ON THE PERFORMANCE ANALYSIS OF THE BEST EFFORT ONLINE POWER CONTROL POLICY FOR ENERGY HARVESTING COMMUNICATIONS

ON THE PERFORMANCE ANALYSIS OF THE BEST EFFORT ONLINE POWER CONTROL POLICY FOR ENERGY HARVESTING COMMUNICATIONS

BY

YE WANG, B.Eng.

A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL & COMPUTER ENGINEERING

AND THE SCHOOL OF GRADUATE STUDIES

OF MCMASTER UNIVERSITY

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF APPLIED SCIENCE

Master of Applied Science (2019)	McMaster University
(Electrical & Computer Engineering)	Hamilton, Ontario, Canada

TITLE:	On the Performance Analysis of the Best Effort Online Power Control Policy for Energy Harvesting Communications
AUTHOR:	Ye Wang B.Eng. (Electronic Information & Optical Engineering), Nankai University, Tianjin, China
SUPERVISOR:	Dr. Jun Chen
NUMBER OF PAGES:	x, 47

To my family and friends

Abstract

Recent advances in energy harvesting techniques have enabled the development of self-sustainable systems that are powered by renewable energy sources in the environment. In this field, designing power control policies that can achieve high throughput is a significant problem. In this thesis, we analyze the long-term average throughput of Best Effort Policy in energy harvesting communications under a Bernoulli energy arrival process. We provide a limit and an order's upper bound of the difference between throughput induced by limited battery and throughput induced by unlimited battery for two Bernoulli cases respectively when battery capacity approaches infinity. Besides, some other energy harvesting processes are considered, their characteristics of throughput are discussed by numerical simulation.

Acknowledgements

Firstly, I would like to thank my supervisor Dr. Jun Chen for his strong support, kind encouragement, and valuable guidance through every stage of my master study. Under his supervision, I learned a lot of academic knowledge and found the direction of my future work.

Secondly, I would also like to thank Dr. Dongmei Zhao and Dr. Sorina Dumitrescu for reviewing my thesis and giving me valuable advice in conducting my thesis.

Thirdly, I would also like to thank my colleague Yaohui Jing for her kind help in research.

Lastly, I am forever grateful to my parents for their love, support, and encouragement and to my friends Rong Wu and Wenjie Zhao for their company and for bringing joy and advise in my life over the past year.

Contents

List of Figures

Notation, Definitions, and Abbreviations

Notation

$\|A\|$	Absolute Value of real number A or Modulus of complex number A
A^*	Complex Conjugate of Complex Number A
$i.i.d$	Independent Identically Distribution
μ	Mean of Energy Arrival Distribution
Θ	Long-term Average Throughput of Energy Harvesting Communications with Limited Battery
Θ_∞	Long-term Average Throughput of Energy Harvesting Communications with Unlimited Battery

Definitions

Energy Arrival Process

The Process of Harvesting Energy from Outside

Power Control Policy

The Scheduling of Energy Allocation from Transmitter

Abbreviations

AWGN	Additive White Gaussian Noise
HSU	Harvest-store-use
HUS	Harvest-use-store
MDP	Markov Decision Process
p.d.f	Probability Distribution Function

Chapter 1

Introduction

Recent advances in energy harvesting techniques have enabled the development of self-sustainable systems that are powered by renewable energy sources in the environment, such as solar energy, wind energy, thermal energy and so on. This technology makes it possible to reduce the dependence of battery and it has received a lot of attention. Unlike conventional communication systems, the power supply of energy harvesting systems is not fixed due to the nature of renewable energy sources, for example, the solar and wind energy are related to weather condition and geographic situation, which indicates that the problems in this field will be challenging.

In the past several years, a lot of work on energy harvesting communications has been done, including the research of energy sources, usage protocol, power control policy and other aspects [10], [27].

For energy sources, mainly, there are four types as solar or light, motion, thermoelectric effect, and electromagnetic radiation. The amount of harvested energy and whether it is controllable or not base on each energy source's own characteristics [9, 18, 20, 21, 24, 30].

Two significant schemes of energy harvesting systems are studied widely as harvest-store-use (HSU) scheme and harvest-use-store (HUS) scheme [19]. The harvest-store-use scheme harvests energy and stores it into the equipped battery before transferring power to the receiver, which is shown in Figure 1.1. On the other side, harvest-use-store scheme use the obtained energy firstly and then store the remaining power into the battery, which is shown in Figure 1.2. Generally, HUS architecture can have a better performance than HSU architecture if the transmitter is equipped with a finite-sized battery because it can reduce the amount of wasted energy caused by overflow in the battery. In our thesis, only the HSU scheme is considered.

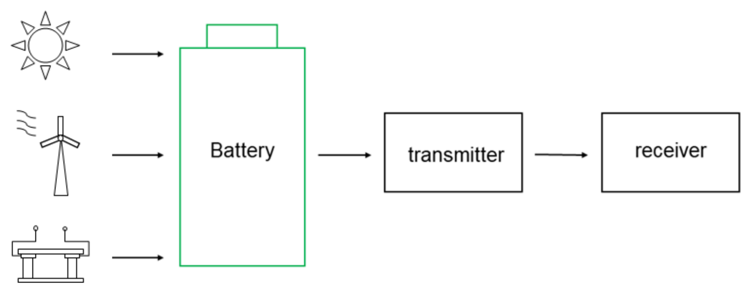


Figure 1.1: Energy Harvesting Communications (HSU)

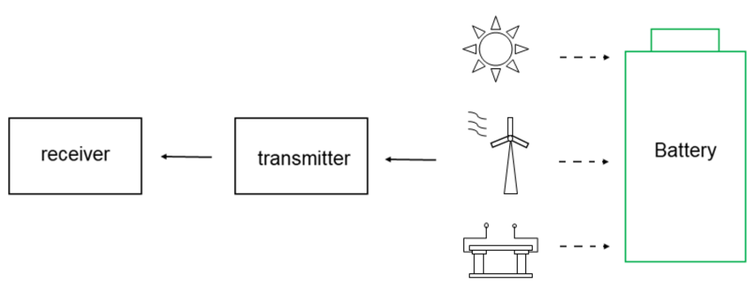


Figure 1.2: Energy Harvesting Communications (HUS)

Power control policy is the energy allocation strategy of energy harvesting system, which determines the energy consumption Y_t at current time slot based on the available knowledge of energy level in the battery B_t and energy arrival E_t. B_t contains past and current information of energy arrivals, we can have a simple expression for HSU scheme like

$$B_t = \min\left(B_{t-1} - Y_{t+1} + E_t,\, m\right). \tag{1.0.1}$$

There are two classes of power control policy, one is offline policy and the other one is online policy. In offline policy, the transmitter is assumed to know the information of all future energy arrivals, in other words, the time slots and amounts of future energy arrivals can be predicted. The energy used from buffer depends on all harvested energy and current energy stored in the battery as

$$Y_t = f_t(B_t, E_{t+1}^{\infty}). \tag{1.0.2}$$

In the offline case, finding the optimal power control policy can be looked like a convex optimization problem whose objective functions and constraint functions can include the information of known future energy arrivals. However, in online policy, future energy arrivals are random and uncertain, so, the transmitter only has causal knowledge of energy arrivals, which can be described by stochastic processes including time-unrelated model as statistical distribution and time-related model as Markov Chain. Here is the formulation of energy consumption in the online case at time slot t

$$Y_t = f_t\left(B_t\right). \tag{1.0.3}$$

Due to the uncertainty of future energy arrivals, neither consuming energy too fast nor too slow is wise. If energy consumption is fast, it is possible that an outage occurs. On the other side, saving too much energy for the future is not always a good choice, an overflow should be considered. Designing a power control policy for the online case can be looked like a Markov Decision Process. Besides, the goal of power control varies, maximizing long-term averaged throughput, minimizing the probability of outage and other objectives will lead to totally different power control policies [2, 5–8, 12–16, 22, 28, 29, 32, 34]. Several examples are summarized as follows.

In [16], An offline energy scheduling is proposed by solving a convex optimization problem that focuses on the maximum throughput associated with a deadline. [34] finds two offline power allocation strategies that can minimize completion time for two cases, where no packet arrival occurs during transmission and packet arrivals exist after the starting of transmission. In [29], authors discuss the offline policies can achieve maximum throughput and minimum mean delay for a point-to-point system equipped with an infinite battery. With a linear reward function, the policy which spends the smaller one of two energy amounts, all available energy in the buffer and required power of data transfer, at every time slot is considered throughput and mean delay optimal. A numerical online optimal policy solved by dynamic programming method is shown in [5], its target is to reach the maximum throughput over a finite time horizon for a limited battery. [5] also uses a numerical method to obtain the offline optimal policy that can maximize throughput for both limited and unlimited buffer over a finite horizon. In [22], authors represent the Fixed Fraction Policy using a fixed proportional p to the energy level in battery. Besides, the throughput optimal policy under the Bernoulli energy harvesting process is obtained by a precise mathematical expression.

In addition, there are some other problems studied in the energy harvesting field, including energy harvesting models, imperfect battery [3, 26, 33] and so on. However, Among existing researches, energy scheduling is a significant key task and throughput is an important evaluation criteria [1, 4, 11, 17, 25, 31]. Therefore, having an understanding and view of the energy harvesting system's long-term average throughput is beneficial for designing power control policies in future work.

