Stochastic Programming Formulations and Structural Properties for Assemble-to-Order Systems

STOCHASTIC PROGRAMMING FORMULATIONS AND STRUCTURAL PROPERTIES FOR ASSEMBLE-TO-ORDER SYSTEMS

BY

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Abstract

Lowering the degree of component commonality may yield a higher type-II service level for a periodic review assemble-to-order system that aims to maximize reward. This is achieved via separating inventories of all the shared components for different products. We investigate the optimal bill-of-materials structure for two-product assemble-to-order systems with arbitrary number of components. The inventory of a shared component can be separated or common between different products. We show that an optimal bill-of-materials can be characterized between the following two extremal configurations: either two products share all common components, or they do not share any common component.

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Notation and abbreviations

n	:	number of componnets
m	:	number of products
i, i'	:	index of component
j	:	index of product
S_i	:	base stock level of component i
c_i	:	unit base stock level cost of component i
L_i	:	lead time of component i
L	:	maximum lead time among all components; that is, $L = \max_i L_i$
w_j	:	time window of product j
k	:	index of period corresponding to the duration $[k, k+1)$; $k = 0$ implies the current period; negative values of k imply previous periods
$x_{j,k}$	:	number of product j assembled in period k
$r_{j,k}$	:	reward for satisfying the demand for product j in period k
$a_{i,j}$	:	number of component i used to assemble one unit of product j; that is, the bill-of-materials (BOM)
B	:	budget, i.e., $\sum_i (c_i\, S_i) \leq B$
$P_{j,k}$	:	demand of product j at period k

P_j	:	demand of product j at the current period; that is, $P_{j,0}$
$D_{i,k}$	:	demand of component i at period k; that is, $D_{i,k} = \sum_j (a_{i,j}\ P_{j,k})$
M	:	number of independent samples
N	:	number of realizations in one sample
l	:	index of sample $l = 1, \ldots, M$
h	:	index of realization $h = 1, \ldots, N$
d	:	number of dedicated components; $d = 0$, (respectively $d = n$) implies a full commonality, (respectively non-commonality, configuration)
x^+	:	the nonnegative part of x; that is, $x^+ = (\lvert x\rvert + x)/2$
MTS	:	make-to-stock.
ATO	:	assemble-to-order.
MTO	:	make-to-order.
ETO	:	engineer-to-order.
FCFS	:	first-come-first-served allocation rule.
BOM	:	bill-of-materials.
LP	:	linear program.
RHS	:	the right hand side.
SAA	:	sample average approximation method.

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Chapter 1

Introduction

Since the 1990s, due to the pressure of high capital costs and the competitive environment in industry, more and more manufacturers have adopted assemble-to-order (ATO) systems to provide a large product variety required by customers and reduce response time without increasing cost. ATO systems play an important role in industry, especially, in electronics industry and automobile industry. Traditional computer manufacturers like IBM and Apple, provide product lines that offer customers few combinations of options but have delivery lead time on the order of weeks or months, whereas Dell Computers allow customers to select among processors, monitors, disk drivers, etc to satisfy customer's wishes at a reasonable price and short assembly and delivery times. Dell's success is heralded as a textbook success story of ATO systems and attracts nearly every manufacturers in the personal computer market to adopt similar ATO systems [1].

In this thesis, we study the inventory allocation problem for a periodic review ATO system with an independent base stock policy and a FCFS allocation rule. We analyze the formulation of Akçay and Xu [2] which jointly optimizes the base stock levels and the component allocation. In particular, we consider two-product stochastic models with arbitrary number of common components and show that either full component commonality or non-component commonality does not work worse than partial component commonality. The works presented in this thesis are published in Deza et al. [3] and Deza et al. [4].

1.1 Types of production environment

Hoekstra and Romme [5] state that the production environment can be classified into four types, make-to-stock (MTS), assemble-to-order (ATO), make-to-order (MTO), and engineered-to-order (ETO). The classification is based on the concept of customer order decoupling point (CODP), a point in the material flow from where customer order-driven activities take place.

In MTS, the product itself has a relatively long cycle time and the manufacturers produce according to a forecast of customer demand. End products are inventoried before they are ordered by customers and customer orders are fulfilled from the existing inventory. Therefore, the key competitiveness is the ability of logistics, rather than lead time [6]. One of the goals of the scheduling policy is to regulate end product inventory [7]. Too large inventories may result in a leftover of inventory and increase holding cost, whereas too small inventories may incur backorder or lost sales cost. Due to mismatch between demand and supply, the main operations issues under MTS environment are inventory planning, lot size determination, and demand forecasting [8].

Strategies like ATO and MTO are developed to reduce the demand-supply mismatches associated with MTS. For example, manufacturers attempt to delay production until they get better demand information [9].

In ATO, manufacturers produce components and generate bill-of-materials (BOM) structuring from these components. When orders arrive, a variety of end products will be assembled using the components inventories and given BOM [6]. However, a problem may arise when customer demand must be backlogged due to lack of some components, while other components remain unused. The main inventory management issues for ATO systems include determining base stock levels without full information on product demands and making component allocation decisions depending on available component inventories and realized product demands [2].

The capability for production customization in MTO systems is greater than that in ATO systems. MTO production is typically used to manufacture single-item or small-batch productions, and offers a higher variety but more expensive products. Components and raw materials are inventoried and most or all the operations necessary to manufacture each specific product are only done after a receipt of a customer order [10]. The customer selects the company due to its reputation in production capability, price, responsiveness and service, therefore the main operations issues are capacity planning, order acceptance or rejection and attaining high due-date adherence [8].

ETO appears to be an extension of MTO. ETO products tend to be highly specialized and technical in nature, thus production output is very low and revenue is based on high profit margin instead of unit sales volume [11]. Each customer order results in a unique set of part numbers, BOM, and routing [10].

Wemmerlöv [12] states that the ATO system is a “graduate” stage of both MTS and MTO systems. An MTS manufacturer, pressed by market considerations, might choose to get into ATO manufacturing to offer a wider variety of products, whereas an MTO manufacturer, due to an expanding volume and a strong similarity among products, might choose to move to ATO manufacturing to satisfy an increased demand and to reduce response time.

1.2 Preliminary

1.2.1 Inventory management

The **inventory** is stocks of goods being held for future use or sale. Manufacturers use inventory management techniques to improve their inventory policy for when and how much to replenish their inventory. Based on the predictability of demand involved, the mathematical inventory models used with this technique can be classified as deterministic models and stochastic models. The **demand** for a product in inventory is the number of units that will need to be withdrawn from inventory for some use like sale during a specific period. When the demand in future periods can be forecast with considerable precision, it is reasonable to use a **deterministic inventory model**, in which all forecasts are always assumed to be completely accurate. However, if demand cannot well predicated, then it is necessary to use a **stochastic inventory model**, in which demands in all periods are random variables with known probability distribution [13].

A major distinction in the way inventories are managed results from the nature of demand for those items [14]. If items are used to produce certain end-products with other raw materials and components, then those items are said to have **dependent demand**. Conversely, **independent demand** is demand for an item that does not depend on the demand for any other items produced by the company. The total amount of raw materials or components needed to assemble cars is dependent demand because it is a function of the number of cars that will be produced, whereas demand for the finished car is an example of independent demand. Independent demand is quite stable once allowances are made for seasonal variations, therefore items having independent demand must be carried on a continual basis. Dependent demand is supposed to be lumpy since large quantities are required at specific time while little or no usage at other times. Consider a manufacturer that produces lawn and garden equipments, some components like bolts and screws, which are used in most of the items, are necessary to have a continual inventory because they are always needed. On the other hand, some components, designed for only one item, are required only when that item is being produced, i.e., once every month, and the demand is zero at rest of the time. Therefore dependent-demand items need only be inventoried just prior to the time they will be required in the production process.

1.2.2 Review period models

The review period can be classified into three types, namely single period models, periodic review models and continuous review models.

In **single period** models, perishable items (fresh fruits, vegetables, seafood and cut flowers) and items with limited useful lives (newspapers and magazines) are ordered. The unsold or unused items will not be carried over from one period to the next, at least not without penalty [14]. One of typical problems is the news-vendor problem, in which a vendor should decide how many newspapers are needed on a given day for his corner newsstand.

Instead of having only a single ordering to meet demand during a selling season, vendors may order products repeatedly at any time during the year, so-called **multiple-period** models. However the orders placed by customers cannot be instantaneously satisfied due to random demand and delivery lead time, thus manufacturers need to hold inventory on hand, perhaps inventory leftover from previous period, to satisfy demand occurring during lead time, protect against uncertainty in demand and balance inventory holding costs and backorder costs [15]. Based on when and how much to order, multiple-period models can further be classified as two types: periodic review models and continuous review models.

In **periodic review** models, the inventory level is reviewed at constant intervals, i.e., at the end of each week, and decisions such as how much to order are made after each review to keep a desired replenishment level. Manufacturers prefer to accept this type of model if they are impossible or inconvenient to frequently review inventory and place order when necessary. In **continuous review** models, the inventory level is reviewed continuously, and order is placed when inventory reaches a reorder point. Manufacturers with computerized inventory systems may choose this type of model [15]. Both review models must have stockout protection until the next order arrives. The periodic review models need protection during an order interval plus a lead time, while the continuous review models need protection only during the lead time. Therefore, the safety-stock in periodic review models is usually higher than that in continuous review models.

1.2.3 Linear programs

Linear programs (LP) are used to formulate a mathematical model to describe the problem of concern and can be used to allocate limited resources among activities in an optimal way [14]. In linear programs, **decision variables**, representing choices available to decision makers in terms of amounts of either inputs or outputs, are denoted by a column vector $x = (x_1, \ldots, x_n)^T$. The values of the decision variables should satisfy certain **constraints** that restrict the alternative available to decision makers. Constraints are either equality constraints having the form $g_i(x) = b_i$ or inequality constraints having the form $g_i(x) \geq b_i$ or $g_i(x) \leq b_i$. Note that $g_i(x)$, called the constraint function, is a given function of the decision variables and b_i, called the right hand constant. $x_j \geq b_j$ or $x_j \leq b_j$, which give restrictions on individual variables, are a special case of constraints and is so-called bound constraints.

If a column vector x satisfies all the constraints, then it is a **feasible solution** for the problem. The **objective function** in the problem is mostly to minimize a cost function or to maximize a profit function. The goal of LP is to find an **optimal solution**, one of the feasible solutions that has the best value for the objective function.

A **linear function** is a function of the form $c_1x_1 + \ldots + c_nx_n$, where $c_1, \ldots, c_n$ are given constants. Therefore a **linear program** is an optimization program in which all the constraint functions and the objective function are linear functions [16].

1.2.4 Two-stage stochastic program

Stochastic programming is an approach to model optimization problems with parameters which are unknown when a decision should be made. Here unknown parameters are assumed to follow some probability distributions, which could be estimated from historical data. The goal is to find some feasible policy for all possible parameter realizations and optimizes the expectation of some function of the decisions.

Two-stage programs are most widely applied and studied stochastic programming models.The basic idea that optimal decision should depend on data available at the time the decisions are made and should not be based on future observations. Inventory model is the classical two-stage stochastic programming and the corresponding optimization problem is written as following [17]

$$\begin{aligned} \max_{S \in \mathbb{R}^n} \quad & \mathbb{E}_\xi \left[Q\left(S, \xi\right) \right] \\ s.t. \quad & AS \le b, \quad S \ge 0, \end{aligned} \tag{1.1}$$

where $Q(S, \xi)$ is the optimal value of the second-stage problem

$$\begin{aligned} \max_{x \in \mathbb{R}^m} \quad & q^T x \\ s.t. \quad & TS + Wx \le v, \quad x \ge 0. \end{aligned} \tag{1.2}$$

where S and x are vectors of the first and second stage decision variables, respectively. The second stage problem depends on the data $\xi = (q, v, T, W)$ where any or all elements can be random.

(1.1) is called the first-stage problem where the ordering decision should be made before demands are known and (1.2) is called the second-stage problem where the component allocation decision is made based on on-hand inventory and realized demands.

In the first-stage, ξ is viewed as a random vector with estimated probability distribution of demand, but sometime the same notation ξ is also used to represent a random vector and its particular realization. If in doubt, we will write the random vector as ξ^h to distinguish from its particular realization.

In the second-stage, a number of random vectors ξ may realize with the second-stage problem data q, v, T and W. For each realization, the second-stage decision x is taken, but typically, the decisions x are not the same under different realizations of ξ. The objective function of (1.1) is the expectation of the second-stage objective $q^T x$ taken over all realizations of the random vector ξ [13] [18].

The standard approach to numerically solve the formulated problem is to assume that random vector ξ has a finite number of possible realizations $\xi^1, \ldots, \xi^N$ with respective probabilities $p_1, \ldots, p_N$. Then the expectation can be written as following

$$\mathbb{E}_\xi \left[Q(S, \xi)\right] = \sum_{h=1}^{N} p_h Q(S, \xi^h). \tag{1.3}$$

Then the two-stage problem can be formulated as one larger programming problem

$$\max_{S, x_1, \ldots, x_N} \quad \sum_{h=1}^{N} p_h q_h^T x_h \tag{1.4}$$

$$s.t. \quad AS \leq b, \quad S \geq 0,$$

$$T_h S + W_h x_h \leq v_h, \quad x_h \geq 0, \quad h = 1, \ldots, N.$$

For each realization $\xi^h = (q_h, T_h, W_h, v_h)$, where $h = 1, \ldots, N$, we solve the above formulation (1.4) and obtain an optimal solution S^* of the first-stage problem and optimal solutions x_h^* of the second-stage problem.

1.2.5 Sample average approximation (SAA) method

Suppose that we can generate a sample $\xi^1, \ldots, \xi^N$ of N realizations of the random vector ξ by using Monte Carlo simulation or historical data. By this we mean that each random vector $\xi^h, h = 1, \ldots, N$, has the same probability distribution as the data vector ξ. Therefore, if each ξ^h is distributed independently of other sample vectors, it is said that the sample is independently identically distributed (iid). Given a sample, we can approximate the expectation function $\mathbb{E}_\xi \left[Q\left(S, \xi\right)\right]$ by averaging values $Q\left(S, \xi^h\right), h = 1, \ldots, N$, and it is so-called **sample average approximation (SAA)**

$$\max \quad \left\{ \frac{1}{N} \sum_{h=1}^{N} Q(x, \xi^h) \right\}. \tag{1.5}$$

Therefore, for a generated sample ξ^h, the SAA problem (1.5) can be considered as the two-stage problem (1.1)-(1.2) with respective realizations ξ^h, each taken with the same probability $p_h = 1/N$ [17].

1.2.6 Mass customization

Mass customization is a strategy for companies to produce standardized goods or services with low cost, but incorporating some degree of customization in the final product or service to satisfy a variety of customer requests. **Delayed differentiation** and **modular design** are typical tactics to make mass customization possible.

In **delayed differentiation**, most part of products are standardized produced but the complete process of production is postponed, and the almost-finished products are held in inventory until customer specifications are know. When customer orders are received, the almost-finished products will be incorporated with customized features.

