045, 0.069)
σ_{RVN}	-	0.3485	0.2768	-	0.3561	0.2706
		[0.010]	[0.016]		[0.011]	[0.019]
		(0.328, 0.368)	(0.243, 0.306)		(0.334, 0.377)	(0.231, 0.306)

The table reports the posterior mean, standard deviation in [] and 0.95 density interval in ().

TABLE 1.8: The estimation results for the SV, SV-RV and DN-SV-RV models.

Moreover, from Table 1.8, the volatility-adjusted mean of overnight returns is lower than that of daytime returns. Back to Table 1.1, the simple mean of overnight returns (0.0317% for SPY and 0.0489% for QQQ) is significantly higher than that of the daytime returns (0.0203% for SPY and 0.0192% for QQQ). However, the mean parameters in the DN-SV-RV model indicate that the overnight return mean ($\mu_N = 0.0524\%$ for SPY and $\mu_N = 0.0762\%$ for QQQ) is not necessarily higher the than daytime return mean ($\mu_D = 0.0882\%$ for SPY and $\mu_D = 0.1236\%$ for QQQ). The stochastic volatility could be a potential explanation of the observed high overnight returns.

Table 1.9 further compares the out-of-sample forecasting of the daytime and overnight return distributions, and we find strong supportive evidence for the overnight realized volatility and our DN-SV-RV model. Take SPY as an example, comparing SV-RV and SV. We see the realized volatility, especially the overnight realized volatility estimated from the high-frequency futures prices, can significantly improve the forecasts of the return distribution. The daytime (overnight) logarithmic predictive Bayes factor of the SV-RV model relative to the SV model is 26.37 (54.99). The logarithmic predictive Bayes factor strongly supports the value of our realized volatility estimated from the high-frequency futures prices. The realized volatility, as a consistent volatility measure, improves the future return distribution forecasting. The improvement shows that our overnight realized volatility not only enhances the daytime realized volatility forecasting in the HAR-RV model, which models the realized volatility process itself, but also matches the corresponding overnight returns. The improvement of the SV-RV model compared to the simple SV model confirms that the overnight realized volatility provides volatility information that helps the SV-RV model identify the real volatility process of the overnight returns and achieve better return prediction.

Moreover, the DN-SV-RV model further exploits the correlation between daytime and overnight volatility process. The daytime (overnight) logarithmic Bayes factor of the DN-SV-RV model over the SV-RV model is 17.05 (38.95) for SPY. From SV-RV to DN-SV-RV, we do not introduce extra data and the SV-RV is a nested version of DN-SV-RV. As discussed earlier, the DN-SV-RV model leads to significantly weaker auto-correlation in the daytime and overnight volatility processes, but strong inter-correlation. Significant forecasting improvements can be achieved by modeling the correlation between daytime and overnight volatility. Again, the consistent results of QQQ support the robustness of our findings, as shown in Table 1.9.

The out-of-sample results confirm that it is necessary to incorporate the overnight (daytime) information for predicting the following daytime (overnight) volatility. The results here are consistent with Section 1.3's findings that overnight (daytime) realized volatility can significantly improve the following daytime (overnight) realized volatility.

	SPY			**QQQ**		
	SV	SV-RV	DN-SV-RV	SV	SV-RV	DN-SV-RV
LPL_D	-1592.67	-1566.30	**-1549.25**	-2125.77	-2105.92	**-2077.65**
LPL_N	-1114.66	-1059.67	**-1020.72**	-1501.02	-1460.56	**-1422.83**
$LPL_D + LPL_N$	-2707.33	-2625.97	**-2569.97**	-3626.79	-3566.48	**-3500.48**

Table 1.9: Out-of-sample camparison of the SV, SV-RV and DN-SV-RV models

1.5 Conclusion

The stock market never sleeps; in fact, "it loves the night". We document that after-trading-hour return and volatility are significant for the U.S. equity market as the overnight volatility comprises about 40% of the daily close-to-close volatility. It is sub-optimal to overlook the overnight volatility.

In this paper, we select equity index future prices to capture the daytime and overnight volatility, as the futures market trades for almost the full day. Using futures high frequency data, we estimate both the daytime and overnight realized volatility. We demonstrate that overnight (daytime) realized volatility will significantly improve the forecasting of the following daytime (overnight) realized volatility. The inter-correlation between daytime and overnight realized volatility is significant and informative for realized volatility prediction. Inspired by the relationship between daytime and overnight volatility, we propose the stochastic volatility model by jointly estimating overnight and daytime returns and their realized volatility, as in the DN-SV-RV model. This model significantly improves both daytime and overnight return density forecasting. The out-of-sample improvements strongly support the value of the overnight realized volatility estimated from the equity futures market. Also, the estimation outcome provides evidence of the non-negligible importance of cross-correlation between daytime and overnight volatility. The evidence suggests that daytime and overnight volatility complement one another. We conclude that volatility clustering is persistent throughout the day and night.

Chapter 2

Estimating the Stochastic Volatility Model with Realized Volatility and Implied Volatility

2.1 Introduction

Heteroscedasticity (Engle 1982; Bollerslev 1986) is a generally accepted feature of stock return series (Nelson 1991; Schwert and Seguin 1990; Stein and Stein 1991). Empirical stochastic volatility (SV) models have been proposed (Hull and White 1987; Heston 1993; Duan 1995) and widely studied to model this heteroscedasticity. The traditional SV models treat the volatility process as an unobserved process. Bayesian estimation of the SV model (Harvey and Shephard 1996; Kim et al. 1998; Jacquier et al. 1994; Jacquier et al. 2004), samples the latent volatility process conditionally on return series alone.

Realized volatility, as shown by Andersen and Bollerslev (1998) and Barndorff-Nielsen and Shephard (2001), is a consistent estimator of the volatility process. Realized volatility, as an *ex-post* measure of volatility, can then be used to improve the estimation of the unobserved volatility. Takahashi et al. (2009) propose a realized stochastic volatility model, whose specification estimates the return and realized volatility simultaneously. Following the work of Shirota et al. (2014), Koopman and Scharth (2013) and Asai et al. (2017) further extend this model by allowing volatility asymmetry and long-term memory, obtaining improvements in volatility forecasts. The realized stochastic model is a non-linear common factor model (Yalcin and Amemiya 2001) in which the return and logarithmic realized volatility series share the latent volatility as a common factor. Like the well-known stochastic volatility model, the return is non-linearly correlated with the latent volatility, while the logarithmic realized volatility is linearly correlated with the latent volatility.

On the other hand, option-implied volatility can be treated as an *ex-ante* measure of the volatility process. Early work (Blair et al. 2001) reveals that implied volatility is more informative and provides more accurate out-of-sample volatility forecasts. The survey paper by Poon and Granger (2003) compares different volatility forecasting methods and concludes that option-implied volatility is the best predictor of return volatility. Christensen and Prabhala (1998) show that implied volatility improves volatility forecasts compared to using past realized volatility. However, implied volatility is an expectation of the real volatility assuming risk-neutrality (see Hao and Zhang 2013 and the CBOE white paper[1]). The risk premium may bias the implied volatility as a volatility measure.

[1] https://www.cboe.com/micro/vix/vixwhite.pdf

The purpose of this paper is to extend the work of Takahashi et al. (2009) by including both realized volatility and implied volatility. We estimate respectively the simple SV model, the SV model with realized volatility (RVSV), the SV model with implied volatility (IVSV), and, finally, the SV model with realized volatility and implied volatility together (RVIVSV). Previous literature that extends heteroscedasticity models with RV or IV specifications typically selects lagged RV terms or IV as explanatory variables in the volatility equation (Koopman et al. 2005; Blair et al. 2001). This paper jointly estimates the return and volatility measures assuming that they share the latent volatility as a common factor. The inclusion of volatility measures should improve the identification of the latent volatility and the one-day ahead return densities forecasting compared to the simple stochastic volatility model.

Realized volatility, as a volatility measure based on high-frequency data, suffers from micro-structure noises. Implied volatility, although relatively stable, has risk premium bias as a measure of the true volatility. Compared with the simple SV model without volatility measures, the IVSV model leads to a smooth latent volatility process while RVSV leads to a rough latent volatility process. Including both the implied volatility and realized volatility as the RVIVSV model, the latent volatility process is almost identical to that of the IVSV model. The in-sample estimation results confirm that volatility measures substantially affect the estimation of the latent volatility. Since volatility measures are linearly related to the latent volatility while return is non-linearly related, volatility measures provide strong information when identifying the latent volatility. This suggests that the accuracy of the volatility measures could significantly affect the forecasting performance of the model.

To examine the out-of-sample forecasting, we focus on the predictive density of the one-day forward equity returns. We compare the proposed SV models with the benchmark SV model using predictive likelihood (Amisano and Giacomini 2007; Bao et al. 2007). The empirical results indicate that incorporating either the implied volatility or realized volatility into the stochastic volatility specification will significantly improve the out-of-sample return density forecasts. For the S&P500 index and some individual stocks, realized volatility out-performs implied volatility. On the other hand, implied volatility dominates realized volatility for two banking stocks whose return processes are highly volatile during our sample period. Finally, when we jointly estimate the return, realized volatility and implied volatility, the predictive performance is not significantly better than RVSV/IVSV and may be worse for specific stocks.

The remainder of this paper is organized as follows. Section 2.2 introduces the model specification with implied volatility for in-sample estimation of out-of-sample prediction. We specify the data used in the study and present estimation results for both the equity index and individual stocks in Section 2.3. Section 2.4 provides the predictive density of the proposed SV model with a comparison to the benchmark SV model. Section 2.5 concludes the paper. We illustrate the derivation of posterior distributions and more empirical results in the Appendix.

2.2 Model

2.2.1 Model Specifications

The original SV model is specified as:

$$
\begin{aligned}
y_t &= \mu + \exp(h_t/2)u_t, & u_t &\sim N(0,1),\\
h_{t+1} &= \alpha + \delta h_t + \sigma_h \eta_t & \eta_t &\sim N(0,1),
\end{aligned}
$$

where y_t denotes daily stock returns. The realized stochastic volatility model (Takahashi et al. 2009) adds realized volatility to the model and jointly estimates the return and realized volatility series as follows:

$$
\begin{aligned}
y_t &= \mu + \exp(h_t/2)u_t, & u_t &\sim N(0,1),\\
\log(RV_t) &= h_t + \sigma_{RV}\epsilon_t^{RV}, & \epsilon_t^{RV} &\sim N(0,1),\\
h_{t+1} &= \alpha + \delta h_t + \sigma_h \eta_t, & \eta_t &\sim N(0,1),
\end{aligned} \tag{2.1}
$$

where RV_t denotes daily realized volatility. This specification is based on the fact that realized volatility is a consistent estimator of quadratic return variation under certain assumptions of stock returns. The realized stochastic volatility model estimates y_t and $\log(RV_t)$ jointly with a common latent volatility factor h_t. We extend the realized stochastic volatility model by including the implied volatility and assuming the common latent volatility process also determines the implied volatility process. The model is specified as follows:

$$y_t = \mu + \exp(h_t/2)u_t, \qquad u_t \sim N(0,1) \tag{2.2}$$

$$\log(RV_t) = a_{RV} + b_{RV}h_t + \sigma_{RV}\epsilon_t^{RV}, \qquad \epsilon_t^{RV} \sim N(0,1) \tag{2.3}$$

$$\log(IV_t) = a_{IV} + b_{IV}h_t + \sigma_{IV}\epsilon_t^{IV}, \qquad \epsilon_t^{IV} \sim N(0,1) \tag{2.4}$$

$$h_{t+1} = \alpha + \delta h_t + \sigma_h\epsilon_t^{h}, \qquad \epsilon_t^{h} \sim N(0,1) \tag{2.5}$$

IV_t denotes the daily implied volatility. This model specification jointly estimates y_t, $\log(RV_t)$ and $\log(IV_t)$ with a common factor h_t. In this chapter, the implied volatility measure we use is the VIX index for y_t, representing S&P500 ETF (SPY) returns, or calculated short-term implied volatility mimicking the construction of VIX for y_t, representing individual stock returns. The data provider has scaled the VIX index and the implied volatility surface. The constant and slope terms in Equation (2.4) will automatically capture the scaling.

We use the unadjusted realized volatility estimator. Given intraday returns $r_{t,i}$, with time intervals $i = 1, ...I$, the unadjusted realized volatility is estimated as $RV_t = \sum_{i=1}^{I} r_i^2$. This estimator suffers from market microstructure noise and non-trading hours limitations. As a result, this simple estimator of realized volatility could be biased and inconsistent (see Bandi and Russell 2008). Takahashi et al. (2009) put the constant term in Equation (2.1) to resolve the potential bias. Maheu and McCurdy (2011) adopt a bias-corrected estimator of realized volatility. In this paper, we adopt the unadjusted estimator of realized volatility and use the ten-minute intraday stock returns, which are less affected by the micro-structure noise, to estimate the realized volatility within exchange trading hours. Then we add the estimated realized volatility within exchange trading hours with squared overnight returns to calculate the daily realized volatility.

On the other hand, implied volatility does not have microstructure noise, but is biased due to the risk premium. To address the potential noise or bias for both realized volatility and implied volatility, we keep the constant and slope terms in Equation (2.1) and (2.4) for empirical study purposes. In our model specification, Equations (2.2) and (2.5) comprise the simple stochastic volatility model (denoted SV hereafter). By including Equation (2.3) or Equation (2.4), we have the realized stochastic volatility model (RSV) or the implied stochastic volatility model (IVSV). If we estimate stock return, realized volatility and implied volatility together, the extended SV model becomes the RVIVSV model with a combined specification from the RSV and the IVSV models.

2.2.2 Latent Volatility Sampling

We apply the Bayesian MCMC method to estimate the proposed and alternative SV models. The latent volatility $h_t, t = 1, 2, ...T$ are parameters to be sampled with the Metropolis-Hastings algorithm. Based on the single move sampler for the simple SV model (Kim et al. 1998), we modify the proposal distribution of h_t to include the RV_t and IV_t processes. For notation simplicity, RV_t and IV_t will denote the logarithm of realized volatility and implied volatility for this section.

Let Θ denote the parameter set, $\Theta = [\mu, a_{RV}, b_{RV}, \sigma_{RV}, a_{IV}, b_{IV}, \sigma_{IV}, \alpha, \delta]'$. The conditional posterior of h_t is:

$$\begin{aligned} p(h_t|h_{-t}, y_t, RV_t, IV_t, \Theta) \;\propto\; & p(y_t|h_t, \Theta)\; p(\log(RV_t)|h_t, \Theta)\; p(\log(IV_t)|h_t, \Theta) \\ & p(h_t|h_{-t}, \Theta), \end{aligned} \tag{2.6}$$

where $p(\log(RV_t)|h_t, \Theta)p(\log(IV_t)|h_t, \Theta)p(h_t|h_{-t}, \Theta)$ on the right hand side of Equation (2.6) is proportional to a normal distribution with mean of μ_t, as in equation (2.8), and variance of σ^2, as in equation (2.9) (derivation details can be found in Appendix section B1.1). The posterior distribution of h_t will be proportional to:

$$\begin{aligned} & p(y_t|h_t, \Theta)p(RV_t|h_t, \Theta)p(IV_t|h_t, \Theta)p(h_t|h_{-t}, \Theta) \\ \propto & \exp(-\frac{h_t}{2}) \exp(-\frac{(y_t - \mu)^2}{2\exp(h_t)}) \exp(-\frac{(h_t - \mu_t)^2}{2\sigma^2}), \end{aligned} \tag{2.7}$$

where

$$\mu_t = \left[\frac{(RV_t - a_{RV})b_{RV}}{\sigma^2_{RV}} + \frac{(IV_t - a_{IV})b_{IV}}{\sigma^2_{IV}} + \frac{\alpha(1-\delta) + \delta(h_{t+1} + h_{t-1})}{\sigma^2_h} \right] \sigma^2 \tag{2.8}$$

$$\sigma^2 = \left(\frac{b^2_{RV}}{\sigma^2_{RV}} + \frac{b^2_{IV}}{\sigma^2_{IV}} + \frac{1+\delta^2}{\sigma^2_h} \right)^{-1}. \tag{2.9}$$

Given μ_t and σ^2, similar to Kim et al. (1998), we have the following proposal distribution of h_t:

$$h_t \sim N(\mu_t + \frac{\sigma^2}{2}[(y_t - \mu)^2 \exp(-\mu_t) - 1], \;\; \sigma^2).$$

It is interesting to compare this proposal distribution with that of the simple SV model. The counterpart specification of Equations (2.7), (2.8) and (2.9) is as follows:

$$
\begin{aligned}
p(h_t|h_{-t}, y_t, \theta) &\propto p(y_t|h_t, \Theta)p(h_{t+1}|h_t, w)p(h_t|h_{t-1}, w) \\
&\propto \exp(-\frac{h_t}{2})\exp(-\frac{(y_t-\mu)^2}{2\exp(h_t)})\exp(-\frac{(ht-\hat{\mu}_t)^2}{2\hat{\sigma}^2}),
\end{aligned} \tag{2.10}
$$

where:

$$
\begin{aligned}
\hat{\mu}_t &= \frac{\alpha(1-\delta)+\delta(h_{t+1}+h_{t-1})}{1+\delta^2}, \\
\hat{\sigma}^2 &= \frac{\sigma_h^2}{1+\delta^2}.
\end{aligned}
$$

Comparing Equation (2.7) with Equation (2.10), in the simple SV model in Equation (2.10), $\hat{\mu}_t$ contains information only from adjacent latent volatility variables h_{t-1} and h_{t+1}. This $\hat{\mu}_t$ could be far from target h_t, especially when period t corresponds to a highly volatile period (e.g. a sudden market shock within the period). In the equity market, this kind of sudden market move is not rare, especially for individual stocks. As a result, the approximation of first order Taylor extension of $\exp(-h_t)$ at $\hat{\mu}_t$ may produce a poor proposal distribution for sampling h_t.

In contrast, μ_t from Equation (2.7) contains information not only from adjacent latent volatility variables, but also from RV_t and IV_t, which directly measure the volatility in period t. Rearranging Equation (2.8), we rewrite μ_t as a weighted average of three normal distributions' means in Equation (2.7). Equation (2.11) shows that this average is weighted by the precision (inverse of variance) of these three normal distributions. Intuitively, μ_t is a weighted average of the means of h_t

derived from RV_t and IV_t and adjacent latent volatility variables (h_{t-1} and h_{t+1}). In this way, we avoid the issue that we discussed for the simple SV model; i.e. when there is a sudden market crash in period t, which indicates high h_t. Although h_{t-1} and h_{t+1} may not be high, RV_t and IV_t are high during a market crash. Therefore, by including the information from RV_t and/or IV_t, μ_t under the RVSV, IVSV and RVIVSV specification is much closer to the target h_t, which leads to a much closer approximation with a first order Taylor extension.

In addition, if we explore Equation (2.11) in more detail, the weight is directly related to the slope and variance parameters in the model. If σ_{IV} is very low compared to σ_{RV} and σ_h, the mean of h_t derived from the implied volatility (Equation (2.4)) will have a large weight, as shown in Equation (2.11). We will come back to this discussion in the empirical results.

$$\mu_t = \frac{1}{W}\left[\frac{RV_t - a_{RV}}{b_{RV}} \cdot \frac{b_{RV}^2}{\sigma_{RV}^2} + \frac{IV_t - a_{IV}}{b_{IV}} \cdot \frac{b_{IV}^2}{\sigma_{IV}^2} + \frac{\alpha(1-\delta) + \delta(h_{t+1} + h_{t-2})}{1+\delta^2} \cdot \frac{1+\delta^2}{\sigma_h^2}\right], \tag{2.11}$$

where

$$W = \frac{b_{RV}^2}{\sigma_{RV}^2} + \frac{b_{IV}^2}{\sigma_{IV}^2} + \frac{1+\delta^2}{\sigma_h^2}.$$

Conditional on $h_t, t = 1, ...T$, other parameters are sampled by Gibbs sampling with conjugate priors. The MCMC steps and the corresponding prior distribution is specified as follows:

- $\mu \mid \alpha, \delta, h, y$. μ is drawn from a normal posterior distribution with a conjugate normal prior $N(0, 100)$.
- $h_t \mid h_{-t}, y_t, RV_t, IV_t, \Theta$. h_t is drawn using the Metropolis-Hastings method with the proposal distribution discussed in the previous section.

- $\alpha, \delta, \sigma_h | h$, Conditional on the latent volatility process h, parameters α, δ, σ_h can be drawn as parameters of the linear regression with conjugate prior distributions: $\alpha \sim N(0, 50)$, $\sigma_h \sim IG(\frac{3}{2}, \frac{0.5}{2})$ and δ follow a truncated normal prior $N(0.95, 5)I_{|\delta|<1}$ for δ for stationary.

- $a_{RV}, b_{RV}, \sigma_{RV} \mid h, RV$. Conditional on the latent volatility process h and the RV process, a_{RV}, b_{RV}, σ_{RV} can be drawn as parameters of the linear regression with conjugate prior distributions: $a_{RV} \sim N(0, 50)$, $b_{RV} \sim N(1, 50)$ and $\sigma_{RV} \sim IG(\frac{3}{2}, \frac{0.5}{2})$.

- $a_{IV}, b_{IV}, \sigma_{IV} \mid h, IV$. Conditional on the latent volatility process h and the RV process, a_{IV}, b_{IV}, σ_{IV} can be drawn as parameters of the linear regression with conjugate prior distributions: $a_{IV} \sim N(0, 50)$, $b_{IV} \sim N(1, 50)$ and $\sigma_{IV} \sim IG(\frac{3}{2}, \frac{0.5}{2})$.

Notice that, for all the parameters, we set flat priors to make sure the posterior sampling results contain little prior information. This is especially important for the σ_{RV} and σ_{IV}, as these two parameters represent the amount of noise contained in the RV and IV as measures of the latent volatility and is critical in sampling the latent volatility process. We set the total number of MCMC iterations to be 100,000 and we keep 1 for every 10 draws. We also set 5,000 burn-in iterations for parameter convergence.

2.3 Data and Estimation Results

We examine the in-sample explanatory and out-of-sample forecast performance of the proposed RVIVSV model using both an equity index and individual stocks. For the equity index, we select the S&P 500 ETF and the corresponding VIX index as the implied volatility measure. For individual stocks, we select six stocks and use their daily individual traded option prices to back out the short-term implied volatility.

2.3.1 S&P 500 and VIX

The S&P 500 index is one of the most commonly used and cited U.S. stock market indices, and its corresponding Standard and Poor's Depository Receipt (SPY) is among the most popular tradable assets among market participants who seek liquidity. We retrieve fourteen years (from the beginning of 2001 to the end of 2014) high-frequency quote records from the TAQ database to calculate unajusted daily return and daily realized volatility. To be more specific, we use the mid-price between the bid and ask prices to calculate five-minutes returns. Then we use the simple unadjusted RV estimator to calculate the daily RV. The mean of RV is 1.50, while the sample variance of the daily return is 1.62 in our sample. The difference between the RV mean and the sample variance is not large, which indicates that the simple unadjusted RV estimator is acceptable. As for the implied volatility of the SPY ETF, we will use the VIX index, which is constructed from S&P 500 options with near to 30-day maturity. As shown in the next section, we follow the construction of VIX to build the implied volatility measure for individual stocks. We present summary statistics for SPY in Table 2.1. The mean of realized volatility is close to the variance of the daily returns, which supports the accuracy of our estimation of realized volatility. The VIX index is scaled to match the standard deviation of the following month. The parameters in equation (2.4) will automatically capture the scaling.

	Return	**RV**	**VIX**
Mean	0.0133	1.5021	20.7702
Variance	1.6235	12.3586	85.7557

TABLE 2.1: The summary statistics for SPY.

To illustrate the dynamics, Figure 2.1 plots the data for SPY. The graphs from top to bottom are (1) daily closing prices of the SPY, (2) daily returns of the SPY, (3) logarithm of daily RV and (4) logarithm of the VIX index. Both RV and VIX can reflect the return volatility, and they share similar dynamics. As our sample covers the 2008 financial crisis, we expect high volatility during the crisis. Both RV and VIX are high during this period. Moreover, from the figure, it appears that the VIX index is more stable than the RV. The daily RV only measures the volatility within a trading day while the VIX index measures the expectation of the future 1-month volatility. The VIX index is smoother than the daily RV since the VIX index is a forward average of the following month's volatility.

