INVESTIGATION OF TWO POLARIMETRIC OPTICAL STRAIN SENSORS
In the memory of my father
INVESTIGATION OF TWO POLARIMETRIC OPTICAL STRAIN SENSORS

By

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Abstract:

This thesis presents a theoretical description and experimental testing of two polarimetric fiber-optic strain sensors. The first of these is an intrinsic sensor in which a single mode optical fiber acts as the sensing element. This sensor exhibited low strain sensitivity and high cross-sensitivity to temperature. The second sensor was of the extrinsic type. A single polarization maintaining fiber was used to deliver and collect light from the sensing element, which was a piece of photoelastic sheet PS-1. This type of sensor exhibited large dynamic range of 1:1350, resolution of 0.59 ps, and strain sensitivity of the normalized signal of 0.00219 ps⁻¹.
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Table of Content:

Descriptive Note ii
Abstract iii
Acknowledgements iv
List of Illustrations and Tables vii
Introduction 1
Chapter 1: Birefringence 3
  1.1. INTRODUCTION 3
  1.2. LINEAR ISOTROPIC MEDIA: MONOCHROMATIC PLANE WAVES 4
  1.3. LINEAR ANISOTROPIC MEDIA: BIREFRINGENCE 7
    1.3.1. Linear Birefringence 11
      1.3.1.1. The Ordinary Wave Solution 12
      1.3.1.2. The Extraordinary Wave Solution 14
      1.3.1.3. Propagation of Light in and Uniaxial Media 17
      1.3.1.4. The Index Ellipsoid for Uniaxial Media 19
    1.3.2. Circular Birefringence 22
Chapter 2: Photoelastic Effect 24
  2.1. INTRODUCTION 24
  2.2. STRESS AND STRAIN 24
    2.2.1. Stress 24
    2.2.2. Strain 26
2.2.3. Thermally Induced Expansion 29

2.3. ELASTICITY – HOOKE’S LAW 29

2.4. PHOTOELASTICITY – MAXWELL-NEUMANN’S LAW 32

Chapter 3: The Experimental Fiber-Optic Strain Sensor Constructions 39

3.1. BASIC PRINCIPLES OF PHOTOELASTIC STRAIN SENSING 39

3.1.1. Birefringence Induced Phase Shift 39

3.1.2. Decoding the Induced Phase Shift 48

3.2. SINGLE MODE FIBER-OPTIC STRAIN SENSOR CONSTRUCTION 55

3.2.1. The Laboratory Setup for the SM Fiber Optic Sensor 55

3.2.2. Induced Phase Change in the Epoxied SM Fiber 58

3.2.3. Experimental Results for SM Fiber-Optic Strain Sensor 65

3.2.4. Thermal Apparent-Strain Sensitivity of SM Fiber-Optic Strain Sensor 75

3.2.4.1. Experimental Results for Thermal Apparent-Strain Sensitivity 78

3.3. STRAIN SENSOR CONSTRUCTION WITH POLARIZATION MAINTAINING FIBERS 83

3.3.1. Difference-Over-Sum Processing 84

3.3.2. Experimental Results for PM Fiber-Optic Strain Sensor 86

Chapter 4: Conclusion and Suggestions for Further Research 95

Appendix A: Symmetry Property of Dielectric Tensor 97

Appendix B: Diagonalization of Hermitian Matrices 98

Appendix C: Stress Tensor 100

References 102
<table>
<thead>
<tr>
<th>Illustration</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fig. 1.2.1</td>
<td>8</td>
</tr>
<tr>
<td>Fig. 1.3.1</td>
<td>11</td>
</tr>
<tr>
<td>Fig. 1.3.2a</td>
<td>13</td>
</tr>
<tr>
<td>Fig. 1.3.2b</td>
<td>14</td>
</tr>
<tr>
<td>Fig. 1.3.3a</td>
<td>16</td>
</tr>
<tr>
<td>Fig. 1.3.3b</td>
<td>17</td>
</tr>
<tr>
<td>Fig. 1.3.4</td>
<td>21</td>
</tr>
<tr>
<td>Fig. 1.3.5</td>
<td>23</td>
</tr>
<tr>
<td>Fig. 2.2.1</td>
<td>24</td>
</tr>
<tr>
<td>Fig. 2.2.2</td>
<td>27</td>
</tr>
<tr>
<td>Fig. 3.1.1</td>
<td>41</td>
</tr>
<tr>
<td>Fig. 3.1.2</td>
<td>46</td>
</tr>
<tr>
<td>Fig. 3.1.3</td>
<td>49</td>
</tr>
<tr>
<td>Fig. 3.1.4</td>
<td>50</td>
</tr>
<tr>
<td>Fig. 3.1.5</td>
<td>50</td>
</tr>
<tr>
<td>Fig. 3.1.6</td>
<td>54</td>
</tr>
<tr>
<td>Fig. 3.2.1</td>
<td>56</td>
</tr>
<tr>
<td>Fig. 3.2.2</td>
<td>57</td>
</tr>
<tr>
<td>Fig. 3.2.3a</td>
<td>63</td>
</tr>
<tr>
<td>Fig. 3.2.3b</td>
<td>64</td>
</tr>
<tr>
<td>Fig. 3.2.3c</td>
<td>65</td>
</tr>
<tr>
<td>Fig. 3.2.4</td>
<td>66</td>
</tr>
<tr>
<td>Fig. 3.2.5</td>
<td>66</td>
</tr>
<tr>
<td>Fig. 3.2.6</td>
<td>68</td>
</tr>
<tr>
<td>Fig. 3.2.7</td>
<td>70</td>
</tr>
<tr>
<td>Fig. 3.2.8</td>
<td>71</td>
</tr>
<tr>
<td>Fig. 3.2.9</td>
<td>71</td>
</tr>
<tr>
<td>Fig. 3.2.10</td>
<td>72</td>
</tr>
<tr>
<td>Fig. 3.2.11a</td>
<td>72</td>
</tr>
<tr>
<td>Fig. 3.2.11b</td>
<td>73</td>
</tr>
<tr>
<td>Fig. 3.2.12</td>
<td>74</td>
</tr>
<tr>
<td>Fig. 3.2.13</td>
<td>77</td>
</tr>
<tr>
<td>Fig. 3.2.14</td>
<td>79</td>
</tr>
<tr>
<td>Fig. 3.2.15</td>
<td>80</td>
</tr>
<tr>
<td>Fig. 3.2.16</td>
<td>81</td>
</tr>
<tr>
<td>Fig. 3.3.1</td>
<td>82</td>
</tr>
</tbody>
</table>
Chapter 1:
Birefringence

1.1 INTRODUCTION

This introductory chapter is concerned with elements of the theory of light propagation in linear homogeneous isotropic and anisotropic media. Results of this theory are very important in order to establish a mathematical model of the optical strain sensors. Starting with Maxwell’s equations and material equations for the medium we obtain appropriate propagation modes.

An electromagnetic wave in the space is described by two field vectors [3], \( \mathbf{E} \) and \( \mathbf{H} \) related by Maxwell’s equations (1.1). To include the effect of the field on materials, it is necessary to introduce second set of vectors \( \mathbf{D} \) and \( \mathbf{B} \) called the electric displacement and the magnetic induction that are given by the material equations (1.2).

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.1a)
\]
\[
\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (1.1b)
\]
\[
\nabla \cdot \mathbf{B} = 0 \quad (1.1d)
\]
\[
\nabla \cdot \mathbf{D} = \rho \quad (1.1c)
\]
\[ \mathbf{D} = \varepsilon \mathbf{E} = \varphi \mathbf{E} + \mathbf{P} \quad (1.2a) \]

\[ \mathbf{B} = \mu \mathbf{H} = \mathbf{P} \mathbf{H} \quad (1.2b) \]

The charge density \( p \) and the current density \( J \) are sources of electromagnetic fields (1.1). In our model of the strain sensor, \( p \) and \( J \) are zero, since we deal with electromagnetic radiation propagating in regions far from the sources. In the material equations (1.2), vectors \( \mathbf{P} \) and \( \mathbf{M} \) are the electric and magnetic polarizations, respectively. Materials used for optical sensors are non-magnetic, and therefore \( \mathbf{M} = 0 \) and \( p \) equals permeability of vacuum \( \mu_0 \) (i.e. \( \varphi = 1 \), \( p = \varepsilon_0 \mu_0 \)). \( \varepsilon \) is permittivity of media, in which the lightwave propagates. It governs how the electrical field \( \mathbf{E} \) is distributed in the material with the respect to electric polarization vector \( \mathbf{P} \). In general, for anisotropic media \( \varepsilon \) is tensor, but for isotropic media it reduces to a scalar. The permittivity can be written as a product of \( \varepsilon_0 \), the permittivity of vacuum, and \( \varepsilon_r \), the relative permittivity, \( \varepsilon = \varepsilon_0 \varepsilon_r \).

### 1.2 LINEAR ISOTROPIC MEDIA: MONOCHROMATIC PLANE WAVES [3]

In isotropic media \( \varepsilon \) is scalar. Additionally we are interested in the case where \( p = 0 \), \( J = 0 \) and \( \mu = 1 \). In order to separate the fields \( \mathbf{E} \) and \( \mathbf{H} \) from the coupled curl-equations (1.1a) and (1.1b) we take the curl of equation (1.1a) and (1.1b) and employ the
divergence equations (1.1c) and (1.1d) and following vector identity for double cross product:

$$\nabla \times (\nabla \times \vec{a}) = \nabla (\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

we get two decoupled wave equations for linear isotropic non-magnetic media (1.3) and (1.4).

\[
\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad (1.3)
\]

\[
\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} \quad (1.4)
\]

These are the standard electromagnetic wave equations and they are satisfied by the functions:

\[
\begin{bmatrix}
\vec{E}_{00}(\vec{r}, t) \\
\vec{H}_{00}(\vec{r}, t)
\end{bmatrix} e^{i(k \cdot \vec{r} - \omega t)}
\]

(1.5)

Here \( k = 2\pi/\lambda \) is the magnitude of the wavevector (\( \lambda \) is wavelength). Taking the amplitudes of the waves in (1.5) to be constant we have the well known expression for the plane waves (surfaces of constant phase are planes):
Comparing (1.3) and (1.4) with the general form of wave equation we identify the velocity of the electromagnetic waves as

\[
u = \frac{c}{n}
\]

(1.7)

where \( c = 1 / (\mu_0 \varepsilon_0)^{1/2} = 2.997930 \times 10^8 \, \text{m/s} \) is speed of light in vacuum and \( n = (\varepsilon_r)^{1/2} \) is the index of refraction for the medium of propagation (with \( \mu_r = 1 \)). By introducing the plane wave solutions (1.6) back into the Maxwell’s equations we can conclude the following about the magnitudes and the directions of the electric field, the magnetic field and the wave vector. The magnitudes of the electric and magnetic fields are related by a coefficient of proportionality, which is called impedance of the medium \( \eta = (\mu / \varepsilon)^{1/2} \),

\[
|\vec{E}| = \sqrt{\frac{\mu}{\varepsilon}} |\vec{H}|
\]

(1.8)

The electric field vector, the magnetic field and wavevector are mutually orthogonal and form a right-handed triad. In terms of unit vectors we have:
\[ \hat{E} \times \hat{H} = \hat{k}. \]  \hspace{1cm} (1.9)

So \( \mathbf{E} \) and \( \mathbf{H} \) lie in the planes of constant phase, which are transverse to the \( \mathbf{k} \) direction and therefore the wave is called a transverse electromagnetic (TEM) wave. Also, the direction of energy flow, defined by Poynting's vector \( \mathbf{S} = \mathbf{E} \times \mathbf{H} \), is the same as the propagating direction of planes of constant phases, defined by \( \mathbf{k} \).

The dispersion relation (1.10) for this case follows from introducing the plane wave solution (1.6) back in the wave equation.

\[ k^2 = \omega^2 \mu \varepsilon \]  \hspace{1cm} (1.10)

1.3. LINEAR ANISOTROPIC MEDIA: BIREFRINGENCE [3], [4], [5]

A basic distinguishing feature of transparent crystals, as far as optical properties are concerned, is the fact that many crystals are anisotropic. This means that polarization produced in the crystal by a given electric field depends on the direction of the applied field in relation to the crystal lattice. A model to illustrate anisotropic polarizability of a crystal is shown in Fig. 1.2.1.
A bound electron is pictured here as attached to a set of fictitious elastic springs. The springs have different stiffnesses for different directions of the electron’s displacement from its equilibrium position in the crystal lattice. So, the displacement of the electron under the influence of an external field \( \mathbf{E} \) depends on the direction of the field as well as its magnitude. This is also true for the resulting polarization \( \mathbf{P} \). Therefore the dependence of \( \mathbf{P} \) on \( \mathbf{E} \) is through the third rank susceptibility tensor, equation (1.11)

\[
\begin{bmatrix}
    P_x \\
    P_y \\
    P_z 
\end{bmatrix} = \varepsilon_0 \begin{bmatrix}
    \chi_{11} & \chi_{12} & \chi_{13} \\
    \chi_{21} & \chi_{22} & \chi_{23} \\
    \chi_{31} & \chi_{32} & \chi_{33}
\end{bmatrix}
\begin{bmatrix}
    E_x \\
    E_y \\
    E_z 
\end{bmatrix}
\]

(1.11)

The corresponding permittivity tensor \( \varepsilon \) is defined in a similar manner as for scalars.
\[
\tilde{\varepsilon} = \varepsilon_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \chi
\] (1.12)

Thus, the relation (1.2a) can be written in the form

\[
D_i = \varepsilon_{ij} E_j
\] (1.13)

where the convention of summation over repeated indices is observed. If we assume that the medium is homogeneous, nonabsorbing, and magnetically isotropic it can be shown (Appendix A) that the dielectric tensor is symmetric (1.14).

\[
\varepsilon_{ij} = \varepsilon_{ji}
\] (1.14)

Based on the property of symmetry of the dielectric tensor (more general, on Hermitian symmetry if the tensor is not real) we can apply the diagonalization theorem from linear algebra (Appendix B). Therefore, we can write the dielectric tensor in the basis of the eigenvectors where it is diagonal with the eigenvalues on the main diagonal.

\[
\tilde{\varepsilon} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}
\] (1.15)
In optics this coordinate system is called principal coordinate system and eigenvalues of the dielectric tensor are called principal dielectric constants. Classification of crystals according to the optical symmetry is given in the Table-1.1.