For example, furniture manufacturers can produce furniture without applying stain and supply a choice of stains to customers. Once the decision is made, the stain can be applied in such a relatively short time that the waiting time of customers is significantly reduced. The advantage of delayed differentiation is to both enable companies to produce high volumes of relatively low-cost standardized products and satisfy customer desire for a wide variety of products and a short waiting time.

Modular design is a form of standardization in which a complex system is decomposed into simple modules in order to organize complex designs and processes more efficiently. These modules are independently created and can be easily replaced or interchanged among different systems. One familiar example of modular design is computer. Typical modular parts include power supply units, processors, motherboard, graphics card and hard drives. According to customer requests, these modular parts are arranged in different configurations and customized computer capabilities will be achieved. Due to grouping the components into modules before customer orders are arrived, it becomes feasible to customize large varieties of high demand products within short customer waiting time. As long as these parts that support the same standardized interface, they can be easily replaceable if they become defective and easily be upgraded. Therefore, the users do not have to buy a new computer. The main issue associated modular design is designing products, assemblies, and components that fulfill various customer demands through the configuration of distinct building modules [14][19].

1.2.7 Supply chain

A **supply chain** is a network of organizations that are involved in converting raw materials into final products and then delivering products to the final customers. This network includes different facilities, functions, activities, information and resources. The facilities associated with the supply chain include warehouses, factories, processing centers, distribution centers, retail outlets and offices. Functions and activities include forecasting, purchasing, inventory management, information management, quality assurance, scheduling, production, distributions, delivery and customer service.

For each organization, there are two components involved in the supply chain, a supply component and a demand component. The supply component starts at the beginning of the chain and ends with the internal operations of the organization while the demand component starts at the point where the organization's output is delivered to its intermediaries and ends with the final customer in the chain. **Supply chain management** is the design and management of process across all organizations with the goal of matching supply and demand as efficiently as possible. When supply chain management is done effectively, it can offer numerous benefits such as lower inventories, lower overall costs, higher productivity, shorter lead times, higher profits, and shorter response time. For example, since the 1980s, Wal-mart began to work directly with manufacturers to cut the cost from intermediaries. Walmart's successful supply chain helped Wal-mart become the largest and most profitable retailer in the world [14].

1.2.8 Component commonality

Component commonality refers to a situation in which several different components are replaced by one component. The introduction of component commonality allows the manufacturer a decrease in safety stock owing to risk pooling and a decrease in total component inventory cost [20]. For example, a manufacturer sells printer in Europe and the United States. As it is known, the European printer should have 220V power supplies while the American printers should have 120V power supplies. If the printer sells very well in Europe but very poorly in the United States, then a shortage of 220V power supplies and a surplus of 120V power supplies will happen. If the manufacturer decides to use a common component, i.e. a universal power supply, to replace these two unique components, the shortage of 220V power supplies for European printers will be fulfilled by the surplus of power supplies for American printers [21].

1.2.9 Other basic definitions

The concept of matching, first introduced by Axsäter [22], is defined for the supply and demand of the same item. In ATO systems, we extend this concept and establish the matching between multiple components and multiple products. This is called **multi-matching**. For example, for a given component i, at period k, a demand of $D_{i,k} = \sum_{j=1}^{m} a_{i,j} P_{j,k}$ units of component i is realized. Then we regard the $\sum_{j=1}^{m} a_{i,j} P_{j,k}$ units of components i as a whole and satisfy this demand with a supply of $\sum_{j=1}^{m} a_{i,j} P_{j,k}$ units of component i. Therefore, a multi-matching between supply and demand is established. Furthermore, we extend such matching to all components.

Lead time is the amount of time taken between order placement and order fulfillment. **Cycle time** is the maximum time allowed at each workstation to complete assigned tasks from start to finish.

The **Type-I service level** measures the proportion of periods in which the demand of a product or the aggregated demand of all products is satisfied. The **Type-II service level** measures the proportion of the demand of a product or the proportion of the aggregated demand of all products that is fulfilled [2].

Economies-of-scale is an important aspect of efficiency in production in which larger businesses can benefit from a reduction in average costs of production as they increase their scale of production. When a business grows and its output increases, although total costs will increase, the cost of producing each unit will fall, and this gives the business a competitive benefit over smaller companies [23].

In case the actual demand will exceed expected demand, it is necessary to hold additional inventory, called **safety stock**, to prevent a stock-out during lead time. Order cycle **service level** is the probability that on-hand inventory will be sufficient to meet demand during lead time. For example, a service level of 95 percent indicates a probability of 95 percent that demand will not exceed supply during lead time. It is obvious that the customer service level increases as the risk of running out of inventory decreases. Since it costs to hold safety stock, it is a challenge to balance the conflicting goals of maximizing service level and minimizing inventory cost [14][24].

Risk pooling is an important concept in supply chain management. **Risk pooling** suggests that demand variability is reduced if one can use centralized inventory rather than decentralized inventory. When demand is aggregated across different locations, it is more likely that higher than average demand from one customer will be offset by lower than average demand from another. This reduction in variability directly leads to a decrease in safety stock and eventually reduces average inventory.

When involved orders are relatively lengthy, it is important to consider the order of processing in terms of expenses associated with orders waiting for processing. **Priority rules** or **allocation rules** are simple heuristics used to determine the order of jobs to be processed. **First-come-first-served (FCFS)** is one of the most common priority rules, in which customer orders within a particular period cannot be allocated by the system until all the earlier orders are satisfied [14].

A **product** in ATO system is an article needed to satisfy customer demand. A **component** is a part of a product and can be used in different products. The relationship between products and components is given by the bill-of-material (BOM). A **Bill-of-material (BOM)** is a list of all of the raw materials, parts, sub-assemblies, and assemblies needed to manufacture one unit of an end-product [14].

When implementing a **base-stock policy**, also known as the order-up-to level, inventory is ordered to keep inventory position, i.e., on-hand inventory plus on-order inventory minus backorders) equaling the base stock level [9].

1.3 Literature review

Component commonality is widely adopted and often preferred in ATO systems in order to offset the reduction of economies-of-scale when providing customized products. The economic impact of component commonality for single period models has been extensively studied. Eynan and Rosenblatt [20] presented three models to compare and analyze the effects of increasing component commonality, and demonstrated that some forms of commonality might not always be beneficial. They also provided conditions for which commonality should be either employed or avoided. Mirchandani and Mishra [25] compared a non-commonality model with two different commonality models – based on whether or not the products are prioritized – for a system with two products and independent uniform demand distributions. They derived theoretical conditions when component commonality is beneficial for this specific system. Both Eynan and Rosenblatt [20] and Mirchandani and Mishra [25] allowed the common component to be more expensive than those it replaces. However, in our formulation, we apply component commonality to the inventory management rather than to the design process. Thus, we assume, like Baker et al.[26] and Gerchak et al. [27], that the costs of the dedicated component and the common component are identical. Baker et al. [26] studied the effect of component commonality on optimal safety stock levels for an ATO system with two end-products and two components. They considered the problem of minimizing safety stock levels while satisfying a service level constraint under independent uniform demand distributions and showed that component commonality induced a reduction in the optimal safety stock levels. Gerchak et al. [27] extended this work by investigating whether the results hold for a system with an arbitrary number of products and a general joint demand distribution.

In contrast to the above works where a single period model is assumed, our commonality study focuses on a multi-period model. Considering a simple multi-period ATO model, Hillier [28] observed that component commonality is not always beneficial. Hillier [28] studied a periodic review ATO system with zero lead times and uniformly distributed demands, and derived a closed-form solution for a cost minimization model with service level constraints. The results demonstrated that, for a multi-period model, the use of a common component is always beneficial if its price does not exceed the price of the replaced components. If the common component is more expensive than the replaced ones, then in contrast to the single period case, it is almost never beneficial to use it. Hillier [21] further extended these results to systems with an arbitrary number of final products and components. Song and Zhao [29] considered a continuous-review ATO system with one common component, two end products, and Poisson distributed demands, and showed that, while component commonality is generally beneficial, its added value depends strongly on the component costs, lead times, and allocation rules.

In the literature reviewed above, minimizing inventory level or inventory cost subject to some service level constraints is commonly used to model ATO systems. However, the problem we consider follows another line of research: component commonality for systems with a given budget for all the components. Jönsson and Silver [30] analyzed the impact of component commonality for an ATO system with two end products and two components, with one being common to both products. Fong et al. [31] pursued the approach of Baker et al. [26] and provided analytical formulations for a commonality problem minimizing the expected shortage subject to a fixed budget constraint and assuming independent Erlang demand distributions. In particular, they observed that the relative reduction in the expected shortage can be substantial when the budget level is high relative to the demand requirements for the end products – even if the component is much more expensive. Note that all these models assume a single period.

Another relevant work is Nonås [32] who formulated a two-stage stochastic program for an ATO system with three products and an arbitrary number of components, and introduced a gradient-based search method to find the optimal inventory levels for a profit maximization problem. The key difference is that we consider a budget constraint.

Akçay and Xu [2] studied a periodic review assemble-to-order (ATO) system with an independent base stock policy and a first-come-first-served (FCFS) allocation rule. They formulated a two-stage stochastic integer nonlinear program where the base stock levels and the component allocation are optimized jointly. They showed that the component allocation problem is an NP-hard multidimensional knapsack problem and proposed an order-based component allocation heuristic rule that commits a component to an order only if it leads to the fulfillment of the order within the committed time window. They concluded that their order-based component allocation rule outperforms the component-based allocation rules, such as the fixed-priority and fair-shared rules, see [33, 34]. Huang and de Kok [35] studied periodic-review ATO systems with linear holding and backlogging costs, installation stock policy, and a FCFS allocation rule. They introduced the concept of multimatching which refers to the coupling of multiple component units and product units. They showed that the FCFS allocation rule decouples the problem of optimal component allocation over time into deterministic period-by-period component allocation optimization problems. Huang [36] evaluated the impact of two non-FCFS allocation rules in a periodic review ATO system with component base stock policy; i.e., the last-come-first-served-within-one-period rule and the product-based-priority-within-time-windows rule. He proposed three benchmark mathematical programming models to test the non-FCFS allocation rules and concluded that both rules cannot only outperform FCFS allocation rule in certain areas, but also better address the differences in customer service requirements. Doğru et al. [37] investigated a continuous review W system and concluded that the FCFS base stock policy is typically suboptimal. They also provided a lower bound for the optimal objective value and developed a policy attaining the lower bound under some symmetry condition for the cost parameters and a so-called *balanced capacity* condition for the solution. Jaarsveld and Scheller-Wolf [38] developed a heuristic algorithm for large scale continuous review ATO systems which improves as the average newsvendor fractiles increase. They showed that, for large scale ATO systems, the best FCFS rule is nearly optimal, and proposed a no-holdback allocation rule which can outperform the best FCFS rule.

1.4 Thesis outline

In Chapter 2, we detail the formulations. The main results are presented in Chapter 3, the proofs are given in Chapter 4, and a few future directions are presented in Chapter 5.

Chapter 2

The stochastic programming model

2.1 Akçay and Xu formulation

2.1.1 ATO system setting

Following the model proposed by Akçay and Xu [2], we assume

(1) a periodic review system,

(2) an independent base stock policy is used for each component,

(3) the product demands are satisfied by a FCFS rule,

(4) the product demands are correlated within each period, while the demands over different periods are independent,

(5) the replenishment lead time for each component is constant,

(6) a product reward is collected if the assembly is completed within the given time window.

In addition, the following sequence of events, illustrated in Figure 2.1, is assumed for each period:

(i) inventory position reviewed (IPR),

(ii) new replenishment order of components placed(NOP),

(iii) earlier component replenishment order arrive(ROA),

(iv) demand realized(DR),

(v) component allocated and product assembled(CAPA),

(vi) associated rewards accounted for(ARA).

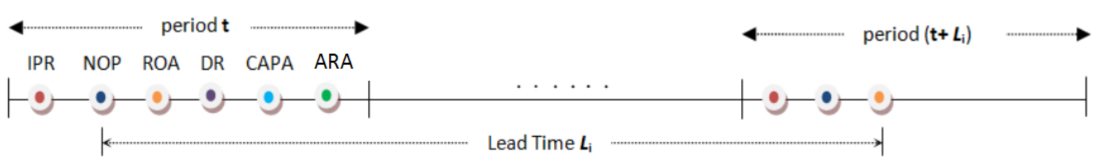

Figure 2.1: Sequence of events and decisions

2.1.2 On-hand inventory

Let $I_{i,t}$ represent the net inventory of component i at the end of period t, $A_{i,t}$ represent the replenishment order of component i arriving at period t, and $D_{i,t}$ represent the demand for component i at period t. Then we can derive the following key equations:

$$I_{i,t-1} + A_{i,t} - D_{i,t} = I_{i,t}, \tag{2.6}$$

$$A_{i,t} = D_{i,t-L_i-1}, \tag{2.7}$$

where L_i is the lead time of component i.

Based on the sequence of events, at any given period t, the replenishment order of component i, $A_{i,t}$, arrives and then the demand of component i, $D_{i,t}$, is realized, see Figure 2.2. Therefore, the net inventory of component i at the end of period t is equal to the net inventory of component i at the end of previous period $t-1$ plus the arrival of the replenishment order of component i at period t minus the amount of component i used to assemble ordered products at period t, that is, $I_{i,t-1} + A_{i,t} - D_{i,t} = I_{i,t}$.

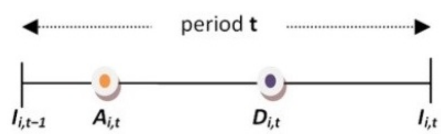

Figure 2.2: $I_{i,t-1} + A_{i,t} - D_{i,t} = I_{i,t}$

At period $t - L_i - 1$, a demand for component i, $D_{i,t-L_i-1}$, is realized. At the next period $t - L_i$, due to the base stock policy, a new replenishment order of component i with the amount of $D_{i,t-L_i-1}$ is placed in order to bring the inventory position back to the base stock level. According to the definition of lead time, this replenishment order, which is placed at period $t - L_i$, will arrive at period t, that is $A_{i,t}$. Therefore, we have $A_{i,t} = D_{i,t-L_i-1}$, see Figure 2.3.

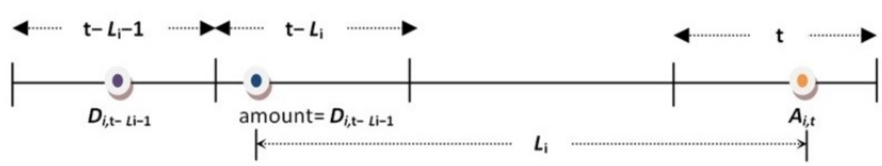

Figure 2.3: $A_{i,t} = D_{i,t-L_i-1}$

Let S_i be the base stock level of component i, and let $D_i[s,t]$ and $A_i[s,t]$ represent the total demand and total replenishment of component i from period s through period t respectively, that is, $D_i[s,t] = \sum_{\mu=s}^{t} D_{i,\mu}$ and $A_i[s,t] = \sum_{\mu=s}^{t} A_{i,\mu}$ for $s \leq t$. Then we wish to point out the following key equations:

$$I_{i,t} = S_i - D_i[t - L_i, t] \tag{2.8}$$

$$I_{i,t+k} = I_{i,t-1} + A_i[t, t+k] - D_i[t, t+k]. \tag{2.9}$$

Suppose that the net inventory of component i at the end of period t is under the base stock control S_i, that is, the inventory position of component i at the beginning of period $t - L_i - 1$ is S_i and there is no replenishment order arrived from period $t - L_i - 1$ through period $t - 1$. The earliest replenishment order of component i arrives at period t, denoted as $A_{i,t}$. The process is illustrated in Figure 2.4.

