PERTURBATIVE FAILURE NEAR HORIZONS: THE RINDLER EXAMPLE

Perturbative Failure Near Horizons:
the Rindler Example

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TITLE: Perturbative Failure Near Horizons: the Rindler Example

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Lay Abstract:

Perturbation theory is an approximation technique widely used in the study of quantum field theory (QFT). In this work it is argued that the predictions of perturbation theory can fail when used on a QFT living in a spacetime that contains an event horizon. This work focuses on perturbative breakdowns that occur after long lengths of time have passed, providing an explicit example in the simplest possible spacetime with an event horizon: Rindler space. It is argued that this breakdown may occur in more complicated settings.

Abstract:

Quantum field theory (QFT) in curved spacetime treats a gravitational field as a classical background upon which quantum corrections may be computed. When couplings are assumed to be small, it is traditionally believed that perturbation theory yields trustworthy predictions about interacting quantum fields in such settings — this work asserts that this is not always the case.

It is argued that perturbative predictions about evolution for very long times near a horizon are subject to problems of secular growth — *ie.* powers of small couplings come systematically together with growing functions of time. Such growth signals a breakdown of naïve perturbative calculations of late-time behaviour, regardless of how small ambient curvatures might be. Evidence is built that such breakdowns should be generic for gravitational fields, particularly those containing horizons.

This work makes use of the Rindler horizon in flat Minkowski space to demonstrate an explicit example of such perturbative breakdown. A loop correction involving an IR/UV interplay is shown to result in a two-point correlation function which exhibits secular growth. This result is shown to parallel a breakdown occurring in finite temperature QFT, where problems of secular growth are known to occur. The problematic correction is then resummed, allowing for trustworthy late-time inferences. We conclude by discussing how this calculation may be relevant for predictions near black hole horizons.

Acknowledgements:

I would like to sincerely thank my supervisor Dr. Cliff Burgess whose deep understanding of Effective Field Theories was the basis for the ideas which fledged this work. His vast knowledge, attention to detail and patience with me and this project have been invaluable.

I would like to express my gratitude to Dr. Markus Rummel for all his revisions and helpful advice throughout the past months. I am thankful to Joshua Hainge, spending late nights together learning and discussing the real-time finite temperature formalisms.

I also graciously thank Andrés Schlief for the time taken to introduce and coherently explain the Schwinger-Keldysh formalism.

Declaration of Academic Achievement:

I, Gregory Paul Kaplanek, declare that this thesis titled, "Perturbative Failure Near Horizons: the Rindler Example" and the work presented in it are my own. Significant contributions from collaborators are as follows:

- The main idea for the entire work, that is to search for secular growth near horizons (as per the motivations outlined in Chapter 1), was due to Dr. Cliff Burgess. In addition to this, the resummation performed in Chapter 4, and the interpretations mentioned in Chapter 5 are due to Dr. Cliff Burgess.

The results of this work, specifically those of Chapters 4 and 5 and Appendix D, have been submitted for publication [1] on July 4$^{\text{th}}$, 2018 to the Journal of High Energy Physics (JHEP).

Dedication:

For Mia and for Ana.

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Chapter 1

Introduction

This work is based on a paper [1] of the same title submitted for publication in July 2018 by C. Burgess, J. Hainge, M. Rummel and myself. Quoting the paper:

> *In this work the argument is made that for quantum fields in the presence of horizons, perturbation theory is like good weather: if you wait long enough it eventually fails.*

1.1 Open Quantum Systems, Thermality and Secular Growth

By definition an *open quantum system* consists of an internal sector we would like to use to make physical predictions with and an external sector, called *the environment*, which is not measured in detail. The most famous and well-understood example of an open quantum system is that of a thermal system. In this case, the heat bath plays the role of the environment where all the details of the physics it's built out of are replaced by a coarse-grained thermodynamic description. By tracing over all of the unmeasured degrees of freedom this allows the environment to be described by a density matrix (rather than a pure state).

If the heat bath is ever-present and always exchanging energy with our system, it is generic that this can cause a breakdown in perturbative methods at late times. If our observables time-evolve with the operator $U(t) = e^{-iHt}$ where $H = H_{\rm free} + \lambda H_{\rm int}$, it is always true that no matter how small the coupling λ there will eventually come a time late enough where is is a bad approximation to compute $U(t)$ in powers of $\lambda H_{\rm int} t$. To be explicit: if we use perturbation theory to compute a physical quantity $\mathcal{Q}(t)$ where

$$\mathcal{Q}(t) = \mathcal{Q}_{\rm free}(t) + \lambda \mathcal{Q}_1(t) + \lambda^2 \mathcal{Q}_2(t) + \dots \ , \tag{1.1}$$

and find that the correction functions $\mathcal{Q}_n(t)$ grow as $t \to \infty$, then the perturbative series will break down in this limit. This is called *secular growth* — no matter how small λ is, there will always come a time that our perturbative series blows up. For open systems in general, this can be understood as the environment's eternal presence allowing for perturbatively small effects to accumulate and become large. This problem is less familiar in particle physics, where the focus is usually on scattering problems — here the wave-packets involved separate from each other in the past and future (so interactions are only briefly turned on).

Some famous examples of thermal systems suffering perturbative breakdown are the hard thermal loop problem from QCD [2, 3] as well as the breakdown of perturbative mean-field methods near a critical point [4]. Both of these examples are associated with infrared (IR) effects due to the presence of massless bosons (or at least, bosons much lighter than the scale set by the temperature). In general, the heat bath causes extra low-energy IR modes to become occupied which can cause the correlation functions describing the bosons to become singular at low-frequencies, giving rise to both IR divergences and secular growth. Mathematically, the fluctuations of the light bosons are stronger at non-zero temperature because of the singular low-energy behaviour of the Bose-Einstein distribution function, *ie.*

in the limit $\beta\omega \ll 1$ we have

$$\frac{1}{e^{\beta\omega}-1} \approx \frac{1}{\beta\omega} - \frac{1}{2} + \mathscr{O}(\beta\omega)\ . \tag{1.2}$$

The good news is that the existence of secular growth doesn't necessarily need to imply that you can't make late time predictions. For example, exponential decay laws of the form $n(t) = n_0 e^{-\Gamma t}$ are posited to hold for times t well after the mean lifetime of the particle (*ie.* for $\Gamma t \gg 1$) even though one usually computes the decay rate Γ using perturbation theory. The evolution of the number of particles satisfies $\frac{dn}{dt} = -\Gamma n$ for all times, which means that exponential decay in $n(t)$ persists no matter how large Γt might be. Observations such as this one suggest that late-time resummations are possible for systems exhibiting secular growth. It is worth mentioning that even when such resummations can be performed, one usually loses analyticity in the small coupling (meaning the resummation cannot be captured by a Taylor series in integer powers of the coupling).

1.2 Quantum Field Theory with Horizons

Quantum field theory (QFT) on spacetimes with horizons contain many surprising features. A famous example is that of Hawking radiation [5, 6], which was one of the first demonstration of the peculiar link between gravity and thermodynamics. Following this, a wealth of associated puzzles regarding information loss continue to be studied (for reviews with extensive references see [7, 8, 9, 10]). It is worth noting that many of these puzzles involve understanding the system *at very late times* (such as the Page time in [11, 12]).

Many of these puzzles rely only on the presence of horizons without large amounts of curvature, which would imply that they are within the domain of validity of effective field theory (EFT) methods. However, because these information loss puzzles continue to persist it is interesting to ask whether perturbative methods might break down near horizons (even when the curvatures are small).

The key feature of a quantum system near a horizon is that there is an *unobserved sector* (behind the horizon) with which an observer interacts with. In addition to this, the spectrum of emitted radiation takes on a black-body form. It is in these ways that quantum systems near horizons resemble thermal systems.

Realizing the thermal nature of these systems it is natural to ask whether perturbation series can break down at late times near horizons. If the problem of secular growth is found to exist for QFTs on spacetimes with horizons this could have interesting implications for the late-time arguments often made about information loss regarding these systems.

1.3 Structure of this Work

In this work we search for a failure of perturbation series via secular growth in the simplest possible setting: a real scalar field in flat Minkowski space. Although this is the first example of a QFT that most students learn about, there are very interesting physics hidden inside this QFT when quantized relative to an observer along a uniformly accelerated worldline.

This accelerating *Rindler observer* perceives the null light-cone as an artificial event horizon: we shall see that this results in the observer experiencing the Minkowski vacuum state as a thermal state. The connection between the Rindler observers and thermality was first reported by Unruh in [13] and has the advantage that it is well-understood and takes place in a flat spacetime (where the curvature is zero).

We argue in this work that Rindler observers moving through the Minkowski vacuum do see instances of secular growth, as one would expect given the Rindler observer's thermal interpretation of the Minkowski vacuum. By specializing to a massless scalar field ϕ subject to a $\lambda\phi^4$ interaction, we find an explicit example of a breakdown of perturbation theory: a $\mathscr{O}(\lambda)$ two-point correlation function which diverges for large proper times near the Rindler horizon. We'll also see that this breakdown does *not* take place for ordinary Minkowski observers.

So as to not detract from the story of the thesis, we refer the reader to the appendices for background information on the quantization of a real scalar field ϕ in ordinary flat Minkowski space. In Appendix A, the quantization of ϕ in arbitrary coordinates is discussed (relative to the notion of time specified by a global timelike Killing vector $\mathcal{K}$). In Appendix B, the usual Minkowski quantization scheme (relative to ordinary Minkowski time x^0) is reviewed, as well as the notion of the multi-particle states emerging from this scheme.

In Chapter 2, we briefly review the objects of the free theory that will be of interest; namely, the Minkowski vacuum and the Feynman propagator. We consider corrections to the time-ordered two-point correlation function in a $\lambda\phi^4$-interacting theory, and examine the behaviour of the so-called tadpole and cactus graphs in the massless limit which are shown to vanish.

The goal of Chapter 3 is to have an overview of how the corresponding real-time QFT at finite temperature works and to compute two-point $\lambda\phi^4$ corrections in such a setting. The main feature of the formalism examined is that a second field (representing the presence of the heat bath) must be introduced so as to discuss the notion of a thermal vacuum state — the result is that the thermal time-ordered two-point correlation functions of the theory take on a matrix structure. The same massless corrections to the two-point function are considered in a $\lambda\phi^4$ theory and are shown to exhibit the problem of secular growth.

In Chapter 4, the field ϕ is quantized relative to an accelerating Rindler observer which is shown to result in thermality. The same $\lambda\phi^4$ corrections from Chapter 2 are computed for the Rindler observer, and it is shown that this results in a secular breakdown *precisely* matching the thermal result from Chapter 3. As a bonus, it is demonstrated that a near-horizon resummation may be performed in this setting, correctly capturing the late-time behaviour of the interacting two-point correlation function.

In Chapter 5, we'll make some conclusions about the calculations presented in this work and briefly explain their relevance to black hole physics.

Chapter 2

The Minkowski Observer's QFT

The purpose of this chapter is to review the QFT of a real scalar field ϕ in ordinary flat Minkowski space so as to compute some simple corrections to the time-ordered two-point correlation function in $\lambda\phi^4$ theory. In particular, we find that the usual choice of counter-term for the massless tadpole and cactus graphs yields contributions that evaluate to zero.

2.1 The Feynman Propagator

In this section we consider the real scalar field ϕ living in Minkowski space with Lagrangian density

$$\mathscr{L}_0[\phi] \;\; = \;\; -\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 \; . \tag{2.1}$$

We use rectangular coordinates $x = (x^0, \mathbf{x})$ and the metric $\eta_{\mu\nu} = \mathrm{diag}(-+++)$. We quantize the field relative to a fiducial observer whose proper time is measured along the flow-lines of the Killing vector $\mathcal{T} = \frac{\partial}{\partial x^0}$ (corresponding to the notion of x^0-time). This procedure is outline extensively in Appendices A and B, so we skip ahead noting that this is the usual 'Minkowski quantization scheme'. Here the quantized field $\hat{\phi}$ is expanded such that

$$\hat{\phi}(x) \;\; = \;\; \int d^3\mathbf{k} \, \Big[\hat{a}_{\mathbf{k}} u_{\mathbf{k}}(x) + \hat{a}^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(x) \Big] \tag{2.2}$$

where the Minkowski modes $\{u_{\mathbf{k}}, u^*_{\mathbf{k}}\}$ (labelled by the mode parameters $\mathbf{k} = (k^1, k^2, k^3) \in \mathbb{R}^3$) take the form

$$u_{\mathbf{k}}(x) = \left(16\pi^3\sqrt{|\mathbf{k}|^2+m^2}\right)^{-\frac{1}{2}} e^{-i\sqrt{|\mathbf{k}|^2+m^2}x^0+i\mathbf{k}\cdot\mathbf{x}} \tag{2.3}$$

$$u^*_{\mathbf{k}}(x) = \left(16\pi^3\sqrt{|\mathbf{k}|^2+m^2}\right)^{-\frac{1}{2}} e^{+i\sqrt{|\mathbf{k}|^2+m^2}x^0-i\mathbf{k}\cdot\mathbf{x}} . \tag{2.4}$$

and the raising and lowering operators $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ satisfy the canonical commutation relations (B.19-B.21)[1].

Importantly, the Minkowski Hamiltonian (B.23) diagonalizes here with

$$\hat{H}_{\mathrm{M}} = \int d^3\mathbf{k}\ \tfrac{1}{2}\sqrt{|\mathbf{k}|^2+m^2}\ \left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\right] , \tag{2.5}$$

and has an energy that is bounded from below, which implies the existence of a state of minimum energy $|0_{\mathrm{M}}\rangle$ called the *Minkowski vacuum*. The usual notion of multi-particle states is built out of the Minkowski vacuum $|0_{\mathrm{M}}\rangle$ by applying combinations of $a^\dagger_{\mathbf{k}}$ operators (for a discussion on this, see Appendix B.2).

An object of crucial importance in this work is the time-ordered expectation value of two field operators *evaluated in the Minkowski vacuum*:

$$G_{\mathrm{F}}(x;y) \equiv \langle 0_{\mathrm{M}}|\mathscr{T}\left(\hat{\phi}(x)\hat{\phi}(y)\right)|0_{\mathrm{M}}\rangle \tag{2.6}$$

This is known as the *Feynman propagator*. The time-ordering operator $\mathscr{T}$ orders the fields according to their Minkowski time-coordinates, so more explicitly we have

$$G_{\mathrm{F}}(x,y) = \Theta(x^0-y^0)\,\langle 0_{\mathrm{M}}|\hat{\phi}(x)\hat{\phi}(y)|0_{\mathrm{M}}\rangle + \Theta(y^0-x^0)\,\langle 0_{\mathrm{M}}|\hat{\phi}(y)\hat{\phi}(x)|0_{\mathrm{M}}\rangle\ , \tag{2.7}$$

where Θ is the Heaviside step function. The Feynman propagator is a Green's function for the operator $\Box - m^2$ in the sense that it satisfies the distributional

[1] It is worth noting that one usually writes the expansion of $\hat{\phi}$ explicitly in terms of plane-waves $e^{\mp i\sqrt{|\mathbf{k}|^2+m^2}x^0 \pm i\mathbf{k}\cdot\mathbf{x}}$ and a Lorentz-invariant measure of the form $\left(16\pi^3\sqrt{|\mathbf{k}|^2+m^2}\right)^{-\frac{1}{2}} d^3\mathbf{k}$ while also enforcing $\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{p}}\right] = 16\pi^3\sqrt{|\mathbf{k}|^2+m^2}\delta^{(3)}(\mathbf{k}-\mathbf{p})$. Although this description is manifestly covariant at every level, we avoid it so as to be consistent with the normalizations set by the Klein-Gordon inner product (A.10) here and later on in the Rindler-Fulling quantization scheme of Chapter 4.

equation [22]

$$(\Box_x - m^2)G_{\mathrm{F}}(x;y) = i\delta^{(4)}(x-y) \tag{2.8}$$

Of course, this is not the only Green's function one can consider, but for applications it is arguably the most useful one — this is because whenever perturbation theory is used, the Dyson series used always ends up being a smattering of integrals over time-ordered products of field operators (which we can always reduce to combinations of G_{F} through Wick's theorem).

Using the expanded form of our field as in (2.2) the above takes the form:

$$G_{\mathrm{F}}(x;y) \;=\; \Theta(x^0-y^0)\int d^3\mathbf{k}\ u_{\mathbf{k}}(x)u^*_{\mathbf{k}}(y) + \Theta(y^0-x^0)\int d^3\mathbf{k}\ u^*_{\mathbf{k}}(x)u_{\mathbf{k}}(y) \tag{2.9}$$

Colloquially, the Feynman propagator is said to propagate positive frequencies to the future, and negative frequencies to the past [20] — in this form, this is evident (taking $x^0 \to \infty$ leaves only positive frequency terms and visa versa). After inserting our expressions for the Minkowski modes we get

$$\begin{aligned} G_{\mathrm{F}}(x;y) \;&=\; \Theta(x^0-y^0)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \frac{e^{-i\sqrt{|\mathbf{k}|^2+m^2}(x^0-y^0)}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\sqrt{|\mathbf{k}|^2+m^2}} \\ &\quad +\Theta(y^0-x^0)\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \frac{e^{+i\sqrt{|\mathbf{k}|^2+m^2}(y^0-x^0)}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{2\sqrt{|\mathbf{k}|^2+m^2}}\ , \end{aligned} \tag{2.10}$$

If we note the integral representation of the Heaviside step function [22]

$$\Theta(t) \;=\; \frac{i}{2\pi}\lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} dk^0\ \frac{e^{-itk^0}}{k^0+i\epsilon}\ , \tag{2.11}$$

then we may write the Feynman propagator in terms of a four-dimensional integral

$$G_{\mathrm{F}}(x;y) \;=\; \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty}\frac{dk^0}{2\pi}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{-i\ e^{-i(x^0-y^0)k^0+i(\mathbf{x}-\mathbf{y})\cdot\mathbf{k}}}{-(k^0)^2+|\mathbf{k}|^2+m^2-i\epsilon}\ . \tag{2.12}$$

Interpreting the integration variables as a four-momentum $k = (k^0,\mathbf{k})$ we can compactly write the above in relativistic notation as

$$G_{\mathrm{F}}(x;y) \;=\; \int\frac{d^4k}{(2\pi)^4}\ \big[-i\Delta_{\mathrm{F}}(k)\big]e^{i(x-y)\cdot k}\ , \tag{2.13}$$

where we have defined the *momentum-space Feynman propagator* Δ_{F} as

$$-i\Delta_{\mathrm{F}}(k) = \frac{-i}{k^2+m^2-i\epsilon}\ , \tag{2.14}$$

and where from here on out, the presence of an ϵ is always understood as in the limit $\epsilon \to 0^+$. In this work we are interested in the position-space representation of $G_{\rm F}$, so performing the $\mathbf{k}$-integration in (2.10) gives

$$G_{\rm F}(x;y) \;=\; \frac{1}{4\pi^2}\frac{m}{\sqrt{(x-y)^2+i\epsilon}}K_1\left(m\sqrt{(x-y)^2+i\epsilon}\right), \tag{2.15}$$

where K_ν refers to the order-ν modified Bessel function of the second kind [22]. We remind ourselves, that with our $\eta_{\mu\nu} = \mathrm{diag}(-+++)$ convention, the four-separation $(x-y)^2$ is negative for timelike separations and positive for spacelike separations. We note the following asymptotic forms for $z \in \mathbb{C}$

$$K_1(z) \;=\; \sqrt{\frac{\pi}{2z}}e^{-z}\left(1+\frac{3}{8z}+\cdots\right) \qquad \text{for } |z|\to\infty,\ \text{if } |\arg(z)| < \tfrac{3\pi}{2} \tag{2.16}$$

$$K_1(z) \;=\; \frac{1}{z}+\left(2\gamma-1+2\ln\left(\frac{z}{2}\right)\right)\frac{z}{4}+\ldots \qquad \text{for } |z|\to 0\ , \tag{2.17}$$

with γ the Euler-Mascheroni constant [25, 26].

With (2.16) we can compute the large-separation limit $|m^2(x-y)^2| \gg 1$ of the Feynman propagator as

$$G_{\rm F}(x;y) \;=\; \frac{1}{\sqrt{32\pi^3}}\frac{\sqrt{m}}{[(x-y)^2+i\epsilon]^{3/4}}e^{-\sqrt{m^2(x-y)^2+i\epsilon}} \tag{2.18}$$

And also important for us, we can use (2.17) to recover the *massless* limit $m \to 0$ of the Feynman propagator

$$G_{\rm F}(x;y)\big|_{m=0} \;=\; \frac{1}{4\pi^2}\frac{1}{(x-y)^2+i\epsilon} \tag{2.19}$$

2.2 ϕ^4 Corrections to the Massless Propagator

We'll now take our free theory and add an interaction. We consider a quartic interacting theory:

$$\mathscr{L}[\phi] \;=\; -\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 - \tfrac{\lambda}{4}\phi^4 \tag{2.20}$$

We suppose the dimensionless coupling is weak in the sense $0 < \lambda \ll 1$. In this work we focus on computing the full interacting time-ordered propagator for this theory

$$G(x;y) \equiv \langle\Omega_{\rm M}|\mathscr{T}\big(\hat{\phi}(x)\hat{\phi}(y)\big)|\Omega_{\rm M}\rangle \ , \tag{2.21}$$

where $|\Omega_{\rm M}\rangle$ is the vacuum of the interacting theory (which is distinct from the Minkowski vacuum $|0_{\rm M}\rangle$ of the free theory). There is really only one reliable way of computing this correlation function: through the use of *perturbation theory* — we give only a brief overview of how this works. The overall Hamiltonian for this theory is $H = H_{\rm M} + H_{\rm int}$, where $H_{\rm M}$ is the Minkowski Hamiltonian (2.5) and the interacting part is

$$H_{\rm int} \equiv \int d^3\mathbf{x}\ \frac{\lambda}{4}\phi(t,\mathbf{x})^4 \ , \tag{2.22}$$

where the integration is again performed over a slice of constant $x^0 = t$. With the above we may consider the field in the *interaction picture* $\hat{\phi}_{\mathcal{I}}$ given by:

$$\phi_{\mathcal{I}}(t,\mathbf{x}) \equiv e^{+iH_{\rm M}(t-t_0)}\phi(t_0,\mathbf{x})e^{-iH_{\rm M}(t-t_0)} \tag{2.23}$$

This has the utility that it can be expressed in a Minkowski mode expansion of the form (2.2). One can then introduce the interacting part of the Hamiltonian $H_{\rm int}$ expressed in the interaction picture:

$$H_{\mathcal{I}} \equiv e^{+iH_{\rm M}(t-t_0)}H_{\rm int}e^{-iH_{\rm M}(t-t_0)} \tag{2.24}$$

Equipped with above, standard manipulations of (2.21) leaves us with the following expression for the full propagator

$$G(x;y) = \lim_{T\to\infty-i\epsilon} \frac{\langle 0_{\rm M}|\mathscr{T}\left(\hat{\phi}_{\mathcal{I}}(x)\hat{\phi}_{\mathcal{I}}(y)e^{-i\int_{-T}^{T}dt\ H_{\mathcal{I}}}\right)|0_{\rm M}\rangle}{\langle 0_{\rm M}|\mathscr{T}\left(e^{-i\int_{-T}^{T}dt\ H_{\mathcal{I}}}\right)|0_{\rm M}\rangle} \ , \tag{2.25}$$

where we take $T \to \infty - i\epsilon$ in a slightly imaginary direction [27] so as to pick out the overlap $\langle\Omega_{\rm M}|0_{\rm M}\rangle$ with the Minkowski vacuum. The power of the expression (2.25) is that we can expand it in powers of the coupling λ, which leaves us with a series containing terms of the form

$$\langle 0_{\rm M}|\mathscr{T}\big(\hat{\phi}_{\mathcal{I}}(x_1)\cdots\hat{\phi}_{\mathcal{I}}(x_n)\big)|0_{\rm M}\rangle \ . \tag{2.26}$$

These can all be reduced to combinations of $G_{\mathrm{F}}(x;y)$ via *Wick's Theorem*. Using the above machinery we are able to compute the first few corrections to the propagator:

$$G(x;y) = G_{\mathrm{F}}(x;y) + G^{(1)}_{\text{Tadpole}}(x;y) + G^{(2)}_{\text{Sunset}}(x;y) + G^{(2)}_{\text{Cactus}}(x;y) + \dots \tag{2.27}$$

In the above $G^{(1)}_{\text{Tadpole}}$ is an $\mathscr{O}(\lambda)$ correction and $G^{(2)}_{\text{Sunset}}(x;y)$ and $G^{(2)}_{\text{Cactus}}(x;y)$ are $\mathscr{O}(\lambda^2)$. Even though there are four more contributions up to this order in λ, these graphs are the only *one-particle irreducible* (1PI) contributions — these are the only ones that we need to consider (more on this when we discuss the self-energy in Chapter 4).

The aforementioned contributions are given by the following integrals:

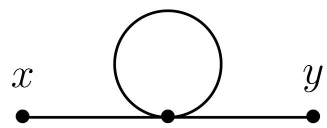
$$G^{(1)}_{\text{Tadpole}}(x;y) = 3(-i\lambda)\int d^4u \; G_{\mathrm{F}}(x;u)G_{\mathrm{F}}(u;u)G_{\mathrm{F}}(u;y) \tag{2.28}$$

$$G^{(2)}_{\text{Sunset}}(x;y) = (-i\lambda)^2\int d^4u \int d^4v \; G_{\mathrm{F}}(x;u)G_{\mathrm{F}}(u;v)^3G_{\mathrm{F}}(v;y) \tag{2.29}$$

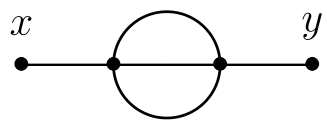
$$G^{(2)}_{\text{Cactus}}(x;y) = \frac{3(-i\lambda)^2}{2}\int d^4u \int d^4v \; G_{\mathrm{F}}(x;u)G_{\mathrm{F}}(u;v)^2G_{\mathrm{F}}(v;v)G_{\mathrm{F}}(u;y) \tag{2.30}$$

It is customary to represent these integrals as *Feynman diagrams*, where in this case:

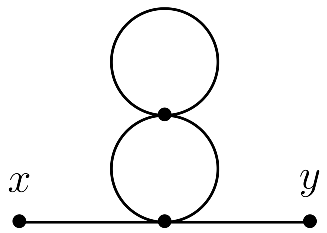
$$G^{(1)}_{\text{Tadpole}}(x;y) = \tag{2.31}$$



$$G^{(2)}_{\text{Sunset}}(x;y) = \tag{2.32}$$



$$G^{(2)}_{\text{Cactus}}(x;y) = \tag{2.33}$$



In this representation it is easy to explain what is meant by these contributions being 1PI — these are the graphs which cannot be split into two Feynman graphs by removing a single line [22]. There are non-1PI graphs that can be written down, such the "double-tadpole" diagram shown in the figure below. The non-1PI graphs

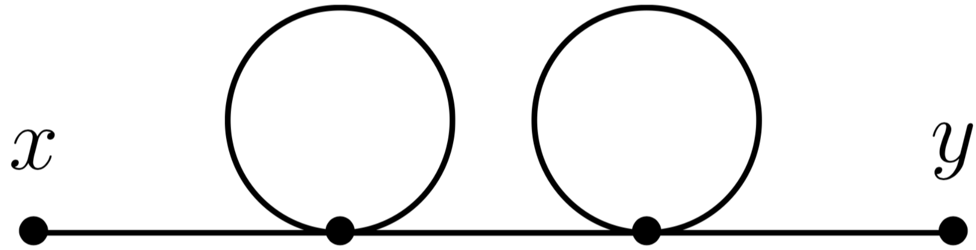


Figure 2.1: An example of a non-1PI Feynman graph. Snipping this graph with some scissors in the middle line will yield two tadpole graphs.

can be ignored when we appropriately resum 1PI diagrams using the self-energy.

