OPTIMIZATION MODELS AND ALGORITHMS FOR PRICING IN E-COMMERCE

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Lay Abstract

The increase in online retail and the improvements in mobile technologies has lead to advantages and opportunities for both customers and retailers. One of these advantages is the ability to keep and efficiently access records of historical orders for both customers and retailers. In addition, online retailing has dramatically decreased the cost of price adjustments and discounts compared to the brick and mortar environment. At the same time, with the increase in online retailing we are witnessing proliferations of online reviews in e-commerce platforms. Given this availability of data and the new capabilities in an online retail environment, there is a need to develop pricing optimization models that integrate all these new features. The overarching motivation and theme of this thesis is to review these opportunities and provide methods and models in the context of retailers' online pricing decisions.

Abstract

With the rise of online retailer giants like Amazon, and enhancements in internet and mobile technologies, online shopping is becoming increasingly popular. This has lead to new opportunities in online price optimization. The overarching motivation and theme of this thesis is to review these opportunities and provide methods and models in the context of retailers' online pricing decisions.

In Chapter 2 a multi-period revenue maximization and pricing optimization problem in the presence of reference prices is formulated as a mixed integer nonlinear program. Two algorithms are developed to solve the optimization problem: a generalized Benders' decomposition algorithm and a myopic heuristic. This is followed by numerical computations to illustrate the efficiency of the solution approaches as well as some managerial pricing insights.

In Chapter 3 a data-driven quadratic programming optimization model for online pricing in the presence of customer ratings is proposed. A new demand function is developed for a multi-product, finite horizon, online retail environment. To solve the optimization problem, a myopic pricing heuristic as well as exact solution approaches are introduced. Using customer reviews ratings data from Amazon.com, a new customer rating forecasting model is validated. This is followed by several analytical and numerical insights.

In Chapter 4 a multinomial choice model is used for customer purchase decision to find optimal personalized price discounts for an online retailer that incorporates customer locations and feedback from their reviews. Closed form solutions are derived for two special cases of this problem. To gain some analytical insights extensive numerical experiments are carried followed by several analytical and numerical insights.

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Notation

List of Notations: Chapter 2

Indices	
t	Period in horizon ($t = 1, 2, \cdots, T$)
k	Iteration of the GBD
Parameters	
a	Estimate of the market size in the linear demand function $D_t = a - bp_t$, $a \geq 0$.
b	Estimate of the price sensitivity parameter in the linear demand function $b \geq 0$.
β_G	Gain parameter, $0 \leq \beta_G \leq \beta_L$.
β_L	Loss parameter, $0 \leq \beta_G \leq \beta_L$.
τ	Gain threshold, $\tau \geq 0$.
ρ	Loss threshold, $\rho \geq 0$.
α	The parameter in exponential smoothing reference price model, $0.1 \leq \alpha \leq 0.4$.
ω_i	Parameters in approximation of reference prices $i = 1, 2, 3$, $\omega_i \geq 0$.
c	Constant unit cost, $c \geq 0$
Decision Variables	
p_t	Price in period t, $p_t \geq 0$.
$Y_{G,t}$	Binary gain indicator
$Y_{L,t}$	Binary loss indicator
$q_{L,k}$	Linearizing binary variable for loss history constraints in (GBD-Master) in iteration k
$q_{G,k}$	Linearizing binary variable for gain history constraints in (GBD-Master) in iteration k

Table 1: List of notations (Chapter 2)

List of Notations: Chapter 3

Symbols	
$r_{i,t}$	Reference price for product i in period t.
$\eta_{i,t}$	Consumer rating for product i during period t.
$D_{i,t}$	Demand for product i in period t.
Indices	
t	Period in horizon ($t = 1, 2, \cdots, T$).
i, j	Products, $i, j \in \{1, 2\}$.
Parameters	
a_i	Estimate of the market size for product i in the linear demand function $D_{i,t} = a_i - b_i p_{i,t}$, $a_i \geq 0$.
b_i	Estimate of the price sensitivity parameter in the linear demand function $b_i \geq 0$.
c_i	Constant unit cost for product i, $c_i \geq 0$.
$\mathcal{R}$	Maximum possible rating.
β_i	Reference price weight parameter for product i, $\beta_i \geq 0$.
$\lambda i, j$	Weight of the difference between ratings of product i and j in demand model for product i.
$\gamma i, j$	Weight of the difference between prices of product i and j in demand model for product i.
α	The parameter in exponential smoothing reference price . model, $0.1 \leq \alpha \leq 0.4$.
ω_k	Parameters in approximation of reference prices $k = 1, 2, 3$, $\omega_i \geq 0$.
θ	The parameter in exponential smoothing consumer rating model, $0.1 \leq \alpha \leq 0.4$.
θ_k	Parameters in approximation of consumer ratings $k = 1, 2, 3$, $\omega_i \geq 0$.
ϕ_i	Reasonable price upper bound for product i.
Decision Variables	
$p_{i,t}$	Price of product i in period t, $p_{i,t} \geq 0$.

Table 2: List of notations (Chapter 3)

List of Notations: Chapter 4

Decision Variables	
$d_{i,j}$	The amount of discount provided to consumer i for product j
Symbols	
$U_{i,j}$	The utility that consumer i enjoys from action j.
$\chi_{i,j}$	The probability that consumer i chooses action j.
$\pi_{i,j}$	The expected profits from selling product j to consumer i.
π_i	The expected profits from consumer i.
$\Pi_{i,j}$	Expected total profits.
Indices	
i	Customers ($i \in I$)
j	Products
Parameters	
$L_{i,j}$	Consumer i's loyalty to product j.
b_i	Consumer i's price sensitivity parameter.
p_j	List price of product j.
c_j	Unit cost of product j, $c_i \geq 0$.
$s_{i,j}$	Cost of shipping product j to consumer i.
β_i	Consumer i's sensitivity to price departures from reference price.
λ_i	Customer i's weight for ratings
η_j	Consumer rating for product j.
r_j	Reference price for product j.

Table 3: List of notations (Chapter 4)

Chapter 1

Introduction

1.1 Motivation

With the rise of online retailer giants like Amazon, and enhancements in internet and mobile technologies, online shopping is becoming increasingly popular. In Canada, about 84% of internet users shopped online reaching $57.4 billion spending in 2018 [58]. That is a large increase compared to $18.9 billion in 2012 [58]. In addition, between 2016 and 2018, Canadian online sales have risen by 58% compared to 5% increase in traditional retail sales [50, 58]. The recent COVID-19 pandemic has further contributed to the rise in online retailing. For example e-commerce sales increased by about 40% in the week of May 26, 2020 compared to that of Feb 24, 2020 [7], and the number of US households that have placed online orders for groceries has doubled to reach 40 million in March 2020, compared to August 2019 [51]. In addition, many businesses are forced to invest and adjust to implement stronger online presence which will likely lead to permanent increase in online sales [42].

The increase in online retail and the improvements in mobile technologies has lead to advantages and opportunities for both customers and retailers. One of these advantages is the ability to keep and efficiently access records of historical orders for both customers and retailers. This can improve retailers' estimate of reference prices. In addition this helps customers to form a more uniform historical-price based reference prices, the price customers use as benchmark for their purchase decisions [61]. This visibility of historical purchase order data for both the customer and the retailer is making it more practical and meaningful to incorporate reference prices in pricing optimization models.

In addition, online retailing has dramatically decreased the cost of price adjustments and discounts compared to the brick and mortar environment. For example, it has been reported that Amazon adjusted prices more than 2.5 million times every day in December 2013 [23]. At the same time, with the increase in online retailing we are witnessing proliferations of online reviews in e-commerce platforms. As an example, TripAdvisor observed a 15% increase in the number of reviews in 2019 reaching 859 million reviews [36]. Also, according to a study, online ratings affect the purchase decision of 93% of customers [49]. It is also reported that customers are willing to pay a higher premium for products with higher quality and that online ratings is a representation of product quality [49]. This speaks to the importance of customer reviews and the need to incorporate its impact in demand function modelling as well as pricing optimization models.

Furthermore, with the increasing popularity of online shopping, we are witnessing an increasing number of websites and browser extensions for online discount codes such as PromoCodes, Wikibuy, RetailMeNot, Slickdeals, Honey, Groupon, and

Ebates. As of 2019, about 60% of online shoppers worldwide looked for online discount codes before making a purchase [14]. For example, there was an average of 68 million monthly visits to Slickdeals website as of March 2019 [14]. Customers adopt various shopping strategies to acquire discounts. For example, some customers fill the virtual shopping cart and "abandon" it realizing that a majority of companies will send discount codes as an incentive to convince them to complete their purchase [15]. In addition, many customers connecting with brands on social media do so in the hope of receiving regular coupons and other promotions [14]. Given the availability of data on prices and the ease of making price discounts online, there is a need to develop pricing optimization models that integrate all these new features.

The overarching motivation and theme of this thesis is to review these opportunities and provide methods and models in the context of retailers' online pricing decisions. In order to provide a unified conceptual framework, a brief overview of the literature in this area is provided in Section 1.2. A summary of contributions and thesis overview is presented in Section 1.3.

1.2 Background and Literature Review

In this section a general overview of the background material necessary for the remainder of this thesis is provided. The remaining chapters of this dissertation consist of work published in or submitted for publication in peer-reviewed journals. There is some overlap between the material in this section and those produced in chapters 2-4. The purpose of this section is to provide a unified background for the remaining chapters of the thesis, and thus, the review of the literature in this chapter is kept brief. With this purpose in mind, an overview of the literature on reference prices, customer ratings, customer choice, shipping fees, and product discounts are provided in sections 1.2.1-1.2.5.

1.2.1 Reference Prices and Demand

Reference price is the price that customers take as a point of reference to decide on whether an observed price is a good deal or not [43]. Many studies have considered modelling and applications of reference prices (e.g., see [8, 27, 30, 43]). We briefly discuss the most relevant studies. A more comprehensive review of this topic can be found in Chapters 2 and 3.

The general literature on reference prices can be grouped into two sections. Some studies have focused on modelling reference prices. To name a few, Briesch et al. [8] concluded that different product categories require different reference price models. Wang [64] studied different formulations of reference prices in a multi-product setting. Nasiry and Popescu explore a memory-based reference price model based on the peak-end rule; that is, the reference price is modelled as a weighted average of the lowest price and the most recent price [46]. Mazumdar et al. present a review of reference price research [43]. They model reference price in a selling period as the weighted average of the price and reference price in the previous period. This model is widely adopted in the literature (e.g., see [2, 43, 61]) and is used throughout this thesis.

To find optimal pricing strategies, another group of studies have considered modelling customers' response and demand function in the presence of reference prices. Yuan and Lee [66] showed that advertised online reference prices influence customers' price perceptions. In these cases, the retailer presents a marked up price (the advertised reference price) and a lower sale price (the observed price). Hsieh and Dye [25] consider an additive reference price effect on demand in an infinite selling horizon for deteriorating goods. Under certain conditions, they show that in loss-neutral and loss-averse markets, optimal prices converge monotonically to a long-term equilibrium price and in loss-seeking market, a high-low pricing policy is optimal [25].

Von Massow and Hassini [61] have considered reference prices to have an additive effect on the linear demand function and introduced thresholds to the model. In particular, they modelled the demand function as

$$D_t = \begin{cases} a - bp_t + \beta_G(r_t - \tau - p_t) & p_t \leq r_t - \tau \\ a - bp_t & r_t - \tau \leq p_t \leq r_t + \rho \\ a - bp_t + \beta_L(r_t + \rho - p_t) & p_t \geq r_t + \rho. \end{cases} \tag{1.2.1}$$

where subscript t denotes the period in the selling horizon; D_t, p_t, r_t, β_G, and β_L denote the demand, price, reference price, gain, and loss parameters, respectively. The term $a - bp_t$ represents the traditional linear demand. Parameters $\tau \geq 0$ and $\rho \geq 0$ denote the gain threshold below the reference price and the loss threshold above the reference price, respectively. Note that in this demand model, reference price effects the demand only if $p_t \leq r_t - \tau$ or $p_t \geq r_t + \rho$. This demand model is adopted in Chapter 2. Anderson et al. [2] have considered a duopoly multiplicative model where the demand for each period is the previous period's demand multiplied by a function of reference price. They have considered a two-firm market of different sizes with the objective to maximize the profits of the smaller firm in different pricing decision scenarios [2]. Casado and Ferrer [10] introduce a customer utility model that estimates price thresholds and its impact on demand elasticity, taking into account customer heterogeneity. Lu et al. [39] studied joint pricing strategies with reference price effect for a monopolistic firm. They considered all four possible combinations of dynamic and static strategies for pricing and advertising effort levels. The study concluded that the dynamic strategies result in higher profits than static strategies [39]. Li and Teng [33] use a multiplicative form of price, reference price, freshness, and displayed stock level to model the demand for perishable goods. They show that price and periodic ending inventory level converge monotonically to a long term equilibrium [33].

1.2.2 Customer Ratings

The bulk of literature on customer rating can be divided into tow main groups. The first group studies the effects of ratings on sales and prices. The other, studies factors affecting the ratings.

Effect of Customer Rating on Sales

Many have studied the effects of customer reviews on sales (e.g., [5, 11, 28, 34]). Anand et al. [1] provide a comprehensive review of this literature. Zhu and Zhang [68] show that on average one point increase in average customer ratings increases the sales of a video game by 4%. Some have focused on the asymmetric effects of positive and negative ratings on sales (e.g., [13, 48]). These studies argue that extreme ratings, positive or negative, have more effect on customer choice than moderate ratings and the magnitude of the effect is asymmetric with negative ratings having a larger impact.

Some studies consider the effects of the volume of reviews on sales (e.g., [6, 18, 41]). Generally these studies argue that the volume of online reviews positively affect product sales. However, Maslowska et al. [41] concluded that customer rating affects the purchase decision and the magnitude of the effect increases as the volume of the reviews increase. They further argue that this effect is increased if the price is relatively high. Others have considered the effect of the variability of customer ratings on sales [4, 60, 69]. Generally it is argued that variability of ratings has a negative effect on sales as customers view variability in ratings as an increased risk. Sun [60], however, concluded that the effect of the variance of ratings on demand depends on the average rating. In particular, a high variance in customer ratings decreases the demand when the average rating is high and increases it when the average rating is low.

De Maeyer [16] provides a review of the literature on online customer reviews and sales. One of the common findings in the literature is that high product ratings reduce customers' price sensitivity and increase the price premium customers are willing to pay. For example, Smith et al. [57] investigate the effects of customer ratings on prices using a large data set of ratings and prices of beer. They conclude that an increase in customer ratings is positively related to an increase in prices.

Wang et al. [62] study the optimal pricing strategy of an online seller in a duopoly market competing with an off-line seller. They show that the online seller's optimal price decreases at the early stages where the amount of information (customer ratings) is low and increases as the amount of information increase.

He and Chen [23] show that it may be beneficial to use low prices for high-quality products at the beginning stages to speed-up the customer learning process.

Factors Affecting Ratings

Many have studied possible factors that affect customer ratings. The vast majority consider factors other than prices (e.g., [3, 12, 24, 35, 65]). Ho et al. [24] model the individuals' rating as a linear combination of experienced performance, the difference between pre and post purchase evaluations, and other control variables. Lin et al. [35] study the effects of free product sampling on product ratings. They conclude that free product sampling increases product ratings on average by 1.1%. They also note that the magnitude of this bias is larger when the product list price is higher.

Very few studies consider prices affecting customer ratings. Shapiro [55] conducted an experimental study and concluded that price can be viewed by some customers as an indication of the quality of the product and thus affect their rating decision. Li and Hitt [34] note that customer reviews are affected not only by quality factors, but are also biased by price effects. Engler et al. [19] argue that online customer ratings represent customers satisfaction. They show that customer satisfaction is explained by their pre-purchase expectation (formed from product rating, price, and brand reputation) and their post-purchase observed performance. Stenzel et al. [59] also consider prices having direct negative effect on customer ratings.

Some have considered customer ratings in a profit maximization problem as a two stage game where customers in the second period/stage observe the price and ratings from customers in the first period. For example, Kuksov and Xie [28] show that the customer rating is decreasing with respect to price. Similarly, Feng et al. [20] show price has a negative effect on reviews.

1.2.3 Customer Choice

Customer choice models are used to estimate customers' willingness to pay and product preference. Some studies have investigated customers' screening rules and probabilities of purchase.

Bucklin and Lattin [9] introduce a probabilistic model for purchase incident and brand choice. Some, such as Gilbride and Allenby [22] and Wang et al. [63], model customers' decision making as a two stage process. In the first stage a "consideration set" is chosen from all available options. Customers' then make a final purchase decision from this set. Lachaab et al. [29] model evolution of customers preferences using Bayesian state space models. They show that customers preferences not only differ across customers, it also changes over time. In particular they find that customers become more price sensitive over time. They explain that this effect may be due to frequent price promotions that reduce customers' reference price. Several studies have used customer choice models to investigate product recommendation systems (e.g., [37, 38, 52, 67]).

1.2.4 Shipping Fees

With the ever growing e-commerce platforms, there has been increasing studies considering the effects of free-shipping policies on customer purchase decisions and ultimately on revenues. Lewis et al. [32] study the impact of shipping fees on customer purchasing behaviour. Leng et al. [31] investigate when shipping fee promotions improve profits and derive several managerial insights for monopoly and duopoly market structures. Ma [40] showed that delivery time does not have a large impact on customer satisfaction but it has a significant impact on purchase intentions. They also noted that customers are willing to pay a premium for quick delivery.

1.2.5 Product Discounts

The literature in product discounts can be divided into three groups. Some have studied advance purchase discounts. Gale and Holmes [21] study the optimal pricing policy for a monopolistic airline and show that if capacity constraints are present, the monopolist must divert peak period demand to off peak period by offering advance purchase discounts. Möller and Watanabe [44] find conditions under which for a monopolist, early price discount or late clearance sales are optimal. Nocke et al. [47] present necessary and sufficient condition under which advance purchase discounts is an optimal strategy when capacity constraints are not present.

Others have studied quantity discounts. For example, Dolan [17] studies motivations for quantity discounts and provides guidelines for quantity discount schedules. The last category addresses product bundle discounts. Sheng et al. [56] study the effects of discounts on the discounted product in bundles. They show that these types of discounts negatively affect customers' evaluations of the discounted product. Janiszewski and Cunha [26] show that the customer evaluation of price discounts in bundled products depend on which product in the bundle is being discounted.

1.3 Contributions and Thesis Overview

The main contribution of this thesis is to close some of the gaps in literature on pricing optimization in an e-commerce context. In doing so, this research also contributes to new models and solution methods that builds on previous literature. Contributions of Chapters 2-4 are summarized below.

In Chapter 2 price optimization in the presence of reference prices with thresholds are studied. The existing literature in this area has mostly relied on dynamic programming. This has allowed for obtaining some structural pricing results, but under stringent conditions. As a results, only small problems were solved computationally due to the "curse of dimensionality" that exists in dynamic programming algorithms. However, with the growth in e-commerce industry there is an increasing need to solve large pricing problems. It is the goal of this chapter to fill this gap by developing an efficient methodology for solving realistic and large scale price optimization problems in the presence of reference prices and thresholds. This chapter builds on the models used in [61] with several significant differences: First, there is a difference in the modelling approach. As we will see, the constraints used in this chapter are necessary and sufficient as opposed to sufficient constraints used in [61] which can exclude some feasible, and possibly optimal, pricing combinations. Second, different solution techniques are introduced to efficiently solve large optimization problems. These include a heuristic approach as well as a Benders' decomposition method. Finally, unlike [61], no optimal price patterns (such as pricing cycles) are assumed.

The focus of Chapter 3 is to consider the effect of customer ratings on optimal prices and vice versa in a multi-period, multi-product environment. The effects of customer reviews, reference price, and cross-price effects on demand have been studied in the literature on revenue management (e.g., [2, 8, 43, 53, 54, 61]). As discussed in Section 1.2.2, very few studies have considered customer ratings as a response to prices. There are no studies that explicitly study customer ratings as a response to prices where prices are decision variables in a revenue optimization problem utilizing reference prices, customer ratings, and cross-price effects. This chapters bridges this gap in the literature and offers some insights on how prices impact customer ratings. In doing so, a new model for forecasting reviews is introduced and validated using Amazon data. Finally, a comprehensive price optimization model that incorporates the impact of reviews and historical prices on optimal prices for multiple products in an online retail environment is developed. A linear demand model that accounts for reference prices, cross-price effects, and customer ratings is used in the price optimization problem. To solve the optimization problem, a heuristic method is introduced and compared with commercial solvers. In addition, conditions under which the heuristic produces close to optimal results are provided.

In previous research it has been shown that discount coupons are efficient tools for price discrimination (e.g., [45]). The current literature consists of studies on discounts, shipping fees, and customer purchase behaviour. However, there are no studies that explicitly study personalised product discount optimization in a multinomial choice model utilizing customer locations, in the form of shipping costs, and product review data. In Chapter 4, a nonlinear programming model is introduced that uses multinomial customer choice utility function. In addition, customers' utility and purchase probabilities from customer choice literature are modified to account for prices, discounts, and ratings. Furthermore, exact solutions for two special cases are provided followed by extensive numerical analysis and insights.

The remainder of this thesis is organized as follows. Three works published in or submitted for publication in peer-reviewed journals are presented in Chapters 2-4 (discussed above). Chapter 5 concludes the thesis and offers remarks on the overall theme and potential avenues for future research.

1.4 Author's Statement of Contribution

I am the author of this thesis and the first author of all works submitted or accepted for publication included in this thesis.

Bibliography

[1] Anand, O., Srivastava, P. R., and Rakshit, A. (2017). Assessment, implication, and analysis of online consumer reviews: A literature review. *Pacific Asia Journal of the Association for Information Systems*, **9**(2).

[2] Anderson, C. K., Rasmussen, H., and MacDonald, L. (2005). Competitive pricing with dynamic asymmetric price effects. *International Transactions in Operational Research*, **12**(5), 509–525.

[3] Archak, N., Ghose, A., and Ipeirotis, P. G. (2011). Deriving the pricing power of product features by mining consumer reviews. *Management Science*, **57**(8), 1485–1509.

[4] Babić Rosario, A., Sotgiu, F., De Valck, K., and Bijmolt, T. H. (2016). The effect of electronic word of mouth on sales: A meta-analytic review of platform, product, and metric factors. *Journal of Marketing Research*, **53**(3), 297–318.

[5] Barbado, R., Araque, O., and Iglesias, C. A. (2019). A framework for fake review detection in online consumer electronics retailers. *Information Processing & Management*, **56**(4), 1234–1244.

[6] Berger, J., Sorensen, A. T., and Rasmussen, S. J. (2010). Positive effects of

negative publicity: When negative reviews increase sales. *Marketing Science*, **29**(5), 815–827.

[7] Berthene, A. (2020). Shoppers buy more online compared with before the pandemic. https://www.digitalcommerce360.com.

[8] Briesch, R. A., Krishnamurthi, L., Mazumdar, T., and Raj, S. P. (1997). A comparative analysis of reference price models. *Journal of Consumer Research*, **24**(2), 202–214.

[9] Bucklin, R. E. and Lattin, J. M. (1991). A two-state model of purchase incidence and brand choice. *Marketing Science*, **10**(1), 24–39.

[10] Casado, E. and Ferrer, J. C. (2013). Consumer price sensitivity in the retail industry: Latitude of acceptance with heterogeneous demand. *European Journal of Operational Research*, **228**(2), 418–426.

[11] Chen, Y. and Xie, J. (2008). Online consumer review: Word-of-mouth as a new element of marketing communication mix. *Management Science*, **54**(3), 477–491.

[12] Chen, Y., Fay, S., and Wang, Q. (2011). The role of marketing in social media: How online consumer reviews evolve. *Journal of Interactive Marketing*, **25**(2), 85–94.

[13] Chevalier, J. A. and Mayzlin, D. (2006). The effect of word of mouth on sales: Online book reviews. *Journal of Marketing Research*, **43**(3), 345–354.

[14] Clement, J. (2019). Digital coupons and deals - statistics & facts. Technical report, Statista.

[15] Daily, L. (2019). Nine tricks for getting the most out of online coupon codes. https://www.washingtonpost.com.

[16] De Maeyer, P. (2012). Impact of online consumer reviews on sales and price strategies: A review and directions for future research. *Journal of Product & Brand Management*, **21**(2), 132–139.

[17] Dolan, R. J. (1987). Quantity discounts: Managerial issues and research opportunities. *Marketing Science*, **6**(1), 1–22.

[18] Duan, W., Gu, B., and Whinston, A. B. (2008). The dynamics of online word-of-mouth and product sales—an empirical investigation of the movie industry. *Journal of Retailing*, **84**(2), 233–242.

[19] Engler, T. H., Winter, P., and Schulz, M. (2015). Understanding online product ratings: A customer satisfaction model. *Journal of Retailing and Consumer Services*, **27**, 113–120.

[20] Feng, J., Li, X., and Zhang, X. (2019). Online product reviews-triggered dynamic pricing: Theory and evidence. *Information Systems Research*, **30**(4), 1107–1123.

[21] Gale, I. L. and Holmes, T. J. (1993). Advance-purchase discounts and monopoly allocation of capacity. *The American Economic Review*, pages 135–146.

[22] Gilbride, T. J. and Allenby, G. M. (2004). A choice model with conjunctive, disjunctive, and compensatory screening rules. *Marketing Science*, **23**(3), 391–406.

[23] He, Q.-C. and Chen, Y.-J. (2018). Dynamic pricing of electronic products with consumer reviews. *Omega*, **80**, 123–134.

[24] Ho, Y.-C., Wu, J., and Tan, Y. (2017). Disconfirmation effect on online rating behavior: A structural model. *Information Systems Research*, **28**(3), 626–642.

[25] Hsieh, T. P. and Dye, C. Y. (2017). Optimal dynamic pricing for deteriorating items with reference price effects when inventories stimulate demand. *European Journal of Operational Research*, **262**(1), 136–150.

[26] Janiszewski, C. and Cunha Jr, M. (2004). The influence of price discount framing on the evaluation of a product bundle. *Journal of Consumer Research*, **30**(4), 534–546.

[27] Kalyanaram, G. and Winer, R. S. (1995). Empirical generalizations from reference price research. *Marketing Science*, **14**(3_supplement), G161–G169.

[28] Kuksov, D. and Xie, Y. (2010). Pricing, frills, and customer ratings. *Marketing Science*, **29**(5), 925–943.

[29] Lachaab, M., Ansari, A., Jedidi, K., and Trabelsi, A. (2006). Modeling preference evolution in discrete choice models: A bayesian state-space approach. *Quantitative Marketing and Economics*, **4**(1), 57–81.

[30] Lattin, J. M. and Bucklin, R. E. (1989). Reference effects of price and promotion on brand choice behavior. *Journal of Marketing Research*, pages 299–310.

[31] Leng, M. and Becerril-Arreola, R. (2010). Joint pricing and contingent free-shipping decisions in b2c transactions. *Production and Operations Management*, **19**(4), 390–405.

[32] Lewis, M., Singh, V., and Fay, S. (2006). An empirical study of the impact

of nonlinear shipping and handling fees on purchase incidence and expenditure decisions. *Marketing Science*, **25**(1), 51–64.

[33] Li, R. and Teng, J. T. (2018). Pricing and lot-sizing decisions for perishable goods when demand depends on selling price, reference price, product freshness, and displayed stocks. *European Journal of Operational Research*.

[34] Li, X. and Hitt, L. M. (2010). Price effects in online product reviews: An analytical model and empirical analysis. *MIS Quarterly*, pages 809–831.

[35] Lin, Z., Zhang, Y., and Tan, Y. (2019). An empirical study of free product sampling and rating bias. *Information Systems Research*, **30**(1), 260–275.

[36] Lock, S. (2020). Number of user reviews and opinions on tripadvisor worldwide 2014-2019s. Technical report, Statista.

[37] Lopes, P. and Roy, B. (2015). Dynamic recommendation system using web usage mining for e-commerce users. *Procedia Computer Science*, **45**, 60–69.

[38] Louca, R., Bhattacharya, M., Hu, D., and Hong, L. (2019). Joint optimization of profit and relevance for recommendation systems in e-commerce. In *Workshop on Recommendation in Multi-stakeholder Environments (RMSE'19), in conjunction with the 13th ACM Conference on Recommender Systems, RecSys*, volume 19, page 2019.

[39] Lu, L., Gou, Q., Tang, W., and Zhang, J. (2016). Joint pricing and advertising strategy with reference price effect. *International Journal of Production Research*, pages 1–21.

[40] Ma, S. (2017). Fast or free shipping options in online and omni-channel retail? the mediating role of uncertainty on satisfaction and purchase intentions. *The International Journal of Logistics Management*.

[41] Maslowska, E., Malthouse, E. C., and Viswanathan, V. (2017). Do customer reviews drive purchase decisions? the moderating roles of review exposure and price. *Decision Support Systems*, **98**, 1–9.

[42] Matusiak, E. (2020). Planning for the post covid-19 retail world. https://www.bdo.ca.

[43] Mazumdar, T., Raj, S., and Sinha, I. (2005). Reference price research: Review and propositions. *Journal of Marketing*, **69**(4), 84–102.

[44] Möller, M. and Watanabe, M. (2010). Advance purchase discounts versus clearance sales. *The Economic Journal*, **120**(547), 1125–1148.

[45] Narasimhan, C. (1984). A price discrimination theory of coupons. *Marketing Science*, **3**(2), 128–147.

[46] Nasiry, J. and Popescu, I. (2011). Dynamic pricing with loss-averse consumers and peak-end anchoring. *Operations Research*, **59**(6), 1361–1368.

[47] Nocke, V., Peitz, M., and Rosar, F. (2011). Advance-purchase discounts as a price discrimination device. *Journal of Economic Theory*, **146**(1), 141–162.

[48] Park, S. and Nicolau, J. L. (2015). Asymmetric effects of online consumer reviews. *Annals of Tourism Research*, **50**, 67–83.

[49] Podium (2017). 2017 state of online reviews. Technical report, Podium.

[50] Post, C. (2019). The 2019 canadian e-commerce benchmark report. Technical report, Canada Post.

[51] Ryan, C. (2020). Online grocers are getting a preview of their future. https://www.wsj.com.

[52] Scholz, M., Dorner, V., Schryen, G., and Benlian, A. (2017). A configuration-based recommender system for supporting e-commerce decisions. *European Journal of Operational Research*, **259**(1), 205–215.

[53] Sethuraman, R., Srinivasan, V., and Kim, D. (1999). Asymmetric and neighborhood cross-price effects: Some empirical generalizations. *Marketing Science*, **18**(1), 23–41.

[54] Shams-Shoaaee, S. S. and Hassini, E. (2019). Price optimization with reference price effects: A generalized Benders' decomposition method and a myopic heuristic approach. *European Journal of Operational Research*.

[55] Shapiro, B. P. (1973). Price reliance: existence and sources. *Journal of Marketing Research*, **10**(3), 286–294.

[56] Sheng, S., Parker, A. M., and Nakamoto, K. (2007). The effects of price discount and product complementarity on consumer evaluations of bundle components. *Journal of Marketing Theory and Practice*, **15**(1), 53–64.

[57] Smith, R. A., McKinney, C. N., Caudill, S. B., and Mixon, F. G. (2016). Consumer ratings and the pricing of experience goods: hedonic regression analysis of beer prices. *Agricultural and Food Economics*, **4**(1), 24.

[58] StatCan (2019). Retail trade. Technical report, Statistics Canada.

[59] Stenzel, A., Wolf, C., Schmidt, P., *et al.* (2020). Pricing for the stars-dynamic pricing in the presence of rating systems. Technical report, University of Bonn and University of Mannheim, Germany.

[60] Sun, M. (2012). How does the variance of product ratings matter? *Management Science*, **58**(4), 696–707.

[61] Von Massow, M. and Hassini, E. (2013). Pricing strategy in the presence of reference prices with thresholds. *Journal of Revenue & Pricing Management*, **12**(4), 339–359.

[62] Wang, H., Zhang, W., and Zheng, L. (2011). Dynamic pricing in b2c based on online product reviews. *Procedia Engineering*, **23**, 270–275.

[63] Wang, H., Wei, Q., and Chen, G. (2013). From clicking to consideration: A business intelligence approach to estimating consumers' consideration probabilities. *Decision Support Systems*, **56**, 397–405.

[64] Wang, R. (2018). When prospect theory meets consumer choice models: Assortment and pricing management with reference prices. *Manufacturing & Service Operations Management*.

[65] Yin, D., Mitra, S., and Zhang, H. (2016). Research note—when do consumers value positive vs. negative reviews? an empirical investigation of confirmation bias in online word of mouth. *Information Systems Research*, **27**(1), 131–144.

[66] Yuan-Shuh Lii, D. and Lee, M. (2005). Consumers' evaluations of online reference price advertisement. *International Journal of Commerce and Management*, **15**(2), 101–112.

[67] Zhao, Q., Zhang, Y., Friedman, D., and Tan, F. (2015). E-commerce recommendation with personalized promotion. In *Proceedings of the 9th ACM Conference on Recommender Systems*, pages 219–226. ACM.

[68] Zhu, F. and Zhang, X. (2006). The influence of online consumer reviews on the demand for experience goods: The case of video games. *ICIS 2006 Proceedings*, page 25.

[69] Zimmermann, S., Herrmann, P., Kundisch, D., and Nault, B. R. (2018). Decomposing the variance of consumer ratings and the impact on price and demand. *Information Systems Research*, **29**(4), 984–1002.

Chapter 2

Price Optimization with Reference Price Effects: A Generalized Benders' Decomposition Method and a Myopic Heuristic Approach

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Price optimization with reference price effects: A generalized Benders' decomposition method and a myopic heuristic approach

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Abstract

We consider a multi-period revenue maximization and pricing optimization problem in the presence of reference prices. We formulate the problem as a mixed integer non program and develop a generalized Benders' decomposition algorithm to solve it. In addition, we propose a myopic heuristic and discuss the conditions under which it produces efficient solutions. We provide analytical results as well as numerical computations to illustrate the efficiency of the solution approaches as well as some managerial pricing insights.

Keywords: Pricing, reference pricing, mixed-integer nonlinear programming, generalized Benders' decomposition.

2.1 Introduction

Developing a pricing strategy is one of the most important operations aspects of any firm in a given market [4, 14, 16, 26]. Marn et al. reported that a 1% improvement in pricing can result in 8% improvement in profits [18]. Studies aimed at finding optimal pricing strategies have led researchers to consider reference prices in the development of a pricing strategy [2, 12, 15, 19]. Reference prices, used widely in the mainstream marketing literature, is the price consumers use to determine whether or not an observed price is a good deal [25].

Kalyanaram and Winer studied the effects of consistent price promotions [12]. They found that the later promotions are not seen as a good deal as the earlier ones, and reverting to the original price can be perceived by customers as a price increase [12]. They also concluded that consumers' responses to gain and loss are asymmetric. Here, gain is defined as when the observed price is below the reference price, and loss is when the observed price is above the reference price [12]. Briesch et al. [2] studied different models of household reference prices and concluded that the model of reference price differs between different product categories.

Mazumdar et al. present a review of reference price research [19]. The authors model reference price based on the "adaptive expectation" model introduced by Nerlove [22]. Specifically, they use the following model

$$r_t = \alpha r_{t-1} + (1-\alpha)p_{t-1} \qquad \alpha \in [0,1]. \tag{2.1.1}$$

where p_t and r_t denote price and reference price in period t, respectively, and α is a memory parameter. They also suggested that α usually ranges between 0.15 and 0.4. This exponential smoothing model for calculating reference prices is widely adopted in the literature (e.g., see [1, 19, 25]) and will be used in this paper.

Nasiry and Popescu explore a memory-based reference price model based on the peak-end rule; that is, the reference price is modelled as a weighted average of the lowest price and the most recent price [21]. This model assumes that consumers can remember the lowest prices indefinitely. This may be a reasonable assumption when transaction frequency is high relative to the length of the horizon. However, this is not a reasonable assumption in general. Also, in their modelling of demand, they did not consider thresholds in the reference price effect. They concluded that behavioural regularities cause prices to converge over time and lead to constant pricing strategies [21].

To find an optimal pricing strategy with the consideration of the reference price, one must first model the demand function. Huang et al. [9] provide a comprehensive survey of demand models. Linear, exponential, and iso-elastic demand functions are a few of many demand functions used in the literature. In this paper, we will assume a piece-wise linear demand function because of the the relative simplicity to estimate its parameters, price dependent demand elasticity, and the lack of unrealistic asymptotic behaviour. For example, when iso-elastic demand functions are used, the demand approaches infinity as the price approaches zero which is an unrealistic behaviour as the market size is finite [9]. Also, the linear demand functions require a finite upper bound on price which reflects the finite upper limit on consumers' acceptable price range (e.g., see [20]). This also limits the search area in the price optimization problem.

