THE CALCULUS OF ALTERNATING DIFFERENTIAL FORMS
AN ALGEBRAIC FOUNDATION OF THE CALCULUS OF
ALTERNATING DIFFERENTIAL FORMS

By

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SCOPE AND CONTENTS: This thesis is concerned with the calculus of alternating differential forms on a manifold. It establishes that this calculus is of a purely algebraic nature by developing its precise analogue for an arbitrary commutative algebra with unit over a field.
PREFACE

The first section of this thesis summarizes the definitions and theorems necessary for an understanding of the algebraic concepts used later on.

The second section is devoted to showing that for any commutative algebra $A$ with unit over a field $K$, there exists a homogeneous derivation $d$ of degree 1 on the regularly graded algebra $A$ of alternating differential forms of $A$. Here, $A$ consists of the direct sum of the $A$-modules $\Lambda$, of alternating multilinear mappings into $A$ of degree $n$ of the $A$-module $D$ of all $K$-derivations on $A$, taken with its Grassmann multiplication.

In the final section, the algebra $\mathcal{A}$, over the real field $\mathbb{R}$ of all differentiable functions on a differentiable manifold is considered. An explicit formula in terms of coordinate functions and coordinate neighbourhoods is obtained for the above mentioned derivation on the algebra $\mathcal{A}$ of alternating differential forms which identifies it with Cartan's exterior derivation on $\mathcal{A}$.
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I ALGEBRAIC PRELIMINARIES

Definition 1: Let \( R \) be a commutative ring and \( M_i, i=1,2,\ldots,n, \) a family of modules over \( R. \) Then a mapping \( \tau \) of \( \prod_{i=1}^{n} M_i \) into a module \( P \) over \( R \) is multilinear if the following conditions are satisfied: (i) for \( x_i \in M_i, \)
\[
\tau(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) = \tau(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) + \tau(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)
\]
and (ii) for \( x \in M \) and \( \alpha \in R, \)
\[
\alpha(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n) = \alpha(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n).
\]

Definition 2: Let \( R \) be a commutative ring and \( M_i, i=1,2,\ldots,n, \) a family of modules over \( R. \) Then a module \( T \) over \( R \) is called the tensor product of the \( M_i \) if \( T \) and the multilinear mapping \( \phi \) of \( \prod_{i=1}^{n} M_i \) into \( T \) have the following property: if \( \phi \) is any multilinear mapping of \( \prod_{i=1}^{n} M_i \) into a module \( P \) over \( R \) then there exists a unique linear mapping \( f \) of \( T \) into \( P \) such that \( f \cdot \gamma = \phi. \)

Note that \( T \) is usually denoted by \( M_1 \otimes \cdots \otimes M_n \) and \( \tau(x_1,\ldots,x_n) \)
by \( x_1 \otimes \cdots \otimes x_n. \)

Remark: In section 10, chapter III of Fundamental Concepts of Algebra, Chevalley shows that such products do exist and proves the following theorem. Let \( M_i, i=1,2,\ldots,n, \) be a family of modules over a commutative ring \( R. \) Let \( \{1,2,\ldots,n\} = \bigcup_{\epsilon \in I} J_\epsilon \) be a partition of \( \{1,2,\ldots,n\} \) into a finite number of mutually disjoint sets \( J_\epsilon. \) Then there exists a unique isomorphism \( \phi \) of \( M_1 \otimes \cdots \otimes M_n \) with \( \bigotimes_{\epsilon \in I}(M_i) \) such that \( \phi(x_1 \otimes \cdots \otimes x_n) = \bigotimes_{\epsilon \in I}(M_i) \) for every \( (x_1,\ldots,x_n) \in \prod_{i=1}^{n} M_i. \) This isomorphism is called the natural or canonical isomorphism.

Definition 3: An algebra \( A \) over a commutative ring \( R \) is called a regularly graded algebra if the following conditions are satisfied: (1) \( A = \bigoplus_{n \geq 0} A_n (\text{direct}), \)
A_n are submodules of A (ii) A has a unit 1 and \( A_0 = R \cdot 1 \cong R \), and (iii) \( A_n \cdot A_m \subseteq A_{n+m} \).

Note that the \( A_n \) are called homogeneous submodules and \( x \in A_n \) is called a homogeneous element of degree \( n \).

Definition 4: Let \( R \) be a commutative ring and \( M \) a module over \( R \). Let \( T \) be an algebra over \( R \) and \( \psi \) a linear mapping of \( M \) into \( T \). Then the pair \( (T, \psi) \) is a tensor algebra on \( M \) if the following condition is satisfied: if \( \phi \) is any linear mapping of \( M \) into an algebra \( A \) over \( R \) there exists a unique homomorphism \( f \) of the algebra \( T \) into \( A \) such that \( f \circ \psi = \phi \).

Remark: In section 3, chapter V, Chevalley shows that such algebras do exist.

Definition 5: If \( A \) is a regularly graded algebra over a commutative ring \( R \) then \( A \) is called anticommutative if it satisfies the following conditions:

(i) if \( a \) and \( a' \) are homogeneous elements of respective degrees \( k \) and \( k' \) then \( a a' = (-1)^{kk'} aa' \) and (ii) if \( a \) is a homogeneous element of odd degree of \( A \) then \( a^2 = 0 \).

Remark: If \( A_1 \) and \( A_2 \) are regularly graded algebras over a commutative ring \( R \) then \( T = A_1 \otimes A_2 \) the tensor product of the \( R \)-modules \( A_1 \) and \( A_2 \) can be made into a regularly graded algebra by the following definition of multiplication \( (a_1 a'_1)(a_2 a'_2) = (-1)^{kk'} a_1 a'_1 a_2 a'_2 \) for \( a_1, a'_1 \in A_1 \) and \( a_2, a'_2 \in A_2 \). (See Chevalley section 4, chapter V).

Definition 6: Let \( M \) be a module over a commutative ring \( R \) then an exterior algebra on \( M \) is an algebra \( E \) over \( R \) and a linear mapping \( \psi \) of \( M \) into \( E \) with the following properties: (i) \( (\psi(x))^2 = 0 \) for every \( x \in M \) and (ii) if \( \phi \) is any linear mapping of \( M \) into an algebra \( A \) over \( R \) such that \( (\phi(x))^2 = 0 \) for every \( x \in M \), then there exists a unique homomorphism \( f \).
Remark 1: If we speak of an exterior algebra E on M without specifying any mapping of M into E, we shall mean that M is a subset of E and that if \( \psi \) is the identity mapping of M into E then \((E, \psi)\) is an exterior algebra on M.

Remark 2: In section 7, chapter V, Chevalley proves the following statements.

1. **Exterior algebras exist.**
2. An exterior algebra is regularly graded and anticommutative.
3. \( \psi \) is a module isomorphism of M with \( \psi(M) \) and the homogeneous elements of degree 1 of E are those of \( \psi(M) \). Thus if M is a subset of E then E has M itself as its homogeneous submodule of degree 1.
4. M generates E.

**Definition 7:** Let A be an algebra over a ring R then a derivation of A is an R-linear mapping \( T: A \rightarrow A \) such that \( T(fg) = (Tf)g + f(Tg) \) for \( f, g \in A \).

**Definition 8:** If A is a regularly graded algebra over a ring R then a homogeneous derivation of degree 1 on A is an R-linear mapping \( T: A \rightarrow A \) such that \( T(fg) = (Tf)g + (-1)^n f(Tg) \) for \( f, g \in A \) where \( f \) is homogeneous of degree \( n \), and such that \( T \) has degree 1 i.e. \( T(A_n) \subseteq A_{n+1} \).

**Definition 9:** Let R be a commutative ring and M an R-module then for \( n=1, 2, \ldots \) the multilinear alternating forms of degree \( n \) on the R-module M are the mappings \( \Theta: M^n \rightarrow L \) where L is an R-module which are (i) multilinear with respect to R and (ii) alternating i.e. \( \Theta(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \varepsilon(\pi) \Theta(x_1, \ldots, x_n) \) for \( x_1, \ldots, x_n \in M \) where \( \pi \) is any permutation and \( \varepsilon(\pi) \) the alternating character of \( \pi \).
Remark: The forms of degree 1 are simply the linear forms.

Now we denote by $\text{Hom}_R(E_n, L)$ the set of $R$-module homomorphisms of $E_n$ into $L$ where $E_n$ is the homogeneous component of degree $n$ of the exterior algebra $E$ on the $R$-module $M$. Clearly $\text{Hom}_R(E_n, L)$ is an additive group with respect to $(\phi \ast \psi)(t) = \phi(t) \ast \psi(t)$ for $\phi, \psi \in \text{Hom}_R(E_n, L)$, and an $R$-module with respect to $(\lambda \phi)(t) = \lambda \phi(t)$ for $\lambda \in R$, $t \in E_n$ and $\phi \in \text{Hom}_R(E_n, L)$. If we denote the set of multilinear alternating forms of degree $n$ on the $R$-module $M$ by $\text{Alt}_R^n(M, L)$ then we clearly have an $R$-module by the same reasoning as above and the following lemma holds.

Lemma 1: The $R$-module $\text{Alt}_R^n(M, L)$ is isomorphic to the $R$-module $\text{Hom}_R(E_n, L)$.

Proof: Suppose $M$ is a module over the commutative ring $R$ and let $(T, \Theta)$ be a tensor algebra over $M$. For any $n \geq 0$, let $\tau_n$ be the multilinear mapping of the product of $n$ modules identical with $M$, defined by $\tau_n(x_1, \ldots, x_n) = \Theta(x_1) \cdots \Theta(x_n)$. Let $T_n$ be the submodule of $T$ generated by the elements $\tau_n(x_1, \ldots, x_n)$ for all $x_1, \ldots, x_n \in M$ then $T = \bigoplus_{n=0}^{\infty} T_n$ (direct) and $T_n$ is a tensor product of $n$ modules identical with the module $M$, with $\tau_n$ the tensor map from $M^n$ to $T_n$ (theorem 6, section 3, chapter V, Chevalley). Then by the definition of tensor product if $\varphi$ is any multilinear mapping of $M^n$ into an $R$-module $L$ then there exists a unique linear mapping $f$ of $T_n$ into $L$ such that $\varphi = f \circ \tau_n$. Consider the correspondence $\varphi \rightarrow f$; to each there corresponds a unique $f$ with $\varphi = f \circ \tau_n$ and to each $f$: $T_n \rightarrow L$ there corresponds $f \circ \tau_n$. Also if $\varphi_1 \rightarrow f_1$ and $\varphi_2 \rightarrow f_2$, then $\varphi_1 \ast \varphi_2 \rightarrow f_1 + f_2$ since $\varphi_1 \ast \varphi_2 = f_1 \circ \tau_n + f_2 \circ \tau_n = (f_1 + f_2) \circ \tau_n$, and $\lambda \varphi \rightarrow \lambda f$ for $\lambda \in R$ since $\lambda(f \circ \tau_n) = (\lambda f) \circ \tau_n$. Thus the correspondence $\varphi \rightarrow f$ is an $R$-module isomorphism.

In the construction of an exterior algebra in section 7, chapter V, Chevalley shows that if $E$ is an exterior algebra on the $R$-module
M then $E$ is of the form $T/F$ where $(T,e)$ is a tensor algebra on $M$ and $F$ is the ideal of $T$ which is generated by the elements $(e(x))^2$ for $x \in M$.

Such elements are homogeneous of degree 2 in $T$ and hence $F$ is a homogeneous ideal (theorem 4, section 2, chapter V, Chevalley). It follows that $E=T/F$ has the structure of a regularly graded algebra (theorem 5, section 2, chapter V, Chevalley). If $\pi$ is the natural mapping of $T$ on $E$ then the module $E_n$ of homogeneous elements of degree $n$ of $E$ is the image under $\pi$ of the module $T_n$ of homogeneous elements of degree $n$ of $T$ i.e.

$$E_n = \pi(T_n) = T_n/F_n$$

where $T_n$ consists of linear combinations of products in $T$ of $n$ elements of $\Theta(M)$, and $F_n$ is generated as an $R$-module by elements of the form $a(e(x))^2b$ where $a = a(y_1, \ldots, y_k), b = b(y_1, \ldots, y_k)$ and $k + l \times 2 = n$.

Now our multilinear mapping $\phi$ from $M^n$ to $L$ is alternating if and only if the corresponding $f$ vanishes on $F_n$ since $\phi(x_1, \ldots, x_n)$

$$= f(e(x_1, \ldots, x_n)) = f(e(x_1) \ldots e(x_n)).$$

If $\bar{\phi}_n$ is the restriction of $\bar{\phi}$ to $T_n$ then the kernel of $\bar{\phi}_n$ is $F_n$. The kernel of $f$ contains $F_n$ when $\phi$ is alternating; hence there exists a unique linear mapping $f'$ from $E$ into $L$ such that $f' \circ \bar{\phi}_n = f$. Hence there is a one-to-one correspondence between the alternating $\phi$ and the $R$-module homomorphisms $f'$ from $E_n$ to $L$, for if $\phi$ is alternating then $f'$ exists with $\phi = f' \circ \pi_n \circ c_n$ and if $f'$ is an $R$-module homomorphism from $E_n$ to $L$ then put $f' \circ \pi_n = f$ and $f$ vanishes on $F_n$ and hence we have $\phi = f \circ c_n$ and $\phi$ is alternating. Also if $\phi_1$ and $\phi_2$ are alternating and $\phi_1 \rightarrow f_1', \phi_2 \rightarrow f_2'$ then $\phi_1 \circ \phi_2 \rightarrow f_1' \circ f_2'$ since $\phi_1 \circ \phi_2 = f_1' \circ \pi_n \circ c_n + f_2' \circ \pi_n \circ c_n = (f_1' + f_2') \circ \pi_n \circ c_n$ and $\phi_1 \rightarrow \phi_1'$ for $\phi_1 \circ c_n$ and $\phi_2 \circ c_n$ for $\phi_2 \circ c_n$. Hence the correspondence $\phi \rightarrow f'$ is an $R$-module isomorphism between the alternating multilinear mappings from $M^n$ to $L$ and the $R$-module homomorphisms from $E_n$ to $L$. i.e. $\operatorname{Alt}_R^n(M, L) \cong \operatorname{Hom}_R(E_n, L)$. 
Let us now consider $A$ a commutative $K$-algebra where $K$ is a field. Let $D$ be the set of $K$-derivations of $A$. Then the set $D$ forms an $A$-module under $(fT)g = f(Tg)$ for $f, g \in A$ and $T \in D$ i.e. $A \times D \to D$ by $(f, T) \to fT$ where $fT : g \to f(Tg)$. First, $fT \in D$ since $fT(gh) = f(T(gh)) = f(Tg)h + g(Th)$

$= ((fT)g)h + g((fT)h)$ for $g, h \in A$ and clearly $fT$ is $K$-linear since $T$ is.

