Vehicle Routing Problem with Interdiction

VEHICLE ROUTING PROBLEM WITH INTERDICTION

BY

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Abstract

In this thesis, we study the role of interdiction in the Vehicle Routing Problem (VRP), which naturally arises in humanitarian logistics and military applications. We assume that in a general network, each arc has a chance to be interdicted. When interdiction happens, the vehicle traveling on this arc is lost or blocked and thus unable to continue the trip. We model the occurrence of interdiction as a given probability and consider the multi-period expected delivery. Our objective is to minimize the total travel cost or to maximize the demand fulfillment, depending on the supply quantity. This problem is called the Vehicle Routing Problem with Interdiction (VRPI). We first prove that the proposed VRPI problems are NP-hard. Then we show some key analytical properties pertaining to the optimal solutions of these problems. Most importantly, we examine Dror and Trudeau's property applied to our problem setting. Finally, we present efficient heuristic algorithms to solve these problems and show the effectiveness through numerical studies.

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Notation and abbreviations

List of Abbreviations

VRP	Vehicle Routing Problem
CVRP	Capacitated Vehicle Routing Problem
SDVRP	Split Delivery Vehicle Routing Problem
VRPI	Vehicle Routing Problem with Interdiction
SEC	Subtour Elimination Constraints
NP-Hard	Non-deterministic Polynomial-time hard
HCP	Hamiltonian Cycle Problem
GRASP	Greedy Random Adaptive Search Procedure
RCL	Restricted Candidate List
CPLEX	IBM/ILOG Optimization Software

List of Notations

V	Set of nodes
A	Set of arcs/directed edges
G	Graph of (V,E)

N	Number of nodes
$\{0\}$	Depot node
i	Indices for demand nodes, $i = 1, \ldots, N$
ij	Indices for arcs, $ij \in A$
d_i	Node demands, $i = 1, \ldots, N$
s_i	Safety stock, $i = 1, \ldots, N$
c_{ij}	cost of arc ij (CVRP)
x_{ij}	arc ij is selected, $x_{ij} = \{0, 1\}$ (CVRP)
K	Number of available vehicles
Q	Vehicle capacity
q_{ij}	Probability of interdiction $ij \in A$
p_{ij}	Probability of no interdiction $ij \in A$
$r(S)$	Minimum number of vehicles to serve subset $S \subseteq V$ (CVRP)
Ω	Set of feasible routes
r	A feasible route originating from the depot
c_r	cost of route r
ϕ_{ir}	probability of reaching node i in r
y_{ir}	amount of supply delivered to i in route r
x_r	route r is selected, $x_r = \{0, 1\}$
a_{ir}	node i is visited by route r, $a_{ir} = \{0, 1\}$
b_{ijr}	route r uses arc (i, j), $b_{ijr} = \{0, 1\}$
y_{ir}	amount delivered to i in r

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Chapter 1

Introduction and Problem Statement

In the face of natural or man-made disasters, a major area of focus is on the distribution of vital emergency supplies. Following the earthquake in Nepal (Sharma and Adkin, 2015), or the ongoing armed conflict in Syria (Danish Refugee Council, 2016), much efforts have been put on humanitarian relief in a way that minimizes human suffering, cost of distributing emergency supplies, and risk of further loss of lives. Typically, we wish to plan out a route for which vehicles may supply those that are in need, however, one problem that often arises in these situations is the possibility that the a supply vehicle becomes damaged or broken down. In natural disasters, these vehicles may be caught by resultant avalanches or tsunamis arising in the aftershocks of an earthquake. Bridges, underpasses, or tracks can be severely weakened from the strains of these disasters and may be on the brink of collapse. In armed conflicts, this can happen due to the malicious action of the enemy, or by land-mine, or crossfire. Even in day to day business operations, supply trucks may breakdown or encounter other traffic obstructions that prohibit them from completing their delivery. Thus, it is clear that the roads in many routing applications are highly unstable and have a chance to be interdicted during use, which can incapacitate or even destroy the vehicles that are traveling upon it. Though the chances for most of these scenarios to happen can be quite low, when these highly unstable networks are repeatedly traveled upon, it is very important to have a routing plan that addresses these long term uncertainties of interdiction.

The goal of our paper is to address the high risks that may be presented within a vehicle routing application. We introduce the Vehicle Routing Problem with Interdiction (VRPI). That is, the Vehicle Routing Problem (VRP) with the possibility of interdiction on any of the arcs in a network.

The first instance of the VRP was proposed by (Dantzig and Ramser, 1959) originally called the Truck Dispatching Problem. It has since been one of the most widely studied integer programming problem in the field of optimization, business, transportation and logistics. In the classic VRP, we are given a number of vehicles (trucks) that start at a depot. The goal of the VRP is to deliver enough goods to satisfy all customer demands such that cost is minimized. For many applications, the distance between two nodes are interpreted as the cost of travel. Other constraints for the VRP include limited truck capacities, and limited fleet size. There are many variations of the VRP as discussed by (Toth and Vigo, 2014), for which new variables can be added to represent values such as network structure, vehicle capacities, inter-route constraints, fleet composition, or transportation type.

For our research, we assume that it is impossible to have guaranteed successful delivery, especially in high risk areas where we might encounter destructive forces of nature or acts of violence. Instead, we assume that it is possible to estimate or predict the chance of interdiction. We use the term *interdiction* to describe when a road in our network is destroyed, damaged, or blocked. Interdiction may occur in many scenarios including military supply distributions, disaster relief systems, communications networks, and so on. When interdiction occurs on a path, all the supply on the vehicle traveling on the path are considered lost. As a result, vehicles will be unable to supply their target nodes in a single delivery period. Our problem setting emerges when it is possible to hedge against that uncertainty across multiple delivery periods so that long term delivery goals are met. This specification is very important because although expected delivery goals will be met, we cannot guarantee sufficient demands for a single delivery period.

Our problem setting focuses on customers (or locations requiring supplies) that have a specified *expected* demand requirement which receive deliveries over multiple periods. Therefore, in the VRPI, we are interested in the expected supply rate of vehicles in a network. Just to contrast, in the original VRP, the expected supply is exactly equal to the amount of supply we send along any route since we assume nothing will happen to our vehicles. In our problem setting, the expected supply will change depending on the route that is selected. Riskier routes will of course be less favorable than safer routes.

The two key situations in this paper are: when the total supply is greater than the total demand, and when the total supply is less than the total demand. In the former case, we consider cost minimization, which is similar to the typical goal of classical VRP models. Cost can be interpreted as the cost incurred by traveling upon that arc, or it can be interpreted as the travel time, where the minimization of the travel time corresponds to accelerated response time. It is noted that a feasible solution corresponding to the classical VRP case may no longer be feasible in the case of interdiction. In the latter case, it becomes critical to fulfill the demands of the customers as much as possible; and cost is no longer the priority. This situation often prevails in the midst of a disaster, where the number of affected people is high and available resources (supplies/vehicles) is limited.

In this paper, we present two models, the VRPI with the objective to minimize cost and the VRPI with the objective to maximize demand fulfillment. These models build upon the Split Delivery Vehicle Routing Problem (SDVRP), which comes with some unique analytical properties. Most notably, is Dror and Trudeau's property (Dror and Trudeau, 1990), which mentions that if the route costs satisfy the triangle inequality, then the optimal SDVRP solution has at most one common demand point among any two routes. In our problem setting, we guarantee that the triangle inequality holds with respect to cost, however, we later mention that the interdiction probabilities do not. This is the key difference that gives our problem different structural properties for the optimal solutions from the typical SDVRP. The development of these properties are examined in more detail later for the VRPI models. We also prove that the proposed VRPI problems are NP-hard. Because of the uniqueness of our problem, we design heuristic algorithms to solve several carefully selected network instances of the VRPI. It is important to mention that for our problems, solving to optimality is much harder because there are no "Benchmark Instances" like in (Uchoa *et al.*, 2017). Optimal solutions were computed through brute force and we later compared these values to the performance of our algorithms in several computational studies.

The remainder of this thesis will be organized as follows. Chapter 2 begins with a review of the literature that is relevant to our problem. Chapter 3 introduces the two models of VRPI and provides some insight to the difficulty of our problem. Chapter 4 brings forth some analytical properties of the optimal solution, some of which are closely related to SDVRP. In Section 5, we describe two heuristic algorithms that are used to solve the VRPI. The experimental results are presented in Chapter 6. Concluding remarks and future research directions are given in Chapter 7.

Chapter 2

Literature review

Before we get into more detail about the VRPI, we must first present the basis for our problem. First of all, we take a look at the class of problems called the Vehicle Routing Problem (VRP).

2.1 The Capacitated Vehicle Routing Problem

To define the VRP, we are given a graph $G = (V, A)$, with N nodes where $V = \{v_0, v_1, v_2, \ldots, v_{N-1}\}$ and $A = \{i, j : (i, j) \in V\}$. v_0 represents the depot and nodes $v_1 \ldots v_{N-1}$ represent the customers. Each customer $i \in 1 \ldots N-1$ will have a demand that we need to satisfy, denoted by d_i, for each demand node $i \in V \backslash \{v_0\}$. Additionally, each arc has an associated travel cost $c_{ij}, \forall (i, j) \in A$ that is incurred upon traveling on that arc. Oftentimes, the cost, c_{ij}, may be treated as the travel time between two nodes in the network. In the VRP, our objective is to find the set of arcs that minimizes the total cost for satisfying all demands in the network. Finally, we have the constraints on fleet size and vehicle capacity, K and Q respectively. This model is typically referred to as the Capacitated Vehicle Routing Problem (CVRP) because of the capacity constraints on each vehicle. By the nature of VRP problems, all vehicles must originate and finish their journey at the depot.

In the optimization problem, the binary decision variables are $x_{ij}, \forall (i, j) \in A$ where

$$x_{ij} = \begin{cases} 1 & \text{if arc (i,j) is in our solution} \\ 0 & \text{otherwise} \end{cases}$$

There are predominantly two families of formulations for the CVRP: The *compact* formulation and the *extensive* formulation (Toth and Vigo, 2014).