It is now that the momentum-space Feynman propagator is very useful. With (2.14) the above integrals (2.28-2.30) simplify to:

$$G^{(1)}_{\text{Tadpole}}(x;y) = 3(-i\lambda)\mathcal{I}_{\text{Top}}(m)\int \frac{d^4p}{(2\pi)^4}\big[-i\Delta_{\text{F}}(p)\big]^2 e^{i(x-y)\cdot p} \tag{2.34}$$

$$G^{(2)}_{\text{Cactus}}(x;y) = \frac{3(-i\lambda)^2}{2}\mathcal{I}_{\text{Top}}(m)\mathcal{I}_{\text{Middle}}(m)\int \frac{d^4p}{(2\pi)^4}\big[-i\Delta_{\text{F}}(p)\big]^2 e^{i(x-y)\cdot p} \tag{2.35}$$

$$G^{(2)}_{\text{Sunset}}(x;y) = (-i\lambda)^2\int \frac{d^4p}{(2\pi)^4}\left[-i\Delta_{\text{F}}(p)\right]^2 \mathcal{I}_{\text{Sunset}}(p;m) \tag{2.36}$$

Where we have introduced the loop integrals:

$$\mathcal{I}_{\text{Top}}(m) \equiv \int \frac{d^4k}{(2\pi)^4}\left[-i\Delta_{\text{F}}(k)\right] \tag{2.37}$$

$$\mathcal{I}_{\text{Middle}}(m) \equiv \int \frac{d^4k}{(2\pi)^4}\left[-i\Delta_{\text{F}}(k)\right]^2 \tag{2.38}$$

$$\mathcal{I}_{\text{Sunset}}(p;m) \equiv \int \frac{d^4q}{(2\pi)^4}\int \frac{d^4k}{(2\pi)^4}\left[-i\Delta_{\text{F}}(k)\right]\left[-i\Delta_{\text{F}}(q)\right]\left[-i\Delta_{\text{F}}(p-k-q)\right] \tag{2.39}$$

Note that only $\mathcal{I}_{\text{Sunset}}(p)$ depends on the momentum p connecting the points x and y. Before examining the loops, we should note that we can very easily evaluate the Fourier transform present in (2.34) and (2.35) by differentiating the Feynman propagator with respect to m^2:

$$\begin{aligned}
\int \frac{d^4p}{(2\pi)^4}\big[-i\Delta_{\rm F}(p)\big]^2 e^{i(x-y)\cdot p} &= \int \frac{d^4p}{(2\pi)^4}\frac{-1}{(p^2+m^2-i\epsilon)^2}e^{i(x-y)\cdot p} && (2.40)\\
&= i\frac{\partial}{\partial(m^2)}\left\{\int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2-i\epsilon}e^{i(x-y)\cdot p}\right\} && (2.41)\\
&= i\frac{\partial}{\partial(m^2)}\left\{\frac{1}{4\pi^2}\frac{mK_1\big(m\sqrt{(x-y)^2+i\epsilon}\big)}{\sqrt{(x-y)^2+i\epsilon}}\right\} && (2.42)\\
&= -\frac{i}{8\pi^2}K_0\big(m\sqrt{(x-y)^2+i\epsilon}\big) && (2.43)
\end{aligned}$$

The loop integrals (2.37-2.39) all diverge in the (UV) as usual in an interacting theory. These infinities must be regularized and then we must renormalize our theory with appropriate counter-terms in the Lagrangian (2.20) to render them finite.

We use dimensional regularization to evaluate these loops. We Wick-rotate the energy variable to obtain a 4-dimensional Euclidean integral, whose dimension we then analytically continue to $D \in \mathbb{C}$ (in doing so, introducing a scale μ with mass dimension 1). An expansion about $D = 4$ then yields the following:

$$\begin{aligned}
\mathcal{I}_{\rm Top} &= \frac{m^2}{8\pi^2}\left[\frac{1}{4-D}+\frac{\gamma-1-\ln(4\pi)}{2}+\frac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right]+\mathscr{O}(4-D) && (2.44)\\
\mathcal{I}_{\rm Middle} &= -\frac{i}{8\pi^2}\frac{1}{4-D}+\frac{i}{16\pi^2}\left[\gamma-\ln(4\pi)+\ln\left(\frac{m^2}{\mu^2}\right)\right]+\mathscr{O}(4-D) && (2.45)
\end{aligned}$$

In this form, the $\frac{1}{4-D}$ poles parametrize our infinities (these are the terms that must be renormalized with counter-terms in the Lagrangian). Omitting terms $\mathscr{O}(4-D)$, we assume we've renormalized our theory appropriately and examine the finite parts of the above loops:

$$\begin{aligned}
\to\ \mathcal{I}^{\rm renormalized}_{\rm Top}(m) &= \frac{m^2}{16\pi^2}\left[\gamma-1-\ln(4\pi)+\ln\left(\frac{m^2}{\mu^2}\right)\right] && (2.46)\\
\to\ \mathcal{I}^{\rm renormalized}_{\rm Middle}(m) &= \frac{i}{16\pi^2}\left[\gamma-\ln(4\pi)+\ln\left(\frac{m^2}{\mu^2}\right)\right] && (2.47)
\end{aligned}$$

We pay special attention to what happens to these loops in the limit $m \to 0$:

$$\begin{aligned}
\lim_{m\to 0}\mathcal{I}^{\rm renormalized}_{\rm Top}(m) &= 0 && (2.48)\\
\lim_{m\to 0}\mathcal{I}^{\rm renormalized}_{\rm Middle}(m) &= \infty && (2.49)
\end{aligned}$$

We see that in the massless version of our theory, the Cactus graph acquires an additional IR divergence in its middle loop (which is 'soft' as it is only logarithmic in nature). However, since $\mathcal{I}^{\text{renormalized}}_{\text{Top}}(0) = 0$ is contained as a multiplicative factor in both $G^{(2)}_{\text{Tadpole}}(x;y)$ and $G^{(1)}_{\text{Cactus}}(x;y)$ we find that both of these graphs *vanish* in the massless limit

$$G^{(1)}_{\text{Tadpole}}(x;y) \;=\; G^{(2)}_{\text{Cactus}}(x;y) \;=\; 0 \tag{2.50}$$

Relative to a Minkowski observer we find that both of these corrections vanish for the massless theory. The first non-zero (1PI) correction in this setting will be the $\mathscr{O}(\lambda^2)$ graph, $G_{\text{sunset}}(x;y)$.

Chapter 3

Real Time QFT at Finite Temperature

We now suppose our real scalar field is in contact with a heat bath held at fixed temperature $T \equiv \frac{1}{\beta}$. In this chapter we wish to explicitly track the time-dependence of the thermal time-ordered two-point correlation functions — we do this using *Thermo field dynamics*. We will find that the thermal corrections analogous to those examined at the end of the previous chapter exhibit secular growth.

3.1 Thermality in Quantum Mechanics

Before we speak of a thermal QFT let's slow things down and discuss the notion of thermality in quantum mechnics with the goal of building some intuition about thermal quantum systems. Let $\mathcal{H}$ be an N-dimensional Hilbert space whose states describe some quantum system, with its dynamics governed by some Hamiltonian $\hat{H}$. This means that there exists an orthonormal energy-eigenbasis $\{|n\rangle\}_{n=1}^{N} \subset \mathcal{H}$ such that:

$$\hat{H}\,|n\rangle = E_n\,|n\rangle \tag{3.1}$$

$$\langle n|m\rangle = \delta_{nm} \tag{3.2}$$

For simplicity we assume that the Hamiltonian is time-independent so that the time-evolution operator takes the form $\hat{U}(t) = e^{-it\hat{H}}$. We also recall the form of the thermal density matrix for a canonical ensemble in thermal equilibrium with a heat bath:

$$e^{-\beta\hat{H}} \tag{3.3}$$

We notice that $\text{Tr}\left[e^{-\beta\hat{H}}\right] = \sum_{n=1}^{N} e^{-\beta E_n} = \mathcal{Z}$ is precisely the partition function, encoding all the thermal statistics for our system. For any observable $\hat{O}$, we have the result

$$\frac{\text{Tr}\left[e^{-\beta\hat{H}}\hat{O}\right]}{\text{Tr}\left[e^{-\beta\hat{H}}\right]} = \frac{\sum_{n=1}^{N} \langle n|\hat{O}|n\rangle\, e^{-\beta E_n}}{\mathcal{Z}} \equiv \langle\hat{O}\rangle_\beta\ , \tag{3.4}$$

where we have introduced the notation $\langle\hat{O}\rangle_\beta$ for the thermal average of the operator $\hat{O}$. We see that we are able to reproduce all of the desired thermal statistics for our finite-dimensional system by manipulating the operator $e^{-\beta\hat{H}}$.

Before moving on we notice the striking similarity between the time-evolution operator $e^{-it\hat{H}}$ and the thermal density matrix $e^{-\beta\hat{H}}$. In some naïve sense $e^{-\beta\hat{H}}$ is the time evolution operator evaluated at the imaginary time $t = -i\beta$. As it turns out, this funny interpretation holds great utility in thermal quantum physics.

3.2 KMS States and Doubling the Degrees of Freedom

Continuing on with our finite dimensional system, consider two observables $\hat{A}$ and $\hat{B}$ written in the Schrödinger picture. According to (3.4) their thermal average is given by:

$$\langle\hat{A}\hat{B}\rangle_\beta = \frac{\text{Tr}\left[e^{-\beta\hat{H}}\hat{A}\hat{B}\right]}{\text{Tr}\left[e^{-\beta\hat{H}}\right]} \tag{3.5}$$

However the above thermal correlation function omits any dependence on time. If we switch to the Heisenberg picture so that the the operators are now explicitly time-dependent with

$$\hat{A}_{\rm H}(t) = e^{+it\hat{H}}\hat{A}e^{-it\hat{H}} \tag{3.6}$$

$$\hat{B}_{\rm H}(t) = e^{+it\hat{H}}\hat{B}e^{-it\hat{H}}\ , \tag{3.7}$$

then the thermal correlation function for Heisenberg operators $\hat{A}_{\rm H}(t_1)$ and $\hat{B}_{\rm H}(t_2)$ at distinct times $t_1 \neq t_2$ is

$$\langle \hat{A}_{\rm H}(t_1)\hat{B}_{\rm H}(t_2)\rangle_\beta \;=\; \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{A}_{\rm H}(t_1)\hat{B}_{\rm H}(t_2)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]}\ . \tag{3.8}$$

The above expression can be manipulated by insertion of $\hat{\mathbb{I}} = e^{+\beta\hat{H}}e^{-\beta\hat{H}}$, identifying β as an imaginary time and then using the cyclic property of the trace [28]:

$$\begin{aligned}
\langle \hat{A}_{\rm H}(t_1)\hat{B}_{\rm H}(t_2)\rangle_\beta &= \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{A}_{\rm H}(t_1)e^{+\beta\hat{H}}e^{-\beta\hat{H}}\hat{B}_{\rm H}(t_2)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]} &(3.9)\\
&= \frac{\mathrm{Tr}\left[e^{+i(+i\beta)\hat{H}}\hat{A}_{\rm H}(t_1)e^{-i(+i\beta)\hat{H}}e^{-\beta\hat{H}}\hat{B}_{\rm H}(t_2)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]} &(3.10)\\
&= \frac{\mathrm{Tr}\left[\hat{A}_{\rm H}(t_1+i\beta)e^{-\beta\hat{H}}\hat{B}_{\rm H}(t_2)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]} &(3.11)\\
&= \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\hat{B}_{\rm H}(t_2)\hat{A}_{\rm H}(t_1+i\beta)\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]} &(3.12)
\end{aligned}$$

We have arrived at the *Kubo-Martin-Schwinger (KMS) condition*:

$$\langle \hat{A}_{\rm H}(t_1)\hat{B}_{\rm H}(t_2)\rangle_\beta = \langle \hat{B}_{\rm H}(t_2)\hat{A}_{\rm H}(t_1+i\beta)\rangle_\beta \tag{3.13}$$

A quantum state which yields correlations between *any* operators $\hat{A}$ and $\hat{B}$ such that (3.13) is satisfied, is called a *KMS state*. This defines a thermal state. In fact, this was the starting point in Schwinger and Martin's paper [29] where they used the above property as the defining property for thermodynamic Green's functions. In principle, were we given some arbitrary system we could check whether the correlation functions of the theory obey the KMS condition to check whether the system is thermal. It turns out that KMS states reproduce all the required thermal statistics, and is more versatile as a definition of thermality. For this reason, an easy way to check for thermality in a system is to check if the KMS conditions is satisfied.

Since we've been referring to a 'KMS state', this naturally begs the question: does there exist a state $|\beta\rangle \in \mathcal{H}$ which allows us to write the thermal average as an expectation value in some 'thermal vacuum state'? This would have the defining property

$$\langle\beta|\hat{O}|\beta\rangle = \langle\hat{O}\rangle_\beta \ . \tag{3.14}$$

Such a state $|\beta\rangle$ would be our KMS state.

The answer to this question is **no**, not if $|\beta\rangle$ is restricted to be in our original Hilbert space $\mathcal{H}$. To see why, suppose there did exist such a $|\beta\rangle \in \mathcal{H}$ and expand it in terms of the orthonormal energy-eigenbasis as $|\beta\rangle = \sum_{n=1}^{N} b_n |n\rangle$. This leads to the expectation value $\langle\beta|\hat{O}|\beta\rangle = \sum_{n,m=1}^{N} b_m^* b_n \langle n|\hat{O}|m\rangle$. Comparing this expectation value to (3.4), we find that (3.14) implies a condition $b_m^* b_n = \delta_{nm}\frac{e^{-\beta E_n}}{\mathcal{Z}}$ — but since the coefficients b_n are simply complex numbers, it's impossible to satisfy this condition. The point is this: as long as we restrict ourselves to states in the original Hilbert space $\mathcal{H}$ we can't define a thermal vacuum state with the property (3.14).

But we're given a hint as to how to define such a state $|\beta\rangle$ since we somehow need to get a trace from an expectation value. We consider augmenting our Hilbert space with an *identical copy of itself* so that we work overall on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$ where the overall Hamiltonian for the system is $\hat{H} \otimes \hat{\mathbb{I}} + \hat{\mathbb{I}} \otimes \hat{H}$ [28]. We now define the state $|\beta\rangle \in \mathcal{H} \otimes \mathcal{H}$ as

$$|\beta\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \sum_{n=1}^{N} e^{-\frac{\beta E_n}{2}} |n\rangle \otimes |n\rangle \tag{3.15}$$

For any observable $\hat{O}$ on our original Hilbert space $\mathcal{H}$ we now can compute the following:

$$
\begin{aligned}
\langle\beta|\hat{O}\otimes\mathbb{I}|\beta\rangle &= \left(\frac{1}{\sqrt{\mathcal{Z}}}\sum_{n=1}^{N} e^{-\frac{\beta E_n}{2}}\langle n|\otimes\langle n|\right)\left(\hat{O}\otimes\mathbb{I}\right)\left(\frac{1}{\sqrt{\mathcal{Z}}}\sum_{m=1}^{N} e^{-\frac{\beta E_m}{2}}|m\rangle\otimes|m\rangle\right) && (3.16)\\
&= \frac{1}{\mathcal{Z}}\sum_{n,m=1}^{N} e^{-\frac{\beta E_n}{2}-\frac{\beta E_m}{2}}\langle n|\hat{O}|m\rangle\langle n|m\rangle && (3.17)\\
&= \frac{1}{\mathcal{Z}}\sum_{n=1}^{N} e^{-\beta E_n}\langle n|\hat{O}|n\rangle && (3.18)\\
&= \langle\hat{O}\rangle_\beta && (3.19)
\end{aligned}
$$

So $|\beta\rangle \in \mathcal{H}\otimes\mathcal{H}$ is a state which reproduces the required thermal statistical averages: all it took was to *double the degrees of freedom* in our problem and the construction was obvious.

The fact that extra degrees of freedom need to be introduced should not come as a complete surprise — there are definitely extra 'hidden' physics that we don't have access to when we assume that we're in thermal equilbrium with a heat bath. This is the coarse-grained thermodynamic description: all of the physics which exchanges energy with our system is lumped into one package and called 'the heat bath'. It's in this abstract way that the second Hilbert space $\mathcal{H}$ represents the thermal system we're interacting and exchanging energy with.

In this finite-dimensional cartoon of thermal physics it may seem that defining a thermal vacuum state $|\beta\rangle$ is somewhat useless — after all, we know exactly how to compute all the relevant thermal averages, so what's the point?

The point is that when dealing with a thermal QFT this doubling of the degrees of freedom is an extremely useful construct. By thinking of the propagators of the theory as an expectation value in some thermal vacuum state $|\beta\rangle$ we are able to use well-established (diagrammatic) techniques familiar from zero-temperature QFT. Defining the thermal vacuum state is straightforward once one doubles the degrees of freedom as we've done here — this means for every field we'd like to do physics with, we need to introduce an extra field.

3.3 The Free Theory in Thermo field Dynamics

In this section we provide a brief overview of Takahashi and Umezawa's *thermo field dynamics* [30], a real time QFT at finite temperature. This is a variant of the Schwinger-Keldysh finite temperature field theory [31, 32] introduced in the early 60's (to which it is physically equivalent). *Real time* finite temperature field theories all involve a doubling of the degrees of freedom analogous to what we saw in the previous section (this means introducing a second field).

It's worth mentioning that simpler *imaginary time* formalisms for finite temperature field theory do exist — namely, the Euclidean Matsubara formalism. Here one dispenses of the second field, but loses explicit time dependence. Imaginary time formalisms are generally more useful for computing thermodynamic quantities (for a wealth of textbook discussions, see [28, 33, 34, 35, 36, 37]).

Here we'll take the real scalar field from Chapter 2, now calling it ϕ_1, and put it in contact with a heat bath held at fixed temperature $\frac{1}{\beta}$. Inspired by the previous section, we'll double the degrees of freedom and make a copy ϕ_2 of the field. This second field is often called the *thermal ghost field* as it is entirely fictitious and there to represent the influence of the heat bath on our *physical field* ϕ_1. Here we treat everything in flat Minkowski space in ordinary rectangular coordinates $x = (x^0, \mathbf{x})$, with the metric $\eta_{\mu\nu}$ and we take the Lagrangian density for the free theory to be:

$$\mathscr{L}_0[\phi_1, \phi_2] \;=\; -\tfrac{1}{2}(\partial^\mu \phi_1)(\partial_\mu \phi_1) - \tfrac{1}{2}m^2\phi_1^2 + \tfrac{1}{2}(\partial^\mu \phi_2)(\partial_\mu \phi_2) + \tfrac{1}{2}m^2\phi_2^2 \tag{3.20}$$

The relative minus sign between the ϕ_1 and ϕ_2 terms in the Lagrangian density yield a desirable propagator structure as we will see in the next section [28]. Varying the action $S = d^4x\ \mathscr{L}[\phi_1, \phi_2](x)$ with respect to each of the fields leads to two equations of motion:

$$\left[-\left(\tfrac{\partial}{\partial x^0}\right)^2 + \boldsymbol{\nabla}^2_{\mathbf{x}} - m^2\right]\phi_1(x^0,\mathbf{x}) \;=\; 0 \tag{3.21}$$

$$\left[-\left(\tfrac{\partial}{\partial x^0}\right)^2 + \boldsymbol{\nabla}^2_{\mathbf{x}} - m^2\right]\phi_2(x^0,\mathbf{x}) \;=\; 0 \tag{3.22}$$

So we simply have two copies of the Klein-Gordon equation. The fields are independent from one another, and lucky for us we already know exactly how to quantize a free real scalar field. As in (2.2), an expansion in terms of the Minkowski modes yields for each of the fields:

$$\hat{\phi}_1(x) \;=\; \int d^3\mathbf{k}\;\left[u_{\mathbf{k}}(x)\hat{a}^{\mathrm{P}}_{\mathbf{k}} + u^*_{\mathbf{k}}(x)\hat{a}^{\mathrm{P}\dagger}_{\mathbf{k}}\right] \tag{3.23}$$

$$\hat{\phi}_2(x) \;=\; \int d^3\mathbf{k}\;\left[u_{\mathbf{k}}(x)\hat{a}^{\mathrm{G}}_{\mathbf{k}} + u^*_{\mathbf{k}}(x)\hat{a}^{\mathrm{G}\dagger}_{\mathbf{k}}\right] \tag{3.24}$$

The raising and lowering operators $\{\hat{a}^{\mathrm{P}},\hat{a}^{\mathrm{P}\dagger}\}$ for the physical field ϕ_1 and $\{\hat{a}^{\mathrm{G}},\hat{a}^{\mathrm{G}\dagger}\}$ for the thermal ghost field ϕ_2 obey their own set of bosonic canonical commutation relations:

$$\begin{aligned}
\left[\hat{a}^{\mathrm{P}}_{\mathbf{k}},\hat{a}^{\mathrm{P}\dagger}_{\mathbf{p}}\right] &= \delta^{(3)}(\mathbf{k}-\mathbf{p}) &\qquad \left[\hat{a}^{\mathrm{G}}_{\mathbf{k}},\hat{a}^{\mathrm{G}\dagger}_{\mathbf{p}}\right] &= \delta^{(3)}(\mathbf{k}-\mathbf{p}) \\
\left[\hat{a}^{\mathrm{P}}_{\mathbf{k}},\hat{a}^{\mathrm{P}}_{\mathbf{p}}\right] &= 0 &\qquad \left[\hat{a}^{\mathrm{G}}_{\mathbf{k}},\hat{a}^{\mathrm{G}}_{\mathbf{p}}\right] &= 0 \\
\left[\hat{a}^{\mathrm{P}\dagger}_{\mathbf{k}},\hat{a}^{\mathrm{P}\dagger}_{\mathbf{p}}\right] &= 0 &\qquad \left[\hat{a}^{\mathrm{G}\dagger}_{\mathbf{k}},\hat{a}^{\mathrm{G}\dagger}_{\mathbf{p}}\right] &= 0
\end{aligned} \tag{3.25}$$

The operators are independent of one another in the sense that

$$\begin{aligned}
\left[\hat{a}^{\mathrm{P}}_{\mathbf{k}},\hat{a}^{\mathrm{G}}_{\mathbf{p}}\right] &= 0 &\qquad \left[\hat{a}^{\mathrm{P}\dagger}_{\mathbf{k}},\hat{a}^{\mathrm{G}}_{\mathbf{p}}\right] &= 0 \\
\left[\hat{a}^{\mathrm{P}}_{\mathbf{k}},\hat{a}^{\mathrm{G}\dagger}_{\mathbf{p}}\right] &= 0 &\qquad \left[\hat{a}^{\mathrm{P}\dagger}_{\mathbf{k}},\hat{a}^{\mathrm{G}\dagger}_{\mathbf{p}}\right] &= 0
\end{aligned} \tag{3.26}$$

This manifests itself in the fields commuting with each other. As a consequence of the above, there exists a symmetric Fock space corresponding to each of the fields; one built out of a vacuum state $|0^{\mathrm{P}}\rangle$ for the physical field, and the other from a vacuum state $|0^{\mathrm{G}}\rangle$ for the thermal ghost field. The overall space state for the system is the tensor product of these two Fock spaces, where the vacuum for the overall system is $|0^{\mathrm{G}}\rangle \otimes |0^{\mathrm{G}}\rangle \equiv |0^{\mathrm{P}},0^{\mathrm{G}}\rangle$.