Secondly, (a) $f(T_1 + T_2) = fT_1 + fT_2$ for $f \in A$ and $T_1, T_2 \in D$ since $f(T_1 + T_2)g = f((T_1 + T_2)g) = f(T_1g + T_2g)$ for $g \in A$ (b) $(f + h)T = fT + hT$ for $f, h \in A$ and $T \in D$ since $((f + h)T)g = (f + h)(Tg) = f(Tg)h + h(Tg)$ for $g \in A$ and (c) $(fT)g = f(Tg)$ for $f, h \in A$ and $T \in D$ since $((fh)T)g = (fh)(Tg) = f((hT)g)$ for $g \in A$. Similarly $D$ is a $K$-module under $(\lambda T)g = \lambda(Tg)$ for $\lambda \in K$, $g \in A$ and $T \in D$ i.e. $K \times D \to D$ by $(\lambda, T) \to \lambda T$ where $\lambda T : g \to \lambda(Tg)$. Also the $K$-module $D$ forms a Lie $K$-algebra with the following definition of multiplication. If $T_1, T_2 \in D$ then define $[T_1, T_2]$ as the mapping from $A$ to itself which has the following effect $[T_1, T_2]f = T_1(f) - T_2(T_1(f))$. In order for $D$ to be a Lie $K$-algebra with this definition of multiplication we have to check the following things:

(a) $[T_1, T_2] : D \to D$ (b) $[T_1, T_2] = -[T_2, T_1]$ and $[T_1, T_2] + [T_2, T_3] + [T_3, T_1] = 0$ and (c) the multiplication is bilinear. (a) $[T_1, T_2]$ is $K$-linear since for $f_1, f_2 \in A$ and $T_1, T_2 \in D$

$[T_1, T_2](f_1 + f_2) = T_1(T_2(f_1 + f_2)) = T_2(T_1(f_1 + f_2))$

$= T_2(T_1(f_1) + T_2(T_1(f_2))) = [T_1, T_2]T_1(f_1) + [T_1, T_2]T_2(f_2)$ and for $f \in A$, $\lambda \in K$ and $T_1, T_2 \in D$

$[T_1, T_2](\lambda f) = \lambda(T_1(T_2(\lambda f))) = \lambda(T_1(\lambda f))$

$= \lambda(T_1(T_2(f))) = \lambda[T_1, T_2](f)$

since $T_1$ and $T_2$ are $K$-linear. Also

$[T_1, T_2](fg) = T_1(T_2(fg)) - T_2(T_1(fg))$

$= T_1((T_2(f) + T_1(f)g)) - T_2((T_1(f)g) + T_1(f))$

$= T_1((T_2(f)g + f(T_2(g))) - T_2((T_1(f)g + f(T_1(g)))$
\[(T_1 (T_2 f)) g + (T_2 f) (T_1 g) \cdot (T_1 f) (T_2 g) = (T_1 f) (T_2 g) + (T_2 g) (T_1 f) + f(T_1 (T_2 g)) - (T_1 f) (T_2 g) - (T_2 f) (T_1 g) - f(T_2 (T_1 g))
\]
\[= (T_1, f) g f ([T_1, T_2] g)
\]

Hence \([T_1, T_2] \in D\).

(b) \([T_1, f] f = (T_1 f) = T_1 (T_2 f) - T_2 (T_1 f)
\]
\[= - (T_2 f) + (T_1 f) = [T_1, T_2] f
\]

for \(f \in A\) and \(T_1, T_2 \in D\). Also for \(f \in A\) and \(T_1, T_2, T_3 \in D\)
\[\left[ [T_1, T_2], T_3 \right] f = \left[ [T_1, T_2], T_3 \right] f + \left[ [T_3, T_2], T_1 \right] f + \left[ [T_3, T_1], T_2 \right] f
\]
\[= \left[ [T_1, T_2], T_3 \right] f - T_2 \left[ [T_1, T_2], T_3 \right] f + \left[ [T_3, T_1], T_2 \right] f - T_1 \left[ [T_3, T_1], T_2 \right] f
\]
\[= 0,
\]

(c) \([T_1, + T_2], T_3] f = (T_1, + T_2), (T_3 f) - T_3 ((T_1, + T_2) f)
\[= [T_1, T_2] f + [T_2, T_1] f
\]

for \(f \in A\) and \(T_1, T_2, T_3 \in D\). Also for \(f \in A, K, K, T_1, T_2 \in D\).
\[\left[ [T_1, T_2], T_3 \right] f = (\lambda T_2 f) - T_2 ((\lambda T_1) f)
\]
\[= (\lambda T_1 f), - T_2 (\lambda f) \cdot (\lambda (T_1 f)) = (\lambda T_1 f), - T_2 (\lambda (T_1 f))
\]
\[= \lambda [T_1, T_2] f
\]

Hence multiplication is bilinear.

**Definition 12:** Let \(A\) be a commutative algebra over a field \(K\) then the alternating differential forms of degree \(n\) of \(A\) are (i) the \(f \in A\) for \(n = 0\) (ii) the alternating multilinear forms of degree \(n\) of the \(A\)-module \(B\) for \(n = 1, 2, \ldots\)

Let the set of alternating differential forms of degree \(n\) of \(A\) be denoted by \(\tilde{A}_n\) and put \(\tilde{A} = \tilde{A}_n \) (direct).
II THE ALGEBRA OF ALTERNATING DIFFERENTIAL FORMS
OF AN ALGEBRA A AND ITS EXTERIOR DERIVATION

In this section we shall show that if \( A \) is a commutative
K-algebra with unit where \( K \) is a field, the \( A \)-module of its alternating
differential forms, \( \widetilde{A} \), can be made into a regularly graded \( A \)-algebra, and
there exists a homogeneous derivation, \( d \), of degree 1 on \( \widetilde{A} \) with the property
that \( dd\phi = 0 \) for \( \phi \in \widetilde{A} \).

In the following we will consider an arbitrary \( A \)-module \( M \)
(to be specialized later to the module of derivations on \( A \)). This module
\( M \) is naturally a \( K \)-module since if we identify \( \mathfrak{z} \in K \) with \( 1 \) where 1 is
the unit in \( A \) then \( A \approx K \). Hence we can consider the exterior algebra \( E \)
of \( M \) considered as a \( K \)-module. Then as in the first section \( \text{Hom}_K(E, A) \)
is the set of all \( K \)-module homomorphisms \( \phi : E \to A \), and it is clearly an
\( A \)-module with respect to \( (f \phi)(t) = f \phi(t) \) for \( f \in A \) and \( t \in E \).

\( \phi \in \text{Hom}_K(E, A) \) is called homogeneous of degree \( n \) if \( \phi/E^n = 0 \)
for all \( m + n \). Put \( H_n = \text{set of all } \phi \text{ which are homogeneous of degree } n \).
\( H_n \) is an \( A \)-submodule for if \( \phi, \psi \in H_n \) then \( \phi + \psi / E^m = \phi/E^m + \psi/E^m = 0 + 0 \) if
\( m + n \) and if \( f \in A \) and \( \phi \in H_n \) then \( f \phi / E^m = f(\phi/E^m) = 0 \) if \( m + n \). Let \( H = \bigoplus H_n \) and
clearly \( H \cap E^0 = 0 \); hence the sum is direct. If \( \phi \in H_n \) then \( \phi(t) = 0 \) for
t \( n+1 \) \( E \), and for \( t \in E_0 \) \( \phi(1) = \phi(1) = \phi(1) - \phi \) and hence if \( \epsilon \in H_0 \) is
the mapping \( \epsilon(f) = f \) then \( \phi = \phi(1) \epsilon \). Thus \( H_0 = A \epsilon \).

\( \phi \in H_n \) is called multilinear in \( A \) if \( \phi(x_1, \ldots, f(x_i), \ldots x_n) \)
= \( f \phi(x_1, \ldots, x_n) \). The set \( G_n \) of all such \( \phi \) is clearly an \( A \)-submodule and
\( G = \bigoplus G_n \) is direct. Furthermore \( G_0 = A \epsilon \) since any \( \epsilon \) for \( f \in A \) is multilinear
\( f \).
and hence $A_\varepsilon \leq G_\varepsilon$, but $G_\varepsilon \leq H_\varepsilon = A_\varepsilon$.

By lemma 1, section I we know that $\text{Alt}_n^r (M, A) \cong \text{Hom}_k (E^n A)$, but clearly $\text{Hom}_k (E^n A) \cong M_n$. Thus $\phi \in \text{Alt}_n^r (M, A) \rightarrow \phi \in \text{Hom}_n (E^n A)$ with $\phi / E^n = 0$ if $m \neq n$ and $\phi^*(x, \ldots, x_n) = \phi(x, \ldots, x_n)$. Now, if $\phi \in \text{Alt}_n^r (M, A) \cong \text{Alt}_n^r (M, A)$ then $\phi(x, \ldots, f(x_i), \ldots, x_n) = f\phi(x, \ldots, x_n)$ which implies that $\phi^* \in G_n$. Conversely if $\phi^* \in G_n$ then $\phi(x, \ldots, f(x_i), \ldots, x_n) = f\phi(x, \ldots, x_n)$ and $\phi$ is $A$-multilinear. Hence $\text{Alt}_n^r (M, A) \cong G_n$.

When we specialize $M$ to $D$, the $A$-module of $K$-derivations of $A$ then $\text{Alt}_n^r (D, A) = \widetilde{A}_n$ and $\tilde{A} = \sum \tilde{A}_n = \sum \text{Alt}_n^r (D, A)$.

1. The Algebra $\text{Hom}_k (E, A)$.

In the following in order to make $\text{Hom}_k (E, A)$ into an associative $A$-algebra with unit we wish to define a law of composition in $\text{Hom}_k (E, A)$ which we will call multiplication.

Consider the algebra $E \otimes E$ which is the tensor product of the exterior algebra $E$ on the $K$-module $M$ with itself. The $K$-module $E \otimes E$ is the direct sum of submodules $E_2 \otimes E_n$. Now consider the mapping $x \mapsto x \otimes 1 + 1 \otimes x$ for $x \in M$ which maps $M$ into $E \otimes E$. Since $(x \otimes 1 + 1 \otimes x)(x \otimes 1 + 1 \otimes x)$

$= x \otimes 1 - 1 \otimes x + 1 \otimes x + x \otimes 1 + 1 \otimes x + 1 \otimes x = 0$, by the definition of an exterior algebra this can be extended to a $K$-homomorphism $U: E \rightarrow E \otimes E$. $U$ is called the analyzing map of $E$. An explicit formula for $U$ is $U(x_1, \ldots, x_n) = \sum_{\sigma} (\sigma \otimes \sigma^*) t(\sigma) \otimes t(\sigma^*)$

where $\sigma = (i_1, \ldots, k \ldots i_k)$ with $i_1 < \ldots < i_k$; $\sigma^*$ is the complementary sequence $(j_1, \ldots, j_{n-k})$ with $j_1 < \ldots < j_{n-k}$; $t(\sigma) = x_{i_1} \cdots x_{i_k}$; $t(\sigma^*) = x_{j_1} \cdots x_{j_{n-k}}$; and

$\eta(\sigma, \sigma^*) = (-1)^{N(\sigma, \sigma^*)}$ where $N(\sigma, \sigma^*)$ is the number of pairs $(i, j)$ with $i < j$.

Remarks: $\eta(\sigma, \sigma^*)$ is the signature of the permutation $(1, \ldots, k, l \ldots n)$.

$\begin{pmatrix} 1, \ldots, k & l, \ldots, n \\ i_1, \ldots, i_k & j_1, \ldots, j_{n-k} \end{pmatrix}$
Let \( \xi : A \circ A \to A \) be the mapping defined by \( \xi(f \circ g) = fg \). \( \xi \) is called the \textit{linearization} of the multiplication in \( A \). For \( \phi, \psi \in \text{Hom}_K(E, A) \) put \( \phi \otimes \psi = \xi(\phi \circ \psi) \). Thus \( \phi \otimes \psi(x_1, \ldots, x_n) = \xi(\eta_{x_1, \ldots, x_n} \phi(t(x)) \psi(t(x))) \), and clearly \( \phi \otimes \psi \) is in \( \text{Hom}_K(E, A) \).

**Theorem 1:** \( \text{Hom}_K(E, A) \) is an associative \( A \)-algebra with unit, with respect to the law of composition \( (\phi \otimes \psi) \to \phi \otimes \psi \), and \( H \) and \( G \) are regularly graded subalgebras with \( H \) anticommutative.

**Proof:** The mapping \( (\phi \otimes \psi) \to \phi \otimes \psi \) is bilinear with respect to \( A \). For if \( \phi, \psi \in \text{Hom}_K(E, A) \) then
\[
(\phi + \phi') \otimes \psi = \xi\left( (\phi + \phi') \circ \psi \right) = \xi\left( \phi \circ \psi \right) + \xi\left( \phi' \circ \psi \right) = \phi \otimes \psi + \phi' \otimes \psi,
\]
and similarly for \( \psi, \phi \in \text{Hom}_K(E, A) \). \( \phi \otimes (\psi + \psi') = \phi \otimes \psi + \phi \otimes \psi' \); also if \( \phi \in A \) then
\[
(\phi \otimes \psi)(x_1, \ldots, x_n) = \xi(\eta_{x_1, \ldots, x_n} \phi(t(x)) \psi(t(x))) = \phi \otimes (\psi(x_1, \ldots, x_n)) = (\phi \otimes \psi)(x_1, \ldots, x_n)
\]
since \( A \) is commutative, and similarly \( \phi \otimes (f \circ \psi) = \phi \otimes (f \circ \psi) \).