<table>
<thead>
<tr>
<th>Optical Symmetry</th>
<th>Crystal System</th>
<th>Elements of $\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>Cubic</td>
<td>$\varepsilon_x = \varepsilon_y = \varepsilon_z$</td>
</tr>
<tr>
<td>Uniaxial</td>
<td>Tetragonal, Hexagonal, Trigonal</td>
<td>$\varepsilon_x = \varepsilon_y \neq \varepsilon_z$</td>
</tr>
<tr>
<td>Biaxial</td>
<td>Triclinic, Monoclinic, Orthorombic</td>
<td>$\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$</td>
</tr>
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</table>

Table-1.1 Classification of crystals according to the optical symmetry

In the term of refractive indices, the dielectric tensor can be written as

$$\tilde{\varepsilon} = \varepsilon_0 \begin{bmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$$

for optically isotropic media, and as

$$\tilde{\varepsilon} = \varepsilon_0 \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

(1.16)

for uniaxial optical media, and finally as

$$\tilde{\varepsilon} = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}$$

for biaxial optical media.
1.3.1. Linear Birefringence In Uniaxial Media

We will now find plane wave solutions for uniaxial media. In this case, the dielectric tensor takes the form of (1.16) and we assume that an electric field of the following form is sustained in a uniaxial medium:

\[ \vec{E}(x,y,z) = \vec{E}_0 e^{i(k x - \omega t)} \quad (1.17) \]

Wave equation (1.3) in this case will be

\[ \vec{k} \times (\vec{k} \times \vec{E}) = -\omega^2 \mu \varepsilon \vec{E} \quad (1.18) \]

Since in uniaxial media \( \varepsilon_x = \varepsilon_y \neq \varepsilon_z \) the solution must be invariant under rotation about the z-axis. Multiplying (1.16) with the appropriate rotational matrix can prove this. In other words we have freedom in choosing angle \( \varphi \) (Fig. 1.3.1).

![Fig. 1.3.1. The coordinate system](image)

To simplify calculations, we will choose \( \varphi = 0 \), which means \( k_y = 0 \) and the wavevector is simply:

\[ \vec{k} = k_x \hat{x} + k_z \hat{z} = k \sin \theta \hat{x} + k \cos \theta \hat{z} \quad (1.20) \]
Having in mind (1.20) we can project (1.18) on the three coordinate axes and solve it as a scalar problem. This yields the following set of equations (1.21), where \( \epsilon_x = \epsilon_y = \epsilon_z \):

\[
\begin{align*}
\hat{1}: \quad & k_x (k_x \epsilon_x - k_y \epsilon_y) = \omega^2 \mu \epsilon_x \\
\hat{2}: \quad & (k_y^2 + k_z^2) \epsilon_y = \omega^2 \mu \epsilon_y \\
\hat{3}: \quad & k_z (k_x \epsilon_x - k_y \epsilon_y) = -\omega^2 \mu \epsilon_x 
\end{align*}
\]

The system of homogeneous equations (1.21) requires for non-trivial solution \( |E| \neq 0 \) that the determinant of the system is zero. After short algebraic transformations the determinant of the system can be written as:

\[
(k_x^2 + k_y^2 + \frac{\omega^2}{\mu})[(k_x^2 - \frac{\omega^2}{\mu})(k_y^2 - \frac{\omega^2}{\mu}) - k_x^2 k_y^2] = 0
\]

1.3.1.1 The Ordinary Wave Solution

The first possible solution of (1.22) is

\[
k_\parallel^2 = k_x^2 + k_y^2 = \frac{\omega^2}{\mu}
\]

Since this dispersion relation is the same as for isotropic media (1.10) we call this solution the ordinary wave. Explicitly the solution is:

\[
\vec{E} = \vec{E}_0 \exp(\vec{k} \cdot \vec{r} - \omega t)
\]
which is a plane wave in the direction of the $y$-axis. To calculate the electric displacement $\mathbf{D}$, magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$, we use (1.13) and Maxwell’s equations. This gives us

$$\mathbf{D} = \varepsilon \mathbf{E} = \hat{y} \varepsilon E_0 e^{i(k \hat{y} - \omega t)}$$  \hspace{1cm} (1.24b)

$$\mathbf{H} = \frac{1}{\omega \mu} \vec{k} \times \vec{E} = \frac{1}{\omega \mu} (-\hat{x}k_z + \hat{z}k_x) E_0 e^{i(k \hat{y} - \omega t)}$$  \hspace{1cm} (1.24c)

$$\mathbf{B} = \mu \mathbf{H} = \frac{1}{\omega} (-\hat{x}k_z + \hat{z}k_x) E_0 e^{i(k \hat{y} - \omega t)}$$  \hspace{1cm} (1.24d)

![Fig.1.3.2a Vectors in the ordinary wave solution](image)

The electric displacement (1.24b) is parallel to electric field (1.24a), just like in isotropic media, and both are perpendicular to vectors of the magnetic field $\mathbf{H}$ and magnetic induction $\mathbf{B}$ (Fig.1.3.2a). The ordinary wave surface for $\varphi = 0$ is graphed in Fig.1.3.2b.
If we allow \( \varphi \neq 0 \), or in the other words if we allow the wavevector \( \mathbf{k} \) to rotate around the \( z \)-axis the dispersion relation would be a sphere in \( k \)-space:

\[
k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon.
\]

1.3.1.2. The Extraordinary Wave Solution

The other possible solution of (1.22) is:

\[
\frac{k_x^2}{\omega^2 \mu \varepsilon_x} + \frac{k_y^2}{\omega^2 \mu \varepsilon_y} = 1 \quad (1.27)
\]
This is a completely new dispersion relation compared to (1.10) and therefore we call this wave the extraordinary wave. If (1.27) is true, then from the system (1.21) we find that $E_y = 0$. For the other two components of the electric field we have from (1.21) and (1.27):

\[ k_x \varepsilon E_x + k_z \varepsilon E_z = 0 \quad (1.28a) \]

\[ \vec{k} \cdot \vec{D} = 0 \quad (1.28b) \]

which is, actually, the condition of orthogonality for the wave vector $k$ and the electric displacement vector $D$. Vectors of the electric field and the electric displacements are not parallel, since $\varepsilon$ is not scalar. $E$ has two non-zero components in $x$- and $z$-directions. The complete solutions are:

\[ \vec{E} = (\hat{x} - \hat{z} \frac{k_x \varepsilon_x}{k_z \varepsilon_z}) E_x 0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (1.29a) \]

\[ \vec{H} = \frac{1}{\omega \mu} \vec{k} \times \vec{E} = \hat{y} \frac{\omega \varepsilon}{k_z} E_x 0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (1.29b) \]

\[ \vec{D} = \varepsilon \vec{E} = (\hat{x} - \hat{z} \frac{k_x \varepsilon_x}{k_z}) E_x 0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (1.29d) \]

\[ \vec{B} = \mu \vec{H} = \hat{y} \frac{\mu \omega \varepsilon}{k_z} E_x 0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (1.29c) \]
The relationship between the vectors in the extraordinary wave solution is depicted in Fig.1.3.3a.

Extraordinary wave surfaces are graphed in Fig.1.3.3b for \( \varphi = 0 \). But for \( \varphi \neq 0 \), or when the wavevector \( \mathbf{k} \) rotates around the \( z \)-axis the wave surface turns into the rotational ellipsoid:

\[
\frac{k_x^2}{\omega^2 \mu \varepsilon_z} + \frac{k_y^2}{\omega^2 \mu \varepsilon_z} + \frac{k_z^2}{\omega^2 \mu \varepsilon} = 1.
\]
1.3.1.3. Propagation of Light in an Uniaxial Medium

Now, we will analyze the propagation of light perpendicular to the optical axis (z-axis), for example let it be in the x-direction:

\[ \vec{k} = \hat{x}k, \quad k_y = k_z = 0 \]

In the first case let the launched light be polarized along the y-direction. Then it matches the ordinary wave (1.24) and travels with wavevector \( k_0 \) without changing the polarization:

\[ \vec{E} = \hat{y}E_0 e^{i(k_0 x - \omega t)} \text{, where} \quad k_0 = \omega \sqrt{\mu \varepsilon} \]

Therefore we interpret this wave as a normal mode for propagation in x-direction. In the
second case let the launched light be polarized along z-direction. Then it matches the extraordinary wave (1.29) and travels with wavevector $k_e$ without changing the polarization:

$$\vec{E} = \hat{z}E_0e^{i(k_e x + \omega t)}, \text{ where } k_e = \omega \sqrt{\mu E_z}$$

Therefore we interpret this wave as a normal mode for propagation in x-direction too. In the third case, if we launched light in x-direction that is arbitrarily polarized we can always write it in the basis of the two normal modes. Since these two components travel with different wavevectors, $k_0$ and $k_e$, they undergo different phase shifts, $k_0d$ and $k_ed$ after propagating a distance $d$. Therefore the state of polarization of the resultant wave will be changing along its paths through the uniaxial medium.

Now, let's analyze propagation of light along the optical axis (z-axis). In this case the wavevector is

$$\vec{k} = \hat{z}k, \quad k_x = k_y = 0$$

and from the equations (1.24) and (1.29) we can conclude that both ordinary and extraordinary waves become degenerate, propagating with the same wavevector $k_d$. The ordinary wave is polarized along the y-direction:

$$\vec{E} = \hat{y}E_0e^{i(k_d x - \omega t)}$$

and the extraordinary wave is polarized along x-direction:
\[ \vec{E} = \hat{\lambda} E_0 e^{i(k_0 z - \omega t)}. \]

In the general case, when the wave propagates in a direction with an angle \( \theta \) to the optical axis, the ordinary wave dispersion relation is

\[
\frac{k^2 \sin^2 \theta + k^2 \cos^2 \theta}{\omega^2 \mu \varepsilon} = 1
\]

and the extraordinary wave dispersion relation is

\[
\frac{k^2 \sin^2 \theta}{\omega^2 \mu \varepsilon} + \frac{k^2 \cos^2 \theta}{\omega^2 \mu \varepsilon} = 1
\]

Therefore, to the two normal modes, (1.24) and (1.29), correspond two different phase velocities and two different indices of refraction depending on angle theta between the optic axis and the wave vector:

\[
u_o = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{(1.31a)}
\]

\[
u_s = \sqrt{\frac{\sin^2 \theta}{\mu \varepsilon} + \frac{\cos^2 \theta}{\mu \varepsilon}} \quad \text{(1.31b)}
\]

1.3.1.4. **Index Ellipsoid for Uniaxial Media**

The dielectric tensor (1.15) is a symmetric tensor of a second rank. Therefore it has six independent components, which can be associated with a second-degree surface - a quadric [4]. For a given dielectric tensor (1.15) we define an impermeability tensor as
An index ellipsoid is defined as (summation is assumed over the repeated indices):

\[ \varepsilon_0 K_{ij} x_i x_j = 1 \]  

(1.32)

\( K_{ij} \) in (1.32) are elements of impermeability tensor and \( x_1 \equiv x \), \( x_2 \equiv y \) and \( x_3 \equiv z \). In the principal coordinate system the dielectric tensor is diagonalized:

\[
\tilde{\varepsilon} = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}
\]

and the index ellipsoid (1.32) becomes

\[
\frac{x^2}{n_x^2} + \frac{y^2}{n_y^2} + \frac{z^2}{n_z^2} = 1
\]  

(1.33)

In the case of uniaxial media \( n_x = n_y = n_o \) and \( n_z = n_e \), so we have:

\[
\frac{x^2}{n_o^2} + \frac{y^2}{n_o^2} + \frac{z^2}{n_e^2} = 1
\]  

(1.34)

The equation of the index ellipsoid (1.34) is graphed in Fig.1.3.4. The angle between the wavevector \( k \) and the optical axis is \( \theta \) and line AB is perpendicular to \( k \). The index of refraction: \( n_e(\theta) \) is equal to the length OB and it depends on the angle \( \theta \), while \( n_o(\theta) = n_o \).
is constant and equal to the length OF. If we introduce \( x = OB' = -n_E(\theta) \cos \theta \), \( y = 0 \) and \( z = OB'' = n_E(\theta) \sin \theta \) into the ellipsoid equation (1.34) we get

\[
\frac{1}{n_E^2(\theta)} = \frac{\cos^2 \theta}{n_O^2} + \frac{\sin^2 \theta}{n_E^2(\theta)}
\]  

(1.35)

For \( \theta = 0^\circ \) – propagation along the extraordinary axis, \( n_E(\theta) = n_O \) and there is no birefringence. For \( \theta = 90^\circ \) – propagation perpendicular to the optical axis, \( n_E(\theta) = n_E \), its
maximum value, and we have maximum birefringence.

In summary, the propagation of light in a uniaxial medium is described in the terms (orthogonal basis) of two normal modes, an ordinary and an extraordinary wave, that are solutions to the eigenvalue problem (1.18). The electric field vector $\mathbf{E}$ and the displacement vector $\mathbf{D}$ for the ordinary wave are always perpendicular to the propagation vector $\mathbf{k}$. The phase velocity for ordinary wave is always $c/n_0$, regardless of the direction of propagation. The displacement vector $\mathbf{D}$ of the extraordinary wave is perpendicular to $\mathbf{k}$, but electric field vector $\mathbf{E}$ is not perpendicular to it in general. The phase velocity of extraordinary wave depends on the angle of propagation $\theta$ as $c/n_E(\theta)$. The electric field vectors of these two waves are mutually orthogonal. If $n_0 < n_E$ the crystal is said to be positive, whereas if $n_0 > n_E$ it is called negative crystal.