We also note that the overall Hamiltonian is

$$
\begin{aligned}
H &= H^{\rm P} - H^{\rm G} && (3.27)\\
H^{\rm P} &= \int d^3\mathbf{k}\frac{\sqrt{|\mathbf{k}|^2+m^2}}{2}\left[\hat{a}^{\rm P}\hat{a}^{\rm P\dagger} + \hat{a}^{\rm P\dagger}\hat{a}^{\rm P}\right] && (3.28)\\
H^{\rm G} &= \int d^3\mathbf{k}\frac{\sqrt{|\mathbf{k}|^2+m^2}}{2}\left[\hat{a}^{\rm G}\hat{a}^{\rm G\dagger} + \hat{a}^{\rm G\dagger}\hat{a}^{\rm G}\right] && (3.29)
\end{aligned}
$$

We'd like to consider what it would look like for our field ϕ_1 to be in thermal equilibrium with a heat bath held at temperature $\frac{1}{\beta}$. We know that the partition function will schematically have the form

$$
\mathcal{Z} = \mathrm{Tr}\left[e^{-\beta\hat{H}^{\rm P}}\right] \ , \quad (3.30)
$$

and from this we'd like to evaluate thermal averages of observables for the physical ϕ_1 field via

$$
\langle\hat{\mathcal{O}}^{\rm P}\rangle = \mathrm{Tr}\left[e^{-\beta\hat{H}^{\rm P}}\hat{\mathcal{O}}^{\rm P}\right] \ . \quad (3.31)
$$

We seek to construct a thermal vacuum state $|\beta\rangle$ which has the property

$$
\langle\beta|\hat{\mathcal{O}}^{\rm P}\otimes\mathbb{I}^{\rm G}|\beta\rangle = \mathrm{Tr}\left[e^{-\beta\hat{H}^{\rm P}}\hat{\mathcal{O}}^{\rm P}\right] \quad (3.32)
$$

The construction is quite involved, and since we only care about an overview of our thermal field theory we skip ahead an say that such a $|\beta\rangle$ *does* exist and can be explicitly constructed. In particular, there exists a set of *thermal* raising and lowering operators

$$
\{\hat{\beta}^{\rm P}_{\mathbf{k}}, \hat{\beta}^{\rm P\dagger}_{\mathbf{k}}, \hat{\beta}^{\rm G}_{\mathbf{k}}, \hat{\beta}^{\rm G\dagger}_{\mathbf{k}}\} \quad (3.33)
$$

which have the special property that they annihilate the thermal vacuum $|\beta\rangle$ with

$$
\begin{aligned}
\hat{\beta}^{\rm P}_{\mathbf{k}}\,|\beta\rangle &= 0 && (3.34)\\
\hat{\beta}^{\rm G}_{\mathbf{k}}\,|\beta\rangle &= 0 \ . && (3.35)
\end{aligned}
$$

The operators $\{\hat{\beta}^{\rm P}_{\mathbf{k}}, \hat{\beta}^{\rm P\dagger}_{\mathbf{k}}, \hat{\beta}^{\rm G}_{\mathbf{k}}, \hat{\beta}^{\rm G\dagger}_{\mathbf{k}}\}$ and $\{\hat{a}^{\rm P}_{\mathbf{k}}, \hat{a}^{\rm P\dagger}_{\mathbf{k}}, \hat{a}^{\rm G}_{\mathbf{k}}, \hat{a}^{\rm G\dagger}_{\mathbf{k}}\}$ are be related to one another through the following Bogoliubov transformation [39]:

$$\hat{\beta}^{\rm P}_{\mathbf{k}} = \frac{1}{\sqrt{1 - e^{-\beta\sqrt{|\mathbf{k}|^2+m^2}}}}\, \hat{a}^{\rm P}_{\mathbf{k}} - \frac{1}{\sqrt{e^{\beta\sqrt{|\mathbf{k}|^2+m^2}} - 1}}\, \hat{a}^{\rm G\dagger}_{\mathbf{k}} \tag{3.36}$$

$$\hat{\beta}^{\rm G}_{\mathbf{k}} = \frac{1}{\sqrt{1 - e^{-\beta\sqrt{|\mathbf{k}|^2+m^2}}}}\, \hat{a}^{\rm G}_{\mathbf{k}} - \frac{1}{\sqrt{e^{\beta\sqrt{|\mathbf{k}|^2+m^2}} - 1}}\, \hat{a}^{\rm P\dagger}_{\mathbf{k}} \tag{3.37}$$

We see that a mixture of raising and lowering operators takes place — this is a manifestation of the fact that the vacuua $|\beta\rangle$ and $|0^{\rm P}, 0^{\rm G}\rangle$ and are inequivalent. It follows from the above information that

$$\langle\beta|\hat{a}^{\rm P\dagger}_{\mathbf{p}}\hat{a}^{\rm P}_{\mathbf{k}}|\beta\rangle = \frac{\delta^{(3)}(\mathbf{p}-\mathbf{k})}{e^{\beta\sqrt{|\mathbf{k}|^2+m^2}} - 1} \tag{3.38}$$

We see that evaluated in the thermal state, the number of modes with momentum $\mathbf{k}$ are suppressed by a Bose-Einstein distribution.

3.4 Thermal Propagators

Using our knowledge from Chapter 2, we can easily write down the time-ordered expectation values $\langle 0^{\rm P}, 0^{\rm G}|\mathscr{T}\big(\phi_a(x)\phi_b(y)\big)|0^{\rm P}, 0^{\rm G}\rangle$. It will be convenient to arrange these objects in a matrix:

$$\mathbb{G}^0(x;y) = \begin{bmatrix} \langle 0^{\rm P}, 0^{\rm G}|\mathcal{T}\big(\phi_1(x)\phi_1(y)\big)|0^{\rm P}, 0^{\rm G}\rangle & \langle 0^{\rm P}, 0^{\rm G}|\mathcal{T}\big(\phi_1(x)\phi_2(y)\big)|0^{\rm P}, 0^{\rm G}\rangle \\ \langle 0^{\rm P}, 0^{\rm G}|\mathcal{T}\big(\phi_2(x)\phi_1(y)\big)|0^{\rm P}, 0^{\rm G}\rangle & \langle 0^{\rm P}, 0^{\rm G}|\mathcal{T}\big(\phi_2(x)\phi_2(y)\big)|0^{\rm P}, 0^{\rm G}\rangle \end{bmatrix} \tag{3.39}$$

Following the treatment from section 2.3, the above takes on the momentum-space representation:

$$\mathbb{G}^0(x;y) = \int \frac{d^4k}{(2\pi)^4} \begin{bmatrix} \frac{-i}{k^2+m^2-i\epsilon} & 0 \\ 0 & \frac{i}{k^2+m^2+i\epsilon} \end{bmatrix} e^{i(x-y)\cdot k}\,, \tag{3.40}$$

where there are obviously no correlations between the fields ϕ_1 and ϕ_2 since we're evaluating in the vacuum $|0^{\rm P}, 0^{\rm G}\rangle$. We note that the (22)-component of the above matrix is the complex conjugate of the (11)-component.

We are however not very interested in the above propagators — instead we seek to compute $\mathcal{T}\big(\phi_1(x)\phi_1(y)\big)$ evaluated in the *thermal vacuum* $|\beta\rangle$. We define:

$$\mathbb{G}^\beta(x;y) = \begin{bmatrix} \langle\beta|\mathcal{T}\big(\phi_1(x)\phi_1(y)\big)|\beta\rangle & \langle\beta|\mathcal{T}\big(\phi_1(x)\phi_2(y)\big)|\beta\rangle \\ \langle\beta|\mathcal{T}\big(\phi_2(x)\phi_1(y)\big)|\beta\rangle & \langle\beta|\mathcal{T}\big(\phi_2(x)\phi_2(y)\big)|\beta\rangle \end{bmatrix} \tag{3.41}$$

We take special interest in the momentum space formulation for these propagators

$$\mathbb{G}^\beta(x;y) = \int \frac{d^4k}{(2\pi)^4} \begin{bmatrix} -i\Delta^\beta_{11}(k) & -i\Delta^\beta_{12}(k) \\ -i\Delta^\beta_{21}(k) & -i\Delta^\beta_{22}(k) \end{bmatrix} e^{i(x-y)\cdot k} \tag{3.42}$$

Using the elementary relations between $\{\hat{a}^{\mathrm{P}}_{\mathbf{k}}, \hat{a}^{\mathrm{P}\dagger}_{\mathbf{k}}, \hat{a}^{\mathrm{G}}_{\mathbf{k}}, \hat{a}^{\mathrm{G}\dagger}_{\mathbf{k}}\}$ and $\{\hat{\beta}^{\mathrm{P}}_{\mathbf{k}}, \hat{\beta}^{\mathrm{P}\dagger}_{\mathbf{k}}, \hat{\beta}^{\mathrm{G}}_{\mathbf{k}}, \hat{\beta}^{\mathrm{G}\dagger}_{\mathbf{k}}\}$ quoted in the previous section, the action of the fields on the state $|\beta\rangle$ can be related to that on $|0^{\mathrm{P}}, 0^{\mathrm{G}}\rangle$ which eventually yields

$$-i\Delta^\beta_{11}(p) = \frac{-i}{k^2+m^2-i\epsilon} + \frac{2\pi\delta\big(k^2+m^2\big)}{e^{\beta|k^0|}-1} \tag{3.43}$$

$$-i\Delta^\beta_{12}(p) = -i\Delta^\beta_{21}(p) = \frac{\pi\delta\big(k^2+m^2\big)}{\sinh\left(\frac{1}{2}\beta|k^0|\right)} \tag{3.44}$$

$$-i\Delta^\beta_{22}(p) = \frac{i}{k^2+m^2+i\epsilon} + \frac{2\pi\delta\big(k^2+m^2\big)}{e^{\beta|k^0|}-1} \tag{3.45}$$

The physical propagator we care about is of course $\Delta^\beta_{11}(p)$. What is striking about this structure is that there is a clear divide between the zero-temperature and finite-temperature contributions to the propagator. There are now also non-trivial correlations between the physical system and the heat bath — this is represents the exchange of energy constantly taking place. Interestingly, in the zero-temperature limit $\beta \to \infty$, the physical propagator $\Delta^\beta_{11}(p)$ becomes precisely the Feynman propagator from Chapter 2 and the correlations with the heat bath turn off.

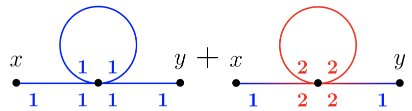
3.5 Interactions in Thermo field Dynamics

We consider once again the quartic interacting theory, except now we must add a corresponding interacting term for the thermal ghost field as well:

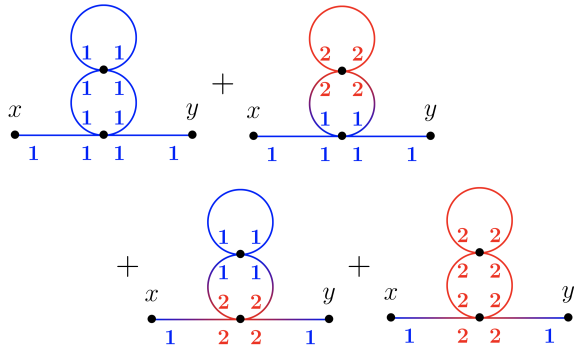
$$\begin{aligned} \mathscr{L}[\phi_1,\phi_2] = {} & -\tfrac{1}{2}(\partial^\mu\phi_1)(\partial_\mu\phi_1) - \tfrac{1}{2}m^2\phi_1^2 - \tfrac{\lambda}{4}\phi_1^4 \\ & +\tfrac{1}{2}(\partial^\mu\phi_2)(\partial_\mu\phi_2) + \tfrac{1}{2}m^2\phi_2^2 + \tfrac{\lambda}{4}\phi_2^4 \end{aligned} \tag{3.46}$$

We see that the fields are only self-interacting at the level of the Lagrangian. However, the two fields interact with each other on account of the non-trivial correlations functions (3.44). We are interested in corrections to the physical propagator G^{β}_{11}, which we again compute using perturbation theory. There are two corrections we're interested in

$$G^{\beta}_{\text{Tadpole}}(x;y)_{11} = \tag{3.47}$$



$$G^{\beta}_{\text{Cactus}}(x;y)_{11} = \tag{3.48}$$



These have the same topology as the diagrams considered in Chapter 2, but we notice the complication that there are *more* diagrams introduced due to the matrix structure of the thermal propagator (a point labelled by a attached to a point labelled by b represents G^{β}_{ab}).

Here we only state results and demonstrate explicit calculations in Appendix C. We consider the *massless* theory $m \to 0$, and compute these corrections with $\mathbf{x} = \mathbf{y}$ and find that asymptotically in the limit $\frac{|x^0 - y^0|}{\beta} \gg 1$ these corrections become

$$G^{\beta}_{\text{Tadpole}}(x^0, \mathbf{x}; y^0, \mathbf{x})_{11} = \frac{1}{32\pi}\frac{\lambda|x^0 - y^0|}{\beta^3} + \text{subdominant} \tag{3.49}$$

$$G^{\beta}_{\text{Cactus}}(x^0, \mathbf{x}; y^0, \mathbf{x})_{11} = \frac{-i}{64\pi}\mathcal{M}\frac{\lambda^2|x^0 - y^0|}{\beta^3} + \text{subdominant} \tag{3.50}$$

This is our first encounter with secular growth. No matter how small we make $0 < \lambda \ll 1$, these functions diverge in the considered limit. In addition to this, the finite part of the middle loop $\mathcal{M}$ of the cactus diagram has a finite part given by

$$\to \mathcal{M}^{\text{renormalized}} = -\frac{i}{8\pi}\frac{1}{\beta m} + \frac{i}{16\pi^2}\ln\left(\frac{4\pi}{\beta^2\mu^2}\right) - \frac{i\gamma}{16\pi^2}\,. \tag{3.51}$$

We see that the initially 'soft' IR divergence encountered in Chapter 2 has been made more severe by the presence of the heat bath — here we see an enhanced *power-law IR divergence.* This is a well-known [40] simple example of a thermal IR problem. The need to resum the IR parts of such graphs is the source of hard thermal-loop effects in [2, 3] which eventually lead to a fractional power dependence on λ — this non-analyticity in λ incidentally also reveals a breakdown of expansions in powers of the coupling.

To re-iterate, the point of this chapter was to provide an overview of thermal field theory with the intent of computing the thermal corrections (3.49) and (3.50). We see that these corrections exhibit secular growth. In the next chapter, it will be shown that a Rindler observer uniformly accelerating through flat Minkowski space perceives the Minkowski vacuum $|0_{\mathrm{M}}\rangle$ as a thermal state (totally analogous to the state $|\beta\rangle$ considered in this chapter). The duality between the physics described by the Rindler observer and the thermal physics of this chapter will be developed in Chapter 4, with the goal of computing the tadpole correction from the point of view of a Rindler observer — interestingly, we find the *same* behaviour (3.49) there.

Chapter 4

The Rindler Observer's QFT

In this chapter the free scalar field ϕ in Minkowski space is quantized relative to a different notion of time — that of a Rindler observer, who is shown to percieve the Minkowski vacuum state as a thermal state. It is shown that the Rindler observer's time-ordered two-point correlation function exhibits secular growth in a manner analogous to that seen in Chapter 3. This is the manifestation of a UV/IR interplay that takes place in how the Rindler observer decides to renormalize his tadpole loop. A near-horizon resummation is then performed which yields the correct late-time behaviour, followed by a discussion on other graphs.

4.1 Rindler Coordinates and Accelerated World-lines

In the previous sections we've been describing Minkowski space using the rectangular coordinates (x^0, x^1, x^2, x^3) where the line-element is simply $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$. Now we consider a transformation in the first two tuples $(x^0, x^1) \mapsto (\eta, \xi)$ where:

$$x^0 = \xi \sinh(\eta) \tag{4.1}$$

$$x^1 = \xi \cosh(\eta) \tag{4.2}$$

In these coordinates the line-element takes the form

$$ds^2 = -\xi^2 d\eta^2 + d\xi^2 + (dx^2)^2 + (dx^3)^2 \; . \tag{4.3}$$

So the metric is now $g_{\mu\nu} = \text{diag}\,(-\xi^2, 1, 1, 1)$ and we have $\sqrt{-\det(g)} = |\xi|$. Letting ξ and η take on any values in $\mathbb{R}$ we notice that these coordinates cover only the following two portions of spacetime:

$$\mathcal{R}_+ = \left\{ \, (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \mid x^1 > |x^0| \right\} \tag{4.4}$$

$$\mathcal{R}_- = \left\{ \, (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \mid x^1 < -|x^0| \right\} \tag{4.5}$$

We call $\mathcal{R}_+$ the *right Rindler wedge* (equivalently, this is the set where $\eta \in \mathbb{R}$ and $\xi \geq 0$) and $\mathcal{R}_-$ the *left Rindler wedge* (where $\eta \in \mathbb{R}$ and $\xi < 0$). The remainder of Minkowski space we partition into three regions; the *future wedge* $\mathcal{F}$, the *past wedge* $\mathcal{P}$, and the *null hyperplane* $\mathcal{N}$:

$$\mathcal{F} = \left\{ \, (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \mid x^0 > |x^1| \right\} \tag{4.6}$$

$$\mathcal{P} = \left\{ \, (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \mid x^0 < -|x^1| \right\} \tag{4.7}$$

$$\mathcal{N} = \left\{ \, (x^0, x^1, x^2, x^3) \in \mathbb{R}^4 \mid |x^0| = |x^1| \right\} \tag{4.8}$$

It's worth noting that taking $\eta \to \pm\infty$ or $\xi \to 0^\pm$ takes one to the null hypersurface $\mathcal{N}$. We notice that the Rindler metric $g_{\mu\nu}$ does not depend on η, so we immediately know that the vector $\mathcal{B} = \frac{\partial}{\partial \eta}$ is a *timelike Killing vector* with components $\mathcal{B}^\mu = \delta^\mu{}_0$ (in Rindler coordinates). We can write this vector in terms of rectangular (x^0, x^1, x^2, x^3)-coordinates where $\mathcal{B} = x^1 \frac{\partial}{\partial x^0} + x^0 \frac{\partial}{\partial x^1}$. In this form, we recognize $\mathcal{B}$ a the generator of boosts (in the 1-direction). We also compute $\mathcal{B}\cdot\mathcal{T} = \eta_{\mu\nu}\mathcal{B}^\mu\mathcal{T}^\nu = -x^1$, which tells us that B is *future-oriented* in $\mathcal{R}_+$ (where $x^1 > 0$) and *past-oriented* in $\mathcal{R}_-$ (where $x^1 < 0$) with respect to $\mathcal{T}$.

The orbits of this vector $\mathcal{B}$ are timelike worldlines of constant ξ, x^2 and x^3. In particular, if we parametrize such a worldline with the proper time τ, picking constants $\xi(\tau) = \frac{1}{a} > 0$, $x^2(\tau) = y \in \mathbb{R}$ and $x^3(\tau) = z \in \mathbb{R}$ we find that

$$\eta(\tau) = a\tau \; . \tag{4.9}$$

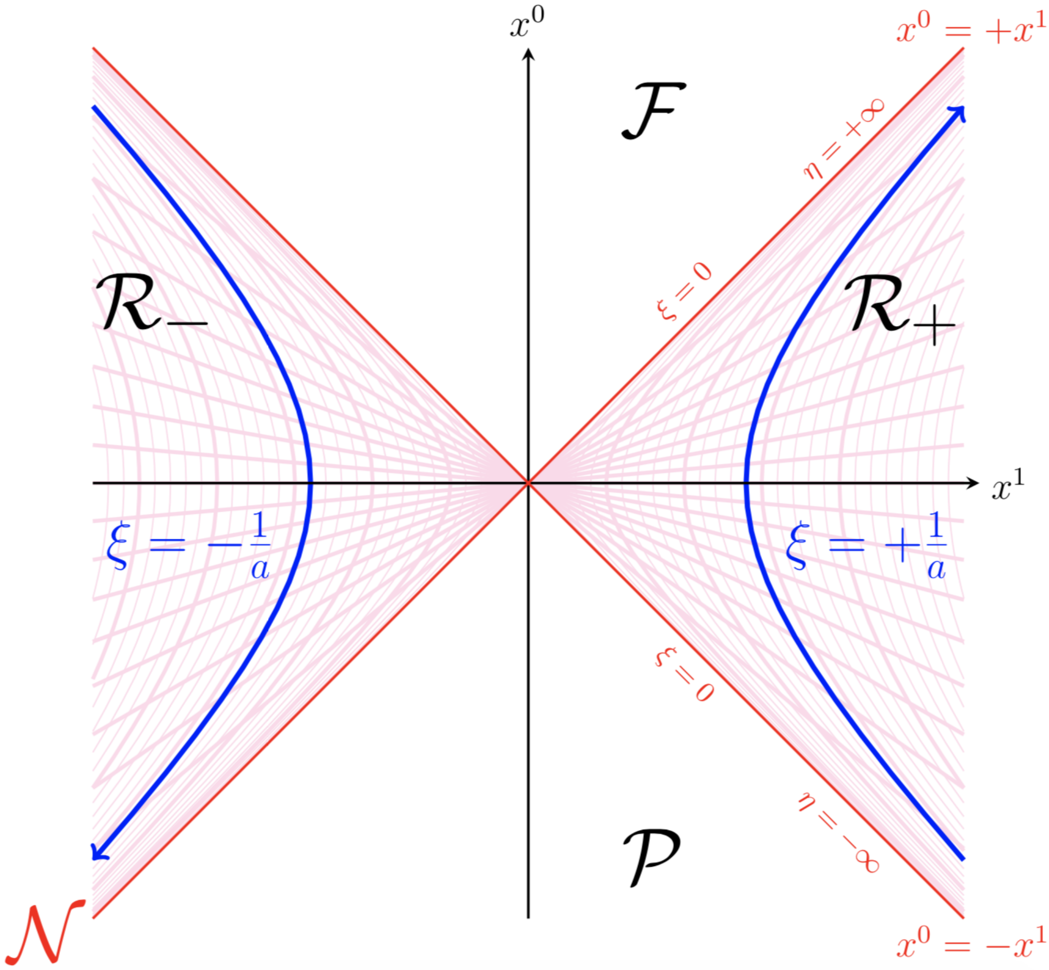


Figure 4.1: In these coordinates only the portion $\mathcal{R}_+ \cup \mathcal{R}_-$ of Minkowski space are covered. Here the lines of constant $\xi = \pm\frac{1}{a}$ describe the worldlines of uniformly accelerated observers. Notice that taking $\tau \to \pm\infty$ causes the observer to approach his perceived event horizon.

This worldline is traced out within the right Rindler wedge $\mathcal{R}_+$. In rectangular coordinates this same worldline takes the form

$$\begin{aligned} x^0(\tau) &= \tfrac{1}{a}\sinh(a\tau) \\ x^1(\tau) &= \tfrac{1}{a}\cosh(a\tau) \\ x^2(\tau) &= y \\ x^3(\tau) &= z \end{aligned} \tag{4.10}$$

If we consider an observer travelling along this worldline $x(\tau)$ at two distinct proper times $\tau = \tau_1 > \tau_2$, we find that the invariant separation between the two points along the trajectory is

$$\big(x(\tau_1) - x(\tau_2)\big)^2 = \tfrac{4}{a^2}\sinh^2\left(\tfrac{a(\tau_1-\tau_2)}{2}\right) \ . \tag{4.11}$$

This is the worldline of an observer who is *accelerated uniformly* with a proper acceleration given by a [14]. We can construct a similar worldline in $\mathcal{R}_-$ by taking $\xi = \frac{1}{a} \mapsto -\frac{1}{a}$ in the above (and keeping all else the same).

The most important feature of the worldlines generated by $\mathcal{B}$ is as follows: an observer travelling the worldline (4.10) in $\mathcal{R}_+$ is *causally separated from* $\mathcal{R}_-$. The null hyperplane $\mathcal{N}$ appears to the observer as a future event horizon. In this sense $\mathcal{R}_+$ and $\mathcal{R}_-$ are separate universes, unable to influence one another.

A slice of constant η that goes through both wedges is a spacelike hypersurface [16], which defines a valid notion of time — this makes $\mathcal{R}_+ \cup \mathcal{R}_-$ a static spacetime with respect to the timelike Killing vector $\mathcal{B}$. Initial data for solving the Klein-Gordon equation can be supplied along sheets of constant η (which goes through both $\mathcal{R}_+$ and $\mathcal{R}_-$) and it is in this sense that we can quantize the field ϕ with respect to η-time. We shall see that this quantization scheme defines a new Rindler vacuum state which is distinct from the Minkowski vacuum $|0_{\rm M}\rangle$.

4.2 Rindler-Fulling Quantization

It will be useful to refer to Appendix A when following the procedure given here. From here on out, we speak in terms of Rindler coordinates $x = (\eta, \xi, x^2, x^3)$ and quantize the field with respect to η-time. In terms of the metric $g_{\mu\nu}$ associated with Rindler coordinates, the Lagrangian density (A.1) becomes

$$\mathscr{L}_0[\phi] = \frac{1}{2|\xi|}\left(\frac{\partial\phi}{\partial\eta}\right)^2 - \frac{|\xi|}{2}\left(\frac{\partial\phi}{\partial\xi}\right)^2 - \frac{|\xi|}{2}\left(\frac{\partial\phi}{\partial x^2}\right)^2 - \frac{|\xi|}{2}\left(\frac{\partial\phi}{\partial x^3}\right)^2 - \frac{|\xi|}{2}m^2\phi^2 \ . \tag{4.12}$$

In these coordinates, the Klein-Gordon equation (A.2) becomes

$$\left[-\tfrac{1}{\xi^2}\left(\tfrac{\partial}{\partial\eta}\right)^2 + \left(\tfrac{\partial}{\partial\xi}\right)^2 + \tfrac{1}{\xi}\tfrac{\partial}{\partial\xi} + \left(\tfrac{\partial}{\partial x^2}\right)^2 + \left(\tfrac{\partial}{\partial x^3}\right)^2 - m^2\right]\phi(\eta,\xi,x^2,x^3) = 0 \tag{4.13}$$

We make the following physical choice in stark contrast to the one made in Chapter 2: *we consider fiducial observers whose proper times are measured along the flow-lines of $\mathcal{B}$.* That is to say, we're going to consider Rindler observers moving along the worldlines discussed in previous section. According to (A.6), the conserved energy associated with this choice is

$$H_{\rm R} = -\int_{\Sigma} d^3\Sigma_\nu \; T^\nu{}_\mu \mathcal{B}^\mu \tag{4.14}$$

We call this object the *Rindler Hamiltonian* (it ends up being the generator of boosts in the 1-direction). By taking the spacelike hypersurface Σ to be a slice of constant η (through both wedges $\mathcal{R}_+ \cup \mathcal{R}_-$) and then noting our formula for the stress energy tensor (A.4) we find that [14]

$$H_{\rm R} = \int_{-\infty}^{\infty} \frac{d\xi}{2|\xi|} \int_{-\infty}^{\infty} dx^2 \int_{-\infty}^{\infty} dx^3 \left[\left(\tfrac{\partial\phi}{\partial\eta}\right)^2 + \xi^2 \left(\left(\tfrac{\partial\phi}{\partial\xi}\right)^2 + \left(\tfrac{\partial\phi}{\partial x^2}\right)^2 + \left(\tfrac{\partial\phi}{\partial x^3}\right)^2 + m^2\phi^2 \right) \right] \tag{4.15}$$

As we shall see, this is *not* equivalent to the Minkowski Hamiltonian and is a conserved charge in the sense that $\frac{d}{d\eta} H_{\rm R} = 0$.