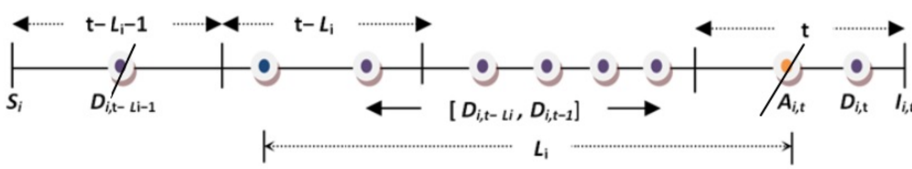

Figure 2.4: $I_{i,t} = S_i - D_i[t - L_i, t]$

Therefore, the derivation of (2.8) is as following:

$$\begin{aligned} I_{i,t} &= S_i - D_{i,t-L_i-1} - D_i[t - L_i, t-1] + A_{i,t} - D_{i,t} \\ &= S_i - D_i[t - L_i, t] \qquad (\because A_{i,t} = D_{i,t-L_i-1}) \end{aligned}$$

Furthermore, we replace t by $t+k$ and rewrite (2.8) as following:

$$I_{i,t+k} = S_i - D_i[t+k-L_i, t+k] \tag{2.10}$$

If we apply (2.6) repeatedly for periods $t, t+1, t+2, \ldots, t+k$, then we have the following balance equations:

$$\begin{aligned}
&I_{i,t} = I_{i,t-1} + A_{i,t} - D_{i,t} \\
&I_{i,t+1} = I_{i,t} + A_{i,t+1} - D_{i,t+1} \\
&\ldots \\
&I_{i,t+k-1} = I_{i,t+k-2} + A_{i,t+k-1} - D_{i,t+k-1} \\
&I_{i,t+k} = I_{i,t+k-1} + A_{i,t+k} - D_{i,t+k}
\end{aligned}$$

Now, we can relate the ending inventory of component i at periods $t-1$ and $t+k$ as follows:

$$\begin{aligned}
I_{i,t+k} &= I_{i,t+k-1} + A_{i,t+k} - D_{i,t+k} \\
&= I_{i,t+k-2} + A_{i,t+k-1} - D_{i+k-1} + A_{i,t+k} - D_{i,t+k} \\
&= \ldots \\
&= I_{i,t-1} + A_i[t,t+k] - D_i[t,t+k]
\end{aligned}$$

Substituting (2.10) into (2.9), we reach the following result:

$$
\begin{aligned}
& I_{i,t-1} + A_i[t, t+k] - D_i[t, t+k] = S_i - D_i[t+k-L_i, t+k] \\
\Longleftrightarrow \quad & I_{i,t-1} + A_i[t, t+k] = S_i - (D_i[t+k-L_i, t+k] - D_i[t, t+k]) \\
\Longleftrightarrow \quad & I_{i,t-1} + A_i[t, t+k] = S_i - D_i[t+k-L_i, t-1] \qquad \text{(see Figure 2.5)}
\end{aligned}
$$

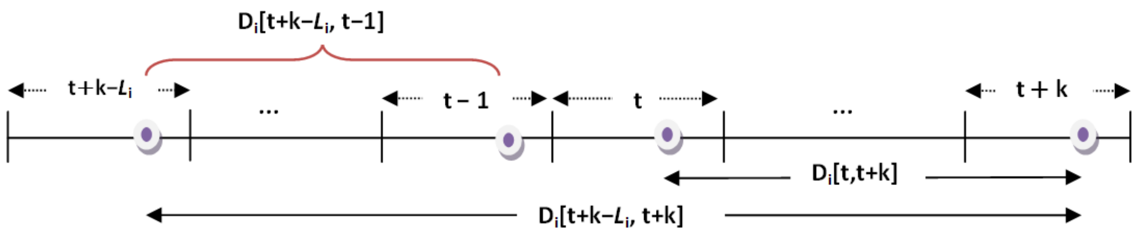

Figure 2.5: $D_i[t+k-L_i, t-1] = D_i[t+k-L_i, t+k] - D_i[t, t+k]$

We observe that $I_{i,t-1} + A_i[t, t+k]$ is the net inventory of component i in period $t+k$ after receiving all replenishment orders from periods t through $t+k$, but before fulfilling any orders arrived after period $t-1$. Since the system uses FCFS to fill orders, the demand $D_{i,t}$ received at period t can not be realized until all earlier customer orders are allocated. Thus, at period $t+k$, provided that no inventory of component i has been allocated to those orders since their arrival, where $k = 0, 1, 2, \ldots, L_i$, the available on-hand inventory of component i used to fulfill the demand at period t is:

$$
(S_i - D_i[t+k-L_i, t-1])^+ = \max\{S_i - D_i[t+k-L_i, t-1], 0\}. \tag{2.11}
$$

In addition, we can drop the time index t from the notation and use $D_i^{L_i-k}$ to represent the stationary version of $D_i[t+k-L_i, t-1]$, and we introduce $O_{i,k}$ as a shorthand notation for the on-hand inventory of component i available at period $t+k$ that can be used for the demand at period t. We update (2.11) as follows:

$$O_{i,k} := \left(S_i - D_i^{L_i-k}\right)^+ := \left(S_i - D_i[t+k-L_i, t-1]\right)^+, \tag{2.12}$$

where $0 \leq k \leq L_i$. Note that when $k = L_i$, the on-hand inventory $\left(S_i - D_i^{L_i-k}\right)^+$ becomes S_i. However, it is still possible that the demand at period t will not be fulfilled by period $t + L_i$. When $D_{i,t} > S_i$, there is $D_{i,t} - S_i$ demand (from period t) unfulfilled at the end of period $t + L_i$. In this case, the demand will be completely fulfilled by period $t + L_i + 1$ due to (2.7).

2.1.3 Formulation

In our model, assembly takes zero time while component lead times are greater than zero. The model is based on a multi-matching approach proposed by Huang [36] and Huang and de Kok [35] where multiple components are matched with multiple products to satisfy demands. In each period within the time window, rewards are collected by satisfying product demands. We recall that the time window is the number of periods between the order receiving period and the order fulfillment period. In particular, a time window equal to 0 means that the demand must be fulfilled within the period the order is received; that is, we must have enough components to satisfy the demand within that period in order to collect reward. The base stocks of the ATO system are constrained by a pre-set overall budget. The approach is based on a two-stage decision model. The first stage consists of determining a base stock level for each component, and the second stage consists of determining products that need to be assembled in each period with respect to some constraints reflecting the inventory availability. The first stage decisions are made before the second stage decisions following a two-stage stochastic programming framework, see Birge and Louveaux [18]. The objective of the approach is to maximize the expected total reward collected from the products assembled within given time windows. Note that while all products are eventually assembled within $L+1$ periods, the reward are collected only within the pre-set time windows.

The second stage corresponds to the allocation problem $\big(Alloc(S,\xi)\big)$, where $S=(S_i)$ is the vector representing base stock levels, $\xi=\{P_{j,k}|j=1,\ldots,m;k=0,-1,\ldots,-L\}$ is the vector representing random demands, and $O_{i,k}$ is the number of component i available at period k. Note that $O_{i,k}=(S_i-D_i^{L_i-k})^+$ for $0\le k\le L_i$ where $D_i^{L_i-k}=\sum_{s=0}^{L_i-k}D_{i,-s}$, and $O_{i,k}=D_{i,0}$ for $L_i+1\le k\le L+1$ are inferred from the base stock policy and a FCFS rule, see Huang [36] and Huang and de Kok [35].

$$\begin{aligned}
\max \quad & \sum_{j=1}^{m}\sum_{k=0}^{w_j}(r_{j,k}\,x_{j,k}) && \big(Alloc(S,\xi)\big)\\
& \sum_{k=0}^{w_j} x_{j,k} \le P_j && j=1,\ldots,m\\
\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\,x_{j,\mu}) &\le O_{i,k} && i=1,\ldots,n, \quad k=0,\ldots,L+1\\
& x_{j,k}\in\mathbb{Z}_+ && j=1,\ldots,m, \quad k=0,\ldots,L+1
\end{aligned}$$

The first set of constraints guarantees that assembly will satisfy customer demand. Please note that $w_j\le L+1$. As the system uses FCFS and base stock policy, all the customer demands will be fulfilled by period $L+1$. Therefore we can conclude that $\sum_{k=0}^{L+1}x_{j,k}=P_j$, where $j=1,\ldots,m$. Recall that the reward is collected if the products is assembled within time window w. Consequently, replacing the constraint $\sum_{k=0}^{L+1} x_{j,k} = P_j$ by $\sum_{k=0}^{w_j} x_{j,k} \leq P_j$ would yield the same optimal reward. From the aspect of operation research, we can treat all the variables $x_{j,k}$, where $k = w + 1, \ldots, L+1$ as slack variables that are added to an inequality constraint to transform it into an equality but have no effect on the objective function. The second set of constraints – called inventory availability constraints – guarantees that assembly could only happen when there are enough component inventories. While an optimal allocation can be computed for a given base stock level S and demand ξ, we still need to determine the optimal base stock levels. Thus, we use the two-stage stochastic integer program $\big(Joint(B)\big)$ where the first stage determines the base stock levels and the second stage maximizes the expectation of the component allocations:

$$\begin{aligned} \max \quad & \mathbf{E}[Alloc(S,\xi)] & & \big(Joint(B)\big) \\ & \sum_{i=1}^{n} (c_i\, S_i) \leq B & & \\ & S_i \in \mathbb{Z}_+ & i = 1, \ldots, n & \end{aligned}$$

We recall in Section 2.1.4 the sample average approximation method used to solve $\big(Joint(B)\big)$.

2.1.4 Sample average approximation method

The sample average approximation (SAA) method, see Kleywegt et al. [39], consists of the following steps:

(i) generate M independent samples for $l = 1, \ldots, M$ with N realizations for each sample. The vector $\xi_l^N = (\xi(\omega_l^1), \xi(\omega_l^2), \ldots, \xi(\omega_l^N))$ represents the N realizations of the l-th sample,

(ii) solve the optimization problem $(INLP)$ for each sample, which is the associated deterministic version of $\big(Joint(B)\big)$. where the objective function is set to $\frac{1}{N}\sum_{h=1}^{N} Alloc(S, \xi(\omega_l^h))$ as described below. Note that $(INLP)$ is non-linear not only due to the integrality constraints but also due to the right hand side of the inventory availability constraints. Let $\hat{S}_l$ denote the optimal base stock levels for $(INLP)$ and $\hat{G}(\hat{S}_l)$ denote its optimal objective value.

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}\sum_{j=1}^{m}\sum_{k=0}^{w_j}(r_{j,k}\, x_{j,k}^h) && && && (INLP)\\
& \sum_{k=0}^{w_j} x_{j,k}^h \le P_j^h && && j = 1,\ldots,m, && h = 1,\ldots,N\\
& \sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}^h) \le O_{i,k}^h && i = 1,\ldots,n, && k = 0,\ldots,L+1, && h = 1,\ldots,N\\
& \sum_{i=1}^{n}(c_i\, S_i) \le B\\
& S_i \in \mathbb{Z}_+ && && && i = 1,\ldots,n\\
& x_{j,k}^h \in \mathbb{Z}_+ && j = 1,\ldots,m, && k = 0,\ldots,L+1, && h = 1,\ldots,N
\end{aligned}
$$

(iii) generate a different sample $\xi^{N'}$ with $N' \gg N$ realizations and compare the performance among all the base stock vectors $\hat{S}_l$ solved in (ii) by solving $\big(Alloc(S, \xi^{N'})\big)$ with $S = \hat{S}_l$. Let $\bar{G}(\hat{S}_l)$ be the new optimal objective value.

(iv) select the optimal base stock vector $\hat{S}^*$ achieving the best performance among all the base stock vectors; that is, $\hat{S}^* = \text{argmax}\{\bar{G}(\hat{S}_l) : l = 1, \ldots, M\}$.

Let $\hat{G}_M = \frac{1}{M}\sum_{l=1}^{M} \hat{G}(\hat{S}_l)$, $\bar{G}_{N'} = \bar{G}(\hat{S}^*)$, and G^* be the optimal objective value of $\big(Joint(B)\big)$. Since $\bar{G}_{N'} \leq G^* \leq \hat{G}_M$ under certain conditions for N, M, N', see Birge and Louveaux [18], $\bar{G}_{N'}$ and $\hat{G}_M$ are, respectively, a lower and an upper bound for G^*. For more details concerning the statistical testing of optimality for the SAA method, and the selection of N, M, and N', see Kleywegt et al. [39]. Note that $O_{i,k} = \big(S_i - D_i^{L_i-k}\big)^+$ is a non-convex function of S_i; and we use the standard Big-M method (see section 2.3) to check whether $\big(S_i - D_i^{L_i-k}\big)$ is positive.

2.2 Impact of modifying the inventory availability constraints

While the (*INLP*) formulation uses a plus sign in the right hand side of the inventory constraints, Akçay and Xu [2] replace $\big(S_i - D_i^{L_i-k}\big)^+$ by $\big(S_i - D_i^{L_i-k}\big)$ in the computational experiments. The obtained new formulation (*ILP*) allows faster computation. Note that the feasible region of (*ILP*) is a subset of the feasible region of (*INLP*). In addition, while relaxing the integrality constraints on the variables would make (*ILP*) convex, (*INLP*) would remain non-convex due to the $\big(S_i - D_i^{L_i-k}\big)^+$ term in the right hand side of the inventory availability constraints. Note that substituting $\big(S_i - D_i^{L_i-k}\big)$ by $\big(S_i - D_i^{L_i-k}\big)^+$ may lead to infeasibility. This issue can be addressed by filtering out samples leading to infeasibility and by assuming sufficiently large budget level; that is, by allowing large base stock levels. We argue that substituting

$\left(S_i - D_i^{L_i-k}\right)$ for $\left(S_i - D_i^{L_i-k}\right)^+$ might yield an intractable sample generation process for the SAA approach for low budget levels.

2.2.1 Impact of modifying the inventory availability constraints on the sample generation

Generating enough samples such that the associated (*ILP*) formulation is feasible could be highly challenging for low budget levels. Note that under the extreme case of setting the budget to zero, the only sample yielding a feasible formulation is the trivial zero sample. Disregarding infeasible ones, we generate samples for (*ILP*) until the required number of samples, or a pre-set number of feasibility tests, is reached. For a given budget, the feasibility check is done by comparing with a computed minimum budget for a sample having a feasible solution. The computed minimum budget is determined from the (*ILP*) minimum base stock levels using Algorithm 1 described below. The non-negativity of the left hand side of the inventory availability constrains implies $(S_i - D_i^{L_i-k}) \geq 0$. Note that while we can generate enough feasible samples for (*ILP*), the mean and variance – i.e., the distribution – of generated sample are impacted and, thus, the SAA method.

