ANALYSIS OF A TWO SERVER POLLING SYSTEM WITH OVERLAPPING SKILLS AND 1-LIMITED SERVICE

ANALYSIS OF A TWO SERVER POLLING SYSTEM WITH OVERLAPPING SKILLS AND 1-LIMITED SERVICE

By

VANEETA KAUR GROVER, B.A., M.A.

A Thesis

submitted to the School of Graduate Studies

in Partial Fulfillment of the Requirements

for the Degree

Masters of Science

McMaster University

MASTER OF SCIENCE (1998)	MCMASTER UNIVERSITY
(Statistics)	Hamilton, Ontario
TITLE:	Analysis of a two server polling system with overlapping skills and 1-limited service
AUTHOR:	Vaneeta Kaur Grover
	M.A., University of Delhi
	B.A., Jesus & Mary College, University of Delhi
SUPERVISER:	Dr. Diwakar Gupta
NUMBER OF PAGES:	xii, 121

ABSTRACT

The main aim of the thesis is to find the optimal division of load in the three queues, i.e. the optimal degree of overlap of skills between the two servers with waiting time in queue as the performance measure. The model under consideration is a polling system with two servers and three queues – two specialized queues, 1 and 2, and a common queue, queue 3. One of the servers cycles between queues 1 and 3 and the other between 2 and 3. The imbedded Markov chain state equations and the functional equations for queue length probability generating functions are formulated. It was not possible to obtain a closed for expression for the exact mean waiting time in the queues by solving the functional equations. So, an attempt has been made to get an approximate closed form expression that could be used to find the optimal division of load in the three queues. Since the results are available only for the symmetric system we first assume the two specialized queues to be identical. But later we relax this assumption and give approximation method for the asymmetric system. The recommended method to approximate the mean waiting time in a queue can be used to determine the optimal allocation of load to the three queues.

ACKNOWLEDGEMENT

I would like to take this opportunity to express my deepest gratitude to my supervisor Dr. Diwakar Gupta for his able guidance and constant encouragement throughout the course of this work.

I am also deeply indebted to Dr. M. Lovrić who gave time from his busy schedule to explain some mathematical concepts and also helped me in understanding and using some computer packages. I am also grateful to Dr. S.G. Mohanty, Dr. N. Balakrishnan and Dr. P.D.M. Macdonald for their constant encouragement. I would also like to thank Dr. H. Takagi and O.J. Boxma for sending me the papers that were not easily available to me.

Last but not least, I would also like acknowledge my parents and sister without whose blessings and help I would not have been able to complete this thesis in such form.

To my parents and sister

TABLE OF CONTENTS

Abstract			**iii**
Chapter 1	**INTRODUCTION**		**1**
Chapter 2	**RELEVANT LITERATURE REVIEW**		**11**
	2.1	MODEL DESCRIPTION	11
	2.2	DEFINITIONS	12
	2.3	NOTATION	14
	2.4	LITERATURE REVIEW	17
Chapter 3	**MODEL DISCUSSION**		**25**
	3.1	FORMULATION OF STEADY STATE FLOW EQUATIONS	26
	3.2	FUNCTIONAL EQUATIONS FOR GENERATING FUNCTIONS	39
	3.3	SYMMETRIC SYSTEM	45
Chapter 4	**THE APPROXIMATIONS**		**51**
	4.1	NOTATION	52
	4.2	MEAN WAITING TIME ANALYSIS	52
	4.3	SYMMETRIC SYSTEM	53
		4.3.1 CYCLE ANALYSIS	53

4.3.2 STABILITY CONDITIONS 56

4.3.3 MEAN RESIDUAL CYCLE LENGTH ANALYSIS 57

4.4 ASYMMETRIC SYSTEM 61

4.4.1 CYCLE ANALYSIS 62

4.4.2 STABILITY CONDITIONS 62

4.4.3 MEAN RESIDUAL CYCLE LENGTH ANALYSIS 63

Chapter 5 DISCUSSION OF RESULTS FOR SYMMETRIC SYSTEM 66

RECOMMENDATIONS 83

Chapter 6 DISCUSSION OF RESULTS OF ASYMMETRIC SYSTEM 84

RECOMMENDATIONS 91

Chapter 7 CONCLUSION 92

7.1 CONCLUSION 92

7.2 TOPICS FOR FURHTER RESEARCH 94

Bibliography 97

Appendix A FORTRAN PROGRAM 100

Appendix B CONTROL STATEMENTS 113

Appendix C INPUT DATA FILE 115

Appendix D OUT PUT FILE 117

LIST OF FIGURES

Figure 1.1	n independent queues	1
Figure 1.2	Single queue n servers system	2
Figure 1.3	Two servers three queue polling model	4
Figure 1.4	A polling model	9
Figure 2.1	Sequence of events at queue 1	16
Figure 2.2	Sequence of events at queue 3 by servers $j, j = 1,2$	16
Figure 3.1	Status of server 1	26
Figure 3.2	Status of server 2	27
Figure 3.3	State-transition diagram for $n_1 > 0$	31
Figure 3.4	State-transition diagram for $n_1 = 0$	31
Figure 5.1	Mean waiting time in queue 1 for $\rho = 0.25$.	70
Figure 5.2	Mean waiting time in queue 1 for $\rho = 0.5$.	70
Figure 5.3	Mean waiting time in queue 1 for $\rho = 0.75$.	71
Figure 5.4	Relative error queue 1 for $\rho = 0.25$.	72
Figure 5.5	Relative error queue 1 for $\rho = 0.5$.	73
Figure 5.6	Relative error queue 1 for $\rho = 0.75$.	73
Figure 5.7	Mean waiting time in queue 3 for $\rho = 0.25$.	75

Figure 5.8	Mean waiting time in queue 3 for $\rho = 0.5$.	75
Figure 5.9	Mean waiting time in queue 3 for $\rho = 0.75$.	76
Figure 5.10	Relative error queue 3 for $\rho = 0.25$.	77
Figure 5.11	Relative error queue 3 for $\rho = 0.5$.	77
Figure 5.12	Relative error queue 3 for $\rho = 0.75$.	78
Figure 6.1	Mean waiting time in queue 3 $\rho = 0.25$.	88
Figure 6.2	Mean waiting time in queue 3 $\rho = 0.5$.	88
Figure 6.3	% error queue 3, $\rho = 0.25$.	89
Figure 6.4	% error queue 3, $\rho = 0.5$.	90
Figure 7.1	Four queues and three servers polling model	94

LIST OF TABLES

Table 3.1	Explanation and representation of x_k.	27
Table 3.2	Explanation of probability $p_j(\overline{x}, x_k)$.	28
Table 4.1	Range for optimal λ_3 in a specified range of b in terms of b_3 and r	57
Table 4.2	Combinations of λ_1, λ_3; b_1, b_3 and $E(C_j)$ to obtain $E(RC_1)$	58
Table 4.3	Combinations of λ_1, λ_3; b_1, b_3; to obtain $E(RC_1)$ and $E(RC_2)$	63
Table 4.3	Combinations of λ_1, λ_3; b_1, b_3 to obtain $E(RC_3)$	64
Table 5.1	Exact mean, simulated mean, standard deviation and 95% confidence interval for the mean, and the relative error for testing the FORTRAN program with SLAM II subroutines.	67
Table 5.2	Comparison of the simulated mean for different values of r.	69
Table 5.3	Comparison of different methods for queue 1.	69
Table 5.4	Relative error between the simulated mean and different methods for queue 1.	72
Table 5.5	Comparison of different methods for queue 3.	74
Table 5.6	Relative error between the simulated mean and different methods for queue 3.	76

Table 5.7 Approximate and simulated mean waiting time; standard deviation and 95% confidence interval for the mean; and relative error for queue 1. 79

Table 5.8 Approximate and simulated mean waiting time, standard deviation of the mean, 95% confidence interval for the mean, and relative error for queue 3. 81

Table 6.1 Approximate and simulated mean waiting time; standard deviation and 95% confidence interval for the mean; and relative error for queue 1. 85

Table 6.2 Approximate and simulated mean waiting time; standard deviation and 95% confidence interval for the mean; and relative error for queue 2. 86

Table 6.3 Approximate mean waiting time using 5 different methods; Simulated mean waiting time; standard deviation and 95% confidence interval for the mean for queue 3. 87

Table 6.4 Relative error between the simulated mean and approximate mean using 5 different methods for queue 3. 89

Table 6.5 Approximate and simulated mean waiting; standard deviation and 95% confidence interval for the mean; and relative error for queue 3. 90

Table 7.1 Summary for queues 1 and 2 93

Table 7.2 Summary for queue 3 93

CHAPTER 1

INTRODUCTION

Optimum utilization of the factors of production – capital, land and labor – has been the goal of the society ever since the industrial revolution. Labor at that time meant blue collar workers only. With the advent of information age, the attention has diverted to white collar workers who dominate the labor market today. Different types of real life situations have been modeled mathematically to find optimum worker configurations. One of the areas deals with queues and one such situation has been discussed below.

With the increase in the variety of customer classes, the basic problem has been

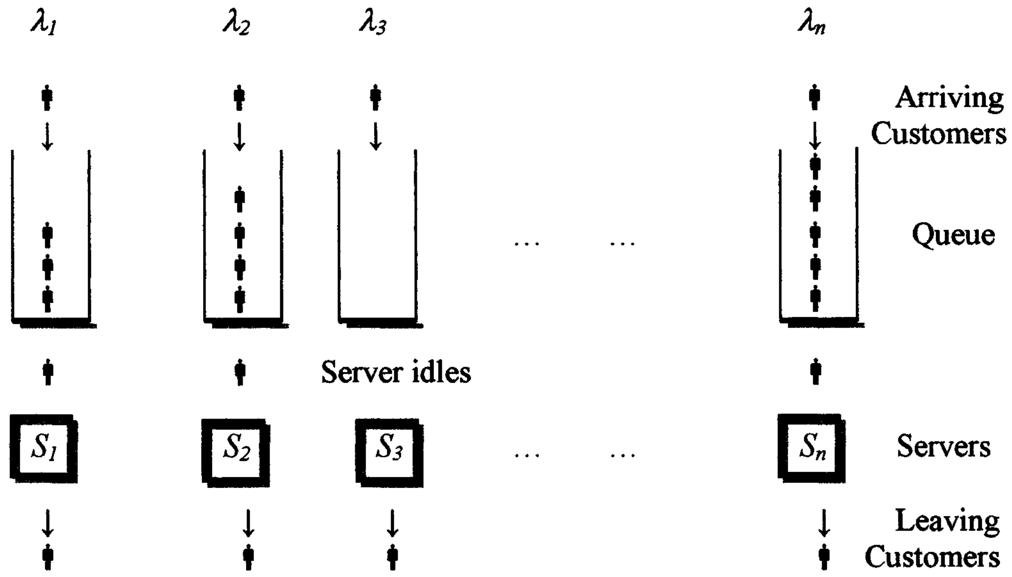

Fig 1.1: n independent queues

that of assigning the required skills to the servers. There are different ways in which this can be accomplished. One of the ways is by having highly specialized servers where each server is capable of handling only one of the customer classes. It results in as many numbers of queues and servers as the number of customer types (Figure 1.1). The server idles if there are no customers in its service class even though there are customers of other classes waiting in the system. This configuration can result in significant differences in the amount of waiting time experienced by each customer type. Therefore, this kind of a system may not be desirable.

In order to improve the system performance, an arriving customer can be allowed to go to the next available server, which is a single queue system with n servers (figure 1.2). Here each server has to be skilled to perform all service types. Smith and Whitt

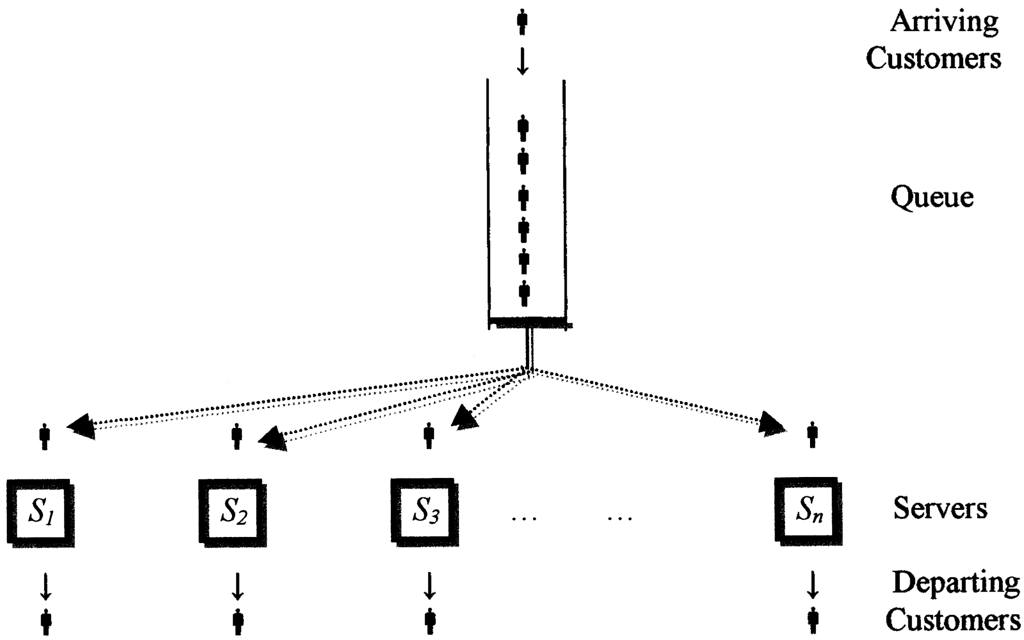

Fig 1.2: Single queue n servers system

(1981) mathematically proved the following result of teletraffic engineering:– having a common queue is more efficient than separate queues. This is so because in case of separate queues, a call can get blocked in a trunk while other trunks are idle. Stidham, (1970) studied the waiting cost function (sum of service cost and waiting cost per unit time) and concluded that single queue system minimizes the waiting cost function. It definitely improves the performance of the system, but at the cost of increased training expenses. It is also possible that the lack of specialization may reduce the service rate. Both of these – high cost of training and low service rate - are not desirable. Moreover, in reality, it may not be feasible for each server to be conversant with each service type. So, this system may not be practical.

Therefore, it would be desirable to allow a certain degree of overlapping and specialization of the service rather than complete specialization or complete flexibility. That is, train the server to perform more than one kind of job but not all kinds of jobs. Thus by having a combination of both i.e., common queue served by several servers and specialized queues served by single server would reduce the training cost, increase efficiency, and balance utilization across all servers. Sheikhzadeh, Benjaafar and Gupta, (1997) showed that in manufacturing systems, pooling, grouping machines together based on the operation, improves the system performance over the specialization, especially in presence of set up times. They also claimed that the chain configuration, limiting the number of parts that can be processed on any individual machine, performs better than completely flexible system once again when set up times are significant.

In this thesis we discuss a similar model with a focus on service system applications. We consider three different types of customer classes (j=1,2,3) and two servers (k=1,2). Each server is specialized to serve only one type of primary customers, (either type 1 or type 2), but both the servers can serve type 3 customers. This kind of system is more practical than either a completely specialized or completely flexible system. The server processes at most one waiting customer each time it visits a queue. This 1-limited service protocol is more suitable for the service sector rather than the manufacturing, because in manufacturing, once a machine has been set up for a particular of kind of job, it is advantageous to exhaust the queue of that kind of job specially in case of high set up costs. The 1-limited service discipline gives equal opportunity to all the queues of the system to receive server attention. It does not allow any queue to monopolize the server. Therefore, this kind of system is preferred in the service sector. The greater the degree of overlap, more flexible the two servers are and more traffic

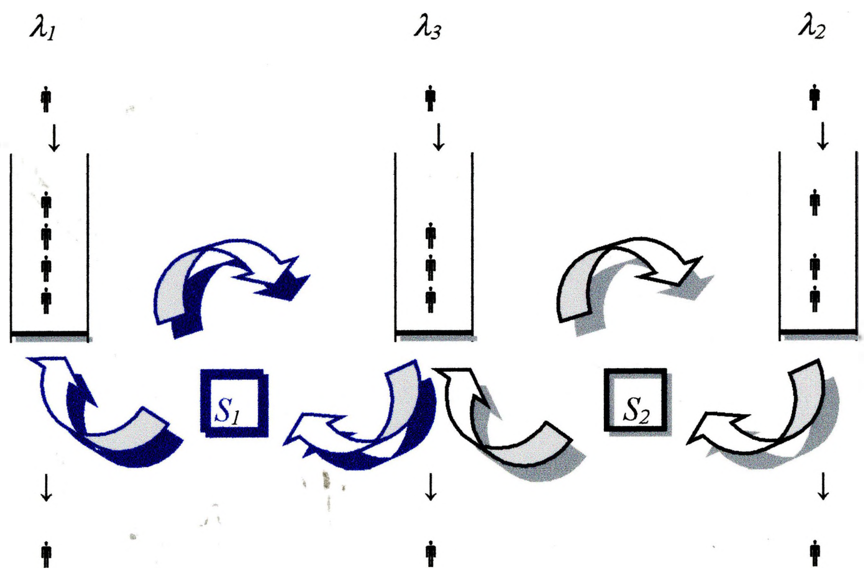

Fig 1.3: Two server three queue polling model.

can be diverted to the common queue. Lower the degree of overlap of skills, more specialized the servers are and less traffic can be sent to the common queue. Intuitively the stability conditions for this type of system would involve the switch over time in addition to the service time and the arrival rate. In order for the system to be stable, the mean number of arrivals during the visits of a server to the queue should be less than 1 (since at most one customer is served each time server visits a queue).

An example of this kind of system is where the service is provided in more than one language. Here the server may either know a single language or be multilingual. Bilingual server systems have been discussed in the past by, for example, Stanford and Grassmann, (1993). They considered a system where there are two customer classes – people speaking majority language and ones speaking minority language - and two server types – unilingual and bilingual. The unilingual servers know only the majority language while bilingual are conversant with both majority and minority languages. The service required by both customer types is the same but, they demand it in different languages. All the arriving customers join the same single queue and proceed to get served by the next available server. If the minority language speaking happens to go to the unilingual server, it is routed to another queue, referred to as the transfer queue, which is served on a non-preemptive priority basis over the entry queue by the bilingual server. Therefore, the customers are not distinguished until the start of their service.

Green, (1984) also discussed a similar model with two types of customers and two types of servers – general servers (type G) and the restricted use servers (type R). Either type of servers – type G as well as type R - can serve the general type of customers. But, R type customers can be served only by R type servers. Both types of customers join the same queue upon arrival and are served on First Come First Serve (FCFS) basis. But if the type R server is unavailable but, type G is available then the general customer, if any waiting in queue, gets served before the type R. On the contrary, if type R server becomes available, the first customer waiting in the queue gets served. Therefore, the customers are not distinguished till the server becomes available.

In our model, we have three separate queues depending on the service kind and the arriving customer joins one of the queues - 1, 2 or 3 - depending on the type of service required on arrival to the system. In other words, the customers are differentiated based on the type of service they require as soon as they enter the system unlike in Stanford and Grassman, (1993) (wait till time of service) or Green, (1984) (wait till server becomes available).

Marsan *et al.* (1990) discuss a polling system with multiple servers. They assume Poison arrivals; service times and switch over times are assumed to be independent, identically distributed random variables with arbitrary probability distribution function. Our model is different from theirs as they assume that each server moves in cyclic order and visits each queue of the system while in our model, server 1 does not go to queue 2 and server 2 does not visit queue 3.

The main idea of carrying out the study is to find the optimal division of load in the three queues. In other words, the aim is to find the optimal degree of overlap of skills between the two servers. The performance measure is the mean waiting time in queue. We formulate the steady state flow equations and the functional equations for queue length probability generating functions. These functional equations give rise to the Boundary Value Problem (Cohen and Boxma, 1983) which is difficult to solve. Therefore, it is not possible to obtain the exact mean waiting time in the queues by solving the functional equations. So, we make an attempt to get an approximate closed form expression that can be used to find the optimal division of load in the three queues. Since previous results (Marsan *et al.*, 1990 and Boxma and Meister, 1987) are available only for the symmetric system, we first assume the two specialized queues to be identical. But later we relax this assumption and give approximation method for the asymmetric system also. In the asymmetric system the rate of arrival and the service rates of specialized queues are different but the total load in both the queues is same. Also, the sum of the loads in the specialized queue is equal to the load in the common queue.

We did the numerical analysis of the methods and conclude that the approximation performs reasonably well (the relative error being with in + 15% to – 15%) except for high server utilization and small switch over times. The reason for the approximations to be not good for high server utilization is that the cycles become highly correlated and this violates one of our assumptions. Small switch over times as well as small server utilization creates a system where the server switches very fast completing thousands of cycles in a very short time leading to an unstable system. The approximations are also not good in some cases of asymmetric traffic. This is in agreement with Boxma and Meister's (1987) observation. They observed that if one or more queues have relatively large arrival rates, these queues become nearly unstable, and so their approximation method does not predict the mean waiting times at the queues with low arrival rate accurately. They also suggest a remedy for this – remove the queues with heavy arrival rate and compensate for its service time in the switch over times. For our model, this remedy would mean removal of one of the two queues – common queue or specialized queue. We applied this remedy too which is our method 7 – single queue with multiple server vacation. This method gives very high errors for queue 1 and so the results are not reported. For queue 3 the proposed method performs better than the single queue model with multiple vacations. Since none of them worked well, giving error between +15% to – 15% for all ρ, we could not use the expressions for the intended optimization of the load in the three queues.

Our model is a polling system where one server cycles between queues 1 and 3 and the other between 2 and 3. A standard polling system consists of multiple customer classes attended by a single server in cyclic order. Customers of class j arrive according to an independent Poisson process of rate λ_j. Each queue has infinite buffer capacity. The server visits each station in some predetermined order. The service times at each queue are independent random variables with general distribution.

Graphically it can be depicted as follows:

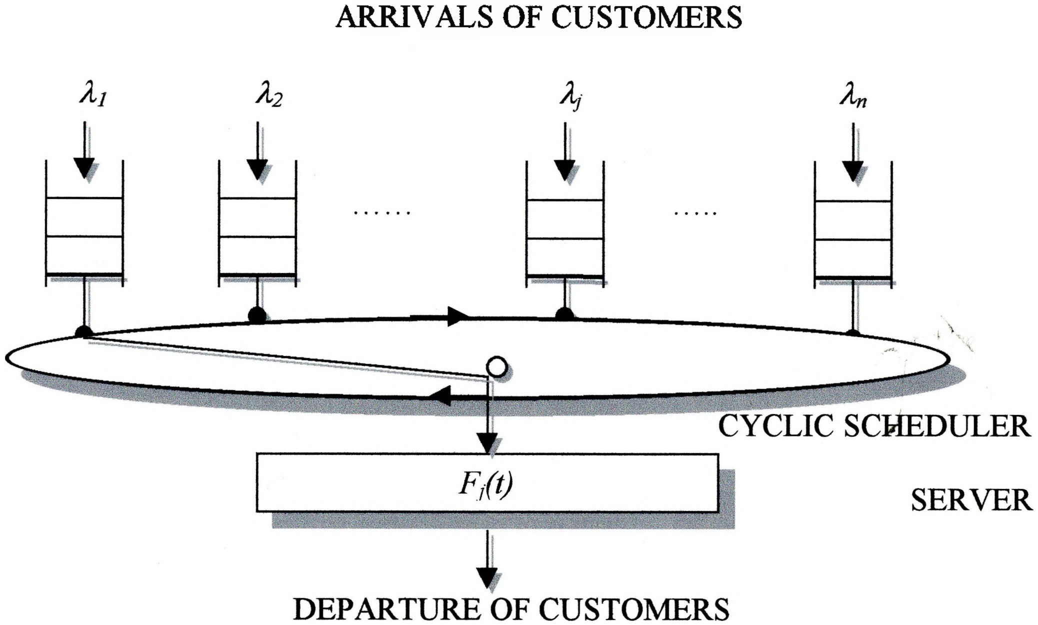

Figure 1.4: A Polling Model

The chapter 2 gives a detailed review of the previous analytical work on analysis of single and multiple servers polling models with 1-limited service. Chapter 3 gives the formulation of steady state flow equations. The discussion about the expressions that approximate the mean waiting time in each queue for symmetric as well as asymmetric systems can be found in chapter 4. The numerical analysis of the different approximation methods proposed and the recommendations for the symmetric system has been recapitulated in chapter 5. The numerical analysis of the different approximation methods proposed in chapter 4 and the recommendations for the asymmetric system has been summarized in chapter 6. Chapter 7 gives the conclusion of the thesis and scope for further study. The Microsoft FORTRAN (1987) program used to compute the estimate for the mean waiting time in the queue for our system has been attached as Appendix A. The Control statements for the SLAM II (1989) are provided in Appendix B. Appendix

C gives the sample input file for the program and Appendix D shows the sample output of the FORTRAN program with SLAM II subroutines. The program was run on Compaq Presario 4504 – Pentium 200.

CHAPTER 2

RELEVANT LITERATURE REVIEW

2.1 MODEL DESCRIPTION

Consider a system consisting of three queues (j = 1, 2, 3) each with infinite buffer capacities. A unit arrives to queue j, j = 1, 2, 3 according to the independent Poisson process with rate λ_j, so that the total arrival rate to the system is given by: $\Lambda = \lambda_1 + \lambda_2 + \lambda_3$. The customer arriving at queue j is referred to as type j customer. There are two servers (k = 1, 2). Server 1 serves queues 1 and 3 only and server 2 serves queues 2 and 3 only. Queues 1 and 2 are referred to as the specialized queues and queue 3 as common queue. Both servers 1 and 2 can be present at queue 3 at the same time serving a customer each. If there is only one unit waiting in queue 3 then the server who reaches queue 3 first servers the waiting unit. The service times of type-j customer are independently and identically distributed according to distribution $F_{S_j}(t)$, having mean b_j, second moment $b_j^{(2)}$ and Laplace-Stieltjes transforms $B_j^*(s)$, j=1,2,3. We assume the service time distribution to be exponential. The service discipline is 1-limited at each station, i.e., each time server arrives at a queue, it serves at the most 1 customer. The queue discipline is FCFS at each queue. The switch over times of the server between queues 1 and 3 and between queues 2 and 3 are independently identically distributed with exponential distribution, first two moments as r_1, r_2 and $r_1^{(2)}$, $r_2^{(2)}$ respectively. The switch over times of the server between queues 3 and 1, and between queues 3 and 2 are independently identically distributed with exponential distribution, having first two moments as r_3 and $r_3^{(2)}$. We also assume that both the servers continuously rove between stations 1 and 3 and between stations 2 and 3 respectively even though their sub-systems may be empty. For each server, the switch over times between queues are assumed to be independent of each other, independent of those of the other server and also independent of the inter arrival times and service times at each queue.

Before proceeding with the discussion of the solution of the model, terms and notations referred to in the thesis are defined followed by the relevant literature review.

2.2 DEFINITIONS

We now define some of the terms that would be used through out the thesis.

- A *continuously roving server* is one that continuously switches among the queues even though there is no one waiting at any of the queues, i.e., irrespective of the system states.
- The *switch point* is the instant (time epoch) at which the server begins to move to the next queue.
- The *switch over time* is the time needed by the server to physically move from queue j to the queue $j+1$. A switch over time may be zero. However, when the server never stops, sum of the switch over times over a server cycle must be strictly positive.

The following definitions are the same as defined by Marsan *et al.* (1990).

- The *arrival instant* is the instant when the server arrives at a queue.
- The *polling instant* is the instant of arrival of a server to a queue where upon arrival the server can serve a waiting customer.

Marsan *et al.* (1990) use these two different epochs because in their model at most S, $S \geq 1$, servers can be present at a queue. If there are S servers serving customers at a queue then the arriving server does not serve any waiting customers in that queue and moves on to the next queue in the cycle. Therefore that server observes the arrival instant but does not observe the polling instant. Hence, every arrival instant is not followed by a polling instant. According to the classical definition, a polling instant is the moment when the server arrives at a queue and is ready to serve any waiting customers. Therefore each time a server arrives at a queue it observes a polling instant but Marsan *et al.* (1990) do not observe the polling instant if the arriving server finds S servers present at that queue. Because of this their definition of polling instant differs from the usual definition. In our system there is no restriction on the maximum number of servers that can be present at a queue. Therefore each time a server arrives at a queue, it servers at most one customer, if any waiting. Hence the arrival instants and the polling instants are the same.

- The *residual j-cycle* is the time from the arrival of a type-j customer until the server returns to the queue j.
- The s*tation cycle time* is the time interval between two consecutive polling instants by any server at queue j.

- The *station-server cycle* is the time between two consecutive polling instants at queue j by server k.
- The, *server-intervisit time* is the time between the consecutive arrival instants of server k at queue j.
- The *intervisit time* is the time between the two consecutive polling instants of any server at queue j.

Figures 2.1 and 2.2, drawn after giving notation illustrate some of the definitions.

2.3 NOTATION

Following notation and system parameter definitions are referred to through out the thesis.

- λ_j : arrival rate to queue j.
- $\Lambda = \sum_j \lambda_j$: total arrival rate to the system.
- $F_j(t)$: service time distribution of type-j customer.
- $B_j^*(s)$: Laplace-Stieltjes transform of service time distribution of type-j customer.
- b_j : mean of service time distribution of type-j customer.
- $b_j^{(2)}$: finite second moment of service time distribution of type-j customer.
- $\rho = \sum_j \rho_j$: total system load where $\rho_j = \lambda_j b_j$.
- r_j : mean of switch over time distribution from queue j.
- $r_j^{(2)}$: finite second moment of switch over time distribution from queue j.