Von Massow and Hassini have considered reference price to have an additive effect on the traditional linear demand function for each period [25]. In particular, they modelled the demand function as

$$D_t = \begin{cases} a - bp_t + \beta_G(r_t - p_t) & p_t \leq r_t \quad \text{Gain} \\ a - bp_t & p_t = r_t \\ a - bp_t + \beta_L(r_t - p_t) & p_t \geq r_t \quad \text{Loss} \end{cases} \tag{2.1.2}$$

$$r_t = \alpha r_{t-1} + (1-\alpha)p_{t-1} \qquad \alpha \in [0,1], \tag{2.1.3}$$

where D_t, p_t, and r_t denote the demand, price, and reference price in period t, respectively. The term $a - bp_t$ represents the traditional linear demand. The constants β_G and β_L denote gain and loss parameters, respectively. Von Massow and Hassini also introduced thresholds to the above model. The rationale for introducing the thresholds is that the consumers only react if the difference between the reference price and price are outside a threshold range, i.e., when the customer perceives the difference in prices to be significant [25]. Letting $\tau \geq 0$ and $\rho \geq 0$ denote the gain threshold below the reference price and the loss threshold above the reference price, respectively, they introduced the following demand function [25]:

$$D_t = \begin{cases} a - bp_t + \beta_G(r_t - \tau - p_t) & p_t \leq r_t - \tau \\ a - bp_t & r_t - \tau \leq p_t \leq r_t + \rho \\ a - bp_t + \beta_L(r_t + \rho - p_t) & p_t \geq r_t + \rho. \end{cases} \tag{2.1.4}$$

Note that in the above demand model, reference price effects the demand only if $p_t \leq r_t - \tau$ or $p_t \geq r_t + \rho$. We adopt this demand model in this paper. Here, parameters a, b, β_G, and β_L can be estimated by multivariate linear regression. In addition, the memory parameter α in (2.1.1) can be estimated as explained by Greenleaf [6]; particularly by varying α and choosing the one that maximizes the explanatory power of (2.1.4). An initial estimate of reference price thresholds τ and ρ can be made by market research experiments and surveys. This estimation can be further improved by varying their values in the initial guess neighbourhood to maximize the explanatory power of (2.1.4).

There is significant evidence in the literature and is commonly adopted that consumers' response to gains and losses are asymmetric. It is further widely accepted that consumers are usually loss-averse; i.e., $\beta_G \leq \beta_L$ (e.g., see [7, 10, 11]). Note that if the price is lower than the reference price (Gain), the demand will be greater than the linear demand and if the price is higher than the reference price (Loss), the demand will be lower than the traditional linear demand. We will further assume that $b \geq \beta_L$. This is reasonable as it means that changes in price affect the traditional linear demand portion more than the reference price part; that is, the overall "trend" of the demand function is linearly decreasing.

Casado and Ferrer [3] introduce a consumer utility model that estimates price thresholds and its impact on demand elasticity, taking into account consumer heterogeneity. They call the interval between gain threshold below the reference price and loss threshold above the reference price as the latitude of acceptance. They conclude that within the latitude of acceptance consumers are less sensitive to price changes and beyond the thresholds there is significant change in elasticity of demand. They also conclude that the threshold values are mostly asymmetric where it is expected that the loss threshold is smaller than the gain threshold. However, in many cases of their study, the opposite was true; that is, the gain threshold was smaller than the loss threshold. They determined that that unexpected outcome was the result of high consumer brand loyalty for the studied products [3]. Lu et al. [17] studied joint pricing strategies with reference price effect for a monopolistic firm. They considered all four possible combinations of dynamic and static strategies for pricing and advertising effort level. The study concluded that the dynamic strategies result in higher profits than static strategies [17].

The existing literature on pricing optimization with reference prices has mostly relied on dynamic programming. This has allowed for obtaining some structural pricing results, but under stringent conditions. It also meant that only small problems were solved computationally, due to the curse of dimensionality of dynamic programming algorithms. However, in practice we noticed the increasing need to solve large dynamic pricing problems. This is largely due to the availability of data in the online shopping world (where customers have access to historical prices that can go back years, such as with amazon) as well as the increasing use of loyalty cards that allow retailers to have better visibility of customers shopping behaviours for extended periods of time. It is thus our goal in this paper to fill this gap by developing an efficient methodology for solving realistic and large scale price optimization problems in the presence of reference prices and thresholds. Our work builds on the models used in [25]. This paper has several significant differences from that of [25]. First, there is a difference in our modelling approach where we introduce a more general non-negativity constraint. As we show in Proposition 2.2.2, the constraint in [25] may unnecessarily exclude some feasible pricing combinations. Second, we propose different solution techniques to efficiently solve larger problems. The solution procedure introduced in [25] uses dynamic programming which runs into the "curse of dimensionality" [23] and therefore is not useful for solving large scale problems. To overcome this computational difficulty, we propose a heuristic approach as well as a Benders' decomposition method to solve larger problems. Finally, unlike [25], in our model and computational experiments we do not assume any optimal price patterns such as pricing cycles.

The remainder of the paper is organized as follows. In Section 2.2 we present the problem and its mixed integer nonlinear (MINLP) formulation for maximizing the total profit for a horizon of length T. we will then introduce a myopic heuristic approach in Section 2.3 and show that in some cases it produces good quality solutions. A modified generalized Benders' decomposition(GBD) method will be developed in Section 2.4 to solve the MINLP for large values of T. We show that the MINLP is a special case of mixed integer cubic problem where fixing the integer variables result in a convex quadratic programming problems. We make use of this fact to design the GBD algorithm and establish some of its analytical properties. Our computational analysis and insight is reported in Section 2.5. Unlike our proposed approaches, current commercial solvers cannot solve the MINLP efficiently for small values of T; that is, they produce feasible solutions and not optimal solutions to the MINLP problem, or produce solutions of the NLP relaxed problem. In addition, commercial solvers cannot provide feasible solutions for larger values of T. Finally, in Section 2.7 we summarize our findings and propose some future directions for research. The proofs of all results are provided in the appendix 2.7 and a list of notations is provided in appendix 2.7.

2.2 The Problem Statement and Formulation

As discussed in Section 2.1, in this paper, we consider a price optimization problem in the presence of reference prices with threshold. In Section 2.2.1 we will describe the problem. This will be followed by MINLP formulation of the problem in Section 2.2.2.

2.2.1 Problem Description

As mentioned in Section 2.1, we adopt the demand model (2.1.4) introduced by Von Massow [24]. The profit in period t denoted by π_t, assuming reference price r_t and a constant per unit cost c, can be shown as

$$\pi_t = (p_t - c)D_t \tag{2.2.1}$$

where D_t is the demand in period t shown in (2.1.4). This can be re-written as

$$D_t = a - bp_t + (\beta_G Y_{G,t} + \beta_L Y_{L,t})(r_t - p_t + \rho Y_{L,t} - \tau Y_{G,t}) \tag{2.2.2}$$

where $Y_{G,t}$ and $Y_{L,t}$ are defined as

$$Y_{G,t} = \begin{cases} 1 & \text{if } p_t \leq r_t - \tau \\ 0 & o.w. \end{cases} \quad , \quad Y_{L,t} = \begin{cases} 1 & \text{if } p_t \geq r_t + \rho \\ 0 & o.w. \end{cases} \tag{2.2.3}$$

Proposition 2.2.1. *As presented in Section 2.1, Mazumdar et al. [19] concluded that $0.15 \leq \alpha \leq 0.4$. Von Massow [24] showed that in this case, the impact of previous prices diminishes very quickly and for a given r_1, we can approximate r_t as*

$$r_2 = \omega_1 p_1 + (1 - \omega_1) r_1 \tag{2.2.4}$$

$$r_3 = \omega_1 p_2 + \omega_2 p_1 + \omega_3 r_1 \tag{2.2.5}$$

$$r_t = \omega_1 p_{t-1} + \omega_2 p_{t-2} + \omega_3 p_{t-3} \qquad \forall t \geq 4 \tag{2.2.6}$$

where

$$\omega_1 = \frac{1-\alpha}{1-\alpha^3} \quad , \quad \omega_2 = \frac{\alpha(1-\alpha)}{1-\alpha^3} \quad , \quad \omega_3 = \frac{\alpha^2(1-\alpha)}{1-\alpha^3}.$$

The price optimization problem is to find optimal prices that maximize the total profit π during the finite horizon $t = 1, 2, \cdots, T$; i.e.,

$$\max_{p_t \in P} \left\{ \pi = \sum_{t=1}^{T} (p_t - c) \left[a - b p_t + (\beta_G Y_{G,t} + \beta_L Y_{L,t})(r_t - p_t + \rho Y_{L,t} - \tau Y_{G,t}) \right] \right\} \tag{2.2.7}$$

where P is a set of feasible prices.

This model aids the retailers in solving their price and revenue optimization problems. It is helpful for products and services for which customers have developed a reference price. These would include loyal customer, that can be identified through loyalty card programs, as well as other customers who may be shopping around for deals and have a good idea about expected product prices. Given that our solution procedures address large scale problems, our model may particularly be attractive to online retailers where it is relatively easier to collect big data about customers and their purchasing behaviours.

2.2.2 MINLP Formulation

In this Section, we re-introduce the MINLP formulation as proposed by Von Massow [24] and adjust its demand non-negativity constraints. For a given r_1 Von Massow [24] introduced the following MINLP:

$$\max_{p,r,Y} \quad \pi = \sum_{t=1}^{T} \pi_t \tag{VonMassow-MINLP}$$

s.t.

$$r_2 = \omega_1 p_1 + (1-\omega_1) r_1 \tag{2.2.8a}$$

$$r_3 = \omega_1 p_2 + \omega_2 p_1 + \omega_3 r_1 \tag{2.2.8b}$$

$$r_t = \omega_1 p_{t-1} + \omega_2 p_{t-2} + \omega_3 p_{t-3} \qquad \forall t \in \{4, 5, \cdots, T\} \tag{2.2.8c}$$

$$(r_t - p_t - \tau) Y_{G,t} \geq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.8d}$$

$$(r_t - p_t - \tau)(1 - Y_{G,t}) \leq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.8e}$$

$$(r_t - p_t + \rho) Y_{L,t} \leq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.8f}$$

$$(r_t - p_t + \rho)(1 - Y_{L,t}) \geq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.8g}$$

$$Y_{G,t} + Y_{L,t} \leq 1 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.8h}$$

$$c \leq p_t \leq \frac{a + \beta_L r_t}{b + \beta_L} \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.8i}$$

$$Y_{G,t}, Y_{L,t} \in \{0, 1\} \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.8j}$$

Constraints (2.2.8a)-(2.2.8c) in (VonMassow-MINLP) define the reference price for periods $2, \ldots, T$, constraints (2.2.8d)-(2.2.8h) define whether we have a gain or loss situation, and constraints (2.2.8i) and (2.2.8j) define bounds on price and binary variables, respectively. The constraints (2.2.8i) are to ensure demand is non-negative. Although constraints (2.2.8i) are sufficient to guarantee the non-negativity of demand, they are not necessary as shown in Proposition 2.2.2.

Proposition 2.2.2. *The constraints* $p_t \leq \frac{a+\beta_L r_t}{b+\beta_L}$ *in* (VonMassow-MINLP) *are not necessary to guarantee the non-negativity of demand.*

In Proposition 2.2.3 we provide sufficient and necessary conditions for the non-negativity of demand.

Proposition 2.2.3. *Let*

$$\chi_t = \beta_G Y_{G,t} + \beta_L Y_{L,t} \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.9}$$

$$\phi_t = \rho Y_{L,t} - \tau Y_{G,t} \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.10}$$

Then the necessary and sufficient conditions to ensure non-negative demand are

$$(b + \chi_t)p_t \leq a + \chi_t(r_1 + \phi_t) \qquad t = 1 \tag{2.2.11}$$

$$(b + \chi_t)p_t - \omega_1\chi_t p_{t-1} \leq a + \chi_t\left((1 - \omega_1)r_1 + \phi_t\right) \qquad t = 2 \tag{2.2.12}$$

$$(b + \chi_t)p_t - \chi_t(\omega_1 p_{t-1} + \omega_2 p_{t-2}) \leq a + \chi_t(\omega_3 r_1 + \phi_t) \qquad t = 3 \tag{2.2.13}$$

$$(b + \chi_t)p_t - \chi_t(\omega_1 p_{t-1} + \omega_2 p_{t-2} + \omega_3 p_{t-3}) \leq a - \chi_t\phi_t \qquad \forall t \in \{4, \cdots, T\} \tag{2.2.14}$$

Using the results from Proposition 2.2.3 we can adjust (VonMassow-MINLP) as follows

$$\max_{p,r,Y} \quad \pi = \sum_{t=1}^{T} \pi_t \tag{MINLP}$$

s.t.

$$r_2 = \omega_1 p_1 + (1-\omega_1) r_1 \tag{2.2.15a}$$

$$r_3 = \omega_1 p_2 + \omega_2 p_1 + \omega_3 r_1 \tag{2.2.15b}$$

$$r_t = \omega_1 p_{t-1} + \omega_2 p_{t-2} + \omega_3 p_{t-3} \qquad \forall t \in \{4, 5, \cdots, T\} \tag{2.2.15c}$$

$$(r_t - p_t - \tau) Y_{G,t} \geq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.15d}$$

$$(r_t - p_t - \tau)(1 - Y_{G,t}) \leq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.15e}$$

$$(r_t - p_t + \rho) Y_{L,t} \leq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.15f}$$

$$(r_t - p_t + \rho)(1 - Y_{L,t}) \geq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.15g}$$

$$Y_{G,t} + Y_{L,t} \leq 1 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.15h}$$

$$(b + \chi_t) p_t \leq a + \chi_t (r_1 + \phi_t) \qquad t = 1 \tag{2.2.15i}$$

$$(b + \chi_t) p_t - \omega_1 \chi_t p_{t-1} \leq a + \chi_t \left((1-\omega_1) r_1 + \phi_t \right) \qquad t = 2 \tag{2.2.15j}$$

$$(b + \chi_t) p_t - \chi_t (\omega_1 p_{t-1} + \omega_2 p_{t-2}) \leq a + \chi_t (\omega_3 r_1 + \phi_t) \qquad t = 3 \tag{2.2.15k}$$

$$(b + \chi_t) p_t - \chi_t (\omega_1 p_{t-1} + \omega_2 p_{t-2} + \omega_3 p_{t-3}) \leq a - \chi_t \phi_t \qquad \forall t \in \{4, \cdots, T\} \tag{2.2.15l}$$

$$p_t \geq 0 \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.15m}$$

$$Y_{G,t}, Y_{L,t} \in \{0, 1\} \qquad \forall t \in \{1, 2, \cdots, T\} \tag{2.2.15n}$$

Note that in (MINLP) the lower bound of prices are set to 0 instead of c. This is done so that we can allow for cases where it may be optimal to make negative profits in a period to increase total profit in during the planning horizon. The demand non-negativity constraints ensure positive demand in each period and thus avoid double negativity (negative demand and negative per unit profit) causing false positive profits. As mentioned in Section 2.1, current commercial MINLP solvers are unable to solve medium and large sizes of problem (MINLP). Therefore there is a need to solve the problem efficiently for medium to long planning horizons. In the next two sections we propose a heuristic and a modified Benders' decomposition methods to solve large instances of problem (MINLP).

2.3 A Myopic Heuristic Approach

In this section, we propose a myopic heuristic to solve the profit maximization problem (MINLP). The main idea of this method is to time-decompose the problem and solve each period's pricing problem independently of other periods. In Proposition 2.3.1 we will show that we can find closed form optimal solutions for each period's pricing problem. Later, in Section 2.5, we show numerically that, in some cases, the heuristic leads to close to optimal solutions for problem (MINLP).

Proposition 2.3.1. *Let*

$$\pi_t = \begin{cases} (p_t - c)\left(a - bp_t + \beta_G(r_t - \tau - p_t)\right) & p_t \le r_t - \tau \\ (p_t - c)\left(a - bp_t\right) & r_t - \tau \le p_t \le r_t + \rho \\ (p_t - c)\left(a - bp_t + \beta_L(r_t + \rho - p_t)\right) & r_t + \rho \le p_t \end{cases} \tag{2.3.1}$$

and

$$z_1 = \tfrac{a+(b+\beta_G)c}{2b+\beta_G} + \tau \qquad z_2 = \tfrac{a+bc}{2b} + \tau$$
$$z_3 = \tfrac{a+bc}{2b} - \rho \qquad z_4 = \tfrac{a+(b+\beta_L)c}{2b+\beta_L} - \rho$$

Then

$$p_t^* = \arg\max_{p_t^{*(i)}} \left\{ \pi_t\big|_{p_t^{*(1)}}, \pi_t\big|_{p_t^{*(2)}}, \pi_t\big|_{p_t^{*(3)}} \right\}$$

maximizes π_t *where*

$$p_t^{*(1)} = \begin{cases} r_t - \tau & \text{if } r_t < z_1 \\ \frac{a+\beta_G(r_t-\tau)+(b+\beta_G)c}{2(b+\beta_G)} & o.w. \end{cases}$$

$$p_t^{*(2)} = \begin{cases} r_t - \tau & \text{if } r_t > z_2 \\ r_t + \rho & \text{if } r_t < z_3 \\ \frac{a+bc}{2b} & o.w. \end{cases}$$

$$p_t^{*(3)} = \begin{cases} r_t + \rho & \text{if } r_t > z_4 \\ \frac{a+\beta_L(r_t+\rho)+(b+\beta_L)c}{2(b+\beta_L)} & o.w. \end{cases}$$

Using Proposition 2.3.1, we define Algorithm 2.3.1.

Algorithm 2.3.1 (Myopic Pricing Heuristic)**.** *Input:* α, β_L, β_G, a, b, c, τ, ρ, p_0, r_0, T.

Initialization:

$$z_1 = \frac{a+(b+\beta_G)c}{2b+\beta_G} + \tau \qquad z_2 = \frac{a+bc}{2b} + \tau$$
$$z_3 = \frac{a+bc}{2b} - \rho \qquad z_4 = \frac{a+(b+\beta_L)c}{2b+\beta_L} - \rho$$

Loop:

$$\textit{For } t = 1 \textit{ to } T \textit{ do}$$

$$r_t = \alpha r_{t-1} + (1-\alpha)p_{t-1}$$

$$p_t^{*(1)} = \begin{cases} r_t - \tau & \textit{if } r_t < z_1 \\ \frac{a+\beta_G(r_t-\tau)+(b+\beta_G)c}{2(b+\beta_G)} & o.w. \end{cases}$$

$$p_t^{*(2)} = \begin{cases} r_t - \tau & \textit{if } r_t > z_2 \\ r_t + \rho & \textit{if } r_t < z_3 \\ \frac{a+bc}{2b} & o.w. \end{cases}$$

$$p_t^{*(3)} = \begin{cases} r_t + \rho & \textit{if } r_t > z_4 \\ \frac{a+\beta_L(r_t+\rho)+(b+\beta_L)c}{2(b+\beta_L)} & o.w. \end{cases}$$

$$p_t^* = \arg\max_{p_t^{*(i)}} \left\{ \pi_t\big|_{p_t^{*(1)}}, \pi_t\big|_{p_t^{*(2)}}, \pi_t\big|_{p_t^{*(3)}} \right\}$$

Note that to increase computational efficiency, we only calculate range identifiers $z_1, \cdots, z_4$ at the initialization step. These values remain constant for all iterations. As we will see in Section 2.5, this approach produces very close to optimal results in some cases, however, in other combinations, the results may be far from optimal. In Section 2.4 we provide an exact algorithm for solving problem (MINLP).

2.4 A Modified Generalized Benders' Decomposition Method

As shown in previous sections, the need to solve (MINLP)) efficiently is well justified. The complexity of the problem stems from its mixed integer nonlinear nature. Multiple attempts have been made to linearize (MINLP), however, the results were not encouraging. In this section, we develop a modified version of the generalized Benders' decomposition (GBD) method [5]. To do so, and for brevity, we will rewrite the problem in matrix form. Define

$$W = \begin{pmatrix} -1 & & & & & & \\ \omega_1 & -1 & & & & & \\ \omega_2 & \omega_1 & -1 & & & & \\ \omega_3 & \omega_2 & \omega_1 & -1 & & & \\ & \omega_3 & \omega_2 & \omega_1 & -1 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & \omega_3 & \omega_2 & \omega_1 & -1 \end{pmatrix} \quad \text{where} \quad \begin{cases} \omega_1 = \frac{1-\alpha}{1-\alpha^3} \\ \omega_2 = \frac{\alpha(1-\alpha)}{1-\alpha^3} \\ \omega_3 = \frac{\alpha^2(1-\alpha)}{1-\alpha^3} \\ \omega_1 + \omega_2 + \omega_3 = 1 \end{cases} \tag{2.4.1}$$

and let

$$p = \begin{pmatrix} p_1 & p_2 & \cdots & p_T \end{pmatrix}^T, \quad d_Y = \begin{pmatrix} d_1 & d_2 & \cdots & d_T \end{pmatrix}^T \tag{2.4.2}$$

where

$$
\begin{aligned}
d_1 &= c\left[\omega_1(\beta_G Y_{G,2} + \beta_L Y_{L,2}) + \omega_2(\beta_G Y_{G,3} + \beta_L Y_{L,3}) + \omega_3(\beta_G Y_{G,4} + \beta_L Y_{L,4})\right] \\
&\quad - \left[a + bc + \beta_G Y_{G,1}(c-\tau) + \beta_L Y_{L,1}(c+\rho) + \beta_G Y_{G,1} + \beta_L Y_{L,1}\right] \\
d_2 &= c\left[\omega_1(\beta_G Y_{G,3} + \beta_L Y_{L,3}) + \omega_2(\beta_G Y_{G,4} + \beta_L Y_{L,4}) + \omega_3(\beta_G Y_{G,5} + \beta_L Y_{L,5})\right] \\
&\quad - \left[a + bc + \beta_G Y_{G,2}(c-\tau) + \beta_L Y_{L,2}(c+\rho) + (1-\omega_1) r_1(\beta_G Y_{G,2} + \beta_L Y_{L,2})\right] \\
d_3 &= c\left[\omega_1(\beta_G Y_{G,4} + \beta_L Y_{L,4}) + \omega_2(\beta_G Y_{G,5} + \beta_L Y_{L,5}) + \omega_3(\beta_G Y_{G,6} + \beta_L Y_{L,6})\right] \\
&\quad - \left[a + bc + \beta_G Y_{G,3}(c-\tau) + \beta_L Y_{L,3}(c+\rho) + \omega_3 r_1(\beta_G Y_{G,3} + \beta_L Y_{L,3})\right] \\
d_t &= c\left[\omega_1(\beta_G Y_{G,t+1} + \beta_L Y_{L,t+1}) + \omega_2(\beta_G Y_{G,t+2} + \beta_L Y_{L,t+2}) + \omega_3(\beta_G Y_{G,t+3} + \beta_L Y_{L,t+3})\right] \\
&\quad - \left[a + bc + \beta_G Y_{G,t}(c-\tau) + \beta_L Y_{L,t}(c+\rho)\right] \qquad \forall t = 4, \cdots, T-3 \\
d_{T-2} &= c\left[\omega_1(\beta_G Y_{G,T-1} + \beta_L Y_{L,T-1}) + \omega_2(\beta_G Y_{G,T} + \beta_L Y_{L,T})\right] \\
&\quad - \left[a + bc + \beta_G Y_{G,T-2}(c-\tau) + \beta_L Y_{L,T-2}(c+\rho)\right] \\
d_{T-1} &= c\left[\omega_1(\beta_G Y_{G,T} + \beta_L Y_{L,T})\right] - \left[a + bc + \beta_G Y_{G,T-1}(c-\tau) + \beta_L Y_{L,T-1}(c+\rho)\right] \\
d_T &= -\left[a + bc + \beta_G Y_{G,T}(c-\tau) + \beta_L Y_{L,T}(c+\rho)\right]
\end{aligned} \tag{2.4.3}
$$

Define the $T \times T$ symmetric sparse matrix $H_Y = [H_{i,j}]$ such that

$$
\begin{aligned}
H_{t,t} &= -2(b + \beta_G Y_{G,t} + \beta_L Y_{L,t}) && \forall t = 1, \cdots, T \\
H_{t,t-1} = H_{t-1,t} &= \omega_1(\beta_G Y_{G,t} + \beta_L Y_{L,t}) && \forall t = 2, \cdots, T \\
H_{t,t-2} = H_{t-2,t} &= \omega_2(\beta_G Y_{G,t} + \beta_L Y_{L,t}) && \forall t = 3, \cdots, T \\
H_{t,t-3} = H_{t-3,t} &= \omega_3(\beta_G Y_{G,t} + \beta_L Y_{L,t}) && \forall t = 4, \cdots, T
\end{aligned} \tag{2.4.4}
$$

Let

$$Y_G = \text{diag}\{Y_{G,1}, Y_{G,2}, \cdots, Y_{G,T}\} \tag{2.4.5}$$

$$Y_L = \text{diag}\{Y_{L,1}, Y_{L,2}, \cdots, Y_{L,T}\} \tag{2.4.6}$$

$$Y = \begin{pmatrix} -Y_G & , & I - Y_G & , & Y_L & , & -(I - Y_L) \end{pmatrix}' \tag{2.4.7}$$

Define the $T \times T$ sparse matrix $B_Y = [B_{i,j}]$ such that

$$B_{t,t} = b + \beta_G Y_{G,t} + \beta_L Y_{L,t} \qquad \forall t = 1, \cdots, T \tag{2.4.8a}$$

$$B_{t,t-1} = -\omega_1(\beta_G Y_{G,t} + \beta_L Y_{L,t}) \qquad \forall t = 2, \cdots, T \tag{2.4.8b}$$

$$B_{t,t-2} = -\omega_2(\beta_G Y_{G,t} + \beta_L Y_{L,t}) \qquad \forall t = 3, \cdots, T \tag{2.4.8c}$$

$$B_{t,t-3} = -\omega_3(\beta_G Y_{G,t} + \beta_L Y_{L,t}) \qquad \forall t = 4, \cdots, T \tag{2.4.8d}$$

and define the $6T \times T$ matrix A_Y as

$$A_Y = \left(\begin{array}{c} YW \\ \hline B_Y \\ \hline -I_T \end{array}\right) \tag{2.4.9}$$

where I_T is the $T \times T$ identity matrix. Let the $6T \times 1$ vector b_Y be defined as

$$b_Y = \begin{pmatrix} Y_G v_G \\ -(I - Y_G) v_G \\ -Y_L v_L \\ (I - Y_L) v_L \\ \hline a + (\beta_G Y_{G,1} + \beta_L Y_{L,1})(r_1 - \rho Y_{L,1} + \tau Y_{G,1}) \\ a + (\beta_G Y_{G,2} + \beta_L Y_{L,2})((1 - \omega_1) r_1 - \rho Y_{L,2} + \tau Y_{G,2}) \\ a + (\beta_G Y_{G,3} + \beta_L Y_{L,3})(\omega_3 r_1 - \rho Y_{L,3} + \tau Y_{G,3}) \\ a - (\beta_G Y_{G,4} + \beta_L Y_{L,4})(\rho Y_{L,4} - \tau Y_{G,4}) \\ \vdots \\ a - (\beta_G Y_{G,T} + \beta_L Y_{L,T})(\rho Y_{L,T} - \tau Y_{G,T}) \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix} \tag{2.4.10}$$

where

$$v_G = \begin{pmatrix} r_1 - \tau & , & (1 - \omega_1) r_1 - \tau & , & \omega_3 r_1 - \tau & , & -\tau & , & \cdots & , & -\tau \end{pmatrix}' \tag{2.4.11}$$

$$v_L = \begin{pmatrix} r_1 + \rho & , & (1 - \omega_1) r_1 + \rho & , & \omega_3 r_1 + \rho & , & \rho & , & \cdots & , & \rho \end{pmatrix}' \tag{2.4.12}$$

The profit maximization problem (MINLP) in matrix form is

$$\max \quad \pi = \frac{1}{2} p^T H_Y p - d_Y^T p + c \sum_{t=1}^{T} (\beta_G Y_{G,t} \tau - \beta_L Y_{L,t} \rho) - c r_1 \left[(\beta_G Y_{G,1} + \beta_L Y_{L,1}) \right.$$

$$+ (1-\omega_1)(\beta_G Y_{G,2} + \beta_L Y_{L,2}) + \omega_3(\beta_G Y_{G,3} + \beta_L Y_{L,3})\Big] \tag{2.4.13}$$

$$\text{s.t.} \quad A_Y p \leq b_Y$$

$$Y_{G,t}, Y_{L,t} \in \{0,1\} \qquad \forall t \in \{1,2,\cdots,T\}$$

This can be re-written as

$$\max_{Y\in\Omega} \quad \Bigg\{ \max_p \left\{ \frac{1}{2} p^T H_Y p - d_Y^T p \quad \text{s.t.} \quad A_Y p \leq b_Y \right\} \tag{2.4.14}$$

$$+ c \sum_{t=1}^{T} (\beta_G Y_{G,t} \tau - \beta_L Y_{L,t} \rho) - c r_1 \Big[(\beta_G Y_{G,1} + \beta_L Y_{L,1})$$

$$+ (1-\omega_1)(\beta_G Y_{G,2} + \beta_L Y_{L,2}) + \omega_3(\beta_G Y_{G,3} + \beta_L Y_{L,3})\Big] \Bigg\}$$

where Ω is the set of feasible values for Y. If Y is fixed to a feasible configuration $\overline{Y}$, the resulting problem is

$$\max_p \left\{ \frac{1}{2} p^T H_{\overline{Y}} p - d_{\overline{Y}}^T p \quad \text{s.t.} \quad A_{\overline{Y}} p \leq b_{\overline{Y}} \right\} + c \sum_{t=1}^{T} (\beta_G \overline{Y}_{G,t} \tau - \beta_L \overline{Y}_{L,t} \rho) \tag{2.4.15}$$

$$- c r_1 \Big[(\beta_G \overline{Y}_{G,1} + \beta_L \overline{Y}_{L,1}) + (1-\omega_1)(\beta_G \overline{Y}_{G,2} + \beta_L \overline{Y}_{L,2}) + \omega_3(\beta_G \overline{Y}_{G,3} + \beta_L \overline{Y}_{L,3})\Big]$$

Lemma 2.4.1 and Proposition 2.4.1 will help us form Benders' subproblem.

Lemma 2.4.1. *If $b \geq \max\{\beta_L, \beta_G\}$ then $H_{\overline{Y}}$ is symmetric negative semi-definite and $H_{\overline{Y}}$ is symmetric negative definite if $b > \max\{\beta_L, \beta_G\}$.*

Proposition 2.4.1. *For $b \geq \max\{\beta_L, \beta_G\}$, the dual of the inner maximization problem in (2.4.14) is*

$$\min \quad u^T b_Y - \frac{1}{2} p^T H_{\overline{Y}} p \tag{2.4.16}$$

$$
\begin{aligned}
s.t. \quad & -H_{\overline{Y}}^T p + A^T u + d_{\overline{Y}} = 0 \\
& u \geq 0
\end{aligned}
$$

We can now define the generalized Benders' decomposition(GBD) subproblem when Y is fixed to a feasible configuration $\overline{Y}$ as

$$
\begin{aligned}
\min_{u,p\geq 0} \quad & u^T b_{\overline{Y}} - \frac{1}{2} p^T H_{\overline{Y}} p + c \sum_{t=1}^{T} (\beta_G \overline{Y}_{G,t} \tau - \beta_L \overline{Y}_{L,t} \rho) \qquad \text{(GBD-Sub)} \\
& - c r_1 \Big[(\beta_G \overline{Y}_{G,1} + \beta_L \overline{Y}_{L,1}) + (1-\omega_1)(\beta_G \overline{Y}_{G,2} + \beta_L \overline{Y}_{L,2}) \\
& + \omega_3 (\beta_G \overline{Y}_{G,3} + \beta_L \overline{Y}_{L,3}) \Big] \\
\text{s.t.} \quad & -H_{\overline{Y}}^T p + A^T u + d_{\overline{Y}} = 0
\end{aligned}
$$

We can also define the relaxed master problem as

$$
\begin{aligned}
\max_{Y_L, Y_G, z} \quad & z \qquad \text{(Master)} \\
\text{s.t.} \quad & z \leq \frac{1}{2} \left(p^{(k)}\right)^T H_Y p^{(k)} - d_Y^T p^{(k)} + \left(u^{(k)}\right)^T (b_Y - A p^{(k)}) \\
& + c \sum_{t=1}^{T} (\beta_G Y_{G,t} \tau - \beta_L Y_{L,t} \rho) - c r_1 \Big[(\beta_G Y_{G,1} + \beta_L Y_{L,1}) \\
& + (1-\omega_1)(\beta_G Y_{G,2} + \beta_L Y_{L,2}) + \omega_3 (\beta_G Y_{G,3} + \beta_L Y_{L,3}) \Big] \quad \forall k \in \{1, \cdots, K\},
\end{aligned}
$$

where $p^{(k)}$ and $u^{(k)}$ are the solutions from the subproblem in iteration $k \in \{1, 2, \cdots, K\}$ with K being the most recent iteration.

In Proposition 2.4.2 we define some valid inequalities to enhance the performance of the GBD algorithm.

Proposition 2.4.2. *Let $Y_{L,t}^{(k)}$ and $Y_{G,t}^{(k)}$ be optimal solutions produced from the master* *problem (Master) in iteration k with $k \in \{1, \cdots, K\}$, $t \in \{1, \cdots, T\}$, and K the most recent (current) iteration. To increase computational efficiency, we add the following constraints to the master problem (Master) so that it does not produce a solution that it has produced before.*

$$\left| \sum_{t=1}^{T} 2^{t-1} \left(Y_{L,t} - Y_{L,t}^{(k)} \right) \right| \geq 1 \tag{2.4.17}$$

$$\left| \sum_{t=1}^{T} 2^{t-1} \left(Y_{G,t} - Y_{G,t}^{(k)} \right) \right| \geq 1 \tag{2.4.18}$$

The constraints above can be replaced by the following linearised constraints

$$\sum_{t=1}^{T} 2^{t-1} \left(Y_{L,t} - Y_{L,t}^{(k)} \right) + M q_{L,k} \geq 1 \tag{2.4.19}$$

$$-\sum_{t=1}^{T} 2^{t-1} \left(Y_{L,t} - Y_{L,t}^{(k)} \right) + M(1 - q_{L,k}) \geq 1 \tag{2.4.20}$$

$$\sum_{t=1}^{T} 2^{t-1} \left(Y_{G,t} - Y_{G,t}^{(k)} \right) + M q_{G,k} \geq 1 \tag{2.4.21}$$

$$-\sum_{t=1}^{T} 2^{t-1} \left(Y_{G,t} - Y_{G,t}^{(k)} \right) + M(1 - q_{G,k}) \geq 1 \tag{2.4.22}$$

where M is a large number (ex. 2^T); $q_{L,k}$ and $q_{G,k}$ are binary variables for all $k \in \{1, \cdots, K\}$.

Using Proposition 2.4.2, we can define the new GBD master problem as

$$\begin{aligned} \max \quad & z && \text{(GBD-Master)} \\ \text{s.t.} \quad & z \leq \frac{1}{2} \left(p^{(k)} \right)^T H_Y p^{(k)} - d_Y^T p^{(k)} + \left(u^{(k)} \right)^T \left(b_Y - A p^{(k)} \right) \end{aligned}$$

$$
\begin{aligned}
& + c\sum_{t=1}^{T}(\beta_G Y_{G,t}\tau - \beta_L Y_{L,t}\rho) - cr_1\Big[(\beta_G Y_{G,1} + \beta_L Y_{L,1}) \\
& + (1-\omega_1)(\beta_G Y_{G,2} + \beta_L Y_{L,2}) + \omega_3(\beta_G Y_{G,3} + \beta_L Y_{L,3})\Big] \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall k \in \{1, \cdots, K\} \\
& Y_{L,t} + Y_{G,t} \leq 1 \qquad\qquad\qquad\qquad \forall t \in \{1, \cdots, T\} \\
& \sum_{t=1}^{T} 2^{t-1}\left(Y_{L,t} - Y_{L,t}^{(k)}\right) + Mq_{L,k} \geq 1 \qquad \forall k \in \{1, \cdots, K\} \\
& \sum_{t=1}^{T} 2^{t-1}\left(Y_{L,t} - Y_{L,t}^{(k)}\right) + M(1 - q_{L,k}) \geq 1 \qquad \forall k \in \{1, \cdots, K\} \\
& \sum_{t=1}^{T} 2^{t-1}\left(Y_{G,t} - Y_{G,t}^{(k)}\right) + Mq_{G,k} \geq 1 \qquad \forall k \in \{1, \cdots, K\} \\
& \sum_{t=1}^{T} 2^{t-1}\left(Y_{G,t} - Y_{G,t}^{(k)}\right) + M(1 - q_{G,k}) \geq 1 \qquad \forall k \in \{1, \cdots, K\} \\
& Y_{L,t}, Y_{G,t} \in \{0, 1\} \qquad\qquad\qquad\qquad \forall t \in \{1, \cdots, T\} \\
& q_{L,k}, q_{G,k} \in \{0, 1\} \qquad\qquad\qquad\qquad \forall k \in \{1, \cdots, K\}
\end{aligned}
$$

We summarize the main steps of our proposed modified GBD approach in Algorithm 2.4.1.

Algorithm 2.4.1 (Modified GBD). *Assume $Y_{start,L,t}$, $Y_{start,L,t}$ be the initial binary values for $t = 1, \cdots, T$ be given. Let ϵ be the convergence criteria.*

Step 1: Let $K = 1$. For all $t \in \{1, \cdots, T\}$, let $\overline{Y}_{L,t} = Y_{start,L,t}$, $\overline{Y}_{G,t} = Y_{start,G,t}$, $Y_{L,t}^{(K)} = \overline{Y}_{L,t}$, $Y_{G,t}^{(K)} = \overline{Y}_{G,t}$, $LB = -\infty$, and $UB = +\infty$.