1.3.2. Circular Birefringence in Uniaxial Media

In this subsection we will briefly describe a phenomenon that can degrade the performance of fiber optic strain sensors. Certain optical media are found to cause a rotation of the plane of polarization of linearly polarized light passing through them, Fig.1.3.5. This phenomenon is known as optical activity or circular birefringence [3], [4], [5]. It can be explained on the basis of the fact that the speed of propagation for right circularly polarized light in the medium is different from that of the left circularly polarized light. In the previous section on linear birefringence the eigenwaves of propagation were linearly polarized plane waves, however in case of optically active media, the dielectric tensor (1.16) contains conjugate imaginary off-diagonal elements.
Fig. 1.3.5. Rotation of plane of polarization by an optically active medium (levorotatory)

(1.36) and the eigenwaves of propagation are right and left circularly polarized waves, and therefore the name.

\[
\hat{\varepsilon} = \begin{bmatrix}
\varepsilon & i\kappa_g & 0 \\
-i\kappa_g & \varepsilon & 0 \\
0 & 0 & \varepsilon_z
\end{bmatrix} \quad (1.36)
\]

The amount of rotation – angle per unit length of travel is called the specific rotatory power. If the sense of rotation of the plane of polarization is to the right, as a right-handed screw pointing in the direction of propagation, the substance is called dextrorotatory or right-handed. If the rotation is to the left, the substance is called levorotatory.
Chapter 2:

Photoelastic Effect

2.1 INTRODUCTION

In this chapter we will further develop the theory needed for the optical strain sensors. First, basic notions in elasticity will be introduced, stress induced strain and thermal expansion, thermally induced strain. Furthermore, they will be connected with birefringence (Chapter 1) through the full derivation of Maxwell - Neumann's photoelastic law.

2.2. STRESS AND STRAIN [8], [9]

2.2.1. Stress

A solid body, in which one part exerts a force on neighboring parts, is said to be in a state of stress. These forces are proportional to the surface area of a volume element (Fig. 2.2.1), and the force per unit area is called the stress.

Fig. 2.2.1. The element of a solid body under the stress.
Components of stress (Fig.2.2.1) \( \sigma_{ii} \) are called normal components and \( \sigma_{ij} \) \((i \neq j)\) are called shear components. Stress is tensile if the normal components are positive and it is compressive if the normal components are negative. If the forces acting on the surface of an element of constant shape and orientation in a solid body are independent of the position of the element in the body, the stress is said to be homogenous.

If all elements of a solid body (Fig.2.2.1) are in the static equilibrium, and if the stress is homogenous (all three components of force on the face pass through the middle of the face), the normal and shear components give no moment. The moment is calculated with respect to the symmetry axis passing through the middle of the face of the element. Since the moment of each normal component \( \sigma_{ii} \) is zero (parallel to the symmetry axis), then (2.1) must hold:

\[
\sigma_{ij} = \sigma_{ji}, \quad i \neq j
\]  

(2.1)

Relation (2.1) continues to hold even for inhomogeneous stresses, when the body is not in statical equilibrium [9]. Components of the stress \( \sigma_{ij} \) form a second-rank tensor (Appendix C). Having in mind the symmetry of the stress tensor as (2.1) we can diagonalize

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}
\]  

(2.2)

it (Appendix B). In a principal coordinate system we can write the stress tensor as in (2.2) where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are principal stresses. In special cases we have uniaxial stress and biaxial stresses. The stress tensor for a uniaxial stress field is given in equation (2.3) and
for the biaxial stress field in given in equation (2.4).

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  
(2.3)

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]  
(2.4)

Triaxial stress is an alternative name for the most general case of stress. Its tensor is given in equation (2.2). The geometric representation of the stress tensor (2.2) is given by the representation quadric in the principal coordinate system:

\[
\sigma_1 x_1^2 + \sigma_2 x_2^2 + \sigma_3 x_3^2 = 1.
\]

If all the \(\sigma\)'s are positive, the representation surface is an ellipsoid, if one is negative the surface is a one-sheet hyperboloid and if two \(\sigma\)'s are negative the surface is a two-sheet hyperboloid.

2.2.2. Strain

In order to specify the state of deformation of a solid body we introduce the notion of strain. A precise definition will be given through the example of one-dimensional strain.
In Fig.2.2.2 are depicted cases of an unstrained one-dimensional solid body (a) and the same body when it is strained (b). Here, distances $AB = \Delta x$, $A'B' = \Delta x + \Delta u$, $OA = x$ and $OA' = x + u$. If the function $u(x)$ is linear, then the body is strained homogeneously. The strain of the section $AB$ is defined as:

$$\frac{A'B' - AB}{AB} = \frac{\Delta u}{\Delta x}$$

The strain at point $A$ is defined as a limit value:

$$e = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{du}{dx}$$

(2.5)

For homogenous strain (2.5) is constant and is equal to the slope of $u(x)$ in the point of interest.

In the case of a three-dimensional solid body, strain is specified with the nine-tensor components:
\[ e_{ij} = \frac{\partial u_i}{\partial x_j}, \quad (i, j = 1, 2, 3) \]  

Here, \( e_{ij} \) are extensions per unit length parallel to the three axes, \( e_{ij} \) (\( i \neq j \)) is rotation of a line element parallel to \( j \)-axis towards \( i \)-axis around the third axis. \( e_{ji} \) is rotation in the opposite direction.

The strain tensor itself is defined as the symmetrical part of \([e_{ij}]\) or,

\[ \varepsilon_{ij} = \frac{e_{ij} + e_{ji}}{2} \]  

The diagonal components of \( \varepsilon_{ij} \) are the stretches or tensile strains. Off diagonal components measure the shear strains. According to the definition (2.7) the strain tensor is symmetric and therefore can be diagonalized (Appendix B).

If referred to the principal coordinate system, the strain tensor is

\[
\begin{bmatrix}
\varepsilon_1 & 0 & 0 \\
0 & \varepsilon_2 & 0 \\
0 & 0 & \varepsilon_3
\end{bmatrix}
\]  

(2.8)

where \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) are principal strains. The defining property of the principal axes is that they are three mutually perpendicular directions in the body, which remain perpendicular during the deformation [8]. As well as stress, the strain tensor (2.8) can be represented with a representation quadric and appropriate ellipsoid.
2.2.3. Thermally induced expansion

Deformation of a solid body due to temperature changes (stress is not present in the system) can be specified by the strain tensor (2.6). For a small temperature change $\Delta T$, which is uniform through the solid body, the deformation is homogenous and it is proportional to all components of the strain tensor (2.9).

$$\varepsilon_{ij} = \alpha_{ij} \Delta T$$  \hspace{1cm} (2.9)

Here, coefficients of proportionality $\alpha_{ij}$ are well known coefficients of thermal expansion. According to (2.9) $[\alpha_{ij}]$ is a tensor and it is symmetrical. In the principal coordinate system (2.9) simplifies to

$$\varepsilon_1 = \alpha_1 \Delta T, \hspace{0.5cm} \varepsilon_2 = \alpha_2 \Delta T, \hspace{0.5cm} \varepsilon_3 = \alpha_3 \Delta T$$  \hspace{1cm} (2.10)

In equation (2.10) $\alpha_i$ ($i = 1, 2, 3$) are principal thermal expansion coefficients.

2.3. ELASTICITY – HOOKE’S LAW [8], [9], [10]

When subjected to stress a solid body changes its shape – it is strained. If the stress is small, below certain limiting value, the elastic limit, the strain is recoverable and the body can return to its original shape. In the case of small elastic deformations, components of a stress tensor are homogenous and linear functions of components of a strain tensor (and vice versa) and stress and strain have same principal directions [8], [9]:
In the equation (2.11) the summation is assumed over the repeated indices \( k, l = 1, 2, 3 \). These equations constitute Hooke’s Law. However, when viscoelastic or plastic phenomena are involved in the deformation of the body, equations (2.11) are not valid. Constants \( s_{ijkl} \) in (2.11a) are called compliances of a crystal and \( c_{ijkl} \) are called stiffness constants. \( s_{ijkl} \) and \( c_{ijkl} \) form fourth rank tensors [9] and both have 81 components. Since stress and strain tensors are symmetric, compliance and stiffness tensors are symmetric with respect to first and second indices and third and fourth indices (2.12). Therefore, compliance and stiffness tensors have at most 36 independent components.

\[
\begin{align*}
\varepsilon_{ij} &= s_{ijkl} \sigma_{kl} \\
\sigma_{ij} &= c_{ijkl} \varepsilon_{kl}
\end{align*}
\]

(2.11a) (2.11b)

\[
\begin{align*}
s_{ijkl} &= s_{jikl}, & s_{ijkl} &= s_{ijlk} \\
c_{ijkl} &= c_{jikl}, & c_{ijkl} &= c_{ijlk}
\end{align*}
\]

(2.12a) (2.12b)

For an elastically deformed body, both strain and stress tensors have the same principal directions, therefore their respective stress and strain ellipsoids, so-called Cauchy’s ellipsoids, are coaxial [10], [11] and therefore stress and strain can each be inferred from presence of other.

Since there are only six independent components of stress and strain tensors and 36 of stiffness and compliance tensors we can introduce a compressed notation (2.13). Both stress and strain components are written as before, but with a single suffix running from 1 to 6 (2.13a) and (2.13b).
\[
\begin{bmatrix}
s_{11} & s_{12} & s_{13} \\
s_{21} & s_{22} & s_{23} \\
s_{31} & s_{32} & s_{33}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
s_1 & s_6 & s_5 \\
s_6 & s_2 & s_4 \\
s_5 & s_4 & s_3
\end{bmatrix}
\quad (2.13a)
\]

\[
\begin{bmatrix}
\varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\varepsilon_1 & \varepsilon_6/2 & \varepsilon_5/2 \\
\varepsilon_6/2 & \varepsilon_2 & \varepsilon_4/2 \\
\varepsilon_5/2 & \varepsilon_4/2 & \varepsilon_3
\end{bmatrix}
\quad (2.13b)
\]

In the compliance tensor the first two suffixes are abbreviated into a single one running from 1 to 6, and the last two are abbreviated in the same way. Factors 2 and 4 are introduced as follows:

\[
s_{ijkl} = s_{mn} \quad \text{when } m \text{ and } n = 1, 2, 3 \quad (2.13c)
\]

\[
2 \, s_{ijkl} = s_{mn} \quad \text{when either } m \text{ or } n = 4, 5, 6 \quad (2.13d)
\]

\[
4 \, s_{ijkl} = s_{mn} \quad \text{when } m \text{ and } n = 4, 5, 6. \quad (2.13d)
\]

The stiffness tensor is abbreviated in the same way, but without factors 2 and 4.

\[
c_{ijkl} = c_{mn} \quad \text{when } i, j, k, l = 1, 2, 3 \text{ and } m, n = 1, 2, ..., 6. \quad (2.13e)
\]

In terms of the new notation, Hooke’s Law can be rewritten as:

\[
\varepsilon_i = s_{ij} \sigma_j \quad (2.14a)
\]

\[
\sigma_i = c_{ij} \varepsilon_j \quad (2.14b)
\]

but here \(i, j = 1, 2, ..., 6\).
In a special case, for isotropic materials in the principal coordinate system, Hooke’s Law can be expressed simply as (2.15). Here, $E$ is Young’s modulus of elasticity and $\nu$ is Poisson’s ratio. Stress and strain have only three components along the principal axes and are written as column matrices.

$$
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E} & -\nu & -\nu \\
\nu & \frac{1}{E} & -\nu \\
\nu & -\nu & \frac{1}{E} \\
\end{bmatrix} \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\end{bmatrix}
$$

(2.15)

2.4. PHOTOELASTICITY – MAXWELL-NEUMANN’S LAW [11], [1]

Beside crystals, birefringence can be observed in certain noncrystalline and initially optically isotropic materials, when they are subjected to a stress field. Under stress, these materials behave like crystals. However, the crystalline nature of the material lasts only during the application of the loads; it vanishes when the loads are removed. This phenomenon is called temporary or artificial birefringence. Sir David Brewster discovered it in 1816.

For an elastically deformed body, both the stress and strain tensors have the same principal directions – therefore their respective stress and strain ellipsoids are coaxial. The mechanically introduced birefringence in an isotropic material has its origin in the physical deformations induced in the body by the strain or stress. Stress or strain cause intermolecular variations of the structure of the body, which alter its optically isotropic character. In the case of elastic deformations, in which the principal axes of the stress and
strain tensor coincide, stress, strain and birefringence can be inferred from the presence of the others. The principal birefringence axes coincide with the principal stress or strain axes. Thus, Fresnel’s refractive index ellipsoid and Cauchy’s stress and strain ellipsoids are coaxial [11]. This result will be used to derive the relationships that exist between the principal stresses or strains and the principal refractive indices. All the following calculations will be referred to common principal axes x, y and z.