2.1.1 Compact Formulation

The compact formulation models the CVRP using a classical network flow method. The formulation using arcs (or directed edges) is given by:

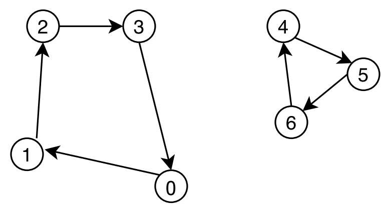
$$
\begin{aligned}
\text{(CVRP–c)} \qquad \min \quad & \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} c_{ij}x_{ij} && && (2.1)\\
\text{s.t.} \quad & \sum_{i=1}^{N-1} x_{ij} = 1 && \forall j = 1,\ldots,n && (2.2)\\
& \sum_{j=1}^{N-1} x_{ij} = 1 && \forall i = 1,\ldots,n && (2.3)\\
& \sum_{j=1}^{N-1} x_{0j} \leq K && && (2.4)\\
& \sum_{j=1}^{N-1} x_{j0} \leq K && && (2.5)\\
& \sum_{j=1}^{N-1}\sum_{j=1}^{N-1} x_{ij} \geq r(S) && \forall S \subseteq N, S \neq \emptyset && (2.6)\\
& x_{ij} \in \{0,1\} && && (2.7)
\end{aligned}
$$

(2.1) is the expression that seeks to minimize the total cost of all the arcs that exist in our network. (2.2) and (2.3) are the constraints to ensure each customer is served. (2.4) and (2.5) make sure that there are no more than K vehicles leaving and arriving back at the depot. This maintains that we do not exceed the total amount of vehicles that are given to us in our fleet. Constraint (2.7) ensures that our decision variable returns an integer value that translates to a logical solution. Finally, constraint (2.6) introduces $r(S)$, which is the minimum number of vehicles needed to serve a subset $S \subseteq V$. This constraint is synonymous to a combination of the Subtour Elimination Constraints (SEC) and capacity constraints because of the restriction imposed on each subset of nodes. The SEC are typically the set of constraints that prevent the problem from having a solution where routes are disconnected from one another. These constraints are more formally represented as follows:

$$\sum_{i,j \in S, i \neq j} x_{ij} \leq |S| - 1, \quad \forall S \subset V, S \neq \emptyset \tag{2.8}$$

In 2.8, we require that each nonempty subset S of nodes contain at most $|S| - 1$ arcs. Without the SEC, it would be possible to have a cycle that is disconnected from the depot. This of course would make little sense because trucks cannot appear out of thin air as shown in the example below.

Figure 2.1: CVRP without SEC



The example in Figure 2.1 shows a CVRP problem where SEC are not considered. If we assume node 0 as the depot, then we clearly see that the vehicle along the cycle $\{4, 5, 6\}$ is invalid because all vehicles must originate from the depot.

Our value for $r(S)$ in constraint (2.6) is obtained by solving the bin packing problem, though a lower bound, given by $\lceil \sum_{i \in S} d_i / Q \rceil$, can also be used.

Typically, the given graph G is a n-complete graph under the assumption that $c_{ij} = c_{ji}$, but the model can be easily altered to fit other cases. For example, if there is no direct route from node i to node j, we can modify the cost to be $c_{ij} = M$, where M is large. Further, if we relax the condition that $c_{ij} = c_{ji}$, we can rewrite the graph using directed paths (arcs).

One consequence of the compact formulation (CVRP–c) is with constraint (2.6).

We notice that as n becomes large, the number of subsets grows exponentially. The requirement of these SEC is the main reasons why the class of VRP problems is NP-Hard (Garey and Johnson, 1979). Many efficient algorithms in practice carefully select a subset of these SEC and solve a relaxed version of the problem.

Alternatively, we can look at what is called the extensive formulation of the CVRP.

2.1.2 Extensive Form

The extensive CVRP formulation originates from the set partitioning model formulated by Balinski and Quandt (1964). Their idea was to define feasible routes as the object of interest and to build an integer programming model to support a carefully chosen set of feasible routes. We define a route to be a sequence of nodes $r = (n_0, n_1, n_2, \ldots, n_s)$ where $n_0 = n_s = 0$ to satisfy the condition that all vehicles must begin and finish at the depot. Note that a route is feasible if $\sum_{i \in r} y_{ir} < Q$, where y_{ir} is the amount of supply delivered to i in route r. We also define Ω to be the set of all feasible routes and we introduce the following binary variables:

$$x_r = \begin{cases} 1 & \text{route } r \text{ is selected} \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ir} = \begin{cases} 1 & \text{node } i \text{ is visited by route } r \\ 0 & \text{otherwise} \end{cases}$$

$$b_{ijr} = \begin{cases} 1 & \text{route } r \text{ uses arc } (i,j) \\ 0 & \text{otherwise} \end{cases}$$

In this formulation, the definition of arc cost is also modified as follows: $c_r = \sum_{(i,j)\in A} b_{ijr}c_{ij}$. Thus the model is given as:

$$
\begin{aligned}
\text{(CVRP–e)} \qquad \min \quad & \sum_{r\in\Omega} c_r x_r && && (2.9)\\
\text{s.t.} \quad & \sum_{r\in\Omega} a_{ir} x_r \geq 1 && \forall i \in V\backslash\{v_0\} && (2.10)\\
& \sum_{r\in\Omega} x_r \leq K && && (2.11)\\
& x_r \in \{0,1\} && \forall r \in \Omega && (2.12)
\end{aligned}
$$

The objective (2.9) seeks to minimize the total costs of all selected routes in the chosen solution. Equation (2.10) guarantees that all customer demands are met by at least one vehicle in the fleet. Often in literature, it is common to model (2.10) as a tight constraint to represent each customer only being visited by a single truck. However, writing this constraint as an inequality relaxes our lower bound and we will see later that it helps with our later models. (2.11) guarantees that we do not schedule more vehicles that are available to us in our fleet. Finally, constraint (2.12) describes the binary decision variables x_r, so that a route must be either used or not. The variable b_{ijr} is not considered in (CVRP–e) because it will only be used in the selection of feasible routes. That is for the extensive form, we are assuming that all routes in Ω are already feasible. Similarly, vehicle capacities, Q, is no longer necessary provided that the Ω is well defined.

We notice immediately that the new formulation (CVRP–e) avoids the problem of having SEC which may occur in the formulation of (CVRP–c). Instead, the model now becomes a problem of finding the set of feasible routes in Ω. Of course, if this problem was solved by brute force, Ω would need to contain every feasible route (which grows exponentially). Advances in research for the CVRP has improved this computation greatly. The most prominent approach, is column generation along with branch & price. The approach of column generation was used by Feillet *et al.* (2004), where dynamic programming was used to generate feasible routes efficiently in (CVRP–e).

The extensive formulation will be the model that we will follow for the remainder of this thesis for the ease of representation.

One unintended consequence of both these CVRP formulations is the restriction on the number of deliveries to a single node. That is, only a maximum of one vehicle may deliver to each node. This restriction may not necessarily be the most logical, because it means that a vehicle must fully satisfy the demand of a node if it must deliver to it. We show in the next section a generalization of this problem where multiple vehicles may visit a single node.

2.2 The Split Delivery Vehicle Routing Problem

One important branch of literature is the Split Delivery Vehicle Routing Problem (SDVRP), which was proposed by Dror and Trudeau (1989). In our previous CVRP formulations, every demand node can only be supplied by a single vehicle. In the SDVRP, a demand node is allowed to be served by multiple vehicles. The reasoning for allowing split deliveries is shown by Dror and Trudeau (1990) to have potential savings in cost. For this model we use the variable y_{ir} to represent the amount of supply delivered to node i along route r. The formulation of the SDVRP is as follows:

$$\text{(SDVRP)} \qquad \min \sum_{r\in\Omega} c_r x_r \tag{2.13}$$

$$\text{s.t.} \quad \sum_{r\in\Omega} x_r \leq K \tag{2.14}$$

$$\sum_{i\in V} y_{ir} \leq Q \qquad \forall r \in \Omega \tag{2.15}$$

$$y_{ir} \leq Q a_{ir} x_r \qquad \forall i \in V, \forall r \in \Omega \tag{2.16}$$

$$\sum_{r\in\Omega} y_{ir} \geq d_i \qquad \forall i \in V \tag{2.17}$$

$$x_r \in \{0,1\} \qquad \forall r \in \Omega \tag{2.18}$$

$$y_{ir} \geq 0 \qquad \forall i \in V, \forall r \in \Omega. \tag{2.19}$$

The constraints (2.13), (2.14), and (2.18) are similar to the constraints (2.9), (2.11), and (2.12) from (CVRP–e). Constraint (2.15) ensures that the amount of supplies that is carried by each vehicle does not exceed its capacity. Constraint (2.16) are the necessary logic constraints to ensure that only trucks (routes) that are selected for use may have positive capacity. Constraint (2.17) are the demand requirements and finally, (2.19) ensures positive values for supply.

In original formulation given by Dror and Trudeau (1989), the SDVRP is modeled using as a compact formulation. We do not show that, but instead we use the much simpler extensive formulation (SDVRP). The work by Dror and Trudeau (1990) focused on the structural properties of the optimal solution. One key property is that in the presence of the triangle inequality, no two routes in the optimal solution can have more than one split demand point in common. This is a very significant result because it gives us some insight to the behavior of the "good" routes. It is also a property that we later generalize to the case of the VRPI in Chapter 4.

The computational complexity of the SDVRP on special networks was shown to be NP-hard by Archetti *et al.* (2011). In their experiments, they characterized four network structures to study the limited fleet SDVRP and unlimited fleet SDVRP. More specifically, the networks that were analyzed were when all the demands appear in, a line, a star, a tree, and a circle. Any network can be reduced to a combination of these four structures, which allows us to conclude that the general instance of the SDVRP with a limited fleet is NP-hard.

The formulation of (SDVRP) given above was used in the applications of humanitarian logistics (Huang *et al.*, 2011), which serves as an important tool in delivering aids to those in disaster or emergency situations. Their work focuses on the distribution of relief supplies according to a few key performance metrics, commonly used in humanitarian logistics: efficiency, efficacy, and equality. The performance metric of efficiency is synonymous to the typical objective of the VRP, which is cost (or travel times). Efficacy refers to the speed and sufficiency of delivery, which they compute using arrival times (not specified in (SDVRP) above). Finally, equity ensures that there is an equal spread in service levels across all demands of the network. For our models, we only consider efficiency and a modified version of efficacy, which considers fulfillment.

The models in the research by Huang *et al.* (2011) address the routing problem in a purely deterministic fashion, where they assume a perfect success rate while traveling across any arc in the network. Several properties were also analyzed in their work regarding the optimal solution structure. Our work is largely based off the work by Huang *et al.* (2011), with the added assumption of interdiction across all arcs.