In what follows the Klein-Gordon inner product (A.10) will be most convenient for our use if we also take Σ to be a slice of constant η. For complex solutions f and h to (4.13) it now takes the form

$$\langle f, h\rangle \;=\; +i \int_{-\infty}^{\infty} \frac{d\xi}{|\xi|} \int_{-\infty}^{\infty} dx^2 \int_{-\infty}^{\infty} dx^3 \left[f^*(x) \frac{\partial h(x)}{\partial\eta} - \frac{\partial f^*(x)}{\partial\eta} h(x) \right] . \tag{4.16}$$

We use this to construct *Rindler modes* which are physically distinct from the Minkowski modes in that they must be positive-frequency (and negative-frequency) with respect to the *Rindler time* η. Because $\mathcal{B}$ has a different orientation in $\mathcal{R}_+$ and $\mathcal{R}_-$ we must define *two* sets of Rindler modes $\{r^+_{\boldsymbol{\omega}}, r^-_{\boldsymbol{\omega}}\}$ (and their two negative frequency counterparts $\{r^{+*}_{\boldsymbol{\omega}}, r^{-*}_{\boldsymbol{\omega}}\}$) which satisfy

$$\pm \frac{\partial}{\partial\eta} r^{(\pm)}_{\boldsymbol{\omega}}(x) \;=\; -i\Omega r^{(\pm)}_{\boldsymbol{\omega}}(x) \tag{4.17}$$

$$\pm \frac{\partial}{\partial\eta} r^{(\pm)*}_{\boldsymbol{\omega}}(x) \;=\; +i\Omega r^{(\pm)*}_{\boldsymbol{\omega}}(x) \tag{4.18}$$

Where we have used that $\mathcal{B}$ is past-oriented in $\mathcal{R}_-$ as well as the shorthand $\boldsymbol{\omega} =$

$(\Omega, \underline{\mathbf{k}}) = (\Omega, k^2, k^3)$ for the mode parameters, where $\Omega > 0$ and $\underline{\mathbf{k}} = (k^2, k^3) \in \mathbb{R}^2$ (note this is *not* the same as $\mathbf{k} \in \mathbb{R}^3$). We define the Rindler modes to be

$$r^{(+)}_{\boldsymbol{\omega}}(x) = \begin{cases} \frac{1}{\sqrt{4\pi^3\Omega}} \frac{1}{\Gamma(i\Omega)} e^{-i\Omega\eta + ik^2x^2 + ik^3x^3} K_{i\Omega}\left(\sqrt{m^2 + |\underline{\mathbf{k}}|^2}\xi\right) & , \ x \in \mathcal{R}_+ \\ 0 & , \ x \in \mathcal{R}_- \end{cases} \tag{4.19}$$

$$r^{(-)}_{\boldsymbol{\omega}}(x) = \begin{cases} 0 & , \ x \in \mathcal{R}_+ \\ \frac{1}{\sqrt{4\pi^3\Omega}} \frac{1}{\Gamma(i\Omega)} e^{+i\Omega\eta + ik^2x^2 + ik^3x^3} K_{i\Omega}\left(\sqrt{m^2 + |\underline{\mathbf{k}}|^2}\xi\right) & , \ x \in \mathcal{R}_- \end{cases} \tag{4.20}$$

We recall that Rindler coordinates only cover $\mathcal{R}_+ \cup \mathcal{R}_-$ (this where we're allowed to quantize our field, since our spacetime is static there with respect to $\mathcal{B}$). Note that $r^{(+)}$ modes vanish in the left wedge and visa versa — this is so that the positive frequency conditions are satisfied in each wedge (also note that these modes diverge as the surface $\mathcal{N}$ is approached with $\xi \to 0$). We can compactly write the above as

$$r^{(\sigma)}_{\boldsymbol{\omega}}(x) = \Theta(\sigma\xi) \frac{1}{\sqrt{4\pi^3\Omega}} \frac{1}{\Gamma(i\Omega)} e^{-i\sigma\Omega\eta + ik^2x^2 + ik^3x^3} K_{i\Omega}\left(\sqrt{m^2 + |\underline{\mathbf{k}}|^2}\xi\right) \tag{4.21}$$

where $\sigma = +$ in $\mathcal{R}_+$ and $\sigma = -$ in $\mathcal{R}_-$. These modes are orthonormal with respect to the inner product (4.16) such that

$$\langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{k}})}, r^{(\tilde{\sigma})}_{(\tilde{\Omega},\underline{\mathbf{p}})} \rangle = \delta^{\sigma}{}_{\tilde{\sigma}} \delta(\Omega - \tilde{\Omega}) \delta^{(2)}(\underline{\mathbf{k}} - \underline{\mathbf{p}}) \tag{4.22}$$

$$\langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{k}})}, r^{(\tilde{\sigma})*}_{(\tilde{\Omega},\underline{\mathbf{p}})} \rangle = 0 \tag{4.23}$$

$$\langle r^{(\sigma)*}_{(\Omega,\underline{\mathbf{k}})}, r^{(\tilde{\sigma})*}_{(\tilde{\Omega},\underline{\mathbf{p}})} \rangle = -\delta^{\sigma}{}_{\tilde{\sigma}} \delta(\Omega - \tilde{\Omega}) \delta^{(2)}(\underline{\mathbf{k}} - \underline{\mathbf{p}}) \tag{4.24}$$

We note that $\{r^{(+)}_{\boldsymbol{\omega}}\}$ is complete only over the right wedge $\mathcal{R}_+$, while $\{r^{(-)}_{\boldsymbol{\omega}}\}$ is complete over only the left wedge. We need *both* sets of modes in order to provide a completeness relation over all of $\mathcal{R}_+ \cup \mathcal{R}_-$. Noting this, we can expand the field ϕ for $x \in \mathcal{R}_+ \cup \mathcal{R}_-$ in terms of the Rindler modes as follows:

$$\begin{aligned} \phi(x) = \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}} \Big[& \langle r^{(+)}_{\boldsymbol{\omega}}, \phi \rangle \, r^{(+)}_{\boldsymbol{\omega}}(x) - \langle r^{(+)*}_{\boldsymbol{\omega}}, \phi \rangle \, r^{(+)*}_{\boldsymbol{\omega}}(x) \\ & + \langle r^{(-)}_{\boldsymbol{\omega}}, \phi \rangle \, r^{(-)}_{\boldsymbol{\omega}}(x) - \langle r^{(-)*}_{\boldsymbol{\omega}}, \phi \rangle \, r^{(-)*}_{\boldsymbol{\omega}}(x) \Big] \end{aligned} \tag{4.25}$$

Imposing that our field is real with $\phi^* = \phi$, this gets put into the form

$$\begin{aligned}\phi(x) \;=\; \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}} \, \Big[& \langle r^{(+)}_{\boldsymbol{\omega}}, \phi\rangle \, r^{(+)}_{\boldsymbol{\omega}}(x) + \langle r^{(+)}_{\boldsymbol{\omega}}, \phi\rangle^* \, r^{(+)*}_{\boldsymbol{\omega}}(x) \\ & + \langle r^{(-)}_{\boldsymbol{\omega}}, \phi\rangle \, r^{(-)}_{\boldsymbol{\omega}}(x) + \langle r^{(-)}_{\boldsymbol{\omega}}, \phi\rangle^* \, r^{(-)*}_{\boldsymbol{\omega}}(x) \Big] \, .\end{aligned} \tag{4.26}$$

We are ready to quantize our theory: we upgrade our expansion coefficients to operators (labelled by $\boldsymbol{\omega} = (\Omega, \underline{\mathbf{k}})$ this time) such that

$$\begin{aligned} \langle r^{(+)}_{\boldsymbol{\omega}}, \phi\rangle &\to \hat{b}^{(+)}_{\boldsymbol{\omega}} & \qquad \langle r^{(-)}_{\boldsymbol{\omega}}, \phi\rangle &\to \hat{b}^{(-)}_{\boldsymbol{\omega}} \\ \langle r^{(+)}_{\boldsymbol{\omega}}, \phi\rangle^* &\to \hat{b}^{(+)\dagger}_{\boldsymbol{\omega}} & \qquad \langle r^{(-)}_{\boldsymbol{\omega}}, \phi\rangle^* &\to \hat{b}^{(-)\dagger}_{\boldsymbol{\omega}} \end{aligned} \tag{4.27}$$

The difference here from the Minkowski scheme is that we have *two* sets of raising and lowering operators — one for each wedge. Our field is now an operator as before

$$\begin{aligned}\hat{\phi}(x) \;=\; \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}} \, \Big[& \hat{b}^{(+)}_{\boldsymbol{\omega}} r^{(+)}_{\boldsymbol{\omega}}(x) + \hat{b}^{(+)\dagger}_{\boldsymbol{\omega}} r^{(+)*}_{\boldsymbol{\omega}}(x) + \\ & + \hat{b}^{(-)}_{\boldsymbol{\omega}} r^{(-)}_{\boldsymbol{\omega}}(x) + \hat{b}^{(-)\dagger}_{\boldsymbol{\omega}} r^{(-)*}_{\boldsymbol{\omega}}(x) \Big]\end{aligned} \tag{4.28}$$

The raising and lowering operators are assumed to obey the canonical commutation relations:

$$\left[\hat{b}^{(\sigma)}_{(\Omega,\underline{\mathbf{k}})}, \hat{b}^{(\sigma)\dagger}_{(\tilde{\Omega},\underline{\mathbf{p}})} \right] = \delta^{\sigma}{}_{\tilde{\sigma}} \delta(\Omega - \tilde{\Omega}) \delta^{(2)}(\underline{\mathbf{k}} - \underline{\mathbf{p}}) \tag{4.29}$$

$$\left[\hat{b}^{(\sigma)}_{(\Omega,\underline{\mathbf{k}})}, \hat{b}^{(\sigma)}_{(\tilde{\Omega},\underline{\mathbf{p}})} \right] = 0 \tag{4.30}$$

$$\left[\hat{b}^{(\sigma)\dagger}_{(\Omega,\underline{\mathbf{k}})}, \hat{b}^{(\sigma)\dagger}_{(\tilde{\Omega},\underline{\mathbf{p}})} \right] = 0 \tag{4.31}$$

In particular, the operators from the left wedge commute with ones from the right wedge. This structure already reminds us of our discussion from Chapter 3. We also have an equal-time commutation relation

$$\left[\hat{\phi}(\eta, \xi, x^2, x^3), \hat{\Pi}(\eta, \tilde{\xi}, \tilde{x}^2, \tilde{x}^3) \right] = i\delta(\xi - \tilde{\xi}) \delta(x^2 - \tilde{x}^2) \delta(x^3 - \tilde{x}^3) \tag{4.32}$$

where classically $\Pi = \frac{\delta \mathscr{L}_0}{\delta(\partial_\eta \phi)} = \frac{1}{|\xi|} \partial_\eta \phi$. This is a manifestation of our choice to quantize the field along the flow lines of $\mathcal{B}$. By expressing the field ϕ as in (4.28) we find that the Rindler modes diagonalize the Rindler Hamiltonian:

$$H_{\rm R} = H_{\rm R}^{(+)} - H_{\rm R}^{(-)} \tag{4.33}$$

$$H_{\rm R}^{(\sigma)} = \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\ \frac{\Omega}{2}\bigg[\ \hat{b}_{\boldsymbol{\omega}}^{(\sigma)}\hat{b}_{\boldsymbol{\omega}}^{(\sigma)\dagger} + \hat{b}_{\boldsymbol{\omega}}^{(\sigma)\dagger}\hat{b}_{\boldsymbol{\omega}}^{(\sigma)}\bigg] \tag{4.34}$$

The minus sign in front of $H_{\rm R}^{(-)}$ is rooted in the fact that $\mathcal{B}$ is past-oriented in $\mathcal{R}_-$ [14]. We also notice that this implies that the energy of the Rindler particles are related to Ω (and not directly to $\underline{\mathbf{k}}$). We finally note that the raising and lowering operators imply the existence of a vacuum state $|0_{\rm R}\rangle$ with the defining property

$$\hat{b}_{\boldsymbol{\omega}}^{(+)}\,|0_{\rm R}\rangle = \hat{b}_{\boldsymbol{\omega}}^{(-)}\,|0_{\rm R}\rangle = 0 \tag{4.35}$$

From here we can build a Fock space of states analogous to section 2.3, where $b_{\boldsymbol{\omega}}^{(+)\dagger}$ creates a particle with mode parameters $\boldsymbol{\omega}$ in $\mathcal{R}_+$ and $b_{\boldsymbol{\omega}}^{(-)\dagger}$ creates a particle with these parameters in $\mathcal{R}_-$. We use the notation

$$|\boldsymbol{\omega};\tilde{\boldsymbol{\omega}}\rangle = b_{\boldsymbol{\omega}}^{(+)\dagger}b_{\tilde{\boldsymbol{\omega}}}^{(-)\dagger}\,|0_{\rm R}\rangle \tag{4.36}$$

In general we can create N particles in $\mathcal{R}_+$ and M particles in $\mathcal{R}_-$ where

$$|\boldsymbol{\omega}_1\cdots\boldsymbol{\omega}_N;\tilde{\boldsymbol{\omega}}_1\cdots\tilde{\boldsymbol{\omega}}_M\rangle = b_{\boldsymbol{\omega}_1}^{(+)\dagger}\cdots b_{\boldsymbol{\omega}_N}^{(+)\dagger}b_{\tilde{\boldsymbol{\omega}}_1}^{(-)\dagger}\cdots b_{\tilde{\boldsymbol{\omega}}_M}^{(-)\dagger}\,|0_{\rm R}\rangle\ . \tag{4.37}$$

We've now quantized ϕ in two ways; with the Minkowski scheme and the Rindler-Fulling scheme. It is a natural question to ask — do these schemes yield equivalent notions of particles? Are the vacua $|0_{\rm M}\rangle$ and $|0_{\rm R}\rangle$ the same?

The answer is **no**. To see why this is, we would like to relate the operators $\{\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^\dagger\}$ to the operators $\{\hat{b}_{\boldsymbol{\omega}}^{(+)}, \hat{b}_{\boldsymbol{\omega}}^{(+)\dagger}, \hat{b}_{\boldsymbol{\omega}}^{(-)}, \hat{b}_{\boldsymbol{\omega}}^{(-)\dagger}\}$. We can do this by examining the two expansions we have derived (for $x \in \mathcal{R}_+ \cup \mathcal{R}_-$)

$$\hat{\phi}(x) = \int d^3\mathbf{k}\ \Big[\hat{a}_{\mathbf{k}}u_{\mathbf{k}}(x) + \hat{a}_{\mathbf{k}}^\dagger u_{\mathbf{k}}^*(x)\Big] \tag{4.38}$$

$$= \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\ \bigg[\ \hat{b}_{\boldsymbol{\omega}}^{(+)}r_{\boldsymbol{\omega}}^{(+)}(x) + \hat{b}_{\boldsymbol{\omega}}^{(+)\dagger}r_{\boldsymbol{\omega}}^{(+)*}(x) + \tag{4.39}$$

$$+\hat{b}_{\boldsymbol{\omega}}^{(-)}r_{\boldsymbol{\omega}}^{(-)}(x) + \hat{b}_{\boldsymbol{\omega}}^{(-)\dagger}r_{\boldsymbol{\omega}}^{(-)*}(x)\bigg]$$

Here the Klein-Gordon inner product provides a mapping between the two schemes: on one hand we know that $\hat{b}_{(\Omega,\underline{\mathbf{p}})}^{(\sigma)} = \langle r_{(\Omega,\underline{\mathbf{p}})}^{(\sigma)}, \hat{\phi}\rangle$ from (4.27), but we can plug the representation of $\hat{\phi}$ given by (4.38) into this inner product giving us

$$\hat{b}^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})} = \int d^3\mathbf{k}\bigg[\langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}, u_{\mathbf{k}}\rangle \, \hat{a}_{\mathbf{k}} + \langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}, u^*_{\mathbf{k}}\rangle \, \hat{a}^\dagger_{\mathbf{k}}\bigg] \tag{4.40}$$

This is known as a *Bogoliubov transformation.* As it stands the expression (4.40) is actually ill-defined — as discussed in Appendix B.2, the coefficients $\langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}, u_{\mathbf{k}}\rangle$ and $\langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}, u^*_{\mathbf{k}}\rangle$ are formally distributions. In order to perform the computation carefully, we would need to supply these objects with test functions in the intermediate steps — this is normally done by introducing wavepackets. We'll skip this treatment however, and simply quote the result [14]

$$\langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}, u_{\mathbf{k}}\rangle = \frac{1}{2\pi}\delta^{(2)}(\underline{\mathbf{p}}-\underline{\mathbf{k}})e^{+\frac{1}{2}\pi\Omega}|\Gamma(i\Omega)|\left(\frac{\Omega}{\sqrt{|\mathbf{k}|^2+m^2}}\right)^{\frac{1}{2}}\left(\frac{\sqrt{|\mathbf{k}|^2+m^2}+k^1}{\sqrt{|\mathbf{k}|^2+m^2}-k^1}\right)^{\frac{i\sigma\Omega}{2}} \tag{4.41}$$

$$\langle r^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}, u^*_{\mathbf{k}}\rangle = \frac{1}{2\pi}\delta^{(2)}(\underline{\mathbf{p}}+\underline{\mathbf{k}})e^{-\frac{1}{2}\pi\Omega}|\Gamma(i\Omega)|\left(\frac{\Omega}{\sqrt{|\mathbf{k}|^2+m^2}}\right)^{\frac{1}{2}}\left(\frac{\sqrt{|\mathbf{k}|^2+m^2}+k^1}{\sqrt{|\mathbf{k}|^2+m^2}-k^1}\right)^{\frac{i\sigma\Omega}{2}} \tag{4.42}$$

Define the set of four 'smeared' operators:

$$\hat{A}^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})} = \int_{-\infty}^{\infty} dp^1 \, \left(2\pi\sqrt{|\mathbf{p}|^2+m^2}\right)^{-\frac{1}{2}}\left(\frac{\sqrt{|\mathbf{p}|^2+m^2}+p^1}{\sqrt{|\mathbf{p}|^2+m^2}-p^1}\right)^{i\sigma\Omega/2} \hat{a}_{\mathbf{p}} \tag{4.43}$$

These can be shown to obey the canonical commutation relations

$$\left[\hat{A}^{(\sigma)}_{(\Omega,\underline{\mathbf{k}})}, \hat{A}^{(\sigma)\dagger}_{(\tilde{\Omega},\underline{\mathbf{p}})}\right] = \delta^{\sigma}{}_{\tilde{\sigma}}\delta(\Omega-\tilde{\Omega})\delta^{(2)}(\underline{\mathbf{k}}-\underline{\mathbf{p}}) \tag{4.44}$$

$$\left[\hat{A}^{(\sigma)}_{(\Omega,\underline{\mathbf{k}})}, \hat{A}^{(\sigma)}_{(\tilde{\Omega},\underline{\mathbf{p}})}\right] = \left[\hat{A}^{(\sigma)\dagger}_{(\Omega,\underline{\mathbf{k}})}, \hat{A}^{(\sigma)\dagger}_{(\tilde{\Omega},\underline{\mathbf{p}})}\right] = 0 \tag{4.45}$$

which implies they form a well-defined notion of particle states. It then follows from (4.41-4.42) that the Bogoliubov transformation (4.40) takes on the form

$$\hat{b}^{(+)}_{(\Omega,\underline{\mathbf{p}})} = \frac{1}{\sqrt{1-e^{-2\pi\Omega}}}\,\hat{A}^{(+)}_{(\Omega,\underline{\mathbf{p}})} + \frac{1}{\sqrt{e^{2\pi\Omega}-1}}\,\hat{A}^{(-)\dagger}_{(\Omega,-\underline{\mathbf{p}})} \tag{4.46}$$

$$\hat{b}^{(-)}_{(\Omega,\underline{\mathbf{p}})} = \frac{1}{\sqrt{1-e^{-2\pi\Omega}}}\,\hat{A}^{(-)}_{(\Omega,\underline{\mathbf{p}})} + \frac{1}{\sqrt{e^{2\pi\Omega}-1}}\,\hat{A}^{(+)\dagger}_{(\Omega,-\underline{\mathbf{p}})} \tag{4.47}$$

At this point we might start getting a little excited — this looks *eerily* familiar. It reminds us of the structure from the Bogoliubov transformation (3.36-3.37) in Chapter 3 relating the thermal and vacuum ladder operators. We note that $\hat{A}^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}$ are also annihilation operators of Minkowski particles since $\hat{A}^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}\,|0_{\mathrm{M}}\rangle = 0$. It is a simple consequence of this that

$$\hat{b}^{(+)}_{(\Omega,\underline{\mathbf{p}})}\,|0_{\mathrm{M}}\rangle \;=\; \frac{1}{\sqrt{e^{2\pi\Omega}-1}}\;\hat{A}^{(-)\dagger}_{(\Omega,-\underline{\mathbf{p}})}\,|0_{\mathrm{M}}\rangle \tag{4.48}$$

$$\hat{b}^{(-)}_{(\Omega,\underline{\mathbf{p}})}\,|0_{\mathrm{M}}\rangle \;=\; \frac{1}{\sqrt{e^{2\pi\Omega}-1}}\;\hat{A}^{(+)\dagger}_{(\Omega,-\underline{\mathbf{p}})}\,|0_{\mathrm{M}}\rangle \tag{4.49}$$

and therefore we find that

$$\langle 0_{\mathrm{M}}|\,\hat{b}^{(\tilde{\sigma})\dagger}_{(\tilde{\Omega},\underline{\mathbf{k}})}\hat{b}^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})}\,|0_{\mathrm{M}}\rangle \;=\; \frac{1}{e^{2\pi\Omega}-1}\;\delta^{\sigma}{}_{\tilde{\sigma}}\delta(\Omega-\tilde{\Omega})\delta^{(2)}(\underline{\mathbf{p}}-\underline{\mathbf{k}})\;. \tag{4.50}$$

We find that the Rindler energies seem to be suppressed by a Bose-Einstein distribution *exactly* as in (3.38). Sometimes the result (4.50) is stated as proof that the Minkowski vacuum is a thermal state relative to a Rindler observer.

The general result, that a Rindler observer experiences the Minkowski vacuum as a genuine thermal state, is known as the *thermalization theorem*, proven in Appendix D. We merely state it here: Consider a Rindler observer confined to a trajectory in $\mathcal{R}_+$ with constant $\xi = +\frac{1}{a}$ (and constant x^2 and x^3) with $\eta(\tau) = a\tau$ (where τ is the observer's proper time). Since the observer is permanently confined to $\mathcal{R}_+$, any local observable $\hat{\mathcal{O}}^{(+)}$ he builds out of ϕ contains only contributions from $r^{(+)}_{\boldsymbol{\omega}}$ (since $r^{(-)}_{\boldsymbol{\omega}}$ vanish in $\mathcal{R}_+$). As a result he measures:

$$\langle 0_{\mathrm{M}}|\hat{\mathcal{O}}^{(+)}|0_{\mathrm{M}}\rangle \;=\; \frac{\mathrm{Tr}^{(+)}\left[e^{-\frac{2\pi}{a}\hat{H}^{(+)}_{\mathrm{R}}}\hat{\mathcal{O}}^{(+)}\right]}{\mathrm{Tr}^{(+)}\left[e^{-\frac{2\pi}{a}\hat{H}^{(+)}_{\mathrm{R}}}\right]} \tag{4.51}$$

where $\hat{H}^{(+)}_{\mathrm{R}}$ is the Rindler Hamiltonian restricted to the right wedge, and the trace is taken over states in the right wedge. In this sense $|0_{\mathrm{M}}\rangle$ is a true thermal state, with the temperature given by the *Unruh temperature*

$$\frac{1}{\beta_{\mathrm{U}}} \equiv \frac{a}{2\pi}\;. \tag{4.52}$$

4.3 The Feynman Propagator as a Thermal Object

Having proven the thermalization theorem in Appendix D, we now know that *any* operator belonging to a Rindler observer evaluated in the Minkowski vacuum yields observables that obey thermal properties. This should therefore be true for the Feynman propagator since

$$G_{\rm F}(x;y) \;\;=\;\; \langle 0_M \big| \mathscr{T}\big(\hat{\phi}(x)\hat{\phi}(y)\big) \big| 0_{\rm M}\rangle \;\;, \tag{4.53}$$

so long as we ensure to put the points x and y in the right Rindler wedge $\mathcal{R}_+$ and force them along the wordline of a Rindler observer. We recall that we can write

$$(x(\tau_x) - y(\tau_y))^2 \;\;=\;\; \frac{4}{a^2}\sinh^2\left(\frac{a(\tau_x-\tau_y)}{2}\right) \tag{4.54}$$

where the trajectory takes place at a Rindler radius of $\xi = +\frac{1}{a}$ and τ_x and τ_y are the proper times of the Rindler observer while they're at the point x and y respectively. Recalling our formula (2.15) we find:

$$G_{\rm F}\big(x(\tau_x);y(\tau_y)\big) \;\;=\;\; \frac{1}{4\pi^2}\frac{mK_1\left(m\sqrt{\frac{4}{a^2}\sinh^2\left(\frac{a(\tau_x-\tau_y)}{2}\right)+i\epsilon}\right)}{\sqrt{\frac{4}{a^2}\sinh^2\left(\frac{a(\tau_x-\tau_y)}{2}\right)+i\epsilon}} \tag{4.55}$$

By noting the relation $\sinh(z+n\pi i) = (-1)^n\sinh(z)$ for all $n\in\mathbb{Z}$, we notice that shifting the proper time τ_y by an amount $+i\frac{2\pi}{a}$ recovers

$$-iG_{\rm F}\big(x(\tau_x);y(\tau_y+i\tfrac{2\pi}{a})\big) \;\;=\;\; -iG_{\rm F}\big(x(\tau_x);y(\tau_y)\big) \tag{4.56}$$

That is, since this function is symmetric under $x \leftrightarrow y$, we've just proven that

$$\langle 0_M \big| \mathscr{T}\big(\hat{\phi}(x(\tau))\hat{\phi}(y(\tau))\big) \big| 0_{\rm M}\rangle = \langle 0_M \big| \mathscr{T}\big(\hat{\phi}(x(\tau))\hat{\phi}(y(\tau+i\beta_{\rm U}))\big) \big| 0_{\rm M}\rangle \;\;, \tag{4.57}$$

meaning the Feynman propagator satisfies the KMS condition (3.13) where the temperature is given by $\frac{1}{\beta_{\rm U}}$ — this should not come as a surprise though, since we've just proven that $|0_{\rm M}\rangle$ is a KMS state (*aka.* a thermal state). From the simplicity of the above calculation, it's obvious why the KMS condition is an appealing way to test whether states are thermal or not.

4.4 Interactions for a Rindler Observer

We move on to consider the quartic interacting theory

$$\mathscr{L}[\phi] \;=\; -\tfrac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \tfrac{1}{2}m^2\phi^2 - \tfrac{\lambda}{4}\phi^4 \;. \tag{4.58}$$

From (2.34) we remind ourselves that the first-order tadpole correction to the two-point correlation function has the form

$$G^{(1)}_{\text{Tadpole}}(x;y) \;=\; 3(-i\lambda)\mathcal{I}_{\text{Top}}(m)\left(\int \frac{d^4p}{(2\pi)^4}\big[-i\Delta_{\text{F}}(p)\big]^2 e^{i(x-y)\cdot p}\right) \tag{4.59}$$

where we had defined

$$\mathcal{I}_{\text{Top}}(m) \;=\; \int \frac{d^4k}{(2\pi)^4}\,[-i\Delta_{\text{F}}(k)] = \langle 0_{\text{M}}|\hat{\phi}^2(v)|0_{\text{M}}\rangle \;. \tag{4.60}$$

We also recall that we were able to explicitly perform the Fourier transform with (2.43) where

$$\int \frac{d^4p}{(2\pi)^4}\big[-i\Delta_{\text{F}}(p)\big]^2 e^{i(x-y)\cdot p} \;=\; -\frac{i}{8\pi^2}K_0\big(m\sqrt{(x-y)^2+i\epsilon}\big) \;. \tag{4.61}$$

We are now interested in computing this correction for the massless theory $m \to 0$ *from the point of view of a Rindler observer* moving along the trajectory (4.54). We re-organize the above information in the following manner

$$G^{(1)}_{\text{Tadpole}}(x;y) \;=\; -\frac{\delta M^2}{8\pi^2}K_0\left(m\sqrt{(x-y)^2+i\epsilon}\right) \tag{4.62}$$

where the δM^2 denotes the renormalized mass shift

$$\delta M^2 = 3\lambda\,\langle 0_{\text{M}}|\hat{\phi}^2(v)|0_{\text{M}}\rangle - \delta Z \tag{4.63}$$

where $\langle 0_{\text{M}}|\hat{\phi}^2(v)|0_{\text{M}}\rangle$ is the UV divergent loop factor from the tadpole graph and δZ represents the choice of mass counter-term in the Lagrangian density that renders δM^2 finite.