The Fresnel’s refractive indices ellipsoid is (1.33).

\[
\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = 1 \tag{2.16}
\]

where \(n_1\), \(n_2\), and \(n_3\) are principal refractive indices. Similarly, Cauchy’s stress ellipsoid is

\[
\sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = 1 \tag{2.17}
\]

where \(\sigma_1\), \(\sigma_2\), and \(\sigma_3\) are principal stresses. As these two ellipsoids are coaxial in the case of elastic stress fields, their planes of circular cross sections must be parallel. These planes coincide with the intersection of each ellipsoid with a concentric sphere; therefore they may be put in the forms (2.18) and (2.19).

\[
\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = A(x^2 + y^2 + z^2) \tag{2.18}
\]

\[
\sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = B(x^2 + y^2 + z^2) \tag{2.19}
\]

If the first member of (2.19) and the second one of (2.18) are multiplied by a constant \(D\), then the resulting equations must coincide. By writing (2.18) and (2.19) in the form
\[
\left(\frac{1}{n_1^2} - AD\right)x^2 + \left(\frac{1}{n_2^2} - AD\right)y^2 + \left(\frac{1}{n_3^2} - AD\right)z^2 = 0
\]
(2.20)

\[
(D\sigma_1 - B)x^2 + (D\sigma_2 - B)y^2 + (D\sigma_3 - B)z^2 = 0
\]
(2.21)

we get the following equations:

\[
\frac{1}{n_1^2} = D\sigma_1 + C
\]
(2.22a)

\[
\frac{1}{n_2^2} = D\sigma_2 + C
\]
(2.22b)

\[
\frac{1}{n_3^2} = D\sigma_3 + C
\]
(2.22c)

where C = AD − B is introduced to simplify the expression. Equations (2.22) interrelate the principal refractive indices and principal stresses through quantities C and D; they are the consequences of the coaxiality of Fresnel’s and Cauchy’s ellipsoids. From Fresnel’s refractive indices ellipsoid (2.16), the refractive index \( n_s \) along a given direction \( s (s_x, s_y, s_z) \) is given by

\[
\frac{1}{n_s^2} = \frac{s_x^2}{n_1^2} + \frac{s_y^2}{n_2^2} + \frac{s_z^2}{n_3^2}
\]
(2.23)

Similarly, the stress, \( \sigma_s \), along the same direction \( s \) is given by Cauchy’s stress ellipsoid (2.17) as

\[
\sigma_s = \sigma_1 s_x^2 + \sigma_2 s_y^2 + \sigma_3 s_z^2
\]
(2.24)

Now, if we introduce (2.22) into (2.23) it yields (2.25).
\[
\frac{1}{n_s^2} = \frac{1}{n^2} + \frac{1}{n^2} + C_2' (\sigma_s + \sigma_r + \sigma_i) + D \sigma_s
\] (2.28\text{a})

or with \(C_1' = C_2' + D\)

\[
\frac{1}{n_s^2} = \frac{1}{n^2} + C_1' \sigma_s + C_2' (\sigma_r + \sigma_i)
\] (2.28\text{b})

We can rewrite (2.28\text{a}) as

\[
\frac{n - n_s}{n + n_s} = \frac{n_s^2 n_s^2}{n + n_s} \left[C_1' \sigma_s + C_2' (\sigma_r + \sigma_i)\right]
\] (2.29)
Since \( n \approx n_s \) we can approximate:

\[
\frac{n^2 n_s^2}{n + n_s} \approx \frac{n^3}{2} \quad (2.30)
\]

We can also define:

\[
C_1 = -\frac{n^3}{2} C_1' \quad \text{and} \quad C_2 = -\frac{n^3}{2} C_2' \quad (2.31a,b)
\]

With (2.31) and (2.30) we can rewrite (2.29) to get (2.32a). By similar procedure as previously mentioned we can get (2.32b) and (2.32c):

\[
\begin{align*}
n_s - n &= C_1 \sigma_s + C_2 (\sigma_t + \sigma_r) \quad (2.32a) \\
n_t - n &= C_1 \sigma_t + C_2 (\sigma_r + \sigma_s) \quad (2.32b) \\
n_r - n &= C_1 \sigma_r + C_2 (\sigma_s + \sigma_t) \quad (2.32c)
\end{align*}
\]

Equations (2.32) are known as Maxwell’s photoelastic law. They give the variation of the refractive indices along the directions \( s, t \) and \( r \) as linear functions of corresponding stresses and the sums of the other two stresses. Constants of proportionality \( C_1 \) and \( C_2 \) are called stress-optical coefficients and they depend on the properties of material of the birefringent body. Maxwell’s photoelastic law (2.32) is formally equivalent to Hooke’s elastic law for isotropic media (2.15), what can be seen if we rewrite (2.32) in matrix form as:
\[
\begin{bmatrix}
  n_s - n \\
  n_t - n \\
  n_r - n
\end{bmatrix} =
\begin{bmatrix}
  C_1 & C_2 & C_2 \\
  C_2 & C_1 & C_2 \\
  C_2 & C_2 & C_1
\end{bmatrix}
\begin{bmatrix}
  \sigma_s \\
  \sigma_t \\
  \sigma_r
\end{bmatrix}
\]  

(2.33)

If we introduce Hooke’s law for isotropic materials (2.15) into (2.33) with replacements: 

\[ s \to 1, \ t \to 2 \text{ and } r \to 3 \]

we obtain

\[
\begin{bmatrix}
  n_1 - n \\
  n_2 - n \\
  n_3 - n
\end{bmatrix} =
\begin{bmatrix}
  b_1 & b_2 & b_2 \\
  b_2 & b_1 & b_2 \\
  b_2 & b_2 & b_1
\end{bmatrix}
\begin{bmatrix}
  \varepsilon_1 \\
  \varepsilon_2 \\
  \varepsilon_3
\end{bmatrix}
\]  

(2.34)

Here, \( n_1, n_2 \) and \( n_3 \) are principal indices of refraction, \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \) are principal strains and constants

\[
C_1 = \frac{1}{E} (b_1 - 2\nu b_2) \]  

(2.35a)

\[
C_2 = \frac{1}{E} (b_2 - \nu b_1 - \nu b_2) \]  

(2.35b)

Equation (2.34) is called Neumann’s photoelastic law. Because in the elastic range of the material, both stress and strain principal directions coincide and stress and strain are linearly related through Hooke’s law equations, (2.33) and (2.34) are referred to as Neumann-Maxwell’s photoelastic law.

For the case in which we are interested only in differences between principal indices of refraction we obtain from (2.32) by cyclical subtractions:
\[ n_1 - n_2 = C_0(\sigma_1 - \sigma_2) \]  
\[ n_2 - n_3 = C_0(\sigma_2 - \sigma_3) \]  
\[ n_3 - n_1 = C_0(\sigma_3 - \sigma_1) \]

with \( C_0 = C_1 - C_2 \).

Similarly from (2.34):

\[ n_1 - n_2 = b_0(\varepsilon_1 - \varepsilon_2) \]  
\[ n_2 - n_3 = b_0(\varepsilon_2 - \varepsilon_3) \]  
\[ n_3 - n_1 = b_0(\varepsilon_3 - \varepsilon_1) \]

with \( b_0 = b_1 - b_2 \). Equations (2.36) and (2.37) are called Wertheim’s stress or strain optical law.

The equations derived in this section constitute fundamental physical laws on which optical strain sensors operate. These equations will be applied in the following chapter to calculate the concrete response of the fiber sensors to the stress input for particular experimental setups.
Chapter 3:

The Experimental Optical Strain Sensor Constructions

3.1 Basic Principles of Optical Elastic Strain Sensing

In this section we will derive the basic equations governing optical strain sensors, based on the theory presented in first two chapters. First, the birefringence induced phase shift will be derived and then a decoding mechanism based on a simple optical setup will be described.

3.1.1. Birefringence Induced Phase Shift [12]

The refractive index of a solid is specified by the index ellipsoid, Section 1.3.1.3:

\[
\left( \frac{1}{n^2} \right)_{x_i, x_j} \equiv \frac{1}{k^2} \quad \hat{\epsilon}_{i,j} = \left[ \begin{array}{ccc} 1 & 2 & 3 \\
\end{array} \right]
\] (3.1)

Here, we have written the impermeability tensor as \((1/n^2)_{ij}\). When the solid is subjected to a mechanical stress the result is a small change in refractive indices as we have shown in equation (2.33), or in terms of strain in equation (2.34). This corresponds to an actual change in the shape, size and orientation of the index ellipsoid. This change is usually specified by the small change in the components of impermeability tensor \(\Delta ((1/n^2)_{ij})\):

\[
\Delta \left( \frac{1}{n^2} \right)_{ij} = \sigma_{ijkl} \quad i,j,k,l = 1,2,3 \quad (3.2a)
\]

\[
\Delta \left( \frac{1}{n^2} \right)_{ij} = P_{ijkl} \quad i,j,k,l = 1,2,3 \quad (3.2b)
\]
If we employ the symmetry of impermeability tensor:

\[
\frac{1}{n^2}_{ij} = \frac{1}{n^2}_{ji},
\]
\[
\Delta \left( \frac{1}{n^2}_{ij} \right) = \Delta \left( \frac{1}{n^2}_{ji} \right),
\]

we can write it in compressed notation where all six components are independent:

\[
\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
B_1 & B_6 & B_5 \\
B_6 & B_2 & B_4 \\
B_5 & B_4 & B_3
\end{bmatrix}
\]

Symmetry in the last two indices of the four-dimensional stress-optic tensor in the case when we can ignore body-torques [9]:

\[
\pi_{ijkl} = \pi_{ijlk},
\]

enables us to write it in compressed notation:

\[
\pi_{nn} = \pi_{ijk}, \quad \text{when } n=1,2,3;
\]
\[
\pi_{nn} = 2\pi_{ijk}, \quad \text{when } n=4,5,6.
\]

The factor 2 appears because of the pairing of the shear stress terms on the right hand side of (3.2a). The induced change in the impermeability tensor is now simply:

\[
\Delta \left( \frac{1}{n^2} \right)_i = \pi_{ij} \sigma_j, \quad i, j = 1,2,...,6.
\]  
(3.3a)
In equation (3.3a), \( \pi_{ij} \) is the stress–optic tensor and \( \sigma_j \) is the stress vector, with components described by compressed notation introduced in (2.13). Similarly, for the strain we would have (3.3b)

\[
\Delta\left(\frac{1}{n^2}\right), \quad p_{ij} \varepsilon_j, \quad i,j \quad 1,2,\ldots,6 \quad (3.3b)
\]

where \( p_{ij} \) is the strain-optic tensor compressed in the following way [9]

\[
p_{mn} = p_{ijkl} \quad \text{for all } m \text{ and } n.
\]

Let's take a look at Fig.3.1.1, where a sample of optically homogeneous material with the index of refraction \( n \) and the stress–optic tensor (3.4) [4], [8] is depicted.

**Fig.3.1.1.** Optically homogeneous material under uniaxial stress field
For the case of a homogenous crystal with refractive index \( n \) when the stress field is absent, the index ellipsoid (3.1) is simply a sphere:

\[
\frac{x^2 + y^2 + z^2}{n^2} = 1
\]  

(3.5)

Under the stress field this sphere transforms to:

\[
\left(\frac{1}{n_x^2}\right)x^2 + \left(\frac{1}{n_y^2}\right)y^2 + \left(\frac{1}{n_z^2}\right)z^2 + \left(\frac{1}{n_{xy}^2}\right)yz + \left(\frac{1}{n_{xz}^2}\right)zx + \left(\frac{1}{n_{yz}^2}\right)xy = 1
\]  

(3.6)

The changes (3.3) of the components of the impermeability tensor in the case of uniaxial stress field – along the x – axis, (Fig.3.1.1), can be explicitly written as:
Now, we can perform matrix multiplication and calculate the components of the impermeability tensor parallel and normal to the stress direction.

\[
\Delta n_{\text{paral}} = \Delta n_1 = -\frac{1}{2} n^3 \pi_{11} \sigma_x
\]  
\[3.8a\]

\[
\Delta n_{\text{norm}} \equiv \Delta n_2 = \Delta n_3 = -\frac{1}{2} n^3 \pi_{12} \sigma_x
\]  
\[3.8b\]

\[
\Delta n_4 = \Delta n_5 = \Delta n_6 = 0
\]  
\[3.8c\]

Here we exploited that the finite difference \( \Delta \) is approximately equal to the derivative and that to a sufficient approximation we may replace \( n_i \) with \( n \):

\[
\Delta \left( \frac{1}{n^2} \right)_i = -\frac{2}{n^3} \Delta n_i, \quad i = 1, 2, ..., 6.
\]
The change of the refractive index in (3.8a), $\Delta n_{\text{paral}}$, corresponds to the polarization direction parallel to the direction of the applied stress, and the change $\Delta n_{\text{norm}}$ in (3.8b) corresponds to the polarization direction perpendicular to direction of stress. It is clear that for light propagating in the $y -$ direction we have a difference in the refractive indices for the two perpendicular polarizations and therefore an induced birefringence:

$$n_{\text{paral}} - n_{\text{norm}} = \Delta n_1 - \Delta n_2 = -\frac{n^3}{2}(\pi_{11} - \pi_{12})\sigma_x$$

(3.9)

The corresponding phase change or retardation is:

$$\delta = -\frac{\pi n^3}{\lambda}(\pi_{11} - \pi_{12})L\sigma_x$$

(3.10)

Here, $L$ is the path length of light in the material (Fig.3.1.1), $\pi = 3.1412\ldots$ and $\lambda$ is the wavelength of light. Equation (3.10) can be rewritten in terms of the stress – optic coefficient $C_\sigma$ and strain – optic coefficient $C_\varepsilon$:

$$\delta = \frac{2\pi}{\lambda} C_\sigma L \sigma_x$$

(3.11a)

$$\delta = -\frac{2\pi}{\lambda} C_\varepsilon L \varepsilon_x$$

(3.11b)

Stress-optic and strain optic coefficients can be related to coefficients $C_0$ in (2.36) and $b_0$ in (2.37) as follows:
The effect of lateral force (stress) is well known to induce birefringence in appropriate materials and in optical fibers as well. When optical fibers are used in telecommunication applications this is parasitic effect. However, this effect can be usefully exploited to measure pressure and strain [13]. Lateral force (stress) is not the only source of induced birefringence in the fibers. Effects like bending, bending under tension, or applied electric and magnetic fields can also induce a phase change between two orthogonal polarizations.

