INTEGRAL EQUATIONS OF THE FIRST KIND

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ON INTEGRAL EQUATIONS OF THE FIRST KIND

AND VARIOUS METHODS OF SOLUTION

By

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A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

October 1962

MASTER OF SCIENCE (1962)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario.

TITLE: On Integral Equations of the First Kind and Various Methods of Solution.

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SUPERVISOR: Professor D. B. Sumner.

NUMBER OF PAGES: v, 49

SCOPE AND CONTENTS: This thesis gives an account of most known methods which can be utilized to solve various integral equations of the first kind.

PREFACE

Integral equations of the second kind have been studied extensively and several general methods of solution have been described by such authors as Hilbert, Fredholm, Schmidt, and Volterra.

No general method of solution is known for equations of the first kind. However, many such equations can be solved by employing special procedures, these being largely dependent on the type of the equation.

In this thesis I propose to give an account of several methods which can be applied to solve some integral equations of the first kind.

The first chapter will deal with the application of orthogonal systems to the solution of integral equations of the first kind.

Two methods to invert the convolution transform will be described in the second chapter. These are the Hirschmann-Widder operator and the Fourier transform.

In the third chapter it will be shown how the theory of functions, in suitable cases, can be applied to the solution of integral equations of the first kind.

ACKNOWLEDGMENTS

I should like to express my deepest gratitude to my research director Dr. D. B. Sumner for his valuable assistance and guidance which made it necessary for him to give up so much of his time during the course of this research.

It is also my pleasure to acknowledge the substantial financial support of the National Research Council and McMaster University.

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CHAPTER I

COMPLETE ORTHOGONAL SYSTEMS

1. Introduction

An equation of the form

$$ay(s) + \int_c^d K(s,t)y(t)\,dt = f(s)$$

where $y(t)$ is unknown and $K(s,t)$ and $f(s)$ given real or complex functions, is called a linear integral equation. It is said to be of the first kind if $a = 0$ and of the second kind if $a \neq 0$.

In the following sections we discuss a method which requires the use of orthogonal systems to reduce an integral equation of the first kind

$$\text{(1.1)} \qquad \int_a^b K(s,t)y(t)\,dt = f(s) \qquad c \leq s \leq d$$

to an infinite system of linear equations with infinitely many unknowns.

An orthogonal system is a system of complex functions

$$g_1(s),\ g_2(s), \ldots \qquad a \leq s \leq b$$

with $g_n(s) \in L^2(a,b)$ for all n, and satisfying the condition

$$\int_a^b g_n(s)\overline{g_m}(s)\,ds = 0 \qquad (n \neq m;\ n,m = 1,2,\ldots)$$

The system is said to be normalized or orthonormal if

$$\int_a^b |g_n(s)|^2 ds = 1 \qquad (n = 1,2,\ldots)$$

Any orthogonal system

(A) $$g_1(s),\ g_2(s),\ldots$$

can be normalized by forming the integrals

$$\int_a^b |g_n(s)|^2 ds \qquad (n = 1,2,\ldots)$$

and considering instead of (A) the functions

$$g_n^*(s) = \frac{g_n(s)}{\left(\int_a^b |g_n(s)|^2 ds\right)^{\frac{1}{2}}} \qquad (n = 1,2,\ldots).$$

In the subsequent work we shall always assume that the orthogonal systems are normalized.

We see immediately that orthogonal systems are always linearly independent. For consider the equation

$$c_1 g_1(s) + c_2 g_2(s) + \ldots + c_n g_n(s) = 0.$$

Multiplying by the complex conjugate $\overline{g_m}(s)$ and integrating we obtain

$$c_m = 0$$

for all m.

Let f(s) be a complex function in $L^2(a,b)$, then the constants

$$f_n = \int_a^b f(s)\overline{g}_n(s)\,ds \qquad (n = 1,2,\ldots)$$

exist and are called expansion coefficients of $f(s)$ with respect to the orthogonal system $(g_n(s))_{n=1}^{\infty}$.

For any $f(s) \in L^2(a,b)$ we have the inequality [4,p.16]

$$\sum_{n=1}^{\infty} |f_n|^2 = \sum_{n=1}^{\infty} \left| \int_a^b f(s)\overline{g}_n(s)\,ds \right|^2 \leq \int_a^b |f(s)|^2\,ds.$$

Since the integral on the right exists we see that $f(s) \in L^2(a,b)$ implies

$$\sum_{n=1}^{\infty} |f_n|^2$$

is convergent.

The above inequality is known as Bessel's inequality. If for any $f(s) \in L^2(a,b)$ we have the equality

$$\sum_{n=1}^{\infty} \left| \int_a^b f(s)\overline{g}_n(s)\,ds \right|^2 = \sum_{n=1}^{\infty} |f_n|^2 = \int_a^b |f(s)|^2\,ds$$

we say that the orthogonal system $(g_n(s))_{n=1}^{\infty}$ is complete or, equivalently, the $L^2(a,b)$ space is complete.

The above equality is known as Parseval's equation.

Equivalent definitions of completeness are

Definition 1 Given an orthogonal system $(g_n(s))_{n=1}^{\infty}$ in $L^2(a,b)$ and a function $f(s) \in L^2(a,b)$, then the orthogonal system is complete if given any $\varepsilon > 0$ there exists an N such that

$$\int_a^b \left| f(s) - \sum_{n=1}^{m} f_n g_n(s) \right|^2 ds < \varepsilon$$

whenever $m > N$.

We say that $f(s)$ is the limit in the mean of the series

$$\sum_{n=1}^{\infty} f_n g_n(s)$$

and write this as

$$f(s) = \text{l.i.m.} \sum_{n=1}^{\infty} f_n g_n(s)$$

or

$$f(s) \sim \sum_{n=1}^{\infty} f_n g_n(s).$$

The function $f(s)$ is unique except for a set of Lebesgue measure zero.

Definition 2 Given an orthogonal system $(g_n(s))_{n=1}^{\infty}$ in $L^2(a,b)$, then the orthogonal system is complete if given $f(s) \in L^2(a,b)$ and

$$\int_a^b f(s)\overline{g_n}(s)\,ds = 0 \qquad (n = 1, 2, \ldots)$$

implies $f(s) = 0$ except for a set of measure zero.

Given two functions $f(s)$, $h(s) \in L^2(a,b)$, then Parseval's equation can be written in the more generalized form

$$\sum_{n=1}^{\infty} \int_a^b f(s)\overline{g_n}(s)\,ds \cdot \int_a^b \overline{h(s)}\, g_n(s)\,ds = \int_a^b f(s)\overline{h(s)}\,ds$$

This follows easily if we apply Parseval's equation on the functions $f(s)+h(s)$ and $f(s)-h(s)$, [4, p. 20].

If we are given a complete orthogonal system $(g_n(s))_{n=1}^{\infty}$ in $L^2(a,b)$, then any $f(s) \in L^2(a,b)$ determines its expansion coefficients f_n with respect to $(g_n(s))_{n=1}^{\infty}$ and the series

$$\sum_{n=1}^{\infty} |f_n|^2$$

is convergent. The converse is also true and is known as the

<u>Riesz-Fischer Theorem:</u> [4, p. 23],

Given a complete orthogonal system $(g_n(s))_{n=1}^{\infty}$ in $L^2(a,b)$ and a system of complex numbers $(y_m)_{m=1}^{\infty}$ such that $\sum_{m=1}^{\infty} |y_m|^2$ is convergent, then there exists a function $y(s) \in L^2(a,b)$ which has y_m as its expansion coefficients and such that

$$y(s) = \text{l.i.m.} \sum_{m=1}^{\infty} y_m g_m(s).$$

The function $y(s)$ is unique almost everywhere i.e. except on a set of Lebesgue measure zero.

After these preliminaries we return to (1.1).

We show first that the problem of solving (1.1) is equivalent to the problem of solving an infinite system of linear equations, and then give a method for constructing the orthogonal system so that the problem of solving the infinite system is simplified.

This method is, in theory, applicable to all integral equations of the first kind in which the given functions $K(s,t)$ and $f(s)$ satisfy the following conditions:

1. $f(s) \in L^2$

2. $x(s) \in L^2 \Rightarrow \int_c^d K(s,t)x(s)ds \in L^2$

3. $y(t) \in L^2 \Rightarrow \int_a^b K(s,t)y(t)dt \in L^2$

4. for any $y(t)$, $x(s) \in L^2$ the equation

$$\int_c^d x(s)ds \int_a^b K(s,t)y(t)dt = \int_a^b y(t)dt \int_c^d K(s,t)x(s)ds$$

is satisfied.

Whenever $K(s,t)$ satisfies conditions 2 to 4 we shall say $K(s,t)$ satisfies condition A.

2. Reduction to an infinite system of linear equations

To reduce (1,1) to an infinite system we consider any complete orthogonal systems $(g_n(s))_{n=1}^{\infty}$ on the interval (c,d) and $(h_m(t))_{m=1}^{\infty}$ on the interval (a,b).