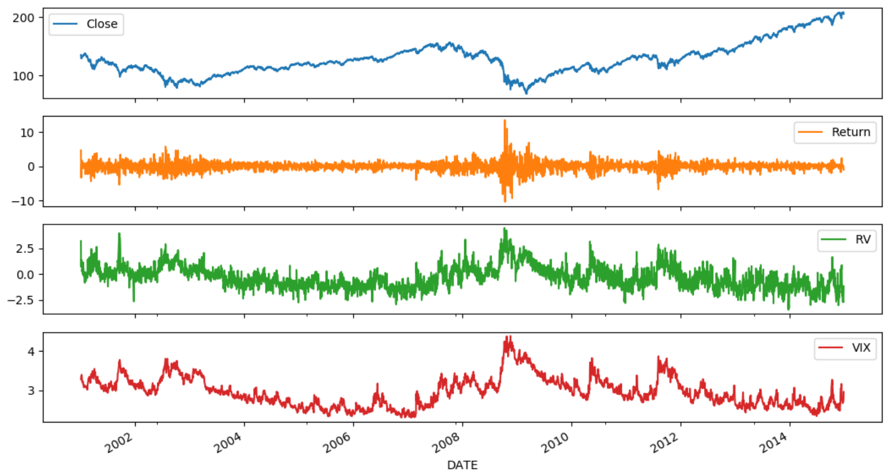

FIGURE 2.1: Data plots for SPY.

Table 2.2 reports the posterior sampling results of the SV, RVSV, IVSV and RVIVSV models on the SPY data for the period January 2nd, 2001 to December 29th, 2014. Comparing the various model estimations in Table 2.1, we have the following: First, compared to the simple SV model, incorporating RV in the SV framework results in a lower δ and higher σ_h, while incorporating IV leads to higher δ and lower σ_h. This result reveals that the hidden volatility process inferred from the IVSV model is more stable than that of the RVSV model. Figure 2.2 plots the hidden volatility process inferred from the RVSV model and the IVSV model and clearly shows that the latent volatility process of the RVSV model is much more volatile. This is consistent with the inference from Figure 2.1, where the RV process itself is more volatile than the IV process. Moreover, the sampling results for common parameters between IVSV and RVIVSV are basically the same, as shown in Table 2.2. Figure 2.3 shows that the latent volatility processes from these two models are almost identical. Once the IV process is included in the model, the IV process will play a dominant role in identifying the latent process compared to RV.

	SV		**RVSV**		**IVSV**		**RVIVSV**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
μ	0.0719	0.0132	0.0839	0.0120	0.0758	0.0129	0.0773	0.0130
	(0.046, 0.098)		(0.060, 0.107)		(0.050, 0.101)		(0.051, 0.103)	
α	-0.0037	0.0036	-0.0105	0.0054	-0.0013	0.0022	-0.0015	0.0023
	(-0.011, 0.003)		(-0.021, -0.000)		(-0.006, 0.003)		(-0.006, 0.003)	
δ	0.9801	0.0044	0.9614	0.0056	0.9915	0.0021	0.9911	0.0022
	(0.971, 0.988)		(0.949, 0.972)		(0.987, 0.996)		(0.987, 0.995)	
σ_h	0.2016	0.0168	0.3017	0.0159	0.1298	0.0040	0.1333	0.0041
	(0.170, 0.237)		(0.272, 0.333)		(0.122, 0.138)		(0.125, 0.141)	
a_{RV}	-	-	-0.1149	0.0240	-	-	-0.2100	0.0239
			(-0.161, -0.068)				(-0.258, -0.163)	
b_{RV}	-	-	0.8898	0.0222	-	-	0.9097	0.0237
			(0.848, 0.934)				(0.866, 0.957)	
σ_{RV}	-	-	0.4607	0.0087	-	-	0.5892	0.0071
			(0.444, 0.478)				(0.575, 0.603)	
a_{IV}	-	-	-	-	3.0039	0.0092	3.0058	0.0091
					(2.986, 3.022)		(2.988, 3.023)	
b_{IV}	-	-	-	-	0.3767	0.0089	0.3750	0.0089
					(0.359, 0.394)		(0.358, 0.393)	
σ_{IV}	-	-	-	-	0.0373	0.0009	0.0367	0.0008
					(0.035, 0.039)		(0.035, 0.038)	

Inside the parentheses is the 95% density interval

Table 2.2: Posterior sampling summary for SPY

From Equation (2.6), μ_t is a weighted average of IV_t, RV_t and the adjacent latent volatility(h_{t-1}, h_{t+1}), and the weights are b_{IV}^2/σ_{IV}^2, b_{RV}^2/σ_{RV}^2 and $(1+\delta^2)/\sigma_h^2$ respectively. For the RVIVSV model, $b_{RV}^2/\sigma_{RV}^2 = 2.3838$, $b_{IV}^2/\sigma_{IV}^2 = 104.4071$ and

$(1+\delta^2)/\sigma_h^2 = 111.5589$. The result indicates that 47.8% of μ_t is from IV_t, 51.1% is from h_{t-1}, h_{t+1} and only 1.1% is from RV_t. For the RVSV model, $b_{RV}^2/\sigma_{RV}^2 = 3.7303$, $(1+\delta^2)/\sigma_h^2 = 21.1407$, and we find that RV_t plays a more important role (15%).

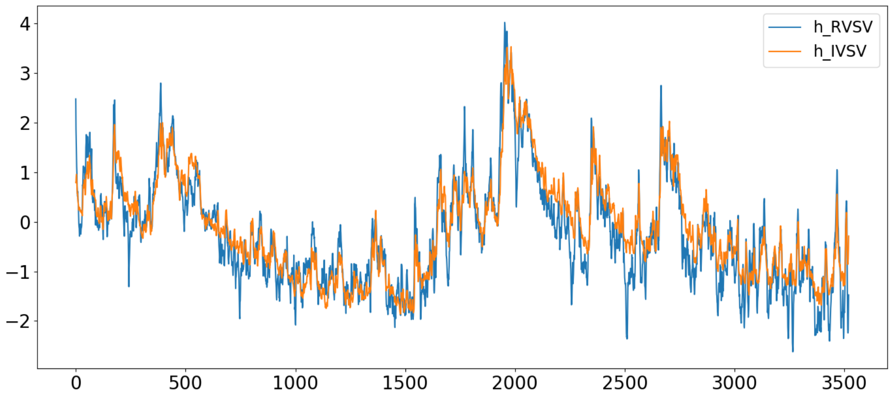

FIGURE 2.2: Smoothed h_t from the RVSV and IVSV models for SPY.

The blue line represents the latent volatility process from the RVSV model, while the orange line represents the IVSV model.

The in-sample estimation results, especially the estimation of the latent volatility process, suggest that the volatility measures have a strong influence. For the equity index (SPY ETF), the RVSV model leads to a rough latent volatility process, while the IVSV model leads to a smooth latent volatility process. Also, in the RVIVSV model, IV (VIX index) plays a more critical role in identifying the latent volatility. The estimated latent volatility process is almost identical to that of the IVSV model.

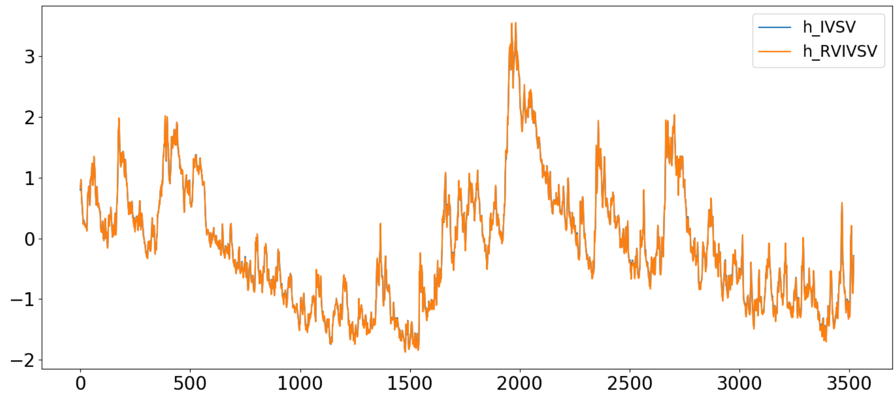

Figure 2.3: Smoothed h_t from the IVSV and RVIVSV models for SPY.

This graph compares the smoothed latent volatility process from the IVSV and RVIVSV models. The blue line (IVSV) and orange line (RVIVSV) are identical.

Considering the non-linear factor model specification, the RVSV(IVSV) model estimates returns and logarithmic realized volatility (implied volatility) jointly with a common latent volatility process. Since $\log(RV_t)$ ($\log(IV_t)$) is linearly correlated with h_t, and y_t is non-linearly correlated, the identification of h_t utilizes the first moment information of volatility measures while only the second moment information of returns is related. So the volatility measures have substantial influence in identifying the latent volatility, and we observe that the estimated latent volatility process share a similar pattern with corresponding volatility measure.

However, when fitting the RVIVSV model with y_t, $\log(RV_t)$ and $\log(IV_t)$ jointly, the estimated latent volatility process seems only to capture the dynamics of $\log(IV_t)$ and attribute a greater portion of variations of $\log(RV_t)$ to noise. As shown in Table 2.2, the variance parameter in Eq. (2.3) in the RVIVSV model ($\sigma_{RV} = 0.59$) is significantly higher than that of Eq. (2.1) in the RVSV model ($\sigma_{RV} = 0.46$). Figure 2.1 has shown that the logarithmic realized volatility and logarithmic implied volatility demonstrate different dynamics. Modelling $\log(RV_t)$ and $\log(IV_t)$ jointly with a single factor h_t will make h_t follow the dynamics of $\log(IV_t)$, which is smoother. The standard SV model assumes AR(1) specification for h_t, which is difficult to capture the rough dynamics of $\log(RV_t)$ process and favors the smoother $\log(IV_t)$ process. Moreover, the daily RV is naturally noisy due to the micro-structure noise. As a result, the RVIVSV model will automatically assign less weight to the daily RV when identifying the latent volatility.

In the next section, we consider individual stocks and examine whether consistent results for the implied volatility persist.

2.3.2 Individual Stocks

To complement the results from the equity index, we select five blue-chip individual stocks: Apple Inc. (NASDAQ: AAPL), Citigroup Inc. (NYSE: C), IBM (NYSE: IBM), Bank of America Corp. (NYSE: BAC), General Electric (NYSE: GE), and one mid-cap stock: JCPenney Company (NYSE: JCP). The five blue-chip stocks (and their options) are among the most liquid assets on the U.S. equity market, given their high trading volume. The daily return, realized volatility and option-implied volatility data on individual stocks cover the period from September 10th, 2003 to December, 29th, 2017. JCP is included in our sample to examine the behavior of the proposed models for a mid-cap company. A consistent result for JCP will enhance the robustness of our conclusion.

We retrieve the high-frequency transaction data for individual stocks from the TAQ database. Similar to the SPY ETF, we calculate ten-minute returns and then the daily RV using the simple unadjusted estimator. The mean of the daily realized volatility and the daily variance of return are presented in Table 2.3 to show the accuracy of the

	AAPL	BAC	C	GE	IBM	JCP
RV Mean	4.6339	8.8964	11.2610	3.5707	1.9186	9.5824
Return Var	4.4899	9.8884	10.8588	3.0751	1.7040	9.0371

Table 2.3: RV mean and sample variance of individual stocks

This table summarizes the comparison between the mean of the daily realized volatility and the variance of daily returns. The difference between them can be the result of micro-structure noise or non-trading hours.

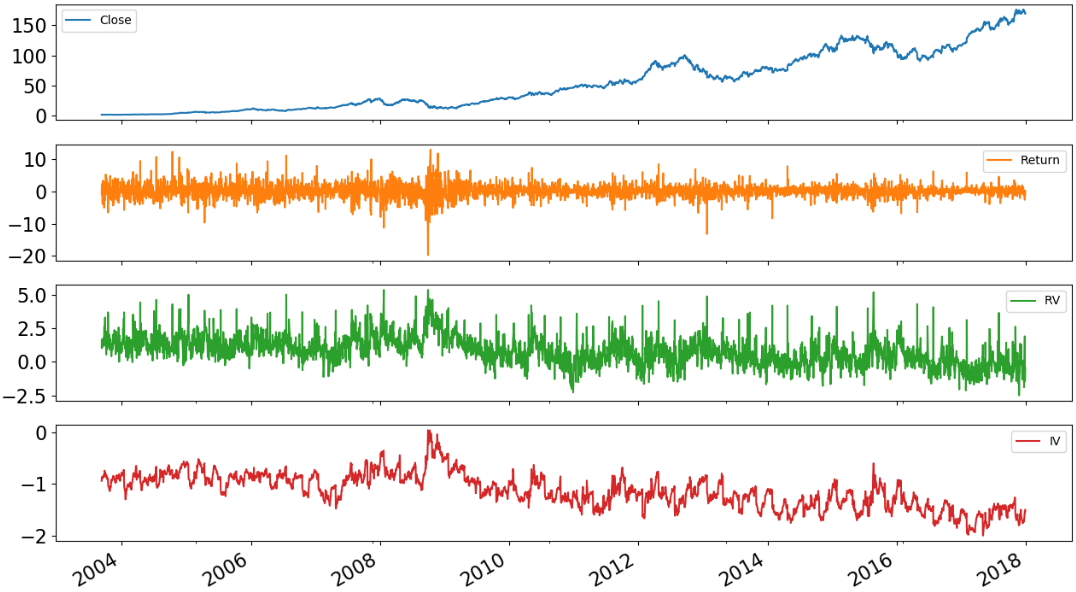

Figure 2.4: Data plots for AAPL

estimated RV. It is clear that for certain stocks (BAC, GE), there can be a significant difference between the mean of the realized volatility and the return variance. Hence, the constant and slope terms in equation (2.3) are necessary to fix the bias of the simple estimator of realized volatility.

As for the implied volatility for individual stocks, we use daily European option trading data from Option Metrics to mimic the construction of VIX. This volatility data set contains traded individual option prices with different maturities and strike prices. The implied volatilities are then backed out using the standard Black-Scholes

European option model. To be consistent with the VIX index for the SPY, we choose implied volatility from call and put options with short-term maturity (30 days) and strike prices close to at-the-money. Then we take the simple average of selected put and call implied volatilities to get the daily IV. Figure 2.4 illustrates the IV process for AAPL as an example. The graphs from top to bottom are (1) daily AAPL close prices, (2) daily returns (scaled by 100) (3) logarithm of daily realized volatility, and (4) logarithm of daily implied volatility. Table 2.3 shows that the mean of daily RVs is quite close to the daily return variance of AAPL, which is a reasonable indicator for the accuracy of RV's estimation. We observe many unusual peaks in the graph for AAPL's RV (with natural logarithm) series, which typically corresponds to the highly volatile trading days. However, its IV process is much more stable and smoother than the RV series. It is common for individual stocks to have sudden high daily realized volatility in our sample. Compared to the SPY, which is a well diversified portfolio, the idiosyncratic risk contained in individual stocks leads the return series to have more unexpected shocks. Consequentially, unusually high h_t for certain periods may cause sampling problems, as mentioned in the previous section.

Similar to the SPY ETF, we estimate four different SV models on the return series of these individual stocks. The estimation sample period is September 10th, 2003 to December 19th, 2017. Table 2.4 presents the posterior sampling results for Apple Inc (AAPL). The simple SV model has a much higher σ_h (0.3088) due to idiosyncratic risk, compared to the SPY results. By introducing realized volatility, the latent volatility process has much higher uncertainty ($\sigma_h = 0.6033$) and lower auto-correlation ($\delta = 0.8342$). However, by introducing implied volatility, the latent volatility process is much more stable and the auto-correlation is higher ($\sigma_h = 0.1177$, $\delta = 0.9867$). Moreover, similar to the SPY ETF results, the latent volatility processes sampled from the RVIVSV and IVSV models, as well as the estimated parameters, are almost identical to each other.

	SV		RVSV		IVSV		RVIVSV	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
μ	0.1619	0.0253	0.1694	0.0210	0.1915	0.0273	0.1921	0.0275
	(0.113, 0.212)		(0.129, 0.211)		(0.138, 0.245)		(0.138, 0.245)	
α	0.0607	0.0125	0.1454	0.0166	0.0160	0.0039	0.0158	0.0040
	(0.038, 0.087)		(0.114, 0.179)		(0.008, 0.024)		(0.008, 0.024)	
δ	0.9428	0.0110	0.8342	0.0155	0.9867	0.0027	0.9868	0.0028
	(0.919, 0.962)		(0.803, 0.863)		(0.981, 0.992)		(0.981, 0.992)	
σ_h	0.3088	0.0299	0.6033	0.0277	0.1177	0.0041	0.1190	0.0041
	(0.254, 0.371)		(0.551, 0.658)		(0.110, 0.126)		(0.111, 0.127)	
a_{RV}	-	-	0.0455	0.0314	-	-	-0.4054	0.0511
			(-0.017, 0.106)				(-0.509, -0.308)	
b_{RV}	-	-	0.8753	0.0229	-	-	0.9950	0.0355
			(0.832, 0.922)				(0.928, 1.068)	
σ_{RV}	-	-	0.4795	0.0164	-	-	0.7898	0.0093
			(0.446, 0.511)				(0.772, 0.808)	
a_{IV}	-	-	-	-	-1.6925	0.0206	-1.6825	0.0198
					(-1.735, -1.654)		(-1.723, -1.645)	
b_{IV}	-	-	-	-	0.4526	0.0140	0.4467	0.0137
					(0.426, 0.481)		(0.421, 0.475)	
σ_{IV}	-	-	-	-	0.0362	0.0008	0.0368	0.0008
					(0.035, 0.038)		(0.035, 0.038)	

Inside the parentheses is the 95% density interval

Table 2.4: Posterior sampling summary for AAPL.

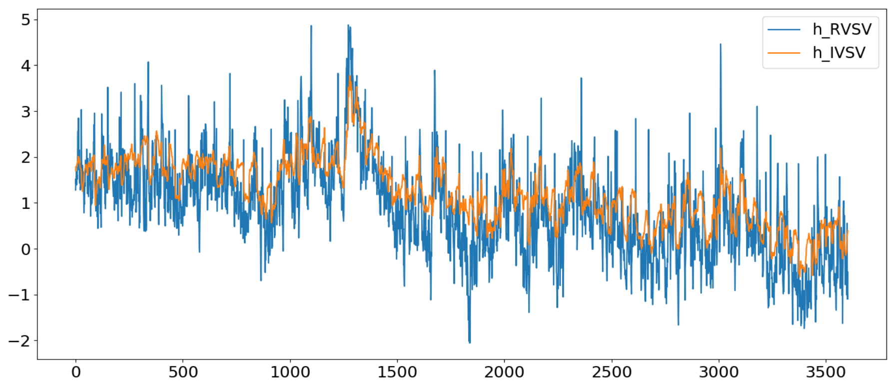

Figure 2.5: Sampled h_t from the RVSV and IVSV models for AAPL.

The blue line (RVSV) and orange line (IVSV) clearly show that the latent volatility process sampled from the RVSV model is much more volatile than that of the IVSV model.

Consistent with the SPY case, we keep the constant (a_{RV}) and slope (b_{RV}) terms in the RV equation. Theoretically, the value of a_{RV} is 0, as there is no bias arising from issues like micro-structure or non-trading hours. The theoretical value of b_{RV} is 1, as RV is consistent and should be equal to the latent volatility asymptotically. The original realized stochastic volatility model specified in Takahashi et al. (2009) sets $a_{RV} = 0$ and $b_{RV} = 1$. However, our estimated coefficients deviate from their theoretical values, where $b_{RV} = 0.8753$ in the RVSV model and $b_{RV} = 0.9950$ in the RVIVSV model. This is a different result from the SPY ETF case, where b_{RV} in both the RVSV and RVIVSV models are very close to one. We find b_{RV} is significantly below 0.9 in the RVSV model, while b_{RV} is still close to one in the RVIVSV model. σ_{RV} in these two models is quite different as well. σ_{RV} in the RVIVSV equation (0.7898) is considerably higher than that of the RVSV model (0.4795). This difference between RVSV and RVIVSV (parameter sampling results for other individual stocks are posted in the Appendix) persists for all of the individual stocks in our sample. Moreover, b_{RV} is below 0.9 for BAC, C, GE, and JCP in both the RVSV and RVIVSV models. After excluding the noise in the RV equation, the part of the RV variation that correlates with the latent volatility is less volatile than the latent volatility as we observe a slope less than one.

From Figure 2.5, consistent with the results for SPY, RVSV leads to a rough latent volatility process with higher $\sigma_h = 0.60$. When further comparing the volatility processes of the SV and IVSV models, as in Figure 2.6, we find that the latent volatility process for IVSV is smoother than that of SV. The $\sigma_h = 0.12$ of IVSV is lower than $\sigma_h = 0.31$ of SV. Compared with the SPY ETF case, the latent volatility process for the SV model with individual stock data is more volatile. However, the IVSV for individual stocks always results in a more stable latent volatility process with much lower σ_h. Finally, the latent processes for RVIVSV and IVSV are very similar in the individual stock cases. Consistent with the previous discussion, including IV in addition to RV leads to higher

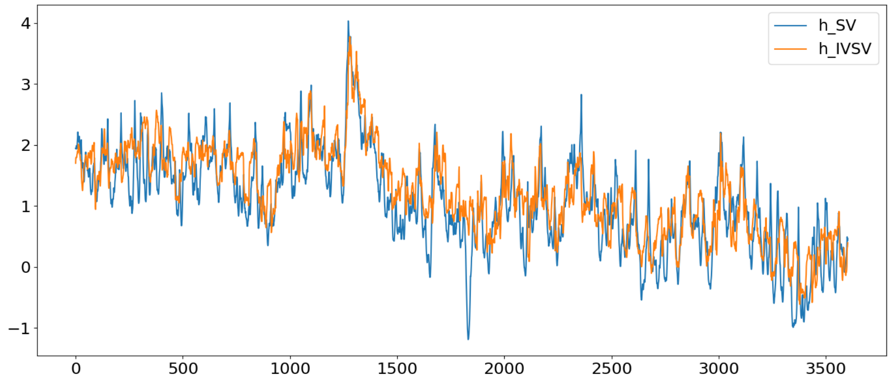

FIGURE 2.6: Volatility process of the SV and IVSV models for AAPL.

σ_{RV} in the RVIVSV model. Consistent with previous discussion for the SPY ETF, jointly modelling y_t, $\log(RV_t)$ and $\log(IV_t)$ with a single common factor h_t leads to a latent volatility process that follows the dynamic of $\log(IV_t)$ and treat $\log(RV_t)$ as a highly noisy process.

The empirical results we have found so far are not unique to Apple Inc. For all the individual stocks in our sample, we find similar patterns of sampled parameters and latent volatility processes using these four models (details can be found in Appendix sections B2 and B3). The results for individual stocks are consistent with the SPY ETF result when we change the implied volatility measure from the VIX index to option implied volatility by our own construction.

To conclude the findings from in-sample results, we find that introducing RV into the SV model leads to a more volatile latent volatility process with a smaller auto-correlation coefficient (δ) and a larger volatility parameter (σ_h) in the latent volatility equation. The result is consistent with Takahashi et al. (2009). However, including the IV process leads to a more stable latent volatility process with a larger auto-correlation coefficient (δ) and a smaller volatility parameter (σ_h). By including both the RV and IV processes, the latent volatility process for RVIVSV is almost identical to that of the IVSV model. Compared to the RV process, the IV process is dominant in sampling the latent volatility process. The empirical results for the RVIVSV model indicate that implied volatility, as an *ex-ante* volatility measure, is more informative than the realized volatility in identifying the latent volatility process.

However, we cannot conclude that implied volatility is better than realized volatility with in-sample results alone. As we have discussed, the implied volatility could be biased as a volatility measure due to the risk premium. The IVSV and IVRVSV models could identify a biased latent volatility process. To evaluate and compare the models, we focus on the one-day ahead return density forecasts in the next section.