The law of composition \( (\phi \otimes \psi) \to \phi \otimes \psi \) for \( \phi, \psi \in \text{Hom}_K(E, A) \) is associative i.e. \( (\phi \otimes \psi) \otimes \eta = \phi \otimes (\psi \otimes \eta) \). For
\[
(\phi \otimes \psi) \otimes \eta = \xi\left( (\phi \otimes \psi) \circ \eta \right) = \xi\left( \phi \circ (\psi \otimes \eta) \right) = \xi\left( \phi \circ (\psi \otimes \eta) \right) \circ \eta \circ \text{Hom}_A(E, A), \quad U = \xi\left( \phi \circ (\psi \otimes \eta) \right) \circ \eta \circ \text{Hom}_A(E, A), \quad U
\]
and
\[
\xi(\phi \otimes (\psi \otimes \eta)) = \xi\left( \phi \circ (\psi \otimes \eta) \right) = \xi\left( \phi \circ (\psi \otimes \eta) \right) \circ \eta \circ \text{Hom}_A(E, A), \quad U = \xi\left( \phi \circ (\psi \otimes \eta) \right) \circ \eta \circ \text{Hom}_A(E, A), \quad U
\]
and
\[
\xi(\phi \otimes \eta) \circ \phi = \xi\left( \phi \circ (\eta \otimes \phi) \right) = \xi\left( \phi \circ (\eta \otimes \phi) \right) \circ \phi \circ \text{Hom}_A(E, A), \quad U = \xi\left( \phi \circ (\eta \otimes \phi) \right) \circ \phi \circ \text{Hom}_A(E, A), \quad U
\]
Now let \( \omega \) be the canonical isomorphism of \( (E \otimes E) \otimes E \) with \( E \circ (E \otimes E) \). For \( t, u, v \in E \), \( \omega(t \otimes (u \otimes v)) = \omega(t \otimes (u \otimes v)) \).

\[
\omega(t \otimes (u \otimes v)) = \omega(t \otimes (u \otimes v)) = \omega(t \otimes (u \otimes v)) \]
and
\[
\omega(\omega(t \otimes (u \otimes v)) \otimes (t \otimes u)) = \omega(t \otimes (u \otimes v)) \otimes (t \otimes u) = \omega(t \otimes (u \otimes v)) \otimes (t \otimes u)
\]
But we can check the following identity \( \omega(\omega(1 \circ E)) \cdot U \)
\[
= (1 \circ U) \cdot U.
\]
It is enough to check that the two mappings have the same
effect on \(x \in H\) since \(\omega\) generates \(E\) and the mappings occurring are algebra homomorphisms.

\[
\omega((UoI) \cdot U(x) = \omega(UoI)(x1 + 1ex)
\]

\[
= \omega(x1 + (1ex)1 + (1ex)x = x1(1ex) + 1ex(x1) + 1ex(1ex).
\]

\[
((IoU) \cdot U(x) = (IoU)(x1 + 1ex) = x1(1ex) + 1ex(x1) + 1ex(1ex).
\]

Hence on \(E\)

\[
\lambda \circ (axI) \cdot \{ (\phi \otimes \psi) \} : (UoI) \cdot U = \lambda \circ (axI) \cdot \{ (\phi \otimes \psi) \} \circ (IoU) \cdot U \quad \text{and} \quad (\phi \otimes \psi) \otimes \psi = \phi(\psi \cdot \psi).
\]

The law of composition \((\phi \cdot \psi) \rightarrow \phi \cdot \psi\) for \(\phi, \psi \in \text{Hom}_K(E, A)\) has a unit \(e\) where \(e \in H_0\) and is the mapping \(e(f) = f\) for \(f \in E_0\). \(\phi \cdot e = \phi\) for \(\phi \in \text{Hom}_K(E, A)\) and \(e(x) = 1\) for \(x \in E_1\). Similarly \(e^{11} = e\).

If \(\phi \in H_m\) and \(\psi \in H_m\) then \(\phi \cdot \psi \in H_{mn}\), and hence \(H\) is closed under \(\wedge\).

For if \(\phi \in H_n\) then \(\phi \cdot E_n = \{0\}\) for all \(n' \neq n\), and if \(\psi \in H_m\) then \(\psi \cdot E_m = \{0\}\) for all \(m' \neq m\). Thus \(\phi \cdot E_n \cdot \psi \cdot E_m = \{0\}\) if \((m', n') \neq (m, n)\). Since \(U(E_p) \subseteq \sum_{n \geq 0} E_{nm} \cdot E_{nm}\), where the gradation of \(E \otimes E\) is the total gradation, then \((\phi \cdot \psi) / E_p = \{0\}\) if \(p \neq m + n\). Therefore \(\phi \cdot \psi \in H_{nm}\) and \(H\) is closed under \(\wedge\). Hence since \(H\) is already an \(A\)-submodule, \(H = \sum_{n \geq 0} H_n \) (direct); and \(H_0 = A \otimes H_0\), \(H\) is a regularly graded algebra.

\(H\) is also anticommutative, for if \(\phi \in H_n\) then \(\phi \cdot \theta = 0\) if \(n\) is odd, and if \(\phi \in H_n\) and \(\psi \in H_m\) then \(\psi \cdot \phi = (-1)^m \phi \cdot \psi\). To prove the first assertion that \(\phi \cdot \theta = 0\) where \(\phi \in H_n\), if \(n\) is odd, it is sufficient to consider

\[
\phi \cdot \theta(x1 \ldots x_n) = \frac{1}{n!} \sum_{\tau} \eta(\tau, \tau^{\star}) \phi(x_{\tau(1)} \ldots x_{\tau(n)}) \theta(x_{\tau(1)} \ldots x_{\tau(n)})
\]

since \(\theta / E_n = \{0\}\) if \(n' \neq n\). This sum has terms of the form \(\eta(\tau, \tau^{\star}) + \eta(\tau^{\star}, \tau)\). Now, \(\eta(\tau, \tau^{\star}) + \eta(\tau^{\star}, \tau) = (-1)^{n \cdot \text{grad}(\tau)} + (-1)^{n \cdot \text{grad}(\tau^{\star})} \) and hence \(n(\tau, \tau^{\star}) + n(\tau^{\star}, \tau) = n^2\).

Therefore \((-1)^{n \cdot \text{grad}(\tau)} = (-1)^{n \cdot \text{grad}(\tau^{\star}) = (-1)^{n \cdot \text{grad}(\tau)}} \) i.e. \(\eta(\tau, \tau^{\star}) = (-1)^{n^2} \eta(\tau^{\star}, \tau)\) and hence \(\eta(\tau, \tau^{\star}) + \eta(\tau^{\star}, \tau) = 0\) since \(n\) is odd. Thus \(\phi \cdot \theta = 0\). In order
to prove the second assertion we consider
\[ \varphi \psi (x_{1} \ldots x_{n} + m) = \sum_{t} (\varphi, \psi) \varphi (x_{1} \ldots x_{n + m}) \psi (x_{1} \ldots x_{n + m}) \]
\[ = \sum_{t} (\varphi, \psi) \varphi (x_{1} \ldots x_{n + m}) \psi (x_{1} \ldots x_{n + m}) . \]
If the length of \( v \) is \( n \) and the length of \( v' \) is \( n \) then \( \psi (v, v') = (-1)^{nm} \psi (v, v') \) since \( N(v, v') + N(v, v') = nm \) and hence \( \psi \psi (x_{1} \ldots x_{n + m}) = (-1)^{nm} \psi (x_{1} \ldots x_{n + m}) \).

If \( p \in G \) and \( \psi \in G \) then \( \psi \psi \) and \( G \) is closed under \( \wedge \).

By a previous assertion we know that \( \psi \psi (x_{1} \ldots x_{n + m}) \) and \( \psi \psi \) is clearly multilinear since \( \varphi \) and \( \psi \) are both multilinear and \( \psi \psi (x_{1} \ldots x_{n + m}) \)
\[ = \sum_{t} (\varphi, \psi) \varphi (t(v)) \psi (t(v')) \].
Thus since we already know that \( G \) is an \( A \)-submodule; \( G = \sum_{n \geq 0} G_{n} \) (direct); and \( G = A \varepsilon \). Then \( G \) is a regularly graded subalgebra.

**Corollary:** Since \( G \in \text{Alt}^{+}(M, A) \) then \( \text{Alt}^{+}(M, A) \) is an \( A \)-algebra, and its law of composition is \( \psi \psi (x_{1} \ldots x_{n + m}) = \sum_{t} (\varphi, \psi) \varphi (t(v)) \psi (t(v')) \) where \( \varphi, \psi \) and \( \psi (v, v') \) are as before and \( t(v) = x_{1} \ldots x_{n} \) with \( i \leq i' \) \( t(v') \)
\[ = x_{1} \ldots j_{1} \ldots x_{j_{m} + k} \] with \( j_{1} \leq \ldots \leq j_{m + k} \).

**Remark:** In the case that \( K \) has characteristic zero \( \psi \psi (x_{1} \ldots x_{n + m}) \)
\[ = \sum_{t} (\varphi, \psi) \varphi (t(v)) \psi (t(v')) \] where \( \varphi \) is a permutation of \( \{ 1, 2, \ldots , n \} \) and \( \psi \) is the alternating character of \( \varphi \). Now \( \psi \psi (x_{1} \ldots x_{n + m}) \)
\[ = \sum_{t} (\varphi, \psi) \varphi (x_{i_{1}} \ldots x_{i_{n + k}}) \psi (x_{i_{1}} \ldots x_{i_{n + k}}) \] say. Suppose \( \tau_{1} \) is the following permutation
\[ \tau_{1} = \left( \begin{array}{cccc} 1 & 2 & \ldots & k \ k + 1 & \ldots & n \end{array} \right) \] and that
\[ \tau_{1} = \left( \begin{array}{cccc} 1 & 2 & \ldots & k \ k + 1 & \ldots & n \end{array} \right) \] and then \( \tau_{1} = \tau_{1} \circ \tau_{1} \).
and \( \tau_{1} = \left( \begin{array}{cccc} 1 & 2 & \ldots & k \ k + 1 & \ldots & n \end{array} \right) \) then \( \tau_{1} = \tau_{1} \circ \tau_{1} \).

Thus
\[ \psi (x_{1} \ldots x_{n + m}) = \psi (x_{1} \ldots x_{n + m}) \psi (x_{1} \ldots x_{n + m}) \]
1. 

\[ \varepsilon(\pi_r) \phi(\epsilon(T') x_{\pi(1)} \ldots x_{\pi(k)}) \psi(\epsilon(u') x_{\pi(k+1)} \ldots x_{\pi(n)}) = \varepsilon(\pi_r) \phi(x_{i_1} \ldots x_{i_k}) \psi(x_{j_1} \ldots x_{j_n}) \]

\[ = \eta(\tau, \tau') \phi(x_{i_1} \ldots x_{i_k}) \psi(x_{j_1} \ldots x_{j_n}) \] 

Hence for fixed \( k \) and fixed \( \pi = (i_1, \ldots, i_k, j_1, \ldots, j_n) \),

\[ \sum_{\pi} \frac{\varepsilon(\pi)}{k! (n-k)!} \phi(x_{\pi(1)} \ldots x_{\pi(k)}) \psi(x_{\pi(k+1)} \ldots x_{\pi(n)}) = \eta(\tau, \tau') \phi(x_{i_1} \ldots x_{i_k}) \psi(x_{j_1} \ldots x_{j_n}) \]

where \( \pi \) is a permutation such that for \( i = 1, \ldots, k \), \( \pi(i) \) is one of the \( i_1, \ldots, i_k \), and since there are \( k ! \) possible permutations \( \pi' \) and \( (n-k)! \) possible permutations \( \pi'' \). Hence

\[ \sum_{\pi} \frac{\varepsilon(\pi)}{k! (n-k)!} \phi(x_{\pi(1)} \ldots x_{\pi(k)}) \psi(x_{\pi(k+1)} \ldots x_{\pi(n)}) = \sum_{\pi} \frac{\varepsilon(\pi)}{k! (n-k)!} \phi(x_{i_1} \ldots x_{i_k}) \psi(x_{j_1} \ldots x_{j_n}). \]

2. The Derivation on \( M \) Determined by an Alternating \( k \)-bilinear \( \Lambda \):

\[ M \otimes M \rightarrow M. \]

Here \( \Lambda \) is an internal law of composition in \( M \).

In order to define the derivation on \( M \) determined by \( \Lambda \) we first consider the following mappings \( \Lambda^* : E \rightarrow E \) by \( \Lambda^*(t) = 0 \) if \( t \in E \), and \( \Lambda^*(xy) = \Lambda(x,y) \) for \( x, y \in M \). \( \Lambda^* = \mu \circ (\Lambda \otimes I) \cdot U \) where \( \mu : E \otimes E \rightarrow E \) is the linearization of multiplication in \( E \). More explicitly \( \Lambda^*(x_1 \ldots x_n) = \mu (\Lambda (x_1 \ldots x_n)) \) where \( \mu = \mu \circ (\Lambda \otimes I) \cdot U \).

\[ \Lambda^* \phi(t) = \overline{\Lambda^* \phi}(t) \]

Hence \( \Lambda^* \phi(t) = \overline{\Lambda^* \phi}(t) \).

since \( \Lambda^*(t) = 0 \) if \( t \) is not of length two and \( \eta = (-1)^{(n-r)} \mu (\Lambda (x_1 \ldots x_n)) \)

since below \( i \) there are \( i-1 \) elements and below \( j \) there are \( j-2 \) elements since \( i \) is removed.

**Lemma 2:** \( \Lambda^* = (\Lambda \otimes I) \cdot U \) where \( I : E \rightarrow E \) is the identity mapping and \( f : E \rightarrow E \) is an extension of the automorphism \( x \rightarrow -x \) of \( M \) to \( E \).