Multiply (1,1) by $\bar{g}_n(s)$ and integrate w.r.t. s to obtain

$$\int_c^d \bar{g}_n(s)ds \int_a^b K(s,t)y(t)dt = \int_c^d \bar{g}_n(s)f(s)ds = f_n \qquad (n=1,2,3\ldots)$$

where f_n is the expansion coefficient of $f(s)$ and $\bar{g}_n(s)$ is the complex conjugate of $g_n(s)$.

If $y(t) \in L^2$ and $K(s,t)$ satisfies condition A

$$\int_a^b y(t)dt \int_c^d K(s,t)\bar{g}_n(s)ds = f_n \qquad (n=1,2\ldots) \tag{2.1}$$

Since $(g_n(s))_{n=1}^{\infty}$ and $(h_m(t))_{m=1}^{\infty}$ are complete we can use Parseval's equation, viz.

$$\int_a^b h(t)\bar{g}(t)\,dt = \sum_{m=0}^{\infty} \int_a^b h(t)\bar{h}_m(t)\,dt \int_a^b \bar{g}(t)h_m(t)\,dt .$$

Letting $h(t) = y(t)$ and

$$\bar{g}(t) = \int_c^d K(s,t)\bar{g}_n(s)\,ds$$

then (2.1) becomes

$$(2.2) \qquad \sum_{m=1}^{\infty} \int_a^b y(t)\bar{h}_m(t)\,dt \int_a^b \left(\int_c^d K(s,t)\bar{g}_n(s)\,ds\right) h_m(t)\,dt = f_n \quad (n=1,2\ldots)$$

The first integral in (2.2) is y_m, the expansion coefficient of $y(t)$ with respect to $(h_m(t))_{m=1}^{\infty}$.

Let the second integral be denoted by a_{nm} i.e.

$$(2.3) \qquad a_{nm} = \int_a^b \left(\int_c^d K(s,t)\bar{g}_n(s)\,ds\right) h_m(t)\,dt$$

then (2.2) becomes

$$(2.4) \qquad \sum_{m=1}^{\infty} y_m a_{nm} = f_n \qquad (n=1,2\ldots)$$

This is an infinite system of linear equations with y_m $(m=1,2..)$ unknown.

For (1.1) to have a solution in L^2 it is necessary that (2.4) have a solution $(y_1, y_2, \ldots)$ with $\sum_{m=1}^{\infty} |y_m|^2$ convergent.

This is also sufficient, for if a solution $(y_1, y_2, \ldots)$ of (2.4) with $\sum_{m=1}^{\infty} |y_m|^2$ convergent is given, then by the Riesz-Fischer Theorem there exists an L^2 function $y(t)$ whose expansion coefficients with respect to $(h_m(t))$ are precisely y_m. For this function $y(t)$ we have then

$$\sum_{m=1}^{\infty} y_m a_{nm} = \sum_{m=1}^{\infty} \int_a^b y(t)\overline{h_m}(t)\,dt \cdot \int_a^b \left(\int_c^d K(s,t)\overline{g_n}(s)\,ds \right) h_m(t)\,dt$$

$$= \int_a^b y(t)\,dt \int_c^d K(s,t)\overline{g_n}(s)\,ds$$

$$= \int_c^d \overline{g_n}(s)\,ds \int_a^b K(s,t)y(t)\,dt$$

but

$$\sum_{n=1}^{\infty} y_m a_{nm} = f_n = \int_c^d f(s)\overline{g_n}(s)\,ds$$

thus $f(s)$ and

$$\int_a^b K(s,t)y(t)\,dt$$

have the same expansion coefficients with respect to $(g_n(s))_{n=1}^{\infty}$. That is

$$f(s) \overset{0}{=} \int_a^b K(s,t)y(t)\,dt,$$

where $\overset{0}{=}$ means "equal almost everywhere".

We sum up our result in

Theorem 2.1 If

$f(s) \in L^2$

$K(s,t)$ satisfies condition A

then (1.1) has a solution $y(t) \in L^2$ if and only if the system

$$\sum_{m=1}^{\infty} a_{nm} y_m = f_n \qquad (n = 1,2,\ldots)$$

has a solution vector $\vec{Y} = (y_1, y_2, \ldots)$ with $\sum_{m=1}^{\infty} |y_m|^2$ convergent.

The solution is then given by

$$y(t) \sim \sum_{m=1}^{\infty} y_m h_m(t).$$

Remark: The question now arises as to when $K(s,t)$ satisfies condition A. It can be shown that [4, pp. 44-47],

Theorem 2.2 $K(s,t)$ satisfies condition A if and only if there exist two complete orthogonal systems $(g_n(s))_{n=1}^{\infty}$ and $(h_m(t))_{m=1}^{\infty}$ such that

a) $g_n(s)$, $h_m(t)$ satisfy condition A for all n,m

b) the coefficient matrix (a_{nm}) defined by (2.4)

is bounded; i.e. there exists a constant M independent of r such that for any ennuples $(x_1, x_2, \ldots x_n)$ and $(y_1, y_2, \ldots, y_n)$ the inequality

$$\left| \sum_{n,m=1}^{r} a_{nm} x_n y_m \right| \leq M \left(\sum_{n=1}^{r} |x_n|^2 \right)^{\frac{1}{2}} \left(\sum_{m=1}^{r} |y_m|^2 \right)^{\frac{1}{2}}$$

is satisfied.

3. Example

Using the method described in the previous section has two disadvantages; one is that it leaves the solution in the form of a series, the second is the difficulty of solving the system (2.4). However, it may happen that with a fortunate choice of the orthogonal systems the matrix in (2.4) is of the form

$$(a_{nm}) = \begin{pmatrix} a_{11} & 0 & 0 \dots \\ 0 & a_{22} & 0 \dots \\ 0 & 0 & a_{33} \\ \dots & \dots & \dots \end{pmatrix} \tag{3.1}$$

if so, then (2.4) is easily solved, for y_m $(m = 1,2,3\dots)$ are then given by

$$y_m = f_m / a_{nm}$$

As an example consider

$$\frac{1}{\pi}\int_0^{\pi} \frac{\sin t\, y(t)\,dt}{\cos t - \cos s} = f(s) \tag{3.2}$$

For the orthogonal systems $(g_n(s))$, $(h_m(t))$ we choose

$$g_0(s) = 1/\pi^{\frac{1}{2}}, \qquad g_n(s) = (2/\pi)^{\frac{1}{2}} \cos ns \qquad (n = 1,2\dots)$$

$$h_m(t) = (2/\pi)^{\frac{1}{2}} \sin mt \qquad (m = 1,2\dots)$$

and find

$$a_{nm} = \int_0^{\pi} (2/\pi)^{\frac{1}{2}} \cos ns\, ds \int_0^{\pi} \frac{(1/\pi) \sin t}{\cos t - \cos s} (2/\pi)^{\frac{1}{2}} \sin mt\, dt \qquad \begin{matrix}(n=1,2\dots)\\(m=1,2\dots)\end{matrix}$$

$$a_{0m} = \int_0^{\pi} (1/\pi)^{\frac{1}{2}} ds \int_0^{\pi} (1/\pi) \frac{\sin t}{\cos t - \cos s} (2/\pi)^{\frac{1}{2}} \sin mt\, dt \qquad (m=1,2\dots)$$

For the inner integral we have

$$\frac{(2/\pi)^{\frac{1}{2}}}{\pi}\int_0^\pi \frac{\sin t \sin mt}{\cos t - \cos s}\, dt = \frac{(2/\pi)^{\frac{1}{2}}}{2\pi}\int_0^\pi \frac{\cos(m-1)t - \cos(m+1)t}{\cos t - \cos s}\, dt .$$

But

$$\frac{1}{\pi}\int_0^\pi \frac{\cos nt}{\cos t - \cos s}\, dt = \frac{\sin ns}{\sin s} \qquad (n = 1,2,\ldots).$$

Therefore

$$\frac{(2/\pi)^{\frac{1}{2}}}{\pi}\int_0^\pi \frac{\sin t \sin mt}{\cos t - \cos s}\, dt = \frac{1}{2}(2/\pi)^{\frac{1}{2}}\frac{\sin(m-1)s - \sin(m+1)s}{\sin s}$$

$$= -(2/\pi)^{\frac{1}{2}}\cos ms.$$

Thus

$$(3.3) \qquad \begin{aligned} a_{nm} &= -\delta_{nm} \qquad && (n,m = 1,2,\ldots) \\ a_{om} &= 0 && (m = 1,2,\ldots). \end{aligned}$$

The matrix (a_{nm}) is bounded. Also the orthogonal systems $(g_n(s))$ and $(h_m(t))$ satisfy condition A. By Theorem 2.2, $K(s,t)$ satisfies condition A. Thus we can apply Theorem 2.1.

Therefore the problem of solving (3.2) is equivalent to solving

$$\sum_{m=1}^{\infty} a_{nm} y_m = f_n \qquad (n = 0,1,2,\ldots)$$

with a_{nm} given by (3.3).