2.4 Predictive Performance Comparison

Besides the in-sample explanation, it is important to compare the out-of-sample performance of volatility forecasting among the various SV specifications. Koopman et al. (2005) and Shirota et al. (2014) use RV as a volatility proxy to test the volatility forecasting performance, as volatility cannot be directly measured. However, this method is potentially problematic since RV itself is an estimator and may contain noise. Moreover, it is meaningless to test the volatility forecast of the IVSV model by comparing it to the future realized volatility. All four models in this paper share a common specification of the return equation. Conditional on h_{t+1}, y_{t+1} follows a normal distribution with constant mean but dynamic variance $e^{h_{t+1}}$. By comparing the density forecasts of y_{t+1}, we also compare the prediction of h_{t+1}, for the reason that an accurate prediction of h_{t+1} leads to better density forecasts of of y_{t+1}.

We examine whether realized volatility or implied volatility improves the return density forecasts by comparing the logarithmic predictive likelihood of one-day ahead returns. Previous work (Maheu and McCurdy 2011) shows that the predictive likelihood is improved by incorporating RV. Given our model specification M, we evaluate the following logarithmic predictive likelihood:

$$\sum_{t=T-\tau-1}^{T-1} \log\left(p_M(y_{t+1}|y_{1:t}, RV_{1:t}, IV_{1:t})\right), \tag{2.12}$$

where $y_{1:t}, RV_{1:t}$ and $IV_{1:t}$ represent the available observations of returns and volatility at time t. So $p_M(y_{t+1}|y_{1:t}, RV_{1:t}, IV_{1:t})$ is the predictive likelihood of one-day ahead return y_{t+1} with available observations at time t according to model M. Given the posterior sampling results $\{\Theta_{M,t}^{(i)}\}_{i=1}^{N}$ by fitting model M with available observations and iid draws $\{\epsilon^{(i)}\}_{i=1}^{N}$ from a standard normal distribution, the predictive likelihood in Equation (2.12) is estimated as:

$$p_M(y_{t+1}|y_{1:t}) \approx \frac{1}{N}\sum_{i=1}^{N} f_N(y_{t+1}|\mu_{M,t}^{(i)},\ \exp(\alpha_{M,t}^{(i)} + \delta_{M,t}^{(i)} h_{M,t}^{(i)} + \sigma_{h}{}_{M,t}^{(i)} \epsilon^{(i)})), \tag{2.13}$$

where $f_N(x|\mu, \sigma^2)$ represents the likelihood function of a normal distribution with mean μ and variance σ^2 evaluated at x and we set $\tau = 3000$, $N = 10000$. For SPY ETF, the predictive window starts January 31st, 2003, and ends December 31st, 2014. For individual stocks, the predictive window starts January 31st, 2006, and ends December 29th, 2017. The predictive window in both cases covers the 2008 financial crisis period. A higher logarithmic predictive likelihood indicates a better predictive performance. To compare two models, M_1 and M_2, we focus on the logarithmic predictive Bayes factor, which is defined as:

$$\sum_{t=T-\tau-1}^{T-1} \log\left(p_{M_1}(y_{t+1}|y_{1:t})\right) - \sum_{t=T-\tau-1}^{T-1} \log\left(p_{M_2}(y_{t+1}|y_{1:t})\right). \tag{2.14}$$

A logarithmic predictive Bayes factor, that is greater than 5, strongly supports model M_1 over M_2. We report the summary of logarithmic predictive likelihood in Table 2.5.

Models	**SPY**	**AAPL**	**BAC**	**C**	**GE**	**IBM**	**JCP**
SV	-3995.31	-5962.00	-6545.27	-7085.19	-5155.67	-4716.11	-7449.77
RVSV	**-3926.60**	**-5912.88**	-6271.46	-6268.22	**-5103.00**	**-4665.63**	**-7366.88**
IVSV	-3958.34	-5920.91	-6248.25	-6257.96	-5146.30	-4697.92	7402.07
RVIVSV	-3954.21	-5920.00	**-6244.87**	**-6254.87**	-5143.54	-4750.48	-7426.97

Table 2.5: The predictive log-likelihood results.

From Table 2.5, the obvious conclusion we can make is that the inclusion of either RV or IV will significantly improve the out-of-sample return prediction. For SPY, the logarithmic Bayes factor of the IVSV model over the benchmark SV model is 36.97 (-3958.34 − -3995.31), which is strong evidence of the superiority of IVSV over SV. However, the RVSV model has a logarithmic Bayes factor of 68.71 (-3926.60 − -3995.31) compared to the simple SV model. Since a logarithmic Bayes factor greater than 5 is already strong evidence, the improvement of RV or IV is substantial compared to the simple SV model. For all individual stocks, we observe consistent improvement from including RV and IV, especially for the two financial stocks, Bank of America (BAC) and Citibank (C). For Citibank, the logarithmic Bayes factor of the IVSV (RVSV) model is 827.23 (816.97). The magnitude of improvement compared to the simple SV model is striking.

As we have discussed, the IVSV (RVSV) model is a non-linear common factor model with two observations, return and IV (RV). The simple SV model can only infer the latent volatility from the second moment of the return series, while IVSV (RVSV) can directly utilize the volatility measure as another observation in addition to the return series. This is important for highly volatile individual stocks like BAC or C since our sample period covers the 2008 financial crisis. The out-of-sample forecasting results confirm that both IV and RV are informative in identifying the real volatility process.

Compared to the benchmark SV model, the IVSV and RVSV exploit additional data with the common factor (the latent volatility h_t) and achieve massive improvements in forecast quality for returns.

Moreover, comparing RVSV and IVSV, we find that RVSV generally outperforms IVSV. Taking the SPY as an example; the logarithmic Bayes factor of RVSV over IVSV is 31.7 (-3926.60 − -3958.3), which is a strong evidence of the superiority of RVSV over IVSV. Except for the two banking stocks, RVSV dominates IVSV in return forecasting. As discussed in the previous section on in-sample empirical results, the RVSV yields a more volatile latent volatility process compared to IVSV and SV. The out-of-sample results favor the volatile latent volatility. In fact, the real volatility process is a rough process with heteroscedasticity and jumps (Andersen et al. 2007; Corsi et al. 2008; Gatheral et al. 2018). The modeling of the volatility process itself is an ongoing research topic. The standard AR(1) specification of the latent volatility process in the SV model is not sufficient to capture all the properties of the volatility process. Gatheral et al. 2018 use fractional Brownian motion to model the volatility process. RV, as a model-free estimator of the real volatility, contains all the documented and undocumented properties of the real volatility process. The common factor model specification of RVSV leads to a latent volatility process that captures the dynamics of the RV process. The out-of-sample forecasting performance of RVSV indicates that the high volatile latent volatility from RVSV is not merely a result of the microstructure noise of RV; it also reflects the properties of the real volatility process. Compared to the smoother latent volatility process from the IVSV, which could be biased due to the risk premium embodied in IV, RVSV better captures the properties of the real volatility and produce better return forecasts.

Unfortunately, jointly estimating the returns, RV and IV, as the RVIVSV, is ineffective in utilizing the information from both RV and IV. From Table 2.5, RVIVSV has better predictive performance only when IVSV has better predictive performance compared to RVSV (on stocks C and BAC). Moreover, RVIVSV is not significantly better than IVSV on these two stocks. The logarithmic Bayes factor of RVIVSV over IVSV is 3.38 (3.09) for BAC (C), which does not support a significant difference between RVIVSV and IVSV. When RVSV is better than IVSV, RVIVSV can be even worse. For IBM, RVIVSV is worse than the benchmark SV model with a logarithmic Bayes factor of -34.37. As we discussed, modeling both RV and IV with a single common latent volatility could be misspecified. RV and IV demonstrate different proprieties, and the inferred latent volatility of RVIVSV is always similar to IV and attributes RV as noises. Modeling return, RV and IV jointly needs further elaborations.

2.5 Conclusion

This paper extends the realized stochastic volatility model by using implied volatility in the SV framework. Besides, we modify the single move sampler of latent volatility h_t to include information from RV and IV. Further extension to a block move sampler of h_t can be quickly conducted.

The estimation results show that there is a significant difference between the latent volatility processes sampled from the simple SV model and RVSV/IVSV. The IVSV model has a relatively stable latent volatility process with stronger auto-correlation and less noise, while the RVSV model has a volatile process with weaker auto-correlation and higher noise. When incorporating both RV and IV into the SV framework simultaneously, implied volatility, rather than realized volatility, plays a dominant role in identifying the latent volatility. As for out-of-sample prediction, RVSV generally obtains better one-day forward return density forecasts according to the logarithmic predictive likelihood. The IVSV model is better for highly volatile stocks like banking stocks during our sample period. The latent volatility process of IVSV could be biased and smoothed since the implied volatility contains the risk premium and reflects the volatility of the following 1-month. As a result, IVSV and IVRVSV are inferior to RVSV in capturing the real volatility process of daily equity returns.

Finally, the RVIVSV model specification fails to effectively utilize information from both RV and IV in out-of-sample predictions. Our results suggest that it is not enough to model equity returns, realized volatility and implied volatility with a single latent factor. Implied volatility is not only affected by the underlying equity market, but also affected by the option market or even by the volatility market like the VIX index, which is a tradable market itself. Further study regarding implied volatility and the realized volatility is necessary for a joint model.

Chapter 3

Improving Asymmetric Stochastic Volatility Models with Ex-post Volatility

3.1 Introduction

The asymmetric stochastic volatility (ASV) model is an important extension of the simple stochastic volatility (SV) model to document the asymmetric correlation between equity returns and volatility. To be specific, equity returns tend to be negatively correlated with current or future volatility changes. Moreover, this asymmetry is stronger for market indexes (aggregate market) than individual stocks (Kim and Kon 1994; Tauchen et al. 1996; Andersen et al. 2001). Harvey and Shephard 1996 propose a asymmetric stochastic volatility model (ASV-HS hereafter), in which the equity return innovation is correlated with future volatility changes. Jacquier et al. 2004 propose an alternative asymmetric stochastic volatility model (ASV-JPR hereafter), in which return innovation is correlated with synchronous volatility changes. Yu 2005 and Men et al. 2017 compare these two model specifications. Yu 2005 show that return series under the ASV-JPR specification is not a martingale. Moreover, Yu 2005 further shows that ASV-HS is superior to ASV-JPR using a model fitting (marginal likelihood) comparison.

The economic interpretation of this observed asymmetry relies on the causal relation between equity returns and volatility. If the impact on returns causes the volatility moves, the return-volatility asymmetry can be explained as the leverage effect (see Black 1976; Christie 1982). A stock price decline leads to a higher leverage ratio of the firm, which results in higher risk or volatility. The ASV-HS specification is consistent with this causality as current innovation in the returns is correlated with future volatility moves. However, if the impact on volatility causes the return changes, the asymmetry can be alternatively interpreted as volatility feedback. An expected increase in volatility leads to a higher required rate of return if volatility risk is priced (see French et al. 1987; Campbell and Hentschel 1992). Previous work has examined and compared these two effects. Figlewski and Wang 2000 argue that the magnitude of the asymmetry could be too large to be explained by the changing of leverage ratios. Andersen et al. 2001 point out that the stronger asymmetry at the aggregate market level compared to individual stocks lends support to the volatility feedback effect rather than the leverage effect. A recent work by Jensen and Maheu 2018 find supportive evidence for volatility feedback effect in monthly data.

The identification of this causality depends on the lead-lag relation between return and volatility. Whether the returns lead the future volatility changes or the inverse. As Bollerslev et al. 2006 point out, the return and volatility innovations appear immediately at daily (or lower) frequencies, and the causality seems indistinguishable. They further investigate the asymmetry with higher frequency data and find that sharp market declines over five-minute intervals lead to a rise in volatility lasting up to several days.

However, as we will show in the Section 3.5, fitting the ASV-HS model with daily and weekly market index returns still yields lead-lag return-volatility correlations that are far from zero.

Besides volatility prediction and stochastic volatility modelling, the asymmetry is also the foundation of analyzing tradable volatility products. Given the asymmetry (inverse return-volatility correlation), investors can utilize volatility products to further hedge the exposure of volatility (Vega) of equity, derivative and mixed portfolios. Alexander et al. 2016 develop the theoretical framework of diversifying risks with volatility products. The existence of volatility products facilitates the contemporary return-volatility moves and thus correlation. In fact, a strong lead-lag return-volatility correlation, which implies predictable volatility changes like those of the ASV-HS model, could potentially violate the market efficiency of volatility products. In the late 1990s, investors could only trade volatility through portfolio of delta hedged options or variance swaps. Carr and Lee 2009 provide an extensive survey of these volatility derivatives. Recent financial innovations allow market participants to trade volatility directly through future contracts or ETFs/ETNs based on a major volatility index e.g. VIX and VSTOXX (see Alexander et al. 2015 for further details). Efficient markets implies that information from the equity return should be fully reflected in its corresponding volatility products in the same trading periods, which implies the contemporaneous return-volatility correlation structure.

The rapid growth of volatility markets and the intensive focus on stochastic volatility modeling requires research to re-examine the structure and magnitude of the return-volatility asymmetry. In this chapter, we use simulation studies to show that estimating the existing ASV models with just daily or lower frequency return data leads to erroneous estimation of the return-volatility relation. To be specific, both the ASV-HS and

ASV-JPR models only reflect the general asymmetry. However, they only represent the asymmetry according to their own return-volatility correlation structure if the true correlation structure is different from their specification. The simulation evidence shows that it is difficult to differentiate the lead-lag or synchronous return-volatility correlations by fitting these existing ASV models with just return data. In other words, significant (far from zero) correlation coefficients from fitting the return-based ASV models, are insufficient to draw the conclusion that the true return-volatility relation is consistent with the corresponding ASV models' correlation structures.

The erroneous correlation could be a result of inadequate volatility measures. In the traditional SV model, the latent volatility process is inferred from the return series according to the corresponding ASV model's specification. So the estimated correlation parameter only reflects correlation of the return series and the inferred latent volatility process, but may be inconsistent with the true return-volatility correlation coefficient. To address this erroneous return-volatility correlation, we propose a simple solution by incorporating accessible volatility measures, such as volatility index (VIX), realized volatility (RV) and bi-power variation (BV). By jointly modeling the equity return and corresponding volatility measures (see Takahashi et al. 2009), the latent volatility process is no longer entirely or endogenously determined by the return series alone. We find that correlation coefficients differ significantly from the original ASV models (ASV-HS and ASV-JPR). Different volatility measures favor different volatility structures. Consistent with our expectations, the forward looking volatility measure (VIX) supports stronger synchronous correlation and weaker lead-lag correlation compared to RV and BV. Out-of-sample forecasts support our conclusion that ASV models with volatility measures greatly improve the out-of-sample return density prediction.

This chapter is organized as follows. Section 3.2 presents the ASV-HS/ASV-JPR models and their corresponding model specifications given volatility measures. Section 3.3 further illustrates how the ASV-HS model could generate pseudo-correlation coefficients and how incorporation of a volatility measure could mitigate this problem. Section 3.4 discusses the data set covered in this chapter. Section 3.5 presents the empirical results of in-sample estimation and out-of-sample return density forecasts. Section 3.6 concludes the chapter.

3.2 Model Specification

3.2.1 Asymmetric Stochastic Volatility Models

The continuous asymmetric stochastic volatility process of the logarithmic equity price S_t and the corresponding centered logarithmic volatility process $\log(\sigma_t^2)$ (see Yu 2005) are:

$$dS_t = \sigma_t \sigma_y dB_t^y, \tag{3.1}$$

$$d\ \log(\sigma_t^2) = \beta \log(\sigma_t^2) dt + \sigma_h dB_t^h. \tag{3.2}$$

There is no constant term in the volatility process (equation. (3.2)) under our specification. However, it is equivalent to having a constant term σ_y in equation (3.1) so that the volatility process is a centered volatility process with an unconditional mean of 0. dB_t^y and dB_t^h are two Brownian processes with $B_t^y dB_t^h = \rho dt$. The consensus on this correlation coefficient is $\rho < 0$.

To estimate the discrete time SV model, we need to discretize this continuous model. We can choose either forward difference ($dt \approx \triangle t = t + 1 - t$) or backward difference ($dt \approx \triangle t = t - t - 1$). If we choose backward difference in the price equation (3.1), i.e. $dS_t \approx S_t - S_{t-1}$ and forward difference in the volatility equation (3.2), i.e. $d\ \log(\sigma_t^2) \approx$

$\log(\sigma_{t+1}^2) - \log(\sigma_t^2)$, the resulting discrete stochastic volatility model will be the ASV-HS model of Harvey and Shephard 1996 in equations (3.3)-(3.5). Current return innovation u_t and future volatility innovation v_{t+1} jointly follow a bi-variate normal distribution.

$$y_t \equiv \log(S_t) - \log(S_{t-1}) = \exp(h_t/2)u_t, \tag{3.3}$$

$$h_{t+1} = \delta h_t + v_{t+1}, \tag{3.4}$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix} \right). \tag{3.5}$$

This specification is consistent with the leverage effect interpretation as we can see from the following uni-variate representation of the ASV-HS model:

$$y_t = \mu + \exp(h_t/2)u_t, \qquad u_t \sim N(0,\ \sigma_y^2), \tag{3.6}$$

$$h_{t+1} = \delta h_t + \frac{\rho\sigma_h}{\sigma_y}u_t + \sqrt{1-\rho^2}\ w_{t+1}, \qquad w_{t+1} \sim N(0,\ \sigma_h^2). \tag{3.7}$$

In equation (3.7), future volatility h_{t+1} can be treated as a dependent variable with current volatility h_t and return innovation u_t as independent variables. The correlation coefficient measures the effect of current return innovation on future volatility. The return-volatility correlation structure of the ASV-HS model is a lead-lag relation in which current equity return is associated with future volatility and is consistent with the leverage effect interpretation of the asymmetry.

If we take backward difference on both the price and volatility equations ($dS_t \approx S_t - S_{t-1}$ and $d\ \log(\sigma_t^2) \approx \log(\sigma_t^2) - \log(\sigma_{t-1}^2)$ for (3.1) and (3.2)), the resulting discrete stochastic volatility model will be the ASV-JPR model of Jacquier et al. 2004 in equations (3.8)-(3.10). Synchronous return innovation u_t and volatility innovation v_t jointly follow a bi-variate normal distribution.

$$y_t \equiv \log(S_t) - \log(S_{t-1}) = \exp(h_t/2)u_t, \tag{3.8}$$

$$h_t = \delta h_{t-1} + v_t, \tag{3.9}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{3.10}$$

Similarly, the following uni-variate representation of the ASV-JPR model shows that there is no lead-lag relation between the return and volatility.

$$y_t = \mu + \exp(h_t/2)u_t, \qquad u_t \sim N(0,\ \sigma_y^2), \tag{3.11}$$

$$h_t = \delta h_{t-1} + \frac{\rho\sigma_h}{\sigma_y}u_t + \sqrt{1-\rho^2}\ w_t, \qquad w_t \sim N(0,\ \sigma_h^2). \tag{3.12}$$

Given equation (3.12), we are unable to identify either volatility feedback or the leverage effect (Men et al. 2017) as the return and volatility innovations are immediate in this model. The return-volatility correlation structure of the ASV-JPR model is consistent with the synchronous asymmetric correlation, especially at lower frequencies, like weekly or even monthly, where we cannot differentiate between return and volatility in the lead-lag relation.

Interestingly, if we take the forward difference in the price equation (3.1) ($dS_t \approx S_{t+1} - S_t$) and the backward difference in the volatility equation (3.2) ($d\ \log(\sigma_t^2) \approx \log(\sigma_t^2) - \log(\sigma_{t-1}^2)$) we obtain the following asymmetric stochastic volatility model, which is consistent with the volatility feedback effect (ASV-VFB hereafter):

$$y_{t+1} \equiv \log(S_{t+1}) - \log(S_t) = \exp(h_{t+1}/2)u_{t+1}, \tag{3.13}$$

$$h_t = \delta h_{t-1} + v_t, \tag{3.14}$$

$$\begin{bmatrix} u_{t+1} \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{3.15}$$

We are not proposing the ASV-VFB model as a reasonable ASV model. We only use this model for data generation purposes as the return-volatility correlation structure is consistent with the volatility feedback assumption given the following uni-variate representation:

$$y_{t+1} = \exp(h_{t+1}/2)(\frac{\rho\sigma_y}{\sigma_h}v_t + \sqrt{1-\rho^2}\ w_{t+1}), \qquad w_{t+1} \sim N(0,\ \sigma_y^2), \tag{3.16}$$

$$h_t = \delta h_{t-1} + v_t, \qquad v_t \sim N(0,\ \sigma_h^2). \tag{3.17}$$

3.2.2 ASV Models with Volatility Measures

If we have a volatility measure denoted as VM_t, we can jointly model the return y_t and the volatility measure VM_t (see Takahashi et al. 2009). Combining the ASV-HS model with a volatility measure leads to the following VM-ASV-HS model specification:

$$y_t = \exp(h_t/2)u_t, \tag{3.18}$$

$$h_{t+1} = \delta h_t + v_{t+1}, \tag{3.19}$$

$$VM_t = a + bh_t + e_t, \qquad e_t \sim N(0, \sigma_{VM}^2), \tag{3.20}$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{3.21}$$

The volatility measure VM_t can be the volatility index(VIX), realized volatility(RV) or bi-power variation(BV). For notation purposes, we will denote them as VIX-ASV-HS, RV-ASV-HS, and BV-ASV-HS respectively. Similarly, we could also extend the ASV-JPR model with a volatility measure as follows:

$$y_t = \exp(h_t/2)u_t, \tag{3.22}$$

$$h_t = \delta h_{t-1} + v_t, \tag{3.23}$$

$$VM_t = a + bh_t + e_t, \qquad e_t \sim N(0, \sigma_{VM}^2), \tag{3.24}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{3.25}$$

Under the state space model specification, we do not need the potential volatility measure VM_t (can be either implied volatility, realized volatility, bi-power variation, etc.) to be equal to the latent volatility. As long as the volatility measure is correlated with the true volatility process, we can use it as extra information to identify the latent volatility h_t. Note that the volatility measure is not necessarily linearly correlated with the latent volatility with normal noise $e_t \sim N(0, \sigma_{VM}^2)$ as in equation (3.20) and (3.24). Investigating the exact non-linear relation between the potential volatility measure and the true latent volatility is beyond the scope of this paper. However, a simple linear relation works well to help us study the asymmetry.

For the balance of this chapter, we use the following specification of VM-ASV-HS:

$$y_t = \exp(h_t/2)u_t, \tag{3.26}$$

$$h_{t+1} = \delta h_t + v_{t+1}, \tag{3.27}$$

$$\widehat{VM}_t = bh_t + e_t, \qquad e_t \sim N(0, \sigma^2_{VM}), \tag{3.28}$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{3.29}$$

Given the specification of the latent volatility, the long run mean $E(h_t) = 0$. However, the volatility measures will have different means. So we use the centered volatility measure: $\widehat{VM}_t = VM_t - \overline{VM}_t$, where $\overline{VM}_t$ is the sample mean of the volatility measure series. Without loss of generality, excluding the constant parameter in the volatility measure equation and centering the volatility measure data improves the identification of the model.

Similarly, the VM-ASV-JPR model we will use for the remaining this chapter is:

$$y_t = \exp(h_t/2)u_t, \tag{3.30}$$

$$h_t = \delta h_{t-1} + v_t, \tag{3.31}$$

$$\widehat{VM}_t = bh_t + e_t, \qquad e_t \sim N(0, \sigma^2_{VM}), \tag{3.32}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{3.33}$$

3.3 Simulation Study

3.3.1 ASV Models with Returns

We use Bayesian MCMC to estimate the ASV-HS, ASV-JPR, VM-ASV-HS and VM-ASV-JPR models. (Details regarding the MCMC steps can be found in Appendix C1.) We set the total number of MCMC iterations to be 100,000 and we keep 1 for every 10 draws. We also set 5,000 burn-in iterations for parameter convergence. To avoid effects of prior choice, we set flat prior distributions as follows:

- δ: Conjugate truncated normal distribution $N(0.98, 4)I_{|\delta|<1}$. A normal distribution with mean 0.98 and variance 4 and truncated for stationary.
- $\sigma_y,\ \sigma_h,\ \rho$: Conjugate Wishart distribution $W(n, S_0^{-1})$, where $n = 3$ and

  $$S_0 = \begin{bmatrix} 0.85^2 & 0 \\ 0 & 0.2^2 \end{bmatrix}.$$

  We set zero correlation in the inverse of the scale matrix. The choices of degree of freedom guarantees the Wishart prior is uninformative.
- b: Conjugate normal distribution $N(1, 10)$.
- σ_{VM}: Conjugate inverse gamma prior distribution $IG(\frac{3}{2}, \frac{0.1}{2})$.

