THE CALCULUS OF ALTERNATING DIFFERENTIAL FORMS

AN ALGEBRAIC FOUNDATION OF THE CALCULUS OF ALTERNATING DIFFERENTIAL FORMS

By

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SCOPE AND CONTENTS: This thesis is concerned with the calculus of alternating differential forms on a manifold. It establishes that this calculus is of a purely algebraic nature by developing its precise analogue for an arbitrary commutative algebra with unit over a field.

PREFACE

The first section of this thesis summarizes the definitions and theorems necessary for an understanding of the algebraic concepts used later on.

The second section is devoted to showing that for any commutative algebra A with unit over a field K, there exists a homogeneous derivation d of degree 1 on the regularly graded algebra $\tilde{A}$ of alternating differential forms of A. Here, $\tilde{A}$ consists of the direct sum of the A-modules $\tilde{A}_n$ of alternating multilinear mappings into A of degree n of the A-module D of all K-derivations on A, taken with its Grassmann multiplication.

In the final section, the algebra $\mathfrak{A}$, over the real field R, of all differentiable functions on a differentiable manifold is considered. An explicit formula in terms of coordinate functions and coordinate neighbourhoods is obtained for the above mentioned derivation on the algebra $\tilde{\mathfrak{A}}$ of alternating differential forms which identifies it with Cartan's exterior derivation on $\mathfrak{A}$.

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I ALGEBRAIC PRELIMINARIES

Definition 1: Let R be a commutative ring and M_i, $i=1,2,\ldots,n$, a family of modules over R. Then a mapping τ of $\prod_{i=1}^{n} M_i$ into a module P over R is multilinear if the following conditions are satisfied: (i) for $x_i \in M_i$, $\tau(x_1,\ldots,x_i+x_i',\ldots,x_n)=\tau(x_1,\ldots,x_i,\ldots,x_n)+\tau(x_1,\ldots,x_i',\ldots,x_n)$ and (ii) for $x \in M$ and $\alpha \in R$, $(x_1,\ldots,\alpha x_i,\ldots,x_n)=\alpha\tau(x_1,\ldots,x_i,\ldots,x_n)$.

Definition 2: Let R be a commutative ring and M_i, $i=1,2,\ldots,n$, a family of modules over R. Then a module T over R is called the tensor product of the M_i if T and the multilinear mapping φ of $\prod_{i=1}^{n} M$ into T have the following property: if φ is any multilinear mapping of $\prod_{i=1}^{n} M_i$ into a module P over R then there exists a unique linear mapping f of T into P such that $f \circ \tau = \varphi$.

Note that T is usually denoted by $M_1 \otimes \ldots \otimes M_n$ and $\tau(x_1,\ldots,x_n)$ by $x_1 \otimes \ldots \otimes x_n$.

Remark: In section 10, chapter III of Fundamental Concepts of Algebra, Chevalley shows that such products do exist and proves the following theorem. Let M_i, $i=1,2,\ldots,n$, be a family of modules over a commutative ring R. Let $\{1,2,\ldots,n\}=\bigcup_{k \in K} J_k$ be a partition of $\{1,2,\ldots,n\}$ into a finite number of mutually disjoint sets J_k. Then there exists a unique isomorphism Φ of $M_1 \otimes \ldots \otimes M_n$ with $\bigotimes_{k \in K}(\bigotimes_{i \in J_k} M_i)$ such that $\Phi(x_1 \otimes \ldots \otimes x_n)=\bigotimes_{k \in K}(\bigotimes_{i \in J_k} x_i)$ for every $(x_1,\ldots,x_n) \in \prod_{i=1}^{n} M_i$. This isomorphism is called the natural or canonical isomorphism.

Definition 3: An algebra A over a commutative ring R is called a regularly graded algebra if the following conditions are satisfied: (i) $A=\sum_{n=0,1,\ldots} A_n$ (direct),

A_n are submodules of A (ii) A has a unit 1 and $A_0 = R \cdot 1 \cong R$, and (iii) $A_n \cdot A_m \subseteq A_{n+m}$.

Note that the A_n are called homogeneous submodules and $x \in A_n$ is called a homogeneous element of degree n.

Definition 4: Let R be a commutative ring and M a module over R. Let T be an algebra over R and ψ a linear mapping of M into T. Then the pair (T, ψ) is a tensor algebra on M if the following condition is satisfied: if φ is any linear mapping of M into an algebra A over R there exists a unique homomorphis f of the algebra T into A such that $f \circ \psi = \varphi$.

Remark: In section 3, chapter V, Chevalley shows that such algebras do exist.

Definition 5: If A is a regularly graded algebra over a commutative ring R then A is called anticommutative if it satisfies the following conditions: (i) if a and a′ are homogeneous elements of respective degrees k and k′ then $a'a = (-1)^{kk'} aa'$ and (ii) if a is a homogeneous element of odd degree of A then $a^2 = 0$.

Remark: If A_1 and A_2 are regularly graded algebras over a commutative ring R then $T = A_1 \otimes A_2$ the tensor product of the R-modules A_1 and A_2 can be made into a regularly graded algebra by the following definition of multiplication $(a_1 \otimes a_2')(a_1' \otimes a_2') = (-1)^{\deg a_1' \deg a_2} a_1 a_1' \otimes a_2 a_2'$ for $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$. (See Chevalley section 4, chapter V).

Definition 6: Let M be a module over a commutative ring R then an exterior algebra on M is an algebra E over R and a linear mapping ψ of M into E with the following properties: (i) $(\psi(x))^2 = 0$ for every $x \in M$ and (ii) if φ is any linear mapping of M into an algebra A over R such that $(\varphi(x))^2 = 0$ for every $x \in M$, then there exists a unique homomorphism f

of E into A such that $f \circ \psi = \phi$.

Remark 1: If we speak of an exterior algebra E on M without specifying any mapping of M into E we shall mean that M is a subset of E and that if ψ_0 is the identity mapping of M into E then (E, ψ_0) is an exterior algebra on M.

Remark 2: In section 7, chapter V, Chevalley proves the following statements.

(1) Exterior algebras exist.

(2) An exterior algebra is regularly graded and anticommutative.

(3) ψ is a module isomorphism of M with $\psi(M)$ and the homogeneous elements of degree 1 of E are those of $\psi(M)$. Thus if M is a subset of E then E has M itself as its homogeneous submodule of degree 1.

(4) M generates E.

Definition 7: Let A be an algebra over a ring R then a derivation of A is an R-linear mapping $T: A \to A$ such that $T(fg)=(Tf)g+f(Tg)$ for $f, g \in A$.

Definition 8: If A is a regularly graded algebra over a ring R then a homogeneous derivation of degree 1 on A is an R-linear mapping $T: A \to A$ such that $T(fg)=(Tf)g+(-1)^n f(Tg)$ for $f, g \in A$ where f is homogeneous of degree n, and such that T has degree 1 i.e. $T(A_n) \subseteq A_{n+1}$.

Definition 9: Let R be a commutative ring and M an R-module then for $n=1,2,\ldots$ the multilinear alternating forms of degree n on the R-module M are the mappings $\theta: M^n \to L$ where L is an R-module which, are (i) multilinear with respect to R and (ii) alternating i.e. $\theta(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \varepsilon(\pi)\theta(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in M$ where π is any permutation and $\varepsilon(\pi)$ the alternating character of π.

Remark: The forms of degree 1 are simply the linear forms.

Now, we denote by $\mathrm{Hom}_R(E_n, L)$ the set of R-module homomorphisms of E_n into L where E_n is the homogeneous component of degree n of the exterior algebra E on the R-module M. Clearly $\mathrm{Hom}_R(E_n, L)$ is an additive group with respect to $(\varphi+\psi)(t)=\varphi(t)+\psi(t)$ for $t\in E_n$ and $\varphi, \psi \in \mathrm{Hom}_R(E_n, L)$, and an R-module with respect to $(\alpha\varphi)(t)=\alpha(\varphi(t))$ for $\alpha\in R$, $t\in E_n$ and $\varphi \in \mathrm{Hom}_R(E_n, L)$. If we denote the set of multilinear alternating forms of degree n on the R-module M by $\mathrm{Alt}_R^n(M, L)$ then we clearly have an R-module by the same reasoning as above and the following lemma holds.

Lemma 1: The R-module $\mathrm{Alt}_R^n(M, L)$ is isomorphic to the R-module $\mathrm{Hom}_R(E_n, L)$.

Proof: Suppose M is a module over the commutative ring R and let (T, Θ) be a tensor algebra over M. For any $n \geq 0$, let τ_n be the multilinear mapping of the product of n modules identical with M, defined by $\tau_n(x_1, \ldots, x_n) = \Theta(x_1) \ldots \Theta(x_n)$. Let T_n be the submodule of T generated by the elements $\tau_n(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in M$ then $T = \sum_{n=0}^{\infty} T_n$ (direct) and T_n is a tensor product of n modules identical with the module M, with τ_n the tensor map from M^n to T_n (theorem 6, section 3, chapter V, Chevalley). Then by the definition of tensor product if φ is any multilinear mapping of M^n into an R-module L then there exists a unique linear mapping f of T_n into L such that $\varphi = f \circ \tau_n$. Consider the correspondence $\varphi \to f$; to each there corresponds a unique f with $\varphi = f \circ \tau_n$ and to each $f: T_n \to L$ there corresponds $f \circ \tau_n$. Also if $\varphi_1 \to f_1$ and $\varphi_2 \to f_2$ then $\varphi_1 + \varphi_2 \to f_1 + f_2$ since $\varphi_1 + \varphi_2 = f_1 \circ \tau_n + f_2 \circ \tau_n = (f_1 + f_2) \circ \tau_n$, and $\alpha\varphi \to \alpha f$ for $\alpha \in R$ since $\alpha(f \circ \tau_n) = (\alpha f) \circ \tau_n$. Thus the correspondence $\varphi \to f$ is an R-module isomorphism.

In the construction of an exterior algebra in section 7, chapter V, Chevalley shows that if E is an exterior algebra on the R-module

M then E is of the form T/F where (T,θ) is a tensor algebra on M and F is the ideal of T which is generated by the elements $(\theta(x))^2$ for $x\in M$. Such elements are homogeneous of degree 2 in T and hence F is a homogeneous ideal (theorem 4, section 2, chapter V, Chevalley). It follows That E=T/F has the structure of a regularly graded algebra (theorem 5, section 2, chapter V, Chevalley). If π is the natural mapping of T on E then the module E_n of homogeneous elements of degree n of E is the image under π of the module T_n of homogeneous elements of degree n of T i.e. $E_n=\pi(T_n)=T_n/F_n$ where T_n consists of linear combinations of products in T of n elements of $\theta(M)$, and F_n is generated as an R-module by elements of the form $a(\theta(x))^2 b$ where $a=\theta(x_1)\ldots\theta(x_k)$, $b=\theta(y_1)\ldots\theta(y_\ell)$ and $k+l+2=n$.

Now our multilinear mapping φ from M^n to L is alternating if and only if the corresponding f vanishes on F_n since $\varphi(x_1,\ldots,x_n)$ $=f(\tau_n(x_1,\ldots,x_n))=f(\theta(x_1)\ldots\theta(x_n))$. If π_n is the restriction of π to T_n then the kernel of π_n is F_n. The kernel of f contains F_n when φ is alternating; hence there exists a unique linear mapping f' from E into L such that $f'\circ\pi_n=f$. Hence there is a one-to-one correspondence between the alternating φ and the R-module homomorphisms, f', from E_n to L, for if φ is alternating then f' exists with $\varphi=f\circ\tau_n=f'\circ\pi_n\circ\tau_n$ and if f' is an R-module homomorphism from E_n to L then put $f'\circ\pi_n=f$ and f vanishes on F_n and hence we have $\varphi=f\circ\tau_n$ and φ is alternating. Also if φ_1 and φ_2 are alternating and $\varphi_1\to f_1'$, $\varphi_2\to f_2'$ then $\varphi_1+\varphi_2\to f_1'+f_2'$ since $\varphi_1+\varphi_2$ $=f_1'\circ\pi_n\circ\tau_n+f_2'\circ\pi_n\circ\tau_n=(f_1'+f_2')\circ\pi_n\circ\tau_n$ and $\alpha\varphi_1\to\alpha f_1'$ for $\alpha\in R$ since $\alpha(f_1'\circ\pi_n\circ\tau_n)$ $=(\alpha f_1')\circ\pi_n\circ\tau_n$. Hence the correspondence $\varphi\to f'$ is an R-module isomorphism between the alternating mutliniear mappings from M^n to L and the R-module homomorphisms from E_n to L i.e. $\mathrm{Alt}_R^n(M,L)\cong\mathrm{Hom}_R(E_n,L)$.

Let us now consider A a commutative K-algebra where K is a field. Let D be the set of K-derivations of A. Then the set D formsan A-module under $(fT)g=f(Tg)$ for $f,g\in A$ and $T\in D$ i.e. $A\times D \to D$ by $(f,T)\to fT$ where $fT:\ g\to f(Tg)$. First, $fT\in D$ since $fT(gh)=f(T(gh))=f[(Tg)h+g(Th)]$ $=((fT)g)h+g((fT)h)$ for g, $h\in A$ and clearly fT is K-linear since T is. Secondly, (a) $f(T_1+T_2)=fT_1+fT_2$ for $f\in A$ and $T_1, T_2\in D$ since $f(T_1+T_2)g$ $=f((T_1+T_2)g) = f(T_1g+T_2g)$ for $g\in A$ (b) $(f+h)T=fT+hT$ for $f,h\in A$ and $T\in D$ since $((f+h)T)g=(f+h)(Tg) = f(Tg)+h(Tg)$ for $g\in A$ and (c) $(fh)T=f(hT)$ for f,h A and $T\in D$ since $((fh)T)g=(fh)(Tg)=f((hT)g)$ for $g\in A$. Similarly D is a K-module under $(\alpha T)g=\alpha(Tg)$ for $\alpha\in K$, $g\in A$ and $T\in D$ i.e. $K\times D\to D$ by $(\alpha,T)\to\alpha T$ where $\alpha T:\ g\to\alpha(Tg)$. Also the K-module D forms a Lie K-algebra with the following definition of multiplication. If $T_1, T_2\in D$ then define $[T_1,T_2]$ as the mapping from A to itself which has the following effect $[T_1,T_2]f=T_1(T_2f)-T_2(T_1f)$. In order for D to be a Lie K-algebra with this definition of multiplication we have to check the following things: (a) $[T_1,T_2]\in D$ (b) $[T_1,T_2]=-[T_2,T_1]$ and $[[T_1,T_2],T_3]+[[T_2,T_3],T_1]$ $+[[T_3,T_1],T_2]=0$ and (c) the multiplication is bilinear. (a) $[T_1,T_2]$ is K-linear since for $f_1,f_2\in A$ and $T_1,T_2\in D$

$[T_1,T_2](f_1+f_2)=T_1(T_2(f_1+f_2))-T_2(T_1(f_1+f_2))$

$=T_1(T_2f_1)+T_1(T_2f_2)-T_2(T_1f_1)-T_2(T_1f_2) = [T_1,T_2]f_1+[T_1,T_2]f_2$ and for $f\in A$, $\alpha\in K$ and $T_1,T_2\in D$

$[T_1,T_2]\alpha f=T_1(T_2(\alpha f))-T_2(T_1(\alpha f))$

$=\alpha T_1(T_2f)-\alpha T_2(T_1f) =\alpha[T_1,T_2]f$

since T_1 and T_2 are K-linear. Also

$[T_1,T_2](fg)=T_1(T_2(fg))-T_2(T_1(fg))$

$=T_1((T_2f)g+f(T_2g))-T_2((T_1f)g+f(T_1g))$

$=(T_1(T_2f))g+(T_2f)(T_1g)+(T_1f)(T_2g)+f(T_1(T_2g))$

$-(T_2(T_1f))g-(T_1f)(T_2g)-(T_2f)(T_1g)-f(T_2(T_1g))$

$=([T_1,T_2]f)g+f([T_1,T_2]g)$

Hence $[T_1,T_2]\in D$.