- RC_j : *residual j-cycle*, the time from the arrival of an arbitrary type- j customer until the server returns to queue j.
- C_j : *station j cycle time*, is the time interval between two consecutive polling instants by any server at queue j.
- $C_{j,k}$: *station-server cycle*, the time between two consecutive polling instants at queue j by server k.
- I_j : *intervisit time*, the time between the two consecutive arrival instants of any server at queue j.
- $I_{j,k}$: *server-intervisit time*, the time between the consecutive arrival instants of server k at queue j.
- $I_{j,k}^{(n)}$: interarrival time of server k at queue j in nth intervisit of server k.

In our system since each time a server arrives at a queue it servers at most one unit, if any waiting in the queue therefore, $I_{j,k} = C_{j,k}$ and $I_j = C_j$.

- R_j : *switch over time*, the time needed by the server to physically move from queue j to queue $j+1$.
- $R \equiv R^{(n)}$: total switch over time in nth intervisit cycle.
- W_j : waiting time in queue j.

Figure 2.1 illustrates the above mentioned definitions for queue 1 which is served by only one server, server 1. (Since queues 1 and 2 are identical, the diagram is exactly same for queue 2).

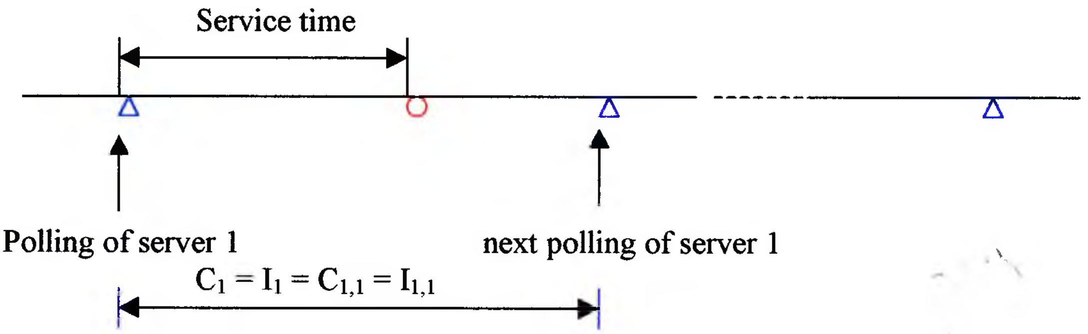

Figure 2.1: Sequence of events at queue 1.

Figure 2.2 illustrates the above mentioned definitions for queue 3 which is served by both the servers, server j, j = 1,2.

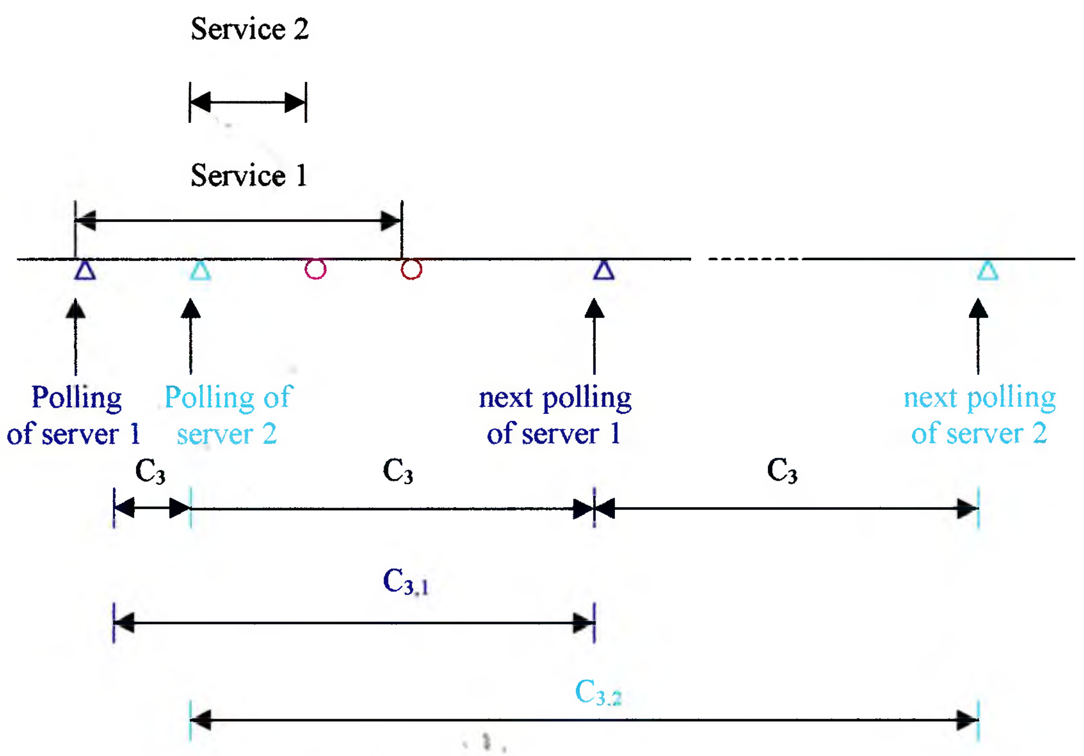

Figure 2.2: Sequence of events at queue 3 by servers j, j = 1, 2.

2.4 LITERATURE REVIEW

There has been a tremendous increase in literature on polling models in the recent years. The main reason for the increase in literature is the fact that multi queue system can be used to model many real life situations with varying degrees of assumptions. The latest development of polling models is multi queue multi server system that can represent a lot of real life and interesting situations including the service sector. Initially the polling models with only two queues and single server were investigated (Takács, 1968; and Eisenberg, 1971) because it was easier to study such systems. As the knowledge proliferated, attempts were made to study more generalized systems, i.e., polling systems with more than two stations (Cooper and Murray, 1969; and Cooper, 1970). Recently polling systems with multiple servers has started getting some attention but because of the complicated structure of the resultant mathematical model only approximate methods have been proposed so far (Morris and Wang, 1984; Marsan *et al.*, 1990; Mei and Borst, 1997). Although, many kinds of generalized systems have been developed over period of time, still even today, two queue systems are studied more often than general ones. The reason for this can be attributed to the fact that the two queue systems are simpler to analyze. A comprehensive literature survey on Polling Models has been carried out by Takagi (1990, 1997).

The 1-limited service discipline, often referred to as alternating service discipline, received attention as early as in late 70s. This kind of system attracted the attention of researchers because of its ability to give equal opportunity to all queues to be served as opposed to exhaustive system where the server empties the queue before moving on to the next queue. This results in an increase in the weighted sum of mean waiting times, pseudo conservation law, in queue for the 1-limited service as compared to the exhaustive service. 1-limited service finds most of its application in the service sector (and is preferred over other disciplines) as opposed to the manufacturing, because it attaches weight to the fairness, i.e. it gives opportunity to all the queues, which is important in case of human customers. But still very little work has been done in this area, especially in the case of asymmetric systems - the main reason being the difficulty in solving mathematical problems it gives rise to. The computation of exact analysis in two-queue case requires either solving singular integral equations, (Eisenberg, 1979) or boundary value problems (Boxma and Groenendijk, 1988; Boxma, 1984; Boxma and Cohen, 1983) from mathematical physics. It is still not generalized to the case with more than two queues.

We now review some of the work that has been done on more than one kind of customer class with cyclic and non-cyclic service discipline in detail.

The model discussed by Green (1985) was described in chapter 1. In order to describe the state of the system following five components are needed.

(i) the number of type R units waiting in queue,

(ii) number of type G customers waiting in queue,

(iii) the number of busy type R servers,

(iv) the number of busy type G servers, and

(v) the number of type G customers who have been passed into service by a type R customer

She reduced the above mentioned five-dimensional state space to two-dimensional Markovian state space –

(i) number of type R customers in service plus the number of customers of either type in the restricted queue (consisting of both type ot customers in FCFS), and

(ii) the number of type G customers in service plus the number of customers in the general queue (consisting of only type G customers who have been passed by a type R customer).

She then approximated the two-dimensional system, infinite in both dimensions, by a two-dimensional Markov process with finite second state variable. She also gave another approximation where she exploited the fact that the queues that have a matrix geometric steady state probability distribution can be solved by iterative techniques (Neuts, 1978).

Stanford and Grassmann (1993), as described in chapter 1, modeled the system as the continuous time Markov chain using matrix-geometric model given by Neuts (1981). They observed that the transition rate matrix had the structure of quasi-birth-and-death process and proposed its solution using iterative procedure as well as by state reduction method of Grassmann.

Eisenberg (1979) studied a two queue, single server model with alternating service discipline and zero switch over times. He transformed the problem of determining the joint queue length distribution at the two queues into the problem of solving a complex Fredholm integral equation of second kind. Boxma and Groenendijk (1988), Boxma (1984), Boxma and Cohen (1983) studied the same model as Eisenberg (1979) did, but they took switch over times into consideration. To determine the joint queue length distribution at the two queues, they all transformed the problem to that of Riemann-Hilbert boundary value problem.

Since it was not very easy to obtain the expressions for the system performance measures, Blanc (1990) proposed an iterative numerical technique based on power series for evaluating queue length distributions of polling systems. He assumed the service times to be exponentially distributed and a Bernoulli service schedule. The Power Series method involves power-series expansion of the state probabilities as functions of one parameter, the traffic intensity, of the system. The coefficients of the power series are calculated iteratively therefore, it is easy to compute additional terms to increase accuracy of the performance measure. Also it requires little effort to obtain performance measures for various values of traffic intensity from the coefficients of the power series. The only limitation this method has is the large amount of computer memory space it requires to store the coefficients of the power series.

Boxma and Meister (1987) obtained approximations for mean waiting time for a single server multi queue symmetric system with non zero switch over times. Assuming that the mean residual cycle lengths $E(RC_j)$ are independent of j (and therefore we drop the subscript j) and that $E(C_j) \approx \dfrac{b_j + E(R)}{1-\rho-\rho_j}$, and using pseudo-conservation law of Watson (1984):

$$\sum_j \rho_j(1-\alpha_j)E(W_j) = \frac{\rho}{2(1-\rho)}\sum_j \lambda_j b_j^{(2)} + \frac{\rho}{2E(R)}\sum_j (r_j^{(2)} - r_j^2) + \frac{E(R)}{2(1-\rho)}\sum_j \rho_j(1+\rho_j), \quad (2.1)$$

where α_j is the probability of finding at least one customer waiting in the queue *j* and is approximated by: $\alpha_j \approx \frac{\lambda_j E(R)}{(1-\rho)}$, they obtained the mean residual cycle as:

$$E(RC) = \frac{1-\rho}{(1-\rho)\rho + \sum_j \rho_j^2} \left(\frac{\rho}{2(1-\rho)} \sum_j \lambda_j b_j^2 + \frac{\rho}{2E(R)} \sum_j (r_j^{(2)} + r_j^2) + \frac{E(R)}{2(1-\rho)} \sum_j \rho_j (1-\rho_j) \right) \tag{2.2}$$

and mean waiting time in queue j as:

$$E(W_j) = \left(\frac{1-\rho+\rho_j}{1-\rho-\lambda_j E(R)} \right) \left(\frac{1-\rho}{(1-\rho)\rho + \sum_j \rho_j^2} \right) \cdot \left(\frac{\rho}{2(1-\rho)} \sum_j \lambda_j b_j^2 + \frac{\rho}{2E(R)} \sum_j (r_j^{(2)} + r_j^2) + \frac{E(R)}{2(1-\rho)} \sum_j \rho_j (1-\rho_j) \right) \tag{2.3}$$

Their approximation does not work that well for high arrival rate or asymmetric traffic. They also suggested a modification in case of high traffic in a queue – remove the queue with high traffic and compensate for its service time by increasing the switch over times.

Since it was hard to solve two queue single server systems with 1-limited service discipline, only approximate expressions were proposed for estimating the mean waiting time in queue for the multi queue multi server systems with 1-limited service policy. Morris and Wang (1984) were one of the first to study the polling model with multiple servers. Assuming m independent servers, they obtained the mean cycle time for bulk arrivals and 1-limited service discipline as:

$$E(C)=\frac{E(R)+\sum_{j=y+1}^{n} G_j b_j}{1-\frac{\sum_{j=1}^{y}\rho_j}{m}} \tag{2.4}$$

where G_j is the bulk size at *j-1*st queue and *y* is the largest integer such that $\lambda_y E(C) < mG_y$. Then the queues *1* through *y* are stable and *y+1* through *n* are unstable. And mean intervisit time for alternating service time discipline as:

$$E(I)=\frac{E(C)}{m}. \tag{2.5}$$

They approximated the mean waiting time in system, $E(S_j^{E(G_j)})$, sojourn time in the bulk service model for 1-limited service discipline as:

$$E(S_j^{E(G_j)})=\frac{E(I^2)}{2E(I)}+\frac{\sum_{z=1}^{\infty}(z-G_j)^+\pi_z}{\lambda_j}+\frac{b_j\sum_{z=1}^{\infty}(\min(z,G_j))^2\pi_z}{\sum_{z=1}^{\infty}\min(z,G_j)\pi_z} \tag{2.6}$$

where π_z is the limiting probability that the queue length just before an arbitrary service epoch. The distribution of the intervisit time was approximated by Gamma-distribution with mean E(C)/m and coefficient of variation as $\sqrt{(m-1)/(m+1)}$. The results show that the approximations do not work that well when the server utilization is very high or the switch over time from one queue to other is very small. They gave the justification that the intervisit time intervals were assumed to be independent identically distributed (iid) while in reality there is a significant negative correlation of consecutive intervisit times in case of queues near saturation. When the switch over times are small, the cycle times of the servers were no longer independent. They also observed that the servers have tendency to congregate if they follow the same schedule.

Marsan, Moraes, Donatelli and Neri (1990) analyzed multi queue multi server system with non zero switch over times. They gave approximate closed form expressions for the estimation of average customer waiting time for two different cases where

1. at most one server (1 X Q), and
2. any number of servers (S X Q)

can simultaneously attend a queue. They independently derived the following expression to compute the mean waiting time in a queue which coincides with the one used by Boxma and Meister (1987)

$$E(W) = \frac{\frac{E(C_j^2)}{2E(C_j)}}{1-\lambda E(C_j)}; \tag{2.7}$$

where $E(C_j)$ and $E(C_j^2)$ are the first two moments of the station cycle C_j. The $E(C_j)$ was given by:

$$E(C_j) = \frac{mr}{S-m\lambda b}. \tag{2.8}$$

But using the same approximation for $E(C_j)$ as Boxma and Meister (1987), they approximated the ratio $E(C_j^2)/E(C_j)$ as:

$$\frac{E(C_j^2)}{E(C_j)} = \frac{S}{S-(m-1)\lambda b}\left(\frac{m\lambda b^{(2)}}{S^2} + \frac{mr}{S-(S-1)\rho}\left(1+\frac{\lambda b}{S}\right) + \left(1-\frac{m\lambda b}{S}\right)\frac{(r^{(2)}-r^2)}{r(S-(S-1)\rho)}\right) \tag{2.9}$$

Thus equations (2.7), (2.8) and (2.9) together giving the mean waiting time in a queue for SXQ policy. The same expressions can easily be obtained for the 1XQ policy by substituting S = 1 in the expressions for SXQ policy.

Unlike Boxma and Meister (1987), and Marsan *et al.* (1990), Mei and Borst (1997) considered a polling model with multiple servers, where at each queue the servers followed a Bernoulli service strategy. They extended the idea of expressing the state probabilities as power series in the load as proposed by Blanc (1990) for single server to multiple servers. They also faced the problem of large memory requirement to store the coefficients of the power series that increase exponentially with the number of queues as well as number of servers. But, as pointed out by them, it can be resolved by removing the coefficients of the state probabilities that would not be needed for further computations and storing those of the relevant performance measures in relatively small arrays.

The analysis of multi queue system with 1-limited service discipline involves mathematical physics, and only approximate solutions are available for multi server systems. Therefore, in this thesis we propose some approximate expressions for the mean waiting time in queue for a system consisting of 2 servers, 3 queues and 1-limited service discipline with non-zero switch over times.

CHAPTER 3

MODEL DISCUSSION

The system under consideration was described in the previous chapter along with the definitions of various terms and notation. In this chapter, we make the mathematical model of the system under study. Generating function technique is a standard technique used to find the exact queue length distribution in polling models and has been widely discussed in the literature since the beginning e.g., see Cooper and Murray (1969). In the next section we formulate the imbedded Markov chain flow balance equations and in the section following that, we develop their generating function equations. The chain is imbedded at instants at which either server finishes serving a customer at any one of the three queues and at the instants at which either server finishes switching to any one of the three queues.

Since the units arrive to queue j (j = 1,2,3) according to the Poisson process with rate λ_j, the probability, $Q_j(u,t)$, that u units arrive at the jth queue in an interval of length t is given by:

$$Q_j(u,t) = \frac{(\lambda_j t)^u}{u!} e^{-\lambda_j t}, \qquad j=1,2,3; \; u=1,2,\ldots \tag{3.1}$$

3.1 FORMULATION OF STEADY STATE FLOW EQUATIONS

We observe the system at a service completion by either server at any one of the three queues and at a switch over completion by either server at any one of the three queues. Let (j,n_1,n_2,n_3,x_k) denote the state of the system at an arbitrary observation epoch where j is the index of the associated queue, n_j is the number of units in queue j, j = 1,2,3, including the ones under service and x_k is a vector of size 4, first two entries represents the queue index attached to server 1 and server 2 respectively and the last two entries represent the status of server 1 and server 2 respectively. The status of the server is represented by either 0 or 1; 0 denoting the server is switching and 1, serving. Figures 3.1 and 3.2 depict the status of the each server.

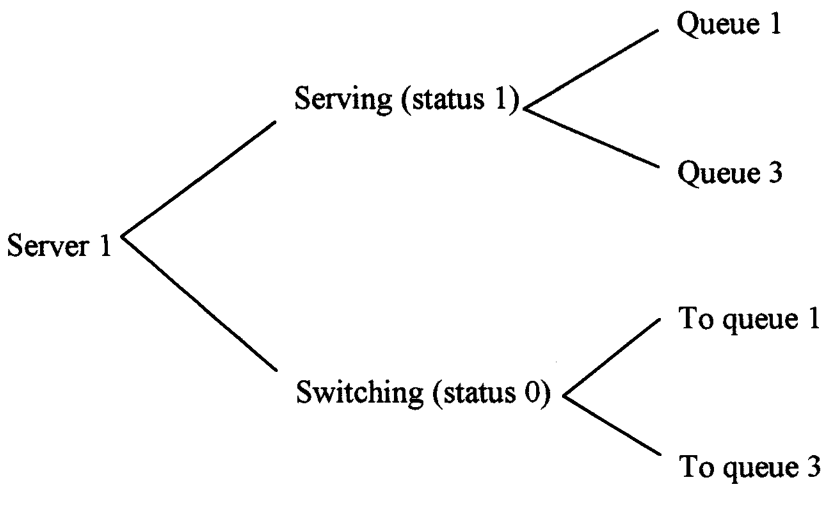

Figure 3.1: Status of server 1

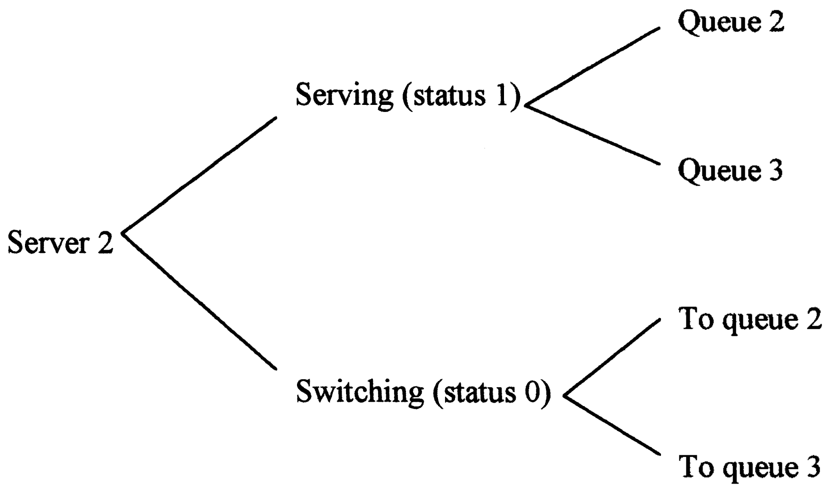

Figure 3.2: Status of server 2

There are 16 possibilities of x_k and are defined as follows:

x_k	Representation	Explanation
x_1	1211	Server 1 is serving in queue 1 and server 2 is serving in queue 2
x_2	1201	Server 1 is switching to queue 1 and server 2 is serving in queue 2
x_3	1210	Server 1 is serving in queue 1 and server 2 is switching to queue 2
x_4	1200	Server 1 is switching to queue 1 and server 2 is switching to queue 2
x_5	1311	Server 1 is serving in queue 1 and server 2 is serving in queue 3
x_6	1301	Server 1 is switching to queue 1 and server 2 is serving in queue 3
x_7	1310	Server 1 is serving in queue 1 and server 2 is switching to queue 3
x_8	1300	Server 1 is switching to queue 1 and server 2 is switching to queue 3
x_9	3211	Server 1 is serving in queue 3 and server 2 is serving in queue 2
x_{10}	3210	Server 1 is serving in queue 3 and server 2 is switching to queue 2
x_{11}	3201	Server 1 is switching to queue 3 and server 2 is serving in queue 2
x_{12}	3200	Server 1 is switching to queue 3 and server 2 is switching to queue 2
x_{13}	3311	Server 1 is serving in queue 3 and server 2 is serving in queue 3
x_{14}	3310	Server 1 is serving in queue 3 and server 2 is switching to queue 3
x_{15}	3301	Server 1 is switching to queue 3 and server 2 is serving in queue 3
x_{16}	3300	Server 1 is switching to queue 3 and server 2 is switching to queue 3

Table 3.1: explanation and representation of x_k.

The server serves at most one waiting customer (if any present) upon arrival to a queue therefore, the system is sub-critical (stable) if and only if the mean number of arrivals at each queue during a cycle is less than 1 (Takagi, 1990). Suppose that equilibrium exists. Let $p_j(n_1,n_2,n_3,x_k)$ be the steady state probability of state (j,n_1,n_2,n_3,x_k), i.e., $p_j(n_1,n_2,n_3,x_k)$ is the steady state joint probability that at queue j event, the status and position of two servers is given by vector x_k and there are n_1, n_2, n_3 customers in queues 1, 2 and 3 respectively including the unit under service. We use $\bar{n}$ to denote (n_1,n_2,n_3). There are 32 possibilities of $p_j(\bar{n},x_k)$ and are defined as follows:

$p_j(\bar{n},x_k)$	EXPLANATION
$p_1(\bar{n},x_1)$	Probability that at the instant when server 1 finished serving a type-1 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 2.
$p_1(\bar{n},x_2)$	Probability that at the instant when server 1 finished switching to queue 1 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 2.
$p_1(\bar{n},x_3)$	Probability that at the instant when server 1 finished serving a type-1 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 2.
$p_1(\bar{n},x_4)$	Probability that at the instant when server 1 finished switching to queue 1 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 2.
$p_1(\bar{n},x_5)$	Probability that at the instant when server 1 finished serving a type-1 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 3.
$p_1(\bar{n},x_6)$	Probability that at the instant when server 1 finished switching to queue 1 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 3.
$p_1(\bar{n},x_7)$	Probability that at the instant when server 1 finished serving a type-1 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 3.
$p_1(\bar{n},x_8)$	Probability that at the instant when server 1 finished switching to queue 1 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 3.

$p_2(\bar{n}, x_1)$	Probability that at the instant when server 2 finished serving a type-2 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 1.
$p_2(\bar{n}, x_2)$	Probability that at the instant when server 2 finished serving a type-2 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 1.
$p_2(\bar{n}, x_3)$	Probability that at the instant when server 2 finished switching to queue 2 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 1.
$p_2(\bar{n}, x_4)$	Probability that at the instant when server 2 finished switching to queue 2 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 1.
$p_2(\bar{n}, x_9)$	Probability that at the instant when server 2 finished serving a type-2 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 3.
$p_2(\bar{n}, x_{10})$	Probability that at the instant when server 2 finished serving a type-2 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 3.
$p_2(\bar{n}, x_{11})$	Probability that at the instant when server 2 finished switching to queue 2 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 3.
$p_2(\bar{n}, x_{12})$	Probability that at the instant when server 2 finished switching to queue 2 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 3.
$p_3(\bar{n}, x_5)$	Probability that at the instant when server 2 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 1.
$p_3(\bar{n}, x_6)$	Probability that at the instant when server 2 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 1.
$p_3(\bar{n}, x_7)$	Probability that at the instant when server 2 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 1.
$p_3(\bar{n}, x_8)$	Probability that at the instant when server 2 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 1.
$p_3(\bar{n}, x_9)$	Probability that at the instant when server 1 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 2.
$p_3(\bar{n}, x_{10})$	Probability that at the instant when server 1 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 2.

$p_3(\bar{n}, x_{11})$	Probability that at the instant when server 1 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 2.
$p_3(\bar{n}, x_{12})$	Probability that at the instant when server 1 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 2.
$p_3^{(1)}(\bar{n}, x_{13})$	Probability that at the instant when server 1 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 3.
$p_3^{(1)}(\bar{n}, x_{14})$	Probability that at the instant when server 1 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 3.
$p_3^{(1)}(\bar{n}, x_{15})$	Probability that at the instant when server 1 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is serving in queue 3.
$p_3^{(1)}(\bar{n}, x_{16})$	Probability that at the instant when server 1 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 2 is switching to queue 3.
$p_3^{(2)}(\bar{n}, x_{13})$	Probability that at the instant when server 2 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 3.
$p_3^{(2)}(\bar{n}, x_{14})$	Probability that at the instant when server 2 finished serving a type-3 unit there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 3.
$p_3^{(2)}(\bar{n}, x_{15})$	Probability that at the instant when server 2 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is serving in queue 3.
$p_3^{(2)}(\bar{n}, x_{16})$	Probability that at the instant when server 2 finished switching to queue 3 there are n_1, n_2 and n_3 customers in queue 1, queue 2 and queue 3 respectively and server 1 is switching to queue 3.

Table 3.2: Explanation of probability $p_j(\bar{n}, x_k)$

The state $(1, \bar{n}, x_1)$, $n_1 \geq 0$, $n_2 \geq 1$, $n_3 \geq 0$ follows the following states:

1. server 1 finishes switching to queue 1 and finds $\ell_1 \geq 1$ units in queue 1 and thus spends a length of time equal to service time of a type-1 unit, server 2 is serving in queue 2, i.e., the state $(1, \bar{\ell}, x_2)$, $\ell_1 \geq 1, \ell_2 \geq 1, \ell_3 \geq 0$, or

2. server 2 finishes switching to queue 2 and finds $\ell_2 \geq 1$ units in queue 2 and thus spends a length of time equal to service time of a type-2 unit, server 1 is serving in queue 1, i.e. the state $(2, \bar{\ell}, x_3)$, $\ell_1 \geq 1, \ell_2 \geq 1, \ell_3 \geq 0$.

The state with $n_1 = 0$, $n_2 \geq 1$, $n_3 \geq 0$, also follows the state $(1, 0, n_2, n_3, x_2)$, i.e., the server 1 finishes switching to queue 1 and finds an empty and, therefore, immediately records a service completion instant, server 2 is serving in queue 2.

i.e.,

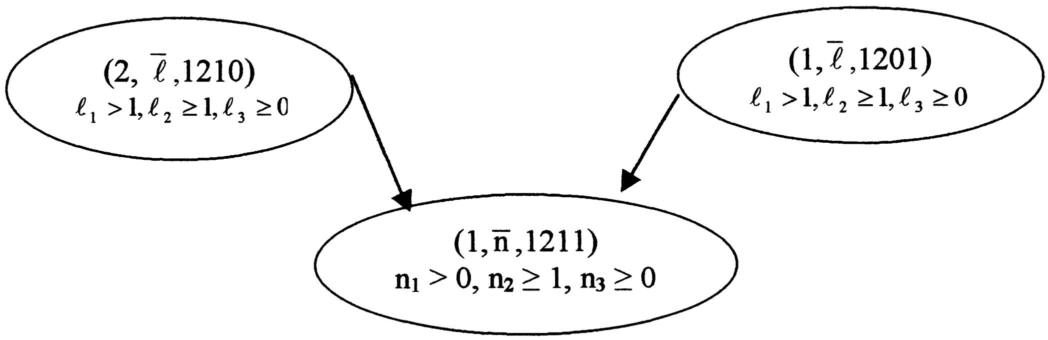

Figure 3.3: State-transition diagram for $n_1 > 0$

and

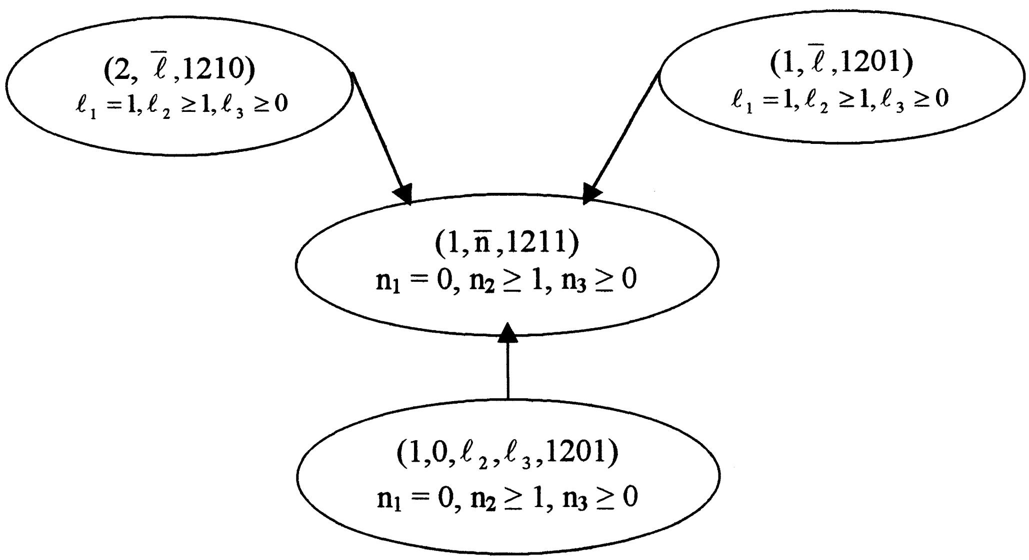

Figure 3.4: State-transition diagram for $n_1 = 0$

Let $I_{\{n_1=0\}}$ denote the indicator function, $I_{\{n_1=0\}} = \begin{cases} 1, & \text{if } n_1 = 0 \\ 0, & \text{otherwise} \end{cases}$.