In this thesis, we will discuss the characteristics of the long-term average throughput of the Best Effort Policy.

Chapter 2

Problem Statement

2.1 Problem Statement

For power control problem, at time slot t, energy arrival is denoted by E_t, energy stored in battery is B_t and energy consumed for data transmission is Y_t. The battery capacity is described as m. A simple sketch of the system model is given by Figure 2.1.

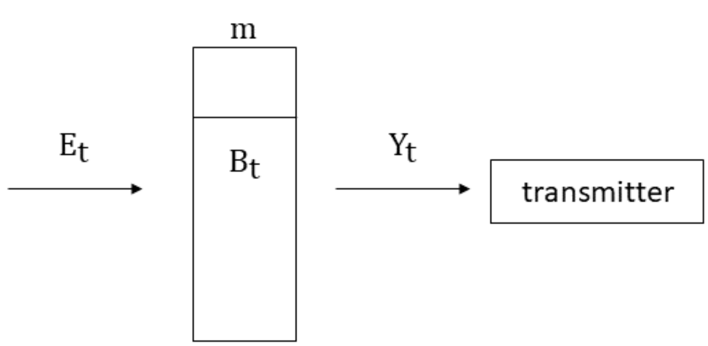


Figure 2.1: Energy Harvesting Communication System Model

Here we consider E_t as $i.i.d$ and system as harvest-store-use (HSU) architecture, which means that at each time slot, we always have $B_t \leq m$ and $Y_t \leq B_t$. The energy harvesting system is capacity-achieving for Additive White Gaussian Noise (AWGN) channel at all signal to noise ratios, in other words, the reward function is Gaussian Channel Capacity formula. We consider the Best Effort Policy, which is an online policy consuming all available energy if B_t is less than the mean of energy arrival μ, otherwise, consuming μ at each time slot. In fact, it can be looked like a combination of the Greedy Policy and the Constant Policy. The Greedy Policy is an online policy that uses up all energy in the battery every time, and it performs well when battery capacity is small. While the Constant Policy keeps a constant consumption as long as the power storage is enough, which is optimal when battery capacity approaches infinity. The Best Effort Policy can combine these two policies to make up their shortfall in different ranges of battery capacity. For transmitter with finite battery, the long-term average throughput is given as

$$\Theta = \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\sum_{t=1}^{n} \frac{1}{2}\log\left(1+\gamma Y_t\right)\right]. \tag{2.1.1}$$

For energy harvesting system equipped with infinite battery, if online energy allocation strategy for consumption is μ at each time slot, the long-term average throughput can achieve [23]

$$\Theta_\infty = \frac{1}{2}\log\left(1+\gamma\mu\right). \tag{2.1.2}$$

Here γ is signal to noise ratio and γ is set as 1 in following discussion.

In this thesis, we analyze the difference between throughput induced by an unlimited battery and a limited battery for the Best Effort Policy if the energy arrival process follows $i.i.d$ Bernoulli distribution. And then, we extend to other distributions by numerical simulation.

2.2 Thesis Structure

This thesis is organized as follows. In the first chapter, introduction and background information of energy harvesting communications are provided. Then, the second chapter includes a statement of the problem and a structure of this thesis. The main results and rough proofs are given in the third chapter. The final chapter is a conclusion. Appendix A and B give detailed proofs of our main results.

Chapter 3

Main Results and Proofs

3.1 Throughput of Bernoulli Energy Arrival Process

In this section, we use Markov Chain to show that if E_t follows $i.i.d$ Bernoulli distribution, when battery capacity approaches infinity, the long-term average throughput of the Best Effort Policy will go to (2.1.2). We provide the limit of $\Theta_\infty - \Theta$ when the mean of Bernoulli energy arrival is one and the upper bound of the order of $\Theta_\infty - \Theta$ in the large m limit when mean of Bernoulli energy arrival is an arbitrary integer.

3.1.1 The Mean of Bernoulli Energy Arrival Process is 1

Firstly, we will prove that under the condition of $\mu = 1$, when m goes to infinity, the probability of no energy in the battery, written as $P(0)$, has

$$\lim_{m \to +\infty} \frac{P(0)}{\frac{1}{m}} = \frac{k-1}{2},$$

which indicates that

$$\lim_{m \to +\infty} \frac{\Theta_\infty - \Theta}{\frac{1}{m}} = \frac{k-1}{4} \log 2. \tag{3.1.1}$$

3.1.1.1 Theorem 1

If energy arrival is assumed as Bernoulli distribution, whose mathematical expression is

$$P(E_t = k) = \frac{1}{k} \qquad P(E_t = 0) = 1 - \frac{1}{k}, \tag{3.1.2}$$

where k is integer and $k \geq 2$.

We can have

$$\lim_{m \to +\infty} \frac{P(0)}{\frac{1}{m}} = \frac{k-1}{2},$$

where $P(0)$ is the probability that energy level in the battery is 0.

3.1.1.2 Proof

Figure 3.1 shows the Markov Decision Process of energy in the battery B_t under Bernoulli energy arrival as (3.1.2). The Best Effort Policy is considered as if $B \geq \mu$ at the current time slot, μ amount of energy can be used; otherwise, all energy in the battery will be spent. In this condition, μ equals to 1, therefore, the policy can be simply looked as energy consumption Y_t is 1 at each time as long as $B_t \neq 0$.

According to the similar process under buffer capacity m, several equations about probabilities of energy level in the battery, from $P(0)$ to $P(m)$, can be obtained, and these equations are related to parameter k. Here $P(i)$ represents the probability of energy amount in the battery is i. After simple calculation, the following equations

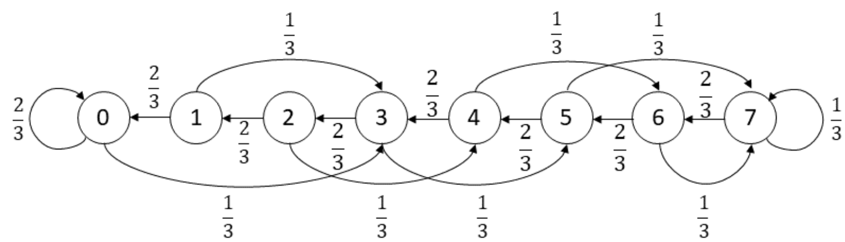


Figure 3.1: Markov Chain of Energy Level in Battery ($k = 3$, $m = 7$)

are gotten:

$$P(k+1) = \frac{1}{k}\big(P(2) + \ldots + P(k+1)\big) \tag{3.1.3}$$

$$P(k+2) = \frac{1}{k}\big(P(3) + \ldots + P(k+2)\big) \tag{3.1.4}$$

$$\cdots$$

$$P(m) = \frac{1}{k}\big(P(m-k+1) + \ldots + P(m)\big). \tag{3.1.5}$$

By equations from (3.1.3) to (3.1.5), $P(2)$, $\ldots$, $P(m)$ can be looked as a linear recurrence and solved by Characteristic Roots Method. $P(2)$, $\ldots$, $P(m)$ can be written as

$$P(n+1) = C_1 + C_2 {x_2}^n + \ldots + C_{k-1} {x_{k-1}}^n,$$

where $C_1 = \frac{2}{k-1}P\left(0\right)$ and $C_2,\ \ldots, C_{k-1} = \frac{1}{k-1}P\left(0\right)$. Here x are roots of

$$\left(k-1\right)x^{k-1} - x^{k-2} - x^{k-3} - \ldots - x - 1 = 0.$$

Through discussion, the absolute values or modulus of x_2, ..., x_{k-1} must be strictly smaller than 1. Since $\sum_{i=0}^{m} P(i) = 1$, we have

$$(1 - \frac{1}{k-1})P\left(0\right) + \sum_{q=1}^{m-1} \frac{2}{k-1}P\left(0\right) + C_2{x_2}^q + \ldots + C_{k-1}{x_{k-1}}^q = 1, \tag{3.1.6}$$

Due to $|x_2| < 1,\ \ldots, |x_{k-1}| < 1$, we can get a result that $P\left(0\right)$ converges to zero when m approaches infinity as

$$\lim_{m\to+\infty} \frac{P\left(0\right)}{\frac{1}{m}} = \frac{1}{\frac{2}{k-1}}.$$

The detailed proof is shown in Appendix A.

3.1.2 The Mean of Bernoulli Energy Arrival Process is j

Secondly, if μ is an arbitrary positive integer, the probability that $B_t = 0$ is denoted by $P\left(0\right)$. For the Best Effort Policy, it can be proved that there is a positive and finite number β such that $\Theta_\infty - \Theta \leq \frac{1}{\beta m}$ for m is sufficiently high by following Theorem 2. Here β is unrelated to battery capacity m.

3.1.2.1 Theorem 2

If energy arrival is assumed as Bernoulli distribution, whose mathematical expression is

$$P\left(E_t = k\right) = \frac{j}{k} \qquad P\left(E_t = 0\right) = 1 - \frac{j}{k}, \tag{3.1.7}$$

where k, j are integer and $k > j > 0$.

There is a positive and finite number α such that $P(0) \leq \frac{1}{\alpha m}$ for m is sufficiently high. Here α is unrelated to battery capacity m.