Algorithm 1 Computing minimum feasible budget

Initialize $maxS \leftarrow zeros(n)$
for any realization h **do**
  for for any component i **do**
    if $D_i^{L_i} > maxS(i)$ **then**
      $maxS(i) \leftarrow D_i^{L_i}$
    end if
  end for
end for
$B = \sum_{i=1}^{n} c_i \times maxS(i)$

2.2.2 Impact of modifying the inventory availability constraints on the SAA method

Following the notation and discussion of Section 2.1.4, let $\bar{G}^{\bullet}_{N'}$, $G^{*}_{\bullet}$, and $\hat{G}^{\bullet}_{M}$ denote respectively the (*ILP*) lower bound, optimal value, and upper bound. Since $x \leq x^{+}$, any feasible solution of (*ILP*) is a feasible solution of (*INLP*). In addition, this inclusion is typically strict as one can set some base stocks to zero to build a solution feasible for (*INLP*) but infeasible for (*ILP*). To ensure a fair comparison, we only consider samples yielding feasible (*ILP*) and (*INLP*) formulations. Since, for a given sample, the optimal objective value for (*INLP*) is at least the one for (*ILP*), we have $G^{\bullet}_{M} \leq \hat{G}_{M}$.

2.3 Big-M method

As stated in section 2.2, the plus sign in the inventory availability constraints plays an important role, especially under low budget levels. Therefore, instead of dropping the plus sign directly, we introduce a standard linearizion technique, the Big-M method, to linearize the $\left(S_i - D_i^{L_i-k}\right)^{+}$ term.

Let M be a big positive number, i.e. $\max_{\omega}\{D_i^{L_i} + \sum_{j=1}^{n}(a_{i,j}P_j)\}$, and $z_{i,k}$ be a binary variable. Then we rewrite the inventory availability constraints, $\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}x_{j,\mu}) \leq \left(S_i - D_i^{L_i-k}\right)^{+}$, as follows:

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}) \leq M\, z_{i,k} \tag{2.13}$$

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}) \leq S_i - D_i^{L_i-k} + M\,(1 - z_{i,k})$$

$$S_i - D_i^{L_i-k} \leq M\, z_{i,k}$$

$$z_{i,k} \in \{0, 1\}.$$

When the value of $z_{i,k}$ is set to 0, (2.13) can be written as

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}) \leq 0 \tag{2.14}$$

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}) \leq S_i - D_i^{L_i-k} + M$$

$$S_i - D_i^{L_i-k} \leq 0,$$

and when the value of $z_{i,k}$ is set to 1, (2.13) will be written as:

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}) \leq M \tag{2.15}$$

$$\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\, x_{j,\mu}) \leq S_i - D_i^{L_i-k}$$

$$S_i - D_i^{L_i-k} \leq M.$$

Now the plus signs in the inventory availability constraints are safe to remove because of the existence of the Big-M. In constraints (2.14), the upper bound of the $\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\ x_{j,\mu})$ term is 0, and since this term is non-negative, we can get a feasible solution, i.e. setting all the base stock levels and assembled products to zeros. The constraints (2.15) can be simplified to $\sum_{\mu=0}^{k}\sum_{j=1}^{m}(a_{i,j}\ x_{j,\mu}) \leq S_i - D_i^{L_i-k}$, in which the plus signs are directly removed.

2.4 Earlier results

In Deza et al. [3], we studied the impact of component commonality on periodic review ATO systems and showed that lowering component commonality may yield a higher type-II service level. The lower degree of component commonality is achieved via separating inventories of the same component for different products. We substantiated this property via computational and theoretical approaches and showed that for low budget levels the use of separate inventories of the same component for different products could achieve a higher reward than with shared inventory. Finally, considering a simple ATO system consisting of one component shared by two products, we characterized the budget ranges such that the use of separate inventories is beneficial, as well as the budget ranges such that component commonality is beneficial. For more details and literature review, please refer to Deza et al. [3].

A natural research question arising from Deza et al. [3] is how to allocate inventories in ATO systems optimally to achieve higher reward. In this thesis, we further study this problem for two-product stochastic models with arbitrary number of common components and show that either full component commonality or non-component commonality does not work worse than the optimal partial component commonality. Components with common function can be replaced by a single one; such universal component is called *common*. A common component is called *dedicated* if it is used to assemble only one product, and *shared* if it is shared by more than one product. A product-specific component that is irreplaceable is called *non-common*.

Chapter 3

Theoretical results for two-product ATO systems

A few additional notations are required in the remainder of the thesis. Let $(BOM_{\circ}^{N})$, $(BOM_{\bullet}^{N})$ and $(BOM_{\textcolor{green}{\bullet}}^{N})$ denote, respectively, non-commonality, full commonality, and partial commonality configurations. Let $x_j^{\circ h}$, $x_j^{\bullet h}$ and $x_j^{\textcolor{green}{\bullet} h}$ denote the number of product j assembled at realization h for, respectively, $(BOM_{\circ}^{N})$, $(BOM_{\bullet}^{N})$ and $(BOM_{\textcolor{green}{\bullet}}^{N})$. Let $S_{j\_i}^{\circ}$ and $S_{j\_i}^{\textcolor{green}{\bullet}}$ denote, respectively, the base stock levels of dedicated component i for product j for $(BOM_{\circ}^{N})$ and $(BOM_{\textcolor{green}{\bullet}}^{N})$. Let $S_{i'}^{\textcolor{green}{\bullet}}$ and $S_{i'}^{\bullet}$ denote, respectively, the base stock levels of common component i' for $(BOM_{\textcolor{green}{\bullet}}^{N})$ and $(BOM_{\bullet}^{N})$. Finally, let c_{j_i} denote the cost of component i for product j.

3.1 Two-product system with full overlap

In the full overlap configuration, product 1 and product 2 use exactly the same set of components. To simplify the analysis, all the product time windows are set to 0 and BOMs are set to 1. In other words, each unit product only contains one unit component, and the reward can be collected only if the assembly happens in the same period of the arrival of the demand.

3.1.1 Non-commonality configuration ($BOM_{\circ}^{N}$)

The non-commonality configuration consists of two products, each comprising n different components, as shown in Table 3.2 where C_i^j denotes dedicated component i used to assemble product j.

	C_1^1	C_1^2	C_2^1	C_2^2	$\dots$	C_n^1	C_n^2
P_1	1	0	1	0	$\dots$	1	0
P_2	0	1	0	1	$\dots$	0	1

Table 3.2: BOM: non-commonality configuration with full overlap

The corresponding SAA formulation ($BOM_{\circ}^{N}$) is as follows:

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\circ h} + r_2\, x_2^{\circ h}) && && \left(BOM_{\circ}^{N}\right)\\
& x_1^{\circ h} \le (S_{1\_i}^{\circ} - D_1^h)^+ && i=1,\dots,n, && h=1,\dots,N\\
& x_2^{\circ h} \le (S_{2\_i}^{\circ} - D_2^h)^+ && i=1,\dots,n, && h=1,\dots,N\\
& x_1^{\circ h} \le P_1^h, \quad x_2^{\circ h} \le P_2^h && && h=1,\dots,N\\
& \sum_{i=1}^{n}(c_{1\_i}\, S_{1\_i}^{\circ} + c_{2\_i}\, S_{2\_i}^{\circ}) \le B && &&\\
& x_1^{\circ h}, x_2^{\circ h}, S_{1\_i}^{\circ}, S_{2\_i}^{\circ} \in \mathbb{Z}_+ && i=1,\dots,n, && h=1,\dots,N
\end{aligned}
$$

3.1.2 Full commonality configuration ($BOM_{\bullet}^{N}$)

In the full commonality configuration, component C_i^1 and C_i^2 in ($BOM_{\circ}^{N}$) are replaced by a common component C_i where $i = 1, \ldots, n$. Therefore there are n common components in total, see Table 3.3.

	C_1	C_2	C_3	$\ldots$	C_n
P_1	1	1	1	$\ldots$	1
P_2	1	1	1	$\ldots$	1

Table 3.3: BOM: full commonality configuration with full overlap

The corresponding SAA formulation ($BOM_{\bullet}^{N}$) is as follows:

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\bullet h} + r_2\, x_2^{\bullet h}) && (BOM_{\bullet}^{N})\\
x_1^{\bullet h} + x_2^{\bullet h} &\le (S_{i'}^{\bullet} - D_1^h - D_2^h)^+ && i' = 1, \ldots, n, \quad h = 1, \ldots, N\\
x_1^{\bullet h} \le P_1^h, \quad x_2^{\bullet h} &\le P_2^h && h = 1, \ldots, N\\
\sum_{i'=1}^{n} c_{i'}\, S_{i'}^{\bullet} &\le B\\
x_1^{\bullet h}, x_2^{\bullet h}, S_{i'}^{\bullet} &\in \mathbb{Z}_+ && i' = 1, \ldots, n, \quad h = 1, \ldots, N
\end{aligned}
$$

3.1.3 Partial commonality configuration ($BOM_{\bullet}^{N}$)

In a partial commonality configuration, let I be a nonempty and strict subset of $\{1, 2, \ldots, n\}$ such that components C_i^1 and C_i^2 in ($BOM_{\circ}^{N}$) are replaced by a common component C_i for $i \in I$. Without loss of generality, we can assume that $1 \notin I$ and $n \in I$, see Table 3.4 where $d = n - |I|$ is the number of dedicated components.

	C_1^1	C_1^2	$\ldots$	C_d^1	C_d^2	C_{d+1}	C_{d+2}	$\ldots$	C_{n-1}	C_n
P_1	1	0	$\ldots$	1	0	1	1	$\ldots$	1	1
P_2	0	1	$\ldots$	0	1	1	1	$\ldots$	1	1

Table 3.4: BOM: partial commonality configuration

The corresponding SAA formulation ($BOM_{\bullet}^N$) is as follows:

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\bullet h} + r_2\, x_2^{\bullet h}) && (BOM_{\bullet}^N) \\
& x_1^{\bullet h} \le (S_{1\_i}^{\bullet} - D_1^h)^+ && i = 1,\ldots,d, \quad h = 1,\ldots,N \\
& x_2^{\bullet h} \le (S_{2\_i}^{\bullet} - D_2^h)^+ && i = 1,\ldots,d, \quad h = 1,\ldots,N \\
& x_1^{\bullet h} + x_2^{\bullet h} \le (S_{i'}^{\bullet} - D_1^h - D_2^h)^+ && i' = d+1,\ldots,n, \quad h = 1,\ldots,N \\
& x_1^{\bullet h} \le P_1^h, \quad x_2^{\bullet h} \le P_2^h && h = 1,\ldots,N \\
& \sum_{i=1}^{d}(c_{1\_i}\, S_{1\_i}^{\bullet} + c_{2\_i}\, S_{2\_i}^{\bullet}) + \sum_{i'=d+1}^{n} c_{i'}\, S_{i'}^{\bullet} \le B \\
& x_1^{\bullet h}, x_2^{\bullet h}, S_{1\_i}^{\bullet}, S_{2\_i}^{\bullet}, S_{i'}^{\bullet} \in \mathbb{Z}_+ && i = 1,\ldots,n, \quad i' = d+1,\ldots,n, \quad h = 1,\ldots,N
\end{aligned}
$$

3.2 Two-product system with partial overlap

In a partial overlap configuration, some components are used only for product 1 or product 2 by design, therefore these components are not allowed to be replaced by common components.

3.2.1 Non-commonality configuration ($BOM^N_\circ$)

The non-commonality configuration consists of two products, product 1 comprising n_1 different components and product 2 comprising n_2 different components.

	C^1_{n+1}	$\ldots$	$C^1_{n_1}$	C^1_1	C^2_1	$\ldots$	C^1_n	C^2_n	C^2_{n+1}	$\ldots$	$C^2_{n_2}$
P_1	1	$\ldots$	1	1	0	$\ldots$	1	0	0	$\ldots$	0
P_2	0	$\ldots$	0	0	1	$\ldots$	0	1	1	$\ldots$	1

Table 3.5: BOM: non-commonality configuration with partial overlap

Let $B^\circ_1 = \sum_{i_1=n+1}^{n_1} c_{1\_i_1}\, S^\circ_{1\_i_1}$, and $B^\circ_2 = \sum_{i_2=n+1}^{n_2} c_{2\_i_2}\, S^\circ_{2\_i_2}$. Then the corresponding SAA formulation ($BOM^N_\circ$) is as follows:

$$\max \quad \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\circ_h} + r_2\, x_2^{\circ_h}) \qquad (BOM^N_\circ)$$

$$x_1^{\circ_h} \le (S^\circ_{1\_i_1} - D^h_1)^+ \qquad i_1 = n+1,\ldots,n_1, \quad h=1,\ldots,N$$

$$x_2^{\circ_h} \le (S^\circ_{2\_i_2} - D^h_2)^+ \qquad i_2 = n+1,\ldots,n_2, \quad h=1,\ldots,N$$

$$x_1^{\circ_h} \le (S^\circ_{1\_i} - D^h_1)^+ \qquad i = 1,\ldots,n, \quad h=1,\ldots,N$$

$$x_2^{\circ_h} \le (S^\circ_{2\_i} - D^h_2)^+ \qquad i = 1,\ldots,n, \quad h=1,\ldots,N$$

$$x_1^{\circ_h} \le P^h_1, \quad x_2^{\circ_h} \le P^h_2 \qquad h=1,\ldots,N$$

$$\sum_{i=1}^{n}(c_{1\_i}\, S^\circ_{1\_i} + c_{2\_i}\, S^\circ_{2\_i}) + B^\circ_1 + B^\circ_2 \le B$$

$$x_1^{\circ_h}, x_2^{\circ_h}, S^\circ_{1\_i}, S^\circ_{2\_i}, \in \mathbb{Z}_+ \qquad i = 1,\ldots,n, \quad h=1,\ldots,N$$

$$S^\circ_{1\_i_1}, S^\circ_{2\_i_2}, \in \mathbb{Z}_+ \qquad i_1 = n+1,\ldots,n_1, \quad i_2 = n+1,\ldots,n_2$$

3.2.2 Full commonality configuration ($BOM_{\bullet}^{N}$)

In the full commonality configuration, component C_i^1 and C_i^2 in ($BOM_{\circ}^{N}$) are replaced by a common component C_i where $i = 1, \ldots, n$. Therefore there are n common components in total, see Table 3.6.