When a fiber of outer cladding radius A is bent with a bending radius R >> A, birefringence is induced in the fiber (a relative change in propagation constants of two orthogonal waves). The normalized resultant birefringence can be expressed as [14]:

\[
B_b \equiv \frac{\beta_y - \beta_x}{\beta} = \frac{n_i^2}{4} (p_{11} - p_{12})(1 + \nu) \left(\frac{A}{R}\right)^2
\]  

(3.12)

The orientation for the coordinate system is given in the Fig.3.1.2. For silica, Poisson’s ratio is \(\nu = 0.16\); and the elements of strain–optic tensor are \(p_{11}=0.121\) and \(p_{12}=0.27\) [14]. Plugging these values in (3.12) we obtain:

\[
B_b = -0.093\left(\frac{A}{R}\right)^2
\]

(3.13)
Pure tensile stress applied to the fiber does not induce birefringence because of the symmetry of such a system. But bending under tension (tensile stress applied to a fiber that is wound on a mandrel of radius R) gives a contribution to birefringence that is in addition to (3.13). The additional birefringence is given by [15]

\[
B_i = \frac{n_1^2}{4} (p_{11} - p_{12}) \frac{(1 + \nu)(2 + 3\nu)}{1 - \nu} \frac{A}{R} S_{zz}
\]  

(3.14)

where \(S_{zz}\) is the axial tensile strain applied externally and the axis orientation is the same as in Fig.3.1.2. The total resultant normalized birefringence is the sum of (3.13) and (3.14). Twisting a fiber around its axis with a uniform twist rate \(2\pi N \text{ (rad/m)}\) (\(N\) is number of turns per meter) will induce a shear stress that leads to circular birefringence,
described in the Section 1.3.2. In previously mentioned cases, the induced birefringence was linear. In this case, eigenmodes are no longer x and y – polarized but right and left – circularly polarized. The induced circular birefringence is given by [16]:

$$\Delta \beta_c = \frac{n_1^2}{2} (p_{11} - p_{12}) 2\pi N = g 2\pi N$$  \hspace{1cm} (3.15)

where $\Delta \beta_c$ is the difference between the propagation constants of the mode circularly polarized in the same sense as the twist and the mode polarized in the opposite sense and $g = -0.14$ (experimentally, [16]). From (3.15) normalized circular birefringence can be obtained:

$$B_c = \frac{\Delta \beta_c}{\beta} \approx \frac{g}{n_1} N\lambda \approx -0.1 N\lambda$$  \hspace{1cm} (3.16)

where $n_1$ is the index of refraction of silica and $\lambda$ is the wavelength of light used.

It is interesting to add that external magnetic and electrical fields can induce birefringence. Transverse (with respect to the fiber axis) electric fields will induce linear birefringence through the Kerr effect [16]:

$$B_k = 2 \times 10^{-22} \frac{m^2}{V^2} E^2$$  \hspace{1cm} (3.17)

where $E$ is the electric field amplitude and the constant of proportionality is the Kerr constant of silica.
A magnetic field applied longitudinally along the fiber axis induces a circular birefringence through the Faraday effect [17]:

$$\Delta \beta_h = 2 \times V_f H$$

(3.18)

where $V_f$ is the Verdet constant of silica ($4.6 \times 10^6$ rad/A [17]) and $H$ is the amplitude of the applied magnetic field.

3.1.2. Decoding the Induced Phase Shift [18]

In the previous section we derived the expression for the strain-induced phase change (3.11) and reviewed the phase change induced by several other mechanical and electromagnetic effects. The induced phase change, and thus the strain or stress, can not be readily "read out" directly, so some kind of optical setup is needed to decode the strain from the phase change of the electromagnetic wave. In such a setup, an optical signal is detected in a photodetector and the phase change is inferred from the observed intensity change. One possible setup is illustrated in Fig.3.1.3. Here, the light propagates axially, along the z-axis, through the cylindrically shaped piece of birefringent material. The uniaxial stress is acting laterally along the y-axis. Suppose that we have launched light that is linearly polarized in the x-y plane with the input polarization at 45° with respect to the x and y directions (3.19a).

$$\vec{E}_{in} = E_0 (\hat{x} + \hat{y}) e^{i(k_z z - \omega t)}$$

(3.19a)
After leaving the material light can be viewed as consisting of two components along x and y-direction with a relative phase retardation \( \delta \) (3.19b)

\[
E_{\text{out}} = E_0(\dot{x} + \dot{y}e^{i\delta})e^{i(kz - \omega t)}
\]  

(3.19b)

The phase retardation is given by (3.11b) and it linearly increases with the strain in the material. This would change the polarization of the light in between the polarizers from linear to elliptical, Fig.3 1.4. Now, if we set the \( x' \)-axis in Fig.3.1.3 to coincide with the direction along which input linear polarization is aligned (\( x'-y' \) coordinate system is \( x-y \) system rotated for angle \( \theta \) of the input polarizer), the component along \( y' \)-direction would be zero for zero strain. The signal would increase with strain until the polarization is circular, then decrease with strain to zero when the output polarization is linear and
orthogonal to the input polarization. In other words we can expect to detect a photosignal at the output that varies sinusoidally with strain if we detect output polarization that is orthogonal to the input polarization.

Fig. 3.1.5 Basic principle for detecting induced phase change in birefringence material
That is why we introduce two crossed polarizers in the setup, Fig.3.1.5. Similarly, if the output polarizer (analyzer) is aligned with the input polarizer, the detected photosignal would decrease from its maximum value, for zero strain, towards zero when appropriate stress is applied. In other words, the detected signal would have cosine shape. Now we can mathematically derive the expression for the detected output intensity of light and confirm predictions of its sinusoidal or cosinusoidal shape.

The input polarizer sets the direction of polarization of the input light in Fig. 3.1.5 along the \(x'\)-directions that the field at \(z=0\) is (3.20a).

\[
\tilde{E}_{in} = E_0 \hat{x}'
\]  

(3.20a)

Since the stress is applied along the \(x\)-axis, the induced eigenaxes will be along the \(y\)- and \(x\)-direction (Section 3.1.1). Therefore, we will project \(E_{in}\) to this coordinate system (3.20b).

\[
\tilde{E}_{in} = E_0 (\hat{x} \cos \theta + \hat{y} \sin \theta)
\]  

(3.20b)

After passing through the thickness \(L\) of the material and just before entering the analyzer the electric field is given by (3.20c).

\[
\tilde{E}(L) = E_0 e^{ikr_{f}L} (\hat{x} \cos \theta + \hat{y} \sin \theta e^{ik(n_xn_y)L})
\]  

(3.20c)

where we have assumed that the induced fast axis is along the \(x\)-direction, and the induced slow axis is along the \(y\)-direction \((n_x=n_{f}<n_y=n_{s}).\) The induced phase change
between two orthogonal polarizations is \( \delta = k(n_s - n_t)\). The effect of the analyzer at \( z=L \) is to project the electric field to the passing axis \( y' \):

\[
\vec{E}_{out} = E_0 e^{i k n_y L} (\hat{x} \cos \theta + \hat{y} \sin \theta e^{i \delta}) \cdot \hat{y}'
\]

\[
= E_0 e^{i k n_y L} (-\sin \theta \cos \theta + \sin \theta \cos \theta e^{i \delta}). \quad (3.21a)
\]

The corresponding output intensity of (3.21a) is given by:

\[
I_{out(y')} = |E_{out}^*|^2
\]

\[
= I_{in}(2 \sin^2 \theta \cos^2 \theta - \sin^2 \theta \cos^2 \theta (e^{i \delta} + e^{-i \delta}))
\]

\[
I_{out(y')} = \frac{I_{in}}{2} \sin^2 2\theta (1 - \cos \delta). \quad (3.21b)
\]

Here, \( I_{in} = E_{in}^* E_{in} \) is the intensity of the light at the input. If the analyzer is rotated by 90°, so the admission axis is aligned with the \( x' \)-axis, the output field of the analyzer at \( z=L \) will be:

\[
\vec{E}_{out} = E_0 e^{i k n_y L} (\hat{x} \cos \theta + \hat{y} \sin \theta e^{i \delta})
\]

\[
= E_0 e^{i k n_y L} (\cos^2 \theta + \sin^2 \theta e^{i \delta}). \quad (3.22a)
\]

Using (2.21a) we can calculate the output intensity for this case:

\[
I_{out(x')} = |E_{out}^*|^2
\]

\[
= I_{in}(\cos^4 \theta + \sin^4 \theta + \sin^2 \theta \cos^2 \theta (e^{i \delta} + e^{-i \delta}))
\]

\[
I_{out(x')} = \frac{I_{in}}{2} (1 + \cos^2 2\theta + \sin^2 2\theta \cos \delta). \quad (3.22b)
\]
Since we have not assumed any absorption in the material, the sum of the intensities in the two polarizations, (3.21b) and (3.22b), yields the input intensity $I_{in}$:

$$I_{in} = I_{out(x')} + I_{out(y')} \quad (3.23a)$$

If we introduce the strain dependence of the induced phase change $\delta$ (3.11b) into the equations (3.21b) and (3.22b) we have:

$$I_{out(y')} = \frac{I_{in}}{2} \sin^2 2\theta (1 - \cos \frac{2\pi x \varepsilon L}{\lambda}) \quad (3.23b)$$

$$I_{out(x')} = \frac{I_{in}}{2} (1 + \cos^2 2\theta + \sin^2 2\theta \cos \frac{2\pi x \varepsilon L}{\lambda}). \quad (3.23c)$$

Strain sensitivity of output signals (3.23b) and (3.23c) depends on the angle $\theta$, the strain-optic coefficient $C_x$, the wavelength $\lambda$ used, and specimen length $L$. The rate of change of intensity with $\delta$ can be maximized by:

- Setting the input polarization to be at angle $\theta$ to $45^\circ + k \times 90^\circ$ (k is integer).
- Using shorter wavelengths
- Using longer specimens
- Using material with larger strain-optic coefficient
We can easily satisfy the first condition, to set the input polarizer at 45° to the input linear polarization. For this case, we have plotted the normalized intensities $I_{\text{out}(x')}/I_{\text{in}}$, from equation (3.21b), and $I_{\text{out}(y')}/I_{\text{in}}$, from equation (3.22b), versus induced phase change $\delta$, in Fig.3.1.6. The solid curve corresponds to (3.21b) and the dashed curve corresponds to (3.22b). For the input angle at 0°, i.e. launching into the eigenmode, there is no effect. For all other input angles the effect is present but it is not optimal.
3.2 SINGLE MODE FIBER OPTIC STRAIN SENSOR CONSTRUCTION

A single mode fiber optic sensor assembled in the laboratory is an intrinsic sensor. In other words, the sensing element is the fiber itself. At the same time, the same fiber is used to bring the light to the strain-sensing region and to lead the useful signal to the detector. Optical fibers are usually made of fused silica, which has low intrinsic birefringence. However, linear birefringence can be induced (Chapter 2) in silica by stress, pressure, external electric field and bending, circular birefringence by twisting and external magnetic field, [17], [18], [19]. Single mode fiber optic sensor construction and induced phase change are analyzed in the following two sections.

3.2.1 The Laboratory Setup for the SM Fiber Optic Sensor

The single mode (SM) fiber optic strain sensor’s experimental setup is illustrated in Fig.3.2.1. The host structures for strain measurements were aluminum bars (width 1.8cm, length 45cm, thickness 0.9cm) and steel bars (width 1.2cm, length 37cm, thickness 0.6cm). They were fixed on one end and on the other positive (compressive) and negative (tensile) stresses were applied with the appropriate screw that went through the drilled hole. The stress was applied along the length of the bars, i.e. perpendicularly to the attached fiber. The lasers used in the experiments were He-Ne ($\lambda = 633\text{nm}$) and diode lasers with 780nm and 820nm wavelengths. Maximum sensitivity was obtained for the input polarizers set at 45° to the bar (horizontal direction). In other words, one of the induced eigenaxes was found to be parallel to the direction of the applied stress. The output polarizer (analyzer) was set to the position at which the maximum depth of modulation of the cosinusoidal signal was obtained, as well, but it also had to satisfy the
following condition: if the analyzer was rotated for 90° in either direction, the output signal had to be an "inverted" sinusoidal. This way it was confirmed that the fiber optic sensor is behaving as theory predicts (3.23).

![Diagram](image)

**Fig.3.2.1 The experimental setup for SM low intrinsic birefringence fiber optic strain sensor**

After passing through the output optics and analyzer the optical signal was detected with the photodetector and transformed to a photovoltage that was recorded with the computer measurement system based on LabView 5.0. The fiber was glued to the host structure with the special epoxy perpendicularly to the straining direction. High quality of fiber optic epoxy insured that the deformation in the bar was transmitted to the fiber in the best possible way, see Fig.3.2.2. The length of the sensing part of the fiber (variable L in 3.11) was therefore equal to the width of the bar. In this region, the fiber was stripped of its jacket in order to make it more sensitive to strain. Any attempt to epoxy the fiber with the jacket yielded no sensitivity to strain. This is not strange since the jacketless fiber is
almost two times more sensitive to pressure than the same fiber with the jacket [19], [20].

Our reference strain gauge was a standard industrial electrical strain gauge OMEGA SG-7/350-LY43. It was attached to the bar, near the fiber, with its own epoxy OMEGA SG-401. The electrical strain gauge was connected to the strain measurement bridge (not shown in the Fig. 3.2.1) which was then connected to the computer.

Ordinary single mode fibers cannot preserve the state of polarization of input light like polarization maintaining fibers (PM). But if the delivering and collecting parts of the SM fiber were kept short enough, so the state of polarization in it is not scrambled, the theory derived in the Section 3.1 could be applied to it. In the actual experiments, lengths of the SM fibers used for sensors were less than 35cm (delivering + sensing + collecting...
part). On the other hand multimode fibers scramble the polarization even after the short distance and that is why we don’t use them for this kind of sensors.