Thus,

$p_1(\bar{n}, x_1) = \Pr\{$there are n_1, n_2, n_3 units in queues 1, 2 and 3 respectively, server 2 is serving a unit in queue 2 | server 1 finished serving a type-1 unit in time t$\}$

$$p_1(\bar{n},x_1) = \sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=1}^{n_2}\sum_{\ell_3=0}^{n_3} p_1(\bar{\ell},x_2)$$

$$\int_0^\infty Q_1(n_1-(\ell_1-1),t)Q_2(n_2-\ell_2,t)Q_3(n_3-\ell_3,t)(1-F_{s_2}(t))dF_{s_1}(t)$$

$$+\sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=1}^{n_2}\sum_{\ell_3=0}^{n_3} p_2(\bar{\ell},x_3)$$

$$\int_0^\infty Q(n_1-(\ell_1-1),t)Q(n_2-\ell_2,t)Q(n_3-\ell_3,t)e^{-b_2 t}dF_{s_1}(t)$$

$$+p_1(0,n_2,n_3,x_2)I_{\{n_1=0\}},$$

$n_1 \geq 0$, $n_2 \geq 1$, $n_3 \geq 0$.

Or,

$$p_1(\bar{n},x_1) = \sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=1}^{n_2}\sum_{\ell_3=0}^{n_3} (p_1(\bar{\ell},x_2)+p_2(\bar{\ell},x_3))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-(\ell_1-1)}}{(n_1-(\ell_1-1))!}e^{-\lambda_1 t}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}e^{-\lambda_2 t}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}e^{-\lambda_3 t}e^{-b_2 t}b_1 e^{-b_1 t}dt$$

$$+p_1(0,n_2,n_3,x_2)I_{\{n_1=0\}},$$

$n_1 \geq 0$, $n_2 \geq 1$, $n_3 \geq 0$.

or,

$$p_1(\overline{n},x_1)=\sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=1}^{n_2}\sum_{\ell_3=0}^{n_3}(p_1(\overline{\ell},x_2)+p_2(\overline{\ell},x_3))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-(\ell_1-1)}}{(n_1-(\ell_1-1))!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}b_1 e^{-(\Lambda+b_1+b_2)t}dt$$

$$+p_1(0,n_2,n_3,x_2)I_{\{n_1=0\}},$$

$n_1 \geq 0$, $n_2 \geq 1$, $n_3 \geq 0$.

Similarly,

$$p_1(\overline{n},x_2)=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=1}^{n_2}\sum_{\ell_3=0}^{n_3}(p_2(\overline{\ell},x_4)+p_3(\overline{\ell},x_9))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_3 e^{-(\Lambda+b_2+r_3)t}dt,$$

$n_1 \geq 0$, $n_2 \geq 1$, $n_3 \geq 0$.

$$p_1(\overline{n},x_3)=\sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_3(\overline{\ell},x_5)+p_1(\overline{\ell},x_4))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-(\ell_1-1)}}{(n_1-(\ell_1-1))!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}b_1 e^{-(\Lambda+b_1+r_3)t}dt$$

$$+p_1(0,n_2,n_3,x_4)I_{\{n_1 0\}},$$

$n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$.

$$p_1(\overline{n},x_4)=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_3(\overline{\ell},x_{10})+p_3(\overline{\ell},x_6))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_3 e^{-(\Lambda+2r_3)t}dt,$$

$n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$.

$$p_1(\bar{n},x_5)=\sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3}(p_3(\bar{\ell},x_7)+p_1(\bar{\ell},x_6))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-(\ell_1-1)}}{(n_1-(\ell_1-1))!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}b_1 e^{-(\Lambda+b_1+b_3)t}dt$$

$$+p_1(0,n_2,n_3,x_6)I_{\{n_1=0\}},$$

$n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 1$.

$$p_1(\bar{n},x_6)=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3}(p_3^{(1)}(\bar{\ell},x_{13})+p_3(\bar{\ell},x_8))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_3 e^{-(\Lambda+b_3+r_3)t}dt,$$

$n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 1$.

$$p_1(\bar{n},x_7)=\sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_1(\bar{\ell},x_8)+p_2(\bar{\ell},x_1))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-(\ell_1-1)}}{(n_1-(\ell_1-1))!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}b_1 e^{-(\Lambda+b_1+r_2)t}dt$$

$$+p_1(0,n_2,n_3,x_8)I_{\{n_1=0\}},$$

$n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$.

$$p_1(\bar{n},x_8)=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_3^{(1)}(\bar{\ell},x_{14})+p_2(\bar{\ell},x_2))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_3 e^{-(\Lambda+r_2+r_3)t}dt,$$

$n_1 \geq 0$, $n_2 \geq 0$, $n_3 \geq 0$.

Clearly,

$$p_1(n_1,0,n_3,x_1)=0;\quad n_1 \geq 0, n_3 \geq 0,$$

$$p_1(n_1,0,n_3,x_2)=0;\quad n_1 \geq 0, n_3 \geq 0,$$

$$p_1(n_1,n_2,0,x_5)=0;\ n_1\ge 0, n_2\ge 0,$$

$$p_1(n_1,n_2,0,x_6)=0;\ n_1\ge 0, n_2\ge 0.$$

The imbedded Markov chain state equations for queue 2 can be written by symmetry and those for queue 3 are as follows:

$$p_3(\overline{n},x_5)=\sum_{\ell_1=1}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3+1}(p_3(\overline{\ell},x_7)+p_1(\overline{\ell},x_6))$$
$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!}b_3 e^{-(\Lambda+b_1+b_3)t}dt$$
$$+p_3(n_1,n_2,0,x_7)I_{\{n_3=0\}},$$

$n_1\ge 1$, $n_2\ge 0$, $n_3\ge 0$.

$$p_3(\overline{n},x_6)=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3+1}(p_3^{(1)}(\overline{\ell},x_{13})+p_3(\overline{\ell},x_8))$$
$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!}b_3 e^{-(\Lambda+r_3+b_3)t}dt$$
$$+p_3(n_1,n_2,0,x_8)I_{\{n_3=0\}},$$

$n_1\ge 0$, $n_2\ge 0$, $n_3\ge 0$.

$$p_3(\overline{n},x_7)=\sum_{\ell_1=1}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_2(\overline{\ell},x_1)+p_1(\overline{\ell},x_8))$$
$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_2 e^{-(\Lambda+b_1+r_2)t}dt,$$

$n_1\ge 1$, $n_2\ge 0$, $n_3\ge 0$.

$$p_3(\overline{n},x_8)=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_3^{(1)}(\overline{\ell},x_{14})+p_2(\overline{\ell},x_2))$$
$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_2 e^{-(\Lambda+r_2+r_3)t}dt,$$

$n_1 \geq 0,\ n_2 \geq 0,\ n_3 \geq 0.$

$$p_3(\bar{n}, x_9) = \sum_{\ell_1=0}^{n_1} \sum_{\ell_2=1}^{n_2} \sum_{\ell_3=1}^{n_3+1} (p_3(\bar{\ell}, x_{11}) + p_2(\bar{\ell}, x_{10}))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!} \frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!} \frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!} b_3 e^{-(\Lambda+b_2+b_3)t} dt$$

$$+ p_3(n_1, n_2, 0, x_{11}) I_{\{n_3=0\}},$$

$n_1 \geq 0,\ n_2 \geq 1,\ n_3 \geq 0.$

$$p_3(\bar{n}, x_{10}) = \sum_{\ell_1=0}^{n_1} \sum_{\ell_2=0}^{n_2} \sum_{\ell_3=1}^{n_3+1} (p_3^{(2)}(\bar{\ell}, x_{13}) + p_3(\bar{\ell}, x_{12}))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!} \frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!} \frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!} b_3 e^{-(\Lambda+r_3+b_3)t} dt$$

$$+ p_3(n_1, n_2, 0, x_{12}) I_{\{n_3=0\}},$$

$n_1 \geq 0,\ n_2 \geq 0,\ n_3 \geq 0.$

$$p_3(\bar{n}, x_{11}) = \sum_{\ell_1=0}^{n_1} \sum_{\ell_2=1}^{n_2} \sum_{\ell_3=0}^{n_3} (p_1(\bar{\ell}, x_1) + p_2(\bar{\ell}, x_{12}))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!} \frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!} \frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!} r_1 e^{-(\Lambda+b_2+r_1)t} dt,$$

$n_1 \geq 0,\ n_2 \geq 1,\ n_3 \geq 0.$

$$p_3(\bar{n}, x_{12}) = \sum_{\ell_1=0}^{n_1} \sum_{\ell_2=0}^{n_2} \sum_{\ell_3=0}^{n_3} (p_3^{(2)}(\bar{\ell}, x_{15}) + p_1(\bar{\ell}, x_3))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!} \frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!} \frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!} r_1 e^{-(\Lambda+r_1+r_3)t} dt,$$

$n_1 \geq 0,\ n_2 \geq 0,\ n_3 \geq 0.$

$$p_3^{(1)}(\bar{n},x_{13}) = \sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=2}^{n_3+1}(p_3^{(2)}(\bar{\ell},x_{14}) + p_3^{(1)}(\bar{\ell},x_{15}))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!} b_3 e^{-(\Lambda+2b_3)t} dt$$

$$+ p_3^{(1)}(n_1,n_2,1,x_{15}) I_{\{n_3=1\}},$$

$n_1 \geq 0,\ n_2 \geq 0,\ n_3 \geq 1.$

$$p_3^{(1)}(\bar{n},x_{14}) = \sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3+1}(p_3^{(1)}(\bar{\ell},x_{16}) + p_2(\bar{\ell},x_9))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!} b_3 e^{-(\Lambda+b_3+r_2)t} dt$$

$$+ p_3^{(1)}(n,n,0,x_{16}) I_{\{n_3=0\}},$$

$n_1 \geq 0,\ n_2 \geq 0,\ n_3 \geq 0.$

$$p_3^{(1)}(\bar{n},x_{15}) = \sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3}(p_1(\bar{\ell},x_5) + p_3^{(2)}(\bar{\ell},x_{16}))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!} r_1 e^{-(\Lambda+b_3+r_1)t} dt,$$

$n_1 \geq 0,\ n_2 \geq 0,\ n_3 \geq 1.$

$$p_3^{(1)}(\bar{n},x_{16}) = \sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_1(\bar{\ell},x_7) + p_2(\bar{\ell},x_{11}))$$

$$\int_0^\infty \frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!} r_1 e^{-(\Lambda+r_1+r_2)t} dt,$$

$n_1 \geq 0,\ n_2 \geq 0,\ n_3 \geq 0.$

$$p_3^{(2)}(\bar{n},x_{13})=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=2}^{n_3+1}(p_3^{(2)}(\bar{\ell},x_{14})+p_3^{(1)}(\bar{\ell},x_{15}))$$

$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!}b_3e^{-(\Lambda+2b_3)t}dt$$

$$+p_3^{(2)}(n_1,n_2,1,x_{14})I_{\{n_3=1\}},$$

$n_1\geq 0,\ n_2\geq 0,\ n_3\geq 1.$

$$p_3^{(2)}(\bar{n},x_{14})=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3}(p_2(\bar{\ell},x_9)+p_3^{(1)}(\bar{\ell},x_{16}))$$

$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_2e^{-(\Lambda+b_3+r_2)t}dt,$$

$n_1\geq 0,\ n_2\geq 0,\ n_3\geq 1.$

$$p_3^{(2)}(\bar{n},x_{15})=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=1}^{n_3+1}(p_3^{(2)}(\bar{\ell},x_{16})+p_1(\bar{\ell},x_5))$$

$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-(\ell_3-1)}}{(n_3-(\ell_3-1))!}b_3e^{-(\Lambda+b_3+r_1)t}dt$$

$$+p_3^{(2)}(n_1,n_2,0,x_{16})I_{\{n_3=0\}},$$

$n_1\geq 0,\ n_2\geq 0,\ n_3\geq 0.$

$$p_3^{(2)}(\bar{n},x_{16})=\sum_{\ell_1=0}^{n_1}\sum_{\ell_2=0}^{n_2}\sum_{\ell_3=0}^{n_3}(p_1(\bar{\ell},x_7)+p_2(\bar{\ell},x_{10}))$$

$$\int_0^\infty\frac{(\lambda_1 t)^{n_1-\ell_1}}{(n_1-\ell_1)!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}r_2e^{-(\Lambda+r_1+r_2)t}dt,$$

$n_1\geq 0,\ n_2\geq 0,\ n_3\geq 0.$

It is evident that,

$$p_3(0,n_2,n_3,x_5)=0;\ \ n_3\geq 0, n_2\geq 0,$$

$$p_3(0,n_2,n_3,x_7)=0;\ \ n_2\geq 0, n_3\geq 0,$$

$$p_3(n_1,0,n_3,x_9)=0;\ n_3\ge 0, n_1\ge 0,$$

$$p_3(n_1,0,n_3,x_{11})=0;\ n_1\ge 0, n_3\ge 0,$$

$$p_3^{(1)}(n_1 n_2,0,x_{13})=0;\ n_1\ge 0, n_2\ge 0,$$

$$p_3^{(1)}(n_1,n_2,0,x_{15})=0;\ n_1\ge 0, n_2\ge 0.$$

$$p_3^{(2)}(n_1,n_2,0,x_{13})=0;\ n_1\ge 0, n_2\ge 0,$$

$$p_3^{(2)}(n_1,n_2,0,x_{14})=0;\ n_1\ge 0, n_2\ge 0.$$

3.2 FUNCTIONAL EQUATIONS FOR GENERATING FUNCTIONS

Define the probability generating function $g_j(\bar{z},x_k)$ as:

$$g_j(\bar{z},x_k)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}p_j(\bar{n},x_k)z_1^{n_1}z_2^{n_2}z_3^{n_3}\ ,\ j=1,2,3;\ k=1,\ldots,16$$

Therefore, $g_1(\bar{z},x_1)$ is given by:

$$g_1(\bar{z},x_1)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}p_1(\bar{n},x_1)z_1^{n_1}z_2^{n_2}z_3^{n_3}$$

$$g_1(\bar{z},x_1)=\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=0}^{\infty}p_1(\bar{n},x_1)z_1^{n_1}z_2^{n_2}z_3^{n_3}+\sum_{n_1=0}^{\infty}\sum_{n_3=0}^{\infty}p_1(n_1,0,n_3,x_1)z_1^{n_1}z_3^{n_3}$$

$$g_1(\bar{z},x_1)=\sum_{n_1=0}^{\infty}\sum_{n_2=1}^{\infty}\sum_{n_3=0}^{\infty}\Big(\sum_{\ell_1=1}^{n_1+1}\sum_{\ell_2=1}^{n_2}\sum_{\ell_3=0}^{n_3}(p_1(\bar{\ell},x_2)+p_2(\bar{\ell},x_3))$$

$$\int_0^{\infty}\frac{(\lambda_1 t)^{(n_1-(\ell_1-1))}}{(n_1-(\ell_1-1))!}\frac{(\lambda_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\frac{(\lambda_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}b_1 e^{-(\Lambda+b_1+b_2)t}dt\Big)z_1^{n_1}z_2^{n_2}z_3^{n_3}$$

$$+\sum_{n_2=1}^{\infty}\sum_{n_3=0}^{\infty}p(0,n_2,n_3,x_2)z_2^{n_2}z_3^{n_3}$$

because $p_1(n_1,0,n_3)=0$.

Changing the order of summation, we get:

$$g_1(\bar{z},x_1)=\sum_{\ell_1=1}^{\infty}\sum_{\ell_2=1}^{\infty}\sum_{\ell_3=0}^{\infty}(p_1(\bar{\ell},x_2)+p_2(\bar{\ell},x_3))z_1^{\ell_1}z_2^{\ell_2}z_3^{\ell_3}$$

$$\int_0^{\infty}(\sum_{n_1=\ell_1-1}^{\infty}\frac{(\lambda_1 z_1 t)^{n_1-(\ell_1-1)}}{(n_1-(\ell_1-1))!}\sum_{n_2=\ell_2}^{\infty}\frac{(\lambda_2 z_2 t)^{n_2-\ell_2}}{(n_2-\ell_2)!}\sum_{n_3=\ell_3}^{\infty}\frac{(\lambda_3 z_3 t)^{n_3-\ell_3}}{(n_3-\ell_3)!}$$

$$\frac{b_1}{z_1}e^{-(\Lambda+b_1+b_2)t}dt)+g_1(0,z_2,z_3,x_2)$$

or

$$g_1(\bar{z},x_1)=g_1(0,z_2,z_3,x_2)+(\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\sum_{\ell_3=0}^{\infty}(p_1(\bar{\ell},x_2)+p_2(\bar{\ell},x_3))z_1^{\ell_1}z_2^{\ell_2}z_3^{\ell_3}$$

$$-\sum_{\ell_1=0}^{\infty}\sum_{\ell_3=0}^{\infty}(p_1(\ell_1,0,\ell_3,x_2)+p_2(\ell_1,0,\ell_3,x_3))z_1^{\ell_1}z_3^{\ell_3}$$

$$+\sum_{\ell_3=0}^{\infty}(p_1(0,0,\ell_3,x_2)+p_2(0,0,\ell_3,x_3))z_3^{\ell_3}$$

$$-\sum_{\ell_2=0}^{\infty}\sum_{\ell_3=0}^{\infty}(p_1(0,\ell_2,\ell_3,x_2)+p_2(0,\ell_2,\ell_3,x_3))z_2^{\ell_2}z_3^{\ell_3}$$

$$+\sum_{\ell_3=0}^{\infty}(p_1(0,0,\ell_3,x_2)+p_2(0,0,\ell_3,x_3))z_3^{\ell_3}$$

$$-\sum_{\ell_3=0}^{\infty}(p_1(0,0,\ell_3,x_2)+p_2(0,0,\ell_3,x_3))z_3^{\ell_3})\int_0^{\infty}\frac{b_1}{z_1}e^{-(\Lambda+b_1+b_2-\sum_{j=1}^{3}\lambda_j z_j)t}dt$$

Since $\forall\ n_3\geq 0$, $p_1(n_1,0,n_3,x_2)=0\ \forall\ n_1\geq 0, n_3\geq 0$; $p_2(0,n_2,n_3,x_3)=0\ \forall\ n_2\geq 0, n_3\geq 0$; and $\int_0^{\infty}\frac{b_1}{z_1}e^{-(\Lambda+b_1+b_2-\sum_{j=1}^{3}\lambda_j z_j)t}dt=\frac{b_1}{z_1(\Lambda+b_1+b_2-\sum_{j=1}^{3}\lambda_j z_j)}$, therefore on rearrangement of terms we get:

$$g_1(\bar{z},x_1)=\frac{b_1}{z_1(\Lambda+b_1+b_2-\sum_{j=1}^{3}\lambda_j z_j)}(g_1(\bar{z},x_2)+g_2(\bar{z},x_3)-g_2(z_1,0,z_3,x_3)$$

$$-g_1(0,z_2,z_3,x_2)(1-\frac{z_1(\Lambda+b_1+b_2-\sum_{j=1}^{3}\lambda_j z_j)}{b_1})).$$

The above equation relates the state end of service completion at queue 1 to the state at the end of previous state. The state end of service completion at queue 1 follows one of the three possibilities:

1. previous state is $(1,\bar{\ell},x_2)$, $\ell_1 \neq 0$;
2. previous state is $(2,\bar{\ell},x_3), \ell_2 \neq 0$;
3. previous state is $(1,0,n_2,n_3,x_2)$. Since queue 1 is empty at the moment server 1 finished switching to queue 1, a service completion event is immediately recorded.

The factor $\frac{b_1}{\Lambda+b_1+b_2-\sum_{j=1}^{3}\lambda_j z_j}$ accounts for the customers that arrive during the service of type-1 customer, and the factor $1/z_1$ accounts for the departure of the served customer from queue 1.

Similarly,

$$g_1(\bar{z},x_2)=\frac{r_3}{\Lambda+r_3+b_2-\sum_{j=1}^{3}\lambda_j z_j}(g_2(\bar{z},x_4)+g_3(\bar{z},x_9)-g_2(z_1,0,z_3,x_4)).$$

$$g_1(\bar{z},x_3)=\frac{b_1}{z_1(\Lambda+b_1+r_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_5)+g_1(\bar{z},x_4)$$

$$-g_1(0,z_2,z_3,x_4)(1-\frac{z_1(\Lambda+b_1+r_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_1})).$$

$$g_1(\bar{z},x_4)=\frac{r_3}{\Lambda+2r_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3(\bar{z},x_{10})+g_3(\bar{z},x_6)).$$

$$g_1(\bar{z},x_5)=\frac{b_1}{z_1(\Lambda+b_1+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_7)+g_1(\bar{z},x_6)-g_3(z_1,z_2,0,x_7)$$

$$-g_1(0,z_2,z_3,x_6)(1-\frac{z_1(\Lambda+b_1+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_1})).$$

$$g_1(\bar{z},x_6)=\frac{r_3}{\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3^{(1)}(\bar{z},x_{13})+g_3(\bar{z},x_8)-g_3(z_1,z_2,0,x_8)).$$

$$g_1(\bar{z},x_7)=\frac{b_1}{z_1(\Lambda+b_1+r_2-\sum_{j=1}^{3}\lambda_j z_j)}(g_1(\bar{z},x_8)+g_2(\bar{z},x_1)$$

$$-g_1(0,z_2,z_3,x_8)(1-\frac{z_1(\Lambda+b_1+r_2-\sum_{j=1}^{3}\lambda_j z_j)}{b_1})).$$

$$g_1(\bar{z},x_8)=\frac{r_3}{\Lambda+r_2+r_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3^{(1)}(\bar{z},x_{14})+g_2(\bar{z},x_2)).$$

Equations follow by symmetry for queue 2, therefore are not mentioned here and for queue 3,

$$g_3(\bar{z},x_5)=\frac{b_3}{z_3(\Lambda+b_1+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_7)+g_1(\bar{z},x_6)-g_1(0,z_2,z_3,x_6)$$

$$-g_3(z_1,z_2,0,x_7)(1-\frac{z_3(\Lambda+b_1+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_6)=\frac{b_3}{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(1)}(\bar{z},x_{13})+g_3(\bar{z},x_8)$$

$$-g_3(z_1,z_2,0,x_8)(1-\frac{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_7)=\frac{r_2}{\Lambda+b_1+r_2-\sum_{j=1}^{3}\lambda_j z_j}(g_2(\bar{z},x_1)+g_1(\bar{z},x_8)-g_1(0,z_2,z_3,x_8)).$$

$$g_3(\bar{z},x_8)=\frac{r_2}{\Lambda+r_3+r_2-\sum_{j=1}^{3}\lambda_j z_j}(g_3^{(1)}(\bar{z},x_{14})+g_2(\bar{z},x_2)).$$

$$g_3(\bar{z},x_9)=\frac{b_3}{z_3(\Lambda+b_2+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_{11})+g_2(\bar{z},x_{10})-g_2(z_1,0,z_3,x_{10})$$

$$-g_3(z_1,z_2,0,x_{11})(1-\frac{z_3(\Lambda+b_2+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_{10})=\frac{b_3}{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(2)}(\bar{z},x_{13})+g_3(\bar{z},x_{12})$$

$$-g_3(z_1,z_2,0,x_{12})(1-\frac{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_{11})=\frac{r_1}{\Lambda+b_2+r_1-\sum_{j=1}^{3}\lambda_j z_j}(g_1(\bar{z},x_1)+g_2(\bar{z},x_{12})-g_2(z_1,0,z_3,x_{12})).$$

$$g_3(\bar{z},x_{12})=\frac{r_1}{\Lambda+r_3+r_1-\sum_{j=1}^{3}\lambda_j z_j}(g_3^{(2)}(\bar{z},x_{15})+g_1(\bar{z},x_3)).$$

$$g_3^{(1)}(\bar{z},x_{13})=\frac{b_3}{z_3(\Lambda+2b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(2)}(\bar{z},x_{14})+g_3^{(1)}(\bar{z},x_{15})+g_3^{(2)}(z,z_2,1,x_{14})$$

$$-g_3^{(1)}(z_1,z_2,1,x_{15})(1-\frac{z_3(\Lambda+2b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3^{(1)}(\bar{z},x_{14})=\frac{b_3}{z_3(\Lambda+r_2+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(1)}(\bar{z},x_{16})+g_2(\bar{z},x_9)$$

$$-g_3^{(1)}(z_1,z_2,0,x_{16})(1-\frac{z_3(\Lambda+r_2+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3^{(1)}(\bar{z},x_{15})=\frac{r_1}{\Lambda+b_3+r_1-\sum_{j=1}^{3}\lambda_j z_j}(g_1(\bar{z},x_5)+g_3^{(2)}(\bar{z},x_{16})-g_3^{(2)}(z_1,z_2,0,x_{16})).$$

$$g_3^{(1)}(\bar{z},x_{16})=\frac{r_1}{\Lambda+r_1+r_2-\sum_{j=1}^{3}\lambda_j z_j}(g_1(\bar{z},x_7)+g_2(\bar{z},x_{11})).$$

$$g_3^{(2)}(\bar{z},x_{13})=\frac{b_3}{z_3(\Lambda+2b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(2)}(\bar{z},x_{14})+g_3^{(1)}(\bar{z},x_{15})+g_3^{(1)}(z,z_2,1,x_{15})$$

$$-g_3^{(2)}(z_1,z_2,1,x_{14})(1-\frac{z_3(\Lambda+2b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3^{(2)}(\bar{z},x_{14})=\frac{r_2}{\Lambda+b_3+r_2-\sum_{j=1}^{3}\lambda_j z_j}(g_2(\bar{z},x_9)+g_3^{(1)}(\bar{z},x_{16})-g_3^{(1)}(z_1,z_2,0,x_{16})).$$

$$g_3^{(2)}(\bar{z},x_{15})=\frac{b_3}{z_3(\Lambda+r_1+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(2)}(\bar{z},x_{16})+g_1(\bar{z},x_5)$$

$$-g_3^{(2)}(z_1,z_2,0,x_{16})(1-\frac{z_3(\Lambda+r_1+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3^{(2)}(\bar{z},x_{16})=\frac{r_2}{\Lambda+r_1+r_2-\sum_{j=1}^{3}\lambda_j z_j}(g_1(\bar{z},x_7)+g_2(\bar{z},x_{11})).$$

It is difficult to solve the above set of equations because of the presence of terms like: $g_1(0,z_2,z_3,x_2)$. Therefore, we look at the symmetric system as we expect it to be less complicated than the asymmetric system.

3.3 SYMMETRIC SYSTEM

We consider a symmetric system, i.e., we assume that $\lambda_1 = \lambda_2 = \lambda$, $b_1 = b_2 = b$, and $r_1 = r_2 = r$. This implies that $g_3^{(1)}(\bar{z},x_{16}) = g_3^{(2)}(\bar{z},x_{16})$. Therefore, we drop the superscripts (1) and (2) from now on. Also, $\Lambda = 2\lambda + \lambda_3$ and $\sum_{j=1}^{3}\lambda_j z_j = \lambda(z_1+z_2)+\lambda_3 z_3$.