First, let's examine the estimation results of fitting the ASV-HS and ASV-JPR model with the full sample daily and weekly S&P500 index returns. The data range from 1957-03 to 2019-06 (15688 daily and 3252 weekly observations). Table 3.1 summarizes the estimation results.

Para	ASV-HS Day Mean	Stdev	ASV-HS Week Mean	Stdev	ASV-JPR Day Mean	Stdev	ASV-JPR Week Mean	Stdev
δ	0.9795	0.0019	0.9404	0.0101	0.9852	0.0015	0.9605	0.0063
	(0.9756, 0.9832)		(0.9194, 0.9587)		(0.9821, 0.9880)		(0.9475, 0.9720)	
σ_y	0.7591	0.0216	1.7653	0.0629	0.8573	0.0310	1.9965	0.0867
	(0.7213, 0.8043)		(1.6409, 1.8908)		(0.8026, 0.9229)		(1.8387, 2.1792)	
σ_h	0.1810	0.0075	0.2693	0.0246	0.1583	0.0064	0.2371	0.0185
	(0.1670, 0.1971)		(0.2246, 0.3200)		(0.1459, 0.1710)		(0.2020, 0.2742)	
ρ	-0.5701	0.0225	-0.5753	0.0437	-0.6004	0.0245	-0.6738	0.0382
	(-0.6129, -0.5258)		(-0.6587, -0.4851)		(-0.6457, -0.5505)		(-0.7431, -0.5935)	

Inside the parentheses is the 95% density interval.

Table 3.1: ASV-HS and ASV-JPR with the full sample S&P500 daily and weekly returns.

Table 3.1 confirms the generally accepted asymmetry as both models lead to correlation coefficients around -0.6. However, the results may be confusing as current returns can be correlated with either current or future volatility from both a daily or weekly perspective. Yu 2005 shows the the ASV-HS model is preferred to ASV-JPR based on the marginal likelihood. However, it is still not sufficient to conclude the correlation coefficient parameter in the ASV-HS model reflects the true return-volatility correlation structure. In fact, we expect the leverage effects to be weak for the weekly returns as the return and volatility changes tend to be synchronous and the lead-lag return-volatility relation should be weaker. However, as shown in the second column of Table 3.1, we still observe a strong lead-lag return-volatility correlation under weekly frequency and the correlation coefficients ρ are basically the same as those of the daily returns. Now we move on to the following simulation study and illustrate how ASV models (both ASV-HS and ASV-JPR) fail to identify the true return-volatility relation.

We generate simulated data according to the following examples of ASV-HS, ASV-JPR, ASV-FB and SV models, which correspond to leverage effect, synchronous correlation, volatility feedback and no correlation. The parameters for data simulation are chosen according to the estimation results of Table 3.1 so that we are simulating data reasonably. Actually, all the simulation results of this section are robust to different parameter settings.

1. To represent the leverage effects, i.e., return leads volatility, we have the following ASV-HS simulation:

$$y_t = \exp(h_t/2)u_t, \tag{3.34}$$

$$h_{t+1} = 0.98h_t + v_{t+1}, \tag{3.35}$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.85^2 & -0.6 \times 0.85 \times 0.2 \\ -0.6 \times 0.85 \times 0.2 & 0.2^2 \end{bmatrix}\right). \tag{3.36}$$

2. To represent the synchronous correlation, i.e., return and volatility changes are contemporaneous, we have the following ASV-JPR simulation:

$$y_t = \exp(h_t/2)u_t, \tag{3.37}$$

$$h_t = 0.98h_{t-1} + v_t, \tag{3.38}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.85^2 & -0.6 \times 0.85 \times 0.2 \\ -0.6 \times 0.85 \times 0.2 & 0.2^2 \end{bmatrix}\right). \tag{3.39}$$

3. To represent the volatility feedback, i.e., volatility leads return, we have the following ASV-VFB simulation:

$$y_{t+1} = \exp(h_{t+1}/2)u_{t+1}, \tag{3.40}$$

$$h_t = 0.98h_{t-1} + v_t, \tag{3.41}$$

$$\begin{bmatrix} u_{t+1} \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.85^2 & -0.6 \times 0.85 \times 0.2 \\ -0.6 \times 0.85 \times 0.2 & 0.2^2 \end{bmatrix}\right). \tag{3.42}$$

4. If there is no return-volatility correlation ($\rho = 0$), all ASV specifications will

degenerate into a symmetric SV model. So we also include the following SV simulation for comparison:

$$y_t = \exp(h_t/2)u_t, \tag{3.43}$$

$$h_t = 0.98h_{t-1} + v_t, \tag{3.44}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.85^2 & 0 \\ 0 & 0.2^2 \end{bmatrix}\right). \tag{3.45}$$

We simulate 4000 return observations of the ASV-HS, ASV-JPR, ASV-VFB and SV examples respectively. Then we fit the ASV-HS and ASV-JPR models with these simulated return series. The estimation results are summarized in Tables 3.2 and 3.3.

	HS Simulation		**JPR Simulation**		**VFB Simulation**		**SV Simulation**	
Para	**Mean**	**True**	**Mean**	**True**	**Mean**	**True**	**Mean**	**True**
δ	0.9793	0.98	0.9757	0.98	0.9811	0.98	0.9717	0.98
	[0.0033]		[0.0044]		[0.0038]		[0.0038]	
	(0.9725, 0.9853)		(0.9665, 0.9836)		(0.9731, 0.9880)		(0.9605, 0.9812)	
σ_y	0.8284	0.85	0.7986	0.85	0.8199	0.85	0.8329	0.85
	[0.0556]		[0.0461]		[0.0614]		[0.0576]	
	(0.7289, 0.9493)		(0.7155, 0.8956)		(0.7103, 0.9442)		(0.7345, 0.9649)	
σ_h	0.2132	0.2	0.1881	0.2	0.1940	0.2	0.2093	0.2
	[0.0139]		[0.0139]		[0.0139]		[0.0155]	
	(0.1873, 0.2416)		(0.1615, 0.2165)		(0.1670, 0.2248)		(0.1806, 0.2413)	
ρ	-0.5856	-0.6	-0.4412	0	-0.3227	0	0.0083	0
	[0.0444]		[0.0564]		[0.0604]		[0.0575]	
	(-0.6676, -0.4934)		(-0.5484, -0.3280)		(-0.4369, -0.2020)		(-0.1031, 0.1195)	

The table reports standard deviation in [] and 0.95 density interval in ().

TABLE 3.2: Fitting the ASV-HS model with simulated data

Table 3.2 shows the results of estimating the ASV-HS model with ASV-HS, ASV-JPR, ASV-VFB and SV simulated returns. The first column confirms that the ASV-HS model can correctly identify the true parameters if the return series is truly generated as the ASV-HS model. This also supports the correctness of our Bayesian MCMC estimation. In addition, fitting the ASV-HS model with a symmetric SV simulated return series further confirms that the model can identify the symmetric SV as a special case of asymmetric SV.

However, when we fit the ASV-HS model with ASV-JPR and ASV-VFB simulated returns, the model erroneously deduces non-zero correlation coefficients (the 95% density intervals also do not include zero). Given the ASV-JPR and ASV-VFB specification, the current return is not related to the future volatility change and we expect estimated ρ close to zero when fitting the ASV-HS model with those simulated returns. However, as we can see from Table 3.2, fitting ASV-HS with ASV-JPR (ASV-VFB) simulation leads to $\rho = -0.44$ ($\rho = -0.32$), which is far from zero. In fact, even the upper bounds of the 95% density intervals are far from zero. The simulation results here show that the estimated correlation coefficient ρ of the ASV-HS model should not be interpreted as a causal relation. In other words, the estimated parameter ρ of the ASV-HS model is not necessarily supportive evidence for the leverage effect. In fact, even if the true asymmetric correlation is volatility changes leading returns (ASV-VFB), the ASV-HS still yields ρ far from zero.

	HS simulation		**JPR simulation**		**VFB simulation**		**SV simulation**	
Para	**Mean**	**True**	**Mean**	**True**	**Mean**	**True**	**Mean**	**True**
δ	0.9833	0.98	0.9785	0.98	0.9835	0.98	0.9720	0.98
	[0.0031]		[0.0039]		[0.0033]		[0.0053]	
	(0.9770, 0.9889)		(0.9704, 0.9857)		(0.9765, 0.9895)		(0.9610, 0.9816)	
σ_y	0.9167	0.85	0.8523	0.85	0.8720	0.85	0.8469	0.85
	[0.0768]		[0.0495]		[0.0626]		[0.0559]	
	(0.7764, 1.0828)		(0.7582, 0.9507)		(0.7576, 0.9963)		(0.7571, 0.9747)	
σ_h	0.1898	0.2	0.1956	0.2	0.1916	0.2	0.2079	0.2
	[0.0131]		[0.0130]		[0.0134]		[0.0157]	
	(0.1660, 0.2172)		(0.1716, 0.2223)		(0.1667, 0.2192)		(0.1786, 0.2400)	
ρ	-0.4451	0	-0.5857	-0.6	-0.5270	0	0.0220	0
	[0.0547]		[0.0442]		[0.0502]		[0.0575]	
	(-0.5487, -0.3347)		(-0.6696, -0.4945)		(-0.6215, -0.4242)		(-0.0923, 0.1386)	

The table reports standard deviation in [] and 0.95 density interval in ().

Table 3.3: The ASV-JPR model with simulated data

Nevertheless, a similar issue exists in the ASV-JPR model. Table 3.3 summarizes the estimation results of fitting the ASV-JPR model with ASV-HS, ASV-JPR, ASV-VFB and SV simulated returns. Similarly, if the return series is generated by the ASV-JPR model or the symmetric SV model, the ASV-JPR model can identify the true parameters correctly. However, the model will also erroneously deduce non-zero correlation coefficients that are far from zero if the true return-volatility correlation structure is either specified as ASV-HS (leverage effect) or ASV-VFB (volatility feedback). Similarly the estimated parameter ρ of the ASV-JPR model does not necessarily serve as evidence of the synchronous return-volatility correlation.

To conclude, both ASV models are consistent with the general asymmetry, i.e. the negative correlation between return and volatility, and both ASV models can also identify the correct correlation coefficient ($\rho = 0$) if there is no asymmetry. However, when there is asymmetry, both ASV models can only reflect the asymmetry according to their own correlation structure even if the true structure is different. This could be misleading when we are trying to interpret the correlation coefficient parameter ρ. Given the simulation study, irrespective of whether the exact return-volatility correlation is return leading volatility (leverage effect), volatility leading return (volatility feedback) or synchronous correlation, both asymmetric specifications (ASV-HS and ASV-JPR) will produce significant correlation coefficients far from zero. As a result, if we are trying to differentiate the exact causal or lead-lag relation between return and volatility, the ASV models with return series alone may lead to a conclusion that is inconsistent with the true return-volatility correlation. An estimated correlation coefficient far from zero would lend little support to the correctness of the model's correlation structure.

Consistent with what we observed in Table 3.1, given weekly frequency, the leverage effect should be weak. However, the asymmetry is still valid and strong for weekly returns. So we expect weaker lead-lag correlation, but stronger synchronous correlation.

Consistent with the simulation results, the second column of Table 3.1 shows that ASV-HS still produce a lead-lag return-volatility correlation coefficient with weekly returns. This indicates that the ASV-HS model still represents the asymmetry as a lead-lag correlation even though the leverage effect should be weak for weekly returns.

This arises from the fact that volatility is not directly observable and the stochastic volatility models treat the volatility process as a latent process. More importantly, all the ASV models will infer the latent process according to their own return-volatility correlation specification. As a result, the correlation between the returns and the inferred latent volatility process could be misleading and inconsistent with the true correlation. Now we will show how this problem could be mitigated if we have volatility measures that can be jointly modeled with the return series.

3.3.2 ASV Models with Returns and Volatility Measures

Given each simulated return series and their corresponding true latent volatility process (the ASV-HS simulation, the ASV-JPR simulation and the ASV-VFB simulation respectively), we can further simulate the corresponding volatility measures that are linearly correlated to the latent volatility process as follows.

$$VM_t^{high} = h_t + e_t, \qquad e_t^{high} \sim N(0,\ 0.2^2), \tag{3.46}$$

$$VM_t^{low} = h_t + e_t, \qquad e_t^{low} \sim N(0,\ 0.02^2). \tag{3.47}$$

For comparison purposes, we simulate two volatility measures that differ in the amount of noise. VM_t^{high} is a volatility measure with more noise as e_t^{high} in equation (3.46) has a standard deviation of 0.2 (same as v_t in the simulated latent volatility process). VM_t^{low} is another volatility measure with less noise as the standard deviation of e_t^{low} in equation (3.47) is only 0.02. Now for each ASV simulated return data, we have two corresponding volatility measures. In total we have six simulated return volatility measure pairs, which are as follows:

1. ASV-VFB returns with low noise volatility measure (VFB Low).
2. ASV-VFB returns with high noise volatility measure (VFB High).
3. ASV-JPR returns with low noise volatility measure (JPR Low).
4. ASV-JPR returns with high noise volatility measure (JPR High).
5. ASV-HS returns with low noise volatility measure (HS Low).
6. ASV-HS returns with high noise volatility measure (HS High).

Now we can fit the VM-ASV-HS model with VFB Low, VFB High, JPR Low and JPR High data sets to check if the volatility measure can help to identify the true return-volatility correlation parameter, which should be zero in these four data sets. The estimation results are summarized in Table 3.4.

As expected, inclusion of a low noise volatility measure will help the model to identify the true correlation parameter. Fitting the VM-ASV-HS model with both VFB Low and JPR Low data sets will lead to correlation parameters (ρ) that are very close to zero (-0.0076 and -0.0786 respectively). Even if we have a high noise volatility measure, ρ will also decrease in magnitude (closer to zero) compared to the result in Table 3.2, where we fit the ASV-HS model with returns alone.

Similarly, we can also fit the VM-ASV-JPR model with VFB Low, VFB High, HS Low and HS High data. The results are summarized in Table 3.5. Consistent with the VM-ASV-HS results, inclusion of a volatility measure with low noise drives the

	VFB Low		VFB High		JPR Low		JPR High	
Para	**Mean**	**True**	**Mean**	**True**	**Mean**	**True**	**Mean**	**True**
δ	0.9831	0.98	0.9788	0.98	0.9773	0.98	0.9754	0.98
	[0.0029]		[0.0033]		[0.0033]		[0.0033]	
	(0.9773, 0.9888)		(0.9723, 0.9853)		(0.9708, 0.9837)		(0.9689, 0.9820)	
σ_y	0.8545	0.85	0.8544	0.85	0.8365	0.85	0.8394	0.85
	[0.0095]		[0.0098]		[0.0093]		[0.0096]	
	(0.8358, 0.8732)		(0.8353, 0.8735)		(0.8188, 0.8551)		(0.8207, 0.8584)	
σ_h	0.1894	0.2	0.2063	0.2	0.1844	0.2	0.1849	0.2
	[0.0050]		[0.0073]		[0.0055]		[0.0068]	
	(0.1798, 0.1995)		(0.1922, 0.2208)		(0.1743, 0.1961)		(0.1719, 0.1982)	
ρ	-0.0076	0	-0.0727	0	-0.0786	0	-0.2780	0
	[0.0175]		[0.0238]		[0.0186]		[0.0273]	
	(-0.0422, 0.0265)		(-0.1203, -0.0272)		(-0.1147, -0.0422)		(-0.3308, -0.2238)	
b	0.9926	1	0.9996	1	1.0095	1	1.0173	1
	[0.0218]		[0.0231]		[0.0259]		[0.0269]	
	(0.9500, 1.0346)		(0.9548, 1.0449)		(0.9542, 1.0576)		(0.9684, 1.0745)	
σ_{VM}	0.0593	0.02	0.2004	0.2	0.0610	0.02	0.2099	0.2
	[0.0032]		[0.0048]		[0.0035]		[0.0043]	
	(0.0530, 0.0659)		(0.1910, 0.2099)		(0.0540, 0.0681)		(0.2014, 0.2183)	

The table reports standard deviation in [] and 0.95 density interval in ().

TABLE 3.4: Estimation results of the VM-ASV-HS models with the simulated return and volatility measure pairs.

correlation coefficient ρ very close to zero. Even high noise volatility measures will also decrease the magnitude of ρ compared to Table 3.3.

Given the volatility measure, the latent volatility process $\{h_t\}_{t=1}^n$ will no longer be fully determined by the return process. With an informative (low noise) volatility measure, a greater portion of the latent volatility will be inferred from the volatility measure. As a result, inclusion of the volatility measure will help to mitigate the issue when estimating the correlation coefficient. Now we will move on to the empirical application with real world data.

	VFB Low		VFB High		HS Low		HS High	
Para	Mean	True	Mean	True	Mean	True	Mean	True
δ	0.9827	0.98	0.9800	0.98	0.9861	0.98	0.9849	0.98
	[0.0029]		[0.0030]		[0.0026]		[0.0026]	
	(0.9770, 0.9883)		(0.9741, 0.9857)		(0.9810, 0.9912)		(0.9797, 0.9900)	
σ_y	0.8597	0.85	0.8613	0.85	0.8370	0.85	0.8391	0.85
	[0.0098]		[0.0099]		[0.0093]		[0.0097]	
	(0.8411, 0.8790)		(0.8424, 0.8809)		(0.8190, 0.8554)		(0.8202, 0.8583)	
σ_h	0.1873	0.2	0.1908	0.2	0.1818	0.2	0.1816	0.2
	[0.0051]		[0.0063]		[0.0045]		[0.0061]	
	(0.1776, 0.1978)		(0.1785, 0.2035)		(0.1732, 0.1908)		(0.1699, 0.1939)	
ρ	-0.0600	0	-0.2726	0	-0.0534	0	-0.2430	0
	[0.0190]		[0.0268]		[0.0191]		[0.0276]	
	(-0.0974, -0.0231)		(-0.3245, -0.2192)		(-0.0909, -0.0166)		(-0.2968, -0.1888)	
b	0.9917	1	1.0063	1	0.9984	1	1.0108	1
	[0.0214]		[0.0216]		[0.0185]		[0.0210]	
	(0.9489, 1.0331)		(0.9655, 1.0523)		(0.9630, 1.0357)		(0.9714, 1.0545)	
σ_{VM}	0.0631	0.02	0.2125	0.2	0.0638	0.02	0.2136	0.2
	[0.0037]		[0.0045]		[0.0037]		[0.0044]	
	(0.0561, 0.0704)		(0.2038, 0.2213)		(0.0567, 0.0711)		(0.2050, 0.2222)	

The table reports standard deviation in [] and 0.95 density interval in ().

TABLE 3.5: Estimation results of the VM-ASV-JPR models with the simulated return and volatility measure pairs.

3.4 Equity Index and Volatility Products

3.4.1 Stock Market Indices and Volatility Measures

We focus on the following four equity indexes and their corresponding volatility measures (IV, RV, BV) from January 2004 to June 2019:

- The S&P 500 Index (SPX) and the corresponding volatility measures: Volatility Index (VIX), Realized Volatility and Bi-power Variation, with a total of 3,899 daily and 808 weekly observations.
- The Russell 2000 Index (RUT) and the corresponding volatility measures: Volatility Index (RVX), Realized Volatility and Bi-power Variation, with a total of 3,899 daily and 808 weekly observations.

- The EURO STOXX 50 (STOXX) and the corresponding volatility measures: Volatility Index (VSTOXX), Realized Volatility and Bi-power Variation, with a total of 3,974 daily and 808 weekly observations.

- The Hong Kong Hang Seng Index (HSI) and the corresponding volatility measures: Volatility Index (VHSI), Realized Volatility and Bi-power Variation, with a total of 3,819 daily and 808 weekly observations.

The CBOE Volatility Index (VIX), closely followed by market participants and global media, is the most recognized measure of the volatility of the U.S. equity market. The VIX index went through a significant change on September 22, 2003. Before September 22, 2003, the index tracked the S&P 100 index whereas the CBOE switched to the S&P 500 index thereafter. The CBOE also adopted a revised methodology to calculate the index. Considering that all other major volatility indices, including the Europe and Hong Kong equity markets, follow a similar method of volatility index calculation, we set our sample period from January 2004 to June 2019 for all equities and their volatility measures. This choice is also consistent with the trading of volatility products as the major tradable volatility products were launched during this period. The S&P 500 index and the Russell 2000 index are highly correlated, they may have similar volatility processes. The inclusion of STOXX and HSI supports the robustness of our empirical results as the European and Hong Kong markets' volatility processes will differ from that of the U.S. market.

We summarize the trading of volatility products related to the equity indices in our sample as follows:

- Chicago Board Options Exchange (CBOE) launched the VIX futures in 2004 and soon became the major volatility product in the market. The VIX option was introduced in 2006. There are several (leveraged) ETN/ETF products with VIX as the underlying index. The major VIX-based ETN, according to trading volume, is the iPath Series B S&P 500 VIX Short-Term Futures ETN (VXX) launched in 2009.

- Chicago Board Options Exchange(CBOE) launched the RVX futures in 2007. It was delisted in 2010 and then relaunched in 2013. However, it was discontinued in 2018. No related ETFs/ETNs are traded.

- Eurex Exchange (EUREX) launched the VSTOXX futures in 2005 and then revised the future contracts in 2009. The revised contracts have 1/10 of the standard contract size. The option of the VSTOXX index was initially listed in 2010. In 2017, EUREX introduced options with the VSTOXX futures as the underlying asset, and the original option with the VSTOXX index as the underlying asset was discontinued. ETN/ETF products based on VSTOXX were introduced in 2009.

- Hong Kong Exchanges and Clearing Limited (HKEx) launched the VHSI futures in 2012.

Given our sample period, VIX-related products are the most popular products in the U.S. equity market, whereas VSTOXX-related products are the most popular volatility products outside the U.S., and provide investors exposure to European market volatility. The trading of volatility indices makes them no longer by-products of equity options and enables investors to profit directly from volatility predictions. Consistent with the well-documented asymmetry, all the volatility indices are negatively correlated with their underlying equity indices. This allows investors to construct further diversified portfolios by holding the equity indices' component companies and the volatility indices. If the volatility market is efficient, the volatility index should include all the current information, including the current underlying equity index returns so that there should be no arbitrage opportunity in the volatility markets by utilizing the leverage effect. However, if strong correlation between current returns and future volatility changes exists, as in the ASV-HS model in Table 3.1 ($\rho = -0.59$ given the full sample estimation results of the ASV-HS model with the S&P 500 daily returns), there could be arbitrage opportunities with volatility products as the volatility changes are highly predictable with current returns. As we see in the following empirical results, including VIX as the volatility measure would lead to stronger contemporaneous return-volatility correlation and weaker lead-lag return-volatility correlation, which is consistent with the market efficiency of the volatility products. Moreover even including RV/BV will lead to a much weaker lead-lag return-volatility correlation compared to the ASV-HS model with return series alone.

3.5 Estimation Results and Forecasts

3.5.1 In-sample Estimation Results

In this section, we summarize the estimation results of fitting the ASV-HS, ASV-JPR, VM-ASV-HS, VM-ASV-JPR with the data sets in Section 3.4. Again, we set the total number of MCMC iterations to be 100,000 and keep 1 for every 10 draws. We also set 5,000 burn-in iterations for parameter convergence. For the following estimations, we choose the prior distributions as follows:

- δ: Conjugate truncated normal distribution $TN(0.95, 4)I_{|\delta|<1}$. Truncated normal distribution for stationary.

- $\sigma_y,\ \sigma_h,\ \rho$: Conjugate Wishart distribution $W(n, S_D^{-1})$ for daily data and $W(n, S_W^{-1})$ for weekly data, where $n = 3$ and

$$S_D = \begin{bmatrix} 0.85^2 & 0 \\ 0 & 0.2^2 \end{bmatrix} \quad and \quad S_W = \begin{bmatrix} 2.0^2 & 0 \\ 0 & 0.3^2 \end{bmatrix}.$$

.

- b: Conjugate normal prior distribution $N(1, 10)$.

- σ_{VM}: Conjugate inverse gamma prior distribution $IG(\frac{3}{2}, \frac{0.5}{2})$

First, we report the estimation results of the ASV-HS and ASV-JPR model with the daily and weekly S&P500 index returns ranging from 2004-01 to 2019-06. The results are summarized in Table 3.6.