	C_{n+1}^1	$\ldots$	$C_{n_1}^1$	C_1	C_2	C_3	$\ldots$	C_n	C_{n+1}^2	$\ldots$	$C_{n_2}^2$
P_1	1	$\ldots$	1	1	1	1	$\ldots$	1	0	$\ldots$	0
P_2	0	$\ldots$	0	1	1	1	$\ldots$	1	1	$\ldots$	1

Table 3.6: BOM: full commonality configuration with partial overlap

Let $B_1^{\bullet} = \sum_{i_1=n+1}^{n_1} c_{1\_i_1} S_{1\_i_1}^{\bullet}$, and $B_2^{\bullet} = \sum_{i_2=n+1}^{n_2} c_{2\_i_2} S_{2\_i_2}^{\bullet}$. Then the corresponding SAA formulation ($BOM_{\bullet}^{N}$) is as follows:

$$
\begin{aligned}
\max \quad & \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\bullet h} + r_2\, x_2^{\bullet h}) && \left(BOM_{\bullet}^{N}\right)\\
x_1^{\bullet h} &\le (S_{1\_i_1}^{\bullet} - D_1^h)^+ && i_1 = n+1, \ldots, n_1, \quad h = 1, \ldots, N\\
x_2^{\bullet h} &\le (S_{2\_i_2}^{\bullet} - D_2^h)^+ && i_2 = n+1, \ldots, n_2, \quad h = 1, \ldots, N\\
x_1^{\bullet h} + x_2^{\bullet h} &\le (S_{i'}^{\bullet} - D_1^h - D_2^h)^+ && i' = 1, \ldots, n, \quad h = 1, \ldots, N\\
x_1^{\bullet h} \le P_1^h, \quad x_2^{\bullet h} &\le P_2^h && h = 1, \ldots, N\\
\sum_{i'=1}^{n} c_{i'}\, S_{i'}^{\bullet} + B_1^{\bullet} + B_2^{\bullet} &\le B\\
x_1^{\bullet h}, x_2^{\bullet h}, S_{i'}^{\bullet} &\in \mathbb{Z}_+ && i' = 1, \ldots, n, \quad h = 1, \ldots, N\\
S_{1\_i_1}^{\bullet}, S_{2\_i_2}^{\bullet}, &\in \mathbb{Z}_+ && i_1 = n+1, \ldots, n_1, \quad i_2 = n+1, \ldots, n_2
\end{aligned}
$$

3.2.3 Partial commonality configuration ($BOM^N_\bullet$)

In a partial commonality configuration, let I be a nonempty and strict subset of $\{1, 2, \ldots, n\}$ such that components C^1_i and C^2_i in ($BOM^N_\circ$) are replaced by a common component C_i for $i \in I$. Without loss of generality, we can assume that $1 \notin I$ and $n \in I$, see Table 3.7 where $d = n - |I|$ is the number of dedicated components.

	C^1_{n+1}	$\ldots$	$C^1_{n_1}$	C^1_1	C^2_1	$\ldots$	C^1_d	C^2_d	C_{d+1}	$\ldots$	C_n	C^2_{n+1}	$\ldots$	$C^2_{n_2}$
P_1	1	$\ldots$	1	1	0	$\ldots$	1	0	1	$\ldots$	1	0	$\ldots$	0
P_2	0	$\ldots$	0	0	1	$\ldots$	0	1	1	$\ldots$	1	1	$\ldots$	1

Table 3.7: BOM: partial commonality configuration with partial overlap

Let $B^\bullet_1 = \sum_{i_1=n+1}^{n_1} c_{1\_i_1} S^\bullet_{1\_i_1}$, and $B^\bullet_2 = \sum_{i_2=n+1}^{n_2} c_{2\_i_2} S^\bullet_{2\_i_2}$. Then the corresponding SAA formulation ($BOM^N_\bullet$) is as follows:

$$
\begin{aligned}
\max \qquad & \frac{1}{N}\sum_{h=1}^{N}(r_1\, x_1^{\bullet h} + r_2\, x_2^{\bullet h}) && \left(BOM_{\bullet}^{N}\right) \\
x_1^{\bullet h} &\le (S^{\bullet}_{1\_i_1} - D_1^h)^+ && i_1 = n+1,\ldots,n_1, \quad h = 1,\ldots,N \\
x_2^{\bullet h} &\le (S^{\bullet}_{2\_i_2} - D_2^h)^+ && i_2 = n+1,\ldots,n_2, \quad h = 1,\ldots,N \\
x_1^{\bullet h} &\le (S^{\bullet}_{1\_i} - D_1^h)^+ && i = 1,\ldots,d, \quad h = 1,\ldots,N \\
x_2^{\bullet h} &\le (S^{\bullet}_{2\_i} - D_2^h)^+ && i = 1,\ldots,d, \quad h = 1,\ldots,N \\
x_1^{\bullet h} + x_2^{\bullet h} &\le (S^{\bullet}_{i'} - D_1^h - D_2^h)^+ && i' = d+1,\ldots,n, \quad h = 1,\ldots,N \\
x_1^{\bullet h} \le P_1^h, \quad x_2^{\bullet h} &\le P_2^h && h = 1,\ldots,N \\
\sum_{i=1}^{d}(c_{1\_i}\, S^{\bullet}_{1\_i} + c_{2\_i}\, S^{\bullet}_{2\_i}) + \sum_{i'=d+1}^{n} c_{i'}\, S^{\bullet}_{i'} + B^{\bullet}_1 + B^{\bullet}_2 &\le B && \\
x_1^{\bullet h}, x_2^{\bullet h} &\in \mathbb{Z}_+ && h = 1,\ldots,N \\
S^{\bullet}_{1\_i}, S^{\bullet}_{2\_i}, S^{\bullet}_{i'} &\in \mathbb{Z}_+ && i = 1,\ldots,n, \quad i' = d+1,\ldots,n \\
S^{\bullet}_{1\_i_1}, S^{\bullet}_{2\_i_2}, &\in \mathbb{Z}_+ && i_1 = n+1,\ldots,n_1, \quad i_2 = n+1,\ldots,n_2
\end{aligned}
$$

3.3 Main theorem

The existence of partial commonality structure makes possible ATO systems more challenging and significantly increases the number of possible BOMs. Theorem 1 states that an optimal BOM can be found by assuming either the full commonality or the non-commonality configuration. Consequently, a search through possibly exponential number of BOMs can be avoided.

Theorem 1. *Given a budget B, let $x_1^{{\color{green}\bullet}h*}$ and $x_2^{{\color{green}\bullet}h*}$ denote the optimal solutions of $(BOM^N_{\color{green}\bullet})$ for $h = 1, \ldots, N$. Then, $x_1^{{\color{green}\bullet}h*}$ and $x_2^{{\color{green}\bullet}h*}$ are feasible solutions in either $(BOM^N_\bullet)$ or $(BOM^N_\circ)$.*

3.4 Examples contrasting and comparing $(BOM^N_\circ)$ and $(BOM^N_\bullet)$

Before proving Theorem 1 in Chapter 4, we wish to provide simple examples illustrating that a feasible allocation for partial commonality can be infeasible for full commonality or non-commonality, and that non-commonality can be beneficial over full commonality under some conditions.

3.4.1 A feasible allocation for partial commonality can be infeasible for full commonality

Due to the plus sign in the $(BOM^N_{\color{green}\bullet})$ and $(BOM^N_\bullet)$ formulations, $x_1^{{\color{green}\bullet}h} \leq (S^{\color{green}\bullet}_{1\_i} - D_1^h)^+$ and $x_2^{{\color{green}\bullet}h} \leq (S^{\color{green}\bullet}_{2\_i} - D_2^h)^+$ do not always imply that $x_1^{\bullet h} + x_2^{\bullet h} \leq (S^\bullet_{i'} - D_1^h - D_2^h)^+$. Assume that in the $(BOM^N_{\color{green}\bullet})$ formulation, $S^{\color{green}\bullet}_{i'} > S^{\color{green}\bullet}_{1\_i} + S^{\color{green}\bullet}_{2\_i}$ and consider the following example:

Partial commonality: Let $S^{\color{green}\bullet}_{1\_i} - D_1^h > 0$, $S^{\color{green}\bullet}_{2\_i} - D_2^h \leq 0$ and $S^{\color{green}\bullet}_{i'} - D_1^h - D_2^h > 0$; then $x_1^{{\color{green}\bullet}h} > 0$ and $x_2^{{\color{green}\bullet}h} = 0$ forms a feasible allocation for partial commonality.

Full commonality: Let $S^\bullet_{i'} = S^{\color{green}\bullet}_{1\_i} + S^{\color{green}\bullet}_{2\_i} < S^{\color{green}\bullet}_{i'}$ and then it is possible to have $S^\bullet_{i'} - D_1^h - D_2^h \leq 0$ for $i' = 1, \ldots, d$. Therefore, $x_1^{\bullet h} = 0$ and $x_2^{\bullet h} = 0$ is the only feasible allocation for full commonality.

3.4.2 A feasible allocation for partial commonality can be infeasible for non-commonality

Assume that in the $(BOM^N_\bullet)$ formulation, $S^\bullet_{i'} < S^\bullet_{1\_i} + S^\bullet_{2\_i}$ and consider the following example:

Partial commonality: Let $S^\bullet_{1\_i} - D^h_1 > 0$, $S^\bullet_{2\_i} - D^h_2 > 0$ and $S^\bullet_{i'} - D^h_1 - D^h_2 > 0$; then $x_1^{\bullet h} > 0$ and $x_2^{\bullet h} > 0$ forms a feasible allocation for partial commonality.

Non-commonality: Let $S^\circ_{1\_i} + S^\circ_{2\_i} = S^\bullet_{i'} < S^\bullet_{1\_i} + S^\bullet_{2\_i}$ and then it is possible to have $S^\circ_{1\_i} - D^h_1 \leq 0$ for $i = d+1, \ldots, n$. Therefore $x_1^{\circ h} = 0$ and $x_2^{\circ h} > 0$ is a feasible allocation for full commonality.

All plus signs in the $(BOM^N_\bullet)$ formulation can be removed for this example. Thus, any feasible allocation for partial commonality is feasible for full commonality; that is, full commonality performs at least as well as non-commonality for such instances.

3.4.3 Non-commonality can be beneficial over full commonality under some condition

Consider an ATO system consisting of 2 components shared by 2 products, and assume that $B = 10, c_1 = c_2 = r_1 = r_2 = 1, N = 2, D^1_1 = 1, D^1_2 = 4, P^1_1 = P^1_2 = 1, D^2_1 = 2, D^2_2 = 3$, and $P^2_1 = P^2_2 = 1$.

Full commonality: For both realizations, 5 units C_1 and 5 units C_2 are used to fulfill previous orders and, at the current period, there is no component available for further assembly. Therefore, $x_1^{\bullet h} = x_2^{\bullet h} = 0$ and the optimal value is 0.

Non-commonality: Let $S^{\circ}_{1\_1} = S^{\circ}_{1\_2} = 2$ and $S^{\circ}_{2\_1} = S^{\circ}_{2\_2} = 3$. For the first realization, 1 unit C_1^1, 1 unit C_2^1 and all 3 units C_1^2 and 1 unit C_2^2 are used to fulfill previous orders. At the current period, there are 1 unit C_1^1 and 1 unit C_2^1 still available. Thus, $x_1^{\circ 1} = 1$. For the second realization, all components are used to fulfill previous orders. Thus, $x_1^{\circ 2} = 0 = x_2^{\circ 2} = 0$ and the objective value is 1.

Chapter 4

Proof of Theorem 1

4.1 Two-product system with full overlap

Let $x_{j,h}$, $y_{j,h}$ and $z_{j,h}$ denote, respectively, a feasible solution for product j in realization h for $(BOM^N_{\circ})$, $(BOM^N_{\textcolor{green}{\bullet}})$ and $(BOM^N_{\bullet})$. In $(BOM^N_{\textcolor{green}{\bullet}})$, due to the symmetry of the structure, we can assume, at optimality, that the base stock levels of the dedicated components for product 1 are equally distributed; that is, $S^{\textcolor{green}{\bullet}}_{1\_i_\alpha}{}^* = S^{\textcolor{green}{\bullet}}_{1\_i_\beta}{}^*$, where $1 \leq i_\alpha \leq i_\beta \leq d$. This is also true for the dedicated components for product 2 and shared components. The base stock levels are independent of the component indexes i and i', and therefore we use the following additional notations in Section 4. Let Y_j and Y denote, respectively, the base stock levels of any dedicated component for product j and any shared component. Recall that a superscripted $*$ indicates an optimal solution. Let Y_j^* denote an optimal base stock level of any dedicated component for product j; that is, $S^{\textcolor{green}{\bullet}}_{1\_i}{}^* = Y_1^*$ and $S^{\textcolor{green}{\bullet}}_{2\_i}{}^* = Y_2^*$ for all i. Finally, let Y^* denote an optimal base stock level of any shared component; that is, $S^{\textcolor{green}{\bullet}}_{i'}{}^* = Y^*$ for all i'.

We have the following assumptions:

1. While proving $y^*_{1,h}$ and $y^*_{2,h}$ are feasible in $(BOM^N_\circ)$, let $S^\circ_{1\_i} = Y^*_1$ and $S^\circ_{2\_i} = Y^*_2$ when $i = 1, \cdots, d$; $S^\circ_{1\_i}+S^\circ_{2\_i} = Y^*$, $S^\circ_{1\_i_\alpha} = S^\circ_{1\_i_\beta}$ and $S^\circ_{2\_i_\alpha} = S^\circ_{2\_i_\beta}$ when $i, i_\alpha, i_\beta = d+1, \cdots, n$. To simplify the notation, let X_j and U_j denote, respectively, the base stock levels of dedicated components for product j for $(BOM^N_\circ)$ when $i = 1, \ldots, d$ and when $i = d+1, \ldots, n$; that is, $X_j = Y^*_j$ and $U_1 + U_2 = Y^*$.

2. While proving $y^*_{1,h}$ and $y^*_{2,h}$ are feasible in $(BOM^N_\bullet)$, let $S^\bullet_{i'} = Y^*_1 + Y^*_2$ when $i' = 1, \cdots, d$; and $S^\bullet_{i'} = Y^*$ when $i' = d+1, \cdots, n$. To simplify the notation, let Z and V denote, respectively, the base stock levels of shared components for $(BOM^N_\bullet)$ when $i' = 1, \ldots, d$ and when $i' = d+1, \ldots, n$; that is, $Z = Y^*_1 + Y^*_2$ and $V = Y^*$.

3. The cost of a shared component is equal to the cost of the dedicated component it replaces. In the full overlap configuration, all components are potential shared components; that is, $c_{1\_i} = c_{2\_i} = c_{i'}$ for all indexes i and i'.