Then the above set of equations reduce to the following:

$$g_1(\bar{z},x_1)=\frac{b}{z_1(\Lambda+2b-\sum_{j=1}^{3}\lambda_j z_j)}(g_1(\bar{z},x_2)+g_2(\bar{z},x_3)-g_2(z_1,0,z_3,x_3)$$

$$-g_1(0,z_2,z_3,x_2)(1-\frac{z_1(\Lambda+2b-\sum_{j=1}^{3}\lambda_j z_j)}{b})).$$

$$g_1(\bar{z},x_2)=\frac{r_3}{\Lambda+r_3+b-\sum_{j=1}^{3}\lambda_j z_j}(g_2(\bar{z},x_4)+g_3(\bar{z},x_9)-g_2(z_1,0,z_3,x_4)).$$

$$g_1(\bar{z},x_3)=\frac{b}{z_1(\Lambda+b+r_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_5)+g_1(\bar{z},x_4)$$

$$-g_1(0,z_2,z_3,x_4)(1-\frac{z_1(\Lambda+b+r_3-\sum_{j=1}^{3}\lambda_j z_j)}{b})).$$

$$g_1(\bar{z},x_4)=\frac{r_3}{\Lambda+2r_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3(\bar{z},x_{10})+g_3(\bar{z},x_6)).$$

$$g_1(\bar{z},x_5)=\frac{b}{z_1(\Lambda+b+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_7)+g_1(\bar{z},x_6)-g_3(z_1,z_2,0,x_7)$$

$$-g_1(0,z_2,z_3,x_6)(1-\frac{z_1(\Lambda+b+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b})).$$

$$g_1(\bar{z},x_6)=\frac{r_3}{\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3(\bar{z},x_{13})+g_3(\bar{z},x_8)-g_3(z_1,z_2,0,x_8)).$$

$$g_1(\bar{z},x_7)=\frac{b}{z_1(\Lambda+b+r-\sum_{j=1}^{3}\lambda_j z_j)}(g_1(\bar{z},x_8)+g_2(\bar{z},x_1)$$

$$-g_1(0,z_2,z_3,x_8)(1-\frac{z_1(\Lambda+b+r-\sum_{j=1}^{3}\lambda_j z_j)}{b})).$$

$$g_1(\bar{z},x_8)=\frac{r_3}{\Lambda+r+r_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3(\bar{z},x_{14})+g_2(\bar{z},x_2)).$$

Equations follow by symmetry for queue 2, therefore are not mentioned here and for queue 3,

$$g_3(\bar{z},x_5)=\frac{b_3}{z_3(\Lambda+b+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_7)+g_1(\bar{z},x_6)-g_1(0,z_2,z_3,x_6)$$

$$-g_3(z_1,z_2,0,x_7)(1-\frac{z_3(\Lambda+b+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_6)=\frac{b_3}{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_{13})+g_3(\bar{z},x_8)$$

$$-g_3(z_1,z_2,0,x_8)(1-\frac{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_7)=\frac{r}{\Lambda+b+r-\sum_{j=1}^{3}\lambda_j z_j}(g_2(\bar{z},x_1)+g_1(\bar{z},x_8)-g_1(0,z_2,z_3,x_8)).$$

$$g_3(\bar{z},x_8)=\frac{r}{\Lambda+r+r_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3(\bar{z},x_{14})+g_2(\bar{z},x_2)).$$

$$g_3(\bar{z},x_9)=\frac{b_3}{z_3(\Lambda+b+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_{11})+g_2(\bar{z},x_{10})-g_2(z_1,0,z_3,x_{10})$$

$$-g_3(z_1,z_2,0,x_{11})(1-\frac{z_3(\Lambda+b+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_{10})=\frac{b_3}{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_{13})+g_3(\bar{z},x_{12})$$

$$-g_3(z_1,z_2,0,x_{12})(1-\frac{z_3(\Lambda+r_3+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_{11})=\frac{r}{\Lambda+b_2+r-\sum_{j=1}^{3}\lambda_j z_j}(g_1(\bar{z},x_1)+g_2(\bar{z},x_{12})-g_2(z_1,0,z_3,x_{12})).$$

$$g_3(\bar{z},x_{12})=\frac{r}{\Lambda+r+r_3-\sum_{j=1}^{3}\lambda_j z_j}(g_3(\bar{z},x_{15})+g_1(\bar{z},x_3)).$$

$$g_3^{(1)}(\bar{z},x_{13})=\frac{b_3}{z_3(\Lambda+2b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(2)}(\bar{z},x_{14})+g_3^{(1)}(\bar{z},x_{15})+g_3^{(2)}(z,z_2,1,x_{14})$$

$$-g_3^{(1)}(z_1,z_2,1,x_{15})(1-\frac{z_3(\Lambda+2b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3^{(1)}(\bar{z},x_{14})=\frac{b_3}{z_3(\Lambda+r+b_3-\sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_{16})+g_2(\bar{z},x_9)$$

$$-g_3(z_1,z_2,0,x_{16})(1-\frac{z_3(\Lambda+r+b_3-\sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3^{(1)}(\bar{z},x_{15}) = \frac{r}{\Lambda + b_3 + r - \sum_{j=1}^{3}\lambda_j z_j}(g_1(\bar{z},x_5) + g_3(\bar{z},x_{16}) - g_3(z_1,z_2,0,x_{16})).$$

$$g_3^{(2)}(\bar{z},x_{13}) = \frac{b_3}{z_3(\Lambda + 2b_3 - \sum_{j=1}^{3}\lambda_j z_j)}(g_3^{(2)}(\bar{z},x_{14}) + g_3^{(1)}(\bar{z},x_{15}) + g_3^{(1)}(z,z_2,1,x_{15})$$

$$- g_3^{(2)}(z_1,z_2,1,x_{14})(1 - \frac{z_3(\Lambda + 2b_3 - \sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3^{(2)}(\bar{z},x_{14}) = \frac{r}{\Lambda + b_3 + r - \sum_{j=1}^{3}\lambda_j z_j}(g_2(\bar{z},x_9) + g_3(\bar{z},x_{16}) - g_3(z_1,z_2,0,x_{16})).$$

$$g_3^{(2)}(\bar{z},x_{15}) = \frac{b_3}{z_3(\Lambda + r + b_3 - \sum_{j=1}^{3}\lambda_j z_j)}(g_3(\bar{z},x_{16}) + g_1(\bar{z},x_5)$$

$$- g_3(z_1,z_2,0,x_{16})(1 - \frac{z_3(\Lambda + r + b_3 - \sum_{j=1}^{3}\lambda_j z_j)}{b_3})).$$

$$g_3(\bar{z},x_{16}) = \frac{r}{\Lambda + 2r - \sum_{j=1}^{3}\lambda_j z_j}(g_1(\bar{z},x_7) + g_2(\bar{z},x_{11})).$$

But once again we get the terms like $g_2(z_1,0,z_3,x_3)$ which are difficult to solve for. We also face problem of cycling, i.e., for example, if substitute the expressions for $g_1(\bar{z},x_2)$ and $g_2(\bar{z},x_3)$ in $g_1(\bar{z},x_1)$; and, $g_3(\bar{z},x_5)$ and $g_1(\bar{z},x_4)$ in $g_2(\bar{z},x_3)$ and so on, we again get the term $g_2(\bar{z},x_3)$, from where we had started. Since even the symmetric system, identical specialized queues, is also difficult to solve for, our next step is, therefore, to consider model approximations. We seek the model approximations instead of numerical since the goal of this research is to find optimal degree of skills overlap between the two servers, we need to obtain closed form expressions for system performance. We discuss such approximations in the next chapter. In chapter 5 and chapter 6 we do the numerical analysis of the different methods proposed in the next chapter along with the conclusions and scope for further study.

CHAPTER 4

THE APPROXIMATIONS

In the last chapter we formulated the imbedded Markov chain state equations and the functional equations for queue length probability generating functions. Because of the difficulty in solving the resulting functional equations, we propose approximations for obtaining the mean waiting time in the system. Since the exact expression for mean waiting time in queue is not known even for a single server system, we follow the approach used by Boxma and Meister (1987), and Marsan *et al.* (1990). Both Boxma and Meister's (1987), and Marsan *et al.* (1990) approximations work best for the symmetric system. Therefore, in order to use their results, we initially assume the two specialized queues to be symmetric, though later we relax this assumption.

We approximate our three queues and two servers system by a two queues single server system. The server in the approximate system is considered to be either twice as fast or of the same speed as the server in the original system. The arrival rates in the single server system is assumed to be either half of the arrival rate, or the same, or twice of the arrival rate in the two servers system. The combination of arrival rate and the service rate in the two queues of the approximating single server system is picked in such a way that the total load carried by each server remains the same as in the original two servers system.

4.1 NOTATION

$\rho^{(1)}$: the proportion of time server 1 is attending to customers

$\rho^{(2)}$: the proportion of time server 1 is attending to customers

In case of the symmetric system, $\rho^{(1)} = \rho^{(2)}$. Let $\rho^{(1)} = \rho^{(2)} = P$. P is the same as ρ used by Boxma and Meister (1987) and Marsan *et al.* (1990).

4.2 MEAN WAITING TIME ANALYSIS

An arbitrary arrival of type-*j* has to wait until the server returns to queue *j* and until all the customers in front of him have been served (Boxma and Meister, 1987). Therefore, mean waiting time is the sum of remaining cycle time (residual *j*-cycle) and the time it has to wait until all units in front of it are served. Since at most one type-*j* customer is served in a complete server cycle, we have mean waiting time in queue *j* as:

$$E(W_j) = E(RC_j) + E(X_j)E(C_j), \quad j = 1,2,3. \tag{4.1}$$

Since the poisson arrivals see time averages (Wollf, 1982), the queue length at queue *j* just before the arrival of a type-*j* customer equals the mean queue length at the queue at any arbitrary instant of time. Therefore, $E(X_j)$ is given by:

$$E(X_j) = \lambda_j E(W_j), \quad j = 1,2,3. \tag{4.2}$$

Substituting for $E(X_j)$ from equation (4.2) in equation (4.1) we get:

$$E(W_j) = \frac{E(RC_j)}{1-\lambda_j E(C_j)}, \qquad j = 1,2,3. \qquad (4.3)$$

4.3 SYMMETRIC SYSTEM

4.3.1 CYCLE ANALYSIS

The proportion of the time a server spends switching in first N cycles is given by (Marsan *et. al*, 1990):

$$\frac{\sum_{n=1}^{N} R^{(n)}}{\sum_{n=1}^{N} I_{j,k}^{(n)}} = \frac{\sum_{n=1}^{N} R^{(n)} \Big/ N}{\sum_{n=1}^{N} I_{j,k}^{(n)} \Big/ N}.$$

Taking limit as $N \rightarrow \infty$, we get:

$$\frac{E(R)}{E(I_{j,k})} = \frac{2r}{E(I_{j,k})}.$$

Since $I_{j,k} = C_{j,k}$, $I_j = C_j$, therefore, the proportion of time server k spends switching is given by:

$$\frac{E(R)}{E(I_{j,k})} = \frac{2r}{E(I_{j,k})} = \frac{2r}{E(C_{j,k})}, \qquad j = 1,2,3.$$

This implies that proportion of time server k spends serving customers is:

$$1 - \frac{2r}{E(C_{j,k})}, \qquad j = 1,2,3.$$

Average arrival rate of work in the system is $2\lambda b + \lambda_3 b_3$. Since the system is symmetric, i.e., the queues 1 and 2 are identical in terms of the arrival and the service rates, and the service rate for both the servers in queue 3 is same, therefore, in the long run both the servers get an equal amount of work, on average. That is, each server's average rate of arrival of work is $\lambda b + \lambda_3 b_3/2$. This is also the average proportion of time each server is serving the customers, P. Since for a system to be stable, the average arrival rate of work must equal the average rate at which work is completed, we have,

$$1 - \frac{2r}{E(C_{j,k})} = P.$$

Or,

$$E(C_{j,k}) = \frac{2r}{1 - P}.$$

The time between consecutive arrival/polling instants of a server *k* at queue *j* is:

$$C_{j,k} = \sum_{j=1}^{M} I_j^{(n)},$$

where *M* is the number of servers different from server *k* that poll station *j* between two consecutive polling instants of server *k*; time between the start and end of the station-server cycle. Since $E(C_j) = E(I_j)$ and since *M* is a stopping time for the point process defined by the sequence of server arrivals at a station, Wald's equation (Cinlar, 1975) gives:

$$E(C_{j,k}) = E(M)E(C_j).$$

For queue *j*, *j*= 1, 2, number of severs different from server *k*, *k* = 1, 2 respectively, that poll the stations between two consecutive polling instants of server *k* is zero (as these are the two specialized queues). Clearly from figure 2.1 it follows that

$$E(C_{1,1}) = E(C_1),$$

$$E(C_{1,2}) = 0,$$

$$E(C_{2,1}) = 0,$$

$$E(C_{2,2}) = E(C_2).$$

For queue 3, only one more different server from server k, k = 1,2, polls it (figure 2.2) therefore,

$$E(C_{3,1}) = E(C_{3,2}) = 2E(C_3).$$

And hence,

$$E(C_1) = E(C_2) = \frac{2r}{1-P}, \tag{4.4}$$

and

$$E(C_3) = \frac{r}{1-P} = \frac{E(R)}{2(1-P)}. \tag{4.5}$$

Using Boxma and Meister's (1987) approach we get a different approximate expression for the cycle length. Define

α_{jk} = Pr(j-cycle contains a type–k service)

or α_{jk} = E(number of type–k services in a j-cycle) (because, a j-cycle contains at most one type–k service)

$$= \lambda_k E(Cj), \qquad j \neq k; \tag{4.6}$$

Since a j-cycle consists of a type-k service, services of customers of other types and the sum of N switch-over times, therefore,

$$E(C_j) = b_j + E(R) + \sum_{j \neq k} \alpha_{jk} b_k$$

substituting for α_{jk} from equation (4.6) in the above equation and rearranging the terms, we get:

$$E(C_j) = \frac{b_j + E(R)}{1 - P + \rho_j}, \tag{4.7}$$

4.3.2 STABILITY CONDITIONS

The necessary and sufficient conditions for stability of a system are given as follows (Kuhen, 1979):

$P + \lambda_j E(R) < 1, j = 1,2;$

and since queue 3 is served by 2 servers therefore, we have 2 equations corresponding with each server as:

$P + \lambda_3 \beta E(R) < 1$

and $P + \lambda_3 (1 - \beta) E(R) < 1$

where α is the proportion of time server 1 spends serving a customer at queue 3. Since the system is symmetric, in the long run both the servers get an equal amount of work, on average therefore, $\beta = ½$ and therefore the stability condition for queue 3 is: $\lambda_3 r + P < 1$.

Hence the conditions for the system to be sub-critical (stable) are:

$$\lambda r + P < 1, \tag{4.8}$$

$$\text{and } \lambda_3 r + P < 1. \tag{4.9}$$

These are the conditions for having finite mean waiting time at a queue. The stability conditions can be interpreted in terms of feasible range of λ_3 for a specific range of b as follows.

Range of b	Feasible range of λ_3	Additional condition
$0 \le b \le b_3 - 2r$	$\lambda_3 < \min\left(1, \frac{2-2r-b}{b_3-2r-b}, \frac{2-b}{b_3+2r-b}\right)$	$b + 2r < \min(2, b_3)$
$b_3 - 2r \le b \le b_3$	$\max\left(0, \frac{b-2+2r}{b-b_3+2r}\right) < \lambda_3 < \min\left(1, \frac{2-b}{b_3+2r-b}\right)$	$b < 2$
$b_3 \le b \le b_3 + 2r$	$\max\left(0, \frac{b-2+2r}{b-b_3+2r}\right) < \lambda_3 < \min\left(1, \frac{2-b}{b_3+2r-b}\right)$	$b_3 < 2$
$b_3 + 2r \le b$	$\max\left(0, \frac{b-2}{b-2r-b_3}, \frac{b-2+2r}{b+2r-b_3}\right) < \lambda_3$	$b_3 + 2r < 2$
$0 \le b$	$\max\left(0, 1-\frac{2}{b}\right) \le \lambda_3 \le \min\left(1, \frac{2}{b_3}\right)$	
$0 \le b$	$\max\left(0, 1-\frac{2}{r}\right) \le \lambda_3 \le \min\left(1, \frac{2}{r}\right)$	

Table 4.1: Feasible range of λ_3 for a specified range of b

4.3.3 MEAN RESIDUAL CYCLE LENGTH ANALYSIS

The mean residual cycle is computed using the Watson's pseudo conservation law given by equation (2.1).

We assume that $E(RC_j)$ is independent of j (Boxma and Meister, 1987). Substituting $E(W_j)$ from equation (4.3) in Watson's pseudo conservation law, equation (2.1), on simplification and rearrangement of terms we get:

$$E(RC) = \frac{(1-P)}{\sum_j \frac{\rho_j(1-P-E(R)\lambda_j)}{1-\lambda_j E(C_j)}} \left(\frac{P}{2(1-P)} \sum_j \lambda_j b_j^{(2)} + \frac{P}{2E(R)} \sum_j r_j^{(2)} + \frac{E(R)}{2(1-P)} \sum_j \rho_j (1+\rho_j) \right) \tag{4.10}$$

Since queues 1 and 2 are identical, only one of them can be considered. With out loss of generality, we consider queue 1 only and the corresponding results for queue 2 are exactly the same.

We use the following combinations of λ_1, λ_3; b_1, b_3; $E(C_j)$ to obtain $E(RC_1)$ and also assume that the switch over times between each queue is exponentially identically distributed with first two moments as r and $r^{(2)}$ respectively. Therefore, $E(R) = 2r$.

$E(C_j)$	$\lambda_1 = \lambda$, $\lambda_3 = \lambda_3/2$. $b_1 = b$, $b_3 = b_3$.	$\lambda_1 = 2\lambda$, $\lambda_3 = \lambda_3$. $b_1 = b/2$, $b_3 = b_3/2$.	$\lambda_1 = \lambda$, $\lambda_3 = \lambda_3$. $b_1 = b$, $b_3 = b_3/2$.
$E(R)/(1-\rho)$	METHOD 1	METHOD 2	METHOD 3
$(b_j+E(R))/(1-\rho+b_j\lambda_j)$	METHOD 4	METHOD 5	METHOD 6

Table 4.2: Combinations of λ_1, λ_3; b_1, b_3 and $E(C_j)$ to obtain $E(RC_1)$

The justification for above approximations is as follows: Methods 1 and 4 look at the system from server 1's perspective. For it, queue 2 does not exist and in long run, it serves approximately half the load in queue 3. The rate of switching between two queues remains the same, r. Methods 2 and 5 assume the total load in the system as same but server as twice as fast in serving the customers but, the mean switch over time remaining the same, r. Methods 3 and 6 assume that the load in queue 1 and 3 remains the same but the server is twice as fast in serving customers in queue 3 and the mean switch over time remain the same, r. The methods 1, 2 and 3 differ from methods 4, 5 and 6 respectively only in $E(C_j)$. Methods 1, 2 and 3 use the $E(C_j)$ as given by equation (4.4) and (4.5) while methods 4, 5 and 6 use $E(C_j)$ as given equation (4.7). We use the same combinations of λ_1, λ_3; b_1, b_3; $E(C_j)$ to obtain $E(RC_3)$ but with $E(R) = \frac{r}{(2-P)}$ instead of $E(R) = 2r$.

We need to make this modification in the switch over times since in high traffic the servers tend to coalesce, and thus stations see batches of servers rather than a uniform distribution of servers along the cycle (Boxma and Meister, 1987). The switch over times are reduced by a factor of 2 in low traffic, but remains the same for $P \rightarrow 1$.

Once $E(RC_j)$ is computed, it is substituted back into equation (4.3) along with the corresponding $E(C_j)$ to get $E(W_j)$ for each queue, $j = 1,2,3$.

METHOD 7: SINGLE QUEUE WITH VACATIONS METHOD:

In order to approximate $E(W_3)$, we consider a system with single server, Poisson arrivals with rate $\lambda_3/2$ and the exponential service with mean b_3. The service times are independent of the arrival times. The server goes on vacation with mean equal to the sum of:

1. sum of mean of three exponential distributions - two switch over times each with mean r, and service time with mean b/2 with probability α (probability that the server finds at least one customer waiting in queue) and
2. 2r with probability 1 - α (probability that no customer is waiting in queue).

i.e.,

$$E(V) = E(R) + b\alpha \tag{4.11}$$

where $\alpha = \frac{E(R)\lambda}{1-P}$,

$E(R) = 2r$

and second moment:

$$E(V^2) = E(R^2) + 2\alpha(b^2 + bE(R)), \tag{4.12}$$

$E(R^2) = 6r^2$.

E(W) is then computed by substituting in the following formula (Takagi, (1991), page 228, equation (6.1b)):

$$E(W) = \frac{E(V^2)}{2E(V)} + \frac{\lambda(b^{(2)} + 2bE(V) + E(V^2)}{2(1 - P - \lambda E(V))} \tag{4.13}$$

where E(V) and $E(V^2)$ are the first two moments of the vacation times. The justification for using this method is that if we look at the system from queue 3's perspective, it is served by 2 identical servers who go away on vacation for the time period equal to either total switch over time in that cycle, with probability α or for the sum of mean of three exponential distributions – two switch over times and one service time at queue 1 or 2 – with probability 1- α. Since both the servers are identical, each server on an average gets half the load of queue 3, therefore, we consider the system with Poisson arrivals with rate $\lambda_3/2$ and the exponential service with mean b_3 where the server goes on vacation to serve queue 1.

Then mean number in queue j, j=1,2,3, $E(L_j)$, is computed using Little's Formula:

$$E(L_j) = \lambda_j E(W_j), \quad j = 1,2,3 \tag{4.12}$$

4.4 ASYMMETRIC SYSTEM

Consider the system where the arrival and the service rates are not identical in the two specialized queues but the total load in both queues is same.

i.e., $\rho_1 = \lambda_1 b_1 = \rho_2 = \lambda_2 b_2$,

and $\rho_3 = \lambda_3 b_3 = \rho_1 + \rho_2$.

We also assume that the total proportion of time each server is busy serving customers in its specialized queue and the common queue is same, i.e., we assume that $\rho^{(1)} = \rho^{(2)} = P$. The load in the common queue is considered as the sum of the loads in the side queues. Clearly the proportion of time each server spends in queue 3 is not the same.

The total amount of time server 1 spends switching to and from queue 1 and queue 3 is: $E(R_1) = r_1 + r_3$. Similarly, the total switch over time for server 2 switching to and from queue 2 and 3 is: $E(R_2) = r_2 + r_3$. Looking at the switch over times from queue 3's perspective we get the switch over time as: $E(R_3) = (r_1 + r_2) / 2 + r_3$.

4.4.1 CYCLE ANALYSIS

Define

α_{jk} = Pr(*j*-cycle contains a type–*k* service)

= E(number of type–*k* services in a *j*-cycle) (because, a *j*-cycle contains at most one type–*k* service)

= $\lambda_k E(Cj)$, $j \neq k$;

(Boxma and Meister (1987)). Since a *j*-cycle consists of a type-*k* service, services of customers of other types and the sum of N switchover times, therefore,

$$E(C_j) = b_j + E(R_j) + \sum_{j \neq k} \alpha_{jk} b_k$$

substituting for α_{jk} from equation (4.6) in the above equation and rearranging the terms, we get:

$$E(C_j) = \frac{b_j + E(R_j)}{1 - P + \rho_j}.$$

4.4.2 STABILITY CONDITIONS

The necessary and sufficient conditions for stability of a system are given as follows (Kuhen, 1979):

$P + \lambda_j E(R_j) < 1, j = 1,2;$

and since queue 3 is served by 2 servers therefore, we have 2 equations corresponding with each server as:

$P + \lambda_3 \beta E(R_3) < 1$

and $P + \lambda_3 (1- \beta) E(R_3) < 1$

where β is the proportion of time server 1 spends serving a customer at queue 3.

Therefore the necessary and sufficient conditions for the system to be sub-critical are:

$$\lambda_1 E(R_1) + P < 1, \tag{4.15}$$

$$\lambda_2 E(R_2) + P < 1, \tag{4.16}$$

and $P + \max(\beta, 1-\beta)\, \lambda_3\ E(R_3) < 1$ (4.17)

4.4.3 MEAN RESIDUAL CYCLE LENGTH ANALYSIS

For the asymmetric system, we propose following approximations:

The average amount of time server 1 is not serving at queue 1 is: $1 - \lambda_1 b_1$ and amount of time both servers are not serving in their respective specialized queues is: $(1 - \lambda_1 b_1) + (1 - \lambda_2 b_2)$. Therefore, amount of load offered at queue 3 to server 1, λ_{31}, is given by:

$$\lambda_{31} = \frac{\lambda_3 (1 - \lambda_1 b_1)}{2 - \lambda_1 b_1 - \lambda_2 b_2}.$$

Similarly, the amount of load offered to server 2 at queue 3, λ_{32}, is given by:

$$\lambda_{32} = \frac{\lambda_3 (1 - \lambda_2 b_2)}{2 - \lambda_1 b_1 - \lambda_2 b_2}.$$

To approximate mean residual cycle length, we substitute following combination of λ_1, λ_3 and r_j; and, E(R) in equation (4.10).

QUEUE	λ_1	b_1	λ_3	b_3	E(R)
1	λ_1	b_1	λ_{31}	b_3	$E(R_1)$
2	λ_2	b_2	λ_{32}	b_3	$E(R_2)$

Table 4.3: Combinations of λ_1, λ_3; b_1, b_3 to obtain $E(RC_1)$ and $E(RC_2)$

For approximating the mean residual cycle time for queue 3, we have 3 methods with following combination of λ_1, λ_3.

$\lambda_1 = (\lambda_1 + \lambda_2)/2$, $b_1 = \frac{\lambda_1 b_1 + \lambda_2 b_2}{\lambda_1 + \lambda_2}$, $\lambda_3 = \lambda_3/2$, $b_3 = b_3$.	$\lambda_1 = \lambda_1 + \lambda_2$, $b_1 = \frac{\lambda_1 b_1 + \lambda_2 b_2}{2(\lambda_1 + \lambda_2)}$, $\lambda_3 = \lambda_3$, $b_3 = b_3/2$.	$\lambda_1 = (\lambda_1 + \lambda_2)/2$, $b_1 = \frac{\lambda_1 b_1 + \lambda_2 b_2}{\lambda_1 + \lambda_2}$, $\lambda_3 = \lambda_3$, $b_3 = b_3/2$.
METHOD 1	METHOD 2	METHOD 3

Table 4.4: Combinations of λ_1, λ_3; b_1, b_3 to obtain $E(RC_3)$

Where $E(R) = r_3/(2-P) + (r_1 + r_2)/(2(2-P))$.

The justification for these methods is same as those for the symmetric case. Substituting the different combinations of λ_1, λ_3 in equation (4.10) we get different values of E(RC). Then once $E(RC_j)$ is computed, it is substituted back into equation (4.3) along with the corresponding $E(C_j)$ to get $E(W_j)$ for each queue j, $j = 1,2,3$.

In addition to these 3 methods, we also have method 4 which is the single sever model with multiple server vacations. In order to approximate $E(W_3)$, we consider a system with single server, Poisson arrivals with rate $\lambda_3/2$ and the exponential service with mean b_3. The service times are independent of the arrival times. The server goes on vacation with mean and second moment given by equations (4.11) and (4.12) respectively with

$$E(R) = \frac{r_1 + r_2}{2} + r_3,$$

and $E(R^2) = 2((\frac{r_1 + r_2}{2})^2 + (\frac{r_1 + r_2}{2})r_3 + r_3^2)$.

E(W) is then computed by substituting in equation (4.13). The justification for using this method remains same as earlier but with different length of vacation period.

Once mean waiting time in queue is known then mean number in queue j, j=1,2,3, $E(L_j)$, is computed using Little's Formula, equation (4.14).

The numerical analysis of the approximation methods for the symmetric system mentioned here is discussed in chapter 5 and those for asymmetric system in chapter 6.

CHAPTER 5

DISCUSSION OF RESULTS FOR SYMMETRIC SYSTEMS

In the previous chapter we proposed a few approximation methods to compute mean waiting time in queue. In this chapter we discuss their numerical analysis and make a few recommendations for the symmetric system.

The approximate values for the mean waiting time in queue are computed using Microsoft EXCEL 97 spread sheet (1997) and are compared with the simulated estimates computed by a program written in Microsoft FORTRAN (version 5.1) with SLAM II (version 4.6 (1986)) subroutines (Appendix A). A sample output of the FORTRAN program with SLAM II subroutines has been attached as Appendix D. The program was run on Compaq Presario 4504 – Pentium 200.

Before using any program, it is very important to test its validity. We tested our program by comparing the simulated values against the mean waiting time in a M/M/1 system with vacations. Having arrival rate of λ and service rate b in the specialized queues and no traffic in the common queue results in two independent M/M/1 systems. The sum of the exponential switch over times is considered as the vacation. The server goes on multiple vacation on finding the queue empty on its return from a vacation. We used the equation (4.10) to compute the mean waiting time in queue for the M/M/1 system with vacations.

The results – exact mean computed using the equation (4.10), simulated mean, standard deviation, 95% confidence interval and the relative error - have been summarized below.

λ	b	ρ	r	Exact mean	Simulated mean	Standard deviation	95% confidence interval	% error
0.5	0.5	0.25	0.0001	0.0835	0.0816	0.3549	(0.0783,0.0850)	-2.276
0.5	1.0	0.5	0.0001	0.5003	0.5198	1.108	(0.4980,0.5416)	3.898
0.5	1.5	0.75	0.0001	2.251	2.218	3.117	(2.157,2.279)	-1.466
0.5	1.8	0.9	0.01	9.108	9.188	9.887	(9.101,9.276)	0.8784
0.5	1.9	0.95	0.01	22.81	22.54	19.29	(22.37,22.71)	-1.184

Table 5.1: Exact mean, simulated mean, standard deviation and 95% confidence interval for the mean, and the relative error for testing the FORTRAN program with SLAM II subroutines.

