EXISTENCE AND REGULARITY OF SOLUTIONS TO SOME SINGULAR PARABOLIC SYSTEMS

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BY

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For Atticus.

Abstract

This thesis continues the work started with my previous supervisor, Dr. Shaohua Chen. In [15], the authors developed some tools that showed the boundedness or blowup of solutions to a semilinear parabolic system with homogeneous Neumann boundary conditions. This system, the so called 'Activator-Inhibitor Model', is of interest as it is used to model biological processes and pattern formation. Similar tools were later adapted to deal with the same parabolic system in [3], in which the authors prove global boundedness of solutions under homogeneous Dirichlet conditions. This new problem is of mathematical interest as the solutions may grow singular near the boundary. Shortly after, a different system was considered in [4], where the authors proved global boundedness of solutions to a system featuring similar singular reaction terms. The goal of this thesis is twofold: first, the tools developed that allow us to tackle these sorts of problems will be demonstrated in detail to showcase its utility; the second is to then use these tools to generalize some of these previous results to a larger class of singular parabolic systems. In doing so, we expand the classical literature found in [14] and other notable works, where nonsingular equations are extensively treated. The motivation for the first should be clear. While there

are numerous bodies of text treating nonsingular problems, there are no collections available dealing with these types of singularities exclusively. This is of practical use to other mathematicians studying partial differential equations. The motivation for the second is, perhaps, more practical. There are a growing number of models found in physics, chemistry and biology that may be generalized to a singular type system. Through allowing those individuals to treat these problems, we may gain valuable insights into the real world and how these processes behave.

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Contents

Chapter 1

Introduction

1.1 Reaction-Diffusion Equations

In 1952, Alan Turing [18] proposed an answer to a curious question: How is it that certain biological systems, which are observed to be initially spatially homogeneous, develop patterns or structure at a later time? Originally, his idea was suggested as an idealized mechanism to account for the phenomena of *morphogenesis*. Put simply, morphogenesis is the biological process that allows various organisms to develop their shape. This includes, for example, the embryonic development of an organism, but can also occur inside tumor cells, or in cell culture. Turing's answer to this phenomena of spontaneous pattern formation is what we now refer to as *Turing instability*. The key aspect of his revolutionary idea was that a system that is linearly stable in the absence of diffusion may become unstable in the presence of diffusion. To investigate this further, consider a general reaction-diffusion equation:

$$\begin{cases} u_t = d\Delta u + cf(u,v), \\ v_t = D\Delta v + cg(u,v), \end{cases} \tag{1.1}$$

where $d, D, c > 0$ are real parameters. Turing realized that systems *with* diffusion may have fundamentally different properties than those without. In particular, he noted that in the absence of diffusion ($d = D = 0$), the equilibria of the above system (ie. the solution when $u_t = v_t = 0$, or the so-called steady state solution) are solutions (u^*, v^*) that satisfy $f(u^*, v^*) = g(u^*, v^*) = 0$. It is clear that these equilibria are independent of the parameters d, D and c. Then, when investigating the time dependent system, it is fairly easy to determine when these equilibria remain linearly stable. If we define the matrix A corresponding to the linearization of (1.1) without diffusion, we obtain

$$A = \begin{bmatrix} f_u(u^*, v^*) & f_v(u^*, v^*) \\ g_u(u^*, v^*) & g_v(u^*, v^*) \end{bmatrix}.$$

The solution to this related linear ordinary differential equation can then be shown to be linearly stable if and only if the eigenvalues of the matrix cA have negative real part. In such cases, small perturbations in the spatial domain will always return to its stable state. Alternatively, in the presence of diffusion ($D, d \neq 0$), the homogeneous steady state may become unstable with respect to small changes in the spatial domain. This was the novel answer proposed by Turing: even with homogeneous initial data, certain reaction-diffusion equations may posses such instabilities in the presence of diffusion, resulting in interesting (and seemingly spontaneous) patterns.

This type of instability is what we are referring to when we say "Turing instability". [6] and [16] serve as good resources for an in depth exploration of diffusive instability.

Although Turing's original work was focused on the process of morphogenesis, his ideas have since been expanded to provide insights into other types of pattern formation found in nature. One may think of the stripes on the tail of a raccoon, the spots on a leopard, or the wide range of dot and stripe patterns found on seashells, to name a few examples. The founding work on this front was a model proposed by Gierer and Meinhardt in 1972 [10], which is now referred to as the Gierer-Meinhardt, or Activator-Inhibitor model. This is where the motivation for the investigation of singular parabolic systems begins.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$ with smooth boundary. The general Activator-Inhibitor model then takes the following form

$$\begin{cases} u_t = d\Delta u - \mu u + \frac{u^p}{v^q} + \rho(x), \\ v_t = D\Delta v - \nu v + \frac{u^r}{v^s}, \qquad x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $d, D, \mu, \nu, p, q, r, s$ are all positive, real valued constants. This system is a special case of the general reaction-diffusion equations described above, and holds interesting properties of its own. Of course, this system as written is its most general form. The original model proposed by Gierer and Meinhardt instead took the particular powers $(p, q, r, s) = (2, 1, 2, 0)$ and proposed homogeneous Neumann boundary conditions (ie. $\partial u/\partial \mathbf{n} = \partial v/\partial \mathbf{n} = 0$, where $\mathbf{n}$ denotes the normal to the boundary). Under these assumptions, Rothe [17] proved the existence and regularity of solutions for $N \leq 3$ in 1984. Since then, many works have been published exploring the existence, regularity, as well as the qualitative behaviour and properties of solutions to the Gierer-Meinhardt system, placing various conditions on the available parameters, initial conditions, and the forcing term $\rho(x)$. One may refer to a few key works highlighting some of these results, namely [15], [5], [13], [2]. All of the literature, at least early on, focused on the system given homogeneous Neumann boundary data. For example, one may consider the results found by Li, Chen and Qin [15] published in 1995. They were able to prove the long time existence, as well as finite time blow up, of solutions given certain conditions on the constants and character of the initial conditions. In order to do so, they considered the following functional:

$$\int_\Omega \frac{u^n(x,t)}{v^m(x,t)} dx, \tag{1.3}$$

where $n \geq m > 0$. Through taking the time derivative of this quantity and integrating by parts, the authors were able to obtain L^p bounds of this quantity, allowing them to then apply semigroup theory to obtain the existence of solutions for all time. Perhaps more interestingly (at least in a mathematical sense), when considering the case when $d = D$, the authors were able to inspect the integral above with $m = \alpha n$ for some $\alpha \in (0, 1)$. Applying the same strategy (deriving in time, integrating by parts and applying some elementary inequalities), they were able to find uniform bounds of this quantity! That is to say, they proved the existence of some $C > 0$, independent of n, so that

$$\int_\Omega \frac{u^n}{v^{\alpha n}} dx \leq C^n, \tag{1.4}$$

and so we in fact have

$$\begin{aligned}\frac{u(x,t)}{v^\alpha(x,t)} &\leq \left\| \frac{u}{v^\alpha} \right\|_{L^\infty} \\ &= \lim_{n\to\infty} \sqrt[n]{\int_\Omega \left(\frac{u}{v^\alpha}\right)^n dx} \\ &\leq C < \infty. \end{aligned} \tag{1.5}$$

Such an a priori bound proves very useful, as this is a bound on a term that looks eerily similar to the nonlinearities appearing in the original system. This is no accident, and has motivated some of the strategies to be used throughout this thesis, as we will soon find out.

1.2 Singular Reaction-Diffusion Equations

The previous problem, while interesting, does not feature the singular behaviour this manuscript is interested in. As it turns out, investigating the problem with homogeneous Dirichlet boundary data is also a highly interesting endeavour, as it presents new mathematical difficulties that the Neumann problem does not posses. The most obvious difficulty is in the singular behaviour of the nonlinearities. That is, our system now takes the form:

$$\begin{cases} u_t = d\Delta u - \mu u + \frac{u^p}{v^q} + \rho(x), \\ v_t = D\Delta v - \nu v + \frac{u^r}{v^s}, & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.6}$$

and we see that for $Q \in \partial\Omega$,

$$\lim_{x \to Q} \frac{u^p(x,t)}{v^q(x,t)} \tag{1.7}$$

is (a priori) undefined. Naturally, some restrictions must be put on the exponents in order to make sense of this quantity, which in turn allows us to make sense of the solution to the system (or even the sense in which we can say a solution exists). Other technical difficulties exist as well. The system above has been referred to as a reaction-diffusion equation and there are existence results for nonlinearities that posses so-called monotone properties. The terms featured in the Gierer-Meinhardt model are non-quasimonotone, and so we cannot apply classical monotone methods. These monotone properties will be discussed in more detail in chapter 3, but here we briefly make note of the fact that the above nonlinearities are not Lipschitz in their arguments (u, v). One may be able to determine values for which these nonlinearities are Hölder continuous in each of their arguments, but this removes the more interesting cases covered in works such as [3]. This removes the case where $q, s > 0$, for a simple but concrete example. Additionally, the use of the comparison principle also fails due to the character of these nonlinearities. Fortunately, one is able to deal with such a system through regularization of the problem and considering bounds similar, but not identical, to those found in the Neumann problem. When we refer to regularizing the problem, we perturb the singular terms in the system by some small, positive parameter ε:

$$\begin{cases} u_t = d\Delta u - \mu u + \frac{u^p}{(v+\varepsilon)^q} + \rho(x), & \\ v_t = D\Delta v - \nu v + \frac{u^r}{(v+\varepsilon)^s}, & x \in \Omega,\ t > 0, \\ u = v = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.8}$$

We denote the solution to this related problem by $(u_\varepsilon, v_\varepsilon)$. Existence of solutions to the regularized problem (1.8) can be shown through application of the existence results to general quasilinear parabolic systems found in [14]. This will not be shown in detail here. We will, however, discuss how the monotone methods mentioned previously *can* be used to show the existence of the regularized problems specific to this manuscript. Once the existence of solutions to the regularized problem is established, the goal is to then obtain a priori bounds on the solutions $(u_\varepsilon, v_\varepsilon)$ in a suitable space. The key fact required is that these bounds remain independent of epsilon. We then hope to extract a subsequence $(u_{\varepsilon_k}, v_{\varepsilon_k})$ that converges (in some sense) to a solution of our original problem.

Taking inspiration from the previous bound obtained, we wish to obtain a similar bound for the Dirichlet problem, but this time with attention given to the boundary behaviour of the nonlinearity. One way to do this would be to bound our nonlinearity by some function that tends to zero as we approach the boundary. That is, if we can show

$$\frac{u_\varepsilon(x,t)}{v_\varepsilon^\alpha(x,t)} \leq Cf(x), \tag{1.9}$$

where $f(x) > 0$ in Ω satisfying

$$\lim_{x\to Q} f(x) = 0, \tag{1.10}$$

for $Q \in \partial\Omega$, it is then necessarily true that

$$\lim_{x\to Q} \frac{u_\varepsilon(x,t)}{v_\varepsilon^\alpha(x,t)} = 0. \tag{1.11}$$

This is exactly the kind of control we require at the boundary of our domain in order to retain control of the nonlinearities appearing in our equations. So, the question now becomes: How might we adjust the integral considered in (1.3) so that we may obtain a bound of the form found in (1.9)? One primordial answer is to consider $\psi(x)$, the solution to the eigenvalue problem

$$\begin{cases} 0 = \Delta\psi(x) + \lambda_1\psi(x), & x \in \Omega, \\ 0 = \psi(x), & x \in \partial\Omega, \end{cases} \tag{1.12}$$

where $\lambda_1 > 0$ denotes the first eigenvalue. We then may construct the following quantity as it relates to the perturbed system:

$$\int_\Omega \frac{u_\varepsilon^n(x,t)}{[v_\varepsilon(x,t)+\varepsilon]^{\alpha n}\,\psi^{\beta n-2}(x)}dx, \tag{1.13}$$

The form of this integral, while potentially arbitrary at first glance, actually makes a lot of practical sense. First, to ensure this integral converges, we choose $\alpha, \beta \in (0,1)$ so that $\alpha+\beta \leq 1$. This integral can then be shown to be well defined by Hopf's boundary lemma. (This is addressed more explicitly in chapter 5). If we can obtain a similar uniform bound,

$$\frac{u_\varepsilon(x,t)}{[v_\varepsilon(x,t)+\varepsilon]^{\alpha}\,\psi^{\beta}(x)} \leq C, \tag{1.14}$$

then we actually have that

$$\frac{u_\varepsilon(x,t)}{v_\varepsilon^\alpha(x,t)} \leq C\psi(x)^\beta. \tag{1.15}$$

(Note that C may depend on t, and the existence of such a C will certainly be dependant on the restrictions put upon (p,q,r,s), but we remind the reader that the primary goal is to find C independent of ε). This bound is of the form found in (1.9) and satisfies the conditions put on f. This in turn allows us to then obtain uniform bounds of the form

$$m\psi(x) \leq u_\varepsilon(x,t), v_\varepsilon(x,t) \leq M\psi(x), \tag{1.16}$$

for some $m, M > 0$, also independent of ε. This is the general philosophy to be taken when dealing with singular parabolic systems in this way.

With a set of tools at hand, attention has then been given to other parabolic problems featuring these types of singular nonlinearities. These problems are primarily motivated by their elliptic counterparts. For example, Ghergu [8], [9], considers the following systems:

$$\begin{cases} 0 = \Delta u + \frac{1}{u^p v^q}, \\ 0 = \Delta v + \frac{1}{u^r v^s}, \quad x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.17}$$

and

$$\begin{cases} 0 = \Delta u + \frac{1}{u^p} + \frac{1}{v^q}, \\ 0 = \Delta v + \frac{1}{u^r} + \frac{1}{u^s}, \quad x \in \Omega, \\ u = v = 0, \quad x \in \partial\Omega, \end{cases} \tag{1.18}$$

where (p, q, r, s) satisfy various size restrictions. Using the method of sub and super solutions to the regularized problem, along with some fixed point arguments, Ghergu was able to prove the existence of solutions of varying regularity. This work then inspired consideration of the following parabolic system:

$$\begin{cases} u_t = \Delta u + \frac{f(x)}{v^p}, \\ v_t = \Delta v + \frac{g(x)}{u^q}, \quad x \in \Omega,\ t > 0, \\ u = v = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.19}$$

In [4], the authors proved the existence of bounded solutions to this problem, provided $p, q \in (0, 1)$ and f, g satisfy certain growth conditions related to the first eigenvalue problem. Under more strict conditions on p, q, these solutions are further shown to be classical. With this context provided, the goal of this thesis can now be more readily stated. The primary goal is to extend the results found in [4] to the following more general systems:

$$\begin{cases} u_t = d\Delta u + \frac{f(x)}{u^p v^q}, \\ v_t = D\Delta v + \frac{g(x)}{u^r v^s}, & x \in \Omega,\ t > 0 \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.20}$$

and

$$\begin{cases} u_t = d\Delta u + \frac{f_1(x)}{u^p} + \frac{f_2(x)}{v^q}, \\ v_t = D\Delta v + \frac{f_3(x)}{u^r} + \frac{f_4(x)}{v^s}, & x \in \Omega,\ t > 0 \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.21}$$

There are a few comparisons that can be made between these systems, (1.19), as well as their elliptic counterparts. First, these systems feature different diffusion coefficients, which entails its own added difficulty. Further, (1.20) is a direct generalization of (1.19), provided that we can set $d = D$ and $p = s = 0$. There is also added difficulty when dealing with products and sums of these singular nonlinearities, in comparison to the standalone singularities appearing in (1.19). In chapter 6 a more in depth exploration of the similarities and differences to other problems is given. In order to tackle these problems in detail, some preliminary results and standard material must be reviewed.

Chapter 2

Preliminaries

In this section, we discuss some of the notation and elementary results referenced throughout this manuscript.

We denote Euclidean space of dimension $N \in \mathbb{N}$ by $\mathbb{R}^N$. $\Omega \subset \mathbb{R}^N$ denotes a bounded domain, always assumed to have a sufficiently smooth boundary. We further denote its boundary by $\partial\Omega$, and its closure by $\bar{\Omega} = \Omega \cup \partial\Omega$. We take $\Omega' \subset\subset \Omega$ to mean Ω' is a subdomain of Ω such that $\overline{\Omega'} \subset \Omega$. When convenient, we will sometimes denote $Q_T = \Omega \times (0, T)$.

We use the usual subscript notation to denote partial derivatives with respect to that variable. (eg. $u_t = \partial u/\partial t$, $f_{x_i} = \partial f/\partial x_i$ and so on.) We write ∇u to mean the gradient of u, the vector of partial derivatives of u; $\Delta u = \nabla \cdot (\nabla u)$ to mean the Laplacian of u, the trace of the Hessian matrix $D^2 u$.

We use the usual notation for various function spaces. $C(\overline{\Omega})$ denotes the space of all continuous functions on $\overline{\Omega}$ with finite associated norm; for $k \in \mathbb{N}$, $C^k(\overline{\Omega})$ denotes the space of all k-times continuously differentiable functions in $\overline{\Omega}$, also with finite associated norm. $C^{k+\theta}(\overline{\Omega})$ denotes the spaces of all k-times continuously differentiable functions on $\overline{\Omega}$ with its k-th derivative being θ-Hölder continuous with exponent $\theta \in (0,1)$, also having finite associated norm. The norms associated with these spaces are defined as follows:

$$\begin{aligned} \|f\|_{C^k(\Omega)} &= \sum_{|a|\leq k} \sup_{\Omega} |D^a f|, \\ \|f\|_{C^{0+\theta}(\Omega)} &= \sup_{\Omega} |f| + \sup_{\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\theta}}, \\ \|f\|_{C^{k+\theta}(\Omega)} &= \sum_{|a|\leq k} \|D^a f\|_{C^{0+\theta}(\Omega)} \end{aligned} \tag{2.1}$$

When $\theta = 1$, we get the usual space of Lipschitz continuous functions. If a function $u \in C^{k+\alpha}(\overline{\Omega'})$ for any $\Omega' \subset\subset \Omega$, we say that $u \in C^{k+\alpha}(\Omega)$.

We extend these definitions in an intuitive way to include time. That is, for $k, s \in \mathbb{N}$, $C^{k,s}(\overline{\Omega} \times [0,T])$ denotes the set of all functions from $\overline{\Omega} \times [0,T] \to \mathbb{R}$ which are k-times differentiable in the *spacial* domain, and s times differentiable in the *time* domain. $C^{k+\theta,s+\tau}(\overline{\Omega} \times [0,T])$ denotes the same space with the k-th derivative in space being θ-Hölder continuous, and the s-th derivative in time being τ-Hölder continuous with $\theta, \tau \in (0,1)$. An in depth coverage of the norms associated with these spaces is more an exercise in typing rather than a mathematical one. Readers are directed to section 1.2.3 of chapter 1 in [20] for further details.

For $1 \leq p \leq \infty$, denote by $L^p(\Omega)$ the set of all Lebesgue measurable functions

$f : \Omega \to \mathbb{R}$ with finite L^p norm. That is, $f \in L^p(\Omega)$ implies that

$$\|f\|_{L^p(\Omega)} \equiv \left(\int_\Omega |f|^p \, dx \right)^{\frac{1}{p}} < \infty, \tag{2.2}$$

when $p \in [1, \infty)$, and

$$\begin{aligned} \|f\|_{L^\infty(\Omega)} &\equiv \operatorname{ess\,sup} \{f\} \\ &\equiv \inf\{\mu \in \mathbb{R} : |\{f > \mu\}| = 0\} < \infty, \end{aligned} \tag{2.3}$$

when $p = \infty$.

Lemma 2.0.1. *Suppose $f \in L^\infty(\Omega)$. Then*

$$\lim_{p \to \infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)} \, . \tag{2.4}$$

For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we denote by $W^{k,p}(\Omega)$ to be the usual Sobolev space: $W^{k,p}(\Omega)$ consists of all Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that f has weak derivatives of all orders up to and including k, and each weak derivative belongs to $L^p(\Omega)$. We consider this space with the usual norm:

$$\|u\|_{W^{k,p}(\Omega)} = \left(\int_\Omega \sum_{|a| \leq k} |D^a u|^p \, dx \right)^{1/p}, \tag{2.5}$$

where the summation is understood to be over all multi-indices a of size up to and including k. We extend this definition to include time in the same way as before: for $k, s \in \mathbb{N}$, $1 \leq p < \infty$, $W_p^{k,s}(\Omega \times (0,T))$ denotes the so-called t-anisotropic Sobolev space. A function belonging to $W_p^{k,s}(\Omega \times (0,T))$ has weak derivatives in space of order up to and including k; weak derivatives in time of up to order s; both of these quantities belonging to L^p in their respective domains. Despite the general definition given, for the purposes of this thesis, we need only consider the space $W_p^{2,1}(\Omega\times(0,T))$ endowed with the norm

$$\|u\|_{W_p^{1,2}(Q_T)} = \left(\int\int_{Q_T} |u_t|^p + \sum_{|k|\le 2} \left|D^k u\right|^p \right)^{1/p}. \tag{2.6}$$

Recall that the sum above is understood to be taken over all multi-indices k of size up to and including 2. Readers are directed to references such as [7], [11], [1], [20] for an extensive summary of Sobolev spaces. We now recall a useful embedding theorem, an analog to the embedding commonly found in the spatial domain only.