	ASV-HS Day		**ASV-HS Week**		**ASV-JPR Day**		**ASV-JPR Week**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9683	0.0043	0.9008	0.0268	0.9803	0.0026	0.9461	0.0113
	(0.9593, 0.9762)		(0.8416, 0.9450)		(0.9749, 0.9851)		(0.9223, 0.9662)	
σ_y	0.8106	0.0373	1.7554	0.1055	1.1421	0.0791	2.2878	0.1977
	(0.7425, 0.8896)		(1.5598, 1.9753)		(1.0070, 1.2850)		(1.9384, 2.6982)	
σ_h	0.2650	0.0172	0.4037	0.0600	0.2213	0.0125	0.3225	0.0353
	(0.2326, 0.3000)		(0.2957, 0.5318)		(0.1981, 0.2461)		(0.2568, 0.3953)	
ρ	-0.7410	0.0320	-0.6855	0.0625	-0.8225	0.0266	-0.8159	0.0449
	(-0.7990, -0.6731)		(-0.7951, -0.5528)		(-0.8689, -0.7637)		(-0.8892, -0.7117)	

Inside the parentheses is the 95% density interval.

TABLE 3.6: Estimation results of ASV models with S&P500 daily and week returns.

For this sample period, we observe stronger asymmetry between return and volatility compared to the full sample results in Table 3.1 for both the ASV-HS and ASV-JPR models. As for the ASV-JPR model, the synchronous return-volatility relation is basically the same for daily or weekly returns. The similarity is expected as the synchronous return-volatility correlation should be similar from a daily or weekly perspective. In contrast, the lead-lag return-volatility (leverage effect) should be weaker for weekly frequency. However, Table 3.1 shows that the posterior mean of ρ from the weekly ASV-HS model is -0.69, which is not significantly different from that of the daily ASV-HS model. Considering the previous simulation study, the correlation coefficient from the weekly ASV-HS model may not truly represent the leverage effect in the weekly results.

Table 3.7 and 3.8 summarize the estimation results of the VM-ASV-HS and VM-ASV-JPR models with the S&P 500 daily returns and the corresponding volatility measures (RV, BV and VIX). As expected, RV and BV have relatively high noise ($\sigma_{VM} = 0.44$ for RV and $\sigma_{VM} = 0.41$ for BV) compared to that of VIX ($\sigma_{VM} = 0.04$ for VIX). Furthermore, the noise of BV is slightly smaller than that of RV since BV is robust from jumps.

Regarding the correlation coefficient parameter ρ, first, we focus on the lead-lag return-volatility correlation (the ASV-HS and VM-ASV-HS models). Including volatility measures leads to significantly weaker (smaller in magnitude) lead-lag correlation. When we use the VIX index as the volatility measure, the correlation coefficient of the VIX-ASV-HS model is -0.23, which is much weaker than that of the ASV-HS model with return data alone ($\rho = -0.74$). RV and BV also lead to significantly weaker correlation, while the difference is not as much as that of VIX. From the perspective of synchronous return-volatility correlation (the ASV-JPR and VM-ASV-JPR models), we find that the VIX-ASV-JPR model yields a correlation coefficient ($\rho = -0.81$), which is basically the same as that of the ASV-JPR model ($\rho = -0.82$) with return data alone. However, the RV-ASV-JPR and BV-ASV-JPR models lead to relatively weaker correlation ($\rho = -0.63$ and $\rho = -0.66$ respectively).

The results here support the fact that asymmetry exists both synchronously and asynchronously (lead-lag correlation) in real world data. Both ASV models will attribute

Para	RV-ASV-HS		BV-ASV-HS		VIX-ASV-HS	
	Mean	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9496	0.0050	0.9509	0.0048	0.9836	0.0026
	(0.9397, 0.9591)		(0.9411, 0.9601)		(0.9786, 0.9886)	
σ_y	0.7629	0.0094	0.7697	0.0094	0.8066	0.0092
	(0.7445, 0.7816)		(0.7518, 0.7885)		(0.7885, 0.8247)	
σ_h	0.3203	0.0137	0.3095	0.0124	0.1647	0.0050
	(0.2942, 0.3479)		(0.2854, 0.3342)		(0.1551, 0.1744)	
ρ	-0.4982	0.0284	-0.4253	0.0299	-0.2275	0.0257
	(-0.5534, -0.4405)		(-0.4833, -0.3658)		(-0.2774, -0.1777)	
b	0.9338	0.0208	0.9277	0.0211	0.3540	0.0079
	(0.8938, 0.9752)		(0.8884, 0.9709)		(0.3394, 0.3699)	
σ_{VM}	0.4436	0.0082	0.4051	0.0081	0.0381	0.0012
	(0.4272, 0.4597)		(0.3892, 0.4211)		(0.0356, 0.0405)	

Table 3.7: The estimation results of ASV-HS models with the S&P500 daily returns and volatility measures.

all the asymmetry to their own return-volatility correlation structure. Inclusion of the volatility measures mitigates the problem. Nevertheless, different volatility measures yield different return-volatility correlations.

The difference is consistent with the fact that VIX is a forward looking volatility measure. Since the leveraged effect is a well-documented phenomenon, the VIX index, as a market-orientated volatility measure, reflects the leverage effect immediately. For example, if the market crashes on day t, the volatility changes corresponding to the leverage effects will be reflected in the VIX index on the same trading day. So, the VIX-ASV-JPR model will have stronger synchronous return-volatility correlation compared to the RV-ASV-JPR and BV-ASV-JPR models as VIX already incorporates the future volatility changes in the same trading day. Consistently, the VIX-ASV-HS model will have weaker lead-lag correlation compared to RV-ASV-JPR and BV-ASV-JPR models since the potential volatility changes from the leverage effect are reflected (or partial reflected) in the VIX index contemporaneously. This also partially explain the predictive

Para	RV-ASV-JPR Mean	RV-ASV-JPR Stdev	BV-ASV-JPR Mean	BV-ASV-JPR Stdev	VIX-ASV-JPR Mean	VIX-ASV-JPR Stdev
δ	0.9613	0.0042	0.9559	0.0044	0.9870	0.0017
	(0.9529, 0.9692)		(0.9471, 0.9643)		(0.9837, 0.9904)	
σ_y	0.7732	0.0094	0.7814	0.0094	0.8166	0.0094
	(0.7549, 0.7916)		(0.7633, 0.7999)		(0.7984, 0.8352)	
σ_h	0.3030	0.0120	0.3232	0.0115	0.1497	0.0037
	(0.2801, 0.3266)		(0.3013, 0.3456)		(0.1424, 0.1568)	
ρ	-0.6252	0.0229	-0.6552	0.0185	-0.8111	0.0078
	(-0.6684, -0.5793)		(-0.6910, -0.6189)		(-0.8261, -0.7953)	
b	0.9789	0.0220	0.9873	0.0211	0.4427	0.0090
	(0.9358, 1.0228)		(0.9469, 1.0292)		(0.4264, 0.4614)	
σ_{VM}	0.4360	0.0077	0.3695	0.0074	0.0283	0.0007
	(0.4210, 0.4512)		(0.3553, 0.3840)		(0.0270, 0.0297)	

TABLE 3.8: The estimation results of ASV-JPR models with the S&P500 daily returns and volatility measures.

power of VIX or implied volatility (IV) on RV as documented by Busch et al. 2011. Since the future RV changes caused by the leveraged effect have already been embedded in current IV changes.

3.5.2 Out-of-sample Forecasts

The in-sample results already shed light on the correlation coefficients of the ASV models. However, as we have shown, forward (VIX) and backward (RV/BV) volatility measures lead to different correlation structures. We need out-of-sample tests to conclude which volatility measure and corresponding correlation structure is better. Considering the return series is our main interest, we calculate the following logarithmic predictive likelihood of the ASV-HS and the VM-ASV-HS models with daily data (see Geweke and Amisano 2010 for details of time series models' comparisons):

$$\sum_{t=T-\tau-1}^{T-1} \log(p_M(y_{t+1}|y_{1:t}, RV_{1:t}, BV_{1:t}, VIX_{1:t})). \tag{3.48}$$

M indicates model specifications, $y_{1:t}, RV_{1:t}, BV_{1:t}$ and $VIX_{1:t}$ represent the available observations of returns and volatility measures at time t. The predictive likelihood measures the predictive performance of one-day ahead return y_{t+1} with available observations at time t according to model M. Given the posterior sampling results $\{\Theta_M^{(i)}\}_{i=1}^N$ by fitting model M with available observations and iid draws $\{\epsilon^{(i)}\}_{i=1}^N$ from a standard normal distribution, the predictive likelihood in Equation (3.48) is estimated as:

$$p_M(y_{t+1}|y_{1:t}, RV_{1:t}, BV_{1:t}, VIX_{1:t}) \approx \frac{1}{N}\sum_{i=1}^{N} f_N(y_{t+1} \mid 0,\ \sigma_{M,y}^{(i)}{}^2 \exp(h_{M,t+1}^{(i)})\ ),$$

where:

$$h_{M,t+1}^{(i)} = \delta_M^{(i)} h_{M,t}^{(i)} + \frac{\rho_M^{(i)}\sigma_{M,h}^{(i)}}{\sigma_{M,y}^{(i)}}\frac{y_t}{\exp(h_{M,t}^{(i)}/2)} + \sqrt{1-\rho_M^{(i)}{}^2}\ \sigma_{M,h}^{(i)}\epsilon^{(i)}.$$

To compare two models M_1 and M_2, the logarithmic predictive Bayes factor of M_1 relative to M_2 is simply defined as:

$$\sum_{t=T-\tau-1}^{T-1} \log(p_{M_1}(y_{t+1}|y_{1:t}, RV_{1:t}, BV_{1:t}, VIX_{1:t})) - \sum_{t=T-\tau-1}^{T-1} \log(p_{M_2}(y_{t+1}|y_{1:t}, RV_{1:t}, BV_{1:t}, VIX_{1:t})). \tag{3.49}$$

A logarithmic predictive Bayes factor greater than 5 lends strong support for model M_1 over model M_2. We choose $\tau = 3000,\ N = 10000$ and report the summary of logarithmic predictive likelihood in Table 3.9:

	S&P500	Russell 2000	STOXX 50	HangSeng Index
ASV-HS	-3929.83	-4907.68	-4712.83	-4907.03
VIX-ASV-HS	-3927.88	**-4874.10**	-4711.21	**-4892.93**
RV-ASV-HS	-3882.43	-4879.67	-4696.74	-4892.93
BV-ASV-HS	**-3865.00**	-4878.29	**-4691.21**	-4897.60

Table 3.9: The predictive log-likelihood of ASV-HS and VM-ASV-HS models

Clearly, volatility measures (both VIX or RV/BV) will improve the out-sample prediction. For the S&P 500 index and the EURO STOXX index, BV-ASV-HS dominates other models whereas VIX does not have a significant improvement compared to the benchmark ASV-HS model. The logarithmic predictive Bayes factor of the VIX-ASV-HS model relative to the ASV-HS model for the S&P 500 index (EURO STOXX 50 index) is 1.95 (1.63). Comparatively, the logarithmic predictive Bayes factor of the BV-ASV-HS model relative to the ASV-HS model for the S&P 500 index (EURO STOXX 50 index) is 64.83 (21.62), which indicates a significant improvement compared to the benchmark ASV-HS model. On the other hand, VIX tends to outperform RV/BV marginally on the Russell 2000 and Hang Seng indices as the log Bayes factor of the VIX-ASV-HS model relative to the BV-ASV-HS model for the Russell 2000 index (Hang Seng index) is 4.19 (4.67). Generally speaking, including volatility measures, either VIX or RV/BV will improve the out-of-sample performance, and BV-ASV-HS tends to have better out-of-sample return density prediction.

The results here further confirm our in-sample findings about the erroneous estimations of the correlation coefficient ρ in the ASV (ASV-HS) models. The in sample estimation results of the ASV-HS model with the daily return series of these four indices show very strong lead-lag correlations ($\rho = -0.74$, -0.77, -0.79 and -0.53 for the S&P 500, Russell 2000, EURO STOXX 50 and Hang Seng indices respectively). The strong lead-lag correlation of the ASV-HS model indicates that the future volatility changes are highly predictable. However, the BV-ASV-HS and RV-ASV-HS lead to much weaker correlations while VIX-ASV-HS has even weaker correlations. In fact, for the Russell 2000 and Hang Seng indices, the VIX-ASV-HS model has correlation coefficients that are close to zero ($\rho = -0.15$, -0.05 respectively). However, VIX-ASV-HS dominates the ASV-HS model for those two indices. It turns out that the strong lead-lag correlation of the ASV-HS model does not necessarily imply a strong prediction of the future true volatility through the leverage effect. Including either VIX or RV/BV will result in a medium or weak lead-lag correlation ($\rho < -0.5$), but dominate ASV-HS in out-of-sample forecasting.

3.6 Conclusion

This paper re-examines asymmetric stochastic volatility models with different return-volatility correlation structures especially under the availability of volatility measures. We find that both ASV-HS and ASV-JPR models only represent the general asymmetry according to their own correlation specification and they attribute all the asymmetric correlation to their own correlation structure. As as result, the estimated correlation coefficient of ASV model with return series alone is not necessarily supportive evidence of the model's correlation structure. In fact, the coefficient ρ only represents the correlation between return and the latent volatility process, which is entirely inferred from the return series according to the model's specification. This estimated coefficient may be inconsistent with the true return-volatility correlation.

This problem can be mitigated if we have volatility measures like RV/BV or VIX. Jointly estimating the volatility measure and the return series will lead to significantly different correlation coefficients compared to the ASV model with return series alone. In addition, we also show that inclusion of volatility measures will greatly improve the out-of-sample predictive performance compared to the benchmark ASV-HS model even if the estimated correlation coefficient of the VM-ASV-HS model implies a much weaker lead-lag return-volatility correlation (leverage effect).

All in all, availability of the volatility measures in recent decades greatly improves the traditional SV/ASV models with return series alone, especially for modelling the asymmetry. In fact, modelling the volatility process itself theoretically and empirically is another popular research area especially for the long memory properties (see e.g. Andersen and Bollerslev 1997, Andersen et al. 2001, Andersen et al. 2003 and Koopman et al. 2005) and the roughness(see e.g. Gatheral et al. 2018, Livieri et al. 2018) of the log-volatility process. Further ASV related research that takes the complexity of the volatility process into consideration would be promising.

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Appendix A

Chapter 1 Supplement

A1 Supplementary Overnight Return Plots

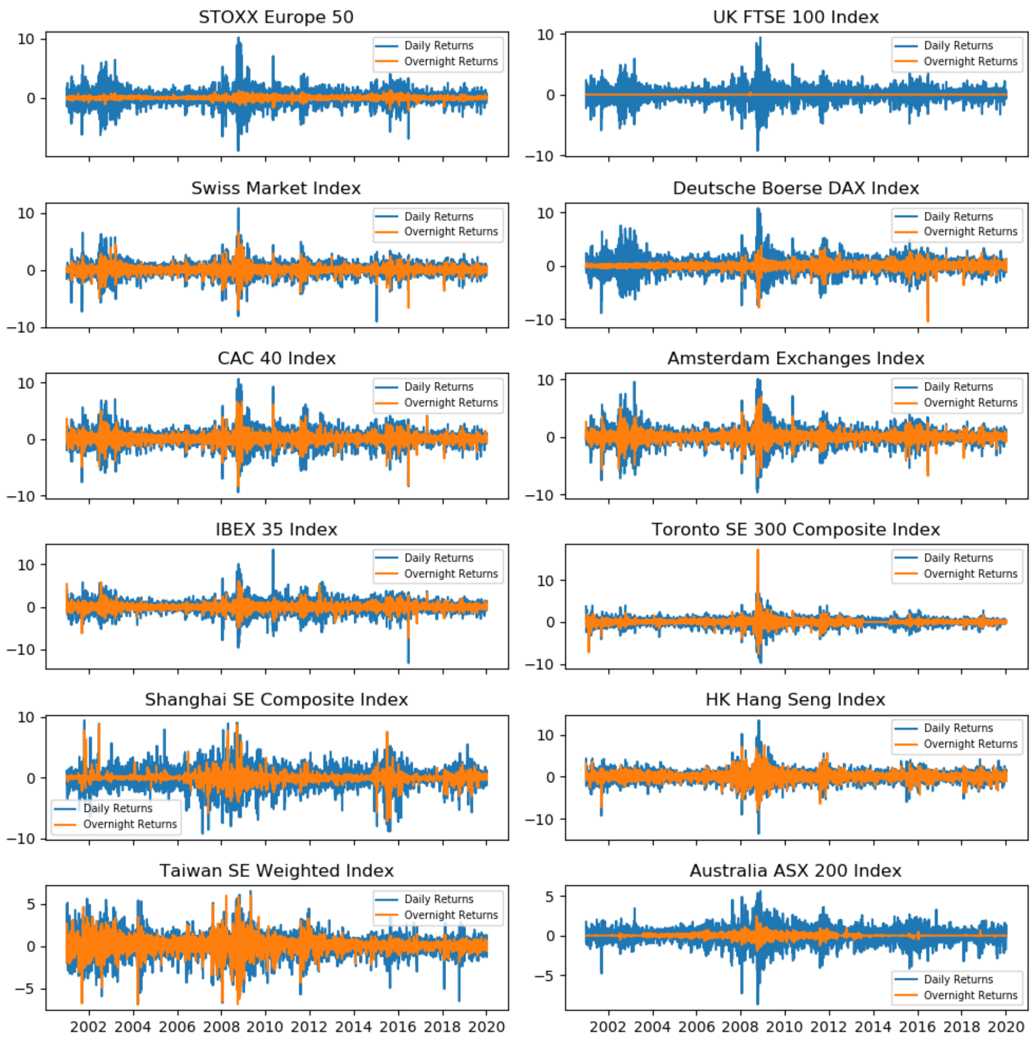

Figure A1.1: Overnight returns of major global market indices.

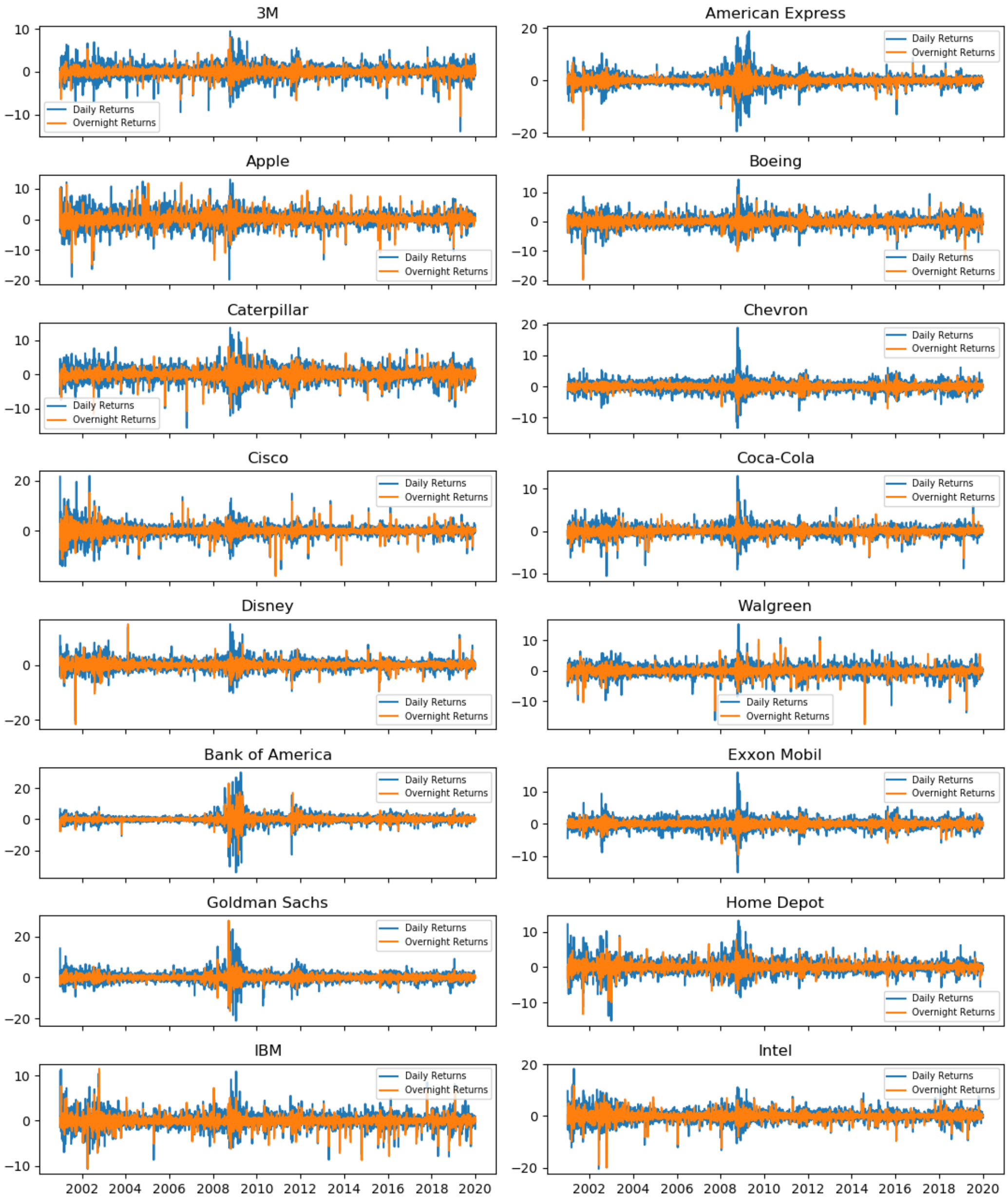

FIGURE A1.2: Overnight return plots of the DJIA.

	RV_t	$RV_{N,t}$	$RV_{D,t}$	BV_t	$BV_{N,t}$	$BV_{D,t}$
Mean	1.1165	0.4002	0.7163	1.0454	0.3469	0.6790
Var	3.0208	0.4878	1.4361	2.6427	0.3974	1.2509
RV_t	1.0000					
$RV_{N,t}$	0.8537	1.0000				
$RV_{D,t}$	0.9528	0.6553	1.0000			
BV_t	0.9877	0.8702	0.9253	1.0000		
$BV_{N,t}$	0.8161	0.9779	0.6136	0.8533	1.0000	
$BV_{D,t}$	0.9581	0.6955	0.9842	0.9549	0.6612	1.0000

Table A1.1: Summary statistics and correlations of QQQ realized volatility measures.

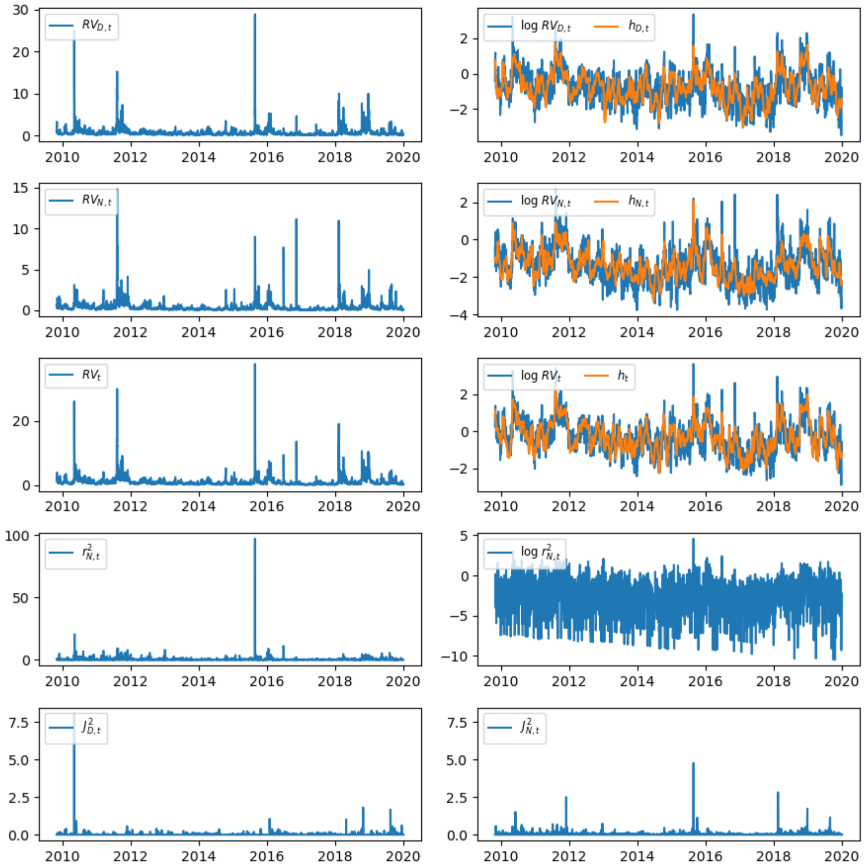

Figure A1.3: QQQ realized volatility measures and squared jumps.