The percent error is calculated as:

$$\%\,\text{error} = \frac{(\text{simulated mean - exact mean})100}{\text{exact mean}}.$$

Table 5.1 shows that the exact mean does not differ significantly from the simulated one at 5% level of significance, except for $\rho = 0.95$ even though it is being underestimated by only 1.18%. This can be attributed to very slow convergence to steady state in heavily loaded queues. Therefore, the program can be used to test the approximate results.

SLAM II (Pritsker, 1986), Simulation Language for Alternative Modeling, is an advanced FORTAN based simulation language. Instead of writing numerous lines of codes, SLAM II furnishes the network symbols for the composition of graphical models that can be easily converted into input statements for direct computer processing. It contains subprograms that support both discrete event and continuous model developments. A discrete change system can be modeled with in either an event orientation or process orientation or both. SLAM II provides a set of standard subprograms to perform common discrete event functions such as event scheduling, file manipulations, statistics collection, and random sample generation.

The process orientation uses network structure that consists of specialized symbols called nodes and branches, which represent elements in process like queues, servers, and decision points. These symbols are combined into a network model that represents the required system.

In the event orientation, it is necessary to define all the events and the possible changes to the system when an event occurs, which are coded as FORTRAN subroutines. SLAM II itself controls the sequence of the codes to be executed.

A continuous model in SLAM II is coded by specifying the differential or difference equations that describe the dynamic behavior of the state variables. The equations are coded in FORTRAN by using a set of special SLAM II storage arrays. SLAM II automatically integrates to calculate the values of state variables within accuracy provided by the modeler. (Pritsker, 1986).

All of the simulated results through out the thesis are sensible to the number of runs and the run length used. The program ran for almost 4 hours before giving the results for $\rho = 0.25$ and $r = 0.0001$ when it was set for 5000 time units. Simulation run length was chosen to be as large as possible under the time and resource constraints.

OBSERVATION: Table 5.2 shows the simulated mean values as well as the corresponding 95% confidence interval for queue 1, ρ =0.5 and ρ_1 of 0.05 (λ_1 = 0.05, b_1 = 1) and for different values of r. Clearly, as $r \to 0$, i.e., as switch over times become very small, even a ten fold change in r has little effect on the mean waiting time. Therefore, for analysis, we look at a reasonable range of r - min 10% of b.

r	Simulated mean	Standard deviation	95% confidence interval
0.0001	0.0263	0.1714	(0.0155,0.0372)
0.001	0.0274	0.1778	(0.0161,0.0387)
0.01	0.0291	0.1819	(0.0175,0.0406)

Table 5.2: Comparison of the simulated mean for different values of r.

The results - the approximate value of mean waiting time in queue 1 using 6 different methods proposed on page number 58, the simulated mean (SIM), standard deviation (S.D.) and the corresponding 95% confidence interval (LCL, UCL) have been summarized in the table below for the server utilization of 0.25, 0.5 and 0.75.

λ_1	b_1	λ_3	b_3	ρ	Approximate mean using method						SIM	S.D.	LCL	UCL
					1	2	3	4	5	6				
0.45	0.5	0.1	0.5	0.25	0.1422	0.2185	0.1382	0.1487	0.2306	0.1446	0.1300	0.4278	0.1269	0.1332
0.35	0.5	0.3	0.5	0.25	0.1076	0.1619	0.0984	0.1045	0.1869	0.1110	0.0954	0.3539	0.0925	0.0983
0.25	0.5	0.5	0.5	0.25	0.0754	0.1116	0.0647	0.0873	0.1362	0.0765	0.0641	0.2786	0.0614	0.0669
0.15	0.5	0.7	0.5	0.25	0.0448	0.0655	0.0359	0.0592	0.0809	0.0432	0.0352	0.1966	0.0327	0.0377
0.05	0.5	0.9	0.5	0.25	0.0149	0.0217	0.0112	0.0216	0.0258	0.0132	0.0109	0.1065	0.0085	0.0132
0.45	1.0	0.1	1.0	0.5	0.7584	1.240	0.7310	0.8049	1.329	0.7763	0.7311	1.246	0.7220	0.7402
0.35	1.0	0.3	1.0	0.5	0.5527	0.8342	0.4916	0.4427	0.9885	0.5671	0.5159	1.041	0.5073	0.5245
0.25	1.0	0.5	1.0	0.5	0.3750	0.5313	0.3056	0.4236	0.6406	0.3542	0.3122	0.7165	0.3051	0.3192
0.15	1.0	0.7	1.0	0.5	0.2167	0.2925	0.1609	0.3087	0.3212	0.1700	0.1509	0.4461	0.1452	0.1565
0.05	1.0	0.9	1.0	0.5	0.0705	0.0919	0.0476	0.1149	0.0822	0.0418	0.0390	0.2056	0.0345	0.0436
0.35	1.5	0.3	1.5	0.75	3.326	14.27	2.918	1.720	17.63	3.508	3.121	4.053	3.088	3.155
0.25	1.0	0.5	2.0	0.75	2.149	4.396	1.435	2.819	4.234	1.344	1.369	2.230	1.347	1.391
0.15	1.5	0.7	1.5	0.75	1.008	1.529	0.7200	1.929	1.412	0.6365	0.4801	0.9076	0.4686	0.4916

Table 5.3: Comparison of different methods for queue 1

Below are the graphs depicting the approximate and simulated mean waiting times in queue 1 for different values of ρ along with 95% confidence interval.

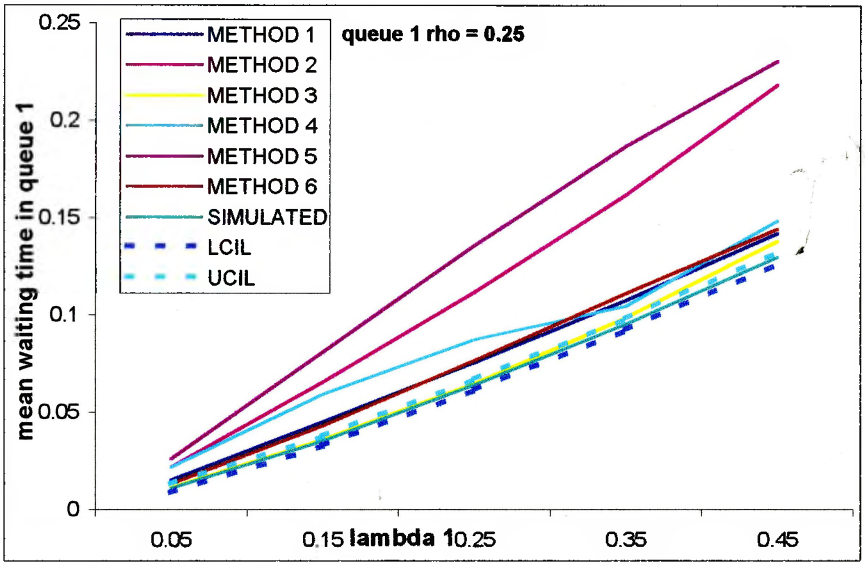

Figure 5.1: mean waiting time in queue 1 for ρ= 0.25.

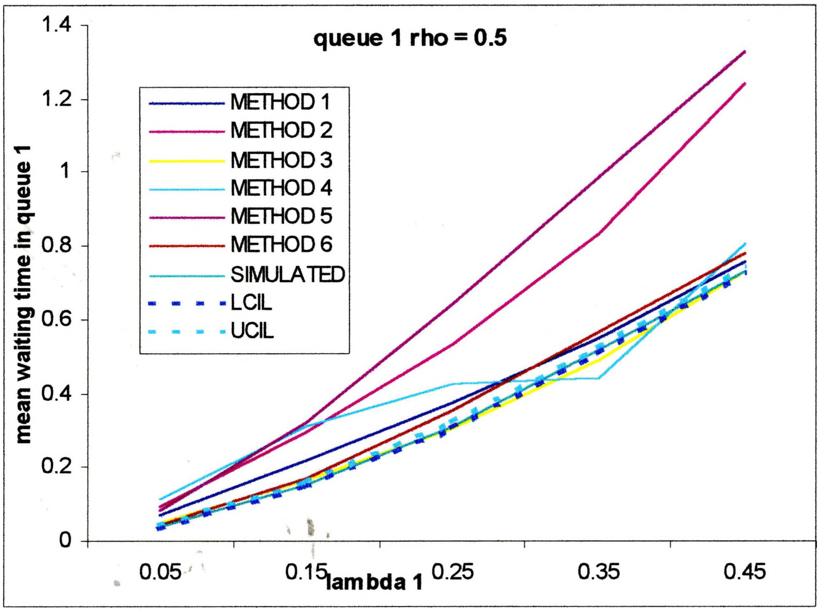

Figure 5.2: mean waiting time in queue 1 for ρ= 0.5.

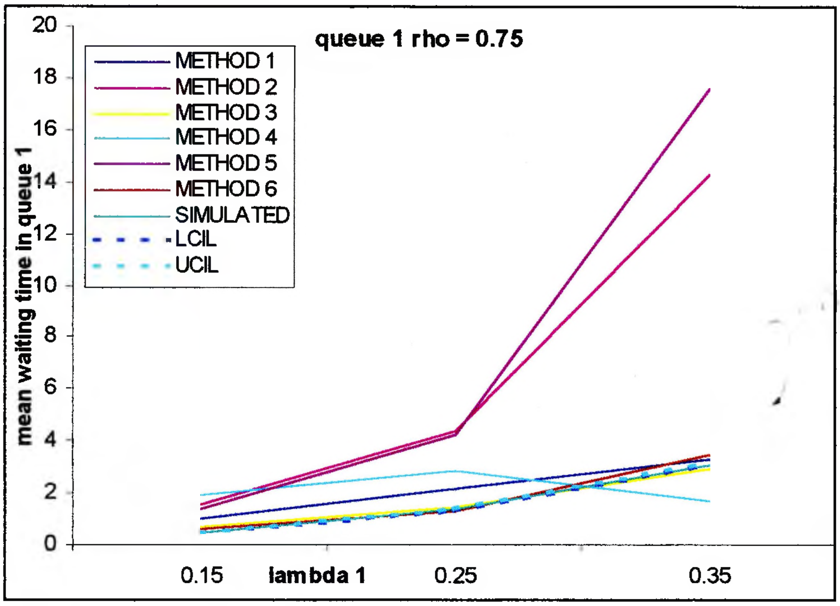

Figure 5.3: mean waiting time in queue 1 for ρ= 0.75.

Table 5.4 gives the percent error of all the 6 methods discussed in chapter 4 for different values of ρ. The Percent error is computed as:

$$\%\text{error} = \frac{(\text{approximate value} - \text{simulated value})100}{\text{simulated value}}$$

λ_1	b_1	λ_3	b_3	ρ	% Error between the simulated mean and approximate mean using method					
					1	2	3	4	5	6
0.45	0.5	0.1	0.5	0.25	9.336	68.04	6.268	14.32	77.32	11.19
0.35	0.5	0.3	0.5	0.25	12.75	69.66	3.126	9.554	95.84	16.33
0.25	0.5	0.5	0.5	0.25	17.59	73.99	0.793	36.07	112.3	19.27
0.15	0.5	0.7	0.5	0.25	27.29	85.97	1.947	68.08	129.6	22.47
0.05	0.5	0.9	0.5	0.25	37.73	99.52	2.953	98.88	138.1	21.37
0.45	1	0.1	1	0.5	3.737	69.65	-0.0160	10.10	81.73	6.181
0.35	1	0.3	1	0.5	7.127	61.69	-4.706	-14.19	91.61	9.916
0.25	1	0.5	1	0.5	20.13	70.19	-2.116	35.70	105.2	13.46
0.15	1	0.7	1	0.5	43.65	93.89	6.623	104.6	112.9	12.71
0.05	1	0.9	1	0.5	80.66	135.4	21.83	194.4	110.5	7.057
0.35	1.5	0.3	1.5	0.75	6.540	357.1	-6.509	-44.89	464.9	12.37
0.25	1	0.5	2	0.75	56.93	221.0	4.767	105.9	209.2	-1.862
0.15	1.5	0.7	1.5	0.75	110.0	218.4	49.98	301.8	194.0	32.58

Table 5.4: Relative error between the simulated mean and different methods for queue 1

Below are the graphs for the relative error corresponding to Table 5.4.

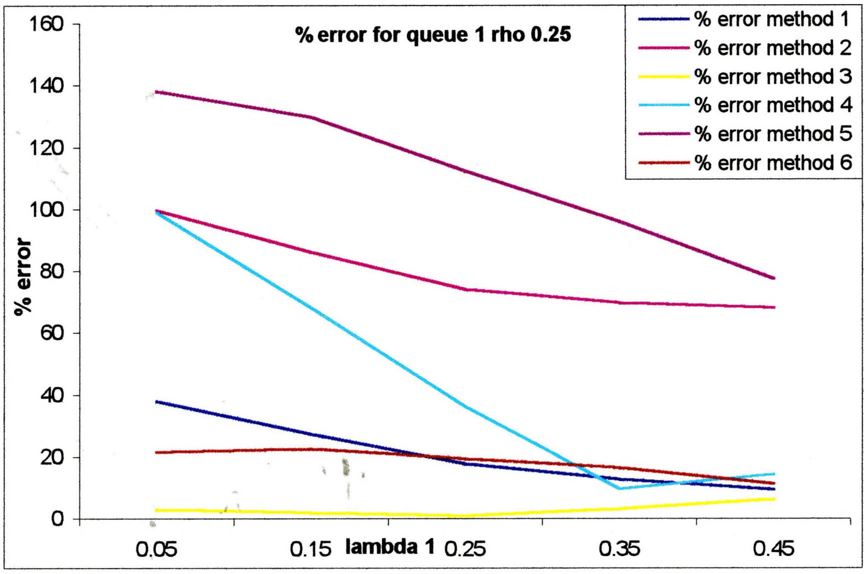

Figure 5.4: Relative error for queue 1, ρ = 0.25

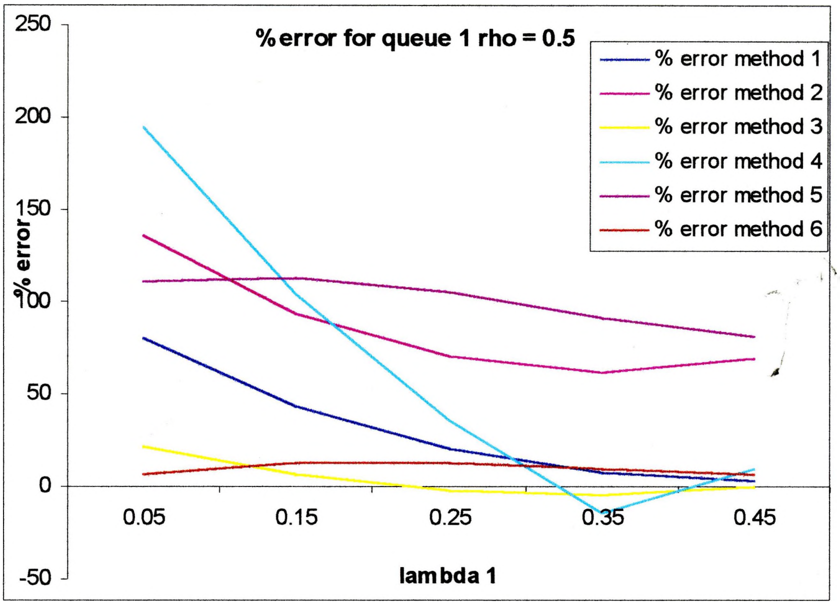

Figure 5.5: Relative error for queue 1, ρ = 0.5

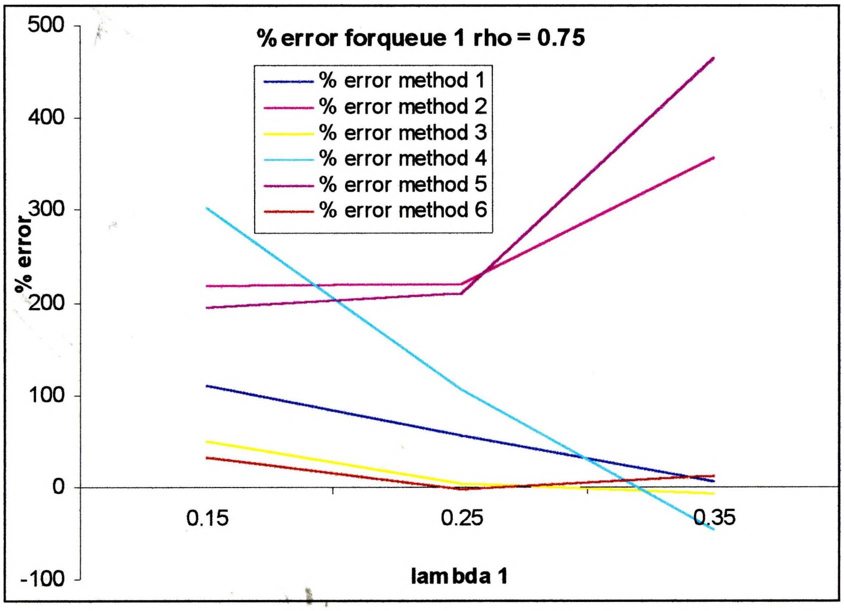

Figure 5.6: Relative error for queue 1, ρ = 0.75

Most of the methods overestimate the mean waiting time in the queue 1. But, the percent error of methods 2 and 5 is very large. Out of methods 1, 3, 4 and 6, clearly method 3 gives the least relative error. Method 3 gives approximations reasonably well for the mean except when ρ is moderate or large and the arrival rate in the other queue is high. When ρ is not moderate or large and the arrival rate in queue 1 is not very small, the relative error is at the most 7% for method 3. Therefore, we use this method for further analysis of queue 1.

Table 5.5 summarizes the results - the approximate value of mean waiting time in queue 3 using 7 different methods of approximations proposed on pages 58 and 59, average, (average of the values computed using method 1 and single queue method), the simulated mean (SIM), standard deviation (S.D.) and the corresponding 95% confidence interval (LCL,UCL) for ρ of 0.25, 0.5 and 0.75.

λ_1	b_1	λ_3	b_3	ρ	Approximate mean using method								SIM	S.D.	LCL	UCL
					Single q	1	2	3	4	5	6	average				
0.45	0.5	0.1	0.5	0.25	0.0013	0.0123	0.0164	0.0238	0.0100	0.0132	0.0194	0.0068	0.0075	0.0873	0.0055	0.0094
0.35	0.5	0.3	0.5	0.25	0.0106	0.0364	0.0483	0.0657	0.0354	0.0480	0.0645	0.0235	0.0245	0.1631	0.0225	0.0266
0.25	0.5	0.5	0.5	0.25	0.0288	0.0607	0.0804	0.1016	0.0674	0.0941	0.1157	0.0447	0.0448	0.2273	0.0426	0.0471
0.15	0.5	0.7	0.5	0.25	0.0554	0.0857	0.1144	0.1326	0.1039	0.1499	0.1690	0.0705	0.0685	0.2947	0.0661	0.0710
0.05	0.5	0.9	0.5	0.25	0.0900	0.1122	0.1520	0.1600	0.1420	0.2123	0.2206	0.1011	0.0921	0.3502	0.0896	0.0947
0.45	1	0.1	1	0.5	0.0131	0.0671	0.0873	0.1297	0.0389	0.0478	0.0758	0.0401	0.0300	0.1815	0.0257	0.0343
0.35	1	0.3	1	0.5	0.0872	0.1976	0.2522	0.3569	0.1667	0.2137	0.3076	0.1424	0.1189	0.3965	0.1139	0.1239
0.25	1	0.5	1	0.5	0.2245	0.3304	0.4231	0.5562	0.3616	0.4904	0.6346	0.2774	0.2674	0.6883	0.2606	0.2742
0.15	1	0.7	1	0.5	0.4172	0.4739	0.6230	0.7443	0.5939	0.8612	1.000	0.4456	0.4525	1.047	0.4439	0.4612
0.05	1	0.9	1	0.5	0.6487	0.6377	0.8812	0.9388	0.8203	1.282	1.343	0.6432	0.6242	1.269	0.6150	0.6335
0.35	1.5	0.3	1.5	0.75	1.154	1.027	1.801	2.009	0.6744	0.9642	1.424	1.091	0.4201	0.8473	0.4094	0.4309
0.25	1	0.5	2	0.75	3.607	1.912	3.186	3.372	2.447	4.450	4.990	2.759	1.811	2.687	1.784	1.837
0.15	1.5	0.7	1.5	0.75	8.985	2.673	5.417	6.323	3.682	9.475	11.10	5.829	3.506	4.870	3.466	3.547

Table 5.5: Comparison of different methods for queue 3

Following are the graphs corresponding to the table 5.5.

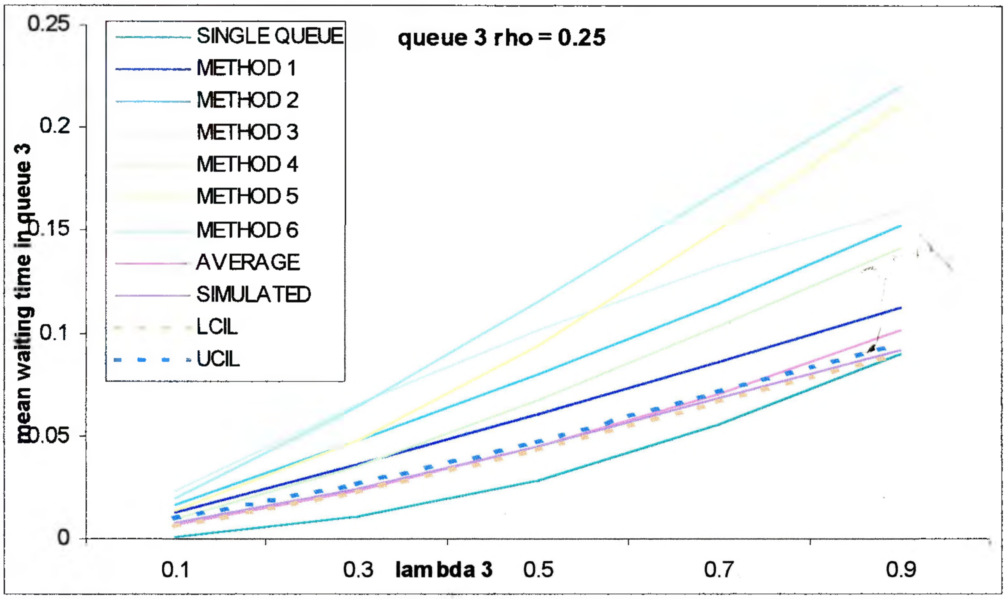

Figure 5.7: mean waiting time in queue 3 for ρ= 0.25.

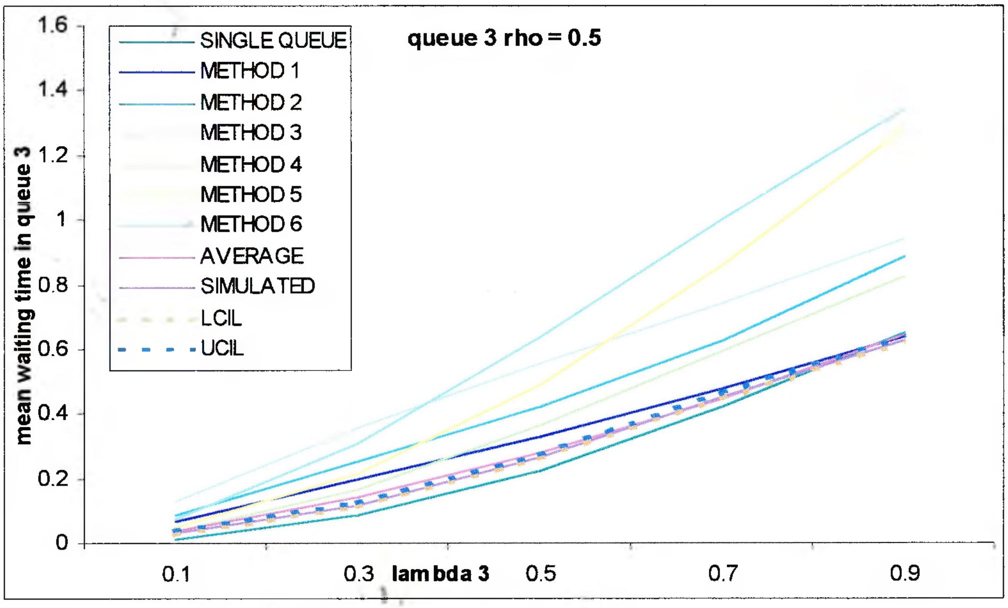

Figure 5.8: mean waiting time in queue 3 for ρ= 0.5.

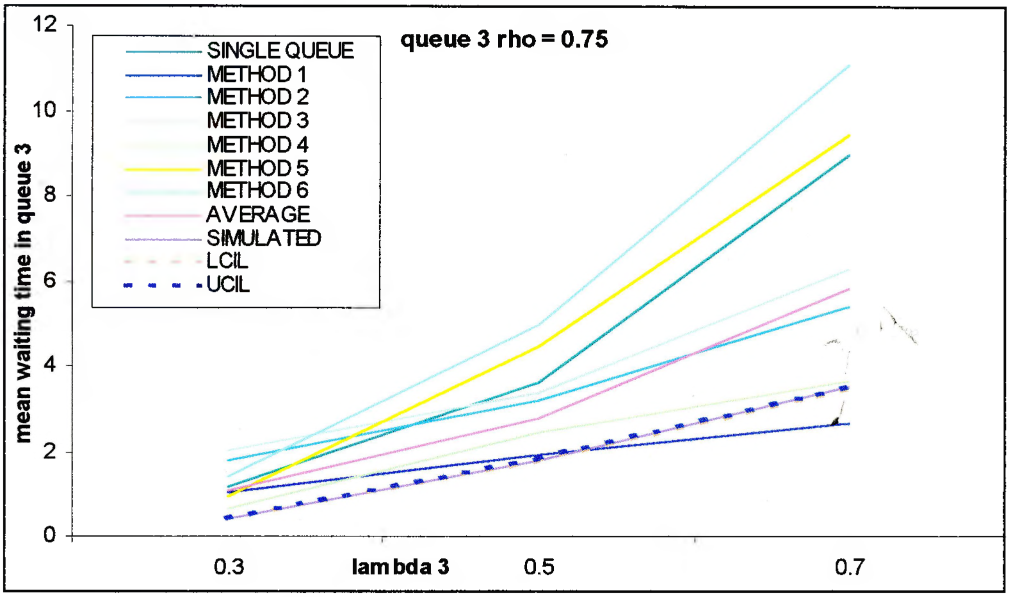

Figure 5.9: mean waiting time in queue 3 for ρ= 0.75.

Table 5.6 gives the percent error of the different methods for computing the mean waiting time in queue 3.

λ_1	b_1	λ_3	b_3	ρ	% Error between the simulated mean and approximate mean using method							
					Single q	1	2	3	4	5	6	average
0.45	0.5	0.1	0.5	0.25	-82.44	64.50	119.4	218.3	33.86	77.43	159.6	-8.966
0.35	0.5	0.3	0.5	0.25	-56.62	48.50	96.78	167.9	44.12	95.50	162.8	-4.059
0.25	0.5	0.5	0.5	0.25	-35.83	35.38	79.48	126.7	50.27	109.9	158.2	-0.2253
0.15	0.5	0.7	0.5	0.25	-19.24	25.01	66.95	93.44	51.55	118.7	146.6	2.881
0.05	0.5	0.9	0.5	0.25	-2.303	21.77	64.96	73.64	54.17	130.5	139.4	9.733
0.45	1.0	0.1	1.0	0.5	-56.26	123.4	191.0	332.3	29.61	59.32	152.4	33.58
0.35	1.0	0.3	1.0	0.5	-26.69	66.17	112.1	200.2	40.20	79.77	158.7	19.74
0.25	1.0	0.5	1.0	0.5	-16.05	23.54	58.21	108.0	35.22	83.38	137.3	3.742
0.15	1.0	0.7	1.0	0.5	-7.808	4.717	37.66	64.47	31.24	90.30	121.0	-1.546
0.05	1.0	0.9	1.0	0.5	3.917	2.155	41.15	50.39	31.41	105.3	115.2	3.036
0.35	1.5	0.3	1.5	0.75	174.8	144.5	328.6	378.3	60.52	129.5	239.0	159.6
0.25	1.0	0.5	2.0	0.75	99.17	5.61	75.94	86.22	35.16	145.7	175.6	52.39
0.15	1.5	0.7	1.5	0.75	156.3	-23.75	54.49	80.34	5.012	170.2	216.7	66.25

Table 5.6: Relative Error between the simulated mean and different methods for queue 3.

Figure 5.10 through Figure 5.12 give the relative error for different methods and different ρ corresponding to Table 5.6.