2.3 Uncertainty in the VRP

Looking at the previous work in the VRP, we notice that there is one key assumption to all of the models. Namely, this is the assumption that vehicles will not encounter any variability in their journey. That is not to say that uncertainty has never been considered in the literature; it is a topic that has been studied extensively, but mostly in a more optimistic manner. The three most widely studied variants are the following:

- The VRP with stochastic demands, first studied by Tillman (1969). In this problem setting, the demands at each customer node in our network have a random (stochastic) volume.
- The VRP with stochastic customers where customers are unknown. This variation was introduced by Bertsimas (1988) and it differs from the first case in that demands are known. The unknown in this scenario is whether or not a customer is present in the network.
- The VRP with stochastic travel time is introduced by Laporte *et al.* (1992) and just as the name suggests, it is the VRP problem considering variable travel times.

In all of the previous studies of uncertainty in the VRP, we notice that there were very few that have actually discuseed the case where vehicles may be lost.

In a recent study in disaster management, Liu *et al.* (2013) discuss a vehicle routing problem, where arcs may be "destroyed". However, in their model, vehicles are not lost when this interdiction occurs. Instead, they are rerouted according to a two-stage model, called the multi-vehicle path decision problem. Furthermore, in contrast to our assumption, they only consider the probabilistic interdiction of a single path, while we believe that interdiction should be considered among all paths.

Finally, the term for interdiction that we have constantly used throughout this thesis, is a concept that is derived from network flow problems. In that model, scholars consider the situation of a leader and a follower. The leader's task is to find the shortest path or maximum flow from origin to destination. The follower's task is to select certain arcs or nodes to interdict to prevent the flow of the leader. This framework leads to a Stackelberg game and bi-level programming models. Cormican *et al.* (1998) and Collado and Papp (2012) apply this framework to illegal material transportation and military operations. In our research, we apply the same concept of interdiction to VRP, however, we are interested in the expected loss incurred by interdiction. Furthermore, by using the expectation of interdiction, we are able to apply the concept of maximum reliability. Roosta (1982) introduced a method to simplify the problem of finding a path through a stochastic network that has the maximum probability. They showed that this problem can be manipulated into the shortest path problem using the following reduction:

$$\max \prod_{p_{ij} \in P} p_{ij} \tag{2.20}$$

$$= \max \log\Big(\prod_{p_{ij} \in P} p_{ij}\Big) \tag{2.21}$$

$$= \max \sum_{p_{ij} \in P} \log p_{ij} \tag{2.22}$$

$$= \min \sum_{p_{ij} \in P} -\log p_{ij} \tag{2.23}$$

p_{ij} is the expected success rate, so our original goal, (2.20) is the expression of finding the set of paths that maximizes expected success rate. We make the reasoning for (2.21) to (2.22) because the logarithmic function is a monotonically increasing function. The concepts that Roosta (1982) developed provide us with a starting point for the development of effective solutions to be later used in our models.

2.4 Contribution

The contribution that we make in this thesis is to introduce a novel formulation of the VRP; namely, the VRP with interdiction. This is a branch of the VRP that has applications in disaster relief, military operations, communications networks and more. The VRPI is primarily an extension to the work done by Huang *et al.* (2011), but with the new idea where there may be delivery failures. Though interdiction has seen several applications in network flow, the inclusion of this idea in the VRP has not yet been considered. With this new VRPI model, we are also able to provide several key properties that will aid us in finding the optimal routing for the VRPI.

Chapter 3

VRPI Formulations

In the classical VRP as we discussed in Chapter 2, we are given a network, G, with N customers and one depot. Each customer i has a specified demand, d_i, and we must find the minimum cost routing plan such that all these demands are met.

In this chapter, we introduce the VRP with interdiction. To model the VRPI, again we have a network $G = (V, A)$, where V is the set of demand nodes plus the depot, and A is the set of arcs connecting two nodes. We assume for our instances, G is a complete graph. Each node apart from the depot has an associated demand, d_i, and each arc has an associated cost, c_{ij}. Our situation arises when demand nodes require a fixed *expected* amount of supplies over multiple delivery periods, where the optimal routing plan will be executed over multiple periods. In this model, we assume that in any single delivery period, arcs are subject to interdiction. Every arc in our network has a chance to be interdicted and when interdiction occurs, all vehicles and supplies carried by the vehicles traveling upon that arc will be lost.

We denote probability that an arc (i, j) is interdicted as q_{ij} $(0 \leq q_{ij} < 1)$, and the probability that (i, j) is not interdicted is $p_{ij} = 1 - q_{ij}$. The values q_{ij} and p_{ij}

are given for each arc (i, j), or we simply assume that these values can be estimated. Moreover, we assume that the interdiction probability of each arc is independent of any other. Since we are optimizing for multiple periods, we use the expectation of delivery across all nodes of our network. This implies that it is possible to encounter insufficient supply in a single period due to interdiction, however, in our model, we are able to safeguard against that situation so that the long term demand requirements are met.

Our models are based on route formulation, similar to the one used by Huang *et al.* (2011). That is, we let Ω be an input parameter containing the set of all feasible routes that begin and end at the depot. Each route, $r \in \Omega$, is identified by the set of nodes that it visits. More formally, $r = \{0, n_1, \cdots, n_k, 0\}$ where $n_1, \cdots, n_k \in V$ and 0 represents the depot. Each route is associated with a cost which can be computed by: $c_r = \sum_{(i,j)\in r} c_{ij}$. We can also compute the probability of reaching n_1 without interdiction in route r as $\phi_{n_1 r} = p_{0n_1}$, the probability of reaching n_2 without interdiction as $\phi_{n_2 r} = p_{0n_1} p_{n_1 n_2}$, and so on. Therefore, we define ϕ_{ir} as the probability of arriving at node i without interdiction in route r (probability of success). Route costs and probabilities of success are computed for each route as an input parameter. A summary of notation is given in Table 3.1.

Next, we examine two performance measures for the VRPI: Cost minimization and fulfillment maximization. The option to use for one model over the other depends on the context of the situation and the amount of supply that is available to us.

When the total supply is greater than the total demand, we have an excess of supply, in which case, it is preferable to minimize with respect to cost. In humanitarian applications, the objective will be to deliver supplies as efficiently as possible. When

Table 3.1: Notations for the VRPI

Indices	
0 :	index of the depot.
i :	index of demand nodes (customers); $i \in V = \{1, \cdots, N\}$.
(i, j) :	index of arcs; $(i, j) \in A$.
r, r_k :	index of routes; $r, r_k \in \Omega$.
Parameters	
d_i :	demand of node i.
c_r :	transportation cost of route r.
ϕ_{ir} :	probability of arriving at node i without interdiction in route r.
a_{ir} :	$a_{ir} = 1$ if node i is in route r; otherwise $a_{ir} = 0$.
K :	number of vehicles.
Q :	vehicle capacity.
Decision Variables	
y_{ir} :	amount of supply delivered to node i in route r.
x_r:	$x_r = 1$ if route r is selected; otherwise $x_r = 0$.

the total supply is less than the total demand, we have a shortage of supplies. In this situation, our goal will be to optimize the distribution of our supplies. We want to make sure that those in need will receive as much supplies as we could possibly give. Here, cost is no longer a primary concern anymore, and as Kaplan (2001) has noted, cost can be treated as a constraint rather than an objective.

3.1 Cost minimization

In the first formulation, we aim at minimizing the cost that is required to satisfy the demands of all nodes. This formulation assumes that the total supply is larger than the total demand.

$$
\begin{aligned}
\text{(VRPI–1)} \qquad \min \quad & \sum_{r\in\Omega} c_r x_r && && (3.1)\\
\text{s.t.} \quad & \sum_{r\in\Omega} x_r \le K && && (3.2)\\
& \sum_{i\in V} y_{ir} \le Q && \forall r \in \Omega && (3.3)\\
& y_{ir} \le Q a_{ir} x_r && \forall i \in V, \forall r \in \Omega && (3.4)\\
& \sum_{r\in\Omega} \phi_{ir} y_{ir} \ge d_i && \forall i \in V && (3.5)\\
& x_r \in \{0,1\} && \forall r \in \Omega && (3.6)\\
& y_{ir} \ge 0 && \forall i \in V, \forall r \in \Omega. && (3.7)
\end{aligned}
$$

The objective (3.1) aims at minimizing the total cost of all selected routes. Ω is the set of all feasible routes. Constraint (3.2) enforces the maximum number of vehicles selected. Constraint (3.3) ensures that the total supplies on a vehicle cannot exceed its capacity Q. Constraint (3.4) enforces the necessary logic constraint that only if a route is selected, the delivery on this route can be positive. Constraint (3.6) guarantees that the expected amount delivered to every node meets their demand. Note that we can add some safety stock s_i if necessary, so that the right hand side of constraint (3.5) becomes $d_i + s_i$. Finally, we require that the decision variables x_r be binary, and y_{ir} be nonnegative in (3.6) and (3.7), respectively.

Even though this model assumes an excess of supply, we cannot always guarantee feasibility. Because we are looking at the expected delivery to nodes, having a risky routing plan (with high interdiction probabilities) could generate infeasible solutions.

Additionally, if the total supply is less than the total demand, this model will be infeasible. In those cases, it is advisable to optimize with respect to fulfillment (VRPI–2), presented below.

3.2 Fulfillment maximization

We also propose a VRPI formulation that aims at maximizing demand fulfillment, which naturally assumes that the total supply is less than the total demand. In humanitarian operations, we might use this model when the performance measure depends on how well we can satisfy demands with limited resources.

$$
\begin{aligned}
\text{(VRPI–2)} \qquad \max \quad & \sum_{i \in V} \sum_{r \in \Omega} \phi_{ir} y_{ir} && && (3.8)\\
\text{s.t.} \quad & \sum_{r \in \Omega} x_r \leq K && && (3.9)\\
& \sum_{i \in V} y_{ir} \leq Q && \forall r \in \Omega && (3.10)\\
& y_{ir} \leq Q a_{ir} x_r && \forall i \in V, \forall r \in \Omega && (3.11)\\
& \sum_{r \in \Omega} \phi_{ir} y_{ir} \leq d_i && \forall i \in V && (3.12)\\
& x_r \in \{0, 1\} && \forall r \in \Omega && (3.13)\\
& y_{ir} \geq 0 && \forall i \in V, r \in \Omega. && (3.14)
\end{aligned}
$$

The objective (3.8) aims at maximizing the expected demand fulfillment for all nodes. Constraints (3.9), (3.10), (3.11), (3.13) and (3.14) are the same as (3.2), (3.3), (3.4), (3.6) and (3.7), respectively. Constraint (3.12) guarantees that we do not oversupply a node so that we can satisfy as much of the other demands as possible.