As discussed in Chapter 2, the Minkowski observer demands the mass shift to be zero in that the sum of his counter-term $\delta Z_{\rm M}$ and the tadpole loop vanish with

$$\delta M^2 \ \rightarrow \ \delta M^2_{\rm M} = 3\lambda \langle 0_{\rm M}|\hat{\phi}^2(v)|0_{\rm M}\rangle - \delta Z_{\rm M} \ = \ 0 \ . \tag{4.64}$$

The Rindler observer also has to choose his own mass counter-term $\delta Z_{\rm R}$ — the natural choice for this observer is to demand that the sum of his counter-term and tadpole graph *evaluated in the Rindler ground state* vanish with

$$\delta Z_{\rm R} - 3\lambda \langle 0_{\rm R}|\hat{\phi}^2(v)|0_{\rm R}\rangle \ = \ 0. \tag{4.65}$$

Assuming the Rindler observer chooses his counter-term $\delta Z_{\rm R}$ according to the above prescription, this means that his choice for the tadpole mass shift *in the Minkowski vacuum* ends up being:

$$\delta M^2 \ \rightarrow \ \delta M^2_{\rm R} = 3\lambda \langle 0_{\rm M}|\hat{\phi}^2(v)|0_{\rm M}\rangle - \delta Z_{\rm R} \tag{4.66}$$

Because of the prescription (4.65) the mass shift for the Rindler observer is therefore given by

$$\delta M^2_{\rm R} \ = \ 3\lambda \langle 0_{\rm M}|\hat{\phi}^2(v)|0_{\rm M}\rangle - 3\lambda \langle 0_{\rm R}|\hat{\phi}^2(v)|0_{\rm R}\rangle \ . \tag{4.67}$$

Evaluating (4.67) is a standard calculation [44, 45] that we evaluate in Appendix E. The *massless* result is

$$\delta M^2_{\rm R} \ = \ \frac{\lambda a^2}{16\pi^2} \tag{4.68}$$

Since we're interested in the massless theory, we note that $K_0(z) \approx -\ln(z)$ in the limit $z \to 0^+$ [25, 26] which implies that in the massless limit $m \to 0^+$ we make the replacement

$$K_0\left(m\sqrt{(x-y)^2+i\epsilon}\right) \rightarrow -\ln\left(\mu\sqrt{(x-y)^2+i\epsilon}\right) \tag{4.69}$$

where μ is a spacetime-independent IR-divergent constant (of mass dimension 1) which depends on the choice of regularization for this IR divergence. We don't particularly care about the value of μ since we are primarily interested in tracking the dependence of the above function on x and y. So far we have

$$G^{(1)}_{\rm Tadpole}(x;y) \ = \ \frac{\lambda a^2}{128\pi^4}\ln\left(\mu\sqrt{(x-y)^2+i\epsilon}\right) \ . \tag{4.70}$$

Parametrizing this in terms of the Rindler proper-times along the trajectory (4.54) we have

$$G^{(1)}_{\text{Tadpole}}\big(x(\tau_x); y(\tau_y)\big) = \frac{\lambda a^2}{128\pi^4}\ln\left(\mu\sqrt{\tfrac{4}{a^2}\sinh^2\left(\tfrac{a(\tau_x-\tau_y)}{2}\right)+i\epsilon}\right), \quad (4.71)$$

which asymptotically in the limit $a|\tau_x - \tau_y| \gg 1$ becomes

$$G^{(1)}_{\text{Tadpole}}\big(x(\tau_x); y(\tau_y)\big) = \frac{\lambda a^3}{256\pi^4}|\tau_x - \tau_y| + \text{subdominant}\ . \quad (4.72)$$

We see that this exactly matches the thermal result, provided that we identify the temperature as $\frac{1}{\beta_{\text{U}}} = \frac{a}{2\pi}$ so that

$$G^{(1)}_{\text{Tadpole}}\big(x(\tau_x); y(\tau_y)\big) = \frac{1}{32\pi}\frac{\lambda|\tau_x - \tau_y|}{\beta_{\text{U}}^3} + \text{subdominant}\ . \quad (4.73)$$

We see that the secular breakdown occurs for large proper times of the Rindler observer — near his perceived event horizon.

Notice also that the Rindler observer's choice to set $\delta Z_{\text{R}} - 3\lambda \langle 0_{\text{R}}|\hat{\phi}(v)^2|0_{\text{R}}\rangle = 0$ means that secular-growth does *not* occur in the Rindler vacuum (for the same reason that the Minkowski observer's choice to set $\delta Z_{\text{M}} - 3\lambda \langle 0_{\text{M}}|\hat{\phi}(v)^2|0_{\text{M}}\rangle = 0$ did not result in secular growth in the Minkowski vacuum). In this sense, secular growth at late Rindler times cannot be avoided for *both* the Minkowski and Rindler vacuua.

4.5 Near-Horizon Resummation of the Tadpole Graph

We've shown that the Rindler renormalization choice results in a secularly-growing correction to the time-ordered two-point correlation function — this indicates a breakdown of perturbation theory. In this section we argue that we can correct this issue in the same way as is done in finite-temperature calculations: by resumming self-energy insertions into the propagator.

In QFT it is convenient to separately consider the sum of all 1PI graphs (which is precisely what we have done in Chapter 2). In momentum space, one omits the two external line factors of $\frac{-i}{p^2-i\epsilon}$ (in our massless theory) and denotes the remaining sum of all 'amputated' 1PI graphs as the *self-energy* $+i\Sigma(p)$ [22]. Because the combination $+i\Sigma(p)$ appears again and again in the full perturbative series, the *full* interacting momentum-space propagator $\Delta_{\rm Full}(p)$ ends up being a geometric series in $\Sigma(p)$ such that

$$-i\Delta_{\rm Full}(p) = \frac{-i}{p^2-i\epsilon} + \sum_{n=1}^{\infty}[+i\Sigma(p)]^n\left[\frac{-i}{p^2-i\epsilon}\right]^{n+1} \tag{4.74}$$

$$= \frac{-i}{p^2-\Sigma(p)-i\epsilon} \tag{4.75}$$

which is related to the full interacting propagator in the obvious way

$$G(x;y) = \int \frac{d^4p}{(2\pi)^4}\,[-i\Delta_{\rm Full}(p)]\,e^{ip\cdot(x-y)}\ . \tag{4.76}$$

See Figure 4.2 for a graphical interpretation of the equation (4.75).

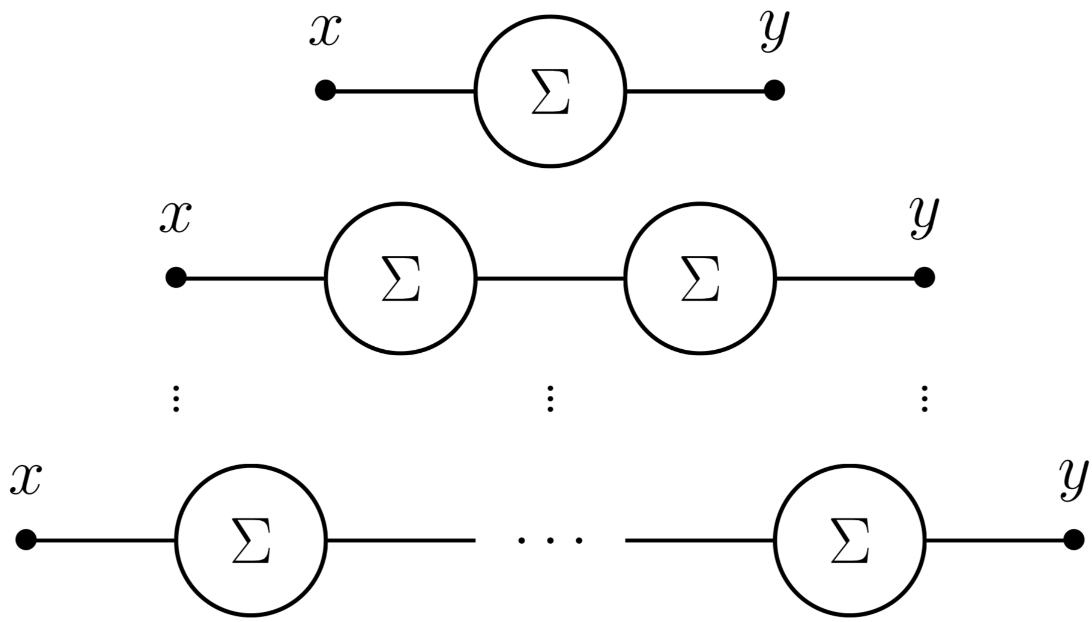


Figure 4.2: The sum of all the 1PI diagrams, the *self-energy*, appears over and over again in the full perturbative series for the full propagator

The basic problem in (4.73) arises because the Rindler observer chooses mass counter-terms that do not completely cancel the self energy $\Sigma(p)\big|_{p=0}$. As a result, perturbation theory is organized in a way which obscures the correct dispersion relation. The position of the pole $\Sigma(p)$ in (4.75) is important in the limit where $|\mathbf{x}-\mathbf{y}|$ and $|x^0-y^0|$ are large and proportional to each other — this is because in this regime the massless propagator (2.10)

$$G_{\rm F}(x;y) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{e^{-i|\mathbf{p}||x^0-y^0|}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{2|\mathbf{p}|} \tag{4.77}$$

is dominated by the contribution of on-shell particles with $p^2=0$. When the particles are on-shell it's always a bad approximation to perturb in $\Sigma(p)$ because it is *the series* (4.74) that is breaking down. No matter how small $\Sigma(p)$ is made (by making λ small), there is always a regime near $p^2=0$ for which our expansion in powers of λ must fail — the secular growth is a manifestation of this.

In this language, we know precisely how to resum the terms that are not small in the perturbative expansion; we must use the full propagator which includes the mass shift $+i\Sigma(p) = -i\delta M_{\rm R}^2$ into the position of the propagator's pole. We can therefore read off the resummed result using the form of the propagator (2.15) where we simply make the replacement $m^2 \mapsto \delta M_{\rm R}^2$

$$\to G_{\rm RS}(x;y) \;=\; \frac{1}{4\pi^2}\sqrt{\frac{\delta M_{\rm R}^2}{(x-y)^2+i\epsilon}}\,K_1\left(\sqrt{\delta M_{\rm R}^2\,(x-y)^2+i\epsilon}\right) \tag{4.78}$$

or explicitly in terms of λ

$$G_{\rm RS}(x;y) \;=\; \frac{1}{16\pi^3}\sqrt{\frac{\lambda}{(x-y)^2+i\epsilon}}\,K_1\left(\tfrac{a}{4\pi}\sqrt{\lambda\,(x-y)^2+i\epsilon}\right)\;. \tag{4.79}$$

The first thing we notice is that for small $|\delta M_{\rm R}^2\,(x-y)^2\,| \ll 1$ the function becomes

$$\begin{aligned} G_{\rm RS}(x;y) \;=\; & \frac{1}{4\pi^2}\frac{1}{(x-y)^2+i\epsilon} \\ & +\frac{\delta M_{\rm R}^2}{16\pi^2}\left[2\gamma - 1 + 2\ln\left(\tfrac{1}{2}\sqrt{\delta M_{\rm R}^4(x-y)^2+i\epsilon}\right)\right] + \mathscr{O}(\delta M_{\rm R}^4) \end{aligned} \tag{4.80}$$

We notice that the $\mathscr{O}(\delta M_{\rm R}^2)$ term here captures the dependence found in (4.73) through the computation of the tadpole graph. In this sense, the secular growth found when expanding in powers of λ was simply a breakdown of the approximation $|\delta M_{\rm R}^2\,(x-y)^2\,| \ll 1$.

This can also be used to determine the late-time behaviour $|\delta M_{\mathrm{R}}^2\,(x-y)^2\,| \gg 1$ for this function. With the asymptotic form (2.18) we find in this limit

$$G_{\mathrm{RS}}(x;y) \approx \frac{1}{\sqrt{32\pi^3}} \frac{(\delta M_{\mathrm{R}}^2)^{1/4}}{((x-y)^2+i\epsilon)^{3/4}} e^{-\sqrt{\delta M_{\mathrm{R}}^2 (x-y)^2 + i\epsilon}} \tag{4.81}$$

$$= \frac{1}{8\pi^2}\sqrt{\frac{a}{2}} \frac{\lambda^{\frac{1}{4}}}{((x-y)^2+i\epsilon)^{3/4}} e^{-\frac{a}{4\pi}\sqrt{\lambda (x-y)^2 + i\epsilon}} \tag{4.82}$$

We notice that for time-like separations $-(\Delta t)^2 \equiv (x-y)^2$ this falls off like $(\Delta t)^{-3/2}$ (the exponential yields oscillatory behaviour rather than suppression in this case), while the original free propagator for the theory falls off like $(\Delta t)^{-2}$ — faster than the resummed result. We have found a way to successfully capture the late time behaviour in this setting.

As a final note, we notice that the late-time limit of the resummed result depends on the coupling as $\propto \lambda^{1/4}$. This is reflecting the fact that the late-time limit *cannot* be captured by a series in integer powers about $\lambda = 0$.

4.6 Other Graphs

Because the Rindler observer chooses to renormalize the top loop of the cactus graph such that $\delta M_{\mathrm{R}}^2 \neq 0$, the cactus graph is now non-zero also. The Rindler observer describes a power-law IR divergence in the middle loop of the cactus graph as a result. If evaluated in position space as in (2.30), the middle loop corresponds to a factor of

$$\text{Middle Loop} \propto \int d^4v\; G_{\mathrm{F}}(u;v)^2 = \frac{1}{16\pi^4}\int \frac{d^4v}{[(u-v)^2+i\epsilon]^2}\,, \tag{4.83}$$

where we are using massless propagators as in (2.19). In this form, the IR divergence encountered corresponds to the failure of this position-space integral to converge in the IR as $v \to \infty$ (for long-distances). Imposing an upper limit L_{max} on the invariant separation on the bounds of the integral (4.83), by power-counting we can see that the divergence is parametrized as $\sim \log(L_{\mathrm{max}}/\mu)$. Once this invariant separation L_{max} is re-expressed in terms of a maximum Rindler time (through a relation like (4.54) with $\tau_x - \tau_y \mapsto \tau_{\max}$) we find that the divergence becomes parametrized as a power-law in $\sim a\tau_{\max}$. At this level, this resembles the IR power-law $\sim \frac{1}{\beta m}$ behaviour encountered in the thermal cactus' middle loop (3.51). Independent of any secular growth, the above suggests the presence of an enhanced IR problem from the point of view of a Rindler observer.

Even though the tadpole and cactus graphs display interesting behaviour, they are non-zero due to the Rindler observer's choice of renormalization that sets $\delta M_{\mathrm{R}}^2 \neq 0$. Do all secular effect disappear if δM_{R}^2 vanishes? To answer this question, more complicated graphs should be considered.

For example, the sunset graph (2.36) from Chapter 2 can be examined. In the massless theory, this correction takes the form

$$G^{(2)}_{\mathrm{Sunset}}(x;y) = \lambda^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathcal{I}_{\mathrm{Sunset}}(p;0)}{(p^2 - i\epsilon)^2} \; . \tag{4.84}$$

where the massless loop integral (2.39) is given by

$$\mathcal{I}_{\mathrm{Sunset}}(p;0) = i \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - i\epsilon)} \frac{1}{(q^2 - i\epsilon)} \frac{1}{\left([p-k-q]^2 - i\epsilon\right)} \; . \tag{4.85}$$

In the small-p limit, this loop integral has the form

$$\mathcal{I}_{\mathrm{Sunset}}(p;0) \approx A + Bp^2 + Cp^2 \ln\left(\frac{p^2}{\mu^2}\right) + \ldots \tag{4.86}$$

where A and B are divergent quantities and C is finite [46, 47]. On dimensional grounds it then follows that the p-independent terms in $\mathcal{I}_{\mathrm{Sunset}}(p;0)$ result in a logarithmic dependence on $(x-y)^2$ for (4.84), while the remaining terms proportional to p^2 yield terms in (4.84) proportional to $(x-y)^{-2}$. This suggest that it is also $\Sigma(0) \sim \mathcal{I}_{\mathrm{Sunset}}(0;0)$ which dominates for late Rindler times for this graph.

Chapter 5

Conclusions

5.1 Conclusions

This work suggests that for QFTs on spacetimes with horizons the usual semi-classical approach to computing quantum corrections can break down when used to predict late-time behaviour. The evidence provided in this thesis comes from the simplest possible QFT with a horizon: an interacting real scalar field probed by accelerated observers within a flat spacetime.

Since secular breakdown arises in thermal systems when very light bosons interact with a heat bath, this implies that a Rindler observer should enjoy a similar description of secular growth (due to his thermal interpretation of the Minkowski vacuum). In this work we find that this is precisely the case: the time-ordered two-point correlation function suffers late-time perturbative breakdown provided that (1) the two points are constrained along the trajectory of the Rindler observer and (2) that the tadpole mass counter-term is chosen so that late-time secular growth does not occur for the Rindler vacuum. Furthermore, the breakdown occurs for late Rindler times: along the perceived event horizon of the Rindler observer.

The required counter-term ensures that the Rindler observer sees a finite acceleration-dependant mass shift. This mass shift exactly matches the thermal mass shift experienced by particles interacting with a heat bath (after identifying the observer's acceleration with the Unruh temperature). Since the secular growth occurs *because* of this finite mass-shift, it is simple to perform a resummation which correctly captures the late-time behaviour: this is done by including the mass-shift in the unperturbed part of the scalar field action. The resulting resummed propagator falls to zero for large Rindler times but does so more slowly than the naïve massless propagator.

5.2 Relevance for Black Hole Physics

Studying a QFT in a curved spacetime involves treating the gravitational field $g_{\mu\nu}$ as a classical background field, upon which the field of interest is quantized. This work suggests that perturbative methods can become problematic in such a setting since the gravitational field behaves like an environment for the quantum fluctuations taking place — when the environment is static (or nearly so) arbitrarily small quantum effects can accumulate over long periods of time resulting in large effects. Since spacetimes with horizons are known to be thermal, generically one would expect secular breakdowns to manifest there.

Perhaps one of the best-studied examples of a spacetime with an event horizon is that of the Schwarzschild solution. Coupled with Hawking's prediction that black holes also produce quantum fields in thermal states (with a Hawking temperature $T_{\rm H} \sim r_{\rm S}^{-1}$ set by the Schwarzschild radius) could mean that similar instances of perturbative breakdown are possible here. The work presented suggests that effects of secular growth in the vicinity of a black hole would be largest for massless bosons (or at the very least, bosons with masses much lighter than $T_{\rm H}$).

In principle, perturbations about a free quantum field prepared in the Hartle-Hawking state should show similar kinds of perturbative breakdown as one approaches future infinity along the event horizon of the black hole. For example, consider a four-point correlation function of the form $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ evaluated at the near-horizon points x_j depicted in the figure below. Such a correlation function could be used to make predictions about mutual entanglement between a Hawking pair at an early time and a much later time. Since these points are near the horizon they are at risk of the secular effects described in this work, and so perturbing about the free field here could be dangerous. In such situations very

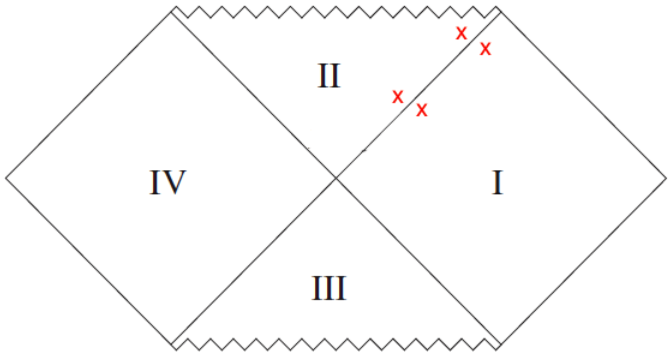


Figure 5.1: The crosses indicate points near the horizon at which a correlation function $\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ could be evaluated

few explicit calculations exist, as the mathematics are less digestable than in our simple Rindler example. It is worth noting that one group claims to have found an explicit example of secularly growing loop corrections in such a setting [48].

One can gain a bit of intuition about the situation near a black-hole by noting the Hadamard form [16] of any curved-space propagator:

$$G^{\mathrm{CS}}(x;y) \sim \frac{1}{4\pi^2\sigma(x;y)} \qquad \text{for } (x-y)^2 \sim 0 \tag{5.1}$$

where $\sigma(x,y)$ is the invariant separation between x and y in generic coordinates. We see that near the light-cone, we recover the massless propagator that we are used to from flat space.

In case the reader is worried that the tadpole diagram hasn't yet been beaten to death: consider it one last time, strictly in position-space:

$$G_{\mathrm{Tadpole}}(x;y) \;\sim\; \delta M^2 \int_{\mathcal{S}} d^4u \; G_{\mathrm{F}}(x;u)G_{\mathrm{F}}(u;y) \tag{5.2}$$

Where $\delta M^2 \sim G_{\rm F}(u;u)$ is a generic mass-shift representing the top loop of the graph topologically. The set $\mathcal{S}$ is an integration region (a *subset* of the overall spacetime) which we use to regulate the divergences that arise. We use massless propagators, and assume that $(x-y)^2 < 0$ so that the separation is timelike. Setting $\Delta = \sqrt{-(x-y)^2}$ we can write the above as (after a boost)

$$G_{\rm Tadpole}(x;y) \;\sim\; \frac{\delta M^2}{16\pi^4}\int_{\mathcal{S}} \frac{d^4u}{[-(\Delta-u^0)^2+|\mathbf{u}|^2+i\epsilon]\,[-(u^0)^2+|\mathbf{u}|^2+i\epsilon]}\ . \tag{5.3}$$

By power-counting, we can see that this integral is logarithmically IR divergent for $u\to\infty$. The integral becomes UV divergent (for $u \to 0$) only in the limit that $\Delta \to 0$. We now let $\mu > 0$ be some scale with mass dimension 1 and make

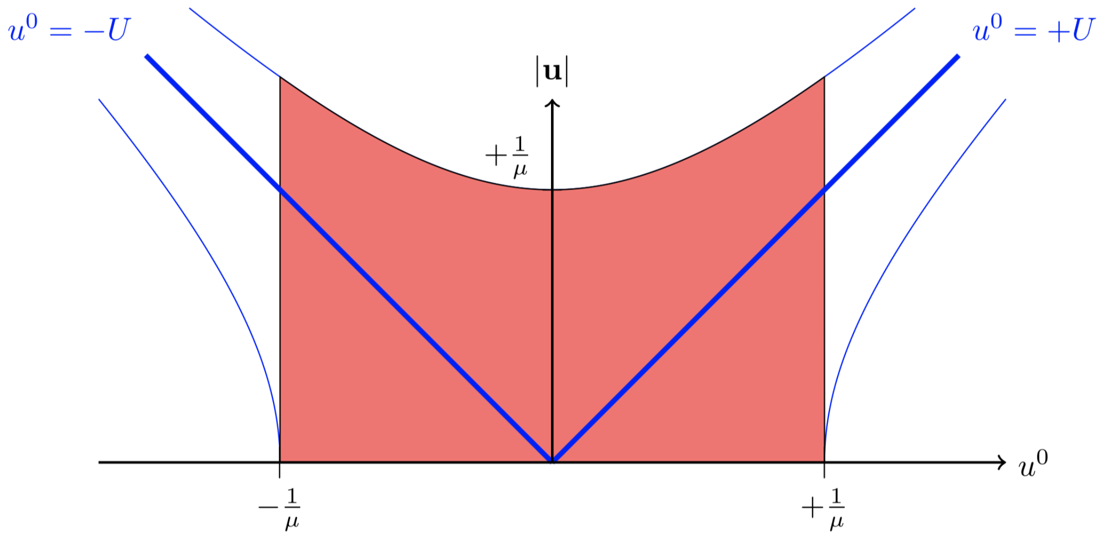


Figure 5.2: The integration region is shown shaded — the blue lines denote the level-set $-(u^0)^2 + |\mathbf{u}|^2 = \mu^2$

the choice for the set $\mathcal{S}$ as shown in Figure 5.2. With this choice, the integral can be easily computed using a Wick rotation in the u^0-variable. If we assume that $\mu\Delta < 1$ the integration results in

$$G_{\rm Tadpole}(x;y) \;\sim\; \frac{\delta M^2}{8\pi^2}\ln\left(\mu\sqrt{\Delta^2+i\epsilon}\right) \;+\; \text{constant}\ . \tag{5.4}$$

With this result we capture the secular growth encountered in Chapter 4 as we take $\Delta\to\infty$. As usual, it's wrong to take this growth literally — in this formulation this is because we assumed $\mu\Delta<1$ before performing the integration.

What is interesting about this result is that we can make $\frac{1}{\mu}$ arbitrarily small, taking the set $\mathcal{S}$ closer and closer to the light-cone. It's in this sense we can see that for an arbitrary spacetime we could compute a similar result simply by using the Hadamard form (5.1) in our integrations (by making $\frac{1}{\mu}$ small enough). Since we're encountering secular growth by integrating in a region arbitrarily close to the light cone, it's feasible that this same feature could be gathered in a more complicated setting.

If this result were true for black-hole spacetimes it would imply that only the near-horizon coincident limit of the propagator matters for secular behaviour. It's feasible that (5.4) capture the features of actual result in a black hole spacetime. Such a calculation more carefully done could also potentially capture the thermal character of black holes since it precisely in the near-horizon limit where Hawking radiation occurs [49].

5.3 Future Work

Since the problem of secular growth encountered in this work in Rindler space are restricted to the level of correlation functions, a future path of pursuit is to consider explicit calculations of time-dependent physical observables as measured by Rindler observers and to investigate whether these can also suffer perturbative problems.

If our conclusions about secular growth are generalizable to spacetimes with non-zero curvature this could have very interesting implications on the black hole information loss problem. Other future work should be to consider explicit calculations of secular growth for quantum fields near black holes.