3.1.2.2 Proof

Battery level state B_t studied by Markov Chain under (3.1.7) is shown in Figure 3.2. Also, the Best Effort Policy is studied in this case.

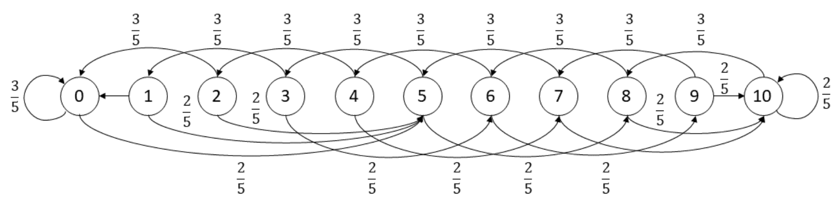


Figure 3.2: Markov Chain of Energy Level in Battery (k = 5, j=2, m = 10)

Similarly, we have some equations of probabilities of battery energy level, $P(i)$, until m as

$$\left(1-\frac{j}{k}\right)P(k+j)=\frac{j}{k}\big(P(j+1)+\ldots+P(k)\big) - \left(1-\frac{j}{k}\right)\big(P(k+1)+\ldots+P(k+j-1)\big) \tag{3.1.8}$$

$$\left(1-\frac{j}{k}\right) P\left(k+j+1\right) = \frac{j}{k}\left(P\left(j+2\right)+\ldots+P\left(k+1\right)\right) \\ -\left(1-\frac{j}{k}\right)\left(P\left(k+2\right)+\ldots+P\left(k+j\right)\right) \qquad (3.1.9)$$

$$\ldots$$

$$\left(1-\frac{j}{k}\right) P\left(m\right) = \frac{j}{k}\left(P\left(m-k+1\right)+\ldots+P\left(m-j\right)\right) \\ -\left(1-\frac{j}{k}\right)\left(P\left(m-j+1\right)+\ldots+P\left(m-1\right)\right). \qquad (3.1.10)$$

According to the above equations, through the Characteristic Roots Method, we can also find the general term formula from $P\left(1+j\right)$ to $P\left(m\right)$ as

$$P\left(n+j\right) = C_1 + C_2{x_2}^n + \ldots + C_{k-1}{x_{k-1}}^n$$

or

$$P\left(n+j\right) = C_1 + C_2{x_2}^n + \ldots + C_l{x_l}^n + \left(C_{l+1} + \ldots + n^{u_1-1}C_{l+u_l}\right){x_{l+1}}^n \\ + \ldots + \left(C_{k-u_w} + \ldots + n^{u_w-1}C_{k-1}\right){x_{l+w}}^n$$

which correspond to the conditions of only $k-1$ number of single roots exist or w number of repeated roots exist for (3.1.11) respectively. Here C_i is independent of m and all x are roots of

$$\left(1-\frac{j}{k}\right)\left(x^{k-1}+\cdots+x^{k-j}\right) - \frac{j}{k}\left(1+x+\cdots+x^{k-j-1}\right) = 0. \qquad (3.1.11)$$

Also, because of $\sum_{i=0}^{m} P\left(i\right) = 1$, by calculation, when m approaches infinity, we can obtain that

$$\frac{k}{k-j}P(0)+C_1(m-j)+H\big(a\cos(m\theta)-b\sin(m\theta)\big)\,t^m P(0)=1$$

or

$$\frac{k}{k-j}P(0)+C_1(m-j)+H\big(a\cos(m\theta)-b\sin(m\theta)\big)\,m^c t^m P(0)=1,$$

where $C_1 \geq \frac{2}{k-j}P(0)$ and H, t, c are real finite numbers. By discussing all possible values of c and t, for large m, we can get the result that

$$0 \leq P(0) \leq \frac{1}{\alpha m},$$

where α is a positive and finite number that is independent of m.

The detailed proof is shown in Appendix B.

3.2 Extension

In this section, we do some numerical simulations of other distributions for the Best Effort Policy.

Figure 3.3 to Figure 3.6 show the simulation results for several different kinds of energy arrival process with different means and variances. Here X_t follows *i.i.d.* In all cases, it is clear that with the increase of battery size m, the throughput of a system equipped with a finite-sized battery converges to optimal throughput induced by an infinite-sized battery. The maximum long-term averaged throughput of the unlimited buffer system is given by (2.1.2), which is independent of battery size.

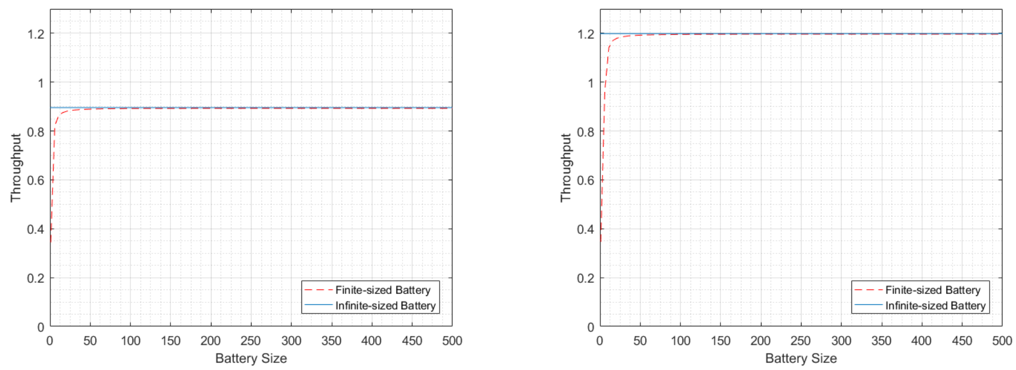
We also consider the influence of energy arrival distribution moments on long-term average throughput curve.

We can find that throughput with limited battery is related to mean and variance. In Figure 3.7, we keep variance is 5. From this figure, there is a positive relationship between μ and the point m that the convergence rate starts decreasing. Then, from Figure 3.8, there is a negative relationship between variance and convergence rate for normal distribution. When $\sigma^2 = 20$, the point that Θ_∞ approaches Θ is larger than the one when $\sigma^2 = 5$. In addition, the throughput curve is smoother if the variance is large, which indicates the rate of convergence is lower.

The skewness and kurtosis of energy arrival distribution affect throughput. However, their effect is relatively small. In Figure 3.3 and Figure 3.9, we assume that mean and variance are same but the skewness and kurosis are different, which shows a similar result in the figures, which shows that these two moments of energy arrival distribution have small influence on performance.

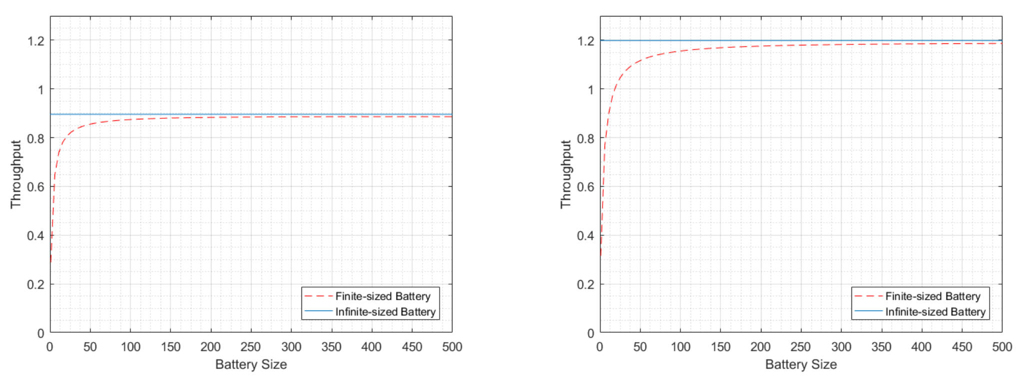
Generally, the shape of p.d.f will affect the curve of throughput, which is related to the four moments in our above discussion. From Figure 3.3a to Figure 3.6a, we can simply say that if p.d.f is monotonically decreasing, the convergence rate of throughput will become highest. While if p.d.f increases firstly and then decreases, the convergence rate will be lowest. For Uniform distribution whose p.d.f is flat, the convergence rate will between those two types. Similar result is shown in Figure 3.3b to Figure 3.6b.

In fact, these results are natural because if the probability of small energy arrival is high, it will be more possible that the energy stored battery is below mean and all energy is used, which indicates the throughput will be lower under a fixed battery capacity.