This system is easily solved since the coefficient matrix is of a simple form. We obtain

$$0 = f_o$$

$$-y_n = f_n \qquad (n=1,2,3...)$$

Thus an L^2 solution exists only if

$$f_o = \int_0^{\pi} f(s)\,ds = 0$$

and $\sum_{n=1}^{\infty} |y_n|^2$ is convergent.

However we know $f \in L^2$, and this implies using Bessel's inequality, that $\sum_{n=0}^{\infty} |f_n|^2$ is convergent. Therefore

$$\sum_{n=1}^{\infty} |y_n|^2 = \sum_{n=1}^{\infty} |-f_n|^2 = \sum_{n=1}^{\infty} |f_n|^2$$

is also convergent.

We sum up our result in

Theorem 3.1 The equation

$$1/\pi \int_0^{\pi} \frac{\sin t}{\cos t - \cos s}\, y(t)\,dt = f(s) \qquad (f(s) \in L^2)$$

has an L^2 solution given by

$$y(t) \sim -(2/\pi)^{\frac{1}{2}} \sum_{n=1}^{\infty} f_n \sin nt$$

only if

$$\int_0^{\pi} f(s)\,ds = 0.$$

4. The homogeneous integral equation

In the above example the matrix was of a simple type because of the fortunate choice of the orthogonal system. Such a choice is not always possible. In that case the orthogonal systems have to be constructured by a somewhat laborious process.

For the construction we begin with the homogeneous equation

$$(4.1)\qquad \int_a^b K(s,t)y(t)\,dt = 0.$$

We assume $K(s,t)$ satisfies condition A. By Theorem 2.1, the equation (4.1) is equivalent to the system

$$(4.2)\qquad \sum_{m=1}^{\infty} a_{nm}y_m = 0 \qquad (n = 1,2,\ldots).$$

Since $K(s,t)$ satisfies condition A this implies (Thm 2.2) that (a_{nm}) is bounded. From matrix theory we quote the following

<u>Theorem 4.1</u> Every system

$$(4.3)\qquad \sum_{m=1}^{\infty} a_{nm}y_m = 0 \qquad (n = 1,2,\ldots)$$

where the matrix (a_{nm}) is bounded, can be replaced by a system

$$(4.4)\qquad \sum_{m=1}^{\infty} b_{nm}y_m = 0 \qquad (n = 1,2,\ldots)$$

with (b_{nm}) an orthogonal matrix.

The system (4.4) has either the single solution $\bar{Y} = \bar{0}$ where $\bar{Y} = (y_1, y_2, \ldots)$ and $\bar{0}$ is the zero vector or there exist finite (or countably) many solution vectors $\bar{Y}_i$ which, together with the vectors $\bar{B}_n = (b_{n1}, b_{n2}, \ldots)$, form a complete orthogonal system of vectors.

Every solution of (4.3) can then be given by a linear form

$$(4.5)\qquad \bar{Y} = c_1\bar{Y}_1 + c_2\bar{Y}_2 + \ldots$$

in which $\sum_i |c_i|^2$ is convergent.

Conversely, every such form (4.5) with $\sum_i |c_i|^2$ convergent is a solution of (4.3).

Returning to the homogeneous equation (4.1) we obtain, using Theorem (4.1), the following result.

Theorem (4.2 The equation

$$(4.1) \qquad \int_a^b K(s,t)y(t)\,dt = 0$$

has either the solution $y(t) \equiv 0$ only, or there exists an orthogonal system $(h_{oi}(t))_{i=1}^{N}$ with N finite or infinite, such that every L^2 solution $y(t)$ of (4.1 can be given by an equivalence

$$(4.6) \qquad y(t) \sim \sum_{i=1}^{N} c_i h_{oi}(t) \qquad \text{(N finite or infinite)}$$

in which $\sum_{i=1}^{N} |c_i|^2$ is convergent.

Conversely, every equivalence (4.6) with $\sum |c_i|^2$ convergent is a solution of (4.1).

Corollary 1 A similar theorem holds for the homogeneous equation

$$(4.7) \qquad \int_c^d K(s,t)x(s)\,ds = 0$$

with solutions $x(s) \equiv 0$ only, or $(\overline{g}_{oi}(s))_{i=1}^{N_1}$ with N_1 finite or infinite.

Corollary 2 Without loss of generality, the orthogonal systems $(h_{oi}(t))$ and $(\overline{g}_{oj}(s))$ can be assumed to be normalized.

5. Construction of the orthogonal system

The orthogonal systems $(h_{oi}(t))$ and $(\overline{g}_{oj}(s))$ mentioned in Theorem 4.2 and Cor. 1 are not complete. For assume $(\overline{g}_{oj}(s))$ complete. This implies that any $x(s) \in L^2$ is a solution of (4.7), in particular

$$x(s) = \begin{cases} 1 & c \leq s_1 \leq s \leq s_2 \leq d \\ 0 & \text{otherwise} \end{cases}$$

is a solution. That is

$$\int_{s_1}^{s_2} K(s,t)\,ds = 0$$

which implies that $K(s,t) \equiv 0$, a contradiction.

Thus (h_{oi}) and $(\overline{g}_{oj})$ are not complete. We complete these orthogonal systems and then use the completed systems to solve (1.1).

To complete $(\overline{g}_{oj}(s))$ we take any function $g_1(s)$ orthogonal to all g_{oj}. $g_1(s)$ exists since with $(\overline{g}_{oj})$ the system (g_{oj}) is not complete. Without loss of generality we can take $g_1(s)$ to be normalized. Define a function $h_1(t)$ by

$$(5.1) \qquad \int_c^d K(s,t)\overline{g}_1(s)\,ds = n_1 \overline{h}_1(t)$$

$n_1 \neq 0$ by definition of $\overline{g}_1(s)$, furthermore $h_1(t)$ is orthogonal to every member of $(h_{oi}(t))$. For let $h_o(t)$ be one such member then

$$\int_a^b h_o(t)\overline{h}_1(t)\,dt = 1/n_1 \int_a^b h_o(t)\,dt \int_c^d K(s,t)\overline{g}_1(s)\,ds$$

$$= 1/n_1 \int_c^d \overline{g}_1(s)\,ds \int_a^b h_o(t)K(s,t)\,dt$$

$$= 0.$$

Next we define $g_2(s)$ by

$$\int_a^b K(s,t)h_1(t)\,dt - n_1 g_1(s) = m_1 g_2(s) \tag{5.2}$$

Here two cases $m_1 = 0$, $m_1 \neq 0$ have to be considered. In the first case $g_2(s)$ is not yet defined by (5.2) and we can take for $g_2(s)$ any function orthogonal to $(g_{oi}(s))$ and $g_1(s)$. If however, the system

$$\{(g_{oi}(s)),\ g_1(s)\}$$

is complete nothing remains to be done.

In the second case (5.2) defines $g_2(s)$ which we can assume to be normalized. Again we claim that $g_2(s)$ is orthogonal to all $g_{oi}(s)$ and $g_1(s)$. To show this we multiply (5.1) by $h_1(t)$, (5.2) by $\overline{g}_1(s)$ integrate and subtract one equation from the other to get

$$\int_c^d m_1 g_2(s)\overline{g}_1(s)\,ds = 0$$

A similar argument shows that

$$\int_c^d g_2(s)\overline{g}_{oi}(s)\,ds = 0$$

for any member of $(g_{oi}(s))$.

We then define $h_2(t)$ by

(5.3) $$\int_c^d K(s,t)\overline{g}_2(s)\,ds - m_1\overline{h}_1(t) = n_2\overline{h}_2(t)$$

The coefficient $n_2 \neq 0$, since $n_2 = 0$ implies, using (5.1), that

$$m_1\overline{g}_1(s) - n_1\overline{g}_2(s)$$

is a solution of (5.1). This however contradicts the orthogonality of g_1 and g_2 to the system (g_{oi}).

As before we can easily see that $h_2(t)$ is orthogonal to $(h_{oi}(t))$ and $h_1(t)$.

We proceed with this construction by defining in general

(5.4) $$\int_a^b K(s,t)h_{i-1}(t)\,dt - n_{i-1}g_{i-1}(s) = m_{i-1}g_i(s)$$

(5.5) $$\int_c^d K(s,t)\overline{g}_i(s)\,ds - m_{i-1}\overline{h}_{i-1}(t) = n_i\overline{h}_i(t)$$

until the system

$$\{(g_{oi}(s)),\ g_1(s),\ g_2(s),\dots\}$$

is complete.

When the above system is complete then

$$\{(h_{oi}(t)),\ h_1(t),\ h_2(t),\ \dots\}$$

is also complete. For by (5.5) it follows that for any function $h(t) \not\equiv 0$ orthogonal to all h_{oi} and h_j

$$\int_c^d \overline{g}_i(s)\,ds \int_a^b K(s,t)h(t)\,dt = 0$$

in particular for $g_i(s) = g_{oi}(s)$.