(b) $[T_1,T_2]f=T_1(T_2f)-T_2(T_1f)$

$=-(T_2(T_1f)-T_1(T_2f))=-[T_2,T_1]f$

for $f\in A$ and $T_1,T_2\in D$. Also for $f\in A$ and $T_1,T_2,T_3\in D$

$[[T_1,T_2],T_3]f+[[T_2,T_3],T_1]f+[[T_3,T_1],T_2]f$

$=[T_1,T_2](T_3f)-T_3([T_1,T_2]f)+[T_2,T_3](T_1f)-T_1([T_2,T_3]f)+[T_3,T_1](T_2f)-T_2([T_3,T_1]f)$

$=T_1(T_2(T_3f))-T_2(T_1(T_3f))-T_3(T_1(T_2f))+T_3(T_2(T_1f))$

$+T_2(T_3(T_1f))-T_3(T_2(T_1f))-T_1(T_2(T_3f))+T_1(T_3(T_2f))$

$+T_3(T_1(T_2f))-T_1(T_3(T_2f))-T_2(T_3(T_1f))+T_2(T_1(T_3f))$

$=0$.

(c) $[T_1+T_2,T_3]f=(T_1+T_2)(T_3f)-T_3((T_1+T_2)f)$

$=[T_1,T_3]f+[T_2,T_3]f$

for $f\in A$ and T_1,T_2 and $T_3\in D$. Also for $f\in A$, $\lambda\in K$, and $T_1,T_2\in D$.

$[\lambda T_1,T_2]f=\lambda T_1(T_2f)-T_2((\lambda T_1)f)$

$=\lambda(T_1(T_2f))-T_2(\lambda(T_1f))=\lambda(T_1(T_2f)-T_2(T_1f))$

$=\lambda[T_1,T_2]f$

Hence multiplication is bilinear.

Definition 12: Let A be a commutative algebra over a field K then the alternating differential forms of degree n of A are (i) the $f\in A$ for $n=0$ (ii) the alternating multilinear forms of degree n of the A-module D for $n=1,2,\ldots$

Let the set of alternating differential forms of degree n of A be denoted by $\tilde{A}_n$ and put $\tilde{A}=\Sigma\tilde{A}_n$ (direct).

II THE ALGEBRA OF ALTERNATING DIFFERENTIAL FORMS OF AN ALGEBRA A AND ITS EXTERIOR DERIVATION

In this section we shall show that if A is a commutative K-algebra with unit where K is a field, the A-module of its alternating differential forms, $\tilde{A}$ can be made into a regularly graded A-algebra, and there exists a homogeneous derivation, d, of degree 1 on $\tilde{A}$ with the property that $dd\varphi=0$ for $\varphi\in\tilde{A}$.

In the following we will consider an arbitrary A-module M (to be specialized later to the module of derivations on A). This module M is naturally a K-module since if we identify $\alpha\in K$ with $\alpha\cdot 1$ where 1 is the unit in A then $A\supseteq K$. Hence we can consider the exterior algebra E of M considered as a K-module. Then as in the first section $\mathrm{Hom}_K(E,A)$ is the set of all K-module homomorphisms $\varphi: E\to A$, and it is clearly an A-module with respect to $(f\varphi)t=f\varphi(t)$ for $f\in A$ and $t\in E$.

$\varphi\in\mathrm{Hom}_K(E,A)$ is called <u>homogeneous</u> of degree n if $\varphi/E_n=0$ for all $m\neq n$. Put H_n = set of all φ which are homogeneous of degree n. H_n is an A-submodule for if $\varphi_1,\varphi_2\in H_n$ then $\varphi_1+\varphi_2/E_m=\varphi_1/E_m+\varphi_2/E_m=0+0$ if $m\neq n$ and if $f\in A$ and $\varphi\in H_n$ then $f\varphi/E_m=f(\varphi/E_m)=0$ if $m\neq n$. Let $H=\sum_{n\geq 0}H_n$ and clearly $H_i\cap\sum_{j=i}H_j=0$; hence the sum is direct. If $\varphi\in H_0$ then $\varphi(t)=0$ for $t\in\sum_{n\geq 1}E_n$, and for $\xi\in E_0=K$ $\varphi(\xi)=\varphi(\xi\cdot 1)=\varphi(1)\cdot\xi$ and hence if $\varepsilon\in H_0$ is the mapping $\varepsilon(\xi)=\xi$ then $\varphi=\varphi(1)\varepsilon$. Thus $H_0=A\varepsilon$.

$\varphi\in H_n$ is called <u>multilinear in A</u> if $\varphi(x_1\ldots f(x_i)\ldots x_n)=f\varphi(x_1\ldots x_n)$. The set G_n of all such φ is clearly an A-submodule and $G=\sum_{n\geq 0}G_n$ is direct. Furthermore $G_0=A\varepsilon$ since any $f\varepsilon$ for $f\in A$ is multilinear

and hence $A_\varepsilon \subseteq G_o$, but $G_o \subseteq H_o = A_\varepsilon$.

By lemma 1, section I we know that $\mathrm{Alt}_K^n(M,A) \cong \mathrm{Hom}_K(E_n,A)$, but clearly $\mathrm{Hom}_K(E_n,A) \cong H_n$. Thus $\varphi \in \mathrm{Alt}_K^n(M,A) \to \varphi^* \in H_n$ with $\varphi^*/E_m = 0$ if $m \neq n$ and $\varphi^*(x_1,\ldots x_n) = \varphi(x_1,\ldots,x_n)$. Now, if $\varphi \in \mathrm{Alt}_A^n(M,A) \subseteq \mathrm{Alt}_K^n(M,A)$ then $\varphi(x_1,\ldots,f(x_i),\ldots,x_n) = f\varphi(x_1,\ldots,x_n)$ which implies that $\varphi^* \in G_n$. Conversely if $\varphi^* \in G_n$ then $\varphi(x_1,\ldots,f(x_i),\ldots,x_n) = f\varphi(x_1,\ldots,x_n)$ and φ is A-multilinear. Hence $\mathrm{Alt}_A^n(M,A) \cong G_n$.

When we specialize M to D, the A-module of K-derivations of A then $\mathrm{Alt}_A^n(D,A) = \tilde{A}_n$ and $\tilde{A} = \sum \tilde{A}_n = \sum \mathrm{Alt}_A^n(D,A)$.

1. The Algebra $\mathrm{Hom}_K(E,A)$.

In the following in order to make $\mathrm{Hom}_K(E,A)$ into an associative A-algebra with unit we wish to define a law of composition in $\mathrm{Hom}_K(E,A)$ which we will call multiplication.

Consider the algebra $E \otimes E$ which is the tensor product of the exterior algebra E on the K-module M with itself. The K-module $E \otimes E$ is the direct sum of submodules $E_n \otimes E_m$. Now consider the mapping $x \to x \otimes 1 + 1 \otimes x$ for $x \in M$ which maps M into $E \otimes E$. Since $(x \otimes 1 + 1 \otimes x)(x \otimes 1 + 1 \otimes x) = x^2 \otimes 1 - x \otimes x + x \otimes x + 1 \otimes x^2 = 0$, by the definition of an exterior algebra this can be extended to a K-homomorphism $U: E \to E \otimes E$. U is called the analyzing map of E. An explicit formula for U is $U(x_1,\ldots x_n) = \sum_\sigma \eta(\sigma,\sigma^*) t(\sigma) \otimes t(\sigma^*)$ where $\sigma = (i_1,\ldots,i_k)$ with $i_1 < \ldots < i_k$; σ^* is the complementary sequence $(j_1,\ldots,j_{n-k})$ with $j_1 < \ldots < j_{n-k}$; $t(\sigma) = x_{i_1}\ldots x_{i_k}$; $t(\sigma^*) = x_{j_1}\ldots x_{j_{n-k}}$; and $\eta(\sigma,\sigma^*) = (-1)^{N(\sigma,\sigma^*)}$ where $N(\sigma,\sigma^*)$ = the number of pairs (i,j) with $i \in \sigma, j \in \sigma^*$ and $i > j$.

Remark: $\eta(\sigma,\sigma^*)$ is the signature of the permutation $\begin{pmatrix} 1\ldots k & k+1\ldots n \\ i_1,\ldots i_k & j_1,\ldots j_{n-k} \end{pmatrix}$.

Let α: $A\otimes A \to A$ be the mapping defined by $\alpha(f\otimes g)=fg$. α is called the linearization of the multiplication in A. For $\phi,\psi\in \mathrm{Hom}_K(E,A)$ put $\phi\wedge\psi=\alpha\circ\phi\otimes\psi\circ U$. Thus $\phi\wedge\psi(x_1,\dots x_n)=\sum_\tau \eta(\tau,\tau^*)\phi(t(\tau))\psi(t(\tau^*))$, and clearly $\phi\wedge\psi$ is in $\mathrm{Hom}_K(E,A)$.

Theorem 1: $\mathrm{Hom}_K(E,A)$ is an associative A-algebra with unit, with respect to the law of composition $(\phi,\psi)\to\phi\wedge\psi$, and H and G are regularly graded subalgebras with H anticommutative.

Proof: The mapping $(\phi,\psi)\to\phi\wedge\psi$ is bilinear with respect to A. For if $\phi_1,\phi_2,\psi\in\mathrm{Hom}_K(E,A)$ then

$(\phi_1+\phi_2)\wedge\psi=\alpha\circ\{(\phi_1+\phi_2)\otimes\psi\}\circ U=\alpha\circ\{(\phi_1\otimes\psi)+(\phi_2\otimes\psi)\}\circ U=\alpha\circ(\phi_1\otimes\psi)\circ U+\alpha\circ(\phi_2\otimes\psi)\circ U=\phi_1\wedge\psi+\phi_2\wedge\psi$.

and similarly for $\psi_1,\psi_2,\phi\in\mathrm{Hom}_K(E,A)$ $\phi\wedge(\psi_1+\psi_2)=\phi\wedge\psi_1+\phi\wedge\psi_2$; also if $f\in A$ then for $\phi,\psi\in\mathrm{Hom}_K(E,A)$, $(f\phi)\wedge\psi=f(\phi\wedge\psi)$ since $(f\phi)\wedge\psi(x_1,\dots x_n)=\sum_\tau\eta(\tau,\tau^*)f\phi(t(\tau))\psi(t(\tau^*))$ $=f\sum_\tau\eta(\tau,\tau^*)\phi(t(\tau))\psi(t(\tau^*))$ since A is commutative, and similarly $\phi\wedge(f\psi)=f(\phi\wedge\psi)$.

The law of composition $(\phi,\psi)\to\phi\wedge\psi$ for $\phi,\psi\in\mathrm{Hom}_K(E,A)$ is associative i.e. $(\phi\wedge\psi)\wedge\theta=\phi\wedge(\psi\wedge\theta)$. For

$(\phi\wedge\psi)\wedge\theta=\alpha\circ\{(\phi\wedge\psi)\otimes\theta\}\circ U=\alpha\circ\left[\{\alpha\circ(\phi\otimes\psi)\circ U\}\otimes\theta\right]\circ U=\alpha\circ\left[\{\alpha\circ(\phi\otimes\psi)\}\otimes\theta\right]\circ(U\otimes I)\circ U$

$=\alpha\circ(\alpha\otimes I)\circ\{(\phi\otimes\psi)\otimes\theta\}\circ(U\otimes I)\circ U$

and

$\phi\wedge(\psi\wedge\theta)=\alpha\circ\{\phi\otimes(\psi\wedge\theta)\}\circ U=\alpha\circ\left[\phi\otimes\{\alpha\circ(\psi\otimes\theta)\circ U\}\right]\circ U=\alpha\circ\left[\phi\otimes\{\alpha\circ(\psi\otimes\theta)\}\right]\circ(I\otimes U)\circ U$

$=\alpha\circ(I\otimes\alpha)\circ\{\phi\otimes(\psi\otimes\theta)\}\circ(I\otimes U)\circ U$. Now let ω be the canonical isomorphism of $(E\otimes E)\otimes E$ with $E\otimes(E\otimes E)$. For $t,u,v\in E$ $\alpha\circ(\alpha\otimes I)\circ\{(\phi\otimes\psi)\otimes\theta\}\{(t\otimes u)\otimes v\}$

$=\alpha\circ(\alpha\otimes I)\{(\phi(t)\otimes\psi(u))\otimes\theta(v)\}=\alpha\{(\phi(t)\psi(u))\otimes\theta(v)\}=\phi(t)\psi(u)\theta(v)$

and

$\alpha\circ(I\otimes\alpha)\circ\{\phi\otimes(\psi\otimes\theta)\}\circ\omega\{(t\otimes u)\otimes v\}=\alpha\circ(I\otimes\alpha)\{\phi(t)\otimes(\psi(u)\otimes\theta(v)\}=\alpha\{\phi(t)\otimes(\psi(u)\theta(v))$

$=\phi(t)\psi(u)\theta(v)$. But we can check the following identity $\omega\circ(U\otimes I)\circ U$ $=(I\otimes U)\circ U$. It is enough to check that the two mappings have the same

effect on $x \in M$ since $\overset{M}{}_n$ generates E and the mappings occuring are algebra homomorphisms.

$\omega \cdot (U \otimes I) \cdot U(x) = \omega (U \otimes I)(x \otimes 1 + 1 \otimes x)$

$= \omega \{(x \otimes 1) \otimes 1 + (1 \otimes x) \otimes 1 + (1 \otimes 1) \otimes x = x \otimes (1 \otimes 1) + 1 \otimes (x \otimes 1) + 1 \otimes (1 \otimes x)$.

$(I \otimes U) \cdot U(x) = (I \otimes U)(x \otimes 1 + 1 \otimes x) = x \otimes (1 \otimes 1) + 1 \otimes (x \otimes 1) + 1 \otimes (1 \otimes x)$.

Hence on E

$\alpha \circ (\alpha \otimes I) \circ \{(\phi \otimes \psi) \otimes \theta\} \circ (U \otimes I) \circ U = \alpha \circ (I \otimes \alpha) \circ \{\phi \otimes (\psi \otimes \theta)\} \circ (I \otimes U) \circ U$ and $(\phi \wedge \psi) \wedge \theta = \phi \wedge (\psi \wedge \theta)$.

The law of composition $(\phi, \psi) \to \phi \wedge \psi$ for $\phi, \psi \in \mathrm{Hom}_K(E, A)$ has a unit ε where $\varepsilon \in H_0$ and is the mapping $\varepsilon(\xi) = \xi$ for $\xi \in E_0$. $\phi \wedge \varepsilon = \phi$ for $\phi \wedge \varepsilon(x_1 \cdots x_n) = \sum_\tau \eta(\tau, \tau^*) \phi(t(\tau)) \varepsilon(t(\tau^*))$ and $\varepsilon(t(\tau^*)) = 0$ unless τ^* is the empty sequence since $\varepsilon \in H_0$. Similarly $\varepsilon \wedge \phi = \phi$.

If $\phi \in H_n$ and $\psi \in H_m$ then $\phi \wedge \psi H_{n+m}$ and hence H is closed under $\wedge$. For if $\phi \in H_n$ then $\phi / E_{n'} = \{0\}$ for all $n' \neq n$, and if $\psi \in H_m$ then $\phi / E_{m'} = \{0\}$ for all $m' \neq m$. Thus $\phi \otimes \psi / E_{n'} \otimes E_{m'} = \{0\}$ if $(m', n') \neq (m, n)$. Since $U(E_p) \subset \sum_{n'+m'=p} E_{n'} \otimes E_{m'}$ where the gradation of $E \otimes E$ is the total gradation, then $(\phi \wedge \psi) / E_p = \{0\}$ if $p \neq m + n$. Therefore $\phi \wedge \psi \in H_{n+m}$ and H is closed under $\wedge$. Hence since H is already an A-submodule; $H = \sum_{n \geq 0} H_n$ (direct); and $H_0 = A\varepsilon$, H is a regularly graded algebra.