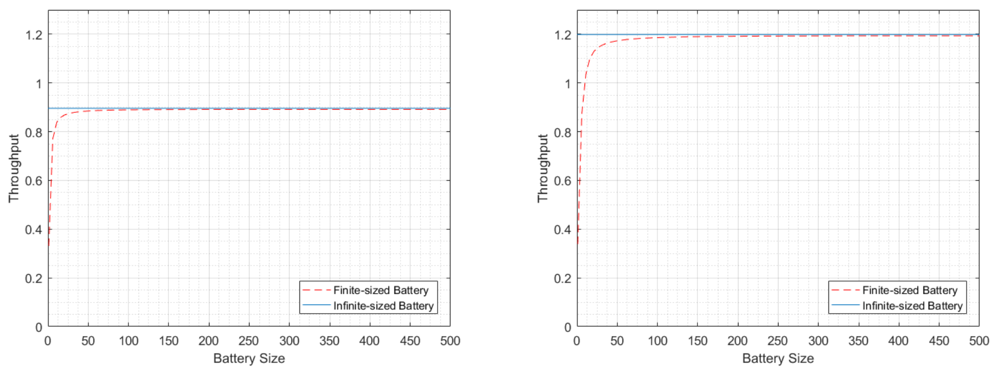
(a) $\mu = 5$ (b) $\mu = 10$

Figure 3.3: Poisson Energy Arrival Process



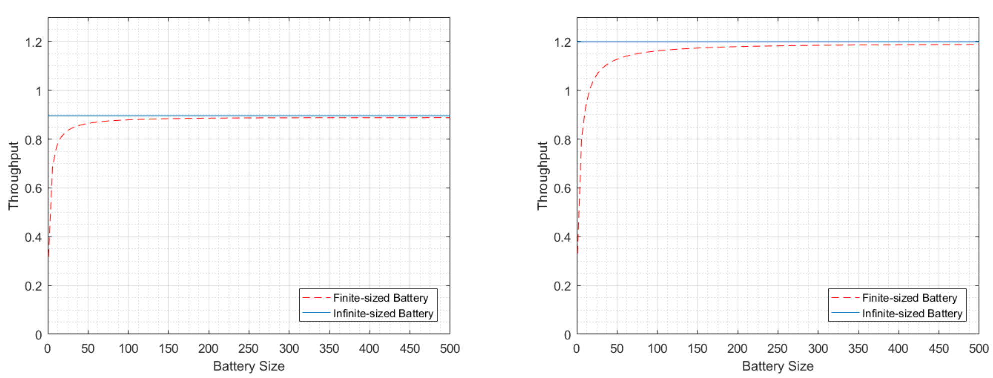
(a) $\mu = 5$ (b) $\mu = 10$

Figure 3.4: Geometric Energy Arrival Process



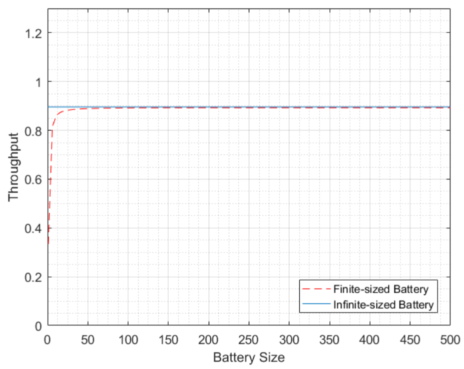
(a) $\mu = 5$ (b) $\mu = 10$

Figure 3.5: Uniform Energy Arrival Process

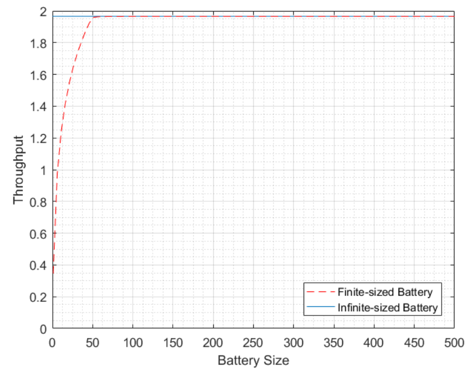


(a) $\mu = 5$ (b) $\mu = 10$

Figure 3.6: Exponential Energy Arrival Process

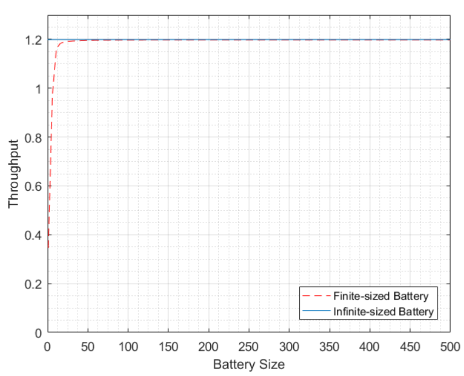


(a) $\mu = 5$

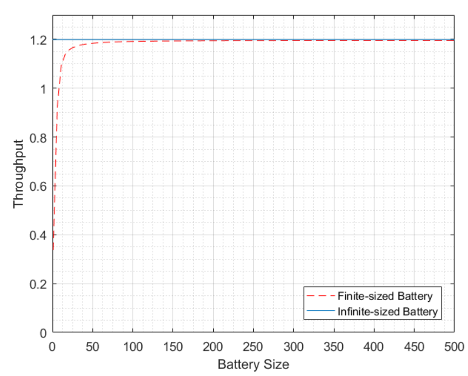


(b) $\mu = 50$

Figure 3.7: Normal Energy Arrival Process with different Means

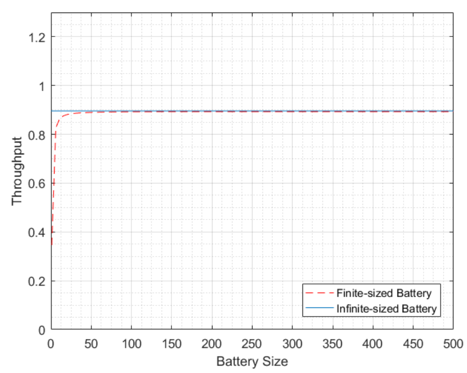


(a) $\sigma^2 = 5$

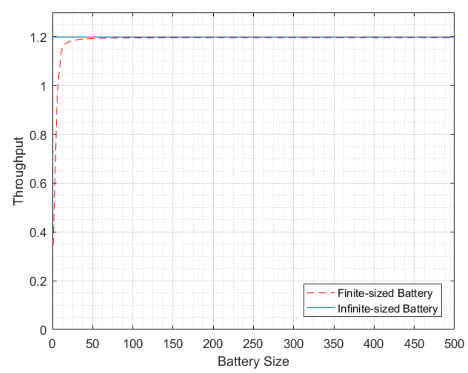


(b) $\sigma^2 = 20$

Figure 3.8: Normal Energy Arrival Process with Different Variances



(a) $\mu = 5$



(b) $\mu = 10$

Figure 3.9: Gamma Energy Arrival Process

Chapter 4

Conclusion

In this thesis, we study the performance of the Best Effort Policy in energy harvesting communications. For the Bernoulli energy arrival process, the difference between throughput values induced by an unlimited battery and large limited battery is calculated. The limit and upper bound of this difference corresponding to two Bernoulli cases are given. Under Bernoulli energy harvesting condition with mean equals to one, we prove that $\lim_{m\to+\infty} \frac{\Theta_\infty - \Theta}{\frac{1}{m}} = \frac{1}{\frac{4}{k-1}} \log 2$, while under Bernoulli energy harvesting condition with mean is an arbitrary positive integer, we can obtain that $\Theta_\infty - \Theta$ is no greater than $\frac{1}{\alpha m}$ when m is sufficiently high. Here α is a positive and finite number that is independent of battery capacity m. Besides, we also provide some numerical throughput results of other energy arrival distributions and discuss their characterizations.

Appendix A

Proof of Theorem 1

According to Markov Chain of B_t, the following equations can be obtained.

$$\frac{1}{k}P\left(0\right)=\left(1-\frac{1}{k}\right)P\left(1\right) \tag{A.0.1}$$

$$P\left(1\right)=\left(1-\frac{1}{k}\right)P\left(2\right) \tag{A.0.2}$$

$$\cdots$$

$$P\left(k-1\right)=\left(1-\frac{1}{k}\right)P\left(k\right) \tag{A.0.3}$$

$$P\left(k\right)=\frac{1}{k}P\left(0\right)+\frac{1}{k}P\left(1\right)+\left(1-\frac{1}{k}\right)P\left(k+1\right) \tag{A.0.4}$$

$$P\left(k+1\right)=\frac{1}{k}P\left(2\right)+\left(1-\frac{1}{k}\right)P\left(k+2\right) \tag{A.0.5}$$

$$\ldots$$

$$P\left(m-1\right)=\frac{1}{k}P\left(m-k\right)+\left(1-\frac{1}{k}\right)P\left(m\right) \tag{A.0.6}$$

$$P\left(m\right)=\frac{1}{k}\left(P\left(m-k+1\right)+\ldots+P\left(m\right)\right). \tag{A.0.7}$$

The equations (A.0.5), ..., (A.0.7) can be written as

$$P\left(k+1\right)=\frac{1}{k}\left(P\left(2\right)+\ldots+P\left(k+1\right)\right) \tag{A.0.8}$$

$$P\left(k+2\right)=\frac{1}{k}\left(P\left(3\right)+\ldots+P\left(k+2\right)\right) \tag{A.0.9}$$

$$\ldots$$

$$P\left(m\right)=\frac{1}{k}\left(P\left(m-k+1\right)+\ldots+P\left(m\right)\right). \tag{A.0.10}$$

Then $P\left(2\right)$, ..., $P\left(m\right)$ can be regarded as linear recurrence, and be expressed by Characteristic Roots Method as

$$P\left(j\right)=C_1{x_1}^j+C_2{x_2}^j+\ldots+C_{k-1}{x_{k-1}}^j$$

if the solutions $x_1,\ \ldots,\ x_{k-1}$ of equation

$$\left(k-1\right)x^{k-1}-x^{k-2}-x^{k-3}-\ldots-x-1=0 \tag{A.0.11}$$

are all single roots. For (A.0.11), there will be one solution as 1 and couples of solutions as conjugate complex pair, besides, in some conditions, other real solutions probably exist. Let the left side of (A.0.11) as $f(x)$, we have

$$f(x) = (x-1)\Big((k-1)\,x^{k-2} + (k-2)\,x^{k-3} + \ldots + 2x + 1\Big)$$

and

$$f'(x) = (k-1)^2\,x^{k-2} - (k-2)\,x^{k-3} - \ldots - 2x - 1.$$

If $f(x)$ and $f'(x)$ are relatively prime for $k \geq 2$, all solutions of (A.0.11) will not be repeated. Therefore, we need to check whether same roots of $f(x) = 0$ and $f'(x) = 0$ exist or not. Obviously, $x = 1$ must not satisfy $f'(x) = 0$ with $k \neq 1$ or 2. Now, the problem is converted to finding whether

$$(k-1)\,x^{k-2} + (k-2)\,x^{k-3} + \ldots + 2x + 1 = 0 \tag{A.0.12}$$

and

$$(k-1)^2\,x^{k-2} - (k-2)\,x^{k-3} - \ldots - 2x - 1 = 0 \tag{A.0.13}$$

have same solution.