This implies that

$$\int_a^b K(s,t)h(t)\,dt = 0$$

Thus $h(t)$ is a member of $\{h_{oi}(t)\}$. This contradicts the definition of $h(t)$.

It may happen that after countably many steps the system

(5.6) $$\{(g_{oi}(s)),\ g_1(s),\ g_2(s), \ldots\}$$

is not complete. In this case we take any function orthogonal to all members of (5.6) and proceed as before.

Since any complete orthogonal systems have only countably infinite many members, the construction process must end after at most countably infinite many steps.

We use the two complete orthogonal systems to solve (1.1).

From (5.4) and (5.5) it follows that

$$a_{ii-1} = \int_c^d \overline{g_i}(s)\,ds \int_a^b K(s,t)h_{i-1}(t)\,dt = m_{i-1}$$

$$a_{ii} = \int_c^d \overline{g_i}(s)\,ds \int_a^b K(s,t)h_i(t)\,dt = n_i$$

$a_{jk} = 0$ otherwise.

Thus the matrix (a_{ij}) has the form

$$(a_{ij}) = \begin{pmatrix} 0 & 0 & 0 & \cdots & & & \\ 0 & n_1 & 0 & \cdots & & & \\ 0 & m_1 & n_2 & 0 & \cdots & & \\ \cdots & 0 & m_2 & n_3 & 0 & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

where we have as many zero rows as members of $(g_{oi}(s))$ and as many zero columns as members of $(h_{oj}(t))$.

The system (2.4) with the above matrix can be solved more easily.

<u>Remark:</u> Under some conditions the above matrix reduces to a diagonal matrix. It can be shown [4, p.169], that if $K(s,t) \in L^2$ and $K(s,t)$ satisfies condition A, then two orthogonal systems can be constructed such that the matrix is a diagonal matrix.

Obviously, this construction method is not very practical, so that most integral equations of the first kind are solved by special methods which are best suited for the particular equation involved.

The next chapters are devoted to some of these special methods.

6. The Fourier transform

The well-known Fourier transform

$$(6.1) \qquad \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} y(t)\, e^{ist}\, dt = f(s) \qquad (f(s) \in L^2)$$

is, from a different point of view, an integral equation of the first kind. For this equation a fortunate choice of orthogonal systems can be made.

Using the Hermitian functions

$$\Phi_n(s) = \frac{(-1)^n}{(2^n n!\sqrt{\pi})^{\frac{1}{2}}} e^{\frac{s^2}{2}} \frac{d^n e^{-s^2}}{ds^n} \qquad (n = 0,1,2,\ldots)$$

we have [8, p. 81],

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{ist}\Phi_n(t)\,dt = i^n\Phi_n(s) \qquad (n = 0,1,2,\ldots)$$

and we obtain for the matrix of the system (2.4)

$$(a_{nm}) = (i^n \delta_{nm}).$$

This matrix is bounded. Also the functions $\Phi_n(s)$ satify condition A, so that by Theorems 2.1 and 2.2 the equation (6.1) is equivalent to the system

$$\sum_{m=0}^{\infty} y_m i^n \delta_{nm} = f_n \qquad (n = 0,1,2,\ldots)$$

and the solution is

$$y(t) \sim \sum_{n=0}^{\infty} (-1)^n f_n \Phi_n(t)$$

if $\sum_{n=0}^{\infty} |(-1)^n f_n|^2$ is convergent, which is true since $f \in L^2$.

Now the expression

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(s)e^{-ist}dt$$

has the same expansion coefficients as $y(t)$. Thus

$$y(t) \stackrel{o}{=} \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} f(s)e^{-ist}dt$$

gives the solution in a closed form.

Besides the Plancherel Theorem for which Titchmarsh gives several proofs [8,p.69], there is another method which makes use of an inversion operator, This method is due to Rooney [9].

Here we consider the slightly more general Fourier transform

$$(6.2) \qquad F(x) = \frac{d}{dx}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} ((e^{ixy}-1)/iy)f(y)\,dy$$

and the operator

$$F_{k,t}(F) = (-ik/t)^{k+1}(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} (x-ik/t)^{-(k+1)}F(x)\,dx$$

where $k = 1,2,\ldots,$. Rooney shows that,

If $f \in L^p(-\infty,\infty)$, $1 \leq p \leq 2$ and F is defined by (6.2) then

a.) $\lim_{k\to\infty} F_{k,t}(F) = f(t)$ at every point $t \neq 0$ in the Lebesgue set of f.

b) $\lim_{k\to\infty} \int_{-\infty}^{\infty} \left|F_{k,t}(F) - f(t)\right|^p dt = 0,$

where the Lebesgue set of f(x) is the set of values of x for which

$$\int_0^h \left|f(x+t) - f(x)\right| dt = o(h) \qquad (h \to 0).$$

The proof is similar to those of the Laplace transforms and the Widder-Post inversion operator. In fact Rooney derives the operator $F_{k,t}$ by changing the Fourier transform into the Laplace transform and then applying the Widder-Post inversion operator.

CHAPTER II

THE CONVOLUTION TRANSFORM

7. Introduction

The integral equations

$$(7.1) \qquad \int_{-\infty}^{\infty} K(s-t)y(t)\,dt = f(s) \qquad -\infty < s < \infty$$

where the kernel is of the form $K(s-t)$ are called convolution transforms. It is s mall exaggeration to say that nearly all the integral transforms are either in this form or can be put into it by a change of variable.

A very useful method for inverting the convolution transform is the operational calculus. The technique consists in treating the operational symbol D (for differentiation) as if it were a number throughout some calculation and finally in restoring to it its original operational meaning.

As an illustration let us deduce a meaning for the operation e^{aD}. Treating D as a number we have formally

$$e^{aD} = \sum_{k=0}^{\infty} \frac{a^k D^k}{k!}$$

Operating on $f(x)$ we obtain

$$e^{aD}f(x) = \sum_{k=0}^{\infty} \frac{a^k D^k f(x)}{k!}.$$

$$= \sum_{k=0}^{\infty} \frac{a^k f^{(k)}(x)}{k!} = f(x+a)$$

We define

$$e^{aD}f(x) = f(x+a) \tag{7.2}$$

even if $f(x)$ is not differentiable.

We apply this operational procedure to obtain an inversion formula for the convolution transform.[5]

8. The inversion operator

Consider the equation

$$\int_{-\infty}^{\infty} K(s-t)y(t)\,dt = f(s) \tag{8.1}$$

Let

$$1/E(s) = \int_{-\infty}^{\infty} K(x)e^{-sx}\,dx \tag{8.2}$$

be the bilateral Laplace transform of $K(u)$. We then have formally

$$\begin{aligned} (1/E(D))y(s) &= \int_{-\infty}^{\infty} K(x)e^{-Dx}\,dx\; y(s) \\ &= \int_{-\infty}^{\infty} K(x)e^{-xD}y(s)\,dx \\ &= \int_{-\infty}^{\infty} K(x)y(s-x)\,dx \\ &= \int_{-\infty}^{\infty} K(s-x)y(x)\,dx \\ &= f(s) \end{aligned} \tag{8.3}$$

Thus

$$[E(D)]^{-1} y(s) = f(s) \tag{8.4}$$

Operating on (8.4) by E(D) we obtain formally

$$y(s) = E(D)f(s) \tag{8.5}$$

our desired inversion formula. Thus the solution of (8.1) would be known if we could interpret the operator E(D). An effective interpretation of E(D) can always be made if E(s) defined by (8.2) is an entire function. E(D) in this case is a differentiation operator. See [5], [14], [15].

In [6] and [7] D.B. Sumner dismussed convolution transforms which admitted an inversion function E(s) which was meromorphic i.e. of the form G(s)/F(s) with G,F entire. G was interpreted as a differentiator, F as an integrator. Thus E(s) consisted of an integrator and a differentiator factor.

In the next section a convolution transform will be discussed which admitts a meromorphic inversion function E(s) where the integrator factor and the differentiator factor can be represented in one step.

9. A convolution transform with an inversion operator of integro-differential type

Consider

$$\int_{-\infty}^{\infty} \log\left|\coth \frac{x-t}{2}\right| y(t)\,dt = f(x) \tag{9.1}$$

The bilateral Laplace transform of K(u) is

$$\int_{-\infty}^{\infty} \log\left|\coth \frac{t}{2}\right| e^{-st}\,dt = 1/E(s). \tag{9.2}$$

But

$$\int_{-\infty}^{\infty} \log\left|\coth t/2\right| e^{-st}\,dt = \int_{-\infty}^{\infty} e^{-st}\log\left|\frac{1+e^{-t}}{1-e^{-t}}\right| dt$$

set $e^{-t} = x$ and integrate by parts. Then (9.2) becomes

$$1/E(s) = \int_0^{\infty} \log\left|\frac{1+x}{1-x}\right| x^{s-1}\,dx$$

$$= \log\left|\frac{1+x}{1-x}\right| x^s/s \Big|_0^{\infty} - 2/s \int_0^{\infty} x^s/(1-x^2)\,dx$$

The first term tends to zero if $-1 < s < 0$, the second becomes

$$-1/s\left[\int_0^{\infty} x^s/(1-x)\,dx + \int_0^{\infty} x^s/(1+x)\,dx\right].$$

Using the well-known integrals [10, p.105],

$$\int_0^{\infty} \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a} \qquad 0 < a < 1$$

$$\int_0^{\infty} \frac{x^{a-1}}{1-x}\,dx = \pi \cot(a\pi) \qquad 0 < a < 1$$

we obtain

$$1/E(s) = \pi/s \tan \pi s/2 \qquad -1 < s < 0$$

Thus the inversion operator E(D) becomes

$$E(D) = (D/\pi) \cot D\pi/2$$

a meromorphic function.