A2 HAR-RV Results for NASDAQ 100

	(1)	(2)	(3)	(4)
μ	-0.0805	-0.0824	-0.1462	-0.1488
	[0.018]	[0.017]	[0.022]	[0.022]
	(-0.115, -0.045)	(-0.116, -0.048)	(-0.190, -0.103)	(-0.192, -0.105)
$\log RV_{D,t-1}$	0.5551	0.4989	0.3258	0.3233
[-1.5ex]	[0.018]	[0.019]	[0.022]	[0.023]
	(0.519, 0.591)	(0.461, 0.537)	(0.282, 0.370)	(0.279, 0.368)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{D,t-i}$	0.2429	0.2870	0.2513	0.2542
	[0.024]	[0.024]	[0.034]	[0.034]
	(0.196, 0.289)	(0.239, 0.335)	(0.186, 0.316)	(0.188, 0.321)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{D,t-i}$	0.1044	0.1117	0.1493	0.1440
	[0.023]	[0.022]	[0.041]	[0.041]
	(0.060, 0.148)	(0.069, 0.155)	(0.068, 0.228)	(0.065, 0.223)
$r_{D,t-1}$	-	-0.1216	-	-0.0526
		[0.014]		[0.014]
		(-0.148, -0.095)		(-0.080, -0.026)
$\log IV_{t-1}$	-	-	1.2131	1.0877
			[0.078]	[0.086]
			(1.060, 1.363)	(0.916, 1.255)
$\frac{1}{4}\sum_{i=2}^{5}\log IV_{t-i}$	-	-	-0.7478	-0.6239
			[0.099]	[0.105]
			(-0.938, -0.557)	(-0.828, -0.415)
$\frac{1}{17}\sum_{i=6}^{22}\log IV_{t-i}$	-	-	-0.1859	-0.1826
			[0.074]	[0.074]
			(-0.333, -0.037)	(-0.327, -0.039)
FMSE	0.3207	0.3116	0.2935	0.2927
LPL	-2129.95	-2093.25	-2016.03	-2010.81
FMSE P.C.	-	-2.84%	-8.48%	-8.73%
LBF	-	36.7	113.92	119.14

Standard deviation in brackets and 95% density interval in parentheses.

TABLE A1.2: Forecasting NASDAQ 100 daytime realized volatility without overnight information.

	(1)	(2)	(3)	(4)	(5)
μ	-0.0616	0.0143	0.0258	-0.0849	-0.0884
	[0.017]	[0.029]	[0.029]	[0.045]	[0.045]
	(-0.094, -0.029)	(-0.043, 0.072)	(-0.030, 0.083)	(-0.172, 0.002)	(-0.176, -0.001)
$\log RV_{D,t-1}$	0.5610	0.3312	0.3685	0.2440	0.2434
	[0.018]	[0.020]	[0.020]	[0.022]	[0.022]
	(0.526, 0.595)	(0.292, 0.371)	(0.329, 0.407)	(0.200, 0.287)	(0.200, 0.286)
$\frac{1}{4}\sum_{i=2}^{5} \log RV_{D,t-i}$	0.2512	0.2097	0.1932	0.2467	0.2448
	[0.023]	[0.035]	[0.033]	[0.035]	[0.035]
	(0.206, 0.296)	(0.142, 0.277)	(0.127, 0.259)	(0.177, 0.317)	(0.178, 0.315)
$\frac{1}{17}\sum_{i=6}^{22} \log RV_{D,t-i}$	0.0974	0.2180	0.2008	0.2240	0.2246
[-1.5ex]	[0.021]	[0.048]	[0.047]	[0.046]	[0.046]
	(0.056, 0.139)	(0.123, 0.313)	(0.109, 0.292)	(0.135, 0.314)	(0.134, 0.316)
$\log RV_{N,t}$	-	0.4282	0.3578	0.2630	0.2585
		[0.021]	[0.021]	[0.022]	[0.022]
		(0.388, 0.468)	(0.317, 0.398)	(0.220, 0.305)	(0.215, 0.301)
$\frac{1}{4}\sum_{i=1}^{4} \log RV_{N,t-i}$	-	-0.1200	-0.0632	-0.0731	-0.0642
		[0.037]	[0.036]	[0.037]	[0.037]
		(-0.193, -0.047)	(-0.133, 0.006)	(-0.146, -0.001)	(-0.138, 0.010)
$\frac{1}{17}\sum_{i=5}^{21} \log RV_{N,t-i}$	-	-0.1624	-0.1495	-0.1398	-0.1467
		[0.046]	[0.044]	[0.050]	[0.050]
		(-0.252, -0.071)	(-0.235, -0.064)	(-0.239, -0.042)	(-0.246, -0.047)
$r_{N,t}$	-0.2705	-	-0.2007	-0.2258	-0.2263
	[0.016]		[0.016]	[0.015]	[0.016]
	(-0.302, -0.239)		(-0.232, -0.170)	(-0.256, -0.196)	(-0.257, -0.195)
$r_{D,t-1}$	-	-	-	-	-0.0410
					[0.013]
					(-0.067, -0.015)
$\log IV_{t-1}$	-	-	-	0.9487	0.8536
				[0.078]	[0.082]
				(0.797, 1.103)	(0.692, 1.016)
$\frac{1}{4}\sum_{i=2}^{5} \log IV_{t-i}$	-	-	-	-0.6185	-0.5285
				[0.094]	[0.098]
				(-0.803, -0.435)	(-0.719, -0.335)
$\frac{1}{17}\sum_{i=6}^{22} \log IV_{t-i}$	-	-	-	-0.0965	-0.0888
				[0.078]	[0.078]
				(-0.249, 0.054)	(-0.242, 0.063)
FMSE	0.2892	0.2762	0.2602	0.2467	0.2464
LPL	-2000.68	-1943.41	-1867.53	-1799.54	-1796.94
FMSE P.C.	-9.82%	-13.88%	-18.86%	-23.07%	-23.17%
LBF	129.27	186.54	262.42	330.41	333.01

Standard deviation in brackets and 95% density interval in parentheses.

TABLE A1.3: Forecasting NASDAQ 100 daytime realized volatility with overnight information.

	(1)	(2)	(3)	(4)
μ	-0.1219	-0.1139	-0.3600	-0.3569
	[0.024]	[0.024]	[0.048]	[0.048]
	(-0.170, -0.073)	(-0.161, -0.067)	(-0.454, -0.265)	(-0.450, -0.262)
$\log RV_{N,t-1}$	0.4874	0.4503	0.2999	0.2670
	[0.019]	[0.020]	[0.022]	[0.022]
	(0.450, 0.525)	(0.412, 0.488)	(0.258, 0.342)	(0.224, 0.310)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{N,t-i}$	0.3521	0.3930	0.2662	0.2946
	[0.025]	[0.025]	[0.035]	[0.034]
	(0.303, 0.400)	(0.343, 0.442)	(0.198, 0.333)	(0.227, 0.362)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{N,t-i}$	0.0769	0.0739	0.1083	0.1092
	[0.022]	[0.021]	[0.045]	[0.044]
	(0.033, 0.119)	(0.032, 0.116)	(0.021, 0.194)	(0.023, 0.194)
$r_{N,t-1}$	-	-0.1439	-	-0.1383
		[0.017]		[0.016]
		(-0.177, -0.110)		(-0.170, -0.108)
$\log IV^{O}_{t-1}$	-	-	1.0568	1.0319
			[0.073]	[0.072]
			(0.914, 1.196)	(0.890, 1.173)
$\frac{1}{4}\sum_{i=2}^{5}\log IV^{O}_{t-i}$	-	-	-0.4573	-0.4079
			[0.093]	[0.094]
			(-0.638, -0.276)	(-0.590, -0.220)
$\frac{1}{17}\sum_{i=6}^{22}\log IV^{O}_{t-i}$	-	-	-0.1817	-0.2001
			[0.082]	[0.081]
			(-0.342, -0.017)	(-0.358, -0.042)
FMSE	0.3113	0.3032	0.2854	0.2778
LPL	-2090.87	-2058.39	-1982.50	-1949.56
FMSE P.C.	-	-2.60%	-8.32%	-10.76%
LBF	-	32.48	108.37	141.31

Standard deviation in brackets and 95% density interval in parentheses.

TABLE A1.4: Forecasting NASDAQ 100 overnight realized volatility without daytime information.

	(1)	(2)	(3)	(4)	(5)
μ	-0.1145	-0.1659	-0.1621	-0.3282	-0.3266
	[0.023]	[0.028]	[0.028]	[0.044]	[0.043]
	(-0.160, -0.069)	(-0.220, -0.111)	(-0.216, -0.108)	(-0.413, -0.243)	(-0.412, -0.242)
$\log RV_{N,t-1}$	0.4858	0.2118	0.2479	0.1784	0.1654
	[0.018]	[0.020]	[0.020]	[0.021]	[0.021]
	(0.450, 0.521)	(0.173, 0.251)	(0.208, 0.287)	(0.137, 0.221)	(0.124, 0.207)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{N,t-i}$	0.3538	0.3225	0.3077	0.2772	0.2848
	[0.024]	[0.032]	[0.031]	[0.035]	[0.034]
	(0.307, 0.400)	(0.260, 0.384)	(0.246, 0.369)	(0.208, 0.346)	(0.218, 0.353)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{N,t-i}$	0.0793	0.2220	0.2107	0.1828	0.1900
	[0.021]	[0.041]	[0.040]	[0.049]	[0.049]
	(0.039, 0.120)	(0.142, 0.300)	(0.132, 0.288)	(0.086, 0.278)	(0.093, 0.284)
$\log RV_{D,t-1}$	-	0.4660	0.3944	0.3322	0.3068
		[0.018]	[0.019]	[0.020]	[0.020]
		(0.430, 0.502)	(0.357, 0.432)	(0.293, 0.371)	(0.266, 0.347)
$\frac{1}{4}\sum_{i=2}^{5}\log RV_{D,t-i}$	-	-0.0583	-0.0213	-0.0839	-0.0604
		[0.033]	[0.032]	[0.033]	[0.033]
		(-0.125, 0.006)	(-0.084, 0.041)	(-0.149, -0.021)	(-0.125, 0.005)
$\frac{1}{17}\sum_{i=6}^{22}\log RV_{D,t-i}$	-	-0.1773	-0.1584	-0.1200	-0.1280
		[0.044]	[0.043]	[0.043]	[0.043]
		(-0.263, -0.091)	(-0.243, -0.074)	(-0.206, -0.037)	(-0.212, -0.043)
$r_{D,t-1}$	-0.2134	-	-0.1232	-0.1364	-0.1413
	[0.012]		[0.012]	[0.012]	[0.012]
	(-0.237, -0.189)		(-0.147, -0.099)	(-0.160, -0.113)	(-0.165, -0.118)
$r_{N,t-1}$	-	-	-	-	-0.0890
					[0.015]
					(-0.119, -0.060)
$\log IV_{t-1}^{O}$	-	-	-	0.6481	0.6530
				[0.071]	[0.069]
				(0.509, 0.791)	(0.516, 0.791)
$\frac{1}{4}\sum_{i=2}^{5}\log IV_{t-i}^{O}$	-	-	-	-0.1844	-0.1726
				[0.085]	[0.086]
				(-0.352, -0.019)	(-0.339, -0.004)
$\frac{1}{17}\sum_{i=6}^{22}\log IV_{t-i}^{O}$	-	-	-	-0.1322	-0.1393
				[0.074]	[0.074]
				(-0.277, 0.011)	(-0.282, 0.007)
FMSE	0.2779	0.2460	0.2365	0.2279	0.2250
LPL	-1948.40	-1797.76	-1749.00	-1703.11	-1688.40
FMSE P.C.	-10.73%	-20.98%	-24.03%	-26.79%	-27.72%
LBF	142.47	293.11	341.87	387.76	402.47

Standard deviation in brackets and 95% density interval in parentheses.

TABLE A1.5: Forecasting NASDAQ 100 overnight realized volatility with daytime information.

A3 Latent Volatility Sampling for DN-SV-RV

We sample the daytime and overnight latent volatility process ($\{h_{D,t}\}_{t=1}^{T}$ and $\{h_{N,t}\}_{t=1}^{T}$) seperately. Conditional on $\{h_{D,t}\}_{t=1}^{T}$, we have:

$$r_{N,t} = \mu_N + \exp(h_{N,t}/2)u_{N,t}, \tag{A.1}$$

$$h_{N,t} = \alpha_N + \beta_N h_{D,t-1} + \delta_N h_{N,t-1} + \sigma_{hN} v_{N,t}, \tag{A.2}$$

$$h_{D,t} = \alpha_D + \beta_D h_{N,t} + \delta_D h_{D,t-1} + \sigma_{hD} v_{D,t}, \tag{A.3}$$

$$\log RV_{N,t} = \xi_N + h_{N,t} + \sigma_{RVN} e_{N,t}. \tag{A.4}$$

Let Θ denote the set of parameters. The posterior distribution of $h_{N,t}$ is:

$$\begin{aligned}
&p(h_{N,t}|r_{N,t}, \log RV_{N,t}, h_{N,-t}, h_D, \Theta)\\
\propto &p(r_{N,t}, \log RV_{N,t}|h_{N,t}, \Theta)\ p(h_{N,t+1}|h_{N,t}, h_{D,t}, \Theta)\ p(h_{D,t}|h_{N,t}, \Theta) p(h_{N,t}|h_{N,t-1}, h_{D,t-1}, \Theta)\\
\propto &\exp\left(-\frac{h_{N,t}}{2}\right) \exp\left(-\frac{(r_{N,t}-\mu_N)^2}{2\exp(h_{N,t})}\right) \exp\left(-\frac{(\log RV_{N,t} - \xi_N - h_{N,t})^2}{2\sigma_{RVN}^2}\right)\\
&\exp\left(-\frac{(h_{N,t+1} - \alpha_N - \beta_N h_{D,t} - \delta_N h_{N,t})^2}{2\sigma_{hN}^2}\right) \exp\left(-\frac{(h_{D,t} - \alpha_D - \beta_D h_{N,t} - \delta_D h_{D,t-1})^2}{2\sigma_{hD}^2}\right)\\
&\exp\left(-\frac{(h_{N,t} - \alpha_N - \beta_N h_{D,t-1} - \delta_N h_{N,t-1})^2}{2\sigma_{hN}^2}\right)
\end{aligned} \tag{A.5}$$

We use the Metropolis-Hastings algorithm with tailored proposals to sample . We need to simulate $h_{D,0}$, $h_{N,0}$ and $h_{N,T+1}$ according to Eq. (A.2) and (A.3).

While conditional on $\{h_{N,t}\}_{t=1}^{T}$, we have

$$r_{D,t} = \mu_D + \exp(h_{D,t}/2)u_{D,t}, \qquad u_{D,t} \sim N(0,1), \tag{A.6}$$

$$h_{D,t} = \alpha_D + \beta_D h_{N,t} + \delta_D h_{D,t-1} + \sigma_{hD} v_{D,t}, \qquad v_{D,t} \sim N(0,1). \tag{A.7}$$

$$h_{N,t+1} = \alpha_N + \beta_N h_{D,t} + \delta_N h_{N,t} + \sigma_{hN} v_{N,t+1}, \qquad v_{N,t+1} \sim N(0,1), \tag{A.8}$$

$$\log RV_{D,t} = \xi_D + h_{D,t} + \sigma_{RVD} e_{D,t}, \qquad e_{D,t} \sim N(0,1). \tag{A.9}$$

The posterior distribution of $h_{D,t}$ is

$$\begin{aligned}
&p(h_{D,t}|r_{D,t}, h_{D,-t}, h_N, \Theta) \\
\propto& p(r_{D,t}, \log RV_{N,t}|h_{D,t}, \Theta)\ p(h_{D,t+1}|h_{D,t}, h_{N,t+1}, \Theta)\ p(h_{N,t+1}|h_{D,t}, \Theta) p(h_{D,t}|h_{D,t-1}, h_{N,t}, \Theta) \\
\propto& \exp\left(-\frac{h_{D,t}}{2}\right) \exp\left(-\frac{(r_{D,t}-\mu_D)^2}{2\exp(h_{N,t})}\right) \exp\left(-\frac{(\log RV_{D,t} - \xi_D - h_{D,t})^2}{2\sigma_{RVD}^2}\right) \\
&\exp\left(-\frac{(h_{D,t+1} - \alpha_D - \beta_D h_{N,t+1} - \delta_D h_{D,t})^2}{2\sigma_{hD}^2}\right) \exp\left(-\frac{(h_{N,t+1} - \alpha_N - \beta_N h_{D,t} - \delta_N h_{N,t})^2}{2\sigma_{hN}^2}\right) \\
&\exp\left(-\frac{(h_{D,t} - \alpha_D - \beta_D h_{N,t} - \delta_D h_{D,t-1})^2}{2\sigma_{hD}^2}\right)
\end{aligned} \tag{A.10}$$

We need to simulate $h_{D,0}$, $h_{N,T+1}$ and $h_{D,T+1}$. If we are fitting the DN-SV-RV model at the market open time, $h_{N,T+1}$ will be available.

Appendix B

Chapter 2 Supplement

B1 Sampling Details

B1.1 Posterior Distribution of Volatility Process

Details of derivation from Equation (2.7) to (2.9):

$$
\begin{aligned}
&p(RV_t|h_t,\Theta)p(IV_t|h_t,\Theta)p(h_t|h_{t-1},\Theta)p(h_{t+1}|h_t,\Theta) \\
&= f_N(RV_t|a_{RV}+b_{RV}h_t,\sigma^2_{RV})f_N(IV_t|a_{IV}+b_{IV}h_t,\sigma^2_{IV})f_N(h_t|\alpha+\delta h_{t-1},\sigma^2_h) \\
&\qquad f_N(h_{t+1}|\alpha+\delta h_t,\sigma^2_h). \\
&\propto f_N\left(h_t\middle|\frac{RV_t-a_{RV}}{b_{RV}},\frac{\sigma^2_{RV}}{b^2_{RV}}\right)f_N\left(h_t\middle|\frac{IV_t-a_{IV}}{b_{IV}},\frac{\sigma^2_{IV}}{b^2_{IV}}\right) \\
&\qquad f_N\left(h_t\middle|\frac{\alpha(1-\delta)+\delta(h_{t+1}+h_{t-1})}{1+\delta^2},\frac{\sigma^2_h}{1+\delta^2}\right).
\end{aligned}
$$

This will lead to the results in Equations (2.8) and (2.9). To get the proposal distribution, we go back to the conditional posterior of h_t.

$$
\begin{aligned}
p(h_t|h_{-t}, r, w) &\propto p(y_t|h_t, w)p(RV_t|h_t, w)p(VIX_t|h_t, w)p(h_t|h_{-t}, w) \\
&\propto f_N(r_t|\mu, \exp(h-t))f_N(h_t|\mu_t, \sigma^2) \\
&\propto \frac{1}{\exp(h_t/2)} \exp\left(-\frac{1}{2\exp(h_t)}(r_t-\mu)^2\right) f_N(h_t|\mu_t, \sigma^2) \\
&\simeq \exp\left(-\frac{h_t}{2} - \frac{1}{2}\exp(-\mu_t)(r_t-\mu)^2(1+\mu_t-h_t)\right)\exp\left(-\frac{(h_t-\mu_t)^2}{2\sigma^2}\right) \\
&\propto f_N(h_t|\bar{\mu_t}, \sigma^2), \qquad \text{(B.1)}
\end{aligned}
$$

where $\bar{\mu_t} = \mu_t + \frac{\sigma^2}{2}\left[(r_t-\mu)^2\exp(-\mu_t) - 1\right]$ and $\sigma^2 = \left(\frac{b_{RV}^2}{\sigma_{RV}^2} + \frac{b_{IV}^2}{\sigma_{IV}^2} + \frac{1+\delta^2}{\sigma_h^2}\right)^{-1}$.

B2 Latent Volatility Sampling Results in Graphs

Notice that the latent volatility processes sampled from IVSV and RVIVSV models are almost identical. So we only post the hidden volatility sampled from SV, IVSV and RVSV here.

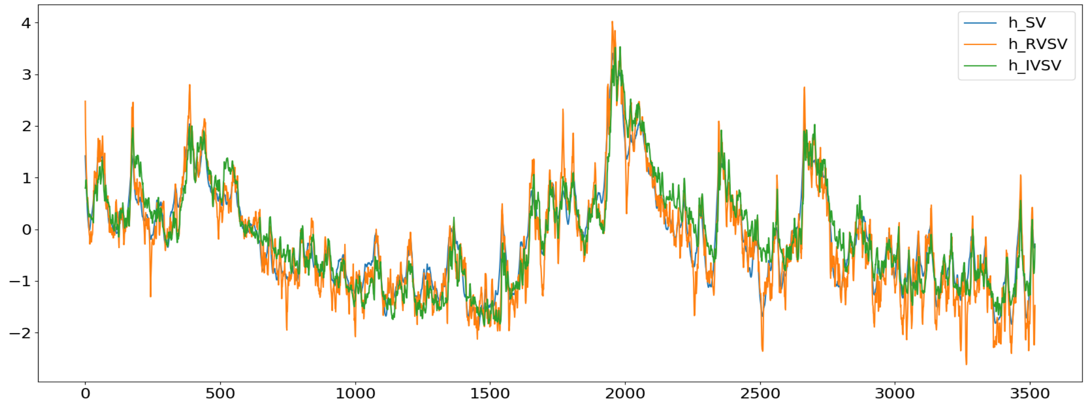

Figure B2.1: Latent volatility process for SPY.

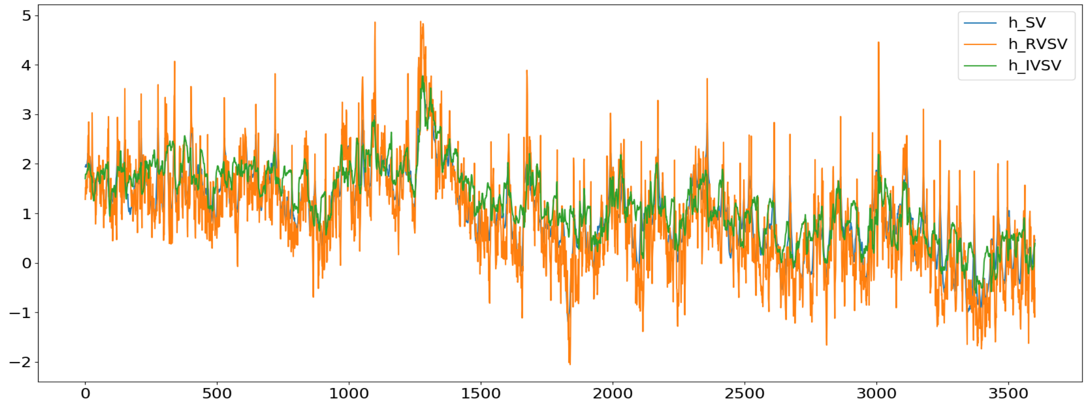

Figure B2.2: Latent volatility process for AAPL

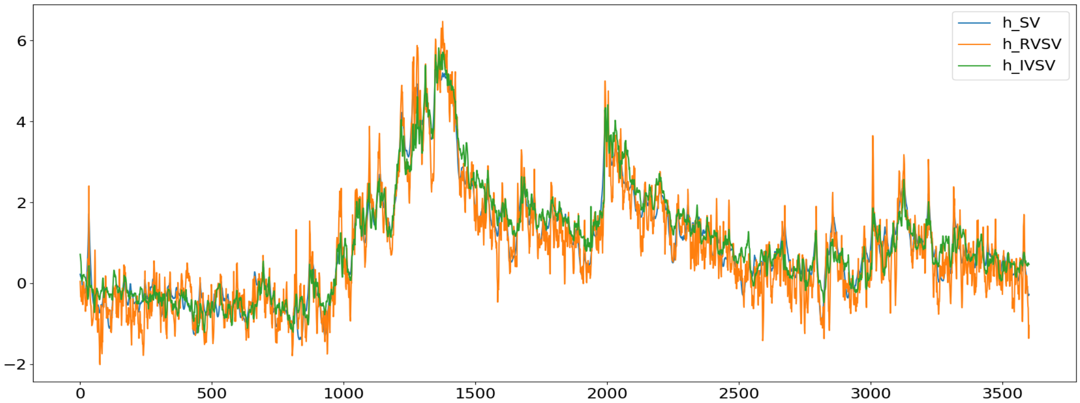

Figure B2.3: Latent volatility process for BAC.