Assuming a same root x_0 for both above equations, by (A.0.13), we can get

$$(k-1)^2\,{x_0}^{k-2} = (k-2){x_0}^{k-3} + \ldots + 2x_0 + 1,$$

then plug this equation into

$$(k-1)^2\,{x_0}^{k-2} - (k-2)\,{x_0}^{k-3} - \ldots - 2x - 0 - 1 = 0,$$

we can have

$$(k-1)^2 {x_0}^{k-2} + (k-1)\, {x_0}^{k-2} = 0.$$

Above equation is satisfied if and only if $k = 0$, $k = 1$ or $x_0 = 0$, which contradicts $k \geq 2$ and $f'(x_0) = 0$. So, we have $f(x)$ and $f'(x)$ are relatively prime, which indicates that $f(x)$ do not have repeated solutions.

Because (A.0.8) starts from $P(2)$, $P(2)$, $\ldots$, $P(m)$ can be represented as

$$P(n+1) = C_1 + C_2 {x_2}^n + \ldots + C_{k-1} {x_{k-1}}^n,$$

where $n = 1, \ \ldots, \ m-1$ and $x_2, \ldots, x_{k-1}$ are solutions except 1 for (A.0.11).

By following equations:

$$P(2) = \frac{1}{k-1}\frac{k}{k-1} P(0) = C_1 + C_2 x_2 + \ldots + C_{k-1} x_{k-1}$$

$$P(3) = \frac{1}{k-1}\left(\frac{k}{k-1}\right)^2 P(0) = C_1 + C_2 {x_2}^2 + \ldots + C_{k-1} {x_{k-1}}^2$$

$$\ldots$$

$$P(k) = \frac{1}{k-1}\left(\frac{k}{k-1}\right)^{k-1} P(0) = C_1 + C_2 {x_2}^{k-1} + \ldots + C_{k-1} {x_{k-1}}^{k-1},$$

we can get $C_1 = \frac{2}{k-1} P(0)$ and $C_2 = C_3 = \ldots = C_{k-1} = \frac{1}{k-1} P(0)$.

Then, we will discuss the absolute values of $x_2, \ldots, x_{k-1}$.

Firstly, real solutions are considered. For (A.0.11), obviously, any positive solutions except 1 can satisfy it, therefore, all values of $x_2, \ldots, x_{k-1}$ are negative. Besides, when $x_i \leq -1$ and $i = 2, \ \ldots, \ k-1$, if $k-2$ is odd, we will have $(k-1)\,x^{k-2} + (k-2)\,x^{k-3} < 0$ and by parity of reasoning, the left side of (A.0.11) must less than 0. On the other side, if $k-2$ is even, we will have $(k-1)\,x^{k-2} + (k-2)\,x^{k-3} > 0$ and by parity of reasoning, the left side of (A.0.11) must larger than 0. These indicates that the absolute values of $x_2, \ldots, x_{k-1}$ should be strictly less than 1.

Secondly, we consider solutions as the conjugate complex pair. Complex roots are assumed as $r(\cos\theta + i\sin\theta)$ and $\cos\theta \neq \pm 1$, $\sin\theta \neq 0$, which satisfy (A.0.12). Assuming

$$S = (k-1)\,x^{k-2} + (k-2)\,x^{k-3} + \ldots + 2x + 1,$$

we will have

$$\begin{aligned} S &= \frac{x^{k-2} + x^{k-3} + \ \ldots \ + x + 1 - (k-1)\,x^{k-1}}{1-x} \\ &= \frac{1 - x^{k-1}}{(1-x)^2} - \frac{(k-1)\,x^{k-1}}{1-x}. \end{aligned} \tag{A.0.14}$$

Let (A.0.14) equals 0, we can obtain

$$kx^{k-1} = (k-1)\,x^k + 1,$$

because $x = 0$ is not a solution of (A.0.11),

$$1 = \frac{k-1}{k}x + \frac{1}{k}x^{-(k-1)}$$

$$1 = \frac{k-1}{k}r(\cos\theta + i\sin\theta) + \frac{1}{k}r^{-(k-1)}\left[\cos\big(-(k-1)\big)\,\theta + i\sin\big(-(k-1)\big)\,\theta\right],$$

which indicates

$$1 = \frac{k-1}{k}r\cos\theta + \frac{1}{k}r^{-(k-1)}\cos\big(-(k-1)\big)\,\theta \tag{A.0.15}$$

and

$$0 = \frac{k-1}{k} r \sin\theta + \frac{1}{k} r^{-(k-1)} \sin\big(-(k-1)\big)\theta. \tag{A.0.16}$$

By (A.0.16),

$$(k-1) r^k = \frac{\sin(k-1)\theta}{\sin\theta}$$

is gotten, and due to

$$\left|\frac{\sin(k-1)\theta}{\sin\theta}\right| \le k-1,$$

we can have

$$r \le 1.$$

Besides, $r = 1$ satisfies (A.0.16) only when $\sin\theta = 0$ or $k = 2$. $\sin\theta = 0$ is in contradiction with our assumption and if $k = 2$, (A.0.11) will only have one solution as 1. Therefore, for all complex solutions, the modulus will strictly less than 1.

Overall, $|x_2| < 1, \ldots, |x_{k-1}| < 1$.

Because of $P(0) + P(1) + \ \ldots\ + P(m) = 1$, we have

$$\left(1 - \frac{1}{k-1}\right) P(0) + \sum_{q=1}^{m-1} \frac{2}{k-1} P(0) + C_2 {x_2}^q + \ldots + C_{k-1} {x_{k-1}}^q = 1. \tag{A.0.17}$$

For each couple of conjugate complex solutions, since their coefficients are same as $\frac{1}{k-1} P(0)$, we have

$$\frac{1}{k-1} P(0) \left(x^q + (x^*)^q\right) = \frac{2}{k-1} P(0) r^q \cos(q\theta)$$

with $r < 1$. Therefore, (A.0.17) can be rewritten as

$$\left(1 - \frac{1}{k-1}\right) P\left(0\right) + \frac{2}{k-1} P\left(0\right)\left(m-1\right) + \frac{2}{k-1} P\left(0\right) \sum_{q=1}^{m-1} \cos\left(q\theta_1\right) {r_1}^q + \ldots = 1$$

Obviously, we have

$$\lim_{m \to +\infty} \frac{P\left(0\right)}{\frac{1}{m}} = \frac{k-1}{2}.$$

Appendix B

Proof of Theorem 2

According to Markov Chain of B_t, the following equations can be obtained.

$$P(0) = \left(1 - \frac{j}{k}\right)\big(P(0) + P(1) + \cdots + P(j)\big) \tag{B.0.1}$$

$$P(1) = \left(1 - \frac{j}{k}\right) P(j+1) \tag{B.0.2}$$

$$\cdots$$

$$P(k-1) = \left(1 - \frac{j}{k}\right) P(j+k-1) \tag{B.0.3}$$

$$P(k) = \frac{j}{k}\big(P(0) + P(1) + \cdots + P(j)\big) + \left(1 - \frac{j}{k}\right) P(k+j) \tag{B.0.4}$$

$$P(k+1) = \frac{j}{k} P(2) + \left(1 - \frac{j}{k}\right) P(k+j+1) \tag{B.0.5}$$

$$\cdots$$

$$P\left(m-j\right)=\frac{j}{k}P\left(m-k\right)+\left(1-\frac{j}{k}\right)P\left(m\right) \tag{B.0.6}$$

$$P\left(m-j+1\right)=\frac{j}{k}\big(P\left(m-k+1\right)\big) \tag{B.0.7}$$

$$\cdots$$

$$P\left(m-1\right)=\frac{j}{k}P\left(m-k+j-1\right) \tag{B.0.8}$$

$$P\left(m\right)=\frac{j}{k}\big(P\left(m-k+j\right)+\cdots+P\left(m\right)\big) \tag{B.0.9}$$