3.2.2. Induced Phase Change in the Epoxied Single Mode Fiber Optics

In this section we will establish a photoelastic model of the fiber epoxied to the host structure. For this purpose it is important to answer the question what does such a glued optical gauge measure? In the present context, the answer to this question involves three inference components. The first inference relates the strain in the host structure, $\varepsilon^h$, to the strain in the epoxy, $\varepsilon^e$. The second inference relates the strain in the epoxy to the strain in the optical fiber’s core. The third inference relates the strain in the optical fiber’s core, $\varepsilon^f$, to the induced phase shift $\delta$. We have explained the third inference through the derivation of Maxwell-Neumann’s photoelastic law in the Chapter 3 and the derivation for the induced phase shift (3.11) in this chapter. Considering the first two inferences we will assume following: the boundary conditions on a surface-mounted optical fiber govern the phase response induced in the fiber by the host structure to which it is bonded. The strain fields in the optical fiber, epoxy and the host structure are assumed to be continuous, and all the bonds are assumed to be without gaps or slips. Therefore, the displacement fields at the boundaries host-epoxy and epoxy-fiber are assumed to be equal. Furthermore, it is assumed that the strain field in the optical fiber is equal to the strain field in the host structure, if the optical fiber was not present. In other words it is assumed that the glued fiber will not significantly alter the strain distribution in the structure itself [20], [21]. We can write this assumption simply as (3.24), denoting all the strains with a single symbol $\varepsilon$: 
\[ \varepsilon = \varepsilon^h = \varepsilon^e = \varepsilon^f \]  

(3.24)

The fiber core is composed of chemically doped fused silica, but dopant concentrations are small enough that the single-mode fiber can be considered homogenous and isotropic [17]. Therefore we will start from Hooke’s law for a homogenous isotropic medium written in the principal coordinate system (2.15) for the fiber glued to the host structure, Fig.3.2.2:

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z
\end{bmatrix} =
\begin{bmatrix}
1 & \nu & \nu \\
\frac{\nu}{E} & \frac{1}{E} & \frac{\nu}{E} \\
\frac{\nu}{E} & \frac{\nu}{E} & \frac{1}{E}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
0 \\
0
\end{bmatrix}
\]

(3.25)

Applied stress \(\sigma\) is uniaxial, along the x-axis. \(E\) and \(\nu\) in equation (3.25) are Young’s modulus of elasticity and Poisson’s ratio of silica. Performing the matrix multiplication in (3.25) yields components of strain in the core of optical fiber.

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z
\end{bmatrix} =
\begin{bmatrix}
\frac{\sigma_x}{E} \\
\frac{\nu\sigma_x}{E} \\
\frac{\nu\sigma_x}{E}
\end{bmatrix}
\]

(3.26)
Now, when the components of strain in the fiber are known, we can apply the photoelastic law written in terms of induced small changes of the impermeability tensor. The strain-optic tensor for initially isotropic material is:

\[
\mathbf{P}_0 = \begin{bmatrix}
  p_{11} & p_{12} & 0 & 0 & 0 \\
  p_{12} & p_{11} & p_{12} & 0 & 0 \\
  p_{12} & p_{11} & 0 & 0 & 0 \\
  0 & 0 & 0 & p_{44} & 0 \\
  0 & 0 & 0 & 0 & p_{44}
\end{bmatrix}
\] (3.27)

with \( p_{44} = (p_{11} - p_{12})/2 \), [4]. Now, we can calculate induced changes in components of the impermeability tensor from the following equation:

\[
\begin{bmatrix}
  \Delta \left( \frac{1}{n^2} \right)_1 \\
  \Delta \left( \frac{1}{n^2} \right)_2 \\
  \Delta \left( \frac{1}{n^2} \right)_3 \\
  \Delta \left( \frac{1}{n^2} \right)_4 \\
  \Delta \left( \frac{1}{n^2} \right)_5 \\
  \Delta \left( \frac{1}{n^2} \right)_6
\end{bmatrix} = \mathbf{P}_0 \begin{bmatrix}
  \frac{\sigma_x}{E} \\
  \frac{-v\sigma_x}{E} \\
  \frac{E}{0} \\
  \frac{-v\sigma_x}{E} \\
  \frac{0}{0}
\end{bmatrix}
\] (3.28)

Denoting the strain \( \varepsilon \) in the previous equation with \( \varepsilon = \varepsilon_x = \sigma_x/E \), it yields:
\[
\Delta \left( \frac{1}{n^2} \right)_i = (p_{11} - 2\nu p_{12})\varepsilon \quad (3.29)
\]

\[
\Delta \left( \frac{1}{n^2} \right)_2 = \Delta \left( \frac{1}{n^2} \right)_3 = [p_{12} - \nu(p_{11} + p_{12})]\varepsilon \quad (3.30)
\]

\[
\Delta \left( \frac{1}{n^2} \right)_4 = \Delta \left( \frac{1}{n^2} \right)_5 = \Delta \left( \frac{1}{n^2} \right)_6 = 0 \quad (3.31)
\]

Using the derivative of \((1/n^2)\) to calculate the finite difference and approximate \(n_i\) with \(n\) (the unperturbed index of refraction):

\[
\Delta \left( \frac{1}{n^2} \right)_i = -\frac{2}{n^3} \Delta n_i, \quad i = 1, 2, ..., 6;
\]

we can calculate induced refractive index changes:

\[
\Delta n_1 = -\frac{n^3}{2}(p_{11} - 2\nu p_{12})\varepsilon \quad (3.32a)
\]

\[
\Delta n_2 = \Delta n_3 = -\frac{n^3}{2}[p_{12} - \nu(p_{11} + p_{12})]\varepsilon \quad (3.32b)
\]

\[
\Delta n_4 = \Delta n_5 = \Delta n_6 = 0 \quad (3.32c)
\]

Finally, the index difference between polarization components that are parallel and perpendicular to the applied uniaxial stress is:

\[
n_{paral} - n_{normal} = \Delta n_1 - \Delta n_2 = -\frac{n^3}{2}(1 + \nu)(p_{11} - p_{12})\varepsilon. \quad (3.33a)
\]
After traveling length $L$ of the sensor, Fig. 3.2.2, the acquired phase difference $\delta$ due to the index difference (3.33a) is:

$$\delta = -\frac{\pi n^3}{\lambda}(1 + \nu)(p_{11} - p_{12})\epsilon L$$

(3.33b)

Properties of silica and the host structure that are of interest are listed in Table 3-1 and Table 3-2. Using these values we can calculate phase sensitivity to strain.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (Young's modulus)</td>
<td>$6.5 \times 10^{10}$ N/m$^2$</td>
</tr>
<tr>
<td>$\nu$ (Poisson's ratio)</td>
<td>0.16</td>
</tr>
<tr>
<td>$p_{11}$ (Strain optic coefficient 1, at $\lambda=0.63\mu m$)</td>
<td>0.121</td>
</tr>
<tr>
<td>$p_{12}$ (Strain optic coefficient 2, at $\lambda=0.63\mu m$)</td>
<td>0.270</td>
</tr>
<tr>
<td>$n$ (Refractive index)</td>
<td>1.46</td>
</tr>
</tbody>
</table>

**Table 3-1: Some useful properties of silica [4]**

In the case of the aluminum bar and wavelength that corresponds to the He-Ne laser, the phase change sensitivity to the strain is:

$$\delta [rad] = 0.048\left[\frac{rad}{\mu strain}\right] \cdot \varepsilon [\mu strain].$$

<table>
<thead>
<tr>
<th>Material</th>
<th>Young's modulus [N/m$^2$]</th>
<th>Width $L$ (sensor length) [cm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum</td>
<td>$7 \times 10^{10}$</td>
<td>1.8</td>
</tr>
<tr>
<td>Steel</td>
<td>$(22 - 25) \times 10^{10}$</td>
<td>1.2</td>
</tr>
</tbody>
</table>

**Table 3-2: Some physical properties of the host structure useful for the sensor**
For this case we will plot the detected normalized intensity in Fig.3.2.3a from equations (3.23b) and (3.23c) with $\theta = 45^\circ$ and previously calculated $\delta$, ($\varepsilon$ is in microstrains).

$$\frac{I}{I_0} = \frac{1}{2}[1 \pm \cos(0.048 \cdot \varepsilon)]$$

(3.34)

In Fig.3.2.3a, with solid line is depicted normalized intensity in the direction parallel to the admission axis of the input polarizer and with the dashed line the intensity detected in the orthogonal direction.

Mathematically, we can view the operation of a photoelastic strain sensor in the following way. The state of polarization of the light can be described by a Jones vector [4] in two dimensional linear vector space. This is so because we can resolve any state of polarization written as a Jones vector in the basis of two orthogonal polarizations (two linear polarizations with orthogonal planes of oscillation, or two circular polarizations
with opposite sense of rotation, with scalar product defined as multiplication of Jones vectors with first vector conjugated). In this vector space, the input polarizer acts like a projection operator with which we can control the plane in which we launch light (or to control a subspace onto which we want to project the field vector of the input light). The photoelastic element acts as a linear operator as well. It has two eigenvectors, depending on the direction of applied stress (or force) – those are the states of polarization that correspond to induced fast and slow axes. If we launch the light into the either of the eigenmodes – the resultant vector (or state of polarization) will be unchanged (except for a phase factor), yielding no information about stress. Otherwise, for launching into the other modes the information about stress will be encoded into the output state of polarization. As the theory derived in this section shows, the best sensitivity is achieved when the input polarization equally excites both eigenmodes, Fig3.2.3a. For the input polarizer position at 18°, the response is depicted in Fig.3.2.3b. Launching into the induced eigenmodes yields no information about the strain in the material, as shown in Fig3.2.3c.

![Calculated normalized output intensity in the two orthogonal polarizations of SM fiber optic strain sensor for input polarizer set at 18° to the induced horizontal eigenaxis of the fiber.](image-url)
3.2.3 Experimental Results for SM Fiber Optic Strain Sensor

The simplest SM strain sensor was assembled as it was shown in Fig.3.2.1. We used Wave Optics FS-SN-4224 single mode fiber and a SHARP LTO27MD diode laser with a wavelength 779nm. The fiber jacket was stripped off and then the fiber was attached to the metal bar with special high temperature fiber optic epoxy, Epoxy Technology EPO-TEK 353ND. We used a pair of standard sheet polarizers in the experiments. The fiber length was kept below 40cm in order not to scramble the polarization in the delivering and collecting parts of the fiber. The lead-in and lead-out sections of the fiber can contribute to system noise, which is another reason to keep them as short as possible. These lengths of the fibers turned out in the experiments to be short enough to have the detected output signal in compliance with the theoretically derived formula (3.23). The input polarizer was set at $\theta = 45^\circ$ to the surface of the bar in order to have maximum sensitivity. The analyzer was set according to the same criterion. The recorded curves for case of increasing the strain followed the (co)sinusoidal law. One of the representative experimental graphs is shown in Fig.3.2.4.
In this case strain was increased from 0 to 1000 microstrains, as measured by the electric strain gauge. A similar curve for the case in which the strain was decreased from 1000 to 0 microstrains is shown in Fig. 3.2.5.

We can immediately spot four problems with the recorded data in these two graphs.
• The minima of the sinusoidal curves are not at the zero laser light level.

• For zero strain, the curves are not starting from the minimum (for sinusoidal curve, when input and output polarizers are crossed) or the maximum (for cosinusoidal curve, when input and output polarizers are aligned parallel).

• In some parts the curves deviate from the theoretically expected (co)sinusoidal shape.

• The phase sensitivity is much less than predicted by our simple model.

The first problem, which actually tells us that we either don't have maximum modulation, can be understood on the basis of following facts. The most likely, the polarization is changed in the lead-in (and lead-out) section before light gets to the epoxied section. Also, some background light is caught in the detector shifting the curve up. This is only roughly, and additionally the bar is moved up and down while we are straining it. That is why we can not maintain this angle constant during the straining process. The second problem, “zero strain shift”, can be explained by the fact that during the curing process unavoidable bias strain is built in the epoxy and that is what we see. The third problem is that both graphs in Fig.3.2.4. and Fig.3.2.5 are deviating from the sinusoidal shape. This is the consequence of the background light caught in the detector and the fact that sometimes the curing process is not completely successful and the fiber slips inside epoxy when they are strained. Same type of fiber was epoxied a couple of times and the measurements were done on them. One of the representative graphs for repeated measurements is Fig.3.2.6. Comparing this graph with the previous ones we can see differences in “built in strain” or in other words, differences in “zero shift” which are due to differences in curing processes.
Fig. 3.2.6: Typical response of SM fiber optic strain sensor

Properties of all the sensor configurations presented are presented in the Table 3-3.

<table>
<thead>
<tr>
<th></th>
<th>Output signal sensitivity [V/µε]</th>
<th>Phase sensitivity [rad/µε]</th>
<th>Resolution [µε]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum host</td>
<td>0.00041</td>
<td>0.00184</td>
<td>3.75</td>
</tr>
<tr>
<td>Steel host</td>
<td>0.00067</td>
<td>0.00103</td>
<td>2.44</td>
</tr>
</tbody>
</table>

**Table 3-3: Properties of the simplest SM fiber optic strain sensors (633nm)**

Output signal sensitivity is expressed in volts per microstrain and phase sensitivity in radians per microstrains. The output signal sensitivity is calculated as the slope of a fit of the linear portion of the sinusoidal curve. The phase sensitivity is calculated as slope of $\sin^{-1}$ or $\cos^{-1}$ of the recorded curves. The sensor can achieve this sensitivity only in the linear part of its sinusoidal characteristics. Preferably, the sensor should be biased so that zero strain coincides with the middle point between minimum and maximum in ordered to have large sensitivities for small strains. Strain resolution – smallest strain that the sensor can measure – is expressed in microstrains. It is calculated
by dividing the maximum error in the recorded signal by the sensitivity. For each point plotted in the graphs, 21 measurements of strain were done, the arithmetic mean value and 1- sigma error bars were calculated. The maximum of the calculated errors for each plotted point in the graph was taken for calculation of sensor resolution. This version of the sensor had a dynamic range of 1:160 in case of aluminum host and somewhat lower for the steel host. Dynamic range was calculated when the linear range of the curve was divided with the resolution.

We see that the experimental values for phase sensitivity to small strains is one order of magnitude smaller than predicted by the coarse theoretical model, Section 3.2.2. This is not an unexpected result since the model assumes many simplifications. The model doesn't account for the fact that the fiber is not directly strained but the strain is transmitted through the epoxy. It has been observed that even when a jacketless fiber is directly strained with line force, the experimental phase sensitivity to strain is 10% less than theoretical prediction, and 50% less for the fiber with the jacket [21]. In our experiments, the epoxied fiber with the jacket was not sensitive to strain at all, so we expected the jacketless fiber to be much less sensitive to strain than the coarse model predicts. In other words, not all the strain is transferred from the host structure to the fiber. This problem can be overcome by calibration. The important fact is that, generally, the (co)sinusoidal law is confirmed in the experiment.