H is also anticommutative, for if $\phi \in H_n$ then $\phi \wedge \phi = 0$ if n is odd, and if $\phi \in H_n$ and $\psi \in H_m$ then $\psi \wedge \phi = (-1)^{nm} \phi \wedge \psi$. To prove the first assertion that $\phi \wedge \phi = 0$ where $\phi \in H_n$, if n is odd, it is sufficient to consider $\phi \wedge \phi(x_1 \cdots x_{2n}) = \sum_\tau \eta(\tau, \tau^*) \phi(x_{\iota_1} \cdots x_{\iota_n}) \phi(x_{j_1} \cdots x_{j_n})$ since $\phi / E_{n'} = \{0\}$ if $n' \neq n$. This sum has terms of the form $(\eta(\tau, \tau^*) + \eta(\tau^*, \tau)) \phi(t(\tau)) \phi(t(\tau^*))$. Now, $\eta(\tau, \tau^*) + \eta(\tau^*, \tau) = (-1)^{N(\tau, \tau^*)} + (-1)^{N(\tau^*, \tau)}$ and here $N(\tau^*, \tau) + N(\tau, \tau^*) = n^2$. Therefore $(-1)^{N(\tau, \tau^*)} = (-1)^{N(\tau, \tau^*) + 2N(\tau^*, \tau)} = (-1)^{n^2 + N(\tau^*, \tau)}$ i.e. $\eta(\tau, \tau^*) = (-1)^{n^2} \eta(\tau^*, \tau)$ and hence here $\eta(\tau, \tau^*) + \eta(\tau^*, \tau) = 0$ since n is odd. Thus $\phi \wedge \phi = 0$. In order

to prove the second assertion we consider

$\varphi\wedge\psi(x_1,\dots x_{n+m}) = \sum_\sigma (\sigma,\sigma^*)\varphi(x_{i_1},\dots x_{i_n})\psi(x_{j_1},\dots x_{j_m})$ and $\psi\wedge\varphi(x_1,\dots x_{n+m})$ $= \sum_\sigma \eta(\sigma^*,\sigma)\psi(x_{j_1},\dots x_{j_m})\varphi(x_{i_1},\dots x_{i_n})$. If the length of σ is n and the length of σ^* is m then $\eta(\sigma,\sigma^*) = (-1)^{nm}\eta(\sigma^*,\sigma)$ since $N(\sigma^*,\sigma) + N(\sigma,\sigma^*) = nm$ and hence $\psi\wedge\varphi(x_1,\dots x_{n+m}) = (-1)^{nm}\varphi\wedge\psi(x_1,\dots x_{n+m})$.

If $\varphi\in G_n$ and $\psi\in G_m$ then $\varphi\wedge\psi\in G_{n+m}$ and G is closed under $\wedge$.

By a previous assertion we know that $\varphi\wedge\psi\in H_{n+m}$, and $\varphi\wedge\psi$ is clearly multilinear since φ and ψ are both multilinear and $\varphi\wedge\psi(x_1,\dots x_n)$ $= \sum_\sigma \eta(\sigma,\sigma^*)\varphi(t(\sigma))\psi(t(\sigma^*))$. Thus since we already know that G is an A-submodule; $G = \sum_{n\geq 0} G_n$ (direct); and $G_0 = A\epsilon$. Then G is a regularly graded subalgebra.

Corollary: Since $G\cong \mathrm{Alt}_A^n(M,A)$ then $\mathrm{Alt}_A^n(M,A)$ is an A-algebra, and its law of composition is $\varphi\wedge\psi(x_1,\dots,x_n) = \sum_\sigma \eta(\sigma,\sigma^*)\varphi(t(\sigma))\psi(t(\sigma^*))$ where σ,σ^* and $\eta(\sigma,\sigma^*)$ are as before and $t(\sigma) = x_{i_1},\dots,x_{i_k}$ with $i_1<\dots i_k$; $t(\sigma^*)$ $= x_{j_1},\dots,x_{j_{n-k}}$ with $j_1<\dots<j_{n-k}$.

Remark: In the case that K has characteristic zero $\varphi\wedge\psi(x_1,\dots x_n)$ $= \sum_{k,\pi} \frac{\epsilon(\pi)}{k!(n-k)!}\varphi(x_{\pi(1)},\dots x_{\pi(k)})\psi(x_{\pi(k+1)},\dots x_{\pi(n)})$ where π is a permutation of $\{1,2,\dots,n\}$ and $\epsilon(\pi)$ is the alternating character of π. Now $\varphi\wedge\psi(x_1,\dots x_n)$ $= \sum_\sigma \eta(\sigma,\sigma^*)\varphi(x_{i_1},\dots x_{i_k})\psi(x_{j_1},\dots x_{j_{n-k}})$ say. Suppose π is the following permutation $\pi = \begin{pmatrix} 1 & \dots & 2 & \dots & k & k+1 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(k) & \pi(k+1) & \dots & \pi(n) \end{pmatrix}$ and that

$\pi_\sigma = \begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n \\ i_1 & i_2 & \dots & i_k & j_1 & \dots & j_{n-k} \end{pmatrix}$; $\pi'(i) = \begin{cases} \pi(\nu) \text{ if } i = i_\nu \\ i \text{ if } i\notin(i_1,\dots,i_k) \end{cases}$

and $\pi''(i) = \begin{cases} i \text{ if } i\in(i_1,\dots,i_k) \\ \pi(\nu) \text{ if } i = j_{\nu-k} \end{cases}$ then $\pi = \pi''\cdot\pi'\cdot\pi_\sigma$ and hence $\epsilon(\pi) = \epsilon(\pi_\sigma)\epsilon(\pi')\epsilon(\pi'')$. Thus

$\epsilon(\pi)\varphi(x_{\pi(1)}\dots x_{\pi(k)})\psi(x_{\pi(k+1)}\dots x_{\pi(n)}) = \epsilon(\pi_\sigma)\epsilon(\pi')\epsilon(\pi'')\varphi(x_{\pi(1)}\dots x_{\pi(k)})\psi(x_{\pi(k+1)}\dots x_{\pi(n)})$.

$= \varepsilon(\pi_\tau)\phi(\varepsilon(\pi')x_{\pi(1)}\cdots x_{\pi(k)})\psi(\varepsilon(\pi'')x_{\pi(k+1)}\cdots x_{\pi(n)}) = \varepsilon(\pi_\tau)\phi(x_{i_1}\cdots x_{i_k})\psi(x_{j_1}\cdots x_{j_{n-k}})$
$= \eta(\nabla,\nabla^*)\phi(x_{i_1}\cdots x_{i_k})\psi(x_{j_1}\cdots x_{j_{n-k}})$. Hence for fixed k and fixed $\nabla = (i_1,\ldots,i_k)$, $\sum_\pi \frac{\varepsilon(\pi)}{k!(n-k)!}\phi(x_{\pi(1)}\cdots x_{\pi(k)})\psi(x_{\pi(k+1)}\cdots x_{\pi(n)}) = \eta(\tau,\tau^*)\phi(x_{i_1}\cdots x_{i_k})\psi(x_{j_1}\cdots x_{j_{n-k}})$ where π is a permutation such that for $i=1,\ldots,k$ $\pi(i)$ is one of the $i_1,\ldots,i_k$, and since there are $k!$ possible permutations π' and $(n-k)!$ possible permutations π''. Hence $\sum_\tau \eta(\tau,\tau^*)\phi(x_{i_1}\cdots x_{i_k})\psi(x_{j_1}\cdots x_{j_{n-k}}) = \sum_{k,\pi} \frac{\varepsilon(\pi)}{k!(n-k)!}\phi(x_{\pi(1)}\cdots x_{\pi(k)})\psi(x_{\pi(k+1)}\cdots x_{\pi(n)})$.

2. The Derivation on H Determined by an Alternating K-bilinear Λ: $M\times M \to M$.

Here Λ is an internal law of composition in M.

In order to define the derivation on H determined by Λ we first consider the following mappings Λ^*: $E \to E$ by $\Lambda^*(t) = 0$ if $t \in E_n$ and $n \neq 2$ and $\Lambda^*(xy) = \Lambda(x,y)$ for $x,y \in M$. $\tilde{\Lambda} = \mu\circ(\Lambda^*\otimes I)\cdot U$ where μ: $E\otimes E \to E$ is the linearization of multiplication in E. More explicity $\tilde{\Lambda}(x_1\cdots x_n) = \mu\circ\Lambda^*\otimes I(\sum_\tau \eta(\tau,\tau^*)t(\tau)\otimes t(\tau^*))$
$= \sum_\tau \eta(\tau,\tau^*)\Lambda^*(t(\tau))t(\tau^*)$. Hence $\tilde{\Lambda}(x_1\cdots x_n) = \sum_{i<j}(-1)^{i+j+1}\Lambda(x_i,x_j)x_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots x_n$ since $\Lambda^*(t(\tau))=0$ if τ is not of length two and $\eta(\tau,\tau^*) = (-1)^{N(\tau,\tau^*)} = (-1)^{i+j+1}$ since below i there are i-1 elements and below j there are j-2 elements since i is removed.

Lemma 2: $U\cdot\tilde{\Lambda} = (\tilde{\Lambda}\otimes I + J\otimes\tilde{\Lambda})\cdot U$ where I: $E \to E$ is the identity mapping and J: $E \to E$ is an extension of the automorphism $x \to -x$ of M_1 to E.

Proof: $U\circ\tilde{\Lambda} = U\cdot\mu\circ(\Lambda^*\otimes I)\cdot U$ by the definition of $\tilde{\Lambda}$. The multiplication in the algebra $E\otimes E$ is a bilinear mapping of $(E\otimes E)\times(E\otimes E)$ into $E\otimes E$; the linearization of this bilinear mapping is a linear mapping of $(E\otimes E)\otimes(E\otimes E)$ into $(E\otimes E)$ which we shall denote by μ_2. If t and u are in E we have $U(\mu(t\otimes u)) = U(tu) = U(t)U(u) = \mu_2(U(t)\otimes U(u))$ whence $U\circ\mu = \mu_2\circ(U\otimes U)$ on $E\otimes E$. Then $U\circ\tilde{\Lambda} = \mu_2\cdot(U\otimes U)\circ(\Lambda^*\otimes I)\cdot U = \mu_2\circ\{(U\circ\Lambda^*)\otimes U\}\cdot U = \mu_2\circ\{(U\circ\Lambda^*)\otimes(I\otimes I)\}\circ(I\otimes U)\,U$.
If $t\in E$ then $\Lambda^*(t)=0$ unless $t\in E_2$ and $\Lambda^*(xy) = (x,y)\in M$ for $x,y\in M$ and hence

$$U\circ\Lambda^*(t)=\begin{cases}0 \text{ if } t\in E\\ (\Lambda^*(xy))\otimes 1+1\otimes(\Lambda^*(xy)) \text{ if } t\in E_2\end{cases}$$

Denote by Θ_1 and Θ_2 respectively the linear mappings $v\to v\otimes 1$ and $v\to 1\otimes v$ of E into $E\otimes E$, then we have

$$U\circ\Lambda^*=(\Theta_1\circ\Lambda^*)+(\Theta_2\circ\Lambda^*)$$

whence

$$U\circ\tilde{\Lambda}=\mu\circ\{(U\circ\Lambda^*)\otimes(I\otimes I)\}\circ(I\otimes U)\circ U=\mu_2\circ\left[\{(\Theta_1\circ\Lambda^*)+(\Theta_2\circ\Lambda^*)\}\otimes(I\otimes I)\right]\circ(I\otimes U)\circ U.$$

Let t, u, and v be in E, then

$$\mu_2\left[\{(\Theta_1\circ\Lambda^*)\otimes(I\otimes I)\}\{t\otimes(u\otimes v)\}\right]=\mu_2\left[(\Lambda^*(t)\otimes 1)\otimes(u\otimes v)\right]=(\Lambda^*(t)u)\otimes v=\{\mu\circ(\Lambda^*\otimes I)(t\otimes u)\}\otimes v$$
$$=\left[\{\mu\circ(\Lambda^*\otimes I)\}\otimes I\right]\left[\{t\otimes u\}\otimes v\right].$$

Denote by ρ the canonical isomorphism of $E\otimes(E\otimes E)$ with $(E\otimes E)\otimes E$ which maps $t\otimes(u\otimes v)$ upon $(t\otimes u)\otimes v$ for t, u and $v\in E$. Then we have

$$\mu_2\circ\{(\Theta_1\circ\Lambda^*)\otimes(I\otimes I)\}\circ(I\otimes U)\circ U=\left[\{\mu\circ(\Lambda^*\otimes I)\}\otimes I\right]\circ\rho\circ(I\otimes U)\circ U.$$

Now, we can check the following identity (*) $\rho\circ(I\otimes U)\circ U=(U\otimes I)\circ U$ by showing that the two mappings have the same effect on $x\in M$ and hence are equal on E since M generates E and all occuring mappings are clearly algebra homomorphisms.

$$\rho\circ(I\otimes U)\circ U(x)=\rho\circ(I\otimes U)(x\otimes 1+1\otimes x)=\rho\left[x\otimes(1\otimes 1)+1\otimes(x\otimes 1+1\otimes x)\right]=(x\otimes 1)\otimes 1+(1\otimes x)\otimes 1+(1\otimes 1)\otimes x$$

$$(U\otimes I)\circ U(x)=U\otimes I(x\otimes 1+1\otimes x)=(x\otimes 1)\otimes 1+(1\otimes x)\otimes 1+(1\otimes 1)\otimes x$$

Hence

$$\mu_2\circ\{(\Theta_1\circ\Lambda^*)\otimes(I\otimes I)\}\circ(I\otimes U)\circ U=\left[\{\mu\circ(\Lambda^*\otimes I)\}\otimes I\right]\circ(U\otimes I)\circ U=\left[\{\mu\circ(\Lambda^*\otimes I)\circ U\}\otimes I\right]\circ U=(\tilde{\Lambda}\otimes I)\circ U$$

For t, u, and $v\in E$

$$\mu_2\left[\{(\Theta_2\circ\Lambda^*)\otimes(I\otimes I)\}\{t\otimes(u\otimes v)\}\right]=\mu_2\left[(1\otimes\Lambda^*(t))\otimes(u\otimes v)\right]=(1\otimes\Lambda^*(t))(u\otimes v)$$

Since $\Lambda^*(t)\in E$, we have $(1\otimes\Lambda^*(t))(u\otimes v)=(-1)^{\deg u}u\otimes\Lambda^*(t)v=1(u)\otimes\Lambda^*(t)v$.

Denote by τ the automorphism of the module $E\otimes(E\otimes E)$ which maps $t\otimes(u\otimes v)$ upon $u\otimes(t\otimes v)$ for t, u, $v\in E$. Then

$$\mu_2\circ\{(\Theta_2\circ\Lambda^*)\otimes(I\otimes I)\}\circ(I\otimes U)\circ U=\left[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}\right]\circ\tau\circ(I\otimes U)\circ U.$$

Denote by ν the automorphism of the module $E\otimes E$ which maps $t\otimes u$ upon $u\otimes t$ if $t, u, v\in E$

$$\{\rho^{-1}\circ(\nu\otimes I)\circ\rho\}\{t\otimes(u\otimes v)\}=\{\rho^{-1}\circ(\nu\otimes I)\}\{(t\otimes u)\otimes v\}=\rho^{-1}\{(u\otimes t)\otimes v\}=u\otimes(t\otimes v)=\tilde{\tau}\{t\otimes(u\otimes v)\}.$$

Hence $\tilde{\tau}=\rho^{-1}\circ(\nu\otimes I)\circ\rho$.