(B.0.5), ..., (B.0.9) can be rewritten as

$$\begin{aligned}\left(1-\frac{j}{k}\right)P\left(k+j\right)=&\ \frac{j}{k}\big(P\left(j+1\right)+\cdots+P\left(k\right)\big)\\ &-\left(1-\frac{j}{k}\right)\big(P\left(k+1\right)+\cdots+P\left(k+j-1\right)\big)\end{aligned} \tag{B.0.10}$$

$$\begin{aligned}\left(1-\frac{j}{k}\right)P\left(k+j+1\right)=&\ \frac{j}{k}\big(P\left(j+2\right)+\cdots+P\left(k+1\right)\big)\\ &-\left(1-\frac{j}{k}\right)\big(P\left(k+2\right)+\cdots+P\left(k+j\right)\big)\end{aligned} \tag{B.0.11}$$

$$\cdots$$

$$\begin{aligned}\left(1-\frac{j}{k}\right)P\left(m\right)=&\frac{j}{k}\left(P\left(m-k+1\right)+\cdots+P\left(m-j\right)\right)\\&-\left(1-\frac{j}{k}\right)\left(P\left(m-j+1\right)+\cdots+P(m-1)\right).\end{aligned}\tag{B.0.12}$$

Similarly, $P(k+j)$, ..., $P(m)$ ca be solved as linear recurrence. According to the characteristic equation:

$$\left(1-\frac{j}{k}\right)\left(x^{k-1}+\cdots+x^{k-j}\right)-\frac{j}{k}\left(1+x+\cdots+x^{k-j-1}\right)=0,\tag{B.0.13}$$

we can find that $x=1$ satisfies (B.0.13) as a single root and all complex solutions should appear as conjugate couples. Two conditions are discussed: characteristic equation only has single roots and has repeated roots.

Firstly, we consider the case that only single roots exist. Assuming x_1 as 1, we have

$$P\left(n+j\right)=C_1+C_2{x_2}^n+\cdots+C_{k-1}{x_{k-1}}^n,\tag{B.0.14}$$

where C_1, ..., C_{k-1} are decided by following $k-1$ equations

$$C_1+C_2x_2+\cdots+C_{k-1}x_{k-1}=\frac{k}{k-j}P\left(1\right)$$

$$\cdots$$

$$C_1+C_2{x_2}^j+\cdots+C_{k-1}{x_{k-1}}^j=\frac{k}{k-j}P\left(j\right)$$

$$C_1 + C_2 {x_2}^{j+1} + \cdots + C_{k-1} {x_{k-1}}^{j+1} = \left(\frac{k}{k-j}\right)^2 P\left(1\right)$$

$$\cdots$$

$$C_1 + C_2 {x_2}^{j+1} + \cdots + C_{k-1} {x_{k-1}}^{j+1} = \left(\frac{k}{k-j}\right)^2 P\left(j\right),$$

$$\cdots$$

where $P\left(j\right) = \frac{j}{k-j} P\left(0\right) - P\left(1\right) - \ldots - P\left(j-1\right)$. It is clear that $C_1, \ldots, C_{k-1}$ are functions of k, j, $P\left(0\right), \ldots, P\left(j-1\right)$ as

$$C_i = A_{i,0} P\left(0\right) + A_{i,1} P\left(1\right) + \cdots + A_{i,j-1} P\left(j-1\right). \tag{B.0.15}$$

By calculation, we have

$$C_1 = \frac{2}{k-j} P\left(0\right) + \frac{j-1}{j} \frac{2}{k-j} P\left(1\right) + \cdots + \frac{1}{j} \frac{2}{k-j} P\left(j-1\right). \tag{B.0.16}$$

According to (B.0.7), …, (B.0.8), we can obtain

$$C_1 + C_2 {x_2}^{m-2j+1} + \cdots + C_{k-1} {x_{k-1}}^{m-2j+1} = \frac{j}{k}\left(C_1 + C_2 {x_2}^{m-k-j+1} + \cdots + C_{k-1} {x_{k-1}}^{m-k-j+1}\right)$$

$$\cdots$$

$$C_1 + C_2 {x_2}^{m-j-1} + \cdots + C_{k-1} {x_{k-1}}^{m-j-1} = \frac{j}{k}\left(C_1 + C_2 {x_2}^{m-k-1} + \cdots + C_{k-1} {x_{k-1}}^{m-k-1}\right),$$

which are still satisfied when m approaches infinity. Plugging (B.0.15) into above equations and by Cramer's Rule, $P(1)$, ..., $P(j-1)$ is related to $P(0)$ as

$$P(n) = \frac{\sum^{z} N^{n}{}_{z} {x_2}^{mh^{n}{}_{z,2}} \cdots {x_{k-1}}^{mh^{n}{}_{z,k-1}}}{\sum^{z} N_z {x_2}^{mh_{z,2}} \cdots {x_{k-1}}^{mh_{z,k-1}}} P(0),$$

where $n = 1,\ 2,\ \ldots,\ j-1$ and . Then, by (A.0.16), coefficients can be represented as

$$C_i = \left(\frac{b_i}{T}\right) P(0), \tag{B.0.17}$$

where

$$b_i = \sum^{z} R_{i,z} {x_2}^{mh_{i,z,2}} \cdots {x_{k-1}}^{mh_{i,z,k-1}}$$

and

$$T = \sum^{z} N_z {x_2}^{mh_{z,2}} \cdots {x_{k-1}}^{mh_{z,k-1}}.$$

Here, $h^{r}{}_{z,2},\ \ldots,\ h^{r}{}_{z,k-1},\ h_{z,2},\ \ldots,\ h_{z,k-1} \geq 0$ and $h^{r}{}_{z,2} + \ldots + h^{r}{}_{z,k-1} + h_{z,2} + \ldots + h_{z,k-1} \leq j-1$, besides, $R^{r}{}_{z}$ and N_z are probably equal to 0. From (B.0.16), it is clear that

$$C_1 \geq \frac{2}{k-j} P(0) \tag{B.0.18}$$

Since the coefficients of conjugate roots are conjugate couples, for complex conjugate solutions, we have

$$Cx^n + C^*(x^*)^n = r^n \left(2a\cos(n\theta) - 2b\sin(n\theta)\right), \tag{B.0.19}$$

where r is modulus of x and $C = a + b\,i$. According to (B.0.1), we can get

$$P\left(j\right) = \frac{j}{k-j} P\left(0\right) - P\left(1\right) - \cdots - P(j-1),$$

which indicates that

$$P\left(0\right) + P\left(1\right) + \cdots + P\left(j\right) = \frac{k}{k-j} P\left(0\right).$$

Due to $P\left(0\right) + P\left(1\right) + \ \ldots\ + P\left(m\right) = 1$ and (B.0.17), following equations are gotten:

$$1 = \left(\frac{k}{k-j}\right) P\left(0\right) + \sum_{q=1}^{m-j} C_1 + C_2 {x_2}^q + \cdots + C_{k-1} {x_{k-1}}^q$$

$$1 = \left(\frac{k}{k-j} + \frac{b_1}{T}\left(m-j\right) + \frac{b_2}{T}\frac{x_2}{1-x_2} - \frac{x_2^{i-j}}{1-x_2}\frac{b_2 x_2^m}{T} + \cdots \right. \\ \left. + \frac{b_{k-1}}{T}\frac{x_{k-1}}{1-x_{k-1}} - \frac{x_{k-1}^{i-j}}{1-x_{k-1}}\frac{b_{k-1} x_{k-1}^m}{T}\right) P\left(0\right), \quad \text{(B.0.20)}$$

where $b_2,\ \ldots,\ b_{k-1}$ and T are exponential terms with m power as above discussion. When m goes to infinity, it is easy to obtain that

$$\lim_{m \to +\infty} \frac{b_2}{T}\frac{x_2}{1-x_2} - \frac{x_2^{i-j}}{1-x_2}\frac{b_2 x_2^m}{T} + \cdots + \frac{b_{k-1}}{T}\frac{x_{k-1}}{1-x_{k-1}} - \frac{x_{k-1}^{i-j}}{1-x_{k-1}}\frac{b_{k-1} x_{k-1}^m}{T} = \\ H\left(a\cos\left(m\theta\right) - b\sin\left(m\theta\right)\right) t^m \quad \text{(B.0.21)}$$

where H, t are finite real number and $\cos\left(m\theta\right)$, $\sin\left(m\theta\right) \in [-1, 1]$. (B.0.20) will be converted to

$$\left(\frac{k}{k-j}+\frac{b_1}{T}\left(m-j\right)+H\left(a\cos\left(m\theta\right)-b\sin\left(m\theta\right)\right)t^m\right)P\left(0\right)=1. \tag{B.0.22}$$

By (B.0.16) and (B.0.1), we have

$$\frac{2}{k-j}\le\frac{b_1}{T}\le\frac{2\left(k-1\right)}{\left(k-j\right)^2} \tag{B.0.23}$$

If $0<|t|<1$ or $H\left(a\cos\left(m\theta\right)-b\sin\left(m\theta\right)\right)=0$, obviously,

$$\lim_{m\to+\infty}H\left(a\cos\left(m\theta\right)-b\sin\left(m\theta\right)\right)t^m=0,$$

which indicates when m approaches infinity, we have

$$P\left(0\right)\le\frac{1}{\frac{k}{k-j}+\frac{2}{k-j}\left(m-j\right)+H\left(a\cos\left(m\theta\right)-b\sin\left(m\theta\right)\right)t^m}$$

$$P\left(0\right)\le\frac{1}{\frac{2}{k-j}m}.$$

If $|t|=1$, (B.0.22) will lead to

$$\lim_{x\to+\infty}P\left(0\right)=\lim_{m\to+\infty}\frac{1}{\frac{k}{k-j}+\frac{b_1}{T}\left(m-j\right)+H\left(a\cos\left(m\theta\right)-b\sin\left(m\theta\right)\right)}.$$

also, we have

$$P\left(0\right)\le\frac{1}{\frac{2}{k-j}m}$$

for m is sufficiently large.