In (VRPI–2), we assume that the total supply is less than the total supply, i.e., there is a lack of supply. If on the contrary, there is an excess of supply, using the demand fulfillment model will give us multiple optimal solutions. In such situations, the solution of (VRPI–2) becomes meaningless, since it would be better to optimize over cost using (VRPI–1). Alternatively, we can add a term $-\epsilon \sum_{r\in\Omega} c_r x_r$ in the objective function of (VRPI–2), where ϵ is a small positive constant. This treatment could allow us to find the most economic route among the routes that have the highest fulfillment.

One important feature of the VRPI formulations is that for the final leg of the journey (from node i back to the depot 0), the interdiction probability has no impact on the solution. This problem could therefore be formulated as the Open Vehicle Routing Problem (OVRP), where vehicles do not need to return to the depot after servicing its last customer (see (Li *et al.*, 2007)). We choose not to adopt this model for the purpose of possibly reusing vehicles. Another reasoning for this is that an empty vehicle has very little reason to be interdicted and therefore has no impact on the fulfillment of nodes.

3.3 Examples

To further clarify our VRPI formulations, we show two examples. First, we have a cost minimization model with the following 5 demand nodes.

Figure 3.1: (VRPI–1) Network Example

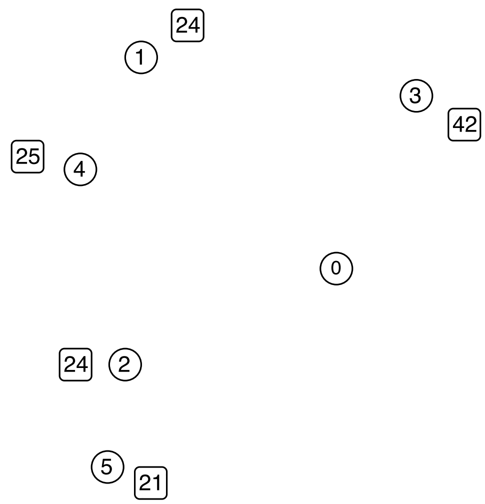


Table 3.2: (VRPI–1) example with 5 nodes

node	x-coordinate	y-coordinate	demand
depot (0)	0	0	
1	-122	132	24
2	-132	-60	24
3	50	108	42
4	-160	62	25
5	-143	-124	21

In Table 3.2, we have a fleet size of $K = 4$ and a vehicle capacity of $Q = 50$. In this network, we generate the interdiction probabilities randomly between $1\% - 5\%$ and we will use distance to represent cost of travel. The interdiction probabilities can be summarized in the following matrix:

$$\text{Interdiction Probabilities : }\begin{bmatrix} \sim & 4\% & 3\% & 1\% & 5\% & 1\% \\ 4\% & \sim & 1\% & 2\% & 2\% & 3\% \\ 3\% & 1\% & \sim & 5\% & 5\% & 5\% \\ 1\% & 2\% & 1\% & \sim & 4\% & 3\% \\ 5\% & 2\% & 5\% & 4\% & \sim & 4\% \\ 1\% & 3\% & 5\% & 3\% & 4\% & \sim \end{bmatrix}$$

The index (i, j) in the above matrix represents the probability of interdiction from traveling between node i and node j.

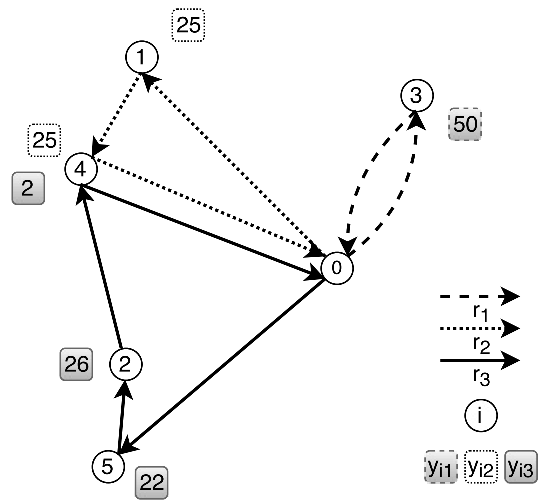
Similarly, the distances between nodes (cost) are represented as follows:

$$\text{Cost Matrix : }\begin{bmatrix} \sim & 180 & 145 & 119 & 172 & 189 \\ 180 & \sim & 192 & 174 & 80 & 257 \\ 145 & 192 & \sim & 248 & 125 & 65 \\ 119 & 174 & 248 & \sim & 215 & 302 \\ 172 & 80 & 125 & 215 & \sim & 187 \\ 189 & 257 & 65 & 302 & 187 & \sim \end{bmatrix}$$

Upon solving this network using (VRPI–1), we get the set of routes illustrated in Figure 3.2. The optimal solution for this network has the value of 1221 and it has the following routes:

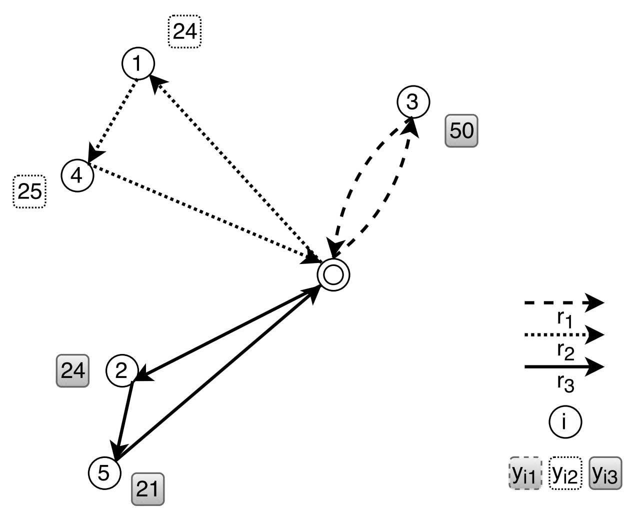
- $r_1 = \{0, 3, 0\}$: $y_{31} = 50$
- $r_2 = \{0, 1, 4, 0\}$: $y_{12} = 25$, $y_{42} = 25$
- $r_3 = \{0, 5, 2, 4, 0\}$: $y_{53} = 21.2$, $y_{23} = 25.5$, $y_{43} = 1.7$

Figure 3.2: (VRPI–1) Example with interdiction



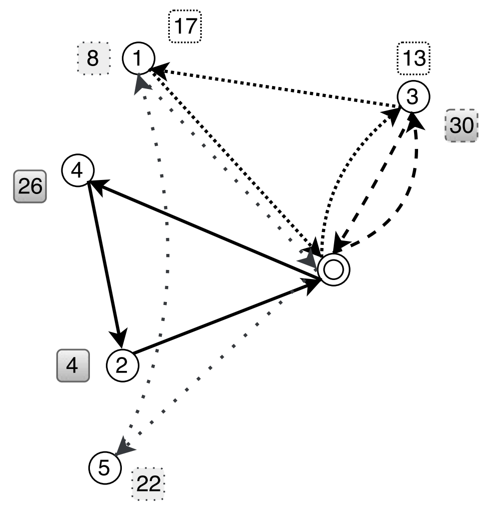
Just to contrast, the same network with no interdictions has an optimal value of 1069 with the routes shown in Figure 3.3

Figure 3.3: (VRPI–1) Example with no Interdiction



For the (VRPI–2), we use the same network, but we adjust it slightly so that the total demand is greater than the total supplies available. In this scenario, we change the vehicle capacity to $Q = 30$. Interdiction rates and costs, demands, and fleet size remain the same. The optimal solution that we obtained is shown in figure 4.5, with a value of 115.981.

Figure 3.4: (VRPI–2) Example



The case for no interdiction for (VRPI–2) is less meaningful since any feasible route will 'maximize' the fulfillment of the nodes. Since we will then have multiple optimal solutions, we can add the term $-\epsilon \sum_{r \in \Omega} c_r x_r$ to the objective of (VRPI–2). We thus encounter the same problem as (VRPI–1), to optimize the secondary objective which is cost.

We explain in Chapter 6 with more detail on how networks were generated and how they were solved.

3.4 Complexity

The classical VRP is already known to be NP-hard since it is shown to be an immediate reduction of the Traveling Salesman Problem introduced by Garey and Johnson

(1979). It was much later that Archetti *et al.* (2011) proved the NP-hardness of SDVRP with a limited fleet. Knowing the complexity of SDVRP, we can use a simple reduction to show the complexity of (VRPI–1).

Theorem 1 *(VRPI–1) is NP-hard.*

Proof: The VRPI for cost minimization is NP-hard since the SDVRP is a special case of (VRPI–1), where the probability of interdiction is 0 across all arcs. ■

The complexity of the (VRPI–2) is not as obvious, since the objective of this model is much different from typical VRP models. However, we show that it is still a difficult problem using a reduction from the Hamiltonian Cycle Problem (HCP), which is known to be NP-complete (Garey and Johnson, 1979). The objective of the HCP is to find a Hamiltonian cycle (a cycle that visits each vertex exactly once in a given network).

Theorem 2 *(VRPI–2) is NP-hard.*

Proof: We use a reduction from the Hamiltonian Cycle Problem (HCP). Consider any arbitrary graph G' with n nodes. A corresponding instance of the (VRPI–2) can be constructed on a complete graph as follows: We set $p_{ij} = 1$ if $(i, j) \in G$, otherwise $p_{ij} = 0.01$ (or an arbitrary very small positive number). We optimize (VRPI–2) using a single vehicle, $K = 1$ with capacity $Q = n$, and n customers each with $d_i = 1$. If and only if the optimal routing to this instance of the (VRPI–2) gives us an objective value of n, the corresponding graph in G' has a Hamiltonian cycle. ■

Chapter 4

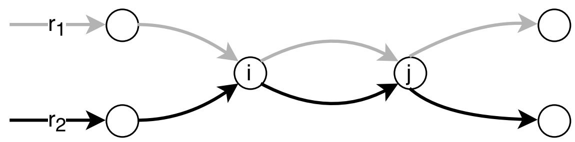
Analysis of the VRPI

In this chapter, we characterize the properties of the optimal solutions of VRPI models.

4.1 Optimal Route Characteristics

Dror and Trudeau (1989) introduced one key property in their introduction of the SDVRP. They proved that when the arc costs in the network satisfy the triangle inequality, then an optimal solution cannot have two routes with more than a single common demand point. Their reasoning for this was that if more than a single demand point is shared among two routes, than we are able to adjust the demands such that the new delivery routes produce a better solution in terms of cost. For example, consider the following network.