Theorem 2.0.2 (t-Anisotropic Sobolev Embedding Theorem)**.** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with appropriately smooth boundary (say $C^{2,\theta}$) and let $1 \le p < \infty$. Then, when $p > (N+2)/2$, it is true that*

$$W_p^{2,1}(Q_T) \hookrightarrow C^{\theta,\theta/2}(\overline{Q}_T), \quad 0 < \theta \le 2 - \frac{N+2}{p}. \tag{2.7}$$

This means that, under the above assumptions,

$$W_p^{2,1}(Q_T) \subset C^{\theta,\theta/2}(\overline{Q}_T), \tag{2.8}$$

and for any $f \in W_p^{2,1}(Q_T)$,

$$\|f\|_{C^{\theta,\theta/2}(Q_T)} \leq C(N,p,Q_T)\,\|f\|_{W_p^{2,1}(Q_T)}\,. \tag{2.9}$$

Readers are directed to Theorem 1.4.1 in [20] for details.

We now introduce the notion of weak and classical solutions. Suppose we have a second order, linear, nonhomogeneous heat equation with Dirichlet boundary condition:

$$\begin{cases} u_t = d\Delta u + f(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{2.10}$$

By classical solution, we mean a function $u : Q_T \to \mathbb{R}$ such that $u \in C^{2+\theta,1+\theta/2}(\Omega \times (0,T)) \cap C^{1+\theta,(1+\theta)/2}(\overline{\Omega} \times [0,T))$ for some $\theta > 0$, and satisfies (2.10) for all $(x,t) \in Q_T$. We further call a classical solution u global if u is a classical solution for all $t \in (0,\infty)$. We call u a weak solution of (2.10) if $u \in L^2(0,T;W_0^{1,2}(\Omega)) \cap L^\infty(\overline{\Omega} \times (0,T))$,

$$f(x,t) \in L^1(Q_T), \tag{2.11}$$

and

$$\int_\Omega u_0 \xi dx + \int_0^T \int_\Omega \left[u\xi_t - d\nabla u \nabla \xi + f(x,t)\xi\right] dxdt = 0, \tag{2.12}$$

for all $\xi \in C^\infty(\overline{\Omega} \times (0,T))$ satisfying $\xi(x,\cdot) = 0$ on $\partial\Omega \times (0,T)$ and $\xi(\cdot,T) = 0$. Recall that $u \in L^2((0,T);W_0^{1,2}(\Omega))$ means that the L^2-norm of $\|u\|_{W^{1,2}(\Omega)}$ with respect to time remains finite:

$$\begin{aligned}
\|u\|_{L^2((0,T);W_0^{1,2}(\Omega))} &= \left(\int_0^T \|u\|_{W^{1,2}(\Omega)}^2 \, dt\right)^{1/2} \\
&= \left(\int_0^T \int_\Omega \left(|u|^2 + |\nabla u|^2\right) dx dt\right)^{1/2} \\
&< \infty. \qquad (2.13)
\end{aligned}$$

This definition above will be expanded to define the weak solutions to a system of equations in a later section. Next, we will review some useful material related to some general, non-singular reaction-diffusion equations.

Chapter 3

Classical Theory

This section will cover the classical theory, where classical in this case refers to the L^p theory of parabolic equations, as well as the monotone methods used for particular systems of reaction-diffusion equations. First, we merely state the L^p theory necessary for the main results section. Second to this, we will include a more in depth investigation of monotone properties (nonincreasing monotone properties for our purposes, but references will be provided for nondecreasing and mixed monotone cases), sub and super solutions to systems, monotonicity of iterations (starting from sub and super solutions), and finally existence results. This will be relevant for this manuscript as we need to be able to show existence for the perturbed system. Ladyzhenskaya [14] may be acceptable as a reference for existence of very general quasilinear systems, but that is employing very complicated procedures for a fairly basic result, considering that our nonlinearities appear as zero-th order terms. Consequently, the monotone methods presented here give a more immediately accessible process to prove the existence of (classical) solutions.

3.1 Parabolic L^p Theory

To start, let us consider the following nonhomogeneous heat equation

$$u_t = d\Delta u + f(x,t), \quad x \in \Omega,\ t > 0. \tag{3.1}$$

Theorem 3.1.1 (L^p Existence)**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $1 < p < \infty$. Then, for any $f \in L^p(Q_T)$, problem* (3.1) *admits a unique strong solution $W_p^{2,1}(Q_T) \cap \overset{\bullet}{W}{}_p^{1,1}(Q_T)$.*

Note: $\overset{\bullet}{W}{}_p^{1,1}(Q_T)$ is understood to be the closure of $\overset{\bullet}{C}{}^\infty(\overline{Q}_T)$ in $W_p^{1,1}(Q_T)$, where $\overset{\bullet}{C}{}^\infty(\overline{Q}_T)$ denotes the class of all infinitely differentiable functions vanishing near $\overline{\Omega} \times \{t = 0\} \cup \partial\Omega \times (0,T)$.

Proof. For the proof, readers are directed to Theorem 9.2.4 in [20]. □

Theorem 3.1.2 (L^p Estimates)**.** *Let $\Omega \subset \mathbb{R}^N$ be a smooth, bounded domain with $1 < p < \infty$ and $u \in W_p^{2,1}(Q_T) \cap \overset{\bullet}{W}{}_p^{1,1}(Q_T)$ satisfy* (3.1) *almost everywhere in Q_T. Then*

$$\left\| D^2 u \right\|_{L^p(Q_T)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^p(Q_T)} \le C \left\| f \right\|_{L^p(Q_T)}, \tag{3.2}$$

where C depends only on N, p, T and Ω. In particular, this means we have that

$$\|u\|_{W^{2,1}_p(Q_T)} \leq C\left(\|f\|_{L^p(Q_T)} + \|u\|_{L^p(Q_T)}\right). \tag{3.3}$$

Proof. The reader is directed to the proof of Theorem 9.2.1 and Remarks 9.2.1 and 9.2.2 found in [20]. □

While this has been only a brief statement, these results will be important later on. The inclusion of Theorem 3.1.1 may be redundant, as a classical solution is automatically a weak (and strong) solution. Regardless, it has been included as Theorem 3.1.2 is stated under the assumption that a strong solution exists. These results allow us to uniformly bound the solutions to our perturbed system by showing $f \in L^p(Q_T)$ independent of ε. As a final note, the reader is also directed to Theorem 6 found in [19] for an alternative approach to obtaining Theorem 3.1.2 above.

3.2 Monotone Methods

We now turn out attention to coupled parabolic systems. To start, we introduce the system we are interested in:

$$\begin{cases} u_t = d\Delta u + f_1(u,v), & \\ v_t = D\Delta v + f_2(u,v), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & \\ v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{3.4}$$

where $d, D > 0$ are real valued constants, $\Omega \subset \mathbb{R}^N$ is a bounded domain with $\partial\Omega$ of class $C^{2,\theta}$ and $f_i \in C^{\theta}(\mathbb{R}^2)$ for some $\theta \in (0,1)$. We further assume that $u_0, v_0 \in C^{2+\theta}(\overline{\Omega})$. Note that, through an application of the barrier function technique, this requirement can actually be weakened to merely requiring that $u_0, v_0 \in C(\overline{\Omega})$ and all of the following conclusions still hold true. Before we discuss existence of solutions to this problem, we must introduce a bit of terminology. Let J_1, J_2 denote open sets of $\mathbb{R}$.

Definition 3.2.1 (Quasimonotone nonincreasing)**.** A function $f_i = f_i(x_1, x_2)$ is said to be *quasimonotone nonincreasing* in $J_1 \times J_2$ if, for any fixed $x_i \in J_i$, f_i is nonincreasing in $x_j \in J_j$, for $i \neq j$.

This further motivates the additional definition for vector valued functions.

Definition 3.2.2. A vector function $\mathbf{f} = (f_1, f_2)$ is said to be quasimonotone nonincreasing if both f_1 and f_2 are quasimonotone nonincreasing in $J_1 \times J_2$.

In particular, if $\mathbf{f}$ is a C^1 function in each of its arguments, the above definition corresponds to the following condition:

$$\frac{\partial f_1}{\partial x_2} \leq 0, \quad \frac{\partial f_2}{\partial x_1} \leq 0, \quad (x_1, x_2) \in J_1 \times J_2. \tag{3.5}$$

Next, we will establish a definition of sub/super solutions to (3.4) so that we can adapt monotone methods usually applied to a scalar equation to a system of equations instead. As usual, we denote the subsolution and supersolution by $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$, respectively. The chosen sub and supersolution must also satisfy the following boundary conditions

$$\begin{cases} (\overline{u}, \overline{v}) \geq (0,0) \geq (\underline{u}, \underline{v}), \quad x \in \partial\Omega,\ t > 0, \\ (\overline{u}, \overline{v}) \geq (u_0, v_0) \geq (\underline{u}, \underline{v}), \quad x \in \overline{\Omega},\ t = 0, \end{cases} \tag{3.6}$$

where we interpret the inequality $(x_1, x_2) \geq (y_1, y_2)$ to mean $x_1 \geq y_1$ and $x_2 \geq y_2$. As with the usual definition of sub and supersolutions, they are defined in terms of differential inequalities. However, we must now give attention to the definition given in relation to the quasimonotone properties of the reaction functions f_i. In more general cases (quasimonotone nondecreasing functions, or mixed quasimonotone functions), the definition of sub and supersolution is different than what is presented here. For details concerning different monotone properties, readers are directed to section 12.2 of chapter 12 in [20].

Definition 3.2.3. A pair of functions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ in $C^{2,1}(Q_T) \cap C(\overline{Q}_T)$ are called ordered subsolution and supersolution to problem (3.4) if they satisfy (3.6), $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$ in $\overline{Q}_T$, and

$$\begin{cases} \overline{u}_t - d\Delta \overline{u} - f_1(\overline{u}, \underline{v}) \geq 0 \geq \underline{u}_t - d\Delta \underline{u} - f_1(\underline{u}, \overline{v}), \\ \overline{v}_t - D\Delta \overline{v} - f_2(\underline{u}, \overline{v}) \geq 0 \geq \underline{v}_t - D\Delta \underline{v} - f_2(\overline{u}, \underline{v}), \quad x \in \Omega, \ t > 0. \end{cases} \tag{3.7}$$

Keen readers may notice that this particular choice of sub and super solutions is not universal in the sense that one will not obtain the existence of solutions to systems featuring *any* reaction function f_1, f_2. Instead, we focus our attention to the case where the reaction terms are monotone nonincreasing. Specifically, we need the above definition in order to prove Lemma 3.2.4 to follow.

We now wish to construct a monotone increasing and a monotone decreasing sequence of functions based on these differential inequalities and the properties of f_i. Define the functions:

$$\begin{cases} F_1(u,v) &= c_1 u + f_1(u,v), \\ F_2(u,v) &= c_2 v + f_2(u,v), \end{cases} \tag{3.8}$$

where the constants c_i come from the one-sided Lipschitz condition

$$\begin{cases} f_1(x_1, \cdot) - f_1(y_1, \cdot) &\geq -c_1(x_1 - y_1), \quad \text{when } x_1 \geq y_1, \\ f_2(\cdot, x_2) - f_2(\cdot, y_2) &\geq -c_2(x_2 - y_2), \quad \text{when } x_2 \geq y_2. \end{cases} \tag{3.9}$$

Notice that for $\mathbf{f} \in C^1$, these conditions are automatically satisfied. Starting from an initial iteration $(u^{(0)}, v^{(0)}) \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, we construct our desired sequence $\{(u^{(k)}, v^{(k)})\}_{k=0}^{\infty}$ through the following iteration process:

$$\begin{cases} u_t^{(k)} - d\Delta u^{(k)} + c_1 u^{(k)} = F_1(u^{(k-1)}, v^{(k-1)}), \\ v_t^{(k)} - D\Delta v^{(k)} + c_2 v^{(k)} = F_2(u^{(k-1)}, v^{(k-1)}), & x \in \Omega,\ t > 0, \\ u^{(k)}(x,0) = u_0(x), \\ v^{(k)}(x,0) = v_0(x), & x \in \Omega, \\ u^{(k)}(x,t) = v^{(k)}(x,t) = 0, & x \in \partial\Omega. \end{cases} \tag{3.10}$$

Through construction, we have decoupled our original nonlinear system into two linear initial-boundary value problems. By the L^p and classical theory of parabolic equations, we know that

$$u^{(1)}, v^{(1)} \in C^{\theta,\theta/2}(\overline{Q}_T), \quad \text{and} \quad u^{(k)}, v^{(k)} \in C^{2+\theta,1+\theta/2}(\overline{Q}_T), \quad k \geq 2. \tag{3.11}$$

Let us clarify this claim. Consider the equation for u. First, we note that our forcing term F_1 is continuous. On our first iteration, $u^{(0)}, v^{(0)} \in C(\overline{Q}_T)$. As a result, the composition $F_1(u^{(0)}, v^{(0)}) \in L^p$ for arbitrary $p > 1$, and so $u^{(1)} \in W_p^{2,1}(Q_T)$ by Theorem 3.1.1. Then, we may choose p sufficiently large so that $u^{(1)} \in C^{\theta,\theta/2}(\overline{Q}_T)$ via Theorem 2.0.2 stated in the preliminaries. Then, for every subsequent iteration, the forcing term now belongs to $C^{\theta,\theta/2}(\overline{Q}_T)$, as we are composing F_1 with a function of slightly higher regularity. Thus, by classical parabolic theory, $F_1 \in C^{\theta,\theta/2}(\overline{Q}_T) \Rightarrow$

$u^{(k)} \in C^{2+\theta,1+\theta/2}(\overline{Q}_T)$, for all $k \geq 2$. (One may refer to Theorem 8.3.7 in chapter 8 of [20]). The same argument holds for $v^{(k)}$.

We now want to ensure that this sequence $\{(u^{(k)}, v^{(k)})\}_{k=0}^{\infty}$ is monotone and converges to a solution of our original problem (3.4). This requires a suitable choice of our initial iteration, which the following lemma provides.

Lemma 3.2.4. *For quasimonotone nonincreasing* (f_1, f_2), *the two sequences* $\{(\overline{u}^{(k)}, \underline{v}^{(k)})\}$ *and* $\{(\underline{u}^{(k)}, \overline{v}^{(k)})\}$, *generated by choosing* $(\overline{u}, \underline{v})$ *or* $(\underline{u}, \overline{v})$ *as our initial iteration in* (3.10) *respectively, posses the monotone property*

$$(\underline{u}^{(k)}, \underline{v}^{(k)}) \leq (\underline{u}^{(k+1)}, \underline{v}^{(k+1)}) \leq (\overline{u}^{(k+1)}, \overline{v}^{(k+1)}) \leq (\overline{u}^{(k)}, \overline{v}^{(k)}) \tag{3.12}$$

for $(x,t) \in \overline{Q}_T$, *for all* $k \geq 0$.

Proof. We will provide the proof for the case when the initial iteration $(\overline{u}^{(0)}, \underline{v}^{(0)}) \equiv (\overline{u}, \underline{v})$ is chosen. The proof of the second case is quite similar. Let

$$\begin{aligned} w^{(0)}(x,t) &= \overline{u}^{(0)}(x,t) - \overline{u}^{(1)}(x,t) = \overline{u}(x,t) - \overline{u}^{(1)}(x,t), \\ z^{(0)}(x,t) &= \underline{v}^{(1)}(x,t) - \underline{v}^{(0)}(x,t) = \underline{v}(x,t) - \underline{v}^{(1)}(x,t), \quad x \in \overline{\Omega},\ t \geq 0. \end{aligned} \tag{3.13}$$

By virtue of the requirements of our sub and supersolution given by (3.6) and (3.7), we see that

$$\begin{cases} w_t^{(0)} - d\Delta w^{(0)} + c_1 w^{(0)} \geq F_1(\overline{u}, \underline{v}) - F_1(u^{(0)}, v^{(0)}) = 0, & x \in \Omega,\ t > 0, \\ w^{(0)} \geq 0, & x \in \partial\Omega,\ t > 0, \\ w^{(0)} \geq 0, & x \in \overline{\Omega},\ t = 0, \end{cases} \tag{3.14}$$

and

$$\begin{cases} z_t^{(0)} - D\Delta z^{(0)} + c_2 z^{(0)} \geq F_2(\overline{u}^{(0)}, \underline{v}^{(0)}) - F_2(\overline{u}, \underline{v}) = 0, & x \in \Omega,\ t > 0, \\ z^{(0)} \geq 0, & x \in \partial\Omega,\ t > 0, \\ z^{(0)} \geq 0, & x \in \overline{\Omega},\ t = 0. \end{cases} \tag{3.15}$$

The maximum principle applied to (3.14) implies that $w^{(0)} \geq 0$ in $\overline{Q}_T$, and so

$$\overline{u}^{(1)}(x,t) \leq \overline{u}^{(0)}(x,t) \tag{3.16}$$

in $\overline{Q}_T$. The same reasoning applied to (3.15) implies that

$$\underline{v}^{(0)}(x,t) \leq \underline{v}^{(1)}(x,t) \tag{3.17}$$

in $\overline{Q}_T$. In a similar fashion, one can also show

$$\overline{v}^{(1)}(x,t) \leq \overline{v}^{(0)}(x,t), \text{ and } \underline{u}^{(0)}(x,t) \leq \underline{u}^{(1)}(x,t), \tag{3.18}$$

in $\overline{Q}_T$. To achieve this, one simply adjusts the role of $w^{(0)}$ and $z^{(0)}$ defined above to obtain (3.15). It remains to show that $\underline{u}^{(1)} \leq \overline{u}^{(1)}$ and $\underline{v}^{(1)} \leq \overline{v}^{(1)}$. We apply a similar technique, but we now consider $w^{(1)} = \overline{u}^{(1)} - \underline{u}^{(1)}$ and $z^{(1)} = \overline{v}^{(1)} - \underline{v}^{(1)}$:

$$
\begin{aligned}
w_t^{(1)} - d\Delta w^{(1)} + c_1 w^{(1)} &\geq F_1(\overline{u}, \underline{v}) - F_1(\underline{u}, \overline{v}), && \text{by (3.7)}, \\
&= c_1(\overline{u} - \underline{u}) + f_1(\overline{u}, \underline{v}) - f_1(\underline{u}, \overline{v}) \\
&\geq c_2(\overline{v} - \underline{v}), && \text{by (3.9)} \\
&\geq 0, \qquad x \in \Omega,\ t > 0, \\
z_t^{(1)} - D\Delta z^{(1)} + c_2 z^{(1)} &\geq F_2(\underline{u}, \overline{v}) - F_2(\overline{u}, \underline{v}), && \text{by (3.7)}, \\
&= c_2(\overline{v} - \underline{v}) + f_2(\underline{u}, \overline{v}) - f_2(\overline{u}, \underline{v}) \\
&\geq c_1(\overline{u} - \underline{u}), && \text{by (3.9)} \\
&\geq 0, \qquad x \in \Omega,\ t > 0, \\
w^{(1)} &= 0, \qquad x \in \partial\Omega,\ t > 0, \\
w^{(1)} &= 0, \qquad x \in \overline{\Omega},\ t = 0, \\
z^{(1)} &= 0, \qquad x \in \partial\Omega,\ t > 0, \\
z^{(1)} &= 0. \qquad x \in \overline{\Omega},\ t = 0,
\end{aligned}
\tag{3.19}
$$

The maximum principle again gives us the desired conclusion:

$$
(\underline{u}^{(0)}, \underline{v}^{(0)}) \leq (\underline{u}^{(1)}, \underline{v}^{(1)}) \leq (\overline{u}^{(1)}, \overline{u}^{(1)}) \leq (\overline{u}^{(0)}, \overline{u}^{(0)}). \tag{3.20}
$$

Using induction, we may then repeat this process (considering $w^{(k)} = \overline{u}^{(k)} - \overline{u}^{(k-1)}$

for example) and using the fact that f_i is monotone to obtain

$$(\underline{u}^{(k)}, \underline{v}^{(k)}) \leq (\underline{u}^{(k+1)}, \underline{v}^{(k+1)}) \leq (\overline{u}^{(k+1)}, \overline{u}^{(k+1)}) \leq (\overline{u}^{(k)}, \overline{u}^{(k)}), \tag{3.21}$$

valid for all $k \geq 1$. This completes the proof. □

Next, we show that this sequence as constructed produces ordered sub and supersolutions.