If $|t| > 1$, according to (B.0.23), $\frac{b_1}{T}m$ should be linearly order of m, obviously, we have

$$\lim_{m \to +\infty} \left(H \left(a\cos(m\theta) - b\sin(m\theta) \right) t^m \right) P(0) = 1,$$

which tells us that

$$\lim_{m \to +\infty} \left(H \left(a\cos(m\theta) - b\sin(m\theta) \right) t^m \right) \geq 0.$$

Then

$$P(0) = \lim_{m \to +\infty} \frac{1}{\frac{k}{k-j} + \frac{b_1}{T}(m-j) + H\left(a\cos(m\theta) - b\sin(m\theta)\right)}$$

and when m is sufficiently large,

$$P(0) \leq \frac{1}{\frac{2}{k-j}m}.$$

In a word, for single root condition, we have

$$P(0) \leq \frac{1}{\alpha m}$$

in the large m limit, here α is a positive and finite number that is unrelated to m.

Next, we consider the condition that repeated roots exist in (B.0.13). We assume there are w repeated roots and their multiplicity is u_i, then $P(j), \ldots, P(m)$ are

$$\begin{aligned} P(n+j) = {} & C_1 + C_2{x_2}^n + \cdots + C_l{x_l}^n + \left(C_{l+1} + \ldots + n^{u_1-1}C_{l+u_l}\right){x_{l+1}}^n \\ & + \cdots + \left(C_{k-u_w} + \ldots + n^{u_w-1}C_{k-1}\right){x_{l+w}}^n \end{aligned} \quad \text{(B.0.24)}$$

where $x_2, \ldots, x_{l+w}$ are solutions of (B.0.13) and $x_{l+1}, \ldots, x_{l+w}$ are repeated solutions.

$C_1, \ldots, C_{k-1}$ are decided by following $k-1$ equations:

$$P\left(1+j\right) = \left(\frac{k}{k-j}\right) P\left(1\right)$$

$$\cdots$$

$$P\left(2j\right) = \left(\frac{k}{k-j}\right) P\left(j\right) = \frac{k}{k-j}\left(\frac{j}{k-j} P\left(0\right) - P\left(1\right) - \cdots - P\left(j-1\right)\right)$$

$$P\left(2j+1\right) = \left(\frac{k}{k-j}\right)^2 P\left(1\right)$$

$$\cdots$$

$$P\left(3j\right) = \left(\frac{k}{k-j}\right) P\left(2j\right) = \left(\frac{k}{k-j}\right)^2 \left(\frac{j}{k-j} P\left(0\right) - P\left(1\right) - \cdots - P\left(j-1\right)\right)$$

$$\cdots$$

Similarly, we can obtain that $C_i = A_{i,1} P\left(0\right) + \cdots + A_{i,j-1} P(j-1)$ and it is a function independent of m. By (B.0.7) to (B.0.8), we can represent C_i as

$$C_i = \left(\frac{b_i}{T}\right) P\left(0\right), \tag{B.0.25}$$

whose

$$b_i = \sum^{z} Q_{i,z} \left(N_{i,z} + M_{i,z} m + \cdots\right)^{h_{i,z,l}} \cdots \left(U_{i,z} + L_{i,z} m + \cdots\right)^{h_{i,z,l+w}}$$
$$x_2{}^{m h_{i,z,2}} \cdots x_{l+w}{}^{m h_{i,z,l+w}}$$

and

$$T=\sum^{z} Q_z(N_z+M_zm+\cdots)^{h_{z,l}}\cdots(U_z+L_zm+\cdots)^{h_{z,l+w}}{x_2}^{mh_{z,2}}\cdots{x_{l+w}}^{mh_{z,l+w}}.$$

It is clear that b_i and T are exponential terms or power functions about m. Here $h_{i,z,2},\ \ldots,\ h_{i,z,l+w},\ h_{z,2},\ \ldots,\ h_{z,l+w}\geq 0$, $h_{i,z,2}+\ldots+h_{i,z,l+w}\leq j-1$ and $h_{z,2}+\ldots+h_{z,l+w}\leq j-1$. Besides, all coefficients are possible to equal to 0. By calculation, we also have

$$\frac{2}{k-j}\leq\frac{b_1}{T}\leq\frac{2\,(k-1)}{(k-j)^2}. \tag{B.0.26}$$

And for x_i as complex number, we still have (B.0.19). Therefore, according to $\sum_{i=0}^{m}P\,(i)=1$, we have

$$\begin{aligned}1=&\left(\frac{k}{k-j}\right)P\,(0)+\sum_{q=1}^{m-j}C_1+C_2{x_2}^q+\cdots+C_l{x_l}^q+\left(C_{l+1}+\cdots+q^{u_1-1}C_{l+u_1}\right)(x_{l+1})^q\\&+\cdots+\left(C_{k-u_w}+\cdots+q^{u_w-1}C_{k-1}\right)(x_{l+w})^q\,,\end{aligned}$$

which indicates

$$\begin{aligned}1=&\left(\frac{k}{k-j}+\frac{b_1}{T}\,(m-j)+\frac{b_2x_2\left(1-{x_2}^{m-j}\right)}{T\,(1-x_2)}+\cdots+\frac{b_{k-u_{l+w}}x_{l+w}\left(1-{x_{l+w}}^{m-j}\right)}{T\,(1-x_{l+w})}\right.\\&+\sum_{q=1}^{m-j}\left(q\frac{b_{l+2}}{T}+\cdots+q^{u_1-1}\frac{b_{l+u_1}}{T}\right)(x_{l+1})^q+\cdots\\&\left.+\left(q\frac{b_{k-u_w}}{T}+\cdots+q^{u_w-1}\frac{b_k-1}{T}\right)(x_{l+w})^q\right)P\,(0)\end{aligned} \tag{B.0.27}$$

In fact, when m approaches infinity, there is

$$\begin{aligned}
&\lim_{x\to+\infty} \frac{b_2 x_2 \left(1 - {x_2}^{m-j}\right)}{T\left(1-x_2\right)} + \cdots + \frac{b_{k-u_{l+w}} x_{l+w} \left(1 - {x_{l+w}}^{m-j}\right)}{T\left(1 - x_{l+w}\right)} \\
&\qquad + \sum_{q=1}^{m-j} \left(q\frac{b_{l+2}}{T} + \cdots + q^{u_1 - 1} \frac{b_{l+u_1}}{T} \right) \left(x_{l+1}\right)^q + \cdots \\
&\qquad + \left(q\frac{b_{k-u_w}}{T} + \cdots + q^{u_w - 1} \frac{b_k - 1}{T} \right) \left(x_{l+w}\right)^q \\
&= H\left(a\cos\left(m\theta\right) - b\sin\left(m\theta\right)\right) m^c t^m, \qquad \text{(B.0.28)}
\end{aligned}$$

where H is finite real number and $\cos\left(m\theta\right)$, $\sin\left(m\theta\right) \in [-1, 1]$. Now, (B.0.27) is converted to

$$\left(\frac{k}{k-j} + \frac{b_1}{T}\left(m-j\right) + H\left(a\cos\left(m\theta\right) - b\sin\left(m\theta\right)\right) m^c t^m \right) P\left(0\right) = 1. \qquad \text{(B.0.29)}$$

If $0 < |t| < 1$ or $H\left(a\cos\left(m\theta\right) - b\sin\left(m\theta\right)\right) = 0$, it is easy to have

$$\lim_{x\to+\infty} H\left(a\cos\left(m\theta\right) - b\sin\left(m\theta\right)\right) m^c t^m = 0.$$

From (B.0.29), we can obtain that

$$P\left(0\right) \le \lim_{x\to+\infty} \frac{1}{\frac{k}{k-j} + \frac{2}{k-j}m},$$

when m approaches infinity, it is clear that

$$P\left(0\right) \le \frac{1}{\frac{2}{k-j}m}.$$

If $|t| > 1$, it is clear that (B.0.29) leads to

$$\lim_{x\to+\infty} H\left(a\cos(m\theta) - b\sin(m\theta)\right) m^c t^m P(0) = 1,$$

which indicates that $\lim_{x\to+\infty} H\left(a\cos(m\theta) - b\sin(m\theta)\right) m^c t^m > 0$ and

$$\lim_{m\to+\infty} \frac{P(0)}{\frac{1}{m}} \leq \frac{1}{\frac{2}{k-j}}.$$

If $|t| = 1$ and $c < 1$, $\frac{2}{k-j}m$ dominates the limit and leads to a same result as condition of $0 < |t| < 1$.

If $|t| = 1$ and $c > 1$, due to

$$\frac{b_1}{T}m \in \left[\frac{2}{k-j}m, \frac{2(k-1)}{(k-j)^2}m\right],$$

the result is same as the condition that $|t| > 1$.

If $|t| = 1$ and $c = 1$, we consider two conditions.