Step 2: Solve the subproblem (GBD-Sub). Let $\pi^{(K)}$, $p^{(K)}$, and $u^{(K)}$ denote the optimal *objective value, p, and u, respectively. If* $\pi^{(K)} > LB$*, let*

$$LB = \pi^{(k)}$$

$$Best(u) = u^{(K)}$$

$$Best(p) = p^{(K)}$$

$$Best(Y_L) = \overline{Y}_L$$

$$Best(Y_G) = \overline{Y}_G$$

Step 3: Solve (GBD-Master)*. Let* UB, $\overline{Y}_L$*, and* $\overline{Y}_G$ *be the optimal values of* z, Y_L*, and* Y_G*, respectively.*

Step 4: If $\frac{UB-LB}{LB} \leq \epsilon$ *or if the master problem is infeasible (all* Y*'s have been considered), "STOP", otherwise let* $K = K + 1$ *and proceed to "Step 2".*

In Propositions 2.4.3 and 2.4.4 we establish the optimality and convergence of our proposed modified GBD method, respectively.

Proposition 2.4.3. *Let* $Y_L^{(k)}$ *and* $Y_G^{(k)}$ *be the optimal solutions from (Master) in iteration* k *and* $p^{(k+1)}$ *be the optimal solutions from (GBD-Sub) in iteration* $k+1$*. If* $Y_L^{(k)}$ *and* $Y_G^{(k)}$ *are the optimal solutions to (MINLP), then* $p^{(k+1)}$ *is also the optimal solution to (MINLP); i.e., the combination* $Y_L^{(k)}$, $Y_G(k)$*, and* $p^{(k+1)}$ *is an optimal solution to (MINLP).*

Proposition 2.4.4. *Algorithm 2.4.1 converges in a finite number of iterations.*

We note that the condition $b > \beta = \max\{\beta_L, \beta_G\}$, implies that the demand price sensitivity is stronger than the demand sensitivity to reference prices threshold, which is reasonable in the real world. Under such reasonable conditions, $H_{\overline{Y}}$ is a symmetric negative semi-definite matrix; that is, the objective function of (GBD-Sub) is convex regardless of the value of T. The feasible region of (GBD-Sub) is also convex as it is formed of linear constraints. Therefore, (GBD-Sub) is always a convex quadratic problem. Note that (GBD-Master) is a linear problem with linear constraints. Therefore, Using common solvers for linear and quadratic problems, we can find close to optimal solutions for the original MINLP for large values of T.

2.5 Numerical Experiments

In this section we will present results from computational experiments. In Section 2.5.1 we describe the data and specification of the computer and software used for computational experiments. We will then illustrate numerical results in Section 2.5.2.

2.5.1 Problem Data

For the purpose of numerical experiments, we used 1296 different combinations of parameter sets shown in Table 2.1. For all of these combinations of parameters commercial solvers including Baron, were only able to solve (MINLP) for horizons with up to 6 periods. Our computations were executed using Matlab R2014b. Each optimization problem in the generalized Benders' decomposition method and complete enumeration was solved by calling GAMS 24.8.5 through Matlab and by using CPLEX 12.8 solver on a machine running Windows Server 2008 R2 Standard with Intel(R) Xeon(R) E5-2640 v2(8 cores, 16 threads) 2.00 gigahertz - 2.50 gigahertz processor and 64 gigabytes RAM.

Parameter	**Values**
a	10
b	$0.6, 0.8, 1$
c	$0.5, 2$
β_L	$0.2, 0.4$
β_G	$0.2, 0.4$
τ	$0.2, 0.5, 0.7$
ρ	$0.2, 0.5, 0.7$
r_1	$2, 2.75$
α	$0.1, 0.2, 0.3, 0.4$

Table 2.1: Parameter Values

To measure the optimality of the heuristic approach explained in Section 2.3 and the generalized Benders' decomposition(GBD) method from Section 2.4, we compare the results and computation time of these methods with a complete enumeration(CE) method where the primal problem (2.4.15) is solved for all possible combinations of $Y_{L,t}$, $Y_{G,t}$ and the best result is chosen. Since the computation time of the complete enumeration method increases exponentially, $O(3^T)$, only the case with $T = 10$ is used. We use these results as a benchmark to measure the accuracy of the other two methods. We further compare the results of the two methods, myopic heuristic(ME) and GBD, for larger values of T.

Let π_i^* denote the optimal objective value of (MINLP) when the method i is used. Define the comparative index $\mathrm{CI}_{i,j}$ as

$$\mathrm{CI}_{i,j} = \frac{\pi_i^* - \pi_j^*}{\pi_j^*}, i, j = \text{CE, BDH, MH, BDL} \tag{2.5.1}$$

where CE, BDH, MH, BDL refer to complete enumeration, Benders' decomposition with a high limit on iterations (1,000), myopic heuristic, and Benders' decomposition with a low limit on iterations (100), respectively.

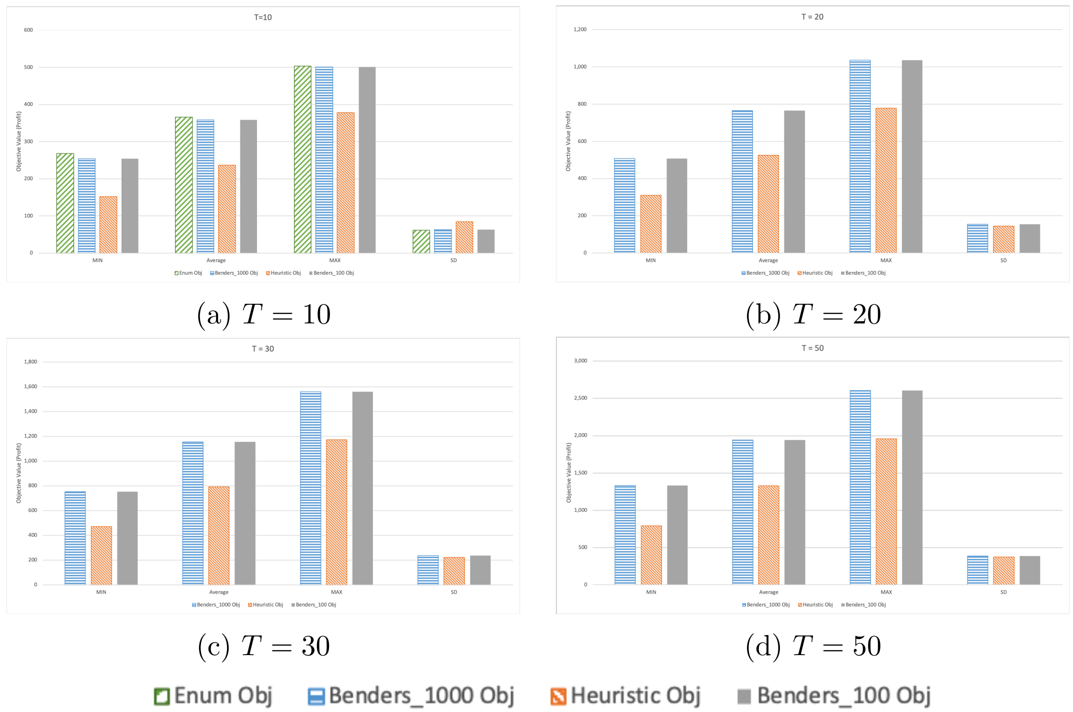
2.5.2 Results

As shown in Table 2.2, results from the Benders' decomposition method are very close to the optimal results obtained by using the complete enumeration method. Figure 2.1 and Figure 2.2 also show this graphically. Note that in Table 2.2 methods CE and MH are only compared with BDH to avoid redundant results. However, we have presented the comparison between BDH and BDL; and therefore CE and MH can be compared with BDL indirectly.

T	$\boldsymbol{i, j}$	**min** $(\text{CI}_{i,j})$	**avg** $(\text{CI}_{i,j})$	**max** $(\text{CI}_{i,j})$	**SD** $(\text{CI}_{i,j})$
10	CE,BDH	0.0034 %	2.2184 %	9.1157 %	2.4832 %
	CE,MH	13.9161 %	69.6544 %	131.0846 %	52.7251 %
	BDL,MH	8.5651 %	54.6538 %	132.5032 %	42.3646 %
	BDL,BDH	0.0000 %	0.0000 %	0.0000 %	0.0000 %
20	BDH,MH	2.6383 %	54.0444 %	127.3851 %	41.7312 %
	BDH,BDL	0.0000 %	0.0000 %	0.0000 %	0.0000 %
30	BDH,MH	9.5835 %	53.7530 %	126.3211 %	41.2386 %
	BDH,BDL	0.0000 %	0.0004 %	0.2402 %	0.0082 %
50	BDH,MH	2.8381 %	54.4323 %	126.5213 %	41.8863 %
	BDH,BDL	0.0000 %	0.0007 %	0.6407 %	0.0181 %

Table 2.2: Optimality of methods

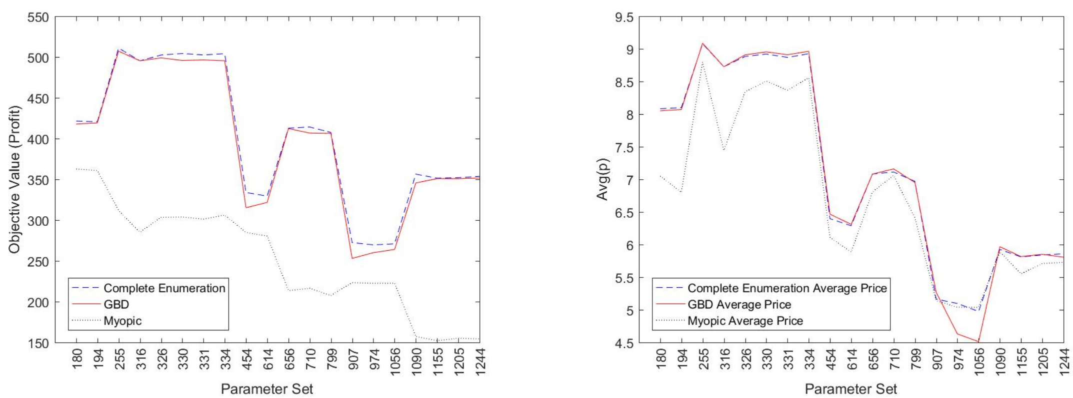
As shown in Figure 2.2 the average prices of the complete enumeration method and the modified GBD method are very close if not identical in most cases. In Figure 2.1, it is illustrated that the results from modified GBD method with 100 and 1000 iteration limits are very similar and both methods' are considerably better than



(a) $T = 10$ (b) $T = 20$

(c) $T = 30$ (d) $T = 50$

Figure 2.1: Summary results



(a) (b)

Figure 2.2: Objective values in (a) and average prices in (b) for $T = 10$ for 20 randomly selected parameter combinations for all three methods.

the myopic heuristic. Table 2.3 shows that the computation time for the complete enumeration is 4-5 hours for a horizon with $T = 10$ compared to less than 4 seconds for all other methods. As expected, the computation time of all of the methods increase as we increase T.

T	Method	Min	Avg	Max	SD
10	CE	9,585.8058	11,788.6143	13,596.2032	1,229.2566
	BDH	0.6956	0.9964	3.9314	0.3315
	MH	0.0003	0.0004	0.0022	0.0001
	BDL	0.6956	0.9964	3.9314	0.3315
20	BDH	0.7537	12.4443	1,909.6366	143.5793
	MH	0.0005	0.0007	0.0028	0.0003
	BDL	0.7475	1.5157	66.7713	4.6907
30	BDH	0.8133	92.0837	5,595.1819	586.3591
	MH	0.0007	0.0010	0.0332	0.0010
	BDL	0.8655	4.8885	129.7584	16.4642
50	BDH	1.0258	821.6214	72,084.6508	4,407.5662
	MH	0.0011	0.0017	0.0054	0.0007
	BDL	1.0628	27.3011	1,839.1405	110.4442

Table 2.3: Computation times of methods in seconds

Taking into account the optimality comparisons in Table 2.2 and the computation times in Table 2.3, when we limit the number of iterations in the modified Benders' decomposition method to 100, the computation time decreases significantly but the optimality only decreases slightly; we therefore can increase the value of the convergence criteria for the GBD method slightly and decrease the computation time significantly. The myopic heuristic approach by far has the shortest computation times and under certain conditions, discussed in Section 2.6.5, it produces very close to optimal results.

T	Method	Min	Avg	Max	SD
10	BDH	2	2.18	7	0.49
	BDL	2	2.18	7	0.49
20	BDH	2	10.44	1,000	78.18
	BDL	2	2.89	100	7.68
30	BDH	2	59.71	1,000	148.24
	BDL	2	6.17	100	17.50
50	BDH	2	453.20	1,000	266.27
	BDL	2	13.45	100	28.88

Table 2.4: Number of iterations for Benders' decomposition methods

2.6 Managerial Insights

Given that our model is general, in that we do not impose any particular conditions on optimal prices, it is not always possible to obtain analytical results on the price patterns. In this section, we will report on additional computational results that will allow us to numerically analyze the changes in total profit, price patterns, and the performance of the heuristic method. In particular, in Section 2.6.1 we will illustrate the effects of gain and loss parameters, β_G and β_L, respectively. Section 2.6.2 outlines the effects of varying threshold parameters τ and ρ. In Section 2.6.3 we discuss the effects of changes in the memory parameter α. Finally, Sections 2.6.4 and 2.6.5 discuss price patterns and the optimality of the heuristic method, respectively.

2.6.1 Effects of Gain and Loss Parameters on Total Profit

The total profits increase as the value of the gain parameter β_G increases. This is an expected results and can easily be proven analytically since the first derivative of π with respect to β_G is non-negative, whenever the price is larger than the cost. In addition, it seems that increasing the value of β_G also increases the magnitude of the effects of the changes in gain threshold in total profit. Table 2.5 illustrates an example of the effects of the gain parameter on total profits. Varying the values of the loss parameter β_L seems to have no effect on the optimal total profit.

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{\beta_L}$	$\boldsymbol{\beta_G}$	$\boldsymbol{\tau}$	$\boldsymbol{\rho}$	$\boldsymbol{\alpha}$	$\boldsymbol{\pi}$
10	1	0.5	0.9	0.3	0.5	0.5	0.2	248.1213
10	1	0.5	0.9	0.6	0.5	0.5	0.2	250.3011
10	1	0.5	0.9	0.9	0.5	0.5	0.2	252.8410
10	1	0.5	0.9	0.3	0.2	0.8	0.2	249.4249
10	1	0.5	0.9	0.6	0.2	0.8	0.2	253.5629
10	1	0.5	0.9	0.9	0.2	0.8	0.2	258.2754
10	1	0.5	0.9	0.3	0.2	0.2	0.3	248.7188
10	1	0.5	0.9	0.6	0.2	0.2	0.3	250.3517
10	1	0.5	0.9	0.9	0.2	0.2	0.3	252.2637

Table 2.5: The effect of changes in gain parameter β_G on profits

2.6.2 Effects of Gain and Loss Thresholds on Total Profit

The loss threshold is viewed as a measure of brand loyalty in some studies (see [25]). It is intuitive that increased brand loyalty results in increased profits; that is, total profit increases as the value of the loss threshold ρ is increased. This is observed in the study of our model and an example of this is shown in Table 2.6. In addition, it seems that the total profits decrease as the value of the gain threshold τ is increases (see Table 2.7).

2.6.3 Effects of the Memory Parameter on Total Profit

We expect the total profits to decrease as the memory parameter α increases. This is because when the value of α increases, the reference prices react more slowly in response to increasing prices which results in lower total profits. An example of this

a	b	c	β_L	β_G	τ	ρ	α	π
10	1	0.5	0.6	0.6	0.2	0.2	0.4	250.2569
10	1	0.5	0.6	0.6	0.2	0.5	0.4	251.8334
10	1	0.5	0.6	0.6	0.2	0.8	0.4	252.5897
10	1	0.5	0.9	0.6	0.2	0.2	0.2	250.4545
10	1	0.5	0.9	0.6	0.2	0.5	0.2	252.5025
10	1	0.5	0.9	0.6	0.2	0.8	0.2	253.5629
10	1	0.5	0.9	0.9	0.2	0.2	0.3	252.2637
10	1	0.5	0.9	0.9	0.2	0.5	0.3	255.6647
10	1	0.5	0.9	0.9	0.2	0.8	0.3	257.6115

Table 2.6: The effect of changes in loss threshold ρ on profits

a	b	c	β_L	β_G	τ	ρ	α	π
10	1	0.5	0.6	0.6	0.2	0.5	0.2	252.5025
10	1	0.5	0.6	0.6	0.5	0.5	0.2	250.3011
10	1	0.5	0.6	0.6	0.8	0.5	0.2	248.6111
10	1	0.5	0.6	0.6	0.2	0.8	0.2	253.5629
10	1	0.5	0.6	0.6	0.5	0.8	0.2	250.9736
10	1	0.5	0.6	0.6	0.8	0.8	0.2	248.8700
10	1	0.5	0.9	0.9	0.2	0.8	0.2	258.2754
10	1	0.5	0.9	0.9	0.5	0.8	0.2	255.1344
10	1	0.5	0.9	0.9	0.8	0.8	0.2	252.0424

Table 2.7: The effect of changes in gain threshold τ on profits

is shown in Tables 2.8 and 2.9. In addition, the marginal change in profits decrease as the memory parameter α increases.

2.6.4 Price Patterns

By examining the different prices, we observed three general patterns: (1) cyclic; (2) constant; (3) constant with price shocks and drops. These patterns were generated based on the data used in Tables 2.1 and 2.10. In practice these observed patterns could apply for a new product that is in its growth phase, occasional disruptions in supply, or high-low pricing. Studying and determining conditions on price patterns

$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{\beta_L}$	$\boldsymbol{\beta_G}$	$\boldsymbol{\tau}$	$\boldsymbol{\rho}$	$\boldsymbol{\alpha}$	$\boldsymbol{\pi}$
10	1	0.5	0.9	0.9	0.5	0.2	0.2	250.3023
10	1	0.5	0.9	0.9	0.5	0.2	0.3	250.1644
10	1	0.5	0.9	0.9	0.5	0.2	0.4	250.0412
10	1	0.5	0.9	0.9	0.5	0.5	0.2	252.8410
10	1	0.5	0.9	0.9	0.5	0.5	0.3	252.4827
10	1	0.5	0.9	0.9	0.5	0.5	0.4	252.2069
10	1	0.5	0.9	0.9	0.5	0.8	0.2	255.1344
10	1	0.5	0.9	0.9	0.5	0.8	0.3	254.2700
10	1	0.5	0.9	0.9	0.5	0.8	0.4	253.4105

Table 2.8: The effect of changes in memory parameter α on profits and changes on its magnitude when loss threshold ρ varies

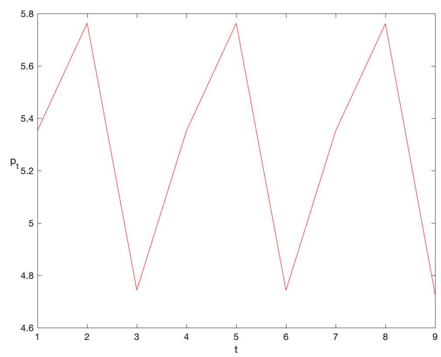
$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{\beta_L}$	$\boldsymbol{\beta_G}$	$\boldsymbol{\tau}$	$\boldsymbol{\rho}$	$\boldsymbol{\alpha}$	$\boldsymbol{\pi}$
10	1	0.5	0.9	0.9	0.2	0.5	0.2	256.0729
10	1	0.5	0.9	0.9	0.2	0.5	0.3	255.6647
10	1	0.5	0.9	0.9	0.2	0.5	0.4	255.3587
10	1	0.5	0.9	0.9	0.5	0.5	0.2	252.8410
10	1	0.5	0.9	0.9	0.5	0.5	0.3	252.4827
10	1	0.5	0.9	0.9	0.5	0.5	0.4	252.2069
10	1	0.5	0.9	0.9	0.8	0.5	0.2	250.7895
10	1	0.5	0.9	0.9	0.8	0.5	0.3	250.4593
10	1	0.5	0.9	0.9	0.8	0.5	0.4	250.1632

Table 2.9: The effect of changes in memory parameter α on profits and changes on its magnitude when gain threshold τ varies

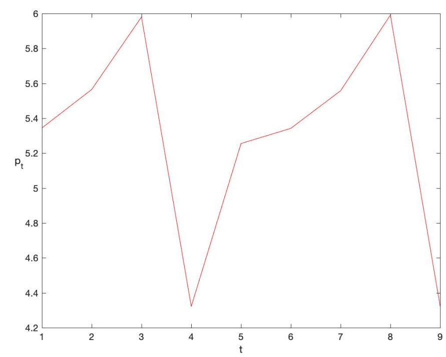
Parameter	**Values**
β_L	$0.3, 0.6, 0.9$
β_G	$0.3, 0.6, 0.9$
τ	$0.2, 0.5, 0.8$
ρ	$0.2, 0.5, 0.8$
α	$0.2, 0.3, 0.4$

Table 2.10: Parameter Values

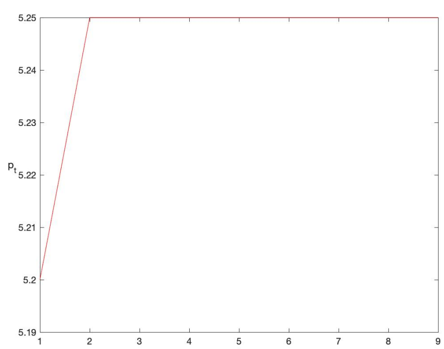
seemed to be dependent on the values of α, β_G, β_L, τ, and ρ. We observed that prices have cyclic pattern when β_G is large. Note that large β_G results in large β_L since we assume loss averse consumers with $\beta_G \leq \beta_L$. This result holds unless ρ is very small (0.2). In addition, when β_G is small (0.3), we only observe cyclic price patterns when τ is also very small. These results are consistent with findings in [13]. In other cases prices generally take constant or constant with price shock and drop patterns. The constant price in these cases are the same as the heuristic solution. Figure 2.3 shows these price patterns.



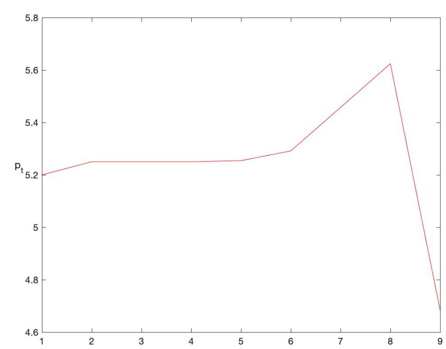
(a)
$(\beta_L, \beta_G, \tau, \rho, \alpha) = (0.9, 0.3, 0.2, 0.5, 0.2)$



(b)
$(\beta_L, \beta_G, \tau, \rho, \alpha) = (0.9, 0.6, 0.8, 0.5, 0.3)$



(c)
$(\beta_L, \beta_G, \tau, \rho, \alpha) = (0.6, 0.3, 0.5, 0.2, 0.3)$



(d)
$(\beta_L, \beta_G, \tau, \rho, \alpha) = (0.3, 0.3, 0.5, 0.2, 0.2)$

Figure 2.3: Price patterns. (a) and (b) illustrate cyclic pasterns. (c) and (d) illustrate constant and constant with price shock and drop patterns, respectively.

2.6.5 Optimality of the Heuristic Method

As discussed in Section 2.5.2, the myopic heuristic approach is very efficient in terms of computation time and, in some cases, produces very close to optimal results. From all the parameter sets used in our computational experiments, the heuristic method's relative error was less than 10% in approximately 5% of cases and less than 20% in 50% of cases. In all cases with low unit cost ($c = 0.5$), the heuristic produced results within 20% of the optimal solution. Table 2.11 summarizes the results from multiple linear regression analysis where the optimality margin is the dependent variable. As shown, all coefficients are statistically significant at 90% confidence level. The coefficients from Table 2.11 indicate that the relative optimality margin of the heuristic method is decreasing in b, c, β_L, and β_G and increasing in the size of the threshold interval; i.e., τ and ρ.

	Estimate	Std. Error	t-value	$Pr(>\lvert t\rvert)$
(Intercept)	-0.17918	0.00196	-91.417	$< 2e-16$
b	0.25748	0.00174	147.985	$< 2e-16$
c	0.22166	0.00038	588.050	$< 2e-16$
β_L	0.02817	0.00288	9.762	$< 2e-16$
β_G	0.00534	0.00295	1.809	0.0706
ρ	-0.01767	0.00111	-15.848	$< 2e-16$
τ	-0.00246	0.00111	-2.206	0.0274

Residual standard error: 0.02282 on 8537 degrees of freedom
Multiple R-squared: 0.9773, Adjusted R-squared: 0.9773
F-statistic: $6.125e+04$ on 6 and 8537 DF, p-value: $< 2.2e-16$

Table 2.11: Multiple linear regression results where the optimality margin of the heuristic is the dependent variable.

Analytically, when β_L and β_G are very small, or equivalently, $\frac{b+\beta_G}{b}$ and $\frac{b+\beta_L}{b}$ are close to 1, we can ignore the reference price effects; that is, we have $\beta_G Y_{G,t}+\beta_L Y_{L,t} \approx 0$

and therefore,

$$\begin{aligned}\pi &= \sum_{t=1}^{T}(p_t - c)\left[a - bp_t + (\beta_G Y_{G,t} + \beta_L Y_{L,t})(r_t - p_t + \rho Y_{L,t} - \tau Y_{G,t})\right] && (2.6.1)\\ &\approx \sum_{t=1}^{T}(p_t - c)\,(a - bp_t) && (2.6.2)\end{aligned}$$

In such a case, heuristic algorithm is approximately optimal. In the extreme case where β_L and β_G are zero, the heuristic method produces optimal results. In summary, we can say that the heuristic procedure is more likely to produce close to optimal results for products that show high customer loyalty and high profit margins.

2.7 Conclusions and Future Research

As we discussed in Section 2.1, the existing literature on pricing optimization mostly relies on dynamic programming which suffer from the curse of dimensionality and thus have been used only to solve small problems. However, in practice, the need to solve large problems is evident. This is specially true in online retail markets due to the large availability of data. In response to this gap in literature, we have proposed a myopic heuristic and a modified generalized Benders' decomposition method to find optimal pricing in a multi-period pricing problem with reference pricing and thresholds. We established analytical results for finding optimal solutions for the approximate heuristics problems as well as the sub problems in the modified GBD approach. We performed numerical computations that show that the heuristic works well for some set of problems. The modified GBD outperforms the heuristic. Furthermore, running it for 100 iterations achieves solutions similar in quality to the case when it is ran for 1000 iterations, but at much lower computational times.

As we seen in Section 2.5 for products with high profit margin and consumer brand loyalty, the heuristic method performs very well. Because using the myopic heuristic does not require any solver licensing and it requires minimal computation time, in these cases it may be preferable to the GBD method. However, in other cases, the GBD outperforms the myopic heuristic significantly. Also, solving the multi-period profit maximization problem can assist managers in deciding about general pricing strategies, such, every day low pricing or high-low pricing.

Our work can be extended in several ways. First, As demand in a market is not always deterministic, it is important to incorporate uncertainty in the demand function. This can be accomplished by adding a stochastic term to the demand function in (2.1.4) and solve the new profit maximization problem. Inventory holding and ordering costs, lead time, and the costs associated with loss of market share as result of shortages also need to be added in future research. Second, the demand function (2.1.4) and the model of reference price (2.1.1) ignore the effect of competition in the reference price and demand. When competition is significant, we need to develop a new reference price function that incorporates the effects of competition. Finally, pricing may be impacted by other factors, such as consumer reviews in on-line retail environments. It is interesting in such environments to consider a demand function that reflects such impacts as well as any cross-impacts.

Acknowledgement

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Appendix A: Proofs

Proof of Proposition 2.2.1. This proof is originally illustrated by VonMassow [24]. We have $r_t = (1-\alpha)p_{t-1} + \alpha r_{t-1}$. Then for a given r_1,

$$r_2 = (1-\alpha)p_1 + \alpha r_1 \tag{2..1}$$

$$r_3 = (1-\alpha)p_2 + \alpha r_2 \tag{2..2}$$

$$= (1-\alpha)p_2 + \alpha\left[(1-\alpha)p_1 + \alpha r_1\right] \tag{2..3}$$

$$= (1-\alpha)p_2 + \alpha(1-\alpha)p_1 + \alpha^2 r_1 \tag{2..4}$$

$$r_4 = (1-\alpha)p_3 + \alpha r_2 \tag{2..5}$$

$$= (1-\alpha)p_3 + \alpha\left[(1-\alpha)p_2 + \alpha(1-\alpha)p_1 + \alpha^2 r_1\right] \tag{2..6}$$

$$= (1-\alpha)p_3 + \alpha(1-\alpha)p_2 + \alpha^2(1-\alpha)p_1 + \alpha^3 r_1 \tag{2..7}$$

$$r_t = (1-\alpha)\sum_{j=0}^{t-1} \alpha^j p_{t-j} + \alpha^{t-1} r_1 \quad \forall\, t \in \{4, \cdots, T\} \tag{2..8}$$

Assuming α is in the closed interval $[0.15, 0.4]$ as noted in [19], the impact of the weights quickly vanish. Ignoring the weights beyond the first three terms for each r_t,

$$r_t \approx (1-\alpha)p_{t-1} + \alpha(1-\alpha)p_{t-2} + \alpha^2(1-\alpha)p_{t-2} \quad \forall\, t \in \{4, \cdots, T\} \tag{2..9}$$

Normalizing the weights above so that they sum to one, we have

$$r_t = \omega_1 p_{t-1} + \omega_2 p_{t-2} + \omega_3 p_{t-3} \qquad \forall\, t \in \{4, \cdots, T\} \tag{2..10}$$

where

$$\omega_1 = \frac{1-\alpha}{1-\alpha^3} \quad , \quad \omega_2 = \frac{\alpha(1-\alpha)}{1-\alpha^3} \quad , \quad \omega_3 = \frac{\alpha^2(1-\alpha)}{1-\alpha^3}.$$

To keep the approximation consistent, we also use these weights for r_2 and r_3; that is,

$$r_2 = \omega_1 p_1 + (1-\omega_1) r_1 \tag{2..11}$$

$$r_3 = \omega_1 p_2 + \omega_2 p_1 + \omega_3 r_1 \tag{2..12}$$

□

Proof of Proposition 2.2.2. Consider a loss situation in period t; that is, $Y_{G,t} = 0$ and $Y_{L,t} = 1$. The demand for such a situation is

$$D_t = a - bp_t + \beta_L(r_t - p_t + \rho) \tag{2..13}$$

The demand non-negativity constraint is therefore

$$p_t \leq \frac{a + \beta_L\,(r_t + \rho)}{b + \beta_L}. \tag{2..14}$$

Since $\frac{a+\beta_L(r_t+\rho)}{b+\beta_L} > \frac{a+\beta_L r_t}{b+\beta_L}$, this constraint is less restrictive than $p_t \leq \frac{a+\beta_L r_t}{b+\beta_L}$. The result follows. □

Proof of Proposition 2.2.3. Demand in period t is given by

$$D_t = a - bp_t + (\beta_G Y_{G,t} + \beta_L Y_{L,t})(r_t - p_t + \rho Y_{L,t} - \tau Y_{G,t}) \tag{2..15}$$

$$= a - (b + \beta_G Y_{G,t} + \beta_L Y_{L,t})p_t + (\beta_G Y_{G,t} + \beta_L Y_{L,t})r_t$$
$$+ (\beta_G Y_{G,t} + \beta_L Y_{L,t})(\rho Y_{L,t} - \tau Y_{G,t})$$

Letting $D_t \geq 0$, we have

$$(b + \beta_G Y_{G,t} + \beta_L Y_{L,t})p_t - (\beta_G Y_{G,t} + \beta_L Y_{L,t})r_t \leq a + (\beta_G Y_{G,t} + \beta_L Y_{L,t})(\rho Y_{L,t} - \tau Y_{G,t}) \tag{2..16}$$

Substituting for r_t will give the results. □

Proof of Proposition 2.3.1. We have

$$\frac{\partial}{\partial p_t}\pi_t = \begin{cases} a - 2bp_t + \beta_G\left(r_t - \tau - 2p_t + c\right) + bc & p_t \leq r_t - \tau \\ a - 2bp_t + bc & r_t - \tau \leq p_t \leq r_t + \rho \\ a - 2bp_t + \beta_L\left(r_t + \rho - 2p_t + c\right) + bc & r_t + \rho \leq p_t \end{cases} \tag{2..17}$$

$$\Rightarrow \frac{\partial^2}{\partial p_t^2}\pi_t = \begin{cases} -2b - 2\beta_G \leq 0 & p_t \leq r_t - \tau \\ -2b \leq 0 & r_t - \tau \leq p_t \leq r_t + \rho \\ -2b - 2\beta_L \leq 0 & r_t + \rho \leq p_t \end{cases} \tag{2..18}$$

Since $\frac{\partial^2}{\partial p_t^2}\pi_t \leq 0$ in 2..18 and π_t is continuous, then the profit function is piecewise concave in each interval and we can find the maximum value as follows.