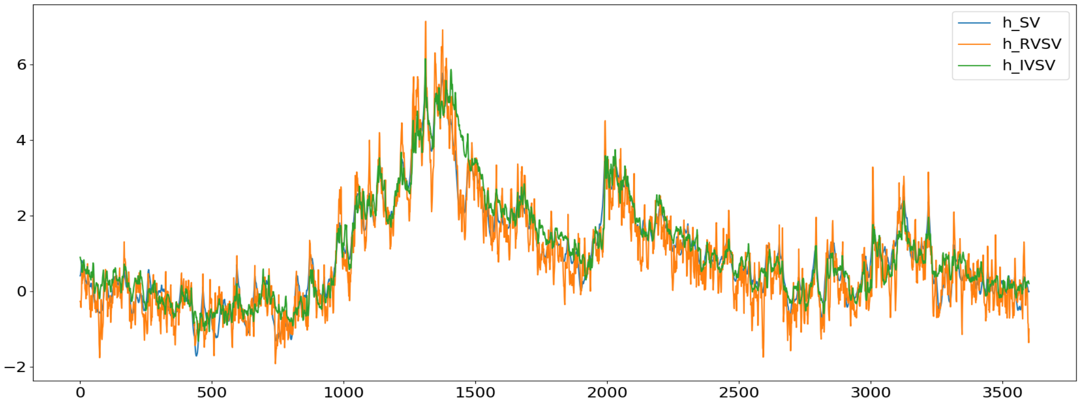

Figure B2.4: Latent volatility process for C.

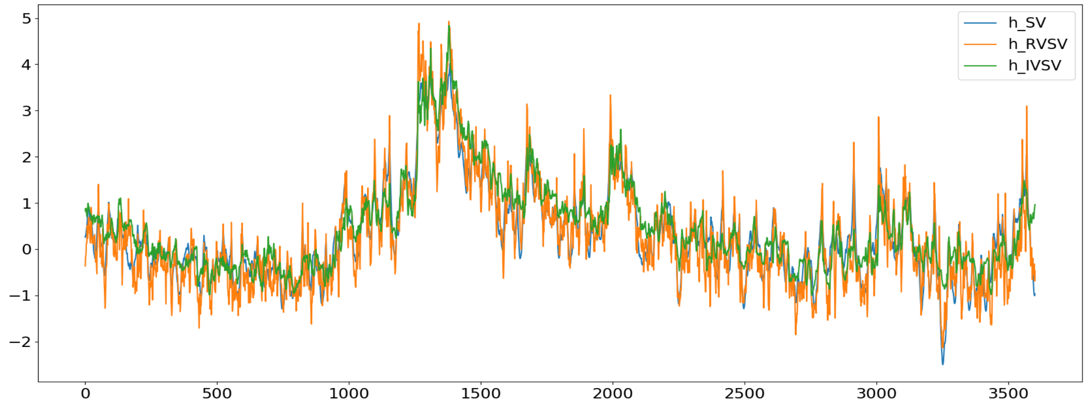

Figure B2.5: Latent volatility process for GE.

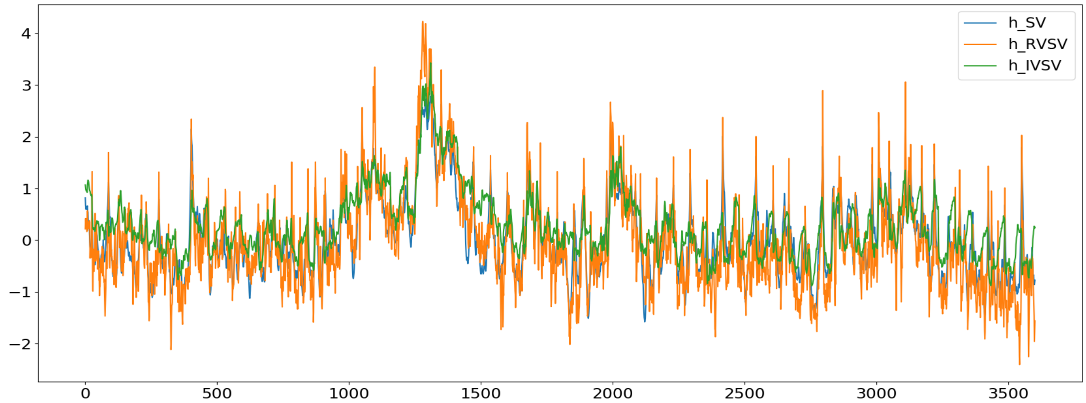

Figure B2.6: Latent volatility process for IBM.

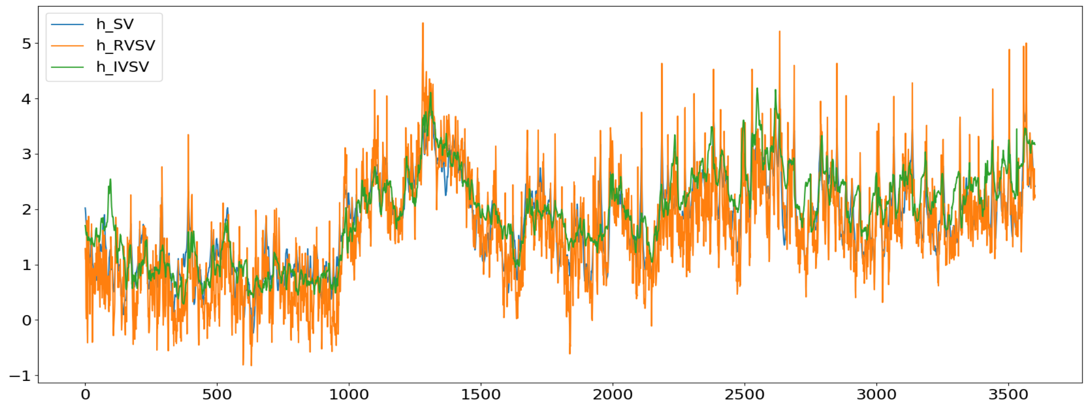

Figure B2.7: Latent volatility process for JCP.

B3 Parameter Sampling Results

TABLE B2.1: Posterior sampling summary for BAC

The posterior estimation results for return series, realized volatility and implied volatility of Bank of American Corp (NYSE: BAC). The parameter set comes from Equations (2.2) to (2.5). The simplest SV model takes the smallest parameter set, and the proposed RVIVSV model takes the full parameter set. The RVSV and IVSV models take a partial parameter set from the full specification. The estimation sample period is September 10th, 2003 to December 19th, 2017.

	SV		**RVSV**		**IVSV**		**RVIVSV**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
μ	0.0421	0.0203	0.0495	0.0181	0.0686	0.0201	0.0688	0.0204
	(0.002, 0.082)*		(0.014, 0.085)		(0.030, 0.108)		(0.029, 0.109)	
α	0.0119	0.0049	0.0349	0.0086	0.0048	0.0027	0.0049	0.0028
	(0.003, 0.022)		(0.019, 0.052)		(-0.001, 0.010)		(-0.001, 0.011)	
δ	0.9872	0.0034	0.9559	0.0060	0.9951	0.0016	0.9950	0.0017
	(0.980, 0.993)		(0.944, 0.967)		(0.992, 0.998)		(0.992, 0.998)	
σ_h	0.2219	0.0189	0.4289	0.0192	0.1332	0.0035	0.1354	0.0035
	(0.187, 0.261)		(0.392, 0.467)		(0.127, 0.140)		(0.129, 0.142)	
a_{RV}	-	-	0.1523	0.0267	-	-	-0.0229	0.0287
			(0.099, 0.204)				(-0.081, 0.032)	
b_{RV}	-	-	0.8616	0.0163	-	-	0.8665	0.0171
			(0.831, 0.894)				(0.833, 0.900)	
σ_{RV}	-	-	0.4454	0.0105	-	-	0.6464	0.0078
			(0.424, 0.466)				(0.631, 0.662)	
a_{IV}	-	-	-	-	-1.7201	0.0122	-1.7159	0.0121
					(-1.744, -1.697)		(-1.740, -1.693)	
b_{IV}	-	-	-	-	0.4131	0.0073	0.4120	0.0072
					(0.399, 0.427)		(0.398, 0.426)	
σ_{IV}	-	-	-	-	0.0399	0.0009	0.0396	0.0009
					(0.038, 0.042)		(0.038, 0.041)	

*Inside the parentheses is the 95% density interval.

Table B2.2: Posterior sampling summary for C

The posterior estimation results for return series, realized volatility and implied volatility of Citigroup Inc (NYSE: C). The parameter set comes from Equations (2.2) to (2.5). The simplest SV model takes the smallest parameter set, and the proposed RVIVSV model takes the full parameter set. The RVSV and IVSV models take a partial parameter set from the full specification. The estimation sample period is September 10th, 2003 to December 19th, 2017.

	SV		**RVSV**		**IVSV**		**RVIVSV**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
μ	0.0376	0.0201	0.0322	0.0184	0.0607	0.0205	0.0606	0.0469
	(-0.02, 0.077)*		(-0.004, 0.068)		(0.020, 0.101)		(0.020, 0.102)	
α	0.0122	0.0050	0.0373	0.0089	0.0042	0.0026	0.0043	0.0027
	(0.003, 0.022)		(0.020, 0.055)		(-0.001, 0.009)		(-0.001, 0.010)	
δ	0.9873	0.0034	0.9543	0.0063	0.9957	0.0015	0.9956	0.0016
	(0.980, 0.994)		(0.942, 0.966)		(0.993, 0.999)		(0.993, 0.999)	
σ_h	0.2215	0.0198	0.4364	0.0211	0.1266	0.0032	0.1280	0.0031
	(0.185, 0.262)		(0.396, 0.479)		(0.120, 0.133)		(0.122, 0.134)	
a_{RV}	-	-	0.1467	0.0283	-	-	-0.0293	0.0293
			(0.091, 0.201)				(-0.088, 0.027)	
b_{RV}	-	-	0.8761	0.0170	-	-	0.8679	0.0169
			(0.843, 0.910)				(0.836, 0.902)	
σ_{RV}	-	-	0.4778	0.0114	-	-	0.6837	0.0081
			(0.455, 0.500)				(0.668, 0.670)	
a_{IV}	-	-	-	-	-1.7055	0.0124	-1.7028	0.0123
					(-1.730, -1.682)		(-1.728, -1.679)	
b_{IV}	-	-	-	-	0.4146	0.0072	0.4141	0.0070
					(0.401, 0.429)		(0.401, 0.428)	
σ_{IV}	-	-	-	-	0.0360	0.0008	0.0358	0.0008
					(0.034, 0.038)		(0.034, 0.037)	

*Inside the parentheses is the 95% density interval.

TABLE B2.3: Posterior sampling summary for GE

The posterior estimation results for return series, realized volatility and implied volatility of General Electric (NYSE: GE). The parameter set comes from Equations (2.2) to (2.5). The simplest SV model takes the smallest parameter set, and the proposed RVIVSV model takes the full parameter set. The RVSV and IVSV models take a partial parameter set from the full specification. The estimation sample period is September 10th, 2003 to December 19th, 2017.

	SV		**RVSV**		**IVSV**		**RVIVSV**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
μ	0.0014	0.0166	0.0026	0.0155	0.0260	0.0179	0.0255	0.0178
	(-0.031, 0.033)*		(-0.028, 0.033)		(-0.009, 0.061)		(-0.009, 0.060)	
α	0.0110	0.0052	0.0149	0.0072	0.0034	0.0022	0.0034	0.0022
	(0.001, 0.021)		(0.001, 0.029)		(-0.001, 0.008)		(-0.001, 0.008)	
δ	0.9651	0.0068	0.9354	0.0080	0.9926	0.0020	0.9925	0.0020
	(0.951, 0.977)		(0.919, 0.951)		(0.989, 0.997)		(0.988, 0.997)	
σ_h	0.2888	0.0242	0.4093	0.0207	0.1196	0.0036	0.1210	0.0037
	(0.243, 0.339)		(0.369, 0.451)		(0.112, 0.127)		(0.114, 0.128)	
a_{RV}	-	-	0.1107	0.0241	-	-	-0.1022	0.0268
			(0.063, 0.157)				(-0.157, -0.051)	
b_{RV}	-	-	0.8759	0.0211	-	-	0.9272	0.0255
			(0.836, 0.918)				(0.879, 0.978)	
σ_{RV}	-	-	0.4735	0.0108	-	-	0.6441	0.0077
			(0.452, 0.495)				(0.629, 0.659)	
a_{IV}	-	-	-	-	-1.7515	0.0111	-1.7472	0.0111
					(-1.774, -1.730)		(-1.770, -1.726)	
b_{IV}	-	-	-	-	0.4245	0.0104	0.4216	0.0105
					(0.401, 0.429)		(0.401, 0.443)	
σ_{IV}	-	-	-	-	0.0379	0.0009	0.0380	0.0009
					(0.036, 0.040)		(0.036, 0.040)	

*Inside the parentheses is the 95% density interval.

Table B2.4: Posterior sampling summary for IBM

The posterior estimation results for return series, realized volatility and implied volatility of IBM (NYSE: IBM). The parameter set comes from Equations (2.2) to (2.5). The simplest SV model takes the smallest parameter set, and the proposed RVIVSV model takes the full parameter set. The RVSV and IVSV models take a partial parameter set from the full specification. The estimation sample period is September 10th, 2003 to December 19th, 2017.

	SV		**RVSV**		**IVSV**		**RVIVSV**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
μ	0.0299	0.0161	0.0463	0.0143	0.0415	0.0178	0.0421	0.0179
	(-0.002, 0.062)*		(0.018, 0.075)		(0.007, 0.077)		(0.008, 0.078)	
α	0.0044	0.0068	-0.0094	0.0094	0.0048	0.0021	0.0044	0.0022
	(-0.009, 0.018)		(-0.028, 0.009)		(0.001, 0.009)		(0.000, 0.009)	
δ	0.9076	0.0146	0.8588	0.0154	0.9826	0.0031	0.9831	0.0031
	(0.877, 0.933)		(0.827, 0.887)		(0.977, 0.988)		(0.977, 0.989)	
σ_h	0.3783	0.0312	0.5058	0.0271	0.1173	0.0045	0.1191	0.007
	(0.320, 0.443)		(0.454, 0.560)		(0.109, 0.126)		(0.110, 0.128)	
a_{RV}	-	-	0.0555	0.0225	-	-	-0.2594	0.0276
			(0.011, 0.099)				(-0.315, -0.207)	
b_{RV}	-	-	0.8130	0.0240	-	-	0.9572	0.0387
			(0.768, 0.862)				(0.884, 1.035)	
σ_{RV}	-	-	0.4813	0.0130	-	-	0.6987	0.0083
			(0.455, 0.506)				(0.683, 0.715)	
a_{IV}	-	-	-	-	-1.7544	0.0113	-1.7467	0.0110
					(-1.777, -1.733)		(-1.769, -1.726)	
b_{IV}	-	-	-	-	0.4351	0.0156	0.4235	0.0154
					(0.406, 0.466)		(0.395, 0.455)	
σ_{IV}	-	-	-	-	0.0369	0.0008	0.0380	0.0009
					(0.035, 0.038)		(0.036, 0.040)	

*Inside the parentheses is the 95% density interval.

Table B2.5: Posterior sampling summary for JCP

The posterior estimation results for return series, realized volatility and implied volatility of JCPenny (NYSE: JCP). The parameter set comes from Equations (2.2) to (2.5). The simplest SV model takes the smallest parameter set, and the proposed RVIVSV model takes the full parameter set. The RVSV and IVSV models take a partial parameter set from the full specification. The estimation sample period is September 10th, 2003 to December 19th, 2017.

	SV		**RVSV**		**IVSV**		**RVIVSV**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
μ	0.0229	0.0363	0.0284	0.0309	0.0541	0.0372	0.0543	0.0374
	(-0.048, 0.095)*		(-0.032, 0.089)		(-0.019, 0.125)		(-0.020, 0.127)	
α	0.1205	0.0256	0.2570	0.0264	0.0146	0.0044	0.0145	0.0044
	(0.075, 0.176)		(0.206, 0.310)		(0.006, 0.023)		(0.006, 0.023)	
δ	0.9317	0.0146	0.8395	0.0158	0.9927	0.0021	0.9926	0.0021
	(0.900, 957)		(0.807, 0.870)		(0.989, 0.997)		(0.989, 0.997)	
σ_h	0.3322	0.0361	0.5628	0.0275	0.0986	0.0032	0.0991	0.0033
	(0.266, 0.406)		(0.510, 0.616)		(0.092, 0.105)		(0.093, 0.106)	
a_{RV}	-	-	0.2248	0.0443	-	-	-0.0849	0.0619
			(0.135, 0.308)				(-0.213, 0.031)	
b_{RV}	-	-	0.8719	0.0228	-	-	0.8876	0.0299
			(0.829, 0.918)				(0.831, 0.949)	
σ_{RV}	-	-	0.4585	0.0156	-	-	0.7261	0.0086
			(0.426, 0.487)				(0.709, 0.743)	
a_{IV}	-	-	-	-	-1.6699	0.0265	-1.6622	0.0264
					(-1.724, -1.621)		(-1.717, -1.613)	
b_{IV}	-	-	-	-	0.4477	0.0127	0.4448	0.0127
					(0.424, 0.474)		(0.421, 0.471)	
σ_{IV}	-	-	-	-	0.0345	0.0007	0.0348	0.0007
					(0.035, 0.036)		(0.033, 0.036)	

*Inside the parentheses is the 95% density interval.

Appendix C

Chapter 3 Supplement

C1 Posterior Sampling

C1.1 ASV-HS and VM-ASV-HS

The ASV-HS model:

$$y_t = \exp(h_t/2)u_t, \tag{C.1}$$

$$h_{t+1} = \delta h_t + v_{t+1}, \tag{C.2}$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{C.3}$$

Denotes the set of the parameters $\{\delta,\ \rho,\ \sigma_y,\ \sigma_h\}$ as Θ, $f_N(x|\mu, \sigma^2)$ denotes the density function of normally distributed variable x with mean μ and variance σ^2. The MCMC steps for the ASV-HS model are:

1. $\delta \mid y,\ h,\ \rho,\ \sigma_y,\ \sigma_h$. Conditional on return series y and latent volatility process h, the univariate representation of the volatility equation is:

$$h_{t+1} - \frac{\rho\sigma_h}{\sigma_y}\frac{y_t}{\exp(h_t/2)} = \delta h_t + \sqrt{1-\rho^2}\ w_{t+1}, w_{t+1} \sim N(0,\ \sigma_h^2). \tag{C.4}$$

   δ can be treated as the coefficient of linear regression with dependent variable $h_{t+1} - \frac{\rho\sigma_h}{\sigma_y}\frac{y_t}{\exp(h_t/2)}$ and independent variable h_t with known variance $(1-\rho^2)\sigma_h^2$. We set conjugate truncated normal prior $p(\delta)\ N(0.98, 4.0)I_{|\delta|<1}$.

2. $\rho,\ \sigma_y,\ \sigma_h \mid y,\ h,\ \delta$. The precision matrix (inverse of the covariance matrix in (C.3)) can be drawn from a Wishart posterior distribution given the conjugate Wishart prior: $W(n, S0)$, where the degree of freedom $n = 3.0$ and

$$S0 = \begin{bmatrix} 0.85^2 & 0.0 \\ 0.0 & 0.2^2 \end{bmatrix}.$$

   .

3. $h_t \mid y,\ h_{-t},\ \Theta$. For $t = 1, 2, ...N$ Sample h_t with the Metropolis-Hasting algorithm. The posterior and proposal distributions will be explained below. Notice that when sampling h_1, we assume $h_0 = 0$, which is the long run mean of h_t. For sampling h_N, we draw h_{N+1} from the following distribution:

$$h_{N+1}|y_N, h_N \sim N(\delta h_N + \frac{\rho\sigma_h}{\sigma_y}\frac{y_N}{\exp(h_N/2)},\ (1-\rho^2)\sigma_h^2).$$

Now we move on to the posterior and proposal distributions of h_t. First, the posterior distribution:

$$
\begin{aligned}
& p(h_t|Y, h_{-t}, \Theta) && \text{(C.5)} \\
& \propto p(y_t|h_t, h_{-t}, \Theta)\ p(h_t|y_{-t}, h_{-t}, \Theta) && \text{(C.6)} \\
& \propto p(y_t|h_t, h_{-t}, \Theta)\ p(h_{t+1}|h_t, \Theta)\ p(h_t|h_{t-1}, y_{t-1}, \Theta) && \text{(C.7)} \\
& \propto \exp(-\frac{h_t}{2}) \exp(-\frac{(\frac{y_t}{\exp(h_t/2)} - \frac{\rho\sigma_y}{\sigma_h}h_{t+1} + \frac{\rho\sigma_y}{\sigma_h}\delta h_t)^2}{2\sigma_y^2(1-\rho^2)}) \exp(-\frac{(h_{t+1}-\delta h_t)^2}{2\sigma_h^2}) \\
& \qquad \exp(-\frac{(h_t - (\delta h_{t-1} + \frac{\rho\sigma_h}{\sigma_y}\ \frac{y_{t-1}}{\exp(h_{t-1}/2)}))^2}{2(1-\rho^2)\sigma_h^2}) && \text{(C.8)} \\
& \propto \exp(-\frac{h_t}{2}) \exp(-\frac{(\frac{y_t}{\exp(h_t/2)} - \frac{\rho\sigma_y}{\sigma_h}h_{t+1} + \frac{\rho\sigma_y}{\sigma_h}\delta h_t)^2}{2\sigma_y^2(1-\rho^2)}) f_N(h_t|\mu_t, \sigma^2), && \text{(C.9)}
\end{aligned}
$$

where:

$$
\mu_t = (\frac{\delta h_{t+1}}{\sigma_h^2} + \frac{\delta h_{t-1} + \frac{\rho\sigma_h}{\sigma_y}\ \frac{y_{t-1}}{\exp(h_{t-1}/2)}}{(1-\rho^2)\sigma_h^2})\sigma^2, \tag{C.10}
$$

$$
\sigma^2 = (\frac{\delta^2}{\sigma_h^2} + \frac{1}{(1-\rho^2)\sigma_h^2})^{-1}. \tag{C.11}
$$

Given the Taylor extension of $\exp(-h_t/2)$ at point $-\mu_t/2$, we have the following proposal distribution:

$$
\begin{aligned}
& p(h_t|Y, h_{-t}, \Theta) \\
& \approx \exp(-\frac{h_t}{2}) \exp(-\frac{(\frac{y_t}{\exp(\mu_t/2)}(1 + \frac{\mu_t}{2} - \frac{h_t}{2}) - \frac{\rho\sigma_y}{\sigma_h}h_{t+1} + \frac{\rho\sigma_y}{\sigma_h}\delta h_t)^2}{2\sigma_y^2(1-\rho^2)}) f_N(h_t|\mu_t, \sigma^2) && \text{(C.12)} \\
& \propto f_N(h_t|\hat{\mu}_t,\ \hat{\sigma}_t^2) f_N(h_t|\mu_t, \sigma^2) && \text{(C.13)} \\
& \propto f_N(h_t|\tilde{\mu}_t, \tilde{\sigma}_t^2), && \text{(C.14)}
\end{aligned}
$$

where

$$\tilde{\mu}_t = (\frac{pq - \sigma_y^2(1-\rho^2)/2}{\sigma_y^2(1-\rho^2)} + \frac{\delta h_{t+1}}{\sigma_h^2} + \frac{\delta h_{t-1} + \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t-1}}{\exp(h_{t-1}/2)}}{(1-\rho^2)\sigma_h^2})\tilde{\sigma}_t^2, \tag{C.15}$$

$$\tilde{\sigma}_t^2 = (\frac{\delta^2}{\sigma_h^2} + \frac{1}{(1-\rho^2)\sigma_h^2} + \frac{q^2}{\sigma_y^2(1-\rho^2)})^{-1}, \tag{C.16}$$

$$p = \frac{y_t}{\exp(\frac{\mu_t}{2})}(1 + \frac{\mu_t}{2}) - \frac{\rho\sigma_y}{\sigma_h}h_{t+1}, \tag{C.17}$$

$$q = \frac{y_t}{2\exp(\frac{\mu_t}{2})} - \frac{\rho\sigma_y}{\sigma_h}\delta. \tag{C.18}$$

If we have a volatility measure series VM_t, for the following VM-ASV-HS specification:

$$y_t = \exp(h_t/2)u_t, \tag{C.19}$$

$$h_{t+1} = \delta h_t + v_{t+1}, \tag{C.20}$$

$$VM_t = a + bh_t + e_t, \qquad e_t \sim N(0, \sigma_{VM}^2), \tag{C.21}$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{C.22}$$

The sampling steps will be:

1. $\delta \mid y,\ h,\ \rho,\ \sigma_y,\ \sigma_h$. As with the ASV-HS model, we set conjugate truncated normal prior $p(\delta)\ N(0.98, 4.0)I_{|\delta|<1}$..
2. $\rho,\ \sigma_y,\ \sigma_h \mid y,\ h,\ \delta$. As with the ASV-HS model, we set the conjugate Wishart prior $W(n, S0)$.