**Proof:** \( \Lambda^* = \mu \circ (\Lambda \otimes I) \cdot U \) by the definition of \( \Lambda^* \). The multiplication in the algebra \( E \otimes E \) is a bilinear mapping of \( (E \otimes E) \cdot (E \otimes E) \) into \( E \otimes E \); the linearization of this bilinear mapping is a linear mapping of \( (E \otimes E) \otimes (E \otimes E) \) into \( E \otimes E \) which we shall denote by \( \mu^* \). If \( t \) and \( u \) are in \( E \) we have

\[ \mu(t \cdot u) = \mu(t \otimes u) = \mu(t \otimes u) = \mu(t \otimes u) \]

whence \( \mu^* = \mu \cdot (U \otimes U) \) on \( E \otimes E \). Then

\[ \Lambda^* = \mu \cdot (U \otimes U) \cdot \Lambda^* \otimes I \cdot U = \mu \cdot \left( (U \otimes U) \cdot \Lambda^* \right) \otimes (I \otimes I) \cdot U. \]

If \( t \in E \) then \( \Lambda^*(t) = 0 \) unless \( t \in E \) and \( \Lambda^*(xy) = (xy)^c M \) for \( x, y \in M \) and hence
Denote by $\Theta_1$ and $\Theta_2$ respectively the linear mappings $v \to v \Theta_1$ and $v \to v \Theta_2$ of $E$ into $E \otimes E$, then we have

$$U \otimes \Lambda^*(t) = \begin{cases} 0 & \text{if } t \in E \\ (\Lambda^*(xy)) \otimes \Theta_1 + \Theta_2(\Lambda^*(xy)) & \text{if } t \in E_2 \end{cases}$$

Denote by $\Theta_1$ and $\Theta_2$ respectively the linear mappings $v \to v \Theta_1$ and $v \to v \Theta_2$ of $E$ into $E \otimes E$, then we have

$$U \otimes \Lambda^*(t) = (\Theta_1 \otimes \Lambda^*) (\Theta_2 \otimes \Lambda^*)$$

whence

$$U \otimes \Lambda^*(t) = \mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \mu_2 \left[ \left( \Theta_2 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U.$$

Let $t$, $u$, and $v$ be in $E$, then

$$\mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \mu_2 \left[ \left( \Theta_2 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U.$$

Denote by $\rho$ the canonical isomorphism of $E \otimes (E \otimes E)$ with $(E \otimes E) \otimes E$ which maps to$(u \otimes v)$ upon $(u \otimes v) \otimes 1$ for $t$, $u$, and $v \in E$. Then we have

$$\mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \mu_2 \left[ \left( \Theta_2 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \rho \cdot (I \otimes U) \cdot U.$$

Now, we can check the following identity $(\ast)$ $\rho \cdot (I \otimes U) \cdot U = (U \otimes I) \cdot U$ by showing that the two mappings have the same effect on $x \in M$ and hence are equal on $E$ since $M$ generates $E$ and all occurring mappings are clearly algebra homomorphisms.

$$\rho \cdot (I \otimes U) \cdot U(x) = \rho \cdot (I \otimes U)(x \otimes 1 + 1 \otimes x) = \rho \left[ x \Theta_1(1 \otimes x) + 1 \otimes (x \otimes 1) \right] = (x \otimes 1) \Theta_1 + (1 \otimes x) \Theta_2 + (1 \otimes 1) \otimes x$$

$(U \otimes I) \cdot U(x) = U \otimes I(x \otimes 1 + 1 \otimes x) = (x \otimes 1) \Theta_1 + (1 \otimes x) \Theta_2 + (1 \otimes 1) \otimes x$

Hence

$$\mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \left[ \mu_2 \left( \Lambda^*(xy) \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \left[ \mu_2 \left( \Lambda^*(xy) \otimes 1 \right) \right] \cdot (I \otimes U) \cdot U$$

For $t$, $u$, and $v \in E$

$$\mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \left[ \mu_2 \left( \Lambda^*(xy) \otimes 1 \right) \right] \cdot (I \otimes U) \cdot U$$

Since $\Lambda^*(t) \in E$, we have $(1 \otimes \Lambda^*(t))(u \otimes v) = u \otimes \Lambda^*(t) \cdot v$. Therefore, we can check the following identity $(\ast)$ $\rho \cdot (I \otimes U) \cdot U = (U \otimes I) \cdot U$ by showing that the two mappings have the same effect on $x \in M$ and hence are equal on $E$ since $M$ generates $E$ and all occurring mappings are clearly algebra homomorphisms.

$$\mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \mu_2 \left[ \left( \Theta_1 \otimes \Lambda^* \right) \otimes \Theta_1 \right] \cdot (I \otimes U) \cdot U = \left[ \mu_2 \left( \Lambda^*(xy) \otimes 1 \right) \right] \cdot (I \otimes U) \cdot U.$$
Denote by $\nu$ the automorphism of the module $E \otimes E$ which maps $tu$ upon $u$ if $t, u, v \in E$

\[
\{p^* (v \otimes l) \} \cdot \{t \otimes (u \otimes v) \} = \{p^* (v \otimes l) \} \cdot \{t \otimes (u \otimes v) \} = u \otimes (t \cdot v) = t \otimes (u \cdot v). \]

Hence $T = p^* (v \otimes l) \cdot p^*$.

Thus

\[
\begin{align*}
\mu_\tau \{ (e \cdot \Lambda^* \otimes (l \otimes l)) \cdot (l \otimes l) \cdot u \\
= & \bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* (v \otimes l) \cdot (l \otimes l) \cdot u \\
= & \bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* (v \otimes l) \cdot (l \otimes l) \cdot u \\
= & \bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* (v \otimes l) \cdot (l \otimes l) \cdot u.
\end{align*}
\]

Now let $K$ be the automorphism of the module $E \otimes E$ which maps any $w \in E$ upon $(-1)^{n \cdot k}$ $w$ then $\nu \cdot U = K \cdot U$ for $\nu \cdot U \cdot (x_1, \ldots, x_n) = \nu \cdot \xi \cdot (T \cdot u \cdot t \cdot (T^\ast))$

\[
= \xi (\tau \cdot l) \cdot (T \cdot u \cdot t \cdot (T^\ast)) \cdot \eta \cdot (T \cdot u \cdot t \cdot (T^\ast)) \cdot \eta \cdot (T \cdot u \cdot t \cdot (T^\ast)) \text{ if } T \text{ is a sequence of length } h; \text{ also } K \cdot U \cdot (x_1, \ldots, x_n) = K \cdot \xi \cdot (T \cdot u \cdot t \cdot (T^\ast))
\]

\[
= \xi (\tau \cdot l) \cdot (T \cdot u \cdot t \cdot (T^\ast)) \cdot \eta \cdot (T \cdot u \cdot t \cdot (T^\ast)) \cdot \eta \cdot (T \cdot u \cdot t \cdot (T^\ast)) \cdot \eta \cdot (T \cdot u \cdot t \cdot (T^\ast)).
\]

The mapping $\mu \cdot (e, \Lambda^*)$ maps $E \otimes E$ upon $\{0\}$ if $q \neq 2$ thus $\bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^*$ maps $(E \otimes E) \cdot E$ upon $\{0\}$ if $q \neq 2$. On the other hand if $t \in E$ and $u \in E$, then $K \cdot (t \cdot u) = (-1)^{2 \cdot p} \cdot (t \cdot u) = J(t) \cdot u$ and it follows that $K \cdot U$ coincides with $(I \otimes I) \cdot U$ on $(E \otimes E) \cdot E$.

Thus we have

\[
\begin{align*}
\bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* \{ (\nu \cdot U) \cdot U \} & = \bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* \{ (K \cdot \nu) \cdot U \} \\
= & \bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* (K \cdot (e, \Lambda^*) \cdot U) = \bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* \{ (l \otimes l) \cdot l \} \cdot (l \otimes l) \cdot U.
\end{align*}
\]

It is clear that $p^* \{ (l \otimes l) \cdot l \} = J \cdot (l \otimes l) \cdot l$ and since $p^* (l \otimes l) \cdot U = (I \otimes U) \cdot U$

then

\[
\begin{align*}
\bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* \{ (l \otimes l) \cdot l \} \cdot (l \otimes l) \cdot U & = \bigcup \{ \mu \cdot (e, \Lambda^*) \} \cdot p^* \{ (l \otimes l) \cdot l \} \cdot (l \otimes l) \cdot U \\
= & J \cdot (l \otimes l) \cdot l \cdot U = J \cdot (l \otimes l) \cdot U.
\end{align*}
\]

Hence $U \cdot \xi = (\xi \otimes l + l \otimes \xi) \cdot U$.

Definition 13: Let $\Delta$ be the following mapping $\Delta : E \to E \otimes E$. In other words
The mapping $d$ has the following properties.

(i) $d$ maps $\text{Hom}_A(E,A)$ into itself.

For $\hat{A}$ is clearly a $K$-endomorphism of $E$ since $\Lambda$ is $K$-bilinear.

(ii) If $\varphi \in H$, then $d\varphi \in H$, hence $d$ maps $H$ into itself.

For if $\varphi \in H$, then $\varphi \in E^*$ if $n' = n$ and hence

$$d\varphi(x_1, \ldots, x_n) = \sum_{i=1}^n (-1)^{i+j} \varphi(\Lambda(x_i, x_j)x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n)$$

unless $m = n + 1$ since $\Lambda(x_i, x_j)x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n$ is $m-1$ elements.

(iii) $d$ is $A$-linear.

For if $f \in A$ then
d

$$d(\varphi f) = d(\Lambda f)$$

and $d_{\Lambda}$ is a homogeneous derivation of degree 1 on $H$.

Proof: For any $\varphi, \psi \in \text{Hom}_A(E,A)$ then

$$d_{\Lambda}(\varphi \psi) = d_{\Lambda}(\varphi \psi) = (\psi \varphi) \circ d_{\Lambda}(\varphi \psi) = (\psi \varphi)(\Lambda x, y) \circ d_{\Lambda}(\varphi \psi).$$

(by lemma 2)

$$= (\psi \varphi)(\Lambda x, y) \circ (\psi \varphi)(\Lambda x, y) = (\psi \varphi)(\Lambda x, y) \circ (\psi \varphi)(\Lambda x, y).$$

For $\varphi \in H$, we have $\varphi_{F} = (-1)^{m} \varphi$, and hence here $d_{\Lambda}(\varphi \psi) = d_{\Lambda}(\varphi \psi) = (-1)^{m} \varphi_{F}$

and $d_{\Lambda}$ is a homogeneous derivation of degree 1 on $H$.

3. The Derivation on $H$ Determined by a $K$-bilinear Mapping $\Delta: M \times A \rightarrow A$

for which $\Delta(x, f g) = \Delta(x, f) g + f \Delta(x, g)$.

Here $\Delta$ is an external law of composition of $M$ with $A$, i.e. makes "$M$ act on $A"$, and the property $\Delta(x, f g) = \Delta(x, f) g + f \Delta(x, g)$ tell us

that the elements of $M$ act as derivations on $A$.

In order to define the derivation determined by $\Delta$ we consider the following two mappings: $\Lambda^*: M \rightarrow A$ i.e. $\Lambda^*$ is the linearization
of $\Delta$, and $P: E \to E_{\cdot} = \mathbb{M}$ by $P/E = \{\circ\}$ if $n \neq 1$ and $P$ is the identity mapping on $E$, i.e., $P$ is the projection of $E$ to $E_{\cdot}$.

**Definition 14:** Let $d_\Delta$ be the following mapping $d_\Delta: \varphi \to \Delta^* (P \varphi \cdot U)$. In other words $d_\Delta \varphi(x, \ldots, x_n) = \Delta^* (P \varphi) \sum_i \varphi(v_i, x_1, \ldots, x_n)$

$$= \Delta^* (\sum_i \varphi(v_i, x_1, \ldots, x_n)P(t(v_i)) \cdot \varphi(t(v_i))) = \sum_i (-1)^{i+1} \Delta(x_i, \varphi(x_1, \ldots, x_n))$$

since if $\varphi$ has length other than 1 then $P$ maps it to zero.

Clearly $d_\Delta \varphi \in \text{Hom}_K(E, A)$ since $\varphi \in \text{Hom}_K(E, A)$ and $\Delta$ is $K$-bilinear; and $d_\Delta \varphi$ has the following two properties.

1. **The mapping $\varphi \to d_\Delta \varphi$ is linear with respect to $K$ though not with respect to $A$.

For $(d_\Delta (f \varphi))(x_1, \ldots, x_n) = \sum_i (-1)^i \Delta(x_i; f \varphi(x_1, \ldots, x_n))$

$$= \sum_i (-1)^i \Delta(x_i; \varphi(x_1, \ldots, x_n)) = (d_\Delta f)(\varphi(x_1, \ldots, x_n))$$

if $f \in K$ since $\Delta$ is $K$-bilinear. Thus $d_\Delta (f \varphi) = f (d_\Delta \varphi)$.