1. **For $\boldsymbol{p_t \leq r_t - \tau}$:**

   $\bar{p_t}^{(1)}$ solves $\frac{\partial}{\partial p_t}\pi_t = 0$ where

$$\bar{p_t}^{(1)} = \frac{a + \beta_G(r_t - \tau) + (b + \beta_G)c}{2\left(b + \beta_G\right)} \tag{2..19}$$

If $\bar{p_t}^{(1)}$ is not realizable in $[0, r_t - \tau]$, i.e., $\bar{p_t}^{(1)} > r_t - \tau$, then π_t is increasing for $p_t \in [0, r_t - \tau]$ and the maximum value of π_t in this interval occurs at the right end point $r_t - \tau$ and so $p_t^{*(1)} = r_t - \tau$. Otherwise, $p_t^{*(1)} = \bar{p_t}^{(1)}$. Note that $\bar{p_t}^{(1)} > r_t - \tau$ if and only if

$$r_t < \frac{a + (b + \beta_G)c}{2b + \beta_G} + \tau \tag{2..20}$$

2. **For $\boldsymbol{r_t - \tau \leq p_t \leq r_t + \rho}$:**

$\bar{p_t}^{(2)}$ solves $\frac{\partial}{\partial p_t}\pi_t = 0$ where

$$\bar{p_t}^{(2)} = \frac{a + bc}{2b} \tag{2..21}$$

If $\bar{p_t}^{(2)} < r_t - \tau$, π_t is decreasing for $r_t - \tau \leq p_t \leq r_t + \rho$ and the maximum value of π_t in this interval occurs at the left end point; i.e., $p_t^{*(2)} = r_t - \tau$. If $\bar{p_t}^{(2)} > r_t + \rho$, π_t is increasing for $r_t - \tau \leq p_t \leq r_t + \rho$ and the maximum value of π_t in this interval occurs at the right end point; i.e., $p_t^{*(2)} = r_t + \rho$. If $r_t - \tau \leq \bar{p_t}^{(2)} \leq r_t + \rho$, then $p_t^{*(2)} = \bar{p_t}^{(2)}$. Note that $\bar{p_t}^{(2)} < r_t - \tau$ if and only if

$$r_t > \frac{a + bc}{2b} + \tau \tag{2..22}$$

and $\bar{p_t}^{(2)} > r_t + \rho$ if and only if

$$r_t < \frac{a + bc}{2b} - \rho \tag{2..23}$$

3. **For $\boldsymbol{p_t \geq r_t + \rho}$:**

$\bar{p_t}^{(3)}$ solves $\frac{\partial}{\partial p_t}\pi_t = 0$ where

$$\bar{p_t}^{(3)} = \frac{a + \beta_L(r_t + \rho) + (b + \beta_L)c}{2\,(b + \beta_L)} \tag{2..24}$$

If $\bar{p_t}^{(3)} < r_t + \rho$, then π_t is decreasing for $r_t + \rho < p_t$ and the maximum value of π_t in this interval occurs at the left end point; i.e., $p_t^{*(3)} = r_t + \rho$. Otherwise, $p_t^{*(3)} = \bar{p_t}^{(3)}$. Note that $\bar{p_t}^{(3)} < r_t + \rho$ if and only if

$$r_t > \frac{a + (b + \beta_L)c}{2b + \beta_L} - \rho \tag{2..25}$$

It is clear that the optimal price p_t^* in each period is given by

$$p_t^* = \arg\max_{p_t^{*(i)}} \left\{ \pi_t\big|_{p_t^{*(1)}}, \pi_t\big|_{p_t^{*(2)}}, \pi_t\big|_{p_t^{*(3)}} \right\} \tag{2..26}$$

□

Proof of Lemma 2.4.1. Let S_i denote the sum of the magnitude of non-diagonal elements in row i of matrix H_Y in (2.4.4) and $\beta = \max\{\beta_G, \beta_L\}$. Then

$$S_1 = \omega_1(\beta_G Y_{G,2} + \beta_L Y_{L,2}) + \omega_2(\beta_G Y_{G,3} + \beta_L Y_{L,3}) + \omega_3(\beta_G Y_{G,4} + \beta_L Y_{L,4}) \le \beta \tag{2..27}$$

$$\begin{aligned} S_2 = &\omega_1 \left[(\beta_G Y_{G,2} + \beta_L Y_{L,2}) + (\beta_G Y_{G,3} + \beta_L Y_{L,3})\right] + \omega_2(\beta_G Y_{G,4} + \beta_L Y_{L,4}) \\ &+ \omega_3(\beta_G Y_{G,5} + \beta_L Y_{L,5}) \end{aligned} \tag{2..28}$$

$$\begin{aligned} S_3 = &\omega_1 \left[(\beta_G Y_{G,3} + \beta_L Y_{L,3}) + (\beta_G Y_{G,4} + \beta_L Y_{L,4})\right] + \omega_2 \left[(\beta_G Y_{G,3} + \beta_L Y_{L,3})\right. \\ &\left.(\beta_G Y_{G,5} + \beta_L Y_{L,5})\right] + \omega_3(\beta_G Y_{G,6} + \beta_L Y_{L,6}) \end{aligned} \tag{2..29}$$

$$\begin{aligned} S_t =& (\beta_G Y_{G,t} + \beta_L Y_{L,t}) + \omega_1(\beta_G Y_{G,t+1} + \beta_L Y_{L,t+1}) + \omega_2(\beta_G Y_{G,t+2} + \beta_L Y_{L,t+2}) \\ &+ \omega_3(\beta_G Y_{G,t+3} + \beta_L Y_{L,t+3}) \quad \forall t = 4, \cdots, T-3 \end{aligned} \tag{2..30}$$

$$\begin{aligned} S_{T-2} =& (\beta_G Y_{G,T-2} + \beta_L Y_{L,T-2}) + \omega_1(\beta_G Y_{G,T-1} + \beta_L Y_{L,T-1}) \\ &+ \omega_2(\beta_G Y_{G,T} + \beta_L Y_{L,T}) \end{aligned} \tag{2..31}$$

$$S_{T-1} = (\beta_G Y_{G,T-1} + \beta_L Y_{L,T-1}) + \omega_1(\beta_G Y_{G,T} + \beta_L Y_{L,T}) \tag{2..32}$$

$$S_T = (\beta_G Y_{G,T} + \beta_L Y_{L,T}) \tag{2..33}$$

$$H_{t,t} = -2(b + \beta_G Y_{G,t} + \beta_L Y_{L,t}) \quad \forall t = 1, \cdots, T \tag{2..34}$$

If $b \geq \beta = \max\{\beta_G, \beta_L\}$, then $|S_t| < |H_{t,t}|$ for all $t = 1, \cdots, T$; that is, $-H_Y$ is a Hermitian diagonally dominant matrix with real positive diagonal entries. A Hermitian diagonally dominant matrix with real non-negative diagonal entries is positive semi-definite [8]. Therefore, $-H_Y$ is a symmetric positive semi-definite matrix; that is, H_Y is a symmetric negative semi-definite matrix. Note that for $b > \beta = \max \beta_G, \beta_L$, H_Y is a symmetric negative definite matrix. □

Proof of Proposition 2.4.1. From Lemma 2.4.1 we know that H_Y is symmetric negative definite. Geoffrion [5] shows that a quadratic programming problem of the form

$$\min_x \left\{ \frac{1}{2} x^T C x - c^T x \quad \text{s.t.} \quad Ax \leq b \right\}, \tag{2..35}$$

where C is a symmetric positive semidefinite matrix, has a dual of the form

$$\max_{\lambda \geq 0, x} \left\{ -\lambda^T b - \frac{1}{2} x^T C x \quad \text{s.t.} \quad C^T x + A^T \lambda - c = 0 \right\} \tag{2..36}$$

Change the minimization problem (2..35) to a maximization problem for the negative of the objective function and let $H_{\overline{Y}} = -C$ and $d_{\overline{Y}} = -c$, and the result follows. □

Proof of Proposition 2.4.2. Consider sequences $(Y_{L,1}, \cdots, Y_{L,T})$, $(Y_{G,1}, \cdots, Y_{G,T})$, $(Y_{L,1}^{(k)}, \cdots, Y_{L,T}^{(k)})$, and $(Y_{G,1}^{(k)}, \cdots, Y_{G,T}^{(k)})$, as numbers in base 2; i.e., let

$$YG = \sum_{t=1}^{T} 2^{t-1} Y_{L,t} \tag{2..37}$$

$$YL = \sum_{t=1}^{T} 2^{t-1} Y_{G,t} \tag{2..38}$$

$$YL^{(k)} = \sum_{t=1}^{T} 2^{t-1} Y_{L,t}^{(k)} \tag{2..39}$$

$$YG^{(k)} = \sum_{t=1}^{T} 2^{t-1} Y_{G,t}^{(k)} \tag{2..40}$$

Two numbers in any base are equal if and only if each pair of corresponding digits are equal; and the results follow. □

Proof of Proposition 2.4.3. The results follow directly from a complete enumeration method; that is, if we fix the values Y_L and Y_G to an optimal combination, the subproblem (GBD-Sub) will produce an optimal solution. □

Proof of Proposition 2.4.4. Because we avoid repeating Y's in the master problem and there are finite number of possible combinations for Y's, the algorithm converges in a finite number of combinations. □

Appendix B: Notation

Indices	
t	Period in horizon $(t = 1, 2, \cdots, T)$
k	Iteration of the GBD
Parameters	
a	Estimate of the market size in the linear demand function $D_t = a - bp_t$, $a \geq 0$.
b	Estimate of the price sensitivity parameter in the linear demand function $b \geq 0$.
β_G	Gain parameter, $0 \leq \beta_G \leq \beta_L$.
β_L	Loss parameter, $0 \leq \beta_G \leq \beta_L$.
τ	Gain threshold, $\tau \geq 0$.
ρ	Loss threshold, $\rho \geq 0$.
α	The parameter in exponential smoothing reference price model, $0.1 \leq \alpha \leq 0.4$.
ω_i	Parameters in approximation of reference prices $i = 1, 2, 3$, $\omega_i \geq 0$.
c	Constant unit cost, $c \geq 0$
Decision Variables	
p_t	Price in period t, $p_t \geq 0$.
$Y_{G,t}$	Binary gain indicator
$Y_{L,t}$	Binary loss indicator
$q_{L,k}$	Linearizing binary variable for loss history constraints in (GBD-Master) in iteration k
$q_{G,k}$	Linearizing binary variable for gain history constraints in (GBD-Master) in iteration k

Table 2.12: List of notations

Bibliography

[1] Anderson, C. K., Rasmussen, H., and MacDonald, L. (2005). Competitive pricing with dynamic asymmetric price effects. *International Transactions in Operational Research*, **12**(5), 509–525.

[2] Briesch, R. A., Krishnamurthi, L., Mazumdar, T., and Raj, S. P. (1997). A comparative analysis of reference price models. *Journal of Consumer Research*, **24**(2), 202–214.

[3] Casado, E. and Ferrer, J. C. (2013). Consumer price sensitivity in the retail industry: Latitude of acceptance with heterogeneous demand. *European Journal of Operational Research*, **228**(2), 418–426.

[4] Dasgupta, S. and Titman, S. (1998). Pricing strategy and financial policy. *Review of Financial Studies*, **11**(4), 705–737.

[5] Geoffrion, A. M. (1971). Duality in nonlinear programming: a simplified applications-oriented development. *SIAM Review*, **13**(1), 1–37.

[6] Greenleaf, E. A. (1995). The impact of reference price effects on the profitability of price promotions. *Marketing science*, **14**(1), 82–104.

[7] Hardie, B. G., Johnson, E. J., and Fader, P. S. (1993). Modeling loss aversion and reference dependence effects on brand choice. *Marketing science*, **12**(4), 378–394.

[8] Horn, R. A. and Johnson, C. R. (1990). *Matrix Analysis*. Cambridge University Press.

[9] Huang, J., Leng, M., and Parlar, M. (2013). Demand functions in decision modeling: A comprehensive survey and research directions. *Decision Sciences*, **44**(3), 557–609.

[10] Kalwani, M. U., Yim, C. K., Rinne, H. J., and Sugita, Y. (1990). A price expectations model of customer brand choice. *Journal of Marketing research*, pages 251–262.

[11] Kalyanaram, G. and Little, J. D. (1994). An empirical analysis of latitude of price acceptance in consumer package goods. *Journal of consumer research*, **21**(3), 408–418.

[12] Kalyanaram, G. and Winer, R. S. (1995). Empirical generalizations from reference price research. *Marketing Science*, **14**(3_supplement), G161–G169.

[13] Kopalle, P. K., Rao, A. G., and Assuncao, J. L. (1996). Asymmetric reference price effects and dynamic pricing policies. *Marketing Science*, **15**(1), 60–85.

[14] Krishnan, T. V., Bass, F. M., and Jain, D. C. (1999). Optimal pricing strategy for new products. *Management Science*, **45**(12), 1650–1663.

[15] Lattin, J. M. and Bucklin, R. E. (1989). Reference effects of price and promotion on brand choice behavior. *Journal of Marketing Research*, pages 299–310.

[16] Lee, E. and Staelin, R. (1997). Vertical strategic interaction: Implications for channel pricing strategy. *Marketing Science*, **16**(3), 185–207.

[17] Lu, L., Gou, Q., Tang, W., and Zhang, J. (2016). Joint pricing and advertising strategy with reference price effect. *International Journal of Production Research*, pages 1–21.

[18] Marn, M. V., Roegner, E. V., and Zawada, C. C. (2003). The power of pricing: Transaction pricing is the key to surviving the current downturn—and to flourishing when conditions improve. *The McKinsey Quarterly*, **1**.

[19] Mazumdar, T., Raj, S., and Sinha, I. (2005). Reference price research: Review and propositions. *Journal of Marketing*, **69**(4), 84–102.

[20] Monroe, K. B. (1971). Measuring price thresholds by psychophysics and latitudes of acceptance. *Journal of marketing research*, pages 460–464.

[21] Nasiry, J. and Popescu, I. (2011). Dynamic pricing with loss-averse consumers and peak-end anchoring. *Operations Research*, **59**(6), 1361–1368.

[22] Nerlove, M. (1958). Adaptive expectations and cobweb phenomena. *The Quarterly Journal of Economics*, **72**(2), 227–240.

[23] Powell, W. B. (2009). What you should know about approximate dynamic programming. *Naval Research Logistics (NRL)*, **56**(3), 239–249.

[24] Von Massow, M. (2011). *Pricing Strategy with Reference Prices*. Ph.D. thesis, McMaster University.

[25] Von Massow, M. and Hassini, E. (2013). Pricing strategy in the presence of reference prices with thresholds. *Journal of Revenue & Pricing Management*, **12**(4), 339–359.

[26] Webster, F. E. (1984). *Industrial Marketing Strategy*, volume 3. Wiley New York, NY.

Chapter 3

Role of Customer Ratings and Historical Prices in Two Products Online Pricing Optimization

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Role of Customer Ratings and Historical Prices in Two Products Online Pricing Optimization

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Abstract

We propose a quadratic programming model for online pricing optimization in the presence of customer reviews ratings. To account for ratings, we develop a new demand function for a multi-product, finite horizon, online retail environment. We validate our new demand model with Amazon.com pricing and customer reviews rating data. To solve the problem we introduce a myopic pricing heuristic as well as exact solution approaches. We provide several analytical results and numerical insights. In addition, we identify optimal prices that result in demand cannibalization and promotional pricing.

Keywords: Pricing, customer rating, e-commerce, reference price, quadratic programming.

3.1 Introduction

With the rise of online retailer giants like Amazon, and the widespread use of Internet and mobile technologies, online shopping is increasingly becoming more popular. For example, between 2016 and 2018, Canadian Internet customer sales have risen by 58%, at a far higher rate than traditional retail sales, where the increase was less than 5% [38, 45]. In addition, a recent Canada Post study has found that 80% of Canadians buy online, with 4 million of them making more than 25 purchases per year [38]. The recent COVID-19 pandemic is only reinforcing this trend as companies, such as Ikea and Homedepot, closed their brick-and-mortar stores and sell online only. With the state of emergency declared in many cities around the world, online shopping is becoming the only buying channel for most customers.

At the same time we are witnessing a proliferations of online reviews in the common e-commerce platforms. For example, TripAdvisor had 859 million reviews and opinions in 2019, representing an increase of 15% (129 million) over the previous year [33]. With a review averaging about 200 characters [39], this represents a text database that is about 57 million pages long! Young online shoppers (18-34 years old who also lead in online shopping) are expecting that each product will have at least about 160 reviews [11]. A recent study by Podium has also found that online reviews impact purchasing decision for 93% of shoppers [37]. Furthermore, the same study found that shoppers do consider the star ratings when deciding on purchases and their threshold for considering purchases is 3.3 stars [37]. It was also reported that customers are willing to pay up to 15% more in order to secure a better experience with the product or service and that a proxy for assessing the quality of that experience is online reviews.

Online shopping has also provided the convenience of keeping an accurate record of historical orders. For example, Amazon provides customers with a searchable database of all their past orders that also includes categories such as “Open Order”, “Buy Again” items and “Cancelled Orders”. The order include several information items such as the date the order was placed, the date it was delivered and the price. The latter makes it easier for a customer to form a reference price from its historical purchases.

The increase in online shopping and reviews, and the sharing and storing of order and product information by online sellers has motivated our research in this paper. We are interested in how can an online retailer use the historical prices, in the form of reference prices, and product review data to optimize its online pricing? In particular, we would like to answer the following research questions: How can we incorporate customer review ratings in a demand function? What should be the retailer's optimal prices? What insights can we obtain in regards to prices that lead to promotions and demand cannibalization?

The effects of customer reviews, reference price, and cross-price effects on demand have been studied in the literature on Revenue Management (e.g., see [2, 7, 35, 40, 41, 49]). The majority of studies consider prices as a response to the market behaviour factors such as customer rating. Although retail firms adjust prices based on market factors, market factors also change in response to prices. Very few studies have considered customer ratings as a response to prices where prices are decision variables in a multi-product revenue optimization problem. We will discuss this further in Section 3.2 where we provide a brief review of the relevant literature. The focus of this paper is to close this gap in the literature and offer some insights on how prices impact customer ratings. We first propose a model for forecasting customer ratings using current prices and ratings. Then, we develop a linear demand model that accounts for reference prices, cross-price effects, and customer ratings to be used in a price optimization problem. Following this, the price optimization problem is developed to study the effect of prices on future ratings and reference prices which in turn affect future demand.

The remainder of this paper is organized as follows. We will provide a review of the relevant literature in Section 3.2. In Section 3.3 we present a model for forecasting customer ratings and use it in a price optimization model. We develop several analytical results relating to the demand function and convexity of the pricing problem. We then introduce a myopic heuristic approach in Section 3.4 and provide closed form solutions for prices. In Section 3.3.2, we present evidence from Amazon data to support some aspects of our modelling efforts. Our computational analysis and insights are reported in Section 3.5. Finally, in Section 3.6 we summarize our findings and propose some future directions for research. The proofs of all results are provided in the Appendix A and a list of notations is provided in Appendix B.

3.2 Related Literature

There is a vast amount of literature on price optimization (e.g., see [12, 25, 29, 52]). We provide a brief overview of the literature on reference prices and cross-price effects in Section 3.2.1. This is followed by a review of relevant literature on customer ratings in Section 3.2.2.

3.2.1 Reference Price Effects

Reference price is the price that customers refer to decide whether an observed price is a good deal or not. There are many studies in the literature that investigate various aspects and applications of reference prices and cross-price effects (e.g., see [7, 24, 28, 35]). Here we only discuss the most relevant studies.

Some studies have focused on modelling the reference prices. Briesch et al. [7] studied different models of household reference prices and concluded that different product categories require different reference price models. Wang [51] studied different formulations of reference prices and the corresponding optimal pricing policies in a multi-product setting. More specifically, the author considered cases where the reference price is the lowest price amongst a set of available products, the assortment variety of the set, and the n^{th} lowest price amongst the set of products. Mazumdar et al. present a review of reference price research [35]. They model reference price in period t as the weighted average of the price and reference price in period $t-1$. This model for calculating reference prices is widely adopted in the literature (e.g., see [2, 35, 41, 49]) and will be used in this paper.

A less number of studies have considered multi-product or multi-firm cases. Anderson et al. [2] have studied a duopoly multiplicative model where the demand for each period is the previous period's demand multiplied by a function of reference price. They have considered a market in which there are only two firms of different sizes operating. Their objective is to maximize profit for the smaller firm using different pricing decision scenarios. Sethuraman et al. [40] study cross-price effects. They conclude that products have higher cross-price effects when their prices are closer to each other. They also note that higher priced products have higher cross-price effect on lower priced products.

3.2.2 Customer Ratings and Reviews

The literature on customer rating can be grouped into three categories. The bulk of the literature on customer ratings is focused on the effects of the ratings on sales (Section 3.2.2) and the factors affecting the ratings (Section 3.2.2). Some studies have considered the effect of the ratings on prices in general or on the optimal pricing problem. Very few of these studies consider the effects of prices on customer ratings (Section 3.2.2).

Effect of Customer Rating on Sales

Many studies have emphasized the effects of customer reviews on sales (eg. [5, 8, 26, 30]). Anand et al. [1] provide a comprehensive review of this literature. Gu et al. [19] study the effects of internal and external word of mouth on retailer sales for high-involvement products and conclude that external word of mouth has a more significant effect. Zhu and Zhang [57] use a nested logit demand model and show that on average one point increase in average customer ratings, increases the sales of a video game by 4%.

Some studies focus on the asymmetric effects of positive and negative ratings on sales. Chevalier and Mayzlin [10] study the relationship between sales rank and customer ratings. They show that extreme negative or positive ratings have more effect on sales. Furthermore, they find that negative ratings have more impact on sales than positive ones. They also studied the effect of positive reviews on two different websites for the same products. They concluded that positive reviews on one website increases sales relative to the other website. Similarly, Park and Nicolau [36] argue that extreme ratings, positive or negative, have more effect on customer choice than moderate ratings. They further note that the magnitude of the effect is asymmetric with negative ratings having a larger impact.

A group of studies consider the effects of the volume of reviews on sales. Duan et al. [14] study customer reviews in the movie industry. They find that the volume of online reviews positively affect product sales. Berger et al. [6] show that negative reviews increase demand for unknown products because they increase publicity. Maslowska et al. [34] study the effects of customer reviews on customer purchase decisions. They conclude that customer rating effects the purchase decision and the magnitude of the effect increases as the volume of the reviews increase; this effect is further increased if the price of the product is relatively high.

Other studies have considered the effect of the variability in customer ratings on sales. Babić Rosario et al. [4] investigate the effects of electronic word of mouth on sales. They note that on average, customer rating is positively correlated with sales. They also, argue that variability of customer rating has a negative effect on sales as customers view variability in ratings as an increased risk. They note that when a composite valence-volume (e.g., total number of five-star rating) is used, valence has a stronger effect on sales than volume consistently across platforms. Sun [47] conclude that the effect of the variance of customer ratings on demand depends on the average rating. In particular, a high variance in customer ratings decreases the demand when the average rating is high and increases it when the average rating is low.

Zimmermann et al. [58] note that both increase in volume and average ratings increase demand. They argue that if the variance of customer ratings is from differences in customer preferences, it decreases demand and increases the optimal price. However, if the variance is as a result of failure risk, it decreases both demand and optimal price. They also note that the risk averse customers prefer a product with lower variance when ratings are the same.

In a broader sense, Kwark et al. [27] study the effects of online customer reviews on different players in the market. They note that quality information increases competition and benefits retailer but hurts manufacturers. Conversely, fit information softens the competition and benefits manufacturers but hurts retailers. Forman et al. [17] study different aspects of customer reviews and concluded that there is evidence that geographic location and other reviewer characteristics affect the decision making of customers.

Li et al. [31] study the effects of numerical customer ratings and sentiments expressed as text in customer reviews on product sales performance. They use a joint sentiment-topic modelling technique to mine customer reviews and sentiments. They conclude that both numerical and textual reviews have significant impact on product sales. They also note that the mean ratings alone is insufficient to capture all effects of customer reviews on sales and that their technique provides a way to extract aspects of products that are important to customers.

Zhang et al. [55] propose a new method for product sales forecasting based on online reviews and macroeconomic indicators. They they combine prospect theory and sentiment analysis resulting a nonlinear autoregressive model. They validate the method by online sales data for three vehicles from 2008 to 2018.

Factors Affecting Customer Ratings

Chen et al. [9] study various marketing variables and their effects on the customer review behaviors. Some studies have considered the effects of price on customer ratings. Kuksov and Xie [26] is one of the very few studies that investigate the effects of prices on customer ratings. They study a two-period, single-product case where customers in the second period observe the rating from customers in the first period. They formulate the profit maximization problem as a two stage game with incomplete information and show that the customer rating is decreasing with respect to price. Shapiro [42] conducted an experimental study and concluded that price can be viewed by some customers as an indication of the quality of the product and thus affect their rating decision. Engler et al. [15] argue that online customer ratings represent customers satisfaction with the product and are not a representation of its quality. They further show that customer satisfaction is explained by their pre-purchase expectation of the product and their post-purchase observed performance of the product. Using path coefficient analysis, they show that there is a positive linear causation relationship between customer ratings and both pre-purchase expectation and post-purchase performance. They further argue that customers form their pre-purchase expectation using previous product rating, price, and brand reputation.They note that the effect of pre-purchase expectation on customer satisfaction is higher than that of post-purchase performance. This may seem counter intuitive. However, assuming constant quality goods, the effect of performance is captured in early reviews.

Ho et al. [21] study the effects of the difference between post-purchase expectation and post-purchase assessment of the product on individuals online feedback. They assume the pre-purchase expectation follows a normal distribution centred at average ratings adjusted for system bias and model the post-purchase evaluation using long-term average ratings adjusted for individual and product random terms. They then model the individual's rating as a linear combination of experienced performance, the difference between pre and post purchase evaluations, and other control variables. They show that their model is 98% accurate for predicting whether or not individuals post a review and 38.6% accurate for the rating they post.

Lin et al. [32] study the effects of free product sampling on product ratings. They conclude that free product sampling increases product ratings on average by 1.1%. They also note that the magnitude of this bias is larger when the product list price is higher.

Yin et al. [54] study the effect of customer bias on how they perceive other reviews helpful. They conclude that customers tend to consider reviews that confirm their initial belief, formed from the aggregate customer reviews, as helpful.

Archak et al. [3] use text mining techniques to identify important product features and corresponding customer opinions from online reviews. They then use these scores to estimate product sales rank using a linear model. They argue that a single scalar product rating is not adequate. In their study they note that reducing a complex review to one number assumes that the product quality is one-dimensional and unless the preferences of a given customer and a given reviewer are identical, a single scalar rating may not provide all necessary information about the purchase decision. This claim holds for each individual review and customer. However, with the reasonable assumption that the pool of reviews are a representative sample of the preferences of the customer base for a given product category, a single rating is a representative estimate of the overall preferences of the customer base and thus can be used in estimation of the aggregate demand.

Effect of Customer Reviews on Prices

De Maeyer [13] provides a review of the literature on online customer reviews and sales. One of the common findings in the literature is that high product ratings reduce customers' price sensitivity and increase the price premium customers are willing to pay. An example of a study in agreement with these findings are provided by Smith et al. [44] where they investigate the effects of customer ratings on prices. They use a large data set from customer ratings and prices of beer and conclude that an increase in customer ratings is positively related to an increase in prices.

Li and Hitt [30] use an analytical model to investigate the effects of customer reviews on firms' optimal pricing and customer welfare in a two-period monopoly and duopoly environments. They note that customer reviews are affected not only by quality factors, but are also biased by price effects. They also note that this is especially true for the commonly used uni-dimensional rating systems. The complexity of the optimization problem is, however, exponential with respect to T where T represents the number of periods in the horizon. Therefore, application of the model to horizons with large T is inefficient.

Wang et al. [50] study the optimal pricing strategy of an online seller in a duopoly market competing with an off-line seller. They argue that purchase price affects customers' positive feedback. They conclude that the online seller's optimal price decreases at the early stages where the amount of information (customer ratings) is low and increases as the amount of information increase.

He and Chen [20] study a dynamic pricing problem where customers arrive following a Poisson process, observe the price, and either make a purchase and leave a review or do not make a purchase and leave the platform. They assume binary reviews; that is, customers leave either a positive or negative feedback. The Bernoulli review outcomes are used to form and update the Bayesian posterior distribution of public belief. Public belief here is defined as publicized belief of whether the product is of high or low quality. They then formulate and solve a dynamic programming pricing problem. They show that it may be beneficial to use low prices for high-quality products at the beginning stages to speed-up the customer learning process. Jiang and Yang [23] study a firm's price and quality decision for a new product using a dynamic, game theoretic model. They show that in equilibrium, a firm with higher cost efficiency chooses higher quality.

Yang and Zhang [53] study a joint pricing and inventory replenishment problem in presence of online customer ratings for the first time in the literature. At the beginning of each period in the horizon, the firm decides its list price and replenishment level based on current stock level and average product rating. Customers who are served immediately (because of available inventory) will have higher probability to leave positive rating than customers who are wait-listed. The net rating in turn affects future demand for the product. They study the profit maximization problem and introduce a dynamic look-ahead heuristic for solving the problem with small optimality gaps.

Later Shin et al. [43] study the revenue maximization problem with customer ratings using fluid approximations. They first consider a one product case where customers' belief of product quality is updated over time. Customers who choose to purchase the product leave ratings with a constant probability. Customers' rating affect future customers' belief of the product quality. The price, together with this belief affects future demand. They consider multiple pricing strategies. They use book sales data from Amazon. As we will see in Section 3.3.2 Some of their results differ than that or ours. In particular, they consider book sales data set where price may signal book quality (this also applies to luxury products). Our model studies customers' rating behaviour for bargain brand products where higher prices play a negative roll in customers' product evaluation.

Huang et al. [22] study a case where a monopolist offers a service to two types of customers (naive and sophisticated) in a M/M/1 queue. They assume that the market size is large enough so that congestion is the major factor in customer consumption utility (service reward minus waiting time). They also assume the rating that customers leave are equal to their consumption utility. Here, congestion is controlled by price and average rating and customer rating is determined by their consumption utility. They show that in such a case, optimal pricing strategy is a cyclic high-low pricing strategy. They note that this model does not work for goods market. Similarly, Zhao and Zhang [56], using dynamic programming, study the optimal quality-price strategy for a service provider.

Very few studies consider the effects of prices on customer ratings. Feng et al. [16] show price has a negative effect on reviews and study a two period profit maximization. They formulate the problem as a two stage game where in the first stage the seller sets a price p_0 for customers with product quality expectation q_0. The price p_0 affects amount of sales and customers' post-purchase enjoyed utility. This post-purchase utility is assumed to be equal to the product rating at the end of first period/stage q_1 (this is also equal to pre-purchase expectation of customers in second stage/period). The rating q_1 in turn affects sales in second stage. They use book price and review data to show that sellers can use customer rating data to develop better pricing strategies. Stenzel et al. [46] also consider prices having direct negative effect on customer ratings. They study a revenue maximization problem considering customer ratings.

As outlined above, in the literature, the effects of reference price, cross-price effects, and effects of customer reviews on demand have been studied. To the best of our knowledge there are no studies that explicitly study customer rating as a response to prices where prices are decision variables in a revenue optimization problem utilizing reference prices, customer rating, and cross-price effects. The main contributions of this paper are:

- Close the gap in the literature by explicitly modelling the impact of customer ratings in a revenue optimization model.
- We propose a new demand function that incorporates reviewers ratings and validate it with Amazon data.
- We propose a comprehensive price optimization model that incorporates the impact of reviews and historical prices on optimal prices for two products in an online retail environment.
- We develop solution procedures for finding optimal prices for two products.
- We identify prices under promotions and demand cannibalization conditions.

3.3 Demand Model and Problem Formulation

In this section we state our assumption, present a demand function and formulate the price optimization problem.

3.3.1 Assumptions

We assume products are in the same price/quality category; that is, high price/quality or low price/quality. Also, for computational simplicity and without the loss of generality, we will use a single scalar rating model. Our model can extended to a multi-dimensional rating system.

The majority of online retailers display average product rating in a star system (for example a five star rating system by Amazon.com). If a product has a high number of reviews however, any recent review will have a negligible effect on the average product rating. In this paper, we assume customers form their pre-purchase rating expectation from recent customer ratings. If we assume reviews arrive following a Poisson distribution, in our approach each review (on average) has the same weight.

3.3.2 Demand Model

We introduce a new demand model inspired by the results from [15] and [30] and building on reference pricing models in [49] and [41]. Let $\eta_{i,t}$ denote the average rating from period t for product i. Inspired from the results from Feng et al. [16], Engler et al. [15], Kuksov and Xie [26], Ho et al. [21], and Stenzel et al. [46], we define

$$\eta_{i,t} = \theta\eta_{i,t-1} + (1-\theta)\mathcal{R}\left(1 - \frac{p_{i,t-1}}{\phi_i}\right) \tag{3.3.1}$$

where $0 \leq \theta \leq 1$ is a parameter, ϕ_i is a reasonable upper bound for $p_{i,t}$, and $\eta_{i,t}$ are in interval $[1, \eta]$ where $\mathcal{R} \geq 1$ is an integer. Here, $\eta_{i,0}$ is estimated using brand reputation and third-party reviewers such as Consumer Report in the United States, Stiftung Warentest in Germany, or Sharp helmet rating in the United Kingdom. In Section 3.3.2, we provide evidence for this customer rating model (3.3.1).

Mazumdar et al. [35] model the reference price in period t for product i as

$$r_{i,t} = \alpha r_{i,t-1} + (1-\alpha) p_{i,t-1} \qquad \alpha \in [0,1]. \tag{3.3.2}$$

where α is the memory parameter and $p_{i,t-1}$ is product i price in period $t-1$. Von Massow [48] showed that the impact of previous prices diminishes very quickly. We can therefore approximate $r_{i,t}$ in a similar manner to [41, 48]:

$$r_{i,2} \approx \omega_1 p_{i,1} + (1-\omega_1) r_{i,1} \tag{3.3.3}$$

$$r_{i,3} \approx \omega_1 p_{i,2} + \omega_2 p_{i,1} + \omega_{i,3} r_{i,1} \tag{3.3.4}$$

$$r_{i,t} \approx \omega_1 p_{i,t-1} + \omega_2 p_{i,t-2} + \omega_3 p_{i,t-3} \qquad \forall t \geq 4 \tag{3.3.5}$$

where

$$\omega_1 = \frac{1-\alpha}{1-\alpha^3}, \omega_2 = \frac{\alpha(1-\alpha)}{1-\alpha^3}, \text{ and } \omega_3 = \frac{\alpha^2(1-\alpha)}{1-\alpha^3}.$$

Using a similar logic, we can approximate $\eta_{i,t}$ for a given $\eta_{i,0}$ as follows:

$$\eta_{i,1} \approx \theta_1 \mathcal{R}\left(1 - \frac{p_{i,1}}{\phi_i}\right) + (1-\theta_1)\,\eta_{i,0} \tag{3.3.6}$$

$$\eta_{i,2} \approx \theta_1 \mathcal{R}\left(1 - \frac{p_{i,2}}{\phi_i}\right) + \theta_2 \mathcal{R}\left(1 - \frac{p_{i,1}}{\phi_i}\right) + \theta_3 \eta_{i,0} \tag{3.3.7}$$

$$\eta_{i,t} \approx \theta_1 \mathcal{R}\left(1-\frac{p_{i,t}}{\phi_i}\right)+\theta_2 \mathcal{R}\left(1-\frac{p_{i,t-1}}{\phi_i}\right)+\theta_3 \mathcal{R}\left(1-\frac{p_{i,t-2}}{\phi_i}\right) \quad \forall t \in \{3,\cdots,T\} \tag{3.3.8}$$

where

$$\theta_1=\frac{1-\theta}{1-\theta^3}, \theta_2=\frac{\theta(1-\theta)}{1-\theta^3}, \text{ and } \theta_3=\frac{\theta^2(1-\theta)}{1-\theta^3}. \tag{3.3.9}$$

Using the above formulations for average product ratings and reference prices, we model the demand for product i in period t as

$$D_{i,t}=a_i-b_i p_{i,t}+\beta_i\left(r_{i,t}-p_{i,t}\right)+\sum_{j\neq i}\lambda_{i,j}\left(\eta_{i,t-1}-\eta_{j,t-1}\right)+\sum_{j\neq i}\gamma_{i,j}\left(p_{j,t}-p_{i,t}\right). \tag{3.3.10}$$

The function $a_i - b_i p_{i,t}$ represents the classical price-sensitive linear demand function for non-negative parameters a_i and b_i. Parameters β_i, $\lambda_{i,j}$, and $\gamma_{i,j}$ are non-negative. It is intuitive that if the customer rating of product j, a substitute for product i, is higher than that of product i, some demand will move from product i to product j, and thus, decrease the demand for product i and vice versa. This is implemented in the demand function by the terms $\lambda_{i,j}\left(\eta_{i,t}-\eta_{j,t}\right)$. A similar argument can be made for the cross-price effects of the products in terms $\gamma_{i,j}\left(p_{j,t}-p_{i,t}\right)$. Parameters a_i, b_i, β_i, $\lambda_{i,j}$, and $\gamma_{i,j}$ can be estimated by multivariate linear regression. In addition, parameters α and θ in (3.3.1) and (3.3.2) can be estimated as explained by Greenleaf [18]; particularly by varying α and θ, and choosing the one that maximizes the explanatory power of (3.3.10).

Demand Model Analysis

In Proposition 3.3.1 we show necessary and sufficient conditions for ensuring non-negative demand in all periods. These conditions are needed in our pricing optimization model in order to guarantee that the integrity of the demand function.

Proposition 3.3.1. *The necessary and sufficient conditions to ensure a non-negative demand in all periods are*

$$p_{1,t} - \left(\frac{\gamma}{b_1 + \beta_1 + \gamma}\right) p_{2,t} - \frac{a_1 + \beta_1 r_{1,t} + \lambda\left(\eta_{1,t-1} - \eta_{2,t-1}\right)}{b_1 + \beta_1 + \gamma} \leq 0 \quad \forall t \in \{1, 2, \cdots, T\} \tag{3.3.11}$$

$$p_{2,t} - \left(\frac{\gamma}{b_2 + \beta_2 + \gamma}\right) p_{1,t} - \frac{a_2 + \beta_2 r_{2,t} + \lambda\left(\eta_{2,t-1} - \eta_{1,t-1}\right)}{b_2 + \beta_2 + \gamma} \leq 0 \quad \forall t \in \{1, 2, \cdots, T\} \tag{3.3.12}$$

In Propositions 3.3.2 and 3.3.3 we show that the demand model (3.3.10) is in agreement with common findings in the literature. As suggested in the literature, increasing customer ratings decreases customers' price sensitivity E_d (e.g.,see [13, 44]). This is shown in Propositions 3.3.2 .

Proposition 3.3.2. *The changes in customers' price sensitivity E_d is inversely proportional to changes in customer ratings, i.e., $\frac{\partial E_d}{\partial \eta} \leq 0$.*

In Proposition 3.3.3 we show that the demand model (3.3.10) is in agreement with findings in the literature that show that higher customer ratings increase the price premium customers are willing to pay (e.g., see [13, 44]).

Proposition 3.3.3. *For a given D, the changes in customer ratings and prices are positively related, i.e., $\frac{\partial p}{\partial \eta} \geq 0$.*

Evidence to Support the Rating Function in the Demand Model

To provide evidence for the customer rating function (3.3.1), we collected price and customer rating data from Amazon.com using tools provided by Helium10. Helium10, founded by a group of Amazon sellers, is a subscription based service that provides tools such as Amazon product research, keyword research, competitor intelligence, and listing optimization for Amazon sellers. They provided us with a limited time academic access to their service to aid us in our research. Below we briefly discuss the data collection and processing. This is followed by the description and analyses of the data.

To find products that are suitable for this paper, we needed to find products that have significant price variations together with significant number of customer reviews. Using Helium10 tools as a Google Chrome extension, we downloaded customer reviews including review text, rating, and date of the review as a CSV file. We then extracted raw price data through source code of the historical price plot and formatted it to JSON standard. Incorporating "jsonlite" and "openxlsx" packages in R, we loaded the data-frame and further processed it to convert Amazon date to real date, merging price and rating data, and aggregating the data-frame to form daily data points.

The data collected consists of price and customer rating data from six different product categories (action cameras, electric shavers, bags, headphones, TV boxes, and batteries). The data set, depending on availability, included 307 to 1588 data points of daily prices and customer ratings for each product. A summary of the results from regression analysis where $\mathcal{R}\left(1 - \frac{p_{t-1}}{\phi}\right)$ and η_{t-1} are the independent variables and η_t is the dependent variable is provided in Table 3.1.