Lemma 3.2.5. *Let* $(\underline{u}, \underline{v})$ *and* $(\overline{u}, \overline{v})$ *be the corresponding sub and supersolutions of problem* (3.4). *Then, the iterations given by Lemma 3.2.4 are ordered sub and supersolutions.*

Proof. We compute directly using (3.9) and the monotone property of (f_1, f_2):

$$\begin{aligned}
\overline{u}_t^{(k)} - d\Delta\overline{u}^{(k)} &= -c_1\overline{u}^{(k)} + F_1(\overline{u}^{(k-1)}, \underline{v}^{(k-1)}) \\
&= c_1\left[\overline{u}^{(k-1)} - \overline{u}^{(k)}\right] + \left[f_1(\overline{u}^{(k-1)}, \underline{v}^{(k-1)}) - f_1(\overline{u}^{(k)}, \underline{v}^{(k-1)})\right] \\
&\quad + \left[f_1(\overline{u}^{(k)}, \underline{v}^{(k-1)}) - f_1(\overline{u}^{(k)}, \underline{v}^{(k)})\right] + f_1(\overline{u}^{(k)}, \underline{v}^{(k)}) \\
&\geq f_1(\overline{u}^{(k)}, \underline{v}^{(k)}), \qquad x \in \Omega,\ t > 0, \\
\underline{v}_t^{(k)} - D\Delta\underline{v}^{(k)} &= -c_2\underline{v}^{(k)} + F_2(\overline{u}^{(k-1)}, \underline{v}^{(k-1)}) \\
&= -c_2\left[\underline{v}^{(k)} - \underline{v}^{(k-1)}\right] - \left[f_2(\overline{u}^{(k-1)}, \underline{v}^{(k)}) - f_2(\overline{u}^{(k-1)}, \underline{v}^{(k-1)})\right] \\
&\quad + \left[f_2(\overline{u}^{(k-1)}, \underline{v}^{(k)}) - f_2(\overline{u}^{(k)}, \underline{v}^{(k)})\right] + f_2(\overline{u}^{(k)}, \underline{v}^{(k)}) \\
&\leq f_2(\overline{u}^{(k)}, \underline{v}^{(k)}), \qquad x \in \Omega,\ t > 0.
\end{aligned} \tag{3.22}$$

An almost identical calculation shows that

$$\begin{aligned} \underline{u}_t^{(k)} - d\Delta \underline{u}^{(k)} &\leq f_1(\underline{u}^{(k)}, \overline{v}^{(k)}), && x \in \Omega,\ t > 0, \\ \overline{v}_t^{(k)} - D\Delta \overline{v}^{(k)} &\geq f_2(\underline{u}^{(k)}, \overline{v}^{(k)}), && x \in \Omega,\ t > 0. \end{aligned} \tag{3.23}$$

Thus, for each $k \geq 1$, we have shown that $(\underline{u}^{(k)}, \underline{v}^{(k)})$ and $(\overline{u}^{(k)}, \overline{v}^{(k)})$ are ordered sub and supersolutions to problem (3.4), and the proof is complete. □

Now we have done the hard work and are able to apply some of these conclusions to obtain some useful existence results. Notice that, through the above construction and calculations, the sequences $(\underline{u}^{(k)}, \underline{v}^{(k)})$ and $(\overline{u}^{(k)}, \overline{v}^{(k)})$ are monotone and uniformly bounded (in k) above and below. Thus, this sequence must converge monotonically to some limit. This limit is our candidate for a solution to problem (3.4). This brings us now to the conclusion of this chapter.

Theorem 3.2.6. *Let $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ be ordered sub and supersolutions to problem* (3.4) *and suppose (f_1, f_2) is quasimonotone nonincreasing satisfying* (3.9)*. Then problem* (3.4) *has a classical solution $(u, v) \in C^{2+\theta, 1+\theta/2}(\overline{\Omega} \times [0, T])$.*

Proof. To reach this conclusion, one may consider $\{(u^{(k)}, v^{(k)})\}_{k\in\mathbb{N}}$ to be either $\{(\overline{u}^{(k)}, \underline{v}^{(k)})\}_{k\in N}$ or $\{(\underline{u}^{(k)}, \overline{v}^{(k)})\}_{k\in\mathbb{N}}$, as chosen in the previous material. Since this sequence was shown to be monotone (in either case), and by the continuity and monotonicity of our functions $F_i(u, v)$, we know the $F_i(u^{(k)}, v^{(k)})$ converges monotonically to some limit $F(u, v)$. Then, by the regularity of solutions to the heat equation (as previously noted),

$$u^{(1)}, v^{(1)} \in C^{\theta,\theta/2}(\overline{\Omega} \times [0,T]), \quad u^{(k)}, v^{(k)} \in C^{2+\theta,1+\theta/2}(\overline{\Omega} \times [0,T]), \tag{3.24}$$

and by the usual Schauder estimates for parabolic equations (see Theorem 7.2.24 in chapter 7 of [20]), we have the following bounds:

$$\begin{gathered}
\left\|u^{(k)}\right\|_{C^{2+\theta,1+\theta/2}(Q_T)} \leq C_1 \left(\|u_0\|_{C^{2+\theta}(\Omega)} + \left\|u^{(k-1)}\right\|_{C^0(Q_T)} + \left\|v^{(k-1)}\right\|_{C^0(Q_T)} \right), \\
\text{and} \\
\left\|v^{(k)}\right\|_{C^{2+\theta,1+\theta/2}(Q_T)} \leq C_2 \left(\|v_0\|_{C^{2+\theta}(\Omega)} + \left\|u^{(k-1)}\right\|_{C^0(Q_T)} + \left\|v^{(k-1)}\right\|_{C^0(Q_T)} \right),
\end{gathered} \tag{3.25}$$

for all $k \geq 2$, where C_1, C_2 depend on θ, Ω, T and the Lipschitz constants associated to f_i, but remain independent of k. Since the sequence $\{u^{(k)}, v^{(k)}\}_{k\in\mathbb{N}}$ has been shown to be monotone, we thus have that $\{u^{(k)}, v^{(k)}\}_{k\in\mathbb{N}}$ is a unformly bounded sequence in $C^{2+\theta,1+\theta/2}(\overline{Q}_T)$. Consequently, the limit $(u,v) \in C^{2+\theta,1+\theta/2}(\overline{Q}_T)$ is a solution to the original problem (3.4). This completes the proof. □

This work has given us the tools necessary to guarantee the existence of a classical solution to the perturbed problem in a later section. There are a few remarks worth noting to conclude this section. First, we again make note of the fact that we require $(u_0, v_0) \in C^{2+\theta}(\overline{\Omega})$, which is quite strong. This is standard in all reference material and makes the above results more immediate, due to the nature of the Schauder estimates used. Specifically, the norm on the initial condition appearing on the right hand side of (3.25) is on the space $C^{2+\theta}$. Consequently, more care and attention must be given in order to obtain the same results for less regular initial data. For the sake of space, we will not go into the details here. Theorem 8.2.3 in [20] or Theorem 6.13 in [11], combined with other relevant results in the same chapters respectively, provide the necessary details. The second, perhaps more subtle, note to make is the dependencies of the constants $C_i, i = 1, 2$ found above. Namely, they both depend on the Lipschitz constant associated to the reaction terms f_i. As one may notice, the reaction terms of problems (1.20) and (1.21) are *not* Lipschitz continuous, but will be made so through the regularization of the singularities, as we will soon see. As a result, the Lipschitz constant associated to the perturbed reaction functions depend explicitly on the small perturbation ε, and thus so will the constants C_1, C_2 found above. The point to be made is that the following results are, indeed, nontrivial. That is, we cannot simply regularize our problem and apply the monotone methods above in a direct way, due to this dependence.

Chapter 4

Some Elliptic Problems

In this section we will state some results for elliptic equations. This is not meant to be exhaustive by any means; we are only concerned with presenting the material necessary for the remaining sections. We first remind ourselves of the first eigenvalue problem mentioned in the introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. We denote by $\phi(x)$ the positive solution to the following problem:

$$\begin{cases} 0 = \Delta\phi(x) + \lambda_1\phi(x), & x \in \Omega, \\ 0 = \phi(x), & x \in \partial\Omega. \end{cases} \tag{4.1}$$

This problem is well understood, and so there are many useful features that we can make note of. First, we know that the first eigenvalue is simple, and $\lambda_1 > 0$. We also know that since $\partial\Omega$ has a smooth boundary, $\phi \in C^2(\Omega) \cap C(\overline{\Omega})$. For further details, one may refer to [7],[11],[1],[20]. We now define a related problem, which happens (by no accident) to feature a singular nonlinearity similar to those found in the parabolic problems we are interested in. Denote by $\psi(x)$ the solution to the following problem:

$$\begin{cases} 0 = \Delta\psi(x) + \psi^{-\sigma}(x), & x \in \Omega, \\ 0 = \psi(x), & x \in \partial\Omega. \end{cases} \tag{4.2}$$

There are some rather interesting results for this problem, depending on where you let the power σ live. For our purposes, we consider the case when $\sigma \in (0,1)$. Under this assumption, Gui & Lin [12] prove that $\psi(x) \in C^{1,1-\sigma}(\overline{\Omega})$ is a positive solution. Further, we have estimates relating $\phi(x)$ and $d(x)$, where $d(x)$ is the distance to the boundary function:

$$\begin{aligned} d(x) &= d(x, \partial\Omega) \\ &\equiv \inf_{y \in \partial\Omega} d(x,y), \quad x \in \overline{\Omega}. \end{aligned} \tag{4.3}$$

Specifically, they show that there exist $\gamma_0, \gamma_1 > 0$ so that

$$\gamma_0 d(x) \leq \psi(x) \leq \gamma_1 d(x). \tag{4.4}$$

This will prove incredibly useful, as we are able to then relate $\phi(x)$ to $\psi(x)$ near the boundary by the smoothness of $\partial\Omega$. We now state a lemma provided in [11].

Lemma 4.0.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded and $\partial\Omega \in C^k$ for $k \geq 2$. Then, there exists a positive constant μ such that $d(x) \in C^k(\Gamma_\mu)$, where $\Gamma_\mu \equiv \{x \in \overline{\Omega} : d(x) < \mu\}$.*

In particular, assuming our boundary is of class C^2, we ensure that $d(x) \in C^2(\Gamma_\mu)$.

The following argument is fairly standard and can be found in [8] or [21], for example. Redefining $d(x)$ outside of this neighbourhood if necessary, we may assume that $d(x) \in C^2(\overline{\Omega})$. Then, since $\phi(x) \in C^2(\overline{\Omega})$, we know that

$$|\phi(x) - \phi(y)| \leq C_0 |x - y| , \tag{4.5}$$

for $x, y \in \overline{\Omega}$. Choosing $y \in \partial\Omega$, recalling that $\phi(y) = 0$, and choosing $x \in \Omega$ sufficiently close to $\partial\Omega$ such that there exists a unique $y \in \partial\Omega$ with $d(x) = |x - y|$, we then have that

$$|\phi(x)| \leq C_0 d(x). \tag{4.6}$$

This argument allows us to show that

$$c_0 d(x) \leq \phi(x) \leq C_0 d(x), \tag{4.7}$$

and so redefining γ_0, γ_1 in (4.4), we are able to write

$$\gamma_0 \phi(x) \leq \psi(x) \leq \gamma_1 \phi(x). \tag{4.8}$$

This allows us to interchange $\psi(x)$ and $\phi(x)$ merely at the cost of some constant multiple of the other. With these facts noted, we are now ready to prove some useful inequalities to be used in the main results section.

Chapter 5

Some Useful Inequalities

In this section, we display some of the key inequalities used to obtain a priori estimates related to the singular equations we are interested in. Many of the proofs have the same flavour, using integration by parts and arranging higher order terms in a particular way. Before we show this, we state a generalized version of Young's inequality which is also very important to this thesis. This inequality, used in conjunction with the subsequent integral inequalities, allows us to obtain the uniform L^k bounds we require.

Lemma 5.0.1 (Generalized Young's Inequality)**.** *Suppose $u(x), v(x), f(x), g(x) > 0$. For any indices $p_1, p_2, q_1, q_2, \alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1$ satisfying $\theta_1 < p_1 < \alpha_1$ (not necessarily positive), and given any constant $c > 0$, we have that*

$$\frac{u^{p_1} f^{p_2}}{v^{q_1} g^{q_2}} \leq c \frac{u^{\alpha_1} f^{\alpha_2}}{v^{\beta_1} g^{\beta_2}} + c^{-(p_1-\theta_1)/(\alpha_1-p_1)} \frac{u^{\theta_1} f^{\theta_2}}{v^{\eta_1} g^{\eta_2}} \ ,$$

where

$$\theta_2 = [p_2(\alpha_1 - \theta_1) - \alpha_2(p_1 - \theta_1)](\alpha_1 - p_1)^{-1},$$
$$\eta_1 = [q_1(\alpha_1 - \theta_1) - \beta_1(p_1 - \theta_1)](\alpha_1 - p_1)^{-1},$$
$$\eta_2 = [q_2(\alpha_1 - \theta_1) - \beta_2(p_1 - \theta_1)](\alpha_1 - p_1)^{-1} \ .$$

Proof. By assumption we may choose $p = (\alpha_1 - \theta_1)/(p_1 - \theta_1) > 1$, $q = (\alpha_1 - \theta_1)/(\alpha_1 - p_1) > 1$ so that $p^{-1} + q^{-1} = 1$. We then apply Young's inequality (similar to the proof of Young's inequality with epsilon) as follows

$$\begin{aligned}
\frac{u^{p_1} f^{p_2}}{v^{q_1} g^{q_2}} &= \frac{c^{1/p}\left(u^{\alpha_1/p} u^{\theta_1/q}\right)\left(f^{\alpha_2/p} f^{p_2-\alpha_2/p}\right)}{c^{1/p}\left(v^{\beta_1/p} v^{q_1-\beta_1/p}\right)\left(g^{\beta_2/p} g^{q_2-\beta_2/p}\right)} \\
&= \left(\frac{c^{1/p} u^{\alpha_1/p} f^{\alpha_2/p}}{v^{\beta_1/p} g^{\beta_2/p}}\right)\left(\frac{c^{-1/p} u^{\theta_1/p} f^{p_2-\alpha_2/p}}{v^{q_1-\beta_1/p} g^{q_2-\beta_2/p}}\right) \\
&\le \frac{\left(\frac{c^{1/p} u^{\alpha_1/p} f^{\alpha_2/p}}{v^{\beta_1/p} g^{\beta_2/p}}\right)^p}{p} + \frac{\left(\frac{c^{-1/p} u^{\theta_1/p} f^{p_2-\alpha_2/p}}{v^{q_1-\beta_1/p} g^{q_2-\beta_2/p}}\right)^q}{q} \\
&\le c\frac{u^{\alpha_1} f^{\alpha_2}}{v^{\beta_1} g^{\beta_2}} + c^{-q/p}\frac{u^{\theta_1} f^{p_2/q-\alpha_2 q/p}}{v^{q_1/q-\beta_1 q/p} g^{q_2/q-\beta_2 q/p}} \\
&= c\frac{u^{\alpha_1} f^{\alpha_2}}{v^{\beta_1} g^{\beta_2}} + c^{-(p_1-\theta_1)/(\alpha_1-p_1)}\frac{u^{\theta_1} f^{\theta_2}}{v^{\eta_1} g^{\eta_2}} \ .
\end{aligned}$$

This completes the proof. □

5.1 Integral Inequalities

For the next few results, it is instructive to consider a linear second order system of equations. In doing so, we highlight the focus, which is to control the diffusion terms found in our later equations. This should also indicate that these inequalities are useful for many parabolic systems, not just systems featuring singular nonlinearities. So, let $(u, v) \in [C^{1,1}(\overline{Q}_T)]^2$ be positive solutions in Q_T satisfying the following:

$$\begin{cases} u_t = d\Delta u + F(x,t), & x \in \Omega, \\ v_t = D\Delta v + G(x,t), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{5.1}$$

where $F, G \in C(\Omega \times [0, T])$ and $d, D > 0$. By Hopf's boundary lemma, since $u, v = 0$ on $\partial\Omega$, we have that $\partial v/\partial \mathbf{n} < 0$ and $\partial u/\partial \mathbf{n} < 0$ on $\partial\Omega$, where $\mathbf{n}$ denotes the outward unit normal vector. The first inequality considered will be useful in obtaining lower bounds for our solutions, and furthermore will demonstrate the tricks to be used in proving the later inequalities.

Lemma 5.1.1. *Let ϕ be the solution of* (4.1) *and u be a solution of* (5.1). *For any $n > 2$, we have that*

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^n}dx \le \lambda_1 n \int_\Omega \frac{\phi^{n+2}}{u^n}dx - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x,t)dx. \tag{5.2}$$

Proof. First, one must ensure these quantities are well defined. Indeed, this quantity is well defined by Hopf's lemma noted above. To see this, let $Q \in \partial\Omega$ and fix $t > 0$.

Then,

$$\begin{aligned}
\lim_{x\to Q}\frac{\phi(x)}{u(x,t)} &= \lim_{x\to Q}\frac{\phi(x)-\phi(Q)}{u(x,t)-u(Q,t)} \\
&= \lim_{x\to Q}\frac{\phi(x)-\phi(Q)}{x-Q}\Bigg/\frac{u(x,t)-u(Q,t)}{x-Q} \\
&= \frac{\partial\phi}{\partial\mathbf{n}}\Bigg/\frac{\partial u}{\partial\mathbf{n}} > 0. \qquad (5.3)
\end{aligned}$$

In other words, our integral as written is well defined up to the boundary of our domain. Also note that since our solutions to (4.1) and (5.1) are smooth, these normal derivatives remain finite. Next, differentiating with respect to t and integrating by parts, we have that:

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^n}dx &= -n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}\Delta u dx - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x,t)dx \\
&= n\int_\Omega \nabla\left(\frac{\phi^{n+2}}{u^{n+1}}\right)\nabla u dx - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x,t)dx \\
&= -n(n+1)\int_\Omega \frac{\phi^{n+2}}{u^{n+2}}\left|\nabla u\right|^2 dx + n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{n+1}}\nabla u\nabla\phi dx \\
&\quad - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x,t)dx. \qquad (5.4)
\end{aligned}$$

Here, we also provide justification for the removal of the boundary term. The same reasoning will apply for the remainder of this thesis, and so often the details will be omitted. In our case above, after integrating by parts the boundary term looks like

$$\begin{aligned}
\int_{\partial\Omega}\frac{\phi^{n+2}}{u^{n+1}}\frac{\partial u}{\partial\mathbf{n}}dS &= \int_{\partial\Omega}\left(\frac{\partial\phi/\partial\mathbf{n}}{\partial u/\partial\mathbf{n}}\right)^{n+1}\phi\frac{\partial u}{\partial\mathbf{n}}dS \\
&= 0, \qquad (5.5)
\end{aligned}$$

since $\phi = 0$ on $\partial\Omega$. By (5.3), we safely remove most powers of the normal derivatives and are left with $\phi(x)$ in the numerator. This ensures that our boundary term is indeed zero. This is important to note in the other integral inequalities to be used as well, as this provides the intuition as to why we require the power on the numerator to be at least 2 higher than the power on the denominator, rather than having equal powers, for example. Continuing now, we use the following identity:

$$\phi^2 |\nabla u|^2 = |\phi\nabla u - u\nabla\phi|^2 + 2u\phi\nabla u\nabla\phi - u^2 |\nabla\phi|^2 . \tag{5.6}$$

We then see that (5.4) can be written instead as

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^n}dx = &- n(n+1)\int_\Omega \frac{\phi^n}{u^{n+2}} |\phi\nabla u - u\nabla\phi|^2 dx - n^2\int_\Omega \frac{\phi^{n+1}}{u^{n+1}}\nabla u\nabla\phi dx \\ &+ n(n+1)\int_\Omega \frac{\phi^n}{u^n} |\nabla\phi|^2 dx - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x,t)dx.\end{aligned} \tag{5.7}$$

If we integrate by parts on the second term of (5.7), we find

$$\begin{aligned}-n^2\int_\Omega \frac{\phi^{n+1}}{u^{n+1}}\nabla u\nabla\phi dx &= n\int_\Omega \phi^{n+1}\nabla\phi\nabla\left(\frac{1}{u^n}\right)dx \\ &= -n\int_\Omega \nabla\left(\phi^{n+1}\nabla\phi\right)\frac{1}{u^n}dx \\ &= -n(n+1)\int_\Omega \frac{\phi^n}{u^n}|\nabla\phi|^2 dx - n\int_\Omega \frac{\phi^{n+1}}{u^n}\Delta\phi dx.\end{aligned} \tag{5.8}$$

We again note that the boundary term vanishes. Combining the above with the rest of (5.7) gives us that

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^n}dx \leq -n\int_\Omega \frac{\phi^{n+1}}{u^n}\Delta\phi dx - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x,t)dx. \tag{5.9}$$

Lastly, we recall (4.1) to conclude that

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^n}dx \leq \lambda_1 n\int_\Omega \frac{\phi^{n+2}}{u^n}dx - n\int_\Omega \frac{\phi^{n+2}}{u^{n+1}}F(x,t)dx. \tag{5.10}$$

This completes the proof. □

The next inequality is similar to that shown above, though the difficulty is increased by the presence of terms from both equations for u and v. Despite this, it proves useful in obtaining uniform bounds in time in the case when $d = D$.

Lemma 5.1.2. *Let ϕ be the solution of* (4.1) *and u, v be solutions of* (5.1) *for $d = D$. Then, for any $\alpha, \beta > 0$ satisfying $\alpha + \beta \leq 1$, it is true that*

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n}} \leq -\alpha n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n}}F(x,t)dx - \beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+1}}G(x,t)dx \\ + \lambda_1(n+2)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n}}dx,\end{aligned} \tag{5.11}$$

for any $n > 0$.