To interpret the operator E(D) we make use of the well-known integral [10, p. 46],

$$\int_0^\infty \frac{\sinh (1+s)t\pi}{\sinh t} \, dt = -\frac{1}{2} \cot \frac{\pi s}{2}$$

After a change of variable $\pi t = z$ we obtain

$$E(s) = s/\pi \cot s\pi/2 \qquad s/\pi^2 \int_0^\infty \frac{e^{-z} e^{-sz} - e^{z} e^{sz}}{\sinh z} \, dz$$

We now make use of the definition of e^{aD} to get

$$\text{(9.2a)} \qquad E(D)f(x) = D/\pi^2 \int_0^\infty \frac{e^{-z} f(x-z) - e^{z} f(x+z)}{\sinh z} \, dz$$

$$= 1/\pi^2 \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{-z} f(x-z)}{\sinh z} \, dz.$$

We claim that

$$\text{(9.3)} \qquad y(x) = 1/\pi^2 \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{-z} f(x-z)}{\sinh z} \, dz = E(D)f(x)$$

To prove this we substitute the expression (obtained from (9.1))

$$f(x-z) = \int_{-\infty}^{\infty} \log \left| \coth \frac{x-z-t}{2} \right| y(t) \, dt$$

in the left side of (9.3) and obtain

$$1/\pi^2 \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{-z}}{\sinh z} dz \int_{-\infty}^{\infty} \log \left| \coth \frac{x-z-t}{2} \right| y(t)\,dt$$

Making use of the Tonelli-Hobson Theorem we can invert the order of integration to obtain

$$(9.4) \qquad 1/\pi^2 \frac{d}{dx} \int_{-\infty}^{\infty} y(t)\,dt \int_{-\infty}^{\infty} \frac{e^{-z}}{\sinh z} \log \left| \coth \frac{x-z-t}{2} \right| dz$$

To obtain the inner integral I in (9.4) we set $x-t = w$. Then

$$I = \int_{-\infty}^{\infty} \frac{e^{-z}}{\sinh z} \log \left| \coth \frac{w-z}{2} \right| dz$$

$$= 2\int_{-\infty}^{\infty} \frac{e^{-z}}{e^{z} - e^{-z}} \log \left| \frac{e^{w} + e^{z}}{e^{w} - e^{z}} \right| dz$$

$$= 2\int_{0}^{\infty} \frac{u}{v^2 - u^2} \log \left| \frac{u+1}{u-1} \right| du$$

where we have made the substitution $e^{w} = v$ and $e^{z} = v/u$

Then

$$I = 2\int_{0}^{1} \frac{u}{v^2 - u^2} \log \left| \frac{1+u}{1-u} \right| du + 2\int_{1}^{\infty} \frac{u}{v^2 - u^2} \log \frac{u+1}{u-1}\, du$$

In the first integral set

$$\frac{1+u}{1-u} = e^{t},$$

in the second set

$$\frac{u+1}{u-1} = e^{t}.$$

Then

$$(9.5) \quad I = 4\int_0^{\infty} \frac{(e^t-1)te^t}{(e^t+1)(v^2(e^t+1)^2-(e^t-1)^2)}\,dt$$

$$+\ 4\int_0^{\infty} \frac{(e^t+1)te^t}{(e^t-1)(v^2(e^t-1)^2-(e^t+1)^2)}\,dt$$

Since both integrands are even functions we have

$$I = 2\int_{-\infty}^{\infty}\left\{\frac{(e^t-1)te^t}{(e^t+1)(v^2(e^t+1)^2-(e^t-1)^2)} + \frac{(e^t+1)te^t}{(e^t-1)(v^2(e^t-1)^2-(e^t+1)^2)}\right\}dt$$

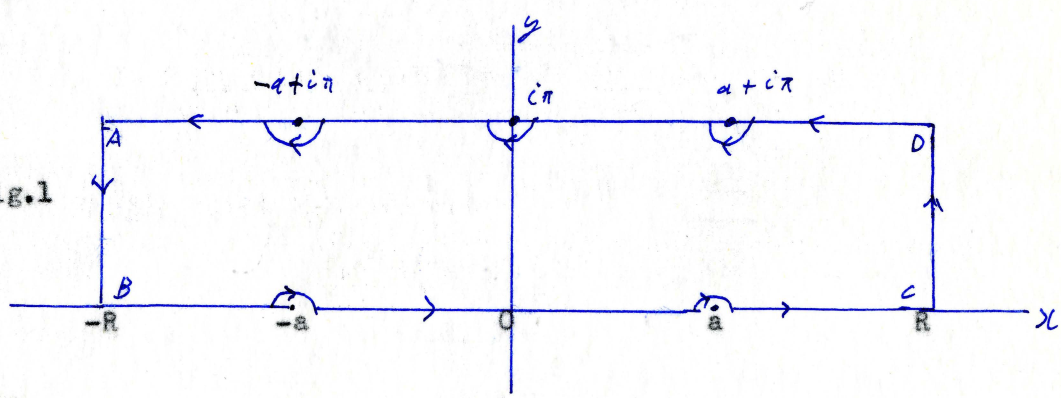
$$= 2J$$

To evaluate the integral J we consider the contour integral

$$(9.6) \quad \oint\left\{\frac{(e^z-1)ze^z}{(e^z+1)(v^2(e^z+1)^2-(e^z-1)^2)} + \frac{(e^z+1)ze^z}{(e^z-1)(v^2(e^z-1)^2-(e^z+1)^2)}\right\}dz$$

around the contour shown in fig. 1.

fig.1



where

$$a = \begin{cases} \log\frac{v+1}{v-1}, & v > 1 \\ \log\frac{1+v}{1-v}, & 0 < v < 1 \end{cases}$$

The first term of (9.6) has simple poles at

$$z = i\pi$$

$$z = \pm\log\frac{1+v}{1-v} \qquad 0 < v < 1$$

$$z = \pm\log\frac{v+1}{v-1} + i\pi \qquad v > 1$$

The second term has simple poles at

$$z = \pm\log\frac{1+v}{1-v} + i\pi \qquad 0 < v < 1$$

$$z = \pm\log\frac{v+1}{v-1} \qquad v > 1$$

Thus we have to consider two cases corresponding to $v > 1$ or $0 < v < 1$.

For $v > 1$ we have poles at

$$z = i\pi \qquad \text{with residue} \qquad -i\pi/2$$

(9.7) $$z = \pm\log\frac{v+1}{v-1} + i\pi \qquad \text{with total residue} \qquad -i\pi/2$$

$$z = \pm\log\frac{v+1}{v-1} \qquad \text{with total residue} \qquad 0$$

For $0 < v < 1$ we have poles at

$$z = i\pi \qquad \text{with residue} \qquad -i\pi/2$$

(9.8) $$z = \pm\log\frac{1+v}{1-v} + i\pi \qquad \text{with total residue} \qquad +i\pi/2$$

$$z = \pm\log\frac{1+v}{1-v} \qquad \text{with total residue} \qquad 0$$

It is easily seen that the integrals along AB and CD are of the order $O(R/e^R)$.

Furthermore, both integrals along BC and DA tend to J, as $R \to \infty$

Thus we have

$$0 = 2J + O(R/e^R) - i\pi(\text{sum of residues})$$

i.e. $$2J = i\pi(\text{sum of residues}) \quad \text{when } R \to \infty$$

Fom (9.7) and (9.8) it easily seen that

$$i\pi(\text{sum of residues}) = \begin{cases} \pi^2 & v > 1 \\ 0 & 0 < v < 1 \end{cases}$$

Thus

$$2J = I = \begin{cases} \pi^2 & v > 1 \\ 0 & 0 < v < 1. \end{cases}$$

Recalling the definition of $v = e^w$, $w = x - t$ we obtain

$$I = \begin{cases} \pi^2 & x > t \\ 0 & x < t \end{cases}$$

For the case $x = t$ i.e. $v = 1$, we return to (9.5) and obtain

$$I = 4\int_0^\infty \frac{t}{e^t - e^{-t}}\, dt = 4\pi^2/8 = \pi^2/2$$

Substituting the values for the inner integral into (9.4) we finally have

$$1/\pi^2 \frac{d}{dx}\int_{-\infty}^{\infty} y(t)\, I\, dt = \frac{d}{dx}\int_{-\infty}^{x} y(t)\, dt$$
$$\stackrel{o}{=} y(x).$$

This proves our claim (9.3); and we see that the operator

$$E(D) = (D/\pi)\cot \pi D/2$$

defined by (9.2a) inverts the equation (9.1).