As we can see in the table, the linear model explains 20% or more of the variation

Product Category	Coefficient of $\mathcal{R}\left(1-\frac{p_{t-1}}{\phi}\right)$	Coefficient of η_{t-1}	p-value for $\mathcal{R}\left(1-\frac{p_{t-1}}{\phi}\right)$	p-value for η_{t-1}	r-squared
Action Camera	0.0179	0.9818	0.0010	$< 1e-16$	0.9848
Action Camera	0.0152	0.9732	0.0129	$< 1e-16$	0.9601
Action Camera	0.0902	0.9329	0.0004	$< 1e-16$	0.8979
Electric Shaver	0.1029	0.8773	0.0349	$< 1e-16$	0.7804
Bag	0.1372	0.8429	0.0914	$< 1e-16$	0.7097
Headphone	0.4188	0.8328	0.0611	$< 1e-16$	0.7046
Headphone	0.2717	0.7842	0.0003	$< 1e-16$	0.6837
Bag	-0.1280	0.7728	0.0478	$< 1e-16$	0.6149
Action Camera	0.4171	0.5534	0.0498	$< 1e-16$	0.3347
Electric Shaver	-0.0368	0.5338	0.0125	$< 1e-16$	0.2936
Battery	0.4981	0.4839	0.0053	$< 1e-16$	0.2785
Headphone	-0.3974	0.4258	0.0007	$< 1e-16$	0.2165
Battery	-0.0733	0.4382	0.0316	$< 1e-16$	0.1982
Electric Shaver	0.3352	0.4023	0.0219	$< 1e-16$	0.1699
Battery	0.1864	0.2932	0.0608	$< 1e-8$	0.1008
Headphone	-0.8554	0.2583	0.0321	$< 1e-8$	0.0755
Headphone	0.0847	0.2610	0.0798	$< 1e-16$	0.0719
TV Box	0.2780	0.2128	0.0659	$< 1e-4$	0.0568

Table 3.1: Summary of linear regression analysis where $\mathcal{R}\left(1-\frac{p_{t-1}}{\phi}\right)$ and η_{t-1} are the independent variables and η_t is the dependent variable

in customer rating for most of the products (with mostly very small p-values). In addition, the coefficients of $\mathcal{R}\left(1-\frac{p_{t-1}}{\phi}\right)$ and η_{t-1} are positive for most products. This supports the idea that, everything else constant, for commodity products, increasing prices, decreases the product rating. Also, these results are in agreement with the results in [15]. In rare cases, we see negative coefficients for $\mathcal{R}\left(1-\frac{p_{t-1}}{\phi}\right)$ which signals that increasing price increases customers' perception of the quality of these particular products. This behaviour has been discussed in literature previously (e.g.,see [42]).

Figure 3.1 visually illustrates the historical prices (blue) together with customer ratings (red) for two products. Also included are the linear and polynomial approximations of the behaviour. These two plots provide strong evidence of the discussed

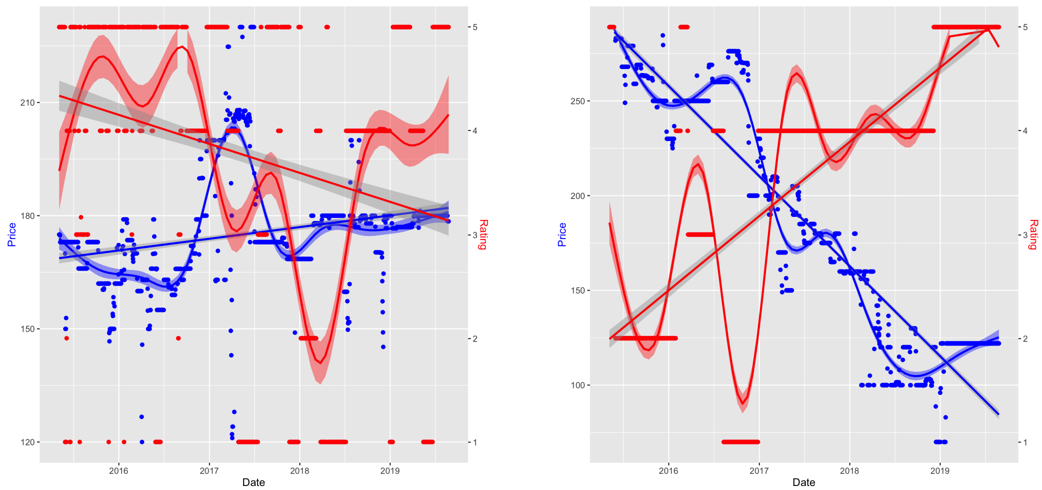


Figure 3.1: Price (blue) and customer rating (red) over time, including linear and polynomial approximations of the behaviour.

relationship between prices and customer reviews in this paper; that is, increasing prices are followed by decreasing reviews and vice versa.

We also looked at the characteristics of the products for which R-squared values in Table 3.1 are high. A graph of the approximation of the R-squared as a polynomial function of product average price and product category average price is illustrated in Figure 3.2. Although the product average prices seems to have small effects on the R-squared values, the largest effect seems to be from the product category average prices; that is, the effects of price and previous customer ratings seem to have a more significant impact on future ratings when the product is from a high price category.

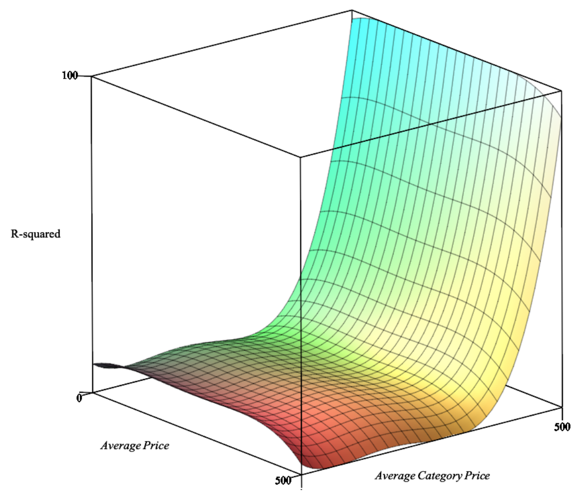


Figure 3.2: The polynomial approximation of the R-squared from linear regression results in Table 3.1 vs. the product average price and product category average price.

3.3.3 Problem Formulation

For constant per unit cost c_i for product i, we can define

$$\pi_{i,t} = (p_{i,t} - c_i)D_{i,t} \tag{3.3.13}$$

$$\Pi_t = \sum_i \pi_{i,t} \tag{3.3.14}$$

$$\Pi = \sum_t \Pi_t \tag{3.3.15}$$

where, $\pi_{i,t}$, Π_t, and Π are profit from product i in period t, total profit in period t from all products, and total profit in the horizon, respectively.

Without loss of generality and for computational simplicity, we consider a two-product case with $\lambda_{1,2} = \lambda_{2,1} = \lambda$ and $\gamma_{1,2} = \gamma_{2,1} = \gamma$. Therefore, we have

$$\Pi = \sum_t \sum_{i=1}^{2} (p_{i,t} - c_i) \left[a_i - b_i p_{i,t} + \beta_i \left(r_{i,t} - p_{i,t} \right) + \lambda \left(\eta_{i,t-1} - \eta_{3-i,t-1} \right) + \gamma \left(p_{3-i,t} - p_{i,t} \right) \right]. \tag{3.3.16}$$

We can now define the total profit maximization problem for a horizon with T periods as the following quadratic programming problem:

$$\max_{p_{i,t}} \quad \Pi \tag{Horizon}$$

s.t.

$$\eta_{i,1} = \theta_1 \mathcal{R} \left(1 - \frac{p_{i,1}}{\phi_i} \right) + (1 - \theta_1)\, \eta_{i,0} \qquad \forall\, i \tag{3.3.17a}$$

$$\eta_{i,2} = \theta_1 \mathcal{R} \left(1 - \frac{p_{i,2}}{\phi_i} \right) + \theta_2 \mathcal{R} \left(1 - \frac{p_{i,1}}{\phi_i} \right) + \theta_3 \eta_{i,0} \qquad \forall\, i \tag{3.3.17b}$$

$$\eta_{i,t} = \theta_1 \mathcal{R} \left(1 - \frac{p_{i,t}}{\phi_i} \right) + \theta_2 \mathcal{R} \left(1 - \frac{p_{i,t-1}}{\phi_i} \right) + \theta_3 \mathcal{R} \left(1 - \frac{p_{i,t-2}}{\phi_i} \right) \qquad \forall\, t \geq 3, \forall\, i \tag{3.3.17c}$$

$$r_{i,2} = \omega_1 p_{i,1} + (1 - \omega_1) r_{i,1} \qquad \forall\, i \tag{3.3.17d}$$

$$r_{i,3} = \omega_1 p_{i,2} + \omega_2 p_{i,1} + \omega_{i,3} r_{i,1} \qquad \forall\, i \tag{3.3.17e}$$

$$r_{i,t} = \omega_1 p_{i,t-1} + \omega_2 p_{i,t-2} + \omega_3 p_{i,t-3} \qquad \forall\, t \geq 4 \forall\, i \tag{3.3.17f}$$

$$p_{i,t} - \left(\frac{\gamma}{b_i + \beta_i + \gamma} \right) p_{j,t} - \frac{a_i + \beta_i r_{i,t} + \lambda \, \left(\eta_{i,t-1} - \eta_{j,t-1} \right)}{b_i + \beta_i + \gamma} \leq 0 \qquad \forall\, t \,,\, i \neq j \in \{1, 2\} \tag{3.3.17g}$$

$$p_{i,t} \geq 0 \qquad \forall\, i, t. \tag{3.3.17h}$$

Constraints (3.3.17a)-(3.3.17c) and (3.3.17d)-(3.3.17f) are for approximation of customer ratings and reference prices, respectively, and constraint (3.3.17g) is to ensure non-negative demands.

In Proposition 3.3.4 we provide sufficient conditions for the convexity of the quadratic optimization problem (Horizon).

Proposition 3.3.4. *Let* $\phi = \min\{\phi_1, \phi_2\}$ *and* $b = \min\{b_1, b_2\}$*. Then, for all* $\lambda \leq \frac{b\phi}{2\mathcal{R}}$*, the total profit optimization problem* (Horizon) *is a convex problem.*

The condition in Proposition 3.3.4 can be interpreted as a requirement that the rating impact of the two products on demand is less than that of the price, which is a reasonable condition to have as one expect that customers often place more weight on prices than ratings.

Problem (Horizon) can be solved using current commercial solvers such as CPLEX. However, these solver often require convexity properties to hold under all parameter combinations. Furthermore, because of licensing costs, it may be preferable to use heuristic methods to solve the problem. In Section 3.4 we use a myopic approach with closed form solutions to maximize the total profit from all products in each period. We then provide numerical results in Section 3.5.

3.4 A Myopic Approach

In this section we use a myopic approach in which we consider maximizing the profit of each period for a two-product case. We define problem (Myopic) for all t as follows:

$$\max_{p_{i,t}} \quad \Pi_t \tag{Myopic}$$

s.t.

$$p_{i,t} - \left(\frac{\gamma}{b_i + \beta_i + \gamma}\right) p_{j,t} - \frac{a_i + \beta_i r_{i,t} + \lambda\left(\eta_{i,t-1} - \eta_{j,t-1}\right)}{b_i + \beta_i + \gamma} \leq 0 \qquad \forall\, i \neq j \in \{1,2\}$$

$$p_{i,t} \geq 0 \qquad \forall\, i \in \{1,2\}$$

In Proposition 3.4.1 we show that (Myopic) is a convex problem.

Proposition 3.4.1. *The myopic optimization problem (Myopic) is a convex problem.*

Proposition 3.4.1 allows us to use Karush–Kuhn–Tucker (KKT) conditions to solve problem (Myopic). To this end we propose Algorithm 3.4.1.

Algorithm 3.4.1. *Let $LB = -\infty$. For each feasible KKT condition for problem (Myopic):*

1. *Solve stationarity and complementary slackness conditions for primal and dual variables.*

2. *Check if primal and dual feasibility conditions hold.*

3. *If all conditions are satisfied, find the objective value Π_t.*

4. *If $LB < \Pi_t$, let $p^*_{i,t} = p_{i,t}$ and $LB = \Pi_t$.*

5. *Move to the next feasible KKT condition and follow the steps above until all cases are considered. The optimal solution is $p^*_{i,t}$ with the objective value LB.*

The validity of Algorithm 3.4.1 is established in Theorem 3.4.1.

Theorem 3.4.1. *Algorithm 3.4.1 finds an optimal solution to problem* (Myopic).

We note that at most there will be 16 KKT solutions and we show in the proof in Appendix A that there are only 11 feasible KKT solutions.

We say product i *cannibalizes* product j when the prices for the products are set so that the demand of product j is captured by product i and therefore the demand for product j diminishes. We are interested in identifying optimal prices that result in demand cannibalization and promotional pricing, defined as under what conditions one product is offered for free.

The following propositions, introduce pricing results under free product promotion and product demand cannibalization strategies. For these propositions, without loss of generality, we assume $\eta_{1,t} \leq \eta_{2,t}$.

Proposition 3.4.2 (Promotional Pricing). *The optimal promotional pricing is as follows:*

(a) *For Product 1 (the product with a lower rating) when Product 2 is free ($p_{2,t} = 0$):*

$$p^*_{1,t} = \arg\max_{p_{1,t}} \left\{ \Pi_t \middle| p_{1,t} \in P_1 \right\} \tag{3.4.2}$$

where

$$P_1 = \left\{ \frac{a_1 + \beta_1 r_1 + (b_1 + \beta_1 + \gamma)\, c_1 + \lambda\,(\eta_1 - \eta_2) - \gamma c_2}{2b_1 + 2\beta_1 + 2\gamma}, \right. \tag{3.4.3}$$

$$\left.\frac{\lambda\left(\eta_1-\eta_2\right)+a_1+\beta_1 r_1}{b_1+\beta_1+\gamma}\right\}.$$

(b) *For Product 2 (the product with a higher rating) when Product 1 is free ($p_{1,t} = 0$):*

$$p^*_{2,t} = \arg\max_{p_{2,t}} \left\{ \Pi_t \middle| p_{2,t} \in P_2 \right\} \tag{3.4.4}$$

where

$$P_2 = \left\{ \frac{a_2+\beta_2 r_2+\left(b_2+\beta_2+\gamma\right)c_2+\lambda\left(\eta_2-\eta_1\right)-\gamma c_1}{2b_2+2\beta_2+2\gamma}, \right. \tag{3.4.5}$$
$$\left. \frac{\lambda\left(\eta_2-\eta_1\right)+a_2+\beta_2 r_2}{b_2+\beta_2+\gamma}, \frac{\lambda\left(\eta_2-\eta_1\right)-a_1-\beta_1 r_1}{\gamma} \right\}.$$

We note that the case where the two products have similar demand properties, except for customer ratings (i.e., $\eta_1 \neq \eta_2$), then the product with the higher customer ratings will always have a higher promotional price.

Proposition 3.4.3 (Cannibalization Pricing). *Under the condition where Product i is cannibalizing Product j ($i, j \in \{1, 2\}$ and $i \neq j$), the optimal prices are as follows:*

$$p_i = \frac{c_i}{2} + \frac{\left(b_j+\beta_j+\gamma\right)\left(\beta_i r_i+a_i\right)+\gamma\left(\beta_j r_j+a_j\right)+\lambda\left(b_j+\beta_j\right)\left(\eta_i-\eta_j\right)}{2\gamma\left(b_i+\beta_i+b_j+\beta_j\right)+2\left(b_j+\beta_j\right)\left(b_i+\beta_i\right)} \tag{3.4.6}$$

$$\begin{aligned} p_j =& \frac{2\beta_j r_j+2a_j+\gamma c_i+2\lambda\left(\eta_j-\eta_i\right)}{2b_j+2\beta_j+2\gamma} \\ &+ \frac{\gamma\left(\beta_i r_i+a_i\right)}{2\gamma\left(b_i+\beta_i+b_j+\beta_j\right)+2\left(b_j+\beta_j\right)\left(b_i+\beta_i\right)} \\ &+ \frac{\gamma^2\left(\beta_j r_j+a_j\right)+\gamma\left(b_j+\beta_j\right)\lambda\left(\eta_i-\eta_j\right)}{2\Big(\gamma\left(b_i+\beta_i+b_j+\beta_j\right)+\left(b_j+\beta_j\right)\left(b_i+\beta_i\right)\Big)\left(b_j+\beta_j+\gamma\right)}. \end{aligned} \tag{3.4.7}$$

We note that the above results in Proposition 3.4.3 can also be used to aid in pricing when one product is in shortage. In such cases, the retailer would like to price in a way to discourage ordering the stockout product. Another possible use of the pricing scenarios in Proposition 3.4.3 is when one product has significantly more inventory than the other product and the retailer would want to price the abundant product in a way to attract the demand for the product that has limited inventory.

3.5 Numerical Analysis and Insights

For the purpose of numerical experiments, we used 28,187 different pair combinations of the parameter sets shown in Table 3.2 for the two-product case.

Parameter	a	b	c	β	r_1	η_0	α	θ	λ	γ
Values	5	1	0.5	0.5	1	1	0.15	0.1	0.2	0.3
		2	1	0.75	2	3	0.45	0.3	0.3	0.5
				1		5		0.5	0.4	1
								0.7	0.5	
								0.8		
								0.9		

Table 3.2: Parameter Values

Let π_i^* denote the optimal objective value of (Horizon) when method i is used. Define the optimality comparative index for method i compared to method j as $\mathrm{CI}_{i,j} = \frac{\pi_i^* - \pi_j^*}{\pi_j^*}$. Let MH and CS refer to “myopic heuristic” and “commercial solver”, respectively. In addition, let subscripts F, IR, and IRC, refer to “full model”, “model ignoring rating effects”, and “model ignoring both Ratings and cross-price effects”, respectively. As shown in Table 3.3 when the myopic approach is used, ignoring customer rating and cross-price effects result in up to approximately 52%, and on average about 2% loss of total profits. This is also shown in Figure 3.5. In Table

$\boldsymbol{i, j}$	**min** $(\mathrm{CI}_{\boldsymbol{i,j}})$	**avg** $(\mathrm{CI}_{\boldsymbol{i,j}})$	**max** $(\mathrm{CI}_{\boldsymbol{i,j}})$
$\mathrm{MH_F}$, $\mathrm{MH_{IR}}$	0.0003%	0.8672%	11.9512%
$\mathrm{MH_F}$, $\mathrm{MH_{IRC}}$	0.0000%	1.9690%	51.5965%
$\mathrm{CS_F}$, $\mathrm{CS_{IR}}$	0.0218%	0.6760%	4.5988%
$\mathrm{CS_F}$, $\mathrm{CS_{IRC}}$	0.0008%	2.8447%	40.2482%
$\mathrm{CS_F}$, $\mathrm{MH_F}$	0.0036%	5.9722%	31.5241%

Table 3.3: Comparing optimality of methods

Method	**min**	**avg**	**max**
$\mathrm{MH_F}$	0.00	0.01	1.29
$\mathrm{CS_F}$	0.99	2.02	121.39

Table 3.4: Computation Time (seconds)

3.3 and Figure 3.4, it is illustrated that when we solve the problem using commercial solvers, ignoring customer ratings and cross price effects can result in approximately 3% on average and up to 40% loss in profits. In addition comparing the two methods $\mathrm{CS_F}$ and $\mathrm{MH_F}$ in Table 3.3 and Figure 3.3, shows that the heuristic method produces very close to optimal results in some cases. We will further discuss the optimality of the myopic heuristic in Section 3.5.4.

The effects of the parameters on total optimal profits, optimality of the results when customer rating is ignored, and optimality of the results when customer rating and cross-price effects are ignored are discussed in Sections 3.5.1-3.5.4, respectively. In the remainder of this section, product one ($i = 1$) has lower initial rating than product two ($i = 2$).

3.5.1 Effect of Parameters on Total Profits

In this section we investigate the effect of parameters on the total optimal profits when the full model is used with commercial solvers. The result of multiple linear

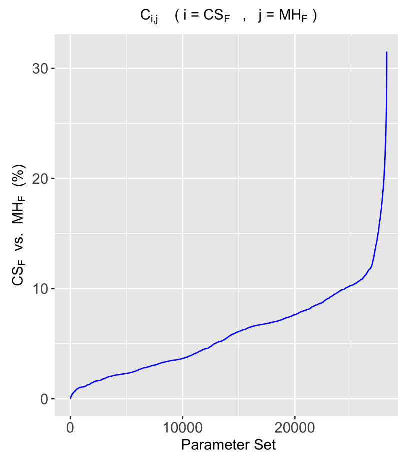
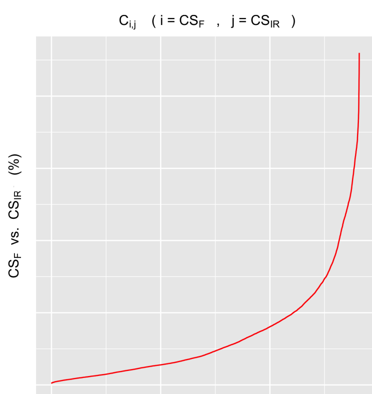
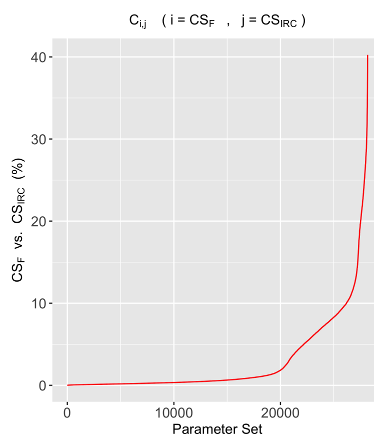


Figure 3.3: Optimality margin of the myopic heuristic method for all parameter sets.



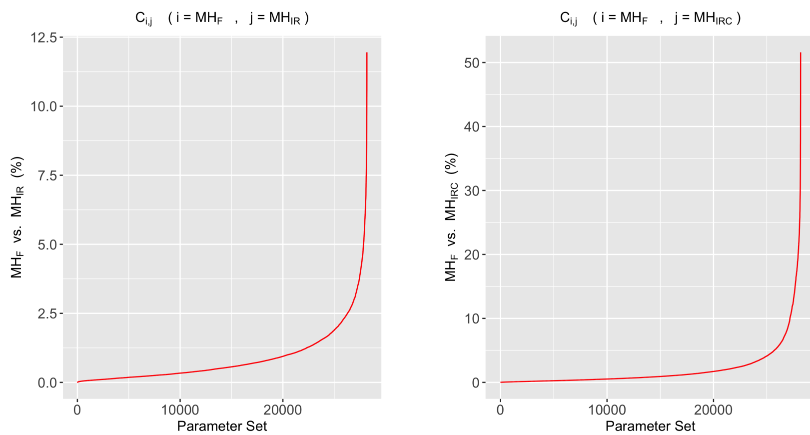
(a) Loss of optimality when customer rating is ignored for all parameter sets.



(b) Loss of optimality when customer rating and cross-price effects are ignored for all parameter sets.

Figure 3.4: Using commercial solvers for solving the problem for the horizon.

regression analysis when total profits is the dependent variable is provided in Table 3.5. As expected, increasing the values of b_i and c_i ($i = 1, 2$) has negative effect on profits. In addition, increasing the values of β_i, the reference price memory parameter α, the customer rating memory parameter θ, and γ also has a negative effect on profits. On the other hand, increasing the values of initial reference prices $r_{i,1}$ and λ has a



(a) Loss of optimality when customer rating is ignored for all parameter sets.

(b) Loss of optimality when customer rating and cross-price effects are ignored for all parameter sets.

Figure 3.5: Using the myopic heuristic method.

positive effect on profits. We will discuss managerial insights from these findings in Section 3.6.

3.5.2 The Effect of Parameters on Profits when Customer Rating is Ignored

In Table 3.3 we showed that ignoring customer rating can cause up to approximately 5% loss of profits when we solve the profit optimization problem for the horizon and up to 12% loss of profits when using the myopic heuristic. In this section we study the effect of parameters on this loss of optimality. Table 3.6 summarizes the multiple linear regression results when the optimality loss of the model ignoring customer ratings is the dependent variable. The coefficients for b_i, c_i, $\eta_{i,0}$ indicate that increasing the value of these parameters for the product with lower initial rating ($i = 1$) and

	Estimate	Std. Error	t-value	$Pr(>\|t\|)$
(Intercept)	167.3304	0.0656	2549.79	$< 2e - 16$
b_1	-26.0614	0.0193	-1348.81	$< 2e - 16$
b_2	-28.1926	0.0196	-1441.17	$< 2e - 16$
c_1	-21.7722	0.0313	-696.18	$< 2e - 16$
c_2	-17.0119	0.0320	-530.93	$< 2e - 16$
β_1	-1.5296	0.0327	-46.75	$< 2e - 16$
β_2	-1.2374	0.0334	-37.05	$< 2e - 16$
$r_{1,1}$	1.1223	0.0189	59.26	$< 2e - 16$
$r_{2,1}$	1.5755	0.0178	88.53	$< 2e - 16$
α	-2.7559	0.0459	-60.03	$< 2e - 16$
θ	-0.3970	0.0250	-15.87	$< 2e - 16$
λ	2.1866	0.0645	33.90	$< 2e - 16$
γ	-0.3265	0.0263	-12.42	$< 2e - 16$
Residual standard error: 1.153 on $28,165$ degrees of freedom				
Multiple R-squared: 0.9977,		Adjusted R-squared: 0.9977		
F-statistic: $1.03e + 06$ on 12 and $28,165$ DF,			p-value: $< 2.2e - 16$	

Table 3.5: Multiple linear regression results where the optimal total profits is the dependent variable.

decreasing them for the product with higher initial rating ($i = 2$) improves the performance (decreases the optimality loss) of the model ignoring customer rating. In addition, as it is expected from the model, increasing the value of λ, the parameter for the weight of customer rating in demand model increases the optimality loss of the model ignoring customer ratings. In addition, increasing the value of γ decreases the optimality loss of the model. This is because increasing γ relative to λ will decrease the relative effect of customer ratings on demand.

	Estimate	Std. Error	t-value	Pr($>|t|$)
(Intercept)	-2.5461	0.0204	-124.96	$< 2e - 16$
b_1	-0.3269	0.0054	-60.07	$< 2e - 16$
b_2	0.7170	0.0055	130.23	$< 2e - 16$
c_1	-0.0774	0.0088	-8.83	$< 2e - 16$
c_2	0.6320	0.0090	70.26	$< 2e - 16$
β_1	-0.0464	0.0092	-5.05	$< 2e - 16$
β_2	-0.0532	0.0094	-5.67	$< 2e - 16$
$r_{2,1}$	-0.0600	0.0046	-12.92	$< 2e - 16$
$\eta_{1,0}$	-0.3912	0.0024	-165.60	$< 2e - 16$
$\eta_{2,0}$	0.3912	0.0024	165.54	$< 2e - 16$
α	-0.0567	0.0129	-4.39	$< 2e - 16$
θ	0.2968	0.0070	42.25	$< 2e - 16$
λ	3.1061	0.0181	171.61	$< 2e - 16$
γ	-0.3099	0.0074	-42.14	$< 2e - 16$
Residual standard error: 0.3242 on $28,164$ degrees of freedom				
Multiple R-squared: 0.7819,			Adjusted R-squared: 0.7818	
F-statistic: $7,768$ on 13 and $28,164$ DF,			p-value: $< 2.2e - 16$	

Table 3.6: Multiple linear regression results where the optimality loss (%) of the model ignoring customer rating is the dependent variable. Here we are using commercial solvers to solve the pricing problem for the horizon.

3.5.3 The Effect of Parameters on Profits when Customer Rating and Cross-Price Effects are Ignored

As we saw in Table 3.3, ignoring customer rating and cross-price effects can result in up to 40% of loss in total profits. Here, we will investigate how parameters affect this loss in profits. Table 3.7 summarizes the multiple linear regression results when the relative optimality loss is the dependent variable. We note that increasing the customer rating memory parameter θ, customer rating weight in demand λ, and the weight of cross-price effects in demand γ, increases the loss of profits. This is expected as increasing these parameters increases the effects of customer rating and cross-price effects on demand. In addition, increasing the initial rating of the product with lower

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	-8.6608	0.1672	-51.79	$< 2e-16$
b_1	-9.0582	0.0442	-204.82	$< 2e-16$
b_2	9.5213	0.0447	212.85	$< 2e-16$
c_1	0.3022	0.0714	4.23	$< 2e-16$
c_2	1.6331	0.0732	22.31	$< 2e-16$
β_1	-1.3652	0.0748	-18.26	$< 2e-16$
β_2	0.7854	0.0763	10.29	$< 2e-16$
$r_{1,1}$	0.5390	0.0433	12.46	$< 2e-16$
$r_{2,1}$	-0.3830	0.0406	-9.42	$< 2e-16$
$\eta_{1,0}$	-0.6890	0.0192	-35.90	$< 2e-16$
$\eta_{2,0}$	0.6873	0.0192	35.81	$< 2e-16$
α	-0.4344	0.1049	-4.14	$< 2e-16$
θ	2.1787	0.0571	38.13	$< 2e-16$
λ	5.5303	0.1474	37.51	$< 2e-16$
γ	3.3453	0.0601	55.71	$< 2e-16$
Residual standard error: 2.634 on 28,163 degrees of freedom				
Multiple R-squared: 0.7196,				Adjusted R-squared: 0.7195
F-statistic: 5,163 on 14 and 28,163 DF,				p-value: $< 2.2e-16$

Table 3.7: Multiple linear regression results where the optimality loss (%) of the model ignoring customer rating and cross-price effects is the dependent variable. Here we are using commercial solvers to solve the pricing problem for the horizon.

initial rating, $\eta_{1,0}$ and decreasing the initial rating of the product with higher initial rating, $\eta_{2,0}$, decreases the loss of profits; that is, decreasing the difference between the initial rating of the products reduces the profit loss. It is also worth mentioning that when the price sensitivity of demand is high for the product with lower initial ratings, b_1, and is low for the product with higher initial ratings, b_2, we can expect lower loss in profits. In addition, increasing product unit costs c_i for $i = 1, 2$ increases the loss of profits.

3.5.4 Optimality of the Myopic Heuristic

The myopic heuristic approach is very efficient in terms of computation time and, in some cases, produces very close to optimal results as shown in Table 3.4. In fact, from all the 28,187 parameter sets used in our computational experiments, the heuristic method's relative error was less than 10% in approximately 87% of cases and less than 5% in approximately 46% of cases. Table 3.8 summarizes the results from multiple linear regression analysis where the optimality margin is the dependent variable. Negative coefficients of b_1, c_1, $\eta_{1,0}$, γ, and $r_{i,1}$ for $i = 1, 2$, show that increasing these parameters results in better performance of the myopic heuristic. On the other hand, the positive coefficients of b_2, c_2, $\eta_{2,0}$, α, θ, λ, and β_i for $i = 1, 2$, indicate increasing optimality error of the myopic approach as these parameters increase.

3.6 Conclusions, Managerial Insights, and Future Research

As discussed in Section 3.1, there are extensive studies that investigate the effects of reference price, cross-price effects, and effects of customer reviews on demand. In terms of the relationship between prices and customer rating in the literature, prices are mostly studied as responses to customer ratings. Although it is known that market behaviour is also a response to prices, there are no studies that explicitly investigate customer rating as a response to prices where prices are decision variables in a revenue optimization context utilizing reference prices, customer rating, and cross-price effects. This paper bridges this gap in the literature and builds a basis for further research in this area.

	Estimate	Std. Error	t value	Pr(> $\|t\|$)
(Intercept)	0.0091	0.0013	6.82	0.0000
b_1	-0.0656	0.0004	-185.24	0.0000
b_2	0.0204	0.0004	57.02	0.0000
c_1	-0.0180	0.0006	-31.42	0.0000
c_2	0.0353	0.0006	60.29	0.0000
β_1	0.0450	0.0006	75.12	0.0000
β_2	0.0363	0.0006	59.36	0.0000
$r_{1,1}$	-0.0033	0.0003	-9.55	0.0000
$r_{2,1}$	-0.0048	0.0003	-14.79	0.0000
$\eta_{1,0}$	-0.0023	0.0002	-15.05	0.0000
$\eta_{2,0}$	0.0023	0.0002	14.96	0.0000
α	0.0093	0.0008	11.03	0.0000
θ	0.0274	0.0005	59.81	0.0000
λ	0.0291	0.0012	24.69	0.0000
γ	0.0066	0.0005	13.70	0.0000
Residual standard error: 0.02109 on 28,163 degrees of freedom				
Multiple R-squared: 0.7048,		Adjusted R-squared: 0.7046		
F-statistic: 4,802 on 14 and 28,163 DF,			p-value: $< 2.2e-16$	

Table 3.8: Multiple linear regression results where the optimality margin of the heuristic is the dependent variable.

We discussed the effect of parameters on total profits in Section 3.5.1. As expected, increasing price sensitivity of demand, unit product cost, and customer sensitivity to reference prices decrease total profits. It was also shown that increasing initial reference prices increases total profits. The negative coefficients for the weight of previous period's customer rating in the model for customer rating (θ) indicate that it is to the retailer's benefit to reduce the value of this parameters directly or try to reduce the effect of previous periods rating indirectly. This may be one factor that the default sorting of reviews on online retailers such as Amazon.com is "top reviews" rather than "recent reviews". In addition, the negative coefficient for the weight of the cross-price effects in demand model (γ) indicate that reducing the effect of cross-price effects increases retailer's total profits. This can be a factor in selecting "relevant products" at the time a customer is observing the product in order to reduce this effect.

In Sections 3.5.2 and 3.5.3 we discussed the effect of parameters on total profits when customer ratings or both customer ratings and cross price effects are ignored. We showed that ignoring customer rating and/or cross-price effects can lead to large profit losses when

- the difference in initial rating of products is high,
- the product with lower initial rating has low demand price sensitivity,
- the product with higher initial rating has high demand price sensitivity,
- the customer reference price memory parameter (α), is low, and
- the customer memory parameter for rating (θ) and the weight of difference in ratings in demand (λ) are high.

It is also expected that if the weight of the cross-price effects in demand is high, ignoring cross-price effects will result in large losses in profits.

We also discussed the performance of the myopic heuristic in Section 3.5.4. We noted that from 28,187 parameter sets used in our computational experiments, the heuristic method's relative error was less than 10% in approximately 87% of cases and less than 5% in approximately 46% of cases. To put the findings in Table 3.8 in perspective, the myopic heuristic performs well when

- the difference in initial ratings is high,

- the product with higher initial rating has high customer price sensitivity and low profit margin, and
- the product with lower initial rating has low customer price sensitivity and high profit margin.

Under the circumstances described above, the retailer may prefer to use the myopic heuristic instead of commercial solvers due to the licensing costs of commercial solvers and the computational performance of the myopic heuristic.

Our work can be extended in several ways. First, as demand in a market is not always deterministic, it is important to incorporate uncertainty in the demand function. This can be accomplished by adding a stochastic term to the demand function in (3.3.10) in a multiplicative or additive manner. Inventory holding and ordering costs, lead time, and the costs associated with loss of market share as a result of shortages also need to be added in future research. Second, in this paper, we only considered a one-retailer, two-product case. This study can be extended to a duopoly market. Also, the model of customer rating can further be expanded to capture the effects of reference price and other factors.