Proof. Since $d = D$, we may rescale the space variables such that $d = D = 1$. Then, for simplicity we first set

$$z_n(t) = \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n}}dx.$$

Consider the following after taking the derivative with respect to time:

$$
\begin{aligned}
z_n'(t) =& -\alpha n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} u_t dx - \beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} v_t dx \\
=& -\alpha n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} \Delta u dx - \beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} \Delta v dx \\
& -\alpha n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} F(x,t) dx - \beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} G(x,t) dx. \qquad (5.12)
\end{aligned}
$$

Looking at only the Laplacian terms and integrating by parts gives us that:

$$
\begin{aligned}
& \alpha n \int_\Omega \nabla \left(\frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} \right) \nabla u dx + \beta n \int_\Omega \nabla \left(\frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} \right) \nabla v dx \\
=& \ \alpha n (n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1} v^{\beta n}} \nabla \phi \nabla u dx + \beta n (n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n} v^{\beta n+1}} \nabla \phi \nabla v dx \\
& - \alpha n (\alpha n + 1) \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2} v^{\beta n}} |\nabla u|^2 dx - \beta n (\beta n + 1) \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+2}} |\nabla v|^2 dx \\
& - 2 \alpha \beta n^2 \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n+1}} \nabla u \nabla v dx \\
\equiv& \sum_{k=1}^{5} I_k. \qquad (5.13)
\end{aligned}
$$

Once again, justification for the removal of one of the boundary terms will be provided. In this case, the first integral in the line above has a boundary that looks like

$$
\begin{aligned}
\int_{\partial\Omega} \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} \frac{\partial u}{\partial \mathbf{n}} dS &= \int_{\partial\Omega} \left(\frac{\phi}{u} \right)^{\alpha n} \left(\frac{\phi}{v} \right)^{\beta n} \phi^{n(1-\alpha-\beta)+1} \left(\frac{\partial u}{\partial \mathbf{n}} \right) dS \\
&= \int_{\partial\Omega} \left(\frac{\partial \phi / \partial \mathbf{n}}{\partial u / \partial \mathbf{n}} \right)^{\alpha n} \left(\frac{\partial \phi / \partial \mathbf{n}}{\partial v / \partial \mathbf{n}} \right)^{\beta n} \phi^{n(1-\alpha-\beta)+1} \left(\frac{\partial u}{\partial \mathbf{n}} \right) dS \\
&= 0, \qquad (5.14)
\end{aligned}
$$

since $\alpha+\beta \leq 1$. Again, it is important to have a dominant power in the numerator, and this condition on α, β ensures this is true for any $n > 0$. Now, consider the following term and integrate by parts:

$$\begin{aligned} J &\equiv -\alpha n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n}}\nabla\phi\nabla u dx \\ &= (n+2)\int_\Omega \frac{\phi^{n+1}\nabla\phi}{v^{\beta n}}\nabla\left(\frac{1}{u^{\alpha n}}\right)dx \\ &= -(n+2)\int_\Omega \nabla\left(\frac{\phi^{n+1}\nabla\phi}{v^{\beta n}}\right)\frac{1}{u^{\alpha n}}dx \\ &= -(n+2)(n+1)\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx - (n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx \\ &\quad + \beta n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n+1}}\nabla\phi\nabla v dx. \end{aligned} \tag{5.15}$$

We notice some similar terms, which combine as follows:

$$\begin{aligned} (I_1 - J) \ + \ (J + I_2) &= 2\alpha n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u + \frac{\beta}{\alpha}u\nabla v\right)dx \\ &- (n+2)(n+1)\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx - (n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx\ . \end{aligned} \tag{5.16}$$

This gives us

$$
\begin{aligned}
\sum_{k=1}^{5} I_k = 2\alpha n(n+2)&\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u+\frac{\beta}{\alpha}u\nabla v\right)dx \\
&-\alpha n(\alpha n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n}}\left|\nabla u\right|^2 dx-\beta n(\beta n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+2}}\left|\nabla v\right|^2 dx \\
&-2\alpha\beta n^2\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n+1}}\nabla u\nabla v dx-(n+2)(n+1)\int_\Omega \frac{\phi^{n}}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
&-(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx. \qquad (5.17)
\end{aligned}
$$

Then we complete the square in a specific way, keeping mind of the lower order terms (in n):

$$
\begin{aligned}
\sum_{k=1}^{5} I_k = 2\alpha n(n+2)&\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u+\frac{\beta}{\alpha}u\nabla v\right)dx \\
&-\alpha^2 n(n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u+\frac{\beta}{\alpha}u\nabla v\right|^2 dx \\
&-\alpha(1-\alpha)n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n}}\left|\nabla u\right|^2 dx-\beta(1-\beta)n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n}v^{\beta n+2}}\left|\nabla v\right|^2 dx \\
&+2\alpha\beta n\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1}v^{\beta n+1}}\nabla u\nabla v dx-(n+2)(n+1)\int_\Omega \frac{\phi^{n}}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
&-(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx. \qquad (5.18)
\end{aligned}
$$

If we recall that $\alpha + \beta \leq 1$, we combine terms $3, 4$ and 5 of (5.18) as follows:

$$\begin{aligned}
&- \alpha(1-\alpha)n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2} v^{\beta n}} |\nabla u|^2 \, dx - \beta(1-\beta)n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+2}} |\nabla v|^2 \, dx \\
&+ 2\alpha\beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n+1}} \nabla u \nabla v dx \\
\leq &- \alpha\beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2} v^{\beta n}} |\nabla u|^2 \, dx - \alpha\beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+2}} |\nabla v|^2 \, dx \\
&+ 2\alpha\beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n+1}} \nabla u \nabla v dx \\
= &- \alpha\beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2} v^{\beta n+2}} |v\nabla u - u \nabla v|^2 \, dx \\
\leq &\; 0. \qquad (5.19)
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\sum_{k=1}^{5} I_k \leq &\; 2\alpha n(n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1} v^{\beta n+1}} \nabla \phi \left(v \nabla u + \frac{\beta}{\alpha} u \nabla v \right) dx \\
&- \alpha^2 n(n+1) \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2} v^{\beta n+2}} \left| v \nabla u + \frac{\beta}{\alpha} u \nabla v \right|^2 dx \\
&- (n+2)(n+1) \int_\Omega \frac{\phi^n}{u^{\alpha n} v^{\beta n}} |\nabla \phi|^2 \, dx - (n+2) \int_\Omega \frac{\phi^{n+1}}{u^{\alpha n} v^{\beta n}} \Delta \phi dx. \qquad (5.20)
\end{aligned}$$

Applying Cauchy's Inequality to the first term of (5.20), we have that:

$$\begin{aligned}
&\left|2\alpha n(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n+1}v^{\beta n+1}}\nabla\phi\left(v\nabla u+\frac{\beta}{\alpha}u\nabla v\right)dx\right| \\
&\le \frac{1}{2}\left(2\alpha^2 n(n+1)\right)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u+\frac{\beta}{\alpha}u\nabla v\right|^2 dx \\
&\quad + \frac{1}{2}\left(\frac{2\alpha^2 n^2(n+2)^2}{\alpha^2 n(n+1)}\right)\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
&= \alpha^2 n(n+1)\int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+2}v^{\beta n+2}}\left|v\nabla u+\frac{\beta}{\alpha}u\nabla v\right|^2 dx \\
&\quad + \frac{n(n+2)^2}{n+1}\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx.
\end{aligned} \tag{5.21}$$

Combining this with our remaining terms leaves us with the following:

$$\begin{aligned}
\sum_{k=1}^{5} I_k &\le -\left[(n+2)(n+1)-\frac{n(n+2)^2}{n+1}\right]\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
&\quad -(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx \\
&= -\frac{(n+2)}{(n+1)}\left[(n+1)^2-n(n+2)\right]\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx \\
&\quad -(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx \\
&= -\frac{(n+2)}{(n+1)}\int_\Omega \frac{\phi^n}{u^{\alpha n}v^{\beta n}}\left|\nabla\phi\right|^2 dx-(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx \\
&\le -(n+2)\int_\Omega \frac{\phi^{n+1}}{u^{\alpha n}v^{\beta n}}\Delta\phi dx.
\end{aligned} \tag{5.22}$$

Finally, applying (4.1) to the last term in the line above leaves us with

$$z_n'(t) \leq -\alpha n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n+1} v^{\beta n}} F(x,t)dx - \beta n \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n+1}} G(x,t)dx \\ + \lambda_1(n+2) \int_\Omega \frac{\phi^{n+2}}{u^{\alpha n} v^{\beta n}} dx. \tag{5.23}$$

This completes the proof. □

This last inequality will be useful in obtaining upper bounds for our solutions. The motivation behind this will become clear in the next section, but the main idea is this: we cannot combine terms u appearing in the numerator with the perturbed terms $u+\varepsilon$ appearing in the denominator in a useful way. Instead, we treat them as their own terms and remove them individually.

Lemma 5.1.3. *Suppose that u is a solution of* (5.1) *and let ψ be the solution of (4.2). For any $\varepsilon > 0$, define $w_\varepsilon = u + \varepsilon$. Then, for any $\alpha, \beta \in (0,1)$ satisfying $\alpha + \beta \leq 1$, we have*

$$\frac{d}{dt} \int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n} \psi^{\beta n-2}} dx \leq n \int_\Omega \frac{u^{n-1}}{{w_\varepsilon}^{\alpha n} \psi^{\beta n-2}} F(x,t)dx - \alpha n \int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n+1} \psi^{\beta n-2}} F(x,t)dx \\ - d(\beta n - 2) \int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n} \psi^{\beta n-1+\sigma}} dx, \tag{5.24}$$

for all $n > 2$.

Proof. First, since $\alpha + \beta \leq 1$, we know that the quantity under consideration is well defined up to the boundary. The idea is similar to that shown in the previous proof, but this time we instead write $u = u^\beta u^{1-\beta}$ and show that u/ψ is defined up to the boundary by Hopf's lemma. Notice that since w_ε is bounded away from zero near $\partial\Omega$ and positive in Ω, we need not worry about this term causing singular behaviour.

Now, similar to the proof of Lemma (5.1.2), we differentiate with respect to time to find

$$
\begin{aligned}
(d^{-1})\frac{d}{dt}\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}dx = & \ n\int_\Omega \frac{u^{n-1}}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}[\Delta u + d^{-1}F(x,t)]dx \\
& - \alpha n\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n+1}\psi^{\beta n-2}}[\Delta w_\varepsilon + d^{-1}F(x,t)]dx \\
= & -n(n-1)\int_\Omega \frac{u^{n-2}}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}\left|\nabla u\right|^2 dx \\
& + 2\alpha n^2\int_\Omega \frac{u^{n-1}}{{w_\varepsilon}^{\alpha n+1}\psi^{\beta n-2}}\nabla u\nabla w_\varepsilon dx \\
& + n(\beta n-2)\int_\Omega \frac{u^{n-1}}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-1}}\nabla u\nabla\psi dx \\
& + \frac{n}{d}\int_\Omega \frac{u^{n-1}}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}F(x,t)dx \\
& - \alpha n(\alpha n+1)\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n+2}\psi^{\beta n-2}}\left|\nabla w_\varepsilon\right|^2 dx \\
& - \alpha n(\beta n-2)\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n+1}\psi^{\beta n-1}}\nabla w_\varepsilon\nabla\psi dx \\
& - \frac{\alpha n}{d}\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n+1}\psi^{\beta n-2}}F(x,t)dx \\
\equiv & \ \sum_{k=1}^{7} I_k.
\end{aligned}
\tag{5.25}
$$

By Lemma 2.5 in [3], it is true that

$$
I_1 + I_2 + I_3 + I_5 + I_6 \leq -(\beta n-2)\int_\Omega \frac{u^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx. \tag{5.26}
$$

(5.25) then becomes

$$
\begin{aligned}
(d^{-1})\frac{d}{dt}\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}dx &\leq I_4 + I_7 - (\beta n - 2)\int_\Omega \frac{u^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx \\
&= \frac{n}{d}\int_\Omega \frac{u^{n-1}}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}F(x,t)dx - \frac{\alpha n}{d}\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n+1}\psi^{\beta n-2}}F(x,t)dx \\
&- (\beta n - 2)\int_\Omega \frac{u^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx. \qquad (5.27)
\end{aligned}
$$

Multiplying both sides by d completes the proof. □

We these inequalities at our disposal, we are now ready to discuss the main results of this thesis.

Chapter 6

Main Results

6.1 Problem Statement and Assumptions

In this section, we are now able to discuss the main results of this thesis. The two systems under consideration are written as follows:

$$\begin{cases} u_t = d\Delta u + \frac{f(x)}{u^p v^q}, \\ v_t = D\Delta v + \frac{g(x)}{u^r v^s}, \quad x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \qquad x \in \Omega \\ u = v = 0, \qquad\qquad x \in \partial\Omega, \end{cases} \tag{6.1}$$

and

$$\begin{cases} u_t = d\Delta u + \frac{f_1(x)}{u^p} + \frac{f_2(x)}{v^q}, \\ v_t = D\Delta v + \frac{f_3(x)}{u^r} + \frac{f_4(x)}{v^s}, \quad x \in \Omega, \ t > 0, \\ u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \qquad x \in \Omega \\ u = v = 0, \qquad x \in \partial\Omega. \end{cases} \tag{6.2}$$

For these results, we continue to consider $\Omega \subset \mathbb{R}^N$, a bounded domain with smooth boundary. For the remainder of this thesis, we consider $d, D > 0$ and $p, q, r, s \in (0,1)$. Of course, further restrictions will be put upon p, q, r, s dependent on the problem, as we will soon see. We further assume that $u_0, v_0 \in C_0^{2+\theta}(\Omega)$, for some $\theta \in (0,1)$, and that there exists a constant $\varepsilon_0 > 0$ so that

$$u_0(x), v_0(x) \geq \varepsilon_0 \phi(x). \tag{6.3}$$

The reason for this assumed regularity of our initial data up to the boundary is two fold; appealing to the theory discussed in chapter 3, we need sufficiently regular initial data in order to obtain the existence of classical solutions to the perturbed problem, and we also need to be able to control the ratio of the initial data with the first eigenfunction of the Laplacian near the boundary of our domain. For simplicity, we choose $C_0^{2+\theta}(\Omega)$ to achieve this, though this condition can be weakened to $C_0^1(\Omega)$ with a barrier function type argument. As an interesting remark, it should be clear that we do not expect a classical solution in the sense that $u, v \in C^{2+\theta, 1+\theta/2}(\overline{\Omega} \times [0,T))$ if $f, g, f_i > 0$ as $x \to \partial\Omega$. We will now expand upon the definition given in chapter

2, lines (2.10) – (2.12), for what we mean by a *weak* and *classical* solution to problems (6.1) and (6.2). We call (u, v) a *weak* solution to problem (6.1) provided $u, v \in L^2(0, T; W_0^{1,2}(\Omega)) \cap L^\infty(\overline{\Omega} \times (0, T))$,

$$\frac{f(x)}{u^p v^q}, \frac{g(x)}{u^r v^s} \in L^1(\Omega \times (0, T)), \tag{6.4}$$

and

$$\begin{cases} \int_\Omega u_0 \xi dx + \int_0^T \int_\Omega \left(u\xi_t - d\nabla u \nabla \xi + \frac{f}{u^p v^q} \xi \right) dxdt = 0, \\ \int_\Omega v_0 \xi dx + \int_0^T \int_\Omega \left(v\xi_t - D\nabla v \nabla \xi + \frac{g}{u^r v^s} \xi \right) dxdt = 0, \end{cases} \tag{6.5}$$

for all $\xi \in C^\infty(\overline{\Omega} \times (0, T))$ with $\xi(x, t) = 0$ on $\partial\Omega \times (0, T)$ and $\xi(x, T) = 0$ in Ω. Furthermore, the solution (u, v) is called *global* if the above conditions are satisfied for every $T > 0$. This distinction is a bit superfluous in the case of this thesis, as every solution will be shown to be global. However, there is a slight distinction to be made when considering *classical* solutions to problem (6.1). We call (u, v) a *classical* solution to problem (6.1) if $u, v \in C^{2+\theta, 1+\theta/2}(\Omega \times (0, T)) \cap C^{1,0}(\overline{\Omega} \times [0, T))$ and satisfy (6.1) pointwise. As with weak solutions, we call (u, v) a *global* classical solution if u, v exist in the classical sense for all $T > 0$. Finally, we call a global classical solution a *globally bounded* classical solution if u, v exist in the classical sense and also belong to $L^\infty(\overline{\Omega} \times [0, \infty))$. Here, the distinction being made is that a global solution may blow up in *infinite* time, while a globally bounded solution remains uniformly bounded for all $T > 0$. These definitions apply to problem (6.2) in the obvious way, where we need only swap the nonlinearities in (6.4) and (6.5).

The general outline of the method to be used in proving the subsequent theorem statements comes in a number of steps:

i. Perturb the equations appropriately by some small quantity $\varepsilon > 0$.

ii. Obtain uniform L^∞ bounds on the perturbed solutions.

iii. Obtain uniform L^k bounds on the nonlinear terms.

iv. Apply standard L^p theory to obtain a weak solution of the original problem.

v. Apply a Sobolev embedding to obtain additional smoothness of solutions.

To expand, step i removes the singular nature of the nonlinearities near the boundary. This allows us to apply the classical theory of chapter 3 in order to obtain the existence of classical solutions to the perturbed problem. Then, step ii applies the material established in chapters 4 and 5 in order to uniformly bound these perturbed solutions, where the uniformity is required to be in ε, though sometimes these bounds are uniform in t as well. It is worth emphasizing that we require the presence of solutions to the elliptic problems, discussed in chapter 4, in our integral inequalities in order to maintain control of our solutions up to the boundary of Ω. This is distinct from the problem under Neumann conditions, in which case this inclusion is not necessary. In step iii, using these uniform bounds, one is then able to show that the nonlinear terms are uniformly bounded in an appropriate L^k space, for some $k > 1$. For steps iv and v, we then apply standard L^p theory and the Sobolev embedding found in chapter 2 in order to obtain the existence of weak, and when possible, classical solutions.

6.2 Theorem Statements

Here we present the theorem statements, the proofs of which will appear in the following sections. The first, simpler case is where the diffusion coefficients are equal. Here, we are able to obtain the existence of globally bounded classical solutions. The two results are as follows.

Theorem 6.2.1 (Existence)**.** *Suppose $d = D$, $p, s \in [0, 1)$ and $q, r \in (0, 1)$ such that $p + q < 1$ and $r + s < 1$. Also suppose that there exists $c_0, C_0 > 0$ so that*

$$\begin{cases} c_0 d^{\theta}(x) \leq f(x) \leq C_0 d^{\tau}(x), \\ c_0 d^{\eta}(x) \leq g(x) \leq C_0 d^{\mu}(x), \end{cases} \tag{6.6}$$

where $0 \leq \mu \leq \eta \leq 1$, $0 \leq \tau \leq \theta \leq 1$ and $d(x)$ denotes the distance to the boundary function defined in chapter 4. Then, there exists at least one global weak solution (u, v) to problem (6.1). Furthermore, if $-1 < N[\tau - (p + q)]$, u is a globally bounded classical solution. If $-1 < N[\mu - (r + s)]$, v is a globally bounded classical solution.

Theorem 6.2.2 (Existence)**.** *Suppose $d = D$, $p, q, r, s \in (0, 1)$. Also suppose that there exists $c_0, C_0 > 0$ and $c \geq 0$ so that*

$$\begin{cases} c d^{\theta_1}(x) \leq f_1(x) \leq C_0 d^{\tau_1}(x), \\ c_0 d^{\theta_2}(x) \leq f_2(x) \leq C_0 d^{\tau_2}(x), \\ c_0 d^{\theta_3}(x) \leq f_3(x) \leq C_0 d^{\tau_3}(x), \\ c d^{\theta_4}(x) \leq f_4(x) \leq C_0 d^{\tau_4}(x), \end{cases} \tag{6.7}$$

where $0 \le \tau_i \le \theta_i \le 1$*,* $i = 1, 2, 3, 4$*. Then, there exists at least one global weak solution* (u, v) *to problem* (6.2)*. Furthermore, if* $-1 < N \min\{(\tau_1 - p), (\tau_2 - q)\}$*,* u *is a globally bounded classical solution. If* $-1 < N \min\{(\tau_3 - r), (\tau_4 - s)\}$*,* v *is a globally bounded classical solution.*

The next two results, treating unequal diffusion coefficients, are slightly weaker in the sense that the classical solutions obtained are not globally bounded.

Theorem 6.2.3 (Extension)**.** *Suppose* $d, D > 0$*, and all hypotheses of Theorem* 6.2.1 *hold. Then there exists at least one global weak solution* (u, v) *to problem* (6.1)*. If* $-1 < N[\tau - (p + q)]$*,* u *is a global classical solution. If* $-1 < N[\mu - (r + s)]$*,* v *is a global classical solution.*

Theorem 6.2.4 (Extension)**.** *Suppose* $d, D > 0$*, and all hypotheses of Theorem* 6.2.2 *hold. Then there exists at least one global weak solution* (u, v) *to problem* (6.1)*. If* $-1 < N \min\{(\tau_1 - p), (\tau_2 - q)\}$*,* u *is a global classical solution. If* $-1 < N \min\{(\tau_3 - r), (\tau_4 - s)\}$*,* v *is a global classical solution.*

Before diving into the proofs of these results, we will first discuss some of the intuition behind the conditions appearing in these theorems, as well as how they compare to other similar systems of equations.