Appendix A

The Free Real Scalar Field In Arbitrary Coordinates

We begin by considering a massive real scalar field ϕ living in ordinary, flat 4-dimensional Minkowski space, whose Lagrangian density is assumed to be

$$\mathscr{L}_0[\phi] \;=\; -\sqrt{-g}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \tfrac{1}{2}m^2\phi^2\right] . \tag{A.1}$$

In the above formulation, $g_{\mu\nu}$ represents any metric describing flat Minkowski space (in arbitrary coordinates) and $g \equiv \det(g_{\mu\nu})$. The parameter m represents the mass of a single-particle excitation of the field. Varying the action $S = \int d^4x\ \mathscr{L}_0\big[\phi\big](x)$ with respect to the field ϕ such that $\frac{\delta S}{\delta \phi} = 0$ leads to the classical equation of motion

$$(\Box - m^2)\phi(x) \;=\; 0 \tag{A.2}$$

otherwise known as the Klein-Gordon (KG) equation. In the above, $\Box$ denotes the d'Alembertian operator in arbitrary coordinates [14]

$$\Box \;=\; \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}g^{\mu\nu}\sqrt{-g}\frac{\partial}{\partial x^\nu} . \tag{A.3}$$

Since we're working in flat Minkowski space we know that there exists a *global timelike Killing vector field* $\mathcal{K}$. In essence, this tells us that we can define a global 'time' coordinate on the manifold, which we'll generally refer to as $t_{\mathcal{K}}$ (a general curved spacetime which admits such a $\mathcal{K}$ is called a *static spacetime*). The metric g can then always be chosen to be independent of $t_{\mathcal{K}}$ and if $\mathcal{K}$ is normalized such that $g_{\mu\nu}\mathcal{K}^\mu\mathcal{K}^\nu = -1$ then the coordinate $t_{\mathcal{K}}$ is simply the proper time measured by a clock having whose worldlines have tangents parallel to $\mathcal{K}$ [15]. We note that any 3-dimensional submanifold of constant $t_{\mathcal{K}}$ is a spacelike hypersurface (whose tangent spaces are everywhere normal to $\mathcal{K}$).

Since we can determine a global notion of time for a given $\mathcal{K}$, we expect that there should be a corresponding conserved quantity: some sort of energy $H_{\mathcal{K}}$. By varying the action $S = \int d^4x\ \mathscr{L}_0\big[\phi\big](x)$ with respect to the metric we get the stress-energy tensor for our scalar field in arbitrary coordinates [16]

$$T_{\mu\nu} \;=\; \partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi - \tfrac{1}{2}g_{\mu\nu}m^2\phi^2\ . \tag{A.4}$$

We recall that these four conserved currents have vanishing (covariant) divergences $\nabla_\nu T^{\mu\nu} = 0$ on account of Noether's theorem, or in arbitrary coordinates (in flat space):

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\bigg(\sqrt{-g}T^{\mu\nu}(x)\bigg) \;=\; 0 \tag{A.5}$$

This, coupled with Killing's equation $\nabla_\nu\mathcal{K}_\mu + \nabla_\mu\mathcal{K}_\nu = 0$ for $\mathcal{K}$ tells us that the 4-vector $K^\mu T_\mu{}^\nu$ is also a conserved current with $\nabla_\nu(K^\mu T_\mu{}^\nu) = 0$. The corresponding conserved charge is

$$H_{\mathcal{K}} \;=\; -\int_\Sigma d\Sigma_\nu\ \mathcal{K}^\mu T_\mu{}^\nu\ , \tag{A.6}$$

which is conserved relative to an observer with world-lines parallel to $\mathcal{K}$. In the above the integration is performed over a spacelike hypersurface Σ and $d^3\Sigma_\nu = \frac{1}{3!}\epsilon_{\nu\rho\sigma\theta}dx^\rho \wedge dx^\sigma \wedge dx^\theta$ is the 3-volume 1-form in arbitrary coordinates [17] (with ϵ the Levi-Civita completely antisymmetric tensor). In practice one tends to use a spacelike hypersurface Σ specified by $\mathcal{K}$ (*ie.* a slice of constant $t_{\mathcal{K}}$).

An important property of $H_{\mathcal{K}}$ is that is independent of the choice of Σ used to integrate it. To see why this is, take two spacelike hypersurfaces Σ and $\tilde{\Sigma}$ and we can conclude that

$$-\int_{\Sigma} d\Sigma_\nu \; \mathcal{K}^\mu T_\mu{}^\nu \;+\; \int_{\tilde{\Sigma}} d\Sigma_\nu \; \mathcal{K}^\mu T_\mu{}^\nu = 0 \; . \tag{A.7}$$

The left-hand side of the above equation is a consistently-oriented surface integral over some 4-volume bounded by the hypersurfaces Σ and $\tilde{\Sigma}$. Assuming that the integrand vanishes sufficiently quickly at the timelike boundary at spatial infinity, we can then use Gauss' theorem to relate the above surface integral to a 4-volume integral over the divergence of the integrand [18, 19] which we know vanishes via $\nabla_\nu(K^\mu T_\mu{}^\nu) = 0$.

In a generic stationary spacetime, $H_{\mathcal{K}}$ can be the *only* conserved charge that there is [15] (this is because Noether's theorem in curved spacetimes requires the existence of a Killing vector field to precisely define what is meant by a 'conserved' charge: it's that the quantity is conserved along world-lines whose tangents parallel the Killing vector). Here we know that there exists a global time coordinate $t_{\mathcal{K}}$ and so (A.6) is a conserved charge in the sense that $\frac{d}{dt_{\mathcal{K}}} H_{\mathcal{K}} = 0$. *This energy is to be understood relative to a fiducial observer whose world-lines are parallel to* $\mathcal{K}$ (which is of unit norm -1) [15].

In this work we'll play with two choices of $\mathcal{K}$ which will lead to two *different* conserved energies of the above form (A.6). We say that a solution f to the Klein-Gordon equation (A.2) is of *positive frequency* with respect to the time specified by $\mathcal{K}$ if it satisfies the eigenvalue relation

$$\mathcal{L}_{\mathcal{K}}(f) = -i\lambda f \tag{A.8}$$

for some $\lambda > 0$. Here $\mathcal{L}_{\mathcal{K}}$ denotes the Lie derivative with respect to $\mathcal{K}$ (when made future-directed), which becomes the directional derivative when expressed in terms of coordinates $\{x^\mu\}$ such that $\mathcal{L}_{\mathcal{K}}(f) = \mathcal{K}^\mu \frac{\partial f}{\partial x^\mu}$ [18]. Similarly, the solution is of *negative frequency* (with respect to the time specified by $\mathcal{K}$) if it satisfies

$$\mathcal{L}_{\mathcal{K}}(f) = +i\lambda f. \tag{A.9}$$

In this work, we will quantize our field ϕ in two different ways — where in each way we sum the field over different *mode functions* which are solutions to (A.2). First we'll write the field in terms of *Minkowski modes*, and later on in terms of *Rindler modes*. We will need to normalize these modes in a consistent manner, and we will also need a way to translate between the two quantization schemes. To facilitate this we introduce the *Klein-Gordon inner product*, defined for any complex-valued solutions f, h to (A.2) as

$$\langle f, h \rangle \;=\; -i \int_\Sigma d^3\Sigma_\mu \; \left[f^*(x) \frac{\partial h(x)}{\partial x_\mu} - \frac{\partial f^*(x)}{\partial x_\mu} h(x) \right], \tag{A.10}$$

where the integration is performed over a spacelike hypersurface Σ as in (A.6) [19]. The utility of the above inner product is that $\langle f, h \rangle$ is, once again, independent of the choice of Σ used to integrate it. This will be powerful when expanding our field ϕ in terms of mode functions in different coordinate systems. More generally, we seek to quantize our theory which means to construct a Hilbert space of particle states for our system out of solutions to (A.2) — this requires the notion of an inner product, where (A.10) is a natural choice [20].

To see why $\langle f, h \rangle$ is invariant under the choice of Σ we note that

$$\nabla_\mu \big[- i f^* (\partial^\mu h) + i (\partial^\mu f^*) h \big] = 0 \tag{A.11}$$

which means that the Klein-Gordon inner product is invariant under choice of Σ for the same reason as (A.6). More fundamentally, it is because the differential operator $\overleftrightarrow{W}^\mu \equiv \vec{\partial}^\mu - \overleftarrow{\partial}^\mu$ hidden inside (A.10) is the Wronskian operator corresponding to the Klein-Gordon operator $\Box - m^2$ [15].

We now point out that calling (A.10) an inner product is dishonest without further explanation. It's easy to see that (A.10) is *not* positive-definite in general — for arbitrary complex solutions f to the Klein-Gordon equation, we find that $\langle f, f \rangle$ can be negative or zero. It is precisely the positive frequency condition (A.8) which saves the day here: when considering the *subspace* of complex solutions to the Klein-Gordon equation which are positive-frequency as in (A.8), then (A.10) is a genuine inner product.

One of the reasons it is difficult to quantize a QFT in a generic curved spacetime is that there are arbitrarily many notions of positive and negative frequency (and so time), and so there is no natural choice of subspace which yields a positive-definite inner product [20].

Appendix B

The Minkowski Quantization Scheme

Here we present some background information on the Minkowski quantization scheme, where the field ϕ is quantized relative to ordinary Minkowski time x^0.

B.1 Minkowski Modes

We use the ordinary Minkowski metric $\eta^{\mu\nu}$ in rectangular coordinates *ie.* with the line-element

$$ds^2 \;=\; \eta_{\mu\nu} dx^\mu dx^\nu \;=\; -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \; . \tag{B.1}$$

We describe particular points in Minkowski space with rectangular coordinates $x = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x})$. In this case we return the familiar Lagrangian

$$\mathscr{L}_0[\phi] \;=\; -\tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \tfrac{1}{2} m^2 \phi^2 \; . \tag{B.2}$$

and the Klein-Gordon equation

$$(\Box_x - m^2)\phi(x) \;=\; \left[-\left(\tfrac{\partial}{\partial x^0}\right)^2 + \boldsymbol{\nabla}^2_{\mathbf{x}} - m^2 \right] \phi(x^0, \mathbf{x}) \;=\; 0 \; . \tag{B.3}$$

Here x^0 is a global time coordinate, and since the metric is x^0-independent it is obvious that $\mathcal{T} = \frac{\partial}{\partial x^0}$ (with components $\mathcal{T}^\mu = \delta^\mu{}_0$) is a global timelike Killing vector field. We now make a very important physical choice: *we consider fiducial observers whose proper times are measured along the flow-lines of $\mathcal{T}$.* According to (A.6) the conserved energy associated with this is:

$$H_{\rm M} \quad = \quad -\int_\Sigma d^3\Sigma_\nu \; \delta^\mu{}_0 T^\nu_\mu \tag{B.4}$$

We call this object the *Minkowski Hamiltonian.* Taking the spacelike hypersurface Σ to be a slice of constant $x^0 = t$ and noting our formula for the stress-energy tensor (A.4), the above simplifies to

$$H_{\rm M} \quad = \quad \int d^3\mathbf{x} \; \left[\frac{1}{2}\left(\frac{\partial\phi(x^0,\mathbf{x})}{\partial x^0}\bigg|_{x^0=t}\right)^2 + \frac{1}{2}\big|\nabla_{\mathbf{x}}\phi(t,\mathbf{x})\big|^2 + \frac{1}{2}m^2\phi(t,\mathbf{x})^2\right] \tag{B.5}$$

We recognize this as the usual Hamiltonian that one studies in a first course on QFT. It is a conserved charge here in the sense that $\frac{d}{dt}H_{\rm M} = 0$. Next we wish to study the dynamics imposed on $\phi(x^0,\mathbf{x})$ by the Klein-Gordon equation (B.3) so that we can later quantize this theory. To do this, we construct two sets of solutions $\{u_{\mathbf{k}}\}$ and $\{u^*_{\mathbf{k}}\}$ where for *any* mode labels $\mathbf{k} = (k^1,k^2,k^3) \in \mathbb{R}^3$ we have:

$$u_{\mathbf{k}}(x) \quad = \quad \left(16\pi^3\sqrt{|\mathbf{k}|^2+m^2}\right)^{-\frac{1}{2}} \; e^{-i\sqrt{|\mathbf{k}|^2+m^2}x^0 + i\mathbf{k}\cdot\mathbf{x}} \tag{B.6}$$

$$u^*_{\mathbf{k}}(x) \quad = \quad \left(16\pi^3\sqrt{|\mathbf{k}|^2+m^2}\right)^{-\frac{1}{2}} \; e^{+i\sqrt{|\mathbf{k}|^2+m^2}x^0 - i\mathbf{k}\cdot\mathbf{x}} \; . \tag{B.7}$$

It is easy to see that $(\Box_x - m^2)u_{\mathbf{k}}(x) = (\Box_x - m^2)u^*_{\mathbf{k}}(x) = 0$. These solutions to (B.3) obey the properties:

$$\frac{\partial}{\partial x^0}u_{\mathbf{k}}(x) \quad = \quad -i\sqrt{|\mathbf{k}|^2+m^2}\,u_{\mathbf{k}}(x) \tag{B.8}$$

$$\frac{\partial}{\partial x^0}u^*_{\mathbf{k}}(x) \quad = \quad +i\sqrt{|\mathbf{k}|^2+m^2}\,u^*_{\mathbf{k}}(x) \tag{B.9}$$

so $u_{\mathbf{k}}$ are of positive-frequency with respect to the Minkowski time x^0 (and $u^*_{\mathbf{k}}$ are of negative-frequency). They are also normalized with respect to the Klein-Gordon inner product such that

$$\langle u_{\mathbf{k}}, u_{\mathbf{p}}\rangle \quad = \quad \delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{B.10}$$

$$\langle u_{\mathbf{k}}, u^*_{\mathbf{p}}\rangle \quad = \quad 0 \tag{B.11}$$

$$\langle u^*_{\mathbf{k}}, u^*_{\mathbf{p}}\rangle \quad = \quad -\delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{B.12}$$

The normalizations in (B.10-B.12) are computed most simply by picking Σ to be a slice of constant $x^0 = t$ in (A.10) so that the Klein-Gordon inner product is now

$$\langle f, h \rangle \;=\; +i \int d^3\mathbf{x} \left[f^*(t,\mathbf{x})\,(\partial_0 h(t,\mathbf{x})) - (\partial_0 f^*(t,\mathbf{x}))\,h(t,\mathbf{x}) \right] . \tag{B.13}$$

We find that the Minkowski modes $\{u_{\mathbf{k}}, u^*_{\mathbf{k}}\}$ are *complete* [21] in that we obtain the following resolution of the identity:

$$+i \int d^3\mathbf{k}\;\left[u^*_{\mathbf{k}}(t,\mathbf{x})\,(\partial_0 u_{\mathbf{k}}(t,\mathbf{y})) - u_{\mathbf{k}}(t,\mathbf{x})\,(\partial_0 u^*_{\mathbf{k}}(t,\mathbf{y}))\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{B.14}$$

where $t \in \mathbb{R}$ is arbitrary. The relation (B.14) allows us to expand the field ϕ in terms of the Minkowski modes (using the form (B.13) of the Klein-Gordon inner product)

$$\phi(x^0,\mathbf{x}) \;=\; \int d^3\mathbf{k} \left[\langle u_{\mathbf{k}}, \phi\rangle\, u_{\mathbf{k}}(x^0,\mathbf{x}) - \langle u^*_{\mathbf{k}}, \phi\rangle\, u^*_{\mathbf{k}}(x^0,\mathbf{x}) \right] \tag{B.15}$$

By imposing that our scalar field is *real* with $\phi^* = \phi$, we find that the expansion coefficients are related to each other through $-\langle u^*_{\mathbf{k}}, \phi\rangle = \langle u_{\mathbf{k}}, \phi\rangle^*$ putting (B.15) into the more convenient form

$$\phi(x) \;=\; \int d^3\mathbf{k} \left[\langle u_{\mathbf{k}}, \phi\rangle\, u_{\mathbf{k}}(x) + \langle u_{\mathbf{k}}, \phi\rangle^*\, u^*_{\mathbf{k}}(x) \right] . \tag{B.16}$$

We are now ready to quantize our theory: we do this by 'upgrading' our expansion coefficients to *operators* which are labelled by the mode parameters $\mathbf{k} \in \mathbb{R}^3$:

$$\langle u_{\mathbf{k}}, \phi\rangle \;\to\; \hat{a}_{\mathbf{k}} \tag{B.17}$$

$$\langle u_{\mathbf{k}}, \phi\rangle^* \;\to\; \hat{a}^\dagger_{\mathbf{k}} \tag{B.18}$$

These are assumed to obey the canonical commutation relations (encapsulating Bose-Einstein statistics, since we have a spinless field):

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{p}}\right] \;=\; \delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{B.19}$$

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{p}}\right] \;=\; 0 \tag{B.20}$$

$$\left[\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{p}}\right] \;=\; 0 \tag{B.21}$$

Now our field is an operator

$$\hat{\phi}(x) \;=\; \int d^3\mathbf{k}\,\left[\hat{a}_{\mathbf{k}} u_{\mathbf{k}}(x) + \hat{a}^{\dagger}_{\mathbf{k}} u^*_{\mathbf{k}}(x)\right] \tag{B.22}$$

and the Minkowski Hamiltonian (B.5) now takes on the simple form

$$\hat{H}_{\mathrm{M}} \;=\; \int d^3\mathbf{k}\;\tfrac{1}{2}\sqrt{|\mathbf{k}|^2+m^2}\;\left[\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}\right] \tag{B.23}$$

It's said that the Minkowski modes *diagonalize* the Minkowski Hamiltonian. It's now also clear that we recover the usual 'equal time' commutation relation

$$\left[\hat{\phi}(x), \hat{\Pi}(y)\right]\Big|_{x^0=y^0} = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\ , \tag{B.24}$$

where classically $\Pi = \frac{\delta\mathcal{L}_0}{\delta(\partial_0\phi)} = \partial_0\phi$ is the momentum field conjugate to our ϕ field [14]. This is equivalent to the relation (B.14).

B.2 Multi-Particle States

To motivate the construction to our Hilbert space of states, we consider an eigenstate $|E\rangle$ of the Minkowski Hamiltonian such that $\hat{H}_{\mathrm{M}}\,|E\rangle = E\,|E\rangle$. Using the form (B.23) we can write the energy E corresponding to this state as

$$E = \langle E|\hat{H}_{\mathrm{M}}|E\rangle \;=\; \int d^3\mathbf{k}\;\tfrac{1}{2}\sqrt{\mathbf{k}^2+m^2}\big[\,\langle E|\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}|E\rangle + \langle E|\hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}|E\rangle\,\big] \;\geq\; 0\ , \tag{B.25}$$

where we assume that the norm of any state in our Hilbert space is positive semi-definite (so that here $\langle E|\hat{a}^{\dagger}_{\mathbf{k}}\hat{a}_{\mathbf{k}}|E\rangle \geq 0$ and $\langle E|\hat{a}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}|E\rangle \geq 0$). This tells us that the energy is bounded from below (*ie.* there exists a state of minimum energy) [23]. It follows from (B.19)-(B.21) that the states $\hat{a}_{\mathbf{k}}\,|E\rangle$ and $\hat{a}^{\dagger}_{\mathbf{k}}\,|E\rangle$ have corresponding energies $E-\sqrt{|\mathbf{k}|^2+m^2}$ and $E+\sqrt{|\mathbf{k}|^2+m^2}$ respectively. This means that acting on an energy eigenstate with $\hat{a}_{\mathbf{k}}$ *lowers* the energy of the state, and acting with $\hat{a}^{\dagger}_{\mathbf{k}}$ *raises* the energy. Since the energy is bounded from below, by successively applying combinations of $\hat{a}_{\mathbf{k}}$ we eventually reach the state of minimum energy $|0_{\mathrm{M}}\rangle$ which cannot be lowered any further [23], defined by the condition

$$\hat{a}_{\mathbf{k}}\,|0_{\mathrm{M}}\rangle = 0\ . \tag{B.26}$$

We call this state $|0_\mathrm{M}\rangle$ the *Minkwoski vacuum* and assume that it is appropriately normalized with $\langle 0_\mathrm{M}|0_\mathrm{M}\rangle = 1$ [22]. From here we can define the Hilbert space of single-particle states $\mathcal{H}$ as being spanned by the vectors

$$|\mathbf{k}\rangle \equiv \hat{a}^\dagger_\mathbf{k} |0_\mathrm{M}\rangle \ . \tag{B.27}$$

There are infinitely many such single-particle states specified by their mode parameters $\mathbf{k} \in \mathbb{R}^3$. They are orthonormal with

$$\langle \mathbf{k}|\mathbf{p}\rangle = \delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{B.28}$$

on account of (B.19). A state $|\mathbf{k}\rangle$ represents a single particle with a corresponding energy $\omega_\mathbf{k}$ (one must normal-order the Minkowski Hamiltonian to see this explicitly). We also notice that this implies that a given state $|\mathbf{k}\rangle$ is not formally normalizable since $\langle \mathbf{k}|\mathbf{k}\rangle = \delta^{(3)}(\mathbf{0})$. This is unsurprising though, as we've already argued that $a^\dagger_\mathbf{k}$ creates a single particle of definite energy and momentum — by the uncertainty principle we cannot know exactly where this particle is located. We should then expect from ordinary quantum mechanics that since the volume of our system is infinite the corresponding wavefunction is a non-normalizable plane-wave [24] — this is precisely what the infinity $\delta^{(3)}(\mathbf{0})$ represents. To obtain a physical, normalizable state we must build a *wave-packet* by superposition, or rather by 'smearing' out the operator $\hat{a}^\dagger_\mathbf{k}$ in states of the form

$$\int d^3\mathbf{k}\ F(\mathbf{k})\ \hat{a}^\dagger_\mathbf{k} |0_\mathrm{M}\rangle \ . \tag{B.29}$$

If the test function F is square-integrable in the sense that $\int d^3\mathbf{k}\ |F(\mathbf{k})|^2 < \infty$ then the state (B.29) is normalizable. We can loosely see then that states involving $\hat{a}^\dagger_\mathbf{k}$ only make sense underneath an integral with a well-behaved test function F, actually making the object $\hat{a}^\dagger_\mathbf{k}$ an *operator-valued distribution* [24]. This holds true for all the related operators, most importantly the quantized field $\hat{\phi}$. We need to be aware of this in this work — most notably, we need to use some properties of distributions in Appendix C and we have to 'smear' some raising and lowering operators in Appendix D when proving the thermalization theorem.

Having established the notion of single-particle states, we can define a *two*-particle state as the state created by hitting the vacuum with *two* raising operators

$$|\mathbf{k}_1\mathbf{k}_2\rangle \;\; \equiv \;\; \hat{a}^\dagger_{\mathbf{k}_1}\hat{a}^\dagger_{\mathbf{k}_2}\,|0_{\rm M}\rangle \tag{B.30}$$

Since $\hat{a}^\dagger_{\mathbf{k}_1}$ and $\hat{a}^\dagger_{\mathbf{k}_2}$ commute, we see that two-particle states are symmetric under the interchange $\mathbf{k}_1 \leftrightarrow \mathbf{k}_2$ and so we conclude that our particles are indeed *bosons*. Consequently, such states are *symmetrized* tensor products of single particle states. These states obey the normalization

$$\langle\mathbf{p}_1\mathbf{p}_2|\mathbf{k}_1\mathbf{k}_2\rangle = \delta^{(3)}(\mathbf{p}_1-\mathbf{k}_1)\delta^{(3)}(\mathbf{p}_2-\mathbf{k}_2) + \delta^{(3)}(\mathbf{p}_1-\mathbf{k}_2)\delta^{(3)}(\mathbf{p}_2-\mathbf{k}_1)\ , \tag{B.31}$$

which again underlines the symmetry under exchange of particles. A general multi-particle state containing N particles we define as

$$|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_N\rangle \;\; \equiv \;\; a^\dagger_{\mathbf{k}_1}a^\dagger_{\mathbf{k}_2}\cdots a^\dagger_{\mathbf{k}_N}\,|0_{\rm M}\rangle \ \ . \tag{B.32}$$

In general, the normalization for an N-particle state with an M-particle state is

$$\langle\mathbf{p}_1\mathbf{p}_2\cdots\mathbf{p}_N|\mathbf{k}_1\mathbf{k}_2\cdots\mathbf{k}_M\rangle = \delta^N_{\ M}\sum_{\mathscr{P}}\prod_{j=1}^N \delta^{(3)}\big(\mathbf{k}_j - \mathbf{p}_{\mathscr{P}(j)}\big)\ , \tag{B.33}$$

where the sum is taken over all permutations $\mathscr{P}$ of the integers $\{1,\ldots,N\}$ [22]. This has the required bosonic symmetry properties, and is the generalization of (B.31).