4.1.1 Case $N = 1$

We first consider the case $N = 1$; that is, only one realization is used in the SAA method. The associated formulations $(BOM^1_\circ)$, $(BOM^1_\bullet)$ and $(BOM^1_{\bullet})$ correspond to a deterministic demand where P^1_1 and P^1_2 represent the demands in the current period for, respectively, product 1 and 2, and D^1_1 and D^1_2 represent the overall demands from all previous periods.

$$
\begin{aligned}
\max \qquad & r_1\, x_{1,1} + r_2\, x_{2,1} && \left(BOM_{\circ}^{1}\right)\\
x_{1,1} &\leq (X_1 - D_1^1)^+ \\
x_{1,1} &\leq (U_1 - D_1^1)^+ \\
x_{2,1} &\leq (X_2 - D_2^1)^+ \\
x_{2,1} &\leq (U_2 - D_2^1)^+ \\
x_{1,1} \leq P_1^1, &\quad x_{2,1} \leq P_2^1 \\
X_1 \sum_{i=1}^{d} c_{1\_i} + X_2 \sum_{i=1}^{d} c_{2\_i} + U_1 \sum_{i=d+1}^{n} c_{1\_i} + U_2 \sum_{i=d+1}^{n} c_{2\_i} &\leq B \\
x_{1,1}, x_{2,1}, X_1, X_2, U_1, U_2 &\in \mathbb{Z}_+
\end{aligned}
$$

$$
\begin{aligned}
\max \qquad & r_1\, z_{1,1} + r_2\, z_{2,1} && \left(BOM_{\bullet}^{1}\right)\\
z_{1,1} + z_{2,1} &\leq (Z - D_1^1 - D_2^1)^+ \\
z_{1,1} + z_{2,1} &\leq (V - D_1^1 - D_2^1)^+ \\
z_{1,1} \leq P_1^1, &\quad z_{2,1} \leq P_2^1 \\
Z \sum_{i'=1}^{d} c_{i'} + V \sum_{i'=d+1}^{n} c_{i'} &\leq B \\
z_{1,1}, z_{2,1}, Z, V &\in \mathbb{Z}_+
\end{aligned}
$$

$$
\begin{aligned}
\max \qquad & r_1\, y_{1,1} + r_2\, y_{2,1} && (BOM_{\bullet}^{1})\\
y_{1,1} &\le (Y_1 - D_1^1)^+ \\
y_{2,1} &\le (Y_2 - D_2^1)^+ \\
y_{1,1} + y_{2,1} &\le (Y - D_1^1 - D_2^1)^+ \\
y_{1,1} \le P_1^1, &\quad y_{2,1} \le P_2^1 \\
Y_1 \sum_{i=1}^{d} c_{1\_i} + Y_2 \sum_{i=1}^{d} c_{2\_i} + Y \sum_{i'=d+1}^{n} c_{i'} &\le B \\
y_{1,1}, y_{2,1}, Y_1, Y_2, Y &\in \mathbb{Z}_+
\end{aligned}
$$

First of all, we want to prove that with the constraint $Y_1^* \sum_{i=1}^{d} c_{1\_i} + Y_2^* \sum_{i=1}^{d} c_{2\_i} + Y^* \sum_{i'=d+1}^{n} c_{i'} \le B$, either the constraint $X_1 \sum_{i=1}^{d} c_{1\_i} + X_2 \sum_{i=1}^{d} c_{2\_i} + U_1 \sum_{i=d+1}^{n} c_{1\_i} + U_2 \sum_{i=d+1}^{n} c_{2\_i} \le B$ or the constraint $Z \sum_{i'=1}^{d} c_{i'} + V \sum_{i'=d+1}^{n} c_{i'} \le B$ holds. The former can be proved by substituting assumptions 1 and 3, while the latter can be proved by substituting assumptions 2 and 3.

Then, to show that $y_{1,1}^*$ and $y_{2,1}^*$ is feasible for either $(BOM_{\bullet}^{1})$ or $(BOM_{\circ}^{1})$, we consider the following three cases.

Case 1: Reward from both product 1 and 2 are 0, i.e. $y_{1,1}^* = 0$ and $y_{2,1}^* = 0$ and the point $y_{1,1}^* = 0$ and $y_{2,1}^* = 0$ is a feasible solution for either $(BOM_{\bullet}^{1})$ or $(BOM_{\circ}^{1})$.

Take $(BOM_{\bullet}^{1})$ as an example:

- $y^*_{1,1} + y^*_{2,1} = 0 \leq (Z - D^1_1 - D^1_2)^+$, this is always true by the definition of +.
- $y^*_{1,1} + y^*_{2,1} = 0 \leq (V - D^1_1 - D^1_2)^+$, this is always true by the definition of +.
- $y^*_{1,1} = 0 \leq P^1_1, \quad y^*_{2,1} = 0 \leq P^1_2$, this is always true because P^1_1 andP^1_2 are both nonnegative.
- $y^*_{1,1}, y^*_{2,1} \in \mathbb{Z}_+$, this is always true because 0 is a nonnegative integer.

Note: If the optimal solution $y^*_{j,h}$ is zero, then the point $y^*_{j,h} = 0$ is feasible for either $(BOM^N_\bullet)$ or $(BOM^N_\circ)$.

Case 2: We get some reward from exactly one of the products.

Case 2.1: Getting reward only from product 1, i.e. $y^*_{1,1} > 0$, and $y^*_{2,1} = 0$. We want to show that the point $y^*_{1,1} > 0$, and $y^*_{2,1} = 0$ is a feasible solution for $BOM^1_\circ$.

$y^*_{2,1} = 0$ is a feasible solution of $(BOM^1_\circ)$. Since $y^*_{1,1}$ is an optimal solution of $(BOM^1_\bullet)$, the following inequalities are valid:

$$y^*_{1,1} \leq (Y^*_1 - D^1_1)^+$$
$$y^*_{1,1} \leq (Y^* - D^1_1 - D^1_2)^+$$

To prove $y^*_{1,1}$ is feasible in $(BOM^1_\circ)$, we need to show that $y^*_{1,1} \leq (X_1 - D^1_1)^+$ and $y^*_{1,1} \leq (U_1 - D^1_1)^+$. Let $U_2 = 0$; that is, all the budget spent on the shared components is used to buy dedicated components for product 1.

$$y^*_{1,1} \leq (Y^*_1 - D^1_1)^+ = (X_1 - D^1_1)^+ \qquad < substitution >$$

$$y^*_{1,1} \leq (Y^* - D^1_1 - D^1_2)^+ = (U_1 - D^1_1 - D^1_2)^+ \leq (U_1 - D^1_1)^+ \qquad < recall\ D^1_2 \geq 0 >$$

Case 2.2: Getting reward only from product 2, i.e. $y^*_{1,1} = 0$, and $y^*_{2,1} > 0$. We want to show that the point $y^*_{1,1} = 0$, and $y^*_{2,1} > 0$ is a feasible solution for $(BOM^1_\circ)$. The proof is the same as for Case 2.1 considering $U_1 = 0$.

Case 3: We get reward from both products 1 and 2, i.e. $y^*_{1,1} > 0$ and $y^*_{2,1} > 0$. We want to show that the point $y^*_{1,1} > 0$ and $y^*_{2,1} > 0$ is a feasible solution for $(BOM^1_\bullet)$.

Since $y^*_{1,1}$ and $y^*_{2,1}$ is an optimal solution of $(BOM^1_\bullet)$, the following inequalities hold:

$$y^*_{1,1} \leq (Y^*_1 - D^1_1)^+$$

$$y^*_{2,1} \leq (Y^*_2 - D^2_1)^+$$

$$y^*_{1,1} + y^*_{2,1} \leq (Y^* - D^1_1 - D^1_2)^+$$

To prove $y^*_{1,1}$ and $y^*_{2,1}$ is feasible in $(BOM^1_\bullet)$, we need to show that $y^*_{1,1} + y^*_{2,1} \leq (Z - D^1_1 - D^1_2)^+$ and $y^*_{1,1} + y^*_{2,1} \leq (V - D^1_1 - D^1_2)^+$.

Since $y^*_{1,1} > 0$ and $y^*_{2,1} > 0$, all the plus signs can be removed.

$$
\begin{aligned}
& y_{1,1}^* \le Y_1^* - D_1^1 \quad \text{and} \quad y_{2,1}^* \le Y_2^* - D_1^2 \\
\Longrightarrow \quad & y_{1,1}^* + y_{2,1}^* \quad \le \quad Y_1^* + Y_2^* - D_1^1 - D_2^1 \\
\Longrightarrow \quad & \qquad\qquad\quad = \quad Z - D_1^1 - D_2^1,
\end{aligned}
$$

and

$$
y_{1,1}^* + y_{2,1}^* \le (Y^* - D_1^1 - D_2^1)^+ = (V - D_1^1 - D_2^1)^+. \qquad < \textit{substitution} >
$$

4.1.2 Case $N = 2$

We consider the case $N = 2$; that is the simplest random demand with only two realizations. We assume that both realizations have probability 0.5 and omit this constant term in the objectives for clarity. In the associated formulations $(BOM_\circ^N)$, $(BOM_\bullet^N)$ and $(BOM_\bullet^N)$ below, superscripts are use to distinguish different realizations. For example, $x_{1,2}, x_{2,2}, D_1^2, D_2^2, P_1^2$, and P_2^2 refer to the second realization.

$$
\begin{aligned}
\max \quad & r_1\, x_{1,1} + r_2\, x_{2,1} + r_1\, x_{1,2} + r_2\, x_{2,2} && \left(BOM^2_\circ\right)\\
& x_{1,1} \le (X_1 - D^1_1)^+\\
& x_{1,1} \le (U_1 - D^1_1)^+\\
& x_{2,1} \le (X_2 - D^1_2)^+\\
& x_{2,1} \le (U_2 - D^1_2)^+\\
& x_{1,2} \le (X_1 - D^2_1)^+\\
& x_{1,2} \le (U_1 - D^2_1)^+\\
& x_{2,2} \le (X_2 - D^2_2)^+\\
& x_{2,2} \le (U_2 - D^2_2)^+\\
& x_{1,1} \le P^1_1, \quad x_{2,1} \le P^1_2, \quad x_{1,2} \le P^2_1, \quad x_{2,2} \le P^2_2\\
& X_1 \sum_{i=1}^{d} c_{1\_i} + X_2 \sum_{i=1}^{d} c_{2\_i} + U_1 \sum_{i=d+1}^{n} c_{1\_i} + U_2 \sum_{i=d+1}^{n} c_{2\_i} \le B\\
& x_{1,1}, x_{2,1}, x_{1,2}, x_{2,2}, X_1, X_2, U_1, U_2 \in \mathbb{Z}_+
\end{aligned}
$$

$$
\begin{aligned}
\max \qquad & r_1\, z_{1,1} + r_2\, z_{2,1} + r_1\, z_{1,2} + r_2\, z_{2,2} && (BOM_{\bullet}^{2}) \\
& z_{1,1} + z_{2,1} \le (Z - D_1^1 - D_2^1)^+ \\
& z_{1,1} + z_{2,1} \le (V - D_1^1 - D_2^1)^+ \\
& z_{1,2} + z_{2,2} \le (Z - D_1^2 - D_2^2)^+ \\
& z_{1,2} + z_{2,2} \le (V - D_1^2 - D_2^2)^+ \\
& z_{1,1} \le P_1^1, \quad z_{2,1} \le P_2^1, \quad z_{1,2} \le P_1^2, \quad z_{2,2} \le P_2^2 \\
& Z \sum_{i'=1}^{d} c_{i'} + V \sum_{i'=d+1}^{n} c_{i'} \le B \\
& z_{1,1}, z_{2,1}, z_{1,2}, z_{2,2}, X_1, X_2, U_1, U_2 \in \mathbb{Z}_+
\end{aligned}
$$

$$
\begin{aligned}
\max \qquad r_1\, y_{1,1} + r_2\, y_{2,1} + r_1\, y_{1,2} + r_2\, y_{2,2} & \qquad \qquad (BOM^2_\bullet) \\
y_{1,1} \leq (Y_1 - D_1^1)^+ & \\
y_{2,1} \leq (Y_2 - D_2^1)^+ & \\
y_{1,1} + y_{2,1} \leq (Y - D_1^1 - D_2^1)^+ & \\
y_{1,2} \leq (Y_1 - D_1^2)^+ & \\
y_{2,2} \leq (Y_2 - D_2^2)^+ & \\
y_{1,2} + y_{2,2} \leq (Y - D_1^2 - D_2^2)^+ & \\
y_{1,1} \leq P_1^1, \quad y_{2,1} \leq P_2^1, \quad y_{1,2} \leq P_1^2, \quad y_{2,2} \leq P_2^2 & \\
Y_1 \sum_{i=1}^{d} c_{1\_i} + Y_2 \sum_{i=1}^{d} c_{2\_i} + Y \sum_{i'=d+1}^{n} c_{i'} \leq B & \\
y_{1,1}, y_{2,1}, y_{1,2}, y_{2,2}, X_1, X_2, U_1, U_2 \in \mathbb{Z}_+ &
\end{aligned}
$$

Case 1: Reward from both product 1 and 2 for both realizations are 0, i.e. $y^*_{1,1} = 0, y^*_{2,1} = 0, y^*_{1,2} = 0$ and $y^*_{2,2} = 0$. We want to show that this point is a feasible solution for either $(BOM^2_\bullet)$ or $(BOM^2_\circ)$.

The proof is the same as for $N = 1$ Case 1.

Case 2: We get some reward from exactly one of the products.

Case 2.1: Getting reward only from product 1.

Case 2.1.1: Getting reward only from the first realization, i.e. $y^*_{1,1} > 0, y^*_{2,1} = 0, y^*_{1,2} = 0$ and $y^*_{2,2} = 0$. We want to show that this point is a feasible solution for $(BOM^2_\circ)$.

The proof is the same as for $N = 1$ Case 2.1.

Case 2.1.2: Getting reward only from the second realization, i.e. $y^*_{1,1} = 0, y^*_{2,1} = 0, y^*_{1,2} > 0$ and $y^*_{2,2} = 0$. We want to show that this point is a feasible solution for $(BOM^2_\circ)$.

The proof is the same as for $N = 1$ Case 2.1 replacing D^1_1 by D^2_1 and D^1_2 by D^2_2.

Case 2.1.3: Getting reward from both realizations, i.e. $y^*_{1,1} > 0, y^*_{2,1} = 0, y^*_{1,2} > 0$ and $y^*_{2,2} = 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y^*_1 + Y^*_2$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y^*_1 + Y^*_2$.