Thus

$$\mu_2\circ\{(\theta_2\circ\Lambda^*)\otimes(I\otimes I)\}\circ(I\otimes U)\circ U$$
$$=[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ(\nu\otimes I)\circ\rho\circ(I\otimes U)\circ U$$
$$=[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ(\nu\otimes I)\circ(U\otimes I)\circ U$$
$$=[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ\{(\nu\circ U)\otimes I\}\circ U.$$

Now let K be the automorphism of the module $E\otimes E$ which maps any $w\in E_p\otimes E_q$ upon $(-1)^{pq}w$ then $\nu\circ U=K\circ U$ for $\nu\circ U(x_1,\dots x_n)=\nu\left[\sum_\tau\eta(\tau,\tau^*)t(\tau)\otimes t(\tau^*)\right]$
$=\sum_\tau(-1)^{h(n-h)}\eta(\tau^*,\tau)t(\tau^*)\otimes t(\tau)$ since $\eta(\tau,\tau^*)=(-1)^{h(n-h)}\eta(\tau^*,\tau)$ if τ is a sequence of length h; also $K\circ U(x_1,\dots x_n)=K\left[\sum_\tau\eta(\tau,\tau^*)t(\tau)\otimes t(\tau^*)\right]$
$=\sum_\tau(-1)^{h(n-h)}\eta(\tau,\tau^*)t(\tau)\otimes t(\tau^*)=\sum_\tau\eta(\tau^*,\tau)t(\tau^*)\otimes t(\tau)$. The mapping $\mu\circ(\Lambda^*\otimes I)$ maps $E_q\otimes E$ upon $\{o\}$ if $q\neq 2$ thus $[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}$ maps $(E\otimes E_q)\otimes E$ upon $\{o\}$ if $q\neq 2$. On the other hand if $t\in E_p$ and $u\in E_2$ then $K(t\otimes u)=(-1)^{2p}t\otimes u$
$=1^2(t)\otimes u$ and it follows that $K\otimes I$ coincides with $(1^2\otimes I)\otimes I$ on $(E\otimes E_2)\otimes E$.

Thus we have

$$[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ\{(\nu\circ U)\otimes I\}\circ U=[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ\{(K\circ U)\otimes I\}\circ U$$
$$=[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ(K\otimes I)\circ(U\otimes I)\circ U=[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ\{(1^2\otimes I)\otimes I\}\circ(U\otimes I)\circ U.$$

It is clear that $\rho^{-1}\circ\{(1^2\otimes I)\otimes I\}=\{1^2\otimes(I\otimes I)\}\circ\rho^{-1}$ and since $\rho^{-1}\circ(U\otimes I)\circ U=(I\otimes U)\circ U$ then

$$[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\rho^{-1}\circ\{(1^2\otimes I)\otimes I\}\circ(U\otimes I)\circ U=[1\otimes\{\mu\circ(\Lambda^*\otimes I)\}]\circ\{1^2\otimes(I\otimes I)\}\circ(I\otimes U)\circ U$$
$$=1^3\otimes\{\mu\circ(\Lambda^*\otimes I)\circ U\}\circ U=1\circ\tilde{\Lambda}\circ U.$$

Hence $U\circ\tilde{\Lambda}=(\tilde{\Lambda}\otimes I+1\otimes\tilde{\Lambda})\circ U$.

Definition 13: Let d_Λ be the following mapping $d_\Lambda:\varphi\to\varphi\circ\tilde{\Lambda}$. In other words

$$d_\wedge\varphi(x_1,\dots x_n)=\sum_{i<j}(-1)^{i+j+1}\varphi(\wedge(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n).$$

The mapping $d_\wedge$ has the following properties.

(i) $d_\wedge$ maps $\mathrm{Hom}_K(E,A)$ into itself.

For $\tilde{\wedge}$ is clearly a K-endomorphism of E since $\wedge$ is K-bilinear.

(ii) If $\varphi\in H_n$ then $d_\wedge\varphi\in H_{n+1}$ hence $d_\wedge$ maps H into itself.

For if $\varphi\in H_n$ then $\varphi/E_{n'}=\{0\}$ if $n'\neq n$ and hence

$$d_\wedge\varphi(x_1,\dots x_m)=\sum_{i<j}(-1)^{i+j+1}\varphi(\wedge(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_m)=0$$

unless $m=n+1$ since $\wedge(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_m$ is m-1 elements.

(iii) $d_\wedge$ is A-linear.

For if $f\in A$ then $d_\wedge(f\varphi)(x_1,\dots x_n)=\sum_{i<j}(-1)^{i+j+1}f\varphi(\wedge(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$

$=f\sum_{i<j}(-1)^{i+j+1}\varphi(\wedge(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n)=f(d_\wedge\varphi)$.

Theorem 2: $d_\wedge$ is a homogeneous derivation of degree 1 on H.

Proof: For any $\varphi,\psi\in\mathrm{Hom}_K(E,A)$ then

$d_\wedge(\varphi\wedge\psi)=(\varphi\wedge\psi)\cdot\tilde{\wedge}=\wedge\circ(\varphi\otimes\psi)\circ U\cdot\tilde{\wedge}=\wedge\circ(\varphi\otimes\psi)\circ(\tilde{\wedge}\otimes I+1\otimes\tilde{\wedge})\cdot U$ (by lemma 2)

$=\wedge\circ\left[\{(\varphi\cdot\tilde{\wedge})\otimes(\psi\circ I)\}+\{(\varphi\cdot 1)\otimes(\psi\cdot\tilde{\wedge})\}\right]\cdot U=\wedge\circ\{(\varphi\cdot\tilde{\wedge})\otimes\psi\}\circ U+\wedge\circ\{(\varphi\cdot 1)\otimes(\psi\cdot\tilde{\wedge})\}\cdot U$

$=(\varphi\circ\tilde{\wedge})\wedge\psi+(\varphi\circ 1)\wedge(\psi\circ\tilde{\wedge})=(d_\wedge\varphi)\wedge\psi+(\varphi\cdot 1)\wedge(d_\wedge\psi)$.

For $\varphi\in H_n$ we have $\varphi\circ 1=(-1)^n\varphi$ and hence here $d_\wedge(\varphi\wedge\psi)=(d_\wedge\varphi)\wedge\psi+(-1)^n\varphi\wedge(d_\wedge\psi)$

and $d_\wedge$ is a homogeneous derivation of degree 1 on H.

3. The Derivation on H Determined by a K-bilinear Mapping Δ: $M\times A\to A$ for which $\Delta(x,fg)=\Delta(x,f)g+f\Delta(x,g)$.

Here Δ is an external law of composition of M with A i.e. makes "M act on A", and the property $\Delta(x,fg)=\Delta(x,f)g+f\Delta(x,g)$ tell us that the elements of M act as derivations on A.

In order to define the derivation determined by Δ we consider the following two mappings: Δ^*: $M\otimes A\to A$ i.e. Δ^* is the linearization

of Δ, and P: $E \to E_1 = M$ by $P/E_n = \{o\}$ if $n \neq 1$ and P is the identity mapping on E_1 i.e. P is the projection of E to E_1.

Definition 14: Let d_Δ be the following mapping d_Δ: $\varphi \to \Delta^* \circ (P \otimes \varphi) \circ U$. In other words $d_\Delta \varphi(x_1 \ldots x_n) = \Delta^* \circ (P \otimes \varphi)\left[\sum_\nabla \eta(\nabla, \nabla^*) t(\nabla) \otimes t(\nabla^*)\right]$
$= \Delta^*\left(\sum_\nabla \eta(\nabla, \nabla^*) P(t(\nabla)) \otimes \varphi(t(\nabla^*))\right) = \sum_i (-1)^{i+1} \Delta(x_i, \varphi(x_1 \ldots \hat{x}_i \ldots x_n))$ since if ∇ has length other than 1 then P maps it to zero.

Clearly $d_\Delta \varphi \in \mathrm{Hom}_K(E, A)$ since $\varphi \in \mathrm{Hom}_K(E, A)$ and Δ is K-bilinear; and $d_\Delta \varphi$ has the following two properties.

(i) The mapping $\varphi \to d_\Delta \varphi$ is linear with respect to K though not with respect to A.

For $(d_\Delta(f\varphi))(x_1 \ldots x_n) = \sum_i (-1)^{i+1} \Delta(x_i, f\varphi(x_1 \ldots \hat{x}_i \ldots x_n)$
$= f\sum_i (-1)^{i+1} \Delta(x_i, \varphi(x_1 \ldots \hat{x}_i \ldots x_n)) = f d_\Delta \varphi(x_1 \ldots x_n)$ if $f \in K$ since Δ is K-bilinear. Thus $d_\Delta(f\varphi) = f(d_\Delta \varphi)$.

(ii) If $\varphi \in H_m$ then $d_\Delta \varphi \in H_{m+1}$.

For if $\varphi \in H_m$ then $\varphi / E_{m'} = \{o\}$ if $m' \neq m$ and hence $d_\Delta \varphi(x_1 \ldots x_n)$
$= \sum_i (-1)^{i+1} \Delta(x_i, \varphi(x_1 \ldots \hat{x}_i \ldots x_n)) = 0$ if $n \neq m+1$.

Theorem 3: d_Δ is a homogeneous derivation of degree 1 on H.

Proof: $d_\Delta(\varphi \wedge \psi) = \Delta^* \circ \{P \otimes (\varphi \wedge \psi)\} \circ U = \Delta^* \circ \left[P \otimes \{\alpha \circ (\varphi \otimes \psi)\} \circ U\right] \circ U = \Delta^* \circ \left[P \otimes \{\alpha \circ (\varphi \otimes \psi)\} \circ (I \otimes U)\right] \circ U$.
Here $\Delta^* \circ \left[P \otimes \{\alpha \circ (\varphi \otimes \psi)\}\right]$ acts on $E \otimes (E \otimes E)$ for
$\Delta^* \circ \left[P \otimes \{\alpha \circ (\varphi \otimes \psi)\}\right]\left[x \otimes \{y \otimes z\}\right] = \Delta^*\{P(x) \otimes \varphi(y)\psi(z)\} = \Delta(P(x), \varphi(y)\psi(z))$
$= \Delta(P(x), \varphi(y))\psi(z) + \varphi(y)\,\Delta(P(x), \psi(z)) = \Delta^*(P(x) \otimes \varphi(y))\psi(z) + \varphi(y)\,\Delta^*(P(x) \otimes \psi(z))$
$= \alpha\left[\{\Delta^* \circ (P \otimes \varphi)\} \otimes \psi\right]\left[(x \otimes y) \otimes z\right] + \alpha\left[\varphi \otimes \{\Delta^* \circ (P \otimes \psi)\}\right]\left[y \otimes (x \otimes z)\right]$ for $x, y, z \in E$.

Thus if ρ maps $E \otimes (E \otimes E)$ to $(E \otimes E) \otimes E$ by $x \otimes (y \otimes z) \to (x \otimes y) \otimes z$ and τ maps $E \otimes (E \otimes E)$ to $E \otimes (E \otimes E)$ by $x \otimes (y \otimes z) \to y \otimes (x \otimes z)$ one finds
$\Delta^* \circ \left[P \otimes \{\alpha \circ (\varphi \otimes \psi)\}\right] = \alpha \circ \left[\{\Delta^* \circ (P \otimes \varphi)\} \otimes \psi\right]\rho + \alpha \circ \left[\varphi \otimes \{\Delta^* \circ (P \otimes \psi)\}\right] \circ \tau$
Hence

$d_\Delta(\varphi,\psi)=\mu\circ\left[\{\Delta^*\circ(P\otimes\varphi)\otimes\psi\right]\circ\rho\circ(I\otimes U)\circ U+\mu\circ\left[\varphi\otimes\{\Delta^*\circ(P\otimes\psi)\}\right]\circ\tau\circ(I\otimes U)\circ U$

$=\mu\circ\left[\{\Delta^*\circ(P\otimes\varphi)\otimes\psi\right]\circ\rho\circ(I\otimes U)\circ U+\mu\circ\left[\varphi\otimes\{\Delta^*\circ(I\otimes\psi)\}\right]\circ\left[I\otimes\{P\otimes I\}\right]\circ\tau\circ(I\otimes U)\circ U$

Now from (*) in lemma 2 we know $\rho\circ(I\otimes U)\circ U=(U\otimes I)\circ U$ and we can check the following identity

(**) $\{I\otimes(P\otimes I)\}\circ\tau\circ(I\otimes U)\circ U=\{\Delta\otimes(P\otimes I)\}\circ(I\otimes U)\circ U$. To show this we first check that $\tau\circ(I\otimes U)\circ U$ has the same effect on $x\in M$ as $(I\otimes U)\circ U$.

$\tau\circ(I\otimes U)\circ U(x)=\tau(I\otimes U)(x\otimes 1+1\otimes x)=(x\otimes(1\otimes 1)+1\otimes(x\otimes 1+1\otimes x))=1\otimes(x\otimes 1)+x\otimes(1\otimes 1)+1\otimes(1\otimes x)$,

and

$(I\otimes U)\circ U(x)=(I\otimes U)(x\otimes 1+1\otimes x)=x\otimes(1\otimes 1)+1\otimes(x\otimes 1)+1\otimes(1\otimes x)$.

Thus $\tau\circ(I\otimes U)\circ U=(I\otimes U)\circ U$ on M since M is a module and hence addition is commutative. But $\tau\circ(I\otimes U)\circ U$ is not an algebra homomorphism for if $x_1, y_1, z_1, x_2, y_2, z_2$ are homogeneous elements in E then

$\tau\left[\{x_1\otimes(y_1\otimes z_1)\}\{x_2\otimes(y_2\otimes z_2)\}\right]=\tau\left[(-1)^{\deg(y_1\otimes z_1)\deg x_2}\,x_1x_2\otimes\{(y_1\otimes z_1)(y_2\otimes z_2)\}\right]$

$=\tau\left[(-1)^{\deg(y_1\otimes z_1)\deg x_2}\,x_1x_2\otimes\{(-1)^{\deg z_1\deg y_2}\,y_1y_2\otimes z_1z_2\}\right]$

$=(-1)^{\deg(y_1\otimes z_1)\deg x_2+\deg z_1\deg y_2}\,y_1y_2\otimes(x_1x_2\otimes z_1z_2)$

$=(-1)^{\deg y_1\deg x_2+\deg z_1\deg x_2+\deg z_1\deg y_2}\,y_1y_2\otimes(x_1x_2\otimes z_1z_1)$ where the gradation in E•E is the total gradation; and

$\tau\{x_1\otimes(y_1\otimes z_1)\}\tau\{x_2\otimes(y_2\otimes z_2)\}=\{y_1\otimes(x_1\otimes z_1)\}\{y_2\otimes(x_2\otimes z_2)\}$

$=(-1)^{\deg(x_1\otimes z_1)\deg y_2}\,y_1y_2\otimes\{(x_1\otimes z_1)(x_2\otimes z_2)\}$

$=(-1)^{\deg(x_1\otimes z_1)\deg y_2}\,y_1y_2\otimes\{(-1)^{\deg z_1\deg x_2}\,x_1x_2\otimes z_1z_2\}$

$=(-1)^{\deg x_1\deg y_2+\deg z_1\deg y_2+\deg z_1\deg x_2}\,y_1y_2\otimes(x_1x_2\otimes z_1z_2)$.