2. **If $\varphi \in H_{m+1}$ then $d_\Delta \varphi \in H_m$.**

For if $\varphi \in H_{m+1}$ then $\varphi / E = \{\circ\}$ if $n \neq m$ and hence $d_\Delta \varphi(x_1, \ldots, x_n)$

$$= \sum_i (-1)^i \Delta(x_i; \varphi(x_1, \ldots, x_n)) = 0 \text{ if } n \neq m + 1.$$

**Theorem 3:** $d_\Delta$ is a homogeneous derivation of degree 1 on $H$.

**Proof:** $d_\Delta (f \varphi \gamma) = \Delta^* \left[ f \circ (\varphi \gamma) \cdot U \right] = \Delta^* \left[ P \otimes_k \left( \circ \left( \varphi \gamma \right) \cdot U \right) \right] = \Delta^* \left[ P \otimes_k \left( \circ \left( \varphi \gamma \right) \cdot (I \otimes U) \right) \right] \cdot U.$

Here $\Delta^* \left[ P \otimes_k \left( \circ \left( \varphi \gamma \right) \right) \right]$ acts on $E \otimes (E \otimes E)$ for

$$\Delta^* \left[ P \otimes_k \left( \circ \left( \varphi \gamma \right) \right) \right] \left[ x \otimes (y \otimes z) \right] = \Delta^* \left[ P \otimes_k \left( \circ \left( \varphi \gamma \right) \cdot (I \otimes U) \right) \right] \left[ x \otimes (y \otimes z) \right].$$

Thus if $f$ maps $E \otimes (E \otimes E)$ to $(E \otimes E) \otimes E$ by $x \otimes (y \otimes z) \to (x \otimes y) \otimes z$ and $\gamma$ maps $E \otimes (E \otimes E)$

to $E \otimes (E \otimes E)$ by $x \otimes (y \otimes z) \to y \otimes (x \otimes z)$ one finds

$$\Delta^* \left[ P \otimes_k \left( \circ \left( \varphi \gamma \right) \right) \right] = \varphi \left[ \Delta^* \left( P \otimes_k \left( \circ \left( \varphi \gamma \right) \right) \right) \right] \cdot \gamma.$$

Hence
\[ d_\Delta (f \cdot g) = \Delta \left[ (\Delta \cdot (P \cdot f) \cdot g) \right] \cdot \rho (\Delta (U) \cdot U) + \Delta \left[ (\Delta \cdot (P \cdot g) \cdot f) \right] \cdot \rho (\Delta (U) \cdot U) \]

Now from (**) in lemma 2 we know \( \rho (\Delta (U) \cdot U) = (\Delta (U) \cdot U) \), and we can check the following identity

\[ (\Delta (U) \cdot U)(x) = (\Delta (U)(x_1 + l \cdot x)) = (x_1(1 + l) + l(x_1 + l)) = x_1(x_1(l_1) + l_1(x_1) + l_1), \]

and

\[ (\Delta (U) \cdot U)(x) = (\Delta (U)(x_1 + l \cdot x)) = x_1(1 + l) + l(x_1) + l(x_1). \]

Thus \( \rho (\Delta (U)) \cdot U = (\Delta (U)) \cdot U \) on \( M \) since \( M \) is a module and hence addition is commutative. But \( \rho (\Delta (U)) \cdot U \) is not an algebra homomorphism for if

\[ x_1, x_2, y_1, z_1, x_1, y_2, z_2 \] are homogeneous elements in \( E \) then

\[ \rho \left[ x_1 \cdot (y_1 \cdot z_1) \right] = \rho \left[ x_1 \cdot (y_1 \cdot z_1) \right] \]

\[ = \rho \left[ x_1 \cdot (y_1 \cdot z_1) \right]. \]

Thus \( \rho (\Delta (U)) \cdot U \) on \( M \) since \( M \) is a module and hence addition is commutative. But \( \rho (\Delta (U)) \cdot U \) is not an algebra homomorphism for if

\[ x_1, x_2, y_1, z_1, x_1, y_2, z_2 \] are homogeneous elements in \( E \) then

\[ \rho \left[ x_1 \cdot (y_1 \cdot z_1) \right] = \rho \left[ x_1 \cdot (y_1 \cdot z_1) \right]. \]

Hence

\[ \rho \left[ x_1 \cdot (y_1 \cdot z_1) \right] = \rho \left[ x_1 \cdot (y_1 \cdot z_1) \right]. \]

However, if \( \rho : x_1(y_1 \cdot z_1) \rightarrow (-1)^{\deg y \cdot \deg z} x_1(y_1 \cdot z_1) \) for homogeneous \( x_1, y_1, z_1 \) then

\[ \rho \left[ x_1(y_1 \cdot z_1) \right] = (-1)^{\deg y \cdot \deg z} x_1(y_1 \cdot z_1) \cdot x_1(y_1 \cdot z_1) \]

Therefore

\[ \rho \left[ x_1(y_1 \cdot z_1) \right] = (-1)^{\deg y \cdot \deg z} x_1(y_1 \cdot z_1), \]

and

\[ \rho \left[ x_1(y_1 \cdot z_1) \right] = (-1)^{\deg y \cdot \deg z} x_1(y_1 \cdot z_1). \]
Also for homogeneous $x, y$ and $z \in \mathbb{E} R(x \otimes y \otimes z) = \begin{cases} 0 & \text{if } \deg y \neq 1 \\ (-1)^{\deg y} x \otimes y \otimes z & \text{if } \deg y = 1 \end{cases}

and hence $I_{\otimes}(\mathbb{E}I) \cdot R = I_{\otimes}(\mathbb{E}I)$. Thus $\{I_{\otimes}(\mathbb{E}I)\} \cdot R = R - \omega(I_{\otimes}U) \cdot U$.

For $\phi \in H$, $\phi \cdot J = (-1)^{\deg \phi}$ and hence $d_A(\phi \cdot \psi) = (d_A \phi) \star (\psi \cdot (-1)^{\deg \phi} \psi)$. Thus $d_A$ is a homogeneous derivation of degree 1 on $H$.

4. Conditions linking $\Lambda$ and $A$.

For $x \in A$ consider $d\Lambda(x \ldots (fx_1) \ldots x_n)$

$$= \sum_{k \leq j} (-1)^{i+j+1} \phi(\Lambda(fx_1 \ldots x_k \ldots x_j) \ldots \hat{x}_j \ldots x_n) \cdot \sum_{i \leq k} (-1)^{i+k+1} \phi(\Lambda(x_k \ldots x_j) \ldots \hat{x}_k \ldots x_n)

+ \sum_{i \leq j} (-1)^{i+j+1} \phi(\Lambda(x_i \ldots x_j) \ldots \hat{x}_i \ldots x_n)$$

where $fx_1$ is contained in $x_1 \ldots \hat{x}_1 \ldots x_n$ in the last sum and hence the last sum is $f \sum_{i \leq j} (-1)^{i+j+1} \phi(\Lambda(x_i \ldots x_j) \ldots \hat{x}_i \ldots x_n)$ if $\phi$ is $A$-multilinear. Hence for $\phi$ $A$-multilinear

$$f d\Lambda(x \ldots x_n) \cdot (f \Lambda(x \ldots x_n) = \sum_{k \leq j} (-1)^{i+j+1} \phi(\Lambda(fx_1 \ldots x_k \ldots x_j) \ldots \hat{x}_j \ldots x_n)

+ \sum_{i \leq j} (-1)^{i+j+1} \phi(\Lambda(fx_i \ldots x_k) \ldots \hat{x}_i \ldots x_n)$.

Now assume that $\Lambda(fx_1, y) = \Lambda(x_1, f(y))$ where $\Delta$ is a $K$-bilinear mapping from $M \times A$ to $A$. 

Therefore

$$= \sum_{k \leq j} (-1)^{i+j+1} \phi(\Lambda(fx_1 \ldots x_k \ldots x_j) \ldots \hat{x}_j \ldots x_n)$$

+ \sum_{i \leq j} (-1)^{i+j+1} \phi(\Lambda(fx_i \ldots x_k) \ldots \hat{x}_i \ldots x_n)$.
i.e. \( f \land (x,y) = \land (fx,y) = \Delta(y,f)x \). Then \( f \land (y,z) = \land (y,fz) = -(f \land (x,y) = \land (fx,y) \)

since \( \land \) is alternating, and thus \( f \land (y,x) = -(f \land (x,fy) \land (fx,y)) \).

Hence

\[
\begin{align*}
\sum (-1)^{k+j} \varphi((\Delta(x,y)\Delta(x,y)x)K_j \ldots K_n x \ldots x = \\
\sum (-1)^{j} \Delta(x,f)xK_j \ldots K_n x \ldots x = \sum (-1)^{j} \Delta(x,f)\varphi(x_1 \ldots \hat{x}_j \ldots x_n)
\end{align*}
\]

since \( \varphi \) is \( A \)-multilinear. Now one has for \( \Delta \), \( \Delta(y,fz) = fg \land (x,y) - \land (fgx,y) \)

\[
= f g \land (x,y) - \land (gx,y) + \land (gx,y) - \land (fgx,y) = \Delta(y,f)g x + \Delta(y,g)\).
\]

Thus it is natural to assume further that \( \Delta(y,f)g = \Delta(y,f)g + \Delta(y,g) \) and

\[
\Delta(fy,g) = \Delta(y,g). \text{ Thus we can form the derivation } d_{\Delta}. \text{ Then }
\]

\[
\begin{align*}
(d_{\Delta}) \varphi(x_1 \ldots x_n) - d \varphi(x_1 \ldots (fx),x_n) = \\
- (d_{\Delta}) \varphi(x_1 \ldots x_n) - (-1) \Delta(x_2,x) \varphi(x_1 \ldots \hat{x}_2 \ldots x_n)
\end{align*}
\]

for

\[
(d_{\Delta}) \varphi(x_1 \ldots x_n) = \sum \eta(v,v^*) d \varphi(t(v)) \varphi(t(v^*)) = \sum (-1)^i \Delta_{\Delta}(x,\varphi(x_1 \ldots \hat{x}_i \ldots x_n))
\]

since \( f = \hat{A} = H \), and so \( d \in H \). Therefore \( (d_{\Delta}) \varphi(x_1 \ldots x_n) = \sum (-1)^i \Delta_{\Delta}(x_i, \varphi(x_1 \ldots \hat{x}_i \ldots x_n)) \) and \( - (d_{\Delta}) \varphi = \sum (-1)^i \Delta_{\Delta}(x_i, \varphi(x_1 \ldots \hat{x}_i \ldots x_n)) \).

If \( \varphi \) is \( A \)-multilinear then

\[
(d_{\Delta}) \varphi(x_1 \ldots (fx),x_n) = (-1)^{k+1} \Delta_{\Delta}(x_2, \varphi(x_1 \ldots \hat{x}_2 \ldots x_n))
\]

\[
+ \sum (-1)^i \Delta_{\Delta}(x_i, \varphi(x_1 \ldots \hat{x}_i \ldots x_n)) = (-1)^{k+1} \Delta_{\Delta}(x_2, \varphi(x_1 \ldots \hat{x}_2 \ldots x_n))
\]

\[
+ \sum (-1)^i \Delta_{\Delta}(x_i, \varphi(x_1 \ldots \hat{x}_i \ldots x_n)) = (-1)^{k+1} \Delta_{\Delta}(x_2, \varphi(x_1 \ldots \hat{x}_2 \ldots x_n))
\]

Thus from \((+)\) and \((++)\) we get \( f d_{\Delta} \varphi(x_1 \ldots x_n) - d \varphi(x_1 \ldots (fx),x_n) \)

\[
+ d \varphi(x_1 \ldots (fx),x_n) = fd_{\Delta} \varphi(x_1 \ldots x_n).
\]
Therefore \( (d_\omega - d_\Delta) \phi(x, \ldots, (fx, \ldots, x_n) = (d_\omega - d_\Delta) \phi(x, \ldots, x_n) \) i.e. \( d_\omega - d_\Delta \)

is multilinear in \( A \). Hence \( (d_\omega - d_\Lambda) \in G \) for \( \phi \in G \). Also \( d_\omega - d_\Delta \) is a homogeneous derivation of degree 1 on \( G \). \( d_\omega \phi(x, \ldots, x_n) = (d_\omega - d_\Lambda) \phi(x, \ldots, x_n) \)

\[ = \sum_{i=1}^n \Delta(x_i) \phi(x, \ldots, x_i, \ldots, x_n) - \sum_{i=1}^n \phi(x, x_i, \ldots, x_i, \ldots, x_n). \]

Now if \( x \in H = A \) then \( d_\Delta f(x) = A(x, f) \) for \( x \in M \) and \( d_\Lambda f = 0 \) for \( d_\Lambda f(x) = f \cdot \Lambda(x) = f \cdot \mu_\Lambda(x) (\Lambda(x)) = 0 \) for all \( x \in M \).

Call the mapping \( f \rightarrow A(x, f) \) of \( \Lambda \) into itself \( D_x \). Then

\[ df(x) = (d_\omega - d_\Lambda) f(x) = d_\Lambda f(x) - d_\Lambda f(x) = A(x, f) = D_x f \]

for \( x \in M \). Thus we have the following two relations:

(i) \[ d_\Lambda df(xy) = d_\Lambda A f(xy) = d_\Lambda \phi(x, y) = D_x D_y f \]

for \( x, y \in M \) and

(ii) \[ d_\Delta df(xy) = d_\Delta A f(xy) = (x, d_\Delta f(y)) - d_\Delta \phi(x) = A(x, y) \]

\[ = A(x, A(y, f)) - A(y, d_\Lambda f(x)) = D_x D_y f - D_x f = [D_x, D_y] f \]

for \( x, y \in M \), where \( [D_x, D_y] \) is the Lie product of the derivations \( D_x, D_y \) on \( A \). Hence \( d_\Delta df(xy) = (d_\omega - d_\Lambda) df(xy) = ([D_x, D_y] - D_\Lambda (x, y)) f \) and thus \( d_\Delta df = 0 \) for all \( f \in A \) if and only if \( D_\Lambda (x, y) = [D_x, D_y] \) for all \( x, y \in M \) i.e. the \( \Lambda \)-composite of \( x, y \in M \) is mapped into the Lie composite of the derivations \( D_x \) and \( D_y \) on \( A \).