Summing up our result we have:

The equation

$$\int_{-\infty}^{\infty} \log \left|\coth \frac{x-t}{2}\right| y(t)\,dt = f(x)$$

has the solution

$$y(x) = 1/\pi^2 \frac{d}{dx} \int_{-\infty}^{\infty} \frac{e^{-s} f(x-s)\,ds}{\sinh s}.$$

10. Inversion of the convolution transform using Fourier transforms

Be-sides the operational method described in section 8 and 9, the Fourier transform can be used to invert the convolution transform. See [8].

Consider

$$(10.1) \qquad \int_{-\infty}^{\infty} k(s-t)y(t)\,dt = f(s)$$

with $y(t)$, $f(s) \in L^2$ and $k(s-t)$ satisfying cond. A.

Multiplying by $1/(2\pi)^{\frac{1}{2}} e^{ius}$ and integrating,

$$1/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} f(s) e^{ius} ds = 1/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{ius} ds \int_{-\infty}^{\infty} k(s-t)y(t)\,dt$$

$$= 1/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{iux} k(x)\,dx \int_{-\infty}^{\infty} e^{iut} y(t)\,dt$$

the inversion of integration being justified by the conditions on k, f, and y.

Thus

$$F(u) = K(u)Y(u)(2\pi)^{\frac{1}{2}} \tag{10.2}$$

where

$$F(u) = 1/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} f(s)\,e^{ius}\,ds$$

is the Fourier transform of f(s) and K(u) and Y(u) are the Fourier transforms of k(s) and y(s) respectively.

From (10.2) we have

$$Y(u) = F(u)/K(u)(2\pi)^{\frac{1}{2}}.$$

But

$$y(t) = 1/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} Y(u)\; e^{-iut}\,du$$

therefore

$$y(t) = 1/(2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} \frac{F(u)}{K(u)(2\pi)^{\frac{1}{2}}}\; e^{-iut}\,du$$

Thus the formal solution is given by

$$y(t) = 1/(2\pi)\int_{-\infty}^{\infty} \frac{F(u)}{K(u)}\, e^{-iut}du \tag{10.3}$$

For this to be an actual solution $K(u)$ has to satisfy special conditions which we summarize in

<u>Theorem 10.1</u> Let $f(s) \in L^2(-\infty, \infty)$, and $k(x) \in L(-\infty, \infty)$. Then in order that there should be a solution $y(t) \in L^2(-\infty, \infty)$, it is necessary and sufficient that $F(u)/K(u)$ should belong to $L^2(-\infty, \infty)$.

For the proof suppose that $f(s)$, $k(x)$, $y(t)$ belong to the given L-classes, and (10.1) holds. Then using Thm.65 in Titchmarsh [8] we find that (10.2) holds and $Y(u) \in L^2$, Hence $F(u)/K(u) \in L^2$.

Conversely, if $F(u)/K(u) \in L^2$ then $y(t)$ defined by (10.3) is in L^2 and by the same theorem in [8], the Fourier transform of the left side of (10.1) is

$$(2\pi)^{\frac{1}{2}}K(u)\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{F(u)}{K(u)} = F(u)$$

Hence (10.1) holds.

<u>Remark:</u> Laplace transforms instead of Fourier transforms can be used. The method is similar to the above method. For more detailed account we refer to Doetsch [16] and Widder [17].

CHAPTER III

THE USE OF FUNCTION THEORY IN SOLVING INTEGRAL EQUATIONS

11. Poisson's integral formula

The integral equation

$$(11.1)\qquad \frac{r}{\pi}\int_0^{\pi}\left\{\frac{\sin(s+t)}{1-2r\cos(s+t)+r^2}+\frac{\sin(s-t)}{1-2r\cos(s-t)+r^2}\right\}y(t)\,dt = f(s)$$

with $0<r<1$ can be solved by using the orthogonal systems [4, p.169]

$$g_n(s) = (2/\pi)^{\frac{1}{2}}\sin ns \qquad (n=1,2,\ldots)$$

$$h_m(t) = (2/\pi)^{\frac{1}{2}}\cos mt, \qquad h_0(t) = 1/\pi \qquad (m=1,2,\ldots)$$

The solution is again given in the form of a series

$$y(t) \sim \sum_{n=1}^{\infty}\frac{f_n}{r^n}\cos nt + y_0$$

with the condition that

$$\sum_{n=1}^{\infty}\left|\frac{f_n}{r^n}\right|^2$$

be convergent. y_0 is an arbitrarily chosen constant.

The equation (11.1) however, can be put in the form

$$(11.2)\qquad \frac{r}{\pi}\int_0^{2\pi}\frac{\sin(s-t)y(t)}{1-2r\cos(s-t)+r^2}\,dt = f(s) \qquad 0<r<1$$

(11.2) has the same denominator as the equation in Poisson's integral formula, viz. [10, p.124]

(11.3) $$u(r,\theta) \quad \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}\, u(R,\varphi)\, d\varphi$$

As it turns out, we can use (11.3) to solve (11.2) and give the solution in a closed form.

We first prove

<u>Theorem 11.1</u> Let $f(z)$ be analytic inside the circle $|z| \leq R$ and let $u(r,\theta)$, $v(r,\theta)$ be the real and imaginary part respectively.

Then for $0 \leq r < R$ we have

$$v(r,\theta) = \frac{rR}{\pi}\int_0^{2\pi} \frac{u(R,\varphi)\sin(\theta-\varphi)\, d\varphi}{R^2-2Rr\cos(\theta-\varphi)+r^2}$$

$$u(r,\theta) = -\frac{rR}{\pi}\int_0^{2\pi} \frac{v(R,\varphi)\sin(\theta-\varphi)\, d\varphi}{R^2-2Rr\cos(\theta-\varphi)+r^2} + \text{constant}$$

Proof: Let $z = re^{i\theta}$ $\quad r < R$

then $f(z)$ is analytic and can be expanded in a series i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (\alpha_n + i\beta_n) r^n e^{in\theta}$$

$$u(r,\theta) + iv(r,\theta)$$

Separating real and imaginary parts, we have

(11.4) $$u(r,\theta) = \sum_{n=0}^{\infty} (\alpha_n \cos n\theta - \beta_n \sin n\theta)r^n$$

(11.5) $$v(r,\theta) = \sum_{n=0}^{\infty} (\alpha_n \sin n\theta + \beta_n \cos n\theta)r^n$$

Both series are uniformly convergent with respect to θ.

Hence we may multiply by cos $n\theta$ or sin $n\theta$ and integrate term by term; and obtain from (11.4)

$$(11.6)\qquad \alpha_n R^n = 1/\pi \int_0^{2\pi} u(R,\varphi)\cos n\varphi\, d\varphi, \quad \alpha_0 = \frac{1}{2\pi}\int_0^{2\pi} u(R,\varphi)\, d\varphi$$

$$(11.7)\qquad \beta_n R^n = -1/\pi \int_0^{2\pi} u(R,\varphi)\sin n\varphi\, d\varphi, \quad \beta_0 = 0$$

from (11.5) we obtain

$$(11.8)\qquad \alpha_n R^n = 1/\pi \int_0^{2\pi} v(R,\varphi)\sin n\varphi\, d\varphi, \qquad \alpha_0 = 0$$

$$(11.9)\qquad \beta_n R^n = 1/\pi \int_0^{2\pi} v(R,\varphi)\operatorname{con} n\varphi\, d\varphi, \quad \beta_0 = \frac{1}{2\pi}\int_0^{2\pi} v(R,\varphi)\, d\varphi$$

We then substitute the values of α_n and β_n as given by (11.6) and (11.7) into equation (11.5) and obtain

$$(11.10)\qquad v(r,\theta) = 1/\pi \sum_{n=1}^{\infty} \frac{r^n}{R^n}\int_0^{2\pi} u(R,\varphi)(\cos n\varphi \sin n\theta - \sin n\varphi \cos n\theta)\, d\varphi$$

$$= 1/\pi \int_0^{2\pi} u(R,\varphi)\left\{\sum_{n=1}^{\infty} \sin n(\theta-\varphi)\ (r/R)^n\right\} d\varphi$$

$$= rR/\pi \int_0^{2\pi} \frac{u(R,\varphi)\sin(\theta-\varphi)}{R^2 - 2rR\cos(\theta-\varphi) + r^2}\, d\varphi$$

the inversion being justified by the uniform convergence of the series.