Since $y^*_{1,1}, y^*_{2,1}, y^*_{1,2}$ and $y^*_{2,2}$ is an optimal point of $BOM^2_\bullet$, the following inequalities hold:

$$y^*_{1,1} \leq (Y^*_1 - D^1_1)^+$$

$$Y^*_2 - D^2_1 \leq 0$$

$$y^*_{1,1} \leq (Y^* - D^1_1 - D^1_2)^+$$

$$y^*_{1,2} \leq (Y^*_1 - D^2_1)^+$$

$$Y^*_2 - D^2_2 \leq 0$$

$$y^*_{1,2} \leq (Y^* - D^2_1 - D^2_2)^+$$

When proving $y^*_{1,1}, y^*_{2,1}, y^*_{1,2}$ and $y^*_{2,2}$ is feasible in $(BOM^2_\circ)$, we assume that $Y^* \geq Y^*_1 + Y^*_2$ and we need to show that

$$y^*_{1,1} \leq (X_1 - D^1_1)^+ \tag{4.16}$$

$$y^*_{1,1} \leq (U_1 - D^1_1)^+ \tag{4.17}$$

$$y^*_{1,2} \leq (X_1 - D^2_1)^+ \tag{4.18}$$

$$y^*_{1,2} \leq (U_1 - D^2_1)^+ \tag{4.19}$$

Since $y^*_{1,1} > 0$ and $y^*_{1,2} > 0$, all the plus signs can be removed. Let $U_2 = Y^*_2$ and thus $U_2 - D^2_1 \leq 0$ and $U_2 - D^2_2 \leq 0$.

$$\begin{aligned}
\text{For (4.16)}: \quad & y^*_{1,1} \leq (Y^*_1 - D^1_1)^+ = (X_1 - D^1_1)^+; \qquad < \text{substitution} > \\
\text{For (4.17)}: \quad & y^*_{1,1} \leq Y^* - D^1_1 - D^1_2 = (U_1 - D^1_1) + (U_2 - D^2_1) \leq U_1 - D^1_1, \\
\text{and} \quad & U_1 - D^1_1 = (Y^* - U_2) - D^1_1 \geq Y^*_1 + Y^*_2 - U_2 - D^1_1 = Y^*_1 - D^1_1 > 0, \\
\text{therefore} \quad & y^*_{1,1} \leq (U_1 - D^1_1)^+. \\
\text{For (4.18)}: \quad & y^*_{1,2} \leq (Y^*_1 - D^2_1)^+ = (X_1 - D^2_1)^+, \qquad < \text{substitution} > \\
\text{For (4.19)}: \quad & y^*_{1,2} \leq Y^* - D^2_1 - D^2_2 = (U_1 - D^2_1) + (U_2 - D^2_2) \leq U_1 - D^2_1, \\
\text{and} \quad & U_1 - D^2_1 = (Y^* - U_2) - D^2_1 \geq Y^*_1 + Y^*_2 - U_2 - D^2_1 = Y^*_1 - D^2_1 > 0, \\
\text{therefore} \quad & y^*_{1,2} \leq (U_1 - D^2_1)^+.
\end{aligned}$$

When proving $y^*_{1,1}, y^*_{2,1}, y^*_{1,2}$ and $y^*_{2,2}$ is feasible in $(BOM^2_\bullet)$, we assume that $Y^* < Y^*_1 + Y^*_2$ and we need to show that

$$y^*_{1,1} \leq (Z - D^1_1 - D^1_2)^+ \tag{4.20}$$

$$y^*_{1,1} \leq (V - D^1_1 - D^1_2)^+ \tag{4.21}$$

$$y^*_{1,2} \leq (Z - D^2_1 - D^2_2)^+ \tag{4.22}$$

$$y^*_{1,2} \leq (V - D^2_1 - D^2_2)^+ \tag{4.23}$$

Based on Assumption 2, we have $V < Z$.

$$\text{For (4.21)}: \quad y^*_{1,1} \leq (Y^* - D^1_1 - D^1_2)^+ = (V - D^1_1 - D^1_2)^+; \qquad < \text{substitution} >$$

$$\text{For (4.20)}: \quad y^*_{1,1} \leq (V - D^1_1 - D^1_2)^+ < (Z - D^1_1 - D^1_2)^+ \leq (Z - D^1_1 - D^1_2)^+;$$

$$\text{For (4.23)}: \quad y^*_{1,2} \leq (Y^* - D^2_1 - D^2_2)^+ = (V - D^2_1 - D^2_2)^+; \qquad < \text{substitution} >$$

$$\text{For (4.22)}: \quad y^*_{1,2} \leq (V - D^2_1 - D^2_2)^+ < (Z - D^2_1 - D^2_2)^+ \leq (Z - D^2_1 - D^2_2)^+.$$

Case 2.2: Getting reward only from product 2.

Case 2.2.1: Getting reward only from the first realization, i.e. $y^*_{1,1} = 0, y^*_{2,1} > 0, y^*_{1,2} = 0$ and $y^*_{2,2} = 0$. We want to show that this point is a feasible solution for $(BOM^2_\circ)$.

The proof is the same as for $N = 1$ Case 2.2.

Case 2.2.2: Getting reward only from the second realization, i.e. $y^*_{1,1} = 0, y^*_{2,1} = 0, y^*_{1,2} = 0$ and $y^*_{2,2} > 0$. We want to show that this point is a feasible solution for $(BOM^2_\circ)$.

The proof is the same as for $N = 1$ Case 2.2 replacing D^1_1 by D^2_1 and D^1_2 by D^2_2.

Case 2.2.3: Getting reward from both realizations, i.e. $y^*_{1,1} = 0, y^*_{2,1} > 0, y^*_{1,2} = 0$ and $y^*_{2,2} > 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y^*_1 + Y^*_2$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y_1^* + Y_2^*$.

The proof is the same as for $N = 2$ Case 2.1.3 setting $U_1 = Y_1^*$.

Case 3: We get reward from both product 1 and 2.

Case 3.1: Getting reward from same realization.

Case 3.1.1: Getting reward from the first realization, i.e. $y^*_{1,1} > 0, y^*_{2,1} > 0, y^*_{1,2} = 0$ and $y^*_{2,2} = 0$. We want to show that this point is a feasible solution for $(BOM^2_\bullet)$.

The proof is the same as for $N = 1$ Case 3.

Case 3.1.2: Getting reward from the second realization, i.e. $y^*_{1,1} = 0, y^*_{2,1} = 0, y^*_{1,2} > 0$ and $y^*_{2,2} > 0$. We want to show that this point is a feasible solution for $(BOM^2_\bullet)$.

The proof is the same as for $N = 1$ Case 2.2 replacing D_1^1 by D_1^2 and D_2^1 by D_2^2.

Case 3.2: Getting reward from different realization.

Case 3.2.1: Getting reward from product 1 in the first realization and from product 2 in the second realization, i.e. $y^*_{1,1} > 0, y^*_{2,1} = 0, y^*_{1,2} = 0$ and $y^*_{2,2} > 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y_1^* + Y_2^*$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y_1^* + Y_2^*$.

The proof is the same as for $N = 2$ Case 2.1.3.

Case 3.2.2: Getting reward from product 1 in the second realization and from product 2 in the first realization, i.e. $y^*_{1,1} = 0, y^*_{2,1} > 0, y^*_{1,2} > 0$ and $y^*_{2,2} = 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y_1^* + Y_2^*$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y_1^* + Y_2^*$.

The proof is the same as for $N = 2$ Case 2.1.3.

Case 3.3: Getting reward from both realizations, but having one $y^*_{j,h} = 0$.

Case 3.3.1: $y^*_{1,1} > 0, y^*_{2,1} > 0, y^*_{1,2} > 0$ and $y^*_{2,2} = 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y^*_1 + Y^*_2$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y^*_1 + Y^*_2$.

Since $y^*_{1,1}, y^*_{2,1}, y^*_{1,2}$ and $y^*_{2,2}$ is an optimal point of $BOM^2_\bullet$, the following inequalities hold:

$$y^*_{1,1} \leq (Y^*_1 - D^1_1)^+$$

$$y^*_{2,1} \leq (Y^*_2 - D^1_2)^+$$

$$y^*_{1,1} + y^*_{2,1} \leq (Y^* - D^1_1 - D^1_2)^+$$

$$y^*_{1,2} \leq (Y^*_1 - D^2_1)^+$$

$$Y^*_2 - D^2_2 \leq 0$$

$$y^*_{1,2} \leq (Y^* - D^2_1 - D^2_2)^+$$

When proving $y^*_{1,1}, y^*_{2,1}, y^*_{1,2}$ and $y^*_{2,2}$ is feasible in $(BOM^2_\circ)$, we assume that $Y^* \geq Y^*_1 + Y^*_2$ and we need to show that

$$y^*_{1,1} \le (X_1 - D^1_1)^+ \tag{4.24}$$

$$y^*_{1,1} \le (U_1 - D^1_1)^+ \tag{4.25}$$

$$y^*_{2,1} \le (X_2 - D^1_2)^+ \tag{4.26}$$

$$y^*_{2,1} \le (U_2 - D^1_2)^+ \tag{4.27}$$

$$y^*_{1,2} \le (X_1 - D^2_1)^+ \tag{4.28}$$

$$y^*_{1,2} \le (U_1 - D^2_1)^+ \tag{4.29}$$

Since $y^*_{1,1} > 0, y^*_{2,1} > 0$ and $y^*_{1,2} > 0$, all the plus signs can be removed. Let $U_2 = Y^*_2$ and thus $U_2 - D^2_2 \le 0$. Based on Assumption 1, we have $U_1 + U_2 \ge Y^*_1 + Y^*_2$ and thus $U_1 \ge Y^*_1$.

$$\begin{aligned}
\text{For (4.24)}: &\quad y^*_{1,1} \le (Y^*_1 - D^1_1)^+ = (X_1 - D^1_1)^+; \qquad < \text{substitution} > \\
\text{For (4.25)}: &\quad y^*_{1,1} \le (Y^*_1 - D^1_1)^+ \le (U_1 - D^1_1)^+; \\
\text{For (4.26)}: &\quad y^*_{2,1} \le (Y^*_2 - D^1_2)^+ = (X_2 - D^1_2)^+; \qquad < \text{substitution} > \\
\text{For (4.27)}: &\quad y^*_{2,1} \le (Y^*_2 - D^1_2)^+ \le (U_2 - D^1_2)^+; \qquad < \text{substitution} > \\
\text{For (4.28)}: &\quad y^*_{1,2} \le (Y^*_1 - D^2_1)^+ = (X_1 - D^2_1)^+; \qquad < \text{substitution} > \\
\text{For (4.29)}: &\quad y^*_{1,2} \le Y^* - D^2_1 - D^2_2 = (U_1 - D^2_1) + (U_2 - D^2_2) \le U_1 - D^2_1, \\
\text{and} &\quad U_1 - D^2_1 = (Y^* - U_2) - D^2_1 \ge Y^*_1 + Y^*_2 - U_2 - D^2_1 = Y^*_1 - D^2_1 > 0, \\
\text{therefore} &\quad y^*_{1,2} \le (U_1 - D^2_1)^+.
\end{aligned}$$

When proving $y^*_{1,1}, y^*_{2,1}, y^*_{1,2}$ and $y^*_{2,2}$ is feasible in $(BOM^2_\bullet)$, we assume that $Y^* < Y^*_1 + Y^*_2$ and we need to show that

$$y^*_{1,1} + y^*_{2,1} \leq (Z - D^1_1 - D^1_2)^+ \tag{4.30}$$

$$y^*_{1,1} + y^*_{2,1} \leq (V - D^1_1 - D^1_2)^+ \tag{4.31}$$

$$y^*_{1,2} \leq (Z - D^2_1 - D^2_2)^+ \tag{4.32}$$

$$y^*_{1,2} \leq (V - D^2_1 - D^2_2)^+ \tag{4.33}$$

Based on Assumption 2, we have $V < Z$.

For (4.31) : $y^*_{1,1} + y^*_{2,1} \leq (Y^* - D^1_1 - D^1_2)^+ = (V - D^1_1 - D^1_2)^+$; < substitution >

For (4.30) : $y^*_{1,1} + y^*_{2,1} \leq (V - D^1_1 - D^1_2)^+ < (Z - D^1_1 - D^1_2)^+ \leq (Z - D^1_1 - D^1_2)^+$;

For (4.33) : $y^*_{1,2} \leq (Y^* - D^2_1 - D^2_2)^+ = (V - D^2_1 - D^2_2)^+$; < substitution >

For (4.32) : $y^*_{1,2} \leq (V - D^2_1 - D^2_2)^+ < (Z - D^2_1 - D^2_2)^+ \leq (Z - D^2_1 - D^2_2)^+$.

Case 3.3.2: $y^*_{1,1} > 0, y^*_{2,1} = 0, y^*_{1,2} > 0$ and $y^*_{2,2} > 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y^*_1 + Y^*_2$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y^*_1 + Y^*_2$.

The proof is the same as for $N = 2$ Case 3.3.1.

Case 3.3.3: $y^*_{1,1} > 0, y^*_{2,1} > 0, y^*_{1,2} = 0$ and $y^*_{2,2} > 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y^*_1 + Y^*_2$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y^*_1 + Y^*_2$.

The proof is the same as for $N = 2$ Case 3.3.1 considering $U_1 = Y^*_1$.

Case 3.3.4: $y^*_{1,1} = 0, y^*_{2,1} > 0, y^*_{1,2} > 0$ and $y^*_{2,2} > 0$. We conclude that this point is feasible in $(BOM^2_\circ)$ when $Y^* \geq Y^*_1 + Y^*_2$, and is feasible in $(BOM^2_\bullet)$ when $Y^* < Y^*_1 + Y^*_2$.

The proof is the same as for $N = 2$ Case 3.3.3.

Case: 3.4: Getting reward from both realizations, i.e. $y^*_{1,1} > 0, y^*_{2,1} > 0, y^*_{1,2} > 0$ and $y^*_{2,2} > 0$. We want to show that this point is a feasible solution for $(BOM^2_\bullet)$. The proof is the same as for $N = 1$ Case 2.

The result for the case $N = 2$ is given in Table 4.8. Let $+$ represent the condition where $Y^* \geq Y^*_1 + Y^*_2$, and $-$ represent the condition where $Y^* < Y^*_1 + Y^*_2$.

	$y^*_{1,1} = 0$, $y^*_{2,1} = 0$	$y^*_{1,1} > 0$, $y^*_{2,1} = 0$	$y^*_{1,1} = 0$, $y^*_{2,1} > 0$	$y^*_{1,1} > 0$, $y^*_{2,1} > 0$
$y^*_{1,2} = 0$, $y^*_{2,2} = 0$	$(BOM^2_\circ)$ or $(BOM^2_\bullet)$	$(BOM^2_\circ)$	$(BOM^2_\circ)$	$(BOM^2_\bullet)$
$y^*_{1,2} > 0$, $y^*_{2,2} = 0$	$(BOM^2_\circ)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$
$y^*_{1,2} = 0$, $y^*_{2,2} > 0$	$(BOM^2_\circ)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$
$y^*_{1,2} > 0$, $y^*_{2,2} > 0$	$(BOM^2_\bullet)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$	$+ :\ (BOM^2_\circ)$ $- :\ (BOM^2_\bullet)$	$(BOM^2_\bullet)$

Table 4.8: Summary for the case $N = 2$

We observe that the relationship between the base stock levels of the shared components Y^* and the sum of the base stock levels of the dedicated components $Y^*_1 + Y^*_2$ plays an important role in deciding the feasibility. This observation is proved to be true for general case in section 4.1.3.