Acknowledgment

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Appendix A: Proofs

Proof of Proposition 3.3.2. Price sensitivity is defined as a percentage change in demand divided by the percentage change in price. For simplicity we show a two product case. The proof is applicable to the general case as well. Let the demand be (as defined in (3.3.10))

$$D_1 = a - bp + \beta\left(r - p\right) + \lambda\left(\eta - \eta'\right) + \gamma\left(p - p'\right) \tag{3..1}$$

where η' and p' be the rating and price of the other product. Everything else constant, increasing price p to $p + \Delta p$ the demand will be

$$D_2 = a - b\left(p + \Delta p\right) + \beta\left(r - \left(p + \Delta p\right)\right) + \lambda\left(\eta - \eta'\right) + \gamma\left(\left(p + \Delta p\right) - p'\right) \tag{3..2}$$

The price sensitivity is then defined as

$$E_d = \frac{\left|\frac{D_2 - D_1}{D_1}\right|}{\left|\frac{\Delta p}{p}\right|} = \frac{\left|\frac{-(b+\beta-\gamma)}{a-bp+\beta(r-p)+\lambda(\eta-\eta')+\gamma(p-p')}\right|}{\left|\frac{\Delta p}{p}\right|} \tag{3..3}$$

Everything else constant, increasing η increases the denominator of the numerator in (3..3) and therefore decreases the price sensitivity; and the result follows. □

Proof of Proposition 3.3.3. For simplicity we show a two product case . Let the demand be (as defined in (3.3.10))

$$D = a - bp + \beta\left(r - p\right) + \lambda\left(\eta - \eta'\right) + \gamma\left(p - p'\right) \tag{3..4}$$

It is clear that increasing η increases D and therefore, the retailer can increase the price to keep D constant; i.e., the customers' are willing to pay higher price premium for the product when ratings are higher. □

Proof of Proposition 3.3.1. Results follow directly by letting $D_{i,t} \geq 0$ for $i \in \{1, 2\}$. □

Proof of Proposition 3.3.4. Let J denote the $2T \times 2T$ Jacobian matrix for the horizon optimization problem (Horizon). Let

$$J = \begin{pmatrix} J^{(1)} & J^{(2)} \\ J^{(3)} & J^{(4)} \end{pmatrix} \tag{3..5}$$

where all $J^{(l)}$ matrices are $T \times T$ diagonal matrices defined as

$$J^{(1)}_{k,k} = -2\left(b_1 + \beta_1 + \gamma\right) \tag{3..6}$$

$$J^{(1)}_{k,k+1} = J1_{k+1,k} = \frac{-\lambda\, \theta_1 \mathcal{R} + \beta_1 \omega_1 \phi_1}{\phi_1} \tag{3..7}$$

$$J^{(1)}_{k,k+2} = J1_{k+2,k} = \frac{-\lambda\, \theta_2 \mathcal{R} + \beta_1 \omega_2 \phi_1}{\phi_1} \tag{3..8}$$

$$J^{(1)}_{k,k+3} = J1_{k+3,k} = \frac{-\lambda\, \theta_3 \mathcal{R} + \beta_1 \omega_3 \phi_1}{\phi_1} \tag{3..9}$$

$$J^{(2)}_{k,k} = J^{(3)}_{k,k} = 2\gamma \tag{3..10}$$

$$J^{(2)}_{k,k+1} = J^{(3)}_{k+1,k} = \frac{\lambda\, \theta_1 \mathcal{R}}{\phi_1} \tag{3..11}$$

$$J^{(2)}_{k,k+2} = J^{(3)}_{k+2,k} = \frac{\lambda\, \theta_2 \mathcal{R}}{\phi_1} \tag{3..12}$$

$$J^{(2)}_{k,k+3} = J^{(3)}_{k+3,k} = \frac{\lambda\, \theta_3 \mathcal{R}}{\phi_1} \tag{3..13}$$

$$J^{(2)}_{k+1,k} = J^{(3)}_{k,k+1} = \frac{\lambda\,\theta_1\mathcal{R}}{\phi_2} \tag{3..14}$$

$$J^{(2)}_{k+2,k} = J^{(3)}_{k,k+2} = \frac{\lambda\,\theta_2\mathcal{R}}{\phi_2} \tag{3..15}$$

$$J^{(2)}_{k+3,k} = J^{(3)}_{k,k+3} = \frac{\lambda\,\theta_3\mathcal{R}}{\phi_2} \tag{3..16}$$

$$J^{(4)}_{k,k} = -2\,(b_2 + \beta_2 + \gamma) \tag{3..17}$$

$$J^{(4)}_{k,k+1} = J1_{k+1,k} = \frac{-\lambda\,\theta_1\mathcal{R} + \beta_2\omega_1\phi_1}{\phi_2} \tag{3..18}$$

$$J^{(4)}_{k,k+2} = J1_{k+2,k} = \frac{-\lambda\,\theta_2\mathcal{R} + \beta_2\omega_2\phi_1}{\phi_2} \tag{3..19}$$

$$J^{(4)}_{k,k+3} = J1_{k+3,k} = \frac{-\lambda\,\theta_3\mathcal{R} + \beta_2\omega_3\phi_1}{\phi_2} \tag{3..20}$$

The largest sum of the magnitude of non-diagonal entries in upper half of the Jacobian matrix $\left[J^{(1)}\ J^{(2)}\right]$ is

$$2\left(\left|\frac{-\lambda\,\theta_1\mathcal{R} + \beta_1\omega_1\phi_1}{\phi_1}\right| + \left|\frac{-\lambda\,\theta_2\mathcal{R} + \beta_1\omega_2\phi_1}{\phi_1}\right| + \left|\frac{-\lambda\,\theta_3\mathcal{R} + \beta_1\omega_3\phi_1}{\phi_1}\right|\right) + 2\gamma + \frac{\lambda\mathcal{R}}{\phi_1} + \frac{\lambda\mathcal{R}}{\phi_2} \tag{3..21}$$

with the magnitude of diagonal entries

$$2\,(b_1 + \beta_1 + \gamma) \tag{3..22}$$

Let $\phi = \min\{\phi_1, \phi_2\}$ and $b = \min\{b_1, b_2\}$ and

$$\lambda \le \frac{b\phi}{2\mathcal{R}} = \frac{2b}{\frac{4\mathcal{R}}{\phi}} \tag{3..23}$$

Then

$$\lambda \leq \frac{2b_1}{\frac{3\mathcal{R}}{\phi_1} + \frac{\mathcal{R}}{\phi_2}} \tag{3..24}$$

$$\Rightarrow 2b_1 \geq \lambda \left(\frac{3\mathcal{R}}{\phi_1} + \frac{\mathcal{R}}{\phi_2} \right) \tag{3..25}$$

$$\Rightarrow 2b_1 \geq 3 \left(\frac{\lambda\mathcal{R}}{\phi_1} \right) + \frac{\lambda mathcal R}{\phi_2} \tag{3..26}$$

$$\Rightarrow 2\left(b_1 + \beta_1 + \gamma\right) \geq 2\left(\beta_1 + \frac{\lambda\mathcal{R}}{\phi_1}\right) + 2\gamma + \frac{\lambda\mathcal{R}}{\phi_1} + \frac{\lambda\mathcal{R}}{\phi_2} \tag{3..27}$$

We know that

$$2\left(\beta_1 + \frac{\lambda\mathcal{R}}{\phi_1}\right) + 2\gamma + \frac{\lambda\mathcal{R}}{\phi_1} + \frac{\lambda\mathcal{R}}{\phi_2} \tag{3..28}$$

$$\geq 2\left(\left| \frac{-\lambda\,\theta_1\mathcal{R} + \beta_1\omega_1\phi_1}{\phi_1} \right| + \left| \frac{-\lambda\,\theta_2\mathcal{R} + \beta_1\omega_2\phi_1}{\phi_1} \right| + \left| \frac{-\lambda\,\theta_3\mathcal{R} + \beta_1\omega_3\phi_1}{\phi_1} \right| \right) + 2\gamma + \frac{\lambda\mathcal{R}}{\phi_1} + \frac{\lambda\mathcal{R}}{\phi_2} \tag{3..29}$$

Therefore

$$2\left(b_1 + \beta_1 + \gamma\right) \tag{3..30}$$

$$\geq 2\left(\left| \frac{-\lambda\,\theta_1\mathcal{R} + \beta_1\omega_1\phi_1}{\phi_1} \right| + \left| \frac{-\lambda\,\theta_2\mathcal{R} + \beta_1\omega_2\phi_1}{\phi_1} \right| + \left| \frac{-\lambda\,\theta_3\mathcal{R} + \beta_1\omega_3\phi_1}{\phi_1} \right| \right) + 2\gamma + \frac{\lambda\mathcal{R}}{\phi_1} + \frac{\lambda\mathcal{R}}{\phi_2}. \tag{3..31}$$

A similar argument follows for the bottom submatrix of the Jacobian; i.e., $\left[J^{(3)} \;\; J^{(4)}\right]$. Therefore $-J$ is a diagonally dominant Hermitian matrix with positive diagonal elements; i.e., $-J$ is a positive semidefinite matrix. Thus, J is a negative semidefinite matrix. This results in problem (Horizon) being a maximization problem with concave objective function and linear constraints; that is, problem (Horizon) is a convex optimization problem. □

Proof of Proposition 3.4.1. We have

$$\Pi_t = \sum_i (p_{i,t} - c_i)\Big[a_i - b_i p_{i,t} + \beta_i \left(r_{i,t} - p_{i,t}\right) + \sum_{j\neq i} \lambda_{i,j} \left(\eta_{i,t-1} - \eta_{j,t-1}\right) + \sum_{j\neq i} \gamma_{i,j} \left(p_{j,t} - p_{i,t}\right)\Big]. \tag{3..32}$$

Then

$$\frac{\partial^2 \Pi_t}{\partial p_{i,t}^2} = -2(b_i + \beta_i + \gamma) \tag{3..33}$$

$$\frac{\partial^2 \Pi_t}{\partial p_{i,t} p_{j,t}} = 2\gamma \qquad \forall\, i \neq j \tag{3..34}$$

with Jacobian matrix

$$J = \begin{pmatrix} -2(b_i + \beta_i + \gamma) & 2\gamma \\ 2\gamma & -2(b_i + \beta_i + \gamma) \end{pmatrix}. \tag{3..35}$$

Therefore $-J$ is a diagonally dominant Hermitian matrix with positive diagonal elements; i.e., $-J$ is a positive semi-definite matrix. Therefore J is a negative semi-definite matrix. This results in problem (3.4) being a maximization problem with concave objective function and linear constraints; that is, problem (3.4) is a convex optimization problem. □

Proof of Theorem 3.4.1. Proposition 3.4.1 allows us to apply KKT conditions to solve problem (Myopic). Below we state the KKT conditions corresponding to problem (Myopic) for each t in horizon:

Primal Feasibility

$$p_{i,t} - \left(\frac{\gamma}{b_i + \beta_i + \gamma}\right) p_{j,t} - \frac{a_i + \beta_i r_{i,t} + \lambda\left(\eta_{i,t-1} - \eta_{j,t-1}\right)}{b_i + \beta_i + \gamma} \leq 0 \quad \forall\, i \neq j \in \{1, 2\} \tag{3..36}$$

$$p_{i,t} \geq 0 \qquad \forall\, i, j \in \{1, 2\} \tag{3..37}$$

Dual Feasibility

$$\mu_i \geq 0 \qquad \forall i \in \{1, 2\} \tag{3..38}$$

$$u_i \geq 0 \qquad \forall i \in \{1, 2\} \tag{3..39}$$

Stationarity

$$\frac{\partial \Pi_t}{\partial p_{i,t}} - \mu_i + \left(\frac{\gamma}{b_j + \beta_j + \gamma}\right) \mu_j + u_i = 0 \quad \forall\, i \neq j \in \{1, 2\} \tag{3..40}$$

Complementary Slackness

$$\mu_i \left[p_{i,t} - \left(\frac{\gamma}{b_i + \beta_i + \gamma}\right) p_{j,t} - \frac{a_i + \beta_i r_{i,t} + \lambda\left(\eta_{i,t-1} - \eta_{j,t-1}\right)}{b_i + \beta_i + \gamma}\right] = 0 \tag{3..41}$$

$$\forall\, i, j \in \{1, 2\}\,,\ i \neq j$$

$$u_i p_{i,t} \geq 0 \qquad \forall i \in \{1, 2\} \tag{3..42}$$

We use the following Lemma to prove the proposition.

Lemma 3..1. *If three or more of μ_1, μ_2, u_1, u_2 are greater than zero, the KKT conditions* (3..41)-(3..42) *are infeasible.*

Proof of Lemma 3..1 is as follows: For any $i \in \{1, 2\}$, from complementary slackness conditions (3..41)-(3..42)

$$\mu_i > 0 \Rightarrow D_{i,t} = 0 \tag{3..43}$$

$$u_i > 0 \Rightarrow p_{i,t} = 0 \tag{3..44}$$

If three or more of dual variables are strictly positive, the resulting case will be one of the following

$$\begin{cases} D_{1,t} = 0 \\ D_{2,t} = 0 \\ p_{1,t} = 0 \end{cases}, \begin{cases} D_{1,t} = 0 \\ D_{2,t} = 0 \\ p_{2,t} = 0 \end{cases}, \begin{cases} D_{1,t} = 0 \\ p_{1,t} = 0 \\ p_{2,t} = 0 \end{cases}, \begin{cases} D_{2,t} = 0 \\ p_{1,t} = 0 \\ p_{2,t} = 0 \end{cases}, \begin{cases} D_{1,t} = 0 \\ D_{2,t} = 0 \\ p_{1,t} = 0 \\ p_{2,t} = 0 \end{cases} \tag{3..45}$$

It is clear that all of these cases are infeasible. We therefore will use the results from this lemma to continue the proof for Proposition 3.4.1. First, we show that there are eleven cases that are not deemed infeasible by Lemma 3..1. From lemma 3..1, we can have either none, one, or two of dual variables to be strictly greater than zero. This therefore is a simple counting problem adding up to eleven cases as follows:

- There are six cases where two of the four dual variables are strictly greater than zero;
- There are four cases where only one of the four dual variables is strictly greater

than zero;

- There is one case where all dual variables are zero.

Closed form solutions for all cases are provided below. Each case can be confirmed by substitution.

- **Case 1:** $(\mu_1 = 0 \; , \; \mu_2 = 0 \; , \; u_1 = 0 \; , \; u_2 = 0)$

$$p_1 = \frac{c_1}{2} + \frac{(b_2 + \beta_2 + \gamma)(\beta_1 r_1 + a_1) + \gamma(\beta_2 r_2 + a_2) + \lambda(b_2 + \beta_2)(\eta_1 - \eta_2)}{2\gamma(b_1 + \beta_1 + b_2 + \beta_2) + 2(b_2 + \beta_2)(b_1 + \beta_1)} \tag{3..46}$$

$$p_2 = \frac{c_2}{2} + \frac{(b_1 + \beta_1 + \gamma)(\beta_2 r_2 + a_2) + \gamma(\beta_1 r_1 + a_1) + \lambda(b_1 + \beta_1)(\eta_2 - \eta_1)}{2\gamma(b_1 + \beta_1 + b_2 + \beta_2) + 2(b_2 + \beta_2)(b_1 + \beta_1)}. \tag{3..47}$$

- **Case 2:** $(\mu_1 = 0 \; , \; \mu_2 = 0 \; , \; u_1 = 0 \; , \; u_2 > 0)$

$$p_1 = \frac{a_1 + \beta_1 r_1 + (b_1 + \beta_1 + \gamma) c_1 + \lambda(\eta_1 - \eta_2) - \gamma c_2}{2b_1 + 2\beta_1 + 2\gamma} \tag{3..48}$$

$$p_2 = 0. \tag{3..49}$$

- **Case 3:** $(\mu_1 = 0 \; , \; \mu_2 = 0 \; , \; u_1 > 0 \; , \; u_2 = 0)$

$$p_1 = 0 \tag{3..50}$$

$$p_2 = \frac{-\gamma c_1 + (b_2 + \beta_2 + \gamma) c_2 + \lambda(\eta_2 - \eta_1) + \beta_2 r_2 + a_2}{2b_2 + 2\beta_2 + 2\gamma}. \tag{3..51}$$

- **Case 4:** $(\mu_1 = 0 \; , \; \mu_2 = 0 \; , \; u_1 > 0 \; , \; u_2 > 0)$

$$p_1 = 0 \tag{3..52}$$

$$p_2 = 0. \tag{3..53}$$

- **Case 5:** ($\mu_1 = 0$, $\mu_2 > 0$, $u_1 = 0$, $u_2 = 0$)

$$p_1 = \frac{c_1}{2} + \frac{(b_2+\beta_2+\gamma)(\beta_1 r_1 + a_1) + \gamma(\beta_2 r_2 + a_2) + \lambda(b_2+\beta_2)(\eta_1-\eta_2)}{2\gamma\,(b_1+\beta_1+b_2+\beta_2) + 2(b_2+\beta_2)(b_1+\beta_1)} \tag{3..54}$$

$$\begin{aligned} p_2 =& \frac{2\beta_2 r_2 + 2a_2 + \gamma c_1 + 2\lambda(\eta_2-\eta_1)}{2b_2+2\beta_2+2\gamma} \\ &+ \frac{\gamma(\beta_1 r_1 + a_1)}{2\gamma(b_1+\beta_1+b_2+\beta_2) + 2(b_2+\beta_2)(b_1+\beta_1)} \\ &+ \frac{\gamma^2(\beta_2 r_2 + a_2) + \gamma(b_2+\beta_2)\lambda(\eta_1-\eta_2)}{2\big(\gamma(b_1+\beta_1+b_2+\beta_2) + (b_2+\beta_2)(b_1+\beta_1)\big)(b_2+\beta_2+\gamma)}. \end{aligned} \tag{3..55}$$

- **Case 6:** ($\mu_1 = 0$, $\mu_2 > 0$, $u_1 = 0$, $u_2 > 0$)

$$p_1 = \frac{-\beta_2 r_2 + \lambda(\eta_1-\eta_2) - a_2}{\gamma} \tag{3..56}$$

$$p_2 = 0. \tag{3..57}$$

- **Case 7:** ($\mu_1 = 0$, $\mu_2 > 0$, $u_1 > 0$, $u_2 = 0$)

$$p_1 = 0 \tag{3..58}$$

$$p_2 = \frac{\lambda(\eta_2-\eta_1) + a_2 + \beta_2 r_2}{b_2+\beta_2+\gamma}. \tag{3..59}$$

- **Case 8:** ($\mu_1 > 0$, $\mu_2 = 0$, $u_1 = 0$, $u_2 = 0$)

$$p_1 = \frac{2\beta_1 r_1 + 2a_1 + \gamma c_2 + 2\lambda(\eta_1-\eta_2)}{2b_1+2\beta_1+2\gamma}$$

$$
\begin{aligned}
&+\frac{\gamma\left(\beta_2 r_2+a_2\right)}{2\gamma\left(b_1+\beta_1+b_2+\beta_2\right)+2\left(b_2+\beta_2\right)\left(b_1+\beta_1\right)}\\
&+\frac{\gamma^2\left(\beta_1 r_1+a_1\right)+\gamma\left(b_1+\beta_1\right)\lambda\left(\eta_2-\eta_1\right)}{2\big(\gamma\left(b_1+\beta_1+b_2+\beta_2\right)+\left(b_2+\beta_2\right)\left(b_1+\beta_1\right)\big)\left(b_1+\beta_1+\gamma\right)} \qquad (3..60)\\
p_2 =&\frac{c_2}{2}+\frac{\left(b_1+\beta_1+\gamma\right)\left(\beta_2 r_2+a_2\right)+\gamma\left(\beta_1 r_1+a_1\right)+\lambda\left(b_1+\beta_1\right)\left(\eta_2-\eta_1\right)}{2\gamma\left(b_1+\beta_1+b_2+\beta_2\right)+2\left(b_2+\beta_2\right)\left(b_1+\beta_1\right)}.
\end{aligned}
$$

(3..61)

- **Case 9:** $(\mu_1 > 0\ ,\ \mu_2 = 0\ ,\ u_1 = 0\ ,\ u_2 > 0)$

$$p_1 =\frac{\lambda\left(\eta_1-\eta_2\right)+a_1+\beta_1 r_1}{b_1+\beta_1+\gamma} \qquad (3..62)$$

$$p_2 =0. \qquad (3..63)$$

- **Case 10:** $(\mu_1 > 0\ ,\ \mu_2 = 0\ ,\ u_1 > 0\ ,\ u_2 = 0)$

$$p_1 =0 \qquad (3..64)$$

$$p_2 =-\frac{\lambda\left(\eta_1-\eta_2\right)+a_1+\beta_1 r_1}{\gamma}. \qquad (3..65)$$

- **Case 11:** $(\mu_1 > 0\ ,\ \mu_2 > 0\ ,\ u_1 = 0\ ,\ u_2 = 0)$

$$p_1 =\frac{\big(\lambda\left(\eta_1-\eta_2\right)+a_1+\beta_1 r_1\big)\left(b_2+\beta_2\right)+\gamma\left(\beta_1 r_1+\beta_2 r_2+a_1+a_2\right)}{\gamma\left(b_1+\beta_1+b_2+\beta_2\right)+\left(b_2+\beta_2\right)\left(b_1+\beta_1\right)} \qquad (3..66)$$

$$p_2 =\frac{\left(b_1+\beta_1\right)\big(\lambda\left(\eta_2-\eta_1\right)+a_2+\beta_2 r_2\big)+\gamma\left(\beta_1 r_1+\beta_2 r_2+a_1+a_2\right)}{\gamma\left(b_1+\beta_1+b_2+\beta_2\right)+\left(b_2+\beta_2\right)\left(b_1+\beta_1\right)}. \qquad (3..67)$$

□

***Proof of Proposition** 3.4.2.* In part (a) we seek for the optimal price for product 1 under free product 2 promotion. Setting the price of product 2 to zero is equivalent to setting the dual variable $u_2 > 0$ in KKT conditions in the proof of Proposition 3.4.1 above. This corresponds to KKT cases 2, 4, 6, and 9. Under the assumption that $\eta_{1,t} \leq \eta_{2,t}$, the price for product 1 in case 6, however becomes negative and therefore infeasible. The prices for product 1 in (3.4.3) refers to KKT cases 2, 4, and 9, and the result follows.

In part (b) we seek for the optimal price for product 2 under free product 1 promotion. Setting the price of product 1 to zero is equivalent to setting the dual variable $u_1 > 0$ in KKT conditions in the proof of Proposition 3.4.1 above. This corresponds to KKT cases 3, 4, 7, and 10. The prices for product 2 in (3.4.5) refers to these cases and the result follows. $\square$

Proof of Proposition 3.4.3. When product i is cannibalizing product j, we must set a price so that the demand for product j becomes zero when the demand for product i is non-zero. This is equivalent to setting $\mu_i = 0$ and $\mu_j > 0$ in KKT cases in the proof of Proposition 3.4.1 above; that is cases 5 or 6 depending on the values of i and j. The prices refer to these cases and the results follow. $\square$

Appendix B: Notation

Symbols	
$r_{i,t}$	Reference price for product i in period t.
$\eta_{i,t}$	Customer rating for product i during period t.
$D_{i,t}$	Demand for product i in period t.
Indices	
t	Period in horizon ($t = 1, 2, \cdots, T$).
i, j	Products, $i, j \in \{1, 2\}$.
Parameters	
a_i	Estimate of the market size for product i in the linear demand function $D_{i,t} = a_i - b_i p_{i,t}$, $a_i \geq 0$.
b_i	Estimate of the price sensitivity parameter in the linear demand function $b_i \geq 0$.
c_i	Constant unit cost for product i, $c_i \geq 0$.
$\mathcal{R}$	Maximum possible rating.
β_i	Reference price weight parameter for product i, $\beta_i \geq 0$.
$\lambda i, j$	Weight of the difference between ratings of product i and j in demand model for product i.
$\gamma i, j$	Weight of the difference between prices of product i and j in demand model for product i.
α	The parameter in exponential smoothing reference price . model, $0.1 \leq \alpha \leq 0.4$.
ω_k	Parameters in approximation of reference prices $k = 1, 2, 3$, $\omega_i \geq 0$.
θ	The parameter in exponential smoothing customer rating model, $0.1 \leq \alpha \leq 0.4$.
θ_k	Parameters in approximation of customer ratings $k = 1, 2, 3$, $\omega_i \geq 0$.
ϕ_i	Reasonable price upper bound for product i.
Decision Variables	
$p_{i,t}$	Price of product i in period t, $p_{i,t} \geq 0$.

Table 3.9: List of notations

Bibliography

[1] Anand, O., Srivastava, P. R., and Rakshit, A. (2017). Assessment, implication, and analysis of online consumer reviews: A literature review. *Pacific Asia Journal of the Association for Information Systems*, **9**(2).

[2] Anderson, C. K., Rasmussen, H., and MacDonald, L. (2005). Competitive pricing with dynamic asymmetric price effects. *International Transactions in Operational Research*, **12**(5), 509–525.

[3] Archak, N., Ghose, A., and Ipeirotis, P. G. (2011). Deriving the pricing power of product features by mining consumer reviews. *Management Science*, **57**(8), 1485–1509.

[4] Babić Rosario, A., Sotgiu, F., De Valck, K., and Bijmolt, T. H. (2016). The effect of electronic word of mouth on sales: A meta-analytic review of platform, product, and metric factors. *Journal of Marketing Research*, **53**(3), 297–318.

[5] Barbado, R., Araque, O., and Iglesias, C. A. (2019). A framework for fake review detection in online consumer electronics retailers. *Information Processing & Management*, **56**(4), 1234–1244.

[6] Berger, J., Sorensen, A. T., and Rasmussen, S. J. (2010). Positive effects of

negative publicity: When negative reviews increase sales. *Marketing Science*, **29**(5), 815–827.

[7] Briesch, R. A., Krishnamurthi, L., Mazumdar, T., and Raj, S. P. (1997). A comparative analysis of reference price models. *Journal of Consumer Research*, **24**(2), 202–214.

[8] Chen, Y. and Xie, J. (2008). Online consumer review: Word-of-mouth as a new element of marketing communication mix. *Management Science*, **54**(3), 477–491.

[9] Chen, Y., Fay, S., and Wang, Q. (2011). The role of marketing in social media: How online consumer reviews evolve. *Journal of Interactive Marketing*, **25**(2), 85–94.

[10] Chevalier, J. A. and Mayzlin, D. (2006). The effect of word of mouth on sales: Online book reviews. *Journal of Marketing Research*, **43**(3), 345–354.

[11] Clement, J. (2019). Number of online product reviews expected by u.s. digital shoppers 2019, by age. Technical report, Statista.

[12] Dasgupta, S. and Titman, S. (1998). Pricing strategy and financial policy. *Review of Financial Studies*, **11**(4), 705–737.

[13] De Maeyer, P. (2012). Impact of online consumer reviews on sales and price strategies: A review and directions for future research. *Journal of Product & Brand Management*, **21**(2), 132–139.

[14] Duan, W., Gu, B., and Whinston, A. B. (2008). The dynamics of online word-of-mouth and product sales—an empirical investigation of the movie industry. *Journal of Retailing*, **84**(2), 233–242.

[15] Engler, T. H., Winter, P., and Schulz, M. (2015). Understanding online product ratings: A customer satisfaction model. *Journal of Retailing and Consumer Services*, **27**, 113–120.

[16] Feng, J., Li, X., and Zhang, X. (2019). Online product reviews-triggered dynamic pricing: Theory and evidence. *Information Systems Research*, **30**(4), 1107–1123.

[17] Forman, C., Ghose, A., and Wiesenfeld, B. (2008). Examining the relationship between reviews and sales: The role of reviewer identity disclosure in electronic markets. *Information Systems Research*, **19**(3), 291–313.

[18] Greenleaf, E. A. (1995). The impact of reference price effects on the profitability of price promotions. *Marketing Science*, **14**(1), 82–104.

[19] Gu, B., Park, J., and Konana, P. (2012). Research note—the impact of external word-of-mouth sources on retailer sales of high-involvement products. *Information Systems Research*, **23**(1), 182–196.

[20] He, Q.-C. and Chen, Y.-J. (2018). Dynamic pricing of electronic products with consumer reviews. *Omega*, **80**, 123–134.

[21] Ho, Y.-C., Wu, J., and Tan, Y. (2017). Disconfirmation effect on online rating behavior: A structural model. *Information Systems Research*, **28**(3), 626–642.

[22] Huang, F., Guo, P., and Wang, Y. (2019). Cyclic pricing when customers queue with rating information. *Production and Operations Management*.

[23] Jiang, B. and Yang, B. (2019). Quality and pricing decisions in a market with consumer information sharing. *Management Science*, **65**(1), 272–285.

[24] Kalyanaram, G. and Winer, R. S. (1995). Empirical generalizations from reference price research. *Marketing Science*, **14**(3_supplement), G161–G169.

[25] Krishnan, T. V., Bass, F. M., and Jain, D. C. (1999). Optimal pricing strategy for new products. *Management Science*, **45**(12), 1650–1663.

[26] Kuksov, D. and Xie, Y. (2010). Pricing, frills, and customer ratings. *Marketing Science*, **29**(5), 925–943.

[27] Kwark, Y., Chen, J., and Raghunathan, S. (2014). Online product reviews: Implications for retailers and competing manufacturers. *Information Systems Research*, **25**(1), 93–110.

[28] Lattin, J. M. and Bucklin, R. E. (1989). Reference effects of price and promotion on brand choice behavior. *Journal of Marketing Research*, pages 299–310.

[29] Lee, E. and Staelin, R. (1997). Vertical strategic interaction: Implications for channel pricing strategy. *Marketing Science*, **16**(3), 185–207.

[30] Li, X. and Hitt, L. M. (2010). Price effects in online product reviews: An analytical model and empirical analysis. *MIS Quarterly*, pages 809–831.

[31] Li, X., Wu, C., and Mai, F. (2019). The effect of online reviews on product sales: A joint sentiment-topic analysis. *Information & Management*, **56**(2), 172–184.

[32] Lin, Z., Zhang, Y., and Tan, Y. (2019). An empirical study of free product sampling and rating bias. *Information Systems Research*, **30**(1), 260–275.

[33] Lock, S. (2020). Number of user reviews and opinions on tripadvisor worldwide 2014-2019s. Technical report, Statista.

[34] Maslowska, E., Malthouse, E. C., and Viswanathan, V. (2017). Do customer reviews drive purchase decisions? the moderating roles of review exposure and price. *Decision Support Systems*, **98**, 1–9.

[35] Mazumdar, T., Raj, S., and Sinha, I. (2005). Reference price research: Review and propositions. *Journal of Marketing*, **69**(4), 84–102.

[36] Park, S. and Nicolau, J. L. (2015). Asymmetric effects of online consumer reviews. *Annals of Tourism Research*, **50**, 67–83.

[37] Podium (2017). 2017 state of online reviews. Technical report, Podium.

[38] Post, C. (2019). The 2019 canadian e-commerce benchmark report. Technical report, Canada Post.

[39] Reviewtrackers (2018). 2018 reviewtrackers online reviews survey. Technical report, Reviewtrackers.

[40] Sethuraman, R., Srinivasan, V., and Kim, D. (1999). Asymmetric and neighborhood cross-price effects: Some empirical generalizations. *Marketing Science*, **18**(1), 23–41.

[41] Shams-Shoaaee, S. S. and Hassini, E. (2019). Price optimization with reference price effects: A generalized Benders' decomposition method and a myopic heuristic approach. *European Journal of Operational Research*.

[42] Shapiro, B. P. (1973). Price reliance: existence and sources. *Journal of Marketing Research*, **10**(3), 286–294.

[43] Shin, D., Vaccari, S., and Zeevi, A. (2019). Dynamic pricing with online reviews. *Available at SSRN 3439836*.

[44] Smith, R. A., McKinney, C. N., Caudill, S. B., and Mixon, F. G. (2016). Consumer ratings and the pricing of experience goods: hedonic regression analysis of beer prices. *Agricultural and Food Economics*, **4**(1), 24.

[45] StatCan (2019). Retail trade. Technical report, Statistics Canada.

[46] Stenzel, A., Wolf, C., Schmidt, P., *et al.* (2020). Pricing for the stars-dynamic pricing in the presence of rating systems. Technical report, University of Bonn and University of Mannheim, Germany.

[47] Sun, M. (2012). How does the variance of product ratings matter? *Management Science*, **58**(4), 696–707.

[48] Von Massow, M. (2011). *Pricing Strategy with Reference Prices*. Ph.D. thesis, McMaster University.

[49] Von Massow, M. and Hassini, E. (2013). Pricing strategy in the presence of reference prices with thresholds. *Journal of Revenue & Pricing Management*, **12**(4), 339–359.

[50] Wang, H., Zhang, W., and Zheng, L. (2011). Dynamic pricing in b2c based on online product reviews. *Procedia Engineering*, **23**, 270–275.

[51] Wang, R. (2018). When prospect theory meets consumer choice models: Assortment and pricing management with reference prices. *Manufacturing & Service Operations Management*.

[52] Webster, F. E. (1984). *Industrial Marketing Strategy*, volume 3. Wiley New York, NY.

[53] Yang, N. and Zhang, R. P. (2018). Dynamic pricing and inventory management in the presence of online reviews. *Available at SSRN 2571705*.

[54] Yin, D., Mitra, S., and Zhang, H. (2016). Research note—when do consumers value positive vs. negative reviews? an empirical investigation of confirmation bias in online word of mouth. *Information Systems Research*, **27**(1), 131–144.

[55] Zhang, C., Tian, Y.-X., Fan, Z.-P., Liu, Y., and Fan, L.-W. (2019). Product sales forecasting using macroeconomic indicators and online reviews: a method combining prospect theory and sentiment analysis. *Soft Computing*, pages 1–14.

[56] Zhao, C. and Zhang, Y. (2019). Dynamic quality and pricing decisions in customer-intensive service systems with online reviews. *International Journal of Production Research*, **57**(18), 5725–5748.

[57] Zhu, F. and Zhang, X. (2006). The influence of online consumer reviews on the demand for experience goods: The case of video games. *ICIS 2006 Proceedings*, page 25.

[58] Zimmermann, S., Herrmann, P., Kundisch, D., and Nault, B. R. (2018). Decomposing the variance of consumer ratings and the impact on price and demand. *Information Systems Research*, **29**(4), 984–1002.

Chapter 4

Optimal Online Personalized Location Based Price Discounts in the Presence of Customer Ratings

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Optimal Online Personalized Location Based Price Discounts in the Presence of Customer Ratings

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Abstract

We use a multinomial choice model for customer purchase decision to find optimal personalized price discounts for an online retailer that incorporates customer locations and feedback from their reviews. We then consider two special cases of this problem and derive closed form solutions. To gain some analytical insights we carried extensive numerical experiments. For a two-customer and two-product case, we find that: (i) the product with lower rating is discounted more frequently than the product with higher rating, (ii) the two products should not be discounted simultaneously for a customer, (iii) a large difference between product ratings increases total profits and decreases (increases) price discounts for the product with a higher (lower) rating, (iv) and, as expected, increased customer product loyalty decreases price discounts.

Keywords: Personalized price discounts, nonlinear programming, customer rating, e-commerce.

4.1 Introduction

With enhancements in mobile technology and the rise of online retailers like Amazon, online shopping is increasingly becoming the shopping method of choice. For example, about 84% of internet users in Canada shopped online reaching $57.4 billion spending in 2018, compared to $18.9 billion in 2012 [32]. The recent COVID-19 pandemic has further contributed to increase in online retailing. The e-commerce sales increased about 40% in the week of May 26, 2020 compared to that of Feb 24, 2020 [3]. In addition, many business are forced to invest and adjust to implement stronger online presence which will likely lead to permanent increase in online sales [24].

One of the advantages of online retailing is the relatively low cost and ease of price adjustments and discounts compared to a brick and mortar environment. For example, it has been reported that Amazon adjusted prices more than 2.5 million times every day in December 2013 [10]. With the increasing popularity of online shopping, we are witnessing a proliferation of discount codes enticing customers to buy. The popularity and importance of online discounts is apparent in the increasing number of websites and browser extensions designed specifically for online discount codes (e.g., PromoCodes, Wikibuy, RetailMeNot, Slickdeals, Honey, Groupon, and

Ebates). For example, there was an average of 68 million monthly visits to Slickdeals website as of March 2019 [6]. Also, about 60% of online shoppers worldwide looked for online discount codes before making a purchase as of June 2019 [6].

In addition, online retailers such as Amazon can provide price discounts instantaneously depending on what the customer has viewed or picked so far and the state of their inventory and logistics costs. For example, if there are existing orders for the area of a new customer's delivery location, the marginal shipping cost absorbed by the retailer for the new customer's order may become negligible (depending on delivery capacity constraints). In such a case it may be beneficial to the retailer to offer a limited time discount coupon to the customer to entice a purchase decision. This, together with the increase in online shopping, reviews, and online discounts has motivated our research. In particular, we are interested in how can an online retailer use the customer locations, in the form of shipping costs, and product review data to optimally design personalized online price discounts for them.

The remainder of this paper is organized as follows. We provide an overview of the relevant literature in Section 4.2 followed by this paper's contributions in Section 4.3. In Section 4.4 we present a customer purchase decision model and formulate the discount pricing problem. Our computational analysis and insights are reported in Section 4.5. Finally, in Section 4.6 we summarize our findings and propose some future directions for research. The proofs of all results are provided in appendix 4.6 and a list of notations is provided in appendix 4.6.

4.2 Related Literature

In this section we will discuss the most relevant literature. The literature on customer choice and product recommendation systems are discussed in Section 4.2.1. Our customer purchase decision models are mainly inspired by this literature. We will then present the literature on location-based pricing and shipping fees, price discounts, and reference prices in Sections 4.2.2, 4.2.3, and 4.2.4, respectively. We will see that some studies consider using discount coupons as ways of price discrimination. However, there are no studies considering customer ratings and shipping costs for optimizing individualized discounts in a multi-product online market environment.

4.2.1 Customer Choice and Product Recommendation Systems

Customer choice models are used to estimate customers' willingness to pay and product preference. The literature in this section is the inspiration for our customer choice model.

Some studies have investigated customers' screening rules and probabilities. Bucklin and Lattin [5] introduce a probabilistic model for purchase incident and brand choice. Gilbride and Allenby [9] model customers' decision making as a two stage process. In the first stage a "consideration set" is chosen from all available options. Customers' then make a final purchase decision from this set; i.e., if a product is not included in the consideration set at the first stage, it has zero probability of being selected. They derive empirical results from an experiment on customers' choice of cameras. They find evidence of a conjunctive screening rule; that is, if an item does not satisfy requirements of a set of attributes, it will no be included in the consideration set. They also conclude that customers use well known attributes as opposed to new attributes in their screening process. Similarly, Wang et al. [34] study customers' purchase process from a "Consider-then-Choose" point of view. They explain that

$$\mathrm{PP} = \mathrm{CP} \times \mathrm{ChP} \tag{4.2.1}$$

where PP is the probability that a customer purchases a product after inspecting it (Purchase Probability), CP is the probability that a product is selected in the consideration set after being inspected (Consideration Probability), and ChP is the probability that a product from customer's consideration set is chosen for purchase (Choice Probability). Their focus is on estimating the CPs. In our study we present a purchase probability model for products already in the "consideration set".

Lachaab et al. [16] model evolution of customers preferences using Bayesian state space models. They show that customers preferences not only differ across customers, it also changes over time. In particular they find that customers become more price sensitive over time. They explain that this effect may be due to frequent price promotions that reduce customers' reference price.

Built on customer choice models, there is a vast amount of research on product recommendation systems. Lopes and Roy [21] explain the importance of recommendation systems in the ever growing e-commerce platforms. They introduce a product recommendation system that relies on customers "click-stream" data to increase customers satisfaction. Zhaao et al. [37] use a training data set from a small set of products using a biding and lottery experiment and machine learning to predict customers' willingness to pay (WTP) for other products. They then incorporate this in a product recommendation mechanism that includes personalized price discounts based on customers' WTP. Scholz et al. [30] introduce a new product recommendation system where unlike other systems it does not assume well-trained customers who are willing to spend time and effort in their selection process. Louca et al. [22] propose a recommendation system model for e-commerce platforms to maximize the sum of the probability of purchase and revenue.