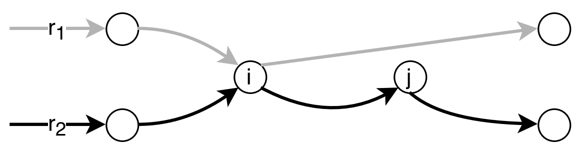
Figure 4.1: Dror's property with 2 split deliveries



In Figure 4.1 we have a routing plan where two demand nodes share two routes. If we assume WLOG that $y_m in = \min y_{i1}, y_{i2}, y_{j1}, y_{j2} = y_{j1}$, then the following changes to deliveries will produce a better solution

$$y^*_{i1} = y_{i1} + y_{j1}$$
$$y^*_{i2} = y_{i2} - y_{j1}$$
$$y^*_{j2} = y_{j2} + y_{j1}$$
$$y^*_{j1} = 0$$

Figure 4.2: Dror's property with new deliveries



This change in delivery values is also illustrated in Figure 4.2

A large part of this property by Dror and Trudeau (1989) is reliant on the fact that costs satisfy the triangle inequality, however when interdiction is involved, things start to change. However, we can first generalize this property as Lemma 1.

Lemma 1 *For (VRPI–1) and (VRPI–2), if* $q_{ij} \equiv \xi$ *for a constant* $0 \leq \xi < 1$ *for all* $(i, j) \in A$*, then there exists an optimal solution where two routes will have at most one shared demand node.*

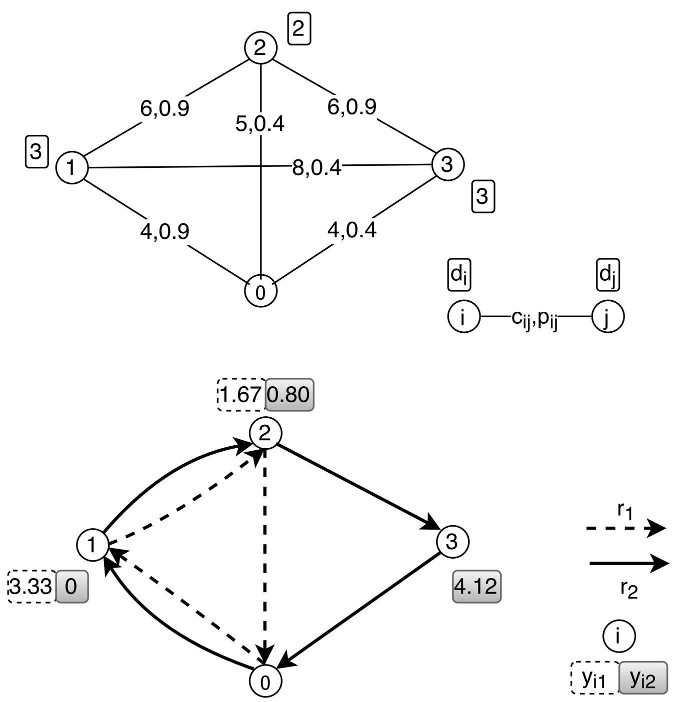
This conclusion is true because when q_{ij} is constant, the triangle inequality will hold and taking a less direct route will not be cost efficient. This is a slightly generalized argument to the one presented by Dror and Trudeau (1990), for which we have provided a sketch above.

Property 1 *For (VRPI–1) and (VRPI–2), two demand nodes may have multiple routes in common in an optimal solution.*

We notice that property 1 is a direct violation of Lemma 1, i.e., Dror & Trudeau's property. An important assumption behind Dror & Trudeau's property is the fact that arc costs in the network satisfy the triangle inequality. While this is also true in VRPI, interdiction probabilities do not satisfy this assumption. In the VRPI, some overlapping routes may have a better probability of success (i.e., probability of no interdiction) since the chance that each arc is interdicted is independent of one another. *Expected delivery* in our models (i.e., probability of no interdiction multiplied by delivery quantity) does not follow the triangle inequality either, so additional routes covering the same nodes may be present in the optimal solution because of the higher probability of success. While a more direct route that bypasses nodes with fulfilled demands could appear to save costs, the expected delivery from taking this route could be too small. In this scenario, an additional vehicle would be required to cover any unmet demands, which is ultimately less optimal. In other words, the cost associated with adding an additional vehicle in our solution to cover these unmet demands, could outweigh the cost of making a less direct route to avoid interdiction. In some instances, adding an additional vehicle can even be infeasible (due to the constraint of vehicle number K). In general, as interdiction probabilities become larger, there is a better chance that we see overlaps between routes, where this property applies.

An example is given in Figure 4.3. In Figure 4.3, we have the goal of minimizing cost. The optimal solution consists of $r_1 = \{0, 1, 2, 0\}$ and $r_2 = \{0, 1, 2, 3, 0\}$, both covering demand nodes 1 and 2. Although r_2 is not supplying node 1 and does not need to travel on arcs $(0, 1)$, $(1, 2)$, the vehicle still makes that detour because otherwise our solution would require 3 vehicles ($p_{0,3} \cdot Q = 2 < 3 = d_3$ and $p_{0,2} \cdot p_{2,3} \cdot Q = 1.8 < 3 = d_3$), which is infeasible since $K = 2$.

Figure 4.3: (VRPI–1) with 3 demand nodes, $Q = 5$, $K = 2$.



For (VRPI–2), it can be easily shown that property 1 is also true. Naturally, arcs with higher probability of successs are more favorable in maximizing fulfillment because the expected delivery will be greater.

Figure 4.3 also illustrates a critical property different from Dror & Trudeau's property.

Property 2 *For (VRPI–1) and (VRPI–2), there exists an optimal solution, where for every pair of nodes, at most one route delivers to both nodes.*

Proof: First, we prove the result for (VRPI–1). We consider two routes 1 and 2, both of which visit two arbitrary demand nodes i and j. For simplicity, we use the following notation:

$$\phi_{i1} = \phi_1, \qquad \phi_{j1} = \phi_2, \qquad \phi_{i2} = \phi_3, \qquad \phi_{j2} = \phi_4,$$
$$y_{i1} = y_1, \qquad y_{j1} = y_2, \qquad y_{i2} = y_3, \qquad y_{j2} = y_4.$$

We assume that demands for i and j are both satisfied, i.e., $\phi_1 y_1 + \phi_3 y_3 \geq d_i$ and $\phi_2 y_2 + \phi_4 y_4 \geq d_j$. The capacity constraints are also satisfied, i.e., $y_1 + y_2 \leq Q$ and $y_3 + y_4 \leq Q$. Assume that in this optimal solution, both nodes have split demands, i.e., y_1, y_2, y_3, $y_4 > 0$.

We consider four cases:

case 1: if $\phi_4 \geq \frac{\phi_2\phi_3}{\phi_1}$ and $y_2 > \frac{\phi_3}{\phi_1} y_3$ we can make the following substitution:

$$y_1^* = y_1 + \frac{\phi_3}{\phi_1} y_3 \qquad y_2^* = y_2 - \frac{\phi_3}{\phi_1} y_3$$
$$y_3^* = 0 \qquad y_4^* = y_4 + y_3.$$

case 2: if $\phi_4 < \frac{\phi_2\phi_3}{\phi_1}$ and $y_4 > \frac{\phi_1}{\phi_3} y_1$ we can make the following substitution:

$$y_1^* = 0 \qquad y_2^* = y_2 + y_1$$
$$y_3^* = y_3 + \frac{\phi_1}{\phi_3} y_1 \qquad y_4^* = y_4 - \frac{\phi_1}{\phi_3} y_1.$$

case 3: if $\phi_4 < \frac{\phi_2\phi_3}{\phi_1}$ and $y_4 \leq \frac{\phi_1}{\phi_3}y_1$ we can make the following substitution:

$$y_1^* = y_1 - \frac{\phi_4}{\phi_2}y_4 \qquad y_2^* = y_2 + \frac{\phi_4}{\phi_2}y_4$$
$$y_3^* = y_3 + y_4 \qquad y_4^* = 0.$$

case 4: if $\phi_4 \geq \frac{\phi_2\phi_3}{\phi_1}$ and $y_2 \leq \frac{\phi_3}{\phi_1}y_3$ we can make the following substitution:

$$y_1^* = y_1 + y_2 \qquad y_2^* = 0$$
$$y_3^* = y_3 - \frac{\phi_2}{\phi_4}y_2 \qquad y_4^* = y_4 + \frac{\phi_2}{\phi_4}y_2.$$

We can verify that all these four cases give feasible solutions. For example, in case 4, capacity constraints are satisfied: $y_1^* + y_2^* = y_1 + y_2 \leq Q$ and $y_3^* + y_4^* = y_3 - \frac{\phi_2}{\phi_4}y_2 + y_4 + \frac{\phi_2}{\phi_4}y_2 = y_3 + y_4 \leq Q$. Demand constraints are also satisfied:

$$\begin{aligned}
\phi_1 y_1^* + \phi_3 y_3^* &= \phi_1(y_1 + y_2) + \phi_3(y_3 - \frac{\phi_2}{\phi_4}y_2) \\
&= \phi_1 y_1 + \phi_3 y_3 + (\phi_1 - \frac{\phi_3\phi_2}{\phi_4})y_2 \\
&\geq \phi_1 y_1 + \phi_3 y_3 \\
&\geq d_i
\end{aligned}$$

and

$$\begin{aligned}
\phi_2 y_2^* + \phi_4 y_4^* &= \phi_2(0) + \phi_4(y_4 + \frac{\phi_2}{\phi_4}y_2) \\
&= \phi_2 y_2 + \phi_4 y_4 \\
&\geq d_j.
\end{aligned}$$

A similar argument applies for the other three cases. In all cases, after applying the substitution, we are left with a solution where no supply is delivered on one of the nodes from one of the routes.

Finally, we check to ensure that the adjusted deliveries do not negatively affect the objective value. In (VRPI–1), it is clear that the objective value $\sum_{r\in\Omega} c_r x_r$ remains unchanged since the selection of routes is still the same. The only modification is the distribution of supplies between the vehicles.