Now suppose that \( \omega \in H \), then \( d_\Lambda (x, y) = \omega (\Lambda(x, y)) \) and

\[ d_\Delta (x, y) = D_x \omega (y) - D_y \omega (x) \]

hence

\[ d_\Lambda (x, y) = (d_\omega - d_\Lambda) \omega (x, y) = D_x \omega (y) - D_y \omega (x) - \omega (\Lambda(x, y)) \]. Therefore we have the following two relations:

(i) \[ d_\Lambda d_\omega (xyz) = (-1)^{2^z+1} d_\omega (\Lambda(x, y) z) + (-1)^{2^z+1} d_\omega (\Lambda(x, z) y) \]

\[ + (-1)^{2^z+1} d_\omega (\Lambda(y, z) x) \]

\[ = d_\Lambda (x, y) + D_x \omega (\Lambda(x, y)) - \omega (\Lambda(x, y) z) \]

\[ - D_x (x, z) + D_y \omega (\Lambda(x, z)) - \omega (\Lambda(x, z) y) \]
\[ +D_{\Lambda(y,z)}\omega(x) = D_x\omega(\Lambda(y,z)) - \omega(\Lambda(y,z), x) \]

for \( x, y, z \in M \) and

\[(ii) \quad \frac{\partial}{\partial y} \omega(xyz) = \Delta(x, \frac{\partial}{\partial y}(yz)) - \Delta(y, \frac{\partial}{\partial y}(zx)) + \Delta(z, \frac{\partial}{\partial y}(xy)) \]
\[= \Delta(x, \frac{\partial}{\partial y}(z)) - \Delta_{x, y}(y) = \omega(\Lambda(y,z), x) \]
\[+ \Delta(z, \frac{\partial}{\partial y}(x)) - \Delta_{x, y}(y) = \omega(\Lambda(z,y), x) \]
\[+ \Delta_{x, y}(z) - \partial z_{x, y}(y) = \partial_{x, y}(z) = \omega(\Lambda(x,z), y) \]
\[= D_{x, y}(z) - D_x D_y (\Lambda(y,z)) - D_x \partial_y (\Lambda(y,z)) + D_{x, y} \partial_z (\Lambda(x,y)) \]

for \( x, y, z \in M \). Thus

\[ d\omega(xyz) = (\frac{\partial}{\partial y} - \frac{\partial}{\partial y}) d\omega(xyz) = \omega(\Lambda(x, y, z), x) + \omega(\Lambda(y, z, x), y) + \omega(\Lambda(z, x, y), z) \]
\[+ \left( [D_x D_y] - D_{\Lambda(x,y)} \right) \omega(z) - \left( [D_x D_z] + D_{\Lambda(x,z)} \right) \omega(y) + \left( [D_z D_y] - D_{\Lambda(y,z)} \right) \omega(x) \]

Hence if \( dd\phi = 0 \) then \( dd\omega = 0 \) for all \( \omega \in \Lambda \), if the composition \( \Lambda \) satisfies the Jacobi-identity i.e. \( (\Lambda, \Lambda) \) is a Lie Algebra over \( K \).

**Theorem 6:** If \( dd\phi = 0 \) for all \( \phi \in H_0 = A \) and \( dd\omega = 0 \) for all \( \omega \in \Lambda \), then \( dd\phi = 0 \) for all \( \phi \in \Lambda \).

**Proof:** If \( \phi \in H_{n-2} \) then \( d\phi \in H_{n-1} \) and \( d\phi \in H_{n-1} \) and thus

\[ d_{\Lambda} \phi(x_1, \ldots, x_{n-1}) = \sum_{i \in j} (-1)^{i+j+r} \phi(\Lambda(x_i, x_j), x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n-1}) \]

and

\[ d_{\Lambda} \phi(x_1, \ldots, x_{n-1}) = \sum_{i \in j} (-1)^{i+j} \Delta(x_i, x_j, \phi(x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{n}) \]

Next,

\[ d\phi = (d_{\Lambda} - d_{\Lambda}) \phi \text{ and } d(d\phi) = d_{\Lambda} d\phi = (d_{\Lambda} d\phi + d_{\Lambda} d\phi) + d_{\Lambda} d\phi \]

First,

\[ d_{\Lambda} d_{\Lambda} \phi(x_1, \ldots, x_n) = \sum_{i} (-1)^{i+1} \Delta(x_i, x_i \phi(x_1, \ldots, \hat{x}_i, \ldots, x_{n}) \]
\[= \sum_{i} (-1)^{i+1} \Delta(x_i, x_i \phi(x_1, \ldots, \hat{x}_i, \ldots, x_{n})) \]
\[+ \sum_{i} (-1)^{i+1} \Delta(x_i, x_i \phi(x_1, \ldots, \hat{x}_i, \ldots, x_{n})) \]
\[= \sum_{i} (-1)^{i+1} \Delta(x_i, x_i \phi(x_1, \ldots, \hat{x}_i, \ldots, x_{n})) \]
\[
\sum_{i, i, j} (-1)^{i+j} D_{x_i} D_{x_j} \phi(x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n) = \sum_{i, j} (-1)^{i+j} \left[ D_{x_i} D_{x_j} \right] \phi(x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n).
\]

Next
\[
d_{\lambda} d_{\lambda} \phi(x, \ldots, x_n) = \sum_{k} (-1)^{k+1} \left[ d_{\lambda} \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
= \sum_{k} (-1)^{k+1} \left[ d_{\lambda} \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
and
\[
d_{\lambda} d_{\lambda} \phi(x, \ldots, x_n) = \sum_{i, j} \left[ d_{\lambda} \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
= \sum_{i, j} \left[ d_{\lambda} \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
Therefore,
\[
d_{\lambda} d_{\lambda} \phi(x, \ldots, x_n) + d_{\lambda} d_{\lambda} \phi(x, \ldots, x_n) = \sum_{i, j} (-1)^{i+j} d_{\lambda} \phi \left( x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right).
\]
Finally,
\[
d_{\lambda} d_{\lambda} \phi(x, \ldots, x_n) = \sum_{i, j} (-1)^{i+j} d_{\lambda} \phi \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
= \sum_{i, j} (-1)^{i+j} \left[ d_{\lambda} \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
+ \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right)
\]
\[
= \sum_{i, j} (-1)^{i+j} \left[ \sum_{k} (-1)^{k+1} \left[ d_{\lambda} \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right) \right] + \sum_{i, j} (-1)^{i+j} \left[ \phi \right] \left( n(x, x_j) x, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n \right).
\]
As before let $I$ be a field and $A$ a commutative $K$-algebra with unit. Let $H$ be the exterior algebra on $M$ considered as a $K$-module. Let $H$ and $G$ be as before.

Consider the mapping $\Lambda: D \times D \to D$ by $(x, y) \mapsto [x, y]$ where $X, Y \in D$ and $[x, y] = x(y) - y(x)$, the Lie product of $X$ and $Y$, for $x, y \in A$. Since $D$ is a Lie $K$-algebra then $\Lambda$ is $K$-bilinear and alternating. Hence as before we have

$$\Lambda([x, y], z) = \Lambda(x, [y, z]) - \Lambda(y, [x, z]).$$
can use $\Lambda$ to determine a homogeneous derivation $d_{\Lambda}$ of degree 1 on $H$ where
for $x_1, \ldots, x_n \in D$ and $\phi \in \text{Hom}_k (E_A)$

$$d_{\Lambda} \phi (x_1, \ldots, x_n) = \sum (-1)^{i+j} \phi \left( \nabla (x_i, x_j) x_1 \ldots \hat{x}_i \ldots {\hat{x}_j} \ldots x_n \right)$$

$$= \sum (-1)^{i+j} \phi \left( \left[ \frac{\partial}{\partial x_i, \partial x_j} \right] x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n \right).$$

Now let us consider the mapping $\Delta : D \times A \rightarrow A$ defined as follows

$$(X, \phi) \mapsto X \phi.$$ 

$\Delta$ is clearly $K$-bilinear since $X$ is a $K$-derivation on $A$. Also $\Delta (x, fg) = x (fg) = (Xf)g + f (Xg) = \Delta (x, f)g + \Delta (x, g)$ since $X$ is a derivation.

Hence as before we can use $\Delta$ to determine a homogeneous derivation $d_{\Delta}$ of degree 1 on $H$ where

$$d_{\Delta} \phi (x_1, \ldots, x_n) = \sum (-1)^{i+j} \phi \left( \nabla (x_i, x_j) x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n \right)$$

$$= \sum (-1)^{i+j} x_i \phi (x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n).$$

Also, as before we have $d = -d_{\Delta}$ a homogeneous derivation of degree 1 on $G$ where

$$d \phi (x_1, \ldots, x_n) = \sum (-1)^{i+j} \phi \left( \left[ \frac{\partial}{\partial x_i, \partial x_j} \right] x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n \right)$$

$$+ \sum (-1)^{i+j} x_i \phi (x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n)$$

for $x_1, \ldots, x_n \in D$ and $\phi \in \text{Hom}_k (E_A)$.

**Theorem 5:** If $\tilde{A}$ is the algebra of alternating differential forms on $A$ there exists a homogeneous derivation $d_{\tilde{A}}$ of degree 1 on $\tilde{A}$ such that $d d \phi = 0$ for all $\phi \in \tilde{A}$ where

$$d \phi (x_1, \ldots, x_n) = \sum (-1)^{i+j} \phi \left( \left[ \frac{\partial}{\partial x_i, \partial x_j} \right] x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n \right)$$

$$+ \sum (-1)^{i+j} x_i \phi (x_1 \ldots \hat{x}_i \ldots \hat{x}_j \ldots x_n).$$

**Proof:** Follows from the fact that $G \cong \tilde{A} \cong \text{Alt}^n (D, A)$. Note that $d f$ for $f \in \tilde{A}$ is called the differential of $f$ and $d \phi$ for $\phi \in \tilde{A}$ of degree greater than 1 is called the exterior differential of $\phi$. Also, the derivation $d$ on the algebra $\tilde{A}$ of alternating differential forms is called the exterior derivation of $\tilde{A}$. 
III ALTERNATING DIFFERENTIAL FORMS ON A MANIFOLD

Our aim in this section is to indicate some of the considerations which motivated the choice of topic for this thesis.

Definition 15: A manifold is a pair $(E, \mathcal{A})$ where $E$ is a connected topological space and $\mathcal{A}$ is a set of continuous real functions on $E$ such that the following conditions are satisfied:

M1. For each $a \in E$ there exist $\xi_1, \ldots, \xi_n \in \mathcal{A}$ such that (i) $\Phi_a: x \mapsto (\xi_1(x), \ldots, \xi_n(x))$ for $x \in E$, maps some open neighbourhood $U$ of a homeomorphically onto an open set in $\mathbb{R}^n$ (real $n$-space), and (ii) for any $f \in \mathcal{A}$, the function $f \cdot \Phi_a^{-1}$ on $\Phi_a(U)$ is (unrestrictedly) differentiable.

M2. $\mathcal{A}$ is maximal (on $\mathcal{C}(E)$, the set of all continuous functions on $E$) with respect to condition M1.

Remark 1: Since $E$ is connected, the $n$ in M1 is the same for all $a \in E$.

Remark 2: In the following differentiable will be taken to mean unrestrictedly differentiable.

Lemma 3: M1 implies that given any $\xi_1, \ldots, \xi_n \in \mathcal{A}$ such that $\Phi_a: x \mapsto (\xi_1(x), \ldots, \xi_n(x))$ maps some open $V \subseteq E$ homeomorphically onto an open set in $\mathbb{R}^n$ then, for any $f \in \mathcal{A}$, $f \cdot \Phi_a^{-1}$ is a differentiable function on $\Phi_a(V)$.

Proof: For any $a \in V$ there exist $\xi_1, \ldots, \xi_n \in \mathcal{A}$ such that $\Phi_a$ maps some open neighbourhood $W \subseteq V$ of a homeomorphically onto an open set in $\mathbb{R}^n$ and such that for any $f \in \mathcal{A}$, the function $f \cdot \Phi_a^{-1}$ on $\Phi_a(U)$ is differentiable. Then $\Phi_a \circ \Phi_a^{-1}$ is a differentiable homeomorphism of $\Phi_a(U)$ onto $\Phi_a(U)$ hence its inverse $\Phi_a^{-1} \circ \Phi_a$ is differentiable. Now, $f \circ \Phi_a^{-1} = (f \cdot \Phi_a^{-1}) (\Phi_a^{-1} \circ \Phi_a)$ and hence since $f \circ \Phi_a^{-1}$ is differentiable so is $f \cdot \Phi_a^{-1}$.
Definition 16: \( \zeta_1, \ldots, \zeta_n \) are called coordinates on the open set \( U \) if they satisfy MI.

Lemma 4: MI and M2 imply M2'. If \( f \in \mathcal{C}(E) \) is such that for any \( x \in E \) there exist coordinates \( \zeta_1, \ldots, \zeta_n \) on an open neighbourhood \( U \) of \( x \) such that \( f \circ \Phi^{-1} \) is differentiable on \( \Phi(U) \) then \( f \in \mathcal{A} \).

Proof: Let \( \mathcal{A}' \subseteq \mathcal{A} \) be the set of all \( f \in \mathcal{C}(E) \) with this property. Clearly \( \mathcal{A}' \) satisfies MI, and hence by M2 \( \mathcal{A}' = \mathcal{A} \).

Corollary: If \( F \) is differentiable on \( \mathbb{R}^k \), then \( F \circ \Phi \in \mathcal{A} \) for any \( \zeta_1, \ldots, \zeta_n \).

Also \( \mathcal{A} \) is a ring with respect to the usual operations and contains the constant functions.

Proof: Take \( x \in E \) and let \( U \) be an open neighbourhood of \( x \) with coordinates \( \zeta_1, \ldots, \zeta_n \in \mathcal{A} \). On \( \Phi(U) \) consider \( (F \circ \Phi) \circ \Phi^{-1} \in \mathcal{C}(\Phi(U)) \). Now \( \Phi \circ \Phi^{-1} \) is differentiable since each \( \zeta_i \) is differentiable on \( \Phi(U) \). Thus \( (F \circ \Phi) \circ \Phi^{-1} \) is differentiable on \( \Phi(U) \), and by Lemma 4 \( F \circ \Phi \in \mathcal{A} \). Also \( \mathcal{A} \) is a ring with respect to the usual operations, for if \( F \) is one of the differentiable functions \( (x, y) \rightarrow xy \) on \( \mathbb{R}^2 \) then \( f, g \in \mathcal{A} \) implies that \( f \circ g = F(f, g) \in \mathcal{A} \) by the first part of the corollary.

Lemma 5: MI and M2' imply M2.

Proof: Let \( \mathcal{A}' \subseteq \mathcal{A} \), \( \mathcal{A}' \) satisfying MI and \( \mathcal{A} \) satisfying both MI and M2. If \( f \in \mathcal{A}' \) and \( \zeta_1, \ldots, \zeta_n \) are coordinates on an open \( U \subseteq E \) then by Lemma 3 \( f \circ \Phi^{-1} \) is differentiable on \( \Phi(U) \) and by M2' \( f \in \mathcal{A} \). Hence \( \mathcal{A}' = \mathcal{A} \) and \( \mathcal{A} \) is maximal with respect to MI. Thus MI and M2' imply M2.