6.3 Comparisons to Other Works

As mentioned in chapter 1, the two systems under consideration are motivated by the elliptic problems given in lines (1.17) and (1.18). These systems have been treated in a few works, namely [8], [21] and [9]. In [8] and [21], the authors consider the system given by (1.17) with positive exponents. In doing so, they maintain the singular behaviour near the boundary. The tools used in these works are primarily methods of sub and super solutions, along with some fixed point arguments. This allows the authors to give more explicit boundary behaviour, which is covered in [21] is great detail. In particular, they obtain some fairly exact estimates of the decay of the solutions near the boundary, depending on the size of the exponents. These decay rates are written as bounds on the solutions in terms of $d(x)$ raised to various powers. In this thesis, a time variable has been introduced, and so it should be clear that the systems are no longer elliptic. In doing so, the methods used in these papers are no longer applicable as prescribed. As discovered in the investigation done in chapter 3, the definition for a sub and supersolution pair is not so obvious for parabolic systems, even when the functions are monotone in their arguments u, v. Despite this, we are able to construct a sub-super solution pair upon perturbing the problem, but this does not give us a uniform bound in ε, and so additional tools are necessary in order to obtain the existence of solutions to the unperturbed problem. Moreover, there are other difficulties introduced in this thesis not addressed by either of these papers. Namely, we consider different diffusion coefficients, as well as increased complexity in the nonlinearities through the appearance of time independent functions. For example, we include f, g in system (6.1), which is not explicitly covered in either [8] or [21]. It should nonetheless be noted that [8] does discuss the ability of the methods used to be applied to more general reaction functions of the form considered in this thesis, as well as more general second order differential operators. Despite this, the author of this thesis believes that these generalizations are nontrivial and deserve further investigation.

There is motivation to study the case of different diffusion coefficients in the parabolic case. When discussing Turing instability, this phenomenon does not actually occur in the cases where $d = D$. As discussed in chapter 1, the diffusion drives the resulting instability, and the unequal diffusion coefficients is key in obtaining such results. For this reason, it is of notable worth to study (6.1) and (6.2) in such generality for purposes of applicability, even outside the study of Turing instability.

When comparing the actual existence results, the elliptic cases treat more general exponents (and consequently more general conditions on the exponents). For example, the exponents may live outside the interval $(0, 1)$, whereas this thesis considers only the case when all exponents live within $(0, 1)$ or $[0, 1)$. From this, their solutions may only be continuous up to the boundary as opposed to once continuously differentiable, as is found in this work. They also discuss nonexistence as well as uniqueness of solutions in some cases. In [9], the author discusses the existence of solutions to problem (1.18), where similar comparisons to problem (6.2) can be made.

In the parabolic case, the most comparable problem is given in line (1.19), which is covered in [4]. The authors discuss the existence and regularity of solutions, similar to the results found in this thesis. However, problems (6.1) and (6.2) are much more general. In fact, both problems can be reduced to problem (1.19) under the assumptions found in the previous hypothesis for existence and regularity. In particular, one is able to take $p = s = 0$ in Theorem 6.2.1 and recover the results found in the main theorem of [4]. One could also take $f_1(x) = f_4(x) = 0$ under the hypothesis found in theorem 6.2.2 in order to recover the same result. But, as previously mentioned, this is made even more general by the inclusion of different diffusion coefficients, covered in Theorem 6.2.3 and Theorem 6.2.4.

Finally, we discuss some of the intuition behind the hypotheses of the theorem statements. In Theorem's 6.2.1 and 6.2.3, we consider $p, s \in [0, 1)$ while $q, r \in (0, 1)$. The reason for this is clear when paying attention to what these exponents are attached to. We wish to include 0 for p and s, as this allows us to generalize problem (6.1) to the results found in [4]. Further, we do not allow r, q to take the value 0 as this would decouple at least one of the equations in the system, and so things are (maybe) not so interesting. Here, we note that the methods used in the following proofs are not directly applicable to the case where $r, q = 0$, though this may be considered under appropriate adjustments to the arguments made. In contrast, we are able to freely consider the case where $p, s = 0$. Notice that we do not have the same coupling in system (6.2), at least in relation to the exponents. On the other hand, we do see the coupling through the functions f_i. In the same way that some exponents can take the value 0 in system (6.1), we allow the lower bound for f_1 and f_4 to be identically 0. In doing so, we do not decouple the equation, and we are also able to then generalize system (6.2) to the results found in [4]. As we will soon find out, these weakened conditions on some aspects of the equations considered are natural to the problems themselves and fall into place in an expected way.

The motivation for the seemingly complicated conditions on the boundary behaviour of the functions appearing in the numerator of our reaction terms comes from works such as [12]. Here, the authors consider the same conditions for a reaction term, but the exponents must take the same value. Explicitly stated, they require the existence of some constants $a, b > 0$ and $\alpha \in [0, 1]$ such that $ad^{\alpha}(x) \leq p(x) \leq bd^{\alpha}(x)$, where $p(x)$, a non-negative measurable function, appears in the numerator of the singular reaction term. In our case(s), we extend this idea in order to include a wider range of functions through considering unequal exponents in the upper and lower bounds. For simplicity, If we restrict ourselves to one spatial dimension some easy examples may be considered to clarify these differences. So, take $\Omega = (0, 1)$ and observe the following reaction functions:

$$\begin{cases} h_1(x) = \sin(\pi x), \\ h_2(x) = x(1 - x), \\ h_3(x) = \sqrt{x}\sqrt{1 - x}, \\ h_4(x) = \sqrt{x}(1 - \sqrt{x}). \end{cases}$$

Admittedly, these are very straightforward examples, but in choosing such examples we demonstrate the differences between one exponent and multiple in an effective way. To start, the function $h_1(x)$ is known to decay linearly near the boundary on Ω, and so we may simply take $d(x) \leq h_1(x) \leq 2\pi d(x)$. Here, we see that $h_1(x)$ fits into both the results here as well as the restriction in [12]. The second and third example, $h_2(x)$ and $h_3(x)$, also fall into both scenarios. In this case, we simply take $2^{-1}d(x) \leq h_2(x) \leq 2d(x)$ and $2^{-1}d^{1/2}(x) \leq h_3(x) \leq 2d^{1/2}(x)$. Notice that in both of these cases, the rate of decay on the bounding function $d(x)$ corresponds to the rate of decay on the reaction function h_i, $i = 2, 3$. So far there are no issues, but what happens if there are different rates of decay at different areas of the boundary? For this example, we consider $h_4(x)$, which decays like $\sqrt{x}$ near $x = 0$ but decays linearly near $x = 1$. This is an example of a function which *does not* fall into the condition of equal powers on the upper and lower bounds, but *does* fall into the hypotheses presented here. For example, one may take $4^{-1}d(x) \leq h_4(x) \leq 4d^{1/2}(x)$. Here, we are able to bound from below by linear decay, while maintaining a bound from above with some nonlinear decay. This covers the linear decay near the right endpoint, but also covers the nonlinear decay near the left endpoint. Alternatively, if the exponents are required to be equal, we cannot control the left endpoint behaviour without sacrificing control of the right endpoint behaviour, and vice versa. Put simply, the restriction that the decay bounds from above and below must be the same rate removes any functions that decay differently at different areas of our boundary. Though this is a one dimensional case, it is easy to visualize how this might look in two dimensions on a disk, for example.

One may wonder how the above examples act with the exponents to be chosen on the nonlinearities with u and v. Given that $N = 1$ and $p+q < 1$ (in Theorem 6.2.1, for example), we see that the condition for the existence of a classical solution u is always satisfied regardless of τ, the exponent on the upper bound of our reaction function. For a more interesting example, one may consider the following two dimensional example generalizing $h_3(x)$ on $\Omega = (0, 1) \times (0, 1)$

$$h(x, y) = \sqrt[a]{x}\sqrt[a]{1-x}\sqrt[a]{y}\sqrt[a]{1-y}, \quad 1 \leq a < \infty. \tag{6.8}$$

Similar to $h_3(x)$ in one dimension, this function decays like $\sqrt[a]{x}$ near all sides of the domain. Consequently, the upper bound can be taken to be $C_0 d^{1/a}(x)$ for some C_0 sufficiently large. Theorem 6.2.1 will guarantee the existence of a positive, global weak solution u for any $p, q \in (0,1)$ satisfying $p+q<1$, but the existence of a classical solution is no longer guaranteed. Indeed, the condition for the existence of a classical solution u becomes

$$-1 < 2\left(\frac{1}{a} - (p+q)\right), \tag{6.9}$$

and so one may see that for any $a \in [1,2]$, u is a classical solution for any $p, q \in (0,1)$ satisfying $p+q<1$. Alternatively, for any $a \in (2,\infty)$, there exist $p, q \in (0,1)$ satisfying $p+q<1$ such that Theorem 6.2.1 does *not* guarantee the existence of a classical solution. For example, take $a=3$. Then, by (6.9), if we choose $p=\frac{1}{2}$, $q=\frac{5}{12}$ such that $p+q=\frac{11}{12}>\frac{5}{6}$, the hypotheses of Theorem 6.2.1 for u to be a classical solution are not satisfied.

Lastly, the conditions ensuring the existence of classical solutions are clear when one notes that we use a Sobolev embedding to obtain the additional regularity. There is nothing very mysterious for this condition; it comes from direct computation once one obtains appropriate L^k bounds, noted in step iii.

6.4 Existence for $d = D$

We begin with the proof of theorems 6.2.1 and 6.2.2, treating the simpler case where $d = D$. In this case, we are able to control our solutions by an appropriate eigenfunction, independent of time. The time independence comes from applying Lemma

5.1.2, which is only valid when $d = D$. Essentially, we were able to bound the higher order terms by a non-positive functional the can be thought of as a parabolic cylinder. When $d \neq D$, this functional turns into what can be thought of as a hyperbolic cylinder, and thus becomes positive for large arguments in particular directions. It is worth noting that the simplicity in this case is in obtaining globally bounded solutions, which is not so clear when $d \neq D$, though the proofs of each are still highly nontrivial.

Proof of Theorem 6.2.1. **Step** i. To start, we consider the associated perturbed system after rescaling space variables such that $d = D = 1$:

$$\begin{cases} u_t = \Delta u + \frac{f(x)}{(u+\varepsilon)^p(v+\varepsilon)^q} \ , \\ v_t = \Delta v + \frac{g(x)}{(u+\varepsilon)^r(v+\varepsilon)^s} \ , \quad x \in \Omega, \ t > 0. \end{cases} \tag{6.10}$$

First, we note that this perturbed system now satisfies the form investigated in section 3. In particular, the reaction terms are now $C^\infty(\mathbb{R}^+)$ in their arguments u, v and monotone non-increasing. To see this, one may note

$$\begin{aligned} \frac{\partial}{\partial u}\left(\frac{f(x)}{(u+\varepsilon)^p(v+\varepsilon)^q}\right) &= -p\frac{f(x)}{(u+\varepsilon)^{p+1}(v+\varepsilon)^q} \\ &\leq 0, \end{aligned} \tag{6.11}$$

for example. Furthermore, we note that the Hölder continuity of the reaction terms in their arguments u, v depend critically on $\varepsilon > 0$ in the sense that we lose this property if $\varepsilon = 0$. To show the existence of solutions to this problem, we must construct a pair of sub and supersolutions satisfying the necessary conditions in Theorem 3.2.6. First, consider the following auxiliary elliptic problem:

$$\begin{cases} 0 = \Delta a(x) + 1, & x \in \Omega, \\ 0 = a(x), & x \in \partial\Omega. \end{cases} \tag{6.12}$$

By the theory of linear elliptic equations (note that this is just Poisson's equation in a smooth, bounded domain), we know a smooth solution to this problem exists. Further, since 0 is a subsolution, the maximum principle implies that $a(x) > 0$ in Ω. Our goal is to use the eigenvalue problem discussed in chapter 4 in conjunction with the elliptic problem above in order to construct an appropriate supersolution to both problems under consideration. The reason to include this additional auxiliary problem is to control the supersolution for arguments less than 1, while the eigenvalue problem will control the arguments greater than 1. This can be seen explicitly in line (6.17) below. First, let's consider the following candidates:

$$\begin{cases} (\underline{u}, \underline{v}) \equiv (0, 0), \\ (\overline{u}, \overline{v}) \equiv \left(c_1\phi(x) + c_2 a(x) + \|u_0\|_{L^\infty},\ c_3\phi(x) + c_4 a(x) + \|v_0\|_{L^\infty}\right), \end{cases} \tag{6.13}$$

where

$$\begin{cases} c_1 = \frac{C_0}{\varepsilon^{p+q} d\lambda_1}, \quad c_2 = \frac{C_0}{\varepsilon^{p+q} d}, \\ c_3 = \frac{C_0}{\varepsilon^{r+s} D\lambda_1}, \quad c_4 = \frac{C_0}{\varepsilon^{r+s} D}. \end{cases} \tag{6.14}$$

Though we have scaled the spatial variables such that $d = D = 1$, we include these parameters in the constants above since we want the existence of solutions to the perturbed problem for general $d, D > 0$.

Now, for any $(\overline{u}, \overline{v}) \geq (0,0)$, one may see that

$$\begin{aligned} \underline{u}_t - d\Delta \underline{u} - f(x)(\underline{u}+\varepsilon)^{-p}(\overline{v}+\varepsilon)^{-q} &= -f(x)\varepsilon^{-p}(\overline{v}+\varepsilon)^{-q} \\ &\leq 0, \end{aligned} \tag{6.15}$$

since $f(x) > 0$ in Ω, and similarly

$$\begin{aligned} \underline{v}_t - D\Delta \underline{v} - g(x)(\overline{u}+\varepsilon)^{-r}(\underline{v}+\varepsilon)^{-s} &= -g(x)\varepsilon^{-s}(\overline{v}+\varepsilon)^{-r} \\ &\leq 0, \end{aligned} \tag{6.16}$$

since $g(x) > 0$ in Ω. Then, since $(0,0) \leq (0,0)$ on $\partial\Omega \times (0,t)$ and $(0,0) \leq (u_0, v_0)$ on $\overline{\Omega} \times \{t = 0\}$ by hypotheses, $(0,0)$ is an appropriate subsolution to the perturbed problem. Taking our defined supersolution above, and noting the bounds on f, g given in our hypotheses, we see that

$$\begin{aligned} \overline{u}_t - d\Delta \overline{u} - f(x)(\overline{u}+\varepsilon)^{-p}(\underline{v}+\varepsilon)^{-q} &\geq dc_1\lambda_1\phi + c_2 d - \frac{C_0}{\varepsilon^{p+q}}\phi^{\tau} \\ &= \frac{C_0}{\varepsilon^{p+q}}\left(\phi - \phi^{\tau} + 1\right) \\ &\geq 0, \end{aligned} \tag{6.17}$$

for all $\phi \geq 0$. Similarly,

$$\begin{aligned} \overline{v}_t - D\Delta\overline{v} - g(x)(\underline{u}+\varepsilon)^{-r}(\overline{v}+\varepsilon)^{-s} &\geq Dc_3\lambda_1\phi + c_4 D - \frac{C_0}{\varepsilon^{r+s}}\phi^{\mu} \\ &= \frac{C_0}{\varepsilon^{r+s}}\left(\phi - \phi^{\mu} + 1\right) \\ &\geq 0, \end{aligned} \tag{6.18}$$

for all $\phi \geq 0$. Finally, since $(0,0) \leq (\overline{u},\overline{v})$ on $\partial\Omega \times (0,t)$ and $(u_0(x), v_0(x)) \leq (\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}) \leq (\overline{u},\overline{v})$ on $\overline{\Omega} \times \{t=0\}$, $(\overline{u},\overline{v})$ satisfy all the necessary conditions to be an appropriate ordered supersolution. By Theorem 3.2.6, we know a positive classical solution to this system exists for $t \in (0,T)$, and we denote it by $(u_\varepsilon, v_\varepsilon)$.

Step ii. The goal is to now find uniform bounds for $(u_\varepsilon, v_\varepsilon)$. To start, let $w_\varepsilon = u_\varepsilon + \varepsilon$ and $z_\varepsilon = v_\varepsilon + \varepsilon$. Lemma (5.1.2), with u replaced by w_ε and v replaced by z_ε gives us the following:

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \leq \lambda_1(n+2)\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx - \alpha n \int_\Omega \frac{\phi^{n+2} f(x)}{w_\varepsilon^{\alpha n+p+1} z_\varepsilon^{\beta n+q}}dx \\ - \beta n \int_\Omega \frac{\phi^{n+2} g(x)}{w_\varepsilon^{\alpha n+r} z_\varepsilon^{\beta n+s+1}}dx \ . \end{aligned} \tag{6.19}$$

Notice the three terms appearing above, with one positive and two negative. Instead of throwing away the two negative terms (which results in a useful bound, but it depends exponentially on t, which we want to remove), we use these two terms to control the positive term. In doing so, we are able to remove explicit dependence on $w_\varepsilon, z_\varepsilon$ on the right hand side of (6.19). This allows us to then integrate in time instead of applying the Grönwall inequality. So, applying Lemma 5.0.1 to the first term of (6.19) gives us the following, valid for any $\delta > 0$ and $n \geq 2$:

$$\frac{\lambda_1(n+2)}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} \leq \alpha n \frac{f(x)}{w_\varepsilon^{\alpha n+p+1} z_\varepsilon^{\beta n+q}} + \lambda_1(n+2)\left(\frac{\alpha n}{\lambda_1(n+2)}\right)^{-\delta} \frac{f^{-\delta}(x)}{w_\varepsilon^{\alpha n-\delta(p+1)} z_\varepsilon^{\beta n-q\delta}} \ , \tag{6.20}$$

where

$$\lambda_1(n+2)\left(\frac{\alpha n}{\lambda_1(n+2)}\right)^{-\delta} \leq \lambda_1(n+2)\left(\frac{2\lambda_1}{\alpha}\right)^{\delta} \equiv c_1(n) \ . \tag{6.21}$$

Combining (6.20) and (6.21) with (6.19) leaves us with the following:

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \leq c_1(n) \int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\beta n-\delta(p+1)} z_\varepsilon^{\beta n-q\delta} f^\delta(x)} dx - \beta n \int_\Omega \frac{\phi^{n+2} g(x)}{w_\varepsilon^{\alpha n+r} z_\varepsilon^{\beta n+s+1}} dx \ . \tag{6.22}$$

If we apply Lemma 5.0.1 to the first term of (6.22), similar to (6.20), we may find the following:

$$\frac{c_1(n)}{w_\varepsilon^{\alpha n-\delta(p+1)} z_\varepsilon^{\beta n-q\delta} f^\delta(x)} = c_1(n) \frac{(w_\varepsilon^{-1})^{\alpha n-\delta(p+1)} g^0(x)}{z_\varepsilon^{\beta n-q\delta} f^\delta(x)} \leq \beta n \frac{(w_\varepsilon^{-1})^{\alpha n+r} g(x)}{z_\varepsilon^{\beta n+s+1}} + c_1(n)\left(\frac{\beta n}{c_1(n)}\right)^{-\frac{(\alpha n-\delta(p+1))}{(r+\delta(p+1))}} \frac{g^{\theta_2}(x)}{z_\varepsilon^{\eta_1} f^{\eta_2}(x)} \ , \tag{6.23}$$

where

$$\begin{aligned}
\theta_2 &= -\frac{[\alpha n - \delta(p+1)]}{r+\delta(p+1)} \ , \\
\eta_1 &= \frac{[(\beta n - q\delta)(\alpha n + r) - (\beta n + s + 1)(\alpha n - \delta(p+1))]}{r+\delta(p+1)} \\
&= \frac{[\beta n r - \alpha n(s+1) - \delta(\alpha n q - \beta n(p+1) + rq - (s+1)(p+1))]}{r+\delta(p+1)} \ , \\
\eta_2 &= \frac{\delta(\alpha n + r)}{r+\delta(p+1)} \ , \qquad (6.24)
\end{aligned}$$

and

$$\begin{aligned}
c_1(n)\left(\frac{\beta n}{c_1(n)}\right)^{-\frac{(\alpha n-\delta(p+1))}{(r+\delta(p+1))}} &\leq \lambda_1(n+2)\left(\frac{2\lambda_1}{\alpha}\right)^{\delta}\left(\frac{(2\lambda_1)^{\delta+1}}{\alpha^{\delta}\beta}\right)^{\frac{(\alpha n-\delta(p+1))}{(r+\delta(p+1))}} \\
&\equiv c_2(n) \ . \qquad (6.25)
\end{aligned}$$

Combining (6.23)-(6.25) with (6.22) then yields

$$\frac{d}{dt}\int_{\Omega}\frac{\phi^{n+2}}{w_{\varepsilon}^{\alpha n} z_{\varepsilon}^{\beta n}}dx \leq c_2(n)\int_{\Omega}\frac{\phi^{n+2}g^{\theta_2}(x)}{z_{\varepsilon}^{\eta_1} f^{\eta_2}(x)}dx. \qquad (6.26)$$

The goal is to now set $\eta_1 = 0$ (by choosing a particular δ) to remove the dependence on z_ε on the right hand side of (6.26) and show that $\delta > 0$, as required. If we set $\eta_1 = 0$ and solve for δ, we find that

$$\delta = \frac{n(\alpha(s+1) - \beta r)}{n(\beta(p+1) - \alpha q) + (s+1)(p+1) - rq} \ . \qquad (6.27)$$

From this, we see that we must require

$$\alpha(s+1) - \beta r > 0, \tag{6.28}$$

$$\beta(p+1) - \alpha q \geq 0, \tag{6.29}$$

so that $\delta > 0$. Before we attempt to choose appropriate values for α, β, we simplify (6.26) further by using that $c_0\phi^\theta \leq f(x)$ and $c_0\phi^\eta \leq g(x)$:

$$\frac{\phi^{n+2}}{f^{\eta_2} g^{-\theta_2}} \leq \frac{1}{c_0^{\eta_2-\theta_2}} \phi^{n+2-\theta\eta_2+\eta\theta_2}. \tag{6.30}$$

Naturally, we require that $n+2-\theta\eta_2+\eta\theta_2 \geq 0$ so ϕ in the integral remains nonsingular as we approach the boundary. Simplifying this, we see

$$\begin{aligned} n+2-\theta\eta_2+\eta\theta_2 &= n+2-\left(\frac{\alpha\theta\delta n + r\theta\delta + \alpha\eta n - \eta\delta(p+1)}{r+\delta(p+1)}\right) \\ &= n\left(\frac{r-\alpha\eta+\delta(p+1-\alpha\theta)}{r+\delta(p+1)}\right) - \frac{\delta(r\theta - \eta(p+1))}{r+\delta(p+1)} + 2. \end{aligned} \tag{6.31}$$

Given that we are to take n large, if the coefficient on n is positive, our exponent will be positive for n sufficiently large. Since $0 < \alpha < 1$, $0 \leq \theta \leq 1$, we have $p+1 > \alpha\theta$ for any $p \in [0,1)$, and so we only require that

$$r - \alpha\eta \geq 0 \tag{6.32}$$

to ensure (6.31) remains positive for sufficiently large n. Define $2^* \equiv q + r$ which may be close, but not equal, to 2. If we choose $\alpha = \frac{r}{2^*}$, $\beta = \frac{q}{2^*}$ so that $\alpha + \beta = 1$, we see that under the hypotheses of the theorem,

$$\begin{aligned} \alpha(s+1) - \beta r &= \frac{r}{2^*}(s+1-q) > 0, \\ \beta(p+1) - \alpha q &= \frac{q}{2^*}(p+1-r) > 0, \\ r - \alpha\eta &= r(1 - \frac{\eta}{2^*}) > 0, \end{aligned}$$

and so (6.28), (6.29), (6.32) are satisfied. Putting all of this together, (6.28) becomes

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \leq c_3(n) \int_\Omega \phi^{\frac{n(r-\alpha\eta+\delta(p+1-\alpha\theta))-\delta(r\theta-\eta(p+1))}{r+\delta(p+1)}+2} dx, \tag{6.33}$$

where

$$c_3(n) = \frac{c_2(n)}{c_0^{\eta_2-\theta_2}}.$$

Integrating (6.33) from 0 to t, we then arrive at

$$\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \leq c_3(n) T \int_\Omega \phi^{\frac{n(r-\alpha\eta+\delta(p+1-\alpha\theta))-\delta(r\theta-\eta(p+1))}{r+\delta(p+1)}+2} dx + \int_\Omega \frac{\phi^{n+2}}{u_0^{\alpha n} v_0^{\beta n}} dx,$$

for arbitrary $T > 0$. Extracting n^{th} roots and taking $n \to \infty$, we finally obtain

$$\begin{aligned} \frac{\phi(x)}{w_\varepsilon^{\frac{r}{2^*}}(x,t) z_\varepsilon^{\frac{q}{2^*}}(x,t)} &\leq \left\| \frac{\phi}{w_\varepsilon^{\frac{r}{2^*}} z_\varepsilon^{\frac{q}{2^*}}} \right\|_\infty \\ &\leq \max \left\{ m_1 \left\| \phi^{\sigma_1} \right\|_\infty , \left\| \frac{\phi}{u_0^{\frac{r}{2^*}} v_0^{\frac{q}{2^*}}} \right\|_\infty \right\} \\ &\equiv M_1 < \infty, \end{aligned} \tag{6.34}$$

where

$$\sigma_1 = \frac{r(1 - \frac{\eta}{2^*}) + \delta(p + 1 - \frac{r\theta}{2^*})}{r + \delta(p+1)} > 0,$$
$$m_1 = \frac{\lambda_1}{c_0^{\frac{\alpha(\delta+1)}{r+\delta(p+1)}}} \cdot \left(\frac{(2\lambda_1)^{\delta+1}}{\alpha^\delta \beta} \right)^{\frac{\alpha}{r+\delta(p+1)}}.$$

Notice that while M_1 depends on many of our given parameters, it remains independent of ε and a maximal time T. Consequently, the bound above holds for all $t \in (0, \infty)$. This result will be helpful in obtaining upper bounds for both w_ε and z_ε, as we will find in the following steps.

In this next step, we will obtain upper bounds for u_ε, independent of ε and T. Referring to Lemma 5.1.3, we replace u with u_ε and see that

$$\begin{aligned} \frac{d}{dt} \int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n} \psi^{\beta n - 2}} dx \leq n \int_\Omega \frac{u_\varepsilon^{n-1} f(x)}{w_\varepsilon^{\alpha n + p} z_\varepsilon^q \psi^{\beta n - 2}} dx - \alpha n \int_\Omega \frac{u_\varepsilon^n f(x)}{w_\varepsilon^{\alpha n + p + 1} z_\varepsilon^q \psi^{\beta n - 2}} dx \\ - (\beta n - 2) \int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n} \psi^{\beta n - 1 + \sigma}} dx \ . \end{aligned} \tag{6.35}$$

Note that these α, β are a new parameters to be determined, independent of the previous choices $r/2^*$ and $q/2^*$. Now, for any $\delta_1 > 1$ we may apply Lemma 5.0.1 to to first term of (6.35) to find that

$$\frac{u_\varepsilon^{n-1}}{w_\varepsilon^{\alpha n+p}} \leq \alpha \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n+p+1}} + \alpha^{-(\delta_1-1)} \frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-\delta_1}} \ ,$$

in which case (6.35) becomes

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \leq \frac{n}{\alpha^{\delta_1-1}}\int_\Omega \frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\alpha n+p+1-\delta_1} z_\varepsilon^q \psi^{\beta n-2}}dx \\ -(\beta n-2)\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx \ . \end{aligned} \tag{6.36}$$

Here is where we use our previously obtained bound, noting the independence of time and ε. If we apply (4.8) and (6.34) to (6.36), we can then remove z_ε^q from the denominator of the positive term above as follows:

$$\begin{aligned} \frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\alpha n+p+1-\delta_1} z_\varepsilon^q \psi^{\beta n-2}} &= \frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\alpha n+p+1-\delta_1-r}\psi^{\beta n-2+2^*}} \left(\frac{\phi}{w_\varepsilon^{\frac{r}{2^*}} z_\varepsilon^{\frac{q}{2^*}}}\right)^{2^*} \left(\frac{\psi}{\phi}\right)^{2^*} \\ &\leq \gamma_1^{2^*} M_1^{2^*} \frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\alpha n+p+1-r-\delta_1}\psi^{\beta n-2+2^*}} \ . \end{aligned} \tag{6.37}$$

If we pair the above with Lemma 5.0.1, we can then obtain

$$\begin{aligned} \frac{M_1^{2^*}\gamma_1^{2^*}n}{\alpha^{\delta_1-1}} \frac{u_\varepsilon^{n-\delta_1}f(x)}{w_\varepsilon^{\alpha n+p+1-r-\delta_1}\psi^{\beta n-2+2^*}} &\leq (\beta n-2)\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}} \\ &\quad + \left(\frac{M_1^{2^*}\gamma_1^{2^*}n}{\alpha^{\delta_1-1}}\right)\left(\frac{(\beta n-2)\alpha^{\delta_1-1}}{M_1^{2^*}\gamma_1^{2^*}n}\right)^{-\frac{n}{\delta_1}+1} \frac{f^{\frac{n}{\delta_1}}}{w_\varepsilon^{\eta_1}\psi^{\eta_2}} \\ &\leq (\beta n-2)\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}} + c_4(n)\frac{f^{\frac{n}{\delta_1}}}{w_\varepsilon^{\eta_1}\psi^{\eta_2}} \ , \end{aligned} \tag{6.38}$$

where

$$\begin{aligned} \eta_1 &= \frac{[(\alpha n + p + 1 - r - \delta_1)(n) - (\alpha n)(n - \delta_1)]}{\delta_1} \\ &= \frac{n(p+1-r) - n\delta_1(1-\alpha)}{\delta_1} \ , \\ \eta_2 &= \frac{[(\beta n - 2 + 2^*)(n) - (\beta n - 1 + \sigma)(n - \delta_1)]}{\delta_1} \\ &= \frac{-n(1 + \sigma - 2^* - \delta_1 \beta)}{\delta_1} - (1 - \sigma) \ , \end{aligned} \tag{6.39}$$

and

$$\begin{aligned} \left(\frac{M_1^{2^*} \gamma_1^{2^*} n}{\alpha^{\delta_1 - 1}} \right) \left(\frac{(\beta n - 2)\alpha^{\delta_1 - 1}}{M_1^{2^*} \gamma_1^{2^*} n} \right)^{-\frac{n}{\delta_1} + 1} &\le (\beta n - 2) \left(\frac{2 M_1^{2^*} \gamma_1^{2^*}}{\alpha^{\delta_1 - 1} \beta} \right)^{\frac{n}{\delta_1}} \\ &\equiv c_4(n). \end{aligned} \tag{6.40}$$

With these computations, (6.36) then becomes

$$\frac{d}{dt} \int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n} \psi^{\beta n - 2}} dx \le c_4(n) \int_\Omega \frac{f^{\frac{n}{\delta_1}}}{w_\varepsilon^{\eta_1} \psi^{\eta_2}} dx. \tag{6.41}$$

The goal is to now remove the dependence on w_ε on the right hand side of (6.41) by setting $\eta_1 = 0$. Doing so, we find

$$\delta_1 = \frac{p + 1 - r}{1 - \alpha}. \tag{6.42}$$

Recall that we require $\delta_1 > 1$. Given that $p + 1 - r > 0$, one may notice that this holds true for $\alpha \in (r, 1)$. (6.41) may then be written as

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le c_4(n)\int_\Omega \frac{f^{\frac{n}{\delta_1}}}{\psi^{\eta_2}}dx. \tag{6.43}$$

We now require the integral on the right hand side of (6.43) to be finite in order to appropriately bound u_ε. Through the strategy above, we have actually removed any dependence our desired bound may have on ε or the solutions u_ε and v_ε. Using that $f(x) \le C_0\psi^\tau$ via (4.4), we further simplify (6.43):

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le C_0^{\frac{n}{\delta_1}}c_4(n)\int_\Omega \psi^{\frac{n}{\delta_1}(1+\sigma+\tau-2^*-\beta\delta_1)+1-\sigma}dx. \tag{6.44}$$

In order for the right hand side to remain finite, we require that

$$-1 < \frac{n}{\delta_1}\left(1+\sigma+\tau-2^*-\beta\delta_1\right)+1-\sigma, \tag{6.45}$$

but as n gets large, this can only be true if $1+\sigma+\tau-2^*-\beta\delta_1 > 0$. In order to show this is true, recall that $2^* = q+r$ and choose $\alpha \in (r, 1)$ with $\beta = 1-\alpha$. achieve this, we choose σ sufficiently close to 1 such that $1+\sigma-2^* > 0$. Now compute the following:

$$\begin{aligned}1+\sigma+\tau-2^*-\beta\delta_1 &\ge 1+\sigma-q-r-(p+1-r)\\ &= \sigma-(p+q)\\ &> 0,\end{aligned} \tag{6.46}$$

which is true for σ sufficiently close to 1, since $p+q<1$ by hypothesis. Consequently, (6.45) is true for any $\tau \in [0,1]$. With this, we may now solve (6.44) to find that

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx &\leq C_0^{\frac{n}{\delta_1}} c_4(n)T\int_\Omega \psi^{\frac{n}{\delta_1}(1+\sigma+\tau-2^*-\beta\delta_1)+1-\sigma}dx + \int_\Omega \frac{u_0^n}{(u_0+\varepsilon)^{\alpha n}\psi^{\beta n-2}}dx \\ &\leq C_0^{\frac{n}{\delta_1}} c_4(n)T\int_\Omega \psi^{\frac{n}{\delta_1}(1+\sigma+\tau-2^*-\beta\delta_1)+1-\sigma}dx + \int_\Omega \frac{u_0^{n(1-\alpha)}}{\psi^{\beta n-2}}dx, \qquad (6.47)\end{aligned}$$

for any $T>0$. We now justify bounding the last term of (6.47) above, particularly near the boundary. First, by the smoothness of $\partial\Omega$, for x sufficiently close to $\partial\Omega$, we know that there exists a unique point $y \in \partial\Omega$ such that $d(x)=|x-y|$. Using (4.4) and the fact that $u_0(x) \in C_0^1(\Omega)$, we then see that

$$\begin{aligned}\frac{u_0(x)}{\psi(x)} &\leq \frac{1}{\gamma_0}\frac{|u_0(x)-u_0(y)|}{|x-y|} \\ &\leq \frac{K_u}{\gamma_0}, \qquad (6.48)\end{aligned}$$

where K_u is a fixed constant. As a result, (6.47) can be written as

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \leq C_0^{\frac{n}{\delta_1}} c_4(n)T&\int_\Omega \psi^{\frac{n}{\delta_1}(\sigma+\tau-p-q)+1-\sigma}dx \\ &+\left(\frac{K_u}{\gamma_0}\right)^{\beta n}\int_\Omega \psi^2 dx. \qquad (6.49)\end{aligned}$$

Extracting n^{th} roots and letting $n \to \infty$, we arrive at

$$\begin{aligned}\frac{u_\varepsilon(x,t)}{w_\varepsilon^\alpha(x,t)\psi^\beta(x)} &\leq \left\| \frac{u_\varepsilon}{w_\varepsilon^\alpha \psi^\beta} \right\|_\infty \\ &\leq \max\left\{ m_2 \left\| \psi^{(\sigma+\tau-p-q)/\delta_1} \right\|_\infty, \left(\frac{K_u}{\gamma_0} \right)^\beta \right\} \\ &\equiv M_2 < \infty, \end{aligned} \tag{6.50}$$

where

$$m_2 = C_0^{1/\delta_1} \left(\frac{2M_1^{2^*} \gamma_1^{2^*}}{\alpha^{\delta_1 - 1}\beta} \right)^{\frac{1}{\delta_1}},$$

independent of ε. Rearranging we finally find that

$$u_\varepsilon(x,t) \leq M_2^{\frac{1}{1-\alpha}} \psi(x), \tag{6.51}$$

and since T was arbitrary, this bound holds for all $t \in (0, \infty)$. We now state the analogous result for v_ε. Many of the technical details will be omitted, as they are essentially the exact same calculations as those done for u_ε previously, but the conditions now relate to the exponents r, s instead. Notice that in (6.34), we have control over a term involving $w_\varepsilon^{-r/2^*}$. In the same way we removed z_ε^{-q} for the equation for u_ε, we are able to remove w_ε^{-r} in the equation for v_ε. Computations yield

$$\frac{d}{dt} \int_\Omega \frac{v_\varepsilon^n}{z_\varepsilon^{\alpha n} \psi^{\beta n - 2}} dx \leq C_0^{\frac{n}{\delta_2}} c_5(n) \int_\Omega \psi^{\frac{n}{\delta_2}(\sigma+\mu-r-s)+1-\sigma} dx, \tag{6.52}$$

where

$$\delta_2 = \frac{s+1-q}{1-\alpha},$$
$$c_5(n) = (\beta n - 2)\left(\frac{2M_1^{2^*}\gamma_1^{2^*}}{\alpha^{\delta_2-1}\beta}\right)^{\frac{n}{\delta_2}}, \tag{6.53}$$

where we now choose $\alpha \in (q,1)$ so that $\delta_2 > 1$. Integrating, and again using the fact that $v_0 \in C_0^1(\Omega)$, we obtain

$$\int_\Omega \frac{v_\varepsilon^n}{z_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \leq C_0^{\frac{n}{\delta_2}} c_5(n) T \int_\Omega \psi^{\frac{n}{\delta_2}(\sigma+\mu-r-s)+1-\sigma} dx$$
$$+ \left(\frac{K_v}{\gamma_0}\right)^{\beta n} \int_\Omega \psi^2 dx, \tag{6.54}$$

for any $T > 0$. Notice that by the same reasoning, we may choose $\beta = 1-\alpha$ and σ sufficiently close to 1 so that the exponent on ψ on the right hand side of (6.52) is positive. Extracting n^{th} roots and letting $n \to \infty$ yields

$$\frac{v_\varepsilon(x,t)}{z_\varepsilon^\alpha(x,t)\psi^\beta(x)} \leq \left\|\frac{v_\varepsilon}{z_\varepsilon^\alpha \psi^\beta}\right\|_\infty$$
$$\leq \max\left\{ m_3 \left\|\psi^{(\sigma+\mu-r-s)/\delta_2}\right\|_\infty, \left(\frac{K_v}{\gamma_0}\right)^\beta \right\}$$
$$\equiv M_3 < \infty, \tag{6.55}$$

where

$$m_3 = C_0^{1/\delta_2}\left(\frac{2M_1^{2^*}\gamma_0^{2^*}}{\alpha^{\delta_2-1}\beta}\right)^{\frac{1}{\delta_2}}.$$

Putting this all together, we have

$$v_\varepsilon(x,t) \leq M_3^{\frac{1}{1-\alpha}} \psi(x), \tag{6.56}$$

uniformly for all $t \in (0,\infty)$. At this point, we have succeeded in obtaining upper bounds for $u_\varepsilon, v_\varepsilon$ independent of ε. The next step is to obtain similar uniform lower bounds. To see this, we apply Lemma 5.1.1 with u replaced by w_ε to see that

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n} dx \leq \lambda_1 n \int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n} dx - n\int_\Omega \frac{\phi^{n+2} f(x)}{w_\varepsilon^{n+p+1} z_\varepsilon^q} dx. \tag{6.57}$$

We then apply Young's Inequality to the first term of (6.57) to obtain

$$\lambda_1 (w_\varepsilon^{-1})^n \leq \frac{(w_\varepsilon^{-1})^{n+p+1} f(x)}{z_\varepsilon^q} + \lambda_1 \left(\frac{1}{\lambda_1}\right)^{-\frac{n}{p+1}} \frac{f^{-\frac{n}{p+1}}(x)}{z_\varepsilon^{-\frac{qn}{p+1}}}.$$

(6.57) then becomes

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n} dx \leq n\lambda_1^{\frac{n+p+1}{p+1}} \int_\Omega \frac{\phi^{n+2} z_\varepsilon^{\frac{nq}{p+1}}}{f^{\frac{n}{p+1}}} dx. \tag{6.58}$$

Using that $c_0\phi^\theta \leq f$, we see that

$$f^{-\frac{n}{p+1}} \leq c_0^{-\frac{n}{p+1}} \phi^{-\frac{\theta n}{p+1}}.$$

Combining this with (6.58), we arrive at

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n} dx \leq n\lambda_1^{\frac{n+p+1}{p+1}} c_0^{-\frac{n}{p+1}} \int_\Omega \phi^{n+2-\frac{\theta n}{p+1}} z_\varepsilon^{\frac{nq}{p+1}} dx. \tag{6.59}$$

Integrating (6.59) and using (6.3), we find

$$\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx \le n\lambda_1^{\frac{n+p+1}{p+1}} c_0^{-\frac{n}{p+1}} T \int_\Omega \phi^{n+2-\frac{\theta n}{p+1}} z_\varepsilon^{\frac{nq}{p+1}} dx + \frac{1}{\varepsilon_0^n}\int_\Omega \phi^2 dx, \tag{6.60}$$

for any $T > 0$. Extracting n^{th} roots and letting $n \to \infty$ yields

$$\begin{aligned}\frac{\phi(x)}{w_\varepsilon(x,t)} &\le \left\|\frac{\phi}{w_\varepsilon}\right\|_\infty \\ &\le \max\left\{m_4 \left\|\phi^{\frac{p+1-\theta}{p+1}} v_\varepsilon^{\frac{q}{p+1}}\right\|_\infty, \varepsilon_0^{-1}\right\} \\ &\equiv M_4 < \infty,\end{aligned} \tag{6.61}$$

where

$$m_4 = \left(\frac{\lambda_1}{c_0}\right)^{\frac{1}{p+1}}.$$

Notice that despite v_ε appearing in our constant M_4, by (6.56), M_4 does indeed remain independent of ε by the previously obtained bounds from above on v_ε. Rearranging, we see that $w_\varepsilon \ge \phi M_4^{-1}$ for all $t > 0$. Repeating this process for v_ε gives us

$$\begin{aligned}\frac{\phi(x)}{v_\varepsilon(x,t)} &\le \left\|\frac{\phi}{v_\varepsilon}\right\|_\infty \\ &\le \max\left\{m_5 \left\|\phi^{\frac{s+1-\eta}{s+1}} w_\varepsilon^{\frac{r}{s+1}}\right\|_\infty, \varepsilon_0^{-1}\right\} \\ &\equiv M_5 < \infty,\end{aligned} \tag{6.62}$$

where

$$m_5 = \left(\frac{\lambda_1}{c_0}\right)^{\frac{1}{s+1}}.$$

Thus, $M_5^{-1}\phi(x) \leq z_\varepsilon(x,t)$ for all $t > 0$.

Step iii. The next step is to use these estimates to show that the nonlinear terms of our system are uniformly bounded in $L^k(\Omega)$ for some $k > 1$. To see this, we compute for any $k_1 \in (1, \frac{1}{p+q})$

$$\begin{aligned}\int_\Omega \left|\frac{f(x)}{w_\varepsilon^p z_\varepsilon^q}\right|^{k_1} dx &\leq c_0^{k_1} \int_\Omega \frac{\phi^{k_1\tau}}{w_\varepsilon^{k_1 p} z_\varepsilon^{k_1 q}} dx \\ &\leq M_6 \int_\Omega \phi^{k_1[\tau-(p+q)]} dx. \end{aligned} \tag{6.63}$$

Similarly, for $k_2 \in (1, \frac{1}{r+s})$

$$\begin{aligned}\int_\Omega \left|\frac{g(x)}{w_\varepsilon^r z_\varepsilon^s}\right|^{k_2} dx &\leq c_0^{k_2} \int_\Omega \frac{\phi^{k_2\mu}}{w_\varepsilon^{k_2 r} z_\varepsilon^{k_2 s}} dx \\ &\leq M_7 \int_\Omega \phi^{k_2[\mu-(r+s)]} dx. \end{aligned} \tag{6.64}$$

These choices in where k_i lives ensures that $-1 < k_1(\tau - (p+q))$ and $-1 < k_2(\mu - (r+s))$ for any $\theta, \mu \in [0,1]$.