The proof for (VRPI–2) is similar. The only adjustment that must be made is if a node receives too much supply and thus violates constraint (3.12). To account for this ($\phi_1 y_1 + \phi_3 y_3 \leq d_i$ and $\phi_2 y_2 + \phi_4 y_4 \leq d_j$), we can simply remove the excess supply from the vehicles. In case 4:

$$\begin{aligned} y_1^* &= y_1 + y_2 & y_2^* &= 0 \\ y_3^* &= min\{y_3 - \frac{\phi_2}{\phi_4} y_2, \frac{d_i - \phi_1 y_1^*}{\phi_3}\} & y_4^* &= min\{y_4 + \frac{\phi_2}{\phi_4} y_2, \frac{d_j}{\phi_4}\}. \end{aligned}$$

The new objective value for this case is $\sum_{i\in V}\sum_{r\in\Omega} \phi_{ir} y_{ir}$. Other parts of the network remain unchanged so we focus on the following four entries:

$$
\begin{aligned}
&\phi_1 y_1^* + \phi_2 y_2^* + \phi_3 y_3^* + \phi_4 y_4^* \\
&= \phi_1 y_1^* + \phi_2(0) + \phi_3(y_3 - \frac{\phi_2}{\phi_4} y_2) + \phi_4(y_4 + \frac{\phi_2}{\phi_4} y_2) \\
&\geq \phi_1 y_1^* + \phi_2(0) + \phi_3(\frac{d_i - \phi_1 y_1^*}{\phi_3}) + \phi_4(\frac{d_j}{\phi_4}) \\
&= \phi_1 y_1^* + d_i - \phi_1 y_1^* + d_j \\
&\geq \phi_1 y_1^* + \phi_1 y_1 + \phi_3 y_3 - \phi_1 y_1^* + \phi_2 y_2 + \phi_4 y_4 \\
&= \phi_1 y_1 + \phi_2 y_2 + \phi_3 y_3 + \phi_4 y_4.
\end{aligned}
$$

With the new delivery values, the objective value for (VRPI–2) is nondecreasing. A similar argument applies to the other cases as well. Thus we have shown that there will always exist an optimal solution where for every pair of nodes, at most one route delivers to both nodes. ■

Property 2 differs from property 1 in that we acknowledge multiple routes may visit several common nodes, however, there exists an optimal solution such that demand split only happens on one of these nodes. For example, two routes r_1 and r_2 may share nodes i, j; if $y_{i2} > 0$ and $y_{j2} > 0$, then either $y_{i1} = 0$ or $y_{j1} = 0$. On the other hand, property 2 can be understood as a generalization of Dror & Trudeau's property. Indeed, in the classic SDVRP, property 2 automatically implies property 1, because when the triangle inequality holds, it is unnecessary to make a detour and visit nodes that have already received deliveries.

Next we formally define the concept of a "detour point".

4.2 Detours

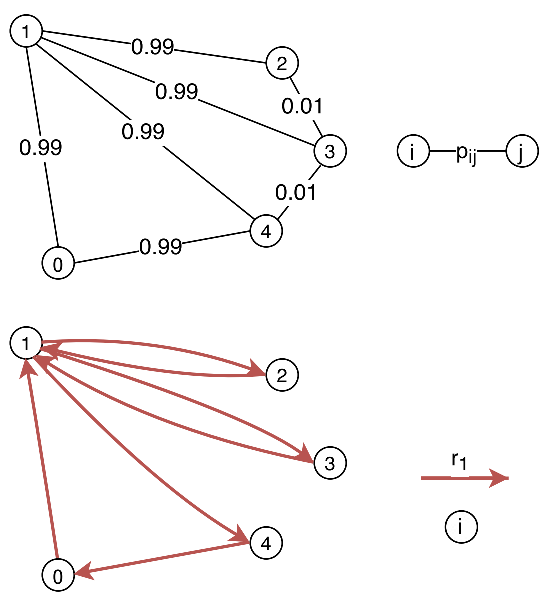
Definition 1 *A route r contains a detour point at node i, if i is visited by r without receiving any supply.*

The use of a detour point is shown in Figure 4.3 and they are frequently used when there are large interdiction probabilities. In fact, one distinguishing property of the VRPI is the following.

Property 3 *For (VRPI–1) and (VRPI–2), an optimal solution may contain several detour points; a single node can be used as a detour point several times.*

An example of property 3 is shown in Figure 4.4 below. Though the interdiction probabilities in the network are quite extreme, such an instance may also occur when the interdiction probabilities on the arcs $(2, 3)$ and $(3, 4)$ are much larger.

Figure 4.4: Multiple detours



Another observation of the VRPI comes from the fact that a single trip to a node can still be insufficient to satisfy the node demand. This can happen in both the case of $Q \geq d_i$ and the case of $Q < d_i$, since the interdiction probability may make the expected delivery unable to satisfy the node demand completely. This observation brings us to our next property:

Property 4 *For (VRPI–1) and (VRPI–2), an optimal solution may use a single route multiple times.*

The presence of multiple routes is illustrated in Figure 4.5. In Figure 4.5, the optimal solution is to use route $r_1 = \{0, 1, 2, 0\}$ twice (i.e., send two vehicles along the route r_1). The supply distributions for the two routes are different and satisfy properties 1 and 2. The route r_1 is chosen twice because it gives us the highest probability of supplying demand node 2 to its desired demand. Indeed, if we were to send the second vehicle directly from the depot to node 2 (i.e., $r_2 = \{0, 2, 0\}$), then our solution would be infeasible since $p_{02}Q + y_{21} < d_2$.

Figure 4.5: (VRPI–1) with 2 demand nodes, $Q = 37$, $K = 2$.

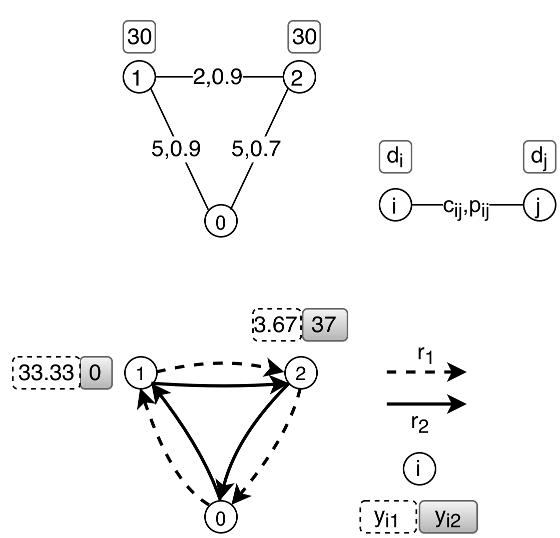
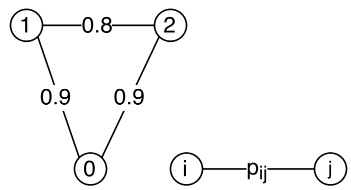


Figure 4.6: Depot as a detour point.



Throughout this paper, we have assumed that the probability of interdiction for each arc is independent of one another. Therefore, with respect to probability of success, the triangle inequality does not hold. One consequence is that we could travel upon the same arcs twice and revisit a node whose demand has already been satisfied (property 1 and property 2), or the usage of multiple highly reliable routes (property 4). Another interesting consequence is the existence of an optimal solution where a vehicle uses the depot as a detour point. This scenario is shown in Figure 4.6.

In Figure 4.6, we can see that, a vehicle who just visits node 1, has a choice of going directly to node 2 (with $p_{12} = 0.8$), or making a detour via the depot to reach node 2 (with $p_{10}p_{02} = 0.81$). Obviously, making the direct trip would incur a higher interdiction probability. So in (VRPI–2), we would choose the detour trip. The routing decision for (VRPI–1) will only choose the direct trip from node 1 to node 2 if the supply left on the vehicle is enough to satisfy the demand at node 2 (i.e., $Q - y_{1r} \geq \frac{d_2}{\phi_{2r}}$). To account for this detour in our model, we could add an auxiliary node with demand 0 to act as the depot. In practice, if such an instance occurs, another option could be to send the required supply to node 1, then replenish the vehicle at the depot before rerouting it. The option to replenish vehicles may however come with an additional cost.

The same applies in general to any sequence with arbitrary number of nodes. In

(VRPI–2), consider a sequence $i_1, i_2, \dots, i_k$, we will use the path $\{i_1, i_2, \dots, i_k\}$ over $\{i_1, i_k\}$ if $p_{i_1 i_2} \dots p_{i_{k-1} i_k} > p_{i_1 i_k}$, although it would be more advantageous in terms of travel cost to make the direct trip from i_1 to i_k due to the triangle inequality on travel costs.

Finally, we have a property describing the number of vehicles in our optimal solution.

Property 5 *For (VRPI–1), we may not minimize the number of vehicles used. However, for (VRPI–2), we are always able to find an optimal solution where we use all vehicles that are available to us.*

We observe that in some instances of (VRPI–1), it is much more cost effective to route a new vehicle from the depot than to use each vehicle to its capacity. Consider for example, a relaxed variation of Figure 4.3 where $K = 5$ (all the other data unchanged). The current solution with 2 routes yield an objective value of $z = c_{r_1} + c_{r_2} = 15 + 20 = 35$. Now consider the following solution $r_1 = \{0, 1, 0\}$, $r_2 = \{0, 2, 0\}$, $r_3 = r_4 = \{0, 3, 0\}$. The objective value of this solution is $c_{r_1} + c_{r_2} + c_{r_3} + c_{r_4} = 8 + 10 + 8 + 8 = 34 < z$.

The second part of property 5 is more obvious since the use of additional vehicles does not decrease our objective value in (VRPI–2), so we are always able to find an optimal solution that uses the maximum allowable number of vehicles.

In the VRPI models, we are optimizing the long term average performance of a single routing plan that is executed repeatedly. Thus, in a single delivery period, it is possible for us to oversupply nodes in a network. The amount of additional stock from the oversupply could be used as a safety stock to account for uncertainties in future deliveries due to interdiction. The same concept applies for both (VRPI–1) and (VRPI–2).

In the next chapter, we discuss the solution methodologies for VRPI models.

Chapter 5

Algorithms

To model the VRPI problems, we use route formulations. Therefore, to solve a VRPI problem to optimality, we have to enumerate all the possible routes are in Ω. However, it is time consuming to do this. Moreover, according to properties 3 and 4, there may exist multiple repeated routes or detour points in the optimal solution, which implies that a much larger set is required for Ω. On the other hand, though we must consider all the possible routes, only a small subset of those routes will appear in the optimal solution, so an efficient procedure would only require a well picked set of routes in Ω. For our numerical studies, we use heuristic algorithms to generate the set of routes that we will use.

The auxiliary model that we use to select "good" routes relies on a metaheuristic framework, called GRASP (Greedy Random Adaptive Search Procedure) as described by Feo and Resende (1995).