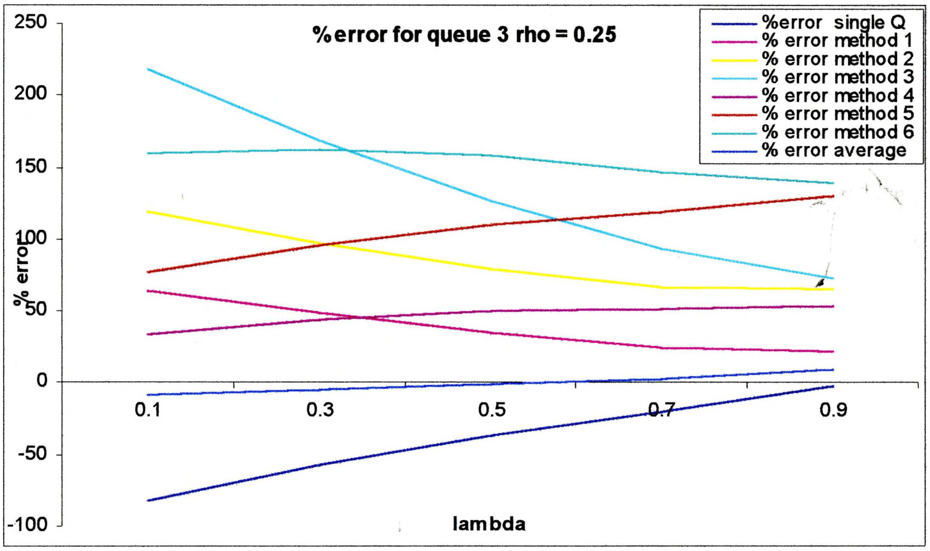

Figure 5.10: Relative error for queue 3, $\rho = 0.25$

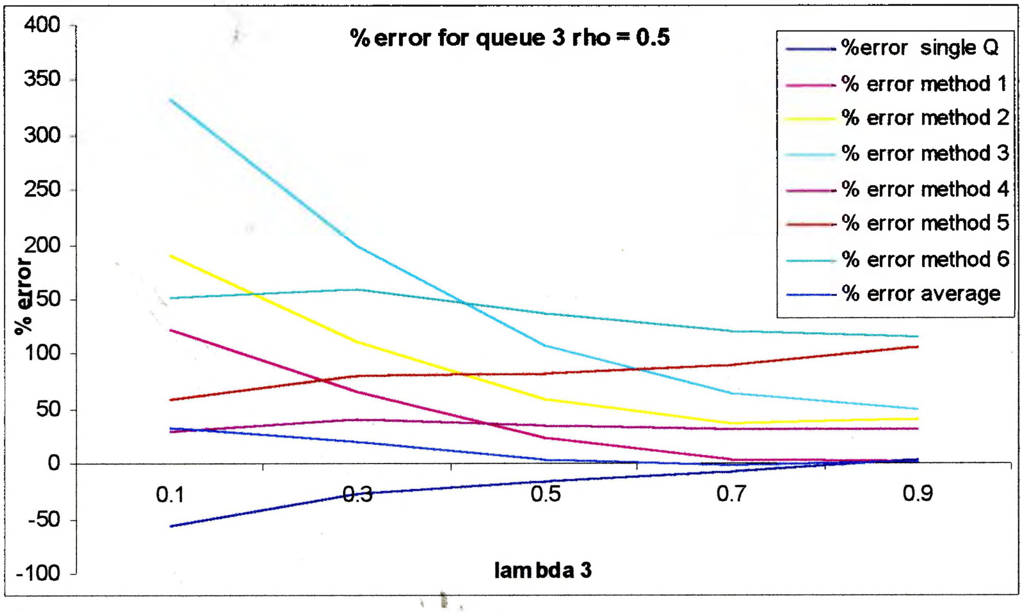

Figure 5.11: Relative error for queue 3, $\rho = 0.5$

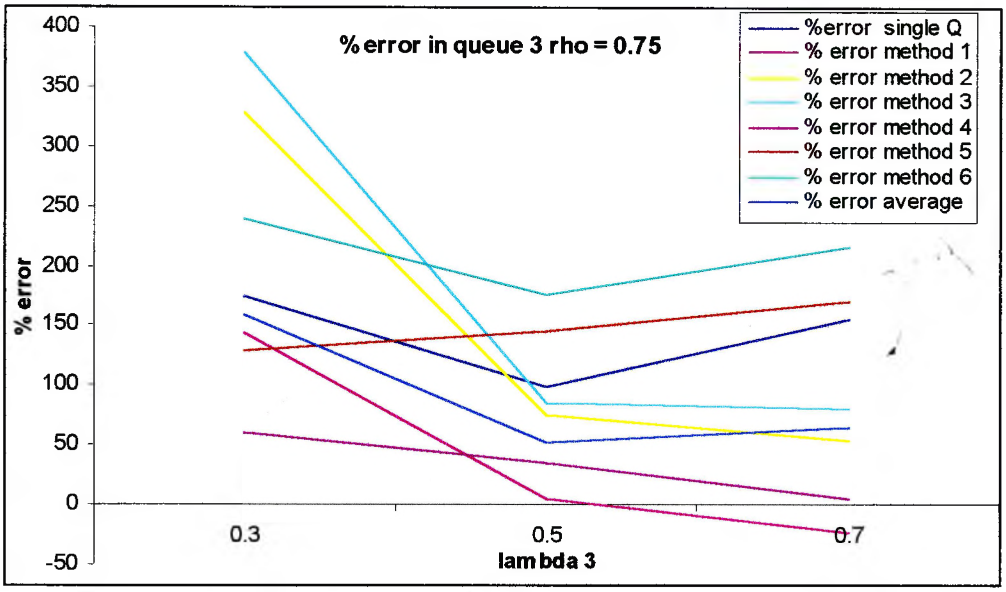

Figure 5.12: Relative error for queue 3, $\rho = 0.75$

All methods except the single queue with multiple vacations method over estimate the mean waiting time in the queue 3. But, the percent error of methods 3 and 6 is very large. In methods 3 and 6, we assume the arrival rates as λ and λ_3 and the server to be twice as fast in the second queue. Out of the rest of the methods, it is evident that taking average of the single queue method and method 1 gives the least relative error. This method (average of the two) gives reasonably good approximation for the mean except when ρ is moderate or large and the arrival rate in the queue 3 is high. When ρ is not moderate or large and the arrival rate in queue 3 is not high, the relative error is at the most 10% in case of the proposed average method. Therefore, we use this method for further analysis of queue 3.

Table 5.7 gives the results for queue 1 as the switch over time and ρ is varied. The last column shows whether the approximated mean falls in the 95% confidence interval or not.

λ_1	b_1	λ_3	b_3	r as % of b	ρ	APPROX	SIM	S.D	LCL	UCL	% error
0.45	1	0.1	1	1	0.5	0.4528	0.4762	1.0475	0.4544	0.4979	-4.897
0.35	1	0.3	1	1	0.5	0.3144	0.3314	0.7926	0.3128	0.3500	-5.126
0.25	1	0.5	1	1	0.5	0.1982	0.2266	0.6281	0.2092	0.2440	-12.53
0.15	1	0.7	1	1	0.5	0.1035	0.1080	0.3715	0.0947	0.1213	-4.182
0.05	1	0.9	1	1	0.5	0.0295	0.0291	0.1819	0.0175	0.0406	1.370
0.45	1	0.1	3	1	0.6	0.8393	0.9945	1.931	0.9746	1.015	-15.61
0.25	1.2	0.5	1.2	1	0.6	0.3556	0.3745	0.8566	0.3626	0.3864	-5.054
0.05	3	0.9	1	1	0.6	0.0895	0.0592	0.2694	0.0507	0.0677	51.11
0.45	1	0.1	5	1	0.7	1.862	2.454	4.071	2.411	2.496	-24.13
0.05	5	0.9	1	1	0.7	0.2575	0.1365	0.4436	0.1226	0.1503	88.6
0.45	1	0.1	6	1	0.75	2.908	3.932	5.869	3.872	3.993	-26.06
0.35	1.5	0.3	1.5	1	0.75	1.434	1.497	2.330	1.469	1.524	-4.161
0.25	1	0.5	2	1	0.75	0.7994	0.8521	1.4598	0.8318	0.8724	-6.184
0.15	1.5	0.7	1.5	1	0.75	0.4622	0.3401	0.7716	0.3262	0.3539	35.93
0.05	6	0.9	1	1	0.75	0.4233	0.1720	0.5152	0.1558	0.1883	146.1
0.45	1.5	0.1	4.5	1	0.9	9.674	9.841	11.44	9.722	9.959	-1.692
0.35	1.5	0.3	2.5	1	0.9	5.240	6.257	7.944	6.163	6.350	-16.24
0.25	1.5	0.5	2.1	1	0.9	3.137	2.758	3.497	2.709	2.807	13.74
0.15	2.5	0.7	1.5	1	0.9	2.174	0.8223	1.407	0.7970	0.8477	164.4
0.05	4.5	0.9	1.5	1	0.9	0.8090	0.1749	0.486	0.16	0.19	362.6
0.45	0.5	0.1	0.5	10	0.25	0.1382	0.1300	0.4278	0.1269	0.1332	6.268
0.35	0.5	0.3	0.5	10	0.25	0.0984	0.0954	0.3539	0.0925	0.0983	3.126
0.25	0.5	0.5	0.5	10	0.25	0.0647	0.0641	0.2786	0.0614	0.0669	0.7928
0.15	0.5	0.7	0.5	10	0.25	0.0359	0.0352	0.1966	0.0327	0.0377	1.947
0.05	0.5	0.9	0.5	10	0.25	0.0112	0.0109	0.1065	0.0085	0.0132	2.953
0.45	0.5	0.1	1.5	10	0.3	0.2766	0.2743	0.6770	0.2673	0.2813	0.8599
0.35	0.6	0.3	0.6	10	0.3	0.1412	0.1383	0.4434	0.1331	0.1435	2.102

0.25	0.6	0.5	0.6	10	0.3	0.0918	0.0927	0.3484	0.0879	0.0976	-1.004
0.15	0.6	0.7	0.6	10	0.3	0.0505	0.0496	0.2345	0.0453	0.0538	1.832
0.05	1.5	0.9	0.5	10	0.3	0.0272	0.0218	0.1510	0.0171	0.0266	24.48
0.45	0.5	0.1	3.5	10	0.4	0.9074	1.038	1.980	1.018	1.059	-12.60
0.35	0.5	0.3	1.5	10	0.4	0.2982	0.3272	0.7693	0.3182	0.3363	-8.868
0.25	0.8	0.5	0.8	10	0.4	0.1714	0.1759	0.5068	0.1689	0.1830	-2.553
0.15	1.5	0.7	0.5	10	0.4	0.1517	0.1360	0.4362	0.1281	0.1438	11.60
0.05	3.5	0.9	0.5	10	0.4	0.0946	0.0667	0.2860	0.0577	0.0757	41.78
0.45	1	0.1	1	10	0.5	0.7310	0.7311	1.246	0.7220	0.7402	-0.0160
0.35	1	0.3	1	10	0.5	0.4916	0.5159	1.041	0.5073	0.5245	-4.706
0.25	1	0.5	1	10	0.5	0.3056	0.3122	0.7165	0.3051	0.3192	-2.116
0.15	1	0.7	1	10	0.5	0.1609	0.1509	0.4461	0.1452	0.1565	6.623
0.05	1	0.9	1	10	0.5	0.0476	0.0390	0.2056	0.0345	0.0436	21.83
0.45	1	0.1	3	10	0.6	2.148	2.335	3.230	2.302	2.369	-8.037
0.35	0.3	0.3	3.3	10	0.6	1.567	2.089	3.314	2.050	2.128	-24.97
0.25	1.2	0.5	1.2	10	0.6	0.5471	0.5437	1.029	0.5294	0.5580	0.613
0.15	3.3	0.7	0.3	10	0.6	0.8792	0.7702	1.437	0.7444	0.7960	14.15
0.05	3	0.9	1	10	0.6	0.1382	0.0830	0.3162	0.0731	0.0930	66.42
0.45	1	0.1	5	10	0.7	24.28	22.43	21.87	22.20	22.65	8.252
0.35	0.5	0.3	3.5	10	0.7	3.035	4.495	6.223	4.422	4.568	-32.48
0.25	1.4	0.5	1.4	10	0.7	1.050	1.045	1.676	1.022	1.068	0.4790
0.35	1.5	0.3	1.5	10	0.75	2.918	3.121	4.053	3.088	3.155	-6.509
0.25	1	0.5	2	10	0.75	1.435	1.369	2.230	1.347	1.391	4.767
0.15	1.5	0.7	1.5	10	0.75	0.7200	0.4801	0.9076	0.4686	0.4916	49.98

Table 5.7: Approximate and simulated average mean waiting time; standard deviation of the mean; 95% confidence interval of the mean; and relative error for queue 1.

Table 5.8 gives the results for queue 3 as the switch over time and ρ is varied.

λ_1	b_1	λ_3	b_3	r as % of b	ρ	APPROX	SIM	S.D	LCL	UCL	% error
0.45	1	0.1	1	1	0.5	0.0285	0.0181	0.1439	0.0089	0.0273	57.54
0.35	1	0.3	1	1	0.5	0.1011	0.0716	0.3241	0.0598	0.0835	41.13
0.25	1	0.5	1	1	0.5	0.1948	0.1491	0.5168	0.1346	0.1636	30.68
0.15	1	0.7	1	1	0.5	0.3096	0.2477	0.7271	0.2305	0.2648	25.00
0.05	1	0.9	1	1	0.5	0.4450	0.3240	0.8842	0.3057	0.3424	37.34
0.45	1	0.1	3	1	0.6	0.0883	0.0369	0.2087	0.0303	0.0434	139.4
0.25	1.2	0.5	1.2	1	0.6	0.3535	0.3087	0.8260	0.2972	0.3202	14.54
0.05	3	0.9	1	1	0.6	0.8345	0.7891	1.748	0.7710	0.8072	5.759
0.45	1	0.1	5	1	0.7	0.2602	0.0815	0.3351	0.0710	0.0921	219.1
0.05	5	0.9	1	1	0.7	1.9456	2.314	4.111	2.271	2.356	-15.91
0.45	1	0.1	6	1	0.75	0.4343	0.1231	0.4405	0.1092	0.1370	252.8
0.35	1.5	0.3	1.5	1	0.75	0.4702	0.2825	0.7303	0.2694	0.2956	66.45
0.25	1	0.5	2	1	0.75	1.2031	1.038	1.870	1.0120	1.064	15.91
0.15	1.5	0.7	1.5	1	0.75	1.4605	1.586	2.709	1.5538	1.618	-7.896
0.05	6	0.9	1	1	0.75	3.2089	3.894	6.413	3.8278	3.961	-17.60
0.45	1.5	0.1	4.5	1	0.9	0.9703	0.1251	0.3970	0.1126	0.1376	675.4
0.35	1.5	0.3	2.5	1	0.9	2.727	0.7266	1.334	0.7027	0.7506	275.3
0.25	1.5	0.5	2.1	1	0.9	5.023	3.682	4.995	3.613	3.752	36.41
0.15	2.5	0.7	1.5	1	0.9	8.801	9.370	12.92	9.218	9.522	-6.071
0.05	4.5	0.9	1.5	1	0.9	32.11	13.01	13.66	12.87	13.15	146.8
0.45	0.5	0.1	0.5	10	0.25	0.0068	0.0075	0.0873	0.0055	0.0094	-8.966
0.35	0.5	0.3	0.5	10	0.25	0.0235	0.0245	0.1631	0.0225	0.0266	-4.059
0.25	0.5	0.5	0.5	10	0.25	0.0447	0.0448	0.2273	0.0426	0.0471	-0.225
0.15	0.5	0.7	0.5	10	0.25	0.0705	0.0685	0.2947	0.0661	0.0710	2.881
0.05	0.5	0.9	0.5	10	0.25	0.1011	0.0921	0.3502	0.0896	0.0947	9.733
0.45	0.5	0.1	1.5	10	0.3	0.0183	0.0163	0.1311	0.0122	0.0205	11.98
0.35	0.6	0.3	0.6	10	0.3	0.0356	0.0354	0.1989	0.0318	0.0390	0.4199
0.25	0.6	0.5	0.6	10	0.3	0.0679	0.0672	0.2909	0.0632	0.0713	1.011
0.15	0.6	0.7	0.6	10	0.3	0.1076	0.1030	0.3775	0.0986	0.1074	4.445
0.05	1.5	0.9	0.5	10	0.3	0.1952	0.2192	0.5988	0.2130	0.2254	-10.94

0.45	0.5	0.1	3.5	10	0.4	0.0800	0.0471	0.2323	0.0398	0.0544	69.76
0.35	0.5	0.3	1.5	10	0.4	0.1270	0.1002	0.3838	0.0933	0.1071	26.77
0.25	0.8	0.5	0.8	10	0.4	0.1414	0.1377	0.4463	0.1315	0.1439	2.638
0.15	1.5	0.7	0.5	10	0.4	0.2442	0.2668	0.7192	0.2584	0.2753	-8.477
0.05	3.5	0.9	0.5	10	0.4	0.8124	0.8428	1.674	0.8255	0.8601	-3.611
0.45	1	0.1	1	10	0.5	0.0401	0.0300	0.1815	0.0257	0.0343	33.58
0.35	1	0.3	1	10	0.5	0.1424	0.1189	0.3965	0.1139	0.1239	19.74
0.25	1	0.5	1	10	0.5	0.2774	0.2674	0.6883	0.2606	0.2742	3.742
0.15	1	0.7	1	10	0.5	0.4456	0.4525	1.047	0.4439	0.4612	-1.546
0.05	1	0.9	1	10	0.5	0.6432	0.6242	1.269	0.6150	0.6335	3.036
0.45	1	0.1	3	10	0.6	0.1464	0.0691	0.2805	0.0602	0.0779	112.0
0.35	0.3	0.3	3.3	10	0.6	0.8398	0.6096	1.323	0.5858	0.6334	37.76
0.25	1.2	0.5	1.2	10	0.6	0.5568	0.5141	1.064	0.4993	0.5289	8.313
0.15	3.3	0.7	0.3	10	0.6	2.3460	2.248	4.342	2.197	2.299	4.348
0.05	3	0.9	1	10	0.6	2.4521	2.273	3.2600	2.2388	2.306	7.902
0.45	1	0.1	5	10	0.7	0.5403	0.1353	0.4204	0.1220	0.1485	299.5
0.35	0.5	0.3	3.5	10	0.7	1.5105	0.9049	1.690	0.8745	0.9352	66.93
0.25	1.4	0.5	1.4	10	0.7	1.3538	1.009	1.710	0.9856	1.033	34.13
0.35	1.5	0.3	1.5	10	0.75	1.0908	0.4201	0.8473	0.4094	0.4309	159.6
0.25	1	0.5	2	10	0.75	2.7594	1.811	2.687	1.784	1.837	52.39
0.15	1.5	0.7	1.5	10	0.75	5.8292	3.506	4.870	3.466	3.547	66.25

Table 5.8: Approximate and simulated average mean waiting time, standard deviation of the mean, 95% confidence interval of the mean and relative error for queue 3.

OBSERVATIONS: For small switch over time, large ρ and low arrival rate in the queue, the proposed approximation methods do not give good results. The reason for the high relative error in these cases is high degree of asymmetry between the two queues. The high arrival rate in a queue makes it nearly unstable and hence the approximation method breaks down. The approximation method, otherwise, works very well, giving relative error on an average of 3.02% for $\rho = 0.25$, 5.66% for $\rho = 0.3$, 5.87% for $\rho = 0.4$, 4.32% for $\rho = 0.5$, 9.64% for $\rho = 0.6$, 7.92% for $\rho = 0.7$, and 16% for $\rho = 0.75$ for queue

1 and 0.13% for $\rho = 0.25$, 1.38% for $\rho = 0.3$, 17.42% for $\rho = 0.4$, 11.7% for $\rho = 0.5$, 8.3% for $\rho = 0.6$, 34% for $\rho = 0.7$ and 66.3% for $\rho = 0.75$ for queue 3. As compared to the approximations proposed by Boxma and Meister (1987), our approximations perform quite good for asymmetric system considering the degree of complexity involved in our model as compared to theirs.

RECOMMENDATIONS

We conclude by proposing method 3 for estimating mean in queue 1, average of single queue model with multiple server vacations and method 1 for queue 3 in case of symmetric system. These methods can be used when either the switch over times are not too small, or ρ is not too large, or the arrival rate in the other queue is not too high, leading to asymmetry. Our approximations perform quite well for symmetric system keeping in mind the degree of complexity involved in our model as compared to Boxma and Meister's single server polling model. Although not included in this study, our methods can be used to determine the optimal allocation of load to the three queues. This will provide an efficient method to perform analysis of design of service systems.

In the next chapter, we report a numerical analysis for the asymmetric system.

CHAPTER 6

DISCUSSION OF RESULTS FOR ASYMMETRIC SYSTEMS

In chapter 5 we analyzed the symmetric system where the two specialized queues are identical. We propose method 3 for estimating mean waiting time in queue 1, and average of method 1 and single queue model for queue 3. In this chapter we consider a system where the two specialized queues are not identical but still have the same load. We conclude the chapter by making the recommendations and suggest topics for further research.

We analyze the system by varying ρ from 0.25 to 0.6 and considering r as 10% of b. We do not consider ρ higher than 0.6 since for higher ρ, r as 10% of b is not feasible (it violates the stability conditions) and, the results for ρ of 0.6 are not so promising.

Table 6.1 summarizes the results – approximate mean waiting time in queue1 using the equation (4.10) and the combination of λ_1, b_1, λ_3 and b_3 as given in table 4.2 on page 58 (APPROX), the simulated mean waiting time in queue1 (SIM), the standard deviation of the mean (S.D.) and the 95% confidence interval (LCL,UCL) of the mean and the relative error (% error) for queue 1 over a range of ρ from 0.25 to 0.6.

λ_1	b_1	λ_2	b_2	λ_3	b_3	ρ	APPROX	SIM	S.D.	LCL	UCL	% error
0.05	2.5	0.45	0.2778	0.5	0.5	0.25	0.0146	0.0458	0.2445	0.0310	0.0606	-68.04
0.15	0.8333	0.35	0.3571	0.5	0.5	0.25	0.0575	0.0486	0.2386	0.0401	0.0571	18.31
0.25	0.5	0.25	0.5	0.5	0.5	0.25	0.0775	0.0664	0.2896	0.0607	0.0721	16.68
0.35	0.3571	0.15	0.8333	0.5	0.5	0.25	0.0970	0.0801	0.3108	0.0728	0.0874	21.08
0.45	0.2778	0.05	2.5	0.5	0.5	0.25	0.1169	0.1117	0.3886	0.1037	0.1197	4.640
0.15	1.0	0.35	0.4286	0.5	0.6	0.3	0.0808	0.0688	0.2908	0.0584	0.0792	17.34
0.25	0.6	0.25	0.6	0.5	0.6	0.3	0.1128	0.0966	0.3582	0.0895	0.1036	16.83
0.35	0.4286	0.15	1.0	0.5	0.6	0.3	0.1429	0.1213	0.3986	0.1119	0.1306	17.81
0.45	0.3333	0.05	3.0	0.5	0.6	0.3	0.1732	0.1676	0.5079	0.1571	0.1781	3.314
0.15	1.333	0.35	0.5714	0.5	0.8	0.4	0.1413	0.1324	0.4258	0.1172	0.1476	6.722
0.25	0.8	0.25	0.8	0.5	0.8	0.4	0.2222	0.1813	0.5208	0.1711	0.1915	22.54
0.35	0.5714	0.15	1.333	0.5	0.8	0.4	0.2907	0.2443	0.5977	0.2303	0.2583	19.02
0.45	0.4444	0.05	4.0	0.5	0.8	0.4	0.3584	0.3316	0.7453	0.3162	0.3470	8.081
0.05	5.0	0.45	0.5556	0.5	1.0	0.5	0.1947	0.6600	1.222	0.6348	0.6853	-6.92
0.15	1.667	0.35	0.7143	0.5	1.0	0.5	0.4229	0.3520	0.7943	0.3388	0.3652	27.42
0.25	1.0	0.25	1.0	0.5	1.0	0.5	0.5862	0.3319	0.7537	0.3171	0.3467	66.52
0.35	0.7143	0.15	1.667	0.5	1.0	0.5	0.7428	0.2092	0.5476	0.1932	0.2251	12.54
0.25	1.2	0.25	1.2	0.5	1.2	0.6	0.8371	0.5733	1.077	0.5522	0.5943	46.02
0.35	0.8571	0.15	2.0	0.5	1.2	0.6	1.313	0.8445	1.428	0.8171	0.8718	55.44
0.45	0.6667	0.05	6.0	0.5	1.2	0.6	1.749	1.563	2.307	1.515	1.610	11.93

Table 6.1: Approximate and simulated average mean waiting time; standard deviation and 95% confidence interval of the mean; and relative error for queue 1.

Observations: The relative error decreases with the increase in the arrival rate in the queue 1, when the arrival rate in the two specialized queues is made approximately equal and for small ρ - the average percent error being approximately –2% for $\rho = 0.25$, 14% for $\rho = 0.3$, 14% for $\rho = 0.4$, 25% for $\rho = 0.5$, and 38% for $\rho = 0.6$. However, our results are not bad as compared with the results of Boxma and Meister (1987) keeping in mind the degree of complexity involved in our model as compared to theirs.

Table 6.2 summarizes the results for queue 2 over a range of ρ from 0.25 to 0.6.

λ_1	b_1	λ_2	b_2	λ_3	b_3	ρ	APPROX	SIM	S.D.	LCL	UCL	% error
0.45	0.2778	0.05	2.5	0.5	0.5	0.25	0.0146	0.0412	0.2089	0.0285	0.0539	-64.44
0.35	0.3571	0.15	0.8333	0.5	0.5	0.25	0.0575	0.0553	0.2631	0.0460	0.0646	3.966
0.25	0.5	0.25	0.5	0.5	0.5	0.25	0.0775	0.0664	0.2896	0.0607	0.0721	16.67
0.15	0.8333	0.35	0.3571	0.5	0.5	0.25	0.0970	0.0859	0.3229	0.0784	0.0935	12.86
0.05	2.5	0.45	0.2778	0.5	0.5	0.25	0.1169	0.1149	0.3899	0.1069	0.1230	1.69
0.35	0.4286	0.15	1.0	0.5	0.6	0.3	0.0808	0.0813	0.3266	0.0697	0.0929	-0.6083
0.25	0.6	0.25	0.6	0.5	0.6	0.3	0.1128	0.0966	0.3582	0.0895	0.1036	16.83
0.15	1.0	0.35	0.4286	0.5	0.6	0.3	0.1429	0.1274	0.4094	0.1178	0.1369	12.17
0.05	3.0	0.45	0.3333	0.5	0.6	0.3	0.1732	0.1735	0.5007	0.1631	0.1838	-0.1875
0.35	0.5714	0.15	1.3333	0.5	0.8	0.4	0.1413	0.1466	0.4618	0.1303	0.1630	-3.641
0.25	0.80	0.25	0.8	0.5	0.8	0.4	0.2222	0.1813	0.5208	0.1711	0.1915	22.54
0.15	1.3333	0.35	0.5714	0.5	0.8	0.4	0.2907	0.2516	0.6312	0.2368	0.2663	15.56
0.05	4.0	0.45	0.4444	0.5	0.8	0.4	0.3584	0.3413	0.7644	0.3255	0.3571	5.015
0.35	0.7143	0.15	1.667	0.5	1.0	0.5	0.1947	0.1994	0.5398	0.1857	0.2131	-2.368
0.25	1.0	0.25	1.0	0.5	1.0	0.5	0.4229	0.3319	0.7537	0.3171	0.3467	27.42
0.15	1.667	0.35	0.7143	0.5	1.0	0.5	0.5862	0.3835	0.8556	0.3672	0.3999	52.84
0.05	5.0	0.45	0.5556	0.5	1.0	0.5	0.7428	0.5923	1.154	0.5684	0.6161	25.42
0.25	1.2	0.25	1.2	0.5	1.2	0.6	0.8371	0.5733	1.077	0.5522	0.5943	46.02
0.15	2.0	0.35	0.8571	0.5	1.2	0.6	1.312	0.8661	1.439	0.8386	0.8936	51.54
0.05	6.0	0.45	0.6667	0.5	1.2	0.6	1.749	1.578	2.260	1.531	1.624	10.87

Table 6.2: Approximate and simulated average mean waiting time; standard deviation and 95% confidence interval of the mean; and relative error for queue 2.

Once again we observe that the relative error increases with the decrease in the arrival rate in queue 2 - average percent error being approximately -6% for ρ = 0.25, 7% for ρ = 0.3, 10% for ρ = 0.4, 26% for ρ = 0.5, and 36% for ρ = 0.6. Like in the symmetric case, here also the approximation gets worse with the increase in asymmetry. As compared with the results of Boxma and Meister (1987), our results are not bad taking into account the degree of complexity involved in our model as compared to theirs.

Table 6.3 summarizes the results for queue 3 for ρ of 0.25 and 0.5.