3. $a,\ b,\ \sigma_{VM}|\ \delta,\ y,\ h,\ \rho,\ \sigma_y,\ \sigma_h$. Conditional on latent volatility process, $a,\ b,\ \sigma_{VM}$ can be sampled as linear regression coefficients and variance parameters with conjugate prior: $p(a) \sim N(-1.0, 100)$, $p(b) \sim N(1.0, 100)$ and $p(\sigma^2_{VM}) \sim IG(\frac{3}{2}, \frac{0.1}{2})$.

4. $h_t \mid y,\ h_{-t},\ \Theta$. For $t = 1, 2, ...N$, sample h_t with the Metropolis-Hasting algorithm as with the ASV-HS model. The posterior and proposal distributions will be explained below.

The posterior of h_t will be:

$$p(h_t|y, VM, h_{-t}, \Theta) \tag{C.23}$$

$$\propto p(y_t|h_t, h_{-t}, \Theta)\ p(h_{t+1}|h_t, \Theta)\ p(h_t|h_{t-1}, y_{t-1}, \Theta)\ p(h_t|VM_t, \Theta) \tag{C.24}$$

$$\propto \exp(-\frac{h_t}{2}) \exp(-\frac{(\frac{y_t}{\exp(h_t/2)} - \frac{\rho\sigma_y}{\sigma_h} h_{t+1} + \frac{\rho\sigma_y}{\sigma_h}\delta h_t)^2}{2\sigma_y^2(1-\rho^2)}) f_N(h_t|\mu_t, \sigma^2), \tag{C.25}$$

where:

$$\mu_t = (\frac{\delta h_{t+1}}{\sigma_h^2} + \frac{\delta h_{t-1} + \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t-1}}{\exp(h_{t-1}/2)}}{(1-\rho^2)\sigma_h^2} + \frac{(VM_t - a)b}{\sigma^2_{VM}})\sigma^2, \tag{C.26}$$

$$\sigma^2 = (\frac{\delta^2}{\sigma_h^2} + \frac{1}{(1-\rho^2)\sigma_h^2} + \frac{b^2}{\sigma^2_{VM}})^{-1}. \tag{C.27}$$

And the proposal distribution:

$$f_N(h_t|\tilde{\mu}_t, \tilde{\sigma}_t^2), \tag{C.28}$$

where:

$$\tilde{\mu}_t = (\frac{pq - \sigma_y^2(1-\rho^2)/2}{\sigma_y^2(1-\rho^2)} + \frac{\delta h_{t+1}}{\sigma_h^2} + \frac{\delta h_{t-1} + \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t-1}}{\exp(h_{t-1}/2)}}{(1-\rho^2)\sigma_h^2} + \frac{(VM_t - a)b}{\sigma_{VM}^2})\tilde{\sigma}_t^2, \quad \text{(C.29)}$$

$$\tilde{\sigma}_t^2 = (\frac{\delta^2}{\sigma_h^2} + \frac{1}{(1-\rho^2)\sigma_h^2} + \frac{q^2}{\sigma_y^2(1-\rho^2)} + \frac{b^2}{\sigma_{VM}^2})^{-1}, \quad \text{(C.30)}$$

$$p = \frac{y_t}{\exp(\frac{\mu_t}{2})}(1 + \frac{\mu_t}{2}) - \frac{\rho\sigma_y}{\sigma_h}h_{t+1}, \quad \text{(C.31)}$$

$$q = \frac{y_t}{2\exp(\frac{\mu_t}{2})} - \frac{\rho\sigma_y}{\sigma_h}\delta. \quad \text{(C.32)}$$

C1.2 ASV-JPR and VM-ASV-JPR

The ASV-JPR specification is

$$y_t = \exp(h_t/2)u_t, \quad \text{(C.33)}$$

$$h_t = \delta h_{t-1} + v_t, \quad \text{(C.34)}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \quad \text{(C.35)}$$

The MCMC steps are similar to those of the ASV-HS model. For $\delta \mid y,\ h,\ \rho,\ \sigma_y,\ \sigma_h$, conditional on return series y and latent volatility process h, the univariate representation of the volatility equation is:

$$h_t - \frac{\rho\sigma_h}{\sigma_y}\frac{y_t}{\exp(h_t/2)} = \delta h_{t-1} + \sqrt{1-\rho^2}\ w_t, \quad w_t \sim N(0,\ \sigma_h^2).$$

The MCMC steps for the parameters and prior settings will be as for the ASV-HS model. The main difference is the posterior of $h_t,\ t = 1, 2, ...N$. To sample h_1, we set $h_0 = 0$, as with the ASV-HS model. To sample h_N, we need h_{N+1} and y_{N+1}, and we can easily draw a sample of $[y_{N+1}, h_{N+1}]$ according to the ASV-JPR model.

The posterior distribution of h_t is:

$$p(h_t|Y, h_{-t}, \Theta) \tag{C.36}$$

$$\propto p(y_t|h_t, h_{-t}, \Theta)\ p(h_{t+1}|h_t, y_{t+1}, \Theta)\ p(h_t|h_{t-1}, \Theta) \tag{C.37}$$

$$\propto \exp(-\frac{h_t}{2}) \exp(-\frac{(\frac{y_t}{\exp(h_t/2)} - \frac{\rho\sigma_y}{\sigma_h}h_t + \frac{\rho\sigma_y}{\sigma_h}\delta h_{t-1})^2}{2\sigma_y^2(1-\rho^2)}) \exp(-\frac{(h_t - \delta h_{t-1})^2}{2\sigma_h^2})$$
$$\exp(-\frac{(h_{t+1} - (\delta h_t + \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t+1}}{\exp(h_{t+1}/2)}))^2}{2(1-\rho^2)\sigma_h^2}) \tag{C.38}$$

$$\propto \exp(-\frac{h_t}{2}) \exp(-\frac{(\frac{y_t}{\exp(h_t/2)} - \frac{\rho\sigma_y}{\sigma_h}h_t + \frac{\rho\sigma_y}{\sigma_h}\delta h_{t-1})^2}{2\sigma_y^2(1-\rho^2)}) f_N(h_t|\mu_t, \sigma^2), \tag{C.39}$$

where:

$$\mu_t = (\frac{\delta h_{t-1}}{\sigma_h^2} + \frac{\delta(h_{t+1} - \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t+1}}{\exp(h_{t+1}/2)})}{(1-\rho^2)\sigma_h^2})\sigma^2, \tag{C.40}$$

$$\sigma^2 = (\frac{1}{\sigma_h^2} + \frac{\delta^2}{(1-\rho^2)\sigma_h^2})^{-1}. \tag{C.41}$$

And the proposal distribution of h_t will be a normal distribution $f_N(h_t|\tilde{\mu}_t, \tilde{\sigma}_t^2)$ where:

$$\tilde{\mu}_t = (\frac{pq - \sigma_y^2(1-\rho^2)/2}{\sigma_y^2(1-\rho^2)} + \frac{\delta h_{t-1}}{\sigma_h^2} + \frac{\delta(h_{t+1} - \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t+1}}{\exp(h_{t+1}/2)})}{(1-\rho^2)\sigma_h^2})\tilde{\sigma}_t^2, \tag{C.42}$$

$$\tilde{\sigma}_t^2 = (\frac{1}{\sigma_h^2} + \frac{\delta^2}{(1-\rho^2)\sigma_h^2} + \frac{q^2}{\sigma_y^2(1-\rho^2)})^{-1}, \tag{C.43}$$

$$p = \frac{y_t}{\exp(\frac{\mu_t}{2})}(1 + \frac{\mu_t}{2}) + \frac{\rho\sigma_y}{\sigma_h}\delta h_{t-1}, \tag{C.44}$$

$$q = \frac{y_t}{2\exp(\frac{\mu_t}{2})} + \frac{\rho\sigma_y}{\sigma_h}. \tag{C.45}$$

For the VM-ASV-JPR model:

$$y_t = \mu + \exp(h_t/2)u_t, \tag{C.46}$$

$$h_t = \delta h_{t-1} + v_t, \tag{C.47}$$

$$VM_t = a + bh_t + e_t, \qquad e_t \sim N(0, \sigma_{VM}^2), \tag{C.48}$$

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim \boldsymbol{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \rho\sigma_y\sigma_h \\ \rho\sigma_y\sigma_h & \sigma_h^2 \end{bmatrix}\right). \tag{C.49}$$

Similar to the VM-ASV-HS and ASV-JPR, we set the same prior and similar MCMC steps for $\delta, \sigma_y, \sigma_h, \rho, a, b, \sigma_{VM}$. The key difference is the posterior of h_t:

The posterior of h_t in the VM-ASV-JPR model is:

$$p(h_t|Y, h_{-t}, \Theta) \tag{C.50}$$

$$\propto p(y_t|h_t, h_{-t}, \Theta)\ p(h_{t+1}|h_t, y_{t+1}, \Theta)\ p(h_t|h_{t-1}, \Theta)p(h_t|VM_t, \Theta) \tag{C.51}$$

$$\propto \exp(-\frac{h_t}{2}) \exp(-\frac{(\frac{y_t}{\exp(h_t/2)} - \frac{\rho\sigma_y}{\sigma_h}h_t + \frac{\rho\sigma_y}{\sigma_h}\delta h_{t-1})^2}{2\sigma_y^2(1-\rho^2)}) f_N(h_t|\mu_t, \sigma^2), \tag{C.52}$$

$$\tag{C.53}$$

where:

$$\mu_t = (\frac{\delta h_{t-1}}{\sigma_h^2} + \frac{\delta(h_{t+1} - \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t+1}}{\exp(h_{t+1}/2)})}{(1-\rho^2)\sigma_h^2} + \frac{(VM_t - a)b}{\sigma_{VM}^2})\sigma^2, \tag{C.54}$$

$$\sigma^2 = (\frac{1}{\sigma_h^2} + \frac{\delta^2}{(1-\rho^2)\sigma_h^2} + \frac{b^2}{\sigma_{VM}^2})^{-1}. \tag{C.55}$$

And the proposal distribution will be a normal distribution $f_N(h_t|\tilde{\mu}_t, \tilde{\sigma}_t^2)$ where:

$$\begin{aligned}\tilde{\mu}_t = (&\frac{pq - \sigma_y^2(1-\rho^2)/2}{\sigma_y^2(1-\rho^2)} + \frac{\delta h_{t-1}}{\sigma_h^2} + \frac{\delta(h_{t+1} - \frac{\rho\sigma_h}{\sigma_y}\frac{y_{t+1}}{\exp(h_{t+1}/2)})}{(1-\rho^2)\sigma_h^2} \\ &+ \frac{(VM_t - a)b}{\sigma_{VM}^2})\tilde{\sigma}_t^2, \end{aligned} \tag{C.56}$$

$$\tilde{\sigma}_t^2 = (\frac{1}{\sigma_h^2} + \frac{\delta^2}{(1-\rho^2)\sigma_h^2} + \frac{q^2}{\sigma_y^2(1-\rho^2)} + \frac{b^2}{\sigma_{VM}^2})^{-1}, \tag{C.57}$$

$$p = \frac{y_t}{\exp(\frac{\mu_t}{2})}(1 + \frac{\mu_t}{2}) + \frac{\rho\sigma_y}{\sigma_h}\delta h_{t-1}, \tag{C.58}$$

$$q = \frac{y_t}{2\exp(\frac{\mu_t}{2})} + \frac{\rho\sigma_y}{\sigma_h}. \tag{C.59}$$

C2 Supplementary Empirical Results

Russell 2000								
	ASV-HS Day		**ASV-HS Week**		**ASV-JPR Day**		**ASV-JPR Week**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9823	0.0035	0.9355	0.0196	0.9869	0.0021	0.9582	0.0114
	(0.9747, 0.9886)		(0.8917, 0.9674)		(0.9824, 0.9908)		(0.9336, 0.9774)	
σ_y	1.1273	0.0552	2.4464	0.1581	1.4964	0.0828	3.0006	0.2680
	(1.0244, 1.2368)		(2.1513, 2.7606)		(1.3520, 1.6524)		(2.5009, 3.5790)	
σ_h	0.1634	0.0140	0.2824	0.0418	0.1605	0.0109	0.2600	0.0318
	(0.1375, 0.1921)		(0.2058, 0.3707)		(0.1381, 0.1820)		(0.2030, 0.3261)	
ρ	-0.7663	0.0395	-0.7024	0.0737	-0.8449	0.0249	-0.8009	0.0551
	(-0.8386, -0.6816)		(-0.8275, -0.5379)		(-0.8889, -0.7912)		(-0.8883, -0.6717)	
HSI								
	ASV-HS Day		**ASV-HS Week**		**ASV-JPR Day**		**ASV-JPR Week**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9868	0.0031	0.9678	0.0132	0.9883	0.0026	0.9707	0.0116
	(0.9802, 0.9923)		(0.9374, 0.9886)		(0.9829, 0.9928)		(0.9452, 0.9912)	
σ_y	1.1231	0.0658	2.5704	0.2054	1.2193	0.0748	2.5739	0.2585
	(1.0046, 1.2551)		(2.1782, 3.0073)		(1.0776, 1.3783)		(1.9622, 3.0374)	
σ_h	0.1305	0.0129	0.1655	0.0307	0.1281	0.0113	0.1639	0.0270
	(0.1071, 0.1585)		(0.1151, 0.2329)		(0.1081, 0.1519)		(0.1156, 0.2229)	
ρ	-0.5346	0.0580	-0.4229	0.1218	-0.6014	0.0493	-0.4967	0.1082
	(-0.6413, -0.4133)		(-0.6346, -0.1587)		(-0.6924, -0.5017)		(-0.6833, -0.2597)	
STOXX50E								
	ASV-HS Day		**ASV-HS Week**		**ASV-JPR Day**		**ASV-JPR Week**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9706	0.0045	0.9424	0.0170	0.9825	0.0027	0.9493	0.0115
	(0.9611, 0.9783)		(0.9029, 0.9696)		(0.9769, 0.9875)		(0.9242, 0.9698)	
σ_y	1.0172	0.0422	2.3265	0.1441	1.2884	0.0743	2.8123	0.2027
	(0.9390, 1.0978)		(2.0555, 2.6235)		(1.1649, 1.4340)		(2.4646, 3.2410)	
σ_h	0.2261	0.0170	0.2565	0.0390	0.1775	0.0127	0.2621	0.0321
	(0.1958, 0.2621)		(0.1870, 0.3434)		(0.1533, 0.2032)		(0.2036, 0.3290)	
ρ	-0.7922	0.0285	-0.7465	0.0707	-0.7918	0.0298	-0.8240	0.0431
	(-0.8433, -0.7316)		(-0.8615, -0.5892)		(-0.8462, -0.7294)		(-0.8942, -0.7263)	

Inside the parentheses is the 95% density interval.

TABLE C3.1: ASV models with the Russell 2000, HSI and STOXX50E indices daily and weekly returns.

Para	RV-ASV-HS Mean	RV-ASV-HS Stdev	BV-ASV-HS Mean	BV-ASV-HS Stdev	VIX-ASV-HS Mean	VIX-ASV-HS Stdev
δ	0.9493	0.0053	0.9498	0.0053	0.9891	0.0022
	(0.9387, 0.9594)		(0.9391, 0.9597)		(0.9848, 0.9933)	
σ_y	1.1292	0.0139	1.1304	0.0138	1.1540	0.0132
	(1.1027, 1.1569)		(1.1035, 1.1575)		(1.1283, 1.1798)	
σ_h	0.2574	0.0126	0.2530	0.0121	0.1159	0.0038
	(0.2334, 0.2830)		(0.2299, 0.2778)		(0.1088, 0.1235)	
ρ	-0.4937	0.0324	-0.4723	0.0330	-0.1546	0.0258
	(-0.5556, -0.4292)		(-0.5366, -0.4074)		(-0.2054, -0.1047)	
b	0.9559	0.0280	0.9828	0.0281	0.3723	0.0093
	(0.9029, 1.0126)		(0.9287, 1.0396)		(0.3537, 0.3902)	
σ_{VM}	0.4631	0.0079	0.4604	0.0081	0.0302	0.0009
	(0.4475, 0.4782)		(0.4445, 0.4763)		(0.0286, 0.0320)	
Para	**RV-ASV-JPR Mean**	**RV-ASV-JPR Stdev**	**BV-ASV-JPR Mean**	**BV-ASV-JPR Stdev**	**VIX-ASV-JPR Mean**	**VIX-ASV-JPR Stdev**
δ	0.9552	0.0048	0.9545	0.0048	0.9907	0.0014
	(0.9455, 0.9642)		(0.9450, 0.9636)		(0.9878, 0.9935)	
σ_y	1.1365	0.0137	1.1390	0.0137	1.1606	0.0134
	(1.1101, 1.1640)		(1.1127, 1.1661)		(1.1348, 1.1875)	
σ_h	0.2739	0.0120	0.2726	0.0119	0.1129	0.0033
	(0.2516, 0.2976)		(0.2503, 0.2960)		(0.1065, 0.1198)	
ρ	-0.6580	0.0210	-0.6679	0.0202	-0.8303	0.0066
	(-0.6989, -0.6163)		(-0.7068, -0.6274)		(-0.8429, -0.8171)	
b	0.9871	0.0274	1.0264	0.0292	0.4498	0.0117
	(0.9350, 1.0422)		(0.9714, 1.0864)		(0.4261, 0.4737)	
σ_{VM}	0.4389	0.0074	0.4304	0.0074	0.0209	0.0004
	(0.4244, 0.4535)		(0.4161, 0.4449)		(0.0201, 0.0218)	

Inside the parentheses is the 95% density interval.

Table C3.2: The estimation results of the ASV models with Russell 2000 daily returns and volatility measures.

	RV-ASV-HS		BV-ASV-HS		VIX-ASV-HS	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9760	0.0039	0.9751	0.0039	0.9923	0.0020
	(0.9681, 0.9834)		(0.9672, 0.9826)		(0.9884, 0.9961)	
σ_y	1.1108	0.0138	1.1138	0.0138	1.1141	0.0127
	(1.0837, 1.1376)		(1.0871, 1.1415)		(1.0894, 1.1395)	
σ_h	0.1748	0.0101	0.1777	0.0097	0.1036	0.0035
	(0.1554, 0.1949)		(0.1592, 0.1970)		(0.0960, 0.1101)	
ρ	-0.2162	0.0381	-0.2101	0.0369	-0.0463	0.0217
	(-0.2913, -0.1425)		(-0.2802, -0.1364)		(-0.0892, -0.0038)	
b	0.8622	0.0259	0.8850	0.0255	0.4253	0.0123
	(0.8131, 0.9127)		(0.8363, 0.9360)		(0.4048, 0.4571)	
σ_{VM}	0.4212	0.0063	0.4223	0.0064	0.0266	0.0007
	(0.4092, 0.4337)		(0.4096, 0.4349)		(0.0252, 0.0279)	
	RV-ASV-JPR		**BV-ASV-JPR**		**VIX-ASV-JPR**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9735	0.0038	0.9727	0.0040	0.9912	0.0017
	(0.9657, 0.9807)		(0.9647, 0.9802)		(0.9879, 0.9945)	
σ_y	1.1125	0.0137	1.1150	0.0136	1.1187	0.0130
	(1.0864, 1.1397)		(1.0887, 1.1421)		(1.0941, 1.1443)	
σ_h	0.1818	0.0097	0.1844	0.0102	0.0940	0.0028
	(0.1638, 0.2018)		(0.1650, 0.2052)		(0.0884, 0.0994)	
ρ	-0.4575	0.0295	-0.4464	0.0293	-0.6070	0.0127
	(-0.5149, -0.3989)		(-0.5022, -0.3884)		(-0.6316, -0.5814)	
b	0.8943	0.0259	0.9192	0.0280	0.5009	0.0131
	(0.8434, 0.9448)		(0.8662, 0.9744)		(0.4763, 0.5298)	
σ_{VM}	0.4131	0.0063	0.4138	0.0064	0.0228	0.0005
	(0.4008, 0.4256)		(0.4014, 0.4264)		(0.0218, 0.0239)	

Inside the parentheses is the 95% density interval.

Table C3.3: The estimation results of the ASV models with HSI daily returns and volatility measures.

Para	RV-ASV-HS Mean	RV-ASV-HS Stdev	BV-ASV-HS Mean	BV-ASV-HS Stdev	VIX-ASV-HS Mean	VIX-ASV-HS Stdev
δ	0.9573	0.0045	0.9626	0.0040	0.9868	0.0023
	(0.9482, 0.9656)		(0.9544, 0.9700)		(0.9822, 0.9913)	
σ_y	1.0010	0.0125	1.0085	0.0125	1.0469	0.0119
	(0.9769, 1.0256)		(0.9843, 1.0330)		(1.0241, 1.0705)	
σ_h	0.2621	0.0120	0.2391	0.0110	0.1280	0.0046
	(0.2398, 0.2866)		(0.2180, 0.2612)		(0.1187, 0.1367)	
ρ	-0.6167	0.0308	-0.6141	0.0309	-0.1995	0.0261
	(-0.6758, -0.5541)		(-0.6726, -0.5517)		(-0.2514, -0.1499)	
b	0.8843	0.0226	0.9170	0.0242	0.3843	0.0115
	(0.8413, 0.9294)		(0.8702, 0.9648)		(0.3650, 0.4090)	
σ_{VM}	0.5039	0.0076	0.4979	0.0073	0.0330	0.0010
	(0.4888, 0.5188)		(0.4838, 0.5120)		(0.0311, 0.0349)	
	RV-ASV-JPR		**BV-ASV-JPR**		**VIX-ASV-JPR**	
Para	**Mean**	**Stdev**	**Mean**	**Stdev**	**Mean**	**Stdev**
δ	0.9667	0.0039	0.9685	0.0036	0.9889	0.0015
	(0.9587, 0.9741)		(0.9610, 0.9753)		(0.9859, 0.9919)	
σ_y	1.0168	0.0126	1.0239	0.0125	1.0567	0.0120
	(0.9925, 1.0420)		(0.9995, 1.0487)		(1.0336, 1.0802)	
σ_h	0.2419	0.0111	0.2348	0.0102	0.1234	0.0034
	(0.2213, 0.2654)		(0.2158, 0.2556)		(0.1169, 0.1303)	
ρ	-0.6765	0.0248	-0.7124	0.0210	-0.8332	0.0063
	(-0.7227, -0.6258)		(-0.7520, -0.6697)		(-0.8451, -0.8206)	
b	0.9623	0.0263	1.0077	0.0282	0.4658	0.0113
	(0.9118, 1.0154)		(0.9556, 1.0641)		(0.4436, 0.4878)	
σ_{VM}	0.4914	0.0072	0.4732	0.0069	0.0221	0.0005
	(0.4773, 0.5056)		(0.4596, 0.4869)		(0.0211, 0.0230)	

Inside the parentheses is the 95% density interval.

Table C3.4: The estimation results of the ASV models with STOXX50E daily returns and volatility measures.