5.1 GRASP

GRASP can be applied in two phases: 1. A construction phase, and 2. Local Search. In the construction phase, a feasible solution is iteratively built. This is followed by the improvement phase where a local search is performed in the neighborhood of the constructed feasible solution. These two steps are performed repeatedly until a stopping criteria is reached. During each iteration of the algorithm the best solution is kept. The random and greedy aspect of GRASP comes in the construction phase where a candidate list is chosen based on a desired degree of randomness or greediness. The pseudocode of their algorithm (Feo and Resende, 1995) is shown below in Algorithm 5.1

Algorithm 1 GRASP pseudo-code

1: BestSolution = $\{\}$
2: **while** Stop criteria not met **do**
3:     $S = \{\}$
4:     **while** element list $\neq \emptyset$ **do**
5:         Make Restricted Candidate list(RCL)
6:         Select random element c in RCL
7:         $S = S \cup \{c\}$
8:         Update RCL
9:     **end while**
10:     **while** Stop criteria not met **do**
11:         Find better solution T in the neighbourhood of S
12:         Compare T with S
13:         Update BestSolution with $\{T, S\}$
14:     **end while**
15: **end while**
16: Return BestSolution

Lines 4-9 of Algorithm 1 refer to the construction phase. This step builds the feasible solution using a Restricted Candidate List (RCL). The RCL is built based on a variable $\alpha = [0, 1]$. An α value of 0 makes the RCL purely random, whereas an

α value of 1 makes the RCL purely greedy.

Figure 5.1: GRASP

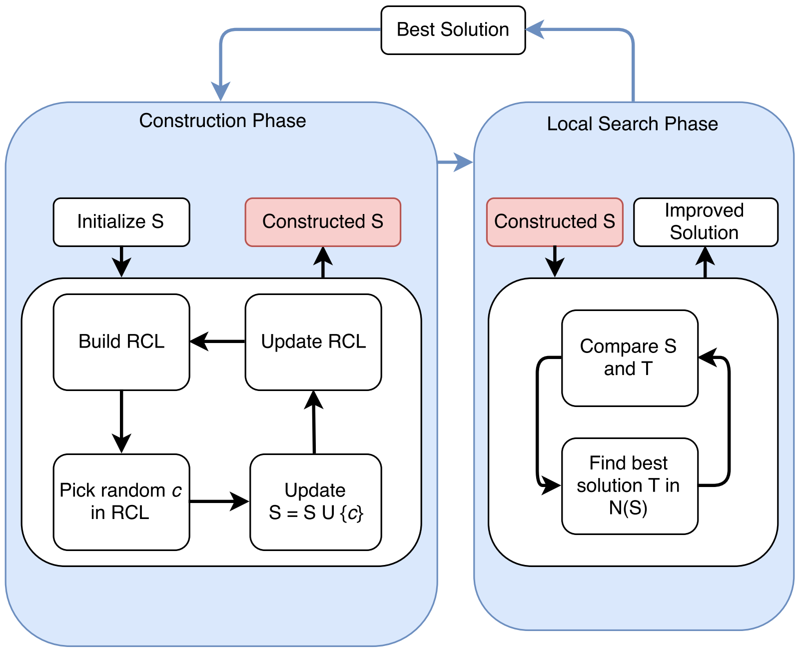


Figure 5.1 illustrates the general GRASP algorithm

In the next section, we adapt the GRASP to our own problem setting.

5.2 Algorithm for (VRPI–1)

We begin with an empty s et of routes ($\Omega = \emptyset$) and as the auxiliary model is solved, new routes are generated. In this auxiliary model, we identify paths that are likely to exist in the optimal solution based on the following two criteria: Paths with a high probability of success, or paths with low travel cost. The algorithms for the two different models are described in more detail below.

Cost minimization

Algorithm 2 Route Generation for (VRPI–1)

```
Initialization: Ω = ∅; generate U, i.e., the set of all initial routes; set γ as the interdiction threshold
while U ≠ ∅ do
    Choose r_k = {0, i, ..., m, 0} ∈ U
    Find node u ∉ r_k such that min_u c_mu
    Let r_k^1 = {0, i, ..., m, u, 0}
    Find node v ∉ r_k such that max_v p_mv
    Let r_k^2 = {0, i, ..., m, v, 0}
    if Σ_{j∈r_k^1} d_j < Q and φ_{u r_k^1} > γ then
        Add route r_k^1 to U
    end if
    if Σ_{j∈r_k^2} d_j < Q and φ_{v r_k^2} > γ and u ≠ v then
        Add route r_k^2 to U
    end if
    Remove route r_k from U and add it to Ω
    Reoptimize U
end while
```

Our heuristic for the cost minimization model, Algorithm 2, has a construction phase and a local improvement phase. We begin with the initial set of K routes, where K is the number of vehicles available to us. The K routes are chosen based on the criteria of having the K lowest interdiction rates. Each route has the form $\{0, i, 0\}$.

The set Ω is the final set of routes that we input into our formulation and the set U is the set of unprocessed routes waiting for improvement. In each iteration of the procedure, we perform what is called the extension step on each of the routes in our set. In the extension step, we take a route and identify a neighboring unvisited node that has minimum cost and a neighboring unvisited node that has minimum chance of interdiction. Completing this step would provide us two new routes.

Afterwards, we check the feasibility of these two new routes and discard those infeasible routes. In this feasibility check step, we check the following two conditions: 1) The sum of all demands of visited nodes is no greater than the capacity. 2) The probability of reaching the final demand node in the route is greater than γ (γ is the stopping condition chosen by the decision-maker). We will usually set this value according to the average probability of success in the network.

The last step is to perform some local improvements for the set U with the newly added routes. This is line 14 of Algorithm 1, where we remove any routes that are dominated by another route. If any two routes contain the same demand nodes, we only keep the route that has a lower cost or higher probability of success. This step ensures a smoother runtime and that we don't need to process unnecessary routes that are potentially less good. A flow chart of Algorithm 2 is presented in Figure 5.2.

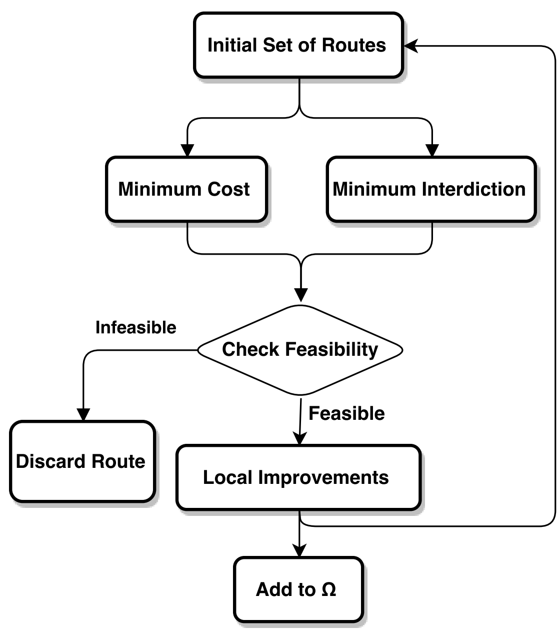


Figure 5.2: Heuristic for (VRPI–1)

5.3 Algorithm for (VRPI–2)

For (VRPI–2), we use the same GRASP framework where we have a construction phase and a local improvement phase.

Fulfillment maximization

Algorithm 3 Route Generation for (VRPI–2)

1: Initialize $\Omega = \emptyset$, $U = \emptyset$, and set ϵ as the threshold value
2: **for** $i \in V$ **do**
3:     Find the most reliable path $\{0, n_1, \ldots, n_k\}$ from 0 to n_k, where $n_k = i$
4:     Add route $\{0, n_1, \ldots, n_k, 0\}$ to Ω and U
5: **end for**
6: **while** $U \neq \emptyset$ **do**
7:     Select route $r = \{0, i, \ldots, m, 0\} \in U$
8:     Find a neighboring node $u \notin r$ with $\max_u\{p_{mu}\}$
9:     Let $r' = \{0, i, \ldots, m, u, 0\}$
10:     **if** $\phi_{u,r'} > \epsilon$ and $\sum_{j \in r'} d_j < Q$ **then**
11:         Add route r' to U
12:     **end if**
13:     Remove r from U and add to Ω
14: **end while**

In (VRPI–2), we notice that the solution that maximizes our objective function consists of the arcs yielding the highest value for $\phi_{ir} y_{ir}$. The choice of y_{ir} is greedy in the sense that larger ϕ_{ir} leads to larger y_{ir}. Thus we turn to ϕ_{ir}, which is our probability of success. We use the idea of maximum reliability, introduced by Roosta (1982) and solved using a variation of the shortest path problem. In Algorithm 3, the initial loop begins by generating a set of routes that maximize the probability of success (or minimize the chance of interdiction). To find the path with maximum probability of success, we compute $-log(p_{ij})$ for each arc (i, j) in our network and solve the shortest path problem (with Dijkstra's algorithm). These paths (computed in lines 2-5) serve as the initial set of routes from which we can make further improvements.

In the subsequent step, when additional capacity is available on a vehicle, we use a greedy route extension step: Picking a neighboring arc with unfulfilled demand that has the smallest chance of interdiction to extend our current path. The route extension step continues until we systematically determine that the path can no longer contain any more demand nodes and/or when a certain threshold, representing highest acceptable interdiction chance, is reached. The set U is used to represent the set of unprocessed routes which can be extended. Initial routes and routes that can no longer be extended are candidates to be placed in set Ω.

A threshold for stopping our route extension is added because routes need not to be inclusive of all nodes. When looking at the general length of a route, optimally, we prefer a route that stops by the major nodes without being too constrained by the chances of interdiction. We also observe that longer routes will have a quickly diminishing capacity, unless the prior nodes in a route are detour points. Furthermore, the accumulation of interdiction probabilities upon a longer route will quickly make such routes less optimal. Therefore, with the exception of routes with detour points, the capacity of a vehicle is diminished by approximately $\frac{d_i}{\phi_{ir}}$ for each subsequent service to demand node i in route r. Another stopping condition that can be applied is a threshold based on a target service level that one must achieve. For example, in a network with a target service level of 90% and constant interdiction probabilities of 0.01 across all arcs, we can expect the optimal routes to contain no more than 10 demand nodes (a route containing 11 nodes will have one customer with $\phi_{ir} = 0.99^{11} < 0.9$). The set of feasible routes is expected to be smaller for a higher service level.

Chapter 6

Numerical Results

The numerical experiments serve two purposes: 1). We evaluate the effectiveness of the proposed models; 2) we test the efficiency of the proposed algorithms.