λ_1	b_1	λ_2	b_2	λ_3	b_3	ρ	Approximate mean using method					SIM	S.D	LCIL	UCIL
							single q	1	2	3	average				
0.05	2.5	0.45	0.278	0.5	0.5	0.25	0.0473	0.1280	0.2211	0.2436	0.0877	0.0691	0.2881	0.0634	0.0748
0.15	0.833	0.35	0.357	0.5	0.5	0.25	0.0303	0.0744	0.1086	0.1304	0.0524	0.0460	0.2286	0.0414	0.0505
0.25	0.5	0.25	0.5	0.5	0.5	0.25	0.0288	0.0674	0.0941	0.1157	0.0481	0.0455	0.2302	0.0423	0.0487
0.35	0.357	0.15	0.833	0.5	0.5	0.25	0.0293	0.0669	0.0932	0.1150	0.0481	0.0486	0.2363	0.0439	0.0533
0.45	0.278	0.05	2.5	0.5	0.5	0.25	0.0357	0.0796	0.1205	0.1429	0.0577	0.0661	0.2787	0.0606	0.0717
0.05	5.0	0.45	0.567	0.5	1.0	0.5	0.4738	0.5735	1.002	1.173	0.5236	0.3396	0.8277	0.3281	0.3511
0.15	1.667	0.35	0.714	0.5	1.0	0.5	0.2435	0.3844	0.5418	0.6882	0.3139	0.2460	0.6839	0.2365	0.2556
0.25	1.0	0.25	1.0	0.5	1.0	0.5	0.2245	0.3616	0.4904	0.6346	0.2930	0.2695	0.6867	0.2600	0.2790
0.35	0.714	0.15	1.667	0.5	1.0	0.5	0.2336	0.3658	0.5014	0.6477	0.2997	0.2308	0.6464	0.2218	0.2398

Table 6.3: Approximate mean waiting time using 5 different methods; simulated mean waiting time; standard deviation and 95% confidence interval of the mean for queue 3.

Below are the graphs for the simulated and the approximate mean waiting time in queue 3 along with 95% confidence interval.

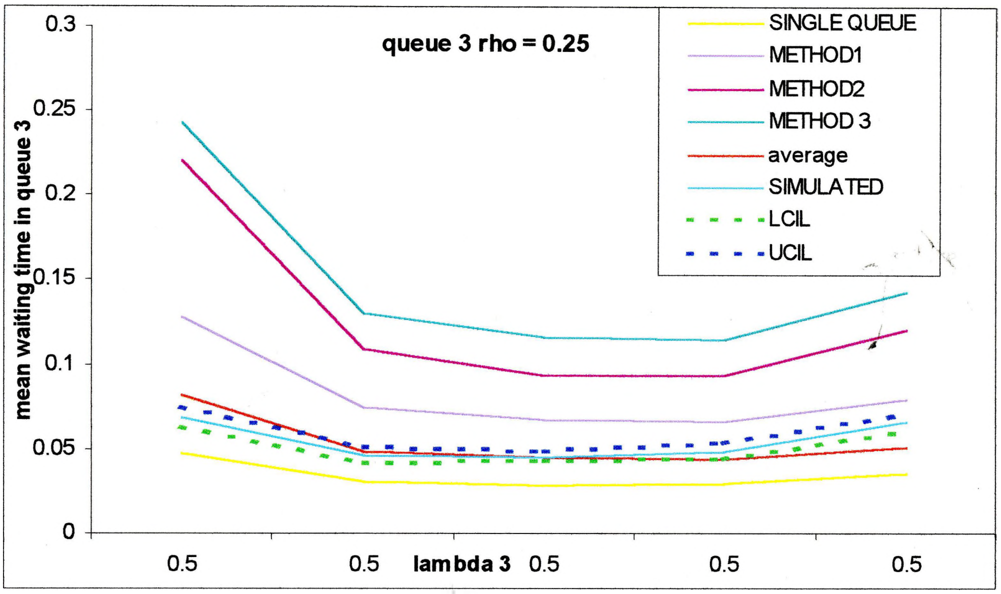

Figure 6.1: Mean waiting time in queue 3, ρ = 0.25.

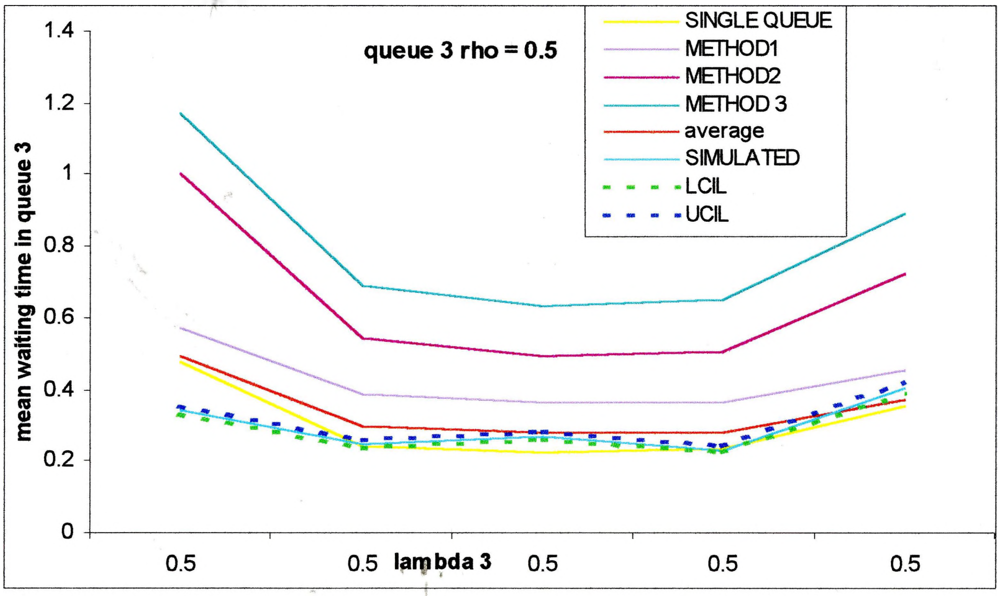

Figure 6.2: Mean waiting time in queue 3, ρ = 0.5.

Table 6.4 gives the relative error of all the methods of approximation.

λ_1	b_1	λ_2	b_2	λ_3	b_3	ρ	% Error between the simulated mean and approximate mean using method				
							single q	1	2	3	Average
0.05	2.5	0.45	0.2778	0.5	0.5	0.25	-31.54	85.37	220.1	252.6	26.92
0.15	0.8333	0.35	0.3571	0.5	0.5	0.25	-34.04	61.98	136.4	183.7	13.97
0.25	0.5	0.25	0.5	0.5	0.5	0.25	-36.83	47.94	106.6	154.2	5.55
0.35	0.3571	0.15	0.8333	0.5	0.5	0.25	-39.80	37.66	91.83	136.6	-1.07
0.45	0.2778	0.05	2.5	0.5	0.5	0.25	-46.02	20.35	82.16	116.1	-12.83
0.05	5.0	0.45	0.5666	0.5	1.0	0.5	39.51	68.85	195.1	245.41	54.18
0.15	1.667	0.35	0.7143	0.5	1.0	0.5	-1.036	56.24	120.2	179.70	27.60
0.25	1.0	0.25	1.0	0.5	1.0	0.5	-16.70	34.18	81.96	135.48	8.74
0.35	0.7143	0.15	1.667	0.5	1.0	0.5	1.200	58.46	117.2	180.61	29.83

Table 6.4: Relative Error between the simulated mean and approximate mean using 5 different methods for queue 3.

Following are the graphs for the relative error for queue 3 for different approximation methods.

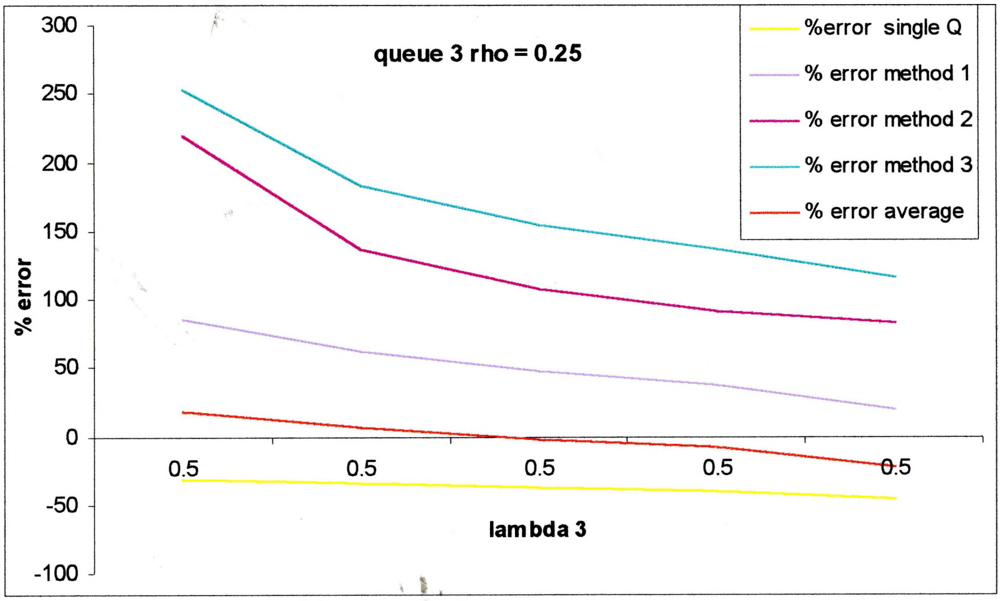

Figure 6.3: Relative error for queue 3, ρ = 0.25.

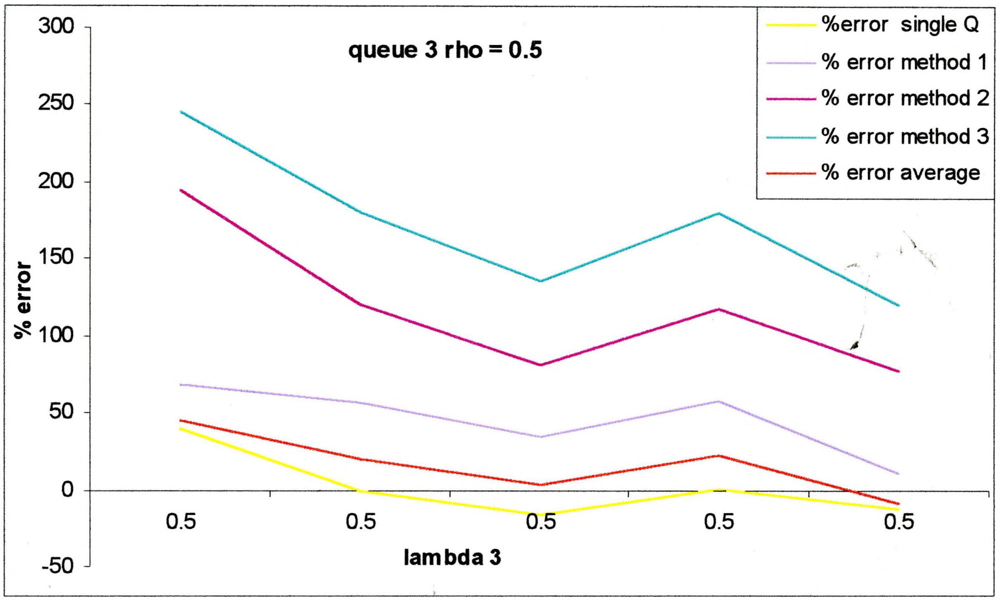

Figure 6.4: Relative error for queue 3, $\rho = 0.5$.

Since the relative error of the average of the method 4 and single queue with multiple vacation gives the least error, therefore, we use this method for further analysis of queue 3.

Table 6.5 summarizes the results for queue 3 over a range of ρ from 0.25 to 0.6.

λ_1	b_1	λ_2	b_2	λ_3	b_3	ρ	APPROX	SIM	S.D	LCIL	UCIL	% Error
0.05	2.5	0.45	0.2778	0.5	0.5	0.25	0.0877	0.0691	0.2881	0.0634	0.0748	26.92
0.15	0.8333	0.35	0.3571	0.5	0.5	0.25	0.0524	0.0460	0.2286	0.0414	0.0505	13.97
0.25	0.5000	0.25	0.5000	0.5	0.5	0.25	0.0481	0.0455	0.2302	0.0423	0.0487	5.55
0.35	0.3571	0.15	0.8333	0.5	0.5	0.25	0.0481	0.0486	0.2363	0.0439	0.0533	-1.07
0.45	0.2778	0.05	2.5	0.5	0.5	0.25	0.0577	0.0661	0.2787	0.0606	0.0717	-12.83
0.05	3.0	0.45	0.3333	0.5	0.6	0.3	0.1283	0.1065	0.3778	0.0990	0.1139	20.55
0.15	1.0	0.35	0.4286	0.5	0.6	0.3	0.0785	0.0701	0.2902	0.0644	0.0759	12.00
0.25	0.6	0.25	0.6	0.5	0.6	0.3	0.0726	0.0672	0.2909	0.0632	0.0713	7.98
0.35	0.4286	0.15	1.0	0.5	0.6	0.3	0.0729	0.0704	0.2921	0.0646	0.0762	3.61

0.05	4.0	0.45	0.4444	0.5	0.8	0.4	0.2590	0.2184	0.5959	0.2067	0.2302	18.56
0.15	1.333	0.35	0.5714	0.5	0.8	0.4	0.1609	0.1424	0.4463	0.1336	0.1512	13.02
0.25	0.8	0.25	0.8	0.5	0.8	0.4	0.1501	0.1377	0.4463	0.1315	0.1439	8.95
0.35	0.5714	0.15	1.333	0.5	0.8	0.4	0.1520	0.1481	0.4601	0.1390	0.1572	2.64
0.05	5.0	0.45	0.5667	0.5	1.0	0.5	0.5236	0.3396	0.8277	0.3281	0.3511	54.18
0.15	1.667	0.35	0.7143	0.5	1.0	0.5	0.3139	0.2460	0.6839	0.2365	0.2556	27.60
0.25	1.0	0.25	1.0	0.5	1.0	0.5	0.2930	0.2695	0.6867	0.2600	0.2790	8.74
0.35	0.7143	0.15	1.667	0.5	1.0	0.5	0.2997	0.2308	0.6464	0.2218	0.2398	29.83
0.05	6.0	0.45	0.6667	0.5	1.2	0.6	1.167	1.021	1.692	0.9873	1.054	14.36
0.15	2.0	0.35	0.8571	0.5	1.2	0.6	0.6322	0.5690	1.139	0.5507	0.5873	11.12
0.25	1.2	0.25	1.2	0.5	1.2	0.6	0.5853	0.5141	1.064	0.4993	0.5289	13.86

Table 6.5: Approximate and simulated average mean waiting time; standard deviation and 95% confidence interval of the mean; and relative error for queue 3.

We again observe that the relative error increases with the increase in asymmetry in the specialized queues - the average percent error being approximately 7% for ρ = 0.25, 11% for ρ = 0.3, 11% for ρ = 0.4, 30% for ρ = 0.5, 13% for ρ = 0.6. But once again, this method is better than the one proposed by Boxma and Meister (1987) considering the degree of complexity involved in our model as compared to theirs.

RECOMMENDATIONS

We conclude that in case of asymmetric system the only proposed method for approximating mean waiting time in queues 1 and 2 give approximations reasonably well. The average of the single queue with multiple server vacations model and the method 1 gives reasonable estimates for the mean waiting time in the queue 3. Therefore these methods can be used to determine the optimal allocation of load to the three queues.

CHAPTER 7

CONCLUSION

7.1 CONCLUSION

The main idea of carrying out this study is to find the optimal degree of overlap between the two servers so as to minimize the mean waiting time in three queues. We formulated the steady state equations. They resulted in a complicated boundary value problem. Therefore, we resorted to the model approximations as discussed in chapter 4.

The numerical analysis of the approximation methods (proposed in chapter 4) for the symmetric system in chapter 5 show that the approximation method 3 and the average of method 1 and the single server with multiple vacations is the best method for approximating the mean waiting time in queues 1 and 2 and queue 3 respectively. The approximations are reasonably well (% error within +15% to −15%) when either the switch over times are not too small, or ρ is not too large, or the arrival rate in the queue is not too small. The following table summarizes the approximate (APPROX) and the simulated (SIM) mean for queues 1 and 2; confidence interval (LCL, UCL); and the relative error for different values of ρ when all the three queues are identical.

ρ	APPROX	SIM	LCL	UCL	% ERROR
0.25	0.0647	0.0641	0.0614	0.0669	0.733
0.30	0.0918	0.0927	0.0879	0.0976	-1.004
0.40	0.1714	0.1759	0.1686	0.1830	-2.553
0.50	0.3056	0.3122	0.3051	0.3192	-2.116
0.60	0.5471	0.5437	0.5294	0.5580	0.613

Table 7.1: Summary for queues 1 and 2

The following table summarizes the approximate (APPROX) and the simulated (SIM) mean for queue 3; confidence interval (LCL, UCL); and the relative error for different values of ρ when all the three queues are identical.

ρ	APPROX	SIM	LCL	UCL	% ERROR
0.25	0.0447	0.0448	0.0423	0.0471	-0.225
0.30	0.0679	0.0672	0.0632	0.0713	1.011
0.40	0.1414	0.1377	0.1315	0.1439	2.638
0.50	0.2774	0.2674	0.2606	0.2742	3.742
0.60	0.5568	0.5141	0.4993	0.5289	8.313

Table 7.2: Summary for queue 3

The numerical analysis of the approximation methods for the asymmetric system in chapter 6 show that the only approximation method for queues 1 and 2; and the average of method 1 and the single server with multiple vacations for queue 3 gives the approximations that are reasonably well once again when either the switch over times are not too small, or ρ is not too large, or the arrival rate in the queue is not too small.

The recommended methods for approximating the mean waiting time in three queues for both symmetric and asymmetric cases perform reasonably well as compared to the results of Boxma and Meiser (1987) which are for single server polling model, a much simpler system compared to ours. Although we conclude our study with the recommendations and feasible range for λ_3 for symmetric system, the recommended methods can be used to obtain an optimal allocation of load to the three queues for both symmetric as well as asymmetric cases.

7.2 TOPICS FOR FURTHER RESEARCH

It would be interesting to extend the model, for example, increasing the number of queues from 3 to 4 and number of servers to 3. Where server 1 is specialized to serve queues 1 and 2 only; server 2, queues 2 and 3 only; and server 3, queues 3 and 4 only, i.e.,

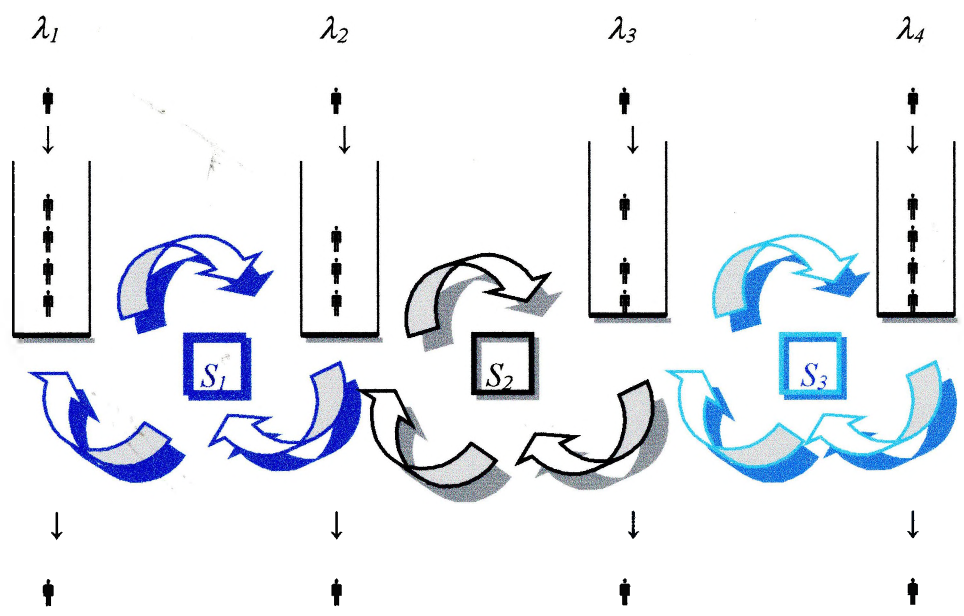

Figure 7.1: Four queues and three servers model

But, since our model – two servers, three queues - itself is not easy to handle, the degree of difficulty in extending it can be enormous. One can generalize our model to n queues and m servers where each server is capable of handling j different kinds of customer classes. But, once again, the degree of complexity involved here will be enormous. It will be interesting to explore the possibility of the application of the mathematical physics to solve the functional equations like Eisenberg (1979), Boxma and Groenendijk (1988), Boxma (1984), and Boxma and Cohen (1983) have done.

We investigated only the 1-limited service type protocol. It would be interesting to study other service protocols, for the two servers three queues system, as mentioned below.

1. *Exhaustive service*: The server serves the queue until there are no more customers left in that queue and only then leaves for next queue in the sequence.
2. *Gated service*: The server serves only those customers who were present in the queue at the server's arrival epoch, then moves to the next queue in the sequence.
3. *Globally Gated service*: The server serves only those customers that were present in queue j, j=1,…,n, at the most recent server's arrival epoch to queue 1 (which is chosen arbitrarily, without loss of generality).
1. *K-Limited service*: The server serves up to K, $K \geq 1$, customers on each visit to the queue. It can further be classified as:
   i. K- gated: It serves either K customers or the number of customers those were present in the queue at the server's arrival epoch, which ever is smaller, then moves to the next queue in the sequence.
   ii. K- exhaustive: It serves either K customers or exhausts the queue, then moves to the next queue in the sequence.

iii. *Decrementing service*: The server moves to next queue in the sequence only if the number of customers left in queue is K customers less than what it found on the arrival epoch. But, if the number of customers at the arrival epoch of the server was less than K, then server exhausts the queue before moving to the next queue in the sequence.

5. *Timer Limited*: The server stays at a queue for a pre-specified length of time or until it empties the queue, which ever occurs first before moving to the next queue in the sequence.

BIBLIOGRAPHY

1. Blanc, J.P.C., (1990), A numerical approach to cyclic-service queueing models. Queueing Systems, 6, 173-188.

2. Boxma, O.J., (1984), Two symmetric queues with alternating service and switching time, Performance'84, E. Gelenbe (editor), Elsevier Science Publishers B.V. (North Holland), Amsterdam, 409-431.

3. Boxma, O.J. and Groendijk, W.P., (1987), Two queues with alternating and switching times. Queueing theory and its applications – Liber Amicorum for J.W. Cohen, O.J. Boxma and R. Syski (editors), Elsevier Science Publishers B.V. (North Holland), Amsterdam, 261-282.

4. Boxma,O.J., Meister,B.W., (1987), Waiting time approximations for cyclic-service systems with switchover times, Performance Evaluation, Vol. 7, No.4, 299-308.

5. Cinlar, E. (1975), Introduction to Stochastic processes, prentice Hall, Eaglewood Cliffs, NJ.

6. Cooper, R.B., (1970), Queues serves in cyclic order: waiting times, The Bell system Technical Journal, Vol.49, No. 3, 399-413.

7. Cooper, R.B. and Murray, G., (1969), Queues serves in cyclic order, The Bell system Technical Journal, Vol.48, No. 3, 675-689.

8. Cohen,J.W., Boxma,O.J., (1983), Boundary Value Problems in Queueing System Analysis, Mathematics studies, 79, North-Holland, Amsterdam.

9. Eisenberg, M., (1979), Two queues with alternating service, SIAM Journal of Applied Mathematics, Vol. 36, No.2, 287-303.

10. Eisenberg, M., (1971), Two queues with changeover times, Operations Research, Vol. 19, No. 2, 386-401.

11. Green, L., (1985), A queueing system with general-use and limited-use servers Operations Research, Vol. 33, 168-182.

12. Kuehn, P.J., (1979), Multiqueue systems with nonexhaustive cyclic service, The Bell System Technical Journal, Vol. 58, No. 3, 671-698.

13. Marsan, A., M., De Moraes, L.F., Donatelli,S., Neri,F., (1990), Analysis of symmetric nonexhaustive polling with multiple servers, Proc. INFOCOM'90, 284-295.

14. Mei, R.D.Van der, Borst, S.C., (1997), Analysis of multiple-server polling systems by means of the power series algorithm, Stochastic Models, Vol. 13, NO. 2, 339-369.

15. Microsoft FORTRAN optimization compiler, (1987), Version 5.1, Microsoft Corporation.

16. Microsoft EXCEL 97, (1997), Microsoft Corporation.

17. Morris,R.T.J, Wang,Y.T., (1984), Some results for multi-queue systems with multiple-cyclic servers, Performance of Computer-Communication Systems, Rudin, H. and Bux,W. (Editors), IFIP/North-Holland, Amsterdam, 245-258.

18. Neuts, M.F., (1981), Matrix-geometric solutions in stochastic models, The John Hopkins University Press, Baltimore.

19. Neuts, M.F., (1978), Markov chains with applications in queueing theory which have matrix-geometric invariant probability vector, Advanced Applied Probability, Vol 10, 185-212.

20. Pritsker, A.A.B., (1986), Introduction to Simulation and SLAM II, John Wiley & Sons, NewYork, and Systems Publishing Corporation, Indiana.

21. Sheikhzadeh, M., Benjaafar, S., and Gupta, D. (1998), Machine sharing in manufacturing systems: total flexibility versus chaining, International Journal of Flexible Manufacturing Systems, Vol. 10, 351-378.

22. SLAM II, (1992), Quick reference manual, Pritsker Corporation, Indianapolis.

23. SLAM II (1989), Simulation Language for Alternative Modeling, SLAM systems 4.6, Prisker & Associates, Inc.

24. Smith, D.R., and Whitt, W. (1981), Resource sharing for efficiency in traffic systems, Bell System Technical Journal, Vol. 60, No. 1, 39-55.

25. Stanford, D.A., and Grassman, W.K., (1993), The billingual server system: A queueing model featuring fully and partially qualified servers, INFOR, Vol. 31, No. 4, 261-277.

26. Stidham, Jr. S., (1970), On the optimality of single-server queuing systems, Operations Research, Vol. 18, 708-732.

27. Takács, L., (1968), Two queues attended by a single server, Operations Research, Vol. 16, No. 3, 639-650.

28. Takagi, H., (1997), Queueing analysis of polling models: progress in 1990-1994, Frontiers in Queueing, Dshalalow (editor), CRC Press Inc., New York, 119-146.

29. Takagi, H., (1991), Queueing Analysis a foundation of performance evaluation, Volume 1: vacation and priority systems, part 1, Elsevier Science Publishers, B.V., (North-Holland).

30. Takagi, H., (1990), Queueing analysis of polling models: an update, Stochastic Analysis of Computer and Communication Systems, Takagi, H. (editor), Elsevier Science Publishers, B.V., (North-Holland), 267-318.

31. Watson, K.S., (1984), Performance evaluation of cyclic server strategies – A survey, Performance'84, Gelenbe, E., (Editor), Elsevier Science Publishers, B.V., (North-Holland), 521-533.