Hence

$\tau\left[\{x_1\otimes(y_1\otimes z_1)\}\{x_2\otimes(y_2\otimes z_2)\}\right]\neq\tau\{x_1\otimes(y_1\otimes z_1)\}\tau\{x_2\otimes(y_2\otimes z_2)\}$. However, if $\mathcal{R}$: $x\otimes(y\otimes z)\longrightarrow(-1)^{\deg x\deg y}\,x\otimes(y\otimes z)$ for homogeneous x, y and $z\in E$ then

$\mathcal{R}\cdot\tau$ is an algebra homomorphism for $\mathcal{R}(\tau\left[\{x_1\otimes(y_1\otimes z_1)\}\{x_2\otimes(y_2\otimes z_2)\}\right])$

$=(-1)^{\deg y_1\deg x_2+\deg z_1\deg x_2+\deg z_1\deg y_2+\deg y_1y_2\deg x_1x_2}\,y_1y_2\otimes(x_1x_2\otimes z_1z_2)$

$=(-1)^{\deg y_1 \deg x_2 + \deg z_1 \deg x_2 + \deg z_1 \deg y_2 + \deg x_1 \deg y_1 + \deg x_2 \deg y_2 + \deg y_1 \deg x_2 + \deg x_1 \deg y_2} y_1 y_2 \otimes (x_1 x_2 \otimes z_1 z_2)$

and

$[\mathcal{R}\circ\tau\{x_1\otimes(y_1\otimes z_1)\}][\mathcal{R}\circ\tau\{x_2\otimes(y_2\otimes z_2)\}]$

$= \left[(-1)^{\deg y_1 \deg x_1}\{y_1\otimes(x_1\otimes z_1)\}\right]\left[(-1)^{\deg y_2 \deg x_2}\{y_2\otimes(x_2\otimes z_2)\}\right]$

$=(-1)^{\deg y_1 \deg x_1 + \deg y_2 \deg x_2 + \deg x_1 \deg y_2 + \deg z_1 \deg y_2 + \deg z_1 \deg x_2} y_1 y_2 \otimes (x_1 x_2 \otimes z_1 z_2)$.

Also for homogeneous x, y and $z \in E$ $I\otimes(P\otimes I)\circ\mathcal{R}(x\otimes(y\otimes z)) = \begin{cases} 0 \text{ if } \deg y \neq 1 \\ (-1)^{\deg x} x\otimes(y\otimes z) \text{ if } \deg y = 1 \end{cases}$

and hence $I\otimes(P\otimes I)\circ\mathcal{R}=J\otimes(P\otimes I)$. Thus $\{I\otimes(P\otimes I)\}\circ\mathcal{R}\circ\mathcal{R}\circ\tau\circ(I\otimes U)\circ U$

$=\{J\otimes(P\otimes I)\}\circ(I\otimes U)\circ U$ and thus (**) holds since the second use of $\mathcal{R}$ annuls the effect of the first use. Thus we have

$d_\Delta(\varphi\wedge\psi)=\alpha\circ\left[\{\Delta^*\circ(P\otimes\varphi)\}\otimes\psi\right]\circ(U\otimes I)\circ U+\alpha\circ\left[\varphi\otimes\{\Delta^*\circ(I\otimes\psi)\}\right]\circ\left[J\otimes\{P\otimes I\}\right]\circ(I\otimes U)\circ U$

$=\alpha\left[\{\Delta^*\circ(P\otimes\varphi)\circ U\}\otimes\psi\right]\circ U+\alpha\circ\left[(\varphi\circ J)\otimes\{\Delta^*\circ(I\otimes\psi)\circ(P\otimes I)\}\right]\circ(I\otimes U)$

$=\alpha\circ\left[\{\Delta^*\circ(P\otimes\varphi)\circ U\}\otimes\psi\right]\circ U+\alpha\circ\left[(\varphi\circ J)\otimes\{\Delta^*\circ(P\otimes\psi)\circ U\}\right]\circ U=(d_\Delta\varphi)\wedge\psi+(\varphi\circ J)\wedge(d_\Delta\psi)$.

For $\varphi\in H_n$, $\varphi\circ J=(-1)^n\varphi$ and hence

$d_\Delta(\varphi\wedge\psi)=(d_\Delta\varphi)\wedge\psi+(-1)^n\varphi\wedge(d_\Delta\psi)$. Thus d_Δ is a homogeneous derivation of degree 1 on H.

4. Conditions linking Λ and Δ.

For $f\in A$ consider $d_\Lambda\varphi(x_1,\dots,(fx_k),\dots x_n)$

$=\sum_{k<j}(-1)^{k+j+1}\varphi(\Lambda(fx_k,x_j)x_1\dots\hat{x}_k\dots\hat{x}_j\dots x_n)+\sum_{i<k}(-1)^{i+k+1}\varphi(\Lambda(x_i,fx_k)x_1\dots\hat{x}_i\dots\hat{x}_k\dots x_n)$

$+\sum_{\substack{i<j\\ i=k}}(-1)^{i+j+1}\varphi(\Lambda(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$ where fx_k is contained in $x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n$ in the last sum and hence the last sum is

$f\sum_{\substack{i<j\\ i\neq k}}(-1)^{i+j+1}\varphi(\Lambda(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$ if φ is A-multilinear. Hence for φ A-multilinear

$f\,d_\Lambda\varphi(x_1,\dots x_k\dots x_n)-d_\Lambda\varphi(x_1,\dots(fx_k)\dots x_n)$

$=\sum_{k<j}(-1)^{k+j+1}\varphi((f\Lambda(x_k,x_j)-\Lambda(fx_k,x_j))x_1\dots\hat{x}_k\dots\hat{x}_j\dots x_n)$

$+\sum_{i<k}(-1)^{i+k+1}\varphi((f\Lambda(x_i,x_k)-\Lambda(x_i,fx_k))x_1\dots\hat{x}_i\dots\hat{x}_k\dots x_n)$. Now assume that

$\Lambda(fx,y)=f\Lambda(x,y)-\Delta(y,f)x$ where Δ is a K-bilinear mapping from $M\times A$ to A

i.e. $f\wedge(x,y)-\wedge(fx,y)=\Delta(y,f)x$. Then $f\wedge(y,x)-\wedge(y,fx)=-(f\wedge(x,y)-\wedge(fx,y)$ since $\wedge$ is alternating, and thus $f\wedge(y,x)-\wedge(y,fx)=-\Delta(y,f)x$.

Hence

$$fd_\wedge\varphi(x_1\dots x_k\dots x_n)-d_\wedge\varphi(x_1\dots(fx_k)\dots x_n)=\sum_{k<j}(-1)^{k+j+1}\varphi((\Delta(x_j,f)x_k)x_1\dots\hat{x}_k\dots\hat{x}_j\dots x_n)$$
$$+\sum_{i<k}(-1)^{i+k+1}\varphi((-\Delta(x_i,f)x_k)x_1\dots\hat{x}_i\dots\hat{x}_k\dots x_n)=\sum_{j\neq k}(-1)^j\Delta(x_j,f)\varphi(x_1\dots\hat{x}_j\dots x_n)$$

since φ is A-multilinear. Now one has for Δ, $\Delta(y,fg)x=fg\wedge(x,y)-\wedge(fgx,y)$

$=fg\wedge(x,y)-f\wedge(gx,y)+f\wedge(gx,y)-\wedge(fgx,y)=\Delta(y,f)gx+f\Delta(y,g)x$. Thus it is natural to assume further that $\Delta(y,fg)=\Delta(y,f)g+f\Delta(y,g)$ and $\Delta(fy,g)=f\Delta(y,g)$. Thus we can form the derivation d_Δ. Then

$$(\dagger)\quad fd_\wedge\varphi(x_1\dots x_k\dots x_n)-d_\wedge\varphi(x_1\dots(fx_k)\dots x_n)$$
$$=-(d_\Delta f)\wedge\varphi(x_1\dots x_n)-(-1)^k\Delta(x_k,f)\varphi(x_1\dots\hat{x}_k\dots x_n)$$

for

$$(d_\Delta f)\wedge\varphi(x_1\dots x_n)=\sum_\nabla\eta(\nabla,\nabla^*)d_\Delta f(t(\nabla))\varphi(t(\nabla^*))=\sum_i(-1)^{i-1}d_\Delta f(x_i)\varphi(x_1\dots\hat{x}_i\dots x_n)$$

since $f\in A=H_0$ and so $d_\Delta f\in H_1$. Therefore $(d_\Delta f)\wedge\varphi(x_1\dots x_n)$

$=\sum(-1)^{i-1+2}\Delta(x_i,f)\varphi(x_1\dots\hat{x}_i\dots x_n)$ and $-(d_\Delta f)\wedge\varphi=\sum_i(-1)^i\Delta(x_i,f)\varphi(x_1\dots\hat{x}_i\dots x_n)$.

If φ is A-multilinear then

$$(\dagger\dagger)\quad d_\Delta\varphi(x_1\dots(fx_k)\dots x_n)=f(-1)^{k+1}\Delta(x_k,\varphi(x_1\dots\hat{x}_k\dots x_n))$$
$$+\sum_{i\neq k}(-1)^{i+1}\Delta(x_i,f\varphi(x_1\dots\hat{x}_i\dots x_n)=f(-1)^{k+1}\Delta(x_k,\varphi(x_1\dots\hat{x}_k\dots x_n)$$
$$+\sum_i(-1)^{i+1}\Delta(x_i,f\varphi(x_1\dots\hat{x}_i\dots x_n))-(-1)^{k+1}\Delta(x_k,f\varphi(x_1\dots\hat{x}_k\dots x_n)$$
$$=f(-1)^{k+1}\Delta(x_k,\varphi(x_1\dots\hat{x}_k\dots x_n))+d_\Delta(f\varphi)(x_1\dots x_n)-(-1)^{k+1}\Delta(x_k,f\varphi(x_1\dots\hat{x}_k\dots x_n))$$
$$=f(-1)^{k+1}\Delta(x_k,\varphi(x_1\dots\hat{x}_k\dots x_n))+\sum_i(-1)^{i+1}\Delta(x_i,f)\varphi(x_1\dots\hat{x}_i\dots x_n)$$
$$+f\sum_i(-1)^{i+1}\Delta(x_i,\varphi(x_1\dots\hat{x}_i\dots x_n))-(-1)^{k+1}\Delta(x_k,f)\varphi(x_1\dots\hat{x}_k\dots x_n)$$
$$-f(-1)^{k+1}\Delta(x_k,\varphi(x_1\dots\hat{x}_k\dots x_n))=(d_\Delta f)\wedge\varphi(x_1\dots x_n)+fd_\Delta\varphi(x_1\dots x_n)$$
$$+(-1)^k\Delta(x_k,f)\varphi(x_1\dots\hat{x}_k\dots x_n).$$

Thus from $(\dagger)$ and $(\dagger\dagger)$ we get $fd_\wedge\varphi(x_1\dots x_n)-d_\wedge\varphi(x_1\dots(fx_k)\dots x_n)$

$+d_\Delta\varphi(x_1\dots(fx_k)\dots x_n)=fd_\Delta\varphi(x_1\dots x_n)$.

Therefore $(d_\Delta - d_\wedge)\varphi(x_1,\dots(fx_i)\dots x_n) = f(d_\Delta - d_\wedge)\varphi(x_1,\dots x_n)$ i.e. $d = d_\Delta - d_\wedge$ is multilinear in A. Hence $(d_\Delta - d_\wedge) \in G$ for $\varphi \in G$. Also $d = d_\Delta - d_\wedge$ is a homogeneous derivation of degree 1 on G. $d\varphi(x_1,\dots x_n) = (d_\Delta - d_\wedge)\varphi(x_1,\dots x_n)$

$$= \sum_i (-1)^{i+1} \Delta(x_i, \varphi(x_1,\dots \hat{x}_i \dots x_n) - \sum_{i<j} (-1)^{i+j+1} \varphi(\wedge(x_i,x_j)x_1,\dots \hat{x}_i \dots \hat{x}_j \dots x_n).$$

Now if $f \in H_0 = A$ then $d_\Delta f(x) = \Delta(x,f)$ for $x \in M$ and $d_\wedge f = 0$ for $d_\wedge f(x) = f\cdot\tilde{\wedge}(x) = f\circ\mu\circ(\wedge^*\otimes I)\circ U(x) = f\circ\mu\cdot(\wedge^*\otimes I)(x\otimes 1 + 1\otimes x) = 0$ for all $x \in M$.

Call the mapping $f \to \Delta(x,f)$ of $\wedge$ into itself D_x. Then $df(x) = (d_\Delta - d_\wedge)f(x) = d_\Delta f(x) - d_\wedge f(x) = d_\Delta f(x) = \Delta(x,f) = D_x f$ for $x \in M$. Thus we have the following two relations:

(i) $d_\wedge df(xy) = d_\wedge d_\Delta f(xy) = d_\Delta f(\wedge(x,y)) = \Delta(\wedge(x,y),f) = D_{\wedge(x,y)} f$

for $x,y \in M$ and

(ii) $d_\Delta df(xy) = d_\Delta d_\Delta f(xy) = (x, d_\Delta f(y)) - \Delta(y, d_\Delta f(x))$

$= \Delta(x,\Delta(y,f)) - \Delta(y,\Delta(x,f)) = D_x D_y f - D_y D_x f = [D_x, D_y] f$

for $x,y \in M$, where $[D_x, D_y]$ is the Lie product of the derivations D_x, D_y on A. Hence $ddf(xy) = (d_\Delta df - d_\wedge df)(xy) = ([D_x, D_y] - D_{\wedge(x,y)})f$ and thus $ddf = 0$ for all $f \in A$ if and only if $D_{\wedge(x,y)} = [D_x, D_y]$ for all $x,y \in M$ i.e. the $\wedge$-composite of $x,y \in M$ is mapped into the Lie composite of the derivations D_x and D_y on A.

Now suppose that $\omega \in H$, then $d_\wedge\omega(xy) = \omega(\wedge(x,y))$ and $d_\Delta\omega(xy) = \Delta(x,\omega(y)) - \Delta(y,\omega(x)) = D_x\omega(y) - D_y\omega(x)$; hence $d\omega(xy) = (d_\Delta - d_\wedge)\omega(xy) = D_x\omega(y) - D_y\omega(x) - \omega(\wedge(x,y))$. Therefore we have the following two relations:

(i) $d_\wedge d\omega(xyz) = (-1)^{1+2+1} d\omega(\wedge(x,y)z) + (-1)^{1+3+1} d\omega(\wedge(x,z)y)$

$+ (-1)^{2+3+1} d\omega(\wedge(y,z)x)$

$= D_{\wedge(x,y)}\omega(z) - D_z\omega(\wedge(x,y)) - \omega(\wedge(\wedge(x,y),z)$

$- D_{\wedge(x,z)}\omega(y) + D_y\omega(\wedge(x,z)) + \omega(\wedge(\wedge(x,z),y)$

$$+D_{\Lambda(y,z)}\omega(x)-D_x\omega(\Lambda(y,z))-\omega(\Lambda(\Lambda(y,z),x)$$

for $x,y,z\in M$ and

(ii) $$d_\Delta d\omega(xyz)=\Delta(x,d\omega(yz))-\Delta(y,d\omega(xz))+\Delta(z,d\omega(xy))$$

$$=\Delta(x,D_y\omega(z)-D_z\omega(y)-\omega(\Lambda(y,z)))$$

$$-\Delta(y,D_x\omega(z)-D_z\omega(x)-\omega(\Lambda(x,z))$$

$$+\Delta(z,D_x\omega(y)-D_y\omega(x)-\omega(\Lambda(x,y)))$$

$$=D_xD_y\omega(z)-D_xD_z\omega(y)-D_z\omega(\Lambda(y,z))-D_yD_x\omega(z)+D_yD_z\omega(x)$$

$$+D_y\omega(\Lambda(x,z))+D_zD_x\omega(y)-D_zD_y\omega(x)-D_z\omega(\Lambda(x,y))$$

for $x,y,z\in M$. Thus

$$dd\omega(xyz)=(d_\Delta d_\Lambda)d\omega(xyz)=\omega(\Lambda(\Lambda(x,y),z))+\omega(\Lambda(\Lambda(y,z),x))+\omega(\Lambda(\Lambda(z,x),y))$$

$$+([D_x,D_y]-D_{\Lambda(x,y)})\omega(z)-([D_x,D_z]+D_{\Lambda(x,z)})\omega(y)+([D_y,D_z]-D_{\Lambda(y,z)})\omega(x).$$

Hence if $ddf=0$ then $dd\omega=0$ for all $\omega\in H_1$ if the composition Λ satisfies the Jacobi-identity i.e. (M,Λ) is a Lie Algebra over K.