4.1.3 General case

We assume for N realizations, each with probability $1/N$. Without loss of generality, we omit this constant term in the objectives. In the associated formulations $(BOM_\circ^N)$, $(BOM_\bullet^N)$ and $(BOM_{\color{green}\bullet}^N)$ below, superscripts are use to distinguish different realizations. For example, $x_{1,h}, x_{2,h}, D_1^h, D_2^h, P_1^h$, and P_2^h refer to the h-th realization.

$$
\begin{aligned}
\max \quad & \sum_{h=1}^{N} (r_1\, x_{1,h} + r_2\, x_{2,h}) && \left(BOM_\circ^N\right)\\
& x_{1,h} \le (X_1 - D_1^h)^+ && h = 1, \ldots, N\\
& x_{1,h} \le (U_1 - D_1^h)^+ && h = 1, \ldots, N\\
& x_{2,h} \le (X_2 - D_2^h)^+ && h = 1, \ldots, N\\
& x_{2,h} \le (U_2 - D_2^h)^+ && h = 1, \ldots, N\\
& x_{1,h} \le P_1^h, \quad x_{2,h} \le P_2^h && h = 1, \ldots, N\\
& X_1 \sum_{i=1}^{d} c_{1\_i} + X_2 \sum_{i=1}^{d} c_{2\_i} + U_1 \sum_{i=d+1}^{n} c_{1\_i} + U_2 \sum_{i=d+1}^{n} c_{2\_i} \le B\\
& x_{1,h}, x_{2,h}, X_1, X_2, U_1, U_2 \in \mathbb{Z}_+ && h = 1, \ldots, N
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum_{h=1}^{N} (r_1\, z_{1,h} + r_2\, z_{2,h}) && && \big(BOM_{\bullet}^{N}\big) \\
z_{1,h} + z_{2,h} &\leq (Z - D_1^h - D_2^h)^+ && h = 1, \ldots, N \\
z_{1,h} + z_{2,h} &\leq (V - D_1^h - D_2^h)^+ && h = 1, \ldots, N \\
z_{1,h} \leq P_1^h, \quad z_{2,h} &\leq P_2^h && h = 1, \ldots, N \\
Z \sum_{i'=1}^{d} c_{i'} + V \sum_{i'=d+1}^{n} c_{i'} &\leq B \\
z_{1,h}, z_{2,h}, Z, V &\in \mathbb{Z}_+ && h = 1, \ldots, N
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \sum_{h=1}^{N} (r_1\, y_{1,h} + r_2\, y_{2,h}) && && \big(BOM_{\bullet}^{N}\big) \\
y_{1,h} &\leq (Y_1 - D_1^h)^+ && h = 1, \ldots, N \\
y_{2,h} &\leq (Y_2 - D_2^h)^+ && h = 1, \ldots, N \\
y_{1,h} + y_{2,h} &\leq (Y - D_1^h - D_2^h)^+ && h = 1, \ldots, N \\
y_{1,h} \leq P_1^h, \quad y_{2,h} &\leq P_2^h && h = 1, \ldots, N \\
Y_1 \sum_{i=1}^{d} c_{1\_i} + Y_2 \sum_{i=1}^{d} c_{2\_i} + Y \sum_{i'=d+1}^{n} c_{i'} &\leq B \\
y_{1,h}, y_{2,h}, Y_1, Y_2, Y &\in \mathbb{Z}_+ && h = 1, \ldots, N
\end{aligned}
$$

For any realization, the optimal assembly decision will fall into one of the four, mutually exclusive, outcomes: $y_{1,h}^* > 0$ and $y_{2,h}^* > 0$; $y_{1,h}^* > 0$ and $y_{2,h}^* = 0$; $y_{1,h}^* = 0$ and $y_{2,h}^* > 0$; and $y_{1,h}^* = 0$ and $y_{2,h}^* = 0$.

Consequently the set of all realizations can be partitioned into four non-overlapping subsets: the subset T^{++} of realizations in which $y^*_{1,h} > 0$ and $y^*_{2,h} > 0$, the subset T^{+0} of realizations in which $y^*_{1,h} > 0$ and $y^*_{2,h} = 0$, the subset T^{0+} of realizations in which $y^*_{1,h} = 0$ and $y^*_{2,h} > 0$, and the subset T^{00} of realizations in which $y^*_{1,h} = 0$ and $y^*_{2,h} = 0$.

According to the definitions of Y^*_1, Y^*_2 and Y^*, the following inequalities are valid. Note that the right hand side of constraints (E_1) to (E_7) are positive, therefore all plus sign can be removed.

$$y^*_{1,h} \leq (Y^*_1 - D^h_1)^+ \qquad h \in T^{++} \qquad (E_1)$$

$$y^*_{2,h} \leq (Y^*_2 - D^h_2)^+ \qquad h \in T^{++} \qquad (E_2)$$

$$y^*_{1,h} + y^*_{2,h} \leq (Y^* - D^h_1 - D^h_2)^+ \qquad h \in T^{++} \qquad (E_3)$$

$$y^*_{1,h} \leq (Y^*_1 - D^h_1)^+ \qquad h \in T^{+0} \qquad (E_4)$$

$$y^*_{1,h} \leq (Y^* - D^h_1 - D^h_2)^+ \qquad h \in T^{+0} \qquad (E_5)$$

$$y^*_{2,h} \leq (Y^*_2 - D^h_2)^+ \qquad h \in T^{0+} \qquad (E_6)$$

$$y^*_{2,h} \leq (Y^* - D^h_1 - D^h_2)^+ \qquad h \in T^{0+} \qquad (E_7)$$

The T^{00} cases being trivial, we just need prove that Theorem 1 holds for realizations in $T^{++} \bigcup T^{+0} \bigcup T^{0+}$.

To obtain an optimal solution, we must satisfy:

$$Y^*_1 = \max_{(g,p)\in(T^{++}\times T^{+0})} \left\{D^g_1 + y^*_{1,g}, D^p_1 + y^*_{1,p}\right\},$$
$$Y^*_2 = \max_{(g,p)\in(T^{++}\times T^{0+})} \left\{D^g_2 + y^*_{2,g}, D^p_2 + y^*_{2,p}\right\},$$
$$Y^* = \max_{(g,p,q)\in(T^{++}\times T^{+0}\times T^{0+})} \left\{D^g_1 + D^g_2 + y^*_{1,g} + y^*_{2,g}, D^p_1 + D^p_2 + y^*_{1,p}, D^q_1 + D^q_2 + y^*_{2,q}\right\}.$$

Clearly, either $Y^* \geq Y_1^* + Y_2^*$ or $Y^* < Y_1^* + Y_2^*$.

Case 1: If $Y^* \geq Y_1^* + Y_2^*$, then the point $y_{1,h}^*$ and $y_{2,h}^*$ is feasible in $\left(BOM_\circ^N\right)$. We need to show that

$$y_{1,h}^* \leq (X_1 - D_1^h)^+ \qquad h \in T^{++} \qquad (F_1)$$

$$y_{2,h}^* \leq (X_2 - D_2^h)^+ \qquad h \in T^{++} \qquad (F_2)$$

$$y_{1,h}^* \leq (U_1 - D_1^h)^+ \qquad h \in T^{++} \qquad (F_3)$$

$$y_{2,h}^* \leq (U_2 - D_2^h)^+ \qquad h \in T^{++} \qquad (F_4)$$

$$y_{1,h}^* \leq (X_1 - D_1^h)^+ \qquad h \in T^{+0} \qquad (F_5)$$

$$y_{1,h}^* \leq (U_1 - D_1^h)^+ \qquad h \in T^{+0} \qquad (F_6)$$

$$y_{2,h}^* \leq (X_2 - D_2^h)^+ \qquad h \in T^{0+} \qquad (F_7)$$

$$y_{2,h}^* \leq (U_2 - D_2^h)^+ \qquad h \in T^{0+} \qquad (F_8)$$

One can check that $(E_1) \Rightarrow (F_1)$, $(E_2) \Rightarrow (F_2)$, $(E_4) \Rightarrow (F_5)$, and $(E_6) \Rightarrow (F_7)$.

Let $U_2 = Y_2^*$, for (F_3):

$$U_1 = Y^* - U_2 \geq Y_1^* + Y_2^* - U_2 = Y_1^*,$$

$$\text{thus} \quad U_1 \geq Y_1^* = \max_{(g,p)\in(T^{++}\times T^{+0})} \left\{D_1^g + y_{1,g}^*, D_1^p + y_{1,p}^*\right\} \geq D_1^h + y_{1,h}^*, \quad h \in T^{++}.$$

$$\text{Therefore} \quad y_{1,h}^* \leq (U_1 - D_1^h)^+, \quad h \in T^{++}.$$

For (F_4):

$$U_2 = Y_2^* = \max_{(g,p)\in(T^{++}\times T^{0+})} \left\{D_2^g + y_{2,g}^*, D_2^p + y_{2,p}^*\right\} \geq D_2^h + y_{2,h}^*, \quad h \in T^{++}$$

Therefore $\quad y_{2,h}^* \leq (U_2 - D_2^h)^+, \quad h \in T^{++}.$

For (F_6):

$$\begin{aligned} U_1 - D_1^h \geq Y_1^* - D_1^h &= \max_{(g,p)\in(T^{++}\times T^{+0})} \left\{D_1^g + y_{1,g}^*, D_1^p + y_{1,p}^*\right\} - D_1^h \\ &\geq D_1^h + y_{1,h}^* - D_1^h = y_{1,h}^*, \quad h \in T^{+0} \end{aligned}$$

Therefore $\quad y_{1,h}^* \leq (U_1 - D_1^h)^+, \quad h \in T^{+0}.$

For (F_8):

$$\begin{aligned} U_2 - D_2^h = Y_2^* - D_2^h &= \max_{(g,p)\in(T^{++}\times T^{0+})} \left\{D_2^g + y_{2,g}^*, D_2^p + y_{2,p}^*\right\} - D_2^h \\ &\geq D_2^h + y_{2,h}^* - D_2^h = y_{2,h}^*, \quad h \in T^{0+} \end{aligned}$$

Therefore $\quad y_{2,h}^* \leq (U_2 - D_2^h)^+, \quad h \in T^{0+}.$

Case 2: If $Y^* < Y_1^* + Y_2^*$, then the point $y_{1,h}^*$ and $y_{2,h}^*$ is feasible in $\left(BOM_\bullet^N\right)$. We need to show that

$$y^*_{1,h} + y^*_{2,h} \leq (Z - D^h_1 - D^h_2)^+ \qquad h \in T^{++} \qquad (G_1)$$

$$y^*_{1,h} + y^*_{2,h} \leq (V - D^h_1 - D^h_2)^+ \qquad h \in T^{++} \qquad (G_2)$$

$$y^*_{1,h} \leq (Z - D^h_1 - D^h_2)^+ \qquad h \in T^{+0} \qquad (G_3)$$

$$y^*_{1,h} \leq (V - D^h_1 - D^h_2)^+ \qquad h \in T^{+0} \qquad (G_4)$$

$$y^*_{2,h} \leq (Z - D^h_1 - D^h_2)^+ \qquad h \in T^{0+} \qquad (G_5)$$

$$y^*_{2,h} \leq (V - D^h_1 - D^h_2)^+ \qquad h \in T^{0+} \qquad (G_6)$$

One can check that $(E_3) \Rightarrow (G_2)$, $(E_5) \Rightarrow (G_4)$, and $(E_7) \Rightarrow (G_6)$.

(E_1) and $(E_2) \Rightarrow (G_1)$: Since $y^*_{1,h} > 0$ and $y^*_{2,h} > 0$, where $h \in T^{++}$, all the plus signs can be removed.

$$\begin{array}{lrcl} & 0 < y^*_{1,h} \leq Y^*_1 - D^h_1 \quad \text{and} & & 0 < y^*_{2,h} \leq Y^*_2 - D^h_2 \\ \Longrightarrow & 0 < y^*_{1,h} + y^*_{2,h} & \leq & Y^*_1 + Y^*_2 - D^h_1 - D^h_2 \\ \Longrightarrow & & = & Z - D^h_1 - D^h_2, \quad h \in T^{++} \end{array}$$

Thus, $y^*_{1,h} + y^*_{2,h} \leq (Z - D^h_1 - D^h_2)^+, \quad h \in T^{++}$.

For (G_3):

$$\begin{aligned}
Z &= Y_1^* + Y_2^* \;>\; Y^* \\
&= \max_{(g,p,q)\in(T^{++}\times T^{+0}\times T^{0+})} \left\{D_1^g + D_2^g + y_{1,g}^* + y_{2,g}^*, D_1^p + D_2^p + y_{1,p}^*, D_1^q + D_2^q + y_{2,q}^*\right\} \\
&\geq D_1^h + D_2^h + y_{1,h}^*, \quad h \in T^{+0}.
\end{aligned}$$

Therefore $\quad y_{1,h}^* < (Z - D_1^h - D_2^h)^+ \leq (Z - D_1^h - D_2^h)^+, \quad h \in T^{+0}.$

For (G_5):

$$\begin{aligned}
Z &= Y_1^* + Y_2^* \;>\; Y^* \\
&= \max_{(g,p,q)\in(T^{++}\times T^{+0}\times T^{0+})} \left\{D_1^g + D_2^g + y_{1,g}^* + y_{2,g}^*, D_1^p + D_2^p + y_{1,p}^*, D_1^q + D_2^q + y_{2,q}^*\right\} \\
&\geq D_1^h + D_2^h + y_{2,h}^*, \quad h \in T^{0+}.
\end{aligned}$$

Therefore $\quad y_{2,h}^* < (Z - D_1^h - D_2^h)^+ \leq (Z - D_1^h - D_2^h)^+, \quad h \in T^{0+}.$

4.2 Two-product system with partial overlap

Given that $x_1^{{\color{green}\bullet}h*} \leq (S_{1\_i_1}^{\color{green}\bullet} - D_1^h)^+$ and $x_2^{{\color{green}\bullet}h*} \leq (S_{2\_i_2}^{\color{green}\bullet} - D_2^h)^+$, where $i_1 = n+1,\ldots,n_1, i_2 = n+1,\ldots,n_2, h = 1,\ldots,N$, we want to prove that either the constrains $x_1^{{\color{green}\bullet}h*} \leq (S_{1\_i_1}^{\circ} - D_1^h)^+$ and $x_2^{{\color{green}\bullet}h*} \leq (S_{2\_i_2}^{\circ} - D_2^h)^+$, or the constraints $x_1^{{\color{green}\bullet}h*} \leq (S_{1\_i_1}^{\bullet} - D_1^h)^+$ and $x_2^{{\color{green}\bullet}h*} \leq (S_{2\_i_2}^{\bullet} - D_2^h)^+$ hold. Obviously, if we set $S_{1\_i_1}^{\color{green}\bullet} = S_{1\_i_1}^{\circ} = S_{1\_i_1}^{\bullet}$ and $S_{2\_i_2}^{\color{green}\bullet} = S_{2\_i_2}^{\circ} = S_{2\_i_2}^{\bullet}$, then the optimal solutions of $\left(BOM_{\color{green}\bullet}^N\right)$, i.e., $x_1^{{\color{green}\bullet}h*}$ and $x_2^{{\color{green}\bullet}h*}$, trivially satisfy these constraints in both $\left(BOM_{\circ}^N\right)$ and $\left(BOM_{\bullet}^N\right)$. Excluding the above constraints, the remaining part is exactly the same as the full overlap configuration, whose result is already proved.

Chapter 5

Conclusion and future work

We show that for two-product periodic ATO systems either full component commonality or non-component commonality performs at least as well as any partial component commonality formulation. Consequently, the size of the optimal BOM search space is cut down from an exponential in n to just 2. A possible future direction is to extend this result to multi-product periodic-review ATO systems. While deriving the same theoretical results may be challenging, one may consider a computational approach. Another future direction could be to apply component commonality considering inventory allocation and component design jointly.

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