32. Wolff, R.W. (1982), Poisson arrivals see time averages, Operations Research, Vol.30, 223-231.

APPENDIX A

FORTRAN PROGRAM

```
SUBROUTINE EVENT(I)
C **** SUBROUTINE EVENT(I), I=1 ORDERS ARRIVAL OF CUSTOMERS
C **** SUBROUTINE EVENT(I), I=2,3 ORDERS SERVICE COMPLETION
C **** FOR SERVER 1 AND 2 RESPECTIVELY.
C **** SUBROUTINE EVENT(I), I=4,5 ORDERS SWITCH OVER
C **** COMPLETION FOR SERVER 1 AND 2 RESPECTIVELY.
C **** SUBROUTINE EVENT(I), I=6 ORDERS CLEARING OF
C **** STATISTICS
C
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
      GO TO (1,2,3,4,5,6),I
1     CALL ARVL
      RETURN
2     CALL ENDSV1
      RETURN
3     CALL ENDSV2
      RETURN
4     CALL SWITCH1
      RETURN
5     CALL SWITCH2
      RETURN
6     CALL CLLEAR
      RETURN
      END
C
C
      SUBROUTINE CLLEAR
      CALL CLEAR
      RETURN
      END
C
C
C *** RL: INTER ARRIVAL TIME IN THE SYSTEM
C *** RMA: MEAN SERVICE TIME FOR QUEUE 1
C *** RMB: MEAN SERVICE TIME FOR QUEUE 2
C *** RMC: MEAN SERVICE TIME FOR QUEUE 3
C *** 1-P: PROBABILITY THAT INCOMING CUSTOMER JOINS QUEUE 3
C *** Q: PROBABILITY THAT INCOMING CUSTOMER JOINS QUEUE 1
C *** STT1: MEAN SWITCH OVER TIME TO QUEUE 3 FROM QUEUE 1
C *** STT2: MEAN SWITCH OVER TIME TO QUEUE 3 FROM QUEUE 2
C *** STT3: MEAN SWITCH OVER TIME TO QUEUE 1 OR 2 FROM
C **** QUEUE 3
C *** TCLR: TIME AT WHICH STATISTICS IS CLEARED
```

```
C
C
C ***** XX(1)=1: SERVER 1 IS BUSY SERVING
C ***** XX(1)=0: SERVER 1 IS SWITCHING
C ***** XX(2)=1: SERVER 2 IS BUSY SERVING
C ***** XX(2)=0: SERVER 2 IS SWITCHING
C ***** XX(3)=1: SERVER 1 IS IN QUEUE 1
C ***** XX(3)=3: SERVER 1 IS IN QUEUE 3
C ***** XX(4)=2: SERVER 2 IS IN QUEUE 2
C ***** XX(4)=3: SERVER 2 IS IN QUEUE 3
C ***** XX(5): UNIFORM RANDOM NUMBER DECIDING THE TYPE OF
C **** ARRIVAL: 1,2 OR 3
C ***** XX(6)=1: SERVER 1 IS BUSY SERVING IN QUEUE 1
C ***** XX(7)=1: SERVER 1 IS BUSY SERVING IN QUEUE 3
C ***** XX(8)=1: SERVER 2 IS BUSY SERVING IN QUEUE 2
C ***** XX(9)=1: SERVER 2 IS BUSY SERVING IN QUEUE 3
C ***** XX(10)=1: SERVER 1 IS SWITCHING FROM QUEUE 1 TO
C ****  QUEUE 3
C ***** XX(11)=1: SERVER 1 IS SWITCHING FROM QUEUE 3 TO
C **** QUEUE 1
C ***** XX(12)=1: SERVER 2 IS SWITCHING FROM QUEUE 2 TO
C **** QUEUE 3
C ***** XX(13)=1: SERVER 2 IS SWITCHING FROM QUEUE 3 TO
C **** QUEUE 2
C
C
      SUBROUTINE ARVL
C **** SUBROUTINE ARRVL SCHEDULES ARRIVAL EVENTS
C
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
      COMMON/UCOM1/NTJ,NJ1,NJ2,NJ3,TLAST1,TLAST2,TNSY
     1,TSYS1,TBD1,RL,RMA,RMB,RMC,P,Q,ST,NJ3S1,NJ3S2,TSYS2
     2,TBD2,STT1,STT2,STT3,TSYS13,TBD13,TLAST13
     3,TSYS23,TBD23,TLAST23,NS13,NS31,NS23,NS32
C *** RL: INTER ARRIVAL TIME IN THE SYSTEM
C *** RMA: MEAN SERVICE TIME FOR QUEUE 1
C *** RMB: MEAN SERVICE TIME FOR QUEUE 2
C *** RMC: MEAN SERVICE TIME FOR QUEUE 3
C *** 1-P: PROBABILITY THAT INCOMING CUSTOMER JOINS QUEUE 3
C *** Q: PROBABILITY THAT INCOMING CUSTOMER JOINS QUEUE 1
C **** EVENT NEXT ARRIVAL, MARK ARRIVAL TIME AND INCREMENT
C **** NTJ
      CALL SCHDL(1,EXPON(RL,1),ATRIB)
      ATRIB(1)=TNOW
```

```
C **** NJT COUNTS TOTAL # OF ARRIVALS
      NTJ=NTJ+1
      SY=UNFRM(0.0,1.0,5)
C **** IF (SY. LT. Q), IT IS A TYPE 1 ARRIVAL, GO TO 3
C **** IF (SY. GT. Q) BUT IF(SY. LT. P), IT IS A TYPE 2
C **** ARRIVAL, GO TO 5
C **** IF (SY. GT. P), IT IS A TYPE 3 ARRIVAL, GO TO 10
      IF(SY.GT.Q) GO TO 2
C ***** INCREMENT NJ1, MARK JOB TYPE, AND GENERATE SERVICE
C **** TIME
      NJ1=NJ1+1
C **** FILE THE ARRIVING CUSTOMER
      CALL FILEM(1,ATRIB)
      RETURN
C **** IF THE JOB IS TYPE 3 THEN
2     IF(SY.GT.P) GO TO 10
C ***** OTHERWISE INCREMENT NJ2, MARK JOB TYPE, AND
C **** GENERATE SERVICE TIME
5     NJ2=NJ2+1
C **** FILE THE ARRIVING CUSTOMER
      CALL FILEM(2,ATRIB)
      RETURN
C **** INCREMENT NJ3, MARK JOB TYPE, AND GENERATE SERVICE
C **** TIME
10    NJ3=NJ3+1
C **** FILE THE ARRIVING CUSTOMER
      CALL FILEM(3,ATRIB)
      RETURN
      END
C
C
      SUBROUTINE ENDSV1
C **** SUBROUTINE ENDSV1 SCHEDULES END OF SERVICE BY SERVER
C **** 1
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
      COMMON/UCOM1/NTJ,NJ1,NJ2,NJ3,TLAST1,TLAST2,TNSY
     1,TSYS1,TBD1,RL,RMA,RMB,RMC,P,Q,ST,NJ3S1,NJ3S2,TSYS2
     2,TBD2,STT1,STT2,STT3,TSYS13,TBD13,TLAST13
     3,TSYS23,TBD23,TLAST23,NS13,NS31,NS23,NS32
      XX(1)=0
      IF(XX(3) .EQ. 1) GO TO 5
      GO TO 10
5     XX(6)=0
      TSYS1=TNOW-ATRIB(1)
```

```
      TBD1=TNOW-TLAST1
      TLAST1=TNOW
      CALL COLCT(TSYS1,1)
      CALL COLCT(TBD1,2)
      XX(10)=1
      NS13=NS13+1
      CALL SCHDL(4,EXPON(STT1,8),ATRIB)
      RETURN
10    XX(7)=0
      TSYS13=TNOW-ATRIB(1)
      TBD13=TNOW-TLAST13
      TLAST13=TNOW
      CALL COLCT(TSYS13,3)
      CALL COLCT(TBD13,4)
      XX(11)=1
      NS31=NS31+1
      CALL SCHDL(4,EXPON(STT3,8),ATRIB)
      RETURN
      END
C
C
      SUBROUTINE ENDSV2
C **** SUBROUTINE ENDSV2 SCHEDULES END OF SERVICE BY SERVER
C **** 2
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
      COMMON/UCOM1/NTJ,NJ1,NJ2,NJ3,TLAST1,TLAST2,TNSY
     1,TSYS1,TBD1,RL,RMA,RMB,RMC,P,Q,ST,NJ3S1,NJ3S2,TSYS2
     2,TBD2,STT1,STT2,STT3,TSYS13,TBD13,TLAST13
     3,TSYS23,TBD23,TLAST23,NS13,NS31,NS23,NS32
      XX(2)=0
      IF(XX(4) .EQ. 2) GO TO 5
      GO TO 10
5     XX(8)=0
      TSYS2=TNOW-ATRIB(1)
      TBD2=TNOW-TLAST2
      TLAST2=TNOW
      CALL COLCT(TSYS2,5)
      CALL COLCT(TBD2,6)
      XX(12)=1
      NS23=NS23+1
      CALL SCHDL(5,EXPON(STT2,9),ATRIB)
      RETURN
10    XX(9)=0
      TSYS23=TNOW-ATRIB(1)
```

```
      TBD23=TNOW-TLAST23
      TLAST23=TNOW
      CALL COLCT(TSYS2,7)
      CALL COLCT(TBD2,8)
      XX(13)=1
      NS32=NS32+1
      CALL SCHDL(5,EXPON(STT3,9),ATRIB)
      RETURN
      END
C
C
      SUBROUTINE SWITCH1
C **** SUBROUTINE SWITCH1 SCHEDULES END OF SWITCH OVER BY
C **** SERVER 1
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
     COMMON/UCOM1/NTJ,NJ1,NJ2,NJ3,TLAST1,TLAST2,TNSY
     1,TSYS1,TBD1,RL,RMA,RMB,RMC,P,Q,ST,NJ3S1,NJ3S2,TSYS2
     2,TBD2,STT1,STT2,STT3,TSYS13,TBD13,TLAST13
     3,TSYS23,TBD23,TLAST23,NS13,NS31,NS23,NS32
4     IF(XX(3) .EQ. 3) GO TO 6
      XX(3)=3
      IFILE=3
      XX(10)=0
      IF(NNQ(3) .EQ. 0) GO TO 10
      XX(1)=1
      XX(7)=1
      TNSY=EXPON(RMC,4)
      NJ3S1=NJ3S1+1
      CALL RMOVE(1,IFILE,ATRIB)
      CALL SCHDL(2,TNSY,ATRIB)
      RETURN
6     XX(3)=1
      IFILE=1
      XX(11)=0
      IF(NNQ(1) .EQ. 0) GO TO 11
      XX(1)=1
      XX(6)=1
C      XX(7)=0
      TNSY=EXPON(RMA,2)
      CALL RMOVE(1,IFILE,ATRIB)
      CALL SCHDL(2,TNSY,ATRIB)
      RETURN
10    XX(11)=1
      NS31=NS31+1
```

```
      CALL SCHDL(4,EXPON(STT3,8),ATRIB)
      RETURN
11    XX(10)=1
      NS13=NS13+1
      CALL SCHDL(4,EXPON(STT1,8),ATRIB)
      RETURN
      END
C
C
      SUBROUTINE SWITCH2
C **** SUBROUTINE SWITCH2 SCHEDULES END OF SWITCH OVER BY
C **** SERVER 2
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
      COMMON/UCOM1/NTJ,NJ1,NJ2,NJ3,TLAST1,TLAST2,TNSY
     1,TSYS1,TBD1,RL,RMA,RMB,RMC,P,Q,ST,NJ3S1,NJ3S2,TSYS2
     2,TBD2,STT1,STT2,STT3,TSYS13,TBD13,TLAST13
     3,TSYS23,TBD23,TLAST23,NS13,NS31,NS23,NS32
4     IF(XX(4) .EQ. 3) GO TO 25
      XX(4)=3
      IFILE=3
      XX(12)=0
      IF(NNQ(3) .EQ. 0) GO TO 30
      XX(2)=1
C      XX(8)=0
      XX(9)=1
      TNSY=EXPON(RMC,4)
      NJ3S2=NJ3S2+1
      CALL RMOVE(1,IFILE,ATRIB)
      CALL SCHDL(3,TNSY,ATRIB)
      RETURN
25    XX(4)=2
      IFILE=2
      XX(13)=0
      IF(NNQ(2) .EQ. 0) GO TO 35
      XX(2)=1
      XX(8)=1
C      XX(9)=0
      TNSY=EXPON(RMB,3)
      CALL RMOVE(1,IFILE,ATRIB)
      CALL SCHDL(3,TNSY,ATRIB)
      RETURN
30    XX(13)=1
      NS32=NS32+1
      CALL SCHDL(5,EXPON(STT3,9),ATRIB)
```

```
      RETURN
35    XX(12)=1
      NS23=NS23+1
      CALL SCHDL(5,EXPON(STT2,9),ATRIB)
      RETURN
      END
C
C
      SUBROUTINE INTLC
C **** SUBROUTINE INTLC INITIALISES THE VARIABLES AND READS
C **** THE DATA
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
      COMMON/UCOM1/NTJ,NJ1,NJ2,NJ3,TLAST1,TLAST2,TNSY
     1,TSYS1,TBD1,RL,RMA,RMB,RMC,P,Q,ST,NJ3S1,NJ3S2,TSYS2
     2,TBD2,STT1,STT2,STT3,TSYS13,TBD13,TLAST13
     3,TSYS23,TBD23,TLAST23,NS13,NS31,NS23,NS32
      NTJ=0
      NJ1=0
      NJ2=0
      NJ3=0
      NJ3S1=0
      NJ3S2=0
      NS13=0
      NS31=0
      NS23=0
      NS32=0
      TLAST1=0.0
      TSYS1=0.0
      TBD1=0.0
      TLAST2=0.0
      TSYS2=0.0
      TBD2=0.0
      TLAST13=0.0
      TSYS13=0.0
      TBD13=0.0
      TLAST23=0.0
      TSYS23=0.0
      TBD23=0.0
      OPEN (53,FILE='DATA.DAT')
      READ (53,*) P,Q,RL,RMA,RMB,RMC,STT1,STT2,STT3,TCLR
      CLOSE (53)
      XX(1)=0
      XX(2)=0
      XX(3)=1
```

```
      XX(4)=2
      XX(6)=0
      XX(7)=0
      XX(8)=0
      XX(9)=0
      XX(10)=1
      XX(11)=0
      XX(12)=1
      XX(13)=0
      ATRIB(1)=0.0
      CALL SCHDL(1,EXPON(RL,1),ATRIB)
      CALL SCHDL(4,0.,ATRIB)
      CALL SCHDL(5,0.,ATRIB)
      CALL SCHDL(6,TCLR,ATRIB)

      RETURN
      END
C
C
      SUBROUTINE OTPUT
C *** SUBROUTINE OTPUT PRINTS THE DESIRED OUT PUT OF THE
C **** PROGRAM
$INCLUDE:'PARAM.INC'
$INCLUDE:'SCOM1.COM'
      COMMON/UCOM1/NTJ,NJ1,NJ2,NJ3,TLAST1,TLAST2,TNSY
     1,TSYS1,TBD1,RL,RMA,RMB,RMC,P,Q,ST,NJ3S1,NJ3S2,TSYS2
     2,TBD2,STT1,STT2,STT3,TSYS13,TBD13,TLAST13
     3,TSYS23,TBD23,TLAST23,NS13,NS31,NS23,NS32
      EXTERNAL TTAVG,TTSTD,TTPRD,CCAVG,CCSTD,CCNUM
     1,FFAVG,FFSTD,FFAWT
C      CLOSE(54)
C      WRITE (NPRNT,*) 'CURRENT TIME', GETTIM
      WRITE(NPRNT,*) 'THE QUEUE DISCIPLINE IS ALTERNATING
     1 PRIORITY'
      THRU=FLOAT(NTJ)/TNOW
      WRITE(NPRNT,*) '#', NTJ, NJ1, NJ2, NJ3
      PTYP1=100.*FLOAT(NJ1)/FLOAT(NTJ)
      PTYP2=100.*FLOAT(NJ2)/FLOAT(NTJ)
      PTYP3=100.*FLOAT(NJ3)/FLOAT(NTJ)
      IF(NJ3 .EQ. 0) GO TO 10
      TYP3S1=100.*FLOAT(NJ3S1)/FLOAT(NJ3)
      TYP3S2=100.*FLOAT(NJ3S2)/FLOAT(NJ3)
      TNUM3=CCNUM(7)+CCNUM(3)
      CILL31=FFAVG(3)-1.959961*FFSTD(3)/SQRT(FLOAT(TNUM3))
      CIUL31=FFAVG(3)+1.959961*FFSTD(3)/SQRT(FLOAT(TNUM3))
```

```
10    WRITE(NPRNT,*) 'PROPORTION OF TYPE 3 AND TYPE 1 JOBS=
     1', P,Q
      WRITE(NPRNT,*) 'NUMBER OF ARRIVING JOBS IS : ', NTJ
      WRITE(NPRNT,*) 'NUMBER OF TYPE 1 JOBS IS : ', NJ1
      WRITE(NPRNT,*) 'NUMBER OF TYPE 2 JOBS IS : ', NJ2
      WRITE(NPRNT,*) 'NUMBER OF TYPE 3 JOBS IS : ', NJ3
      WRITE(NPRNT,*) 'PERCENT OF TYPE 1 JOBS IS : ', PTYP1
      WRITE(NPRNT,*) 'PERCENT OF TYPE 2 JOBS IS : ', PTYP2
      WRITE(NPRNT,*) 'PERCENT OF TYPE 3 JOBS IS : ', PTYP3
      WRITE(NPRNT,*) 'NUMBER OF SWITCHES FROM Q1 TO Q3 IS :
     1', NS13
      WRITE(NPRNT,*) 'NUMBER OF SWITCHES FROM Q3 TO Q1 IS :
     1', NS31
      WRITE(NPRNT,*) 'NUMBER OF SWITCHES FROM Q2 TO Q3 IS :
     1', NS23
      WRITE(NPRNT,*) 'NUMBER OF SWITCHES FROM Q3 TO Q2 IS :
     1', NS32
      NS1=NS13+NS31
      NS2=NS23+NS32
      WRITE(NPRNT,*) 'TOTAL NUMBER OF SWITCHES BY SERVER 1
     1IS : ', NS1
      WRITE(NPRNT,*) 'TOTAL NUMBER OF SWITCHES BY SERVER 2
     1IS : ', NS2
      WRITE(NPRNT,*) 'PERCENT OF SWITCHES FROM Q1 TO Q3 IS
     1: ', NS13/FLOAT(NS1)
      WRITE(NPRNT,*) 'PERCENT OF SWITCHES FROM Q3 TO Q1 IS
     1: ', NS31/FLOAT(NS1)
      WRITE(NPRNT,*) 'PERCENT OF SWITCHES FROM Q2 TO Q3 IS
     1: ',
     1NS23/FLOAT(NS2)
      WRITE(NPRNT,*) 'PERCENT OF SWITCHES FROM Q3 TO Q2 IS
     1: ', NS32/FLOAT(NS2)
      IF(NJ3 .EQ. 0) GO TO 20
      WRITE(NPRNT,*) 'NUMBER OF TYPE 3 JOBS SERVED BY
     1SERVER 1 : ', NJ3S1
      WRITE(NPRNT,*) 'NUMBER OF TYPE 3 JOBS SERVED BY
     1SERVER 2 ', NJ3S2
      WRITE(NPRNT,*) 'PERCENT OF TYPE 3 JOBS SERVED BY
     1SERVER 1 ', TYP3S1
      WRITE(NPRNT,*) 'PERCENT OF TYPE 3 JOBS SERVED BY
     1SERVER 2 : ', TYP3S2
20    WRITE(NPRNT,*) 'AVERAGE THROUGHPUT IS : ',THRU
      WRITE(NPRNT,101)
      WRITE(NPRNT,*)
101   FORMAT('1',1X,'THE PERFORMANCE STATISTICS,')
```

```
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'AVERAGE NUMBER, STANDARD DEVIATION OF
     1LENGTH AND AVERAGE WAITING TIME IN THE THREE QUEUES
     2IS:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,110) FFAVG(1),FFSTD(1),FFAWT(1)
      WRITE(NPRNT,110) FFAVG(2),FFSTD(2),FFAWT(2)
      WRITE(NPRNT,110) FFAVG(3),FFSTD(3),FFAWT(2)
110   FORMAT(/,5X,E10.4,5X,E10.4,5X,E10.4)
      CILL11=FFAVG(1)-1.959961*FFSTD(1)/
     1SQRT(FLOAT(CCNUM(1)))
      CIUL11=FFAVG(1)+1.959961*FFSTD(1)/
     1SQRT(FLOAT(CCNUM(1)))
      CILL21=FFAVG(2)-1.959961*FFSTD(2)/
     1SQRT(FLOAT(CCNUM(5)))
      CIUL21=FFAVG(2)+1.959961*FFSTD(2)/
     1SQRT(FLOAT(CCNUM(5)))
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'TIME INTEGRATED AVERAGE, STANDARD
     1DEVIATION  AND TIME PERIOD FOR STATISTICS ON:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'TOTAL SEREVER 1 UTILISATION'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(1),TTSTD(1),TTPRD(1)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'TOTAL SEREVER 2 UTILISATION'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(2),TTSTD(2),TTPRD(2)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'SERVER 1 UTILISATION IN QUEUE 1'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(3),TTSTD(3),TTPRD(3)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'SERVER 1 UTILISATION IN QUEUE 3'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(4),TTSTD(4),TTPRD(4)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'SERVER 2 UTILISATION IN QUEUE 2'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(5),TTSTD(5),TTPRD(5)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'SERVER 2 UTILISATION IN QUEUE 3'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(6),TTSTD(6),TTPRD(6)
      WRITE(NPRNT,*)
```

```
      WRITE(NPRNT,*) 'SERVER 1 SWITCHING FROM QUEUE 1 TO
     1QUEUE 3'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(7),TTSTD(7),TTPRD(7)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'SERVER 1 SWITCHING FROM QUEUE 3 TO
     1QUEUE 1'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(8),TTSTD(8),TTPRD(8)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'SERVER 2 SWITCHING FROM QUEUE 2 TO
     1QUEUE 3'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(9),TTSTD(9),TTPRD(9)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'SERVER 2 SWITCHING FROM QUEUE 3 TO
     1QUEUE 2'
      WRITE(NPRNT,*)
      WRITE(NPRNT,150) TTAVG(10),TTSTD(10),TTPRD(10)
150   FORMAT(5X,E10.4,5X,E10.4,5X,E10.4)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) 'AVERAGE VALUE, STANDARD DEVIATION AND
     1NUMBER OF OBSERVATIONS FOR'
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)  'TIME IN SYSTEM FOR TYPE 1:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(1),CCSTD(1),CCNUM(1)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)  'TIME BETWEEN DEPARTURES FOR TYPE 1:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(2),CCSTD(2),CCNUM(2)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)'TIME IN SYSTEM FOR TYPE 3 WHEN SERVED
     1BY S1:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(3),CCSTD(3),CCNUM(3)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)'TIME BETWEEN DEPARTURES FOR TYPE 3
     1WHEN QUEUE 3 IS SERVED BY SERVER 1:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(4),CCSTD(4),CCNUM(4)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)  'TIME IN SYSTEM IN QUEUE 2:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(5),CCSTD(5),CCNUM(5)
```

```
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)'TIME BETWEEN DEPARTURES FOR TYPE 2:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(6),CCSTD(6),CCNUM(6)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)'TIME IN SYSTEM FOR TYPE 3 WHEN SERVED
     1BY S2:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(7),CCSTD(7),CCNUM(7)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*)'TIME BETWEEN DEPARTURES FOR TYPE 3
     1WHEN QUEUE 3 IS SERVED BY SERVER 2:'
      WRITE(NPRNT,*)
      WRITE(NPRNT,140) CCAVG(8),CCSTD(8),CCNUM(8)
      WRITE(NPRNT,*)
140   FORMAT(/,5X,E10.4,5X,E10.4,5X,E10.4)
      WRITE(NPRNT,*)
      WRITE(NPRNT,*) '95% CONFIDENCE INTERVALS ARE: '
      WRITE(NPRNT,*) '( ',CILL11,' , ',CIUL11,' )'
      WRITE(NPRNT,*) '( ',CILL21,' , ',CIUL21,' )'
      WRITE(NPRNT,*) '( ',CILL31,' , ',CIUL31,' )'
      RETURN
      END
```

APPENDIX B

CONTROL STATEMENTS

```
GEN,VANEETA,ONE LIMITED,06/04/1998,1,Y,Y,Y/Y,Y,Y/1,72;
LIMITS,3,2,10000;
SEEDS,12345(1),23457(2),34567(3);
SEEDS,45679(4),56789(5),67891(6);
SEEDS,78913(7),89123(8),91235(9);
STAT,1,TSYS1;
STAT,2,TBD1;
STAT,3,TSYS13;
STAT,4,TBD13;
STAT,5,TSYS2;
STAT,6,TBD2;
STAT,7,TSYS23;
STAT,8,TBD23;
TIMST,XX(1),UTILISATION 1;
TIMST,XX(2),UTILISATION 2;
TIMST,XX(6),UTILISATION Q1;
TIMST,XX(7),UTILISATION Q3S1;
TIMST,XX(8),UTILISATION Q2;
TIMST,XX(9),UTILISATION Q3S2;
TIMST,XX(10),SWITCH FQ1TQ3;
TIMST,XX(11),SWITCH FQ3TQ1;
TIMST,XX(12),SWITCH FQ2TQ3;
TIMST,XX(13),SWITCH FQ3TQ2;
TIMST,NNQ(1),NO. IN QUEUE 1,10/0/1;
TIMST,NNQ(2),NO. IN QUEUE 2,10/0/1;
TIMST,NNQ(3),NO. IN QUEUE 3,10/0/1;
INITIALIZE,,20000,Y/50;
FIN;
```

APPENDIX C

DATA FILE

```
0.9
0.45
1.0
1.0
1.0
1.0
0.1
0.1
0.1
100.0
```

APPENDIX D

OUT PUT

1

S L A M I I S U M M A R Y R E P O R T

SIMULATION PROJECT ONE LIMITED BY VANEETA

DATE 6/ 4/1998 RUN NUMBER
1 OF 1

CURRENT TIME .2000E+05
STATISTICAL ARRAYS CLEARED AT TIME .1000E+03

STATISTICS FOR VARIABLES BASED ON OBSERVATION

	MEAN VALUE	STANDARD DEVIATION	COEFF. OF VARIATION	MINIMUM VALUE	MAXIMUM VALUE	NO.OF OBS
TSYS1	.262E+01	.243E+01	.927E+00	.586E-02	.205E+02	8967
TBD1	.222E+01	.213E+01	.958E+00	.215E-01	.200E+02	8967
TSYS13	.131E+01	.111E+01	.853E+00	.273E-01	.772E+01	951
TBD13	.209E+02	.201E+02	.960E+00	.192E+00	.160E+03	951
TSYS2	.253E+01	.223E+01	.881E+00	.182E-01	.162E+02	8982
TBD2	.222E+01	.213E+01	.961E+00	.273E-01	.220E+02	8982
TSYS23	.189E+01	.177E+01	.935E+00	.313E-01	.117E+02	956
TBD23	.236E+01	.218E+01	.924E+00	.125E+00	.163E+02	956

STATISTICS FOR TIME-PERSISTENT VARIABLES

	MEAN VALUE	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE	TIME INTERVAL	CURRENT VALUE
UTILISATION 1	.500	.500	.00	1.00	19900.000	.00
UTILISATION 2	.503	.500	.00	1.00	19900.000	.00
UTILISATION Q1	.452	.498	.00	1.00	19900.000	.00
UTILISATION Q3S1	.048	.214	.00	1.00	19900.000	.00
UTILISATION Q2	.455	.498	.00	1.00	19900.000	.00
UTILISATION Q3S2	.048	.213	.00	1.00	19900.000	.00
SWITCH FQ1TQ3	.252	.434	.00	1.00	19900.000	1.00
SWITCH FQ3TQ1	.249	.432	.00	1.00	19900.000	.00
SWITCH FQ2TQ3	.248	.432	.00	1.00	19900.000	.00
SWITCH FQ3TQ2	.249	.433	.00	1.00	19900.000	1.00
NO. IN QUEUE 1	.728	1.295	.00	10.00	19900.000	.00
NO. IN QUEUE 2	.688	1.211	.00	11.00	19900.000	1.00
NO. IN QUEUE 3	.030	.180	.00	3.00	19900.000	.00

FILE STATISTICS

FILE NUMBER	LABEL/TYPE	AVERAGE LENGTH	STANDARD DEVIATION	MAXIMUM LENGTH	CURRENT LENGTH	AVERAGE WAIT TIME
1		.728	1.295	10	0	1.616
2		.688	1.211	11	1	1.525
3		.030	.180	3	0	.309
4	CALENDAR	3.000	.000	3	3	.250

1 **TIME-PERSISTENT HISTOGRAM NUMBER11**
NO. IN QUEUE 1

```
CELL RELA   UPPER
TIME FREQ  CELL LIM 0         20        40        60        80      100
                    +    +    +    +    +    +    +    +    +    +    +
****  .64  .000E+00 +********************************                 +
****  .18  .100E+01 +*********                               C        +
****  .08  .200E+01 +****                                        C    +
887.  .04  .300E+01 +**                                             C +
476.  .02  .400E+01 +*                                               C+
260.  .01  .500E+01 +*                                               C+
144.  .01  .600E+01 +                                                 C
 75.  .00  .700E+01 +                                                 C
 27.  .00  .800E+01 +                                                 C
 10.  .00  .900E+01 +                                                 C
  2.  .00  .100E+02 +                                                 C
  0.  .00    INF    +                                                 C
---                 +    +    +    +    +    +    +    +    +    +    +
****                0         20        40        60        80      100
```

STATISTICS FOR TIME-PERSISTENT VARIABLES

	MEAN VALUE	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE	TIME INTERVAL	CURRENT VALUE
NO. IN QUEUE 1	.728	1.295	.00	10.00	19900.000	.00

```
1                       **TIME-PERSISTENT HISTOGRAM NUMBER12**
                                    NO. IN QUEUE 2

CELL RELA   UPPER
TIME FREQ  CELL LIM 0         20        40        60        80       100
                    +    +    +    +    +    +    +    +    +    +    +
****  .64  .000E+00 +********************************                 +
****  .19  .100E+01 +*********                                C       +
****  .08  .200E+01 +****                                         C   +
886.  .04  .300E+01 +**                                             C +
440.  .02  .400E+01 +*                                               C+
207.  .01  .500E+01 +*                                                C
 96.  .00  .600E+01 +                                                 C
 43.  .00  .700E+01 +                                                 C
 18.  .00  .800E+01 +                                                 C
  8.  .00  .900E+01 +                                                 C
  4.  .00  .100E+02 +                                                 C
  2.  .00    INF    +                                                 C
---                 +    +    +    +    +    +    +    +    +    +    +
****                0         20        40        60        80       100
```

STATISTICS FOR TIME-PERSISTENT VARIABLES

	MEAN VALUE	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE	TIME INTERVAL	CURRENT VALUE
NO. IN QUEUE 2	.688	1.211	.00	11.00	19900.000	1.00

```
1                       **TIME-PERSISTENT HISTOGRAM NUMBER13**
                                    NO. IN QUEUE 3

CELL RELA   UPPER
TIME FREQ  CELL LIM 0         20        40        60        80       100
                    +    +    +    +    +    +    +    +    +    +    +
****  .97  .000E+00 +*************************************************+
522.  .03  .100E+01 +*                                                C
 31.  .00  .200E+01 +                                                 C
  2.  .00  .300E+01 +                                                 C
  0.  .00  .400E+01 +                                                 C
  0.  .00  .500E+01 +                                                 C
  0.  .00  .600E+01 +                                                 C
  0.  .00  .700E+01 +                                                 C
  0.  .00  .800E+01 +                                                 C
  0.  .00  .900E+01 +                                                 C
  0.  .00  .100E+02 +                                                 C
  0.  .00    INF    +                                                 C
---                 +    +    +    +    +    +    +    +    +    +    +
****                0         20        40        60        80       100
```

STATISTICS FOR TIME-PERSISTENT VARIABLES

	MEAN VALUE	STANDARD DEVIATION	MINIMUM VALUE	MAXIMUM VALUE	TIME INTERVAL	CURRENT VALUE
NO. IN QUEUE 3	.030	.180	.00	3.00	19900.000	.00