The following lemmas will prove results about the ring \( \mathcal{A} \) and the Lie \( \mathbb{R} \)-algebra \( \mathcal{D} \) of derivations on \( \mathcal{A} \), which will lead up to a result about the exterior derivation, \( d \), on \( \mathcal{A} \) the \( \mathcal{A} \)-algebra of alternating differential forms of \( \mathcal{A} \).
Definition 17: If \( U \subseteq \mathbb{E} \) is an open set and \( f \) a function on \( U, f \) is called differentiable if for any coordinates \( z_1, \ldots, z_n \in \mathbb{E} \) on any \( V \subseteq U, f \circ \vec{z}_i^{-1} \) is differentiable on \( \vec{z}_i(V) \).

Remark: Clearly \( f \circ \vec{z}_i \) implies \( f/U \) is differentiable for any open \( U \subseteq \mathbb{E} \).

Lemma 6: If \( U \subseteq \mathbb{E} \) is open and \( a \in U \) then there exists an \( \vec{z}_i \) such that \( \vec{z}_iV = 1 \) and \( \vec{z}_iW = 0 \) for some open neighbourhood \( V \) of \( a \) and open \( W \) with \( V \subseteq W \) and \( \overline{W} \subseteq U \), where \( \overline{W} \) is the closure of \( W \).

Proof: To obtain the desired \( \vec{z}_i \) we use the fact that for any \( a < c < b \) there exist differentiable \( f \) on \( R \) with \( f(x) = 0 \) for \( x \in [a, b] \) and \( f(x) = 1 \) for \( x \in [a, b] \).

Since \( a \subseteq \mathbb{E} \), there exist \( z_1, \ldots, z_n \subseteq \mathbb{E} \) such that \( \vec{z}_i \) maps some open neighbourhood \( X \) of \( a \) homeomorphically onto an open set in \( \mathbb{E} \) and such that for any \( \vec{z}_i \) the function \( f \circ \vec{z}_i \) on \( \vec{z}_i(X) \) is differentiable. Let \( U' = U \setminus X \). Now take a closed box consisting of \( (y_1, \ldots, y_n) \) with \( c_i < y_i < b_i \) and within \( \vec{z}_i(U') \), and containing \( \vec{z}_i(a) \). Then take \( c_i', y_i', b_i' \) where \( c_i < c_i' \) and \( b_i < b_i' \). Let \( W = \{ x | c_i' < \vec{z}_i(x) < b_i' \} \) and \( W = \{ x | c_i < \vec{z}_i(x) < b_i \} \). Then \( \overline{W} = \{ x | c_i < \vec{z}_i(x) < b_i \} \) and we have the required \( V \) and \( W \). Now we take \( f_i \) on \( R \) such that \( f_i \) is differentiable; \( f_i(x) = 0 \) for \( x \in [a, b] \); and \( f_i(x) = 1 \) for \( x \in [a, b] \). Put \( e = \prod_{i=1}^{n} f_i \cdot \vec{z}_i \) and \( e \) clearly has the required properties.

Corollary 1: For each \( a \subseteq \mathbb{E} \), there exists an open neighbourhood \( V \subseteq U \) of \( a \) such that for any differentiable \( f \) on \( U \) there exists an \( f^e \in \mathbb{E} \) with \( f^e/V = f/V \).

Proof: Take \( e, V \) and \( W \) as above and define \( f^e = \begin{cases} f \circ \vec{z}_i & \text{on } U \\ 0 & \text{on } U \end{cases} \) differentiable on \( U \) and on \( W \) (here = 0) hence by lemma 4 \( f^e \in \mathbb{E} \). Clearly \( f^e/V = f/V \) since here \( e = 1 \).

Corollary 2: For each \( a \subseteq \mathbb{E} \) there exists a neighbourhood \( V \subseteq U \) of \( a \) and
with $e/\gamma - 1$ such that for any $f, g \in \mathcal{L}$ and $h$ differentiable on $U$ with $f/\gamma = \sum (g, \gamma/)h$, there exists $h^* \in \mathcal{L}$ with $h^* /\gamma = h /\gamma$ and $e = \sum g, h^*$.

Proof: Everything except $e = \sum g, h^*$ follows from lemma 6 and corollary 1.

By corollary 1 $h^* = \sum g$ on $U$, Therefore $\sum g, h^* = \sum g, \gamma$/U on $U$

$= \left\{ \begin{array}{ll}
0 & \text{on } \gamma/ \gamma \in U \\
0 & \text{on } \gamma/ \gamma = e/\gamma \in U
\end{array} \right.$

and $\sum g, h^* = e/\gamma$.

Lemma 7: Let $T$ be a derivation on $\mathcal{L}$; let $\xi_1, \ldots, \xi_n \in \mathcal{L}$ be coordinates on the open set $U \subseteq \mathcal{L}$; and let $f \in \mathcal{L}$ then $Tf/\gamma = \sum \left( \xi_1(T, \xi_1, U) \sum (f, \gamma)/U \right)$ where, for differentiable $g$ on $U$, $\frac{\partial}{\partial \xi_i} = (g \cdot \xi_i) \odot \xi_i$ where $(g \cdot \xi_i)$ is the partial derivative with respect to the $i$th coordinate.

Proof: Put $U' = \frac{\xi}{\xi_i} (U) \subseteq \mathcal{L}$ and let $F$: $U' \rightarrow \mathcal{L}$ be such that $f/\gamma = F \cdot \frac{\xi}{\xi_i}/U$.

Let $a, b \in U$ be a fixed point and $w \in U$ be such that $w' = \frac{\xi}{\xi_i}(w)$ is a sphere with centre $a' = \frac{\xi}{\xi_i}(a)$. Then for any $x' = \frac{\xi}{\xi_i}(x), x \in w, F(x' - F(a')) = \int df$ for any path $a \rightarrow x$ from $a'$ to $x$ by the Fundamental Theorem of Calculus.

Calling the coordinates in $\mathcal{L} \xi_1, \ldots, \xi_n$, one has $\int df = \sum \int df_i(\xi_1, \ldots, \xi_n)dv_i$ where the $F_i$ are the partial derivatives with respect to the coordinates.

Taking $P$ as the straight line segment from $a'$ to $x'$ in parametric representation $v_i = \xi_i(a) - t(\xi_i(a) - \xi_i(x)), 0 \leq t \leq 1$, one gets

$$f(x) - f(a) = \sum \int f_i(\xi_1, \ldots, \xi_n)dv_i,$$

Now for each $i$, $F_i$ is a differentiable function on $U$ and hence $G_i$, given by

$$G_i(\ldots, \xi_j, \ldots) = \int f_i(\ldots, \xi_j(a) - t(\xi_j(a) - \xi_j(x)), \ldots)dt$$

for $(\xi_1, \ldots, \xi_n) \in U$ is a differentiable function on $U'$. Its composite $G = \xi_i$, given by

$$x \rightarrow \int f_i(\ldots, \xi_j(a) - t(\xi_j(a) - \xi_j(x)), \ldots)dt$$

is therefore a differentiable function on $U$ to be denoted by $g_i$. It
follows that

\[ f - f(a)/U = \sum_i (\xi_i - \xi_i(a))/Ug_i. \]

Now, by corollary 2 of lemma 6 there exists a neighbourhood \( V \cong U \) of \( a \) such that \( e(f-f(a)) \equiv \sum_i (\xi_i - \xi_i(a))g_i \).

Acting \( T \) on this identity one gets

\[ (Te)(f- f(a) + e(Tf) = \sum_i (T\xi_i)g_i + (\xi_i - \xi_i(a))(Tg_i). \]

since \( Tf(a) = 0 \) since \( f(a) \) is constant. At the point \( a \in V \) one therefore has \( (Tf)(a) = \sum_i (T\xi_i)(a)g_i(a) \). Now

\[ g_i(a) = g_i(a) = \int_0^1 f_i'(\xi_1, \ldots, \xi_i(a), \ldots) dt = F_i(\xi_1, \ldots, \xi_i(a), \ldots). \]

Denoting the mapping \( x \rightarrow F_i' = f_i'/f_i \) of \( U \) into \( R \) by \( \frac{\partial f_i}{\partial \xi_i} \) where \( F_i = (f_i' f_i) \), one finally has for any \( a \in U \)

\[ (Tf)(a) = \sum_i (T\xi_i)(a) \frac{\partial f_i}{\partial \xi_i} (a) \]

i.e. \( Tf/\xi = \sum_i (T\xi_i)(a) \frac{\partial f_i}{\partial \xi_i} (a) \).

Lemma 8: Let \( \xi_1, \ldots, \xi_n \) be coordinates on an open set \( U \in E \). If \( a \in U \), let

\( V \) and \( e \) be as in lemma 6 and define \( X_i: \mathcal{A} \rightarrow A \) by \( X_i = (\frac{\partial}{\partial \xi_i}(f/\xi_i)/V) \).

i.e. \( X_i f = \left\{ \begin{array}{l} \frac{\partial}{\partial \xi_i}(f/\xi_i)/V) \text{ on } U \\ 0 \text{ on } \mathbb{C} \end{array} \right. \) and \( \frac{\partial}{\partial \xi_i}(f/\xi_i)/V) = (f/\xi_i) \cdot f_i' \) (Note that in particular \( X_i f/\xi_i/V = \frac{\partial}{\partial \xi_i}(f/\xi_i)/V \).) These \( X_i \) are derivations.

Proof: \( X_i \) is an \( R \)-linear mapping from \( \mathcal{A} \) to \( \mathcal{A} \) for if \( f_1, f_2 \in \mathcal{A} \)

(i) \( X_i(f_1 + f_2) = \left\{ \begin{array}{l} \frac{\partial}{\partial \xi_i}(f_1 + f_2)/\xi_i)/V) \text{ on } U \\ 0 \text{ on } \mathbb{C} \end{array} \right. \) on \( U \)

\[ \{ e(\xi_1) \cdot f_1' \text{ on } \mathbb{C} U \}

and for \( \alpha \in \mathcal{A} \) and \( a \in R \)

(ii) \( X_i(\alpha f) = \left\{ \begin{array}{l} \alpha (\frac{\partial}{\partial \xi_i}(f)/\xi_i)/V) \text{ on } U \\ 0 \text{ on } \mathbb{C} \end{array} \right. \) on \( U = \alpha X_i f \)

Also for \( f, g \in \mathcal{A} \)

\[ X_i(fg) = \left\{ \begin{array}{l} e(\frac{\partial}{\partial \xi_i}(f)/\xi_i)/V) \text{ on } U \\ 0 \text{ on } \mathbb{C} \end{array} \right. \) on \( U = \{ e(\frac{\partial}{\partial \xi_i}(g)/\xi_i)/V) \text{ on } U \\ 0 \text{ on } \mathbb{C} \end{array} \right. \) on \( U \)

Also for \( f, g \in \mathcal{A} \)
\[
\{ e((x \cdot \frac{d}{dS_i}) (g \cdot \frac{d}{dS_i}) + (x \cdot \frac{d}{dS_i}) (g \cdot \frac{d}{dS_i}) \cdot \frac{d}{dS_i} g) \text{ on } U \\
0 \text{ on } U \\
\}
\]
\[
\{ e \frac{\partial}{\partial S_i} (f/U)(g/U)+(f/U)g \frac{\partial}{\partial S_i} (g/U) \text{ on } U \\
0 \text{ on } U \\
\}
\]
\[
= (x_i f)g + f(x_i g)
\]

Corollary 1: For any \( T \in \mathcal{O} \) we have \( eT - \Sigma S_i X_i \).

Proof: From lemma 7 one has \( (eT)f/U - \Sigma (T S_i)X_i f/V \) and \( (eT)f/U = 0 \) for all \( f \in A \). On the other hand \( \Sigma (T S_i)X_i f = \left\{ \begin{array}{ll} \Sigma (T S_i) & (f/U) \\
0 & \text{on } U \end{array} \right. \)

and hence the assertion is true.

Corollary 2: If \( f \in A \) and \( f/V \) is constant then \( Tf/V = 0 \) for any \( T \in \mathcal{O} \).

Proof: \( Tf/V = eTf/V = \Sigma (T S_i)X_i f/V \) by corollary 1. Hence it is enough to show that if \( f/V \) is constant \( X_i f/V = 0 \). Now, by definition \( X_i f/V = (\frac{\partial}{\partial S_i} (f/U))/V \), but \( \frac{\partial}{\partial S_i} (f/U) = (f/U \cdot \frac{d}{dS_i}) \cdot \frac{d}{dS_i} \). Hence \( f \cdot \frac{d}{dS_i} \) is constant on \( F_S(V) \), and hence its \( i \)th derivative is zero. Thus \( X_i f/V = 0 \).

Corollary 3: For any \( f \in A \) \( \text{edf} = \Sigma (X_i f) d S_i \).

Proof: For \( T \in \mathcal{O} \), \( df(T) = Tf \) and thus
\[
\text{edf}(T) = e(Tf) = \Sigma (T S_i)X_i f = \Sigma (X_i f) d S_i(T).
\]

Lemma 9: Suppose that \( \omega_1, \ldots, \omega_k \) are linear forms on \( \mathcal{O} \). Then for \( T_1, \ldots, T_k \in \mathcal{O} \) we have

\[
\omega_1 \wedge \cdots \wedge \omega_k (T_1, \ldots, T_k) = \det (\omega_i(T_j))
\]

where \( \wedge \) is the law of composition described in section II.

Proof: (By induction) Clearly if \( k = 1 \) then \( \omega_i(T) = \det (\omega)(T) \).