<u>Theorem 4: If $ddf=0$ for all $f\in H_0=A$ and $dd\omega=0$ for all $\omega\in H_1$ then $dd\varphi=0$ for all $\varphi\in H$.</u>

Proof: If $\varphi\in H_{n-2}$ then $d_\Lambda\varphi\in H_{n-1}$ and $d_\Delta\varphi\in H_{n-1}$ and thus

$$d_\Lambda\varphi(x_1\dots x_{n-1})=\sum_{i<j}(-1)^{i+j+1}\varphi(\Lambda(x_i,x_j)x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_{n-1})$$

and

$$d_\Delta\varphi(x_1\dots x_{n-1})=\sum_i(-1)^{i+1}\Delta(x_i,\varphi(x_1\dots\hat{x}_i\dots x_n).$$

Now,

$$d\varphi=(d_\Delta-d_\Lambda)\varphi \text{ and } d(d\varphi)=d_\Delta d_\Delta\varphi-(d_\Delta d_\Lambda\varphi+d_\Lambda d_\Delta\varphi)+d_\Lambda d_\Lambda\varphi.$$

First,

$$d_\Delta d_\Delta\varphi(x_1\dots x_n)=\sum_i(-1)^{i+1}\Delta(x_i,d_\Delta\varphi(x_1\dots\hat{x}_i\dots x_n)$$

$$=\sum_i(-1)^{i+1}\Delta(x_i,\sum_{j<i}(-1)^{j-1}\Delta(x_j,\varphi(x_1\dots\hat{x}_j\dots\hat{x}_i\dots x_n)))$$

$$+\sum_i(-1)^{i+1}\Delta(x_i,\sum_{i<j}(-1)^{j-2}\Delta(x_j,\varphi(x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n)))$$

$$=\sum_{j,i<j}(-1)^{i+j}\,D_{x_j}D_{x_i}\varphi(x_1\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$$

$$-\sum_{i,\,i<j}(-1)^{i+j}\,D_{x_i}D_{x_j}\varphi(x,\dots\hat{x}_i\dots\hat{x}_j\dots x_n)=\sum_{i,\,i<j}(-1)^{i+j+1}\left[D_{x_i},D_{x_j}\right]\varphi(x,\dots\hat{x}_i\dots\hat{x}_j\dots x_n).$$

Next

$$d_\Delta d_\wedge\varphi(x,\dots x_n)=\sum_k(-1)^{k+1}\Delta(x_k,d_\wedge\varphi(x,\dots\hat{x}_k\dots x_n)$$

$$=\sum_k(-1)^{k+1}\Delta(x_k,\sum_{i<j<k}(-1)^{i+j-3}\varphi(\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_j\dots\hat{x}_k\dots x_n))$$

$$+\sum_k(-1)^{k+1}\Delta(x_k,\sum_{k<i<j}(-1)^{i+j-5}\varphi(\wedge(x_i,x_j)x,\dots\hat{x}_k\dots\hat{x}_i\dots\hat{x}_j\dots x_n))$$

$$+\sum_k(-1)^{k+1}\Delta(x_k,\sum_{i<k<j}(-1)^{i+j-4}\varphi(\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_k\dots\hat{x}_j\dots x_n))$$

and

$$d_\wedge d_\Delta\varphi(x,\dots x_n)=\sum_{i<j}(-1)^{i+j+1}(d_\Delta\varphi)(\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$$

$$=\sum_{i<j}(-1)^{i+j+1}\Big[\sum_{i<j<k}(-1)^{k-2}\Delta(x_k,\varphi(\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_j\dots\hat{x}_k\dots x_n))$$

$$+\sum_{k<i<j}(-1)^{k}\Delta(x_k,\varphi(\wedge(x_i,x_j)x,\dots\hat{x}_k\dots\hat{x}_i\dots\hat{x}_j\dots x_n))$$

$$+\sum_{i<k<j}(-1)^{k-1}\Delta(x_k,\varphi(\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_k\dots\hat{x}_j\dots x_n))$$

$$+\Delta(\wedge(x_i,x_j),\varphi(x,\dots\hat{x}_i\dots\hat{x}_j\dots x_n))\Big].$$

Therefore,

$$d_\Delta d_\wedge\varphi(x,\dots x_n)+d_\wedge d_\Delta\varphi(x,\dots x_n)=\sum_{i<j}(-1)^{i+j-1}D_{\wedge(x_i,x_j)}\varphi(x,\dots\hat{x}_i\dots\hat{x}_j\dots x_n).$$

Finally,

$$d_\wedge d_\wedge\varphi(x,\dots x_n)=\sum_{i<j}(-1)^{i+j+1}d_\wedge\varphi(\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$$

$$=\sum_{i<j}(-1)^{i+j+1}\Big[\sum_{i<j<k}(-1)^{k-3}\varphi(\wedge(\wedge(x_i,x_j),x_k)x,\dots\hat{x}_i\dots\hat{x}_j\dots\hat{x}_k\dots x_n)$$

$$+\sum_{k<i<j}(-1)^{k-1}\varphi(\wedge(\wedge(x_i,x_j),x_k)x,\dots\hat{x}_k\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$$

$$+\sum_{i<k<j}(-1)^{k-2}\varphi(\wedge(\wedge(x_i,x_j),x_k)x,\dots\hat{x}_i\dots\hat{x}_k\dots\hat{x}_j\dots x_n)$$

$$+\sum_{k<l<i<j}(-1)^{k+l+1}\varphi(\wedge(x_k,x_l)\wedge(x_i,x_j)x,\dots\hat{x}_k\dots\hat{x}_l\dots\hat{x}_i\dots\hat{x}_j\dots x_n)$$

$$+\sum_{i<j<k<l}(-1)^{k+l+1}\varphi(\wedge(x_k,x_l)\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_j\dots\hat{x}_k\dots\hat{x}_l\dots x_n)$$

$$+\sum_{k<i<l<j}(-1)^{k+l}\varphi(\wedge(x_k,x_l)\wedge(x_i,x_j)x,\dots\hat{x}_k\dots\hat{x}_i\dots\hat{x}_l\dots\hat{x}_j\dots x_n)$$

$$+\sum_{i<k<j<l}(-1)^{k+l}\varphi(\wedge(x_k,x_l)\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_k\dots\hat{x}_j\dots\hat{x}_l\dots x_n)$$

$$+\sum_{k<i<j<l}(-1)^{k+l+1}\varphi(\wedge(x_k,x_l)\wedge(x_i,x_j)x,\dots\hat{x}_k\dots\hat{x}_i\dots\hat{x}_j\dots\hat{x}_l\dots x_n)$$

$$+\sum_{i<k<l<j}(-1)^{k+l+1}\varphi(\wedge(x_k,x_l)\wedge(x_i,x_j)x,\dots\hat{x}_i\dots\hat{x}_k\dots\hat{x}_l\dots\hat{x}_j\dots x_n)\Big]$$

$$=\sum_{i<j}(-1)^{i+j+1}\Big[\sum_{i<j<k}(-1)^{k-1}\varphi\{\wedge(\wedge(x_i,x_j),x_k)+\wedge(\wedge(x_j,x_k),x_i)$$

$$-\Lambda(\Lambda(x_i,x_k),x_j)x_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n\}]$$

$$+\sum_{k<l<i<j}(-1)^{i+j+k+l}\phi(\Lambda(x_k,x_l)\Lambda(x_i,x_j)x_1\cdots\hat{x}_k\cdots\hat{x}_l\cdots\hat{x}_i\cdots\hat{x}_j\cdots x_n)$$

$$+\sum_{k<l<a<j}(-1)^{k+l+i+j}\phi(\Lambda(x_i,x_j)\Lambda(x_k,x_l)x_1\cdots\hat{x}_k\cdots\hat{x}_l\cdots\hat{x}_i\cdots\hat{x}_j\cdots x_n)$$

$$+\sum_{k<i<l<j}(-1)^{i+j+k+l+1}\phi(\Lambda(x_k,x_l)\Lambda(x_i,x_j)x_1\cdots\hat{x}_k\cdots\hat{x}_i\cdots\hat{x}_l\cdots\hat{x}_j\cdots x_n)$$

$$+\sum_{k<i<l<j}(-1)^{i+j+k+l}\phi(\Lambda(x_i,x_j)\Lambda(x_k,x_l)x_1\cdots\hat{x}_k\cdots\hat{x}_i\cdots\hat{x}_l\cdots\hat{x}_j\cdots x_n)$$

$$+\sum_{k<i<j<l}(-1)^{i+j+k+l}\phi(\Lambda(x_k,x_l)\Lambda(x_i,x_j)x_1\cdots\hat{x}_k\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_l\cdots x_n)$$

$$+\sum_{k<i<j<l}(-1)^{i+j+k+l}\phi(\Lambda(x_i,x_j)\Lambda(x_k,x_l)x_1\cdots\hat{x}_k\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_l\cdots x_n)$$

$$=\sum_{i<j<k}(-1)^{i+j+k}\phi\{\Lambda(\Lambda(x_i,x_j),x_k)+\Lambda(\Lambda(x_j,x_k),x_i)+\Lambda(\Lambda(x_k,x_i),x_j)x_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n\}$$

since $\Lambda(x_k,x_l)\Lambda(x_i,x_j)=(-1)\Lambda(x_i,x_j)\Lambda(x_k,x_l)$.

Hence

$$d(d\phi)(x_1\cdots x_n)=\sum_{i,i<j}(-1)^{i+j-1}([D_{x_i},D_{x_j}]-D_{\Lambda(x_i,x_j)})\phi(x_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots x_n)$$

$$+\sum_{i<j<k}(-1)^{i+j+k}\phi\{\Lambda(\Lambda(x_i,x_j),x_k)+\Lambda(\Lambda(x_j,x_k),x_i)$$

$$+\Lambda(\Lambda(x_k,x_i),x_j)x_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n\}.$$

If $ddf=0$ for all $f\in A$ then one has for $\omega\in H_1$ $dd\omega(x_1x_2x_3)=\omega\{\Lambda(\Lambda(x_1,x_2),x_3)+\Lambda(\Lambda(x_2,x_3)$

$+\Lambda(\Lambda(x_3,x_1),x_2)$. Furthermore if for any $\phi\in\mathrm{Hom}(E,A)$, $x_1,\ldots,x_n\in M$, $n\geq 3$ and $i<j<k$, $w_{x_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n}\in\mathrm{Hom}(M,A)$ is given by $x\longrightarrow\phi(xx_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n)$ one has under the assumption $ddf=0$ for $f\in A$

$$dd\phi(x_1\cdots x_n)=\sum_{i<j<k}(-1)^{i+j+k}(dd\omega_{x_1\cdots\hat{x}_i\cdots\hat{x}_j\cdots\hat{x}_k\cdots x_n})(x_ix_jx_k)$$

which is zero if $dd\omega=0$ for $\omega\in H_1$.

5. The Case M=D, the A-module of K-derivations of A.

As before let K be a field and A a commutative K-algebra with unit. Let M=D, the A-module of K-derivations of A and E_Λ be the exterior algebra on M considered as a K-module. Let H and G be as before.

Consider the mapping Λ: $D\times D\longrightarrow D$ by $(X,Y)\longrightarrow[X,Y]$ where $X,Y\in D$ and $[X,Y]f=X(Yf)-Y(Xf)$, the Lie product of X and Y, for $f\in A$. Since D is a Lie K-algebra then Λ is K-bilinear and alternating. Hence as before we

can use λ to determine a homogeneous derivation d_λ of degree 1 on H where for $X_1,\dots,X_n \in D$ and $\varphi \in \mathrm{Hom}_K(E,A)$

$$d_\lambda\varphi(X_1,\dots,X_n)=\sum_{i<j}(-1)^{i+j+1}\varphi(\lambda(X_i,X_j)X_1\dots\hat{X}_i\dots\hat{X}_j\dots X_n)$$
$$=\sum_{i<j}(-1)^{i+j+1}\varphi([X_i,X_j]X_1\dots\hat{X}_i\dots\hat{X}_j\dots X_n).$$

Now let us consider the mapping Δ: $D\times A \longrightarrow A$ defined as follows $(X,f)\longrightarrow Xf$. Δ is clearly K-bilinear since X is a K-derivation on A. Also $\Delta(X,fg)=X(fg)=(Xf)g+f(Xg)=\Delta(X,f)g+f\Delta(x,g)$ since X is a derivation. Hence as before we can use Δ to determine a homogeneous derivation d_Δ of degree 1 on H where for $X_1,\dots,X_n \in D$ and $\varphi \in \mathrm{Hom}_K(E,A)$ $d_\Delta\varphi(X_1,\dots,X_n)$ $=\sum_i(-1)^{i+1}\Delta(X_i,\varphi(X_1\dots\hat{X}_i\dots X_n))=\sum_i(-1)^{i+1}X_i\varphi(X_1\dots\hat{X}_i\dots X_n)$. Also, as before we have $d=d_\Delta-d_\lambda$ a homogeneous derivation of degree 1 on G where $d\varphi(X_1\dots X_n)$ $=(d_\Delta-d_\lambda)\varphi(X_1\dots X_n)=\sum_{i<j}(-1)^{i+j+1}\varphi([X_i,X_j]X_1\dots\hat{X}_i\dots\hat{X}_j\dots X_n)$ $+\sum_i(-1)^{i+1}X_i\varphi(X_1\dots\hat{X}_i\dots X_n)$ for $X_1,\dots,X_n \in D$ and $\varphi \in \mathrm{Hom}_K(E,A)$.

Theorem 5: If $\tilde{A}$ is the algebra of alternating differential forms on A there exists a homogeneous derivation, d, of degree 1 on $\tilde{A}$ such that $dd\theta=0$ for all $\theta\in\tilde{A}$ where $d\theta(X_1,\dots,X_n)=\sum_{i<j}(-1)^{i+j+1}\varphi([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_n)$ $+\sum_i(-1)^{i+1}X_i\varphi(X_1,\dots,\hat{X}_i,\dots,X_n)$.

Proof: Follows from the fact that $G\cong\sum \mathrm{Alt}_A^n(D,A)$.

Note that df for $f\in\tilde{A}_1$ is called the differential of f and $d\varphi$ for $\varphi\in\tilde{A}$ of degree greater than 1 is called the exterior differential of φ. Also, the derivation d on the algebra $\tilde{A}$ of alternating differential forms is called the exterior derivation of $\tilde{A}$.

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III ALTERNATING DIFFERENTIAL FORMS ON A MANIFOLD

Our aim in this section is to indicate some of the considerations which motivated the choice of topic for this thesis.

Definition 15: A manifold is a pair $(E, \mathcal{A})$ where E is a connected topological space and $\mathcal{A}$ is a set of continuous real functions on E such that the following conditions are satisfied:

M1. For each $a \in E$ there exist $\xi_1, \ldots, \xi_n \in \mathcal{A}$ such that (i) $\Phi_\xi : x \to (\xi_1(x), \ldots, \xi_n(x))$ for $x \in E$, maps some open neighbourhood U of a homeomorphically onto an open set in R^n (real n-space), and (ii) for any $f \in \mathcal{A}$, the function $f \circ \Phi_\xi^{-1}$ on $\Phi_\xi(U)$ is (unrestrictedly) differentiable.

M2. $\mathcal{A}$ is maximal (on $\mathcal{C}(E)$, the set of all continuous functions on E) with respect to condition M1.

Remark 1: Since E is connected, the n in M1 is the same for all $a \in E$.

Remark 2: In the following differentiable will be taken to mean unrestrictedly differentiable.

Lemma 3: M1 implies that given any $\zeta_1, \ldots, \zeta_n \in \mathcal{A}$ such that $\Phi_\zeta : x \to (\zeta_1(x), \ldots, \zeta_n(x))$ maps some open V E homeomorphically onto an open set in R^n then, for any $f \in \mathcal{A}$, $f \circ \Phi_\zeta^{-1}$ is a differentiable function on $\Phi_\zeta(V)$.