This gives the first part of our theorem. For the second part we substitute the values of α_n and β_n as given in (11.8) and (11.9) into (11.4). This will give

$$(11.11) \qquad u(r,\theta) = -rR/\pi \int_0^{2\pi} \frac{v(R,\varphi)\sin(\theta-\varphi)}{R^2-2Rr\cos(\theta-\varphi)+r^2}\, d\varphi + 1/2\pi \int_0^{2\pi} v(R,\varphi)\, d\varphi$$

where the second term can be chosen arbitrarily.

Equation (10.10) corresponds to the intgral equation (11.2) with $R = 1$, $y(t) = v(1,\theta)$, and $f(s) = v(r,\theta)$.

Hence the solution of (11.2) is

$$y(s) = -r/\pi \int_0^{2\pi} \frac{\sin(s-t)f(t)}{1-2r\cos(s-t)+r^2}\, dt \quad + \text{constant.}$$

Theorem 11.1 also shows that $y(t)$ and $f(t)$ are harmonic conjugates. Furthermore, it gives asolution for the more general integral equation

$$rR/\pi \int_0^{2\pi} \frac{\sin(s-t)y(t)}{R^2-2rR\cos(s-t)+r^2}\, dt \;=\; f(s) \qquad 0 < r < R.$$

12. A generalized Abel's integral equation

In this section we shall describe a function-theoretical method used by Carleman [11]. to solve the equation

$$(12.1) \qquad \int_0^1 \frac{y(t)\,dt}{|x-t|^\alpha} = f(x) \qquad 0 < x < 1, \quad 0 < \alpha < 1$$

This integral equation differs from Abel's equation viz.

$$(12.2) \qquad \int_0^x \frac{y(t)}{(x-t)^\alpha}\,dt = f(x) \qquad 0 < x < 1, \quad 0 < \alpha < 1$$

in that (12.2) has a variable upper limit of integration.

The solution of (12.2), which we assume to be known is 13

$$y(x) = \frac{\sin \alpha\pi}{\pi} \frac{d}{dx} \int_0^x f(s)(x-s)^{\alpha-1} ds.$$

We first consider the homogeneous equation which is written as the sum of two integrals; one from 0 to x the other from x to 1. Both integrals are shown to be identicaly equal to zero. From this we deduce that the only solution is $y = 0$.

To solve the nonhomogeneous equation we consider an auxiliary function $\Phi(x)$ containing the unknown function $y(x)$. $\Phi(x)$ is continued analytically into the complex plane and transformed into a Riemann boundary value problem. From the known solution of this problem we obtain the solution of (12.1).

A Riemann boundary value problem, or sometimes called a non-homogeneous Hilbert problem [19, p. 72], is the following problem.

Let R be a connected region, bounded by a smooth contour L. Find the sectionally holomorphic function $\Phi(z)$ having finite degree at infinity and satisfying on L the limiting condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t)$$

where G(t), g(t) are functions on L satisfying the Hölder condition and $G(t) \neq 0$ on L.

$\Phi^+(t), \Phi^-(t)$ are the limits of $\Phi(z)$ as z approaches L from the left and right respectively.

A function $\Phi(z)$ is said to be of finite degree, if in the expansion of $\Phi(z)$ in the neighbourhood of the point at infinity.

$$\Phi(z) = \sum_{j=-\infty}^{\infty} a_j z^j$$

there are only a finite number of terms with positive powers of z.

The general solution of the Riemann problem, which we assume to be known, is given by [19, p. 78],

$$\Phi(z) = \frac{X(z)}{2i\pi} \int_L \frac{g(t)\,dt}{X^+(t)(t-z)} + X(z)P(z)$$

where P(z) is an arbitrary polynomial and X(z) is a solution of the homogeneous Hilbert problem, i. e. $g(t) = 0$.

Following these preliminaries we return to the equation (12.1).

Consider the homogeneous equation

$$(12.3)\qquad \int_0^1 \frac{y(t)\,dt}{|x-t|^\alpha} = 0 = \int_0^x \frac{y(t)\,dt}{(x-t)^\alpha} + \int_x^1 \frac{y(t)\,dt}{(t-x)^\alpha}$$

and the function

$$(12.4)\qquad F(x) = \int_0^1 \frac{y(t)\,dt}{(x-t)^\alpha}$$

analytic for $x > 1$. $y(t)$ is assumed to be absolutely integrable and such that $y(t)/|x-t|^\alpha \in L(0,1)$.

For x real and $x < 1$ define

$$F(x+i0) = \lim_{\varepsilon \to 0} F(x+i\varepsilon),$$

$$F(x-i0) \quad \lim_{\varepsilon \to 0} F(x-i\varepsilon), \qquad (\varepsilon > 0),$$

From this definitionwe obtain the relations

$$(12.5a)\qquad F(x+i0) = \int_0^x \frac{y(t)}{(x-t)^\alpha}\,dt + e^{-\alpha\pi i}\int_x^1 \frac{y(t)\,dt}{(t-x)^\alpha} \qquad (0 < x < 1)$$

$$(12.5b)\qquad F(x+i0) = e^{-\alpha\pi i}\int_0^1 \frac{y(t)}{(t-x)^\alpha}\,dt \qquad (x < 0)$$

$$(12.6a)\qquad F(x-i0) = \int_0^x \frac{y(t)}{(x-t)^\alpha}\,dt + e^{\alpha\pi i}\int_x^1 \frac{y(t)}{(t-x)}\,dt \qquad (0 < x < 1)$$

$$(12.6b) \qquad F(x-i0) = e^{\alpha\pi i} \int_0^1 \frac{y(t)}{(t-x)^{\alpha}}\,dt \qquad (x < 0)$$

We claim that $F(x+i0)$ is real if $0 < x < 1$. To show this we use the reflection principle i.e. [12, p. 265].

If $F(z)$ is analytic in a domain D, D symmetric about the x-axis and if $\overline{F(z)} = F(\bar{z})$ then $F(z)$ is real whenever z is real.

Let $D = \{x+iy \mid 0 < x < 1,\ -\varepsilon < y < \varepsilon\}$ From (12.5a), (12.6a) we have

$$\overline{F(x+iy)} = F(x-iy) = F(\overline{x+iy})$$

It remains to show that $F(z)$ is analytic in D. $F(z)$ is analytic for $y \neq 0$. To prove $F(z)$ analytic on the segment $(0,1)$ of the x-axis consider

$$(12.7) \qquad G(x) = \int^x F(s)\,ds \qquad \frac{1}{1-\alpha} \int_0^1 (x-t)^{1-\alpha} y(t)\,dt$$

Since $|x-t|^{1-\alpha}$ is bounded and

$$\int^x |y(t)|\,dt$$

is absolutely continuous we see that $G(x+i\varepsilon)$ is uniformly convergent in the interval $0 < x < 1$ to $G(x+i0)$. Since $G(x+i\varepsilon)$ is analytic for $\varepsilon > 0$ we see that G is analytic on the interval $(0,1)$. By (12.7) we have that F is analytic on the interval $(0,1)$.

Thus $F(x+i0)$ is real. Similarly $F(x-i0)$ is real. This implies that

$$\int_x^1 \frac{y(t)}{(t-x)^{\alpha}}\,dt = 0.$$

Thus (12.2) reduces to

$$\int_0^x \frac{y(t)}{(x-t)^{\alpha}}\,dt = 0$$

But this is Abel's equation with the solution $y(t) \overset{\circ}{=} 0$.

We sum up our result in

Theorem 12.1 The homogeneous equation

$$\int_0^1 \frac{y(t)}{|x-t|^{\alpha}}\,dt = 0 \qquad 0 < x < 1, \quad 0 < \alpha < 1$$

where $y(t)/|x-t|^{\alpha}$ is assumed to be Lebesgue integrable on the interval $(o,1)$, has the only solution

$$y(t) \overset{\circ}{=} 0.$$

To solve the non-homogeneous equation (12.1) we again consider the function (12.3).

From (12.5a,b), (12.6a,b) and (12.1) we obtain the relations

(12.8) $$F(x+i0) = e^{-2i\alpha\pi} F(x-io) \qquad (x < 0)$$

(12.9) $$F(x+i0) = -e^{-i\alpha\pi} F(x-i0) + (1+e^{-i\alpha\pi}) f(x), (0 < x < 1)$$

To obtain (12.8) multiply (12.6b) by $e^{-2i\alpha\pi}$ and subtract the result from (12.5b). A similar calculation gives (12.9).