Appendix C

Corrections to the Thermal Propagator

Here we consider the massless theory of section 3.5. The momentum-space thermal propagators are:

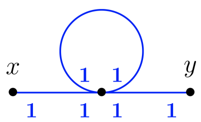
$$-i\Delta^{\beta}_{11}(p) = \frac{-i}{-p_0^2+|\mathbf{p}|^2-i\epsilon} + \frac{2\pi\delta(-p_0^2+|\mathbf{p}|^2)}{e^{\beta|p_0|}-1} \tag{C.1}$$

$$-i\Delta^{\beta}_{12}(p) = -i\Delta^{\beta}_{21}(p) = \pi\text{csch}\left(\tfrac{\beta|p_0|}{2}\right)\delta(-p_0^2+|\mathbf{p}|^2) \tag{C.2}$$

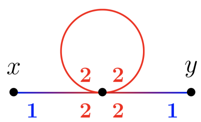
$$-i\Delta^{\beta}_{22}(p) = \frac{i}{-p_0^2+|\mathbf{p}|^2+i\epsilon} + \frac{2\pi\delta(-p_0^2+|\mathbf{p}|^2)}{e^{\beta|p_0|}-1} \tag{C.3}$$

The tadpole correction to the (physical) 11-propagator is a sum over two diagrams; the external points are fixed to be of type 1 and the vertices must be varied over types 1 *and* 2 [50, 51]:

$$G^{\beta}_{\text{Tadpole}}(x;y)_{11} =$$



$$+$$



(C.4)

$$= -3i\lambda \int \frac{d^4p}{(2\pi)^4}\left[-i\Delta^{\beta}_{11}(p)\right]^2 e^{ip\cdot(x-y)} \int \frac{d^4k}{(2\pi)^4}\left[-i\Delta^{\beta}_{11}(k)\right] \tag{C.5}$$
$$+ 3i\lambda \int \frac{d^4p}{(2\pi)^4}\left[-i\Delta^{\beta}_{12}(p)\right]^2 e^{ip\cdot(x-y)} \int \frac{d^4k}{(2\pi)^4}\left[-i\Delta^{\beta}_{22}(k)\right]$$

The loop integrals evaluate to

$$\left[\int \frac{d^4k}{(2\pi)^4}\left[-i\Delta^{\beta}_{11}(k)\right]\right] = \left[\int \frac{d^4k}{(2\pi)^4}\left[-i\Delta^{\beta}_{22}(k)\right]\right]^* = \mathcal{I}_{\text{Top}} + \mathcal{T}_{\text{Top}} \tag{C.6}$$

Where:

$$\mathcal{I}_{\text{Top}} = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{i}{k_0^2 - |\mathbf{k}|^2 + i\epsilon} \to 0 \tag{C.7}$$

$$\mathcal{T}_{\text{Top}} = \int_{-\infty}^{\infty} dk_0 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\delta(k_0^2 - |\mathbf{k}|^2)}{e^{\beta|k_0|} - 1} = \frac{1}{2\pi^2\beta^2} \int_0^{\infty} d\chi \, \frac{\chi}{e^{\chi} - 1} = \frac{1}{12\beta^2} \tag{C.8}$$

We set $\mathcal{I}_{\text{Top}} = 0$ since we already know from Chapter 2 that this loop vanishes in the massless limit and the novel thermal integral evaluates trivially. Now (C.6) simplifies to:

$$G^{\beta}_{\text{Tadpole}}(x; y)_{11} = \frac{i\lambda}{4\beta^2} \int \frac{d^4p}{(2\pi)^4} \left([-i\Delta_{12}(p)]^2 - [-i\Delta_{11}(p)]^2 \right) e^{ip\cdot(x-y)} \tag{C.9}$$

Using the explicit forms (C.1) and (C.2) the function in the brackets becomes:

$$\begin{aligned} [-i\Delta_{12}(p)]^2 - [-i\Delta_{11}(p)]^2 &= \frac{1}{(-p_0^2 + |\mathbf{p}|^2 - i\epsilon)^2} + \frac{4\pi i}{e^{\beta|p_0|} - 1} \frac{\delta(-p_0^2 + |\mathbf{p}|^2)}{-p_0^2 + |\mathbf{p}|^2 - i\epsilon} \\ &\quad + \pi^2 \left[\text{csch}^2\left(\tfrac{\beta|p_0|}{2}\right) - \frac{4}{(e^{\beta|p_0|} - 1)^2} \right] \left[\delta(-p_0^2 + |\mathbf{p}|^2)\right]^2 \end{aligned} \tag{C.10}$$

For any $\chi > 0$ we have the identity $\text{csch}^2(\frac{\chi}{2}) - \frac{4}{(e^{\chi}-1)^2} = \frac{4}{e^{\chi}-1}$ and so the above simplifies to:

$$\begin{aligned} \big[- i\Delta_{12}(p)\big]^2 - \big[- i\Delta_{11}(p)\big]^2 &= \frac{1}{(-p_0^2 + |\mathbf{p}|^2 - i\epsilon)^2} + \frac{4\pi i}{e^{\beta|p_0|} - 1} \frac{\delta(-p_0^2 + |\mathbf{p}|^2)}{-p_0^2 + |\mathbf{p}|^2 - i\epsilon} \\ &\quad + \frac{4\pi^2}{e^{\beta|p_0|} - 1} \left[\delta(-p_0^2 + |\mathbf{p}|^2)\right]^2 \end{aligned} \tag{C.11}$$

At this point we use the regularization $\frac{\delta(z)}{z - i\epsilon} - i\pi \left[\delta(z)\right]^2 = -\frac{1}{2}\delta'(z)$ (where δ' is the derivative of the Dirac delta) [35] and we can write (C.9) as:

$$G^{\beta}_{\text{Tadpole}}(x; y)_{11} = \frac{i\lambda}{4\beta^2} \int \frac{d^4p}{(2\pi)^4} \left[\tfrac{1}{(-p_0^2 + |\mathbf{p}|^2 - i\epsilon)^2} - \tfrac{2\pi i \delta'(-p_0^2 + |\mathbf{p}|^2)}{e^{\beta|p_0|} - 1} \right] e^{ip\cdot(x-y)} \tag{C.12}$$

For any even function $f(-\omega) = f(\omega)$ we can write its 1D Fourier transform in terms of a cosine Fourier transform as follows:

$$\int_{-\infty}^{\infty} d\omega \; f(\omega) e^{-it\omega} = 2\int_0^{\infty} d\omega \; f(\omega)\cos(t\omega) \tag{C.13}$$

Furthermore for any radial function $g(\boldsymbol{\sigma}) = G(|\boldsymbol{\sigma}|)$, we can write its 3D Fourier transform as:

$$\int \frac{d^3\boldsymbol{\sigma}}{(2\pi)^3} G(|\boldsymbol{\sigma}|) e^{-i\mathbf{r}\cdot\boldsymbol{\sigma}} = \frac{1}{2\pi^2|\mathbf{r}|} \int_0^{\infty} d\Sigma \; \Sigma \sin(|\mathbf{r}|\Sigma) G(\Sigma) \tag{C.14}$$

So in terms of the dimensionless variables $\Sigma = \beta|\mathbf{p}|$ and $\omega = \beta p_0$ (C.12) becomes

$$\begin{aligned} G^{\beta}_{\text{Tadpole}}(x;y)_{11} &= \frac{i\lambda}{8\pi^3\beta|\mathbf{x}-\mathbf{y}|}\int_0^{\infty} d\omega\ \cos\left(\tfrac{(x^0-y^0)\omega}{\beta}\right) \\ &\quad\times \int_0^{\infty} d\Sigma\ \Sigma\sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\left[\tfrac{1}{(-\omega^2+\Sigma^2-i\epsilon)^2} - \tfrac{2\pi i\delta'(-\omega^2+\Sigma^2)}{e^{\omega}-1}\right] \end{aligned} \tag{C.15}$$

where we've used the scaling property $\delta'(a\chi) = \frac{\delta'(\chi)}{|a|^2}$ (immediately following from $\delta(a\chi) = \frac{\delta(\chi)}{|a|}$). Next we take the limit $\epsilon \to 0^+$ using the regularization $\frac{1}{(z-i\epsilon)^2} = \text{Pf}\left[\frac{1}{z^2}\right] - i\pi\delta'(z)$ [38] giving

$$\begin{aligned} G^{\beta}_{\text{Tadpole}}(x;y)_{11} &= \frac{i\lambda}{8\pi^3\beta|\mathbf{x}-\mathbf{y}|}\int_0^{\infty} d\omega\ \cos\left(\tfrac{(x^0-y^0)\omega}{\beta}\right) \\ &\quad\times \int_0^{\infty} d\Sigma\ \Sigma\sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\text{Pf}\left[\tfrac{1}{(-\omega^2+\Sigma^2)^2}\right] \\ &\quad+\frac{\lambda}{8\pi^2\beta|\mathbf{x}-\mathbf{y}|}\int_0^{\infty} d\omega\ \cos\left(\tfrac{(x^0-y^0)\omega}{\beta}\right)\coth\left(\tfrac{\omega}{2}\right) \\ &\quad\times \int_0^{\infty} d\Sigma\ \Sigma\sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\delta'(-\omega^2+\Sigma^2) \end{aligned} \tag{C.16}$$

where we've made use of the identity $1 + \frac{2}{e^{\omega}-1} = \coth\left(\frac{\omega}{2}\right)$. The pseudo-function $\text{Pf}\left[\frac{1}{z^2}\right]$ is the regularization of $\frac{1}{z^2}$ [38] obeying $\int_{-\infty}^{\infty} dz\text{Pf}\left[\frac{1}{z^2}\right] f(z) = \mathcal{PV}\int_{-\infty}^{\infty} dz\frac{f'(z)}{z}$ (where $\mathcal{PV}$ is the Cauchy principal value) [52]. We make use of a partial fraction expansion for the first Σ-integral in (C.16) giving:

$$\begin{aligned} \int_0^{\infty} d\Sigma\ \Sigma\sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\text{Pf}\left[\tfrac{1}{(-\omega^2+\Sigma^2)^2}\right] &= \frac{1}{4\omega}\int_0^{\infty} d\Sigma\ \sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\text{Pf}\left[\tfrac{1}{(\Sigma-\omega)^2}\right] \\ &\quad-\frac{1}{4\omega}\int_0^{\infty} d\Sigma\ \sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\text{Pf}\left[\tfrac{1}{(\Sigma+\omega)^2}\right] \end{aligned} \tag{C.17}$$

$$\begin{aligned} &= \frac{|\mathbf{x}-\mathbf{y}|}{4\omega\beta}\,\mathcal{PV}\int_0^{\infty} d\Sigma\ \cos\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\tfrac{1}{\Sigma-\omega} \\ &\quad-\frac{|\mathbf{x}-\mathbf{y}|}{4\omega\beta}\,\mathcal{PV}\int_0^{\infty} d\Sigma\ \cos\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\tfrac{1}{\Sigma+\omega} \end{aligned} \tag{C.18}$$

$$= -\frac{\pi|\mathbf{x}-\mathbf{y}|}{4\omega\beta}\sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\omega}{\beta}\right) \tag{C.19}$$

Next we make note of the well-known identity $\delta(-\omega^2+\Sigma^2) = \frac{\delta(\Sigma-\omega)+\delta(\Sigma+\omega)}{2|\omega|}$ which may be differentiated to yield [52]:

$$\delta'(-\omega^2+\Sigma^2) = \frac{\delta'(\Sigma-\omega)+\delta'(\Sigma+\omega)}{4|\omega|\Sigma} \tag{C.20}$$

With this and the rule $\int_{-\infty}^{\infty} dz f(z)\delta'(z) = -\int_{-\infty}^{\infty} dz f'(z)\delta(z)$ the second Σ-integral in (C.16) may be integrated giving (recalling that $\omega > 0$):

$$\begin{aligned}
\int_0^\infty d\Sigma\ \Sigma \sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\delta'(-\omega^2+\Sigma^2) &= \frac{1}{4\omega}\int_0^\infty d\Sigma\ \sin\left(\tfrac{|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\left[\delta'(\Sigma-\omega)+\delta'(\Sigma+\omega)\right] &\text{(C.21)}\\
&= -\frac{|\mathbf{x}-\mathbf{y}|}{4\omega\beta}\int_0^\infty d\Sigma\ \cos\left(\tfrac{T|\mathbf{x}-\mathbf{y}|\Sigma}{\beta}\right)\left[\delta'(\Sigma-\omega)+\delta'(\Sigma+\omega)\right] &\text{(C.22)}\\
&= -\frac{|\mathbf{x}-\mathbf{y}|}{4\omega\beta}\cos\left(\tfrac{|\mathbf{x}-\mathbf{y}|\omega}{\beta}\right) &\text{(C.23)}
\end{aligned}$$

Putting (C.19) and (C.23) all together into (C.16) leaves us with:

$$\begin{aligned}
G^\beta_{\text{Tadpole}}(x;y)_{11} &= -\frac{i\lambda}{32\pi^2\beta^2}\int_0^\infty d\omega\ \frac{\sin\left(\frac{|\mathbf{x}-\mathbf{y}|\omega}{\beta}\right)\cos\left(\frac{(x^0-y^0)\omega}{\beta}\right)}{\omega} \\
&\quad -\frac{\lambda}{32\pi^2\beta^2}\int_0^\infty d\omega\ \frac{\coth\left(\frac{\omega}{2}\right)\cos\left(\frac{|\mathbf{x}-\mathbf{y}|\omega}{\beta}\right)\cos\left(\frac{(x^0-y^0)\omega}{\beta}\right)}{\omega}
\end{aligned} \tag{C.24}$$

And setting $\mathbf{x}=\mathbf{y}$ we have:

$$G^\beta_{\text{Tadpole}}(x^0,\mathbf{x};y^0,\mathbf{x})_{11} = -\frac{\lambda}{32\pi^2\beta^2}\int_0^\infty d\omega\ \frac{\coth\left(\frac{\omega}{2}\right)}{\omega}\cos\left(\tfrac{(x^0-y^0)\omega}{\beta}\right) \tag{C.25}$$

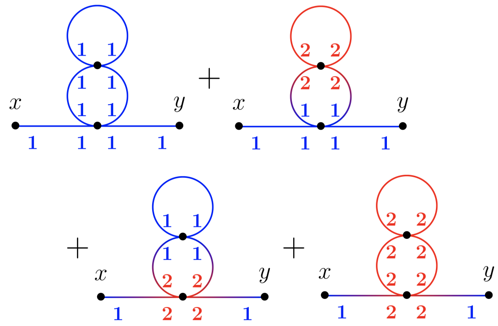
We interpret the singular distribution being cosine Fourier transformed in the following manner:

$$\frac{\coth\left(\frac{\omega}{2}\right)}{\omega} = \frac{2}{\omega^2} + \left[\frac{\coth\left(\frac{\omega}{2}\right)}{\omega} - \frac{2}{\omega^2}\right] \tag{C.26}$$

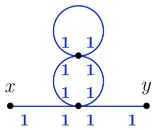
This distribution has one singularity of the form $\frac{2}{\omega^2}$ and the remainder function $\frac{\coth\left(\frac{\omega}{2}\right)}{\omega} - \frac{2}{\omega^2}$ has absolutely integrable N^{th} derivatives for all $N \geq 1$ on the real line, and also falls to zero at $\omega \to \infty$. This means that the asymptotic form of the function in the limit $\frac{x^0-y^0}{\beta} \to \infty$ is governed by the Fourier cosine transform of $\frac{2}{\omega^2}$ which is $-\pi\frac{|x^0-y^0|}{\beta}$ [53] giving the result:

$$G^\beta_{\text{Tadpole}}(x^0,\mathbf{x};y^0,\mathbf{x})_{11} = \frac{1}{32\pi}\frac{\lambda|x^0-y^0|}{\beta^3} + \text{ subdominant} \tag{C.27}$$

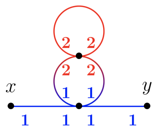
The computation of the massless cactus (11)-correction consists of the sum of the four diagrams

$$G^{\beta}_{\text{Cactus}}(x;y)_{11} \;=\;$$  (C.28)

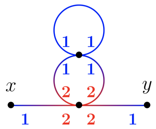
We note that the structure of these diagrams are very similar to those of the tadpole — the top loop evaluates as in (C.6) for each of these in the same way as for the tadpoles, which yields the simplification



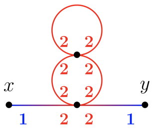
$$= -\frac{3}{2}\frac{\lambda^2}{12\beta^2}\int\frac{d^4p}{(2\pi)^4}[-i\Delta^{\beta}_{11}(p)]^2 e^{ip\cdot(x-y)}\int\frac{d^4k}{(2\pi)^4}[-i\Delta_{11}(k)]^2 \tag{C.29}$$



$$= +\frac{3}{2}\frac{\lambda^2}{12\beta^2}\int\frac{d^4p}{(2\pi)^4}[-i\Delta^{\beta}_{11}(p)]^2 e^{ip\cdot(x-y)}\int\frac{d^4k}{(2\pi)^4}[-i\Delta_{12}(k)]^2 \tag{C.30}$$



$$= +\frac{3}{2}\frac{\lambda^2}{12\beta^2}\int\frac{d^4p}{(2\pi)^4}[-i\Delta^{\beta}_{12}(p)]^2 e^{ip\cdot(x-y)}\int\frac{d^4k}{(2\pi)^4}[-i\Delta_{12}(k)]^2 \tag{C.31}$$



$$= -\frac{3}{2}\frac{\lambda^2}{12\beta^2}\int\frac{d^4p}{(2\pi)^4}[-i\Delta^{\beta}_{12}(p)]^2 e^{ip\cdot(x-y)}\int\frac{d^4k}{(2\pi)^4}[-i\Delta_{22}(k)]^2 \tag{C.32}$$

where we have also used the symmetry $\Delta^{\beta}_{12} = \Delta^{\beta}_{21}$. Summing all of the diagrams and factoring yields

$$\begin{aligned} G^{\beta}_{\text{Cactus}}(x;y)_{11} \;=\; & -\frac{\lambda^2}{8\beta^2}\,\mathcal{M}\int\frac{d^4p}{(2\pi)^4}\left[-i\Delta^{\beta}_{11}(p)\right]^2 e^{ip\cdot(x-y)} \\ & -\frac{\lambda^2}{8\beta^2}\,\mathcal{M}^*\int\frac{d^4p}{(2\pi)^4}\left[-i\Delta^{\beta}_{12}(p)\right]^2 e^{ip\cdot(x-y)} \end{aligned} \tag{C.33}$$

where we have defined the middle loop integral $\mathcal{M}$ as

$$\mathcal{M} \;\equiv\; \int\frac{d^4k}{(2\pi)^4}\left(\left[-i\Delta_{11}(k)\right]^2 - \left[-i\Delta_{12}(k)\right]^2\right) \tag{C.34}$$

which is simply a loop integral over the function encountered in (C.12). This simplifies to the following

$$\mathcal{M} = \mathcal{I}_{\text{Middle}} + \mathcal{T}_{\text{Middle}} \tag{C.35}$$

$$\mathcal{I}_{\text{Middle}} = -\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k_0^2 - |\mathbf{k}|^2 + i\epsilon)^2} \tag{C.36}$$

$$\mathcal{T}_{\text{Middle}} = -\int \frac{d^4k}{(2\pi)^4} \frac{2\pi i\delta'(k_0^2 - |\mathbf{k}|^2)}{e^{\beta|k_0|} - 1} \tag{C.37}$$

where $\mathcal{I}_{\text{Middle}}$ from (2.47) is IR divergent even after renormalization. Re-instating the mass m as an IR cutoff we have

$$\mathcal{I}_{\text{Middle}} = -\frac{i}{8\pi^2}\frac{1}{4-D} + \frac{i}{16\pi^2}\left[\gamma - \ln(4\pi) + \ln\left(\frac{m^2}{\mu^2}\right)\right] + \mathscr{O}(4-D) \tag{C.38}$$

The second integral $\mathcal{T}_{\text{Middle}}$ is also IR divergent (although UV finite). Re-instating a mass m as an IR cutoff the integral can be manipulated into the form

$$\mathcal{T}_{\text{Middle}} = -\frac{i}{4\pi^2}\int_0^\infty \frac{d\zeta}{\sqrt{\zeta^2 + (m\beta)^2}} \frac{1}{e^{\sqrt{\zeta^2+(m\beta)^2}} - 1} \tag{C.39}$$

This integral cannot be evaluated analytically. In the limit $m \to 0$ this integral has the asymptotic form [41]

$$\mathcal{T}^{\text{Cactus}}(m) = -\frac{i}{8\pi}\frac{1}{\beta m} - \frac{i}{8\pi^2}\ln\left(\frac{\beta m}{4\pi}\right) - \frac{i\gamma}{8\pi^2} + \mathscr{O}\big((\beta m)^2\big) \tag{C.40}$$

Now summing $\mathcal{M} = \mathcal{I}_{\text{Middle}} + \mathcal{T}_{\text{Middle}}$ yields the result

$$\mathcal{M} = -\frac{i}{8\pi^2}\frac{1}{4-D} - \frac{i}{8\pi}\frac{1}{\beta m} + \frac{i}{16\pi^2}\ln\left(\frac{4\pi}{\beta^2\mu^2}\right) - \frac{i\gamma}{16\pi^2} \tag{C.41}$$

It is now obvious that $\mathcal{M}^* = -\mathcal{M}$. After renormalization we keep the finite parts of the graph such that

$$\to \mathcal{M}^{\text{renormalized}} = -\frac{i}{8\pi}\frac{1}{\beta m} + \frac{i}{16\pi^2}\ln\left(\frac{4\pi}{\beta^2\mu^2}\right) - \frac{i\gamma}{16\pi^2} \tag{C.42}$$

Keeping the finite part (C.42) the overall cactus correction is now

$$G^\beta_{\text{Cactus}}(x;y)_{11} = \frac{\lambda^2}{8\beta^2}\mathcal{M}\int \frac{d^4p}{(2\pi)^4}\left(\frac{1}{(p_0^2-|\mathbf{p}|^2+i\epsilon)^2} + \frac{2\pi i\delta'(p_0^2-|\mathbf{p}|^2)}{e^{\beta|p_0|}-1}\right) e^{ip\cdot(x-y)} \tag{C.43}$$

The function begin Fourier-transformed is identical to that of the tadpole computation. Setting $\mathbf{x} = \mathbf{y}$ and considering the limit $\frac{|x^0-y^0|}{\beta} \gg 1$ we find

$$G^\beta_{\text{Cactus}}(x^0,\mathbf{x};y^0,\mathbf{x})_{11} = \frac{-i}{64\pi}\mathcal{M}\frac{\lambda^2|x^0-y^0|}{\beta^3} + \text{ subdominant} \tag{C.44}$$

Appendix D

Wave-Packets and the Thermalization Theorem

In Chapter 4 it was argued that the raising operators for the right Rindler wedge $\mathcal{R}_+$ evaluate in the Minkowski vacuum as

$$\langle 0_{\rm M}| \, \hat{b}^{(\tilde{\sigma})\dagger}_{(\tilde{\Omega},\underline{\mathbf{k}})} \hat{b}^{(\sigma)}_{(\Omega,\underline{\mathbf{p}})} \, |0_{\rm M}\rangle \;\; = \;\; \frac{1}{e^{2\pi\Omega}-1} \, \delta^{\sigma}{}_{\tilde{\sigma}} \delta(\Omega-\tilde{\Omega}) \delta^{(2)}(\underline{\mathbf{p}}-\underline{\mathbf{k}}) \; . \tag{D.1}$$

One might be tempted to say that this is evidence enough that the Minkowski vacuum is experienced as a thermal vacuum, but as it stands this is simply an interesting algebraic relation. In this appendix we provide a proof for the thermalization theorem which shows this definitively.

D.1 Wave-Packets

We'd like to properly prove that the Minkowski vacuum state is experienced as a thermal state for a Rindler observer accelerating through $\mathcal{R}_+$. That is to say, we'd like to prove that for *any* observable $\hat{\mathcal{O}}^{(+)}$ constructed out of the right wedge operators $\hat{b}^{(+)}_{(\tilde{\Omega},\underline{\mathbf{k}})}$ and $\hat{b}^{(+)\dagger}_{(\tilde{\Omega},\underline{\mathbf{k}})}$ that we have the property

$$\langle 0_{\rm M}| \, \hat{\mathcal{O}}^{(+)} \, |0_{\rm M}\rangle = \frac{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}\hat{\mathcal{O}}^{(+)}\right]}{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}\right]} \tag{D.2}$$

where $\hat{\rho}^{(+)}$ is a yet-to-be-determined thermal density matrix, and the trace is taken over states in the right wedge. Taking the trace in the above relation is not trivial since multi-particle states are non-normalizable; if one tries to construct an $n_{\mathbf{k}}$-particle state, all with the *same* momentum $\mathbf{k}$ then we find the norm of this state is infinite. For example, a state of the form $|\mathbf{kk}\rangle = \hat{a}^{\dagger}_{\mathbf{k}}\hat{a}^{\dagger}_{\mathbf{k}}|0\rangle$ has the normalization

$$\langle \mathbf{pp}|\mathbf{kk}\rangle = 2\delta^{(3)}(\mathbf{p}-\mathbf{k})^2 \ , \tag{D.3}$$

which is complete non-sense. We would like a countable, *normalizable* basis of states in order to evaluate the trace in (D.2). Wave-packets provide a simple way to do this[1].

There are many ways to form wave-packets, but here we follow Takagi's choice [14] (which is originally that of Hawking's in [5]). We take $\mathcal{E} > 0$ to be of mass dimension 1, and define the set of functions $\{f_{MN} : \mathbb{R} \to \mathbb{C}\}$ for any integers $M, N \in \mathbb{Z}$ as

$$f_{MN}(K) = \begin{cases} \frac{1}{\sqrt{\mathcal{E}}} e^{-2\pi i N \frac{K}{\mathcal{E}}} & , \text{ for } (M-\frac{1}{2})\mathcal{E} < K < (M+\frac{1}{2})\mathcal{E} \\ 0 & , \text{ otherwise} \end{cases} \tag{D.4}$$

Understanding K to be a one-dimensional momentum, we see that the wavepacket is localized in momentum-space around $K \sim M\mathcal{E}$ with a width $\sim \mathcal{E}$. Fourier transforming f_{MN} to position-space yields a wave-packet localized around $\sim \frac{2\pi N}{\mathcal{E}}$ with width $\sim \frac{2\pi}{\mathcal{E}}$. This is why we need two subscripts M and N on the wave-packet: to parametrize a spread in both position-space *and* momentum-space

These functions are obviously square-integrable, and form an orthonormal set in the sense that

$$\int_{-\infty}^{\infty} dK \ f_{MN}(K) f_{IJ}(K) = \delta^{M}_{\ I}\delta^{N}_{\ J} \ . \tag{D.5}$$

[1] It is worth noting that one can also put the spacetime in a box of volume $L_1 \times L_2 \times L_3$ so that there are finitely-many energy states. Since one can shrink the spacings between energy states by making the box extremely large, the effect is that the system in a box is indistinguishable from the infinite volume system (the box volume can be made so large that a physical measurement cannot resolve the finite energy spacings). Here however we note that Rindler coordinates do not enjoy a translational-invariance in the ξ-coordinate — this makes the box normalization rather awkward in this situation which is why we use wave-packets instead[14].

They are also complete in that they span the vector space $L^2(\mathbb{R})$, satisfying

$$\sum_{M,N=-\infty}^{\infty} f^*_{MN}(K) f^*_{MN}(\tilde{K}) = \delta(K - \tilde{K}) \; . \tag{D.6}$$

Note that by restricting the subscript M to non-negative integers ($\mathbb{Z}^+$), the domain of the function f_{MN} is restricted to the positive real line $[0, \infty)$ (provided that a replacement $\mathcal{E} \to \tilde{\mathcal{E}}$ is made where $\tilde{\mathcal{E}}$ is dimensionless, since Ω is dimensionless). These restricted wave-packets are still orthonormal and complete.