Firstly, if $\frac{b_1}{T}m + H\left(a\cos(m\theta) - b\sin(m\theta)\right) m^c t^m = 0$ in the large m limit. Since all powers of m in (B.0.28) is integer, and we assume that $H\left(a\cos(m\theta) - b\sin(m\theta)\right) m^c t^m$ is the largest term when m approaches infinity, which indicates that $P(j+1) + \ldots + P(m)$ approaches to zero with m or it is a constant independent of m. For the condition of $\lim_{x\to+\infty} P(j+1) + \ldots + P(m) = 0$, it is same to condition $|t| < 1$. Then considering $P(j+1) + \ldots + P(m)$ will be a constant which is unrelated to m, by (B.0.1) to (B.0.3) shows that if $m = 2j$, $P(j+1) + \cdots + P(2j) = \frac{k}{k-j}\left(P(1) + \ldots + P(j)\right)$. While if $m = 3j$, we have $P(2j+1) + \cdots + P(3j) = \frac{k}{k-j}\left(P(j+1) + \ldots + P(2j)\right) \neq 0$ as long as any $P(1), \ \ldots, \ P(j)$ is nonzero. According to (B.0.1), this is obvious. So,

$P(j+1)+\ldots+P(m)$ is not a constant independent of m and this condition is impossible.

Secondly, if $\frac{b_1}{T}m+H\left(a\cos(m\theta)-b\sin(m\theta)\right)m^c t^m \neq 0$ in the large m limit. (B.0.29) will be converted to

$$\left(\frac{k}{k-j}+\alpha m\right)P(0)=1,$$

and when m approaches infinite,

$$P(0)\leq\frac{1}{\alpha m},$$

where α is a positive and finite number that is unrelated to battery capacity m.

In conclusion, Theorem 2 is proved.

Bibliography

[1] B. T. Bacinoglu and E. Uysal-Biyikoglu. Finite-horizon online transmission scheduling on an energy harvesting communication link with a discrete set of rates. *Journal of Communications and Networks*, 16(3):1–18, 2014.

[2] X. Chen, W. Ni, X. Wang, and Y. Sun. Provisioning quality-of-service to energy harvesting wireless communications. *IEEE Communications Magazine*, 53(4):102–109, 2015.

[3] M. Gregori and M. Payar. On the optimal resource allocation for a wireless energy harvesting node considering the circuitry power consumption. *IEEE Transactions on Wireless Communications*, 13(11):5968–5984, 2014.

[4] P. He, L. Zhao, S. Zhou, and Z. Niu. Recursive waterfilling for wireless links with energy harvesting transmitters. *IEEE Transactions on Vehicular Technology*, 63(3):1232–1241, 2014.

[5] C. K. Ho and R. Zhang. Optimal energy allocation for wireless communications with energy harvesting constraints. *IEEE Transactions on Signal Processing*, 60(9):4808–4818, 2012.

[6] C. Huang, R. Zhang, and S. Cui. Optimal power allocation for outage probability minimization in fading channels with energy harvesting constraints. *IEEE Transactions on Wireless Communications*, 13(2):1074–1087, 2014.

[7] M. Kashef and A. Ephremides. Optimal packet scheduling for energy harvesting sources on time varying wireless channels. *Journal of Communications and Networks*, 14(2):121–129, 2012.

[8] P. S. Khairnar and N. B. Mehta. Discrete-rate adaptation and selection in energy harvesting wireless systems. *IEEE Transactions on Wireless Communications*, 14(1):219–229, 2014.

[9] S. Kim, R. Vyas, J. Bito, K. Niotaki, A. Collado, A. Georgiadis, and M. M. Tentzeris. Ambient rf energy-harvesting technologies for self-sustainable standalone wireless sensor platforms. *Proceedings of the IEEE*, 102(11):1649–1666, 2014.

[10] M.-L. Ku, W. Li, Y. Chen, and K. J. R. Liu. Advances in energy harvesting communications: Past, present, and future challenges. *IEEE Communications Surveys & Tutorials*, 18(2):1384–1412, 2016.

[11] H. Li, J. Xu, R. Zhang, and S. Cui. A general utility optimization framework for energy-harvesting-based wireless communications. *IEEE Communications Magazine*, 53(4):79–85, 2015.

[12] W. Li, M.-L. Ku, Y. Chen, and K. J. R. Li. On outage probability for stochastic energy harvesting communications in fading channels. *IEEE Signal Processing Letters*, 22(11):1893–1897, 2015.

[13] S. Mao, M. H. Cheung, and V. W. S. Wong. Joint energy allocation for sensing and transmission in rechargeable wireless sensor networks. *IEEE Transactions on Vehicular Technology*, 63(6):2862–2875, 2014.

[14] M. Nourian, A. S. Leong, and S. Dey. Optimal energy allocation for kalman filtering over packet dropping links with imperfect acknowledgments and energy harvesting constraints. *IEEE Transactions on Automatic Control*, 59(8):2128–2143, 2014.

[15] F. M. Ozcelik, G. Uctu, and E. Uysal-Biyikoglu. Minimization of transmission duration of data packets over an energy harvesting fading channel. *IEEE Communications Letters*, 16(12):1968–1971, 2012.

[16] O. Ozei, K. Tutuncuoglu, J. Yang, S. Ulukus, and A. Yener. Transmission with energy harvesting nodes in fading wireless channels: Optimal policies. *IEEE Journal on Selected Areas in Communications*, 29(8):1732–1743, 2011.

[17] B. Pol, D. Gunduz, and M. Dohler. A learning theoretic approach to energy harvesting communication system optimization. *IEEE Transactions on Wireless Communications*, 12(4):1872–1882, 2013.

[18] R. V. Prasad, S. Devasenapathy, V. S. Rao, and J. Vazifehdan. Reincarnation in the ambiance: Devices and networks with energy harvesting. *IEEE Communications Surveys & Tutorials*, 16(1):195–213, 2013.

[19] R. Rajesh, V. Sharma, and P. Viswanath. Capacity of gaussian channels with energy harvesting and processing cost. *IEEE Transactions on Information Theory*, 60(5):2563–2575, 2014.

[20] D. Ramasur and G. Hancke. A wind energy harvester for low power wireless sensor networks. In *2012 IEEE International Instrumentation and Measurement Technology Conference Proceedings*. IEEE, 2012.

[21] R. K. Sathiendran, R. R. Sekaran, B. Chandar, and B. S. A. G. Prasad. Wind energy harvesting system powered wireless sensor networks for structural health monitoring. In *2014 International Conference on Circuits, Power and Computing Technologies*. IEEE, 2014.

[22] D. Shaviv and A. Özgür. Universally near optimal online power control for energy harvesting nodes. *IEEE Journal on Selected Areas in Communications*, 34(12):3620–3631, 2016.

[23] R. Srivastava and C. E. Koksal. Basic performance limits and tradeoffs in energy-harvesting sensor nodes with finite data and energy storage. *IEEE/ACM Transactions on Networking*, 21(4):1049–1062, 2013.

[24] S. Sudevalayam and P. Kulkarni. Energy harvesting sensor nodes: Survey and implications. *IEEE Communications Surveys & Tutorials*, 13(3):443–461, 2010.

[25] K. Tutuncuoglu and A. Yener. Optimum transmission policies for battery limited energy harvesting nodes. *IEEE Transactions on Wireless Communications*, 11(3):1180–1189, 2012.

[26] K. Tutuncuoglu, A. Yener, and S. Ulukus. Optimum policies for an energy harvesting transmitter under energy storage losses. *IEEE Journal on Selected Areas in Communications*, 33(3):467–481, 2015.

[27] S. Ulukus, A. Yener, E. Erkip, O. Simeone, M. Zorzi, P. Grover, and K. Huang. Energy harvesting wireless communications: A review of recent advances. *IEEE Journal on Selected Areas in Communications*, 33(3):360–381, 2015.

[28] R. Vaze, R. Garg, and N. Pathak. Dynamic power allocation for maximizing throughput in energy-harvesting communication system. *IEEE/ACM Transactions on Networking*, 22(5):1621–1630, 2013.

[29] S. Vinod, M. Utpal, J. Vinay, and G. Shery. Optimal energy management policies for energy harvesting sensor nodes. *IEEE Transactions on Wireless Communications*, 9(4):1326–1336, 2010.

[30] R. J. Vullers, R. van Schaijk, H. J. Visser, J. Penders, and C. V. Hoof. Energy harvesting for autonomous wireless sensor networks. *IEEE Solid-State Circuits Magazine*, 2(2):29–38, 2010.

[31] Z. Wang, V. Aggarwal, and X. Wang. Power allocation for energy harvesting transmitter with causal information. *IEEE Transactions on Communications*, 62(11):4080–4093, 2014.

[32] S. Wei, W. Guan, and K. J. R. Liu. Power scheduling for energy harvesting wireless communications with battery capacity constraint. *IEEE Signal Processing Letters*, 14(8):4640–4653, 2015.

[33] J. Xu and R. Zhang. Throughput optimal policies for energy harvesting wireless transmitters with non-ideal circuit power. *IEEE Journal on Selected Areas in Communications*, 32(2):322–332, 2013.

[34] J. Yang and S. Ulukus. Optimal packet scheduling in an energy harvesting communication system. *IEEE Transactions on Communications*, 60(1):220–230, 2012.