Proof: For any $a \in V$ there exist $\xi_1, \ldots, \xi_n \in \mathcal{A}$ such that Φ_ξ maps some open neighbourhood $U \subseteq V$ of a homeomorphically onto an open set in R^n and such that for any $f \in \mathcal{A}$, the function $f \circ \Phi_\xi^{-1}$ on $\Phi_\xi(U)$ is differentiable. Then $\Phi_\zeta \circ \Phi_\xi^{-1}$ is a differentiable homeomorphism of $\Phi_\xi(U)$ onto $\Phi_\zeta(U)$ hence its inverse $\Phi_\xi \circ \Phi_\zeta^{-1}$ is differentiable. Now, $f \circ \Phi_\zeta^{-1} = (f \circ \Phi_\xi^{-1})(\Phi_\xi \circ \Phi_\zeta^{-1})$ and hence since $f \circ \Phi_\xi^{-1}$ is differentiable so is $f \circ \Phi_\zeta^{-1}$.

Definition 16: $\xi_1, \ldots, \xi_n \in \mathcal{A}$ are called coordinates on the open set U if they satisfy M1.

Lemma 4: M1 and M2 imply M2′. If $f \in \mathcal{C}(E)$ is such that for any $a \in E$ there exist coordinates $\xi_1, \ldots, \xi_n \in \mathcal{A}$ on an open neighbourhood U of a such that $f \circ \Phi_\xi^{-1}$ is differentiable on $\Phi_\xi(U)$ then $f \in \mathcal{A}$.

Proof: Let $\mathcal{A}' \supseteq \mathcal{A}$ be the set of all $f \in \mathcal{C}(E)$ with this property. Clearly $\mathcal{A}'$ satisfies M1, and hence by M2 $\mathcal{A}' = \mathcal{A}$.

Corollary: If F is differentiable on R^k, then $F \circ \Phi_\zeta \in \mathcal{A}$ for any $\zeta_1, \ldots, \zeta_k$. Also $\mathcal{A}$ is a ring with respect to the usual operations and contains the constant functions.

Proof: Take $a \in E$ and let U be an open neighbourhood of a with coordinates $\xi_1, \ldots, \xi_n \in \mathcal{A}$. On $\Phi_\xi(U)$ consider $(F \circ \Phi_\zeta) \circ \Phi_\xi^{-1} = F \circ (\Phi_\zeta \circ \Phi_\xi^{-1})$. Now $\Phi_\zeta \circ \Phi_\xi^{-1}$ is differentiable since each $\zeta \circ \Phi_\xi^{-1}$ is differentiable on $\Phi_\xi(U)$. Thus $(F \circ \Phi_\zeta) \circ \Phi_\xi^{-1}$ is differentiable on $\Phi_\xi(U)$, and by lemma 4 $F \circ \Phi_\zeta \in \mathcal{A}$. Also $\mathcal{A}$ is a ring with respect to the usual operations, for if F is one of the differentiable functions $(x,y) \to x \pm y$ on R^2 then $f, g \in \mathcal{A}$ implies that $f \pm g = F(f,g) \in \mathcal{A}$ by the first part of the corollary.

Lemma 5: M1 and M2′ imply M2.

Proof: Let $\mathcal{A}' \supseteq \mathcal{A}$, $\mathcal{A}'$ satisfying M1 and $\mathcal{A}$ satisfying both M1 and M2. If $f \in \mathcal{A}'$ and $\xi_1, \ldots, \xi_n \in \mathcal{A}$ are coordinates on an open $U \subseteq E$ then by lemma 3 $f \circ \Phi_\xi^{-1}$ is differentiable on $\Phi_\xi(U)$ and by M2′ $f \in \mathcal{A}$. Hence $\mathcal{A}' = \mathcal{A}$ and $\mathcal{A}$ is maximal with respect to M1. Thus M1 and M2′ imply M2.

The following lemmas will prove results about the ring $\mathcal{A}$ and the Lie R-algebra $\mathcal{D}$ of derivations on $\mathcal{A}$, which will lead upto a result about the exterior derivation, d, on $\tilde{\mathcal{A}}$ the $\mathcal{A}$-algebra of alternating differential forms of $\mathcal{A}$.

Definition 17: If $U \subseteq E$ is an open set and f a function on U, f is called differentiable if for any coordinates $\xi_1, \ldots, \xi_n \in \mathfrak{A}$ on any $V \subseteq U$, $f \circ \Phi_\xi^{-1}$ is differentiable on $\Phi_\xi(V)$.

Remark: Clearly $f \in \mathfrak{A}$ implies f/U is differentiable for any open $U \subseteq E$.

Lemma 6: If $U \subseteq E$ is open and $a \in U$ then there exists an $e \in \mathfrak{A}$ such that $e|V = 1$ and $e|\complement W = 0$ for some open neighbourhood V of a and open W with $V \subseteq W$ and $\bar{W} \subseteq U$, where $\bar{W}$ is the closure of W.

Proof: To obtain the desired e we use the fact that for any $a < a_1 < b_1 < b$ there exist differentiable f on R with $f(x) = 0$ for $x \notin]a, b[$ and $f(x) = 1$ for $x \in [a_1, b_1]$.

Since $a \in E$ there exist $\xi_1, \ldots, \xi_n \in \mathfrak{A}$ such that $\Phi_\xi : x \to (\xi_1(x), \ldots, \xi_n(x))$ for $x \in E$, maps some open neighbourhood X of a homeomorphically onto an open set in R^n and such that for any $f \in \mathfrak{A}$ the function $f \circ \Phi_\xi^{-1}$ on $\Phi_\xi(X)$ is differentiable. Let $U' = U \cap X$. Now take a closed box consisting of $(y_1, \ldots, y_n)$ with $c_i \leq y_i \leq b_i$ and within $\Phi_\xi(U')$, and containing $\Phi_\xi(a)$. Then take $c_i' < y_i < b_i'$ where $c_i < c_i'$ and $b_i' < b_i$. Let $W = \{x \mid c_i < \xi_i(x) < b_i\}$ and $V = \{x \mid c_i' < \xi_i(x) < b_i'\}$. Then $\bar{W} = \{x \mid c_i \leq \xi_i(x) \leq b_i\}$ and we have the required V and W. Now we take f_i on R such that f_i is differentiable; $f_i(x) = 0$ for $x \notin]c_i, b_i[$; and $f_i(x) = 1$ for $x \in [c_i', b_i']$. Put $e = \prod_{i=1}^{n} f_i \circ \xi_i$ and e clearly has the required properties.

Corollary 1: For each $a \in U$, there exists an open neighbourhood $V \subseteq U$ of a such that for any differentiable f on U there exists an $f^* \in \mathfrak{A}$ with $f^*/V = f/V$.

Proof: Take e, V and W as above and define $f^* = \begin{cases} ef \text{ on } U \\ 0 \text{ on } \complement U \end{cases}$. f^* is differentiable on U and on $\complement W$ (here $=0$) hence by lemma 4 $f^* \in \mathfrak{A}$. Clearly $f^*/V = f/V$ since here $e = 1$.

Corollary 2: For each $a \in U$ there exists a neighbourhood $V \subseteq U$ of a and

$e \in \mathfrak{A}$ with $e/V=1$ such that for any $f,g \in \mathfrak{A}$ and h_i differentiable on U with $f/U = \sum (g_i/U)h_i$ there exists $h_i^* \in \mathfrak{A}$ with $h_i^*/V = h_i/V$ and $ef = \sum g_i h_i^*$.

Proof: Everything except $ef = \sum g_i h_i^*$ follows from lemma 6 and corollary 1. By corollary 1 $h_i^* = \begin{cases} eh_i \text{ on } U \\ 0 \text{ on } \complement U \end{cases}$. Therefore $\sum g_i h_i^* = \begin{cases} \sum g_i eh_i \text{ on } U \\ 0 \text{ on } \complement U \end{cases}$

$= \begin{cases} e \sum g_i h_i = ef \text{ on } U \\ 0 = ef \text{ on } \complement U \end{cases}$ and $\sum g_i h_i^* = ef$.

Lemma 7: Let T be a derivation on $\mathfrak{A}$; let $\xi_1, \ldots, \xi_n \in \mathfrak{A}$ be coordinates on the open set $U \subseteq E$; and let $f \in \mathfrak{A}$ then $Tf/U = \sum_i (T\xi_i/U) \frac{\partial}{\partial \xi_i}(f/U)$ where, for differentiable g on U, $\frac{\partial g}{\partial \xi_i} = (g \circ \bar{\Phi}_\xi^{-1})_i \circ \Phi_\xi$ where $(g \circ \bar{\Phi}_\xi^{-1})_i$ is the partial derivative with respect to the ith coordinate.

Proof: Put $U' = \Phi_\xi(U) \subseteq R^n$ and let $F: U' \to R$ be such that $f/U = F \circ \bar{\Phi}_\xi / U$. Let $a \in U$ be a fixed point and $W \subseteq U$ be such that $W' = \Phi_\xi(W)$ is a sphere with centre $a' = \Phi_\xi(a)$. Then for any $x' = \Phi_\xi(x)$, $x \in W$, $F(x')-F(a') = \int_P dF$ for any path $P \subseteq U$ from a' to x by the Fundamental Theorem of Calculus. Calling the coordinates in R^n $v_1, \ldots, v_n$ one has $\int_P dF = \sum_i \int_P F_i(v_1, \ldots, v_n)dv_i$ where the F_i are the partial derivatives with respect to the coordinates. Taking P as the straight line segement from a' to x' in parametric representation $v_i = \xi_i(a) - t(\xi_i(a) - \xi_i(x))$, $0 \le t \le 1$, one gets

$$f(x)-f(a) = \sum_i (\xi_i(x) - \xi_i(a)) \int_0^1 F_i(\ldots, \xi_j(a) - t(\xi_j(a) - \xi_j(x)), \ldots)dt$$

Now for each i, F_i is a differentiable function on U' and hence G_i, given by

$$G_i(\ldots, v_j, \ldots) = \int_0^1 F(\ldots, \xi_j(a) - t(\xi_j(a) - v_j), \ldots)dt$$

for $(v_1, \ldots, v_n) \in U$ is a differentiable function on U'. Its composite $G_i \circ \bar{\Phi}_\xi$, given by

$$x \to \int_0^1 F_i(\ldots, \xi_j(a) - t(\xi_j(a) - \xi_j(x)), \ldots)dt$$

is therefore a differentiable function on U to be denoted by g_i. It

follows that

$$f-f(a)/U=\sum_i(\xi_i-\xi_i(a))/Ug_i.$$

Now, by corollary 2 of lemma 6 there exists a neighbourhood $V\subset U$ of a; $e\in\mathcal{A}$ with $e/V=1$; and $g_i^*\in\mathcal{A}$ with $g_i^*/V=g_i/V$ such that $e(f-f(a))=\sum_i(\xi_i-\xi_i(a))g_i^*$. Acting T on this identity one gets

$$(Te)(f-f(a)+e(Tf)=\sum_i\{(T\xi_i)g_i^*+(\xi_i-\xi_i(a))(Tg_i^*)$$

since $Tf(a)=0$ since $f(a)$ is constant. At the point a V one therefore has $(Tf)(a)=\sum_i(T\xi_i)(a)g_i^*(a)$. Now

$$g_i^*(a)=g_i(a)=\int_0^1 F_i(\ldots,\xi_j(a),\ldots)dt=F_i(\ldots,\xi_j(a),\ldots).$$

Denoting the mapping $x\to F_i\circ\Phi_\xi(x)$ of U into R by $\frac{\partial f}{\partial\xi}$ where $F_i=(f\circ\Phi_\xi^{-1})_i$ one finally has for any $a\in U$

$$(Tf)(a)=\sum_i(T\xi_i)(a)\,\frac{\partial f}{\partial\xi_i}(a)$$

$$\text{i.e.}\quad Tf/U=\sum_i(T\xi_i/U)\,\frac{\partial}{\partial\xi_i}(f/U).$$

Lemma 8: Let $\xi_1,\ldots,\xi_n$ be coordinates on an open set $U\subset E$. If $a\in U$, let V and e be as in lemma 6 and define $X_i:\mathcal{A}\to\mathcal{A}$ by $X_if=(\frac{\partial}{\partial\xi_i}f/U)^*$ i.e. $X_if=\begin{cases}e\,\frac{\partial}{\partial\xi_i}(f/U) & \text{on } U\\ 0 & \text{on } \complement U\end{cases}$ and $\frac{\partial}{\partial\xi_i}(f/U)=(f\circ\Phi_\xi^{-1})_i\circ\Phi_\xi$ (Note that in particular $X_if/V=\frac{\partial}{\partial\xi_i}(f/U)/V$.) These X_i are derivations.

Proof: X_i is an R-linear mapping from $\mathcal{A}$ to $\mathcal{A}$ for if $f_1,f_2\in\mathcal{A}$

(i) $$X_i(f_1+f_2)=\begin{cases}e\,\frac{\partial}{\partial\xi_i}(f_1+f_2/U) & \text{on } U\\ 0 & \text{on } \complement U\end{cases}=\begin{cases}e(f_1+f_2)\circ\Phi_\xi^{-1})_i\circ\Phi_\xi & \text{on } U\\ 0 & \text{on } \complement U\end{cases}$$

$$=\begin{cases}e(f_1\circ\Phi_\xi^{-1})_i\circ\Phi_\xi+e(f_2\circ\Phi_\xi^{-1})_i\circ\Phi_\xi & \text{on } U\\ 0 & \text{on } \complement U\end{cases}=X_if_1+X_if_2$$

and for $f\in\mathcal{A}$ and $\alpha\in R$

(ii) $$X_i(\alpha f)=\begin{cases}e(\alpha f\circ\Phi_\xi^{-1})_i\circ\Phi_\xi & \text{on } U\\ 0 & \text{on } \complement U\end{cases}=\alpha X_if$$

Also for $f,g\in\mathcal{A}$

$$X_i(fg)=\begin{cases}e\,\frac{\partial}{\partial\xi_i}(f/U\,g/U) & \text{on } U\\ 0 & \text{on } \complement U\end{cases}=\begin{cases}e(fg\circ\Phi_\xi^{-1})_i\circ\Phi_\xi & \text{on } U\\ 0 & \text{on } \complement U\end{cases}$$

$$= \begin{cases} e((f\circ\bar{\Phi}_\xi^{-1})_i\,(g\circ\bar{\Phi}_\xi^{-1})+(f\circ\bar{\Phi}_\xi^{-1})(g\circ\bar{\Phi}_\xi^{-1})_i)\circ\bar{\Phi}_\xi \text{ on } U \\ 0 \quad \text{on} \quad \complement U \end{cases}$$

$$= \begin{cases} e\ \frac{\partial}{\partial\xi_i}(f/U)(g/U)+(f/U)e\ \frac{\partial}{\partial\xi_i}(g/U) \text{ on } U \\ 0 \quad \text{on} \quad \complement U \end{cases}$$

$$= (X_i f)g+f(X_i g)$$

Corollary 1: For any $T\in\mathfrak{D}$, we have $eT=\sum T\xi_i X_i$.

Proof: From lemma 7 one has $(eT)f/U=\sum(T\xi_i/U)e\frac{\partial}{\partial\xi_i}(f/U)$; also $(eT)f/\complement U=0$ for all $f\in\mathfrak{A}$. On the other hand $\sum(T\xi_i)X_i f=\begin{cases}\sum(T\xi_i)e\frac{\partial}{\partial\xi_i}(f/U) \text{ on } U \\ 0 \quad \text{on} \quad \complement U\end{cases}$

and hence the assertion is true.

Corollary 2: If $f\in\mathfrak{A}$ and f/V is constant then $Tf/V=0$ for any $T\in\mathfrak{D}$.

Proof: $Tf/V=eTf/V=\sum(T\xi_i)X_i f/V$ by corollary 1. Hence it is enough to show that if f/V is constant $X_i f/V=0$. Now, by definition $X_i f/V=(\frac{\partial}{\partial\xi_i}(f/U))/V$, but $\frac{\partial}{\partial\xi_i}(f/U)=(f/U\circ\bar{\Phi}_\xi^{-1})_i\circ\Phi_\xi$. Hence $f\circ\bar{\Phi}_\xi^{-1}$ is constant on $\Phi_\xi(V)$, and hence its i th derivative is zero. Thus $X_i f/V=0$.