We also constructed and investigated a SM fiber-optic strain sensor that employed a retroreflection configuration. This sensor is depicted in Fig.3.2.7.
The motivation for this approach was twofold. First, to make the sensor smaller, with only one lead-in lead-out fiber. Second, to increase sensitivity by effectively doubling the sensor length, variable L in (3.33). A mirror was formed on one end of the fiber (enlarged part in Fig.3.2.7) by deposition of Aluminum. This sensor was harder to assemble and test because of a very small returned signal. The recorded signal from this sensor is shown in Fig.3.2.8, where we can see part of the well-known sinusoidal response curve. In these measurements some background light is caught in the detectors. The noise level was too high yielding unacceptable resolution of 8 5με with even doubled sensitivity. Therefore the SM retroreflection configuration was abandoned, but not the idea of retroreflection itself.
It is not easy to equally excite both eigenmodes by launching the light at 45°, Fig.3.2.9, the sensor moves because of applied stress to the bar and in real applications the host structure can vibrate causing changes in the launch angle and therefore degrading
the performance of the sensor. If circularly polarized light is launched, both induced eigenmodes are again equally excited, but a unique input polarization angle cannot be defined because of symmetry. Therefore, the output signal in this case should be independent of the "critical input angle", Fig. 3.2.10.

The introduction of a 90° phase difference between the two launching polarizations will
also have the consequence, theoretically, of shifting the zero-strain bias to the most sensitive point on the curve, Fig.3.2.3a. It means that for the small strains we would have maximum sensitivity. Some of the experimentally recorded curves for circular input polarization are shown in Fig.3.2.11a and Fig.3.2.11b. In this experiment, curves were recorded for two orthogonal positions of the analyzer. As expected, the behavior is similar to theoretical predictions, Fig.3.2.3a, except that the curve is shifted for the amount of strain corresponding to phase shift of 90°. In this case, the host structure was the aluminum bar. The linear portion of the curve in Fig.3.2.11b was used for the calculation below. This part of the curve along with the linear fit is, for example, depicted in Fig.3.2.12.
By using input circular polarization we have recorded a strain curve whose shape is much closer to what we have expected theoretically and therefore, technical characteristics of the sensor were improved as well. In this example (Aluminum bar, He-Ne laser) the fit to a linear function given in (3.35).

$$y[V] = 0.0806V + 0.0018 \frac{V}{\mu e} x[\mu e]$$  \hspace{1cm} (3.35)

We have achieved a resolution of 2.76 microstrains and dynamic range of 1:217. The sensitivity of 0.00181 V/\(\mu e\), or in terms of phase 0.00204 \(\text{rad}/\mu e\) was achieved. As before, phase sensitivity is one order of magnitude smaller than theoretically expected, due to the simplicity of the theoretical model. Also, these results show improvements in resolution, dynamic range and phase sensitivity compared to the experiments with input linear polarization, but not substantial improvements.
3.2.4. Thermal Apparent-Strain Sensitivity of Single Mode Fiber-Optic Strain Sensor

All the experiments in Section 3.2.3 were carried out in an isothermal environment. However, the sensor is known to exhibit an unwanted cross-sensitivity to temperature. Here we do not consider the unadhered temperature sensitivity but rather thermal apparent-strain sensitivity of a sensor bonded to an unconstrained structure. A fiber-optic strain sensor is said to generate a thermally induced apparent-strain reading if the state of polarization of light is altered when it is adhered to an unconstrained structure subject to a temperature change. The ideal is a zero apparent-strain sensitivity sensor that generates constant strain reading if it is adhered to a heated unconstrained structure and that generates output that is proportional to thermal stress if adhered to fully constrained thermally loaded structure [22].

In order to account mathematically for thermally induced strain $\varepsilon_i$ we have to modify Hooke's Law (2.15) as follows:

$$\varepsilon_i = s_{ij} \sigma_j + \varepsilon_i$$  \hspace{1cm} (3.36)

Here, $s_{ij}$ is the compliance tensor as defined in the Section 2.3. The first term in (3.36) is stress related strain and we will denote it as $\varepsilon^s_i \equiv s_{ij} \sigma_j$. The second term in (3.36) is thermal apparent-strain and we will denote it as:

$$\varepsilon_{\text{appar-}i} \equiv \varepsilon_{\text{measured-}i} - \varepsilon^s_i$$  \hspace{1cm} (3.37)

Within the first-order model [23], [24] of an adhered fiber, the following three assumptions will be made:
• The coupling tensor between the fiber (superscript $f$) and the host structure (superscript $h$) is of the form (3.24), (3.25):

$$\varepsilon_i^f = z_{ij} \varepsilon_j^h$$  \hspace{1cm} (3.38)

$$z_{ij} = \begin{bmatrix}
1 & 0 & 0 \\
-\nu^f & 0 & 0 \\
-\nu^f & 0 & 0
\end{bmatrix}$$  \hspace{1cm} (3.39)

Here, $\nu$ is Poisson’s ratio, $i = 1$ is the fiber direction and $i = 2, 3$ are perpendicular directions.

• Thermal mass of the fiber is negligible compared to that of the aluminum or steel host structure.

• The strain and temperature sensitivities are linear and without cross sensitivities.

None of these assumptions are correct over a large strain or temperature range but it is possible to apply them locally.

Phase sensitivity to strain in a fiber-optic sensor in an isothermal environment is

$$g = \frac{\partial \phi}{\partial \varepsilon_i^f} |_{T = \text{const}}$$  \hspace{1cm} (3.40)

Free fiber temperature sensitivity is

$$f = \frac{\partial \phi}{\partial T} |_{\varepsilon_i^f = \text{const}}$$  \hspace{1cm} (3.41)
In order to account only for intrinsic fiber properties we can normalize \( f \) and \( g \) by the length of sensing part of the fiber \( t \), \( G = g/t \) and \( F = f/t \). Finally, we can define thermal apparent-strain sensitivity as

\[
\frac{\varepsilon_{\text{apparent}}}{\Delta T}
\]

In a physical situation where stress induced strain is not present, \( \varepsilon_{\text{apparent}} = 0 \), equation (3 36) yields that measured strain is equal to thermal apparent-strain,

\[
\varepsilon_{\text{apparent}} = \varepsilon_{\text{measured}}
\]

This is the situation of unconstrained host structure. The net induced optical phase change, \( \Delta \phi \), is the superposition of the phase change induced due to stress, \( \Delta \phi^{\text{stress}} \), and temperature induced phase changes, \( \Delta \phi^{\text{temperature}} \)

\[
\Delta \phi = \Delta \phi^{\text{stress}} + \Delta \phi^{\text{temperature}}
\]  

(3 42)

This superposition is illustrated in Fig.3.2.13. \( \alpha_{1}^{\ell} \) is the thermal expansion coefficient of

---

**Fig.3.2.13.** Unconstrained thermal expansion of a free and an adhered fiber optics.
the fiber (silica), and $\alpha_1^h$ is the thermal expansion coefficient of the host structure (steel, aluminum). We will denote the difference in thermal expansion coefficients of the host and the fiber with $\alpha_1^\Delta$:

Substitution of (3.40) and (3.41) in (3.42) yields the expression for phase change (3.43),

$$\Delta \phi = f \Delta T + g \alpha_1^\Delta \Delta T$$  \hspace{1cm} (3.43)

When the phase change is converted to a measured strain change and having in mind that for an unconstrained structure measured and apparent-strains are the same, we arrive at the equation for thermal apparent-strain sensitivity:

$$\frac{\varepsilon_{\text{appar}}}{\Delta T} = F \frac{G}{G} + \alpha_1^\Delta$$  \hspace{1cm} (3.44)

### 3.2.4.1. Experimental Results For Thermal Apparent-Strain Sensitivity

In order to determine the thermal apparent-strain sensitivity, the following experiments were conducted. All the experiments were done on the aluminum and the steel host structures with thermal expansion coefficients $\alpha^{\text{al}} = 22 \, \mu e/K$ and $\alpha^{\text{steel}} = 12 \, \mu e/K$, respectively. The epoxy used to glue the fiber to the structure (EPO-TEK 353ND) had thermal expansion coefficient $\alpha^{\text{epoxy}} = 54 \, \mu e/K$ below $T_g$ ($= 150^\circ C$) and $\alpha^{\text{epoxy}} = 160 \, \mu e/K$ above $T_g$. In all the conducted experiments, the free fiber temperature sensitivity was zero as it was expected due to symmetry of the fibers.

$$F = 0 \frac{\text{rad}}{\text{K} \cdot \text{m}}$$
Further experiments were conducted on fibers adhered to the unconstrained aluminum and steel structure in order to determine the experimental thermal apparent strain sensitivity theoretically described by (3.34). The whole setup was heated with simple industrial elastic heating tape, Fig. 3.2.14. The heating tape was driven with the varistor, so the temperature could be increased gradually, at will. Measurement of each point was stationary, i.e. the temperature, optical signal and electrical strain gauge signal were recorded after the equilibrium was reached. For measuring the temperature, an OMEGA 44034 thermistor was used. An example of recorded optical signal for the thermally loaded bar is shown in Fig. 3.2.15.
For the aluminum host structure, the measured thermal apparent strain sensitivity was

$$\frac{\varepsilon_{\text{appar}}^{\text{Al}}}{\Delta T} = 54.2 \frac{\mu \varepsilon}{\circ C}$$

and for the steel host structure

$$\frac{\varepsilon_{\text{appar}}^{\text{Steel}}}{\Delta T} = 82 \frac{\mu \varepsilon}{\circ C}$$

Both results are in discrepancy with theoretical prediction of the simple model (3.44).

The simple model predicts, in the case $F = 0$, a thermal apparent-strain sensitivity equal to the difference between the thermal coefficient of the host structure and the fiber. Since the thermal expansion coefficient of silica is $0.6 \mu \varepsilon/\circ C$, the model predicts a thermal apparent-strain sensitivity of $21.4 \mu \varepsilon/\circ C$ for an aluminum host and $11.4 \mu \varepsilon/\circ C$ for a steel
host. The simple model did not included thermal expansion coefficient of epoxy which is 54\(\mu\)e/°C, Fig.3.2 16

![Diagram](https://example.com/diagram.png)

**Fig.3.2.16. Interaction of thermal expansion coefficients**

Thermal expansion of the epoxy is constrained by smaller thermal expansion of the host structure. Therefore the epoxy will compressively stress the fiber (the cross section will turn to the ellipse in Fig.3.2 16.). Roughly, compression for the case of aluminum and steel host would be 22\(\mu\)e/°C - 54\(\mu\)e/°C = -32\(\mu\)e/°C and 12\(\mu\)e/°C - 54\(\mu\)e/°C = -42\(\mu\)e/°C, respectively. Although, this is very rough estimate we can see that the compression for steel host is larger than for aluminum host which is consistent with the experimental results.

This cosensitivity is well known [25] and it could be exploited to measure both strain and temperature [26]. However, it is preferred to diminish influence of temperature in strain sensors. It could be done with complicated signal processing [26]. In literature can be found simpler methods like precise coating of the fiber sensor [27], [28]
3.3 STRAIN SENSOR CONSTRUCTION WITH POLARIZATION MAINTAINING FIBER OPTIC

The design that we constructed for a polarization maintaining fiber optic strain sensor is an extrinsic type of sensor. The PM fiber is used only to deliver and collect light, while the sensing element is a piece of photoelastic sheet PS-1. This type of sensor is similar to the retro reflection SM fiber optic strain sensor (Section 3.2.3) except it has one additional feature: "difference over sum" detection. In the next section we will describe in more detail the advantages of "difference over sum" detection. The design of an all-fiber version of the PM fiber optic sensor is illustrated in Fig.3.3.1. Linearly polarized light is launched into one of the eigenmodes of the PM fiber. Using two polarizing beam splitter couplers and PM fiber, linearly polarized light is brought to the

![Fig.3.3.1. All-fiber configuration of PM fiber optic strain sensor](image-url)
sensor head, and collimated with the GRIN lens. The sensing element is PS-II photoelastic sheet to which the GRIN lens is epoxied. The bottom surface, the one that is glued to the host structure, of PS-II is reflecting. Linearly polarized output from the PM fiber is set at 45° to the strain-induced fast or slow axis in the photoelastic sheet in order to have maximum depth of modulation, Fig. 3.2.3a. The direction of the induced slow axis is parallel to the direction of applied uniaxial stress. The light makes two passes through the photoelastic sheet, and returns into the PM fiber elliptically polarized. The elliptical polarization is analyzed in the polarizing beam splitter couplers (like in the output polarizer – analyzer in Fig. 3.1.5, Section 3.1.2). In other words, we can separately detect components of the elliptically polarized as projections onto the basis of the two eigenmodes of the PM fiber, denoted as parallel and cross (to the input linear polarization) polarizations. We can also monitor power fluctuations in the laser in one output of the system, Fig. 3.3.1. One of the outputs is not in use and therefore is terminated with glycerin in order not to have any back reflections that would degrade the system performance.