Armed with these functions, we define what we call the *Rindler wave-packets*

$$R^{(\sigma)}_{\mathbf{NM}}(x) \equiv \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}} \; F_{M_1N_1}(\Omega) F_{M_2N_2}(k^2) F_{M_3N_3}(k^3) r^{(\sigma)}_{(\Omega,\underline{\mathbf{k}})}(x) \tag{D.7}$$

where $\mathbf{M} = (M_1, M_2, M_3) \in \mathbb{Z}^+ \times \mathbb{Z} \times \mathbb{Z}$ and $\mathbf{N} = (N_1, N_2, N_3) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. The Rindler wave-packets are orthonormal with respect to the Klein-Gordon inner product

$$\langle R^{(\sigma)}_{\mathbf{MN}}, R^{(\tilde{\sigma})}_{\mathbf{IJ}} \rangle \;\; = \;\; \delta^{\sigma}{}_{\tilde{\sigma}} \delta^{\mathbf{M}}_{\mathbf{I}} \delta^{\mathbf{N}}_{\mathbf{J}} \tag{D.8}$$

where $\delta^{\mathbf{M}}_{\mathbf{I}}$ is a short-hand for $\delta^{\mathbf{M}}_{\mathbf{I}} = \delta^{M_1}_{I_1} \delta^{M_2}_{I_2} \delta^{M_3}_{I_3}$. In addition to this, we notice that we can use these orthonormality conditions to invert (D.7) in the following manner

$$r^{(\sigma)}_{\boldsymbol{\omega}}(x) \;\; = \;\; \sum_{\mathbf{M},\mathbf{N}} F^*_{M_1N_1}(\Omega) F^*_{M_2N_2}(k^2) F^*_{M_2N_2}(k^3) R^{(\sigma)}_{\mathbf{MN}}(x) \; . \tag{D.9}$$

By then by defining the 'smeared' operators

$$\hat{B}_{\mathbf{MN}} \;\; = \;\; \int_{-\infty}^{\infty} d\Omega \int d^2\underline{\mathbf{k}} \; F^*_{M_1N_1}(\Omega) F^*_{M_2N_2}(k^2) F^*_{M_3N_3}(k^3) \hat{b}^{(\sigma)}_{\boldsymbol{\omega}} \; , \tag{D.10}$$

we are able to re-write our field expansions. The field (4.39) in terms of Rindler wave-packets becomes:

$$\begin{aligned} \hat{\phi}(x) \;\; = \;\; \sum_{\mathbf{MN}} \Big[& \hat{B}^{(+)}_{\mathbf{MN}} R^{(+)}_{\mathbf{MN}}(x) + \hat{B}^{(+)\dagger}_{\mathbf{MN}} R^{(+)*}_{\mathbf{MN}}(x) \\ & + \hat{B}^{(-)}_{\mathbf{MN}} R^{(-)}_{\mathbf{MN}}(x) + \hat{B}^{(-)\dagger}_{\mathbf{MN}} R^{(-)*}_{\mathbf{MN}}(x) \Big] \end{aligned} \tag{D.11}$$

It is worth noting that the new 'smeared' operators also satisfy the canonical commutation relations

$$\left[\hat{B}^{(\sigma)}_{\mathbf{MN}}, \hat{B}^{(\tilde{\sigma})\dagger}_{\mathbf{MN}}\right] = \delta^{\sigma}{}_{\tilde{\sigma}}\delta^{\mathbf{M}}_{\mathbf{I}}\delta^{\mathbf{N}}_{\mathbf{J}} \tag{D.12}$$

$$\left[\hat{B}^{(\sigma)}_{\mathbf{MN}}, \hat{B}^{(\tilde{\sigma})}_{\mathbf{MN}}\right] = \left[\hat{B}^{(\sigma)\dagger}_{\mathbf{MN}}, \hat{B}^{(\tilde{\sigma})\dagger}_{\mathbf{MN}}\right] = 0 \tag{D.13}$$

So we can define a family of 'smeared' wave-packet states. For example, for a wave-packet state characterized by **MN** in the right wedge we write as

$$|1^{(+)}_{\mathbf{MN}}; 0^{(-)}\rangle \equiv \hat{B}^{(+)\dagger}_{\mathbf{MN}}\,|0_{\mathrm{R}}\rangle \ . \tag{D.14}$$

We can simultaneously create a particle in the left wedge bywriting:

$$|1^{(+)}_{\mathbf{MN}}; 1^{(-)}_{\mathbf{IJ}}\rangle \equiv \hat{B}^{(+)\dagger}_{\mathbf{MN}}\hat{B}^{(-)\dagger}_{\mathbf{IJ}}\,|0_{\mathrm{R}}\rangle \ . \tag{D.15}$$

We can generally define a state containing n wave-packets (all in the state **MN**) in the right wedge and m wave-packets (all in the state **IJ**) in the left wedge as:

$$|n^{(+)}_{\mathbf{MN}}; m^{(-)}_{\mathbf{IJ}}\rangle = \frac{\left(\hat{B}^{(+)\dagger}_{\mathbf{MN}}\right)^n}{\sqrt{n!}}\frac{\left(\hat{B}^{(-)\dagger}_{\mathbf{IJ}}\right)^m}{\sqrt{m!}}\,|0_{\mathrm{R}}\rangle \tag{D.16}$$

These are extremely useful in that they are *normalizable states* with

$$\langle n^{(+)}_{\mathbf{MN}}; m^{(-)}_{\mathbf{IJ}}|\tilde{n}^{(+)}_{\tilde{\mathbf{M}}\tilde{\mathbf{N}}}; \tilde{m}^{(-)}_{\tilde{\mathbf{I}}\tilde{\mathbf{J}}}\rangle = \delta^{n}{}_{\tilde{n}}\delta^{m}{}_{\tilde{m}}\delta^{\mathbf{M}}{}_{\tilde{\mathbf{M}}}\delta^{\mathbf{N}}{}_{\tilde{\mathbf{N}}}\delta^{\mathbf{I}}{}_{\tilde{\mathbf{I}}}\delta^{\mathbf{J}}{}_{\tilde{\mathbf{J}}} \ . \tag{D.17}$$

These are the states we're going to be working with when proving the Thermalization theorem.

We end this section by noting that the Hamiltonian $H^{(+)}_{\mathrm{R}}$ (4.34) takes on a very pleasing form when expressed in terms of the smeared operators $\hat{B}^{(+)}_{\mathbf{MN}}$ and $\hat{B}^{(+)\dagger}_{\mathbf{MN}}$. We find that:

$$\hat{H}^{(+)}_{\mathrm{R}} = \sum_{\mathbf{MN}} \Omega_{\mathbf{M}}\hat{B}^{(+)\dagger}_{\mathbf{MN}}\hat{B}^{(+)}_{\mathbf{MN}} \tag{D.18}$$

$$\Omega_{\mathbf{M}} = M_1\tilde{\mathcal{E}} \tag{D.19}$$

Note that the 'energy' $\Omega_{\mathbf{M}} = M_1\tilde{\mathcal{E}}$ only depends on the first label M_1 and the dimensionless parameter $\tilde{\mathcal{E}}$ appearing the restricted version of the wave-packet (recall that $\tilde{\mathcal{E}}$ approximately represents where the energy of the wave-packet is localized).

D.2 The Thermalization Theorem

Having developed all the required equipment, here we prove the thermalization theorem (first given by Sewell in [43]). First we define the operator

$$\hat{\rho}^{(+)}_{\mathbf{MN}} = \exp\left(-2\pi\Omega_{\mathbf{M}}\hat{B}^{(+)\dagger}_{\mathbf{MN}}\hat{B}^{(+)}_{\mathbf{MN}}\right) \tag{D.20}$$

which is specific to a particular mode $\mathbf{MN}$. Note that according to (D.18) we have

$$\prod_{\mathbf{MN}} \hat{\rho}^{(+)}_{\mathbf{MN}} \;=\; e^{-2\pi\hat{H}^{(+)}_{\mathrm{R}}} \tag{D.21}$$

We take the trace of the above operator *taken over the* $\mathbf{MN}$ *states in the right Rindler wedge*:

$$\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\right] \;=\; \sum_{n=0}^{\infty}\langle n^{(+)}_{\mathbf{MN}};0^{(-)}\big|e^{-2\pi\Omega_{\mathbf{M}}\hat{B}^{(+)\dagger}_{\mathbf{MN}}\hat{B}^{(+)}_{\mathbf{MN}}}\big|n^{(+)}_{\mathbf{MN}};0^{(-)}\rangle \tag{D.22}$$

We can explicitly compute this as

$$\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\right] \;=\; \sum_{n=0}^{\infty} e^{-2\pi n\Omega_{\mathbf{M}}} \;=\; \frac{1}{1-e^{-2\pi\Omega_{\mathbf{M}}}} \tag{D.23}$$

Next we consider an arbitrary operator $\hat{\mathcal{O}}^{(+)}$ which consists only modes $\mathcal{R}_+$ *ie.* it is written in terms of only the operators $\hat{B}^{(+)\dagger}_{\mathbf{MN}}$ and $\hat{B}^{(+)}_{\mathbf{MN}}$. We compute a similar trace:

$$\begin{aligned}\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\hat{\mathcal{O}}^{(+)}\right] \;&=\; \sum_{n=0}^{\infty}\langle n^{(+)}_{\mathbf{MN}};0^{(-)}\big|e^{-2\pi H^{(+)}_{\mathrm{R}}}\big|n^{(+)}_{\mathbf{MN}};0^{(-)}\rangle && \text{(D.24)}\\ &=\; \sum_{n=0}^{\infty} e^{-2\pi n\Omega_{\mathbf{M}}}\,\langle n^{(+)}_{\mathbf{MN}};0^{(-)}\big|\hat{\mathcal{O}}^{(+)}\big|n^{(+)}_{\mathbf{MN}};0^{(-)}\rangle && \text{(D.25)}\end{aligned}$$

Putting this together we write:

$$\frac{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\hat{\mathcal{O}}^{(+)}\right]}{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\right]} = \left(1-e^{-2\pi\Omega_{\mathbf{M}}}\right)\sum_{n=0}^{\infty} e^{-2\pi n\Omega_{\mathbf{M}}}\,\langle n^{(+)}_{\mathbf{MN}};0^{(-)}\big|\hat{\mathcal{O}}^{(+)}\big|n^{(+)}_{\mathbf{MN}};0^{(-)}\rangle \tag{D.26}$$

Next we define the operator

$$\hat{Q}_{\mathbf{MN}} \;=\; e^{-\pi\Omega_{\mathbf{M}}}\hat{B}^{(+)\dagger}_{\mathbf{MN}}\hat{B}^{(-)\dagger}_{(M_1,-M_2,-M_3)(N_1,-N_2,-N_3)}\ , \tag{D.27}$$

where we take note of the sign swap taking place on the labels of the second operator (this is a consequence of the Bogoliubov relations (4.46-4.47) in the way of the sign switch $\underline{\mathbf{k}} \leftrightarrow -\underline{\mathbf{k}}$ there). By successive application of the operator $\hat{Q}_{\mathbf{MN}}$

to the Rindler vacuum it follows that

$$e^{\hat{Q}_{\mathbf{MN}}}\left|0_{\mathrm{R}}\right\rangle=\sum_{n=0}^{\infty}e^{-\pi n\Omega_{\mathbf{M}}}\left|n^{(+)}_{\mathbf{MN}};n^{(-)}_{(M_1,-M_2,-M_3)(N_1,-N_2,-N_3)}\right\rangle \tag{D.28}$$

Taking the square norm of the above state gives

$$\left\langle 0_{\mathrm{R}}\right|e^{\hat{Q}^{\dagger}_{\mathbf{MN}}}e^{\hat{Q}_{\mathbf{MN}}}\left|0_{\mathrm{R}}\right\rangle=\sum_{n=0}^{\infty}e^{-2\pi n\Omega_{\mathbf{M}}}=\frac{1}{1-e^{-2\pi\Omega_{\mathbf{M}}}}\ . \tag{D.29}$$

Therefore, we find that the state

$$\sqrt{1-e^{-2\pi\Omega_{\mathbf{M}}}}e^{\hat{Q}_{\mathbf{MN}}}\left|0_{\mathrm{R}}\right\rangle \tag{D.30}$$

is normalized with length 1. We note that the Bogoliubov relations (4.46-4.47) can be manipulated to yield

$$\left(\hat{b}^{(+)}_{(\Omega,\underline{\mathbf{k}})}-e^{-\pi\Omega}\hat{b}^{(-)\dagger}_{(\Omega,-\underline{\mathbf{k}})}\right)\left|0_{\mathrm{M}}\right\rangle \;=\; 0 \tag{D.31}$$

$$\left(\hat{b}^{(-)}_{(\Omega,-\underline{\mathbf{k}})}-e^{-\pi\Omega}\hat{b}^{(+)\dagger}_{(\Omega,\underline{\mathbf{k}})}\right)\left|0_{\mathrm{M}}\right\rangle \;=\; 0 \tag{D.32}$$

These relations in turn imply

$$\left(\hat{B}^{(+)}_{\mathbf{MN}}-e^{-\pi\Omega_{\mathbf{M}}}B^{(-)\dagger}_{(M_1,-M_2,-M_3)(N_1,-N_2,-N_3)}\right)\left|0_{\mathrm{M}}\right\rangle \;=\; 0 \tag{D.33}$$

$$\left(\hat{B}^{(-)}_{(M_1,-M_2,-M_3)(N_1,-N_2,-N_3)}-e^{-\pi\Omega_{\mathbf{M}}}\hat{B}^{(+)\dagger}_{\mathbf{MN}}\right)\left|0_{\mathrm{M}}\right\rangle \;=\; 0 \tag{D.34}$$

By applying the operators shown in (D.33-D.34) onto the state (D.30) we make the conclusion that

$$\left|0_{\mathrm{M}}\right\rangle \;=\; \sqrt{1-e^{-2\pi\Omega_{\mathbf{M}}}}e^{\hat{Q}_{\mathbf{MN}}}\left|0_{\mathrm{R}}\right\rangle\ , \tag{D.35}$$

This is true mode-by-mode **MN**. In a more appealing formulation:

$$\left|0_{\mathrm{M}}\right\rangle \;=\; \sqrt{1-e^{-2\pi\Omega_{\mathbf{M}}}}\sum_{n=0}^{\infty}e^{-n\pi\Omega_{\mathbf{M}}}\left|n^{(+)}_{\mathbf{MN}};n^{(-)}_{(M_1,-M_2,-M_3)(N_1,-N_2,-N_3)}\right\rangle\ . \tag{D.36}$$

Interestingly this has the form of a coherent state familiar from the theory of superfluids, where particles are pairwise correlated [14]. In this case we see that particles in $\mathcal{R}_+$ are correlated with particles in $\mathcal{R}_-$ in wavepacket states that have the same average 'energy' labelled by M_1, but opposite average momenta (labelled by M_2 and M_3). What is striking here is that the particles $\mathcal{R}_+$ and $\mathcal{R}_-$ are *causally separated* and yet they still correlate with one another! (maybe this reminds us of the situation in Chapter 3 pertaining to Δ_{12}^{β}?)

We now state the thermalization theorem: We begin by considering the trajectory of a Rindler observer in $\mathcal{R}_+$ — that is, a trajectory of constant $\xi \equiv +\frac{1}{a}$ (and constant x^2 and x^3) where η is now related to the proper time τ by

$$\eta = a\tau \ . \tag{D.37}$$

Recalling that the Rindler modes have the dependence

$$r^{\sigma}_{\boldsymbol{\omega}}(x) \propto e^{-i\sigma\Omega\eta} = e^{-i\sigma a\Omega\tau} \tag{D.38}$$

Now $E_{\mathbf{M}} \equiv = a\Omega_{\mathbf{M}}$ is the proper (average) energy of the Rindler particle in the state $\mathbf{MN}$ as measured by the Rindler observer. The consequence of this is that where ever in our computation a dimensionless $\Omega_{\mathbf{M}}$ appeared, we must now make the replacement

$$\Omega_{\mathbf{M}} \ \mapsto \frac{E_{\mathbf{M}}}{a} \tag{D.39}$$

Therefore, we find that the Minkowski vacuum is written as

$$|0_{\mathrm{M}}\rangle = \sqrt{1 - e^{-\frac{2\pi}{a}E_{\mathbf{M}}}} \sum_{n=0}^{\infty} e^{-n\frac{\pi}{a}E_{\mathbf{M}}} \, |n^{(+)}_{\mathbf{MN}}; n^{(-)}_{(M_1,-M_2,-M_3)(N_1,-N_2,-N_3)}\rangle \ . \tag{D.40}$$

and the earlier considered combination of traces becomes

$$\frac{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\hat{\mathcal{O}}^{(+)}\right]}{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\right]} = \left(1 - e^{-\frac{2\pi}{a}\Omega_{\mathrm{M}}}\right) \sum_{n=0}^{\infty} e^{-\frac{2\pi}{a}nE_{\mathbf{M}}} \, \langle n^{(+)}_{\mathbf{NM}}; 0^{(-)} \big| \hat{\mathcal{O}}^{(+)} \big| n^{(+)}_{\mathbf{NM}}; 0^{(-)}\rangle \tag{D.41}$$

Something special happens now; because the Rindler observer is accelerating in $\mathcal{R}_+$ he is causally separated from $\mathcal{R}_-$. As a result his local observables constructed out of the field $\hat{\phi}(x)$ can only contain contributions from the modes $r^{(+)}_{\boldsymbol{\omega}}(x)$ (or equivalently $R^{(+)}_{\mathbf{MN}}(x)$). Therefore, his local observable is *always* just a function of the operators $\hat{B}^{(+)}_{\mathbf{MN}}$ and $\hat{B}^{(+)\dagger}_{\mathbf{MN}}$ — in other words, it is an arbitrary operator like $\hat{\mathcal{O}}^{(+)}$ described earlier. In particular, this means

$$\left[\hat{\mathcal{O}}^{(+)}, \hat{B}^{(-)}_{(M_1,-M_2,-M_3)(N_1,-N_2,-N_3)}\right] = 0 \tag{D.42}$$

And so we find that:

$$\langle 0_{\rm M}|\hat{\mathcal{O}}^{(+)}|0_{\rm M}\rangle = \left(1 - e^{-\frac{2\pi}{a}\Omega_{\mathbf{M}}}\right)\sum_{n=0}^{\infty} e^{-\frac{2\pi}{a}nE_{\mathbf{M}}}\, \langle n^{(+)}_{\mathbf{MN}}; 0^{(-)}\big|\hat{\mathcal{O}}^{(+)}\big|n^{(+)}_{\mathbf{MN}}; 0^{(-)}\rangle \tag{D.43}$$

Therefore, we see that the Rindler observer measures

$$\langle 0_{\rm M}|\hat{\mathcal{O}}^{(+)}|0_{\rm M}\rangle = \frac{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\hat{\mathcal{O}}^{(+)}\right]}{\mathrm{Tr}^{(+)}\left[\hat{\rho}^{(+)}_{\mathbf{MN}}\right]} \tag{D.44}$$

Where we have the operator $\hat{\rho}^{(+)}_{\mathbf{MN}} = \exp\left(-\frac{2\pi}{a}E_{\mathbf{M}}\hat{B}^{(+)\dagger}_{\mathbf{MN}}\hat{B}^{(+)}_{\mathbf{MN}}\right)$. Since the above relation is true mode-by-mode **MN** the above is sometimes written as:

$$\langle 0_{\rm M}|\hat{\mathcal{O}}^{(+)}|0_{\rm M}\rangle = \frac{\mathrm{Tr}^{(+)}\left[e^{-\frac{2\pi}{a}\hat{H}^{(+)}_{\rm R}}\hat{\mathcal{O}}^{(+)}\right]}{\mathrm{Tr}^{(+)}\left[e^{-\frac{2\pi}{a}\hat{H}^{(+)}_{\rm R}}\right]} \tag{D.45}$$

We see that the Rindler observer perceives the Minkowski vacuum as a *genuine thermal state* (since the above is true for *any* of his operators) with a temperature $\frac{1}{\beta_{\rm U}} \equiv \frac{a}{2\pi}$.

Appendix E

The Rindler Observer's Tadpole Loop

As promised in section (4.4), here we complete the computation the result (4.67)

$$\delta M_{\rm R}^2 \;\equiv\; 3\lambda\, \langle 0_{\rm M}|\hat{\phi}(v)^2|0_{\rm M}\rangle - 3\lambda\, \langle 0_{\rm R}|\hat{\phi}(v)^2|0_{\rm R}\rangle \tag{E.1}$$

which is the Rindler's observer's choice for the renormalized tadpole loop. We consider the *massless* theory here.

We assume that $v = v(\tau)$ parametrizes the worldline of a Rindler observer in $\mathcal{R}_+$ with

$$v(\tau) \;=\; \big(\eta(\tau), \xi(\tau), x^2(\tau), x^3(\tau)\big) \;=\; \big(a\tau, \tfrac{1}{a}, y, z\big)\ , \tag{E.2}$$

where $a > 0$ and $y, z \in \mathbb{R}$ are constants and τ is the proper time of the observer. Evaluating the Rindler modes (4.19) and (4.20) along this trajectory yields

$$r^{(+)}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big) \;=\; \frac{1}{\sqrt{4\pi^3\Omega}}\frac{1}{\Gamma(i\Omega)}e^{-i\Omega a\tau + ik^2 y + ik^3 z}K_{i\Omega}\left(\frac{|\underline{\mathbf{k}}|}{a}\right) \tag{E.3}$$

$$r^{(-)}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big) \;=\; 0 \tag{E.4}$$

and as a result the field expansion (4.28) takes the particular form

$$\hat{\phi}\big(v(\tau)\big) = \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\ \left[\ \hat{b}^{(+)}_{(\Omega,\underline{\mathbf{k}})} r^{(+)}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big) + \hat{b}^{(+)\dagger}_{(\Omega,\underline{\mathbf{k}})} r^{(+)*}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big)\right]\ . \tag{E.5}$$

Equipped with this expansion, it is elementary to compute the expectation value in the Rindler vacuum in (E.1):

$$\langle 0_{\rm R}|\hat{\phi}(v)^2|0_{\rm R}\rangle \;=\; \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\; \left| r^{(+)}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big)\right|^2 \tag{E.6}$$

This quantity is divergent as it stands. To compute the other expectation value in the Minkowski vacuum, we note the following identities

$$\langle 0_{\rm M}|\hat{b}^{(+)}_{(\Omega,\underline{\mathbf{k}})}\hat{b}^{(+)}_{(\tilde{\Omega},\underline{\mathbf{p}})}|0_{\rm M}\rangle \;=\; 0 \tag{E.7}$$

$$\langle 0_{\rm M}|\hat{b}^{(+)\dagger}_{(\Omega,\underline{\mathbf{k}})}\hat{b}^{(+)}_{(\tilde{\Omega},\underline{\mathbf{p}})}|0_{\rm M}\rangle \;=\; \frac{1}{e^{2\pi\Omega}-1}\delta(\Omega-\tilde{\Omega})\delta^{(2)}(\underline{\mathbf{k}}-\underline{\mathbf{p}}) \tag{E.8}$$

$$\langle 0_{\rm M}|\hat{b}^{(+)}_{(\Omega,\underline{\mathbf{k}})}\hat{b}^{(+)\dagger}_{(\tilde{\Omega},\underline{\mathbf{p}})}|0_{\rm M}\rangle \;=\; \frac{1}{1-e^{-2\pi\Omega}}\delta(\Omega-\tilde{\Omega})\delta^{(2)}(\underline{\mathbf{k}}-\underline{\mathbf{p}}) \tag{E.9}$$

$$\langle 0_{\rm M}|\hat{b}^{(+)\dagger}_{(\Omega,\underline{\mathbf{k}})}\hat{b}^{(+)\dagger}_{(\tilde{\Omega},\underline{\mathbf{p}})}|0_{\rm M}\rangle \;=\; 0 \tag{E.10}$$

These follow as an immediate consequence of the Bogoliubov relation (4.46). From the above, it follows that

$$\langle 0_{\rm M}|\hat{\phi}(v)^2|0_{\rm M}\rangle \;=\; \int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\; \left| r^{(+)}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big)\right|^2 \left[\frac{1}{e^{2\pi\Omega}-1}+\frac{1}{1-e^{-2\pi\Omega}}\right] \tag{E.11}$$

Putting the above two expectation values together into (E.1), along with the identity $\frac{1}{e^\alpha-1}+\frac{1}{1-e^{-\alpha}}-1=\frac{2}{e^\alpha-1}$ we find that

$$\delta M_{\rm R}^2 \;=\; 3\lambda\int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\; \left| r^{(+)}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big)\right|^2 \frac{2}{e^{2\pi\Omega}-1}\;. \tag{E.12}$$

Plugging in (E.3) for $r^{(+)}_{(\Omega,\underline{\mathbf{k}})}\big(v(\tau)\big)$ the above becomes

$$\delta M_{\rm R}^2 \;=\; \frac{3\lambda}{2\pi^3}\int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\; \frac{1}{\Omega|\Gamma(i\Omega)|^2}\left|K_{i\Omega}\left(\frac{|\underline{\mathbf{k}}|}{a}\right)\right|^2 \frac{1}{e^{2\pi\Omega}-1} \tag{E.13}$$

We note that for $\alpha,\beta\in\mathbb{R}$ the function $K_{i\alpha}(\beta)$ is real-valued [26] and the Gamma function satisfies the identity [54]

$$\big|\Gamma(i\alpha)\big| \;=\; \frac{\pi}{\alpha\sinh(\pi\alpha)}\;. \tag{E.14}$$

This simplifies the integral (E.13) to

$$\delta M_{\rm R}^2 \;=\; \frac{3\lambda}{2\pi^4}\int_0^\infty d\Omega \int d^2\underline{\mathbf{k}}\; \frac{\sinh(\pi\Omega)}{e^{2\pi\Omega}-1}K_{i\Omega}\left(\frac{|\underline{\mathbf{k}}|}{a}\right)^2\;. \tag{E.15}$$

Using the identity $\frac{\sinh\left(\frac{\alpha}{2}\right)}{e^\alpha-1}=\frac{1}{2}e^{-\alpha/2}$ and then switching to two-dimensional polar coordinates $(k^2, k^3) = (aR\cos\theta, aR\sin\theta)$ in the $\underline{\mathbf{k}}$-variables, (E.15) becomes

$$\delta M_{\rm R}^2 \;=\; \frac{3\lambda a^2}{2\pi^3}\int_0^\infty d\Omega \int_0^\infty dR\; Re^{-\pi\Omega} K_{i\Omega}\left(R\right)^2 \ . \tag{E.16}$$

The R-integral can be evaluated using formula (6.521.3) from [55]:

$$\int_0^\infty dR\; RK_{i\Omega}(R)^2 \;=\; \frac{\pi\Omega}{2\sinh\left(\pi\Omega\right)} \tag{E.17}$$

The integral (E.16) now becomes

$$\delta M_{\rm R}^2 \;=\; \frac{3\lambda a^2}{4\pi^2}\int_0^\infty d\Omega\; \frac{\Omega e^{-\pi\Omega}}{\sinh\left(\pi\Omega\right)} \tag{E.18}$$

The remaining integral evaluates to the numerical factor $\int_0^\infty d\Omega\ \frac{\Omega e^{-\pi\Omega}}{\sinh(\pi\Omega)} \;=\; \frac{1}{12}$ leaving us with the result quoted in (4.68):

$$\delta M_{\rm R}^2 \;=\; \frac{\lambda a^2}{16\pi^2} \tag{E.19}$$

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