Corollary 3: For any $f\in\mathfrak{A}$ $edf=\sum(X_i f)d\xi_i$.

Proof: For $T\in\mathfrak{D}$, $df(T)=Tf$ and thus

$edf(T)=e(Tf)=(eT)f=\sum(T\xi_i)(X_i f)=\sum(X_i f)d\xi_i(T)$.

Lemma 9: Suppose that $\omega_1,\dots,\omega_k$ are linear forms on $\mathfrak{D}$. Then for $T_1,\dots,T_k\in\mathfrak{D}$ we have

$$\omega_1\wedge\dots\wedge\omega_k(T_1,\dots,T_k) = \det\begin{pmatrix}\omega_1(T_1)\dots\omega_1(T_k)\\ \vdots\quad\quad\vdots\\ \omega_k(T_1)\dots\omega_k(T_k)\end{pmatrix}$$

where $\wedge$ is the law of composition described in section II.

Proof: (By induction) Clearly if k=1 then $\omega_1(T_1)=\det(\omega_1(T_1))$. Assume that for n

$$\omega_1\wedge\dots\wedge\omega_n(T_1,\dots T_n) = \det\begin{pmatrix}\omega_1(T_1)\dots\omega_1(T_n)\\ \vdots\quad\quad\vdots\\ \omega_n(T_1)\dots\omega_n(T_n)\end{pmatrix}.$$

Then $\det \begin{pmatrix} \omega_1(T_1) & \omega_1(T_2) & \cdots & \omega_1(T_n) & \omega_1(T_{n+1}) \\ \omega_2(T_1) & \omega_2(T_2) & \cdots & \omega_2(T_n) & \omega_2(T_{n+1}) \\ \vdots & \vdots & & \vdots & \vdots \\ \omega_n(T_1) & \omega_n(T_2) & \cdots & \omega_n(T_n) & \omega_n(T_{n+1}) \\ \omega_{n+1}(T_1) & \omega_{n+1}(T_2) & \cdots & \omega_{n+1}(T_n) & \omega_{n+1}(T_{n+1}) \end{pmatrix}$

$= (-1)^{1+1}\ \omega_1(T_1)\ \det \begin{pmatrix} \omega_2(T_2) & \cdots & \omega_2(T_{n+1}) \\ \vdots & & \vdots \\ \omega_{n+1}(T_2) & \cdots & \omega_{n+1}(T_{n+1}) \end{pmatrix}$

$+ (-1)^{1+2}\ \omega_1(T_2)\ \det \begin{pmatrix} \omega_2(T_1) & \omega_2(T_3) & \cdots & \omega_2(T_{n+1}) \\ \vdots & \vdots & & \vdots \\ \omega_{n+1}(T_1) & \omega_{n+1}(T_3) & \cdots & \omega_{n+1}(T_{n+1}) \end{pmatrix} + \cdots$

$+ (-1)^{n+1}\ \omega_1(T_n)\ \det \begin{pmatrix} \omega_2(T_1) & \cdots & \omega_2(T_{n-1}) & \omega_2(T_{n+1}) \\ \vdots & & \vdots & \vdots \\ \omega_{n+1}(T_1) & \cdots & \omega_{n+1}(T_{n-1}) & \omega_{n+1}(T_{n+1}) \end{pmatrix}$

$+ (-1)^{n+2}\ \omega_1(T_{n+1})\ \det \begin{pmatrix} \omega_2(T_1) & \cdots & \omega_2(T_n) \\ \vdots & & \vdots \\ \omega_{n+1}(T_1) & \cdots & \omega_{n+1}(T_n) \end{pmatrix}$

$= (-1)^2\ \omega_1(T_1)(\omega_2 \wedge \cdots \wedge \omega_{n+1})(T_2, \ldots, T_{n+1})$

$+ (-1)^3\ \omega_1(T_2)(\omega_2 \wedge \cdots \wedge \omega_{n+1})(T_1, T_3, \ldots, T_{n+1}) + \cdots$

$+ (-1)^{n+1}\ \omega_1(T_n)(\omega_2 \wedge \cdots \wedge \omega_{n+1})(T_1, \ldots, T_{n-1}, T_{n+1})$

$+ (-1)^{n+2}\ \omega_1(T_{n+1})(\omega_2 \wedge \cdots \wedge \omega_{n+1})(T_1, \ldots, T_n)$.

Now recall from section II, if $\phi, \psi \in \mathrm{Hom}_K(E, A)$ then

$\phi \wedge \psi(x_1 \cdots x_n) = \sum \eta(\sigma, \sigma^*)\phi(t(\sigma))\phi(t(\sigma^*))$ and hence here

$(\varphi \wedge (\omega_2 \wedge \cdots \wedge \omega_{n+1}))(T_1, T_2, \ldots, T_{n+1})$

$= (-1)^0\ \omega_1(T_1)(\omega_2 \wedge \cdots \wedge \omega_{n+1})(T_2, \ldots, T_{n+1})$

$+ (-1)^1\ \omega_1(T_2)(\omega_2 \wedge \cdots \wedge \omega_{n+1})(T_1, T_3, \ldots, T_{n+1}) + \cdots$

$+ (-1)^{n-1}\ \omega_1(T_n)(\omega_2 \wedge \cdots \wedge \omega_{n+1})(T_1, \ldots, T_{n-1}, T_{n+1})$

$$+ (-1)^n \omega_1(T_{n+1})(\omega_2 \wedge \dots \wedge \omega_{n+1}(T_1, T_2, \dots, T_n))$$

since ω_1 is a linear form and hence will take everything but a sequence consisting of one of the T_i into zero. Hence

$$\omega_1 \wedge \dots \wedge \omega_{n+1}(T_1, \dots, T_{n+1}) = \det \begin{pmatrix} \omega_1(T_1) & \dots & \omega_1(T_{n+1}) \\ \vdots & & \vdots \\ \omega_{n+1}(T_1) & \dots & \omega_{n+1}(T_{n+1}) \end{pmatrix}.$$

Lemma 10: If $\xi_1, \dots, \xi_n$ are coordinates on the open set $U \subseteq E$ then for any $T_1, \dots, T_k \in \mathfrak{D}$ and any alternating differential form Θ of degree k

$$\Theta(T_1, \dots, T_k)/U = \sum_{i_1 < \dots < i_k} h_{i_1 \dots i_k} \, d\xi_{i_1} \wedge \dots \wedge d\xi_{i_k}(T_1, \dots, T_k)/U$$

with differentiable $h_{i_1 \dots i_k}$ on U.

Proof: For any $a \in U$, let an open neighbourhood $V \subseteq U$, and $e \in \mathfrak{D}$ be as considered earlier in lemma 6. Then

$$e^k \Theta(T_1, \dots, T_k) = \Theta(eT_1, \dots, eT_k) = \Theta\left(\sum_i T_1 \xi_i X_i, \dots, \sum_i T_k \xi_i X_i\right)$$

$$= \sum_{i_1 < \dots < i_k} \det \begin{pmatrix} T_1\xi_{i_1} & \dots & T_1\xi_{i_k} \\ \vdots & & \vdots \\ T_k\xi_{i_1} & \dots & T_k\xi_{i_k} \end{pmatrix} \Theta(X_{i_1}, \dots X_{i_k}).$$

since Θ is multilinear and alternating. For,

$$\Theta\left(\sum_{i_1}(T_1\xi_{i_1})X_{i_1}, \dots, \sum_{i_k}(T_k\xi_{i_k})X_{i_k}\right) = \sum_{i_1, \dots, i_k}(T_1\xi_{i_1}) \dots (T_k\xi_{i_k})\Theta(X_{i_1}, \dots, X_{i_k})$$

$$= \sum_{i_1 < \dots < i_k} \left\{ \sum_{\pi} \varepsilon(\pi) T_1\xi_{i_{\pi(1)}} \dots T_k\xi_{i_{\pi(k)}} \right\} \Theta(X_{i_1}, \dots, X_{i_k})$$

and $\sum_{\pi} \varepsilon(\pi) T_1\xi_{i_{\pi(1)}} \dots T_k\xi_{i_{\pi(k)}}$ is the determinant of the matrix

$$\begin{pmatrix} T_1\xi_{i_1} & \dots & T_1\xi_{i_k} \\ \vdots & & \vdots \\ T_k\xi_{i_1} & \dots & T_k\xi_{i_k} \end{pmatrix}.$$ Thus

$$e^k \Theta(T_1, \dots, T_k) = \sum_{i_1 < \dots < i_k} \Theta(X_{i_1}, \dots, X_{i_k}) d\xi_{i_1} \wedge \dots \wedge d\xi_{i_k}(T_1, \dots, T_k)$$

Hence since $e/V = 1$ we have

$$\Theta(T_1, \dots, T_k)/V = \sum_{i_1 < \dots < i_k} \Theta(X_{i_1}, \dots, X_{i_k}) d\xi_{i_1} \wedge \dots \wedge d\xi_{i_k}(T_1, \dots, T_k)/V.$$

The $\Theta(X_{i_1}, \dots, X_{i_k}) \in \mathfrak{H}$ depend in their definition on the $V \subseteq U$. Let $V' \subseteq U$ and

and $e' \in \mathcal{A}$ be of the same nature as e and V and let X'_i be correspondingly defined, i.e. $X'_i f = \begin{cases} e' \frac{\partial}{\partial \xi_i}(f/U) & \text{on } U \\ 0 & \text{on } \complement U \end{cases}$. It obviously follows that $eX'_i = e' X_i$. Hence

$$e^k \theta(X'_{i_1}, \ldots, X'_{i_k}) = \theta(eX'_{i_1}, \ldots, eX'_{i_k}) = \theta(e'X_{i_1}, \ldots, e'X_{i_k}) = e'^k \theta(X_{i_1}, \ldots, X_{i_k}).$$

Hence $\theta(X'_{i_1}, \ldots, X'_{i_k}) = \theta(X_{i_1}, \ldots, X_{i_k})$ on $V \cap V'$. Now, one has for each suitable open $V \subseteq U$ a function $h^{(v)}_{i_1 \ldots i_k} \in \mathcal{A}$, namely the $\theta(X_{i_1}, \ldots, X_{i_k})$. These give rise to a function $h_{i_1 \ldots i_k}$ on U by the definition $h_{i_1 \ldots i_k}(x) = h^{(v)}_{i_1 \ldots i_k}(x)$ if $x \in V$. If $x \in V'$ also then by what we just proved $h^{(v)}_{i_1 \ldots i_k}(x) = h^{(v')}_{i_1 \ldots i_k}(x)$ and hence since the V's cover U we have the required $h_{i_1 \ldots i_k}$ defined, for $h_{i_1 \ldots i_k}$ is clearly differentiable on U since $h_{i_1 \ldots i_k}/V = h^{(v)}_{i_1 \ldots i_k}$ which is differentiable on V and this is true for all V.

Corollary: For $k > n$ the forms of degree k are zero.

Theorem 6: For any alternating differential form θ of degree k and any open set U on which there exist coordinated functions $\xi_1, \ldots, \xi_n \in \mathcal{A}$ $d\theta(T_1, \ldots, T_{k+1})/U = \sum \frac{\partial}{\partial \xi_i} h_{i_1 \ldots i_k} d\xi_i \wedge d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_k}(T_1, \ldots, T_{k+1})/U$ where the functions $h_{i_1 \ldots i_k}$ are as described in lemma 10.

Proof: For any $a \in U$ let an open neighbourhood $V \subseteq U$ and $e \in \mathcal{A}$ be as considered earlier in lemma 6. Then

$$\theta(T_1, \ldots, T_k)/U = \sum_{i_1 < \ldots < i_k} h_{i_1 \ldots i_k} d\xi_{i_1} \ldots d\xi_{i_k}(T_1, \ldots, T_k)/U.$$

Now,

$$(*) \qquad e\theta = \sum_{i_1 < \ldots < i_k} h^*_{i_1 \ldots i_k} d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_k}$$

since on $\complement U$ both e and $h^*_{i_1 \ldots i_k}$ are zero and on U $h^*_{i_1 \ldots i_k} = e\, h_{i_1 \ldots i_k}$. Acting the exterior derivation, d, on (*) we get

$$(**) \qquad de \wedge \theta + ed\theta = \sum_{i_1 < \ldots < i_k} dh^*_{i_1 \ldots i_k} \wedge d\xi_{i_1} \wedge \ldots \wedge d\xi_{i_k}$$

since e is homogeneous of degree zero and since $dd\xi_i = 0$ for all i. Now

$$de\wedge\theta(T_1,\ldots,T_{k+1}) = \sum_i (-1)^{i+1} (T_i e)\theta(T_1,\ldots,\hat{T}_i,\ldots,T_{k+1}).$$

But e/V is constant and hence by corollary 2, lemma 8, $T_i e/V = 0$. Thus $de\wedge\theta(T_1,\ldots,T_{k+1})/V = 0$. Also, $ed\theta(T_1,\ldots,T_{k+1})/V = d\theta(T_1,\ldots,T_{k+1})/V$. Hence

$$(de\wedge\theta + ed\theta)(T_1,\ldots,T_{k+1})/V = d\theta(T_1,\ldots,T_{k+1})/V.$$

Now $d\theta(T_1,\ldots,T_{k+1})/V = ed\theta(T_1,\ldots,T_{k+1})/V$

$$= \sum_{i_1<\cdots<i_k} edh^*_{i_1\cdots i_k}\ d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k}(T_1,\ldots,T_{k+1})/V.$$

Let us consider $\sum_{i_1<\cdots<i_k} edh^*_{i_1\cdots i_k}\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k} = \sum_{i<i_1<\cdots<i_k}(X_i h^*_{i_1\cdots i_k}\, d\xi_i\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k})$ by corollary 3, lemma 8. If we restrict this to V we have

$$\sum_{i<i_1<\cdots<i_k}(X_i h^*_{i_1\cdots i_k})d\xi_i\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k}(T_1,\ldots,T_{k+1})/V$$

$$= \sum_{i<i_1<\cdots<i_k}\frac{\partial}{\partial\xi_i}h_{i_1\cdots i_k}\, d\xi_i\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k}(T_1,\ldots,T_{k+1})/V$$

since on V, $X_i h^*_{i_1\cdots i_k} = X_i h_{i_1\cdots i_k} = \frac{\partial}{\partial\xi_i} h_{i_1\cdots i_k}$. Hence

$$d\theta(T_1,\ldots,T_{k+1})/V = \sum_{i<i_1<\cdots<i_k}\frac{\partial}{\partial\xi_i} h_{i_1\cdots i_k}\, d\xi_i\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k}(T_1,\ldots,T_{k+1})/V.$$

Thus since the V's cover U we have

$$d\theta(T_1,\ldots,T_{k+1})/U = \sum_{i_1<\cdots<i_k}\frac{\partial}{\partial\xi_i} h_{i_1\cdots i_k}\, d\xi_i\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k}(T_1,\ldots,T_{k+1})/U.$$

Corollary: If d′ is any homogeneous derivation of degree 1 on $\tilde{\mathcal{A}}$ with the properties that $d'f = df$ and $d'df = 0$ for $f\in\mathcal{A}=\tilde{\mathcal{A}}$ then $d = d'$.

Proof: If d′ is a derivation with the prescribed properties then the same proof as above with d′ in place of d shows that

$$d'\theta(T_1,\ldots,T_{k+1})/U = \sum_{i_1<\cdots<i_k}\frac{\partial}{\partial\xi_i} h_{i_1\cdots i_k}\, d\xi_i\wedge d\xi_{i_1}\wedge\cdots\wedge d\xi_{i_k}(T_1,\ldots,T_{k+1})/U.$$

Remark: Hence the algebraically defined derivation d is the same as the analytically defined exterior differentiation of Cartan. (See Theory of Lie Groups by Chevalley.)

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