We now consider the auxiliary function

(12.10) $$\Phi(x) = F(x)\left[x(x-1)\right]^{\frac{\alpha-1}{2}}.$$

From (12.5a,b), (12.6a,b) we obtain the relations

(12.11) $$\Phi(x+i0) = \Phi(x-i0) \qquad (x<0)$$

(12.12) $$\Phi(x+i0)-\Phi(x-i0) = -2i\cos\frac{\alpha\pi}{2}\, f(x)\left[x(x-1)\right]^{\frac{\alpha-1}{2}} \qquad (0<x<1).$$

A Riemann boundary value problem with solution

(12.130 $$\Phi(x) = -1/\pi \,\cos\frac{\alpha\pi}{2}\int_0^1 \frac{f(s)}{s-x}\left[s(s-1)\right]^{\frac{\alpha-1}{2}}\,ds.$$

From (12.5a), (12.6a) we have

(12.14) $$\int_x^1 \frac{y(t)\,dt}{(t-x)^{\alpha}} = \frac{1}{2i\sin\alpha\pi}\left(F(x-i0)-F(x+i0)\right)$$

(12.15) $$\int_0^x \frac{y(t)}{(x-t)^{\alpha}}\,dt = \frac{1}{2i\sin\alpha\pi}\left(e^{i\alpha\pi}F(x+i0)-e^{-i\alpha\pi}F(x-i0)\right)$$

From (12.10) it follows that

$$F(x+i0) = i\left[x(1-x)\right]^{\frac{1-\alpha}{2}} e^{-i\frac{\alpha\pi}{2}}\Phi(x+i0)$$

$$F(x-i0) = -i\left[x(1-x)\right]^{\frac{1-\alpha}{2}} e^{i\frac{\alpha\pi}{2}}\Phi(x-i0).$$

Substitute these into (12.14) and (12.15) to get

(12.16) $$\int_x^1 \frac{y(t)\,dt}{(t-x)^{\alpha}} = \frac{-1}{2\sin\alpha\pi}\left(e^{i\frac{\alpha\pi}{2}}\Phi(x-i0)+e^{-i\frac{\alpha\pi}{2}}\Phi(x+i0)\right)\left[x(1-x)\right]^{\frac{1-\alpha}{2}},$$

(12.17) $$\int_0^x \frac{y(t)}{(x-t)^{\alpha}}\,dt = \frac{1}{2\sin\alpha\pi}\left(e^{i\frac{\alpha\pi}{2}}\Phi(x+i0)+e^{-i\frac{\alpha\pi}{2}}\Phi(x-i0)\right)\left[x(1-x)\right]^{\frac{1-\alpha}{2}}.$$

But (12.17) is Abel's equation, where the solution is known to be

$$(12.18) \qquad y(x) = \frac{1}{2\pi}\frac{d}{dx}\int_0^x \frac{1}{(x-t)^{1-\alpha}}\left(e^{i\frac{\alpha\pi}{2}}\Phi(t+i0)+e^{-i\frac{\alpha\pi}{2}}\Phi(t-i0)\right)\left[t(1-t)\right]^{\frac{1-\alpha}{2}}dt$$

Let Γ_x be a contour in the comlex t-plane which begins at the point $x \in (0,1)$, goes counter-clockwise about the origin and returns to x without cutting the positive real axis.

It is easily seen that (12.18) can be written

$$(12.19) \qquad y(x) = \frac{1}{2i\pi}\frac{d}{dx}\int_{\Gamma_x} \frac{1}{(t-x)^{1-\alpha}}\left[t(t-1)\right]^{\frac{1-\alpha}{2}}\Phi(t)\,dt.$$

This follows if we imagine the t-plane cut along the positive real axis and chose for $\left[t(t-1)\right]^{\frac{1-\alpha}{2}}$ and $(t-x)^{\alpha-1}$ those branches which, for real $t > 1$, are real on the upper edge of the real axis.

Substituting (12.13) into (12.19) we obtain

$$(12.20) \qquad y(x) = \frac{i\cos\frac{\alpha\pi}{2}}{2\pi^2}\frac{d}{dx}\int_{\Gamma_x}\frac{\left[t(t-1)\right]^{\frac{1-\alpha}{2}}}{(t-x)^{1-\alpha}}\,dt\int_0^1\frac{f(s)}{s-t}\left[s(1-s)\right]^{\frac{\alpha-1}{2}}ds.$$

as the solution for (12.17).

We claim that (12.20) is also the solution of (12.1). To verify this, we assume that f(s) is analytic on the interval (0,1). Let D be a region enclosing the interval (0,1) and such that f(s) remains regular in D.

Let g be the curve shown in fig. 2.

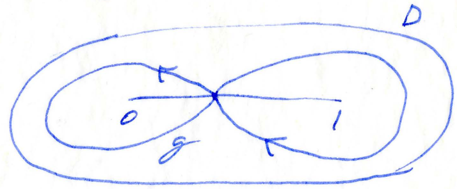


fig. 2.

The inner integral in (12.20) can then be written (for t outside g) as

$$(12.21)\qquad \int_0^1 \frac{f(s)}{s-t}\left[s(1-s)\right]^{\frac{\alpha-1}{2}} ds = \frac{i}{2\cos\frac{\alpha\pi}{2}} \int_g \left[s(s-1)\right]^{\frac{\alpha-1}{2}} \frac{f(s)}{s-t} ds$$

For t inside g we have

$$(12.22)\qquad \int_0^1 \frac{f(s)}{s-t}\left[s(1-s)\right]^{\frac{\alpha-1}{2}} ds = \frac{i}{2\cos\frac{\alpha\pi}{2}} \int_g \left[s(s-1)\right]^{\frac{\alpha-1}{2}} \frac{f(s)}{s-t} ds$$

$$\pm \frac{\pi}{\cos\frac{\alpha\pi}{2}} f(t)\left[t(t-1)\right]^{\frac{\alpha-1}{2}}$$

The first term on the right side is regular in the neighbourhood of $t=0$ and $t=1$. $\Phi(t)$ in this case, can be continued analytically across the interval (0,1). From the behaviour of $\Phi(t)$ near $t=0$ and $t=1$ as shown in (12.22) we see that there exists solutions for (12.16) and (12.17).

Let L be a closed contour enclosing the interval (0,1). The integral

$$(12.23)\qquad \int_L \frac{\left[t(t-1)\right]^{\frac{1-\alpha}{2}}}{(t-x)^{1-\alpha}} \Phi(t)\, dt$$

does not change its value if we deform L without crossing (0,1).

If we let L tend to an infinitely large circle then (12.23) tends to a constant k independent of x.

If we let L shrink into the double line (0,1) (that $\phi(t)$ satisfies the necessary conditions follows from (12.22),) we have

$$\int_0^x \frac{[t(1-t)]^{\frac{1-\alpha}{2}}}{(x-t)^{1-\alpha}} \left(\phi(t-i0)e^{-i\frac{\alpha\pi}{2}} + \phi(t+i0)e^{i\frac{\alpha\pi}{2}}\right)dt$$

$$-\int_x^1 \frac{[t(1-t)]^{\frac{1-\alpha}{2}}}{(t-x)^{1-\alpha}} \left(\phi(t-i0)e^{i\frac{\alpha\pi}{2}} + \phi(t+i0)e^{-i\frac{\alpha\pi}{2}}\right)dt = K(\text{constant})$$

Differentiating, we obtain

$$\frac{d}{dx}\int_0^x \frac{[t(1-t)]^{\frac{1-\alpha}{2}}}{(x-t)^{1-\alpha}} \left(\phi(t-i0)e^{-i\frac{\alpha\pi}{2}} + \phi(t+i0)e^{i\frac{\alpha\pi}{2}}\right)dt$$

$$= \frac{d}{dx}\int_x^1 \frac{[t(1-t)]^{\frac{1-\alpha}{2}}}{(t-x)^{1-\alpha}} \left(\phi(t-i0)e^{i\frac{\alpha\pi}{2}} + \phi(t+i0)e^{-i\frac{\alpha\pi}{2}}\right)dt.$$

This implies that (12.16) and (12.17) have the same solution y(x) where y(x) is given by (12.20).

Adding (12.16) and (12.17) we have

$$\int_0^1 \frac{y(t)\,dt}{|x-t|^{\alpha}} = \frac{i}{2\cos\frac{\alpha\pi}{2}}\left(\phi(x+i0)-\phi(x-i0)\right)[x(1-x)]^{\frac{1-\alpha}{2}},$$

Making use of (12.12) we finally obtain

$$\int_0^1 \frac{y(t)\,dt}{|x-t|^{\alpha}} = f(x)$$

We sum up our result in

Theorem 12.2 The equation

$$\int_0^1 \frac{y(t)\,dt}{|x-t|^{\alpha}} = f(x) \qquad 0< x <1, \quad 0<\alpha< 1$$

has the unique solution

$$y(x) = \frac{i\cos\frac{\alpha\pi}{2}}{2\pi^2}\frac{d}{dx}\int_{\Gamma_x}\frac{[t(t-1)]^{\frac{1-\alpha}{2}}}{(t-x)^{1-\alpha}}\,dt\int_0^1\left[\frac{s(1-s)}{s-t}\right]^{\frac{\alpha-1}{2}} f(s)\,ds$$

where Γ_x is a closed curve cutting the positive real axis at x only.

Remark: This method of analytic continuation and transforming into a Riemann boundary value problem can be applied to more general kernels of the form $1/(t-x)$, $\log|t-x|$, and $|t-x|^{-\alpha}(0<\alpha<1)$ and some of these combinations. See Gakhov, [18, pp.118-132].

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