Assume that for \( n \)
\[
\omega_1 \wedge \cdots \wedge \omega_n (T_1, \ldots, T_n) = \det (\omega_i(T_j))
\]

and for \( n+1 \)
\[
\omega_1 \wedge \cdots \wedge \omega_n \wedge \omega_{n+1} (T_1, \ldots, T_{n+1})
\]
Then \[
\begin{pmatrix}
\omega_1(T_1) & \omega_1(T_2) & \cdots & \omega_1(T_n) & \omega_1(T_{n+1}) \\
\omega_2(T_1) & \omega_2(T_2) & \cdots & \omega_2(T_n) & \omega_2(T_{n+1}) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\omega_n(T_1) & \omega_n(T_2) & \cdots & \omega_n(T_n) & \omega_n(T_{n+1}) \\
\omega_{n+1}(T_1) & \omega_{n+1}(T_2) & \cdots & \omega_{n+1}(T_n) & \omega_{n+1}(T_{n+1}) \\
\end{pmatrix}
\]

\[
= (-1)^{n+1} \omega_1(T_1) \det
\begin{pmatrix}
\omega_2(T_2) & \cdots & \omega_2(T_{n+1}) \\
\vdots & \ddots & \vdots \\
\omega_n(T_n) & \cdots & \omega_n(T_{n+1}) \\
\omega_{n+1}(T_{n+1}) \\
\end{pmatrix}
+ (-1)^{n+2} \omega_1(T_2) \det
\begin{pmatrix}
\omega_2(T_2) & \cdots & \omega_2(T_{n+1}) \\
\vdots & \ddots & \vdots \\
\omega_n(T_n) & \cdots & \omega_n(T_{n+1}) \\
\omega_{n+1}(T_{n+1}) \\
\end{pmatrix}
+ \cdots
\]

\[
+ (-1)^{n+1} \omega_1(T_n) \det
\begin{pmatrix}
\omega_2(T_2) & \cdots & \omega_2(T_{n+1}) \\
\vdots & \ddots & \vdots \\
\omega_n(T_n) & \cdots & \omega_n(T_{n+1}) \\
\omega_{n+1}(T_{n+1}) \\
\end{pmatrix}
+ \cdots
\]

\[
+ (-1)^{n+2} \omega_1(T_{n+1}) \det
\begin{pmatrix}
\omega_2(T_2) & \cdots & \omega_2(T_{n+1}) \\
\vdots & \ddots & \vdots \\
\omega_n(T_n) & \cdots & \omega_n(T_{n+1}) \\
\omega_{n+1}(T_{n+1}) \\
\end{pmatrix}
\]

\[
= (-1)^n \omega_1(T_1) (\omega_2 \cdots \omega_{n+1} (T_2 \cdots T_{n+1}))
+ (-1)^n \omega_1(T_2) (\omega_2 \cdots \omega_{n+1} (T_1 \cdots T_{n+1}))
+ \cdots
\]

Now recall from section II, if \(\phi \psi \in \text{Hom}_k (E_2 \Lambda)\) then

\[
\phi \psi (x_1 \cdots x_n) = \frac{1}{n!} \eta(s) \sigma^2 \psi(t(-1)) \psi(t(q^2))
\]

and hence here

\[
(\omega_2 \cdots \omega_{n+1} (T_2 \cdots T_{n+1}) (T_1 \cdots T_{n+1})
= (-1)^n \omega_1(T_1) (\omega_2 \cdots \omega_{n+1} (T_2 \cdots T_{n+1}))
+ (-1)^n \omega_1(T_2) (\omega_2 \cdots \omega_{n+1} (T_1 \cdots T_{n+1}))
+ \cdots
+ (-1)^n \omega_1(T_n) (\omega_2 \cdots \omega_{n+1} (T_1 \cdots T_{n-1} T_n))
\]
\[ (-1)^n \omega_1(T_{n+1}) \omega_2 \cdots \omega_n (T_1, T_2, \ldots, T_n) \]
since \( \omega_i \) is a linear form and hence will take everything but a sequence consisting of one of the \( T_i \) into zero. Hence

\[ \omega_1 \cdots \omega_n (T_1, \ldots, T_{n+1}) = \det \begin{pmatrix} \omega_1(T_1) & \ldots & \omega_1(T_{n+1}) \\ \vdots & \ddots & \vdots \\ \omega_n(T_1) & \ldots & \omega_n(T_{n+1}) \end{pmatrix}. \]

**Lemma 10:** If \( \xi_1, \ldots, \xi_m \) are coordinates on the open set \( U \subseteq \mathbb{R} \) and any alternating differential form \( \Theta \) of degree \( k \)

\[ \Theta(T_1, \ldots, T_k) / U = \sum_{\sigma} h_{i_1 \ldots i_k} d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k} (T_1, \ldots, T_k) / U \]

with differentiable \( h_{i_1 \ldots i_k} \) on \( U \).

**Proof:** For any \( \sigma \in \mathcal{A} \), let an open neighbourhood \( \mathcal{V} \subseteq U \), and \( e \in \mathcal{D} \) be as considered earlier in lemma 6. Then

\[ \Theta(T_1, \ldots, T_k) = \Theta(eT_1, \ldots, eT_k) = \Theta(T_1 \xi_1 x_i, \ldots, T_k \xi_k x_i) \]

\[ = \sum_{i_1 < \cdots < i_k} \det \begin{pmatrix} T_1 & \cdots & T_k \\ \xi_{i_1} & \cdots & \xi_{i_k} \end{pmatrix} \Theta(X_1, \ldots, X_k). \]

since \( \Theta \) is multilinear and alternating. For,

\[ \Theta(\sum_{i_1 < \cdots < i_k} T_1 \xi_1 x_i, \ldots, \sum_{i_1 < \cdots < i_k} T_k \xi_k x_i) = \sum_{i_1 < \cdots < i_k} (T_1 \xi_1 \cdots T_k \xi_k) \Theta(X_1, \ldots, X_k) \]

\[ = \sum_{i_1 < \cdots < i_k} \left\{ \sum_{\sigma} \epsilon(\sigma) (T_1 \xi_1, \ldots, T_k \xi_k) \right\} \Theta(X_1, \ldots, X_k) \]

and \( \sum_{\sigma} \epsilon(\sigma) (T_1 \xi_1, \ldots, T_k \xi_k) \) is the determinant of the matrix

\[ \begin{pmatrix} T_1 \xi_1 & \cdots & T_k \xi_k \\ \vdots & \ddots & \vdots \\ T_k \xi_1 & \cdots & T_k \xi_k \end{pmatrix}. \]

Thus

\[ e \Theta(T_1, \ldots, T_k) = \sum_{i_1 < \cdots < i_k} \Theta(X_1, \ldots, X_k) d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k} (T_1, \ldots, T_k) / U. \]

Hence since \( e / \mathcal{V} = 1 \) we have

\[ \Theta(T_1, \ldots, T_k) / \mathcal{V} = \sum_{i_1 < \cdots < i_k} \Theta(X_1, \ldots, X_k) d\xi_{i_1} \wedge \cdots \wedge d\xi_{i_k} (T_1, \ldots, T_k) / \mathcal{V}. \]

The \( \Theta(X_1, \ldots, X_k) \) depend in their definition on the \( \mathcal{V} \subseteq U \). Let \( \mathcal{V} \subseteq U \) and
and € ∈ $\mathcal{A}$ be of the same nature as $a$ and $V$ and let $X'_i$ be correspondingly defined, i.e. $X'_i = \begin{cases} \frac{\partial f}{\partial x_i} & \text{on } U \\ 0 & \text{on } \phi \cap U \end{cases}$. It obviously follows that $eX'_i = e'X_i$. Hence

$$e^k \theta (X'_{i_1}, \ldots, X'_{i_k}) = \theta (eX'_{i_1}, \ldots, eX'_{i_k}) = \theta (e'X'_{i_1}, \ldots, e'X'_{i_k}) = e^k \theta (X'_{i_1}, \ldots, X'_{i_k}).$$

Hence $\theta (X'_{i_1}, \ldots, X'_{i_k}) = \theta (X_{i_1}, \ldots, X_{i_k})$ on $V \cap W'$. Now, one has for each suitable open $V \subseteq U$ a function $h_{i_1, \ldots, i_k} \in \mathcal{A}$, namely the $\theta (X'_{i_1}, \ldots, X'_{i_k})$. These give rise to a function $h_{i_1, \ldots, i_k}$ on $U$ by the definition $h_{i_1, \ldots, i_k} (x) = h_{i_1, \ldots, i_k} (x)$ if $x \in V$. If $x \in V'$ also then by what we just proved $h_{i_1, \ldots, i_k} (x) = h_{i_1, \ldots, i_k} (x)$ and hence since the $V$'s cover $U$ we have the required $h_{i_1, \ldots, i_k}$ defined, for $h_{i_1, \ldots, i_k}$ is clearly differentiable on $U$ since $h_{i_1, \ldots, i_k} / V = h_{i_1, \ldots, i_k}$ which is differentiable on $V$ and this is true for all $V$.

**Corollary:** For $k > n$ the forms of degree $k$ are zero.

**Theorem 6:** For any alternating differential form $\theta$ of degree $k$ and any open set $U$ on which there exist coordinated functions $S_1, \ldots, S_n \in \mathcal{A}$

$$d\theta (t, \ldots, t_{k-1}) / U = \sum_{i_1 \leq \ldots \leq i_k} \frac{\partial \theta}{\partial x_i} (t_1, \ldots, t_k) S_{i_1} \wedge \ldots \wedge S_{i_k} (t_1, \ldots, t_k) / U$$

where the functions $h_{i_1, \ldots, i_k}$ are as described in lemma 6.

**Proof:** For any $a \in U$ let an open neighbourhood $V \subseteq U$ and $e \in \mathcal{A}$ be as considered earlier in lemma 6. Then

$$\theta (t, \ldots, t_{k-1}) / U = \sum_{i_1 \leq \ldots \leq i_k} h_{i_1, \ldots, i_k} dS_{i_1} \ldots dS_{i_k} (t_1, \ldots, t_k) / U.$$

Now,

$$\theta (t, \ldots, t_{k-1}) / U = \sum_{i_1 \leq \ldots \leq i_k} h_{i_1, \ldots, i_k} dS_{i_1} \ldots dS_{i_k} (t_1, \ldots, t_k) / U.$$

since on $a \cap U$ both $e$ and $h_{i_1, \ldots, i_k}$ are zero and on $U h_{i_1, \ldots, i_k} = e h_{i_1, \ldots, i_k}$.

Acting the exterior derivation, $d$, on (*) we get

$$(*) \quad d\theta = \sum_{i_1 \leq \ldots \leq i_k} h_{i_1, \ldots, i_k} dS_{i_1} \ldots dS_{i_k} (t_1, \ldots, t_k),$$

since $e$ is homogeneous of degree zero and since $ddS_{i} = 0$ for all $i$. Now
\[
\text{de}(\theta(T_1, \ldots, T_{k+1})) = \sum (-1)^{i_1} (T_{i_1} \theta)(T_1, \ldots, \hat{T}_{i_1}, \ldots, T_{k+1}).
\]

But \(e/V\) is constant and hence by corollary 2, lemma 8, \(T_i e/V = 0\). Thus \(\text{de}(\theta(T_1, \ldots, T_{k+1}))/V = 0\). Also, \(\text{ed}(\theta(T_1, \ldots, T_{k+1}))/V = \text{de}(\theta(T_1, \ldots, T_{k+1}))/V\).

Hence
\[
(\text{de} \wedge \theta + \text{ed}) (T, \ldots, T_{k+1}) / V = \text{de}(\theta(T_1, \ldots, T_{k+1}))/V.
\]

Now \(\text{ed}(\theta(T_1, \ldots, T_{k+1}))/V = \text{ed}(\theta(T_1, \ldots, T_{k+1}))/V\)
\[
= \sum_{i_1, \ldots, i_k} \text{ed} h_{i_1 \ldots i_k} \text{ds}^1 \wedge \ldots \wedge \text{ds}^k (T_1, \ldots, T_{k+1}) / V.
\]

Let us consider
\[
\sum_{i_1, \ldots, i_k} \text{ed} h_{i_1 \ldots i_k} \text{ds}^1 \wedge \ldots \wedge \text{ds}^k (T_1, \ldots, T_{k+1}) / V.
\]

by corollary 3, lemma 8. If we restrict this to \(V\) we have
\[
\sum_{i_1, \ldots, i_k} \text{ed} h_{i_1 \ldots i_k} \text{ds}^1 \wedge \ldots \wedge \text{ds}^k (T_1, \ldots, T_{k+1}) / V
\]
\[
= \sum_{i_1, \ldots, i_k} \frac{\partial}{\partial s_{i_k}} h_{i_1 \ldots i_{k-1}} \text{ds}^1 \wedge \ldots \wedge \text{ds}^k (T_1, \ldots, T_{k+1}) / V.
\]

since on \(V\), \(X h_{i_1 \ldots i_k} = X h_{i_1 \ldots i_{k-1}} \text{ds}^1 \wedge \ldots \wedge \text{ds}^k (T_1, \ldots, T_{k+1}) / V\).

Thus since the \(V\)'s cover \(U\) we have
\[
\text{ed}(\theta(T_1, \ldots, T_{k+1}))/U = \sum_{i_1, \ldots, i_k} \frac{\partial}{\partial s_{i_k}} h_{i_1 \ldots i_{k-1}} \text{ds}^1 \wedge \ldots \wedge \text{ds}^k (T_1, \ldots, T_{k+1}) / U.
\]

Corollary: If \(d'\) is any homogeneous derivation of degree 1 on \(\mathfrak{a}\) with the properties that \(d'f = df\) and \(d'df = 0\) for \(f \in \mathfrak{a}\) then \(d = d'\).

Proof: If \(d'\) is a derivation with the prescribed properties then the same proof as above with \(d'\) in place of \(d\) shows that
\[
\text{d'}(\theta(T_1, \ldots, T_{k+1}))/U = \sum_{i_1, \ldots, i_k} \frac{\partial}{\partial s_{i_k}} h_{i_1 \ldots i_{k-1}} \text{ds}^1 \wedge \ldots \wedge \text{ds}^k (T_1, \ldots, T_{k+1}) / U.
\]

Remark: Hence the algebraically defined derivation \(d\) is the same as the analytically defined exterior differentiation of Cartan. (See Theory of Lie Groups by Chevalley.)
BIBLIOGRAPHY