3.3.1 Difference-Over-Sum Processing

The main advantage of the PM fiber configuration over the SM retro reflection configuration is that we can detect signals in both eigenmodes and process them in order to obtain a signal that is independent of power fluctuations in the laser and linear for small strains. We will show this in the following short calculation. The returned signal in the parallel polarization is given by (3.34) with the length of the sensor doubled (L in this equation is thickness of the PS-II sheet).

\[
I_{\text{out, parallel}} = \frac{I_{\text{in}}}{2} \left(1 + \cos \frac{4\pi C_{\text{PS-I}} L E}{\lambda} \right)
\]  
(3.47)
We can continue to calculate the output signal in orthogonal polarization (plotted in Fig.3.2.3a) as it was done in Section 3.1.2., but it is easier to employ the fact that the two signals are shifted by 180 degrees and trigonometric formula:

$$\cos(\alpha + 180^\circ) = -\cos \alpha.$$ 

Therefore, we have for the output signal in the cross polarization

$$I_{out,\text{cross}} = \frac{I_{in}}{2}(1 - \cos \frac{4\pi C_e t \varepsilon}{\lambda}).$$  \hspace{1cm} (3.48)

Difference-over-sum of signals in (3.47) and (3.48) is given by

$$\frac{I_{out,\text{parallel}} - I_{out,\text{cross}}}{I_{out,\text{parallel}} + I_{out,\text{cross}}} = \cos \frac{4\pi C_e t \varepsilon}{\lambda}.$$ \hspace{1cm} (3.49)

As it can be seen from (3.49), the quantity $I_{in}$ drops out of the equation and the input signal independent of the power fluctuations in the laser output. Furthermore, if the additional strain bias is built into the PS-1 sheet, equivalent to $90^\circ$, the cosine turns into sine and for the small strains the transfer characteristic is linear (3.50):

$$\frac{I_{out,\text{parallel}} - I_{out,\text{cross}}}{I_{out,\text{parallel}} + I_{out,\text{cross}}} = \sin \frac{4\pi C_e t \varepsilon}{\lambda} \approx \frac{4\pi C_e t \varepsilon}{\lambda}.$$ \hspace{1cm} (3.50)

Difference-over-sum processing can be easily achieved with commercially available electronic circuits.
3.3.2 Experimental Results For PM Fiber Optic Strain Sensor

We employed a modified experimental setup for PM fiber optic strain sensor. It is a less expensive equivalent to the system in Fig.3.3.1. It contains one simple beam splitter, one cube polarizing beam splitter, and one PM fiber, as shown in Fig.3.3.2. The parallel polarized return signal is detected in photodetector D-2, while the cross polarized return signal is detected in D-1. The sensing element, PS-1 birefringence sheet, was fixed in the test frame (Fig.3.3.3) that enabled precise application of strain. An electrical strain gauge, ESG, was directly epoxied to the PS-1 sheet and used as reference gauge. The straining frame was used because at the time enough good hardware was not available to couple the light from the sensor head epoxied with the PS-1 to the host structure (Fig.3.3.1) back to the PM fiber which is, of course, single mode.
First we experimentally recorded the back-reflected light intensity vs. mirror distance in the setup, Fig.3.3.2., without the PS-1 sheet and the cube beam splitter polarizer. The result is of interest for the all-fiber sensor and it is shown in Fig.3.3.4.
We used an 850nm PM fiber, terminated with a 0.25 pitch GRIN lens and an 810nm-diode laser in the experiment. The next step was to check out the degree of polarization at the output of the PM fiber. For that purpose we used two identical Newport beam splitter polarizers. The polarizers gave 37dB and 33dB linear polarization in the air. With the PM fiber and the polarizers we had 27dB and 23dB. For the strain experiment we used the better cube beam splitter. In the experiments we had 50 measurements to average for each point in the graphs for the two signals D1 and D2. Based on 50 measurements for each point in the both graphs 1-sigma errors were calculated, E1 and E2. First experiments were conducted with the two lock-in amplifiers and the chopper placed as in Fig.3.3.2. In this case raw data for the two polarizations, D1 and D2, are in Fig.3.3.5a. and Fig.3.3.5b. respectively.

![Graph](image_url)

**Fig.3.3.5a. Signal D1 in the experiment with the chopper and lock-in amplifier**
Fig.3.3.5b. Signal D2 in the experiment with the chopper and lock-in amplifier

Since both curves have shapes close to the theoretical prediction up to 1700 microstrains this interval of strains will be used to calculate the resolution and the dynamic range.

For raw signal D1 in the interval 154-1025 microstrains, the sinusoidal curve can be replaced with its linear fit with coefficient of correlation equal to 0.9916. The slope of the linear fit is 0.00193V/µε. Average 1-sigma error of the measurements was 0.00159V (arithmetic mean of 1-sigma errors of all recorded and plotted points in Fig.3.3.5a.), which corresponds to 0.82 microstrains resolution. These data correspond to 1:1062 dynamic range. However, for the 1025-2450-microstrain interval, which deviates from the theoretical curve, the same procedure yields 1.2 microstrains resolution and 1:1080 dynamic range.
Since it is obvious that detectors for signals D1 and D2 were not equally calibrated, normalization of both signals D1, D2 had to be done before the difference-over-sum was calculated (3.51).

\[ D_{d-o-s} = \frac{D_{n1} - D_{n2}}{D_{n1} + D_{n2}} \]  

(3.51)

Here D_{n1} and D_{n2} are signals D1 and D2 after normalization that was done in the following way. First, the zero offsets were subtracted off. Then, the results were divided with the maximum values in order to yield the curves, D_{n1} and D_{n2}, which are bounded between 1 and 0. Accordingly, scaling of the 1-sigma errors E1, E2 to E_{n1} and E_{n2} with appropriate constants (maximum values of D1 and D2 respectively) had to be done before error of difference-over-sum, E_{d-o-s}, was calculated with the error propagation formula [31]. When this formula is applied to our case, error for difference-over-sum, E_{d-o-s}, is given by (3.52).

\[ E_{d-o-s} = \sqrt{\left(\frac{\partial D_{d-o-s}}{\partial D_{n1}}\right)^2 (E_{n1})^2 + \left(\frac{\partial D_{d-o-s}}{\partial D_{n2}}\right)^2 (E_{n2})^2} \]  

(3.52)

When we introduce (3.51) in (3.52) we obtain explicit formula for the error (3.53).

\[ E_{d-o-s} = \frac{2}{(D_{n1} + D_{n2})^2} \sqrt{(D_{n2})^2 (E_{n1})^2 + (D_{n1})^2 (E_{n2})^2} \]  

(3.53)

Graph D_{d-o-s} vs. strain with E_{d-o-s} error is plotted in the Fig.3.3.5c.
For difference-over-sum signal $D_{d-o-s}$ in the interval 154-951 microstrains, the sinusoidal curve can be replaced with its linear fit with coefficient of correlation equal to 0.9915. The slope of the linear fit is $-0.00219 \ \mu \varepsilon^{-1}$. Average $E_{d-o-s}$ is 0.00129 (arithmetic mean of $E_{d-o-s}$ of points in graph in Fig.3.3.5c), which corresponds to 0.59 microstrains resolution and 1:1350 dynamic range. However, for the region 1188-2400 microstrains, which deviates from the theoretical curve, the same procedure yields 0.86 microstrains resolution and 1:1409 dynamic range.

The same experiments were repeated without using a lock-in amplifier. The signal in the parallel polarization, D2 is graphed in Fig.3.3.6a. and for the cross polarization in the Fig.3.36b.
After similar signal processing was done for the difference-over-sum, the following results are obtained: the difference-over-sum, $D_{d-o-s}$, signal is graphed in Fig.3.3.7. In the
189-1122 microstrain region, the sinusoidal curve can be replaced with its linear fit with coefficient of correlation equal to -0.987. The slope of the linear fit is 0.0021 \( \mu e^{-1} \). Average \( E_{d-o-s} \) is 0.00189, which corresponds to 0.9 microstrains resolution. These data correspond to 1:1010 dynamic range. However, for the region 1196-2351 microstrains, which deviates little bit from the theoretical curve, the same procedure yields 1.18 microstrains resolution and 1:1005 dynamic range.

**Fig.3.3.7. Difference-over-sum vs. strin in the experiment without the chopper and lock-in amplifier**

Without using lock-in amplifiers, some background light was detected, which increased noise and therefore degraded resolution and dynamic range. A practical sensor would not use a chopper and lock-in, but it would be all-fiber, as in Fig.3.3.1, and
consequently detect less background light and therefore have better resolution and
dynamic range than was demonstrated in this experiment.
Chapter 4:

Conclusion and Suggestions for Further Research

This thesis has theoretically and experimentally investigated two designs of polarimetric fiber-optic strain sensors. In the first configuration investigated, the sensing element was single mode optical fiber that was used at the same time to deliver and collect light. This sensor was made in two versions. First version had the light going through with delivering and collecting arm. In order to increase its low sensitivity and dynamic range retro-reflection version was made with the light going twice through the sensing region of the SM fiber. In this case one end of the SM fiber was metalized. Neither versions of this sensor showed good enough sensitivity, resolution and dynamic range. The SM fiber optic sensor also showed significant cross sensitivity to temperature. An optical common-mode rejection technique for suppressing thermal apparent-strain sensitivity was also investigated in this version of sensor and did not yield promising results. Therefore the second approach was undertaken. In this configuration polarization maintaining fiber was used to deliver and collect light and the sensing element was a piece of photoelastic sheet. This was also a retro-reflection sensor, since the mirror in the setup reflected the light back to the PM fiber. The returning signals in the two orthogonal polarizations were detected and processed giving a result that was independent of noise in the laser. Therefore this sensor had excellent resolution and dynamic range. Due to the experimental setup that was used for the second sensor it was not possible to test it for thermal apparent-strain sensitivity. But as for the other approaches undertaken in this
laboratory sensors using symmetric bulk optic elements like cubes and prisms are expected to have minimal thermal apparent-strain sensitivity [32].

As a suggestion for further research to improve the sensor performance, particularly dynamic range, it is important to eliminate ambiguity about the part of the sinusoidal curve we are using for the sensor, "fringe" ambiguity. In the literature a dual wavelength method is often used [33], [34], [35]. Also, instead of the bulk optic beam splitters it is recommended to use polarizing beam splitter fiber coupler and make the sensor all-fiber. This would reduce the level of background light and consequently reduce the noise, making the sensor characteristics similar to those reached in the experiment with chopper and lock-in amplifier.
Appendix A:

Symmetry Property of the Dielectric Tensor [3]

If we assume that the medium is homogenous, nonabsorbing, and magnetically isotropic, the density of energy stored in electric field and its time derivative are:

\[ W = \frac{1}{2} \bar{E} \cdot \bar{D} = \frac{1}{2} E_i \varepsilon_{ij} E_j \ldots \ldots (A.1) \]

\[ \dot{W} = \frac{1}{2} \varepsilon_{ij} (\dot{E}_i E_j + E_i \dot{E}_j) \ldots \ldots (A.2) \]

where we have used (1.14). On the other hand, according to the Poynting’s Theorem the net power flow into a unit volume in a lossless medium is by using (1.14):

\[ -\nabla \cdot (\bar{E} \times \bar{H}) = \bar{E} \cdot \dot{\bar{D}} + \bar{H} \cdot \dot{\bar{B}} = E_i \varepsilon_{ij} \dot{E}_j + \dot{H}_i \dot{B}_j \ldots \ldots (A.3) \]

The first term in (A.3) corresponds to electrical field flux, which must be equal to the expression given by (A.2). Therefore we have:

\[ \varepsilon_{ij} \dot{E}_i \dot{E}_j = \frac{1}{2} \varepsilon_{ij} (\dot{E}_i E_j + \dot{E}_i E_j) \ldots \ldots (A.4) \]

From (A.4) immediately follows

\[ \varepsilon_{ij} = \varepsilon_{ji} \ldots \ldots (A.5) \]

This means that the dielectric tensor is symmetric and has only six independent elements. As we have just seen, this symmetry is consequence of the definition (1.13) and
assumption that tensor is real, i.e. the medium is nonabsorbing. In the case of absorbing medium condition (A.5) will turn to request for dielectric tensor to be Hermitian.

\[ \varepsilon_{y} = \varepsilon^{*\mu} \ldots \ldots (A.6) \]

As a conclusion, this means that conservation of energy of electromagnetic wave requires that the dielectric tensor be Hermitian.
Appendix B:

Diagonalization of Hermitian Matrices

In the literature on functional analysis and linear algebra can be found the Diagonalization Theorem. It states [6]: “Any Hermitian matrix $A$ may be diagonalized by a unitary transformation; i.e., there exist a unitary matrix $U$ such that transformed matrix $U^*AU$ is a diagonal matrix (the diagonal elements of which are the eigenvalues of $A$)”. A matrix $A$ is Hermitian if $A_{ij} = A_{ji}^*$ and this condition reduces to simple symmetry, $A_{ij} = A_{ji}$, for real matrices. The proof of this theorem can be found, for example, in [7].
Appendix C:

Stress Tensor [8], [9]

It can be easily proven that components of stress $\sigma_{ij}$ form second-rank tensor if they relate components of two vectors $l_i$ and $f_j$ as

$$f_i = \sigma_{ij} l_j \quad \quad \quad (C.1)$$

where the summation is assumed over the repeated indices.

For any small surface area, within the stressed body, $\delta S$ (surface ABC in Fig.A.C 1), we will spot unit vector normal to the surface $\mathbf{n}$ and the vector of force transmitted through the surface $\mathbf{F} = \mathbf{p}\delta S$. If the surface element (normal $\mathbf{n}$) takes the different orientation we will calculate how does the force over unit surface $\mathbf{p}$ change.

![Fig. A.C.1. Proof that components of stress form tensor](image-url)
Resolving forces parallel to each of the axes we have

\[ p_i \cdot \text{surf} \,(ABC) = \sigma_{11} \cdot \text{surf} \,(BOC) + \sigma_{12} \cdot \text{surf} \,(AOC) + \sigma_{13} \cdot \text{surf} \,(AOB) \quad (C.2) \]

Or writing in short

\[
\begin{align*}
p_1 &= \sigma_{11} \cdot l_1 + \sigma_{12} \cdot l_2 + \sigma_{13} \cdot l_3 \\
p_2 &= \sigma_{21} \cdot l_1 + \sigma_{22} \cdot l_2 + \sigma_{23} \cdot l_3 \\
p_3 &= \sigma_{31} \cdot l_1 + \sigma_{32} \cdot l_2 + \sigma_{33} \cdot l_3
\end{align*}
\]  
(C.3)

In (C.3) \( p_i \) are components of vector \( \mathbf{p} \) and \( l_i \) are components of vector whose intensity is equal to \( \text{surf}(ABC) \) and projections on planes perpendicular to the three axes are \( \text{surf}(BOC) / \text{surf}(ABC) \), \( \text{surf}(AOC) / \text{surf}(ABC) \) and \( \text{surf}(AOB) / \text{surf}(ABC) \). From (C.3) we see that components of stress satisfy relation of type (C.1) and we conclude that stress is a tensor of second rank.
References:


