Time Consistent Behaviour and Discount Rates

TIME CONSISTENT BEHAVIOUR AND DISCOUNT RATES

BY

OUMAR SOULE MBODJI, M.Sc.

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AUTHOR:	Oumar Soule Mbodji M.Sc., (Mathematics) Ecole Normale Superieure, Lyon, France
SUPERVISOR:	Dr. Traian Pirvu
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To my wife Arub and my parents Souleymane and Aminata

Abstract

Decisions such as saving, investing, policymaking, have consequences in multiple time periods and are called intertemporal. These choices require decision-makers to trade-off costs and benefits at different points in time. Time preference is the preference for immediate gratification or utility over delayed gratification. The discount rate is a tool used to measure this psychological phenomenon.

This thesis considers the problem of an individual maximizing his utility from consumption and final wealth when his discount rate is not constant. The question we answer is the following: if we allow the individual to update his decisions, will he stick to his original strategy or will he switch?

We show that there are cases in which the individual's strategy keeps changing thus his behaviour becomes time inconsistent. In **Chapter 1**, we introduce two notions to solve this inconsistency problem: The agent can pre commit i.e. he does not change his original optimal strategy. The agent can also plan for his future changes of strategy and adopt time consistent strategies also known as subgame perfect strategies. We also review the existing literature on time discounting and time consistency.

Chapter 2 considers the time consistency in the expected utility maximization problem. The risk preference is of the Constant Relative Risk Aversion (CRRA) type, the time preference is specified by a non constant discount rate and we allow the

volatility of the stock price to be stochastic. We show that the determination of one quantity: the utility weighted discount rate completely characterizes the individual's subgame perfect strategies.

Chapter 3 is about equilibrium pricing in a model populated by several economic agents in a complete financial market. These agents are investing, saving and consuming and want to maximize their expected utility of consumption and final wealth. We allow the economic agents to differ in their risk preferences, beliefs about the future of the economy and in their time preferences (non constant discount rates). Since the optimal strategies are time inconsistent, the equilibrium is computed by using the time 0 optimal (precommitment) strategies for the market clearing conditions.

Chapter 4 considers the same model as chapter 2. We solve the equilibrium problem when time consistent strategies are used for the market clearing conditions. We limit the study to two economic agents. The subgame perfect equilibrium is compared to the optimal equilibrium of Chapter 3.

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Notation and abbreviations

For i, j two integers, denote $\delta_{i,j}$ the Kronecker symbol:

$$\delta_{i,j} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \tag{0.1}$$

$T > 0$ is the finite time horizon of the agent considered.

$f(t, s)$ will represent the discount function between t and $s \geq t$.

$\rho_f(t, s)$ is the forward discount rate between t and $s \geq t$: $\rho_f(t, s) = -\frac{\frac{\partial f(t,s)}{\partial s}}{f(t,s)}$.

$\rho_b(t, s)$ is the backward discount rate between t and $s \geq t$: $\rho_b(t, s) = \frac{\frac{\partial f(t,s)}{\partial t}}{f(t,s)}$.

$U_\gamma(x) = \frac{x^\gamma}{\gamma}$ defined for $\gamma < 1, \gamma \neq 0$ is the utility function at x.

$p = \frac{1}{1-\gamma}$ is the inverse of the relative risk aversion.

$\{W_t, 0 \leq t \leq T\}$ is a Brownian motion.

r is an interest rate, θ_S is the market price of risk, σ_S is the volatility of the stock (in a complete market).

For a process $\{Y_s\}_{t \leq s \leq T}$, $Y_s^* = \sup_{t \leq u \leq s} |Y_u|$ is the maximum process of $|Y|$.

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Chapter 1

Introduction

The theory of utility dates back from the 18th century. Swiss mathematicians Cramer and Bernouilli introduced it to model satisfaction. This theory predicts that an economic agent will base his decisions on the expected utility derived from a decision and not on the expected value of such a decision. For example, the St-Petersburg paradox gives an infinite expected value but a finite expected utility. The paradox is as follows: the St Petersbourg game is played by flipping a fair coin until it comes up heads, and the total number of flips, n, determines the prize, which equals 2^n. The expected value is

$$EV = \frac{1}{2} \times 2 + \frac{1}{4} \times 4 + \cdots + \frac{1}{2^n} \times 2^n + ... = \infty$$

Daniel Bernouilli writes regarding the paradox: *The determination of the value of an item must not be based on the price, but rather on the utility it yields ... There is no doubt that a gain of one thousand ducats is more significant to the pauper than to a rich man though both gain the same amount.*

The paradox is solved using a log utility function. The player gets utility $\log x$ from having x dollars so the expected utility

$$EU = \frac{1}{2} \times \log 2 + \frac{1}{4} \times \log 4 + \cdots + \frac{1}{2^n} \times \log(2^n) + ... < \infty$$

Most choices require decision-makers to trade-off costs and benefits at different points in time. Decisions with consequences in multiple time periods are referred to as intertemporal choices. Decisions about savings, work effort, education, nutrition, exercise, and health care are all intertemporal choices.

The theory of discounted utility is the most widely used framework for analysing intertemporal choices. This framework has been used to describe actual behaviour (positive economics) and it has been used to prescribe socially optimal behaviour (normative economics). Descriptive discounting models capture the property that most economic agents prefer current rewards to delayed rewards of similar magnitude. Such time preferences have been ascribed to a combination of mortality effects, impatience effects, and salience effects.

However, mortality effects alone cannot explain time preferences, since mortality rates for young and middle-aged adults are at least 100 times too small to generate observed discounting patterns.

The most widely used discounting model is the discounted utility model (DU) and assumes that total utility can be decomposed into a weighted sum - or weighted integral of utility quantities in each period of time. (Ramsey, 1928) assumes a constant discount rate; he wrote:

This is the only assumption (the discount rate is constant) we can make, without contradicting our fundamental hypothesis that successive generations are actuated by the same system of preferences.

The DU model specifies a decision maker's intertemporal preferences over consumption profiles $(c_t, \cdots, c_T)$. A person's intertemporal utility function can be described by the following special functional form:

$$U^t(c_t, \cdots, c_T) = \sum_{k=0}^{T-t} D(k)u(c_{t+k})$$

where $D(k) = (\frac{1}{1+\rho})^k$ is the discount function between time k and time $k+1$.

As cited in (Samuelson, 1937), $u(c_{t+k})$ is often interpreted as the person's cardinal instantaneous utility function - his well-being in period $t+k$ - and $D(k)$ is often interpreted as the person's discount function - the relative weight that he attaches, in period t, to his well-being in period $t+k$. ρ represents the individual's pure rate of time preference (his discount rate), which is meant to reflect the collective effects of the "psychological" motives discussed earlier.

It is well-known that doing away with the assumption that the discount rate is constant will create time-inconsistency. The constant ρ will now be replaced by a period dependent constant ρ_k. We briefly review the concept of time-inconsistency.

Let us begin with a simple example. Suppose time is discrete. An individual considers entering an activity (for instance, running) which has some cost and benefit:

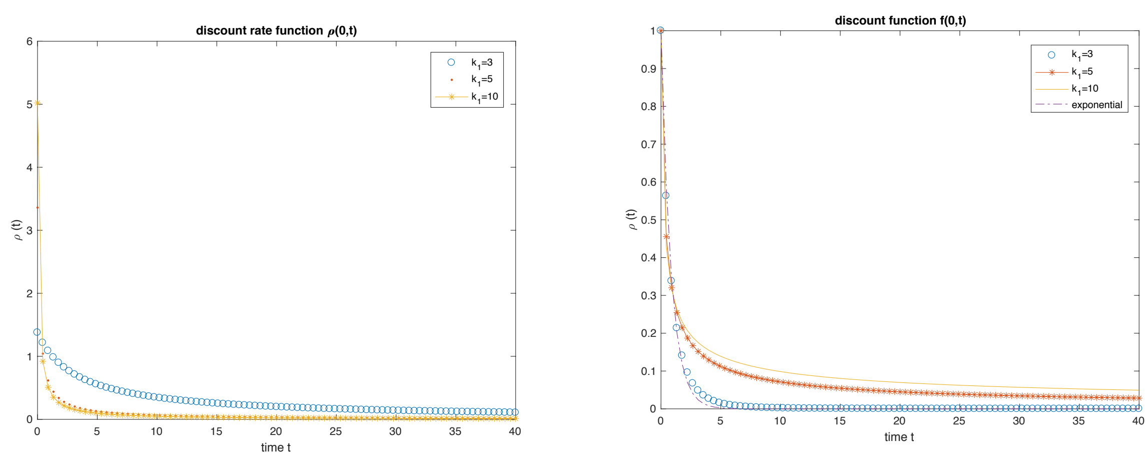
- If he starts today, she will suffer -1 today (pain), but gain +2 tomorrow (health).
- He has a non-constant discount rate: a stream u_t is valued today ($t=0$) at

$$u_0 + \frac{1}{2}\sum_{t=1}^{\infty} \rho^t u_t \quad \text{for some } \rho \in (\frac{1}{2}, 1)$$

- Starting today yields a utility of $-1+\rho < 0$.
- Starting tomorrow yields a utility of $\frac{(-1+2\rho)}{2} > 0$.
- So he decides today to start tomorrow. Unfortunately, when tomorrow comes, it becomes today, and he decides again to start the next day. Thus, the optimal strategy is time inconsistent and this is due to the fact that the discount rate of the individual varies (the discount rate equals $\frac{\rho}{2}$ for the first period and ρ subsequently.)

By now, there is substantial evidence that people discount the future at a non-constant rate. More precisely, there is experimental evidence (see (Frederick *et al.*,

2002) for a review) that people are more sensitive to a given time delay if it occurs earlier: for instance, I might prefer to get two oranges in 21 days than one orange in 20 days, but also prefer to get one orange right now than two oranges tomorrow. This is known as **the common difference effect**, and would not occur if I discounted future utilities at a constant rate. Individual behaviour is best described by *hyperbolic discounting*, where the discount function is $h(t) = (1+at)^{-\frac{b}{a}}$, with $a, b > 0$. The corresponding discount rate is $\rho(t) = \frac{b}{1+at}$, which starts from $\rho(0) = b$ and decreases to zero. Because of its empirical support, hyperbolic discounting has received a lot of attention in the areas of: microeconomics, macroeconomics and behavioural finance.



(a) Hyperbolic discount rates $\rho(0,t) = \rho(t) = \frac{k_2}{1+k_1 t}$

(b) Discount function as a function of time $f(t) = (1+k_1 t)^{-\frac{k_2}{k_1}}$

Figure 1.1: The parameters are: $k_1 = 0.2, 5, 10$ and k_2 is chosen such that the discount function between time 0 and time t equals 0.3.

(Ebert *et al.*, 2017) introduce the notion of weighted discount functions. They show that in the presence of many agents, each discounting their utility at a different rate, the aggregate discount function a weighted average of the individual discount functions. If the agents have a continuous distribution, we obtain an integral and if the agents have a discrete distribution, we get a discrete sum. They also show that hyperbolic discounting and pseudo-exponential discounting are special cases of weighted discount functions. (Ebert *et al.*, 2017) states: *"greater group diversity results in a more elevated group discount function so that more diverse groups discount outcomes at any future time by less."*

When faced with non constant discount rates, a strategy that might be optimal at time 0 might not be so at a later time: time inconsistency bites in. In that situation, an agent has 2 choices:

1. He can commit to follow the strategy that is optimal at time 0 all the way to the horizon (final time) T. This strategy will be called pre commitment or optimal strategy.

2. He can allow an updating of his future preferences, and follow a time consistent strategy also known as a subgame perfect strategy.

Review literature on time consistency in financial economics within a deterministic setting.

We mention here, among other works, (Loewenstein and Prelec, 1992), (Laibson, 1997) and (Barro, 1999).

Time inconsistent behavior was first analyzed by (Strotz, 1955), and this line of research has been pursued by many others (see (Pollak, 1968), (Phelps and Pollak, 1968), Peleg and (Peleg and Yaari, 1973), (Goldman, 1980), (Laibson, 1997), (Barro, 1999), (Krusell and Smith, 2003)), mostly in the framework of planning a discrete-time economy with production (Ramsey's problem). More recently, the problem has been taken up again by (Karp, 2005), (Karp, 2007), (Karp, 2008), Karp and Lee (Karp and Lee, 2003). (Luttmer and Mariotti, 2003), (Ekeland and Lazrak, 2008), (Ekeland and Lazrak, 2010), always within the framework of economic growth planning.

It is by now well established that time-consistent strategies are Stackelberg equilibria of a leader-follower game among successive selves (today's self has divergent interests from tomorrow's). Consider an agent whose preferences vary with the time t. A subgame perfect strategy is constructed as follows: at each infinitesimal interval of time $[t, t + dt]$, there is an agent A_t that has the time t preferences of the agent. Agent A_t can only act between those two times and chooses an optimal strategy $\bar{\mathbf{u}}(t)$ assuming that the agents $A_s,\ s \geq t + dt$ follow the strategy $\bar{\mathbf{u}}(s)$. Thus, a subgame perfect strategy is a Stackelberg equilibria in which the decision maker only commits infinitesimally.

Review literature on time consistency in financial economics within a stochastic paradigm.

(Merton, 1969) studies the optimal investment/consumption problem over a finite horizon: the goal is to maximize expected utility of consumption and terminal wealth. Merton uses a constant (psychological) discount rate. (Ekeland and Pirvu, 2008) has been the first to have considered the Merton problem with non-constant discount rates, thus introducing time inconsistency to the Merton problem. (Ekeland *et al.*, 2012) considers the pensioner's problem where one agent is consuming, paying an insurance premium and in case of death before maturity, derives a final utility of bequest. A death rate modelling the probability of death is introduced. They consider the limiting case when the decision-maker can commit only during an infinitesimal amount of time. The leader-follower game among successive agents formulation is used to define subgame perfect strategies (one for each infinitesimal time interval $[t, t+dt]$ as detailed above). They showed the existence of subgame perfect strategies in the case of an investor who has a CRRA utility $U(c) = \frac{1}{\gamma} c^{\gamma}, \gamma < 1$ and a general discount function. They show that this paradigm could explain the consumption puzzle. The fraction of consumption to wealth in the economy is humped shaped instead of being monotonous as explained by the Merton problem. (Bjork *et al.*, 2014) looks at the mean variance problem which is also time inconsistent. The time inconsistency in the mean variance problem does not originate from non constant discount rates but it comes from the non linear expectation.

Khapko (Khapko, 2015) considers a one agent equilibrium problem in a Lucas type economy and studies the subgame perfect equilibrium interest rate, market price of risk and stock price parameters.

(Bjork *et al.*, 2016) extends the formulation of (Ekeland *et al.*, 2012) with a more general criterion. They show that every subgame perfect strategy is the optimal (pre commitment) strategy for another objective function.

Methodology Review.

Ekeland, Mbodji, Pirvu (Ekeland *et al.*, 2012) solves the Merton problem with non constant discount rates when the stock price follows a Geometric Brownian Motion (GBM). The subgame perfect strategies are characterized in terms of a certain "value function", which is shown to satisfy a Hamilton Jacobi Bellman equation with an additional non local integral term. Assuming utilities to be (CARA), they assume the value function $V(t,x)$ is of the form $a(t)U_\gamma(x)$ where x represents the wealth of the agent at time t. They decouple time and space and reduce the problem to solving a linear ODE on $a(t)$.

(Bjork *et al.*, 2016) considers the more general functional:

$$J(t,x,\mathbf{u}) = \mathbb{E}_{t,x}\left[\int_t^T C\big(t,x,X_s^{\mathbf{u}},\mathbf{u}(X_s^{\mathbf{u}})\big)ds + F(t,x,X_T^{\mathbf{u}})\right] + G(t,x,\mathbb{E}_{t,x}[X_T^{\mathbf{u}}]) \quad (1.1)$$

$\mathbf{u}$ is a well behaved control (consumption and investment), x is the value at time t of the diffusion X that satisfies the SDE

$$dX_s^{\mathbf{u}} = \mu(s,\mathbf{u}_s,X_s^{\mathbf{u}})ds + \sigma(s,\mathbf{u}_s,X_s^{\mathbf{u}})dW(s)\ ,\ s \geq t \quad (1.2)$$

and initial condition $X_t^{\mathbf{u}} = x$.

(Bjork *et al.*, 2016) show that there are three main sources of time inconsistency.

1. The integral term $C\big(t,x,X_s^{\mathbf{u}},\mathbf{u}(X_s^{\mathbf{u}})\big)$ is allowed to depend on the initial point (t,x).

2. F depends on t,x.

3. The function G depends on t,x and may also be non linear in the last argument $\mathbb{E}_{t,x}[X_T^{\mathbf{u}}]$.

The Merton problem with non constant discount rates as considered in (Ekeland *et al.*, 2012) falls in the first category. The mean variance problem as defined in

(Bjork *et al.*, 2014) is $\sup_\pi J(t,x,\pi)$ where

$$J(t,x,\pi) = \mathbb{E}_t[X_T^\pi] - \frac{\lambda}{2}(\mathbb{E}_t[(X_T^\pi)^2] - (\mathbb{E}_t X_T^\pi)^2)$$

for a certain $\lambda > 0$. The wealth process X^π satisfies the SDE

$$dX_t^\pi = X_t^\pi(r + \sigma\theta\pi(t))dt + \sigma\pi(t)X_t^\pi dW_t$$

where π is the fraction of wealth invested in the stock market and σ, θ, r are constants. It falls into the second and third category.

In (Bjork *et al.*, 2016) and (Ekeland *et al.*, 2012), the value function is shown to satisfy a Hamilton Jacobi Bellman (HJB) equation with a non-local term called extended HJB. It can be shown that the existence of a sufficiently regular value function V satisfying the extended HJB implies the existence of a subgame perfect strategy : the consumption to wealth ratio and the investment to wealth ratio can be expressed in terms of V and its first and second spatial derivatives. The difficulty, of course, is to prove that time-consistent strategies exist or equivalently solve the extended HJB for the value function. (Yong, 2012) considers the well posedness of the extended HJB when the diffusion term σ does not depend on the control $\mathbf{u}$ and shows the existence of a subgame perfect (time consistent) strategy.

In the next chapter, we will formally define subgame perfect strategies.

Chapter 2

Subgame Perfect Strategies

2.1 Introduction

The investment/consumption problem in a stochastic context was considered by (Merton, 1969) and (Merton, 1971). Merton's model consists in a risk-free asset with constant rate of return and one or more stocks, the prices of which are driven by a geometric Brownian motion. The horizon T is prescribed, the portfolio is self-financing, and the investor seeks to maximize the expected utility of inter-temporal consumption and of final wealth. Merton provides a closed form solution when the utilities are of constant relative risk aversion (CRRA) or constant absolute risk aversion (CARA) type. It turns out that for (CRRA) type utilities, the fraction of wealth invested in the risky asset is constant through time.

(Samuelson, 1937) was the first to introduce the model of constant discounted utility. This model has been widely accepted as a normative and descriptive model of intertemporal choice.

The aim of this chapter is to revisit these problems in the case when the psychological time discount rate is not constant.

Our motivation and methodology.

In this chapter, we extend the approach pioneered by (Ekeland *et al.*, 2012) to allow for a stock price with stochastic volatility. A special case considered is the constant elasticity of variance (CEV).

Merton, in his pioneering article (Merton, 1971) (constant discount rate) shows that the investor maximizing its utility chooses to invest a constant portion of its wealth equal to the Sharpe ratio divided by the relative risk aversion.

Surprisingly, when the stock price follows a GBM as in (Ekeland *et al.*, 2012), the subgame perfect investment to wealth ratio is the same as the one found by Merton.

The CRRA form of the utility function coupled with the GBM model for the stock price introduces a myopic behaviour for the investment strategy. In that case, the investment-wealth ratio for the subgame perfect agent is the same as the one for the precommitment agent; it is also independent of the discount function.

In this chapter, we have introduced stochastic volatility in the stock dynamics to study the difference in the investment strategies between a subgame perfect agent and an optimal agent (that pre commits). Thus, we allow the interest rate, the stock's drift and volatility to depend on the current stock price.

The methodology developed in (Ekeland *et al.*, 2012) is employed to characterize the subgame perfect strategies (also known as the subgame perfect strategies) through the value function approach. The value function is characterized by an integral equation with a nonlocal term; given the special form of the utility function (which is of power type), an ansatz is provided for the value function.

The novelty in this chapter is *the utility weighted discount rate* $Q_t^{\pi,c}$ induced by (π, c) : it depends on the control strategy, so it is a random (time and state dependent) quantity. In the case of exponential discounting it equals the constant (psychological) discount rate. We decouple on one hand time and space, and wealth on the other hand and show that the utility weighted discount rate induced by the subgame perfect strategy $(\bar{\pi}, \bar{c})$ can be found independently of the subgame perfect strategy. It turns out to be the fixed point of some operator. We reduce the extended HJB of the value function to a linear PDE. Moreover, our methodology is amenable to numerical treatments so one can visualize the subgame perfect strategies arising from different choices of discounting, interest rate and stock price models. The numerical scheme we employ is based on a fixed point that is obtained by iterating a contraction map. This technique will give us a general method of finding subgame perfect strategies when the interest rate, the market price of risk and the stock volatility are well behaved e.g adapted to the Brownian filtration, bounded and sufficiently regular.

The novel findings in this chapter are:

1. We prove the existence of a solution to the extended HJB which was an open question in (Bjork *et al.*, 2016). Our fixed point methodology could be extended to more general utility functions that are not of the power type.

2. The subgame perfect strategy equals the optimal (pre commitment) strategy for the optimization criterion in which the utility weighted discount rate (induced by the subgame perfect strategy) is used as discount rate.

3. The utility weighted discount rate (induced by the subgame perfect strategy) equals the fixed point of a certain operator that is independent of the strategies.

4. We have found two ways to identify what type of strategy a certain investor uses. When compared to the economic predictions for the agent that is following an optimal strategy, the subgame perfect agent has:

   - a higher consumption in the short term.
   - a lower consumption in the medium to long term.

   These are two indications of an agent that is following a subgame perfect strategy.

Organization of the chapter: The remainder of this chapter is organized as follows. In section 2.2, we describe the model and formulate the objective. Section 2.3 introduces the notion of subgame perfect strategies. Section 2.4 introduces the value function. Section 2.5 presents the main result. Section 2.6 deals with the utility weighted discount rate and Section 2.7 compares pre commitment optimal strategies to sub game perfect strategies . We then wrap up our findings in the Conclusion. Appendix 1 contains various proofs.

2.2 The Model

2.2.1 The Financial Market

Consider a financial market consisting of a savings account and one stock (the risky asset). The inclusion of more risky assets can be achieved by notational changes. We assume a benchmark deterministic time horizon T. The stock price per share follows an exponential Brownian motion

$$dS_t = S_t \left[\mu_S(t, S_t)\, dt + \sigma_S(t, S_t)\, dW(t)\right], \quad 0 \leq t \leq T \tag{2.1}$$

where $\{W(t)\}_{t\geq 0}$ is a $1-$dimensional Brownian motion on a filtered probability space, $(\Omega, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$. The filtration $\{\mathcal{F}_t\}$ is the completed filtration generated by $\{W(t)\}$. The savings account accrues interest at the riskless rate $r(t, S_t)$.

Let us denote

$$\theta_S \triangleq \frac{\mu_S - r}{\sigma_S} \tag{2.2}$$

the market price of risk.

We place ourselves in a Markovian setting. The stock mean rate of return μ_S and volatility σ_S will be functions of the running time t and the stock price $S(t)$. We make the following assumption on r, θ_S, σ_S:

Assumption 2.1 (Standing Assumption). Suppose that

1. $\sigma_S, r, \theta_S, S\frac{\partial r}{\partial S}, S\frac{\partial \theta_S}{\partial S}$ are bounded in $[0,T] \times (0,\infty)$ and uniformly Lipschitz continuous in (t,S) in compact subsets of $[0,T] \times (0,\infty)$.

2. $t \mapsto \sigma_S(t,S)$ is uniformly continuous with respect to $(t,S) \in [0,T] \times (0,\infty)$, $S \mapsto S\frac{\partial \sigma_S}{\partial S}$ is uniformly bounded and continuous with respect to (t,S) in $[0,T] \times (0,\infty)$. And there is a positive constant σ_0 such that $\sigma_S \geq \sigma_0$.

There is one agent who is continuously investing in the stock, is using the money market, and consuming. At every time t, the agent chooses $\pi(t)$, the ratio of wealth invested in the risky asset and $c(t)$ the ratio of wealth consumed. Given an adapted process $\{\pi(t), c(t)\}_{0 \leq t \leq T}$, the equation describing the dynamics of wealth $X^{\pi,c}(t)$ is given by :

$$\begin{aligned} dX^{\pi,c}(t) &= [r(t) - c(t) + \sigma_S(t)\theta_S(t)\pi(t)]X^{\pi,c}(t)dt + \pi(t)\sigma_S(t)X^{\pi,c}(t)dW_t \quad (2.3) \\ X^{\pi,c}(0) &= x_0 \end{aligned}$$

the initial wealth x_0 being exogenously specified.

2.2.2 Time preferences and risk preferences

As seen in the introduction, the time preference reflects the economic agent's preference for immediate utility over delayed utility. We now define discount functions and discount rates.

Definition 2.2. A discount function $h : D = \{0 \leq t \leq s \leq T\} \to \mathbb{R}$ is a C^1, positive function satisfying $h(t,t) = 1$.

For a discount function h, we define the backward discount rate as

$$\rho_b(t,s) = \frac{\partial h(t,s)}{\partial t} \times \frac{1}{h(t,s)} \tag{2.4}$$

The forward discount rate is

$$\rho_f(t,s) = -\frac{\partial h(t,s)}{\partial s} \times \frac{1}{h(t,s)} \tag{2.5}$$

In the case $h(t,s) = H(s-t)$ for a certain C^1 function H on $[0,T]$, we get: $\rho_h(t,s) = \rho_b(t,s) = -\frac{H'(s-t)}{H(s-t)}$. If it is obvious from the context, we just write $\rho(t,s)$ and call it the discount rate. For any continuous function $y : D \to (0,\infty)$, denote $||y|| := \sup_{(t,s)\in D} |y(t,s)|$ the norm sup of y.

Remark 2.3. We take a discount form to be of the general form $h(t,s)$ because, as noted in (Pirvu and Zhang, 2014) and (Ekeland *et al.*, 2012), we have to account for stochastic time horizons T which could be the time of death of the agent. In that case, the discount function has to be transformed and will take the general form $h(t,s)$. We can normalize by dividing $h(t,s)$ by $h(t,t)$.

We define next the agent's *risk preferences*. An economic agent will have satisfaction $U(C)$ from consuming an amount C. We assume U belongs to the class of constant relative risk aversion (CRRA) utilities.

$$U(x) = U_\gamma(x) = \frac{x^\gamma}{\gamma}, \gamma < 1, \gamma \neq 0, x > 0 \tag{2.6}$$

2.2.3 The intertemporal utility

Let us now define the admissible strategies.

Definition 2.4. An $\mathbb{R}^2$-valued stochastic process $\{\pi(t), c(t)\}_{0\leq t\leq T}$ is called an admissible strategy process if:

1. it is progressively measurable with respect to the sigma algebra $\sigma(\{W(t)\}_{0\leq t\leq T})$,
2. it satisfies
$$c(t) \geq 0 \ \text{ for all } t \text{ almost surely and } X^{\pi,c}(T) \geq 0, \text{ almost surely} \tag{2.7}$$
3. moreover, we require that
$$\mathbb{E} \sup_{0\leq s\leq T} |c(s)X^{\pi,c}(s)|^{\gamma} < \infty \ \ , \ \ \mathbb{E} \sup_{0\leq s\leq T} |X^{\pi,c}(s)|^{\gamma} < \infty, \ \text{ a.s.} \tag{2.8}$$

The last set of inequalities are purely technical and are satisfied for e.g. bounded strategies.

In order to evaluate the performance of an investment-consumption strategy the couple uses an expected utility criterion. For an admissible strategy process $\{\pi(s), c(s)\}_{s\geq 0}$ and its corresponding wealth process $\{X^{\pi,c}(s)\}_{s\geq 0}$, we denote the intertemporal utility by

$$\begin{aligned} J(t,S,x,\pi,c) \ &= \ \mathbb{E}\bigg[\int_t^T h(t,s)U_\gamma(c(s)X^{\pi,c}(s))\,ds + h(t,T)U_\gamma(X^{\pi,c}(T)) \\ &\qquad \bigg| \ S(t) = S; X^{\pi,c}(t) = x\bigg] \end{aligned} \tag{2.9}$$

A natural objective for the decision maker is to maximize the above expected utility criterion. If we define $\hat{V}(t_0, t, S, x)$ as the optimal value function starting at time t_0,

$$\hat{V}(t_0; t_0, S, x) = \sup_{(\pi,c)\ admissible} J(t_0, S, x, \pi, c) \tag{2.10}$$

Then for $t \geq t_0 : \hat{V}(t_0; t, S, x)$ satisfies the HJB

$$\frac{\partial \hat{V}}{\partial t} + \sup_{(\pi,c)\text{ admissible}} \left[\mathcal{A}^{\pi,c}\hat{V} + U_\gamma(xc(t)) \right] - \rho_h(t_0, t)\hat{V}(t_0; t, S, x) = 0 \tag{2.11}$$

The derivation of (2.11) is standard.

The optimal strategy $(\hat{\pi}, \hat{c})$ is the one realizing the sup in (2.11). However, because the discount function h is not exponential, the discount rate $\rho_h(t_0, t) = -\frac{\frac{\partial h(t_0,t)}{\partial s}}{h(t_0,t)}$ is not constant. Therefore, the solution $\hat{V}$ of the HJB (2.11) depends on the starting point t_0 and so does the optimal strategy starting at t_0. Therefore, every time we change the starting point t_0, we get a different strategy. Time inconsistency will bite, that is, a strategy that will be considered optimal at time 0 will not be considered so at later times, so it will not be implemented unless the decision-maker at time 0 can constrain his successive selves to follow the time 0 - optimal strategy (the precommitment strategy) at all times $0 \leq t \leq T$.

The decision-maker could implement two types of strategies. He could *precommit* at time $t_0 = 0$ to follow the optimal strategy and stay with it until time T. Or he could implement a time consistent strategy that takes into account the fact that the decision maker's preferences will change in the future. This is the object of the next section.

2.3 Subgame perfect strategies

We now introduce a special class of time consistent strategies, which can also be called a *subgame perfect strategy.* That is, we consider that the decision-maker at time t can commit his successors up to time $t+\epsilon$, with $\epsilon \to 0$, and we seek strategies which are optimal to implement right now conditioned on them being implemented in the future. This is made precise in the following formal definition.

Definition 2.5. An admissible trading strategy $\{\bar{\pi}(s), \bar{c}(s)\}_{0\leq s\leq T}$ is a *subgame perfect strategy* if there exists a map $G = (G_\pi, G_c) : [0,T] \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \times [0,\infty) \times \mathbb{R}$ such that for any $t \in [0,T], S, x > 0$

$$\liminf_{\epsilon\downarrow 0} \frac{J(t,S,x,\bar{\pi},\bar{c}) - J(t,S,x,\pi_\epsilon,c_\epsilon)}{\epsilon} \geq 0, \tag{2.12}$$

where:

$$\bar{\pi}(s) = G_\pi(s, S(s), \bar{X}(s)), \quad \bar{c}(s) = G_c(s, S(s), \bar{X}(s)) \tag{2.13}$$

and the wealth process $\bar{X}(s) := X^{\bar{\pi},\bar{c}}(s)$ is a solution of the stochastic differential equation (SDE):

$$\begin{aligned} d\bar{X}(s) &= \bar{X}(s)[r(s) + \sigma_S(s)\theta_S(s)G_\pi(s,S(s),\bar{X}(s)) - G_c(s,S(s),\bar{X}(s))]ds \\ &\quad +\sigma_S(s)G_\pi(s,S(s),\bar{X}(s))\bar{X}(s)dW(s) \end{aligned} \tag{2.14}$$

The process $\{\pi_\epsilon(s), c_\epsilon(s)\}_{s\in[t,T]}$ mentioned above is another investment-consumption strategy defined by

$$\pi_\epsilon(s) = \begin{cases} G_\pi(s, S(s), X_\epsilon(s)), & s \in [t,T]\backslash E_{\epsilon,t} \\ \pi(s), & s \in E_{\epsilon,t}, \end{cases} \tag{2.15}$$

$$c_\epsilon(s) = \begin{cases} G_c(s, S(s), X_\epsilon(s)), & s \in [t,T]\backslash E_{\epsilon,t} \\ c(s), & s \in E_{\epsilon,t}, \end{cases} \tag{2.16}$$

with $E_{\epsilon,t} = [t, t+\epsilon]$, and $\{\pi(s), c(s)\}_{s\in E_{\epsilon,t}}$ is any strategy for which $\{\pi_\epsilon(s), c_\epsilon(s)\}_{s\in[t,T]}$ is an admissible policy. X_ϵ is defined on $[t+\epsilon, T]$ by the SDE:

$$\begin{aligned} dX_\epsilon(s) &= [r(s) - c_\epsilon(s) + \sigma_S(s)\theta_S(s)\pi_\epsilon(s)]X_\epsilon(s)ds + \pi_\epsilon(s)\sigma_S(s)X_\epsilon(s)dW(s) \\ X_\epsilon(t+\epsilon) &= X^{\pi,c}(t+\epsilon). \end{aligned} \tag{2.17}$$

In other words, time consistent strategies are Markov strategies that penalize deviations during an infinitesimally small time interval. Note that:

- this does not mean that unilateral deviations during a finite interval of time are penalized as well: it is possible that deviating from the policy between t_1 and t_2 will be to the advantage of all the decision-makers operating between t_1 and t_2.

- however, in the absence of a firm commitment, if a Markov strategy is not a time consistent strategy, then it certainly will not be implemented, for at some point, a lone decision-maker will deviate, thereby compromising all the plans laid by his predecessors.

So time-consistency in the sense of Definition 2.5 is a minimal requirement for rationality: subgame perfect strategies are the only Markov strategies that are likely to be implemented.

2.4 The Value Function for subgame perfect strategies

Dynamic programming is a very convenient way of writing a large set of dynamic problems in financial economics. Most properties of this tool are well established and understood. In dynamic programming, we introduce an object called the *value function* that is obtained by evaluating a certain functional at our candidate solutions. The solutions of the dynamic programming problem are then the solutions of a certain equation called HJB and can be expressed entirely in terms of the value function and its derivatives. In optimization problems, the value function is the optimal value an agent can derive from his maximization process. The paper (Ekeland *et al.*, 2012) uses the value function methodology to characterize subgame perfect strategies. We will see that the value function can be written as a function $V(t, S, x)$ of time t, stock price S and wealth x and this allows us to find subgame perfect strategies in a feedback form. For fixed t, S, x, the strategy $(\bar{\pi}, \bar{c})$ can be expressed as deterministic functions of V and its derivatives with respect to S and x. We start with a definition.

Definition 2.6. Let $V : [0, T] \times (0, \infty)^2 \to \mathbb{R}, (t, S, x) \mapsto V(t, S, x)$ be a $C^{1,2,2}$ function that is concave in the x variable. Suppose that $\{\bar{\pi}(s), \bar{c}(s)\}_{s \in [0,T]}$ are subgame perfect strategies with the corresponding map

$$\bar{\pi}(s) = G_\pi(s, S(s), \bar{X}(s)) \quad , \quad \bar{c}(s) = G_c(s, S(s), \bar{X}(s)), \tag{2.18}$$

where

$$G_\pi(t,S,x) = -\frac{\theta_S(t,S)\frac{\partial V}{\partial x}(t,S,x) + S\sigma_S(t,S)\frac{\partial^2 V}{\partial S \partial x}}{x\sigma_S(t,S)\frac{\partial^2 V}{\partial x^2}(t,S,x)} \quad , \quad G_c(t,S,x) = \frac{1}{x}\left(\frac{\partial V}{\partial x}(t,S,x)\right)^{\frac{1}{\gamma-1}} \tag{2.19}$$

and $\bar{X}(s)$ is the wealth process given by:

$$\begin{aligned} d\bar{X}(s) &= [r(s) + \sigma_S(s)\theta_S(s)G_\pi(s, S(s), \bar{X}(s)) - G_c(s, S(s), \bar{X}(s))]\bar{X}(s)ds \\ &\quad + \sigma_S(s)G_\pi(s, S(s), \bar{X}(s))\bar{X}(s)dW(s). \end{aligned} \tag{2.20}$$

We shall say that V is a value function if for all $(t, S, x) \in [0, T] \times (0, \infty)^2$, we have:

$$V(t, S, x) = J(t, S, x, \bar{\pi}, \bar{c}) \tag{2.21}$$

The economic interpretation is very natural: if one applies the Markov strategy associated with V by the relations [(2.18), (2.19), (2.20), (2.21)] and computes the corresponding value of the investor's criterion starting from $S_t = S,\ X_t = x$ at time t, one gets precisely $V(t, S, x)$. In other words, this is fundamentally a fixed-point characterization. However, it is mathematically quite complicated, so we will take advantage of the special form of the utility function to put the problem in a simpler light. We begin with the definition of the infinitesimal generator.

Definition 2.7. For an admissible policy (π, c) with corresponding wealth process $X^{\pi,c}$ and $(t, S, x) \mapsto v(t, S, x)$ a continuous function of 3 variables C^1 in t and C^2 in

S, x, define the operator $\mathcal{A}^{\pi,c}$ by:

$$\mathcal{A}^{\pi,c}v(t,S,x) = (r + \sigma_S\theta_S\pi - c)x\frac{\partial v}{\partial x}(t,S,x) + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 v}{\partial S^2}(t,S,x) +$$

$$\frac{1}{2}\sigma_S^2 x^2\pi^2\frac{\partial^2 v}{\partial x^2}(t,S,x) + \sigma_S^2 S\pi x\frac{\partial^2 v}{\partial S\partial x}(t,S,x) + S\mu_S\frac{\partial v}{\partial S}(t,S,x) \tag{2.22}$$

Basically, by Ito's lemma, we see that (2.22) is saying that

$$\frac{\partial v(t,S,x)}{\partial t} + \mathcal{A}^{\pi,c}v(t,S,x) = \frac{d}{dt}\mathbb{E}_t[dv(t,S_t,X^{\pi,c}(t))]$$

Thus, $\mathcal{A}^{\pi,c}v(t,S,x)$ measures the average variation of v when an infinitesimal time dt passes and the agent follows the strategy (π, c).

In the next section, we give the main results of this chapter.

2.5 Main Results

2.5.1 The extended HJB

Theorem 2.8 (Extended HJB)**.** *Let $V : [0,T] \times \mathbb{R} \times (0,\infty) \to \mathbb{R}$ be a $C^{1,2,2}$ function.*

Suppose $(\bar{\pi}, \bar{c})$ is an admissible Markovian policy and that

- *V solves the extended Hamilton Jacobi Bellman equation :*

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,S,x) + \sup_{(\pi,c)\ admissible}\left\{\mathcal{A}^{\pi,c}V(t,S,x) + U_\gamma(xc(t))\right\} \\ = \ &\mathbb{E}_t\left[\int_t^T \frac{\partial h(t,s)}{\partial t}U_\gamma(\bar{c}(s)X^{\bar{\pi},\bar{c}}(s))ds + \frac{\partial h(t,T)}{\partial t}U_\gamma(X^{\bar{\pi},\bar{c}}(T))\right] \end{aligned} \tag{2.23}$$

along with the boundary condition $V(T,S,x) = U_\gamma(x)$.

- $(\bar{\pi},\bar{c})$ *satisfies:*

$$(\bar{\pi},\bar{c}) = \arg\max\{\mathcal{A}^{\pi,c}V(t,S,x) + U_\gamma(xc(t)); (\pi,c) \quad \textit{admissible} \ \} \tag{2.24}$$

Then V *is a value function. Moreover* $(\bar{\pi},\bar{c})$ *given by* (G_π, G_c) *of* (2.21) *is a subgame perfect strategy (conform Definition 2.5).*

The proof of Theorem 2.8 will be given in Appendix 1. The following proposition gives the subgame perfect strategies in terms of the value function.

Proposition 2.9. *If the extended HJB* (2.23) *has a* $C^{1,2,2}$ *solution* V*, then the subgame perfect strategies are given by:*

$$\bar{c} \ = \ G_c(t,S,x) = \frac{V_x^{\frac{1}{\gamma-1}}}{x} \tag{2.25}$$

$$\bar{\pi} \ = \ G_\pi(t,S,x) = -\frac{\theta_S V_x + \sigma_S S V_{Sx}}{\sigma_S x V_{xx}} \tag{2.26}$$

The proof comes from a simple calculation of the first order conditions for the quantity

$$(\bar{\pi},\bar{c}) = \arg\max_{(\pi,c)\ admissible} \{\mathcal{A}^{\pi,c}V + U_\gamma(xc)\}$$

and will be detailed in Appendix 1. Next, we define a strategy dependent discount rate that we call utility weighted discount rate.

2.5.2 The utility weighted discount rate

Definition 2.10 (Utility Weighted Discount Rate)**.** Let (π, c) be an admissible strategy. The utility weighted discount rate corresponding to (π, c) is defined as the process

$$Q_t^{\pi,c} = \frac{\mathbb{E}_t^{\mathbb{P}} \int_t^T \frac{\partial h(t,s)}{\partial t} U_\gamma(c(s)X^{\pi,c}(s))ds + \frac{\partial h(t,T)}{\partial t} U_\gamma(X_T^{\pi,c})}{\mathbb{E}_t^{\mathbb{P}} \int_t^T h(t,s)U_\gamma(c(s)X^{\pi,c}(s))ds + h(t,T)U_\gamma(X_T^{\pi,c})} \tag{2.27}$$

If $(\pi, c) = (\bar{\pi}, \bar{c})$ then $Q_t^{\bar{\pi},\bar{c}}$ is called the subgame perfect utility weighted discount rate. When the context is clear, $Q_t^{\bar{\pi},\bar{c}}$ will just be called utility weighted discount rate.

Remark 2.11*.* The intuition behind $Q_t^{\pi,c}$ is as follows:

the right hand side of the extended HJB (2.23) is $Q_t^{\bar{\pi},\bar{c}} \times V(t, S, x)$. Our goal is to compute $Q_t^{\bar{\pi},\bar{c}}$. If we do so, we will be able to solve the extended HJB as a usual HJB. In the exponential discounting case $h(t, s) = \exp(-\rho(s - t))$, the quantity $Q_t^{\pi,c}$ simplifies to

$$Q_t^{\pi,c} = \frac{\mathbb{E}_t[\int_t^T \rho e^{-\rho(s-t)} U(c_s X_s^{\pi,c})ds + \rho e^{-\rho(T-t)} U(X_T^{\pi,c})]]}{\mathbb{E}_t[\int_t^T e^{-\rho(s-t)} U(c_s X_s^{\pi,c})ds + e^{-\rho(T-t)} U(X_T^{\pi,c})]} = \rho.$$

In general, $Q_t^{\pi,c}$ behaves like an average discount rate thus the name utility weighted discount rate. In what follows, we will only be concerned with the quantity $Q_t^{\bar{\pi},\bar{c}}$ (when $\pi = \bar{\pi}$ and $c = \bar{c}$).

We have found a way to decouple the extended HJB. We will show that there is a function $\mathbb{Q}(t, S)$ such that $Q_t^{\bar{\pi},\bar{c}} = \mathbb{Q}(t, S_t)$. We can calculate $\mathbb{Q}$ as the fixed point of a certain operator that depends only on the parameters of the model : $\gamma, h, T, \theta_S, r, \sigma_S$. Thus, $\mathbb{Q}(t, S)$ can be completely determined without calculating $\bar{\pi}, \bar{c}$. Knowing $\mathbb{Q}$ is equivalent to knowing the subgame perfect strategy $(\bar{\pi}, \bar{c})$. We show the two implications in what follows.

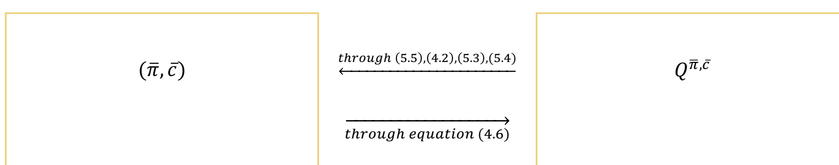
First implication Suppose we know $Q_t^{\bar{\pi},\bar{c}} = \mathbb{Q}(t, S_t)$. We see it by rewriting the extended HJB (2.23) as

$$\frac{\partial V}{\partial t} + \sup_{(\pi,c)\text{ admissible}} \left\{ \mathcal{A}^{\pi,c} V(t,S,x) + U_\gamma(xc(t)) \right\} = Q_t^{\bar{\pi},\bar{c}} V(t,S,x) \tag{2.28}$$

$$\frac{\partial V}{\partial t} + \sup_{(\pi,c)\text{ admissible}} \left\{ \mathcal{A}^{\pi,c} V(t,S,x) + U_\gamma(xc(t)) \right\} = \mathbb{Q}(t,S) V(t,S,x) \tag{2.29}$$

We obtain a classical HJB that can be solved through the standard techniques. Having found V, we use equation (2.26), (2.26). to calculate $(\bar{\pi}, \bar{c})$.

Second implication Suppose we know $(\bar{\pi}, \bar{c})$. Then we can calculate $Q_t^{\bar{\pi},\bar{c}}$ by using the equation (2.27). Due to the special form of the utility function $U_\gamma(x)$, we can see after some calculations that $Q_t^{\bar{\pi},\bar{c}}$ is a function of t, S and is independent of x. The following diagram illustrates the point:



One to one correspondence between $(\bar{\pi}, \bar{c})$ and $Q^{\bar{\pi},\bar{c}}$.

Figure 2.2: range of $\mathbb{Q}(t, S)$

Define p the inverse of the relative risk aversion:

$$p := \frac{1}{1-\gamma} \tag{2.30}$$

In what follows, unless we specify otherwise, we will write

$$\rho = \rho_b : (t,s) \in D \to \mathbb{R}, (t,s) \mapsto \frac{\frac{\partial h(t,s)}{\partial t}}{h(t,s)}. \tag{2.31}$$

Next, we define a space of functions in which we want to find the fixed point $\mathbb{Q}(t,S)$.

Definition 2.12. For $\delta > 0$, let $\mathbb{B}_\delta$ be the space of functions $y : [0,T] \times (0,\infty) \to \mathbb{R}$ such that $(t,S) \mapsto y(t,S)$ and $(t,S) \mapsto \frac{\partial y(t,S)}{\partial S}$ are continuous and

for all $(t,S) \in [0,T] \times (0,\infty)$, $\quad |y(t,S)| \leq ||\rho||$ and $S\left|\frac{\partial y(t,S)}{\partial S}\right| \leq \delta$.

For $y \in \mathbb{B}_\delta$, define

$$||y||_{C([0,T];C^1(0,\infty))} := \sup_{(t,S)\in[0,T]\times(0,\infty)} |y(t,S)| + \sup_{(t,S)\in[0,T]\times(0,\infty)} S\left|\frac{\partial y(t,S)}{\partial S}\right| \tag{2.32}$$

With this structure, $(\mathbb{B}_\delta, ||\quad||_{C([0,T];C^1(0,\infty))})$ is a complete set. This allows us to define an operator F on the elements of $\mathbb{B}_\delta$. By solving the fixed point problem $F[y] = y$, we will get the utility weighted discount rate and this will allow us to solve the extended HJB (2.23).

Definition 2.13. For y an element of $\mathbb{B}_\delta$, define the operators:

$$F_1[y](t,S) = \mathbb{E}_t^{\mathbb{P}} \int_t^T \frac{\partial h(t,s)}{\partial t} \exp\left(p\gamma \int_t^s (r + \frac{\theta_S^2}{2} - y)du + p\gamma \int_t^s \theta_S dW_u\right) ds$$
$$+ \frac{\partial h(t,T)}{\partial t} \exp\left(p\gamma \int_t^T (r + \frac{\theta_S^2}{2} - y)du + p\gamma \int_t^T \theta_S dW_u\right) \tag{2.33}$$

$$F_0[y](t,S) = \mathbb{E}_t^{\mathbb{P}} \int_t^T h(t,s) \exp\left(p\gamma \int_t^s (r + \frac{\theta_S^2}{2} - y)du + p\gamma \int_t^s \theta_S dW_u\right) ds$$
$$+ h(t,T) \exp\left(p\gamma \int_t^T (r + \frac{\theta_S^2}{2} - y)du + p\gamma \int_t^T \theta_S dW_u\right) \tag{2.34}$$

and

$$F[y](t,S) = \frac{F_1[y](t,S)}{F_0[y](t,S)} \tag{2.35}$$

In order to have existence and uniqueness results for the extended HJB (2.23).

Theorem 2.14. *There exists $\delta > 0$ that depends only on $\gamma, T, h, r, \theta_S, \sigma_S$ such that $F(\mathbb{B}_\delta) \subset \mathbb{B}_\delta$. Furthermore, we can choose δ such that F has a unique fixed point $\mathbb{Q} \in \mathbb{B}_\delta$ i.e.*

$$F[\mathbb{Q}](t,S) = \mathbb{Q}(t,S) \text{ for all } (t,S) \in [0,T] \times (0,\infty). \tag{2.36}$$

The proof will be given in the Appendix 1.

Having defined $\mathbb{Q}$, we can obtain a characterization of V through a linear PDE. This is the object of the next subsection.

2.5.3 Characterization of the value function through a linear PDE

Proposition 2.15. *There exists a unique bounded solution v to the linear parabolic PDE :*

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(t,S) + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 v(t,S)}{\partial S^2} + p\left[\gamma r + \frac{\gamma p \theta_S^2}{2} - \mathbb{Q}\right] v(t,S) \\ &\quad + (r + p\sigma_S\theta_S) S \frac{\partial v(t,S)}{\partial S} + 1 \qquad (2.37) \\ v(T,S) &= 1 \end{aligned}$$

$\mathbb{Q}$ is the unique fixed point of F as previously defined and is an element of $\mathbb{B}_\delta$. The function v is in $C^{1,2}([0,T]\times(0,\infty))$ and $S\frac{\partial v(t,S)}{\partial S}$ is also bounded. Moreover, v has the stochastic representation:

$$v(t,S) = \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T e^{\int_t^s p(\gamma r + \frac{\gamma p \theta_S^2}{2} - \mathbb{Q})(u,\bar{S}_u)du} ds + e^{\int_t^T p(\gamma r + \frac{\gamma p \theta_S^2}{2} - \mathbb{Q})(u,\bar{S}_u)du} | \bar{S}_t = S\right] \qquad (2.38)$$

where $\bar{S}_u$ satisfies the SDE

$$\bar{S}_u = S + \int_t^u (r + p\sigma_S\theta_S(v,\bar{S}_v))\bar{S}_v dv + \int_t^u \sigma_S(v,\bar{S}_v)\bar{S}_v dW(v) \qquad (2.39)$$

We can rewrite the PDE (2.37) in a form that will remind us of the heat equation with non constant diffusion coefficient. By changing variables $S = e^z$, we get $v(t,S) = \tilde{v}(t,z)$ where

$$\frac{\partial v}{\partial S}(t,S) = \frac{1}{S}\frac{\partial \tilde{v}}{\partial z} \quad , \quad \frac{\partial^2 v}{\partial S^2}(t,S) = -\frac{1}{S^2}\frac{\partial \tilde{v}}{\partial z} + \frac{1}{S^2}\frac{\partial^2 \tilde{v}}{\partial z^2}$$

Write $\tilde{\mathbb{Q}}(t,z) := \mathbb{Q}(t,e^z)$ and define $\tilde{r}, \tilde{\theta_S}, \tilde{\sigma_S}$ similarly. The PDE (2.34) can be rewritten as :

$$\begin{aligned} 0 \quad = \quad & \frac{\partial \tilde{v}}{\partial t}(t,z) + \frac{\tilde{\sigma_S}^2}{2}\Big(\frac{\partial^2 \tilde{v}(t,z)}{\partial z^2} - \frac{\partial \tilde{v}(t,z)}{\partial z}\Big) + p\left[\gamma \tilde{r} + \frac{\gamma p \tilde{\theta_S}^2}{2} - \tilde{\mathbb{Q}}\right]\tilde{v}(t,z) \\ & + (\tilde{r} + p\tilde{\sigma_S}\tilde{\theta_S})\frac{\partial \tilde{v}(t,z)}{\partial z} + 1 \quad , \quad \tilde{v}(T,z) = 1 \end{aligned}$$

This leads to the following proposition:

Proposition 2.16. *Define $\tilde{v}(t,z) = v(t,e^z)$ for $z \in \mathbb{R}$. $\tilde{v}$ satisfies the PDE:*

$$\begin{aligned} & \frac{\partial \tilde{v}}{\partial t}(t,z) + \frac{\tilde{\sigma_S}^2}{2}\frac{\partial^2 \tilde{v}(t,z)}{\partial z^2} + \big(\tilde{r} + p\tilde{\sigma_S}\tilde{\theta_S} - \frac{\tilde{\sigma_S}^2}{2}\big)\frac{\partial \tilde{v}(t,z)}{\partial z} \\ & + p\left[\gamma \tilde{r} + \frac{\gamma p \tilde{\theta_S}^2}{2} - \tilde{\mathbb{Q}}\right]\tilde{v}(t,z) + 1 = 0 \\ & \tilde{v}(T,z) = 1 \end{aligned} \tag{2.40}$$

For the proof of proposition 2.15, it suffices to apply Friedman (Friedman (1975) , Chapter 6, Theorem 4.6) to the non degenerate linear parabolic PDE of $\tilde{v}$. From $\tilde{v}$ we get v. We now present the most important result of this chapter.

Theorem 2.17. *The following holds:*

$$\forall\ t \in [0,T], \quad \mathbb{Q}(t,S_t) = Q_t^{\bar{\pi},\bar{c}}. \tag{2.41}$$

- *The extended HJB equation (2.23) has a unique $C^{1,2,2}$ solution V which is a* *value function of the form* $V(t,S,x) = v(t,S)^{1-\gamma}\frac{x^\gamma}{\gamma}$.

- v *is the unique bounded solution of the semi linear PDE*

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(t,S) + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 v(t,S)}{\partial S^2} + p\left[\gamma r + \frac{\gamma p \theta_S^2}{2} - \mathbb{Q}(t,S)\right] v(t,S) \qquad (2.42) \\ &+ (r + p\sigma_S\theta_S) S \frac{\partial v(t,S)}{\partial S} + 1, \qquad v(T,S) = 1 \end{aligned}$$

The wealth process $\bar{X}_s$, *consumption-wealth ratio* $\bar{c}(s)$, *investment-wealth ratio* $\bar{\pi}(s)$ *are given by:*

$$d\bar{X}(s) = (r(s) - \bar{c}(s) + \sigma_S(s)\bar{\pi}(s)\theta_S(s))\bar{X}(s)ds + \sigma_S(s)\bar{\pi}(s)\bar{X}(s)dW(s) \quad (2.43)$$

$$\bar{c}(t,S) = \frac{1}{v(t,S)} \quad , \quad \bar{\pi}(t,S) = \frac{\theta_S}{\sigma_S(1-\gamma)} + \frac{S\frac{\partial v}{\partial S}}{v} \qquad (2.44)$$

Theorem 2.17 will be proved in the appendix 1.

Remark 2.18. (Yong, 2012) shows that there exists a subgame perfect equilibrium strategy when the diffusion term in $dX^{\pi,c}(t)$ does not depend on the control. His proof uses a fixed point formulation by means of a contraction operator in a Banach space. Our proof is similar: we establish the existence of a utility weighted discount rate $\mathbb{Q}(t,S)$ by means of a contraction operator. However our result is more general than Yong's (Yong, 2012) since the diffusion term $\sigma_S(t,S_t)\pi(t)X^{\pi,c}(t)$ does depend on the control.

2.6 Further interpretation of the utility weighted discount rate $\mathbb{Q}(t, S)$

We end with a corollary to theorem 2.17 that summarizes our findings about the utility weighted discount rate.

Corollary 2.19. *The subgame perfect strategy is equal to the precommitment strategy of an agent that discounts all future times $t \leq T$ with the utility weighted discount rate $\mathbb{Q}(t, S_t)$ instead of the psychological discount rate $\rho(0, t)$.*

Bjork, Khapko, Murgoci (Bjork *et al.*, 2016) have shown that for every subgame perfect strategy, there is a corresponding time inconsistent strategy that is the optimal strategy of a modified problem. Corollary 2.19 gives the same kind of interpretation.

We can obtain bounds for $\mathbb{Q}(t, S)$ in order to improve our intuition. Let

$$h(t, s) = f(s - t) \text{ for } s \geq t.$$

and let $\alpha(t, s, S)$ and $\rho(t, s)$ denote the quantities

$$\alpha(t, s, S) = \mathbb{E}_t^{\mathbb{P}}\left[\exp\left(p\gamma \int_t^s (r + \frac{\theta_S^2}{2} - y)du + p\gamma \int_t^s \theta_S dW_u\right)\right]$$
$$\rho(t, s) = -\frac{\frac{\partial h(t,s)}{\partial s}}{h(t, s)} = -\frac{f'(s-t)}{f(s-t)} := R(s-t)$$

$\mathbb{Q}(t,S)$ can be written as

$$\mathbb{Q}(t,S) = \frac{\int_t^T \rho(t,s)h(t,s)\alpha(t,s,S)ds + \rho(t,T)h(t,T)\alpha(t,T,S)}{\int_t^T h(t,s)\alpha(t,s,S)ds + h(t,T)\alpha(t,T,S)}$$
$$\mathbb{Q}(t,S) = \frac{\int_t^T R(s-t)f(s-t)\alpha(t,s,S)ds + R(T-t)f(T-t)\alpha(t,T,S)}{\int_t^T f(s-t)\alpha(t,s,S)ds + f(T-t)\alpha(t,T,S)}$$

Therefore

$$\inf_{x\in[0,T-t]} R(x) \le \mathbb{Q}(t,S) \le \sup_{x\in[0,T-t]} R(x) \tag{2.45}$$

In contrast, if the agent was following an optimal policy with pre commitment starting at $t=0$, then its discount rate at time s would be $\rho(0,s) = -\frac{f'(s)}{f(s)} = R(s)$. In the case of the generalized hyperbolic discounting, $f(x) = (1+ax)^{-\frac{b}{a}}.\exp(-\rho x)$ with positive constants a and b and non negative constant ρ. Then

$$R(x) = -\frac{f'(x)}{f(x)} = \rho + \frac{b}{1+ax} \tag{2.46}$$

is decreasing in x. Thus

$$R(T-t) \le \mathbb{Q}(t,S) \le R(0) \tag{2.47}$$

The following graphs show the bounds for the utility weighted discount rate $\mathbb{Q}$:

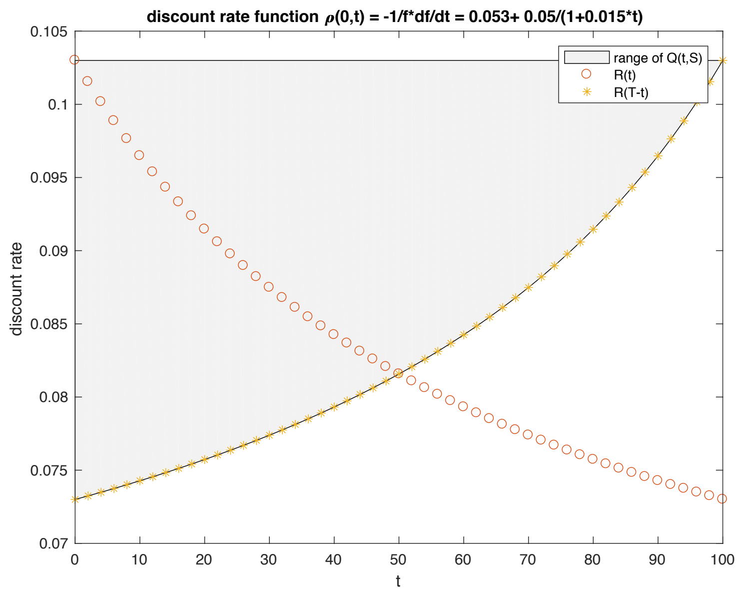


Figure 2.3: The range of $\mathbb{Q}(t, S)$ for hyperbolic discounting is given by the shaded area. The decreasing function is the discount rate $R(t) = \rho(0, t)$.

2.7 Comparison between the sub game perfect and the optimal precommitment strategies .

A rational agent with non constant discount rate has 2 choices:

- He commits to follow the time 0 - optimal strategy also called precommitment strategy at time 0 all the way to time T.
- He does not make commitments and allows her future self to deviate from the time 0 - optimal strategy. As noted in the introduction, his strategy should be subgame perfect.

In this section, we compare the 2 strategies. We consider the same setting where the agent has relative risk aversion $1 - \gamma$. An admissible strategy $(\hat{\pi}, \hat{c})$ is called time

0 - optimal if

$$J(0,S,x,\hat{\pi},\hat{c}) = \sup_{(\pi,c)\in\mathcal{A}_0} J(0,S,x,\pi,c). \tag{2.48}$$

In that case, the value function $\hat{V}$ is $\hat{V}(t,S,x) = J(t,S,x,\hat{\pi},\hat{c})$. As noted before, $(\hat{\pi},\hat{c})$ will also be called precommitment strategy.

The subgame perfect strategy is denoted $(\bar{\pi},\bar{c})$ and the corresponding value function is

$$\bar{V}(t,S,x) = J(t,S,x,\bar{\pi},\bar{c}). \tag{2.49}$$

We will see in the following result that $\bar{V}$ and $\hat{V}$ have essentially the same form:

Proposition 2.20. *For all $(t,S,x)\in[0,T]\times(0,\infty)^2$:*

$$\hat{V}(t,S,x) = \hat{v}(t,S)^{1-\gamma}U_\gamma(x) \text{ and } \bar{V}(t,S,x) = \bar{v}(t,S)^{1-\gamma}U_\gamma(x). \tag{2.50}$$

$\hat{v}$ and $\bar{v}$ satisfy the linear parabolic PDEs:

$$\frac{\partial\bar{v}(t,S)}{\partial t} + \frac{\sigma_S^2 S^2}{2}\bar{v}_{SS} + p\left[\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q}(t,S)\right]\bar{v}(t,S) + (r + p\sigma_S\theta_S)S\bar{v}_S(t,S) + 1 = 0 \tag{2.51}$$

$$\frac{\partial\hat{v}(t,S)}{\partial t} + \frac{\sigma_S^2 S^2}{2}\hat{v}_{SS} + p\left[\gamma r + \frac{\gamma p\theta_S^2}{2} - \rho(0,t)\right]\hat{v}(t,S) + (r + p\sigma_S\theta_S)S\hat{v}_S(t,S) + 1 = 0 \tag{2.52}$$

with final condition $\hat{v}(T,S) = \bar{v}(T,S) = 1$.

The optimal strategies $(\hat{\pi},\hat{c})$ and subgame perfect strategies $(\bar{\pi},\bar{c})$ are given by:

$$\hat{c}(t,S) = \frac{1}{\hat{v}(t,S)} \quad , \quad \bar{c}(t,S) = \frac{1}{\bar{v}(t,S)} \tag{2.53}$$

$$\hat{\pi}(t,S) = \frac{\theta_S(t,S)}{\sigma_S(t,S)} + \frac{pS\hat{v}_S(t,S)}{\hat{v}(t,S)} \quad , \quad \bar{\pi}(t,S) = \frac{\theta_S(t,S)}{\sigma_S(t,S)} + \frac{pS\bar{v}_S(t,S)}{\bar{v}(t,S)} \tag{2.54}$$

We can see that $\hat{v}$ and $\bar{v}$ differ only in the discount rate term. The time 0 - optimal strategy agent discounts the future (time t) at the rate $\rho(0,t)$ while the subgame perfect agent discounts the future at the rate $\mathbb{Q}(t, S_t)$.

Case where there is no intermediate consumption

We now consider an agent that only have utility of final wealth. Let $D = \{(t,s) \in [0,T] \times [0,T] \mid t \leq s\}$ and $h : D \to (0,\infty)$ be a discount function i.e. positive and C^1 on D. Suppose furthermore that $h(t,s)$ could be written as $H(s-t)$.

We consider two strategies: the first one maximizes utility of final wealth. The investor pre-commits at time $t = 0$. The fraction of wealth invested in the risky asset at time t is $\pi(t)$ and $\hat{\pi}$ denotes the optimal π. The criterion is

$$J(t, S, x; \pi) = \mathbb{E}_t^{\mathbb{P}}\big[\frac{h(0,T)}{h(0,t)} U_\gamma(X_T^\pi)\big] = \mathbb{E}_t^{\mathbb{P}}\big[\frac{H(T)}{H(t)} U_\gamma(X_T^\pi)\big]. \tag{2.55}$$

Note that

$$\sup_{\pi \in \mathcal{A}_t} J(t, S, x, \pi) = \frac{H(T)}{H(t)} \sup_{\pi \in \mathcal{A}_t} \mathbb{E}_t\big[U_\gamma(X_T^\pi)\big] \tag{2.56}$$

The problem is time consistent since the factor $\frac{H(T)}{H(t)}$ does not affect the optimization.

We conclude that the precommitment strategy coincides with the subgame perfect strategy. It is the intermediate consumption that introduces the time inconsistency.

2.8 Numerical analysis

$\mathbb{Q}$ is defined as the fixed point of a non linear operator F. We can compute it numerically and use it to calculate $\bar{v}(t, S)$ by Monte Carlo simulation.

We consider 2 cases for the parameters of our model.

1. Constant volatility model. It is a model where all the coefficients are suppose to be constant: $\sigma_S(s) = \sigma, \theta_S(s) = \theta, r(s) = r$. This model is the first one considered in the paper (Ekeland *et al.*, 2012).

2. Constant elasticity of variance model (CEV).

The CEV model is a model where the instantaneous volatility specified to be a power function of the underlying spot price $\sigma_S(S) = \alpha S^{\beta}$ where $\alpha > 0$ is the volatility scale parameter and β is the elasticity parameter of the local volatility : $\beta = \frac{1}{\sigma_S(S)} \times \frac{\partial \sigma_S}{\partial S}$.

For $\beta = 0$, we retrieve the Black Scholes Merton model. For $\beta = -\frac{1}{2}$, we retrieve the square root model of Cox and Ross.

In the remainder, we suppose $\beta < 0$. As explained in (Linetsky and Mendoza, 2010), the spot volatility is a decreasing function of the asset price. The stock price volatility increases as the stock price declines. This shows the leverage effect in equity markets. When we compute the option prices with this model of volatility, we also get an implied volatility skew. That is what makes this model attractive in the finance world. However, (Delbaen and Shirakawa, 2002) shows that there always exists arbitrage in such markets.

Note that the volatility could go to infinity when the stock price goes to zero. We want to avoid those anomalies since we are not concerned with defaults. We fix a minimum σ_m and a maximum σ_M for the volatility σ_S. Suppose the volatility at time 0 is known equal to σ_0 and the stock price is S_0. We choose $\alpha = \sigma_0 S_0^{-\beta}$ and we can write $\sigma_S(S) = \sigma_0(\frac{S}{S_0})^{\beta}$.

and $\sigma_S(S)$ is constant and equals σ_M if $\sigma_S(S) \geq \sigma_M$ and $\sigma_S(S)$ is constant and equals σ_m if $\sigma_S(S) \leq \sigma_m$. Numerically, we choose the parameters presented in the table

Parameters	$r(t,S)$	β	σ_S	$\theta_S(t,S)$	σ_m	σ_M
Values	0.05	-0.4	$0.3(\frac{S}{10})^{\beta}$	$6\sigma_S(t,S)$	0.15	0.45

We take $\gamma = -5$ and

$$h(t,s) = H(s-t) := (1+\alpha_1(s-t))^{-\frac{\beta_1}{\alpha_1}} \exp(-\rho_1(s-t))$$

with $\alpha_1 = 1.0, \beta_1 = 0.02, \rho_1 = 0.02$. The superscript "PC" represents the precommitment optimal strategies while "TC" represents time consistent (subgame perfect) strategies.

For the constant volatility model, we have chosen the market parameters $(r, \theta_S, \sigma_S) = (0.05, 0.2777, 0.30)$.

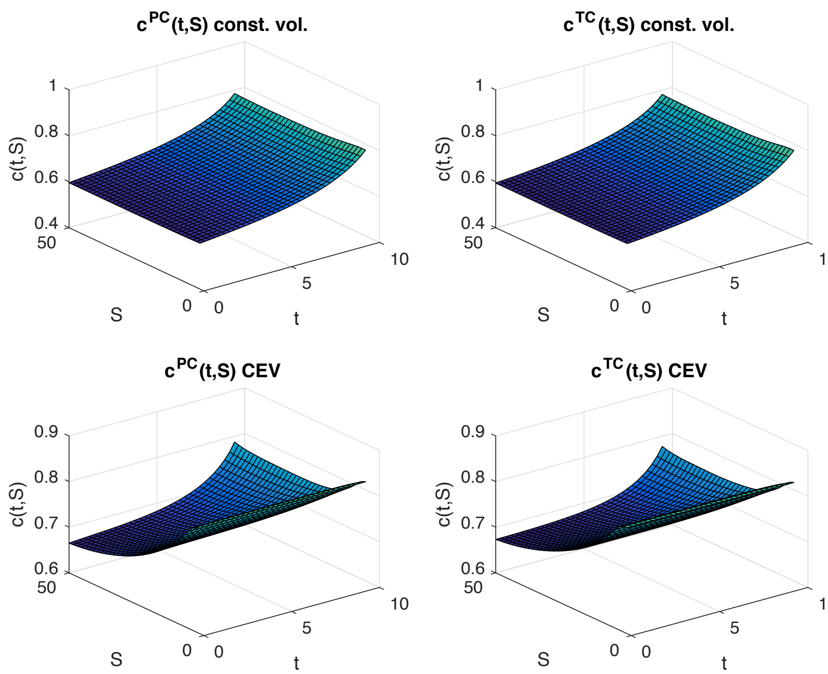


Figure 2.4: Fraction of consumption $c(t, S)$ for different volatilities and strategies

Remark While the discount rate is typically decreasing (see (Frederick *et al.*, 2002)) e.g. $\rho(0, t) = \frac{b}{1+at}$ for hyperbolic discount , the utility weighted discount rate tends to increase over time. (Frederick *et al.*, 2002) show that there is empirical evidence for increasing discount rates. However this phenomenon was not fully understood. Subgame perfect strategies will allow us to have equivalent discount rates that are increasing. However, the utility weighted discount rate does not have the same interpretation as the psychological discount rate ρ. It is the discount rate of the agent if he were to follow an equivalent optimal strategy.

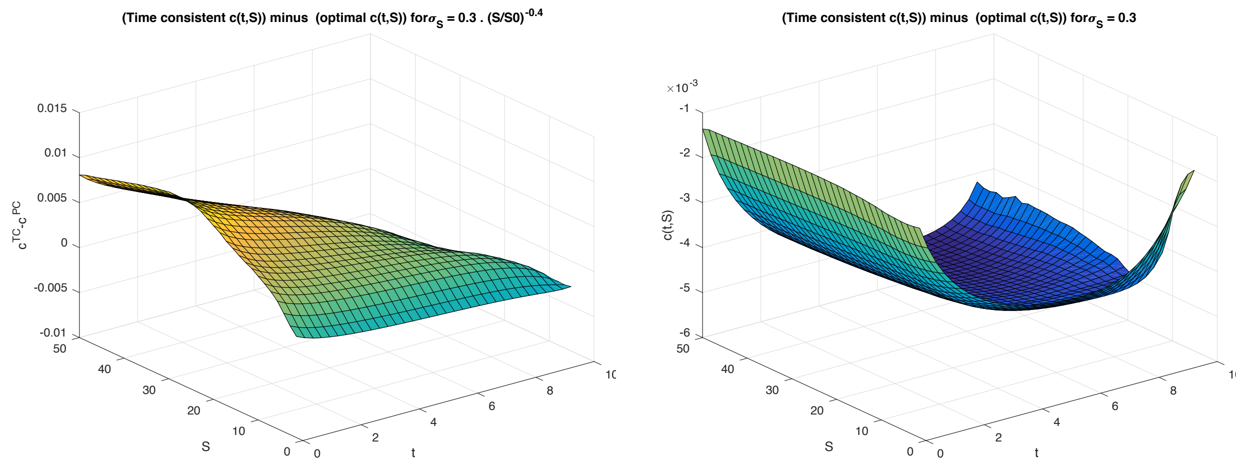


Figure 2.5: Study of $\hat{c}(t,S)$, $\bar{c}(t,S)$ for $\gamma = -5$

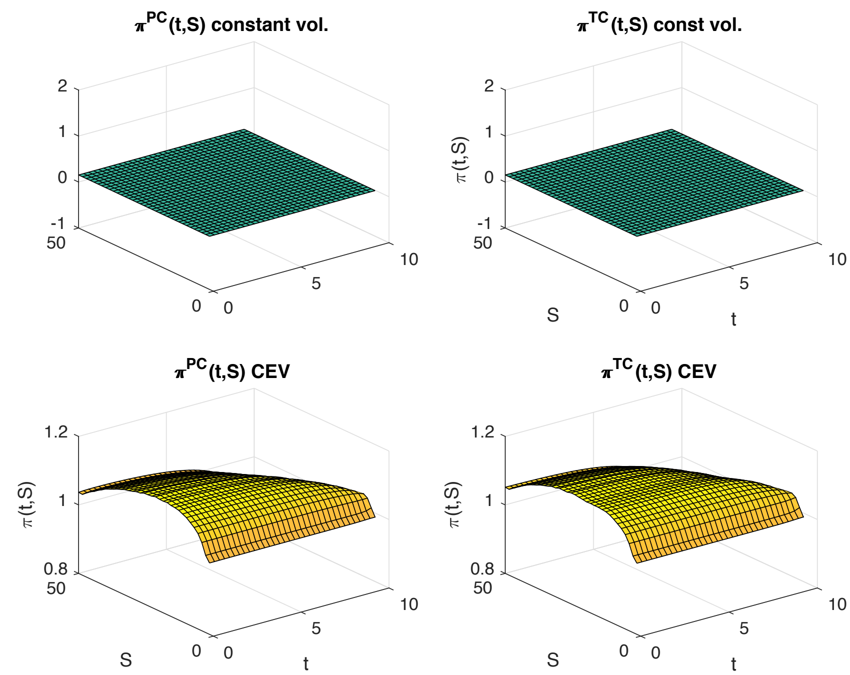
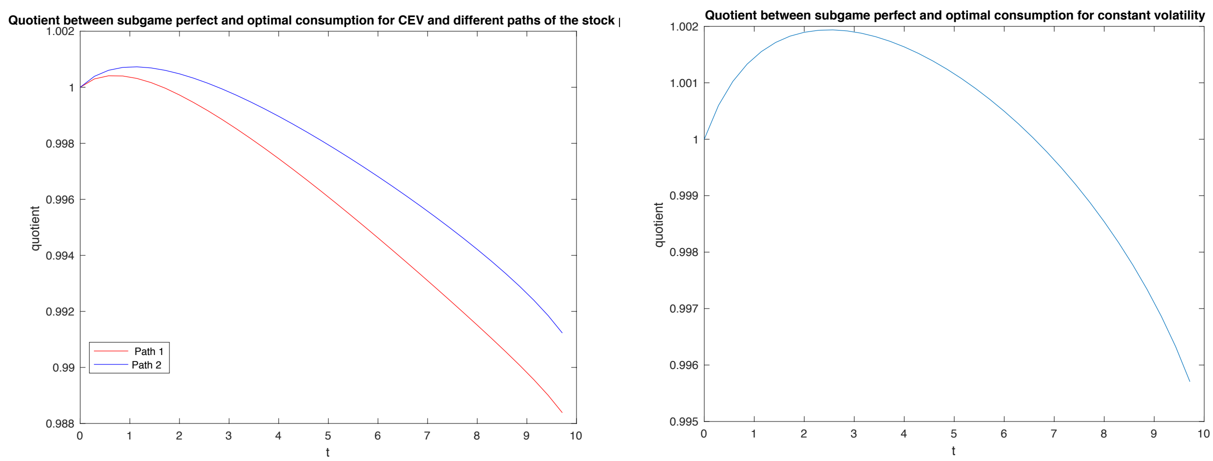


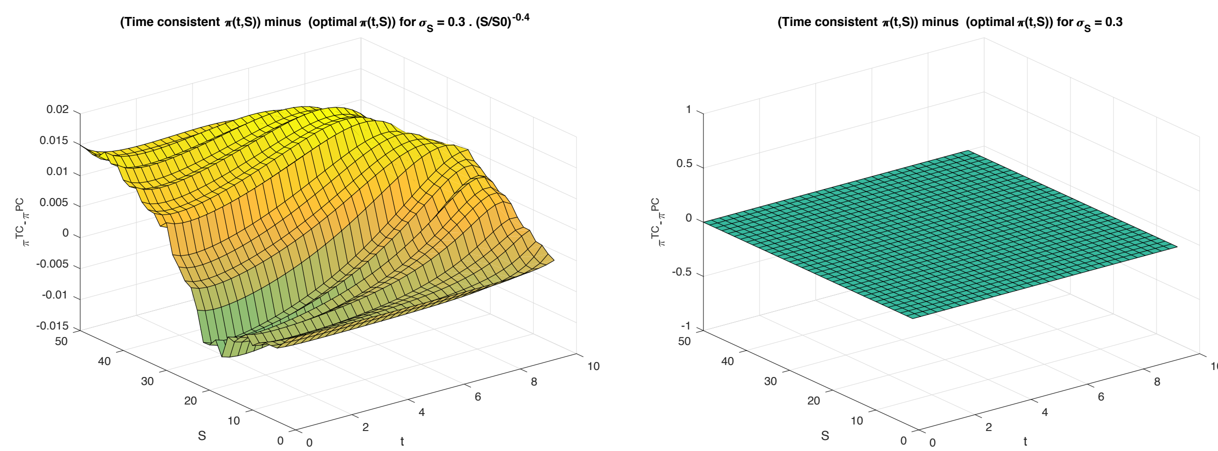
Figure 2.6: Comparing π for different volatilities and strategies : $\pi^{PC} = \hat{\pi} = \bar{\pi} = \pi^{TC}$ when the parameters r, θ_S, σ_S are all constants. Theoretically, we get $\bar{\pi} = \hat{\pi} = \frac{\theta_S}{\sigma_S(1-\gamma)}$ is independent of S.



(a) Graph of the consumption quotients for CEV type of volatility and 2 different paths for the stock.

(b) Graph of the consumption quotients for constant volatility (the quotient is independent of the stock price).

Figure 2.7: Study of $\frac{\bar{c}(s)\bar{X}(s)}{\hat{c}(s)\hat{X}(s)}$ for $\gamma = -5$



(a) $\bar{\pi}(t, S) - \hat{\pi}(t, S)$ in the case of CEV. The difference does not have a fixed sign but changes with the stock price S

(b) $\bar{\pi}(t, S) - \hat{\pi}(t, S)$ for constant volatility. Actually, $\bar{\pi} = \hat{\pi}$ if r, θ_S, σ_S are all constant.

Figure 2.8: Study of $\hat{\pi}(t, S), \bar{\pi}(t, S)$ for $\gamma = -5$

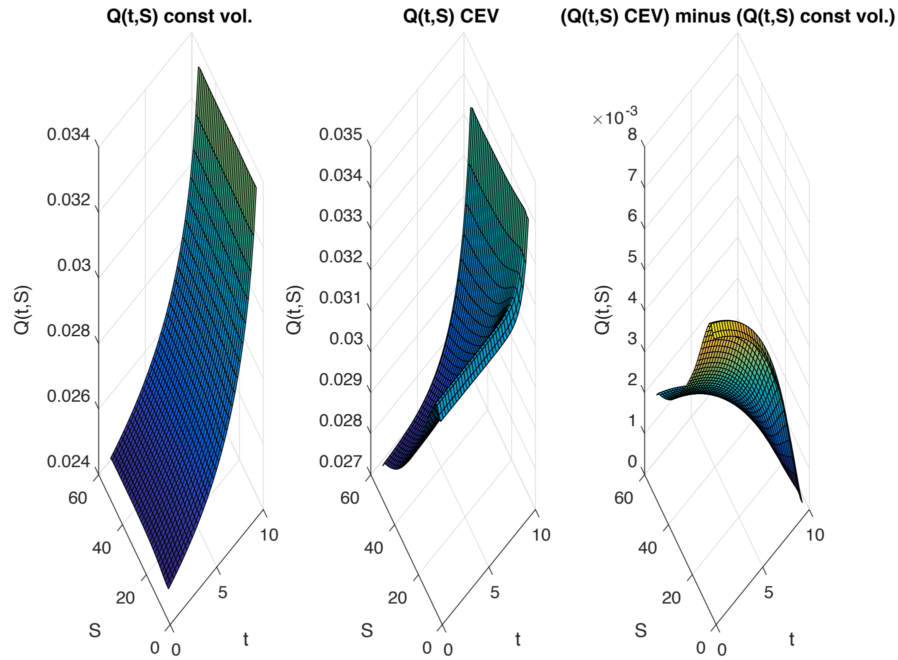
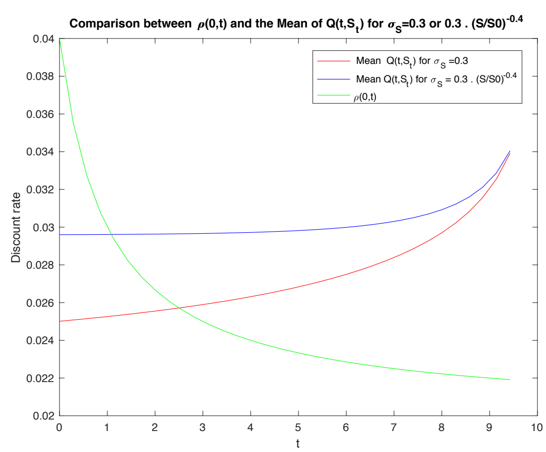
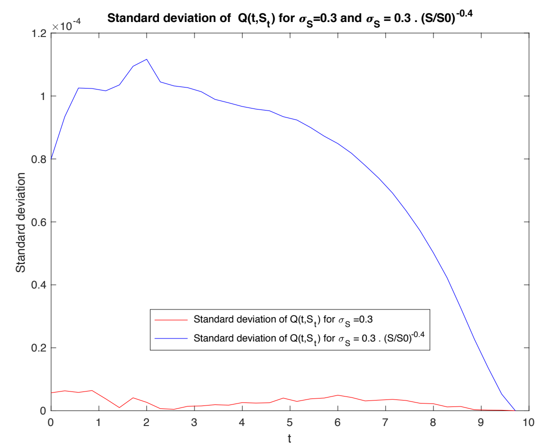


Figure 2.9: $\mathbb{Q}(t, S)$ for different volatility functions



(a) Mean $\mathbb{Q}(t, S_t)$ for different volatilities. The discount function $\rho(0, t)$ is a decreasing function of time t however the utility weighted discount rate $\mathbb{Q}(t, S_t)$ tends to increase with time.



(b) Standard deviation of $\mathbb{Q}(t, S_t)$ for different volatility functions. The curve corresponding to a constant volatility is almost flat indicating that $\mathbb{Q}(t, S_t)$ is independent of the variable S.

Figure 2.10: Study of $\mathbb{Q}$ for $\gamma = -5$

Conclusion

We consider an agent who discounts the future utility of consumption and final wealth at a non constant rate. It is well known that this leads to time inconsistency. A strategy that might be optimal at time s from the agent's perspective at time 0 might not be optimal at time $t < s$. The agent has two choices: The first one is to pre commit and follow the time 0 optimal strategy all the way to the end. The second one is to allow his future selves (himself with his future time dependent preferences) some level of control. We have thus introduced the subgame perfect strategies.

We show that solving the time consistent utility maximization problem is equivalent to solving the extended HJB and this is done in Theorem 2.8. The verification theorem gives us a way to check easily that a candidate solution is time consistent.

We can show that the variables t, S (the current time and stock price) and x (the current wealth of the agent) can be separated and the problem is reduced to solving a PDE with a non local term. This PDE can be further reduced to a linear PDE by computing the utility weighted discount rate $\mathbb{Q}$ as a fixed point of a certain operator. The optimal strategy maximizes a criterion with discount rate $\rho(0,t) = \frac{-1}{h(0,t)} \frac{\partial h(0,t)}{\partial s}$. The time consistent strategy maximizes a criterion with discount rate $\mathbb{Q}(t, S_t)$. For the time consistent strategy, the discount rate is $\mathbb{Q}(t, S_t)$ where S_t is the stock level at the current time t. As stated in corollary 2.19, the time consistent strategy coincides with the time inconsistent strategy where the (deterministic) discount rate $\rho(0,t)$ by a stock price dependent one $\mathbb{Q}(t, S_t)$.

US consumption data has shown that the relative risk aversion should be specified to a value close to 6.0. When the parameters of the economy are constant (constant r, θ_S, σ_S) and the relative risk aversion is bigger than one, we find that:

- In the short term (between 0 and 3 years for the constant volatility graph ; between 0 and 1 year for the CEV graph), the subgame perfect agent has a higher consumption.

- In the medium to long term (between 3 and 10 years for the constant volatility graph ; between 1 and 10 years for the CEV graph), the subgame perfect agent has a lower consumption.

The subgame perfect agent discounts the medium term and the long term at a higher discount rate than the short term. Thus, he consumes more in the short term than the optimal agent. The subgame perfect agent also consumes less in the medium and long term than the comparable optimal agent.

These contributions could inspire a statistical analysis of the discount rates (see Frederick *et al.* (2002) for a review of discount rates and time preferences). That would allow us to identify which agents are already following time consistent strategies.

Another study we need to address is the calibration of the stock parameters $\mu_S(t, S)$, $\sigma_S(t, S)$, $\theta_S(t, S)$ and the money market parameters $r(t, S)$ in terms of the market data.

Chapter 3

Optimal Equilibrium with Heterogeneous Agents

The effects of heterogeneity on the interest rate, stock price and stock volatility are well understood in the setting of an endowment economy when the agents are maximizing their individual utilities from consumption and the psychological time rates of discounting is constant. In this chapter, we study the equilibrium in a Lucas type of economy when the discount rates of the individual agents are not constant and depend on the time the optimization starts. We compare our results with the ones obtained when each agent has a constant discount rate. The heterogeneous economy with time varying discount rates behaves in the long run as the heterogeneous economy where each agent uses his/her asymptotic discount rate.

3.1 Introduction

This chapter presents an equilibrium model in a pure exchange economy with heterogeneous investors that may differ in their beliefs, risk aversions and time preference rates.

At the aggregate level, as in (Cvitanic *et al.*, 2012), we analyze properties of the equilibrium market price of risk, of the risk free rate, of the bond prices, and of the stock price and volatility. We compare equilibrium characteristics to the characteristics in the homogeneous economies populated by one class of agents only. We also consider asymptotic properties of the equilibrium parameters. Heterogeneity implies that investors value differently the states of the world.

As opposed to (Cvitanic *et al.*, 2012), we do not use a martingale methodology to solve for equilibrium due to the fact that the discount rates depend on the initial point t where the optimization starts. Instead, we use a Hamilton Jacobi Bellman (HJB) approach. (Jouini *et al.*, 2010) considers the case where each agent has a different stochastic discount rate (varying with time and state of the world) but each agent has the same constant relative risk aversion. Thus, our results are a combination of (Jouini *et al.*, 2010) and (Cvitanic *et al.*, 2012).

The novelty in this chapter is three fold:

- we allow the discount rates to be non constant.
- we allow the psychological discount rate $\rho_i(t,s)$ of agent i between time t and time s to depend on the initial point t.

This chapter is organized as follows: section 2 introduces the model, admissibility conditions and discusses complete market properties. In section 3, we solve the equilibrium problem by maximizing individual criterions. We introduce the value function separately for each agent and derive an HJB equation. In section 4, we study the long run behaviour of the economy. In particular, we answer questions such as: which agent determines the long term yield, which one determines the stock price in the long run? Proofs are reported in the appendix.

3.2 The Model

We consider a continuous-time Arrow-Debreu economy with a finite horizon T, in which heterogeneous agents maximize their expected utility from future consumption.

3.2.1 The Market setup

Uncertainty is described by a one-dimensional, standard Brownian motion $\{W_t, t \in [0,T]\}$. There is a single consumption good and we denote by ϵ the aggregate dividend or endowment process. We make the assumption that ϵ satisfies the following stochastic differential equation

$$d\epsilon_t = \mu\epsilon_t dt + \sigma\epsilon_t dW_t \tag{3.1}$$

where the mean growth rate μ and the volatility σ are constants. There are I (types of) agents indexed by $i = 1, \cdots, I$. Agent i has wealth $X^i(t)$ at time t and a given initial wealth $X^i(0) = x_i$.

Agents have different expectations about the future of the economy. More precisely, agents disagree about the mean growth rate. We denote by μ_i the mean growth rate anticipated by agent i. The quantity

$$\delta_i := \frac{\mu_i - \mu}{\sigma} \tag{3.2}$$

denotes agent i's error in her perception of the growth of the economy normalized by its risk. We introduce the probability measure $\mathbb{P}^i$ defined by its density with respect to $\mathbb{P}$

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = \exp(\delta_i W_T - \frac{\delta_i^2 T}{2}) \tag{3.3}$$

For $0 \leq t \leq s \leq T$, denote

$$Z_i(t,s) = e^{\delta_i(W_s - W_t) - \frac{1}{2}\delta_i^2(s-t)} \tag{3.4}$$

$$W_t^i = W_t - \delta_i t \tag{3.5}$$

W_t^i is a Brownian motion for $\mathbb{P}^i$. From agent i's point of view, the aggregate endowment process satisfies the following stochastic differential equation :

$$d\epsilon_t = \mu_i \epsilon_t dt + \sigma \epsilon_t dW_t^i \tag{3.6}$$

Note that agents are persistent in their mistakes: the probability measures $\mathbb{P}^i$ may represent erroneous beliefs as well as behavioural biases like optimism (corresponding to $\delta_i > 0$) or pessimism ($\delta_i < 0$). Taking δ_i constant may seem incompatible with learning. However, we consider the case with constant parameters as an approximation of the situation where all the parameters are stochastic and where learning is regularly compensated by new shocks on the drift μ (Jouini *et al.*, 2010).

In our setting, there are three possible sources of heterogeneity among agents: heterogeneity in beliefs, heterogeneity in risk aversion and heterogeneity in time preference rates.

We assume that markets are complete which means that all Arrow-Debreu securities can be traded. In order to deal with asset pricing issues, we suppose that agents can continuously trade in a riskless asset and in risky stocks. We let S^0 denote the riskless asset price process with dynamics

$$dS_t^0 = r_t S_t^0 dt \tag{3.7}$$

the parameter r_t denoting the risk free rate at time t. Since there is only one source of risk, all risky assets have the same instantaneous Sharpe ratio and it suffices to focus on one specific risky asset. We consider the asset S whose dividend process is given by the total endowment of the economy ϵ_t and we denote respectively by μ_S

and σ_S its drift and volatility.

$$\begin{cases} dS_t = S_t \left[\mu_S dt + \sigma_S dW_t\right] - \epsilon_t dt \\ S_T = \epsilon(T) \end{cases} \tag{3.8}$$

We let

$$\theta_S := \frac{\mu_S - r}{\sigma_S} \tag{3.9}$$

denote the asset's Sharpe ratio or equivalently the market price of risk. The parameters r, μ_S and σ_S are to be determined endogenously in equilibrium.

It will be useful to introduce the adjusted market price of risk for agent i:

$$\phi_i := \theta_S + \delta_i \tag{3.10}$$

3.2.2 Portfolio and Consumption Policies

We will make the following assumptions in order to characterize the equilibrium. Then we will verify that the assumed hypotheses are true.

Assumption 3.1.

1. The interest rate process $(r(t))_{t\in[0,T]}$, the market price of risk process $(\theta_S(t))_{t\in[0,T]}$ and the stock volatility process $(\sigma_S(t))_{t\in[0,T]}$ are adapted and bounded, uniformly in $(t,\omega) \in [0,T] \times \Omega$.

2. For $t \in [0,T]$, $r(t), \sigma_S(t), \theta_S(t)$ can be written $r(t) = r(t, W_t), \theta_S(t) = \theta_S(t, W_t)$ for deterministic functions $r(t,w), \theta_S(t,w)$ of $(t,w) \in [0,T] \times \mathbb{R}$. Furthermore, r, θ_S are C^1 in their domain of definition.

Each agent i may choose a portfolio process $\pi^i(t)$ and a nonnegative consumption process $c^i(t), 0 \le t \le T$. For every such pair (π^i, c^i), the corresponding wealth process

$X^i := X^{\pi^i,c^i}$ has initial value $X^i(0) = x_i$ and obeys the self-financing equation

$$dX^i(t) = \pi^i(t)X^i(t)\left(\frac{dS_t + \epsilon_t dt}{S_t}\right) + X^i(t)(1-\pi^i(t))r_t dt - c^i(t)X^i(t)dt$$

This can be rewritten as

$$\begin{aligned} dX^i(t) &= \pi^i(t)X^i(t)(\mu_S dt + \sigma_S dW(t)) + X^i(t)(1-\pi^i(t))r_t dt - c^i(t)X^i(t)dt \\ dX^i(t) &= (r(t) - c^i(t) + \sigma_S\pi^i(t)\theta_S(t))X^i(t)dt + \sigma_S\pi^i(t)X^i(t)dW_t \end{aligned} \tag{3.11}$$

Let us now define the admissible strategies.

Definition 3.2. An $\mathbb{R}^2$-valued process $\{(\pi^i(t), c^i(t))\}_{0\le t\le T}$ is called an admissible strategy process for agent i with corresponding wealth process $X^i = X^{\pi^i,c^i}$ if:

- it is progressively measurable with respect to the sigma algebra $\sigma(\{W_t\}_{0\le t\le T})$.
- $c^i(t) \ge 0$, $X^{\pi^i,c^i}(t) \ge 0$ for all t, a.s. Furthermore, $c^i(t)$ and $\sigma_S(t)\pi^i(t)$ are uniformly bounded.
- Moreover, we require that for all $t \in [0,T]$, $x_i \ge 0$, $\mathbb{E}_t[\sup_{t\le s\le T} |U_i(c^i(s)X^{\pi^i,c^i}(s)|] < \infty$, $\mathbb{E}_t[|U_i(X^i(T)|] < \infty$ where $X^{\pi^i,c^i}(t) = x_i$ almost surely.

It is often useful to consider the dynamics of $X^i = X^{\pi^i,c^i}$ in the $\mathbb{P}^i$ probability space. Recall that $dW^i_t = dW_t - \delta_i dt$. In terms of $\mathbb{P}^i$, the wealth dynamics are given by

$$dX^i(s) = (r(s) - c^i(s) + \sigma_S\pi^i(s)\phi_i(s))X^i(s)ds + \sigma_S\pi^i(s)X^i(s)dW^i_s \tag{3.12}$$

Definition 3.3. Agent i has discount function f_i and utility function U_i with constant relative risk aversion $1-\gamma_i$ with $\gamma_i \neq 0, \gamma_i < 1$ i.e.

$$U_i(x_i) = \frac{x_i^{\gamma_i}}{\gamma_i}, \quad \forall x_i > 0 \tag{3.13}$$

We now introduce another source of heterogeneity : the time preferences. As noted in the Introduction chapter, time preference denotes the preference for immediate utility over delayed utility (see (Frederick *et al.*, 2002)).

Definition 3.4. Agent i's discount function: f_i is a measure of decision-maker i's impatience. f_i is defined on the domain $D := \{(t,s), 0 \leq t \leq s \leq T\}$ and satisfies the following:

1. $f_i(t,t) = 1$.

2. There exists $f_0 > 0$ such that $f_i(t,s) \geq f_0$.

3. f_i is continuously differentiable at every point $(t,s) \in D$.

The (forward) discount rate is the quantity

$$\rho_i(t,s) = -\frac{\partial f_i(t,s)}{\partial s} \times \frac{1}{f_i(t,s)}\ ,\ 0 \leq t \leq s \leq T. \tag{3.14}$$

We will need the following additional conditions to prove the existence of a pre commitment optimal equilibrium.

Assumption 3.5. Discount rate variation

For each $i \in \{1, \cdots, I\}$, the discount rate ρ_i is bounded and has bounded derivatives. $\rho_i(t,s)$, $\frac{\partial \rho_i(t,s)}{\partial t}$ and $\frac{\partial \rho_i(t,s)}{\partial s}$ are bounded by a constant independent of t, s.

We now study the existence of an equilibrium where the agents trade with each other and the markets clear.

3.3 The Equilibrium

The endowment of the economy $\epsilon(s)$ is given exogenously. We consider an asset $S(s)$ that pays a dividend equal to $\epsilon(s)$. In the setting of (Cvitanic *et al.*, 2012), we study the effects of heterogeneity on a complete economy in which agents trade, consume and invest.

3.3.1 The optimization problem

In this section and the next, we fix $t \in [0, T]$ as the time where the agents begin to maximize their utility of consumption and final wealth.

Definition 3.6 (Admissibility)**.** The portfolio/consumption process ratio pair (π^i, c^i) for the ith agent is admissible on $[t, T]$ if the process $\{(\pi^i_{t+s}, c^i_{t+s})\}_{0\le s\le T-t}$ is admissible (see Definition 3.2).

We call $\mathcal{A}^i_t$ the set of admissible strategies on $[t, T]$ (see definition above) which satisfy

$$\mathbb{E}^{\mathbb{P}^i}\left[\int_t^T f_i(t,u)\max(0, -U_i(c^i(u)X^i(u)))du + f_i(t,T)\max(0, -U_i(X^i_T)) \right. \\ \left. \mid X^i_t = x_i\ ,\ W_t = w\right] < \infty \tag{3.15}$$

For every $i = 1, \cdots, I$, agent i wants to maximize the expected utility

$$\mathbb{E}^{\mathbb{P}^i}\left[\int_t^T f_i(t,u)U_i(c^i(u)X^i(u))du + f_i(t,T)U_i(X_T^i) \mid X_t^i = x_i\ ,\ W_t = w\right] \tag{3.16}$$

The maximization is done over the set of strategies $(\pi^i, c^i) \in \mathcal{A}_t^i$.

For $s \geq t$, consider the criterion J^i that represents the continuation expected utility on $[s,T]$:

$$\begin{aligned} J^i(t,s,w,x_i,\pi^i,c^i) &= \mathbb{E}^{\mathbb{P}^i}\Bigg[\int_s^T \frac{f_i(t,u)}{f_i(t,s)}U_i(c^i(u)X^i(u))du \\ &\quad + \frac{f_i(t,T)}{f_i(t,s)}U_i(X_T^i)|X_s^i = x_i\ ,\ W_s = w\Bigg] \end{aligned} \tag{3.17}$$

A pair $(\hat{\pi}^i, \hat{c}^i)$ that achieves the supremum of (3.17) over such pairs is called an *optimal strategy*. Depending on the context, we will also call $(\hat{\pi}^i, \hat{c}^i)$ a *time t pre commitment strategy*. The time t is the time at which the agent decides to start the optimization. After committing to a strategy, the agent does not change it over the remaining time $[t,T]$.

Remark 3.7. If consumption at time $u \geq s$ is discounted using the factor

$$\frac{f_i(t,u)}{f_i(t,s)} = \exp\left(\int_s^u -\rho_i(t,v)dv\right).$$

then, the criterion becomes:

$$\begin{aligned} J^i(t,s,w,x_i,\pi^i,c^i) = \mathbb{E}^{\mathbb{P}^i}\Bigg[\int_s^T e^{\int_s^u -\rho_i(t,v)dv}U_i(c^i(u)X^i(u))du \\ + e^{\int_s^T -\rho_i(t,v)dv)}U_i(X_T^i)|X_s^i = x_i\ ,\ W_s = w\Bigg] \end{aligned} \tag{3.18}$$

The time t is again the time at which the agent decides to start the optimization. The value function at time s, state $W_s = w$ and wealth $X_s^i = x_i$ is

$$V^i(t,s,w,x_i) := \sup_{(\pi^i,c^i)\in\mathcal{A}_s^i} J^i(t,s,w,x_i,\pi^i,c^i) \tag{3.19}$$

where the sup is taken over the pairs $(\pi^i, c^i) \in \mathcal{A}_s^i$. For reasons that will be obvious later, we adopt the Hamilton-Jacobi-Bellman (HJB) methodology instead of the martingale one. Notice that

$$\begin{aligned} V^i(t,t,w,x_i) &= \sup_{(\pi^i,c^i)\in\mathcal{A}_t^i} \mathbb{E}^{\mathbb{P}^i}\Big[\int_t^T f_i(t,u)U_i(c^i(u)X^i(u))du \qquad (3.20)\\ &\qquad + f_i(t,T)U_i(X_T^i)|X_t^i = x_i, W_t = w\Big] \\ &= \sup_{(\pi^i,c^i)\in\mathcal{A}_t^i} J^i(t,t,w,x_i,\pi^i,c^i) \qquad (3.21)\end{aligned}$$

which is exactly the criterion we want to maximize.

3.3.2 The Definition of the Equilibrium

We fix the time t which is the time at which the agents start their optimization. We are concerned with defining an equilibrium on $[t,T]$.

Definition 3.8 (Equilibrium)**.** An equilibrium $(r, \theta_S, \sigma_S, \hat{\pi}^i, \hat{c}^i)$ consists of an interest rate r, a market price of risk θ_S, a stock volatility σ_S, investment and consumption processes π^i_s, c^i_s such that markets clear, i.e.

$$\sum_{i=1}^{I} \hat{c}^i(s)\hat{X}^i(s) = \epsilon(s) \tag{3.22}$$

$$\hat{X}^{tot}(s) := \sum_i \hat{X}^i(s) = S(s) \tag{3.23}$$

$$\hat{X}^{\pi}(s) := \sum_i \hat{\pi}^i(s)\hat{X}^i(s) = S(s) \tag{3.24}$$

for all $s \in [t, T]$ and $(\hat{\pi}^i, \hat{c}^i)$ maximizes agent i's inter temporal optimization program i.e.

$$(\hat{\pi}^i, \hat{c}^i) \in \arg \sup_{(\pi^i, c^i) \in \mathcal{A}^i_t} J^i(t, t, w, x_i, \pi^i, c^i) \tag{3.25}$$

(3.22) states that the aggregate consumption equals the endowment. (3.23) states that the aggregate wealth equals the stock price. (3.24) states that the aggregate investment in the stock equals the stock price (there is only one unit of stock in the whole economy).

We can show that (3.24) is equivalent to { (3.23) with the additional initial condition $\hat{X}^{tot}(t) = S(t)$}.

Note that the market clearing conditions imply that the money market clears. Agent i invests

$$\hat{X}^i(s) - \hat{\pi}^i(s)\hat{X}^i(s)$$

in the money market and we have:

$$\sum_i (\hat{X}^i(s) - \hat{\pi}^i(s)\hat{X}^i(s)) = S(s) - S(s) = 0.$$

3.3.3 Solving Agents Optimization Problem

The following theorem is a verification theorem for the HJB. It says that if we find a candidate solution that satisfies the HJB then it is actually the value function.

Theorem 3.9 (Verification Theorem). *Suppose that we have functions $y^i(t; s, w, x_i)$, $c_y(t; s, w, x_i)$ and $\pi_y(t; s, w, x_i)$ such that:*

- $(\pi_y, c_y) \in \mathcal{A}_t^i$.
- *If g is one of the functions y^i, $\frac{\partial y^i}{\partial s}$, $\frac{\partial y^i}{\partial w}$, $x_i \frac{\partial y^i}{\partial x_i}$ and (π^i, c^i) is any admissible strategy then*
$$\mathbb{E}_{s,w,x_i}^{\mathbb{P}^i}[\sup_{s \le u \le T} |g(t; u, W_u, X_u^{\pi^i,c^i})|^2] < \infty \tag{3.26}$$
and y^i satisfies the HJB
$$\begin{cases} \sup_{\pi^i,c^i} \left\{ y_s^i + x_i \left(r + \sigma_S \theta_S \pi^i - c^i \right) y_x^i + \frac{1}{2}(\pi^i \sigma_S x_i)^2 y_{xx}^i + \frac{1}{2} y_{ww}^i \right. \\ \left. + \pi^i \sigma_S x_i y_{xw}^i - \rho_i(t,s) y^i + \delta_i(\sigma_S \pi^i x_i y_x^i + y_w^i) + U_i(c^i x_i) \right\} = 0 \\ y^i(t, T, w, x_i) = U_i(x_i) \end{cases} \tag{3.27}$$
- *For each fixed (s, x), the supremum in the expression (3.27) is attained by the choice $\pi_y(t, s, w, x_i), c_y(t, s, w, x_i)$.*

Then, the following holds:

The optimal value function V^i to the control problem is given by

$$y^i(t, s, w, x_i) = J^i(t, s, w, x_i, \pi_y, c_y) = V^i(t, s, w, x_i) \tag{3.28}$$

The proof is given in Appendix 2.

In what follows, we denote p_i the inverse of the relative risk aversion:

$$p_i = \frac{1}{1-\gamma_i} \tag{3.29}$$

Because of the CRRA form of the utility function U_i, we can look to separate the variables w, x_i. We look for $V^i(t,s,w,x_i)$ of the form

$$V^i(t,s,w,x_i) = a_i(t,s,w)U_i(x_i)$$

This is the content of the following result.

Theorem 3.10. *The value function is given by*

$$V^i(t,s,w,x_i) = a_i(t,s,w_i)U_i(x_i) \tag{3.30}$$

and a time t optimal strategy $(\hat{\pi}^i, \hat{c}^i)$ is :

$$\hat{c}^i(s) = a_i(t,s,W_s)^{-p_i} \quad ; \quad \sigma_S(s)\hat{\pi}^i(s) = p_i\left(\phi_i(t,s,W_s) + \frac{\frac{\partial a_i}{\partial w}}{a_i}(t,s,W_s)\right) \tag{3.31}$$

for $s \in [t,T]$. The wealth process $\hat{X}^i$ associated with the optimal strategy $(\hat{\pi}^i, \hat{c}^i)$ satisfies the SDE:

$$d\hat{X}^i(s) = (r(s) - \hat{c}^i(s) + \sigma_S(s)\hat{\pi}^i(s)\theta_S(s))\hat{X}^i(s)ds + \sigma_S(s)\hat{\pi}^i(s)\hat{X}^i(s)dW(s) \tag{3.32}$$

The function

$$v_i(t,s,w) := a_i(t,s,w)^{p_i} \tag{3.33}$$

satisfies the linear parabolic PDE:

$$0 = \frac{\partial v_i}{\partial s} + \frac{1}{2}\frac{\partial^2 v_i}{\partial w^2} + (\delta_i + \gamma_i p_i \phi_i)\frac{\partial v_i}{\partial w} + p_i(\gamma_i r + \frac{\gamma_i p_i \phi_i^2}{2} - \rho_i(t,s))v_i + 1 \quad (3.34)$$

$$v_i(t,T,w) = 1$$

and the optimal strategy $(\hat{\pi}^i, \hat{c}^i)$ *with starting time* t *is given by*

$$\hat{c}^i(t,s,W_s) = \frac{1}{v_i(t,s,W_s)} \quad ; \quad \sigma_S \hat{\pi}^i(t,s,W_s) = p_i\phi_i + \sigma_S \frac{\frac{\partial v_i}{\partial w}}{v_i}(t,s,W_s) \qquad (3.35)$$

The proof is given in Appendix 2.

Since $\rho_i(t,s)$ depends in general on t, we see that the linear PDE (3.34) has a linear coefficient

$$p_i\left[\gamma_i r + \frac{\gamma_i p_i \phi_i^2}{2} - \rho_i(t,s)\right] v_i$$

that depends on the starting point t. If we fix two starting points $t_1 < t_2$ for the optimization, the strategies $\hat{\pi}^i(t_1,s,W_s)$, $\hat{c}^i(t_1,s,W_s)$ does not coincide with $\hat{\pi}^i(t_2,s,W_s)$, $\hat{c}^i(t_2,s,W_s)$ on $s \in [t_2,T]$. Therefore, the strategy is time inconsistent.

In the case of exponential discounting, $\rho_i(t,s) = \rho_i = constant$, the t-dependence of the PDE disappears and the strategy becomes time consistent.

In what follows, we fix $t = 0$, i.e. each agent i is optimizing its expected utility of consumption and final wealth starting at time $t = 0$. $r(0,s,w)$ will simply be noted $r(s,w)$, other similar quantities will follow the same simplified notation.

In the sequel, we assume that at time s, the market price of risk $\theta_S(s)$ and the interest rate $r(s)$ and the stock volatility are given and bounded. Since the criterion J^i is an integral of terms of the form $U_i(\hat{c}^i(s)\hat{X}^i(s))$, we want to find a simple expression for $\hat{c}^i(s)\hat{X}^i(s)$. This is the content of the next proposition.

Proposition 3.11. *The following SDE holds:*

$$d\log(\hat{c}^i(s)\hat{X}^i(s)) = p_i\left(r + \frac{\phi_i^2}{2} - \delta_i\phi_i(s,W_s) - \rho_i(0,s)\right)ds + p_i\phi_i(s,W_s)dW(s) \tag{3.36}$$

The remarkable thing is that the expression in the right side of (3.36) does not depend on v_i.

3.3.4 Equilibrium in the homogeneous economies

We start by considering the equilibrium characteristics that would prevail in an economy made of agent i only or that would prevail in our economy if all the initial endowment was concentrated on agent i.

We denote by S_i, r_i, μ_{iS}, σ_{iS}, θ_{iS} the equilibrium stock price, interest rate, stock drift, stock volatility and market price of risk.

$$\theta_{iS}(v) := \frac{\mu_{iS}(v) - r_i(v)}{\sigma_{iS}(v)} \tag{3.37}$$

There is only one agent i so that the commodity clearing condition becomes:

$$\hat{c}^i(s)\hat{X}^i(s) = \epsilon(s)$$

Using the SDE (3.36) for $d\log(\hat{c}^i(s)\hat{X}^i(s))$ obtained previously, we get

$$\begin{aligned} d\log\epsilon_s &= d\log(\hat{c}^i(s)\hat{X}^i(s)) \\ &= p_i(r_i(s) + \frac{(\theta_{iS}+\delta_i)^2}{2} - \delta_i(\theta_{iS}+\delta_i) - \rho_i(0,s))ds + p_i(\theta_{iS}+\delta_i)dW(s) \quad (3.38) \\ &= (\mu - \frac{\sigma^2}{2})ds + \sigma dW(s) \quad (3.39) \end{aligned}$$

Comparing the ds terms and the $dW(s)$ terms in equations (3.38), (3.39) yields:

Proposition 3.12. *In the homogeneous economy where only agent i is present, the market price of risk is θ_{iS}, the interest rate is $r_i(s)$, the stock volatility is σ_{iS} . They are given by the following expressions:*

$$\theta_{iS} = \sigma(1-\gamma_i) - \delta_i \quad , \quad \sigma_{iS} = \sigma \quad (3.40)$$

$$r_i(s) = (1-\gamma_i)\mu_i - \sigma^2(1-\gamma_i)(1-\frac{\gamma_i}{2}) + \rho_i(0,s) \quad (3.41)$$

Remark 3.13. We note that the equilibrium $\theta_{iS}, \sigma_{iS}, r_i$ are the same as those found by (Cvitanic *et al.*, 2012), except that we replace the constant discount rate ρ_i by a time dependent discount rate $\rho_i(0,s)$.

Now, we introduce pricing kernels. Recall that the fundamental theorem of asset pricing in finance suggests that the price of any asset is its discounted expected value of future payoff specifically under risk-neutral measure or valuation. The present value of 1$ in t years is

$$\mathbb{E}^{\mathbb{P}}[M_t] = \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^t r_u du}]$$

where $\mathbb{P}$ is the physical probability measure and $\mathbb{Q}$ is the risk-neutral measure. The pricing kernel in the homogeneous economy of agent i is defined as

$$M_i(s) = \exp\left(\int_0^s (-r_i(u) - \frac{\theta_{iS}^2}{2})du - \int_0^s \theta_{iS} dW(u)\right) \tag{3.42}$$

The next proposition gives another expression for $M_i(s)$.

Proposition 3.14. *The process $M_i(s)$ can be written as:*

$$M_i(s) = f_i(0,s)\exp\left(-\big((1-\gamma_i)\mu_i + (1-\gamma_i)(1-\frac{\gamma_i}{2})\sigma^2 + \frac{\theta_{iS}^2}{2}\big)s - \theta_{iS}W_s\right) \tag{3.43}$$

3.3.5 Equilibrium in the heterogeneous economy

We are now in an heterogeneous economy (the real one) where all agents are present. Define the pricing kernel as

$$M(s) = \exp(-\int_0^s (r_u + \frac{\theta_S(u)^2}{2})du - \int_0^s \theta_S(u)dW_u) \tag{3.44}$$

where $\theta_S(u)$ and $r(u)$ are the equilibrium market price of risk and interest rate to be found later.

Proposition 3.15. *We have the following:*

$$\hat{c}^i(s)\hat{X}^i(s) = \frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon(0)}\epsilon(s)\left(\frac{M_i(s)}{M(s)}\right)^{p_i} \tag{3.45}$$

The proof can be found in Appendix 2.

The clearing condition for the consumption (3.27) becomes:

$$\sum_i \frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon(0)}\left(\frac{M_i(s)}{M(s)}\right)^{p_i} = 1 \tag{3.46}$$

From equation (3.46), we can deduce that $M(s)$ has to be chosen as the solution $y > 0$ of the equation

$$\sum_{i=1}^{I}\left((\frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon(0)})^{\frac{1}{p_i}}\frac{M_i(s)}{y}\right)^{p_i} = 1$$

We make this result more precise by starting with a definition.

Definition 3.16. For $a_1, \cdots, a_I > 0$, define $F(a_1, \cdots, a_I)$ as the unique solution y of

$$\sum_{i=1}^{I}(\frac{a_i}{y})^{p_i} = 1. \tag{3.47}$$

For $s \in [0, T]$ and $i \in \{1, \cdots, I\}$, define

$$a_i(s) := (\frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon(0)})^{\frac{1}{p_i}}M_i(s) \tag{3.48}$$

and

$$a(s) = (a_1(s), \cdots, a_I(s)). \tag{3.49}$$

Recall that the optimal consumption for agent i is $\hat{c}^i(s)\hat{X}^i(s)$. Define the risk weighted fraction of consumption for agent i as

$$\omega_i(s) = \frac{\frac{p_i\hat{c}^i(s)\hat{X}^i(s)}{\epsilon(s)}}{\sum_j \frac{p_j\hat{c}^j(s)\hat{X}^j(s)}{\epsilon(s)}} \tag{3.50}$$

In (Cvitanic *et al.*, 2012), the quantity ω_i is called relative level of absolute risk tolerance of agent i. The implicit function theorem yield that F is C^∞ in its domain of definition and $\{F(a(s)), s \in [0, T]\}$ is an adapted stochastic process. This is the content of the next proposition.

Proposition 3.17 (Pricing kernel and consumption fractions). *The pricing kernel $M(s)$ in the heterogeneous economy is given by*

$$M(s) = F(a(s)) \tag{3.51}$$

The risk weighted fraction of consumption for agent i can be written explicitly as:

$$\omega_i(s) = \frac{p_i \left(\frac{a_i(s)}{F(a(s))}\right)^{p_i}}{\sum_j p_j \left(\frac{a_j(s)}{F(a(s))}\right)^{p_j}} \tag{3.52}$$

Proposition 3.18 (Equilibrium interest rate and market price of risk). *The pricing kernel $M(s)$ satisfies the SDE*

$$dM(s) = -M(s)(r(s)ds + \theta_S(s)dW(s)) \tag{3.53}$$

The equilibrium interest rate r and market price of risk θ_S are given by:

$$\theta_S(s) = \sum_i \omega_i(s)\theta_{iS} \tag{3.54}$$

$$\begin{aligned} r(s) &= \sum_i \omega_i(s) r_i(s) + \frac{1}{2}\sum_i \omega_i(1-p_i)\theta_{iS}^2 - (\sum_i \omega_i\theta_{iS})(\sum_j (1-p_j)\omega_j(s)\theta_{jS}) \\ &+\frac{1}{2}(\sum_i \omega_i(s)\theta_{iS})^2(\sum_j (1-p_j)\omega_j(s)) \end{aligned} \tag{3.55}$$

The weights $\omega_i(s)$ are between 0 and 1. Therefore, in the heterogeneous ecomomy, the equilibrium market price of risk is between the minimum and the maximum market prices of risk in the homogeneous economies.

$$\min_i \theta_{iS} \leq \theta_S(s) \leq \max_i \theta_{iS} \tag{3.56}$$

The interest rate is also bounded. However, the aggregate interest rate could be smaller or bigger than all of the individual interest rates of the homogeneous economies.

3.3.6 Study of the equilibrium stock price

We want to determine the equilibrium stock price $S(s)$ and stock volatility $\sigma_S(s)$.

The homogeneous case

The stock price S_i in the homogeneous economy where only agent i is given by

$$S_i(s) = \frac{1}{M_i(s)}\mathbb{E}_s^{\mathbb{P}}\left[\int_s^T M_i(v)\epsilon(v)dv + M_i(T)\epsilon(T)\right]. \tag{3.57}$$

We are just saying that the price of the asset that pays the dividend stream $(\epsilon_t)_{0\leq s\leq T}$ is the discounted sum of the future dividends. Since, at time s, one dollar that will be received at time $u \geq s$ is worth $\frac{M_i(u)}{M_i(s)}$, we get the formula (3.57).

Proposition 3.19 (Stock Price in the homogeneous economy)**.** *The stock price in the homogeneous economy for agent i is*

$$S_i(s) \quad = \quad \epsilon(s)\int_s^T \frac{f_i(0,v)}{f_i(0,s)}s_i(u,v)dv + \epsilon(s)\frac{f_i(0,T)}{f_i(0,s)}s_i(s,T) \tag{3.58}$$

where

$$s_i(u,v) := \exp\left(\gamma_i(\mu_i - \frac{\sigma^2(1-\gamma_i)}{2})(v-u)\right). \tag{3.59}$$

The heterogeneous case

Using Ito's Lemma, and the market clearing conditions, it is easy to see that

$$d(S(t)M(t)) = -M(t)\epsilon(t)dt + M(t)(\sigma_S(t) - \theta_S(t))S(t)dW(t)$$

Thus, $S(t)M(t) + \int_0^t M(u)\epsilon(u)du$ is a $\mathbb{P}$ martingale. This leads to:

Proposition 3.20 (Stock price in the heterogeneous economy)**.** *The stock price in the heterogeneous economy is given by the discounted dividend*

$$S(s) = \frac{1}{M(s)}\mathbb{E}_s^{\mathbb{P}}\left[\int_s^T M(u)\epsilon(u)du + M(T)\epsilon(T)\right] \tag{3.60}$$

We have the following inequalities:

$$\min_i S_i(s) \leq S(s) \leq \max_i S_i(s). \tag{3.61}$$

The stock volatility is calculated in the following.

Proposition 3.21. *Stock price volatility*

The stock volatility σ_S is given by

$$\sigma_S(s) \quad = \quad \sigma + \frac{\mathbb{E}_s^{\mathbb{P}}\left[\int_s^T (\theta_S(s) - \theta_S(u))M_u\epsilon_u du + (\theta_S(s) - \theta_S(T))M_T\epsilon_T\right]}{\mathbb{E}_s^{\mathbb{P}}\left[\int_s^T M_u\epsilon_u du + M_T\epsilon_T\right]} \tag{3.62}$$

In particular,

$$\sigma + \min_i \theta_{iS} - \max_i \theta_{iS} \leq \sigma_S \leq \sigma + \max_i \theta_{iS} - \min_i \theta_{iS} \tag{3.63}$$

The drift of the stock is given by

$$\mu_S = \sigma_S \theta_S + r \tag{3.64}$$

The proof is based on a straight forward modification of the one found in (Cvitanic *et al.*, 2012).

3.3.7 The existence of the equilibrium

We have seen that the interest rate, market price of risk and stock volatility are bounded. All that is left is to show that r, θ_S have bounded derivatives. This is the content of the next result.

Proposition 3.22. *The quantities* $\frac{\partial r}{\partial w}$, $\frac{\partial \theta_S}{\partial w}$, $\frac{\partial r}{\partial s}$, $\frac{\partial \theta_S}{\partial s}$ *are bounded independently of* $(s,w) \in [0,T] \times \mathbb{R}$.

Proposition 3.23. *The parabolic PDE* (3.34) *has a unique bounded solution* v_i *which is explicitly given by the Feynman Kac formula:*

$$v_i(s,w) = \mathbb{E}_s^{\mathbb{P}}\left[\int_s^T e^{\int_s^z p_i(\gamma_i r + \frac{\gamma_i p_i \phi_i^2}{2} - \rho_i(0,u))du} dz + e^{\int_s^T p_i(\gamma_i r + \frac{\gamma_i p_i \phi_i^2}{2} - \rho_i(0,u))du}\right] \tag{3.65}$$

where r, ϕ_i *are evaluated at the point* (u, Y_u) *and* Y_u *is the unique solution of the SDE on* $[s,T]$*:*

$$Y_u = W_u + \int_s^u (\delta_i + \gamma_i p_i \phi_i(Y_v))dv; Y_s = W_s = w$$

In particular, v_i *is uniformly bounded from below by a positive constant. Furthermore,* $\frac{\partial v_i(s,w)}{\partial w}$ *and* $\frac{\partial^2 v_i(s,w)}{\partial w^2}$ *are uniformly bounded.*

Proof [Proposition 3.23] From proposition 3.22, we know that the coefficients of PDE (3.34) are continuously differentiable uniformly in s, w and they have bounded derivatives. Therefore, there is a unique bounded solution v_i to the PDE (3.34).

The fact that the first and second derivatives of v_i are bounded comes from PDE theory (see (Friedman, 1975) , Theorem 4.6).

□

The following theorem states that an equilibrium exists.

Theorem 3.24. *The equilibrium exists. The optimal strategy* $(\hat{\pi}^i, \hat{c}^i)$ *is given by* (3.35) *and the value function is* $V^i(s, w, x_i) = v_i(s, w)^{1-\gamma_i} U_i(x_i)$ *and* v_i *is given by expression* (3.65).

In the next section, we study the behaviour of the economy in the long run.

3.4 Study of the asymptotic behaviour

In this section, we study the long run behaviour of the equilibrium. We find out which agent dominates the consumption and which determines the long run interest rate and bond yield.

3.4.1 Asymptotic survival

(Yan, 2008) studied the natural selection phenomenon in a complete market. The concept is similar to the one found in biology. The rational expectations assumption is that investors act rationally all the time. So (Yan, 2008) studies a heterogeneous economy with two investors, one rational and the other one with an incorrect belief. (Yan, 2008) addresses the following two questions:

- First, which investor will survive in the long run?
- Second, if the investor with incorrect beliefs cannot survive, what is the time-span of the selection process?

We start with two assumptions. The first one is more general and the second one is more restrictive.

Assumption 3.25. Suppose

$$\frac{1}{T_0}\int_0^{T_0} \rho_i(0,s)ds \to \bar{\rho}_i \tag{3.66}$$

as $\quad T_0, T \to \infty, T_0 \leq T.$

Assumption 3.26.

$$\lim_{s\to\infty} \rho_i(0,s) = \bar{\rho}_i \tag{3.67}$$

Unless explicitly specified, we suppose assumption 3.67 holds. If we can show that the investor with incorrect beliefs cannot survive that would be a justification for the rational expectations hypothesis.

We will see that the agent's survival is dependent on one number: the survival rate which is a function of risk aversion, long run time preference and belief.

Definition 3.27. We say that agent i survives iff

$$\lim_{s\to\infty} \omega_i(s) = 1 \;\; a.s. \tag{3.68}$$

and that it becomes extinct or is driven out of the market iff

$$\lim_{s\to\infty} \omega_i(s) = 0 \;\; a.s. \tag{3.69}$$

Define the fraction of consumption for agent i

$$\eta_i(t) = \frac{\hat{c}^i(t)\hat{X}^i(t)}{\epsilon(t)} \tag{3.70}$$

Then $\eta_i(t) = \eta_{i0}(\frac{M_i(t)}{M(t)})^{p_i}$. Fix two indices $i \neq k$ and $s \in [0, T]$,

$$\begin{aligned}
&\frac{1}{s}(\log \eta_k(s)^{\frac{1}{p_k}} - \log \eta_i(s)^{\frac{1}{p_i}}) = \frac{1}{s}\Bigg(\frac{1}{p_k}\log(p_k\eta_{k0}) + \int_0^s (r(u) + \frac{\theta_S^2}{2} - r_k(u) - \frac{\theta_{kS}^2}{2})du \\
&+ \int_0^s (\theta_S(u) - \theta_{kS})dW_u\Bigg) - \frac{1}{s}\Bigg(\frac{1}{p_i}\log(p_i\eta_{i0}) + \int_0^s (r(u) + \frac{\theta_S^2}{2} - r_i(u) - \frac{\theta_{iS}^2}{2})du \\
&+ \int_0^s (\theta_S(u) - \theta_{iS})dW_u\Bigg) \\
&\sim_{s\to\infty} \frac{1}{s}\int_0^s (r_i(u) + \frac{\theta_{iS}^2}{2} - r_k(u) - \frac{\theta_{kS}^2}{2})du + (\theta_{iS} - \theta_{kS})\frac{W_s}{s}
\end{aligned}$$

By the Law of Iterated Logarithms, as $s \to \infty$,

$$\frac{W_s}{s} \to 0 \tag{3.71}$$

and

$$\frac{1}{s}(\log \eta_k(s)^{\frac{1}{p_k}} - \log \eta_i(s)^{\frac{1}{p_i}}) \to \bar{\rho}_i + \frac{\theta_{iS}^2}{2} - \bar{\rho}_k - \frac{\theta_{kS}^2}{2}. \tag{3.72}$$

Suppose

$$\bar{\rho}_i + \frac{\theta_{iS}^2}{2} < \bar{\rho}_k + \frac{\theta_{kS}^2}{2}$$

Then

$$\lim_{s\to\infty}\frac{1}{s}(\log\eta_k(s)^{\frac{1}{p_k}}-\log\eta_i(s)^{\frac{1}{p_i}})<0$$

i.e.

$$\lim_{s\to\infty}\frac{\eta_k(s)^{\frac{1}{p_k}}}{\eta_i(s)^{\frac{1}{p_i}}}=0 \tag{3.73}$$

This motivates the following definition:

Definition 3.28 (Survival index)**.** The survival index of agent i is defined as

$$\kappa_i=\bar{\rho}_i+\frac{\theta_{iS}^2}{2} \tag{3.74}$$

We add the following assumption

Assumption 3.29. All the κ_i are are assumed to be different.

Let i_K be the agent with the smallest survival index κ_i.

$$i_K:=\arg\min_i\kappa_i \tag{3.75}$$

Then for all $j\neq i_K$,

$$\lim_{s\to\infty}\frac{\omega_j(s)^{\frac{1}{p_j}}}{\omega_{i_K}(s)^{\frac{1}{p_{i_K}}}}=0$$

Since $\eta_{i_K}(s)\leq 1$,

$$\eta_j(s)^{\frac{1}{p_j}}\leq\frac{\eta_j(s)^{\frac{1}{p_j}}}{\eta_{i_K}(s)^{\frac{1}{p_{i_K}}}}\to_{s\to\infty}0 \tag{3.76}$$

We conclude that

$$\forall j\neq i_K,\quad\lim_{s\to\infty}\eta_j(s)=0 \tag{3.77}$$

and since

$$\eta_{i_K} = 1 - \sum_{j \neq i_K} \eta_j$$

we get

$$\lim_{s \to \infty} \eta_{i_K}(s) = 1 \tag{3.78}$$

and since $\omega_i = \frac{p_i \eta_i}{\sum_j p_j \eta_j}$, the asymptotic behaviour of ω_i is the same as the one for η_i. This is the content of the following proposition.

Proposition 3.30. *Agent i_K is the only one surviving in the long run $s, T \to \infty$, while $s \leq T$.*

$$\forall i \in \{1, \cdots, I\}, \quad \lim_{s \to \infty} \omega_i(s) = \delta_{i_K, i} \tag{3.79}$$

In the following $f_i(t, s) := f_i(s - t)$ is a function of one variable $\tau = s - t$.

Proposition 3.31 (exponential discounting)**.** *The discount function for agent i is $f_i(t, s) = \exp(-\rho_i(s - t))$.*

$$\kappa_i = \rho_i + \frac{\theta_{iS}^2}{2} \tag{3.80}$$

is agent i's survival index. This is compatible with the definition in (Cvitanic et al.*, 2012). When $s \to \infty$, the agent with the lowest survival index survives and all the others vanish.*

Proposition 3.32 (hyperbolic and generalized hyperbolic discounting)**.** *The hyperbolic discount function is*

$$f_i(t, s) = f_i(t, s; a_i) := \frac{1}{1 + a_i(s - t)} \tag{3.81}$$

and the generalized hyperbolic function is

$$f_i(t,s) = f_i(t,s;a_i,b_i) := (1 + a_i(s-t))^{-\frac{b_i}{a_i}} \tag{3.82}$$

We only deal with the second family since it generalizes the first.

$$\kappa_i = \lim_{s\to\infty} -\frac{f_i'}{f_i}(s) + \frac{\theta_{iS}^2}{2} = \lim_{s\to\infty} \frac{b_i}{1+a_i s} + \frac{\theta_{iS}^2}{2} = \frac{\theta_{iS}^2}{2}. \tag{3.83}$$

Again, only the agent with the lowest asymptotic survival index survives. That is the agent with the lowest $|\theta_{iS}|$. Agent i_K is not necessarily the agent with the most accurate expectation about the future of the economy.

Proposition 3.33 (generalized hyperbolic discounting with exponential rate)**.**

$$f_i(t,s) = f_i(t,s;\rho_i,a_i,b_i) := (1 + a_i(s-t))^{-\frac{b_i}{a_i}} \exp(-\rho_i(s-t)). \tag{3.84}$$

The survival index is :

$$\kappa_i = \lim_{s\to\infty} \frac{b_i}{1+a_i s} + \rho_i + \frac{\theta_{iS}^2}{2} = \rho_i + \frac{\theta_{iS}^2}{2}. \tag{3.85}$$

Again, only the agent with the lowest asymptotic survival index survives. That is the agent with the lowest $\rho_i + \frac{\theta_{iS}^2}{2}$.

In all the different discount functions above, the agent i_K has the lowest survival rate.

Next, we want to study the long-term (asymptotic) interest rate.

Definition 3.34. The long term interest rate of agent i in its homogeneous economy is

$$\bar{r}_i = \lim_{s\to\infty} r_i(s) = (1-\gamma_i)\mu_i - \sigma^2(1-\gamma_i)(1-\frac{\gamma_i}{2}) + \bar{\rho}_i. \tag{3.86}$$

If we assume all the $\bar{r}_i$ are different, let

$$\bar{r}_{i_r} := \inf_i \bar{r}_i. \tag{3.87}$$

The index i_r is constant and determined by the agent with the smallest long-run interest rate i.e.

$$i_r := \arg\min_{i\in\{1,\cdots,I\}}(1-\gamma_i)\mu_i - \sigma^2(1-\gamma_i)(1-\frac{\gamma_i}{2}) + \bar{\rho}_i. \tag{3.88}$$

Proposition 3.35. *Under assumption 3.29*

$$\lim_{s\to\infty} \frac{\hat{c}^i(s)\hat{X}^i(s)}{\epsilon(s)} = \delta_{i,i_K} \quad a.s. \tag{3.89}$$

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s r(u)du = \lim_{s\to\infty} \frac{1}{s}\int_0^s r_{i_K}(u)du = \bar{r}_{i_K} \tag{3.90}$$

Under the more restrictive assumption 3.67:

The equilibrium long term interest rate is equal to the long term interest rate of the individual agent i_K in its homogeneous economy.

$$\lim_{s\to\infty} r(s) = \lim_{s\to\infty} r_{i_K}(s) = \bar{r}_{i_K} \quad a.s. \tag{3.91}$$

Note that this is not necessarily equal to the lowest long term interest rate of each *agent. The indices i_r and i_K could be different.*

Next, we want to study the long term yield and the bond price for long maturities.

3.4.2 Determination of the bond price and the long yield

Most cash flows are calculated using interest rates from liquid debt instruments such as Treasury bonds. The graphic that depicts the relationship between bond yields of the same credit quality and their maturities is called a yield curve. We consider a bond in this market that matures at time $T_0 \in [t, T]$.

The bond price at time t with maturity $T_0 \geq t$ is given by

$$B(t, T_0) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp(-\int_t^{T_0} r(s)ds)\right] := \exp(-Y(t, T_0)(T_0 - t)) = \mathbb{E}_t^{\mathbb{P}}\left[\frac{M_{T_0}}{M_t}\right] \quad (3.92)$$

where $\mathbb{Q}$ is the risk neutral measure $Y(t, T_0)$ is the yield between time t and T_0. Let $Y_i(t, T_0)$ be the yield in the homogeneous economy populated by agent i between time t and time T_0. r_i is deterministic thus, the bond price in the homogeneous economy is

$$B_i(t, T_0) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp(-\int_t^{T_0} r_i(s)ds)\right] := \exp(-Y_i(t, T_0)(T_0 - t)) \quad (3.93)$$

and the yield is

$$Y_i(t, T_0) = \frac{1}{T_0 - t}\int_t^{T_0} r_i(s)ds \quad (3.94)$$

We want to study the behaviour of the yield curve as $t \to \infty$. We will see that the long term yield is determined by the agent i_r with the lowest long term rate.

Proposition 3.36. *The long term yield is determined by the agent i_r with the lowest asymptotic (long term) average interest rate $\bar{r}_i$. Fix $t > 0$. We have the following asymptotic result for $T_0 \leq T, T_0, T \to \infty$:*

$$Y(t, T_0) \sim \frac{1}{T_0} \int_0^{T_0} r_{i_r}(u) du \sim \bar{r}_{i_r} \tag{3.95}$$

The agent i_r that determines the long term yield could be different from the one that survives and dominates the long term consumption (i.e. i_K).

These results show that the asymptotic behaviour of the equilibrium with non constant discount rates is the same as the one with constant discount rates when the constant discount rates are set to equal the asymptotic average discount rate $\bar{\rho}_i$ of each agent. We can therefore give the same results as in (Cvitanic *et al.*, 2012). The proofs can be found in that paper.

3.5 Conclusion

The equilibrium can be completely solved by studying the individual agent's HJB. We get the equilibrium interest rate, market price of risk, stock volatility by writing the clearing conditions for the consumption and the stock.

Shreve, Karatzas have solved the equilibrium problem completely by considering the dual problem. In this chapter, I show that the equilibrium can be solved directly via the HJB equations. (Cvitanic *et al.*, 2012) follow the martingale approach as well.

If s is the current time and $T \geq s$ is the horizon, we can give the long term behaviour of the economy by taking $T \to \infty$ and $s \to \infty$ while maintaining $s \leq T$. We assume that $\rho_i(0, s)$ the discount rate of agent i between time 0 and time $s \in [0, T]$ converges as s grows to infinity to a constant called the asymptotic average discount rate of agent i. There is only one surviving agent and it is not necessarily the agent who has, at all times, the lowest discount rate. It is the agent with the lowest survival index as defined in this chapter. The survival index is a function of the long term discount rate, the risk aversion and the belief (optimism or pessimism) of the agent.

Chapter 4

Subgame Perfect Equilibrium with Two Heterogeneous Agents

4.1 Introduction

In Chapter 2, we have seen that non constant discount rates introduce time inconsistent behaviour. This dilemma could be solved by supposing each agent precommits to follow the optimal strategy set at time $t = 0$. In Chapter 3, we solved the equilibrium when the agents have different risk aversions, discount rates and beliefs about the future of the economy. We then characterized the equilibrium in terms of the homogeneous equilibria in which a given agent is the only one present in the economy.

However, this approach requires some mechanism to enforce that precommitment. Left to themselves, the agents will keep changing their strategies. In this chapter, we study the subgame perfect equilibrium also known as time consistent equilibrium in presence of heterogeneity. The agents differ in their time preferences, relative risk aversion and beliefs about the future of the economy. To simplify the study, we consider an economy with only two agents (or two types of agents). We show that the equilibrium characteristics for the subgame perfect strategies are the same as the ones obtained in the equilibrium for the pre commitment optimal strategies of Chapter 3, except that the time dependent discount rate of agent i, $\rho_i(0, s)$ is replaced by the utility weighted discount rate $\mathbb{Q}_i(s)$ of agent i.

4.2 The Model

As in chapter 3 , we are given a heterogeneous economy with $I = 2$ agents that have the following characteristics:

Exogenous parameters: There is only one source of uncertainty represented by the Brownian motion $W(t)$. The endowment (dividend) is $\epsilon(t)$ and satisfies the SDE

$$d\epsilon_t = \epsilon_t(\mu dt + \sigma dW_t) \tag{4.1}$$

For $i \in \{1, 2\}$, agent i has initial wealth x_i, discount function $f_i(t, s)$ which is the discounting between time t and time s. She believes the growth rate of the dividend is μ_i instead of μ. Call δ_i the quantity

$$\delta_i := \frac{\mu_i - \mu}{\sigma} \tag{4.2}$$

as introduced by Yan (2008).

Endogenous parameters

Agent i's utility for a given consumption stream (c_s^i) starting at time t is then given by

$$\mathbb{E}_t^{\mathbb{P}^i}\left[\int_t^T f_i(t, s)U_i(c_s^i)ds + f_i(t, T)U_i(X_T^i)\right] \tag{4.3}$$

where $\mathbb{E}^{\mathbb{P}^i}$ denotes the expectation operator from agent i's perspective, X_T^i is the final wealth of agent i, $U_i(x)$ is the utility function of agent i:

$$U_i(x) = \frac{x^{\gamma_i}}{\gamma_i}, x > 0, \gamma_i < 1, \gamma_i \neq 0. \tag{4.4}$$

$\mathbb{P}^i$ is the probability whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is

$$\frac{d\mathbb{P}^i}{d\mathbb{P}} = \exp\left(\delta_i W_T - \frac{\delta_i^2}{2}T\right) \tag{4.5}$$

For $0 \leq t \leq s \leq T$, define the two quantities

$$Z_i(t,s) = \exp(\delta_i(W_s - W_t) - \frac{\delta_i^2}{2}(s-t)) \tag{4.6}$$

W^i is the Brownian motion corresponding to $\mathbb{P}^i$.

$$W_t^i = W_t - \delta_i t \tag{4.7}$$

We assume that markets are complete which means that all Arrow-Debreu securities can be traded. In order to deal with asset pricing issues, we suppose that agents can continuously trade in a riskless asset and in risky stocks. We let S^0 denote the riskless asset price process with dynamics

$$dS_t^0 = r_t S_t^0 dt \tag{4.8}$$

the parameter r denoting the risk free rate at time t . Since there is only one source of risk, all risky assets have the same instantaneous Sharpe ratio and it suffices to focus on one specific risky asset. We consider the asset S whose dividend process is given by the total endowment of the economy and we denote respectively by μ_S and σ_S its drift and volatility.

$$dS_t = S_t\left[\mu_S dt + \sigma_S dW_t\right] - \epsilon_t dt \tag{4.9}$$

We let

$$\theta_S := \frac{\mu_S - r}{\sigma_S} \tag{4.10}$$

denote the asset's Sharpe ratio or equivalently the market price of risk. The parameters r, μ_S and σ_S are to be determined endogenously in equilibrium.

It will be useful to introduce the adjusted market price of risk for agent i:

$$\phi_i := \theta_S + \delta_i. \tag{4.11}$$

Denote by $\pi^i(t), c^i(t)$ the fraction of wealth invested in the risky asset and the fraction of wealth consumed at time t. Define the wealth process $X^i := X^{\pi^i, c^i}$ by the SDE:

$$dX^i(s) = (r(s) - c^i(s) + \sigma_S(s)\pi^i(s)\theta_S(s))X^i(s)ds + \sigma_S(s)\pi^i(s)X^i(s)dW(s) \tag{4.12}$$

or in terms of W^i:

$$dX^i(s) = (r(s) - c^i(s) + \sigma_S(s)\pi^i(s)\phi_i(s))X^i(s)ds + \sigma_S(s)\pi^i(s)X^i(s)dW^i(s) \tag{4.13}$$

The notion of admissible strategies is similar to the one given in chapter 2. We give the definition here:

Definition 4.1. An $\mathbb{R}^2$-valued process $\{(\pi^i(t), c^i(t))\}_{0\le t\le T}$ is called an admissible strategy process if

- it is progressively measurable with respect to the sigma algebra $\sigma(\{W_t\}_{t\ge0})$.
- $c^i(t) \ge 0$, $X^i(t) \ge 0$ for all t, a.s.

- Moreover, we require that for all $t \in [0,T]$, $x_i \geq 0$,

$$\mathbb{E}_t^{\mathbb{P}^i}[\sup_{t \leq s \leq T} |U_i(c^i(s)X^i(s)|] < \infty \ , \ \mathbb{E}_t^{\mathbb{P}^i}[|U_i(X^i(T)|] < \infty$$

  where $X^i(t) = x_i$ almost surely.

Let $\mathcal{A}^i$ be the set of all admissible strategies (π^i, c^i).

Question: If each agent follows a subgame perfect strategy, is there an equilibrium?

Response: Under fairly general conditions, there is an equilibrium. Next, we make precise the definition of an equilibrium.

4.3 The Equilibrium Problem

Subgame perfect strategies are defined as in Chapter 2.

4.3.1 The Definition of the Equilibrium

Definition 4.2. An equilibrium $(r(s), \theta_S(s), \sigma_S(s), \bar{\pi}^i(s), \bar{c}^i(s))$ consists of an interest rate r, a market price of risk θ_S, a stock volatility σ_S, investment and consumption processes $\bar{\pi}^i(s), \bar{c}^i(s)$ such that markets clear, i.e. :

- (i) Each agent i chooses a subgame perfect strategy $(\bar{\pi}^i, \bar{c}^i)$ (as defined in Chapter 2) with wealth process $\bar{X}^i$.

- (ii) The commodity market clears :

$$\sum_{i=1}^{2} \bar{c}^i(s)\bar{X}^i(s) = \epsilon(s). \tag{4.14}$$

- (iii) The stock market clears:

$$\bar{X}^1(s) + \bar{X}^2(s) = S(s) \tag{4.15}$$

- (iv) The money market clears:

$$\bar{\pi}^1(s)\bar{X}^1(s) + \bar{\pi}^2(s)\bar{X}^2(s) = S(s) \tag{4.16}$$

where $\bar{X}^i(s) := X^{\bar{\pi}^i,\bar{c}^i}(s)$ is the wealth process associated to the subgame perfect strategy $(\bar{\pi}^i, \bar{c}^i)$.

We will begin with some definitions.

Definition 4.3. In what follows,

$$p_i := \frac{1}{1-\gamma_i} \tag{4.17}$$

is the inverse of the relative risk aversion. Define ω_2 as the risk weighted fraction of consumption of agent 2. It is defined as:

$$\omega_2(s) = \frac{p_2\bar{c}^2(s)\bar{X}^2(s)}{p_1\bar{c}^1(s)\bar{X}^1(s) + p_2\bar{c}^2(s)\bar{X}^2(s)} \tag{4.18}$$

ω_2 is stochastic, and hence, the resulting allocation of aggregate consumption between investors is, in general, not Pareto-optimal. (Basak and Cuoco, 1998) have solved the equilibrium problem of 2 agents that maximize their utility of inter temporal consumption and final wealth. One of the best treatment on the subject is found in (Chabakauri, 2013) . He demonstrates that the process ω_2 serves as a convenient state variable in terms of which the equilibrium parameters can be expressed.

In what follows, we look for strategies that have a feedback form. In the spirit of (Ekeland *et al.*, 2012), we restrict our search to Markovian strategies. π^i and c^i are deterministic functions of the current wealth x_i, current time t and current agent 2's risk weighted fraction of consumption ω_2. The indirect utility for strategy (π^i, c^i) is

$$J^i(t,\omega_2,x_i,\pi^i,c^i) = \mathbb{E}_t^{\mathbb{P}^i}\left[\int_t^T f_i(t,s)U_i(c^i(s)X^i(s))ds + f_i(t,T)U_i(X_T^i)\right] \tag{4.19}$$

where the expectation is conditioned on $X^i(t) = x_i$ and $\omega_2(t) = \omega_2$.

The value function V^i corresponding to the subgame perfect strategy of agent i will be denoted $V^i(t,\omega_2,x_i)$. For fixed $t \in [0,T]$, $\omega_2 \in [0,1]$ and $x_i > 0$

$$\begin{aligned} V^i(t,\omega_2,x_i) &= \mathbb{E}^{\mathbb{P}^i}\Bigg[\int_t^T f_i(t,s)U_i(\bar{c}^i(s)\bar{X}^i(s))ds \\ &\quad + f_i(t,T)U_i(\bar{X}_T^i)|\omega_2(t) = \omega_2, X_t^i = x_i\Bigg] \end{aligned} \tag{4.20}$$

We assume that ω_2 is given by:

$$d\omega_2(s) = -\omega_2(s)(\mu_{\omega_2}ds + \sigma_{\omega_2}dW_s) = -\omega_2(s)(\mu_{\omega_2 i}ds + \sigma_{\omega_2}dW_s^i) \tag{4.21}$$

with the parameters $\mu_{\omega_2}, \sigma_{\omega_2}$ to be determined at equilibrium and

$$\mu_{\omega_2 i} = \mu_{\omega_2} + \delta_i\sigma_{\omega_2} \tag{4.22}$$

4.3.2 The Agents Subgame Perfect Strategies

The extended HJB was derived in Chapter 2. We give the equation directly without going through the proof again.

$$\begin{aligned}\frac{\partial V^i}{\partial t} + \sup_{(\pi^i,c^i)\text{ admissible}} \{\mathcal{A}^{\pi^i,c^i}V^i + U_i(c^i x_i)\} \\ = \mathbb{E}_t^{\mathbb{P}^i}\left[\int_t^T \frac{\partial f_i(t,s)}{\partial t} U_i(\bar{c}^i(s)\bar{X}^i(s))ds + \frac{\partial f_i(t,T)}{\partial t} U_i(\bar{X}^i(T))\right]\end{aligned} \tag{4.23}$$

and

$$\begin{aligned}\mathcal{A}^{\pi^i,c^i}V^i = (r(t) - c^i(t) + \sigma_S(t)\pi^i(t)\phi_i(t))x_i V_x^i + \frac{1}{2}(\sigma_S\pi^i x_i)^2 V_{xx}^i - \omega_2\mu_{\omega_2 i}V_{\omega_2}^i \\ +\frac{1}{2}(\omega_2\sigma_{\omega_2})^2 V^i_{\omega_2\omega_2} - \sigma_S\pi^i\omega_2\sigma_{\omega_2}x_i V^i_{\omega_2 x}\end{aligned} \tag{4.24}$$

Definition 4.4. Define the utility weighted discount rate $\mathbb{Q}_i$ at time t as:

$$\mathbb{Q}_i(t) = \frac{\mathbb{E}_t^{\mathbb{P}^i}\left[\int_t^T \frac{\partial f_i(t,s)}{\partial t} U_i(\bar{c}^i(s)\bar{X}^i(s))ds + \frac{\partial f_i(t,T)}{\partial t} U_i(\bar{X}^i(T))\right]}{\mathbb{E}_t^{\mathbb{P}^i}\left[\int_t^T f_i(t,s) U_i(\bar{c}^i(s)\bar{X}^i(s))ds + f_i(t,T) U_i(\bar{X}^i(T))\right]} \tag{4.25}$$

In what follows, we make the following assumption:

Assumption 4.5. $\mathbb{Q}_i$ is known and is a deterministic function of $t, \omega_2(t)$. Furthermore, the map

$$\begin{cases} [0,T]\times[0,1] \to \mathbb{R} \\ (t,\omega_2) \mapsto \mathbb{Q}_i(t,\omega_2) \qquad \text{is } C^1. \end{cases}$$

The extended HJB becomes:

$$\frac{\partial V^i}{\partial t} + \sup_{\pi^i, c^i}\{\mathcal{A}^{\pi^i, c^i} V^i + U_i(c^i x_i)\} = \mathbb{Q}_i(t, \omega_2) V^i. \tag{4.26}$$

Taking the first order conditions in the extended HJB, we get:

$$\bar{c}^i = \frac{(x_i V^i_x)^{\frac{1}{\gamma_i - 1}}}{x_i} \quad ; \quad \sigma_S \bar{\pi}^i = \frac{1}{x_i V^i_{xx}}(\omega_2 \sigma_{\omega_2} V^i_{\omega_2 x} - \phi_i V^i_x) \tag{4.27}$$

We want to find V^i of a certain form, and, as in Chapter 3, we can apply a verification theorem to conclude that we have found the value function for agent i. This is the content of the next subsection.

4.3.3 A Degenerate Linear Parabolic PDE

Suppose an equilibrium r, θ_s, σ_S have been found.

Ansatz: We look for V^i of the form:

$$V^i(t, \omega_2, x_i) \quad = \quad v_i(t, \omega_2)^{1-\gamma_i} U_i(x_i) \tag{4.28}$$

where $v_i \in C^{1,2}([0, T] \times [0, 1])$.

We get the following expressions for the subgame perfect strategies in terms of v_i :

$$\sigma_S \bar{\pi}^i = -\omega_2 \sigma_{\omega_2} \frac{\frac{\partial v_i}{\partial \omega_2}}{v_i} + p_i \phi_i \quad ; \quad \bar{c}^i = \frac{1}{v_i} \tag{4.29}$$

Note that a parabola of the form $y(x) = \frac{1}{2} a x^2 + b x$ with $a < 0$ has a maximum at $x_0 = -\frac{b}{a}$ and the maximum is $y(x_0) = -\frac{b^2}{2a}$. Thus:

$$\frac{\partial V^i}{\partial t} + \frac{1}{2}(x_i\sigma_S\bar{\pi}^i)^2V^i_{xx} + (r-\bar{c}^i)x_iV^i_x + \frac{(\bar{c}^ix_i)^{\gamma_i}}{\gamma_i} - \frac{1}{2V^i_{xx}}(\omega_2\sigma_{\omega_2}V^i_{\omega_2 x} - \phi_iV^i_x)^2$$
$$-\omega_2\mu_{\omega_2 i}V^i_{\omega_2} + \frac{1}{2}(\omega_2\sigma_{\omega_2})^2V^i_{\omega_2\omega_2} = \mathbb{Q}_iV^i \tag{4.30}$$

Plugging the expressions (4.29) back into (4.30), we get after lengthy calculations that can be found in the appendix, the following proposition:

Proposition 4.6. *v_i is solution of a second order parabolic PDE:*

$$\frac{\partial v_i}{\partial t} + \frac{1}{2}\omega_2^2\sigma_{\omega_2}^2\frac{\partial^2 v_i}{\partial\omega_2^2} - \omega_2(\mu_{\omega_2 i} + \gamma_ip_i\phi_i\sigma_{\omega_2})\frac{\partial v_i}{\partial\omega_2} \tag{4.31}$$
$$+p_i(\gamma_ir + \frac{\gamma_ip_i\phi_i^2}{2} - \mathbb{Q}_i)v_i + 1 = 0$$

Note that the above PDE is degenerate : the second order matrix is non negative but is not positive. Thus, a fine study near the points of degeneracy is necessary. We will show later that by a change of variables we can eliminate the degeneracy.

Proposition 4.7. *The dynamics of the subgame perfect consumption of agent i are given by the SDE*

$$d\log(\bar{c}^i(t)\bar{X}^i(t)) = p_i(r + \frac{\theta_S^2 - \delta_i^2}{2} - \mathbb{Q}_i)dt + p_i\phi_idW_t \tag{4.32}$$

Again, we see that v_i does not appear explicitly in the expression of the subgame perfect consumption. The proof is given in Appendix 3.

Next, we give the dynamics of $\omega_2(t)$.

Proposition 4.8. *The process $\omega_2(t)$ defined in equation (4.18) satisfies the SDE:*

$$d\omega_2(t) = \omega_1(t)\omega_2(t)(p_1\omega_2 + p_2\omega_1)\big((\mathbb{Q}(t,\omega_2) + \Phi_0(\omega_2))dt + (\theta_{1S} - \theta_{2S})dW_t\big) \quad (4.33)$$

where $\omega_1 = 1 - \omega_2$ and Φ_0 is a polynomial in ω_2 explicity given by

$$\Phi_0(x) = \delta_1^2 - \delta_1\delta_2 + (\mu - \frac{\sigma^2}{2})(\gamma_2 - \gamma_1) + \frac{(\theta_{1S} - \theta_{2S})^2}{2}\big[p_1(2x(1-x) - x^2) + p_2((1-x)^2 - 2x(1-x))\big] \quad (4.34)$$

i.e.

$$\mu_{\omega_2} = -\omega_1(t)(p_1\omega_2 + p_2\omega_1)\big(\mathbb{Q}(t,\omega_2) + \Phi_0(\omega_2)\big) \quad (4.35)$$

$$\sigma_{\omega_2} = -\omega_1(t)(p_1\omega_2 + p_2\omega_1)(\theta_{1S} - \theta_{2S}) \quad (4.36)$$

The proof appears in Appendix 3.

In the next section, we study the homogeneous economy where only agent i is present.

4.3.4 Equilibrium in the homogeneous economies

Just as in Chapter 3, we start by considering the equilibrium characteristics that would prevail in an economy made of agent i only or that would prevail in the economy if all the initial endowment was concentrated on agent i.

We denote by S_i, r_i, μ_{iS}, σ_{iS}, θ_{iS} the equilibrium stock price, interest rate, stock drift, stock volatility and market price of risk in the homogeneous economy.

$$\theta_{iS}(v) := \frac{\mu_{iS}(v) - r_i(v)}{\sigma_{iS}(v)} \quad (4.37)$$

There is only one agent i so that the commodity clearing condition becomes:

$$\bar{c}^i(s)\bar{X}^i(s) = \epsilon(s) \tag{4.38}$$

Using the SDE (4.32) for $d\log(\bar{c}^i(s)\bar{X}^i(s))$ obtained previously, we get

$$\begin{aligned} d\log\epsilon_s &= d\log(\bar{c}^i(s)\bar{X}^i(s)) \\ &= p_i(r_i(s) + \frac{\theta_{iS}^2 - \delta_i^2}{2} - q_i(s))ds + p_i(\theta_{iS} + \delta_i)dW(s) \quad (4.39) \\ &= (\mu - \frac{\sigma^2}{2})ds + \sigma dW(s) \quad (4.40) \end{aligned}$$

Comparing the ds terms and the $dW(s)$ terms in equations (4.39), (4.40) yields:

$$\begin{cases} p_i(r_i(s) + \frac{\theta_{iS}^2 - \delta_i^2}{2} - q_i(s)) = \mu - \frac{\sigma^2}{2} \\ p_i(\theta_{iS} + \delta_i) = \sigma \end{cases}$$

Thus, we have the following proposition:

Proposition 4.9. *In the homogeneous economy where only agent i is present, the market price of risk is θ_{iS}, the interest rate is $r_i(s)$, the stock volatility is σ_{iS} . They are given by the following expressions:*

$$\theta_{iS} = \sigma(1 - \gamma_i) - \delta_i \quad , \quad \sigma_{iS} = \sigma \tag{4.41}$$

$$r_i(s) = (1 - \gamma_i)\mu_i - \sigma^2(1 - \gamma_i)(1 - \frac{\gamma_i}{2}) + q_i(s) \tag{4.42}$$

Remark 4.10. We note that the equilibrium $\theta_{iS}, \sigma_{iS}, r_i$ are the same as those found by (Cvitanic *et al.*, 2012), except that we replace the constant discount rate ρ_i by $q_i(t)$:

the utility weighted discount rate of the homogeneous economy of agent i.

The pricing kernel in the homogeneous economy of agent i is defined as

$$M_i(s) = \exp\left(\int_0^s (-r_i(u) - \frac{\theta_{iS}^2}{2})du - \int_0^s \theta_{iS} dW(u) \right) \tag{4.43}$$

$\mathbb{Q}_i(t, \omega_2)$ is replaced by $q_i(t)$ if there is only one agent i in the economy i.e. $\mathbb{Q}_1(t, 0) = q_1(t)$ and $\mathbb{Q}_2(t, 1) = q_2(t)$.

Proposition 4.11. *The utility weighted discount rate in the homogeneous economy of agent i is given by*

$$q_i(t) = \frac{\int_t^T \frac{\partial f_i(t,s)}{\partial t} \exp(\gamma_i k_i(s-t))ds + \frac{\partial f_i(t,T)}{\partial t} \exp(\gamma_i k_i(T-t))}{\int_t^T f_i(t,s) \exp(\gamma_i k_i(s-t))ds + f_i(t,T) \exp(\gamma_i k_i(T-t))} \tag{4.44}$$

where

$$k_i := \mu_i + \frac{\sigma^2(\gamma_i - 1)}{2} \tag{4.45}$$

The stock price S_i in the homogeneous economy where only agent i is present is given by:

$$S_i(t) = \frac{1}{M_i(t)} \mathbb{E}_t \left[\int_t^T M_i(v)\epsilon(v)dv + M_i(T)\epsilon(T) \right] \tag{4.46}$$

The next proposition computes $S_i(t)$ explicitly:

Proposition 4.12 (Stock Price in the homogeneous economy)**.** *The stock price in the homogeneous economy for agent i is*

$$S_i(t) \quad = \quad \epsilon(t) \int_t^T s_i(t,v)dv + \epsilon(t)s_i(t,T) \tag{4.47}$$

where

$$s_i(t,v) := \exp\left(\int_t^v \gamma_i\big(\mu_i - \frac{\sigma^2(1-\gamma_i)}{2}\big) - q_i(u)du\right) \tag{4.48}$$

Furthermore, we have the relations:

$$S_1(t) = \epsilon(t)v_1(t,0) \quad ; \quad S_2(t) = \epsilon(t)v_2(t,1) \tag{4.49}$$

The proof is in Appendix 3.

We can compare the subgame perfect and optimal equilibria parameters in the homogeneous economy.

4.3.5 Comparison between the subgame perfect and optimal homogeneous economy equilibria

We are back into the homogeneous economy of agent i. We see that the equilibrium market price of risk θ_{iS} is the same for both the subgame perfect and the optimal equilibrium.

The interest rate for the subgame perfect equilibrium is obtained by replacing the discount rate $\rho_i(0,s)$ by $q_i(s)$. The keyword "opt" will denote the optimal equilibrium and "sub" will denote the subgame perfect equilibrium. We choose the following parameters:

$k_1 = 0.3$, $k_2 = -\frac{k_1 \log(0.3)}{\log(1+k1)}$, $f_i(t,s) = (1+k_1(s-t))^{-\frac{k_2}{k_1}}$, $\delta_i = 0.25$, $T = 40$. We see that away from the time horizon T, $q_i(t)$ is almost constant. The subgame perfect agent behaves approximately as if he was someone optimizing his total utility but with a constant discount rate.

In figure a), we plot the price dividend ratio $R(t)$ for subgame perfect and optimal

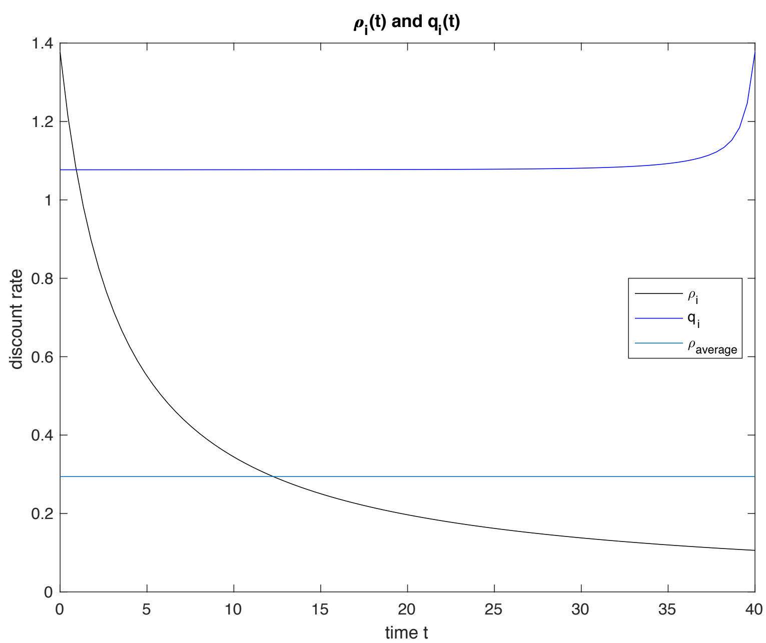


Figure 4.11: $q_i(t)$ and $\rho_i(0,t)$. The discount function $\rho_i(0,t)$ is a decreasing function of time t however the utility weighted discount rate $q_i(t)$ tends to increase with time. *We notice that for the subgame perfect equilibrium, away from the final time T, the utility weighted discount rate $q_i(t)$ is almost constant.*

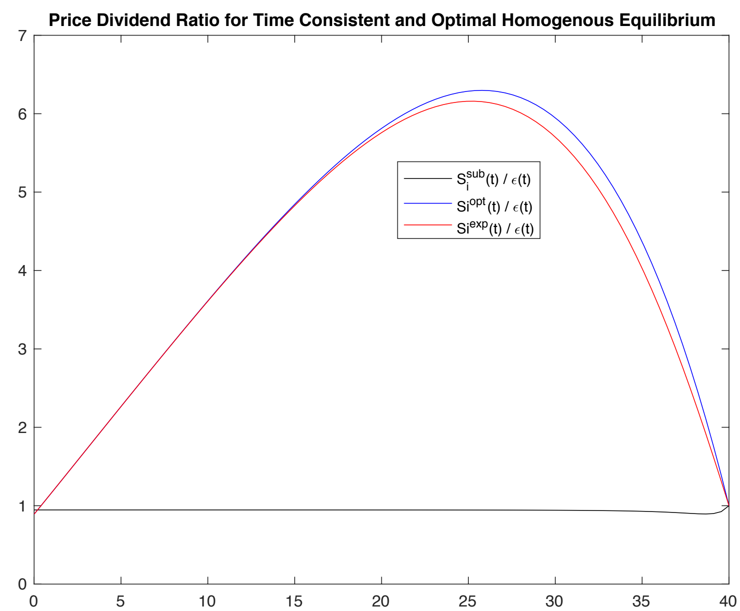


Figure 4.12: Price dividend ratio $R(t)$ as a function of time t.

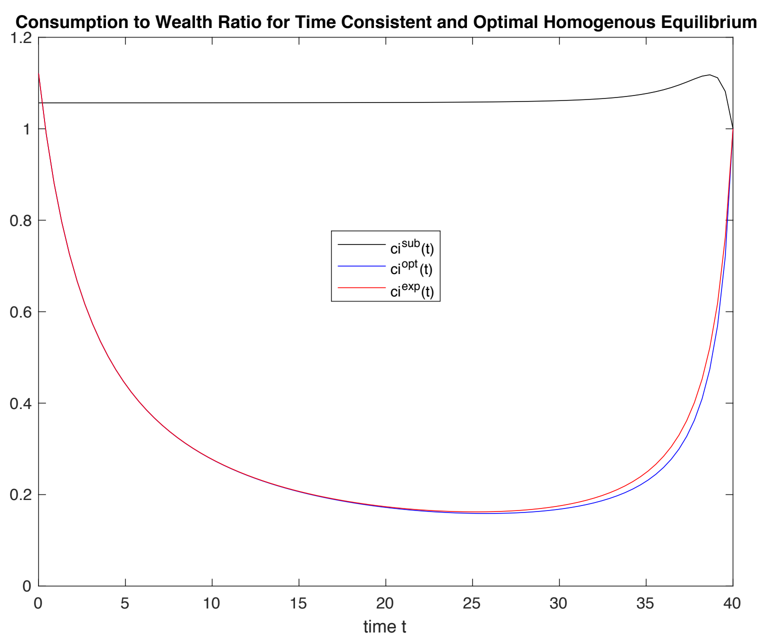


Figure 4.13: Consumption to wealth ratio $\bar{c}^i(t)$ and $\hat{c}^i(t)$ as a function of time t.

strategies. We compare the results to the case when the discount function is of the form $\exp(-\bar{\rho}t$ with $\bar{\rho} = \rho_{average} = \frac{1}{T}\int_0^T \rho_i(0,t)dt$ being the average discount rate of agent i.

In figure b) we see that the consumption rates are not monotonous. However, *we notice that for the subgame perfect equilibrium, away from the final time T, the price dividend ratio and the consumption rate are almost constant.*

4.3.6 Equilibrium in the heterogeneous economy

We now return to the real heterogeneous economy.

$$d\log(\bar{c}^i(t)\bar{X}^i(t) - d\log\epsilon_t = d\log(\bar{c}^i(t)\bar{X}^i(t)) - (p_i(r_i + \frac{\theta_{iS}^2 - \delta_i^2}{2} - q_i)dt - p_i(\theta_{iS} + \delta_i)dW_t$$
$$= p_i(r - r_i + \frac{\theta_S^2 - \theta_{iS}^2}{2} + q_i - \mathbb{Q}_i)dt + p_i(\theta_S - \theta_{iS})dW_t$$

Recall $\mathbb{Q}_i$ is the utility weighted discount rate of agent i. Integrating between 0 and t , we get:

$$\frac{\bar{c}^i(t)\bar{X}^i(t)}{\bar{c}^i(0)\bar{X}^i(0)} = \frac{\epsilon_t}{\epsilon_0}\left(\frac{M_i(t)}{M(t)}\exp(\int_0^t (q_i(u) - \mathbb{Q}_i(u))du\right)^{p_i} \tag{4.50}$$

where

$$M(t) = \exp(\int_0^t \big(-r(u) - \frac{\theta_S(u)^2}{2}\big)du - \int_0^t -\theta_S(u)dW(u)) \tag{4.51}$$

is the pricing kernel in the heterogeneous economy.

The market clearing condition

$$\sum_i \bar{c}^i(t)\bar{X}^i(t) = \epsilon_t$$

becomes

$$\sum_{i=1}^2 \frac{\bar{c}^i(0)\bar{X}^i(0)}{\epsilon(0)}\left(\frac{M_i(t)}{M(t)}\exp(\int_0^t (q_i(u) - \mathbb{Q}_i(u))du)\right)^{p_i} = 1 \tag{4.52}$$

or in the notation of Chapter 3,

$$M(t) = F\left((\frac{\bar{c}^1(0)\bar{X}^1(0)}{\epsilon(0)})^{\frac{1}{p_1}} M_1(t)e^{\int_0^t q_1(u) - \mathbb{Q}_1(u)du}, (\frac{\bar{c}^2(0)\bar{X}^2(0)}{\epsilon(0)})^{\frac{1}{p_2}} M_2(t)e^{\int_0^t q_2(u) - \mathbb{Q}_2(u)du} \right) \tag{4.53}$$

Next, we characterize the equilibrium interest rate and market price of risk.

Proposition 4.13. *The equilibrium interest rate r and market price of risk θ_S are given by:*

$$\theta_S(t) = \sum_{i=1}^{2} \omega_i(t)\theta_{iS} \tag{4.54}$$

$$r(t) = \sum_i \omega_i(t)\big(\mu(1-\gamma_i) - \frac{(1+p_i)\theta_S^2(t)}{2} - \delta_i p_i \theta_S(t) + \frac{(1-p_i)\delta_i^2}{2} + \mathbb{Q}_i(t,\omega_2(t))\big) \tag{4.55}$$

The proof will be given in Appendix 3. Notice that

$$r_i(s) = r_{i0} + q_i(s) \tag{4.56}$$

where

$$r_{i0} := (1-\gamma_i)\mu_i - \sigma^2(1-\gamma_i)(1-\frac{\gamma_i}{2}), \tag{4.57}$$

In the subgame perfect equilibrium, all we are doing is replacing the discount rate $\rho_i(0,s) = -\frac{\frac{\partial f_i(0,s)}{\partial s}}{f_i(0,s)}$ with the term $\mathbb{Q}_i(s)$. Again, the subgame perfect equilibrium is the same as the optimal equilibrium when we replace the discount rate of each agent by the utility weighted discount rate of that agent.

We can rewrite r as

$$r(s) = \sum_{i=1}^{2} \omega_i(s)(r_{i0} + \frac{\theta_{iS}^2 - \theta_S^2 - p_i(\theta_S - \theta_{iS})^2}{2} + \mathbb{Q}_i) \tag{4.58}$$

We see that the equilibrium parameters r, θ_S are completely determined in terms of t, ω_2 once $\mathbb{Q}_1, \mathbb{Q}_2$ are known and have sufficient regularity. We will show later that

S_t the equilibrium stock price depends on those quantities as well.

4.4 The Main Result

The first theorem establishes the existence of agent i's utility weighted discount rate $\mathbb{Q}_i$. We start with a definition.

Definition 4.14. Let

$$||\rho|| := \max(||\rho_1||, ||\rho_2||) \tag{4.59}$$

where

$$||\rho_i|| := \max_{0\le t\le s\le T} |\rho_i(t,s)| = \max_{0\le t\le s\le T} \left| \frac{\frac{\partial f_i(t,s)}{\partial t}}{f_i(t,s)} \right| \tag{4.60}$$

Define $\mathbb{B}$ the space of functions $\psi(t,x)$ from $[0,T]\times[0,1]\to\mathbb{R}$ that are C^1 in t,x and such that ψ and $\frac{\partial\psi}{\partial x}$ are bounded.

For $\nu>0$, define the subset $\mathbb{B}_\nu$ of $\mathbb{B}$:

$$\mathbb{B}_\nu = \{\psi \in \mathbb{B} \ s.t.\ ||\psi|| \le 2||\rho|| \text{ and } ||\frac{\partial\psi}{\partial x}|| \le \nu\} \tag{4.61}$$

where $||\ ||$ denotes the sup norm over all $(t,\omega_2)\in[0,T]\times[0,1]$.

For $y\in\mathbb{B}$, we define the process $\omega_2^y(s)$ by the SDE

$$d\omega_2^y(s) = \omega_2^y\omega_1^y\big(p_1\omega_2^y + p_2\omega_1^y(s)\big)\Big[(y(s,\omega_2^y(s)) + \Phi_0(\omega_2^y(s))ds + (\theta_{1S}-\theta_{2S})dW(s)\Big] \tag{4.62}$$

where

$$\omega_1^y(s) = 1 - \omega_2^y(s) \tag{4.63}$$

$$\Phi_0(x) = \delta_1^2 - \delta_1\delta_2 + \mu(\gamma_2 - \gamma_1) \tag{4.64}$$
$$+\frac{(\theta_{1S} - \theta_{2S})^2}{2}[p_1(2x(1-x) - x^2) + p_2((1-x)^2 - 2x(1-x))]$$

Call

$$\theta(x) = (1-x)\theta_{1S} + x\theta_{2S} \tag{4.65}$$

$$\begin{aligned}\alpha(x) := (1-x)&\left(\frac{\mu}{p_1} - \frac{p_1\theta(x)^2}{2} - \delta_1 p_1\theta(x) + \frac{(1-p_1)\delta_1^2}{2}\right) \\ +x&\left(\frac{\mu}{p_2} - \frac{p_2\theta(x)^2}{2} - \delta_2 p_2\theta(x) + \frac{(1-p_2)\delta_2^2}{2}\right)\end{aligned} \tag{4.66}$$

α is a polynomial of degree 3 in x with constant coefficients. Define

$$\begin{cases} a_1^y(u) = \frac{p_1\delta_1^2}{2} + p_1\gamma_1(\alpha(\omega_2^y(u)) - y(u, \omega_2^y(u))\omega_2^y(u)) \\ b_1^y(u) = \gamma_1 p_1\theta(\omega_2^y(u)) + p_1\delta_1 \\ a_2^y(u) = \frac{p_2\delta_2^2}{2} + p_2\gamma_2(\alpha(\omega_2^y(u)) + y(u, \omega_2^y(u))\omega_1^y(u)) \\ b_2^y(u) = \gamma_2 p_2\theta(\omega_2^y(u)) + p_2\delta_2 \end{cases} \tag{4.67}$$

Now, for $i = 1, 2$: consider the operators F_{1i}, F_{2i}, F_i acting on $y \in \mathbb{B}$ in the following way:

$$\forall t \in [0, T], \omega_2 \in [0, 1] : F_i[y](t, \omega_2) = \frac{F_{1i}[y](t, \omega_2)}{F_{0i}[y](t, \omega_2)} \tag{4.68}$$

where

$$F_{1i}[y](t,\omega_2) := \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T \frac{\partial f_i(t,s)}{\partial t} e^{\int_t^s a_i^y(u)du + \int_t^s b_i^y(u)dW_u} ds + \frac{\partial f_i(t,T)}{\partial t} e^{\int_t^T a_i^y(u)du + \int_t^T b_i^y(u)dW_u}\Big] \tag{4.69}$$

$$F_{0i}[y](t,\omega_2) := \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T f_i(t,s) e^{\int_t^s a_i^y(u)du + \int_t^s b_i^y(u)dW_u} ds + f_i(t,T) e^{\int_t^T a_i^y(u)du + \int_t^s b_i^y(u)dW_u}\Big] \tag{4.70}$$

with the conditional expectation calculated with $\omega_2^y(t) = \omega_2$. Finally, define the operator

$$F[y](t,\omega_2) = F_1[y](t,\omega_2) - F_2[y](t,\omega_2) \tag{4.71}$$

We start with an assumption:

Assumption 4.15. Suppose $\theta_{1S} \neq \theta_{2S}$.

Proposition 4.16. *For $y \in \mathbb{B}$, the process $\{\omega_2^y(s), s \geq t\}$ of (4.62) is well defined:*

$$d\omega_2^y(s) = \omega_2^y \omega_1^y(s)\big(p_1\omega_2^y + p_2\omega_1^y(s)\big)\Big[(y(s,\omega_2^y(s)) + \Phi_0(\omega_2^y(s))ds + (\theta_{1S} - \theta_{2S})dW(s)\Big] \tag{4.72}$$

and $\omega_2^y(s) \in [0,1]$ for all $s \geq t$.

We are now ready to give the main theorem.

Theorem 4.17. *There exists $\nu > 0$ that depends only on $\sigma, \gamma_i, \delta_i, T, f_i$ such that the operator F has a fixed point $\mathbb{Q} \in \mathbb{B}_\nu$ i.e.*

$$\forall (t,\omega_2) \in [0,T] \times [0,1] : F[\mathbb{Q}](t,\omega_2) = \mathbb{Q}(t,\omega_2) \tag{4.73}$$

Theorem 4.17 will be proved in appendix 3.

Theorem 4.18. *The function* $\mathbb{Q}_i : (t, \omega_2) \mapsto F_i[\mathbb{Q}](t, \omega_2)$ *is an element of* $\mathbb{B}$. *Furthermore, for any initial value* $\omega_{20} \in [0, 1]$, *the SDE*

$$\omega_2(0) = \omega_{20} \tag{4.74}$$

$$d\omega_2(s) = \omega_1\omega_2(p_1\omega_2(s) + p_2(1 - \omega_2(s)))[\mathbb{Q}(s, \omega_2(s)) + \Phi_0(\omega_2(s))ds + (\theta_{1S} - \theta_{2S})dW(s)] \tag{4.75}$$

has a unique solution defined for $s \in [0, T]$. *The function* $\Phi_0(x)$ *above represents a polynomial of degree 3 with constant coefficients given by* (4.64) . *Furthermore,*

$$\omega_2(s) \in [0, 1] \quad \forall s \in [0, T] \ \ a.s.$$

This theorem will be proved in Appendix 3. The following proposition gives an explicit PDE for v_i in terms of t, ω_2, $\mathbb{Q}_1$ and $\mathbb{Q}_2$:

Proposition 4.19. *There exists a unique bounded solution of the PDE:*

$$\begin{aligned} &\frac{\partial v_i}{\partial t} + \frac{1}{2}((\omega_1\omega_2(p_1\omega_2 + p_2\omega_1)(\theta_{1S} - \theta_{2S}))^2 \frac{\partial^2 v_i}{\partial \omega_2^2} \\ &+\omega_1\omega_2(p_1\omega_2 + p_2\omega_1)(k_{0i} + \mathbb{Q}(t, \omega_2))\frac{\partial v_i}{\partial \omega_2} + K_{1i}v_i + 1 = 0 \qquad (4.76) \\ &v_i(T, \omega_2) = 1 \qquad (4.77) \end{aligned}$$

where $\omega_1 = 1 - \omega_2$ *and*

$$\begin{aligned} k_{0i}(\omega_2) \quad &= \frac{1}{2}(\theta_{2S}(p_2\omega_1^2 - p_1\omega_2^2 + 2(p_1 - p_2)\omega_1\omega_2)(\theta_{2S} - \theta_{1S})^2 \qquad (4.78) \\ &+\gamma_i p_i \phi_i(\theta_{1S} - \theta_{2S}) + (\gamma_2 - \gamma_1)(\mu_i - \frac{\sigma^2}{2}) + \frac{\delta_1^2 - \delta_2^2}{2} + \delta_i(\delta_2 - \delta_1) \end{aligned}$$

$$K_{11}(t,\omega_2) = p_1\gamma_1(\alpha(\omega_2) + \frac{\omega_2}{2}\beta(\omega_2)) - p_1\gamma_1\omega_2\mathbb{Q}(t,\omega_2) - \mathbb{Q}_1(t,\omega_2) \quad (4.79)$$

$$K_{12}(t,\omega_2) = p_2\gamma_2(\alpha(\omega_2) - \frac{\omega_1}{2}\beta(\omega_2)) + p_2\gamma_2\omega_1\mathbb{Q}(t,\omega_2) - \mathbb{Q}_2(t,\omega_2) \quad (4.80)$$

where and θ, α , β are three polynomials on $q \in \mathbb{R}$ given by:

$$\begin{aligned}
\theta(q) &= (1-q)\theta_{1S} + q\theta_{2S} \\
\alpha(q) &= q(\frac{\mu}{p_2} + \frac{\delta_2^2 - (\theta(q))^2}{2}) + (1-q)(\frac{\mu}{p_1} + \frac{\delta_1^2 - \theta(\omega_2)^2}{2}) \\
\beta(q) &= p_1(\theta(q)+\delta_1)^2 - p_2(\theta(q)+\delta_2)^2
\end{aligned}$$

Theorem 4.20. *Make the assumption 4.15. Then the subgame perfect equilibrium exists. The utility weighted discount rate of agent i is given by*

$$\mathbb{Q}_i := F_i[\mathbb{Q}] \quad (4.81)$$

The value function for agent i is given by

$$V^i(t,\omega_2,x_i) = v_i(t,\omega_2)^{1-\gamma_i}\frac{x_i^{\gamma_i}}{\gamma_i} \quad (4.82)$$

where v_i is given in Proposition 4.19. The risk weighted consumption fraction of agent 2 is $\omega_2(t)$ given in equations (4.74), (4.75) of Theorem 4.18. The risk weighted consumption fraction of agent 1 is

$$\omega_1(t) = 1 - \omega_2(t)$$

The subgame perfect consumption of agent i is given by

$$\bar{c}^2(t)\bar{X}^2(t) = \frac{p_1\omega_2(t)\epsilon_t}{p_2\omega_1(t)+p_1\omega_2(t)} \quad , \quad \bar{c}^1(t)\bar{X}^1(t) = \frac{p_2\omega_1(t)\epsilon_t}{p_2\omega_1(t)+p_1\omega_2(t)} \tag{4.83}$$

Furthermore, $\mathbb{Q}$, $\mathbb{Q}_1$, $\mathbb{Q}_2$, r *and* θ_S *are deterministic functions of the parameters* (t,ω_2). *The equilibrium interest rate is* $r(t)$, *the market price of risk* $\theta_S(t)$ *are given by:*

$$\theta_S(t) = \sum_{i=1}^{2}\omega_i(t)\theta_{iS} \tag{4.84}$$

$$r(t) = \sum_{i=1}^{2}\omega_i(t)\left(\frac{\mu}{p_i} - \frac{1+p_i}{2}\theta_S^2 - \delta_i p_i\theta_S + \frac{1-p_i}{2}\delta_i^2 + \mathbb{Q}_i\right) \tag{4.85}$$

The consumption - wealth ratio is

$$\bar{c}^i(t) = \frac{1}{v_i(t,\omega_2(t))} \tag{4.86}$$

the investment - wealth ratio is

$$\bar{\pi}^i(t) = \omega_1\omega_2(p_1\omega_2+p_2\omega_1)(\theta_{1S}-\theta_{2S})\frac{\frac{\partial v_i(t,\omega_2(t))}{\partial\omega_2}}{v_i} + p_i\phi_i(t) \tag{4.87}$$

where $\phi_i(t) := \theta_S(t)+\delta_i$. *The subgame perfect wealth process* $\bar{X}^i(t)$, *stock price-dividend ratio* $R(t)$ *are given by:*

$$\bar{X}^1(t) = \frac{p_2\omega_1(t)v_1(t,\omega_2)}{p_2\omega_1(t)+p_1\omega_2(t)}\epsilon_t \quad , \quad \bar{X}^2(t) = \frac{p_1\omega_2(t)v_2(t,\omega_2)}{p_2\omega_1(t)+p_1\omega_2(t)}\epsilon_t \tag{4.88}$$

$$R(t) := \frac{S(t)}{\epsilon(t)} = \frac{p_2\omega_1(t)v_1(t,\omega_2)}{p_2\omega_1(t)+p_1\omega_2(t)} + \frac{p_1\omega_2(t)v_2(t,\omega_2)}{p_2\omega_1(t)+p_1\omega_2(t)} \tag{4.89}$$

The volatility of the stock price σ_S is

$$\sigma_S(t) = \sigma + \omega_1(t)\omega_2(t)(p_2\omega_1(t) + p_1\omega_2(t))\frac{\frac{\partial R(t,\omega_2(t))}{\partial \omega_2}}{R}(\theta_{1S} - \theta_{2S}) \tag{4.90}$$

The theorem above will be proved in Appendix 3.

Motivation behind the definition of the operator F

Note that since

$$d\log(\bar{c}^i(t)\bar{X}^i(t)) = p_i(r + \frac{\phi_i^2}{2} - \delta_i\phi_i - \mathbb{Q}_i)dt + p_i\phi_i dW_t,$$

A simple integration between 0 and t leads to:

$$\bar{c}^i(t)\bar{X}^i(t) = \bar{c}^i(0)\bar{X}^i(0)\exp\left(\int_0^t p_i(r + \frac{\phi_i^2}{2} - \delta_i\phi_i - \mathbb{Q}_i)du + \int_0^t p_i\phi_i dW_u\right) \tag{4.91}$$

We can write

$$\mathbb{E}_t^{\mathbb{P}}\left[Z_i(t,s)U_i(\frac{\bar{c}^i(s)\bar{X}^i(s)}{\bar{c}^i(t)\bar{X}^i(t)})\right] = \mathbb{E}_t^{\mathbb{P}}\left[e^{\int_t^s(-\frac{\delta_i^2}{2}+p_i\gamma_i(r+\frac{\phi_i^2}{2}-\delta_i\phi_i-\mathbb{Q}_i))du+(p_i\gamma_i\phi_i(u)-\delta_i)dW_u}\right]$$

So

$$\mathbb{Q}_i(t,\omega_2) = \frac{\int_t^T \frac{\partial f_i(t,s)}{\partial t}\mathbb{E}_t^{\mathbb{P}}\left[Z_i(t,s)U_i(\frac{\bar{c}^i(s)\bar{X}^i(s)}{\bar{c}^i(t)\bar{X}^i(t)})\right]ds + \frac{\partial f_i(t,T)}{\partial t}\mathbb{E}_t^{\mathbb{P}}\left[Z_i(t,T)U_i(\frac{\bar{c}^i(T)\bar{X}^i(T)}{\bar{c}^i(t)\bar{X}^i(t)})\right]}{\int_t^T f_i(t,s)\mathbb{E}_t^{\mathbb{P}}\left[Z_i(t,s)U_i(\frac{\bar{c}^i(s)\bar{X}^i(s)}{\bar{c}^i(t)\bar{X}^i(t)})\right]ds + f_i(t,T)\mathbb{E}_t^{\mathbb{P}}\left[Z_i(t,T)U_i(\frac{\bar{c}^i(T)\bar{X}^i(T)}{\bar{c}^i(t)\bar{X}^i(t)})\right]}$$

$$\mathbb{Q}_i(t,\omega_2) = \frac{\int_t^T \frac{\partial f_i(t,s)}{\partial t}e^{\int_t^s a_i(u)du+\int_t^s b_i(u)dW_u}ds + \frac{\partial f_i(t,T)}{\partial t}e^{\int_t^T a_i(u)du+\int_t^T b_i(u)dW_u}\Big]}{\int_t^T f_i(t,s)e^{\int_t^s a_i(u)du+\int_t^s b_i(u)dW_u}ds + f_i(t,T)e^{\int_t^T a_i(u)du+\int_t^s b_i(u)dW_u}\Big]}$$

where:

$$a_1(u) = -\frac{\delta_1^2}{2} + p_1\gamma_1(r + \frac{\phi_1^2}{2} - \delta_1\phi_1 - \mathbb{Q}_1) \quad ; \quad a_2(u) = -\frac{\delta_2^2}{2} + p_2\gamma_2(r + \frac{\phi_2^2}{2} - \delta_2\phi_2 - \mathbb{Q}_2)$$
$$b_1(u) = \delta_1 + p_1\gamma_1\phi_1 \quad ; \quad b_2(u) = \delta_2 + p_2\gamma_2\phi_2$$

Noting that $\delta_i\phi_i - \frac{1+p_i}{2}\phi_i^2 = \frac{1-p_i}{2}\delta_i^2 - \frac{1+p_i}{2}\theta_S^2 - \delta_i p_i\theta_S$, we get

$$r + \frac{\theta_S^2}{2} - \mathbb{Q}_1 = \sum_i \omega_i(\mu(1-\gamma_i) - \frac{p_i\theta_S^2}{2} - \delta_i p_i\theta_S + \frac{(1-p_i)\delta_i^2}{2}) + \omega_2(\mathbb{Q}_2 - \mathbb{Q}_1)$$
$$r + \frac{\theta_S^2}{2} - \mathbb{Q}_2 = \sum_i \omega_i(\mu(1-\gamma_i) - \frac{p_i\theta_S^2}{2} - \delta_i p_i\theta_S + \frac{(1-p_i)\delta_i^2}{2}) + \omega_1(\mathbb{Q}_1 - \mathbb{Q}_2)$$

Thus, since $\mathbb{Q} = \mathbb{Q}_1 - \mathbb{Q}_2$, we get:

$$a_1(u) = -\frac{\delta_1^2}{2} + p_1\gamma_1\sum_i \omega_i(\mu(1-\gamma_i) - \frac{p_i\theta_S^2}{2} - \delta_i p_i\theta_S + \frac{(1-p_i)\delta_i^2}{2}) - p_1\gamma_1\omega_2\mathbb{Q}$$
$$a_2(u) = -\frac{\delta_2^2}{2} + p_2\gamma_2\sum_i \omega_i(\mu(1-\gamma_i) - \frac{p_i\theta_S^2}{2} - \delta_i p_i\theta_S + \frac{(1-p_i)\delta_i^2}{2}) + p_2\gamma_2\omega_1\mathbb{Q}$$
$$b_1(u) = \delta_1 + p_1\gamma_1(\theta_S + \delta_1) \quad ; \quad b_2(u) = \delta_2 + p_2\gamma_2(\theta_S + \delta_2)$$

We see that the knowledge of the function $\mathbb{Q}$ will allow us to compute $\mathbb{Q}_1$ and $\mathbb{Q}_2$ and thus all the parameters of the equilibrium. We get the expressions for a_i^y, b_i^y by replacing $\mathbb{Q}$ by $y \in \mathbb{B}_\nu$ in the expression above.

4.5 Conclusion

We have shown that there is an equilibrium in an heterogeneous economy with 2 agents where each agent follows a subgame perfect strategy. This result is the most important result of this thesis.

In her paper, Asset Pricing with Dynamically Inconsistent Agents Khapko (2015), Mariana Khapko shows that there exists an equilibrium interest rate, market price of risk and stock price in an economy with one agent: a representative agent that follows rather general dynamics and utility function.

Our result shows the existence of a 2 agents equilibrium without resorting to a representative agent. We see that solving the subgame perfect problem is equivalent to determining the utility weighted discount rate.

Summary and Future work

In Chapter 2, we have introduced the time consistent / time inconsistent problem. Sub game perfect strategies are discussed and we show how they are more relevant than optimal strategies in certain situations. This chapter establishes a general framework for the later chapters.

In Chapter 3, we study the effects of heterogeneity on the equilibrium. We determine the interest rate and long term yield in function of the parameters of the problem. We show that in the long run (investment horizon $T \to \infty$), only the agent with the lowest survival index survives.

In Chapter 4, we study the effects of heterogeneity in an economy in which each agent follows a subgame perfect strategy. We have limited the study to two agents. We showed the importance of the utility weighted discount rate for solving the problem. We finished by providing a detailed comparison between optimal strategies and subgame perfect strategies. A numerical scheme was provided in Appendix 1.

In future work, we could study the asymptotic behaviour of the utility weighted discount rate $\mathbb{Q}$. Another line of work would be to use such a representation to create a martingale theory similar to the one that exists for optimal strategies. Finally, we could study the existence of time consistent strategies when the utility function has a more general form.

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Appendices

Appendix 0: Preliminary results

Stochastic Integration Results

Consider the following N dimensional SDE:

$$
\begin{aligned}
& dX^{t,x}(s) = \mu(s, X^{t,x}(s))ds + \sigma(s, X^{t,x}(s))dW(s); t \leq s \leq T \qquad (5.1)\\
& X^{t,x}(t) = x
\end{aligned}
$$

with $X^{t,x}(s) = (X_1^{t,x}(s), \cdots, X_N^{t,x}(s))$ an N dimensional process, W is an $\mathbb{F}_t$ adapted Brownian motion and $\mu = (\mu_1, \cdots, \mu_N)$, $\sigma = (\sigma_1, \cdots, \sigma_N)$ are continuous functions of t, x with bounded first derivatives $\frac{\partial \mu}{\partial x_i}$, $\frac{\partial \sigma}{\partial x_i}$.

Proposition 5.21. *The SDE* (5.99) *has a unique solution* $X^{t,x}(s)$. *Furthermore, the derivative processes* $D_{k,i}(s) := \frac{\partial X_i^{t,x}(s)}{\partial x_k}$ *exist and satisfy the SDE:*

$$
D_{k,i}(s) = \delta_{i,k} + \int_t^s \sum_{j=1}^N \frac{\partial \mu_i}{\partial x_j}(X_u^{t,x}) D_{k,j}(u) du + \int_t^s \sum_{j=1}^N \frac{\partial \sigma_i}{\partial x_j}(X_u^{t,x}) D_{k,j}(u) dW(u) \qquad (5.2)
$$

Proposition 5.22. *Fix t such that $0 \leq t \leq T$ and let $p \geq 2$. For a process Y_s defined for $s \in [t,T]$, let*

$$Y_s^* := \sup_{t \leq u \leq s} |Y_u|. \tag{5.3}$$

Suppose that a process X with values in $\mathbb{R}^n$ is defined on $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ via the equation

$$X_s = \zeta + J_s + \int_t^s \mu_u du + \int_t^s \sigma_u dW_u \tag{5.4}$$

with initial condition $X_t = \zeta \in \mathbb{L}^p$ independent of $\mathcal{F}$, J_s is $\mathcal{F}_s$ adapted and $J_t = 0$. Then, there is a constant $K_1 = K_1(p,T)$ such that

$$\mathbb{E}_t[(X^*)_s^p] \leq K_1 \left(\mathbb{E}[|\zeta|^p] + \mathbb{E}_t[(J^*)_s^p] + \mathbb{E}_t \left[\int_t^s (|\mu_u|^p + |\sigma_u|^p) du \right] \right) \tag{5.5}$$

If we also have $X_t = x$ for a deterministic x and $\mu_u = \mu(u, X_u)$ and $\sigma_u = \sigma(u, X_u)$ for two Lipschitz functions in the x variable, i.e.

$$|\mu(u,x_2) - \mu(u,x_1)| \leq C_\mu |x_2 - x_1| \ \ ; \ \ |\sigma(u,x_2) - \sigma(u,x_1)| \leq C_\sigma |x_2 - x_1| \tag{5.6}$$

Suppose also $\mu(u,0)$ and $\sigma(u,0)$ are bounded uniformly, then there is a constant $K > 0$ independent of t, s, X such that: then we get the estimate

$$\mathbb{E}_t[(X^*)_s^p] \leq K \left(|x|^p + \mathbb{E}_t[(J^*)_s^p] + \sup_{t \leq u \leq s} (|\mu(u,0)|^p + |\sigma(u,0)|^p)(s-t) \right) e^{2^{p-1}(C_\mu^p + C_\sigma^p)(s-t)} \tag{5.7}$$

Proof We write

$$X_s^{*p} \leq 4^{p-1}(|\zeta|^p + J_s^{*p} + \sup_{t\leq u\leq s}\{|\int_t^u \mu_v dv|^p + |\int_t^u \sigma_v dW(v)|^p\})$$

By Jensen's inequality:

$$\sup_{t\leq u\leq s}|\int_t^u \mu_v dv|^p \leq (\int_t^s |\mu_v|dv)^p \leq (s-t)^{p-1}\int_t^s |\mu_v|^p dv$$

and using the Burkholder-Davis-Gundy inequality followed by Jensen's inequality:

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s}\{|\int_t^u \sigma_v dW(v)|^p\} \leq C_p \mathbb{E}_t^{\mathbb{P}}[\int_t^s \sigma_v^2 dv])^{\frac{p}{2}} \leq C_p(s-t)^{\frac{p}{2}-1}\mathbb{E}_t^{\mathbb{P}}[\int_t^s |\sigma_v|^p dv]$$

This ends the proof of the first inequality. If σ and μ are Lipschitz in the x variable, we write $|\sigma_u|^p \leq (|\sigma(u,0)| + C_\sigma|X_u|)^p \leq 2^{p-1}(|\sigma(u,0)|^p + C_\sigma^p|X_u|)^p$ and a similar inequality : $|\mu_u|^p \leq 2^{p-1}(|\mu(u,0)|^p + C_\mu^p|X_u|)^p$ we see that

$$\mathbb{E}_t[(X^*)_s^p] \leq K_1\mathbb{E}_t^{\mathbb{P}}\left(|x|^p + (J^*)_s^p + 2^{p-1}\int_t^s |\mu(u,0)|^p + |\sigma(u,0)|^p du + 2^{p-1}\int_t^s (C_\mu^p + C_\sigma^p)|X_u|^p du\right)$$

And by Gronwall's inequality, we conclude:

$$\mathbb{E}_t[(X^*)_s^p] \leq K_1\left(|x|^p + \mathbb{E}_t[(J^*)_s^p] + 2^{p-1}\sup_{t\leq u\leq s}\{|\mu(u,0)|^p + |\sigma(u,0)|^p\}(s-t)\right)e^{2^{p-1}(C_\mu^p + C_\sigma^p)(s-t)}$$

Appendix 1: Subgame Perfect Strategies

Proof (Proposition 2.9)

Let $(\bar{\pi},\bar{c}) = \arg\max_{\pi,c}\{\mathcal{A}^{\pi,c}V + U_\gamma(xc)\}$. We recall that since V is a value function, V is concave in the variable x. By studying the convexity of the expression $\mathcal{A}^{\pi,c}V + U_\gamma(xc)$ as a function of 2 variables π, c, we can see that the critical points realize the maximum of the expression. The first order conditions for c give: $xU'_\gamma(cx) - xV_x = 0$, therefore

$$\bar{c} = \frac{V_x^{\frac{1}{\gamma-1}}}{x}$$

Similarly, the first order condition for π gives:

$$\bar{\pi} = -\frac{\theta_S V_x + S\sigma_S V_{Sx}}{\sigma_S x V_{xx}}$$

This ends the proof. $\square$

Proof (Theorem 2.8) Suppose that $V \in C^{1,2,2}$ is concave in x, satisfies (2.23) and $(\bar{\pi}, \bar{c})$ satisfies (2.26). We want to show that V is a value function and $(\bar{\pi}, \bar{c})$ is a sub game perfect strategy. First, we have to show that $V(t, S, x) = J(t, S, x, \bar{\pi}, \bar{c})$. As before, $\bar{X}$ represents the process $X^{\bar{\pi},\bar{c}}$. Dynkin's theorem states that the process

$$V(s, S_s, \bar{X}_s) - \int_0^s \big(\frac{\partial V}{\partial t}(u, S_u, \bar{X}_u) + \mathcal{A}^{\bar{\pi},\bar{c}}V(u, S_u, \bar{X}_u)\big)du \text{ is a } \mathbb{P}\text{-martingale.}$$

Therefore

$$\mathbb{E}_t[V(T, S_T, \bar{X}_T)] = V(t, S, x) + \mathbb{E}_t\left[\int_t^T \frac{\partial V}{\partial t}(u, S_u, \bar{X}_u) + \mathcal{A}^{\bar{\pi},\bar{c}}V(u, S_u, \bar{X}_u)du\right] \quad (5.8)$$

Let the function δ be defined by

$$\delta(t,s,S,x) = \begin{cases} \mathbb{E}_t^{\mathbb{P}}[U_\gamma(\bar{c}(s)\bar{X}(s))|\bar{X}(t)=x, S_t=S] & \text{if } t \le s < T \\ \mathbb{E}_t^{\mathbb{P}}[U_\gamma(\bar{X}(T))|\bar{X}(t)=x, S_t=S] & \text{if } s=T \end{cases} \tag{5.9}$$

By using (2.23) in the RHS of expression (5.99), we get:

$$\begin{aligned} RHS &= V(t,S,x) + \mathbb{E}_t\Big[\int_t^T \frac{\partial V}{\partial t}(s,S_s,\bar{X}_s) + \mathcal{A}^{\bar{\pi},\bar{c}}V(s,S_s,\bar{X}_s)ds\Big] \\ &= V(t,S,x) + \mathbb{E}_t\big[\int_t^T -U_\gamma((xF_c)(s,S_s,\bar{X}_s) + \mathbb{E}_s\big[\int_s^T \frac{\partial h}{\partial t}(s,u)U_\gamma(xF_c(u,S_u,\bar{X}_u)du \\ &+\frac{\partial h}{\partial t}(s,T)U_\gamma(\bar{X}_T)\big]ds\big] \qquad (5.10) \\ &= V(t,S,x) + \int_t^T (-\delta(t,s,S,x) + \int_s^T \frac{\partial h}{\partial t}(s,u)\delta(t,u,S,x)du + \frac{\partial h}{\partial t}(s,T)\delta(t,T,S,x)\big)ds \qquad (5.11) \end{aligned}$$

The last equality comes from the law of iterated conditional expectations. We then use the relation :

$$\begin{aligned} &\frac{\partial}{\partial s}\big(\int_s^T h(s,u)\delta(t,u,S,x)du + h(s,T)\delta(t,T,S,x)\big) \\ &= -\delta(t,s,S,x) + \int_s^T \frac{\partial h}{\partial t}(s,u)\delta(t,u,S,x)du + \frac{\partial h}{\partial t}(s,T)\delta(t,T,S,x) \end{aligned}$$

so that

$$\begin{aligned}
& V(t,S,x) + \int_t^T \frac{\partial}{\partial s}\Big(\int_s^T h(s,u)\delta(t,u,S,x)du + h(s,T)\delta(t,T,S,x)\Big) ds \\
= \;& V(t,S,x) + h(T,T)\delta(T,T,S,x) - \int_t^T h(t,u)\delta(t,u,S,x)du - h(t,T)\delta(t,T,S,x) \\
& = V(t,S,x) + U_\gamma(x) - \int_t^T h(t,u)\delta(t,u,S,x)du - h(t,T)\delta(t,T,S,x)
\end{aligned}$$

Since

$$\mathbb{E}_t[V(T,S_T,\bar{X}_T)] = \mathbb{E}_t[U_\gamma(\bar{X}_T)] \quad ,$$

$$V(t,S,x) \;=\; \int_t^T h(t,u)\delta(t,u,S,x)du + h(t,T)\delta(t,T,S,x) = J(t,S,x,\bar{\pi},\bar{c}) \tag{5.12}$$

This shows that V satisfies $V(t,S,x) = J(t,S,x,\bar{\pi},\bar{c})$. The next step is to show that $(\bar{\pi},\bar{c})$ is a sub game perfect strategy. For this, we study the lim inf of the quotient $\frac{J(t,S,x,\bar{\pi},\bar{c})-J(t,S,x,\pi_\epsilon,c_\epsilon)}{\epsilon}$.

$$\begin{aligned}
& \frac{J(t,S,x,\bar{\pi},\bar{c}) - J(t,S,x,\pi_\epsilon,c_\epsilon)}{\epsilon} \\
= \;& \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}\Big[\int_t^T h(t,u)\left(U_\gamma(\bar{X}(u)\bar{c}(u)) - U_\gamma(X_\epsilon(u)c_\epsilon(u))\right) du \Big] \qquad (5.13) \\
+ \;& \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}[h(t,T)(U_\gamma(\bar{X}(T)) - U_\gamma(X_\epsilon(T)))]
\end{aligned}$$

$$\frac{J(t,S,x,\bar{\pi},\bar{c}) - J(t,S,x,\pi_\epsilon,c_\epsilon)}{\epsilon} = K_1(\epsilon) + K_2(\epsilon) + K_3(\epsilon) \tag{5.14}$$

where the K_1, K_2, K_3 are obtained by introducing intermediate terms in the expression above.

$$
\begin{aligned}
K_1(\epsilon) &= \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}\left[\int_t^{t+\epsilon} h(t,u)\left(U_\gamma(\bar{X}(u)\bar{c}(u)) - U_\gamma(X_\epsilon(u)c_\epsilon(u))\right) du\right] \\
K_2(\epsilon) &= \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}\left[\int_{t+\epsilon}^{T} (h(t+\epsilon,u) - h(t,u))(U_\gamma(X_\epsilon(u)c_\epsilon(u)) - U_\gamma(\bar{X}(u)\bar{c}(u))du\right] \\
&+ \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}\Big[(h(t+\epsilon,T) - h(t,T))(-U_\gamma(X_\epsilon(T)) + U_\gamma(\bar{X}_T))\Big] \\
K_3(\epsilon) &= \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}\left[\int_{t+\epsilon}^{T} h(t+\epsilon,u)\left(U_\gamma(\bar{X}(u)\bar{c}(u)) - U_\gamma(X_\epsilon(u)c_\epsilon(u))\right) du\right] \\
&+ \frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}\left[h(t,T)(U_\gamma(\bar{X}(T)) - U_\gamma(X_\epsilon(T)))\right]
\end{aligned}
$$

It is easy to see that

$$
\lim_{\epsilon\to 0} K_1(\epsilon) = h(t,t)(U_\gamma(\bar{c}(t)x) - U_\gamma(c(t)x)) = U_\gamma(\bar{c}(t)x) - U_\gamma(c(t)x). \tag{5.15}
$$

$$
\begin{aligned}
&K_2(\epsilon) = \frac{1}{\epsilon}\int_{t+\epsilon}^{T}(h(t+\epsilon,u) - h(t,u))\mathbb{E}^{\mathbb{P}}_{t,S,x}\left[(U_\gamma(X_\epsilon(u)c_\epsilon(u)) - U_\gamma(\bar{X}(u)\bar{c}(u))du\right] \\
+ \quad &\frac{1}{\epsilon}\mathbb{E}^{\mathbb{P}}_{t,S,x}\left[(h(t+\epsilon,T) - h(t,T))(U_\gamma(\bar{X}(T)) - U_\gamma(X_\epsilon(T)))\right] \qquad (5.16) \\
&:= I_1(\epsilon) + I_2(\epsilon)
\end{aligned}
$$

where $I_1 = I_1(\epsilon)$ and $I_2 = I_2(\epsilon)$ are given by:

$$
I_1 = \frac{1}{\epsilon}\int_{t+\epsilon}^{T}(h(t+\epsilon,u) - h(t,u))\mathbb{E}_t\Bigg[\Bigg(\bigg(\frac{X_\epsilon(t+\epsilon)c_\epsilon(t+\epsilon)}{\bar{X}(t+\epsilon)\bar{c}(t+\epsilon)}\bigg)^\gamma - 1\Bigg)U_\gamma(\bar{c}(u)\bar{X}(u))du\Bigg] \tag{5.17}
$$

$$I_2 = \frac{1}{\epsilon}(h(t+\epsilon, T) - h(t,T)) \times \mathbb{E}_t \left[\left(\left(\frac{X_\epsilon(t+\epsilon)}{\bar{X}(t+\epsilon)} \right)^\gamma - 1 \right) U_\gamma(\bar{X}(T)) \right] \tag{5.18}$$

We can calculate an upper bound for the integral term I_1:

$$|I_1(\epsilon)| \leq \epsilon \int_{t+\epsilon}^T \left| \frac{\partial h(t_u^\epsilon, u)}{\partial t} \right| \times \mathbb{E}^{\mathbb{P}}_{t,S,x} \left[\left| \left(\frac{X_\epsilon(t+\epsilon)c_\epsilon(t+\epsilon)}{\bar{X}(t+\epsilon)\bar{c}(t+\epsilon)} \right)^\gamma - 1 \right| |U_\gamma(\bar{c}(u)\bar{X}(u))| \right] du$$

where $t_u^\epsilon \in [t, t+\epsilon]$ and by hypothesis $g(t,u) := \sup_{t_0 \in [t,t+1]} \left| \frac{\partial h(t_0,u)}{\partial t} \right|$ is integrable on $[t,T]$. Therefore

$$|I_1(\epsilon)| \leq \epsilon \int_{t+\epsilon}^T g(t,u) \times \mathbb{E}^{\mathbb{P}}_{t,S,x} \left[\left| \left(\frac{X_\epsilon(t+\epsilon)c_\epsilon(t+\epsilon)}{\bar{X}(t+\epsilon)\bar{c}(t+\epsilon)} \right)^\gamma - 1 \right| |U_\gamma(\bar{c}(u)\bar{X}(u))| \right] du$$

Since $X_\epsilon(t+\epsilon) \to x$ and $\bar{X}(t+\epsilon) \to x$, the integrand goes to

$$g(t,u)\mathbb{E}^{\mathbb{P}}_{t,S,x} \left[\left| \left(\frac{c(t)}{\bar{c}(t)} \right)^\gamma - 1 \right| |U_\gamma(\bar{c}(u)\bar{X}(u))| \right]$$

and by the dominated convergence theorem,

$$I_1(\epsilon) \to 0 \text{ when } \epsilon \to 0. \tag{5.19}$$

For the same reasons, $I_2 \to 0$ when $\epsilon \to 0$. Thus,

$$K_2(\epsilon) \to 0 \text{ as } \epsilon \to 0. \tag{5.20}$$

$$
\begin{aligned}
K_3(\epsilon) &= \frac{1}{\epsilon}\mathbb{E}_t^{\mathbb{P}}\big[V(t+\epsilon, S(t+\epsilon), \bar{X}(t+\epsilon)) - V(t+\epsilon, S(t+\epsilon), X_\epsilon(t+\epsilon))\big] \\
&= \frac{1}{\epsilon}\mathbb{E}_t^{\mathbb{P}}\big[V(t+\epsilon, S(t+\epsilon), \bar{X}(t+\epsilon)) - V(t, S, x) \\
&\quad + V(t, S, x) - V(t+\epsilon, S(t+\epsilon), X_\epsilon(t+\epsilon))\big] \\
&= \frac{1}{\epsilon}\mathbb{E}_t^{\mathbb{P}}\bigg[\int_t^{t+\epsilon} dV(u, S(u), \bar{X}(u)) - \frac{1}{\epsilon}\mathbb{E}_{t,S,x}^{\mathbb{P}}\int_t^{t+\epsilon} dV(u, S(u), X_\epsilon(u)))\bigg] \\
&= \frac{1}{\epsilon}\mathbb{E}_t^{\mathbb{P}}\bigg[\int_t^{t+\epsilon}(V_t + \mathcal{A}^{\bar{\pi},\bar{c}}V(u, S_u, \bar{X}_u))du - \int_t^{t+\epsilon}(V_t + \mathcal{A}^{\pi,c}V(u, S_u, X_u))du\bigg]
\end{aligned}
$$

$$
K_3(\epsilon) \rightarrow_{\epsilon\rightarrow 0} [V_t + \mathcal{A}^{\bar{\pi},\bar{c}}V(t, S, x)] - [V_t + \mathcal{A}^{\pi,c}V(t, S, x)] \tag{5.21}
$$

Combining the limits, we get

$$
\begin{aligned}
&\liminf_{\epsilon\rightarrow 0} \frac{J(t, S, x, \bar{\pi}, \bar{c}) - J(t, S, x, \pi_\epsilon, c_\epsilon)}{\epsilon} \\
&\quad = (U_\gamma(\bar{c}(t)x) + \mathcal{A}^{\bar{\pi},\bar{c}}V(t, S, x)) - (U_\gamma(c(t)x) + \mathcal{A}^{\pi,c}V(t, S, x)) \geq 0
\end{aligned} \tag{5.22}
$$

This ends the proof. □

Proof [Theorem 2.14]

We want to show the following: *There exists $\delta > 0$ such that the operator F defines a contraction on the space $\mathbb{B}_\delta$.*

In the following, K will be a positive constant that could vary from line to line.

Call $\rho(t, s) = \frac{\partial h(t,s)}{\partial t}/h(t, s)$ for $t, s \in [0, T]$. ρ is the (backward) discount rate and is bounded by $||\rho||$. Recall that

$$
\mathbb{B}_\delta := \{y \in C([0, T]; C^1(0, \infty)) \mid \forall t, S \in [0, T]\times(0, \infty), |y(t, S)| \leq ||\rho|| \;\&\; S|\frac{\partial y(t, S)}{\partial S}| \leq \delta\}. \tag{5.23}
$$

The proof will be structured in 6 Steps. In Step 1, we fix δ an arbitrary positive constant. We show that there is $\epsilon_0 > 0$ such that for $(t, S) \in [T - \epsilon_0, T] \times (0, \infty)$, $|F[y](t, S)| \leq ||\rho||$ and $S \left|\frac{\partial F[y](t,S)}{\partial S}\right| \leq \delta$.

In Step 2, we show there exists $\epsilon_1 \in (0, \epsilon_0)$ such that for $(t, S) \in [T - \epsilon_1, T] \times (0, \infty)$:

$$|F[y_2](t, S) - F[y_1](t, S)| \leq \frac{1}{2}|y_2(t, S) - y_1(t, S)| \tag{5.24}$$

In Step 3, we show there exists $\epsilon_2 \in (0, \epsilon_1)$ such that for $(t, S) \in [T - \epsilon_2, T] \times (0, \infty)$:

$$\left|S\frac{\partial F[y_2](t, S)}{\partial S} - S\frac{\partial F[y_1](t, S)}{\partial S}\right| \leq \frac{1}{2}\left|S\frac{\partial y_2(t, S)}{\partial S} - S\frac{\partial y_1(t, S)}{\partial S}\right| \tag{5.25}$$

In Step 4, we conclude that the restriction of F to the functions $y : [T - \epsilon_2, T] \times (0, \infty) \to \mathbb{R}$ is a contraction. We invoke Theorem 5 in Suzuki and Takahashi (1996) to conclude that F has a fixed point $\mathbb{Q}^1$ defined on $[T - \epsilon_2, T] \times (0, \infty)$.

In Step 5, we show that if ϵ_2 is small enough, we could find a fixed point $\mathbb{Q}^k$ on domains of the form $[T - k\epsilon_2, T - (k - 1)\epsilon_2] \times (0, \infty)$.

In Step 6, we specify a value for the constant δ. We construct a fixed point $\mathbb{Q}$ for the operator F defined over the whole interval $[0, T] \times (0, \infty)$ by using the fixed points $\mathbb{Q}^k$ of Step 5.

Let us start with the first step.

Step 1

Recall that for $y \in \mathbb{B}_\delta$:

$$F_0[y](t, S) = \mathbb{E}_t\bigg[\int_t^T h(t, s)e^{\int_t^s p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^s p\gamma\theta_S dW_u}ds \tag{5.26}$$
$$+h(t, T)e^{\int_t^T p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^T p\gamma\theta_S dW_u}\bigg]$$

$$F_1[y](t,S) = \mathbb{E}_t\Bigg[\int_t^T \frac{\partial h(t,s)}{\partial t} e^{\int_t^s p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^s p\gamma\theta_S dW_u} ds \qquad (5.27)$$
$$+\frac{\partial h(t,T)}{\partial t} e^{\int_t^T p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^T p\gamma\theta_S dW_u}\Bigg]$$

If $y \in \mathbb{B}_\delta$:

$$|F_1[y](t,S)| \le \mathbb{E}_t\Bigg[\int_t^T |\frac{\partial h(t,s)}{\partial t}| e^{\int_t^s p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^s p\gamma\theta_S dW_u} ds$$
$$+|\frac{\partial h(t,T)}{\partial t}| e^{\int_t^T p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^T p\gamma\theta_S dW_u}\Bigg] \qquad (5.28)$$

The integrand is smaller than

$$||\rho|| h(t,s) e^{\int_t^s p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^s p\gamma\theta_S dW_u}$$

and the second term in (5.120) is smaller than

$$||\rho|| h(t,T) e^{\int_t^T p\gamma(r+\frac{\theta_S^2}{2}-y)du+\int_t^T p\gamma\theta_S dW_u}$$

Adding the two upper bounds, we get:

$$|F_1[y](t,S)| \le ||\rho|| F_0[y](t,S) \qquad (5.29)$$

So $|F[y](t,S)| \le ||\rho|| \quad \forall\ t,S$. Now, we show that $F[y] \in C([0,T]; C^1((0,\infty))$ and get a bound of its S derivative. Define $a(t,S)$ and $b(t,S)$ as the du and dW_u terms appearing inside the exponent in the expressions (5.117), (5.118).

$$a := \gamma p(r + \frac{\theta_S^2}{2}) \quad ; \quad b := \gamma p \theta_S \qquad (5.30)$$

By using (Friedman, 1975), Chapter 5, Theorem 5.5 ,

$$D^{t,S}(u) := \frac{\partial S^{t,S}(u)}{\partial S} \tag{5.31}$$

satisfies

$$dD_u^{t,S} = D_u^{t,S}(\alpha_u du + \beta_u dW_u) \tag{5.32}$$

$$\text{where } \alpha(t,S) = \frac{\partial(S\mu_S(t,S))}{\partial S}; \beta(t,S) = \frac{\partial(S\sigma_S(t,S))}{\partial S} \tag{5.33}$$

are two bounded functions in t, S. $D^{t,S}$ has the closed form:

$$D^{t,S}(s) = \exp\left(\int_t^s \left(\alpha_u - \frac{\beta_u^2}{2}\right)du + \int_t^s \beta_u dW_u\right) \tag{5.34}$$

$$\begin{aligned}\frac{\partial F_0[y](t,S)}{\partial S} &= \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T h(t,s)\big(\int_t^s (\frac{\partial a}{\partial S} - \gamma p\frac{\partial y}{\partial S})D_u^{t,S}du + \int_t^s \frac{\partial b_u}{\partial S}D_u^{t,S}dW_u\big)e^{Z^{y;t,S}(s)}ds \\ &+h(t,T)\big(\int_t^T (\frac{\partial a}{\partial S} - \gamma p\frac{\partial y}{\partial S})D_u^{t,S}du + \int_t^T \frac{\partial b_u}{\partial S}D_u^{t,S}dW_u\big)e^{Z^{y;t,S}(T)}\Big]\end{aligned}$$

where for $s \geq t$,

$$Z^{y;t,S}(s) := \int_t^s (a(u,S_u) - \gamma p y(u,S_u))du + \int_t^s b(u,S_u)dW_u. \tag{5.35}$$

Recall that $p = \frac{1}{1-\gamma}$,

$$\begin{aligned}\frac{\partial F_0[y](t,S)}{\partial S} &= \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T h(t,s)\big(\int_t^s (\frac{\partial a}{\partial S} - \gamma p\frac{\partial y}{\partial S})D_u^{t,S}du + \int_t^s \frac{\partial b_u}{\partial S}D_u^{t,S}dW_u\big)e^{Z^{y;t,S}(s)}ds \\ &+h(t,T)\big(\int_t^T (\frac{\partial a}{\partial S} - \gamma p\frac{\partial y}{\partial S})D_u^{t,S}du + \int_t^T \frac{\partial b_u}{\partial S}D_u^{t,S}dW_u\big)e^{Z^{y;t,S}(T)}\Big]\end{aligned} \tag{5.36}$$

For the same reasons as inequality (5.120), we have the inequality

$$|\frac{\partial F_1[y]}{\partial S}| \leq ||\rho|||\frac{\partial F_0[y]}{\partial S}| \tag{5.37}$$

We can now get an upper bound for $S\left|\frac{\partial F[y](t,S)}{\partial S}\right|$:

$$\begin{aligned} S\left|\frac{\partial F[y](t,S)}{\partial S}\right| &= S\left|\frac{\frac{\partial F_1[y]}{\partial S}}{F_0[y]} - \frac{\frac{\partial F_0[y]}{\partial S}F_1[y]}{F_0^2[y]}\right| = S\left|\frac{\frac{\partial F_1[y]}{\partial S}}{F_0[y]} - F[y]\frac{\frac{\partial F_0[y]}{\partial S}}{F_0[y]}\right| \\ S\left|\frac{\partial F[y](t,S)}{\partial S}\right| &\leq 2||\rho||S\frac{|\frac{\partial F_0[y]}{\partial S}|}{F_0[y]} \end{aligned} \tag{5.38}$$

and by the Cauchy Schwarz inequality:

$$\begin{aligned} &\mathbb{E}_t^{\mathbb{P}}\left|\int_t^s D_u^{t,S}(\frac{\partial a_u}{\partial S} - \gamma p\frac{\partial y_u}{\partial S})du + \int_t^s \frac{\partial b_u}{\partial S}D_u^{t,S}dW_u) \times \exp(Z^{y;t,S}(s))\right| \\ &\leq \sqrt{\mathbb{E}_t^{\mathbb{P}}\Big(\int_t^s D_u^{t,S}(\frac{\partial a_u}{\partial S} - \gamma p\frac{\partial y_u}{\partial S})du + \int_t^s \frac{\partial b_u}{\partial S}D_u^{t,S}dW_u\Big)^2}\sqrt{\mathbb{E}_t^{\mathbb{P}}[\exp(2Z^{y;t,S}(s))]} \end{aligned}$$

Note that $S_u|\frac{\partial a_u}{\partial S}|$, $S_u|\frac{\partial y_u}{\partial S}|$ and $S_u|\frac{\partial b_u}{\partial S}|$ are bounded independently of t, S, u. For example, $S_u|\frac{\partial b_u}{\partial S}| = |\gamma|pS_u|\frac{\partial \theta_S(u,S_u)}{\partial S}| \leq |\gamma|p||S\frac{\partial \theta_S}{\partial S}||$ and $S_u|\frac{\partial y_u}{\partial S}| \leq \delta$. We get:

$$\begin{aligned} S|\frac{\partial F_0[y](t,S)}{\partial S}| \quad &\leq \quad K(1+||S\frac{\partial y}{\partial S}||)\sqrt{\mathbb{E}_t^{\mathbb{P}}\Big(\int_t^s \frac{S}{S_u}D_u^{t,S}du + \int_t^s \frac{S}{S_u}D_u^{t,S}dW_u\Big)^2} \\ &\quad \times e^{|\gamma|p||y||(s-t)} \\ \leq K(1+||S\frac{\partial y}{\partial S}||) \quad &\sqrt{(\mathbb{E}_t^{\mathbb{P}}\Big[2\big(\int_t^s \frac{S}{S_u}D_u^{t,S}du\big)^2 + 2\big(\int_t^s \frac{S}{S_u}D_u^{t,S}dW_u\big)^2\Big]}e^{|\gamma|p||y||(s-t)} \\ \leq K(1+||S\frac{\partial y}{\partial S}||) \quad &\sqrt{\mathbb{E}_t^{\mathbb{P}}\Big[(s-t)\int_t^s (\frac{S}{S_u}D_u^{t,S})^2du + \int_t^s (\frac{S}{S_u}D_u^{t,S})^2du\Big]} \times e^{|\gamma|p||y||(s-t)} \end{aligned}$$

$$S|\frac{\partial F_0[y](t,S)}{\partial S}| \le c(1+||S\frac{\partial y}{\partial S}||)\sqrt{s-t}\times e^{|\gamma|p||y||(s-t)} \tag{5.39}$$

where K is a positive constant independent of t, S. We can also have a lower bound for $F_0[y]$: We assume that $||y|| \le ||\rho||$ and $||S\frac{\partial y}{\partial S}|| \le \delta$ for a certain $\delta > 0$ to be fixed below.

$$F_0[y](t,S) = \mathbb{E}_t^{\mathbb{P}}\Bigg[\int_t^T h(t,s)e^{\int_t^s (a_u-\gamma py(u))du+\int_t^s b_u dW_u}ds \tag{5.40}$$

$$+h(t,T)e^{\int_t^T (a_u-\gamma py(u))du+\int_t^T b_u dW_u}\Bigg]$$

$$\ge \mathbb{E}_t^{\mathbb{P}}\Bigg[\int_t^T h(t,s)e^{\int_t^s -|\gamma|p||\rho||-||a||+\frac{b_u^2}{2}du}ds + h(t,T)e^{\int_t^T (-|\gamma|p||\rho||-||a||+\frac{b_u^2}{2})du}\Bigg] \tag{5.41}$$

$$F_0[y](t,S) \ge m(t)$$

where

$$m(t) = \int_t^T h(t,s)e^{\int_t^s -|\gamma|p||\rho||-||a||du}ds + h(t,T)e^{\int_t^T (-|\gamma|p||\rho||-||a||)du} \tag{5.42}$$

This shows that

$$\left|\frac{\partial F[y](t,S)}{\partial S}\right| \le 2||\rho||\frac{|\frac{\partial F_0[y]}{\partial S}|}{F_0[y]} \le \frac{2||\rho||K(1+||\frac{\partial y}{\partial S}||)\sqrt{T-t}\times e^{|\gamma|p||y||(T-t)}}{m(t)}$$

If we choose $0 < \epsilon_0 < T$ such that

$$\sqrt{\epsilon_0}\frac{2||\rho||K(1+|\delta)\times e^{|\gamma|p||\rho||T}}{\inf\{m(t), 0\le t\le T\}} \le \delta. \tag{5.43}$$

Then for $(t,S) \in [T-\epsilon_0, T]\times(0,\infty)$, $|F[y](t,S)| \le ||\rho||$ and $S\left|\frac{\partial F[y](t,S)}{\partial S}\right| \le \delta$.

This ends the proof of Step 1.

Step 2 We want to find an upper bound for the quantity $|F[y_2](t,S)-F[y_1](t,S)|$. Let $y_1, y_2 \in \mathbb{B}_\delta$:

$$
\begin{aligned}
|F[y_2](t,S) - F[y_1](t,S)| &= \left|\frac{F_1[y_2](t,S)}{F_0[y_2](t,S)} - \frac{F_1[y_1](t,S)}{F_0[y_1](t,S)}\right| \\
\leq & \quad \left|\frac{F_1[y_2](t,S) - F_1[y_1](t,S)}{F_0[y_2](t,S)} + \frac{F_1[y_1](t,S)(F_0[y_2] - F_0[y_1])}{F_0[y_1](t,S)F_0[y_2](t,S)}\right| \\
\leq & \quad \frac{2||\rho|||F_0[y_2] - F_0[y_1]|(t,S)}{F_0[y_2](t,S)}
\end{aligned}
$$

We can now get an upper bound for $|F_0[y_2](t,S) - F_0[y_1](t,S)|$

$$
\begin{aligned}
&|F_0[y_2](t,S) - F_0[y_1](t,S)| = \mathbb{E}_t \int_t^T h(t,s) e^{\int_t^s a_u du + b_u dW_u} \times \\
&|e^{\int_t^s -\gamma p y_2(u) du} - e^{\int_t^s -\gamma p y_1(u) du}| ds \\
&+ h(t,T) e^{\int_t^T a_u du + b_u dW_u} \times |e^{\int_t^T -\gamma p y_2(u) du} - e^{\int_t^T -\gamma p y_1(u) du}| \qquad (5.44)
\end{aligned}
$$

The mean value theorem, applied to the exponential gives

$$
\left|e^{\int_t^s -\gamma p y_2(u) du} - e^{\int_t^s -\gamma p y_1(u) du}\right| \leq |\gamma| p e^{|\gamma||p||\rho||T} \int_t^s |y_2(u) - y_1(u)| du \qquad (5.45)
$$

$$
\begin{aligned}
&|F[y_2](t,S) - F[y_1](t,S)| \leq \frac{1}{m(t)} \mathbb{E}_t \Big[\int_t^T h(t,s) e^{\gamma p \int_t^s a_u du + b_u dW_u} \times |\gamma| p e^{|\gamma||p||\rho||T} \qquad (5.46) \\
&\int_t^s |y_2(u) - y_1(u)| du ds + h(t,T) e^{\gamma p \int_t^s a_u du + b_u dW_u} \times |\gamma| p e^{|\gamma||p||\rho||T} \int_t^T |y_2(u) - y_1(u)| du \Big] \\
&\leq K(T-t) \max_{u,x \in [t,T] \times (0,\infty)} |y_2(u,x) - y_1(u,x)|
\end{aligned}
$$

where K is a constant that is independent of t, S. We can choose $\epsilon_1 \in (0, \epsilon_0]$ such that $K\epsilon_1 \leq \frac{1}{2}$. Call F^1 the operator $F|_{C([T-\epsilon_1,T]\times(0,\infty))}$.

Then for all $(t,S) \in [T-\epsilon_1, T] \times (0,\infty)$:

$$||F^1[y_2] - F^1[y_1]||_{C([T-\epsilon_1,T]\times(0,\infty))} \leq \frac{1}{2}||z-y||_{C([T-\epsilon_1,T]\times(0,\infty))} \tag{5.47}$$

This ends the proof of Step 2.

Step 3

We want to find an upper bound for the quantity $|\frac{\partial F_0[y_2](t,S)}{\partial S} - \frac{\partial F_0[y_1](t,S)}{\partial S}|$.

$$\begin{aligned}
&\left|\frac{\partial F[y_2](t,S)}{\partial S} - \frac{\partial F[y_1](t,S)}{\partial S}\right| = \left|\frac{\frac{\partial F_1[y_2]}{\partial S}}{F_0[y_2]} - \frac{\frac{\partial F_0[y_2]}{\partial S}F_1[y_2]}{F_0^2[y_2]} - \frac{\frac{\partial F_1[y_1]}{\partial S}}{F_0[y_1]} + \frac{\frac{\partial F_0[y_1]}{\partial S}F_1[y_1]}{F_0[y_1]^2}\right| \\
&\leq \left|\frac{\frac{\partial F_1[y_2]}{\partial S} - F[y_2]\frac{\partial F_0[y_2]}{\partial S}}{F_0[y_2]} - \frac{\frac{\partial F_1[y_1]}{\partial S} - F[y_1]\frac{\partial F_0[y_1]}{\partial S}}{F_0[y_1]}\right| \\
&\leq ||\rho||\frac{|F_0[y_2]-F_0[y_1]|}{F_0[y_2]} + \frac{1}{F_0[y_2]}(|\frac{\partial F_1[y_2]}{\partial S} - \frac{\partial F_1[y_1]}{\partial S}| + |F[y_2]-F[y_1]||\frac{\partial F_0[y_2]}{\partial S}| \\
&+|F[y_1]||\frac{\partial F_0[y_2]}{\partial S} - \frac{\partial F_0[y_1]}{\partial S}|)
\end{aligned}$$

$$S\left|\frac{\partial F_0[y_2](t,S)}{\partial S} - \frac{\partial F_0[y_1](t,S)}{\partial S}\right| \leq \int_t^T h(t,s)\Delta(t,s,S)ds + h(t,T)\Delta(t,T,S)$$

$$\begin{aligned}
S\left|\frac{\partial F_1[y_2](t,S)}{\partial S} - \frac{\partial F_1[y_1](t,S)}{\partial S}\right| &\leq \int_t^T |\frac{\partial h(t,s)}{\partial t}|\Delta(t,s,S)ds + |\frac{\partial h(t,T)}{\partial t}|\Delta(t,T,S) \\
S\left|\frac{\partial F_1[y_2](t,S)}{\partial S} - \frac{\partial F_1[y_1](t,S)}{\partial S}\right| &\leq ||\rho||S\left|\frac{\partial F_0[y_2](t,S)}{\partial S} - \frac{\partial F_0[y_1](t,S)}{\partial S}\right| \qquad (5.48)
\end{aligned}$$

where

$$
\begin{aligned}
&\Delta(t,s,S) := \Big|\mathbb{E}_t\Big[\Big(\int_t^s S D_u^{t,S}\big(\frac{\partial a(u)}{\partial S} - \gamma p \frac{\partial y_2}{\partial S}(u)\big)du + \int_t^s S\frac{\partial b_u}{\partial S} D_u^{t,S} dW_u\Big) e^{Z^{y_2;t,S}(s)} \\
- \quad & \Big(\int_t^s S D_u^{t,S}(\frac{\partial a}{\partial S} - \gamma p \frac{\partial y_1}{\partial S}(u))du + S\int_t^s \frac{\partial b_u}{\partial S} D_u^{t,S} dW_u\Big) e^{Z^{y_1;t,S}(s)}\Big]\Big| \\
&\leq \Delta_1(t,s,S) + \Delta_2(t,s,S) \\
&\Delta_1(t,s,S) := \Big|\mathbb{E}_t\Big[\int_t^s -\gamma p \frac{S}{S_u}(S_u \frac{\partial y_2}{\partial S} - S_u\frac{\partial y_1}{\partial S})(u)) D_u^{t,S} du e^{Z^{y_2;t,S}(s)}\Big]\Big| \\
&\leq |\gamma| p \max_{(u,x)\in(t,T)\times(0,\infty)} |x\frac{\partial y_2(u,x)}{\partial S} - x\frac{\partial y_1(u,x)}{\partial S}| \times \mathbb{E}_t^{\mathbb{P}}\big[\int_t^s \frac{S}{S_u} D_u^{t,S} du \times e^{Z^{y_2;t,S}(s)}\big] \\
\leq \quad & |\gamma| p \max_{(u,x)\in(t,T)\times(0,\infty)} |\frac{\partial y_2(u,x)}{\partial S} - \frac{\partial y_1(u,x)}{\partial S}| \\
&\sqrt{\mathbb{E}_t^{\mathbb{P}}\Big[\Big(\int_t^s \frac{S}{S_u} D_u^{t,S} du\Big)^2\Big]}\sqrt{\mathbb{E}_t\Big[\exp(2Z^{y_2;t,S}(s))\Big]} \\
&\Delta_1(t,s,S) \leq c(s-t) \max_{(u,x)\in(t,T)\times(0,\infty)} x|\frac{\partial y_2(u,x)}{\partial S} - \frac{\partial y_1(u,x)}{\partial S}|
\end{aligned}
$$

Similarly,

$$
\Delta_2(t,s,S) := S\Big|\mathbb{E}_t\Big[\Big(\int_t^s (\frac{\partial a}{\partial S} - \gamma p\frac{\partial y_1}{\partial S}) D_u^{t,S} du + \int_t^s \frac{\partial b_u}{\partial S} D_u^{t,S} dW_u\Big)\big(e^{Z_s^{y_2;t,S}} - e^{Z_s^{y_1;t,S}}\big)\Big]\Big|
$$

Noting that by the mean value theorem applied to the exponential function:

$$
\begin{aligned}
\big|e^{Z_s^{y_2;t,S}} - e^{Z_s^{y_1;t,S}}\big| &= e^{\int_t^s a_u du + b_u dW_u}\big|e^{-\gamma p\int_t^s y_2(u)du} - e^{-\gamma p \int_t^s y_1(u)du}\big| \\
&\leq e^{\int_t^s a_u du + b_u dW_u} \times |\gamma| p e^{|\gamma|p||\rho||T} \int_t^s |y_2(u) - y_1(u)| du \\
&\leq K(s-t) \max_{(u,x)\in(t,T)\times(0,\infty)} |y_2(u,x) - y_1(u,x)| e^{\int_t^s b_u dW_u}
\end{aligned}
$$

and

$$\Delta_2(t,s,S) \leq K(1+\delta)(s-t) \max_{(u,x)\in(t,T)\times(0,\infty)} |y_2(u,x) - y_1(u,x)| \quad (5.49)$$

$$\begin{aligned} S\left|\frac{\partial F[y_2](t,S)}{\partial S} - \frac{\partial F[y_1](t,S)}{\partial S}\right| &\leq c(1+\delta)\sqrt{T-t}\Big(\max_{(u,x)\in[t,T]\times(0,\infty)}|y_2(u,x) - y_1(u,x)| \\ &+ \max_{(u,x)\in[t,T]\times(0,\infty)} x|\frac{\partial y_2}{\partial S}(u,x) - \frac{\partial y_1}{\partial S}(u,x)|\Big) \quad (5.50) \end{aligned}$$

We can choose $\epsilon_2 \in (0, \epsilon_1)$ such that $K(1+\delta)\sqrt{\epsilon_2} \leq \frac{1}{2}$. Since

$$\mathbb{E}_t\left[\int_t^T e^{\gamma p \int_t^s a_u du + b_u dW_u}\right] \leq e^{|\gamma| p \frac{(||a|| + \frac{||b||^2}{2})(s-t)}{2}}$$

and $m(t)$ is bounded below by a positive constant, we can further choose ϵ_2 such that

$$\epsilon_2 \sup_{t\in[0,T]} \frac{|\gamma| p e^{|\gamma|p||\rho||T}}{m(t)} \mathbb{E}_t\left[\int_t^T h(t,s) e^{\gamma p \int_t^s a_u du + b_u dW_u} ds + h(t,T) e^{\gamma p \int_t^T a_u du + b_u dW_u}\right] \leq \frac{1}{2} \quad (5.51)$$

Then for all $(t,S) \in [T-\epsilon_2, T] \times (0,\infty)$:

$$||F[y_2] - F[y_1]||_{C([T-\epsilon_2,T];C^1(0,\infty))} \leq \frac{1}{2}||y_2 - y_1||_{C([T-\epsilon_2,T];C^1(0,\infty))} \quad (5.52)$$

This ends the proof of Step 3.

Step 4

If we call $\mathbb{B}^1_\delta := \{y|_{[T-\epsilon_2,T]\times(0,\infty)} \quad | \quad y \in \mathbb{B}_\delta\}$ the restriction of $\mathbb{B}_\delta$ to $[T-\epsilon_2, T] \times$

$(0, \infty)$, then we can conclude that $F(\mathbb{B}^1_\delta) \subset \mathbb{B}^1_\delta$.

Since the space of functions $C([T-\epsilon, T]; C^1(0,\infty))$ is complete and $\mathbb{B}^1_\delta$ is a closed subset of that space, $\mathbb{B}^1_\delta$ is a complete set. We use theorem 5 in (Suzuki and Takahashi, 1996) that we repeat below:

Theorem 5.23. *Let X be a linear normed space and let D be a convex subset of X. Then D is complete if and only if every contractive mapping from D into itself has a fixed point in D.*

$\mathbb{B}^1_\delta$ is both convex and complete, so we can apply the theorem. The operator $y \mapsto F[y]$ has a unique fixed point $\mathbb{Q}^1$ in $\mathbb{B}^1_\delta$. This ends the proof of Step 4. We can repeat the argument on $[T-2\epsilon_2, T-\epsilon_2]$. So on, until we reach 0. So we get $\mathbb{Q} : [0,T] \times (0,\infty) \to \mathbb{R}$ that coincides with the constructed fixed points on each interval $[T-k\epsilon_2, T-(k-1)\epsilon_2]$ and the problem is well posed. Let us do it in detail.

Step 5 Define

$$\begin{aligned}
\mathbb{B}^2_\delta = \{ & y \in C([T-2\epsilon_2, T-\epsilon_2]; C^1(0,\infty)) \text{ such that} \\
& y(T-\epsilon_2, S) = \mathbb{Q}^1(T-\epsilon_2, S); \frac{\partial y}{\partial S}(T-\epsilon_2, S) = \frac{\partial \mathbb{Q}}{\partial S}(T-\epsilon_2, S) \text{ and} \\
& \forall\, (t,S) \in [T-2\epsilon_2, T-\epsilon_2] \times (0,\infty), \quad |y(t,S)| \le ||\rho|| \quad \& \ S|\frac{\partial y}{\partial S}(t,S)| \le \delta \}
\end{aligned}$$

We will see below that δ can be chosen such that $F^2(\mathbb{B}^2_\delta) \subset \mathbb{B}^2_\delta$ and that F^2 has a unique fixed point $\mathbb{Q}^2$ in $\mathbb{B}^2_\delta$.

Let us define the operator F^2 on the space $C([T-2\epsilon_2, T-\epsilon_2]; C^1(0,\infty))$ by:

$$F^2[y](t,S) = \frac{F^2_1[y](t,S)}{F^2_0[y](t,S)}$$

and

$$F_0^2[y](t,S) = \begin{cases} F_0[\mathbb{Q}](t,S), & \text{if } \ T-\epsilon_2 < t \leq T \\ \\ F_0[y](t,S), & \text{if } \ T-\epsilon_2 \leq t \leq T \end{cases} \tag{5.53}$$

$$F_1^2[y](t,S) = \begin{cases} F_1[\mathbb{Q}](t,S), & \text{if } \ T-\epsilon_2 < t \leq T \\ \\ F_1[y](t,S), & \text{if } \ T-\epsilon_2 \leq t \leq T \end{cases} \tag{5.54}$$

For $y \in C([T-2\epsilon_2, T-\epsilon_2]; C^1(0,\infty))$ and $t \in [T-2\epsilon_2, T]$, define the function $y^2(t,S)$ by

$$y^2(t,S) = \begin{cases} \mathbb{Q}(t,S), & \text{if } \ T-\epsilon_2 < t \leq T \\ \\ y(t,S), & \text{if } \ T-2\epsilon_2 \leq t \leq T-\epsilon_2 \end{cases} \tag{5.55}$$

Now, as before, fix two functions $y_1, y_2 \in \mathbb{B}_\delta^2$, we have by the same kind of estimates as 5.137:

$$\begin{aligned} |F^2[y_2](t,S) - F^2[y_1](t,S)| \ &\leq \ \frac{|\gamma| p e^{|\gamma||p||\rho||T}}{m(t)} \mathbb{E}_t \Bigg[\int_t^T h(t,s) e^{\gamma p \int_t^s a_u du + b_u dW_u} \times \\ \int_t^s |y_2^2(u) - y_1^2(u)| du ds \ &+ \ h(t,T) e^{\gamma p \int_t^T a_u du + b_u dW_u} \times \int_t^T |y_2^2(u) - y_1^2(u)| du \Bigg] \end{aligned}$$

Noticing that $\int_t^s |y_2(u) - y_1(u)| du = \begin{cases} 0 \ \text{ if } \ \ t > T-\epsilon_2 \\ \\ \int_t^{T-\epsilon_2} |y_2^2(u) - y_1^2(u)| du \quad \text{if} \quad T-2\epsilon_2 \leq t \leq T-\epsilon_2 \end{cases}$

We get the estimate for $T-2\epsilon_2 \leq t \leq T-\epsilon_2$:

$$\begin{aligned}
&|F^2[y_2](t,S) - F^2[y_1](t,S)| \leq \frac{|\gamma|pe^{|\gamma|p||\rho||T}}{m(t)}\mathbb{E}_t\Bigg[\int_t^T h(t,s)e^{\gamma p\int_t^s a_u du + b_u dW_u} \times \\
&\int_t^{T-\epsilon_2}|y_2^2(u) - y_1^2(u)|duds + h(t,T)e^{\gamma p\int_t^T a_u du + b_u dW_u} \times \int_t^{T-\epsilon_2}|y_2^2(u) - y_1^2(u)|du\Bigg] \\
&\leq \frac{|\gamma|pe^{|\gamma|p||\rho||T}}{m(t)}\mathbb{E}_t\Bigg[\int_t^T h(t,s)e^{\gamma p\int_t^s a_u du + b_u dW_u}ds + h(t,T)e^{\gamma p\int_t^T a_u du + b_u dW_u}\Bigg] \\
&\times\epsilon_2 \max_{(u,x)\in[T-2\epsilon_2,T-\epsilon_2]\times(0,\infty)}|y_2(u,x) - y_1(u,x)|
\end{aligned}$$

And since ϵ_2 was chosen so that
$\frac{|\gamma|pe^{|\gamma|p||\rho||T}}{m(t)}\mathbb{E}_t\Bigg[\int_t^T h(t,s)e^{\gamma p\int_t^s a_u du + b_u dW_u}ds + h(t,T)e^{\gamma p\int_t^T a_u du + b_u dW_u}\Bigg] \times \epsilon_2 \leq \frac{1}{2}$, we get:

$$|F^2[y_2](t,S) - F^2[y_1](t,S)| \leq \frac{1}{2}\max_{(u,x)\in[T-2\epsilon_2,T-\epsilon_2]\times(0,\infty)}|y_2(u,x) - y_1(u,x)| \tag{5.56}$$

and a similar estimate for the derivative:

$$S\Big|\frac{\partial F^2[y_2](t,S)}{\partial S} - \frac{\partial F^2[y_1](t,S)}{\partial S}\Big| \leq \frac{1}{2}\max_{(u,x)\in[T-2\epsilon_2,T-\epsilon_2]\times(0,\infty)}x\Big|\frac{\partial y_2}{\partial S}(u,x) - \frac{\partial y_1}{\partial S}(u,x)\Big| \tag{5.57}$$

Since the space of functions $C([T-\epsilon,T];C^1(0,\infty))$ is complete and $\mathbb{B}_\delta^2$ is a closed subset of that space, $\mathbb{B}_\delta^2$ is a complete set. $\mathbb{B}_\delta^2$ is also convex.

By using theorem 5 (Suzuki and Takahashi, 1996), F^2 has a unique fixed point $\mathbb{Q}^2 \in \mathbb{B}_\delta^2$.

We can keep doing the same in $[T-3\epsilon_2, T-2\epsilon_2]$ until 0 and define F^i and $\mathbb{Q}^i$ (we suppose $N_0 = \frac{T}{\epsilon_2}$ is an integer. This ends the proof of Step 5.

Step 6

We define $\mathbb{Q}(t,S) = \mathbb{Q}^i(t,S)$ if $T - i\epsilon \leq t \leq T - (i-1)\epsilon_2$.

Then $\mathbb{Q}(t,S) = F[\mathbb{Q}](t,S)$ for all $(t,S) \in [0,T] \times (0,\infty)$.

All that is left is to choose $\delta > 0$ that works. We define the sequence

$$\phi_0 = 0, \quad \phi_{n+1} = F[\phi_n]$$

We can take

$$\delta = 2||\phi_1||_{C([0,T],C^1(0,\infty))}. \tag{5.58}$$

For $y \in \mathbb{B}^2_\delta$, we have by the triangle inequality:

$$\begin{aligned}
||F^2[y]||_{C([T-2\epsilon_2,T],C^1(0,\infty))} &\leq ||F^2[y] - F^2[0]||_{C([T-2\epsilon_2,T],C^1(0,\infty))} + ||F^2[0]||_{C([T-2\epsilon_2,T],C^1(0,\infty))} \\
&\leq \frac{1}{2}||y||_{C([T-2\epsilon_2,T],C^1(0,\infty))} + ||F^2[0]||_{C([T-2\epsilon_2,T],C^1(0,\infty))} \\
&\leq \frac{\delta}{2} + ||F^2[0]||_{C([T-2\epsilon_2,T],C^1(0,\infty))} = \frac{\delta}{2} + \frac{\delta}{2} = \delta
\end{aligned}$$

Thus, this choice of δ insures that the set $\mathbb{B}^2_\delta$ is stable under the operator F^2. The successive operators $F^k, k \geq 3$ will also leave the set $\mathbb{B}^k_\delta$ stable. We conclude that $\forall n \geq 0, \phi_n \in \mathbb{B}_\delta$.

This ends the proof that $\mathbb{Q}$ is in $C([0,T];C^1(0,\infty))$ and looking at the form of the t-derivative, we see that it is continuous in t,S. $\mathbb{Q}$ is C^1 in (t,S).

Proof [Theorem 2.17]

We want to prove the following: $\forall t \in [0,T],\ Q_t^{\bar{\pi},\bar{c}} = \mathbb{Q}(t,S_t)$. We proceed in 4 Steps. In Step 1, we define the function v that solves the PDE (2.42). Then we define the strategy

$$(\bar{\pi}, \bar{c}) = \left(\frac{p\theta_S}{\sigma_S} + \frac{Sv_s}{v} \ , \ \frac{1}{v(t,S)} \right)$$

and $\bar{X} := X^{\bar{\pi},\bar{c}}$.

In Step 2, we calculate the quantity $\bar{c}(s)\bar{X}(s)$ in a simple form.

In Step 3, we show that $\mathbb{Q}(t, S_t) = Q_t^{\bar{\pi},\bar{c}}$.

In Step 4, we show that

$$\bar{V}(t,S,x) = \mathbb{E}_t \left[\int_t^T h(t,s) U_\gamma(\bar{c}(s)\bar{X}(s)) ds + h(t,T) U_\gamma(\bar{c}(T)\bar{X}(T)) \right]$$

coincides with $V(t,S,x) = v(t,S)^{1-\gamma} U_\gamma(x)$.

In Step 5, we show that $\bar{V}$ satisfies the extended HJB equation (2.23). We conclude that $(\bar{\pi}, \bar{c})$ is a subgame perfect strategy with corresponding value function $\bar{V}$.

We start with the proof of the first step.

Step 1

We remind that

$$F[y](t,S) = \frac{F_1[y](t,S)}{F_0[y](t,S)} \quad , \quad \mathbb{Q}(t,S) = F[\mathbb{Q}](t,S)$$

We define the function v as the unique bounded $C^{1,2}$ solution of the PDE (2.42):

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t} + \frac{\sigma_S^2 S^2}{2} \frac{\partial^2 v}{\partial S^2} + p \left[\gamma r + \frac{\gamma p \theta_S^2}{2} - \mathbb{Q} \right] v + (r + p\sigma_S\theta_S) S \frac{\partial v}{\partial S} + 1 \\ & v(T,S) = 1 \end{aligned}$$

The existence of v as the bounded solution of the PDE (2.42) is a direct consequence of Friedman (Friedman (1975) , Chapter 6, Theorem 4.6).

Define the strategy $(\bar{\pi}, \bar{c})$ by:

$$\bar{c}(t,S) = \frac{1}{v(t,S)}\ ,\ \bar{\pi}(t,S) = p\frac{\theta_S}{\sigma_S} + \frac{Sv_s}{v} \tag{5.59}$$

and the corresponding wealth process $\bar{X}(s) = X^{\bar{\pi},\bar{c}}(s)$.

$$\frac{d\bar{X}(t)}{\bar{X}(t)} = (r + \sigma_S\theta_S\bar{\pi} - \bar{c})(t,S_t)dt + \sigma_S\bar{\pi}(t,S_t)dW_t \tag{5.60}$$

Step 2

We find a nicer expression for the consumption $\bar{c}(s)\bar{X}(s)$:

$$\begin{aligned}
&d\log(\bar{c}(t)\bar{X}(t)) = d\log(\frac{1}{v}) + d\log(\bar{X}(t)) = -\frac{dv}{v} + \frac{1}{2}(\frac{dv}{v})^2 + \frac{d\bar{X}}{\bar{X}} - \frac{1}{2}\left(\frac{d\bar{X}}{\bar{X}}\right)^2 \\
&= \Bigg[r - v^{-1} + \sigma_S\theta_S(\frac{p\theta_S}{\sigma_S} + \frac{S}{v}\frac{\partial v}{\partial S}) - \frac{\sigma_S^2}{2}(\frac{p\theta_S}{\sigma_S} + \frac{S}{v}\frac{\partial v}{\partial S})^2 \\
&-\frac{1}{v}\frac{\partial v}{\partial t} - \frac{\sigma_S^2S^2}{2v}\frac{\partial^2 v}{\partial^2 S} - \frac{S\mu_S S}{v}\frac{\partial v}{\partial S} + \frac{1}{2}(\frac{\sigma_S S}{v}\frac{\partial v}{\partial S})^2\Bigg]dt + p\theta_S dW(t) \\
&= \Bigg[-\frac{1}{v}\frac{\partial v}{\partial t} - \frac{\sigma_S^2S^2}{2v}\frac{\partial^2 v}{\partial^2 S} + \frac{S}{v}\frac{\partial v}{\partial S}(\sigma_S\theta_S - \frac{p\theta_S\sigma_S^2}{\sigma_S} - \mu_S) \\
&+r - v^{-1} + \sigma_S\theta_S\frac{p\theta_S}{\sigma_S} - \frac{\sigma_S^2}{2}(\frac{p\theta_S}{\sigma_S})^2\Bigg]dt + p\theta_S dW_t \\
&= -\frac{1}{v}\Bigg[\frac{\partial v}{\partial t} + \frac{\sigma_S^2S^2}{2}\frac{\partial^2 v}{\partial^2 S} - S\frac{\partial v}{\partial S}(\sigma_S\theta_S - \frac{p\theta_S\sigma_S^2}{\sigma_S} - \mu_S) - rv + 1 \\
&-\sigma_S\theta_S\frac{p\theta_S}{\sigma_S}v + v\frac{\sigma_S^2}{2}(\frac{p\theta_S}{\sigma_S})^2\Bigg]dt + p\theta_S dW_t
\end{aligned}$$

And since $\frac{\partial v}{\partial t} + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 v}{\partial S^2} + 1 = -p\left(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q}\right)v - (r + p\sigma_S\theta_S)S\frac{\partial v}{\partial S}$,

we get:

$$d\log(\bar{c}(t)\bar{X}(t)) = -\frac{1}{v}\Bigg[-p\left(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q}\right)v - (r + p\sigma_S\theta_S)S\frac{\partial v}{\partial S}$$
$$-S\frac{\partial v}{\partial S}(\sigma_S\theta_S - \frac{p\theta_S\sigma_S^2}{\sigma_S} - \mu_S) - rv - \sigma_S\theta_S\frac{p\theta_S}{\sigma_S}v + v\frac{\sigma_S^2}{2}(\frac{p\theta_S}{\sigma_S})^2\Bigg]dt + p\theta_S dW_t$$

The coefficient of $\frac{\partial v}{\partial S}$ inside the square bracket is :

$$-(r + p\sigma_S\theta_S)S - S(\sigma_S\theta_S - \frac{p\theta_S\sigma_S^2}{\sigma_S} - \mu_S) = S(\mu_S - r - \sigma_S\theta_S) = 0$$

since $\mu_S - r = \sigma_S\theta_S$.

The coefficient of v inside the square bracket is:

$$-p\left(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q}\right) - r - \sigma_S\theta_S\frac{p\theta_S}{\sigma_S} + \frac{\sigma_S^2}{2}(\frac{p\theta_S}{\sigma_S})^2$$
$$= -p(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q}) - r - p\theta_S^2 + \frac{p^2\theta_S^2}{2}) = -p\left(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q}\right)$$

Thus

$$d\log(\bar{c}(t)\bar{X}(t)) = p(r + \frac{\theta_S^2}{2} - \mathbb{Q})dt + p\theta_S dW(t) \tag{5.61}$$

Therefore:

$$\bar{c}(s)\bar{X}(s) = \bar{c}(t)\bar{X}(t)\exp\left(\int_t^s p(r_u + \frac{\theta_S^2(u)}{2} - \mathbb{Q}_u)du + \int_t^s p\theta_S dW(u)\right) \tag{5.62}$$

This ends the proof of Step 2.

Step 3 We get from Step 2,

$$\mathbb{Q}(t,S_t) = F[\mathbb{Q}](t,S_t) = \frac{\mathbb{E}_t\Big[\int_t^T \frac{\partial h(t,s)}{\partial t} U_\gamma(\bar{c}(s)\bar{X}(s))ds + \frac{\partial h(t,T)}{\partial t} U_\gamma(\bar{c}(T)\bar{X}(T))\Big]}{\mathbb{E}_t\Big[\int_t^T h(t,s)U_\gamma(\bar{c}(s)\bar{X}(s))ds + h(t,T)U_\gamma(\bar{c}(T)\bar{X}(T))\Big]}$$

$$\mathbb{Q}(t,S_t) = Q_t^{\bar{\pi},\bar{c}}$$

Remark 5.24. Note that $\bar{c}(T) = \frac{1}{v(T,S)} = 1$. Intuitively, the agent consumes everything at the final time T since the utility of final wealth is the same as that for consumption.

Step 4 Define

$$\bar{V}(t,S,x) = \mathbb{E}_t\Big[\int_t^T h(t,s)U_\gamma(\bar{c}(s)\bar{X}(s))ds + h(t,T)U_\gamma(\bar{c}(T)\bar{X}(T))\Big] \tag{5.63}$$

$$\begin{aligned}\bar{V}(t,S,x) &= \mathbb{E}_t\Big[\int_t^T h(t,s)U_\gamma\big(\frac{x}{v(t,S)}e^{\int_t^s p(r+\frac{\theta_S^2}{2}-\mathbb{Q})du+p\theta_S dW_u}\big)ds \\ &+h(t,T)U_\gamma\Big(\frac{x}{v(t,S)}e^{\int_t^T p(r+\frac{\theta_S^2}{2}-\mathbb{Q})du+p\theta_S dW_u}\Big)\Big] = Z(t,S)U_\gamma\big(\frac{x}{v(t,S)}\big) \\ \text{with } Z(t,S) &= \mathbb{E}_t\Big[\int_t^T h(t,s)e^{\int_t^s p\gamma(r+\frac{\theta_S^2}{2}-\mathbb{Q})du+p\gamma\theta_S dW(u)}ds \\ &+h(t,T)e^{\int_t^T p\gamma(r+\frac{\theta_S^2}{2}-\mathbb{Q})du+p\gamma\theta_S dW(u)}\Big]\end{aligned}$$

From the expression of $Z(t,S)$, we notice that its exponential term has a bounded coefficient in du and in $dW(u)$ so $Z(t,S)$ is uniformly bounded on $[0,T]\times(0,\infty)$. Furthermore, $Z(t,S)^{1-\gamma}$ is exactly the denominator $F_0[\mathbb{Q}]$ in the expression of $F[\mathbb{Q}]$.

We prove next that $Z = v$. Let

$$\alpha(t,s,S) = \mathbb{E}_t\Big[e^{\int_t^s p\gamma(r+\frac{\theta_S^2}{2}-\mathbb{Q})du+p\gamma\theta_S dW(u)}\Big] \tag{5.64}$$

$$e^{\int_0^t p\gamma(r+\frac{\theta_S^2}{2}-\mathbb{Q})du+p\gamma\theta_S dW(u)}\delta(t,s,S_t) \text{ is a } \mathbb{P}\text{-martingale}$$

By Ito's lemma

$$\alpha_t + \frac{\sigma_S^2 S^2}{2}\alpha_{SS} + (S\mu_S + \gamma p S\sigma_S\theta_S)\alpha_S + \gamma p(r + \frac{p\theta_S^2}{2} - \mathbb{Q})\alpha(t,s,S) = 0 \tag{5.65}$$

Using the fact that $\mu_S - r = \sigma_S\theta_S$, we get

$$\alpha_t + \frac{\sigma_S^2 S^2}{2}\alpha_{SS} + (r + p\theta_S\sigma_S)S\alpha_S + \gamma p(r + \frac{p\theta_S^2}{2} - \mathbb{Q})\alpha = 0. \tag{5.66}$$

Since $Z(t,S) = \int_t^T h(t,s)\alpha(t,s,S)ds + h(t,T)\alpha(t,T,S)$ we get:

$$Z_t = -\alpha(t,t,S) + \int_t^T \frac{\partial h(t,s)}{\partial t}\alpha(t,s,S)ds + \int_t^T h(t,s)\alpha_t(t,s,S)ds + \frac{\partial h(t,T)}{\partial t}\alpha(t,T,S)$$
$$+h(t,T)\alpha_t(t,T,S)$$

$$Z_S = \int_t^T h(t,s)\alpha_S(t,s,S)ds + h(t,T)\alpha_S(t,T,S)$$

$$Z_{SS} = \int_t^T h(t,s)\alpha_{SS}(t,s,S)ds + h(t,T)\alpha_{SS}(t,T,S)$$

Therefore

$$
\begin{aligned}
&\frac{\partial Z}{\partial t} + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 Z}{\partial S^2} + p\left[\gamma r + \frac{\gamma p \theta_S^2}{2} - \mathbb{Q}\right] Z + (r + p\sigma_S\theta_S)S\frac{\partial Z}{\partial S} + 1 \\
&= \int_t^T \frac{\partial h(t,s)}{\partial t}\alpha(t,s,S)ds + \frac{\partial h(t,T)}{\partial t}\alpha(t,T,S) \\
&+ \int_t^T h(t,s)\left[\alpha_t + \frac{\sigma_S^2 S^2}{2}\alpha_{SS} + (r + p\sigma_S\theta_S)S\alpha_S + p(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q})\alpha(t,s,S)\right]ds \\
&+h(t,T)\left[\alpha_t + \frac{\sigma_S^2 S^2}{2}\alpha_{SS} + (r + p\sigma_S\theta_S)S\alpha_S + p(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q})\alpha(t,T,S)\right] \\
&= \int_t^T \frac{\partial h(t,s)}{\partial t}\alpha(t,s,S)ds + \frac{\partial h(t,T)}{\partial t}\alpha(t,T,S) \\
&+ \int_t^T -\alpha(t,s,S)\mathbb{Q}(t,S)ds - \alpha(t,T,S)\mathbb{Q}(t,S) \\
&= 0
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{\partial Z}{\partial t} + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 Z}{\partial S^2} + p\left[\gamma r + \frac{\gamma p \theta_S^2}{2} - \mathbb{Q}\right] Z + (r + p\sigma_S\theta_S)S\frac{\partial Z}{\partial S} + 1 = 0 \\
&\text{and } Z(T,S) = 1
\end{aligned}
$$

By uniqueness of a bounded solution of the above linear parabolic PDE, $Z = v$ and

$$
\bar{V}(t,S,x) = Z(t,S)U_\gamma\left(\frac{x}{v(t,S)}\right) = v(t,S)^{1-\gamma}U_\gamma(x). \tag{5.67}
$$

Step 5 :

We want to show that $\bar{V}$ satisfies the extended HJB.

$$\mathcal{A}^{\pi,c}\bar{V} = \tag{5.68}$$
$$(r+\sigma_S\theta_S\pi - c)x\frac{\partial \bar{V}}{\partial x}(t,S,x) + \frac{\sigma_S^2 S^2}{2}\frac{\partial^2 \bar{V}}{\partial S^2}(t,S,x)$$
$$+\frac{1}{2}\sigma_S^2 x^2\pi^2\frac{\partial^2 \bar{V}}{\partial x^2}(t,S,x) + \sigma_S^2 S\pi x\frac{\partial^2 \bar{V}}{\partial S\partial x}(t,S,x) + (\mu_S+\gamma p\theta_S)S\frac{\partial \bar{V}}{\partial S}(t,S,x)$$

A direct calculation gives: $\bar{V}_t + \mathcal{A}^{\bar{\pi},\bar{c}}\bar{V} + U_\gamma(x\bar{c}) = Q_t^{\bar{\pi},\bar{c}}\bar{V}(t,S,x)$ and $\bar{V}$ satisfies

$$\bar{V}_t + \sup_{\pi,c}\{\mathcal{A}^{\pi,c}\bar{V} + U_\gamma(xc)\} = Q_t^{\bar{\pi},\bar{c}}\bar{V}(t,S,x) \tag{5.69}$$

$(\bar{\pi},\bar{c})$ is an admissible Markovian policy and $\bar{V}$ solves the extended Hamilton Jacobi Bellman equation (2.23). Therefore, $(\bar{\pi},\bar{c})$ is a sub game perfect strategy (by Theorem 2.8). This ends the proof of the theorem.

□

Remark 5.25. We can prove that $v(t,S)$ is bounded by invoking Feynman Kac's formula. This yields equation (2.38):

$$\begin{aligned} v(t,S) &= \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T e^{\int_t^s p(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q})(u,\bar{S}_u)du} ds \\ &\quad + e^{\int_t^T p(\gamma r + \frac{\gamma p\theta_S^2}{2} - \mathbb{Q})(u,\bar{S}_u)du} | \bar{S}_t = S\Big] \end{aligned}$$

where $\bar{S}_u$ satisfies the SDE

$$\bar{S}_u = S + \int_t^u (r + p\sigma_S\theta_S(v,\bar{S}_v))\bar{S}_v dv + \int_t^u \sigma_S(v,\bar{S}_v)\bar{S}_v dW(v) \tag{5.70}$$

Proof [Proposition 2.20] *HJB for optimal strategies with non constant discount rate:*

The proof is similar to the HJB for v_i in Chapter 3 where we take $\gamma_i = \gamma$ and $\delta_i = 0$. For the sake of completion, we are giving it here. Consider the criterion:

$$J(t, S, x, \pi, c) = \mathbb{E}_t \left[\int_t^T \frac{f(0,s)}{f(0,t)} U_\gamma(c_s X_s^{\pi,c}) ds + \frac{f(0,T)}{f(0,t)} U_\gamma(X_T^{\pi,c}) \right] \tag{5.71}$$

for (π, c) an admissible policy. For $v = v(t, S, x)$ with enough regularity,

$$\mathcal{A}^{\pi,c} v(t, S, x) := S\mu_S v_S + \frac{\sigma_S^2 S^2}{2} v_{SS} + (r + \sigma_S \theta_S \pi - c) x v_x + \frac{(\sigma_S \pi x)^2}{2} v_{xx} + Sx\sigma_S^2 \pi v_{Sx}. \tag{5.72}$$

Define the optimal value function as

$$\hat{V}(t, S, x) = \sup_{(\pi,c)\ admissible} J(t, S, x, \pi, c) \tag{5.73}$$

We give an outline of the proof, it is similar to the proof when the discount rate is constant.

Fix (π, c) another admissible strategy. Fix a small time $h > 0$ such that $t+h \leq T$. We consider two strategies:

1. Follow $(\hat{\pi}, \hat{c})$ on the time interval $[t, T]$.

2. Follow (π, c) on the interval $[t, t+h]$ and $(\hat{\pi}, \hat{c})$ on the interval $[t+h, T]$

Note that if we follow strategy two, at time $t+h$, the stock price is S_{t+h} and the wealth is $X_{t+h}^{\pi,c}$. The first strategy yields the optimal criterion $J(t, S, x, \hat{\pi}, \hat{c}) = \hat{V}(t, S, x)$. The second strategy yields a criterion equals to

$$\mathbb{E}_t\left[\int_t^{t+h}\frac{f(0,s)}{f(0,t)}U_\gamma(c_sX_s^{\pi,c})ds+\int_{t+h}^T\frac{f(0,s)}{f(0,t)}U_\gamma(\hat{c}_sX_s^{\hat{\pi},\hat{c}})ds+\frac{f(0,T)}{f(0,t)}U_\gamma(X_T^{\hat{\pi},\hat{c}})\right]$$
$$=\mathbb{E}_t\left[\int_t^{t+h}\frac{f(0,s)}{f(0,t)}U_\gamma(c_sX_s^{\pi,c})ds+\frac{f(0,t+h)}{f(0,t)}\hat{V}(t+h,S_{t+h},X_{t+h}^{\pi,c})\right]$$

Since the first strategy is optimal, it is better than the second one. Therefore, the following inequality holds:

$$\hat{V}(t,S,x)\geq\mathbb{E}_t\left[\int_t^{t+h}\frac{f(0,s)}{f(0,t)}U_\gamma(c_sX_s^{\pi,c})ds+\frac{f(0,t+h)}{f(0,t)}\hat{V}(t+h,S_{t+h},X_{t+h}^{\pi,c})\right] \tag{5.74}$$

We can apply Ito's lemma to get:

$$\hat{V}(t+h,S_{t+h},X_{t+h}^{\pi,c})=\hat{V}(t,S,x)+\int_t^{t+h}(\hat{V}_t+\mathcal{A}^{\pi,c}\hat{V}(u,S_u,X_u^{\pi,c}))du$$
$$+(\sigma_S\pi x\hat{V}_x+S\sigma_S\hat{V}_S)dW(u) \tag{5.75}$$

And assuming enough integrability for the $dW(u)$ term, its expectation is zero and :

$$\mathbb{E}_t[\hat{V}(t+h,S_{t+h},X_{t+h}^{\pi,c})]=\hat{V}(t,S,x)+\mathbb{E}_t\left[\int_t^{t+h}(\hat{V}_t+\mathcal{A}^{\pi,c}\hat{V}(u,S_u,X_u^{\pi,c}))du\right] \tag{5.76}$$

$$\hat{V}(t,S,x)\geq\mathbb{E}_t\Bigg[\int_t^{t+h}\frac{f(0,s)}{f(0,t)}U_\gamma(c_sX_s^{\pi,c})ds+\frac{f(0,t+h)}{f(0,t)}\hat{V}(t,S,x)$$
$$+\int_t^{t+h}(\hat{V}_t+\mathcal{A}^{\pi,c}\hat{V}(u,S_u,X_u^{\pi,c}))du\Bigg] \tag{5.77}$$

Dividing by h, we obtain

$$\mathbb{E}_t\left[\frac{1}{h}\int_t^{t+h}\frac{f(0,s)}{f(0,t)}U_\gamma(c_s X_s^{\pi,c}) + \frac{f(0,t+h)}{f(0,t)}(\hat{V}_t + \mathcal{A}^{\pi,c}\hat{V}(s,S_s,X_s^{\pi,c}))ds \right.$$
$$\left. + \frac{\frac{f(0,t+h)-f(0,t)}{h}}{f(0,t)}\hat{V}(t,S,x)\right] \leq 0 \tag{5.78}$$

Assuming enough regularity to allow us to take the limit within the expectation and using the fundamental theorem of integral calculus:

$$U_\gamma(c(t)x) + \hat{V}_t + \mathcal{A}^{\pi,c}\hat{V}(t,S,x) + \frac{\frac{\partial f(0,t)}{\partial s}}{f(0,t)}\hat{V}(t,S,x) \leq 0 \tag{5.79}$$

The last inequality becomes an equality when $c = \hat{c}$ and $\pi = \hat{\pi}$. Thus,

$$\hat{V}_t + \sup_{(\pi,c)\text{ admissible}}\{U_\gamma(c(t)x) + \mathcal{A}^{\pi,c}\hat{V}(t,S,x)\} + \frac{\frac{\partial f(0,t)}{\partial s}}{f(0,t)}\hat{V}(t,S,x) = 0 \tag{5.80}$$

This is the Hamilton Jacobi Bellman equation for the optimal value function. We have looked for $\hat{V}(t,S,x)$ of the form

$$\hat{V}(t,S,x) = \hat{v}(t,S)^{1-\gamma}U_\gamma(x) \tag{5.81}$$

All that is left is to check if $\hat{c},\hat{\pi},\hat{V}$ satisfy the integrability conditions assumed in the derivation of the HJB equation. We refer the reader to (Zariphopoulou, 1999) for a full proof of those facts. Note that

$$\hat{V}(0,S,x) = \mathbb{E}_0\left[\int_0^T f(0,s)U_\gamma(\hat{c}_s\hat{X}_s)ds + f(0,T)U_\gamma(\hat{X}_T)\right]$$

and

$$\bar{V}(0,S,x) = \mathbb{E}_0\left[\int_0^T f(0,s)U_\gamma(\bar{c}_s\bar{X}_s)ds + f(0,T)U_\gamma(\bar{X_T})\right],$$

so in order to compare sub game perfect and optimal strategies we must compare $\hat{V}(0,S,x)$ and $\bar{V}(0,S,x)$.

$\square$

Numerical Scheme

Consider the probability $\bar{\mathbb{P}}$ with density $\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\int_0^T p\gamma\theta_S(u)dW_u - \frac{1}{2}\int_0^T (p\gamma\theta_S(u))^2 du \right)$

$$\mu_1 = r + \frac{p\theta_S^2}{2} \tag{5.82}$$

$$\mu_2 = \frac{\partial}{\partial S}(S\mu_S(t,S)) \quad , \quad \sigma_2 = \frac{\partial}{\partial S}(S\sigma_S(t,S)) \tag{5.83}$$

The stock has the dynamics

$$dS_u = S_u(r + p\sigma_S\theta_S(u,S_u)du + \sigma_S(u,S_u)d\bar{W}_u) \tag{5.84}$$

The derivative of the stock $D_s^{t,S} := \frac{\partial S_s}{\partial S_t}$ satisfies

$$D_s^{t,S} = \exp\left(\int_t^s (\mu_2 - \frac{\sigma_2^2}{2})du + \int_t^s \sigma_2(u,S_u)d\bar{W}_u \right) \tag{5.85}$$

Define the two expected values appearing in the expressions of $\mathbb{Q}$ and $\frac{\partial\mathbb{Q}}{\partial S}$.

$$\alpha(t,s,S) = \mathbb{E}_t^{\bar{\mathbb{P}}}\left[e^{\int_t^s p\gamma(\mu_1 - \mathbb{Q}(u,S_u))du} \right] \tag{5.86}$$

$$\delta(t,s,S) = \mathbb{E}_t^{\bar{\mathbb{P}}}\left[\int_t^s (\frac{\partial\mu_1}{\partial S} - \frac{\partial\mathbb{Q}}{\partial S})D_u^{t,S}du \times e^{\int_t^s p\gamma(\mu_1 - \mathbb{Q}(u,S_u))du} \right] \tag{5.87}$$

We discretize time and space at the points

$\{t_n = T - n\ dt\ ,\ n = 0,\cdots,N\}$; $\{S_i = i\ dS\ ,\ i = 0,\cdots,M\}$.

Let $Q_{n,i}$ and $R_{n,i}$ be an approximation for $\mathbb{Q}(t_n,S_i)$ and $\frac{\partial\mathbb{Q}}{\partial S}(t_n,S_i)$: $Q_{n,i} \sim \mathbb{Q}(t_n,S_i)$, $R_{n,i} \sim \frac{\partial\mathbb{Q}}{\partial S}(t_n,S_i)$. $v_{n,i} \sim v(t_n,S_i)$. Finally $\alpha_{n,j,i} \sim \alpha(t_n,t_j,S_i)$, $\delta_{n,j,i} \sim \delta(t_n,t_j,S_i)$.

We start with $t_0 = T$ and the conditions $Q_{0,i} = \frac{\frac{\partial h}{\partial t}(T,T)}{h(T,T)}$, $R_{0,i} = 0$.

We use the relations

$$\mathbb{Q}(t,S) = \frac{\int_t^T \frac{\partial h}{\partial t}(t,s)\alpha(t,s,S)ds + \frac{\partial h}{\partial t}(t,T)\alpha(t,T,S)}{\int_t^T h(t,s)\alpha(t,s,S)ds + h(t,T)\alpha(t,T,S)} \tag{5.88}$$

$$\frac{\partial \mathbb{Q}}{\partial S}(t,S) = \frac{\int_t^T (\frac{\partial h}{\partial t}(t,s) - h(t,s)\mathbb{Q}(t,S))\delta(t,s,S)ds + (\frac{\partial h}{\partial t}(t,T) - h(t,T)\mathbb{Q}(t,S))\delta(t,T,S)}{\int_t^T h(t,s)\alpha(t,s,S)ds + h(t,T)\alpha(t,T,S)} \tag{5.89}$$

We approximate the integrals using a Riemann approximation and the conditional expectations are calculated using Monte Carlo simulations.

For example, we get

$$Q_{n,i} = \frac{dt \sum_{j=0}^n \frac{\partial h}{\partial t}(t_n,t_j)\alpha_{n,j,i} + \frac{\partial h}{\partial t}(t_n,t_0)\alpha_{n,0,i}}{dt \sum_{j=0}^n h(t_n,t_j)\alpha_{n,j,i} + h(t_n,t_0)\alpha_{n,0,i}} \tag{5.90}$$

This fixed point equation is calculated by getting an initial guess for $Q_{n,i}$, then we iterate the right hand side until the error is small enough.

Appendix 2: Optimal Equilibrium with Heterogeneous Agents

Proof [Theorem 3.9]

Let y be given as in Theorem 3.9. We want to prove equality (3.28).

If $z(t;.,.)$ is in $\mathcal{C}^{1,2,2}([t,T]\times(0,\infty)\times\mathbb{R})$ and (π,c) is an admissible strategy, then

$$\mathcal{A}^{\pi,c}z(t,s,x) := x\left(r+\sigma_S\phi_i\pi - c\right)z_x + \frac{1}{2}(\pi\sigma_S x)^2 z_{xx} + z_{ww} + \pi\sigma_S x z_{xw}. \tag{5.91}$$

Let (π^i, c^i) be an admissible strategy and let $X^i := X^{\pi^i,c^i}$ be the associated wealth process. We know that by Itô's formula

$$\begin{aligned} d(f_i(t,s)Z_i(t,s)y^i(t;s,W_s,X_s^i)) &= f_iZ_i(\delta_i y^i + \pi^i\sigma_S X_s^i y_x^i + y_w^i)dW_s \\ +f_iZ_i\Bigg(y_s^i + \mathcal{A}^{\pi^i,c^i}y^i(t,s,X_s^i) &+ \frac{\frac{\partial f_i(t,s)}{\partial s}}{f_i(t,s)}y^i + \delta_i(\sigma_S\pi^i x_i y_x^i + y_w^i)\Bigg)ds \end{aligned}$$

Integrating this equation between s and T, we get:

$$\begin{aligned} &f_i(t,T)Z_i(t,T)y^i(t;T,W_T,X_T^i) - f_i(t,s)Z_i(t,s)y^i(t;s,W_s,X_s^i) \\ &\quad= \int_s^T f_iZ_i\Bigg(y_s^i + x_i\big(r+\sigma_S\phi_i\pi^i - c^i\big)y_x^i + \frac{1}{2}(\pi^i\sigma_S x_i)^2 y_{xx}^i + \frac{1}{2}y_{ww}^i + \pi^i\sigma_S x_i y_{xw}^i \\ &\quad+\frac{\frac{\partial f_i(t,s)}{\partial s}}{f_i(t,s)}y^i + \delta_i(\sigma_S\pi^i x_i y_x^i + y_w^i)\Bigg)(t;u,W_u,X_u^i)du \\ &\quad+\int_s^T f_iZ_i\left[\delta_i y^i + \pi^i\sigma_S X_s^i y_x^i + y_w^i\right](t;u,W_u,X_u^i)dW_u \end{aligned}$$

In light of the HJB equation (3.27), we get

$$f_i(t,T)Z_i(t,T)y^i(t;T,W_T,X^i_T) - f_i(t,s)Z_i(t,s)y^i(t;s,W_s,X^i_s) \leq -\int_s^T f_i(t,u)Z_i(t,u)$$
$$U_i(c^i(u)X^i(u))du + \int_s^T f_i(t,u)Z_i(t,u)(\delta_i y^i + \pi^i\sigma_S x_i y^i_x + y^i_w)(t;u,W_u,X^i_u)dW_u$$

Taking expectations and using the integrability assumptions, we get:

$$\mathbb{E}^{\mathbb{P}}_{s,w,x_i}\left[f_i(t,T)Z_i(t,T)y^i(t;T,W^i_T,X^i_T) - f_i(t,s)Z_i(t,s)y^i(t;s,W_s,X^i_s)\right]$$
$$\leq \mathbb{E}^{\mathbb{P}}_{s,w,x_i}\left[\int_s^T -f_i(t,u)Z_i(t,u)U_i(c^i(u)X^i(u))du\right]$$

Since the policy (π^i, c^i) is admissible,

$$\mathbb{E}^{\mathbb{P}}_{s,w,x_i}\int_s^T Z_i(t,u)\big|f_i(t,u)U_i(c^i(u)X^i(u))\big|du < \infty$$

and by hypothesis, $y^i(t;T,W^i_T,X^i_T) = U_i(X^i_T)$,

$$\begin{aligned} f_i(t,s)Z_i(t,s)y^i(t;s,w,x_i) \ \geq\ & \mathbb{E}^{\mathbb{P}}_{s,w,x_i}\bigg[\int_s^T f_i(t,u)Z_i(t,u)U_i(c^i(u)X^i(u))du \\ & + f_i(t,T)Z_i(t,T)U_i(X^i_T)\bigg] \end{aligned} \tag{5.92}$$

We get the reverse inequality by considering the admissible policy (π_y, c_y) defined in Theorem 3.9 and the corresponding wealth process X^y.

$$\begin{aligned}
& \mathbb{E}^{\mathbb{P}}_{s,w,x_i}\left[f_i(t,T)Z_i(t,T)y^i(t;T,W_T,X_T^{\pi_y,c_y}) - f_i(t,s)Z_i(t,s)y^i(t;s,w,x_i)\right] \\
= \; & \mathbb{E}^{\mathbb{P}^i}_{s,w,x_i}\left[\int_s^T -f_i(t,u)U_i(c_y(u)X_u^y)du\right]
\end{aligned}$$

$$\begin{aligned}
f_i(t,s)y^i(t;s,w_i,x_i) &= \mathbb{E}^{\mathbb{P}^i}_{s,w_i,x_i}\left[\int_s^T f_i(t,u)U_i(c_y(u)X_u^y)du + f_i(t,T)U_i(X_T^y)\right] \\
&= f_i(t,s)J^i(t,s,w,x_i,\pi_y,c_y) \\
&\leq \sup_{\pi^i,c^i} f_i(t,s)J^i(t,s,w,x_i,\pi^i,c^i) = f_i(t,s)V^i(t;s,w,x_i)
\end{aligned}$$

Therefore,

$$y^i(t;s,x) = J^i(t,s,w,x_i,\pi_y,c_y) = V^i(t,s,w,x_i)$$

□

Proof [Theorem 3.10]

We search for a value function V^i of the form:

$$V^i(t,s,w_i,x_i) = a_i(t,s,w_i)U_i(x_i)$$

for a function a_i to be determined.

The first order conditions of the HJB equation (3.27) give the optimal consumption and portfolio fraction invested in the stock:

$$\hat{c}^i = \frac{(V_x^i)^{\frac{1}{\gamma_i - 1}}}{x_i} = a_i(t,s,w_i)^{-p_i} \tag{5.93}$$

$$\hat{\pi}^i = -\frac{\sigma_S \phi_i x_i V_x^i + \sigma_S x_i V_{xw}^i}{\sigma_S^2 x_i^2 V_{xx}^i} = \frac{\phi_i}{\sigma_S(1-\gamma_i)} + \frac{\frac{\partial a_i}{\partial w}}{a_i(1-\gamma_i)\sigma_S} \tag{5.94}$$

By plugging $\hat{c}^i$ and $\hat{\pi}^i$ in the SDE (3.32):

$$d\hat{X}^i(s) = \left(r(s) - a_i^{-p_i} + p_i\sigma_S(\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i})\right)\hat{X}^i(s)ds + p_i(\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i})\hat{X}^i(s)dW(s) \tag{5.95}$$

Noticing that the function

$$x \mapsto y(x) = \alpha x^2 + \beta x$$

where $\alpha < 0$ has a maximum equal to $\frac{-\beta^2}{2\alpha}$, we find that the sup in the HJB equation (3.27) is equal to:

$$\begin{aligned} 0 \;=\; & V_s^i + \left(r + p_i\sigma_S(\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i}) - a_i^{-p_i}\right) x_i V_x^i + \frac{p_i^2}{2}(\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i})^2 x_i^2 V_{xx}^i + \frac{1}{2}V_{ww}^i \\ & + p_i(\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i}) x_i V_{wx}^i + \frac{\frac{\partial f_i(t,s)}{\partial s}}{f_i(t,s)} V^i + U_i\left(a_i^{-p_i} x_i\right) + \delta_i V_w^i \end{aligned}$$

If we plug in the expression

$$V^i(t,s,w_i,x_i) = a_i(t,s,w_i)\frac{x_i^{\gamma_i}}{\gamma_i}$$

we get PDE for a_i:

$$
\begin{aligned}
0 \;=\; & \frac{\partial a_i}{\partial s} + \left(r + p_i \sigma_S (\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i}) - a_i^{-p_i} \right) \gamma_i a_i + \frac{p_i^2 (\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i})^2}{2} \gamma_i (\gamma_i - 1) a_i + \frac{1}{2} \frac{\partial^2 a_i}{\partial w^2} \\
& + \gamma_i p_i (\phi_i + \frac{\frac{\partial a_i}{\partial w}}{a_i}) \frac{\partial a_i}{\partial w} + \frac{\frac{\partial f_i(t,s)}{\partial s}}{f_i(t,s)} a_i + a_i^{-\gamma_i p_i} + \delta_i \frac{\partial a_i}{\partial w}
\end{aligned}
$$

This PDE simplifies to

$$
\begin{aligned}
0 \;=\; & \frac{\partial a_i}{\partial s} + \frac{1}{2} \frac{\partial^2 a_i}{\partial w^2} + \left[\gamma_i r + \frac{\gamma_i p_i \phi_i^2}{2} + \frac{\frac{\partial f_i(t,s)}{\partial s}}{f_i(t,s)} \right] a_i \\
& + \frac{\gamma_i p_i a_i}{2} (\frac{\frac{\partial a_i}{\partial w}}{a_i})^2 + (\delta_i + \gamma_i p_i \phi_i) \frac{\partial a_i}{\partial w} + \frac{a_i^{-\gamma_i p_i}}{p_i}
\end{aligned}
$$

Plugging the expression $a_i = v_i^{p_i}$ in the preceding PDE, we get after some calculations PDE (3.34). Note that the calculations are similar to the ones in (Zariphopoulou, 1999).

□

Proof [Proposition 3.11]

The wealth process of agent i satisfies

$$
d\hat{X}^i(s) = \big(r(s) + \phi_i (p_i \phi_i + \frac{\frac{\partial v_i}{\partial w}}{v_i}) - v_i^{-1} \big) \hat{X}^i(s) ds + \big(p_i \phi_i + \frac{\frac{\partial v_i}{\partial w}}{v_i} \big) \hat{X}^i(s) dW_s^i .
$$

Thus,

$$\begin{aligned}
d\log(\hat{c}^i(s)\hat{X}^i(s)) &= d\log(v_i(s)^{-1}\hat{X}^i(s)) = d\log(\hat{X}^i(s)) - d\log(v_i(s)) \\
&= \left(r - v_i^{-1} + \sigma_S\left(p_i\phi_i + \frac{\frac{\partial v_i}{\partial w}}{v_i}\right) - \frac{1}{2}\left(p_i\phi_i + \frac{\frac{\partial v_i}{\partial w}}{v_i}\right)^2\right)ds \\
&+\big(p_i\phi_i + \frac{\frac{\partial v_i}{\partial w}}{v_i}\big)dW_s + \left(\frac{1}{2}\left(\frac{\frac{\partial v_i}{\partial w}}{v_i}\right)^2 - \frac{\frac{\partial v_i}{\partial s} + \frac{1}{2}\frac{\partial^2 v_i}{\partial w^2}}{v_i}\right)ds - \frac{\frac{\partial v_i}{\partial w}}{v_i}dW_s
\end{aligned}$$

The terms in $(\frac{\frac{\partial v_i}{\partial w}}{v_i})^2$ cancel out. Using the identity $\gamma_i p_i + 1 - p_i = 0$ and the PDE for v_i, the ds term becomes

$$p_i\left(r + \frac{\phi_i^2}{2} - \delta_i\phi_i - \rho_i(0,s)\right)$$

The dW_s term is equal to $p_i\phi_i(s)$. We conclude that :

$$d\log(\hat{c}^i(s)\hat{X}^i(s)) = p_i\left(r + \frac{\phi_i^2}{2} - \delta_i\phi_i(s,W_s) - \rho_i(0,s)\right)ds + p_i\phi_i(s,W_s)dW(s)$$

□

Proof [Proposition 3.12] We have:

$$\begin{aligned}
0 = d\log(\hat{c}^i(s)\hat{X}^i(s)) - d\log\epsilon(s) &= p_i\left(r_i + \frac{(\theta_{iS}+\delta_i)^2}{2} - \delta_i(\theta_{iS}+\delta_i) + \frac{\frac{\partial f_i(0,s)}{\partial s}}{f_i(0,s)}\right)ds \\
&\quad + p_i(\theta_{iS}+\delta_i)dW(s) - (\mu - \frac{\sigma^2}{2})ds - \sigma dW(s)
\end{aligned}$$

The drift and volatility of the right hand side of the previous equality should therefore be zero. We obtain:

$$\begin{aligned}
\theta_{iS} &= \sigma(1-\gamma_i) - \delta_i \\
r_i(s) &= (1-\gamma_i)(\mu - \frac{\sigma^2}{2}) - \frac{\frac{\partial f_i(0,s)}{\partial s}}{f_i(0,s)} - \frac{(\theta_{iS}+\delta_i)^2}{2} + (1-\gamma_i)(\mu_i - \mu) \\
r_i(s) &= (1-\gamma_i)\mu_i - \sigma^2(1-\gamma_i)(1-\frac{\gamma_i}{2}) - \frac{\frac{\partial f_i(0,s)}{\partial s}}{f_i(0,s)}
\end{aligned}$$

This is the same result as (Cvitanic *et al.*, 2012) with the specification $f_i(t,s) = \exp(-\rho_i(s-t))$. □

Proof [Proposition 3.14]

The pricing kernel is

$$\begin{aligned}
M_i(s) &= \exp\left(\int_0^s (-r_i(v) - \frac{\theta_{iS}^2}{2})dv - \int_t^s \theta_{iS} dW(v)\right) \\
&= \exp\left(\int_0^s (\frac{\frac{\partial f_i(0,v)}{\partial s}}{f_i(0,v)} - (1-\gamma_i)\mu_i - \sigma^2(1-\gamma_i)(1-\frac{\gamma_i}{2}) - \frac{\theta_{iS}^2}{2})dv \right. \\
&\quad \left. - \int_0^s \theta_{iS} dW(v)\right) \\
M_i(s) &= f_i(0,s)\exp\left(\Big(-(1-\gamma_i)\mu_i - (1-\gamma_i)(1-\frac{\gamma_i}{2})\sigma^2 - \frac{\theta_{iS}^2}{2}\Big)s + \theta_{iS}W_s\right)
\end{aligned}$$

□

Proof [Proposition 3.17]

$$\begin{aligned} d\log\left(\epsilon(s)\left(\frac{M_i(s)}{M(s)}\right)^{p_i}\right) &= (\mu-\frac{\sigma^2}{2})ds+\sigma dW_s+p_i(-r_ids-\theta_{iS}dW_s-\frac{\theta_{iS}^2}{2}ds) \\ &\quad +p_i(rds+\theta_S dW_s+\frac{\theta_S^2}{2}ds) \qquad (5.96) \\ d\log\left(\epsilon(s)\left(\frac{M_i(s)}{M(s)}\right)^{p_i}\right) &= \left(p_i(r-r_i)+\frac{p_i(\theta_S^2-\theta_{iS}^2)}{2}+\mu-\frac{\sigma^2}{2}\right)ds \\ &\quad +\left(\sigma+p_i(\theta_S-\theta_{iS})\right)dW(s) \qquad (5.97) \end{aligned}$$

Using (3.36) and (5.187), we get

$$d\log(\hat{c}^i(s)\hat{X}^i(s))-d\log\left(\epsilon(s)\left(\frac{M_i(s)}{M(s)}\right)^{p_i}\right)=\Big(p_i\big(r+\frac{\phi_i^2}{2}-\delta_i\phi_i(s,W_s)-\rho_i(0,s)\big)$$
$$-p_i(r-r_i)-\frac{p_i}{2}(\theta_S^2-\theta_{iS}^2)-(\mu-\frac{\sigma^2}{2})\Big)ds+(p_i\phi_i-\sigma-p_i(\theta_S-\theta_{iS}))dW_s$$

Recall that

$$\phi_i=\theta_S+\delta_i$$

so the dW term is

$$p_i(\theta_S+\delta_i)-\sigma-p_i\theta_S+p_i\theta_{iS}=0$$

The ds term is equal to

$$\begin{aligned} & p_i\big(r+\frac{\phi_i^2}{2}-\delta_i\phi_i(s,W_s)-\rho_i(0,s)\big)-p_i(r-r_i)-\frac{p_i}{2}(\theta_S^2-\theta_{iS}^2)-(\mu-\frac{\sigma^2}{2}) \\ &=p_i\left[r+\frac{\phi_i^2}{2}-\delta_i\phi_i-\rho_i(0,s)-(r-r_i)+\frac{\theta_{iS}^2-\theta_S^2}{2}-(1-\gamma_i)(\mu-\frac{\sigma^2}{2})\right] \end{aligned}$$

and using the expression

$$\frac{\phi_i^2}{2} - \delta_i \phi_i = \frac{(\phi_i - \delta_i)^2 - \delta_i^2}{2} = \frac{\theta_S^2 - \delta_i^2}{2}$$

the ds term equals

$$r_i(s) + \frac{\theta_{iS}^2 - \delta_i^2}{2} - \rho_i(0, s) - (1 - \gamma_i)(\mu - \frac{\sigma^2}{2}) = 0$$

by the definition of $r_i(s)$, the above term is zero. Thus

$$d \log(\hat{c}^i(s) \hat{X}^i(s)) - d \log \left(\epsilon(s) \left(\frac{M_i(s)}{M(s)} \right)^{p_i} \right) = 0$$

In other words, by integrating between 0 and s, we get (3.46). □

Proof [Proposition 3.17]

For $a = (a_1, \ldots, a_I) \in (0, \infty)^I$ and $y > 0$, call $\phi(a, y)$ the quantity

$$\phi(a, y) = \sum_i (\frac{a_i}{y})^{p_i} - 1. \tag{5.98}$$

For fixed $a \in (0, \infty)^I$,

$$\phi(a, .) : y \mapsto \sum_{i=1}^{I} (\frac{a_i}{y})^{p_i} - 1$$

is strictly decreasing and goes to ∞ at 0 and goes to -1 at $y \to \infty$ so there is a unique y such that $\phi(a, y) = 0$.

We call $F(a)$ the unique y such that $\phi(a, y) = 0$.

$$\phi(a, F(a)) = 0. \tag{5.99}$$

Note that

$$\phi_y = -\sum_i p_i a_i^{p_i} y^{-p_i-1} < 0 \tag{5.100}$$

The implicit function theorem proves that F is smooth in $a = (a_1, \cdots, a_I)$.

$$M(s) = F\Big((\frac{\hat{c}^1(0)\hat{X}^1(0)}{\epsilon(0)})^{\frac{1}{p_1}} M_1(s), \cdots, (\frac{\hat{c}^I(0)\hat{X}^I(0)}{\epsilon(0)})^{\frac{1}{p_I}} M_I(s) \Big)$$

is well defined. □

Proof [Proposition 3.18]

We can differentiate the previous equality with respect to a_i to get:

$$\phi_{a_i} + \phi_y F_{a_i}|_{y=F(a)} = 0 \tag{5.101}$$

so

$$F_{a_i} = -\frac{\phi_{a_i}}{\phi_y(a, F(a))} = -(\frac{\partial \phi}{\partial y})^{-1} \frac{\partial \phi}{\partial a_i} = (\sum_j p_j a_j^{p_j} y^{-p_j-1})^{-1} p_i a_i^{p_i-1} y^{-p_i}$$

$$F_{a_i}(a_1, \cdots, a_I) = (\sum_j p_j (a_j^{p_j} F^{-p_j-1})^{-1} p_i a_i^{p_i-1} F^{-p_i} = \frac{p_i a_i^{p_i} F^{-p_i} F}{a_i \sum_j p_j a_j^{p_j} F^{-p_j}}$$

Recall

$$a_i(s) = (\frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon(0)})^{\frac{1}{p_i}} M_i(s) \tag{5.102}$$

$$a(s) = (a_1(s), \cdots, a_I(s)) \tag{5.103}$$

and $M(s) = F(a(s))$. Finally, recall $\omega_i(s)$ the quantity

$$\omega_i(s) = \frac{p_i \left(\frac{a_i(s)}{F(a(s))} \right)^{p_i}}{\sum_j p_j \left(\frac{a_j(s)}{F(a(s))} \right)^{p_j}}. \tag{5.104}$$

We can see that

$$F_{a_i}(a) = \frac{F(a)}{a_i} \frac{p_i \left(\frac{a_i}{F(a)} \right)^{p_i}}{\sum_j p_j \left(\frac{a_j}{F(a)} \right)^{p_j}} \tag{5.105}$$

so, taking

$$\frac{\partial}{\partial a_j} \log F_{a_i}$$

$$\frac{F_{a_i a_j}}{F_{a_i}} = \frac{F_{a_j}}{F} - \frac{\delta_{i,j}}{a_i} + p_i\left(\frac{\delta_{i,j}}{a_i} - \frac{F_{a_j}}{F}\right) - \frac{\sum_k \frac{\partial}{\partial a_j}\left(\sum_k p_k \left(\frac{a_k}{F(a)} \right)^{p_k}\right)}{\sum_k p_k \left(\frac{a_k}{F(a)} \right)^{p_k}} \tag{5.106}$$

$$\frac{\frac{\partial}{\partial a_j} \left(\frac{a_k}{F(a)} \right)^{p_k}}{\left(\frac{a_k}{F(a)} \right)^{p_k}} = \frac{\partial}{\partial a_j}(p_k \log a_k - p_k \log F(a)) = p_k\left(\frac{\delta_{j,k}}{a_k} - \frac{F_{a_j}}{F(a)}\right) \tag{5.107}$$

$$\frac{F_{a_i a_j}}{F_{a_i}} = -\frac{-\delta_{i,j}}{a_i} + (1-p_i)\frac{F_{a_j}}{F} + \frac{p_i \delta_{i,j}}{a_i} - \frac{\sum_k p_k^2\left(\frac{\delta_{j,k}}{a_k} - \frac{F_{a_k}}{F(a)}\right) \left(\frac{a_k}{F(a)} \right)^{p_k})}{\sum_k p_k \left(\frac{a_k}{F(a)} \right)^{p_k})} = (1-p_i)\frac{\omega_j}{a_j} + \frac{p_i \delta_{i,j}}{a_i} -$$

$$\frac{p_j^2}{a_j}\frac{\left(\frac{a_j}{F(a)}\right)^{p_j}}{\sum_k p_k\left(\frac{a_k}{F(a)}\right)^{p_k}}+\frac{\frac{F_{a_j}}{F}\sum_k p_k^2\left(\frac{a_k}{F(a)}\right)^{p_k}}{\sum_k p_k\left(\frac{a_k}{F(a)}\right)^{p_k}}-\frac{\delta_{i,j}}{a_i}$$

$$\begin{aligned} a_i a_j F_{a_i a_j} &= a_i a_j F_{a_i}\left[(1-p_i)\frac{\omega_j}{a_j}+\frac{p_i\delta_{i,j}}{a_i}-\frac{p_j\omega_j}{a_j}-\frac{\delta_{i,j}}{a_i}+\frac{\omega_j}{a_j}\sum_k p_k\omega_k\right] \\ &= F\omega_i[(1-p_j)\omega_j+\frac{p_i\delta_{i,j}a_j}{a_i}-p_j\omega_j+\omega_j\sum_k p_k\omega_k] \\ &\qquad timeconsistent \end{aligned}$$

By Itô's lemma,

$$dM(s)=dF(a(s))=\sum_i F_{a_i}da_i+\frac{1}{2}\sum_{i,j}F_{a_i a_j}da_i da_j$$

Remembering that $a_i(s)$ is a multiple of $M_i(s)$ so

$$da_i(s)=a_i(s)(-r_i(s)ds-\theta_{iS}dW_s)$$

Thus

$$\begin{aligned} dM(s) &= \sum_i F_{a_i}da_i+\frac{1}{2}\sum_{i,j}F_{a_i a_j}da_i da_j \\ &= \sum_i a_i F_{a_i}(-r_i(s)ds-\theta_{iS}dW_s)+\frac{1}{2}\sum_{i,j}a_i a_j F_{a_i a_j}\theta_{iS}\theta_{jS}ds \\ &= \sum_i F\omega_i(-r_i(s)ds-\theta_{iS}dW_s) \\ &+\frac{1}{2}\sum_{i,j}\theta_{iS}\theta_{jS}F\omega_i[(1-p_j)\omega_j+\frac{p_i\delta_{i,j}a_j}{a_i}-p_j\omega_j+\omega_j\sum_k p_k\omega_k]ds \end{aligned}$$

The $dW(s)$ term is

$$\sum_i F\omega_i(-\theta_{iS})$$

and the ds - term is

$\sum_i F\omega_i(-r_i) + \frac{1}{2}\sum_{i,j}\theta_{iS}\theta_{jS}F\omega_i[(1-p_j)\omega_j + \frac{p_i\delta_{i,j}a_j}{a_i} - p_j\omega_j + \omega_j\sum_k p_k\omega_k]$
$= \frac{F}{2}\sum_i \omega_i(p_i-1)\theta_{iS}^2 + \frac{F}{2}\sum_{i,j}\theta_{iS}\theta_{jS}\omega_i\omega_j(1-p_i-p_j+\sum_k p_k\omega_k) - F\sum_k \omega_k r_k$
$= -F\sum_k \omega_k r_k + \frac{F}{2}\sum_i \omega_i(p_i-1)\theta_{iS}^2 - \sum_k \omega_k\theta_k\sum_i(1-p_i)\omega_i\theta_{iS} + \frac{(\sum_k \omega_k\theta_{kS})^2}{2}(1-\sum_k p_k\omega_k)$

Remembering that

$$\sum_k \omega_k = 1$$

and

$$F(a(s)) = M(s),$$

$dM(s)$ becomes:

$$\begin{aligned} dM(s) = -M(s)\sum_k \omega_k(s)\theta_{kS}dW(s) - M(s)\Bigg[\sum_k \omega_k r_k + \frac{1}{2}\sum_k \omega_k(p_k-1)\theta_{kS}^2 \\ + \sum_k \omega_k\theta_k\sum_i(1-p_k)\omega_k\theta_{kS} + \frac{(\sum_k \omega_k\theta_{kS})^2}{2}(\sum_k(1-p_k)\omega_k)\Bigg]ds \end{aligned} \tag{5.108}$$

and since

$$dM(s) = M(s)(-r(s)ds - \theta_S(s)dW(s)) \tag{5.109}$$

we get the expressions for r and θ_S by equating the ds and dW terms . □

Proof [Proposition 3.22]

Since θ_S is linear in the ω_i's and r is a polynomial of degree 3 in the ω_i's,

$i = 1, \cdots, I$, it suffices to find a uniform bound for $\frac{\partial \omega_i(s)}{\partial w}$ and $\frac{\partial \omega_i(s)}{\partial s}$.

Note that

$$\omega_i(s) = \frac{p_i \left(\frac{a_i(s)}{F(a(s))} \right)^{p_i}}{\sum_j p_j \left(\frac{a_j(s)}{F(a(s))} \right)^{p_j}}$$

and taking the derivative, we get:

$$\frac{\partial \omega_i}{\partial a_j} = \omega_i \left[p_i \Big(\frac{\delta_{i,j}}{a_j} - \frac{\omega_j}{a_j} \Big) - \sum_k p_k (\delta_{j,k} - \omega_j) \frac{\omega_k}{a_j} \right]$$

$$\begin{aligned} \frac{\partial \omega_i}{\partial w} &= \sum_j \frac{\partial \omega_i}{\partial a_j} \frac{\partial a_j}{\partial w} = \sum_j \omega_i (-\theta_{jS} a_j) \left[p_i \Big(\frac{\delta_{i,j}}{a_j} - \frac{\omega_j}{a_j} \Big) - \sum_k p_k (\delta_{j,k} - \omega_j) \frac{\omega_k}{a_j} \right] \\ &= \omega_i \left[p_i (\theta_S - \theta_{iS}) - \sum_j p_j \omega_j (\theta_S - \theta_{jS}) \right] \end{aligned} \tag{5.110}$$

$$\begin{aligned} \frac{\partial \omega_i}{\partial s} &= \sum_j \frac{\partial \omega_i}{\partial a_j} \frac{\partial a_j}{\partial s} = \sum_j \omega_i \Big(-r_j - \frac{\theta_{jS}^2}{2}\Big) a_j \left[p_i \Big(\frac{\delta_{i,j}}{a_j} - \frac{\omega_j}{a_j} \Big) - \sum_k p_k (\delta_{j,k} - \omega_j) \frac{\omega_k}{a_j} \right] \\ &= \sum_j \omega_i \Big(-r_j - \frac{\theta_{jS}^2}{2}\Big) \left[p_i \big(\delta_{i,j} - \omega_j \big) - \sum_k p_k (\delta_{j,k} - \omega_j) \omega_k \right] \end{aligned} \tag{5.111}$$

Since $\omega_j \in [0, 1]$ for all j, $\frac{\partial \omega_i}{\partial w}$ and $\frac{\partial \omega_i}{\partial s}$ are bounded. This ends the proof. □

Proof [Theorem 3.24]

From the expression

$$
\begin{aligned}
\hat{c}^i(s)\hat{X}^i(s) &= \frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon(0)}\epsilon(s)\left(\frac{M_i(s)}{M(s)}\right)^{p_i} \\
&= \hat{c}^i(0)\hat{X}^i(0)e^{\int_0^s(\mu-\frac{\sigma^2}{2}+r(u)-r_i(u)+\frac{\theta_S^2-\theta_{iS}^2}{2})du+\int_0^s(\sigma+\theta_S(u)-\theta_{iS})dW_u}
\end{aligned}
$$

and the fact that all the terms in the exponent are uniformly bounded, we conclude that

$$
\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}^i}\sup_{t\le s\le T}[|U_i(\hat{c}^i(s)\hat{X}^i(s))|] \qquad (5.112)\\
&= \frac{|\hat{c}^i(t)\hat{X}^i(t)|^{\gamma_i}}{|\gamma_i|}\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\le s\le T} e^{\int_t^s -\frac{\delta_i^2}{2}+\gamma_i(\mu-\frac{\sigma^2}{2}+r(u)-r_i(u)+\frac{\theta_S^2-\theta_{iS}^2}{2})du+\int_t^s(\delta_i+\gamma_i(\sigma+\theta_S(u)-\theta_{iS})dW_u}\right]\\
&= \frac{|\hat{c}^i(t)\hat{X}^i(t)|^{\gamma_i}}{|\gamma_i|}\mathbb{E}_t^{\mathbb{P}}\left[\sup_{t\le s\le T} e^{\int_t^s\left(-\frac{\delta_i^2}{2}+\gamma_i(\mu-\frac{\sigma^2}{2}+r(u)-r_i(u)+\frac{\theta_S^2-\theta_{iS}^2}{2})+\frac{(\delta_i+\gamma_i(\sigma+\theta_S(u)-\theta_{iS}))^2}{2}\right)du}\right.\\
&\qquad\left.\times E_i(t,s)\right]
\end{aligned}
$$

where

$$
E_i(t,s) := \exp\Big(\int_t^s(\delta_i+\gamma_i(\sigma+\theta_S(u)-\theta_{iS}))dW_u - \frac{1}{2}\int_t^s(\delta_i+\gamma_i(\sigma+\theta_S(u)-\theta_{iS}))^2du\Big) \qquad (5.113)
$$

The term in the exponential in equation (5.184) is bounded by a constant $K \ge 0$ independent of u.

$$
\Big|-\frac{\delta_i^2}{2}+\gamma_i(\mu-\frac{\sigma^2}{2}+r(u)-r_i(u)+\frac{\theta_S^2-\theta_{iS}^2}{2})+\frac{(\delta_i+\gamma_i(\sigma+\theta_S(u)-\theta_{iS}))^2}{2}\Big| \le K \qquad (5.114)
$$

Note that $\mathbb{E}_t^{\mathbb{P}}[E_i(t,s)] = 1$, thus

$$\mathbb{E}_t^{\mathbb{P}^i} \sup_{t \le s \le T} [|U_i(\hat{c}^i(s)\hat{X}^i(s))|] \le \frac{1}{|\gamma_i|}\left(\frac{x_i}{v_i(t,w)}\right)^{\gamma_i} e^{K(T-t)} < \infty. \tag{5.115}$$

From the expression

$$\hat{X}^i(T) = \hat{c}^i(T)\hat{X}^i(T) = \hat{c}^i(t)\hat{X}^i(t) e^{\int_t^T (\mu - \frac{\sigma^2}{2} + r(u) - r_i(u) + \frac{\theta_S^2 - \theta_{iS}^2}{2})du + \int_t^T (\sigma + \theta_S(u) - \theta_{iS}) dW_u}$$

we conclude similarly that $\mathbb{E}_t^{\mathbb{P}^i}[|U_i(\hat{X}^i(T))|] < \infty$.

By the preceding proposition,

$$\hat{c}^i(t,w) = \frac{1}{v_i(t,w)}$$

is positive and uniformly bounded.

$$\sigma_S \hat{\pi}^i(t,w) = p_i\phi_i + \sigma_S \frac{\frac{\partial v_i}{\partial w}}{v_i}(t,w)$$

is also uniformly bounded. Therefore the dt term and the dW_t term in the expression of $\frac{d\hat{X}^i(t)}{\hat{X}^i(t)}$ are all bounded. The same is true for the coefficients of $\frac{dS(t)}{S(t)}$, so both SDEs have strong solutions with continuous paths. We end by verifying the hypotheses of the HJB verification theorem 3.9.

$$x_i V_x^i \;=\; v_i^{p_i} x_i^{\gamma_i}\;, \quad V_s^i = v_i^{p_i - 1} x_i^{\gamma_i} \frac{p_i}{\gamma_i}\frac{\partial v_i}{\partial s}\;, \quad V_w^i = v_i^{p_i-1} x_i^{\gamma_i} \frac{p_i}{\gamma_i}\frac{\partial v_i}{\partial w}$$

Simce v_i and its time and space derivatives are uniformly bounded, we see that all the integrability conditions are satisfied. V^i is indeed the value function for agent i

and this ends the proof of the existence of the equilibrium. □

Proof [Proposition 3.19]

We have:

$$S_i(u) = \frac{1}{M_i(u)}\mathbb{E}_u\left[\int_u^T M_i(v)\epsilon(v)dv + M_i(T)\epsilon(T)\right]$$

To simplify, we omit the final term in the calculations.

$$\begin{aligned}
S_i(u) &= \epsilon(u)\mathbb{E}_u\left[\int_u^T \exp\left(\int_u^v\left(-r_i(y) - \frac{\theta_{iS}^2}{2} + \mu - \frac{\sigma^2}{2}\right)dy + \int_u^v(\sigma - \theta_{iS})dW(y)\right)dv\right] \\
S_i(u) &= \epsilon(u)\int_u^T \exp\left(\int_u^v\left(-r_i(y) - \frac{\theta_{iS}^2}{2} + \mu - \frac{\sigma^2}{2} + \frac{(\sigma\gamma_i + \delta_i)^2}{2}\right)dy\right)dv
\end{aligned}$$

And the argument in the integrand is

$$\begin{aligned}
-r_i(y) - \frac{\theta_{iS}^2}{2} + \mu - \frac{\sigma^2}{2} + \frac{(\sigma\gamma_i + \delta_i)^2}{2} &= -(-\gamma_i(\mu - \frac{\sigma^2}{2}) - \frac{(\sigma\gamma_i + \delta_i)^2 + \delta_i^2}{2} - \frac{\frac{\partial f_i(t,s)}{\partial s}}{f_i(t,s)}) \\
&= -\rho_i(t,s) + \gamma_i(\mu_i - \frac{\sigma^2}{2})
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
S_i(u) &= \epsilon(u)\int_u^T \exp\left(\int_u^v\left(\frac{\frac{\partial f_i(t,s)}{\partial s}}{f_i(t,s)} + \gamma_i(\mu_i - \frac{\sigma^2}{2})\right)ds\right)dv \\
S_i(u) &= \epsilon(u)\int_u^T \frac{f_i(t,v)}{f_i(t,u)}\exp\left(\gamma_i(\mu_i - \frac{\sigma^2}{2})(v-u)\right)dv
\end{aligned}$$

□

Proof [Proposition 3.21]

Let

$$R(t) = \frac{S(t)}{\epsilon(t)}$$

be the price-dividend ratio.

$$R(t) = \frac{1}{M(t)\epsilon(t)}\mathbb{E}_t\left[\int_t^T M_u \epsilon_u du + M_T \epsilon_T\right] \tag{5.116}$$

Write $R(t) = R(t, W_t)$. Then, taking the differentiable in the expression

$$S(t) = R(t)\epsilon(t)$$

and equating the dW terms, we get

$$\sigma_S(t)S(t) = \frac{\partial R(t)}{\partial w}\epsilon_t + \sigma R(t)\epsilon_t \tag{5.117}$$

So

$$\sigma_S(t) = \sigma + \frac{\frac{\partial R(t)}{\partial w}}{R(t)} \tag{5.118}$$

Recall that

$$a_i(s) = (\frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon(0)})^{1-\gamma_i} M_i(s)$$

and

$$M(s) = F(M_1(s), \cdots, M_I(s))$$

by the Chain Rule:

$$\frac{\partial M}{\partial w} = \sum_i \frac{\partial F}{\partial a_i}(a) \times \frac{\partial a_i}{\partial w} = \sum_i \frac{F\omega_i}{a_i}(-\theta_{iS}a_i) \tag{5.119}$$

$$= -\sum_i \theta_{iS}\omega_i F(a) = -M(u)\theta_S(u) \tag{5.120}$$

$$\frac{\partial \epsilon(u)}{\partial w} = \sigma\epsilon(u) \tag{5.121}$$

so that

$$\frac{\partial(M_u\epsilon_u)}{\partial w} = (\sigma - \theta_S(u))M(u)\epsilon(u) \tag{5.122}$$

Taking the derivative under the expectation sign, we get:

$$\frac{\partial}{\partial w}\mathbb{E}_t\left[\int_t^T M_u\epsilon_u du + M_T\epsilon_T\right] = \mathbb{E}_t\left[\int_t^T M_u\epsilon_u(\sigma - \theta_S(u))du + M_T\epsilon_T(\sigma - \theta_S(T))\right] \tag{5.123}$$

On the other hand:

$$\frac{\partial}{\partial w}\mathbb{E}_t\left[\int_t^T M_u\epsilon_u ds + M_T\epsilon_T\right] = R(t)M(t)\epsilon(t)\Big(-\theta_S(t) + \sigma + \frac{\frac{\partial R(t)}{\partial w}}{R(t)}\Big)$$
$$= R(t)M(t)\epsilon(t)(-\theta_S(t) + \sigma_S(t)) \tag{5.124}$$

Thus

$$\sigma_S(t) - \theta_S(t) = \frac{1}{R(t)M(t)\epsilon(t)}\mathbb{E}_t\left[\int_t^T M_u\epsilon_u(\sigma - \theta_S(u))du + M_T\epsilon_T(\sigma - \theta_S(T))\right] \tag{5.125}$$

$$\sigma_S(t) \;=\; \sigma + \frac{\mathbb{E}_t\left[\int_t^T(\theta_S(t)-\theta_S(u))M_u\epsilon_u du + (\theta_S(t)-\theta_S(T))M_T\epsilon_T\right]}{\mathbb{E}_t\left[\int_t^T M(u)\epsilon_u du + M(T)\epsilon_T\right]} \tag{5.126}$$

□

Proof [Proposition 3.35] We use

$$\lim_{s\to\infty}\omega_i(s) = \delta_{i,i_K}$$

in the expression

$$\begin{aligned}r(s) &= \sum_i \omega_i(s)r_i(s) + \frac{1}{2}\sum_i \omega_i(1-p_i)\theta_{iS}^2 - (\sum_i \omega_i\theta_{iS})(\sum_j(1-p_j)\omega_j(s)\theta_{jS}) \\ &+\frac{1}{2}(\sum_i \omega_i(s)\theta_{iS})^2(\sum_j(1-p_j)\omega_j(s))\end{aligned}$$

The limit of the expression in the right hand side is

$$\begin{aligned}&\lim_{s\to\infty} r_{i_K}(s) + \frac{(1-p_{i_K})\theta_{i_K S}^2}{2} - \theta_{i_K S}^2(1-p_{i_K}) + \frac{\theta_{i_K S}^2}{2}(1-p_{i_K}) \\ =\;& \lim_{s\to\infty} r_{i_K}(s)\end{aligned}$$

For the consumption ratio, we use the fact that for all j,

$$\omega_j(s) = \frac{p_j\hat{c}^j(s)\hat{X}^j(s)}{\sum_k p_k\hat{c}^k(s)\hat{X}^k(s)}$$

Thus

$$\frac{p_i\hat{c}^i(s)\hat{X}^i(s)}{\epsilon(s)} = \frac{(1-\gamma_i)\omega_i(s)}{\sum_j(1-\gamma_j)\omega_j(s)} \tag{5.127}$$

And passing to the limit, yields (5.184) □

The following lemma gives tight bounds for the pricing kernel M in terms of the homogenous pricing kernels M_i.

Lemma 5.26. *Let $\Gamma \geq 1$ be such that $\Gamma p_i > 1$ for all i and $\gamma \leq 1$ be such that $\gamma p_i \leq 1$ for all i. Let*

$$\eta_{i0} := \frac{\hat{c}^i(0)\hat{X}^i(0)}{\epsilon_0} \tag{5.128}$$

Let $M(s)$ be defined by (3.53), then:

$$(\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\gamma}} M_i(s)^{\frac{1}{\gamma}})^{\gamma} \leq M(s) \leq (\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\Gamma}} M_i(s)^{\frac{1}{\Gamma}})^{\Gamma} \tag{5.129}$$

We also have

$$\min_i M_i(s) \leq M(s) \leq \max_i M_i(s) \tag{5.130}$$

A simple proof can be found in Cvitanic *et al.* (2012).

Proof [Proposition 3.36]

To simplify we take $T_0 = T =$ maturity of the bond.

Note that

$$\begin{aligned}
\mathbb{E}_t[M_{iT}] &= \mathbb{E}_t[\exp(\int_0^T (-r_i(u) - \frac{\theta_{iS}^2}{2} du - \int_0^T \theta_{iS} dW(u))] \\
&= \exp\big(\int_0^t -r_i(u)du - \theta_{iS}W_t\big)\exp\big(\int_t^T -r_i(u)du\big)
\end{aligned}$$

And using the inequality for M_T in the lemma,

$$(\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\gamma}}[M_{iT}]^{\frac{1}{\gamma}})^{\gamma} \leq M_T \leq (\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\Gamma}} M_{iT}^{\frac{1}{\Gamma}})^{\Gamma}$$

Jensen's inequality implies that

$$\begin{aligned}
&(\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\gamma}}\mathbb{E}_t[M_{iT}]^{\frac{1}{\gamma}})^{\gamma} \leq \mathbb{E}_t[M_T] \leq (\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\Gamma}}\mathbb{E}_t[M_{iT}]^{\frac{1}{\Gamma}})^{\Gamma}\\
&\frac{\gamma}{T}\log(\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\gamma}}(\exp(\int_0^t -r_i(u)du - \theta_{iS}W_t)\exp(\int_t^T -r_i(u)du))^{\frac{1}{\gamma}}) \leq \frac{1}{T}\log\mathbb{E}_t[M_T]\\
&\leq \frac{\Gamma}{T}\mathbb{E}_t\log(\sum_i \eta_{i0}^{\frac{1-\gamma_i}{\Gamma}}(\exp(\int_0^t -r_i(u)du - \theta_{iS}W_t)\exp(\int_t^T -r_i(u)du))^{\frac{1}{\Gamma}})
\end{aligned}$$

and taking the equivalents of the two bounds when $T \to \infty$, we get:

$$\frac{1}{T}\log\mathbb{E}_t[M_T] \sim -\frac{1}{T}\int_0^T r_{i_r}(u)$$

where $r_{i_r} := \inf_i r_i$.

$$B(t,T) = \frac{1}{M_t}\mathbb{E}_t[M_T] = \exp(-Y(t,T)(T-t))$$

and

$$Y(t,T) = -\frac{1}{T-t}(\log(\mathbb{E}_t[M_T] - \log(M_t)) \sim \frac{1}{T}\int_0^T r_{i_r}(u)du$$

as $T \to \infty$, t remaining fixed.

If in addition, we suppose that

$$\lim_{s\to\infty}\rho_i(0,s) = \bar{\rho}_i$$

then it is clear that the above limit is

$$\lim_{T \to \infty} Y(t,T) = \lim_{s \to \infty} r_{i_r}(s) := r_{i_r}(\infty)$$

Appendix 3: Subgame Perfect Equilibrium with Heterogeneous Agents

Proof [Proposition 4.6]

Using equation (4.30), we get:

$$\frac{\partial V^i}{\partial t} + \frac{1}{2}(x_i\sigma_S\bar{\pi}^i)^2 V^i_{xx} + (r-\bar{c}^i)x_iV^i_x + \frac{(\bar{c}^ix_i)^{\gamma_i}}{\gamma_i} - \frac{1}{2V^i_{xx}}(\omega_2\sigma_{\omega_2}V^i_{\omega_2 x} - \phi_iV^i_x)^2$$
$$-\omega_2\mu_{\omega_2 i}V^i_{\omega_2} + \frac{1}{2}(\omega_2\sigma_{\omega_2})^2V^i_{\omega_2\omega_2} = \mathbb{Q}_iV^i \tag{5.131}$$

We use the ansatz:

$$V^i(t,\omega_2,x_i) = v_i(t,\omega_2)^{1-\gamma_i}\frac{x_i^{\gamma_i}}{\gamma_i}$$

and the equations (4.29) to get:

$$\sigma_S\bar{\pi}^i = -\omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial\omega_2}}{v_i} + p_i\phi_i \quad ; \quad \bar{c}^i = \frac{1}{v_i} \tag{5.132}$$

This leads to

$$\frac{\partial v_i}{\partial t}v_i(t,\omega_2)^{-\gamma_i}\frac{(1-\gamma_i)x_i^{\gamma_i}}{\gamma_i} + \frac{1}{2}(-\omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial\omega_2}}{v_i} + p_i\phi_i)^2 v_i^{1-\gamma_i}(\gamma_i-1)x_i^{\gamma_i} + (r-\frac{1}{v_i})v_i^{1-\gamma_i}x_i^{\gamma_i}$$
$$+\frac{x_i^{\gamma_i}}{\gamma_i}v_i^{-\gamma_i} - \frac{1}{2v_i^{1-\gamma_i}(\gamma_i-1)x_i^{\gamma_i}}((1-\gamma_i)\omega_2\sigma_{\omega_2}v_i^{-\gamma_i}x_i^{\gamma_i}\frac{\partial v_i}{\partial\omega_2} - \phi_ix_i^{\gamma_i}v_i^{1-\gamma_i})^2$$
$$-\omega_2\mu_{\omega_2 i}(1-\gamma_i)v_i^{-\gamma_i}\frac{x_i^{\gamma_i}}{\gamma_i}\frac{\partial v_i}{\partial\omega_2} + \frac{1}{2}(\omega_2\sigma_{\omega_2})^2[(\gamma_i-1)v_i^{-\gamma_i-1}x_i^{\gamma_i}(\frac{\partial v_i}{\partial\omega_2})^2 + (1-\gamma_i)v_i^{-\gamma_i}\frac{\partial^2 v_i}{\partial\omega_2^2}\frac{x_i^{\gamma_i}}{\gamma_i}]$$
$$= \mathbb{Q}_iv_i^{1-\gamma_i}\frac{x_i^{\gamma_i}}{\gamma_i}$$

By multiplying the expression above by $v_i(t,\omega_2)^{\gamma_i}x_i^{-\gamma_i}\frac{\gamma_i}{1-\gamma_i}$, we get:

$$\begin{aligned}&\frac{\partial v_i}{\partial t}+\frac{1}{2}(-\omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial \omega_2}}{v_i}+p_i\phi_i)^2v_i(-\gamma_i)+(r-\frac{1}{v_i})v_i\frac{\gamma_i}{1-\gamma_i}\\&(1-\gamma_i)+\frac{1}{2(1-\gamma_i)^2v_i^{1-2\omega_i}}((1-\gamma_i)\omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial \omega_2}}{v_i}-\phi_i)^2v_i^{2-2\gamma_i}\\&-\omega_2\mu_{\omega_2 i}\frac{\partial v_i}{\partial \omega_2}+\frac{1}{2}(\omega_2\sigma_{\omega_2})^2[-\gamma_i\frac{(\frac{\partial v_i}{\partial \omega_2})^2}{v_i}+\frac{\partial^2 v_i}{\partial \omega_2^2}]=\mathbb{Q}_iv_i\frac{1}{1-\gamma_i}\end{aligned}$$

The terms in $\frac{(\frac{\partial v_i}{\partial \omega_2})^2}{v_i}$ add up to zero. Rearranging everything, we get (4.31) i.e. :

$$\begin{aligned}&\frac{\partial v_i}{\partial t}+\frac{1}{2}\omega_2^2\sigma_{\omega_2}^2\frac{\partial^2 v_i}{\partial \omega_2^2}-\omega_2(\mu_{\omega_2 i}+\gamma_ip_i\phi_i\sigma_{\omega_2})\frac{\partial v_i}{\partial \omega_2}\\&+p_i(\gamma_ir+\frac{\gamma_ip_i\phi_i^2}{2}-\mathbb{Q}_i)v_i+1=0\end{aligned}$$

$\square$

Proof [Proposition 4.7]

We prove that $d\log(\bar{c}^i(t)\bar{X}^i(t))=p_i(r+\frac{\phi_i^2}{2}-\delta_i\phi_i-\mathbb{Q}_i)dt+p_i\phi_idW_t$

By use of the Itô formula on $v_i(t,\omega_2(t))$:

$$dv_i=(\frac{\partial v_i}{\partial t}-\omega_2\mu_{\omega_2 i}\frac{\partial v_i}{\partial \omega_2}+\frac{1}{2}\omega_2^2\sigma_{\omega_2}^2\frac{\partial^2 v_i}{\partial \omega_2^2})dt-\omega_2\sigma_{\omega_2}\frac{\partial v_i}{\partial \omega_2}dW_t^i$$

and

$$\frac{d\bar{X}^i(t)}{\bar{X}^i(t)}=\left(r-\frac{1}{v_i}+\phi_i(p_i\phi_i-\omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial \omega_2}}{v_i})\right)dt+(p_i\phi_i-\omega_2\sigma_{\omega_2}\frac{\partial v_i}{\partial \omega_2})dW_t^i$$

Therefore,

$$\begin{aligned}
&d\log(\bar{c}^i(t)\bar{X}^i(t)) = d\log(\frac{\bar{X}^i(t)}{v_i(t)}) = d\log\bar{X}^i(t) - d\log(v_i(t))\\
&= -\frac{dv_i}{v_i} + \frac{1}{2}(\frac{dv_i}{v_i})^2 + \frac{d\bar{X}^i}{\bar{X}^i} - \frac{1}{2}(\frac{d\bar{X}^i}{\bar{X}^i})^2\\
&= \Bigg(r - \frac{1}{v_i} + \phi_i(p_i\phi_i - \omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial\omega_2}}{v_i}) - \frac{(p_i\phi_i - \omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial\omega_2}}{v_i})^2}{2} + \frac{(\omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial\omega_2}}{v_i})^2}{2}\\
&-\frac{\frac{\partial v_i}{\partial t} - \omega_2\mu_{\omega_2 i}\frac{\partial v_i}{\partial\omega_2} + \frac{1}{2}\omega_2^2\sigma_{\omega_2}^2\frac{\partial^2 v_i}{\partial\omega_2^2}}{v_i}\Bigg)dt + (p_i\phi_i - \omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial\omega_2}}{v_i})dW_t^i + \frac{\omega_2\sigma_{\omega_2}\frac{\partial v_i}{\partial\omega_2}}{v_i}dW_t^i
\end{aligned}$$

And using the PDE for v_i in the expression above, we obtain after some calculations:

$$d\log(\bar{c}^i(t)\bar{X}^i(t)) = p_i(r + \frac{\phi_i^2}{2} - \mathbb{Q}_i)dt + p_i\phi_i dW_t^i$$

or in terms of the Brownian motion W:

$$d\log(\bar{c}^i(t)\bar{X}^i(t)) = p_i(r + \frac{\theta_S^2 - \delta_i^2}{2} - \mathbb{Q}_i)dt + p_i\phi_i dW_t$$

$\square$

Proof [Proposition 4.8]

We have

$$\omega_2(t) = \frac{p_2\bar{c}^2(t)\bar{X}^2(t)}{p_1\bar{c}^1(t)\bar{X}^1(t) + p_2\bar{c}^2(t)\bar{X}^2(t)} = \frac{p_2 y_t}{p_1(1-y_t) + p_2 y_t} := g(y_t)$$

Solving for y_t, we get:

$$y_t = \frac{p_1\omega_2}{p_1\omega_2 + p_2\omega_1} \tag{5.133}$$

Since $g'(y_t) = \frac{p_2p_1}{(p_1(1-y_t)+p_2y_t)^2} = p_2p_1(\frac{\omega_2(t)}{p_2y_t})^2$ and $g''(y_t) = -\frac{2p_1p_2(p_2-p_1)}{(p_1(1-y_t)+p_2y_t)^3} = 2p_1p_2(p_2 -$

$p_1)((\frac{\omega_2(t)}{p_2 y_t})^3)$

An application of Ito's lemma gives:

$$\begin{aligned} d\omega_2(t) &= g'(y_t)dy_t + \frac{1}{2}g''(y_t)(dy_t)^2 \\ &= \frac{p_1}{p_2}(\frac{\omega_2(t)}{y_t})^2 p_2 y_t(\alpha_2 dt + (\theta_S - \theta_{2S})dW_t) + \frac{p_1(p_2-p_1)}{p_2^2}(\frac{\omega_2(t)}{y_t})^3 p_2^2 y_t^2(\theta_S - \theta_{2S})^2 dt \end{aligned}$$

Noting that $\theta_S - \theta_{2S} = \omega_1(\theta_{1S} - \theta_{2S})$

$$\begin{aligned} d\omega_2(t) &= \frac{p_1}{p_2}(\frac{p_1\omega_2 + p_2\omega_1}{p_1})^2 p_2 y_t(\alpha_2 dt + \omega_1(\theta_{1S} - \theta_{2S})dW_t) \\ &+ \frac{p_1(p_2-p_1)}{p_2^2}(\frac{p_1\omega_2 + p_2\omega_1}{p_1})^3 p_2^2 y_t^2 \omega_1^2(\theta_{1S} - \theta_{2S})^2 dt \\ d\omega_2(t) &= (p_1\omega_2 + p_2\omega_1)\omega_2(\alpha_2 dt + \omega_1(\theta_{1S} - \theta_{2S})dW_t) \\ &+ (p_1\omega_2 + p_2\omega_1)(p_2 - p_1)\omega_2^2\omega_1^2(\theta_{1S} - \theta_{2S})^2 dt \end{aligned}$$

Note that r is of the form $r = \sum_i \omega_i(\beta_i(\omega_2) + \mathbb{Q}_i)$ where β_i is a polynomial in ω_2 with constant coefficients.

We also note that

$\alpha_2 = r - \mathbb{Q}_2 - \beta_2 = \omega_1(\beta_1 + \mathbb{Q}_1) + \omega_2(\beta_2 + \mathbb{Q}_2) - \mathbb{Q}_2 - \beta_2 = \omega_1(\beta_1 - \beta_2 + \mathbb{Q})$.

$$\alpha_2 = \omega_1\big(\mathbb{Q} + \frac{(1-p_2)\phi_2^2 - (1-p_1)\phi_1^2}{2} + (\gamma_2 - \gamma_1)(\mu - \frac{\sigma^2}{2}) + \frac{p_2(\theta_S - \theta_{2S})^2 - p_1(\theta_S - \theta_{1S})^2}{2}\big) \tag{5.134}$$

Thus ω_2 has an SDE of the form

$$d\omega_2(s) = (p_1\omega_2 + p_2\omega_1)\omega_1\omega_2[(\mathbb{Q}(s, \omega_2(s)) + \Phi_0(\omega_2))ds + (\theta_{1S} - \theta_{2S})dW_s] \tag{5.135}$$

where Φ_0 is a polynomial with constant coefficients and of degree at most 2.

$$\begin{aligned}&\Phi_0(\omega_2) = \delta_1^2 - \delta_1\delta_2 + \mu(\gamma_2 - \gamma_1)\\&+\frac{(\theta_{1S} - \theta_{2S})^2}{2}[p_1(2\omega_1\omega_2 - \omega_2^2) + p_2(\omega_1^2 - 2\omega_1\omega_2)]\end{aligned}$$

□

Proof [Proposition 4.11]

Noticing that in the homogenous economy of agent i: $\vec{c}^i(s)\bar{X}^i(s) = \epsilon(s)$ and replacing it in the expression (4.25) for $\mathbb{Q}_i$, we get

$$q_i(t) = \frac{\mathbb{E}_t^{\mathbb{P}} \int_t^T \frac{\partial f_i(t,s)}{\partial t} Z_i(s)\epsilon(s)^{\gamma_i} ds + \frac{\partial f_i(t,T)}{\partial t} Z_i(T)\epsilon(T)^{\gamma_i}}{\mathbb{E}_t^{\mathbb{P}} \int_t^T f_i(t,s) Z_i(s)\epsilon(s)^{\gamma_i} ds + f_i(t,T) Z_i(T)\epsilon(T)^{\gamma_i}}$$

Using the fact that

$$\epsilon(s)^{\gamma_i} Z_i(s) = \epsilon(t)^{\gamma_i} Z_i(t) e^{(\gamma_i(\mu - \frac{\sigma^2}{2}) - \frac{\delta_i^2}{2})(s-t) + (\delta_i + \gamma_i\sigma)(W_s - W_t)}$$

we get

$$\mathbb{E}_t^{\mathbb{P}}[\epsilon(s)^{\gamma_i} Z_i(s)] = \epsilon(t)^{\gamma_i} Z_i(t) e^{(\gamma_i(\mu - \frac{\sigma^2}{2}) - \frac{\delta_i^2}{2})(s-t) + \frac{(\delta_i + \gamma_i\sigma)^2}{2}(s-t)} = \epsilon(t)^{\gamma_i} Z_i(t) e^{k_i(s-t)}$$

and this yields the result. □

Proof [Proposition 4.12]

We write

$$\frac{M_i(v)\epsilon(v)}{M_i(t)\epsilon(t)} = \exp\left(\int_t^v -r_i(u) - \frac{\theta_{iS}^2}{2} + \mu - \frac{\sigma^2}{2} du + \int_t^v (\sigma - \theta_{iS}) dW(u) \right)$$

And using (4.49), we get

$$\mathbb{E}_t^{\mathbb{P}}\left[\frac{M_i(v)\epsilon(v)}{M_i(t)\epsilon(t)}\right] = \exp\left(\int_t^v \gamma_i(\mu_i - \frac{\sigma^2(1-\gamma_i)}{2} - q_i(u))du\right) := s_i(t,v)$$

Equation (4.49) comes from the clearing conditions for the commodity and stock: For example in agent 1's homogenous economy

$$\bar{c}^1(t)\bar{X}^1(t) = \epsilon(t)$$

and

$$S_1(t) = \bar{X}^1(t) = \frac{\bar{c}^1(t)\bar{X}^1(t)}{\bar{c}^1(t)} = \epsilon(t)v_1(t,0)$$

Similarly

$$S_2(t) = \epsilon(t)v_2(t,1)$$

□

Proof [Proposition 4.13] Let

$$y_t := \frac{\bar{c}^2(t)\bar{X}^2(t)}{\epsilon(t)} \quad ; \quad Y_t = \frac{\bar{c}^1(t)\bar{X}^1(t)}{\epsilon(t)} = 1 - y_t \tag{5.136}$$

Since

$$\bar{c}^2(t)\bar{X}^2(t) = \bar{c}^2(0)\bar{X}^2(0)\exp\left(\int_0^t p_2(r + \frac{(1-p_2)}{2}\phi_2^2 - \mathbb{Q}_2)du + \int_0^t p_2\phi_2(u)dW(u)\right) \tag{5.137}$$

we get:

$$y(t) = y(0)\exp\left(\int_0^t p_2(r + \frac{1-p_2}{2}\phi_2^2 - \mathbb{Q}_2 - (\mu - \frac{\sigma^2}{2}))du + \int_0^t (p_2\phi_2(u) - \sigma)dW(u)\right) \tag{5.138}$$

Noting that

$$p_2\phi_2(u) - \sigma = p_2(\theta_2 + \delta_2 - \sigma(1-\gamma_2)) = p_2(\theta_S(u) - \theta_{2S})$$

This implies that

$$\begin{aligned} dy_t &= p_2 y_t\left[\Big(r + \frac{(1-p_2)\phi_2^2}{2} - \mathbb{Q}_2 - (1-\gamma_2)(\mu - \frac{\sigma^2}{2}) + \frac{p_2(\theta_S - \theta_{2S})^2}{2}\Big)dt + (\theta_S(t) - \theta_{2S})dW(t)\right] \\ dy_t &= p_2 y_t[\alpha_2(t)dt + (\theta_S - \theta_{2S})dW(t)] \end{aligned}$$

Similarly

$$\begin{aligned} dY_t &= p_1 Y_t\left[\Big(r + \frac{(1-p_1)\phi_1^2}{2} - \mathbb{Q}_1 - (1-\gamma_1)(\mu - \frac{\sigma^2}{2}) + \frac{p_1(\theta_S - \theta_{1S})^2}{2}\Big)dt + (\theta_S(t) - \theta_{1S})dW(t)\right] \\ dY_t &= p_1 Y_t[\alpha_1(t)dt + (\theta_S - \theta_{1S})dW(t)] \end{aligned}$$

Adding the two expressions $y_t + Y_t = 1$, we get

$$p_1 Y_t \alpha_1(t) + p_2 y_t \alpha_2(t) = 0$$

$$p_1 Y_t(\theta_S - \theta_{1S}) + p_2 y_t(\theta_S - \theta_{2S}) = 0$$

Thus

$$\theta_S(t) = \frac{p_1 Y_t}{p_1 Y_t + p_2 y_t}\theta_{1S} + \frac{p_2 y_t}{p_1 Y_t + p_2 y_t}\theta_{2S} = \omega_1(t)\theta_{1S} + \omega_2(t)\theta_{2S}$$

We also have

$0 = \frac{p_1 Y_t}{p_1 Y_t + p_2 y_t}\alpha_1(t) + \frac{p_2 y_t}{p_1 Y_t + p_2 y_t}\alpha_2(t) = \omega_1(t)\alpha_1(t) + \omega_2(t)\alpha_2(t) = \sum_i \omega_i\big(r + \frac{(1-p_i)\phi_i^2}{2} - \mathbb{Q}_i - (1-\gamma_i)(\mu - \frac{\sigma^2}{2}) + \frac{p_i(\theta_S - \theta_{iS})^2}{2}\big)$

And solving for r yields:

$$r = \sum_i \omega_i\big(\frac{(p_i-1)\phi_i^2}{2} + \mathbb{Q}_i + (1-\gamma_i)(\mu - \frac{\sigma^2}{2}) - \frac{p_i(\theta_S - \theta_{iS})^2}{2}\big) \tag{5.139}$$

□

Proof [Proposition 4.16]

Consider the SDE (5.231) defined for $s \geq t$ by:

$$d\omega_2^y(s) = \omega_2^y\omega_1^y(s)\big(p_1\omega_2^y + p_2\omega_1^y(s)\big)\Big[(y(s,\omega_2^y(s)) + \Phi_0(\omega_2^y(s))ds + (\theta_{1S} - \theta_{2S})dW(s)\Big] \tag{5.140}$$

with initial condition $\omega_2^y(t) = \omega_2$ and call

$$x_s^y := \log\left(\frac{\omega_2^y(s)}{1-\omega_2^y(s)}\right) \tag{5.141}$$

we get by Ito's lemma:

$dx_s^y = d\log\omega_2^y(s) - d\log\omega_1^y(s) = \frac{d\omega_2^y(s)}{\omega_2^y(s)} - \frac{1}{2}(\frac{d\omega_2^y(s)}{\omega_2^y(s)})^2 - \frac{d\omega_1^y(s)}{\omega_1^y(s)} + \frac{1}{2}(\frac{d\omega_1^y(s)}{\omega_1^y(s)})^2 = (p_1\omega_2^y + p_2\omega_1^y)\Big[\big(y(s,\omega_2^y(s)) + \Phi_0(\omega_2^y(s)) + (\omega_2^y - \omega_1^y)(p_1\omega_2^y + p_2\omega_1^y)\frac{(\theta_{1S}-\theta_{2S})^2}{2}\big)ds + (\theta_{1S} - \theta_{2S})dW_s\Big]$

Let

$$q_2(x) := \frac{1}{1+e^{-x}} \text{ and } q_1(x) := 1 - q_2(x) \tag{5.142}$$

Equation (5.232) implies

$$\omega_i^y(s) = q_i(x_s^y)$$

Thus,

$$dx_s^y = (p_1q_2+p_2q_1)\Big[\big(y(s,q_2(x_s^y))+\Phi_0(q_2)+(q_2-q_1)(p_1q_2+p_2q_1)\frac{(\theta_{1S}-\theta_{2S})^2}{2}\big)ds+(\theta_{1S}-\theta_{2S})dW_s\Big]$$

where all the expressions are evaluated at the point (s, x_s^y). Noting that

$$q_2'(x) = -q_1'(x) = q_1(x)q_2(x)$$

it is clear that the coefficients of the SDE have a bounded x derivative and are continuous with respect to t. This shows that the SDE for x_s^y has a unique solution for $t \leq s \leq T$. Furthermore $\omega_2^y(s) = q_2(x_s^y) \in [0,1]$. □

Proof [Theorem 4.18]

The process $\omega_2(s) = \omega_2^{\mathbb{Q}}(s)$ satisfies the conditions of Proposition 4.16. Therefore, $\omega_2(s)$ exists for $s \in [0,T]$ and stays in the interval $[0,1]$.

An examination of the proof of Theorem 4.17 reveals that $\mathbb{Q}_i := F_i[\mathbb{Q}] \in \mathbb{B}$.

This ends the proof. □

Proof [Proposition 4.19]

To obtain the PDE (4.76), we start with equation (4.31):

$$\begin{aligned}&\frac{\partial v_i}{\partial t} + \frac{1}{2}\omega_2^2\sigma_{\omega_2}^2\frac{\partial^2 v_i}{\partial \omega_2^2} - \omega_2(\mu_{\omega_2 i} + \gamma_i p_i \phi_i \sigma_{\omega_2})\frac{\partial v_i}{\partial \omega_2}\\ &+p_i(\gamma_i r + \frac{\gamma_i p_i \phi_i^2}{2} - \mathbb{Q}_i)v_i + 1 = 0\end{aligned}$$

Recall the equations (4.35), (4.36)

$$\begin{aligned}
\mu_{\omega_2} &= -\omega_1(t)(p_1\omega_2 + p_2\omega_1)\big(\mathbb{Q}(t,\omega_2) + \Phi_0(\omega_2)\big) \\
\sigma_{\omega_2} &= -\omega_1(t)(p_1\omega_2 + p_2\omega_1)(\theta_{1S} - \theta_{2S})
\end{aligned}$$

and that $\mu_{\omega_2 i} = \mu_{\omega_2} + \delta_i\sigma_{\omega_2}$. We replace r and θ_S obtained in Proposition 4.13. We get after rearranging:

$$\begin{aligned}
\frac{\partial v_i}{\partial t} + \frac{1}{2}((\omega_1\omega_2(p_1\omega_2 + p_2\omega_1)(\theta_{1S} - \theta_{2S}))^2\frac{\partial^2 v_i}{\partial \omega_2^2} \\
+\omega_1\omega_2(p_1\omega_2 + p_2\omega_1)(k_{0i} + \mathbb{Q}(t,\omega_2))\frac{\partial v_i}{\partial \omega_2} + K_{1i}v_i + 1 = 0 \\
v_i(T,\omega_2) = 1
\end{aligned}$$

We first prove the existence and uniqueness of a bounded solution to the PDE

$$\begin{cases}
\frac{\partial v_i}{\partial t} + \frac{1}{2}((\omega_1\omega_2(p_1\omega_2 + p_2\omega_1)(\theta_{1S} - \theta_{2S}))^2\frac{\partial^2 v_i}{\partial \omega_2^2} \\
+\omega_1\omega_2(p_1\omega_2 + p_2\omega_1)(k_{0i} + \mathbb{Q}(t,\omega_2))\frac{\partial v_i}{\partial \omega_2} + K_{1i}v_i + 1 = 0 \\
v_i(T,\omega_2) = 1
\end{cases} \tag{5.143}$$

where $\omega_1 = 1 - \omega_2$ and (in terms of the equilibrium parameters):

$$K_{1i}(t,\omega_2) = p_i(\gamma_i r(t,\omega_2) + \frac{\gamma_i p_i \phi_i^2}{2} - \mathbb{Q}_i(t,\omega_2)) \tag{5.144}$$

$$\begin{aligned} k_{0i}(\omega_2) \quad &= \frac{1}{2}(\theta_{2S}(p_2\omega_1^2 - p_1\omega_2^2 + 2(p_1 - p_2)\omega_1\omega_2)(\theta_{2S} - \theta_{1S})^2 \qquad (5.145) \\ &+\gamma_i p_i \phi_i(\theta_{1S} - \theta_{2S}) + (\gamma_2 - \gamma_1)(\mu_i - \frac{\sigma^2}{2}) + \frac{\delta_1^2 - \delta_2^2}{2} + \delta_i(\delta_2 - \delta_1) \\ &K_{1i}(t,\omega_2) = k_{1i}(\omega_2) + p_i\gamma_i(\omega_1\mathbb{Q}_1 + \omega_2\mathbb{Q}_2(t,\omega_2)) - p_i\mathbb{Q}_i \qquad (5.146) \end{aligned}$$

and

$$k_{11} = p_1\gamma_1 \textstyle\sum_j \omega_j(\frac{\mu}{p_j} + \frac{\delta_j^2 - \theta_S^2}{2}) + \frac{\omega_2}{2}(p_1(\theta_{1S}\omega_1 + \theta_{2S}\omega_2 + \delta_2)^2 - p_2(\theta_{1S}\omega_1 + \theta_{2S}\omega_2 + \delta_1)^2)$$

$$k_{12} = p_2\gamma_2 \textstyle\sum_j \omega_j(\frac{\mu}{p_j} + \frac{\delta_j^2 - \theta_S^2}{2}) - \frac{\omega_1}{2}(p_1(\theta_{1S}\omega_1 + \theta_{2S}\omega_2 + \delta_2)^2 - p_2(\theta_{1S}\omega_1 + \theta_{2S}\omega_2 + \delta_1)^2)$$

The above PDE is parabolic, however it has a degeneracy at $\omega_2 = 0$ and $\omega_2 = 1$. Consider the change of variables:

$$x = \log\left(\frac{\omega_2}{1 - \omega_2}\right) \qquad (5.147)$$

Thus

$$\omega_1 = \frac{e^{-x}}{1 + e^{-x}} := q_1(x) \quad ; \quad \omega_2 = \frac{1}{1 + e^{-x}} := q_2(x) \qquad (5.148)$$

Take

$$\bar{v}_i(t,x) := v_i(t, q_2(x)) \qquad (5.149)$$

We get:

$$\begin{aligned} \frac{\partial \bar{v}_i(t,x)}{\partial x} &= q_1(x)q_2(x)\frac{\partial v_i(t,\omega_2)}{\partial \omega_2} \\ \frac{\partial^2 \bar{v}_i(t,\omega_2)}{\partial x^2} &= q_1(x)q_2(x)\left[q_1 q_2 \frac{\partial^2 v_i(t,\omega_2)}{\partial \omega_2^2} + (1 - 2q_2)\frac{\partial v_i(t,\omega_2)}{\partial \omega_2}\right] \end{aligned}$$

The PDE for $\bar{v}_i$ is a uniformly parabolic PDE with bounded coefficients that are uniformly Lipschitz continuous in (t,x) thus it has a unique bounded solution. The

PDE for $\bar{v}_i$ is given by:

$$0 = \frac{\partial \bar{v}_i}{\partial t} + \frac{1}{2}(p_1 q_2(x) + p_2 q_1(x))^2(\theta_{1S} - \theta_{2S})^2 \frac{\partial^2 \bar{v}_i}{\partial x^2} + K_{1i}(t, q_2(x))\bar{v}_i + 1 \tag{5.150}$$
$$+\left[(p_1\omega_2 + p_2\omega_1)(k_{0i} + \mathbb{Q}(t, q_2(x))) - \frac{1 - 2q_2(x)}{2}(p_1 q_2 + p_2 q_1(x))^2(\theta_{1S} - \theta_{2S})^2\right]\frac{\partial \bar{v}_i}{\partial x}$$
$$\bar{v}_i(T, x) = 1 \tag{5.151}$$

□

Proof [Theorem 4.20]

From Proposition 4.19, we see that $v_i(t, \omega_2)^{1-\gamma_i} U_i(x_i)$ is a value function solution of the extended HJB (4.23). The consumption to wealth ratio $\bar{c}^i$ and the investment to wealth ratio $\bar{\pi}^i$ are given by:

$$\bar{c}^i = \frac{1}{v_i} \;\; ; \;\; \bar{\pi}^i = p_i \phi_i - \omega_2 \sigma_{\omega_2} \frac{\frac{\partial v_i}{\partial \omega_2}}{v_i}$$

And since

$$\sigma_{\omega_2} = -(p_1\omega_2 + p_2\omega_1)\omega_1(\theta_{1S} - \theta_{2S}),$$
$$\bar{\pi}^i = p_i\phi_i - \omega_2\sigma_{\omega_2}\frac{\frac{\partial v_i}{\partial \omega_2}}{v_i} = p_i\phi_i + (p_1\omega_2 + p_2\omega_1)\omega_1\omega_2(\theta_{1S} - \theta_{2S})\frac{\frac{\partial v_i}{\partial \omega_2}}{v_i}$$

From the definition of $\omega_1(t)$, we have

$$\omega_1(t) = \frac{p_1 \bar{c}^1(t)\bar{X}^1(t)}{p_1\bar{c}^1(t)\bar{X}^1(t) + p_2\bar{c}^2(t)\bar{X}^2(t)}$$

so $\bar{c}^1(t)\bar{X}^1(t)(p_1 - p_1\omega_1(t)) = p_2\omega_1(t)\bar{c}^2(t)\bar{X}^2(t)$. Thus

$$\bar{c}^1(t)\bar{X}^1(t) = \frac{p_2\omega_1(t)}{p_1\omega_2(t)}\bar{c}^2(t)\bar{X}^2(t)$$

and by the commodity clearing condition

$$\bar{c}^1(t)\bar{X}^1(t) + \bar{c}^1(t)\bar{X}^1(t) = \epsilon(t)$$

we get

$$\frac{p_2\omega_1(t) + p_1\omega_2(t)}{p_1\omega_2(t)}\bar{c}^2(t)\bar{X}^2(t) = \epsilon(t).$$

This proves (5.200).

The expressions for $\bar{X}^i(t)$ are obtained by writing $\bar{X}^i(t) = v_i(t, \omega_2(t))\bar{c}^i(t)\bar{X}^i(t)$.

To obtain $S(t)$, we write the clearing condition $S(t) = \bar{X}^1(t) + \bar{X}^2(t)$.

The price dividend ratio is $R(t) = \frac{S(t)}{\epsilon(t)}$. We notice that $R(t)$ is a deterministic function of $t, \omega_2(t)$. We can

$$dS(t) = d(R(t, \omega_2(t))\epsilon(t)) \tag{5.152}$$

and identifying the $dW(t)$ terms in the expression above:

$$\sigma_S(t) = \sigma + \frac{\frac{\partial R}{\partial \omega_2}}{R}$$

$\square$

Proof [Theorem 4.17 : There exists $\nu > 0$ such that the operator $y \mapsto F[y]$ defines a contraction on the space $\mathbb{B}_\nu$.]

Recall

$$\rho_i(t,s) = \frac{\frac{\partial f_i(t,s)}{\partial t}}{f_i(t,s)} \quad \forall \ 0 \leq t \leq s \leq T.$$

Recall that for $\nu > 0$:

$\mathbb{B}_\nu := \{y \in C([0,T]\times[0,1]) \mid y$ is C^1 in t, ω_2 and $\forall (t,\omega_2) \in [0,T]\times[0,1]$, $|y(t,\omega_2)| \leq 2||\rho||$ & $|\frac{\partial y(t,\omega_2)}{\partial \omega_2}| \leq \nu\}$.

Define the polynomials

$$G_0(x) = x(1-x)(p_1 x + p_2(1-x)) \quad ; \quad g_0(x) = G_0'(x) \tag{5.153}$$

$$\Phi_0(x) = \delta_1^2 - \delta_1\delta_2 + \mu(\gamma_2-\gamma_1) + \frac{(\theta_{1S}-\theta_{2S})^2}{2}[p_1(2x(1-x)-x^2) + p_2((1-x)^2 - 2x(1-x)) \tag{5.154}$$

$$\phi_0(x) = \Phi_0'(x) \tag{5.155}$$

Recall that $\omega_2^y(s)$ is given by the SDE

$$d\omega_2^y(s) = G_0(\omega_2^y)(y(s,\omega_2^y) + \Phi_0(\omega_2^y))ds + G_0(\omega_2^y)(\theta_{1S}-\theta_{2S})dW_s \tag{5.156}$$

Let $D^{t,y}(s)$ be the derivative process $\frac{\partial \omega_2^{t,y}(s)}{\partial \omega_2(t)}$. It satisfies the SDE

$$dD^{t,y}(s) = [G_0.(\frac{\partial y}{\partial \omega_2} + \phi_0) + g_0.(y+\Phi_0)(s,\omega_2^{t,y}(s))]D^{t,y}(s)ds + (\theta_{1S}-\theta_{2S})g_0(\omega_2^{t,y}(s))D^{t,y}(s)dW_s \tag{5.157}$$

$D^{t,y}(s)$ is given by the explicit formula

$$D^{t,y}(s) = \exp\left(\int_t^s G_0.(\frac{\partial y}{\partial \omega_2} + \phi_0) + g_0.(y+\Phi_0) - \frac{(\theta_{1S}-\theta_{2S})^2 g_0^2}{2} du + \int_t^s (\theta_{1S}-\theta_{2S})g_0 \, dW_u\right) \tag{5.158}$$

We can also write the second derivative process

$$E^{t,y}(s) = \frac{\partial D^{t,y}(s)}{\partial \omega_2} \tag{5.159}$$

It is given by the linear SDE

$$dE^{t,y}(s) = [\frac{\partial \alpha(s)}{\partial \omega_2} D^{t,y}(s)^2 + \alpha(s) E^{t,y}(s)]ds + (\theta_{1S} - \theta_{2S})[g_0'(\omega_2^{t,y}(s)) D^{t,y}(s)^2 + g_0(\omega_2^{t,y}(s)) E^{t,y}(s)]dW_s \tag{5.160}$$

where $E^{t,y}(t) = 0$ and

$$\alpha := G_0(\omega_2).(\frac{\partial y(t,\omega_2)}{\partial \omega_2} + \phi_0(\omega_2)) + g_0(\omega_2).(y(t,\omega_2) + \Phi_0(\omega_2))$$

is the drift of the SDE defining $D^{t,y}(s)$.

We can write

$$a_1^y(u) = -p_1\gamma_1 y(u)\omega_2^y(u) + c_1(\omega_2^y(u))$$

$$a_2^y(u) = p_2\gamma_2 y(u)(1-\omega_2^y(u)) + c_2(\omega_2^y(u))$$

$$b_i^y(u) = \gamma_i p_i \theta^y(u) + p_i\delta_i = p_i\delta_i + \gamma_i p_i \theta_{1S} + \gamma_i p_i(\theta_{2S} - \theta_{1S})\omega_2^y(u)$$

where c_i is a polynomial in ω_2^y of degree 3 and constant coefficients and is independent of y. Define

$$Z_i^{t,y}(s) := \int_t^s a_i^y(u)du + \int_t^s b_i^y(u)dW_u \tag{5.161}$$

$$L_i^{t,y}(s) := \int_t^s (\frac{\partial a_i^y(u)}{\partial \omega_2} D^{t,y}(u)du + \int_t^s \frac{\partial b_i^y(u)}{\partial \omega_2} D^{t,y}(u)dW_u \tag{5.162}$$

$$E_i^{t,y}(s) := \frac{\partial L_i^{t,y}(s)}{\partial \omega_2} = \int_t^s \frac{\partial^2 a_i^y(u)}{\partial \omega_2^2}(D^{t,y}(u))^2 + \frac{\partial a_i^y(u)}{\partial \omega_2} E^{t,y}(u)du + \int_t^s p_1\gamma_1(\theta_{1S} - \theta_{2S})E^{t,y}(u)dW_u \tag{5.163}$$

$$F_{0i}[y](t,\omega_2) = \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T f_i(t,s)e^{Z_i^{t,y}(s)}ds + f_i(t,T)e^{Z_i^{t,y}(T)}\Big] \tag{5.164}$$

$$F_{1i}[y](t,\omega_2) = \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T \frac{\partial f_i(t,s)}{\partial t}e^{Z_i^{t,y}(s)}ds + \frac{\partial f_i(t,T)}{\partial t}e^{Z_i^{t,y}(T)}\Big] \tag{5.165}$$

In what follows, we omit the terms $f_i(t,T)e^{Z_i^{t,y}(T)}$, $\frac{\partial f_i(t,T)}{\partial t}e^{Z_i^{t,y}(T)}$ in order to simplify the exposition. We have:

$$\frac{\partial F_{0i}[y](t,\omega_2)}{\partial \omega_2} = \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T f_i(t,s)e^{Z_i^{t,y}(s)}L_i^{t,y}(s)ds\Big] \tag{5.166}$$

$$\frac{\partial^2 F_{0i}[y](t,\omega_2)}{\partial \omega_2^2} = \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T f_i(t,s)e^{Z_i^{t,y}(s)}\big((L_i^{t,y}(s))^2 + E_i^{t,y}(s)\big)ds\Big] \tag{5.167}$$

Preliminary inequalities

For a, b, c three non negative numbers and $p \geq 1$, we have (Jensen):

$$(a+b+c)^p \leq 3^{p-1}(a^p + b^p + c^p) \tag{5.168}$$

Lemma 5.27. *For $p \geq 1$, there is $K_2 > K_1 > 0$ independent of t, s, y but dependent on p such that:*

$$\mathbb{E}_t^{\mathbb{P}}\big[\sup_{t \leq u \leq s} e^{pZ_i^{t,z}(u)}\big] \leq K_2 \tag{5.169}$$

$$\mathbb{E}_t^{\mathbb{P}}\big[\inf_{t \leq u \leq s} e^{pZ_i^{t,z}(u)}\big] \geq K_1 \tag{5.170}$$

For $p \geq 2$: There is $K > 0$ independent of t, s, y such that

$$\mathbb{E}_t^{\mathbb{P}}\big[(D^{t,y})_s^{*p}\big] \leq e^{K(1+\nu)(s-t)} \tag{5.171}$$

$$\mathbb{E}_t^{\mathbb{P}}\left[(L_i^{t,y})_s^{*p}\right] \leq K(1+\nu)^p e^{K(1+\nu)^p(s-t)}(s-t) \tag{5.172}$$

$$\mathbb{E}_t^{\mathbb{P}}[(E_i^{t,y})^{*p}(s)] \leq K(1+\nu)^p(1+\nu+\bar{\nu})^p(s-t)e^{K(1+\nu)^p(s-t)} \tag{5.173}$$

Proof [Lemma 5.27] Let us fix $y \in \mathbb{B}_\nu$. By definition $|y(u)| \leq ||\rho||$. Since $\omega_2^y(u) \in [0,1]$ we can easily see that both $a_i^y(u)$ and $b_i^y(u)$ are bounded by constants $||a_i||$ and $||b_i||$.

Let us use the fact that

$$R_i^{t,y}(s) := e^{Z_i^{t,y}(s)}$$

satisfies the SDE

$$dR_i^{t,y}(s) = R_i^{t,y}(s)(a_i^{t,y}(s) + \frac{(b_i^{t,y}(s))^2}{2})ds + b_i^{t,y}(s)dW(s) \tag{5.174}$$

Proposition 5.22 yields the existence of K_2 independent of t, s such that:

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s} e^{pZ_i^{t,y}(u)}] \leq K_2 e^{K_2(s-t)} \leq K_2 e^{K_2 T} \tag{5.175}$$

The proof of inequality (5.261) is similar if we consider the process $e^{-pZ_i^{t,y}(s)}$.

Now, let us fix $t \leq u \leq s \leq T$ and $y \in \mathbb{B}_\nu$. To prove the inequality (5.262), we use the SDE

$$dD^{t,y}(s) = [G_0.(\frac{\partial y}{\partial \omega_2}+\phi_0)+g_0.(y+\Phi_0)(s,\omega_2^{t,y}(s))]D^{t,y}(s)ds+(\theta_{1S}-\theta_{2S})g_0(\omega_2^{t,y}(s))D^{t,y}(s)dW_s$$

$$\left|G_0.(\tfrac{\partial y}{\partial \omega_2}+\phi_0)+g_0.(y+\Phi_0)\big)dv\right| \leq K(1+\nu)\ ,\ \left|(\theta_{1S}-\theta_{2S})g_0(\omega_2^{t,y}(s))\right| \leq K$$

for a certain constant K independent of t, s, ν, Proposition 5.22 yields

$$\mathbb{E}_t^{\mathbb{P}}\left[(D^{t,y})_s^{*p}\right] \leq Ke^{K(1+\nu)^p(s-t)}$$

The proof of inequality (5.263) is similar. Recall

$$L_i^{t,y}(s) := \int_t^s (\frac{\partial a_i^y(u)}{\partial \omega_2} D^{t,y}(u)du + \int_t^s \frac{\partial b_i^y(u)}{\partial \omega_2} D^{t,y}(u)dW_u$$

$$L_1^{t,y}(s) = \int_t^s (c_1'(\omega_2^y) - p_1\gamma_1(\omega_2^y \frac{\partial y}{\partial \omega_2} + y))D^{t,y}(u)du + \int_t^s p_1\gamma_1(\theta_{2S} - \theta_{1S})D^{t,y}(u)dW_u \tag{5.176}$$

so that by Proposition 5.22,

$$\mathbb{E}_t^{\mathbb{P}}\left[(L_1^{t,y})_s^{*p}\right] \leq K\mathbb{E}_t^{\mathbb{P}} \int_t^s \left|(c_1'(\omega_2^y) - p_1\gamma_1(\omega_2^y \frac{\partial y}{\partial \omega_2} + y))D^{t,y}(u)\right|^p + \left|p_1\gamma_1(\theta_{2S} - \theta_{1S})D^{t,y}(u)\right|^p du$$

We can see that

$$\mathbb{E}_t^{\mathbb{P}}\left[(L_1^{t,y})_s^{*p}\right] \leq K(1+\nu)^p e^{K(1+\nu)^p(s-t)}(s-t) \tag{5.177}$$

A similar inequality can be derived for $L_2^{t,y}$.

The proof of inequality 5.264 is also similar:

$$E_i^{t,y}(s) = \int_t^s \frac{\partial^2 a_i^y(u)}{\partial \omega_2^2}(D^{t,y}(u))^2 + \frac{\partial a_i^y(u)}{\partial \omega_2} E^{t,y}(u)du + \int_t^s p_1\gamma_1(\theta_{1S} - \theta_{2S})E^{t,y}(u)dW_u$$

Taking

$$J_s = \int_t^s \frac{\partial^2 a_i^y(u)}{\partial \omega_2^2}(D^{t,y}(u))^2 du$$

we get by Proposition 5.22:

$$\mathbb{E}_t^{\mathbb{P}}[(E_i^{t,y})^{*p}(s)] \le K_1\bigg(\mathbb{E}_t^{\mathbb{P}}[J_s^{*p}] + K(1+\nu)^p\mathbb{E}_t^{\mathbb{P}}\bigg[\int_t^s |E^{t,y}(u)|^p du\bigg]\bigg)$$

By Jensen's inequality

$$\begin{aligned}&\mathbb{E}_t^{\mathbb{P}}[J_s^{*p}] \le (s-t)^{p-1}\mathbb{E}_t^{\mathbb{P}}[\sup_{t\le u\le s}\int_t^u |\frac{\partial^2 a_i^y}{\partial\omega_2^2}(v)|^p.(D^{t,y}(v))^{2p}dv]\\ &\le K(1+\nu+\bar{\nu})^p(s-t)^p \times Ke^{K(1+\nu)(s-t)}\end{aligned}$$

Again, by applying Proposition 5.22 successively:

$$\begin{aligned}\mathbb{E}_t^{\mathbb{P}}[(E^{t,y})^{*p}(s)] \le K_1(\mathbb{E}_t^{\mathbb{P}}[(J_E^{t,y})_s^{*p}] + 2^{p-1}(s-t)(||\alpha||^p + |(\theta_{1S}-\theta_{2S})g_0|^p)\\ \times e^{2^{p-1}(s-t)(||\alpha||^p+|(\theta_{1S}-\theta_{2S})g_0|^p)})\end{aligned}$$

where

$$J_E^{t,y}(s) = \int_t^s \frac{\partial\alpha(u)}{\partial\omega_2}D^{t,y}(u)^2du + \int_t^s(\theta_{1S}-\theta_{2S})g_0'(\omega_2^{t,y}(s))D^{t,y}(u)^2dW_u \qquad (5.178)$$

and

$$\mathbb{E}_t^{\mathbb{P}}[(J_E^{t,y})_s^{*p}] \le K(1+\nu+\bar{\nu})^p(s-t)^p \times Ke^{K(1+\nu)(s-t)}$$

Since

$$||\frac{\partial^2 a_i^y}{\partial\omega_2}|| \le K(1+\nu+\bar{\nu})$$

we get:

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{P}}[(E^{t,y})^{*p}(s)] &\leq K_1(K(1+\nu+\bar{\nu})^p(s-t)^p \times Ke^{K(1+\nu)(s-t)} \\
&+2^{p-1}(s-t)(||\alpha||^p + |(\theta_{1S}-\theta_{2S})g_0|^p) \times e^{2^{p-1}(s-t)(||\alpha||^p+|(\theta_{1S}-\theta_{2S})g_0|^p)}) \\
\mathbb{E}_t^{\mathbb{P}}[(E^{t,y})^{*p}(s)] &\leq K(1+\nu+\bar{\nu})^p(s-t)e^{K(1+\nu)^p(s-t)}
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}[(E_i^{t,y})^{*p}(s)] \leq K_1\Bigg(\mathbb{E}_t^{\mathbb{P}}[J_s^{*p}] + K(1+\nu)^p\mathbb{E}_t^{\mathbb{P}}\Bigg[\int_t^s |E^{t,y}(u)|^p du\Bigg]\Bigg) \\
&\leq K_1\Bigg(K(1+\nu+\bar{\nu})^p(s-t)^p \times Ke^{K(1+\nu)(s-t)} + K(1+\nu)^p(1+\nu+\bar{\nu})^p(s-t)^2e^{K(1+\nu)^p(s-t)}\Bigg) \\
&\mathbb{E}_t^{\mathbb{P}}[(E_i^{t,y})^{*p}(s)] \leq K(1+\nu)^p(1+\nu+\bar{\nu})^p(s-t)e^{K(1+\nu)^p(s-t)}
\end{aligned}$$

□

Definition of the norm $||.||_{t_0,t_1}$ Suppose z is a continuous bounded function defined on $[0,T]\times[0,1]$, then $||z||_{t_0,t_1}$ will denote the following supremum:

$$||z||_{t_0,t_1} = \sup_{(u,x)\in[t_0,t_1]\times[0,1]} |z(u,x)|$$

Lemma 5.28. *Let $y, z \in \mathbb{B}_\nu$ and $p \geq 1$. There is $K > 0$ independent of t, s, y:*

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s} |\omega_2^{t,z}(u) - \omega_2^{t,y}(u)|^p] \leq K||z-y||_{t,s}^p(s-t)e^{K(1+\nu)^p(s-t)} \tag{5.179}$$

Proof [Lemma 5.28]

Fix $y, z \in \mathbb{B}_\nu$. We suppose $\omega_2^{t,y}(t) = \omega_2^{t,z}(t)$. From the SDE (5.247),

$\omega_2^{t,z}(s)-\omega_2^{t,y}(s) = \int_t^s (G_0.z+G_0.\Phi_0(u,\omega_2^{t,z}(u))du-(G_0.y+G_0.\Phi_0(u,\omega_2^{t,y}(u)))du+\int_t^s (\theta_{1S}-\theta_{2S})(G_0(\omega_2^{t,z}(u))-G_0(\omega_2^{t,y}(u)))dW(u)$

The term in du (by the mean value theorem) is smaller than

$$\begin{aligned}
&|G_0.\Phi_0(u,\omega_2^{t,z}(u))-G_0.\Phi_0(u,\omega_2^{t,y}(u))|+|G_0.z(u,\omega_2^{t,z}(u))-G_0.z(u,\omega_2^{t,y}(u))|\\
&+|G_0.z(u,\omega_2^{t,y}(u))-G_0.y(u,\omega_2^{t,y}(u))|\\
&\leq K(1+\nu)|\omega_2^{t,z}(u)-\omega_2^{t,y}(u)|+K||z-y||_{t,u}
\end{aligned}$$

The $dW(u)$ term is smaller than

$$|G_0(\omega_2^{t,z}(u))-G_0(\omega_2^{t,y}(u))| \leq K|\omega_2^{t,z}(u)-\omega_2^{t,y}(u)| \tag{5.180}$$

Thus, Proposition 5.22 yields:

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s}|\omega_2^{t,z}(u)-\omega_2^{t,y}(u)|^p]\\
&\leq K\mathbb{E}_t^{\mathbb{P}}\Big[(1+\nu)^p\int_t^s|\omega_2^{t,z}(v)-\omega_2^{t,y}(v)|^p dv+||z-y||_{t,s}^p(s-t)\Big]
\end{aligned}$$

And by Gronwall's inequality:

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s}|\omega_2^{t,z}(u)-\omega_2^{t,y}(u)|^p] \leq K||z-y||_{t,s}^p(s-t)e^{K(1+\nu)^p(s-t)}$$

□

Lemma 5.29. *If $y, z \in \mathbb{B}_\nu$, we get:*

$$\mathbb{E}_t^{\mathbb{P}}[|e^{Z_1^{t,z}(s)} - e^{Z_1^{t,y}(s)}|] \leq K(1+\nu)\sqrt{s-t}||z-y||_{t,s}e^{K(1+\nu)^2(s-t)} \tag{5.181}$$

For $p \geq 2$

$$\mathbb{E}_t^{\mathbb{P}}\big[\sup_{t\leq u\leq s}|e^{Z_i^{t,z}(u)} - e^{Z_i^{t,y}(u)}|^p\big] \leq K(1+\nu)^p||z-y||_{t,s}^p(s-t)^{\frac{1}{2}}e^{K(1+\nu)^p(s-t)} \tag{5.182}$$

If furthermore $y, z \in \mathbb{B}_{\nu,\bar{\nu}}$, we get:

$$\mathbb{E}_t^{\mathbb{P}}\big[\sup_{t\leq u\leq s}|D^{t,z}(u)-D^{t,y}(u)|^p\big] \leq K\left(||z-y||_{t,s}^p+||\frac{\partial z}{\partial\omega_2}-\frac{\partial y}{\partial\omega_2}||_{t,s}^p\right)(1+\nu+\bar{\nu})^pe^{K\nu(s-t)}(s-t) \tag{5.183}$$

and

$$\mathbb{E}_t^{\mathbb{P}}\sup_{t\leq u\leq s}[|L_i^{t,z}(u) - L_i^{t,y}(u)|^p] \leq K(1+\nu+\bar{\nu})^{2p}\sqrt{s-t}\times\left(||z-y||^p + ||\frac{\partial z}{\partial\omega_2} - \frac{\partial y}{\partial\omega_2}||_{t,s}^p\right) \tag{5.184}$$

Proof [Lemma 5.29] We first prove inequality 5.272. We consider the case $i = 1$, the case $i = 2$ is similar. We use the mean value theorem to get the estimate

$$|e^{Z_1^{t,z}(s)} - e^{Z_1^{t,y}(s)}| \leq e^{M_1(s)}|Z_1^{t,z}(s) - Z_1^{t,y}(s)| \text{ with } M_1(s) \in [Z_1^{t,y}(s), Z_1^{t,z}(s)] \tag{5.185}$$

Taking the expectation and using Cauchy Schwarz's inequality yields:

$$\mathbb{E}_t^{\mathbb{P}}[|e^{Z_1^{t,z}(s)} - e^{Z_1^{t,y}(s)}|] \leq (\mathbb{E}_t^{\mathbb{P}}[e^{2M_1(s)}])^{\frac{1}{2}} \times (\mathbb{E}_t^{\mathbb{P}}[|Z_1^{t,z}(s) - Z_1^{t,y}(s)|^2])^{\frac{1}{2}} \tag{5.186}$$

The first term $(\mathbb{E}_t^{\mathbb{P}}[e^{2M_1(s)}])^{\frac{1}{2}}$ is bounded by a constant K and the second term is bounded by

$(\mathbb{E}_t^{\mathbb{P}}[2|\int_t^s a_1^{t,z}(u)-a_1^{t,y}(u)du|^2+2|\int_t^s b_1^{t,z}(u)-b_1^{t,y}(u)dW(u)|^2])^{\frac{1}{2}} \leq 2(s-t)\mathbb{E}_t^{\mathbb{P}}[\int_t^s |a_1^{t,z}(u)-a_1^{t,y}(u)|^2du+2\int_t^s |b_1^{t,z}(u)-b_1^{t,y}(u)|^2du)^{\frac{1}{2}} \leq \big(K||z-y||_{t,s}+K(1+\nu)(\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s}|\omega_2^z(u)-\omega_2^y(u)|^2])^{\frac{1}{2}}\big)\sqrt{s-t} \leq \big(K||z-y||_{t,s}+K(1+\nu)\sqrt{s-t}K||z-y||_{t,s}e^{K(1+\nu)^2(s-t)}\big)\sqrt{s-t}$

$$\mathbb{E}_t^{\mathbb{P}}[|e^{Z_1^{t,z}(s)}-e^{Z_1^{t,y}(s)}|] \leq K(1+\nu)\sqrt{s-t}||z-y||_{t,s}e^{K(1+\nu)^2(s-t)}$$

Now, we prove inequality 5.273.

$$\begin{aligned} e^{Z_1^{t,z}(u)}-e^{Z_1^{t,y}(u)} &= \int_t^u (a_1^z+\frac{(b_1^z)^2}{2})e^{Z_1^{t,z}(v)}-(a_1^y+\frac{(b_1^y)^2}{2})e^{Z_1^{t,y}(v)}dv \\ &+ \int_t^u (e^{Z_1^{t,z}(v)}b_1^z(v)-e^{Z_1^{t,y}(v)}b_1^y(v))dW(v) \end{aligned}$$

Thus, by Proposition 5.22

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s}|e^{Z_1^{t,z}(u)}-e^{Z_1^{t,y}(u)}|^p] \leq \mathbb{E}_t^{\mathbb{P}}[\int_t^s |(a_1^z+\frac{(b_1^z)^2}{2})e^{Z_1^{t,z}(v)}-(a_1^y+\frac{(b_1^y)^2}{2})e^{Z_1^{t,y}(v)}|^p dv$$
$$+\int_t^u |e^{Z_1^{t,z}(v)}b_1^z(v)-e^{Z_1^{t,y}(v)}b_1^y(v)|^p dv]$$

Using the expressions

$$a_1^y(u)=-p_1\gamma_1 y(u)\omega_2^y(u)+c_1(\omega_2^y(u)); b_1^y(u)=p_1\delta_1+\gamma_1p_1\theta_{1S}+\gamma_1p_1(\theta_{2S}-\theta_{1S})\omega_2^y(u)$$

and the triangular inequality, we get that the first integrand is smaller than

$$(K(1+\nu)|\omega_2^z(v)-\omega_2^y(v)|+K||z-y||_{t,v})^p \times \max(e^{pZ_1^{t,z}(v)},e^{pZ_1^{t,y}(v)})+K|e^{Z_1^{t,z}(v)}-e^{Z_1^{t,z}(v)}|^p$$

The second integrand is similarly smaller than

$$(K|\omega_2^z(v) - \omega_2^y(v)|)^p \times \max(e^{pZ_1^{t,z}(v)}, e^{pZ_1^{t,y}(v)}) + K|e^{Z_1^{t,z}(v)} - e^{Z_1^{t,z}(v)}|^p$$

Thus, using the fact that

$$\mathbb{E}_t^{\mathbb{P}}[e^{pZ_1^{t,y}(v)}] \leq K$$

and $\mathbb{E}_t^{\mathbb{P}}[|\omega_2^z(v) - \omega_2^y(v)|^p \times e^{pZ_1^{t,y}(v)}] \leq (\mathbb{E}_t^{\mathbb{P}}[|\omega_2^z(v) - \omega_2^y(v)|^{2p}])^{\frac{1}{2}} \times (\mathbb{E}_t^{\mathbb{P}}[e^{2pZ_1^{t,y}(v)}])^{\frac{1}{2}} \leq K(v-t)^{\frac{1}{2}}||z-y||_{t,v}^p e^{K(1+\nu)^{2p}(v-t)}$.

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s} |e^{Z_1^{t,z}(u)} - e^{Z_1^{t,y}(u)}|^p \\
&\leq \mathbb{E}_t^{\mathbb{P}}\left[\int_t^s (K(1+\nu)^p|\omega_2^z(v) - \omega_2^y(v)|^p + K||z-y||_{t,v}^p + K|e^{Z_1^{t,z}(v)} - e^{Z_1^{t,y}(v)}|^p)dv\right]
\end{aligned}$$

And using Lemma 5.28, we get

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s} |e^{Z_1^{t,z}(u)} - e^{Z_1^{t,y}(u)}|^p \\
&\leq \mathbb{E}_t^{\mathbb{P}}\left[\int_t^s (K(1+\nu)^p||z-y||_{t,v}^p(v-t)^{\frac{1}{2}}e^{K(1+\nu)^p(v-t)} + K||z-y||_{t,v}^p + K|e^{Z_1^{t,z}(v)} - e^{Z_1^{t,y}(v)}|^p)dv\right] \\
&\leq K(s-t)||z-y||_{t,s}^p(1+\nu)^p e^{K(1+\nu)^{2p}(s-t)} + K\mathbb{E}_t^{\mathbb{P}}\left[\int_t^s |e^{Z_1^{t,z}(v)} - e^{Z_1^{t,y}(v)}|^p dv\right]
\end{aligned}$$

And by Gronwall's inequality:

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s} |e^{Z_1^{t,z}(u)} - e^{Z_1^{t,y}(u)}|^p] \leq K(s-t)||z-y||_{t,s}^p(1+\nu)^p e^{K(1+\nu)^{2p}(s-t)} e^{K(s-t)}$$

which is equivalent to inequality (5.273).

We give a proof of the estimate (5.274) of $|D^{t,z}(u) - D^{t,y}(u)|$.

$$D^{t,z}(u) - D^{t,y}(u) = \int_t^u (G_0.(\frac{\partial z}{\partial \omega_2} + \phi_0) + g_0.(z + \Phi_0))D^{t,z}(v)$$
$$-(G_0.(\frac{\partial y}{\partial \omega_2} + \phi_0) + g_0.(y + \Phi_0))D^{t,y}(v)dv + \int_t^u (\theta_{1S} - \theta_{2S})(g_0(\omega_2^{t,z}(v)) - g_0(\omega_2^{t,y}(v)))dW_v$$

Thus, the Proposition 5.22 yields:

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t \le u \le s} |D^{t,z}(u) - D^{t,y}(u)|^p]$$
$$\le \mathbb{E}_t^{\mathbb{P}}\Bigg[\int_t^s |(G_0.(\frac{\partial z}{\partial \omega_2} + \phi_0) + g_0.(z + \Phi_0))D^{t,z}(v) - (G_0.(\frac{\partial y}{\partial \omega_2} + \phi_0) + g_0.(y + \Phi_0))D^{t,y}(v)|^p dv$$
$$+ \int_t^s |(\theta_{1S} - \theta_{2S})(g_0(\omega_2^{t,z}(v)) - g_0(\omega_2^{t,y}(v)))|^p dv\Bigg]$$

The dv term is smaller than

$$|(G_0.(\frac{\partial z}{\partial \omega_2} + \phi_0) + g_0.(z + \Phi_0))D^{t,z}(v) - (G_0.(\frac{\partial y}{\partial \omega_2} + \phi_0) + g_0.(y + \Phi_0))D^{t,y}(v)|^p$$
$$\le K \max(D^{t,z}(v)^p, D^{t,y}(v)^p)\Bigg(||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,v} + ||z - y||_{t.v}$$
$$+(1 + \nu + \bar{\nu})|\omega_2^{t,z}(v) - \omega_2^{t,y}(v)|\Bigg)^p + K(1 + \nu)^p |D^{t,z}(v) - D^{t,y}(v)|^p$$

The $dW(v)$ term is smaller than

$$K|\omega_2^{t,z}(v) - \omega_2^{t,y}(v)|^p$$

Using the Cauchy Schwarz inequality:

$\mathbb{E}_t^{\mathbb{P}}[D^{t,z}(v)^p \times |\omega_2^{t,z}(v) - \omega_2^{t,y}(v)|^p] \le (\mathbb{E}_t^{\mathbb{P}}[D^{t,z}(v)^{2p}])^{\frac{1}{2}} \times (\mathbb{E}_t^{\mathbb{P}}[|\omega_2^{t,z}(v) - \omega_2^{t,y}(v)|^{2p}])^{\frac{1}{2}} \le e^{K(1+\nu)(v-t)} \times K||z - y||_{t,s}^p (v - t)^{\frac{1}{2}} e^{K(1+\nu)^{2p}(v-t)}$

Similarly to the previous proof, Gronwall's inequality yields:

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s}|D^{t,z}(u)-D^{t,y}(u)|^p] \leq K(s-t)e^{K(1+\nu)^{2p}(s-t)}(1+\nu+\bar{\nu})^p\left(||z-y||_{t,s}^p+||\frac{\partial z}{\partial \omega_2}-\frac{\partial y}{\partial \omega_2}||_{t,s}^p\right) \tag{5.187}$$

The next inequality is proven in a similar faction:

$$\begin{aligned}|L_1^{t,z}(s) - L_1^{t,y}(s)| &\leq \int_t^s (\frac{\partial a_1^z(u)}{\partial \omega_2}D^{t,z}(u) - \frac{\partial a_1^y(u)}{\partial \omega_2}D^{t,y}(u))du \\ &+ \int_t^s (\frac{\partial b_1^z(u)}{\partial \omega_2}D^{t,z}(u) - \frac{\partial b_1^y(u)}{\partial \omega_2}D^{t,y}(u))dW_u\end{aligned}$$

$$\begin{aligned}\left|\frac{\partial a_1^z(u)}{\partial \omega_2}D^{t,z}(u) - \frac{\partial a_1^y(u)}{\partial \omega_2}D^{t,y}(u)\right| &\leq \left|(\frac{\partial a_1^z(u)}{\partial \omega_2} - \frac{\partial a_1^y(u)}{\partial \omega_2})D^{t,z}(u)\right| + \left|\frac{\partial a_1^y(u)}{\partial \omega_2}(D^{t,z}(u) - D^{t,y}(u))\right| \\ \leq K(1+\nu)|D^{t,z}(u) - D^{t,y}(u)| &+ K(1+\nu+\bar{\nu})D^{t,z}(u)\left[|\omega_2^z(u) - \omega_2^y(u)| + ||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,u}\right]\end{aligned}$$

Using inequalities (5.274), (5.270), we conclude

$$\mathbb{E}_t^{\mathbb{P}}\sup_{t\leq u\leq s}[|L_1^{t,z}(u) - L_1^{t,y}(u)|^p] \leq K(1+\nu+\bar{\nu})^{2p}(s-t)\times\left(||z-y||_{t,s}^p + ||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,s}^p\right)$$

The proof for $i=2$ is similar. □

Fix $y, z \in \mathbb{B}_\nu$. Since $F = F_1 - F_2$, by the triangular inequality:

$|F[z](t,\omega_2) - F[y](t,\omega_2)| \leq |F_1[z] - F_1[y](t,\omega_2)| + |F_2[z] - F_2[y](t,\omega_2)|$

and

$$\begin{aligned}|F_i[z](t,\omega_2) - F_i[y](t,\omega_2)| &= \left|\frac{F_{1i}[z]}{F_{0i}[z]} - \frac{F_{1i}[y]}{F_{0i}[y]}(t,\omega_2)\right| \\ \leq &\quad \left|\frac{F_{1i}[z](t,\omega_2) - F_{1i}[y](t,\omega_2)}{F_{0i}[z](t,\omega_2)} + \frac{F_{1i}[y](t,\omega_2)(F_{0i}[z] - F_{0i}[y])}{F_{0i}[y](t,\omega_2)F_{0i}[z](t,\omega_2)}\right|\end{aligned}$$

Thus

$$|F_i[z](t,\omega_2) - F_i[y](t,\omega_2)| \leq \frac{2||\rho_i|||F_{0i}[z] - F_{0i}[y]|(t,\omega_2)}{F_{0i}[z](t,\omega_2)}$$

and

$$\begin{aligned} &|F_{0i}[z](t,\omega_2) - F_{0i}[y](t,\omega_2)| \\ &\leq \mathbb{E}_t^{\mathbb{P}}\bigg[\int_t^T f_i(t,s)|e^{Z_i^{t,z}(s)} - e^{Z_i^{t,y}(s)}|ds + f_i(t,T)|e^{Z_i^{t,z}(T)} - e^{Z_i^{t,y}(T)}|\bigg] \end{aligned}$$

And since by Lemma 5.29

$$F_{0i}[z](t,\omega_2) \geq K_1\bigg(\int_t^T f_i(t,s)ds + f_i(t,T)\bigg) \tag{5.188}$$

we get:

$$|F_i[z] - F_i[y](t,\omega_2)| \leq \frac{2||\rho_i||\bigg(\int_t^T f_i(t,s)ds + f_i(t,T)\bigg)\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq s\leq T}|e^{Z_i^{t,z}(u)} - e^{Z_i^{t,y}(u)}|]}{K_1\bigg(\int_t^T f_i(t,s)ds + f_i(t,T)\bigg)}$$

$$|F_i[z] - F_i[y](t,\omega_2)| \leq \frac{2||\rho_i|\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq s\leq T}|e^{Z_i^{t,z}(u)} - e^{Z_i^{t,y}(u)}|]}{K_1} \tag{5.189}$$

Since $F = F_1 - F_2$, we get

$$|F[z] - F[y](t,\omega_2)| \leq \frac{4||\rho||K||z-y||_{t,T}(1+\nu)e^{K\nu(T-t)}(T-t)}{K_1} \tag{5.190}$$

$$\frac{\partial F_i[y](t,\omega_2)}{\partial\omega_2} = \frac{\frac{\partial F_{1i}[y]}{\partial\omega_2}}{F_{0i}[y]} - \frac{\frac{\partial F_{0i}[y]}{\partial\omega_2}F_{1i}[y]}{F_{0i}^2[y]} = \frac{\frac{\partial F_{1i}[y]}{\partial\omega_2}}{F_{0i}[y]} - F_i[y]\frac{\frac{\partial F_{0i}[y]}{\partial\omega_2}}{F_{0i}[y]}$$

Thus

$$\Big|\frac{\partial F_i[z](t,\omega_2)}{\partial\omega_2} - \frac{\partial F_i[y](t,\omega_2)}{\partial\omega_2}\Big|$$
$$= \Bigg|\frac{\frac{\partial F_{1i}[z]}{\partial\omega_2}}{F_{0i}[z]} - \frac{\frac{\partial F_{1i}[y]}{\partial\omega_2}}{F_{0i}[y]} - F_i[z]\frac{\frac{\partial F_{0i}[z]}{\partial\omega_2}}{F_{0i}[z]} + F_i[y]\frac{\frac{\partial F_{0i}[y]}{\partial\omega_2}}{F_{0i}[y]}\Bigg|$$

Again after some manipulations and using the triangular inequality, we get:

$$\begin{aligned}&\Big|\frac{\partial F_i[z](t,\omega_2)}{\partial\omega_2} - \frac{\partial F_i[y](t,\omega_2)}{\partial\omega_2}\Big| \qquad (5.191)\\ &\leq \Big|\frac{\partial F_{0i}[z]}{\partial\omega_2} - \frac{\partial F_{0i}[y]}{\partial\omega_2}\Big|\frac{2||\rho_i||}{F_{0i}[z]} + \frac{|\frac{\partial F_{0i}[y]}{\partial\omega_2}|}{F_{0i}[y]}\Big(\frac{2||\rho_i||}{F_{0i}[z]}|F_{0i}[z] - F_{0i}[y]| + |F_i[z] - F_i[y]|\Big)\end{aligned}$$

In what follows, we estimate $|\frac{\partial F_{0i}[y]}{\partial\omega_2}|$ and $\Big|\frac{\partial F_{0i}[z]}{\partial\omega_2} - \frac{\partial F_{0i}[y]}{\partial\omega_2}\Big|$.

$$\begin{aligned}|\frac{\partial F_{0i}[y](t,\omega_2)}{\partial\omega_2}| \quad &= |\mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T f_i(t,s)e^{Z_i^{t,y}(s)}L_i^{t,y}(s)ds\Big]|\\ &\leq \int_t^T f_i(t,s)\sqrt{\mathbb{E}_t^{\mathbb{P}}[e^{2Z_i^{t,y}(s)}]\mathbb{E}_t^{\mathbb{P}}[L_i^{t,y}(s)^2]}ds\\ &\leq \int_t^T f_i(t,s)\sqrt{\mathbb{E}_t^{\mathbb{P}}[e^{2Z_i^{t,y}(s)}]\mathbb{E}_t^{\mathbb{P}}[L_i^{t,y}(s)^2]}ds\\ &\leq \int_t^T f_i(t,s)\sqrt{K.K(1+\nu)^2e^{K(1+\nu)^2(s-t)}(s-t)}ds\\ |\frac{\partial F_{0i}[y](t,\omega_2)}{\partial\omega_2}| \quad &\leq K(1+\nu)\sqrt{T-t}.e^{K(1+\nu)^2(T-t)}.\Big(\int_t^T f_i(t,s)ds + f_i(t,T)\Big)\end{aligned}$$

In the last inequality, we have added the final term that appears in the definition of $F_{0i}[y](t,\omega_2)$. Thus

$$|\frac{\partial F[y](t,\omega_2)}{\partial\omega_2}| \leq \frac{4||\rho||K(1+\nu)e^{K(1+\nu)^2(T-t)}\sqrt{T-t}}{K_1}||z-y||_{t,T} \qquad (5.192)$$

This concludes the estimation of $|\frac{\partial F_{0i}[y](t,\omega_2)}{\partial \omega_2}|$.
Estimation of $\left|\frac{\partial F_{0i}[z]}{\partial \omega_2} - \frac{\partial F_{0i}[y]}{\partial \omega_2}\right|$.

$$\left|\frac{\partial F_{0i}[z]}{\partial \omega_2} - \frac{\partial F_{0i}[y]}{\partial \omega_2}\right| \le \mathbb{E}_t^{\mathbb{P}}\Big[\int_t^T f_i(t,s)\big|e^{Z_i^{t,z}(s)}L_i^{t,z}(s) - e^{Z_i^{t,y}(s)}L_i^{t,y}(s)\big|ds\Big] \tag{5.193}$$

The integrand term

$$\mathbb{E}_t^{\mathbb{P}}\Big[\big|e^{Z_i^{t,z}(s)}L_i^{t,z}(s) - e^{Z_i^{t,y}(s)}L_i^{t,y}(s)\big|\Big] \le \mathbb{E}_t^{\mathbb{P}}\Big[e^{Z_i^{t,z}(s)}|L_i^{t,z}(s) - L_i^{t,y}(s)| + |L_i^{t,y}(s)||e^{Z_i^{t,z}(s)} - e^{Z_i^{t,y}(s)}|\Big]$$
$$\le \Big(\mathbb{E}_t^{\mathbb{P}}[e^{2Z_i^{t,z}(s)}]\mathbb{E}_t^{\mathbb{P}}[|L_i^{t,z}(s) - L_i^{t,y}(s)|^2]\Big)^{\frac{1}{2}} + \Big(\mathbb{E}_t^{\mathbb{P}}[|L_i^{t,y}(s)|^2]\mathbb{E}_t^{\mathbb{P}}[|e^{Z_i^{t,z}(s)} - e^{Z_i^{t,y}(s)}|^2]\Big)^{\frac{1}{2}}$$

and by Lemmas 5.27 and 5.28, we have

$$\mathbb{E}_t^{\mathbb{P}}\Big[\big|e^{Z_i^{t,z}(s)}L_i^{t,z}(s) - e^{Z_i^{t,y}(s)}L_i^{t,y}(s)\big|\Big]$$
$$\le K(1+\nu+\bar{\nu})^2 e^{K(1+\nu)^4(s-t)}(s-t)^{\frac{1}{2}}\Big(||z-y||_{t,s} + ||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,s}\Big) \tag{5.194}$$

Taking the integral between t and T and adding the final term, we get

$$\left|\frac{\partial F_{0i}[z]}{\partial \omega_2} - \frac{\partial F_{0i}[y]}{\partial \omega_2}\right| \le (1+\nu+\bar{\nu})^2 e^{K(1+\nu)^2(T-t)}(T-t)^{\frac{1}{2}} \times (||z-y||_{t,T} + ||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,T}) \tag{5.195}$$

We conclude that

$$\left|\frac{\partial F_{i}[z]}{\partial \omega_2} - \frac{\partial F_{i}[y]}{\partial \omega_2}\right| \le K(1+\nu+\bar{\nu})^2 e^{K(1+\nu)^4(T-t)}(T-t)^{\frac{1}{2}} \times (||z-y||_{t,T} + ||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,T}) \tag{5.196}$$

Since $F = F_1 - F_2$

$$\left|\frac{\partial F[z]}{\partial \omega_2} - \frac{\partial F[y]}{\partial \omega_2}\right| \leq K(1+\nu+\bar{\nu})^2 e^{K(1+\nu)^4(T-t)}(T-t)^{\frac{1}{2}} \times \left(||z-y||_{t,T} + ||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,T}\right)$$

Notice that for all $y \in \mathbb{B}_\nu$, $|F[y](t, \omega_2)| \leq 2||\rho||$ for all $(t, \omega_2) \in [0, T] \times [0, 1]$. For $y \in \mathbb{B}_{\nu,\bar{\nu}}$, if we look at (5.263), (5.264), (5.261),(5.260), it is easy to see that we can obtain an estimate for $|\frac{\partial^2 F[y]}{\partial \omega_2^2}|$

$$|\frac{\partial^2 F[y]}{\partial \omega_2^2}| \leq K(1+\nu)(1+\nu+\bar{\nu})(T-t)^{\frac{1}{2}} e^{K(1+\nu)^2(T-t)} \tag{5.197}$$

Let

$$\nu := 1 + 4 \sup_{(t,\omega_2)\in[0,T]\times[0,1]} |\frac{\partial F[0]}{\partial \omega_2}| \tag{5.198}$$

and

$$\bar{\nu} := 1 + \sup_{(t,\omega_2)\in[0,T]\times[0,1]} |\frac{\partial^2 F[0]}{\partial \omega_2^2}| \tag{5.199}$$

We fix K to be the maximum between 1 and K^2 where K is a constant number that appears in the various inequalities written above.

We choose $\epsilon_0 \in (0, T)$ such that

$$K(1+\nu)(1+\nu+\bar{\nu})\sqrt{\epsilon_0} e^{K(1+\nu)^2 T} \leq \bar{\nu} \tag{5.200}$$

We choose $\epsilon_1 \in (0, \epsilon_0]$ such that

$$\frac{4||\rho||K(1+\nu)e^{K(1+\nu)T}\sqrt{\epsilon_1}}{K_1} \leq \frac{1}{4} \tag{5.201}$$

and $\epsilon_2 \in (0, \epsilon_1]$ such that

$$K(1+\nu+\bar{\nu})^2 e^{K(1+\nu)^4 T} \epsilon_2^{\frac{1}{2}} \leq \frac{\nu}{4} \tag{5.202}$$

Then

1. $F[y] \in \mathbb{B}_{\nu,\bar{\nu}}$ for all $y \in \mathbb{B}_{\nu,\bar{\nu}}$.

2. For all $y, z \in \mathbb{B}_{\nu,\bar{\nu}}$,

$$||\frac{\partial F[z]}{\partial \omega_2} - \frac{\partial F[y]}{\partial \omega_2}||_{T-\epsilon_2,T} + ||F[z]-F[y]||_{T-\epsilon_2,T} \leq \frac{1}{2}(||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{T-\epsilon_2,T} + ||z-y||_{T-\epsilon_2,T}) \tag{5.203}$$

If we call $\mathbb{B}^1_\nu := \{y|_{[T-\epsilon_2,T]\times[0,1]} \mid y \in \mathbb{B}_\nu\}$ the restriction of $\mathbb{B}_\nu$ to $[T-\epsilon_2, T] \times [0,1]$ and similarly , $\mathbb{B}^1_{\nu,\bar{\nu}} := \{y|_{[T-\epsilon_2,T]\times[0,1]} \mid y \in \mathbb{B}_{\nu,\bar{\nu}}\}$ the restriction of $\mathbb{B}_{\nu,\bar{\nu}}$ to $[T-\epsilon_2, T] \times [0,1]$,

then we can conclude that $F^1(\mathbb{B}^1_{\nu,\bar{\nu}}) \subset \mathbb{B}^1_{\nu,\bar{\nu}}$. Define the norm

$$||y||_{\mathbb{B}^1_\nu} := ||y||_{[T-\epsilon_2,T]} + ||\frac{\partial y}{\partial \omega_2}||_{[T-\epsilon_2,T]} \tag{5.204}$$

Wrapping up :

We construct a fix point for F^1 in $\mathbb{B}^1_\nu$:

Define (ψ_n) a sequence of functions in $\mathbb{B}_{\nu,\bar{\nu}}$ by:

$$\psi_0(t,\omega_2) = 0 \quad ; \quad \psi_{n+1}(t,\omega_2) = F[\psi_n](t,\omega_2) \;\; \forall (t,\omega_2) \in [T-\epsilon_2, T] \times [0,1]$$

A simple recursion on n shows that

$$||F^1[\psi_{n+1}] - F^1[\psi_n]||_{\mathbb{B}^1_\nu} \leq \frac{1}{2^n}||\psi_1 - \psi_0||_{\mathbb{B}^1_\nu} \tag{5.205}$$

Therefore (ψ_n) converges in $\mathbb{B}^1_\nu$ to a function Ψ^1. A uniform limit of $\bar{\nu}$-Lipschitz is $\bar{\nu}$-Lipschitz, therefore $\frac{\partial \Psi^1(t,\omega_2)}{\partial \omega_2}$ is actually $\bar{\nu}$-Lipschitz in the ω_2 variable. Furthermore, Ψ^1 is the unique fix point in $\mathbb{B}^1_\nu$: if $\bar{\Psi}^1$ is another fix point of the operator F^1 then we would have

$$||\Psi^1 - \bar{\Psi}^1||_{\mathbb{B}^1_\nu} = ||F^1[\Psi^1] - F^1[\bar{\Psi}^1]||_{\mathbb{B}^1_\nu} \leq \frac{1}{2}||\Psi^1 - \bar{\Psi}^1||_{\mathbb{B}^1_\nu}$$

so that $||\Psi^1 - \bar{\Psi}^1||_{\mathbb{B}^1_\nu} = 0$ i.e. $\Psi^1 = \bar{\Psi}^1$.

We can repeat the argument on $[T - 2\epsilon_2, T - \epsilon_2] \times [0, 1]$. So on, until we reach 0. Let us do it carefully.

Define the operator F^2_i on the space $C([T - 2\epsilon_2, T - \epsilon_2]; C^1[0, 1])$ by

$$F^2_i[y](t, \omega_2) = \frac{F^2_{1i}[y](t, \omega_2)}{F^2_{0i}[y](t, \omega_2)}$$

$$F^2[y] := F^2_1[y] - F^2_2[y]$$

and

$$F^2_{0i}[y](t, \omega_2) := \mathbb{E}_t\left[\int_t^T f_i(t, s) e^{\int_t^s a_i^{y^2}(u)du + \int_t^s b_i^{y^2}(u)dW_u} ds + f_i(t, T) e^{\int_t^T a_i^{y^2}(u)du + b_i^{y^2}(u)dW_u}\right]$$

$$F^2_{1i}[y](t, \omega_2) := \mathbb{E}_t\left[\int_t^T \frac{\partial f_i(t, s)}{\partial t} e^{\int_t^s a_i^{y^2}(u)du + \int_t^s b_i^{y^2}(u)dW_u} ds + \frac{\partial f_i(t, T)}{\partial t} e^{\int_t^T a_i^{y^2}(u)du + b_i^{y^2}(u)dW_u}\right]$$

and

$$y^2(t,\omega_2) = \begin{cases} \Psi^1(t,\omega_2), \text{ if } T-\epsilon_2 < t \leq T \\ y(t,\omega_2), \text{ if } T-2\epsilon_2 \leq t \leq T-\epsilon_2 \end{cases} \tag{5.206}$$

Define

$$\frac{\partial y^2}{\partial \omega_2}(t,\omega_2) = \begin{cases} \frac{\partial \Psi^1}{\partial \omega_2}(t,\omega_2), \text{ if } T-\epsilon_2 < t \leq T \\ \frac{\partial y}{\partial \omega_2}(t,\omega_2), \text{ if } T-2\epsilon_2 \leq t \leq T-\epsilon_2 \end{cases} \tag{5.207}$$

We will omit again the final term $f_i(t,T)e^{\int_t^T a_i^{y^2}(u)du + b_i^{y^2}(u)dW_u}$ to simplify the calculations.

Let $\mathbb{B}^2_\nu := \{y \in C([T-2\epsilon_2, T-\epsilon_2]\times[0,1] \mid y \text{ is } C^1 \text{ in } \ t,\omega_2 \ \text{ and such that } |\frac{\partial y}{\partial \omega_2}| \leq \nu\}$ and $\mathbb{B}^2_{\nu,\bar{\nu}} := \{y \in C([T-2\epsilon_2, T-\epsilon_2]\times[0,1] \mid y \text{ is } C^1 \text{ in } \ t, C^2 \text{ in } \omega_2 \ \text{ and such that } |\frac{\partial y}{\partial \omega_2}| \leq \nu, |\frac{\partial^2 y}{\partial \omega_2^2}| \leq \bar{\nu}\}$

Since $z^2 = y^2$ on $[T-\epsilon_2, T]$, we have

$$||z-y||_{t,T-\epsilon_2} = ||z^2-y^2||_{t,T} \tag{5.208}$$

$$||\frac{\partial z}{\partial \omega_2} - \frac{\partial y}{\partial \omega_2}||_{t,T-\epsilon_2} = ||\frac{\partial z^2}{\partial \omega_2} - \frac{\partial y^2}{\partial \omega_2}||_{t,T} \tag{5.209}$$

we only need to estimate

$|F^2[z](t,\omega_2) - F^2[y](t,\omega_2)|$ and $|\frac{\partial F^2[z](t,\omega_2)}{\partial \omega_2} - \frac{\partial F^2[y](t,\omega_2)}{\partial \omega_2}|$ for $t \in [T-2\epsilon_2, T-\epsilon_2)$.

By observing (5.184) that gives an estimate for $|F_i[z](t,\omega_2) - F_i[y](t,\omega_2)|$, we see that

$$|F^2[z^2](t,\omega_2) - F^2[y^2](t,\omega_2)| \leq \frac{4||\rho||}{K_1}\mathbb{E}_t^{\mathbb{P}}\big[\sup_{i=1,2;t\leq s\leq T}|e^{Z_i^{t,z^2}(s)} - e^{Z_i^{t,y^2}(s)}|\big]$$

If $(t,s) \in [T-2\epsilon_2, T-\epsilon_2]^2$, we can use the fact that $|s-t| \leq \epsilon_2$ to get estimates when y, z is replaced by y^2, z^2.

The only additional case to check is $(t,s) \in [T-2\epsilon_2, T-\epsilon_2] \times [T-\epsilon_2, T]$.

We first find an estimate for $|\omega_2^{t,z^2}(u) - \omega_2^{t,y^2}(u)|$ for $u \in [T-\epsilon_2, s]$.

We use the SDE for ω_2^{t,y^2} and ω_2^{t,z^2} to get

$$\begin{aligned}
|\omega_2^{t,z^2}(u) - \omega_2^{t,y^2}(u)| \quad &= |\omega^{t,z^2}(T-\epsilon_2) - \omega_2^{t,y^2}(T-\epsilon_2) \\
&+ \int_{T-\epsilon_2}^{u} G_0.(\Phi_0 + z^2(v, \omega_2^{z^2}(v))) - G_0.(\Phi_0 + y^2(v, \omega_2^{y^2}(v)))dv \\
&+ \int_{T-\epsilon_2}^{u} (\theta_{1S} - \theta_{2S})(G_0(\omega_2^{z^2}(v)) - G_0(\omega_2^{y^2}(v))dW(v)|
\end{aligned}$$

As before , an application of Gronwall's inequality yields for $p \geq 1$,

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{u\in[T-\epsilon_2,s]} |\omega_2^{t,z^2}(u) - \omega_2^{t,y^2}(u)|^p] \leq K\mathbb{E}_t^{\mathbb{P}}\left[|\omega_2^{t,z^2}(T-\epsilon_2) - \omega_2^{t,y^2}(T-\epsilon_2)|^p e^{K(1+\nu)^p(s-(T-\epsilon_2))}\right] \tag{5.210}$$

And since by (5.270)

$$\mathbb{E}_t^{\mathbb{P}}\left[|\omega_2^{t,z^2} - \omega_2^{t,y^2}(T-\epsilon_2)|^p\right] \leq K||z-y||_{t,T-\epsilon_2}^p (T-\epsilon_2-t) e^{K(1+\nu)^p(T-\epsilon_2-t)} \tag{5.211}$$

we get:

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{P}}[\sup_{u\in[T-\epsilon_2,s]} |\omega_2^{t,z^2}(u) - \omega_2^{t,y^2}(u)|^p] \leq K\mathbb{E}_t^{\mathbb{P}}\left[|\omega_2^{t,z^2} - \omega_2^{t,y^2}(T-\epsilon_2)|^p e^{K(1+\nu)^p(s-(T-\epsilon_2))}\right] \\
&\leq Ke^{K(1+\nu)^p(s-(T-\epsilon_2))} K||z-y||_{t,T-\epsilon_2}^p (T-\epsilon_2-t) e^{K(1+\nu)^p(T-\epsilon_2-t)} \\
&\leq K^2 e^{K(1+\nu)^p(s-t)} ||z-y||_{t,T-\epsilon_2}^p \epsilon_2
\end{aligned}$$

We conclude that

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{u\in[t,s]} |\omega_2^{t,z^2}(u) - \omega_2^{t,y^2}(u)|^p] \leq Ke^{K(1+\nu)^p(s-t)}\epsilon_2||z-y||^p_{t,T-\epsilon_2} \tag{5.212}$$

We then have

$$\mathbb{E}_t^{\mathbb{P}}[\sup_{t\leq u\leq s} |e^{Z_1^{t,z}(u)} - e^{Z_1^{t,y}(u)}|^p] \leq K(1+\nu)^p||z-y||^p_{t,T-\epsilon_2}\epsilon_2^{\frac{1}{2}}e^{K(1+\nu)^p(s-t)}$$

Thus

$$|F^2[z](t,\omega_2) - F^2[y](t,\omega_2)| \leq \frac{4||\rho||}{K_1}\mathbb{E}_t^{\mathbb{P}}\big[\sup_{i=1,2;t\leq s\leq T} |e^{Z_i^{t,z^2}(s)} - e^{Z_i^{t,y^2}(s)}|\big]$$
$$\leq \frac{4||\rho||}{K_1}K(1+\nu)||z-y||_{t,T-\epsilon_2}\epsilon_2^{\frac{1}{2}}e^{K(1+\nu)^2T} \leq \frac{1}{4}||z-y||_{t,T-\epsilon_2}$$

because ϵ_2 was chosen small enough to have

$$\frac{4||\rho||}{K_1}K(1+\nu)\epsilon_2^{\frac{1}{2}}e^{K(1+\nu)T} \leq \frac{1}{4}$$

The same is true for the derivative:

$$|\frac{\partial F^2[z]}{\partial\omega_2}(t,\omega_2) - \frac{\partial F^2[y]}{\partial\omega_2}(t,\omega_2)| \leq \frac{1}{4}\left(||z-y||_{t,T-\epsilon_2} + ||\frac{\partial z}{\partial\omega_2} - \frac{\partial y}{\partial\omega_2}||_{t,T-\epsilon_2}\right)$$

Adding the two estimates we get

$$||F^2[z] - F^2[y]||_{\mathbb{B}^2_\nu} \leq \frac{1}{2}||z-y||_{\mathbb{B}^2_\nu}$$

Again, we get a contraction from $\mathbb{B}^2_\nu$ to $\mathbb{B}^2_\nu$, thus we get a fixed point for the operator F^2. We call the fixed point Ψ as before and now we have Ψ^2 defined on $[T-2\epsilon_2, T-\epsilon_2]\times[0,1]$ and Ψ^1 defined on $[T-\epsilon_2, T]\times[0,1]$. We have to verify that the two functions coincide on the set $\{T-\epsilon_2\}\times[0,1]$.

From the expressions

$$F^2_{0i}[y](t,\omega_2) := \mathbb{E}_t\bigg[\int_t^T f_i(t,s)e^{\int_t^s a_i^{y^2}(u)du+\int_t^s b_i^{y^2}(u)dW_u}ds + f_i(t,T)e^{\int_t^T a_i^{y^2}(u)du+b_i^{y^2}(u)dW_u}\bigg]$$
$$F^2_{1i}[y](t,\omega_2) := \mathbb{E}_t\bigg[\int_t^T \frac{\partial f_i(t,s)}{\partial t}e^{\int_t^s a_i^{y^2}(u)du+\int_t^s b_i^{y^2}(u)dW_u}ds + \frac{\partial f_i(t,T)}{\partial t}e^{\int_t^T a_i^{y^2}(u)du+b_i^{y^2}(u)dW_u}\bigg]$$

We see that

$$F^2_{0i}[\Psi^2](T-\epsilon_2,\omega_2) = F^1_{0i}[\Psi^1](T-\epsilon_2,\omega_2); F^2_{1i}[\Psi^2](T-\epsilon_2,\omega_2) = F^1_{1i}[\Psi^1](T-\epsilon_2,\omega_2)$$

since $\Psi^2(t,\omega_2) = \Psi^1(t,\omega_2)$ for $\omega_2 \in [0,1]$. Thus we conclude that

$$F[\Psi^2](T-\epsilon_2,\omega_2) = (\frac{F_{11}}{F_{01}} - \frac{F_{12}}{F_{02}})(T-\epsilon_2,\omega_2) = F[\Psi^1](T-\epsilon_2,\omega_2)$$

The same is true for the derivative

$$\frac{\partial F[\Psi^2](T-\epsilon_2,\omega_2)}{\partial\omega_2} = \frac{\partial F[\Psi^1](T-\epsilon_2,\omega_2)}{\partial\omega_2}$$

We can continue this process and construct Ψ^k a fixed point of F^k for $t \in [T-k\epsilon_2, T-(k-1)\epsilon_2]$.

We call $\mathbb{Q}$ the function defined on $[0,T]\times[0,1]$ that coincides with Ψ^k on the set $[T-k\epsilon_2, T-(k-1)\epsilon_2]\times[0,1]$.

By looking at the definition of $\mathbb{Q} = F[\mathbb{Q}]$ as a quotient of two integrals over t, we conclude that $\mathbb{Q}$ is actually C^1 in the time variable. $\mathbb{Q} \in \mathbb{B}_\nu$ is a fixed point for F.

$$\frac{\partial \mathbb{Q}(t, \omega_2)}{\partial \omega_2} = \frac{\partial F[\mathbb{Q}](t, \omega_2)}{\partial \omega_2}$$

is also well defined, continuous in the t variable and $\bar{\nu}$-Lipschitz in the ω_2 variable.

This ends the proof of the Theorem. $\square$