4.2.2 Location-Based Pricing and Shipping Fees

Hotelling [12] was among the first to consider location based pricing. Many have studied variations and applications of Hotelling's model. For example, Anderson [2] studied a generalization of the Hotelling model of spatial competition with quadratic transportation costs. Hernandez [11] studied the impact of spatial differentiation on relative prices in a duopoly market where both firms offer two products of high and low quality. Others have studied retailers' location selection decisions. For example, Ledered and Hurter [18] study the problem of setting location and price schedules. Zhu and Singh [38] study the factors affecting entry and location selection decisions of Wal-Mart, Kmart, and Target. Among their conclusions, they find that the majority of these firm's profits are form populations that are closer to their stores. In addition, they found that these retailers have negative effects on each other when they are in close proximity.

There has been many studies considering the effects of free-shipping policies on customer purchase decisions and ultimately on revenues. Lewis et al. [20] study the impact of shipping fees on customer purchasing behaviour. They find that while shipping fees significantly affect order frequency and amount, customer responses to shipping fees are heterogeneous; i.e., the market can be divided in different segments each with different response levels. They also conclude that free shipping when offered to all orders, increases order frequency but leads to smaller order sizes. When free shipping is offered for orders larger than a threshold amount, it has minimal effects on order frequency but leads to larger orders. Leng et al. [19] investigate when shipping fee promotions improves profits and derive several managerial insights for monopoly and duopoly market structures. They also note that shipping strategies that offer free shipping for orders larger than a threshold result in higher shipping costs absorbed by the retailer. Amazon.com introduced such a shipping policy in January 2002 and has gradually reduced the threshold amount until 2006. As a result, its shipping costs (as a percentage of sales) have increased from 0.61% in 2001 to 2.96% in 2006 [19]. Ma [23] investigated the effects of delivery times and shipping charges on customers' satisfaction and purchase intention. They concluded that delivery time does not have a large impact on customer satisfaction but it has a significant impact on purchase intentions. They also noted that customers are willing to pay a premium for quick delivery. Lastly, they showed that when shipping is free, customers do not differentiate between lengthy or medium delivery time, i.e., they only notice fast or lengthy delivery.

Although some of the studies mentioned above consider customers' location in deriving optimal pricing strategies, they are different from our study in two major aspects. First, these studies consider a single price for the product for all customers where in our study, we consider personalized price discounts for customers. Second, these studies consider the transportation costs to be absorbed by customers and therefore affect demand by changing the total cost to customers. In this paper, we assume a free-shipping policy for all products, and thus, all transportation costs are absorbed by the retailer and does not affect demand (for example, consider free prime shipping on Amazon).

4.2.3 Price Discounts

The main research in product discounts can be divided in three categories. Some have studied advance purchase discounts. Gale and Holmes [8] study the optimal pricing policy for a monopolistic airline. They show that if capacity constraints are present, the monopolist must divert peak period demand to an off peak period by offering advance purchase discounts. Möller and Watanabe [26] study early price discount and late clearance sales for a monopoly that faces a market with heterogeneous customers and individual demand uncertainty. They find conditions under which each or a combination of these strategies are optimal. Nocke et al. [29] investigate the monopolist's optimal pricing strategy where capacity constraints are not present. They present necessary and sufficient condition under which advance purchase discounting is an optimal strategy. They argue that advance purchase discounts can be used as a tool for price discrimination between customers.

Others have studied quantity discounts. For example, Dolan [7] studies motivations for quantity discounts and provides guidelines for quantity discount schedules. Munson and Hu [27] provide procedures for all-units and incremental quantity discount schedules under different strategic purchasing configurations. Viswanathan and Qinan [33] study quantity and volume discounts and develop methods for determining optimal discount policies in a single-supplier, single-retailer (buyer) environment. Here the volume discounts are based on the volume of retailer's total annual orders.

Yang [36] studied the optimal quantity discount, pricing, and ordering policy for a perishable good where demand is price sensitive.

The last category addresses product bundle discounts. Sheng et al. [31] study the effects of discounts on the discounted product in bundles. They show that these types of discounts negatively affect customers' evaluations of the discounted product. They also show that the magnitude of this effect is reduced by increasing the complementarity of products. Janiszewski and Cunha [13] show that the customer evaluation of price discounts in bundled products depend on which product in the bundle is being discounted. Johnson et al. [14] study the the amount and timing of discount coupons tailored by households using Bayesian estimation for household preference parameters.

4.2.4 Reference Prices

Reference price, in marketing literature, is defined as the price customers use as benchmark to compare with observed prices for making purchase decisions. There are many studies in the literature that investigate various aspects and applications of reference prices (e.g., see [4, 15, 17, 25]). Mazumdar et al. present a review of reference price research [25] and model reference price in period t as the weighted average of the price and reference price in period $t-1$. Briesch et al. [4] studied different models of household reference prices and concluded that different product categories require different reference price models. Wang [35] studied different formulations of reference prices and the corresponding optimal pricing policies in a multi-product setting. Anderson et al. [1] have considered price optimization problem in a duopoly market in presence of reference prices.

As outlined above, the effects of customer choice, free-shipping policies, price discounts, and reference prices have been studied extensively. It also noted in studies such as [28] that price discount coupons can be used as an efficient tool for price discrimination. To the best of our knowledge there are no studies that explicitly study personalised product discount optimization in a multinomial choice model utilizing customer locations, in the form of shipping costs, and product review data.

4.3 Contributions

As discussed in Section 4.2, the current literature consists of studies on discounts, location-based pricing and shipping fees, and customer purchase behaviour. Below, we have listed the main contributions of this paper.

- Linking customers' location to price discounts and customer ratings.
- We introduce a nonlinear programming model that uses multinomial customer choice utility function. In addition, the customers' utility and purchase probabilities from customer choice literature is modified to account for prices, discounts, and ratings.
- We provide exact solutions for two special cases.
- We conduct extensive numerical analysis followed by insights.

In addition, this paper contributes by filling the gap in literature where there are no studies that study personalised price discount optimization in a multinomial choice model utilizing customer locations, in the form of shipping costs, and product review data.

4.4 Problem Statement and Formulation

Assume there is a set of customers I, each considering to purchase one of n substitute products. Let action j be defined as

$$j = \begin{cases} 0 & \text{customer chooses not to purchase any product} \\ 1, \cdots, n & \text{customer chooses to purchase product } j \in \{1, \cdots, n\} \end{cases} \tag{4.4.1}$$

Let $U_{i,j}$ denote the utility that customer i expects to enjoy from action j. Define

$$U_{i,j} = \begin{cases} L_{i,j} - b_i \left(p_j - d_{i,j}\right) + \beta_i \Big(r_j - \left(p_j - d_{i,j}\right)\Big) + \sum\limits_{k \neq j} \lambda_i \left(\eta_j - \eta_k\right) & j \in \{1, \cdots, n\} \\ 0 & j = 0 \end{cases} \tag{4.4.2}$$

where p_j, r_j, and η_j denote price, reference price, and rating of product j, respectively, and $L_{i,j}$, b_i, β_i, and λ_i denote customer i's loyalty to product j, price sensitivity, sensitivity to departures from reference price, and ratings' weight, respectively. We assume the customer enjoys a zero utility from not purchasing any of the products; i.e., $U_{i,0} = 0$ for all customers. Note that the utility function is in agreement with common sense and general literature in this area (e.g., [5]). As we can see in (4.4.2), the utility that customer i enjoys from purchasing product j is increasing in the customer's loyalty to the product/brand. Considering the rating terms, the customer expects to enjoy a higher utility from a product that has a higher rating than other substitute products. In addition, the utility decreases as net price increases. Defining "gain" ("loss") as when net price is less (greater) than reference price, customer's utility increases (decreases) in a gain (loss) situation.

Let $\chi_{i,j}$ denote the probability that customer i chooses action j, then consistent with the multinomial choice model (e.g., see [5]), define

$$\chi_{i,j} = \frac{e^{U_{i,j}}}{\sum_{k=0}^{n} e^{U_{i,k}}}. \tag{4.4.3}$$

Let c_j, $s_{i,j}$, and $d_{i,j}$ denote unit cost of product j, shipping cost of product j to customer i, and the amount of price discount for customer i for product j, respectively. Then the expected profits from customer i can be written as

$$\pi_i = \sum_{j=0}^{n} \chi_{i,j} \left(p_j - c_j - d_{i,j} - s_{i,j}\right) \tag{4.4.4}$$

The expected total profits from all customers can then be written as

$$\Pi = \sum_{i} \pi_i \tag{4.4.5}$$

$$= \sum_{i} \sum_{j=0}^{n} \chi_{i,j} \left(p_j - c_j - d_{i,j} - s_{i,j}\right) \tag{4.4.6}$$

We can now formulate the individualized discount optimization problem as

$$\max_{d_{i,j}} \quad \Pi \tag{4.4.7a}$$

s.t.

$$d_{i,j} \le p_j - c_j - s_{i,j} \qquad \forall\, i\,,\, j \tag{4.4.7b}$$

$$d_{i,j} \ge 0 \qquad \forall\, i\,,\, j \tag{4.4.7c}$$

where constraints (4.4.7b) ensure non-negative profits from all customers.

In Section 4.4.1 we will present solution methods for 4.4.7 for a case where there is one product and two customers in the system. We will then expand this case in Section 4.4.2 where there are two products and two customers in the system and provide numerical analysis in Section 4.5.

4.4.1 Case 1: One Product and Two Customers

In this section we consider the case where there are two customers and one product in the system. The utility that customer i enjoys from taking action j is then defined as

$$U_{i,j} = \begin{cases} L_{i,j} - b_i\left(p_j - d_{i,j}\right) + \beta_i\Big(r_j - \left(p_j - d_{i,j}\right)\Big) + \lambda_i \eta_j & \text{for } j = 1 \\ 0 & \text{o.w.} \end{cases} \tag{4.4.8}$$

The probability that customer i chooses action j can be written as

$$\chi_{i,j} = \frac{\exp\left\{U_{i,j}\right\}}{\sum\limits_{k=1}^{n} \exp\left\{U_{i,k}\right\}} \tag{4.4.9}$$

$$= \begin{cases} \dfrac{\exp\left\{L_{i,j} - b_i\left(p_j - d_{i,j}\right) + \beta_i\Big(r_j - \left(p_j - d_{i,j}\right)\Big) + \lambda_i \eta_j\right\}}{1 + \exp\left\{L_{i,j} - b_i\left(p_j - d_{i,j}\right) + \beta_i\Big(r_j - \left(p_j - d_{i,j}\right)\Big) + \lambda_i \eta_j\right\}} & \text{for } j = 1 \\ \dfrac{1}{1 + \exp\left\{L_{i,j} - b_i\left(p_j - d_{i,j}\right) + \beta_i\Big(r_j - \left(p_j - d_{i,j}\right)\Big) + \lambda_i \eta_j\right\}} & \text{for } j = 0 \end{cases} \tag{4.4.10}$$

Proposition 4.4.1. *Problem (4.4.7) is a convex maximization problem for one-product, two-customers case (maximization problem with concave objective function and linear constraints).*

Proposition 4.4.1 allows as to solve problem (4.4.7) using Karush–Kuhn–Tucker (KKT) conditions. To that end we propose Algorithm 4.4.1 to solve problem (4.4.7).

Algorithm 4.4.1. *Let $LB = -\infty$ and iteration counter $k = 1$. For each feasible KKT condition for problem (4.4.7):*

1. *Solve stationarity and complementary slackness conditions for primal and dual variables.*

2. *Check if primal and dual feasibility conditions hold.*

3. *If all conditions are satisfied, find the corresponding objective value Π_k.*

4. *If $LB < \Pi$, let $d^*_{i,j} = d_{i,j}$ and $LB = \Pi$.*

5. *Let $k = k + 1$, move to the next feasible KKT case and follow the steps above until all cases are considered. The optimal solution is $d^*_{i,j}$ with the objective value LB.*

The validity of Algorithm 4.4.1 is established in Theorem 4.4.1.

Theorem 4.4.1. *Algorithm 4.4.1 finds an optimal solution to problem (4.4.7).*

We note that at most there will be 16 KKT solutions and we show in the proof in Appendix A that there are only 9 feasible KKT solutions.

4.4.2 Case 2: Two Products and Two Customers

In this section we consider the case where there are two customers and two products in the system. The utility that customer $i \in \{1, 2\}$ enjoys from taking action j is then defined as

$$U_{i,j} = \begin{cases} L_{i,j} - b_i\left(p_j - d_{i,j}\right) + \beta_i\Big(r_j - \left(p_j - d_{i,j}\right)\Big) + \lambda_i\left(\eta_j - \eta_{3-j}\right) & \text{for } j \in \{1,2\} \\ 0 & \text{o.w.} \end{cases} \tag{4.4.11}$$

Let

$$\widetilde{\mathrm{NP}}_i = \max\left\{p_1 - c_1 - s_{i,1}, p_2 - c_2 - s_{i,2}\right\} \qquad i \in \{1,2\}. \tag{4.4.12}$$

The quantity $\widetilde{\mathrm{NP}}_i$ can be thought of as the maximum marginal profit for product i.

In Proposition 4.4.2 we establish conditions for the convexity of problem (4.4.7).

Proposition 4.4.2. *For a two-product two-customer case problem (4.4.7) is a convex maximization problem if*

$$\max\left\{\left(b_i + \beta_i\right)\widetilde{NP}_i \middle| i \in 1,2\right\} \leq 2. \tag{4.4.13}$$

One interpretation of condition (4.4.13), is that it applied to products that have low profit margins such as in the grocery industry. However, we note that condition (4.4.13) is not necessary for convexity, i.e., problem (4.4.7) may be convex even if condition (4.4.13) is violated.

Proposition 4.4.1 allows as to solve problem (4.4.7) using KKT conditions. To that end we propose Algorithm 4.4.2 for solving problem (4.4.7).

Algorithm 4.4.2. *Let $LB_i = -\infty$ and iteration counter $k_i = 1$ or $i \in \{1,2\}$. For each feasible KKT condition for problem (4.4.7):*

1. *Solve stationarity and complementary slackness conditions for primal and dual variables.*

2. *Check if primal and dual feasibility conditions hold*

3. *If all conditions are satisfied, find the corresponding objective value* $\Pi_i^{(k)}$

4. *If* $LB_i < \Pi_i^{(k)}$*, let* $d^*_{i,j} = d_{i,j}$ *and* $LB_i = \Pi_i^{(k)}$ *The optimal solution is* $d^*_{i,j}$ *with the objective value* LB_i*.*

The validity of Algorithm 4.4.2 is established in Theorem 4.4.2.

Theorem 4.4.2. *Algorithm 4.4.2 finds an optimal solution to problem* (4.4.7)*.*

We note that at most there will be 16 KKT solutions and we show in the proof in Appendix A that there are only 9 feasible KKT solutions.

4.5 Numerical Results and Insights

For the purpose of numerical experiments we consider the two-customers two-products case discussed in Section 4.4.2. We use 1,927,680 data points defined by combinations of the parameter sets shown in Table 4.1.

Note that for the purposes of numerical experiments, the customer ratings for product one is equal or higher than that of product two; i.e., for all parameter set combinations $\eta_1 \geq \eta_2$. In Table 4.2 we present the breakdown of the proportion of times that customers receive discounts. The two most interesting observations are that the product with lower rating is discounted more frequently and that the two products were never discounted simultaneously for a customer. In Sections 4.5.1 and 4.5.2 we will discuss the effects of parameters on profits and discount values,

$L_{i,j}$	5.00	10.00	15.00	20.00
$s_{i,j}$	0.50	0.75	1.00	
b_i	0.40	0.50	0.60	0.70
β_i	0.20			
λ_i	0.50	0.75	1.00	
p_j	3.00	5.00		
r_j	4.00			
η_j	1.00	3.00	5.00	
c	2.00			

Table 4.1: Parameter Values

respectively. In Section 4.6 we will present managerial implications of our analysis and suggestions for future research.

Proportion of times a customer received a discount	33.7 %
Proportion of times customer one received a discount	24.3 %
Proportion of times customer 1 received a discount for product 1	8.9 %
Proportion of times customer 1 received a discount for product 2	15.4 %
Proportion of times customer 1 received a discount for both products	0.0 %

Table 4.2: Proportion of discount offerings.

4.5.1 Effect of Parameters on Profits

In this section we discuss the effect of parameters on profits. More specifically, we will discuss the results from four multiple linear regressions where total profit, profit from customer 1, profits from selling product 1 to customer 1, and profit from selling product 2 to customer 1 are the dependent variables.

The results of multiple linear regression where total profits and total profit from customer 1 are the dependent variable are shown in Tables 4.3 and 4.4, respectively.

As expected, the shipping costs $s_{i,j}$ have negative impact on profits. The coefficients of customer brand loyalty parameters $L_{i,j}$ show that profits increase when customers have higher loyalty to the product with the higher rating and lower loyalty to the product with the lower rating. In addition, from the coefficients of product ratings η_j, it is suggested that increasing the difference between product ratings increases profits.

Although the objective is to maximize total profits, it is beneficial to look at profits from each product separately. In Tables 4.5 and 4.6 we show the results of multiple linear regression where profits from selling product 1 and product 2 to customer 1, respectively. As we expect, shipping costs have negative impact on profits. The coefficients of customer loyalty parameters L and product ratings η have opposite signs for the two products; i.e., increasing the customer loyalty parameter for the product with a higher rating and decreasing it for the product with a lower rating, increases the profits from the product with a higher rating and decreases the profits from the product with a lower rating. A similar pattern is observed for customer ratings. More specifically, when the difference between product ratings increases, the profits from the product with a higher rating increases; conversely, increasing the difference between product ratings decreases the profits from the product with a lower rating. It is also worth mentioning that the coefficient of the shipping costs for product one $s_{1,1}$ seem to have no effect on the profit from product 2 (Table 4.6).

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-4.1027	0.0068	-602.85	$< 2e-16$
$L_{1,1}$	0.0017	0.0001	15.12	$< 2e-16$
$L_{1,2}$	-0.0006	0.0001	-5.67	$1.44e-08$
$L_{2,1}$	0.0017	0.0001	15.13	$< 2e-16$
$L_{2,2}$	-0.0006	0.0001	-5.65	$1.65e-08$
$s_{1,1}$	-0.6585	0.0030	-220.80	$< 2e-16$
$s_{1,2}$	-0.3908	0.0030	-131.05	$< 2e-16$
$s_{2,1}$	-0.6561	0.0030	-220.01	$< 2e-16$
$s_{2,2}$	-0.3875	0.0030	-129.94	$< 2e-16$
p_1	1.2367	0.0006	2003.71	$< 2e-16$
p_2	0.7128	0.0006	1154.96	$< 2e-16$
η_1	0.0597	0.0005	126.00	$< 2e-16$
η_2	-0.0597	0.0005	-126.00	$< 2e-16$
λ_1	0.0395	0.0027	14.44	$< 2e-16$
λ_2	0.0395	0.0027	14.45	$< 2e-16$

Residual standard error: 0.8492 on $1,927,665$ degrees of freedom

Multiple R-squared: 0.749, Adjusted R-squared: 0.749

F-statistic: $4.11e+05$ on 14 and 1927665 DF, p-value: $< 2.2e-16$

Table 4.3: Regression results where the total profit (Π) is the dependent variable.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-1.5196	0.0038	-400.73	$< 2e-16$
$L_{1,1}$	0.0466	0.0001	746.38	$< 2e-16$
$L_{1,2}$	-0.0459	0.0001	-734.01	$2.13e-15$
$s_{1,1}$	-0.6444	0.0017	-382.74	$< 2e-16$
$s_{1,2}$	-0.0055	0.0017	-3.30	$< 2e-16$
b_1	-0.0119	0.0035	-3.39	$8.11e-09$
λ_1	0.1265	0.0015	82.04	$< 2e-16$
p_1	0.5614	0.0004	1591.72	$< 2e-16$
p_2	0.0616	0.0004	174.95	$< 2e-16$
η_1	0.1180	0.0003	442.06	$< 2e-16$
η_2	-0.1180	0.0003	-442.06	$< 2e-16$

Residual standard error: 0.5567 on $1,927,669$ degrees of freedom

Multiple R-squared: 0.6384, Adjusted R-squared: 0.6384

F-statistic: $3.403e+05$ on 10 and $1,927,669$ DF, p-value: $< 2.2e-16$

Table 4.4: Regression results where the total profit from customer 1 (π_1) is the dependent variable.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-1.5196	0.0038	-400.73	$< 2e-16$
$L_{1,1}$	0.0466	0.0001	746.38	$< 2e-16$
$L_{1,2}$	-0.0459	0.0001	-734.01	$< 2e-16$
$s_{1,1}$	-0.6444	0.0017	-382.74	$< 2e-16$
$s_{1,2}$	-0.0055	0.0017	-3.30	0.000981
b_1	-0.0119	0.0035	-3.39	0.000704
λ_1	0.1265	0.0015	82.04	$< 2e-16$
p_1	0.5614	0.0004	1591.72	$< 2e-16$
p_2	0.0616	0.0004	174.95	$< 2e-16$
η_1	0.1180	0.0003	442.06	$< 2e-16$
η_2	-0.1180	0.0003	-442.06	$< 2e-16$

Residual standard error: 0.4786 on $1,927,669$ degrees of freedom

Multiple R-squared: 0.6752, Adjusted R-squared: 0.6752

F-statistic: $4.006e + 05$ on 10 and $1,927,669$ DF, p-value: $< 2.2e - 16$

Table 4.5: Regression results where the profit from customer 1, product 1 ($\pi_{1,1}$) is the dependent variable.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-0.5369	0.0034	-156.32	$< 2e-16$
$L_{1,1}$	-0.0449	0.0001	-757.55	$< 2e-16$
$L_{1,2}$	0.0453	0.0001	764.59	$< 2e-16$
$s_{1,2}$	-0.3710	0.0016	-232.73	$< 2e-16$
b_1	-0.0119	0.0033	-3.57	0.000354
λ_1	-0.0870	0.0015	-59.52	$< 2e-16$
p_1	0.0566	0.0003	169.58	$< 2e-16$
p_2	0.2942	0.0003	881.43	$< 2e-16$
η_1	-0.0882	0.0003	-348.41	$< 2e-16$
η_2	0.0882	0.0003	348.41	$< 2e-16$

Residual standard error: 0.4537 on $1,927,670$ degrees of freedom

Multiple R-squared: 0.5178, Adjusted R-squared: 0.5178

F-statistic: $2.3e+05$ on 9 and $1,927,670$ DF, p-value: $< 2.2e-16$

Table 4.6: Regression results where the profit from customer 1, product 2 ($\pi_{1,2}$) is the dependent variable.

4.5.2 Effect of Parameters on Discounts

In this section we present the effects of parameters on optimal discount values. In Tables 4.7 and 4.8 we show the results from multiple linear regression where the discount provided to customer 1 for product 1 ($d_{1,1}$) and the discount provided to customer 1 for product 2 ($d_{1,2}$) are dependent variables.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-0.0874	0.0011	-79.39	$< 2e - 16$
$L_{1,1}$	-0.0053	0.0000	-292.05	$< 2e - 16$
$L_{1,2}$	0.0053	0.0000	289.36	$< 2e - 16$
$s_{1,1}$	-0.0649	0.0005	-132.72	$< 2e - 16$
$s_{1,2}$	0.0952	0.0005	194.89	$< 2e - 16$
b_1	0.1434	0.0010	140.23	$< 2e - 16$
λ_1	-0.0065	0.0004	-14.56	$< 2e - 16$
p_1	0.0519	0.0001	506.38	$< 2e - 16$
p_2	-0.0383	0.0001	-374.26	$< 2e - 16$
η_1	-0.0109	0.0001	-140.63	$< 2e - 16$
η_2	0.0109	0.0001	140.63	$< 2e - 16$

Residual standard error: 0.139 on $1,927,669$ degrees of freedom

Multiple R-squared: 0.2468, Adjusted R-squared: 0.2468

F-statistic: $6.318e + 04$ on 10 and $1,927,669$ DF, p-value: $< 2.2e - 16$

Table 4.7: Multiple linear regression results where the discount provided to customer 1 for product 1 ($d_{1,1}$) is the dependent variable.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-0.1671	0.0014	-119.26	$< 2e-16$
$L_{1,1}$	0.0054	0.0000	232.69	$< 2e-16$
$L_{1,2}$	-0.0054	0.0000	-235.28	$< 2e-16$
$s_{1,1}$	0.1449	0.0006	233.05	$< 2e-16$
$s_{1,2}$	-0.1423	0.0006	-229.04	$< 2e-16$
b_1	0.2382	0.0013	183.15	$< 2e-16$
λ_1	0.0105	0.0006	18.38	$< 2e-16$
p_1	-0.0723	0.0001	-554.92	$< 2e-16$
p_2	0.0969	0.0001	745.11	$< 2e-16$
η_1	0.0137	0.0001	139.07	$< 2e-16$
η_2	-0.0137	0.0001	-139.07	$< 2e-16$
Residual standard error: 0.1768 on $1,927,669$ degrees of freedom				
Multiple R-squared: 0.3622, Adjusted R-squared: 0.3622				
F-statistic: $1.095e+05$ on 10 and $1,927,669$ DF, p-value: $< 2.2e-16$				

Table 4.8: Multiple linear regression results where the discount provided to customer 1 for product 2 ($d_{1,2}$) is the dependent variable.

The effect of customer loyalty and product rating are similar in sign; that is, increasing customer loyalty to a product and/or product rating, decreases the optimal discount value for the product and increases the discounts for the other (competing) product. Similar effect is observed with shipping costs. We also observe that increasing price of a product, increases the discount for that product and decreases the discount for the other (competing) product. It is also worth noting that the customer price sensitivity parameter b has a positive effect on discounts.

4.6 Conclusions and Future Research

As discussed in Section 4.1, online shopping is becoming increasingly popular and the growth in mobile technology has made it easy and cheap to adjust prices frequently, target marketing messages, and personalize discounts. In addition, we are witnessing a proliferation of discount codes through retailers and third party websites such as Wikibuy, RetailMeNot, and Slickdeals. We presented the four main research areas that are relevant to this paper in Section 4.2, namely, customer choice and product recommendation systems, location based pricing and shipping fees, price discounts, and reference prices. We find that there is a gap in the literature on studies that consider a personalized discount pricing optimization as a tool for price discrimination taking into account customer location and product ratings in a multinomial choice model framework. In our study we assumed a free-shipping policy where all shipping costs are absorbed by the retailer.

In Section 4.4 we presented the problem statement and formulation together with closed form solutions for special cases. We used 1,927,680 data points for numerical analysis in Section 4.5. The following are the summary of our findings and managerial implications:

- The product with lower rating is discounted more frequently than the product with higher rating.
- The two products were never discounted simultaneously for a customer.
- As expected, the shipping costs $s_{i,j}$'s have negative impact on profits.
- Profits increase when customers have higher loyalty to the product with the higher rating and lower loyalty to the product with the lower rating; i.e., the

larger gap between product preference. Managers may consider targeted marketing to achieve this larger gap.

- Increasing the difference between product ratings increases profits. This can be achieved by adjusting default product review sorting that biases the average ratings.

- The optimal discount value for a product-customer pair decreases as customer's loyalty to the product and the competing product increases and decreases, respectively. Similar effect is observed with shipping costs.

- The optimal discount value for a product-customer pair increase as the product price and the competing product price increase and decreases, respectively.

Our work can be extended in several ways. First, the work can be expanded to a multi-period horizon where reference prices and product ratings change as a response to net prices. Second, inventory holding and ordering costs, lead time, and the costs associated with loss of market share as a result of shortages can be added in future research. Third, in this paper, we considered a one-retailer case. This study can be extended to a duopoly market.

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Appendix A: Proofs

Proof of Proposition 4.4.1. Let J denote the Jacobian matrix defined as

$$J = \begin{pmatrix} \frac{\partial^2 \Pi}{\partial d_{1,1}^2} & \frac{\partial^2 \Pi}{\partial d_{1,1} \partial d_{2,1}} \\ \frac{\partial^2 \Pi}{\partial d_{2,1} \partial d_{1,1}} & \frac{\partial^2 \Pi}{\partial d_{2,1}^2} \end{pmatrix} \tag{4..1}$$

We have

$$J_{1,1} = \frac{-\left(b_1 + \beta_1\right)}{\left(\exp\left\{U_{1,1}\right\} + 1\right)^3 \left(\exp\left\{U_{2,1}\right\} + 1\right)} \Big[\left(2 + \left(b_1 + \beta_1\right) NP_1\right) \exp\left\{2U_{1,1} + U_{2,1}\right\} \tag{4..2}$$

$$+ \left(2 - \left(b_1 + \beta_1\right) NP_1\right) \exp\left\{U_{1,1} + U_{2,1}\right\} + \left(2 + \left(b_1 + \beta_1\right) NP_1\right) \exp\left\{2U_{1,1}\right\}$$

$$+ \left(2 - \left(b_1 + \beta_1\right) NP_1\right) \exp\left\{U_{1,1}\right\}\Big]$$

$$J_{1,2} = 0 \tag{4..3}$$

$$J_{2,1} = 0 \tag{4..4}$$

$$J_{2,2} = \frac{-\left(b_2 + \beta_2\right)}{\left(\exp\left\{U_{2,1}\right\} + 1\right)^3 \left(\exp\left\{U_{1,1}\right\} + 1\right)} \Big[\left(2 + \left(b_2 + \beta_2\right) NP_2\right) \exp\left\{2U_{2,1} + U_{1,1}\right\} \tag{4..5}$$

$$+ \left(2 - \left(b_2 + \beta_2\right) NP_2\right) \exp\left\{U_{2,1} + U_{1,1}\right\} + \left(2 + \left(b_2 + \beta_2\right) NP_2\right) \exp\left\{2U_{2,1}\right\}$$

$$+ \left(2 - \left(b_2 + \beta_2\right) NP_2\right) \exp\left\{U_{2,1}\right\}\Big]$$

where NP_i is the unit net profit from selling the product to customer i; i.e.,

$$NP_i = p_1 - d_{i,1} - s_{i,1} - c_1 \tag{4..6}$$

For problem (4.4.7) to be a convex problem, matrix J must be a negative semi-definite matrix. Since J is a diagonal matrix, we must show that the diagonal entries of J are non-positive. The term $\frac{-(b_1+\beta_1)}{\left(\exp\{U_{1,1}\}+1\right)^3\left(\exp\{U_{2,1}\}+1\right)}$ in (4..3) is negative. We must then show

$$J_{1,1} \leq 0 \tag{4..7}$$

$$\Leftrightarrow \left(2 + (b_1 + \beta_1)\, NP_1\right) \exp\left\{2U_{1,1} + U_{2,1}\right\} + \left(2 - (b_1 + \beta_1)\, NP_1\right) \exp\left\{U_{1,1} + U_{2,1}\right\} \tag{4..8}$$

$$+ \left(2 + (b_1 + \beta_1)\, NP_1\right) \exp\left\{2U_{1,1}\right\} + \left(2 - (b_1 + \beta_1)\, NP_1\right) \exp\left\{U_{1,1}\right\} \geq 0$$

$$\Leftrightarrow \left(2 + (b_1 + \beta_1)\, NP_1\right) \left[\exp\left\{2U_{1,1} + U_{2,1}\right\} + \exp\left\{2U_{1,1}\right\}\right] \tag{4..9}$$

$$+ \left(2 - (b_1 + \beta_1)\, NP_1\right) \left[\exp\left\{U_{1,1} + U_{2,1}\right\} + \exp\left\{U_{1,1}\right\}\right] \geq 0$$

$$\Leftrightarrow \left(2 + (b_1 + \beta_1)\, NP_1\right) \exp\left\{2U_{1,1}\right\} \left(1 + \exp\left\{U_{2,1}\right\}\right) \tag{4..10}$$

$$+ \left(2 - (b_1 + \beta_1)\, NP_1\right) \exp\left\{U_{1,1}\right\} \left(1 + \exp\left\{U_{2,1}\right\}\right) \geq 0$$

$$\Leftrightarrow \left(2 + (b_1 + \beta_1)\, NP_1\right) \exp\left\{2U_{1,1}\right\} + \left(2 - (b_1 + \beta_1)\, NP_1\right) \exp\left\{U_{1,1}\right\} \geq 0 \tag{4..11}$$

$$\Leftrightarrow \left(2 + (b_1 + \beta_1)\, NP_1\right) \exp\left\{U_{1,1}\right\} + 2 - (b_1 + \beta_1)\, NP_1 \geq 0 \tag{4..12}$$

$$\Leftrightarrow 2\left(1 + \exp\left\{U_{1,1}\right\}\right) + (b_1 + \beta_1)\, NP_1 \left(\exp\left\{U_{1,1}\right\} - 1\right) \geq 0 \tag{4..13}$$

Since $U_{1,1} \geq 0$, then

$$\exp\left\{U_{1,1}\right\} - 1 \geq 0 \tag{4..14}$$

$$\Rightarrow 2\left(1 + \exp\left\{U_{1,1}\right\}\right) + (b_1 + \beta_1)\, NP_1 \left(\exp\left\{U_{1,1}\right\} - 1\right) \geq 0 \tag{4..15}$$

$$\Rightarrow J_{1,1} \leq 0 \tag{4..16}$$

Similarly, we can show that $J_{2,2} \leq 0$, and the results follow. □

Proof of Theorem 4.4.1. The following are the KKT conditions used to solve problem (4.4.7)

Primal Feasibility

$$d_{i,j} \leq p_j - c_j - s_{i,j} \qquad \forall\, i \in \{1,2\}\,,\, j=1 \tag{4..17a}$$

$$d_{i,j} \geq 0 \qquad \forall\, i \in \{1,2\}\,,\, j=1 \tag{4..17b}$$

Dual Feasibility

$$u_{i,j} \geq 0 \qquad \forall\, i \in \{1,2\}\,,\, j=1 \tag{4..17c}$$

$$v_{i,j} \geq 0 \qquad \forall\, i \in \{1,2\}\,,\, j=1 \tag{4..17d}$$

Stationarity

$$\frac{\partial \Pi}{\partial d_{i,j}} - u_{i,j} + v_{i,j} = 0 \qquad \forall\, i \in \{1,2\}\,,\, j=1 \tag{4..17e}$$

Complementary Slackness

$$u_{i,j}\left[d_{i,j} - \big(p_j - c_j - s_{i,j}\big)\right] = 0 \qquad \forall\, i \in \{1,2\}\,,\, j=1 \tag{4..17f}$$

$$v_{i,j} d_{i,j} = 0 \qquad \forall\, i \in \{1,2\}\,,\, j=1 \tag{4..17g}$$

We will use the following Lemma to prove the theorem.

Lemma 4..1. *If $u_{i,j} > 0$ and $v_{i,j} > 0$ for any $i \in \{1,2\}$ and $j = 1$, the KKT conditions (4..17f) - (4..17g) are infeasible.*

Proof of Lemma 4..1 is as follows:

For a pair (i, j), let $u_{i,j} > 0$ and $v_{i,j} > 0$. From complementary slackness conditions (4..17f)-(4..17g), we must have

$$u_{i,j}\left[d_{i,j} - \left(p_j - c_j - s_{i,j}\right)\right] = 0 \qquad (4..18)$$

$$v_{i,j}d_{i,j} = 0 \qquad (4..19)$$

Since $u_{i,j} > 0$ and $v_{i,j} > 0$, we must therefore have

$$d_{i,j} = p_j - c_j - s_{i,j} \qquad (4..20)$$

$$d_{i,j} = 0 \qquad (4..21)$$

This is in contradiction with the assumption that $p_j - c_j - s_{i,j} > 0$ for all (i, j) and the results from Lemma 4..1 follows.

From Lemma 4..1 we will show that there are nine possible feasible cases with closed form solutions for KKT conditions (4..17e)-(4..17g). We know that there are the following three possible cases for a given pair (i, j)

- $u_{i,j} = 0$ and $v_{i,j} = 0$
- $u_{i,j} > 0$ and $v_{i,j} = 0$
- $u_{i,j} = 0$ and $v_{i,j} > 0$

Since $j \in \{1, 2\}$ for $i = 1$, we have nine feasible solutions with the following closed form solutions. The solutions are verified by direct substitution.