Step iv. Since we know $(u_\varepsilon, v_\varepsilon)$ are classical solutions, they are also strong solutions. From this, we apply the L^p estimates found in Theorem 3.1.2 to find that $(u_\varepsilon, v_\varepsilon)$ are uniformly bounded in $[W_{k_1}^{2,1}(\Omega \times (0,T))] \times [W_{k_2}^{2,1}(\Omega \times (0,T))]$. Thus, a subsequence $(u_{\varepsilon_i}, v_{\varepsilon_i})$ can be extracted that converges to a weak solution of (6.1).

Step v. If we have the additional information that $-1 < N(\tau - (p+q))$, we may choose $k_1 > N$ so that $-1 < k_1(\tau - (p+q))$, and (6.63) remains true. Then, by Lemma 2.0.2, $u_\varepsilon \to u \in C^{1+\kappa,1/2+\kappa/2}(\overline{Q}_T)$ for some $\kappa \in (0,1)$. If we then fix $\Omega' \subset\subset \Omega$ and define $Q'_T = \Omega' \times (t_*, t^*)$ for any $0 < t_* < t^* < \infty$, it is easy to see that our nonlinear term $f(x)u^{-p}v^{-q} \in C^{\kappa,\kappa/2}(\overline{Q'}_T)$ for some $\kappa \in (0,1)$. By the classical theory of parabolic equations (see Theorem 8.3.7 in [20], for example), we then have that $u \in C^{2+\kappa,1+\kappa/2}(\overline{Q'}_T)$, and so $u \in C^{2+\kappa,1+\kappa/2}(Q_T)$, by definition. Consequently, since $T > 0$ was arbitrary, $u \in C^{2+\kappa,1+\kappa/2}(\Omega \times (0,\infty)) \cap C^{1+\kappa,(1+\kappa)/2}(\overline{\Omega} \times [0,\infty))$ is a classical solution. Similarly, if $-1 < N(\mu - (r+s))$, then v is a classical solution. This completes the proof. □

Proof of Theorem 6.2.2. Making the same perturbation and rescaling as before, system (6.2) becomes

$$\begin{cases} u_t = \Delta u + \frac{f_1(x)}{(u+\varepsilon)^p} + \frac{f_2(x)}{(v+\varepsilon)^q}\ , \\ v_t = \Delta v + \frac{f_3(x)}{(u+\varepsilon)^r} + \frac{f_4(x)}{(v+\varepsilon)^s}\ , \qquad x \in \Omega,\ t > 0. \end{cases} \tag{6.65}$$

We know solutions to this perturbed system exist, and we denote that $(u_\varepsilon, v_\varepsilon)$. Indeed, the following can be shown to be sub and supersolutions to the system through direct computation:

$$\begin{cases} (\underline{u}, \underline{v}) & \equiv (0,0), \\ (\overline{u}, \overline{v}) & \equiv \left(c_1\phi(x) + c_2 a(x) + \|u_0\|_{L^\infty}\ ,\ c_3\phi(x) + c_4 a(x) + \|v_0\|_{L^\infty}\right), \end{cases} \tag{6.66}$$

where, in contrast to the previous constants used, we now take

$$\begin{cases} c_1 = \frac{C_0}{d\lambda_1}\left(\frac{1}{\varepsilon^p} + \frac{1}{\varepsilon^q}\right), \quad c_2 = \frac{C_0}{d}\left(\frac{1}{\varepsilon^p} + \frac{1}{\varepsilon^q}\right), \\ c_3 = \frac{C_0}{D\lambda_1}\left(\frac{1}{\varepsilon^r} + \frac{1}{\varepsilon^s}\right), \quad c_4 = \frac{C_0}{D}\left(\frac{1}{\varepsilon^r} + \frac{1}{\varepsilon^s}\right). \end{cases} \tag{6.67}$$

We leave this computation to the reader, as it is essentially the exact same as the previous result. With the substitution $w_\varepsilon = u_\varepsilon + \varepsilon$, $z_\varepsilon = v_\varepsilon + \varepsilon$, an application of Lemma 5.1.2 gives us

$$\begin{aligned} \frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \leq & -\alpha n \int_\Omega \frac{\phi^{n+2} f_1(x)}{w_\varepsilon^{\alpha n+p+1} z_\varepsilon^{\beta n}} dx - \alpha n \int_\Omega \frac{\phi^{n+2} f_2(x)}{w_\varepsilon^{\alpha n+1} z_\varepsilon^{\beta n+q}} dx \\ & - \beta n \int_\Omega \frac{\phi^{n+2} f_3(x)}{w_\varepsilon^{\alpha n+r} z_\varepsilon^{\beta n+1}} dx - \beta n \int_\Omega \frac{\phi^{n+2} f_4(x)}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n+s+1}} dx \\ & + \lambda_1(n+2) \int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx \\ \leq & -\alpha n \int_\Omega \frac{\phi^{n+2} f_2(x)}{w_\varepsilon^{\alpha n+1} z_\varepsilon^{\beta n+q}} dx - \beta n \int_\Omega \frac{\phi^{n+2} f_3(x)}{w_\varepsilon^{\alpha n+r} z_\varepsilon^{\beta n+1}} dx \\ & + \lambda_1(n+2) \int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx. \end{aligned} \tag{6.68}$$

Notice that we have thrown away two terms above. Motivated by the previous result, we actually only require two negative terms to completely control the positive term for all $t > 0$. For any $\delta > 0$, applying Lemma 5.0.1 to the positive term of (6.68) yields

$$\lambda_1(n+2)\frac{f_2^0}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} \leq \alpha n \frac{f_2}{w_\varepsilon^{\alpha n+1} z_\varepsilon^{\beta n+q}} + \lambda_1(n+2)\left(\frac{\alpha n}{\lambda_1(n+2)}\right)^{-\delta} \frac{f_2^{-\delta}}{w_\varepsilon^{\alpha n-\delta} z_\varepsilon^{\beta n-q\delta}}, \tag{6.69}$$

where

$$\lambda_1(n+2)\left(\frac{\alpha n}{\lambda_1(n+2)}\right)^{-\delta} \le \lambda_1(n+2)\left(\frac{2\lambda_1}{\alpha}\right)^{\delta} \equiv c_1(n). \tag{6.70}$$

Combining (6.69) and (6.70) with (6.68) leaves us with

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \le c_1(n)\int_\Omega \frac{\phi^{n+2} f_2^{-\delta}}{w_\varepsilon^{\alpha n-\delta} z_\varepsilon^{\beta n-q\delta}}dx - \beta n\int_\Omega \frac{\phi^{n+2} f_3(x)}{w_\varepsilon^{\alpha n+r} z_\varepsilon^{\beta n+1}}dx. \tag{6.71}$$

Applying Lemma 5.0.1 to the positive term of (6.71) yields

$$c_1(n)\frac{(w_e^{-1})^{\alpha n-\delta} f_3^0}{z_\varepsilon^{\beta n-q\delta} f_2^\delta} \le \beta n \frac{(w_\varepsilon^{-1})^{\alpha n+r}}{z_\varepsilon^{\beta n+1} f_2^0} + c_1(n)\left(\frac{\beta n}{c_1(n)}\right)^{-(\alpha n-\delta)/(\delta+r)} \frac{(w_\varepsilon^{-1})^0 f_3^{\theta_0}}{z_\varepsilon^{\eta_1} f_2^{\eta_2}}, \tag{6.72}$$

where

$$\begin{aligned}
\theta_0 &= -\frac{(\alpha n-\delta)}{\delta+r}, \\
\eta_1 &= \frac{[(\beta n-q\delta)(\alpha n+r)-(\beta n+1)(\alpha n-\delta)]}{\delta+r} \\
&= \frac{-n(\alpha-\beta r)+\delta(\beta-\alpha q+1-rq)}{\delta+r}, \\
\eta_2 &= \frac{\delta(\alpha n+r)}{\delta+r},
\end{aligned} \tag{6.73}$$

and

$$c_1(n)\left(\frac{\beta n}{c_1(n)}\right)^{-(\alpha n-\delta)/(\delta+r)} \leq \lambda_1(n+2)\left(\frac{2\lambda_1}{\alpha}\right)^{\delta}\left(\frac{(2\lambda_1)^{\delta+1}}{\alpha^{\delta}\beta}\right)^{(\alpha n-\delta)/(\delta+r)}$$
$$\equiv c_2(n). \tag{6.74}$$

(6.71) can then be written as

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \leq c_2(n)\int_\Omega \frac{\phi^{n+2} f_3^{\theta_0}}{z_\varepsilon^{\eta_1} f_2^{\eta_2}}dx. \tag{6.75}$$

We now apply the same trick used in the previous proof. With the freedom to choose δ, we set $\eta_1 = 0$ and find

$$\delta = \frac{n(\alpha-\beta r)}{n(\beta-\alpha q)+1-rq}.$$

If we choose $\alpha = \frac{r}{2^*}$ and $\beta = \frac{q}{2^*}$, where $2^* \equiv r+q$ as defined previously, we see that

$$\alpha - \beta r = \frac{r}{2^*}(1-q) > 0,$$
$$\beta - \alpha q = \frac{q}{2^*}(1-r) > 0,$$

and so $\delta > 0$ as required. Under the hypotheses of our theorem, we also know $c_0\phi^{\theta_i}(x) \leq f_i(x)$, and so (6.75) can be further simplified as

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}}dx \leq \frac{c_2(n)}{c_0^{[\alpha n(\delta+1)-\delta(1-r)]/(\delta+r)}}\int_\Omega \phi^{n+2+\theta_0\theta_3-\eta_2\theta_2}dx. \tag{6.76}$$

As in the previous proof, we now ensure that the integral on the right hand side remains finite. Indeed, we require $n + 2 + \theta_0\theta_3 - \eta_2\theta_2 \geq 0$, and so we confirm this in the following calculations. For n sufficiently large, we have that

$$\begin{aligned} n + 2 + \theta_0\theta_3 - \eta_2\theta_2 &= n + 2 - \frac{[\theta_2\delta(\alpha n + r) + \theta_3(\alpha n - \delta)]}{(\delta + r)} \\ &= \frac{[n(\delta + r - \theta_2\delta\alpha - \theta_3\alpha) - \delta(r\theta_2 + \theta_3)]}{(\delta + r)} + 2 \\ &= \frac{[n(r - \theta_3\alpha + \delta(1 - \theta_2\alpha)) - \delta(r\theta_2 + \theta_3)]}{(\delta + r)} + 2 \\ &> 0, \end{aligned} \tag{6.77}$$

since $\delta(1 - \alpha\theta_2) > 0$ and $r - \alpha\theta_3 = r(1 - \theta_3/2^*) > 0$. Consequently, (6.76) remains finite. Integrating in time and recalling that $\varepsilon_0\phi \leq u_0, v_0$ gives us

$$\begin{aligned} \int_\Omega \frac{\phi}{w_\varepsilon^{\alpha n} z_\varepsilon^{\beta n}} dx &\leq \frac{c_2(n)T}{c_0^{[\alpha n(\delta+1) - \delta(1-r)]/(\delta+r)}} \int_\Omega \phi^{n+2+\theta_0\theta_3-\eta_2\theta_2} dx + \int_\Omega \frac{\phi^{n+2}}{u_0^{\alpha n} v_0^{\beta n}} dx \\ &\leq \frac{c_2(n)T}{c_0^{[\alpha n(\delta+1) - \delta(1-r)]/(\delta+r)}} \int_\Omega \phi^{n+2+\theta_0\theta_3-\eta_2\theta_2} dx + \frac{1}{\varepsilon_0^n} \int_\Omega \phi^2 dx, \end{aligned} \tag{6.78}$$

for any $T > 0$. Extracting n^{th} roots and letting $n \to \infty$ gives us our desired result:

$$\begin{aligned} \frac{\phi(x)}{w_\varepsilon^{\frac{r}{2^*}}(x,t) z_\varepsilon^{\frac{q}{2^*}}(x,t)} &\leq \left\| \frac{\phi}{w_\varepsilon^{\frac{r}{2^*}} z_\varepsilon^{\frac{q}{2^*}}} \right\|_\infty \\ &\leq \max\{m_1 \left\|\phi^{\sigma_1}\right\|_\infty, \varepsilon_0^{-1}\} \\ &\equiv M_1 < \infty, \end{aligned} \tag{6.79}$$

where

$$\begin{aligned}\sigma_1 &= \frac{r(1-\frac{\theta_3}{2^*})+\delta(1-\frac{r\theta_2}{2^*})}{\delta+r} > 0,\\ m_1 &= \left(\frac{(2\lambda_1)^{\delta+1}}{\alpha^\delta \beta c_0^{\delta+1}}\right)^{\alpha/(\delta+r)}.\end{aligned} \tag{6.80}$$

The next step is to use this uniform bound to obtain an upper bound for u_ε. This process will be similar to the previous proof. For new α, β, Lemma 5.1.3 with u replaced by u_ε yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \leq n\int_\Omega \frac{u_\varepsilon^{n-1}f_1}{w_\varepsilon^{\alpha n+p}\psi^{\beta n-2}}dx + n\int_\Omega \frac{u_\varepsilon^{n-1}f_2}{w_\varepsilon^{\alpha n}z_\varepsilon^q\psi^{\beta n-2}}dx\\ -\alpha n\int_\Omega \frac{u_\varepsilon^n f_1}{w_\varepsilon^{\alpha n+p+1}\psi^{\beta n-2}}dx - \alpha n\int_\Omega \frac{u_\varepsilon^n f_2}{w_\varepsilon^{\alpha n+1}z_\varepsilon^q\psi^{\beta n-2}}dx\\ -(\beta n-2)\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx.\end{aligned} \tag{6.81}$$

For any $\delta_1, \delta_2 > 1$, applying Lemma 5.0.1 to terms 1 and 2 of (6.81) gives us

$$\frac{u_\varepsilon^{n-1}}{w_\varepsilon^{\alpha n+p}} \leq \alpha\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n+p+1}} + \frac{1}{\alpha^{\delta_1-1}}\cdot\frac{u_\varepsilon^{n-\delta_1}}{w_\varepsilon^{\alpha n+p+1-\delta_1}}, \tag{6.82}$$

and

$$\frac{u_\varepsilon^{n-1}}{w_\varepsilon^{\alpha n}} \leq \alpha\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n+1}} + \frac{1}{\alpha^{\delta_2-1}}\cdot\frac{u_\varepsilon^{n-\delta_2}}{w_\varepsilon^{\alpha n+1-\delta_2}}. \tag{6.83}$$

Combining (6.82) and (6.83) with (6.81) leaves us with

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le \frac{n}{\alpha^{\delta_3-1}}\int_\Omega \frac{u_\varepsilon^{n-\delta_3}f_1}{w_\varepsilon^{\alpha n+p+1-\delta_3}\psi^{\beta n-2}}dx + \frac{n}{\alpha^{\delta_4-1}}\int_\Omega \frac{u_\varepsilon^{n-\delta_4}f_2}{w_\varepsilon^{\alpha n+1-\delta_4}z_\varepsilon^q\psi^{\beta n-2}}dx$$
$$-(\beta n-2)\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx. \tag{6.84}$$

We now use (6.79) and (4.8) to remove the dependence on z_ε in line (6.84) as follows

$$\frac{u_\varepsilon^{n-\delta_2}f_1}{w_\varepsilon^{\alpha n+1-\delta_2}z_\varepsilon^q\psi^{\beta n-2}} = \frac{u_\varepsilon^{n-\delta_2}f_1}{w_\varepsilon^{\alpha n+1-r-\delta_2}\psi^{\beta n-2+2^*}}\left(\frac{\phi}{w_\varepsilon^{\frac{r}{2^*}}z_\varepsilon^{\frac{q}{2^*}}}\right)^{2^*}\left(\frac{\psi}{\phi}\right)^{2^*}$$
$$\le M_1^{2^*}\gamma_1^{2^*}\frac{u_\varepsilon^{n-\delta_2}f_1}{w_\varepsilon^{\alpha n+1-r-\delta_2}\psi^{\beta n-2+2^*}}. \tag{6.85}$$

Thus, (6.84) becomes

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le \frac{n}{\alpha^{\delta_1-1}}\int_\Omega \frac{u_\varepsilon^{n-\delta_1}f_1}{w_\varepsilon^{\alpha n+p+1-\delta_1}\psi^{\beta n-2}}dx$$
$$+\frac{M_1^{2^*}\gamma_1^{2^*}n}{\alpha^{\delta_2-1}}\int_\Omega \frac{u_\varepsilon^{n-\delta_2}f_2}{w_\varepsilon^{\alpha n+1-r-\delta_2}\psi^{\beta n-2+2^*}}dx$$
$$-(\beta n-2)\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx. \tag{6.86}$$

We now apply Lemma 5.0.1 again to the first term of (6.86):

$$\left(\frac{n}{\alpha^{\delta_1-1}}\right)\frac{u_\varepsilon^{n-\delta_1}f_1}{w_\varepsilon^{\alpha n+p+1-\delta_1}\psi^{\beta n-2}} \le \left(\frac{(\beta n-2)}{2}\right)\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}$$
$$+\left(\frac{n}{\alpha^{\delta_1-1}}\right)\left(\frac{\alpha^{\delta_1-1}(\beta n-2)}{2n}\right)^{-\frac{n}{\delta_1}+1}\frac{f_1^{\frac{n}{\delta_1}}}{w_\varepsilon^{\eta_1}\psi^{\eta_2}}, \tag{6.87}$$

where

$$\eta_1 = \frac{n(p+1) - n(1-\alpha)\delta_1}{\delta_1},$$
$$\eta_2 = \frac{-n(1+\sigma-\beta\delta_1)}{\delta_1} - (1-\sigma),$$

7.31 and

$$\left(\frac{n}{\alpha^{\delta_1-1}}\right)\left(\frac{\alpha^{\delta_1-1}(\beta n-2)}{2n}\right)^{-\frac{n}{\delta_1}+1} \leq \left(\frac{(\beta n-2)}{2}\right)\left(\frac{4}{\alpha_1^{\delta_1-1}\beta}\right)^{\frac{n}{\delta_3}} \equiv c_3(n). \tag{6.88}$$

Doing the usual trick, if we set $\eta_1 = 0$, we find that

$$\delta_1 = \frac{p+1}{1-\alpha} > 1,$$

for any $\alpha \in (0,1)$. Combining all of these estimates with (6.86) gives us

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \leq c_3(n)\int_\Omega f_1^{\frac{n}{\delta_1}}\psi^{\frac{n(1+\sigma-\beta\delta_1)}{\delta_1}+(1-\sigma)}dx + \frac{M_1^{2^*}\gamma_1^{2^*}n}{\alpha^{\delta_2-1}}\int_\Omega \frac{u_\varepsilon^{n-\delta_2}f_2}{w_\varepsilon^{\alpha n+1-r-\delta_2}\psi^{\beta n-2+2^*}}dx - \frac{(\beta n-2)}{2}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}}dx. \tag{6.89}$$

We now repeat this process. Applying Lemma 5.0.1 to the second term of (6.89) yields

$$\left(\frac{M_1^{2^*}\gamma_1^{2^*}n}{\alpha^{\delta_2-1}}\right)\frac{u_\varepsilon^{n-\delta_2}f_2}{w_\varepsilon^{\alpha n+1-r-\delta_2}\psi^{\beta n-2+2^*}} \leq \left(\frac{(\beta n-2)}{2}\right)\frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-1+\sigma}} + \left(\frac{M_1^{2^*}\gamma_1^{2^*}n}{\alpha^{\delta_2-1}}\right)\left(\frac{\alpha^{\delta_2-1}(\beta n-2)}{2M_1^{2^*}\gamma_1^{2^*}n}\right)^{-\frac{n}{\delta_2}+1}\frac{f_2^{\frac{n}{\delta_2}}}{w_\varepsilon^{\eta_1}\psi^{\eta_2}}, \tag{6.90}$$

where

$$\begin{aligned}\eta_1 &= \frac{n(1-r)-n\delta_2(1-\alpha)}{\delta_2},\\ \eta_2 &= \frac{-n(1+\sigma-2^*-\beta\delta_2)}{\delta_2}-(1-\sigma),\end{aligned} \tag{6.91}$$

and

$$\left(\frac{M_1^{2^*}\gamma_1^{2^*}n}{\alpha^{\delta_2-1}}\right)\left(\frac{\alpha^{\delta_2-1}(\beta n-2)}{2M_1^{2^*}\gamma_1^{2^*}n}\right)^{-\frac{n}{\delta_2}+1} \leq \left(\frac{(\beta n-2)}{2}\right)\left(\frac{4M_1^{2^*}\gamma_1^{2^*}}{\alpha^{\delta_2-1}\beta}\right)^{\frac{n}{\delta_4}} \equiv c_4(n).$$