Case 1: $u_{1,1} = 0$, $u_{2,1} = 0$, $v_{1,1} = 0$, $v_{2,1} = 0$

$$d_{1,1} = \frac{1}{\beta_1 + b_1}\Big[(p_1 - c - s_1)(b_1 + \beta_1) - 1 \tag{4..22}$$
$$- \text{LambertW}\Big(\exp\big\{(r_1 - c - s_1)\beta_1 + (-c - s_1) b_1 + \lambda_1\eta_1 + L_1 - 1\big\}\Big)\Big]$$

$$d_{2,1} = \frac{1}{\beta_2 + b_2}\Big[(p_1 - c - s_2)(b_2 + \beta_2) - 1 \tag{4..23}$$
$$- \text{LambertW}\Big(\exp\big\{(r_1 - c - s_2)\beta_2 + (-c - s_2) b_2 + \lambda_2\eta_1 + L_2 - 1\big\}\Big)\Big]$$

Case 2: $u_{1,1} = 0$, $u_{2,1} = 0$, $v_{1,1} = 0$, $v_{2,1} > 0$

$$d_{1,1} = \frac{1}{\beta_1 + b_1}\Big[(p_1 - c - s_1)(b_1 + \beta_1) - 1 \tag{4..24}$$
$$- \text{LambertW}\Big(\exp\big\{(r_1 - c - s_1)\beta_1 + (-c - s_1) b_1 + \lambda_1\eta_1 + L_1 - 1\big\}\Big)\Big]$$

$$d_{2,1} = 0 \tag{4..25}$$

Case 3: $u_{1,1} = 0$, $u_{2,1} = 0$, $v_{1,1} > 0$, $v_{2,1} = 0$

$$d_{1,1} = 0 \tag{4..26}$$

$$d_{2,1} = \frac{1}{\beta_2 + b_2}\Big[(p_1 - c - s_2)(b_2 + \beta_2) - 1 \tag{4..27}$$
$$- \text{LambertW}\Big(\exp\big\{(r_1 - c - s_2)\beta_2 + (-c - s_2) b_2 + \lambda_2\eta_1 + L_2 - 1\big\}\Big)\Big]$$

Case 4: $u_{1,1} = 0$, $u_{2,1} = 0$, $v_{1,1} > 0$, $v_{2,1} > 0$

$$d_{1,1} = 0 \tag{4..28}$$
$$d_{2,1} = 0$$

Case 5: $u_{1,1} = 0$, $u_{2,1} > 0$, $v_{1,1} = 0$, $v_{2,1} = 0$

$$d_{1,1} = \frac{1}{b_1 + \beta_1} \Big[(p_1 - c - s_1)(b_1 + \beta_1) - 1 \tag{4..29}$$

$$-\text{LambertW}\left(\exp\left\{ (r_1 - c - s_1)\beta_1 + (-c - s_1) b_1 + \lambda_1 \eta_1 + L_1 - 1 \right\} \right) \Big]$$

$$d_{2,1} = p_1 - c - s_2 \tag{4..30}$$

Case 6: $u_{1,1} = 0$, $u_{2,1} > 0$, $v_{1,1} > 0$, $v_{2,1} = 0$

$$d_{1,1} = 0 \tag{4..31}$$

$$d_{2,1} = p_1 - c - s_2 \tag{4..32}$$

Case 7: $u_{1,1} > 0$, $u_{2,1} = 0$, $v_{1,1} = 0$, $v_{2,1} = 0$

$$d_{1,1} = p_1 - c - s_1 \tag{4..33}$$

$$d_{2,1} = \frac{1}{b_2 + \beta_2} \Big[(p_1 - c - s_2)(b_2 + \beta_2) - 1 \tag{4..34}$$

$$- \text{LambertW}\left(\exp\left\{ (r_1 - c - s_2)\beta_2 + (-c - s_2) b_2 + \lambda_2 \eta_1 + L_2 - 1 \right\} \right) \Big]$$

Case 8: $u_{1,1} > 0$, $u_{2,1} = 0$, $v_{1,1} = 0$, $v_{2,1} > 0$

$$d_{1,1} = p_1 - c - s_1 \tag{4..35}$$

$$d_{2,1} = 0 \tag{4..36}$$

Case 9: $u_{1,1} > 0$, $u_{2,1} > 0$, $v_{1,1} = 0$, $v_{2,1} = 0$

$$d_{1,1} = p_1 - c - s_1 \tag{4..37}$$

$$d_{2,1} = p_1 - c - s_2 \tag{4..38}$$

□

Proof of Proposition 4.4.2. Let J denote the Jacobian matrix. The Jacobian matrix is of the form

$$J = \begin{pmatrix} J^{(1,1)} & J^{(1,2)} \\ J^{(2,1)} & J^{(2,2)} \end{pmatrix} \tag{4..39}$$

where

$$J^{(1,1)} = \begin{pmatrix} \frac{\partial^2 \Pi}{\partial d_{1,1}^2} & \frac{\partial^2 \Pi}{\partial d_{1,1} \partial d_{2,1}} \\ \frac{\partial^2 \Pi}{\partial d_{2,1} \partial d_{1,1}} & \frac{\partial^2 \Pi}{\partial d_{2,1}^2} \end{pmatrix} \qquad J^{(1,2)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4..40}$$

$$J^{(2,1)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad J^{(1,1)} = \begin{pmatrix} \frac{\partial^2 \Pi}{\partial d_{1,2}^2} & \frac{\partial^2 \Pi}{\partial d_{1,2} \partial d_{2,2}} \\ \frac{\partial^2 \Pi}{\partial d_{2,2} \partial d_{1,2}} & \frac{\partial^2 \Pi}{\partial d_{2,2}^2} \end{pmatrix} \tag{4..41}$$

Let

$$\mathrm{NP}_{i,j} = p_i - c_i - s_{i,j} - d_{i,j} \tag{4..42}$$

$$B_i = b_i + \beta_i \tag{4..43}$$

We have

$$\max\left\{B_i\widetilde{\mathrm{NP}}_i \middle| i\in 1,2\right\}\leq 2 \tag{4..44}$$

$$\Rightarrow B_1\widetilde{\mathrm{NP}}_1\leq 2 \tag{4..45}$$

$$\tag{4..46}$$

Then

$$B_1\mathrm{NP}_{1,1}\Big(\exp\left\{U_{1,2}\right\}-\exp\left\{U_{1,1}\right\}+1\Big)+B_1\mathrm{NP}_{1,2}\Big(\exp\left\{U_{1,1}\right\}-\exp\left\{U_{1,2}\right\}\Big) \tag{4..47}$$

$$\leq B_1\widetilde{\mathrm{NP}}_1\Big(\exp\left\{U_{1,1}\right\}+\exp\left\{U_{1,2}\right\}+1\Big)\leq 2\Big(\exp\left\{U_{1,1}\right\}+\exp\left\{U_{1,2}\right\}+1\Big) \tag{4..48}$$

$$\Rightarrow B_1\mathrm{NP}_{1,1}\Big(\exp\left\{U_{1,2}\right\}+1\Big)^2+B_1\mathrm{NP}_{1,2}\exp\left\{U_{1,1}\right\}\Big(\exp\left\{U_{1,2}\right\}+1\Big) \tag{4..49}$$

$$\leq 2\Big(\exp\left\{U_{1,2}\right\}+1\Big)\Big(\exp\left\{U_{1,1}\right\}+\exp\left\{U_{1,2}\right\}+1\Big) \tag{4..50}$$

$$+\Big(\exp\left\{U_{1,2}\right\}+1\Big)\Big(B_1\mathrm{NP}_{1,1}\exp\left\{U_{1,1}\right\}+B_1\mathrm{NP}_{1,2}\exp\left\{U_{1,2}\right\}\Big) \tag{4..51}$$

$$\Rightarrow B_1\mathrm{NP}_{1,1}\Big(\exp\left\{U_{1,2}\right\}+1\Big)^2+B_1\mathrm{NP}_{1,2}\exp\left\{U_{1,1}\right\}\exp\left\{U_{1,2}\right\} \tag{4..52}$$

$$\leq 2\Big(\exp\left\{U_{1,2}\right\}+1\Big)\Big(\exp\left\{U_{1,1}\right\}+\exp\left\{U_{1,2}\right\}+1\Big) \tag{4..53}$$

$$+\Big(\exp\left\{U_{1,2}\right\}+1\Big)\Big(B_1\mathrm{NP}_{1,1}\exp\left\{U_{1,1}\right\}+B_1\mathrm{NP}_{1,2}\exp\left\{U_{1,2}\right\}\Big) \tag{4..54}$$

$$\Rightarrow(-2)\Big[\exp\left\{2U_{1,2}\right\}+\exp\left\{U_{1,1}+U_{1,2}\right\}+2\exp\left\{U_{1,2}\right\}+\exp\left\{U_{1,1}\right\}+1\Big] \tag{4..55}$$

$$-B_1\mathrm{NP}_{1,1}\Big[\exp\left\{U_{1,1}\right\}\Big(\exp\left\{U_{1,2}\right\}+1\Big)-\exp\left\{U_{1,2}\right\}\Big(\exp\left\{U_{1,2}\right\}+2\Big)-1\Big] \tag{4..56}$$

$$- B_1\mathrm{NP}_{1,2}\exp\left\{U_{1,2}\right\}\left(-\exp\left\{U_{1,1}\right\}+\exp\left\{U_{1,2}\right\}+1\right)\leq 0 \tag{4..57}$$

$$\Rightarrow\left[-2+\left(\mathrm{NP}_{1,1}-\mathrm{NP}_{1,2}\right)B_1\right]\exp\left\{U_{1,1}+2U_{1,2}\right\} \tag{4..58}$$

$$+\left[-2+\left(\mathrm{NP}_{1,2}-\mathrm{NP}_{1,1}\right)B_1\right]\exp\left\{2U_{1,1}+U_{1,2}\right\} \tag{4..59}$$

$$+\left[-4+\left(2\mathrm{NP}_{1,1}-\mathrm{NP}_{1,2}\right)B_1\right]\exp\left\{U_{1,1}+U_{1,2}\right\} \tag{4..60}$$

$$+B_1\mathrm{NP}_{1,1}\left(\exp\left\{U_{1,1}\right\}-\exp\left\{2U_{1,1}\right\}\right)-2\exp\left\{U_{1,1}\right\}-2\exp\left\{2U_{1,1}\right\}\leq 0 \tag{4..61}$$

$$\Rightarrow\frac{B_1}{\left(\exp\left\{U_{1,1}\right\}+\exp\left\{U_{1,2}\right\}+1\right)^3}\Bigg\{\left[-2+\left(\mathrm{NP}_{1,1}-\mathrm{NP}_{1,2}\right)B_1\right]\exp\left\{U_{1,1}+2U_{1,2}\right\} \tag{4..62}$$

$$+\left[-2+\left(\mathrm{NP}_{1,2}-\mathrm{NP}_{1,1}\right)B_1\right]\exp\left\{2U_{1,1}+U_{1,2}\right\} \tag{4..63}$$

$$+\left[-4+\left(2\mathrm{NP}_{1,1}-\mathrm{NP}_{1,2}\right)B_1\right]\exp\left\{U_{1,1}+U_{1,2}\right\} \tag{4..64}$$

$$+B_1\mathrm{NP}_{1,1}\left(\exp\left\{U_{1,1}\right\}-\exp\left\{2U_{1,1}\right\}\right)-2\exp\left\{U_{1,1}\right\}-2\exp\left\{2U_{1,1}\right\}\Bigg\} \tag{4..65}$$

$$=J_{1,1}^{(1,1)}\leq 0 \tag{4..66}$$

Because $U_{i,j}\geq 0$, we have

$$\exp\left\{U_{i,j}\right\}\geq 1 \tag{4..67}$$

This will result in

$$\det\left(J^{(1,1)}\right)>0 \tag{4..68}$$

Therefore, $J^{(1,1)}$ is a negative semi-definite matrix. Similarly, we can show that $J^{(2,2)}$

is also a negative semi-definite matrix. This results in J being a negative semi-definite matrix, and the results follow. □

Proof of Theorem 4.4.2. Below are the KKT conditions we will use to solve problem (4.4.7):

Primal Feasibility

$$d_{i,j} \leq p_j - c_j - s_{i,j} \qquad \forall\, i \in \{1,2\}\,,\, j \in \{1,2\} \tag{4..69a}$$

$$d_{i,j} \geq 0 \qquad \forall\, i \in \{1,2\}\,,\, j \in \{1,2\} \tag{4..69b}$$

Dual Feasibility

$$u_{i,j} \geq 0 \qquad \forall\, i \in \{1,2\}\,,\, j \in \{1,2\} \tag{4..69c}$$

$$v_{i,j} \geq 0 \qquad \forall\, i \in \{1,2\}\,,\, j \in \{1,2\} \tag{4..69d}$$

Stationarity

$$\frac{\partial \Pi}{\partial d_{i,j}} - u_{i,j} + v_{i,j} = 0 \qquad \forall\, i \in \{1,2\}\,,\, j \in \{1,2\} \tag{4..69e}$$

Complementary Slackness

$$u_{i,j}\left[d_{i,j} - \left(p_j - c_j - s_{i,j}\right)\right] = 0 \qquad \forall\, i \in \{1,2\}\,,\, j \in \{1,2\} \tag{4..69f}$$

$$v_{i,j} d_{i,j} = 0 \qquad \forall\, i \in \{1,2\}\,,\, j \in \{1,2\} \tag{4..69g}$$

Since KKT conditions (4..69) are independent when decomposed by i, KKT conditions (4..69) can be decomposed by customer i and solved separately. We will use the following Lemma to prove the theorem.

Lemma 4..2. *If $u_{i,j} > 0$ and $v_{i,j} > 0$ for any $i, j \in \{1, 2\}$, the KKT conditions (4..69f)-(4..69g) are infeasible.*

Proof of Lemma 4..2 is is similar to that of Lemma 4..1.

From Lemma 4..2, we know that there are the following three possible cases for a given pair (i, j)

- $u_{i,j} = 0$ and $v_{i,j} = 0$
- $u_{i,j} > 0$ and $v_{i,j} = 0$
- $u_{i,j} = 0$ and $v_{i,j} > 0$

Since $j \in \{1, 2\}$ for for each value of i, we have nine feasible solutions with the following closed form solutions. The solutions are verified by direct substitution. **Case 1:** $u_{i,1} = 0$, $v_{i,1} = 0$, $u_{i,2} = 0$, $v_{i,2} = 0$

$$
\begin{aligned}
d_{i,1} = \frac{1}{\beta_i + b_i}\Bigg[& \Big(p_1 - s_{i,1} - c_1\Big)\left(b_i + \beta_i\right) - 1 \\
& - \mathrm{LambertW}\Bigg(\Bigg(1 + \exp\Bigg\{ \Big(-s_{i,1} + s_{i,2} - c_1 + c_2 + r_1 - r_2\Big)\,\beta_i \\
& \quad + \Big(-s_{i,1} + s_{i,2} - c_1 + c_2\Big)\, b_i + 2\left(\eta_1 - \eta_2\right)\lambda_i + L_{i,1} - L_{i,2}\Bigg\}\Bigg) \\
& \quad \times \exp\Bigg\{ \Big(-c_2 + r_2 - s_{1,2}\Big)\,\beta_1 + \lambda_1\left(\eta_2 - \eta_1\right) - c_2 b_1 - s_{1,2} b_1 + L_{1,2} - 1\Bigg\}\Bigg)\Bigg]
\end{aligned}
\tag{4..70}
$$

$$
\begin{aligned}
d_{i,2} = \frac{1}{\beta_i + b_i}\Bigg[& \Big(p_2 - s_{i,2} - c_2\Big)\left(b_i + \beta_i\right) - 1 \\
& - \mathrm{LambertW}\Bigg(\Bigg(1 + \exp\Bigg\{ \Big(-s_{i,1} + s_{i,2} - c_1 + c_2 + r_1 - r_2\Big)\,\beta_i
\end{aligned}
\tag{4..71}
$$

$$+ \left(-s_{i,1} + s_{i,2} - c_1 + c_2\right) b_i + 2\left(\eta_1 - \eta_2\right)\lambda_i + L_{i,1} - L_{i,2}\Big\}\Big)$$

$$\times \exp\Big\{ \left(-c_2 + r_2 - s_{1,2}\right)\beta_1 + \lambda_1\left(\eta_2 - \eta_1\right) - c_2 b_1 - s_{1,2} b_1 + L_{1,2} - 1\Big\}\Big)\Big]$$

Case 2: $u_{1,1} = 0$, $u_{2,1} = 0$, $v_{1,1} = 0$, $v_{2,1} > 0$

$$d_{i,1} = \frac{1}{\left(\exp\left\{L_{i,2} - b_i p_2 + \beta_i\left(r_2 - p_2\right) + \lambda_i\left(\eta_2 - \eta_1\right)\right\} + 1\right)\left(b_i + \beta_i\right)} \tag{4..72}$$

$$\Big[\left(-\exp\left\{L_{i,2} - b_i p_2 + \beta_i\left(r_2 - p_2\right) + \lambda_i\left(\eta_2 - \eta_1\right)\right\} - 1\right)$$

$$\mathrm{LambertW}\Big(\left(\exp\left\{L_{i,2} - b_i p_2 + \beta_i\left(r_2 - p_2\right) + \lambda_i\left(\eta_2 - \eta_1\right)\right\} + 1\right)^{-1}$$

$$\exp\Big\{ \left(\exp\left\{L_{i,2} - b_i p_2 + \beta_i\left(r_2 - p_2\right) + \lambda_i\left(\eta_2 - \eta_1\right)\right\} + 1\right)^{-1}$$

$$\Big(\left(-s_{i,1} + s_{i,2} - c_1 + c_2 - p_2\right)\left(b_i + \beta_i\right) + r_1\beta_i + \lambda_i\left(\eta_1 - \eta_2\right) + L_{i,1} - 1\Big)$$

$$\exp\left\{L_{i,2} - b_i p_2 + \beta_i\left(r_2 - p_2\right) + \lambda_i\left(\eta_2 - \eta_1\right)\right\}$$

$$+ \left(-s_{i,1} - c_1 + r_1\right)\beta_i + \left(-s_{i,1} - c_1\right) b_i + \lambda_i\left(\eta_1 - \eta_2\right) + L_{i,1} - 1\Big\}\Big)$$

$$+ \Big(\left(-s_{i,1} + s_{i,2} - c_1 + c_2 + p_1 - p_2\right)\left(b_i + \beta_i\right) - 1\Big)$$

$$\exp\left\{L_{i,2} - b_i p_2 + \beta_i\left(r_2 - p_2\right) + \lambda_i\left(\eta_2 - \eta_1\right)\right\}$$

$$+ \left(-s_{i,1} - c_1 + p_1\right)\left(b_i + \beta_i\right) - 1\Big]$$

$$d_{i,2} = 0$$

Case 3: $u_{1,1} = 0$, $u_{2,1} = 0$, $v_{1,1} > 0$, $v_{2,1} = 0$

$$d_{i,1} = \frac{1}{b_i + \beta_i}\Bigg[- \text{LambertW}\left(\frac{\exp\Big\{\big(-s_{i,1} - c_1\big)\left(b_i + \beta_i\right) + r_1\beta_i + \lambda_i\left(\eta_1 - \eta_2\right) + L_{i,1} - 1\Big\}}{\exp\Big\{\big(-s_{i,2} - c_2\big)\left(b_i + \beta_i\right) + r_2\beta_i + \lambda_i\left(\eta_2 - \eta_1\right) + L_{i,2}\Big\} + 1} \right)$$

$$+ \big(p_1 - s_{i,1} - c_1\big)\left(b_i + \beta_i\right) - 1\Bigg] \tag{4..73}$$

$$d_{i,2} = p_2 - s_{i,2} - c_2 \tag{4..74}$$

Case 4: $u_{1,1} = 0$, $u_{2,1} = 0$, $v_{1,1} > 0$, $v_{2,1} > 0$

$$d_{i,1} = 0 \tag{4..75}$$

$$d_{i,2} = \frac{1}{\Big(\exp\big\{L_{i,1} - b_ip_1 + \beta_i\left(r_1 - p_1\right) + \lambda_i\left(\eta_1 - \eta_2\right)\big\} + 1\Big)\left(b_i + \beta_i\right)} \tag{4..76}$$

$$\Bigg[\Big(- \exp\big\{L_{i,1} - b_ip_1 + \beta_i\left(r_1 - p_1\right) + \lambda_i\left(\eta_1 - \eta_2\right)\big\} - 1\Big)$$

$$\text{LambertW}\Bigg(\Big(\exp\big\{L_{i,1} - b_ip_1 + \beta_i\left(r_1 - p_1\right) + \lambda_i\left(\eta_1 - \eta_2\right)\big\} + 1\Big)^{-1}$$

$$\exp\Bigg\{ \Big(\exp\big\{L_{i,1} - b_ip_1 + \beta_i\left(r_1 - p_1\right) + \lambda_i\left(\eta_1 - \eta_2\right)\big\} + 1\Big)^{-1}$$

$$\Big(\big(-s_{i,2} + s_{i,1} - c_2 + c_1 - p_1\big)\left(b_i + \beta_i\right) + r_2\beta_i + \lambda_i\left(\eta_2 - \eta_1\right) + L_{i,2} - 1\Big)$$

$$\exp\big\{L_{i,1} - b_ip_1 + \beta_i\left(r_1 - p_1\right) + \lambda_i\left(\eta_1 - \eta_2\right)\big\}$$

$$+ \big(-s_{i,2} - c_2 + r_2\big)\beta_i + \big(-s_{i,2} - c_2\big)b_i + \lambda_i\left(\eta_2 - \eta_1\right) + L_{i,2} - 1\Bigg\}\Bigg)$$

$$+ \Big(\big(-s_{i,2} + s_{i,1} - c_2 + c_1 + p_2 - p_1\big)\left(b_i + \beta_i\right) - 1\Big)$$

$$\exp\big\{L_{i,1} - b_ip_1 + \beta_i\left(r_1 - p_1\right) + \lambda_i\left(\eta_1 - \eta_2\right)\big\}$$

$$+ \left(-s_{i,2} - c_2 + p_2\right)\left(b_i + \beta_i\right) - 1\Bigg]$$

Case 5: $u_{1,1} = 0$, $u_{2,1} > 0$, $v_{1,1} = 0$, $v_{2,1} = 0$

$$d_{i,1} = 0 \tag{4..77}$$

$$d_{i,2} = 0 \tag{4..78}$$

Case 6: $u_{1,1} = 0$, $u_{2,1} > 0$, $v_{1,1} = 0$, $v_{2,1} > 0$

$$d_{i,1} = 0 \tag{4..79}$$

$$d_{i,2} = p_2 - s_{i,2} - c_2 \tag{4..80}$$

Case 7: $u_{1,1} = 0$, $u_{2,1} > 0$, $v_{1,1} > 0$, $v_{2,1} = 0$

$$d_{i,1} = p_i - s_{i,1} - c_i \tag{4..81}$$

$$d_{i,2} = \frac{1}{b_i + \beta_i}\Bigg[\left(p_2 - s_{i,2} - c_2\right)\left(b_i + \beta_i\right) - 1 - \mathrm{LambertW}\left(\frac{\exp\left\{\left(-c_2 - s_{i,2}\right)\left(b_i + \beta_i\right) + r_2\beta_i + \lambda_i\left(\eta_2 - \eta_1\right) + L_{i,2} - 1\right\}}{\exp\left\{\left(-s_{i,1} - c_1\right)\left(b_i + \beta_i\right) + r_1\beta_i + \lambda_i\left(\eta_1 - \eta_2\right) + L_{i,1}\right\} + 1}\right)\Bigg] \tag{4..82}$$

Case 8: $u_{1,1} = 0$, $u_{2,1} > 0$, $v_{1,1} > 0$, $v_{2,1} > 0$

$$d_{i,1} = p_1 - s_{i,1} - c_1 \tag{4..83}$$

$$d_{i,2} = 0 \tag{4..84}$$

Case 9: $u_{1,1} > 0$, $u_{2,1} = 0$, $v_{1,1} = 0$, $v_{2,1} = 0$

$$d_{i,1} = p_1 - s_{i,1} - c_1 \tag{4..85}$$

$$d_{i,2} = p_2 - s_{i,2} - c_2 \tag{4..86}$$

$$\tag{4..87}$$

□

Appendix B: Notation

Decision Variables	
$d_{i,j}$	The amount of discount provided to customer i for product j
Symbols	
$U_{i,j}$	The utility that customer i enjoys from action j.
$\chi_{i,j}$	The probability that customer i chooses action j.
$\pi_{i,j}$	The expected profits from selling product j to customer i.
π_i	The expected profits from customer i.
$\Pi_{i,j}$	Expected total profits.
Indices	
i	Customers ($i \in I$)
j	Products
Parameters	
$L_{i,j}$	Customer i's loyalty to product j.
b_i	Customer i's price sensitivity parameter.
p_j	List price of product j.
c_j	Unit cost of product j, $c_i \geq 0$.
$s_{i,j}$	Cost of shipping product j to customer i.
β_i	Customer i's sensitivity to price departures from reference price.
λ_i	Customer i's weight for ratings
η_j	Customer rating for product j.
r_j	Reference price for product j.

Table 4.9: List of notations

Bibliography

[1] Anderson, C. K., Rasmussen, H., and MacDonald, L. (2005). Competitive pricing with dynamic asymmetric price effects. *International Transactions in Operational Research*, **12**(5), 509–525.

[2] Anderson, S. P. (1988). Equilibrium existence in the linear model of spatial competition. *Economica*, pages 479–491.

[3] Berthene, A. (2020). Shoppers buy more online compared with before the pandemic. https://www.digitalcommerce360.com.

[4] Briesch, R. A., Krishnamurthi, L., Mazumdar, T., and Raj, S. P. (1997). A comparative analysis of reference price models. *Journal of Consumer Research*, **24**(2), 202–214.

[5] Bucklin, R. E. and Lattin, J. M. (1991). A two-state model of purchase incidence and brand choice. *Marketing Science*, **10**(1), 24–39.

[6] Clement, J. (2019). Digital coupons and deals - statistics & facts. Technical report, Statista.

[7] Dolan, R. J. (1987). Quantity discounts: Managerial issues and research opportunities. *Marketing Science*, **6**(1), 1–22.

[8] Gale, I. L. and Holmes, T. J. (1993). Advance-purchase discounts and monopoly allocation of capacity. *The American Economic Review*, pages 135–146.

[9] Gilbride, T. J. and Allenby, G. M. (2004). A choice model with conjunctive, disjunctive, and compensatory screening rules. *Marketing Science*, **23**(3), 391–406.

[10] He, Q.-C. and Chen, Y.-J. (2018). Dynamic pricing of electronic products with consumer reviews. *Omega*, **80**, 123–134.

[11] Hernandez, M. A. (2011). Nonlinear pricing and competition intensity in a hotelling-type model with discrete product and consumer types. *Economics letters*, **110**(3), 174–177.

[12] Hotelling, H. (1990). Stability in competition. In *The Collected Economics Articles of Harold Hotelling*, pages 50–63. Springer.

[13] Janiszewski, C. and Cunha Jr, M. (2004). The influence of price discount framing on the evaluation of a product bundle. *Journal of Consumer Research*, **30**(4), 534–546.

[14] Johnson, J., Tellis, G. J., and Ip, E. H. (2013). To whom, when, and how much to discount? a constrained optimization of customized temporal discounts. *Journal of Retailing*, **89**(4), 361–373.

[15] Kalyanaram, G. and Winer, R. S. (1995). Empirical generalizations from reference price research. *Marketing Science*, **14**(3_supplement), G161–G169.

[16] Lachaab, M., Ansari, A., Jedidi, K., and Trabelsi, A. (2006). Modeling preference evolution in discrete choice models: A bayesian state-space approach. *Quantitative Marketing and Economics*, **4**(1), 57–81.

[17] Lattin, J. M. and Bucklin, R. E. (1989). Reference effects of price and promotion on brand choice behavior. *Journal of Marketing Research*, pages 299–310.

[18] Lederer, P. J. and Hurter Jr, A. P. (1986). Competition of firms: Discriminatory pricing and location. *Econometrica: Journal of the Econometric Society*, pages 623–640.

[19] Leng, M. and Becerril-Arreola, R. (2010). Joint pricing and contingent free-shipping decisions in b2c transactions. *Production and Operations Management*, **19**(4), 390–405.

[20] Lewis, M., Singh, V., and Fay, S. (2006). An empirical study of the impact of nonlinear shipping and handling fees on purchase incidence and expenditure decisions. *Marketing Science*, **25**(1), 51–64.

[21] Lopes, P. and Roy, B. (2015). Dynamic recommendation system using web usage mining for e-commerce users. *Procedia Computer Science*, **45**, 60–69.

[22] Louca, R., Bhattacharya, M., Hu, D., and Hong, L. (2019). Joint optimization of profit and relevance for recommendation systems in e-commerce. In *Workshop on Recommendation in Multi-stakeholder Environments (RMSE'19), in conjunction with the 13th ACM Conference on Recommender Systems, RecSys*, volume 19, page 2019.

[23] Ma, S. (2017). Fast or free shipping options in online and omni-channel retail? the mediating role of uncertainty on satisfaction and purchase intentions. *The International Journal of Logistics Management*.

[24] Matusiak, E. (2020). Planning for the post covid-19 retail world. https://www.bdo.ca.

[25] Mazumdar, T., Raj, S., and Sinha, I. (2005). Reference price research: Review and propositions. *Journal of Marketing*, **69**(4), 84–102.

[26] Möller, M. and Watanabe, M. (2010). Advance purchase discounts versus clearance sales. *The Economic Journal*, **120**(547), 1125–1148.

[27] Munson, C. L. and Hu, J. (2010). Incorporating quantity discounts and their inventory impacts into the centralized purchasing decision. *European Journal of operational research*, **201**(2), 581–592.

[28] Narasimhan, C. (1984). A price discrimination theory of coupons. *Marketing Science*, **3**(2), 128–147.

[29] Nocke, V., Peitz, M., and Rosar, F. (2011). Advance-purchase discounts as a price discrimination device. *Journal of Economic Theory*, **146**(1), 141–162.

[30] Scholz, M., Dorner, V., Schryen, G., and Benlian, A. (2017). A configuration-based recommender system for supporting e-commerce decisions. *European Journal of Operational Research*, **259**(1), 205–215.

[31] Sheng, S., Parker, A. M., and Nakamoto, K. (2007). The effects of price discount and product complementarity on consumer evaluations of bundle components. *Journal of Marketing Theory and Practice*, **15**(1), 53–64.

[32] StatCan (2019). Retail trade. Technical report, Statistics Canada.

[33] Viswanathan, S. and Wang, Q. (2003). Discount pricing decisions in distribution channels with price-sensitive demand. *European Journal of Operational Research*, **149**(3), 571–587.

[34] Wang, H., Wei, Q., and Chen, G. (2013). From clicking to consideration: A business intelligence approach to estimating consumers' consideration probabilities. *Decision Support Systems*, **56**, 397–405.

[35] Wang, R. (2018). When prospect theory meets consumer choice models: Assortment and pricing management with reference prices. *Manufacturing & Service Operations Management*.

[36] Yang, P.-C. (2004). Pricing strategy for deteriorating items using quantity discount when demand is price sensitive. *European Journal of Operational Research*, **157**(2), 389–397.

[37] Zhao, Q., Zhang, Y., Friedman, D., and Tan, F. (2015). E-commerce recommendation with personalized promotion. In *Proceedings of the 9th ACM Conference on Recommender Systems*, pages 219–226. ACM.

[38] Zhu, T. and Singh, V. (2009). Spatial competition with endogenous location choices: An application to discount retailing. *QME*, **7**(1), 1–35.

Chapter 5

Conclusion and Future Research

As discussed in Chapter 1, the main focus of this thesis is to bridge some of the gaps in the literature on pricing optimization in an e-commerce context. In doing so, this research significantly contributes to new models and solution methods discussed in Chapters 2-4.

Chapter 2 focused on price optimization in the presence of reference prices with threshold. The existing literature in this area has mostly relied on dynamic programming. This has lead to solving small sized problems due to the "curse of dimensionality" that is inevitable in dynamic programming algorithms. With the rise of e-commerce giants like Amazon, the growing popularity of online retail, and the availability of data, the need to solve large problems is evident. In response to this need, in Chapter 2 a myopic heuristic and a modified generalized Benders' decomposition (GBD) method for finding optimal pricing in a multi-period pricing problem with reference pricing and thresholds has been proposed. This was followed by establishing analytical results for finding optimal solutions for the heuristic algorithm. In terms of optimality of results, the modified GBD outperforms the heuristic. It was also shown that running the GBD for 100 iterations achieves solutions similar in quality to the case when it is ran for 1000 iterations, but at much lower computational times. Numerical experiments show that the heuristic method works well for products with high profit margin and customer brand loyalty. Because using the myopic heuristic does not require any solver licensing and it requires minimal computational time, in these cases it may be preferable to the GBD method. However, in other cases, the GBD outperforms the myopic heuristic significantly. Also, solving the multi-period profit maximization problem can assist managers in deciding about general pricing strategies, such as every day low pricing or high-low pricing.

The focus of Chapter 3 was to explicitly study customer rating as a response to prices where prices are decision variables in a revenue optimization problem utilizing reference prices, customer ratings, and cross-price effects. A new model for forecasting reviews was introduced and then validated using Amazon data. This was followed by implementing a linear demand model that accounts for reference prices, cross-price effects, and customer ratings in a comprehensive price optimization model for multiple products in an online retail environment. In addition, a heuristic solution method was introduced and compared with commercial solvers. Computational experiments and the effect of parameters on total profits were then studied. It was shown that ignoring customer rating and/or cross-price effects can lead to large profit losses when

- the difference in initial rating of products is high,
- the product with lower initial rating has low demand price sensitivity,
- the product with higher initial rating has high demand price sensitivity,
- the customer reference price memory parameter (α), is low, and

- the customer memory parameter for rating (θ) and the weight of difference in ratings in demand (λ) are high.

It was also discussed that it is to the retailer's benefit to reduce the weight of previous period's customer rating directly or indirectly. This may be one factor that the default sorting of reviews on online retailers such as Amazon.com is "top reviews" rather than "recent reviews". In addition, reducing the effect of cross-price effects increases retailer's total profits. This can lead to further research for selecting "relevant products" at the time a customer is observing the product. It was also noted that form 28,187 parameter sets used in our computational experiments, the heuristic method's relative error was less than 10% in approximately 87% of cases and less than 5% in approximately 46% of cases. Finally, conditions under which the heuristic method produces close to optimal results and may be preferable over commercial solvers was presented. As discussed, the myopic heuristic performs well when

- the difference in initial ratings is high,
- the product with higher initial rating has high customer price sensitivity and low profit margin, and
- the product with lower initial rating has low customer price sensitivity and high profit margin.

Under such circumstances, a retailer may prefer to use the myopic heuristic instead of commercial solvers due to the licensing costs of commercial solvers and the computational performance of the myopic heuristic.

Chapter 4 focused on studying personalised price discount optimization in a binary choice model utilizing customer locations and product review data. To do so, customers' utility and purchase probabilities from customer choice literature was modified to account for prices, discounts, and ratings. Following this, closed form solutions were derived for special cases of the price discount optimization problem. For the purpose of computational experiments, 1,927,680 data points were used. The following are the summary of our findings and managerial implications:

- The product with a lower rating is discounted more frequently than the product with a higher rating.
- The two products were never discounted simultaneously for a customer.
- As expected, the shipping costs have negative impact on profits.
- Profits increase when customers have higher loyalty to the product with the higher rating and lower loyalty to the product with the lower rating; i.e., the larger gap between product preference. Managers may consider targeted marketing to achieve this larger gap.
- Increasing the difference between product ratings increases profits. This can be achieved by adjusting default product review sorting that biases the average ratings.
- The optimal discount value for a product-customer pair decreases as customer's loyalty to the product and the competing product increases and decreases, respectively. Similar effect is observed with shipping costs.
- The optimal discount value for a product-customer pair increases as the product price and the competing product price increases and decreases, respectively.

5.1 Future Research Directions

In this section, future research directions for each chapter are discussed. This is followed by a general discussion of extensions in the overall thesis theme.

5.1.1 Chapter 2

This work can be extended in several ways. First, by incorporating uncertainty in the demand function. This can be accomplished by adding a stochastic term to the demand function. Inventory holding and ordering costs, lead time, and the costs associated with loss of market share as result of shortages also need to be added in future research. Second, the demand function and the model of reference price ignore the effect of competition in the reference price and demand. A new reference price function can be introduced that incorporates the effects of competition.

5.1.2 Chapter 3

Similar to work in Chapter 2, uncertainty can be incorporated in the demand function. This can be accomplished by adding a stochastic term to the demand function in a multiplicative or additive manner. Inventory holding and ordering costs, lead time, and the costs associated with loss of market share as a result of shortages can also be considered in future research. In addition, this study can be extended to a duopoly market. Also, the model of customer rating can further be expanded to capture the effects of reference price and other factors.

5.1.3 Chapter 4

The work in this chapter can be expanded to a multi-period horizon where reference prices and product ratings change as a response to net prices. Second, inventory holding and ordering costs, lead time, and the costs associated with loss of market share as a result of shortages can be added in future research. Third, this study can be extended to a duopoly market.

5.1.4 General Directions for Future Research

The overall theme of the thesis is to achieve optimal pricing, directly or indirectly. In Chapter 2 this was achieved in the presence of reference prices with thresholds. This idea was then expanded in Chapter 3 by modelling and adding the effects of customer ratings. In Chapter 4 the pricing problem was considered indirectly by using discount coupons as a way of price discrimination considering individual customers' preferences and location while taking into account the product rating. As such a natural extension to the research in this thesis is to consider a multi-product, duopoly market in presence of reference prices, product ratings in a finite multi-period horizon where prices and discounts are decision variables. In Chapter 4 it was assumed that shipping fees are fully absorbed by the retailer. The problem can further be expanded by considering the free-shipping order threshold as a decision variable. In addition, this problem can be considered from a courier company's point view; that is, how should a shipping company price its services provided to a retailer while taking into account retailer's optimal pricing and free shipping policies to achieve competitive advantage. Finally, an interesting extension of our model in Chapter 4 is to consider the problem of integrating pricing decisions with routing decisions.