Again, setting $\eta_1 = 0$ and solving for δ_2 yields

$$\delta_2 = \frac{1-r}{1-\alpha} > 1,$$

for $\alpha \in (r, 1)$. Thus, (6.89) becomes

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le c_3(n)\int_\Omega f_1^{\frac{n}{\delta_1}}\psi^{\frac{n(1+\sigma-\beta\delta_1)}{\delta_1}+(1-\sigma)}dx + c_4(n)\int_\Omega f_2^{\frac{n}{\delta_2}}\psi^{\frac{n(1+\sigma-2^*-\beta\delta_2)}{\delta_2}+(1-\sigma)}dx. \tag{6.92}$$

Similar to the previous proof, choose $\alpha \in (r, 1)$ with $\beta = 1-\alpha$. Direct computation gives $1+\sigma-\beta\delta_1 = \sigma - p > 0$ and $1+\sigma-2^*-\beta\delta_2 = \sigma - q > 0$ for σ sufficiently close to 1, and so both integrals on the right hand side (6.92) are indeed finite. Integrating this result in time gives us

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le{}& c_3(n)T\int_\Omega f_1^{\frac{n}{\delta_1}}\psi^{\frac{n(\sigma-p)}{\delta_1}+(1-\sigma)}dx \\ &+ c_4(n)T\int_\Omega f_2^{\frac{n}{\delta_2}}\psi^{\frac{n(\sigma-q)}{\delta_2}+(1-\sigma)}dx \\ &+ \int_\Omega \frac{u_0^{n(1-\alpha)}}{\psi^{\beta n-2}}dx,\end{aligned} \tag{6.93}$$

and if we refer again to (6.49), we can further simplify this as

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le{}& c_3(n)T\int_\Omega f_1^{\frac{n}{\delta_1}}\psi^{\frac{n(\sigma-p)}{\delta_1}+(1-\sigma)}dx \\ &+ c_4(n)T\int_\Omega f_2^{\frac{n}{\delta_2}}\psi^{\frac{n(\sigma-q)}{\delta_2}+(1-\sigma)}dx \\ &+ \left(\frac{K_u}{\gamma_0}\right)^{\beta n}\int_\Omega \psi^2 dx,\end{aligned} \tag{6.94}$$

for all $t \in (0, \infty)$. Extracting n^{th} roots and letting $n \to \infty$ gives our desired bound:

$$\begin{aligned}\frac{u_\varepsilon(x,t)}{w_\varepsilon^\alpha(x,t)\psi^\beta(x)} &\leq \left\| \frac{u_\varepsilon}{w_\varepsilon^\alpha \psi^\beta} \right\|_\infty \\ &\leq \max \left\{ m_3 \left\| f_1^{\frac{1}{\delta_1}} \psi^{\frac{\sigma-p}{\delta_1}} \right\|_\infty, \; m_4 \left\| f_2^{\frac{1}{\delta_2}} \psi^{\frac{\sigma-q}{\delta_2}} \right\|_\infty, \left(\frac{K_u}{\gamma_0} \right)^\beta \right\} \\ &\equiv M_2 < \infty, \end{aligned} \tag{6.95}$$

where

$$\begin{aligned} m_3 &= \left(\frac{4}{\alpha^{\delta_1 - 1}\beta} \right)^{\frac{1}{\delta_1}}, \\ m_4 &= \left(\frac{4 M_1^{2^*} \gamma_1^{2^*}}{\alpha^{\delta_2 - 1}\beta} \right)^{\frac{1}{\delta_2}}, \end{aligned}$$

independent of ε. Rearranging this, we obtain

$$u_\varepsilon(x,t) \leq M_2^{\frac{1}{1-\alpha}} \psi(x), \tag{6.96}$$

uniformly for all $t \in (0, \infty)$, for any $\varepsilon > 0$. The exact same procedure yields the following upper bound for v_ε:

$$v_\varepsilon(x,t) \leq M_3^{\frac{1}{1-\alpha}} \psi(x), \tag{6.97}$$

also independent of ε, for all $t \in (0, \infty)$. We will now use these estimates to obtain a lower bound for u_ε. To start, Lemma 5.1.1 with u replaced by w_ε gives us

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx \leq \lambda_1 n \int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx - n\int_\Omega \frac{\phi^{n+2}f_1}{w_\varepsilon^{n+p+1}}dx - n\int_\Omega \frac{\phi^{n+2}f_2}{w_\varepsilon^{n+1}z_\varepsilon^q}. \tag{6.98}$$

For any $\delta > 0$, applying Lemma 5.0.1 to the first term of (6.98) yields

$$\frac{f_1^0}{w_\varepsilon^n} \leq \lambda_1 \frac{f_1}{w_\varepsilon^{n+p+1}} + \lambda_1^{-\delta}\frac{f_1^{-\delta}}{w_\varepsilon^{n-\delta(p+1)}}. \tag{6.99}$$

With this, (6.98) then becomes

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx \leq \lambda_1^{-\delta} n \int_\Omega \frac{\phi^{n+2}f_1^{-\delta}}{w_\varepsilon^{n-\delta(p+1)}} - n\int_\Omega \frac{\phi^{n+2}f_2}{w_\varepsilon^{n+1}z_\varepsilon^q}. \tag{6.100}$$

Applying Lemma 5.0.1 again to the first term of (6.100) gives us

$$\lambda_1^{-\delta}\frac{(w_\varepsilon^{-1})^{n-\delta(p+1)}f_2^0}{f_1^\delta} \leq \frac{(w_\varepsilon^{-1})^{n+1}f_2}{z_\varepsilon^q} + \left(\lambda_1^\delta\right)^{-\frac{[n-\delta(p+1)]}{\delta(p+1)+1}} \frac{f_2^{\theta_0}}{z_\varepsilon^{\eta_1} f_1^{\eta_2}}, \tag{6.101}$$

where

$$\begin{aligned} \theta_0 &= -\frac{[n-\delta(p+1)]}{\delta(p+1)+1}, \\ \eta_1 &= -\frac{q[n-\delta(p+1)]}{\delta(p+1)+1} < 0, \\ \eta_2 &= \frac{\delta(n+1)}{\delta(p+1)+1}. \end{aligned} \tag{6.102}$$

Combining this to (6.100) gives us

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx \leq \lambda_1^{-\delta}\left(\frac{1}{\lambda_1^\delta}\right)^{\frac{n-\delta(p+1)}{\delta(p+1)+1}} n\int_\Omega \frac{\phi^{n+2}f_2^{\theta_0}}{z_\varepsilon^{\eta_1}f_1^{\eta_2}}dx. \tag{6.103}$$

From our hypothesis, we know that $c_0\phi^{\theta_i}(x) \leq f_i(x)$, and so we further simplify the above as

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx \leq \frac{\lambda_1^{-\delta}}{c_0^{\eta_2-\theta_0}}\left(\frac{1}{\lambda_1^\delta}\right)^{\frac{n-\delta(p+1)}{\delta(p+1)+1}} n\int_\Omega \frac{\phi^{n+2+\theta_0\theta_2-\eta_2\theta_1}}{z_\varepsilon^{\eta_1}}dx. \tag{6.104}$$

We now need to ensure that the integral on the right hand side is finite. One can see that, since z_ε is bounded by the previous result, and $\eta_1 < 0$, this will converge provided that

$$n + 2 + \theta_0\theta_2 - \eta_2\theta_1 \geq 0,$$

and so we compute:

$$\begin{aligned} n + 2 + \theta_0\theta_2 - \eta_2\theta_1 &= n + 2 - \frac{[\theta_2(n-\delta(p+1)) + \theta_1\delta(n+1)]}{\delta(p+1)+1} \\ &= \frac{n[1-\theta_2+\delta(p+1-\theta_1)] + \delta(\theta_2(p+1)-\theta_1)}{\delta(p+1)+1} + 2 \\ &> 0, \end{aligned} \tag{6.105}$$

since $1-\theta_2 \geq 0$ and $p+1-\theta_1 > 0$ for any $\theta_1, \theta_2 \in [0,1]$. Integrating in time and applying our assumption that $\varepsilon_0\phi(x) \leq u_0(x)$, we see that

$$\begin{aligned}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx &\leq \frac{\lambda_1^{-\delta}}{c_0^{\eta_2-\theta_0}}\left(\frac{1}{\lambda_1^\delta}\right)^{\frac{n-\delta(p+1)}{\delta(p+1)+1}} nT\int_\Omega \frac{\phi^{n+2+\theta_0\theta_2-\eta_2\theta_1}}{z_\varepsilon^{\eta_1}} \\ &\quad + \int_\Omega \frac{\phi^{n+2}}{u_0^n}dx \\ &\leq \frac{\lambda_1^{-\delta}}{c_0^{\eta_2-\theta_0}}\left(\frac{1}{\lambda_1^\delta}\right)^{\frac{n-\delta(p+1)}{\delta(p+1)+1}} nT\int_\Omega \frac{\phi^{n+2+\theta_0\theta_2-\eta_2\theta_1}}{z_\varepsilon^{\eta_1}} \\ &\quad + \varepsilon_0^{-n}\int_\Omega \phi^2 dx. \end{aligned} \tag{6.106}$$

Extracting n^{th} roots and letting $n \to \infty$ yields

$$\begin{aligned}\frac{\phi(x)}{w_\varepsilon(x,t)} &\leq \left\|\frac{\phi}{w_\varepsilon}\right\|_\infty \\ &\leq \max\{m_5\left\|\phi^{\frac{[1-\theta_2+\delta(p+1-\theta_1)]}{\delta(p+1)+1}} z_\varepsilon^{\frac{q}{\delta(p+1)+1}}\right\|_\infty,\ \varepsilon_0^{-1}\} \\ &\equiv M_3 < \infty, \end{aligned} \tag{6.107}$$

where

$$m_5 = \left(\frac{1}{c_0^{\delta+1}\lambda_1^\delta}\right)^{1/[\delta(p+1)+1]},$$

independent of ε, uniformly for all $t \in (0,\infty)$. Consequently, we have obtained the bound $M_3^{-1}\phi(x) \leq w_\varepsilon(x,t)$. Applying the same technique yields $M_4^{-1}\phi(x) \leq z_\varepsilon(x,t)$. Our next step is to show that the nonlinear terms of our perturbed system are uniformly bounded in $L^k(\Omega)$. To see this, let $k_1 \in (1, \min\{\frac{1}{p}, \frac{1}{q}\})$, and apply

Minkowski's Inequality as follows:

$$\begin{aligned}\left\|\frac{f_1(x)}{w_\varepsilon^p}+\frac{f_2(x)}{z_\varepsilon^q}\right\|_{L^{k_1}}dx &\leq \left(\int_\Omega\left|\frac{f_1(x)}{w_\varepsilon^p}\right|^{k_1}dx\right)^{\frac{1}{k_1}}+\left(\int_\Omega\left|\frac{f_2(x)}{z_\varepsilon^q}\right|^{k_1}dx\right)^{\frac{1}{k_1}}\\ &\leq C_0M_3^p\left(\int_\Omega \phi^{k_1(\tau_1-p)}dx\right)^{\frac{1}{k_1}}+C_0M_4^q\left(\int_\Omega \phi^{k_1(\tau_2-q)}dx\right)^{\frac{1}{k_1}}\\ &< \infty, \end{aligned} \tag{6.108}$$

where our choice in k_1 ensures that $-1 < k_1(\tau_1 - p)$ and $-1 < k_1(\tau_2 - q)$ for any $\tau_1, \tau_2 \in [0, 1]$, and so (6.108) is true. Similarly, for any $k_2 \in (1, \min\{\frac{1}{r}, \frac{1}{s}\})$, it is true that

$$\left\|\frac{f_3(x)}{w_\varepsilon^r}+\frac{f_4(x)}{z_\varepsilon^s}\right\|_{L^{k_2}} < \infty. \tag{6.109}$$

By Theorem 3.1.2, $(u_\varepsilon, v_\varepsilon)$ are uniformly bounded in $\left[W^{2,1}_{k_1}(\Omega\times(0,T))\right]\times\left[W^{2,1}_{k_2}(\Omega\times(0,T))\right]$, and so we may extract a subsequence $(u_{\varepsilon_i}, v_{\varepsilon_i})$ which converges to a weak solution of (6.2).

We now argue similar to the previous proof. If $-1 < N\min\{(\tau_1 - p), (\tau_2 - q)\}$, we may choose $k_1 > N$ so that (6.108) is true. By Lemma 2.0.2, we then have that $u_\varepsilon \to u \in C^{1+\kappa,1/2+\kappa/2}(\overline{Q}_T)$ for some $\kappa \in (0, 1)$, and so u is a classical solution. Similarly, if $-1 < N\min\{(\tau_3 - r), (\tau_4 - s)\}$, we can again choose $k_2 > N$ so that (6.109) is true, and $v_\varepsilon \to v \in C^{1+\kappa,1/2+\kappa/2}(\overline{Q}_T)$, and v is a classical solution. This completes the proof. □

6.5 Existence for $d \neq D$

We now prove the existence results for $d \neq D$ when $d, D > 0$, but the results are only valid locally in time.

Proof of Theorem 6.2.3. The proof of this theorem will be highly condensed, as we mainly refer to the results of Theorem 6.2.1. Considering the same perturbed system, we notice that for any $d > 0$, Lemma 5.1.1 gives us that

$$\frac{d}{dt}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx \leq \lambda_1 dn \int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx, \tag{6.110}$$

and so by the Gronwall inequality and the assumption that $\varepsilon_0\phi(x) \leq u_0(x)$, we find

$$\begin{aligned}\int_\Omega \frac{\phi^{n+2}}{w_\varepsilon^n}dx &\leq e^{n\lambda_1 dT}\int_\Omega \frac{\phi^{n+2}}{u_0^n}dx \\ &\leq \varepsilon_0^{-n}e^{n\lambda_1 dT}\int_\Omega \phi^2 dx,\end{aligned} \tag{6.111}$$

and so extracting n^{th} roots yields $\phi(x) \leq \varepsilon_0^{-1}e^{\lambda_1 dT}w_\varepsilon(x,t)$, for all $\varepsilon > 0$ and all $t \in (0,T)$. The same procedure gives us that $\phi(x) \leq \varepsilon_0^{-1}e^{\lambda_1 DT}z_\varepsilon(x,t)$. This time dependent lower bounds will replace the uniform in time bound given by (6.34) in the proof of Theorem 6.2.1. Next, Lemma 5.1.3 gives us that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-2}}dx \leq{}& n\int_\Omega \frac{u^{n-1}f(x)}{{w_\varepsilon}^{\alpha n+p}z_\varepsilon^q\psi^{\beta n-2}}dx - \alpha n\int_\Omega \frac{u^n f(x)}{{w_\varepsilon}^{\alpha n+p+1}z_\varepsilon^q\psi^{\beta n-2}}dx \\ &- d(\beta n-2)\int_\Omega \frac{u^n}{{w_\varepsilon}^{\alpha n}\psi^{\beta n-1+\sigma}}dx,\end{aligned} \tag{6.112}$$

and referring to (6.35) – (6.41), the same procedure yields

$$\frac{d}{dt}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx \le c_4(n,T)\int_\Omega \frac{f^{\frac{n}{\delta_1}}}{w_\varepsilon^{\eta_1}\psi^{\eta_2}}dx, \tag{6.113}$$

where

$$\begin{aligned}\eta_1 &= \frac{n(p+1)-\delta_1 n(1-\alpha)}{\delta_1},\\ \eta_2 &= -\frac{n}{\delta_1}[1+\sigma-q-\beta\delta_1]-(1-\sigma),\\ c_4(n,T) &= d(\beta n-2)\left(\frac{2\gamma_1^q e^{\lambda_1 DqT}}{d\alpha^{\delta_1-1}\beta\varepsilon_0^q}\right)^{\frac{n}{\delta_1}}.\end{aligned} \tag{6.114}$$

Notice that the constant $c_4(n,T)$ now depends non-trivially on our maximal existence time T. Similar to the previous process, setting $\eta_1=0$ yields

$$\delta_1 = \frac{p+1}{1-\alpha} > 1,$$

for any $\alpha\in(0,1)$. In this case, we may choose α,β such that $\alpha+\beta=1$. Then,

$$\begin{aligned}1+\sigma-q-\beta\delta_1 &= 1+\sigma-q-p-1\\ &= \sigma-(p+q)>0,\end{aligned} \tag{6.115}$$

for σ sufficiently close to 1. As a result, the right hand side of (6.113) is finite, in which case we may solve the differential inequality to obtain

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^n}{w_\varepsilon^{\alpha n}\psi^{\beta n-2}}dx &\leq c_4(n,T)T\int_\Omega f^{\frac{n}{\delta_1}}\psi^{\frac{n}{\delta_1}(\sigma-p-q)+1-\sigma}dx + \int_\Omega \frac{u_0^{n(1-\alpha)}}{\psi^{\beta n-2}}dx \\ &\leq c_4(n,T)T\int_\Omega f^{\frac{n}{\delta_1}}\psi^{\frac{n}{\delta_1}(\sigma-p-q)+1-\sigma}dx + \left(\frac{K_u}{\gamma_0}\right)^{\beta n}\int_\Omega \psi^2 dx. \quad (6.116)\end{aligned}$$

Extracting n^{th} roots and taking $n\to\infty$, we have

$$\begin{aligned}\frac{u_\varepsilon}{w_\varepsilon^\alpha\psi^\beta} &\leq \left\|\frac{u_\varepsilon}{w_\varepsilon^\alpha\psi^\beta}\right\|_{L^\infty} \\ &\leq \max\left\{m_1\left\|f^{\frac{1}{\delta_1}}\psi^{\frac{\sigma-p-q}{\delta_1}}\right\|_{L^\infty}, \left(\frac{K_u}{\gamma_0}\right)^\beta\right\} \\ &\equiv M_1<\infty, \quad (6.117)\end{aligned}$$

where

$$m_1 = \left(\frac{2\gamma_1^q e^{\lambda_1 DqT}}{d\alpha^{\delta_1-1}\beta\varepsilon_0^q}\right)^{\frac{1}{\delta_1}}.$$

The same procedure yields

$$\begin{aligned}\frac{v_\varepsilon}{z_\varepsilon^\alpha\psi^\beta} &\leq \left\|\frac{v_\varepsilon}{z_\varepsilon^\alpha\psi^\beta}\right\|_{L^\infty} \\ &\leq M_2, \quad (6.118)\end{aligned}$$

ndependent of ε, for all $t\in(0,T)$. With these estimates, one can see that (6.63) and (6.64) are true, independent of ε, for all $t\in(0,T)$, and so there exists a subsequence $(u_{\varepsilon_i}, v_{\varepsilon_i})$ converging to a global weak solution of problem (6.1). The same argument used to obtain additional regularity in Theorem 6.2.1 now holds, and the proof is complete. □

Proof of Theorem 6.2.4. Similar to the proof of Theorem 6.2.3, we may obtain lower bounds on our solutions, independent of ε for all $t \in (0, T)$. We may then use these estimates to obtain upper bounds for our perturbed solutions. Taking ε to zero yields the existence of a global weak solution. With the additional assumptions on our exponents, we are then able to obtain the existence of a global classical solutions. This completes the proof. □

Chapter 7

Conclusion

In this thesis, we have investigated various parabolic systems featuring singular nonlinearities. This includes a generalization of [4], as well as some additional results. As we have seen, this introduces many mathematical difficulties. In particular, classical methods are unsuccessful in proving the existence of solutions. This is somewhat unsurprising, as the classical methods require the nonlinearity to be, at the very least, Hölder continuous up to the boundary. Through the tools developed in previous works, such as [15], [2], [3], [4] and others, we are able to prove the existence of both weak and classical solutions. The key differences between these founding works and the work featured here is primarily the perspective taken on what a priori bounds we actually wish to obtain. In [4], for example, bounds are obtained on a particular functional for some general exponents $\alpha, \beta \in (0, 1)$. Here, more attention is given to what exactly these exponents could be in relation to the nonlinearities as they appear in the respective systems. This seems to give more clarity in what the purpose is of the preliminary bounds obtained. For example, when considering the case where the coupling of equations appear as $u^{-p}v^{-q}$ and $u^{-r}v^{-s}$, it makes intuitive sense to attempt to remove v^{-q} from the equation for u, and u^{-r} from the equation for v. Hence the motivation to choose α and β dependent on r and q, respectively.

Through the development of these tools and doing this research, the various works available concerning singular equations has also become more evident. This concerns both elliptic and parabolic equations. Despite this, there are still very few results concerning singular systems, especially in the parabolic case. This motivates further development of these tools in order to handle singular equations where the singularity appears in a higher order term. For example, one may consider the Keller-Segel system in a smooth, bounded domain, written as

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right), \\ v_t = \Delta v - v + u, \qquad x \in \Omega,\ t > 0, \end{cases} \tag{7.1}$$

where $\chi > 0$ is some constant. Under homogeneous Dirichlet boundary conditions, this system may become singular near the boundary, but this time the singularity appears under a gradient term. This system is much more difficult to deal with, primarily due to diffusion of both u and v appearing in the equation for u. This makes it more difficult to obtain integral inequalities used extensively in proving the results found within this thesis. Consequently, there is future works to be done.

Finally, these tools may be further developed to obtain additional results related to singular parabolic systems. This could include a more detail exploration of the boundary behaviour, such as that found in [8], [9] or [21]. One could further investigate the properties of solutions when the exponents are allowed to take values in $(1, 2)$, such as less regularity of solutions (up to the boundary) and nonexistence results. This could also include finite time blowup results, as well as bounded solutions for all time when $d \neq D$, or even weakening conditions on the initial data and the functions f, g appearing in denominator of our nonlinearities.

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