In our experiments, we test networks of 5, 6, and 7 demand nodes. Though these networks are small, larger instances would not be computable and the solutions that we obtain would not be easily interpreted. For brevity, we only present the results for the 7 node networks. Each node in a selected network is scattered randomly along a 200×200 square grid. The depot is located at the center of the grid and the travel cost between two nodes is set to be the Euclidean distance of the two nodes rounded to the nearest integer.

We set the parameters as: $Q = 50$ or 100, $K = 4$ or 6, and $q_{ij} = U[0.01, 0.05]$ or $U[0.01, 0.10]$ or $U[0.01, 0.20]$, where $U[a, b]$ denotes a uniform distribution between a and b. The 12 instances that we have generated for the 7 node network are summarized in the Table 6.1, where column "Ins. Num." presents the instance number and column "Int. Prob." represents the interdiction probability.

Table 6.1: Selected Network Instances for VRPI Models

Ins. Num.	Q	K	Int. Prob.
1	50	4	U[0.01,0.05]
2	50	4	U[0.01,0.10]
3	50	4	U[0.01,0.20]
4	50	6	U[0.01,0.05]
5	50	6	U[0.01,0.10]
6	50	6	U[0.01,0.20]
7	100	4	U[0.01,0.05]
8	100	4	U[0.01,0.10]
9	100	4	U[0.01,0.20]
10	100	6	U[0.01,0.05]
11	100	6	U[0.01,0.10]
12	100	6	U[0.01,0.20]

All of the models and algorithms are coded in C++ using the IBM ILOG CPLEX Concert Technology (CPLEX 12.2.0). The computations are performed on a 1.60 GHz 64-bit Intel Core i5 CPU with 8GB RAM.

6.1 VRPI Route Structure

For the (VRPI–1), we create instances where the total supply is greater than the total demand; and for the (VRPI–2), we create instances where the total supply is less than the total demand. By solving the VRPI models to optimality, we make the following observations.

Observation 1 *High interdiction, low cost arcs.*

Arcs with a relatively high interdiction probability but low cost will appear near the rear of the route.The reasoning behind this observation is that as a route traverses risky arcs, the accumulation of interdiction probabilities makes the overall probability of success of reaching all subsequent nodes lower. If the risky arc in our network is selected much later in our route, then only nodes that are visited after the risky arc will be affected. Similarly, if we want to maximize the fulfillment in (VRPI–2), then we surely want to minimize the number of nodes that are affected by a single risky arc. Thus, if the selection of a risky arc is inevitable, it is preferable to locate this arc near the rear of a route.

Observation 2 *Low demand nodes.*

Nodes with relatively low demands are visited last in an optimal route. As our route lengthens, the cumulative interdiction rate grows, therefore the expected supply for the nodes served at the rear of the route will be less than if the node was one of the first ones served. Another way to look at it takes into account the expected loss. When delivering a larger amount of supply through a risky path, the expected loss will be higher compared to delivering a smaller quantity through the same path. Correspondingly, nodes with higher demand will typically appear near the front of a route. The expected probability of successs is much more likely to be higher at the front of a route. This observation holds for both (VRPI–1) and (VRPI–2).

Observation 3 *Split deliveries.*

A route experiences split delivery if the supply of the vehicle along that route cannot completely satisfy the demand of a node along the route. We notice that in all our experiments, the number of split deliveries is at most 2. This observation was recorded in both experiments with (VRPI–1) and (VRPI–2). It is conjectured that this observation holds for all instances in general because the loss incurred by interdiction will deter the use of some routes. Particularly those routes where split deliveries occur.

Observation 4 *Repeated routes and detour points.*

Although we show with Property 4 that the presence of multiple routes is possible, in our generated instances of select networks, the optimal solution will rarely contain repeated routes. This observation provides us with a simplification to the problem by not including multiple copies of routes in the set Ω. Additionally, we have shown that there may exist solutions where there is a detour point in Property 2. However, the use of a single detour point, or even multiple detour points is rather uncommon in (VRPI–1). One factor that might account for this observation is the fact that our interdiction probabilities are uniformly distributed. In (VRPI–2), however, the use of detour points is much more common. This is likely because the objective function does not include traveling costs.

Observation 5 *Size of routes.*

We observe that in the instances that we have generated, a single route will often have a length of at most $\frac{Q}{\min_{i\in V}\{d_i : d_i > 0\}}$. In general, model (VRPI–1) has a preference for shorter routes because the cumulative interdiction probabilities along with the traveling costs make it less likely for demands to be completely satisfied on a long route. In (VRPI–2), the size of the routes tends to reach the maximum length (simply based on the absence of route costs and the nature of the CPLEX solver).

However, in this solution, the nodes at the rear of the route will not be served and we can interpret solution in a way such that these nodes are removed from the route. Applying this soft upper bound on the size of routes will allow us to find a good estimation of the optimal solution within a shorter time.

All of these observations allow us to have a better understanding of the optimal route structure of the VRPI models as they are valid for most of the instances.

6.2 Solution of (VRPI–1)

In (VRPI–1), we generate the demands using a random uniform distribution such that $d_i = U[10, 30]$ for instances with $Q = 50$ (Instances 1-6) and $d_i = U[30, 70]$ when $Q = 100$ (Instances 7-12).

Five unique 7-node networks are created within the specifications for each instance and we record the averages of these five networks. We compare the computational time and objective value by our heuristic algorithm with those of the CPLEX solver. Because we use all possible permutations of routes, the CPLEX solver finds the optimal objective value. Below is a summary of the experiments.

Table 6.2: Computational Results for (VRPI–1)

Ins. Num.	Opt. Time	Heu. Time	Opt. Gap
1	553.28	0.79	1.60 %
2	969.20	0.72	1.45 %
3	979.64	0.57	2.06 %
4	886.86	0.69	2.32 %
5	905.03	0.62	2.21 %
6	800.26	0.91	2.75 %
7	233.01	0.87	2.06 %
8	237.92	0.59	3.72 %
9	204.03	0.58	3.87 %
10	250.05	0.55	3.89 %
11	274.06	0.63	3.48 %
12	384.87	0.58	3.75 %

In Table 6.2, column "Opt. Time" represents the solution time of the CPLEX solver. Column "Heu. Time" represents the solution time of the heuristic algorithm. All the times are counted in seconds. Column "Opt. Gap" represents the relative percentage gap of the solution found by the heuristic compared with the optimal objective value.

From the numerical results for (VRPI–1) we notice first that there is generally a $1 - 4\%$ gap between our heuristic solution and the optimal solution. Our heuristic produces values that are closer to the optimal when the vehicles in our fleet has a smaller capacity. When vehicles have an increased capacity, there are more options when creating a longer route, thus making it more difficult for our heuristic algorithm to select the most optimal route. We also notice that in networks where large interdiction probabilities may appear (up to 20%), there is a larger optimality gap from the results of our algorithm. This is most likely explained by the fact that in selecting a less optimal starting route segment, large interdiction rates will be amplified as the route becomes longer.

6.3 Solution of (VRPI–2)

Experiments conducted for (VRPI–2) are similar to those of (VRPI–1). Note that we adjust the demands slightly so that the total supply can be less than the total demand. We set $d_i = U[50, 70]$ for instances with $Q = 50$ (Instances 1-6) and $d_i = U[90, 110]$ when $Q = 100$ (Instances 7-12).

In some cases of the (VRPI–2), we see multiple optimal solutions, so we added the following term $-\epsilon \sum_{r \in \Omega} c_r x_r$ in the objective function, as described in Section 3. ϵ set as 0.001.

Table 6.3: Computational Results for (VRPI–2)

Ins. Num.	Opt. Time	Heu. Time	Opt. Gap
1	611.548	4.057	0.00%
2	662.932	4.831	0.00%
3	479.906	4.975	0.00%
4	532.270	4.783	0.00%
5	686.542	4.994	0.03%
6	463.330	5.763	0.06%
7	537.890	5.697	0.01%
8	418.320	5.978	0.01%
9	414.344	6.003	0.10%
10	541.864	5.342	0.06%
11	536.809	4.872	0.11%
12	557.597	4.750	0.12%

Table 6.3 presents the computational results for (VRPI–2). In the first three instances (when the supply is much less than the demand), our heuristic algorithm is able to find the optimal solution in all generated configurations of the network. As the size of the supply in the network increases, we notice that there are some instances where the heuristic algorithm gives us a less optimal solution in a few of the 5 generated networks. Among all the instances, the optimality gap between the solution found by Algorithm **??** and the optimal solution never exceeds 0.2%, which demonstrates that the truncation of routes in Algorithm **??** tend not to dispose the

optimal solution. Additionally, the time to compute the optimal solution is approximately 100 times slower than the time to compute the heuristic solution and there is little difference between the computational time between any two instances of the (VRPI–2).

Further experimental results are presented in Appendix A

Chapter 7

Conclusions

This paper establishes the general framework for the VRPI. The proposed models are an extension of SDVRP with applications in humanitarian relief or military operations. These models are used in the vehicle routing planning phase and we are concerned of the long term average performance of a single routing plan. Thus, in a single delivery period, it is possible for us to oversupply nodes in a network. The amount of additional stock from the oversupply could be used as a safety stock to account for uncertainties in future deliveries due to interdiction.

Although the VRPI is built upon the SDVRP, they do not share the same properties. For example, in the SDVRP, there is always an optimal solution where routes can have at most one shared demand node, whereas in the VRPI, this property is relaxed, so routes can have multiple shared demand nodes. However, it is still true in the VRPI that there is an optimal solution where only one of these routes will deliver to multiple nodes. The properties exhibited by the VRPI give us key insights into the structure of optimal routes in these problems. In the VRPI, it was necessary to introduce the concept of a detour node and this is observed in the optimal solution of many experimental results. We further introduced a two-phase greedy heuristic algorithm, based on GRASP to solve small instances of the VRPI. Compared with the commercial solver CPLEX, we found that our algorithm perform much more efficiently within 4% optimality gap in (VRPI–1) and within $1\% \backsim 4\%$ optimality gap in (VRPI–2).

One direction for future research for the VRPI includes analysis of the optimal trade-off between cost and interdiction probabilities. This will give us more insight on how to make an optimal problem reduction in larger cases. Additionally, exact algorithms based on cutting plane and branch-and-price could be considered. Finally, in this paper, we have only investigated the static models of the VRPI. A dynamic version of the VRPI is a good direction for future work. Such a version would involve transmitting routing information in an ongoing manner and rerouting vehicles as interdiction occurs.

Appendix A

Results for (VRPI–1)

The experiments in Table A.1 consist of 5 demand nodes and have a uniformly demand range of $20 - 50$. The experiments in Table A.2 consist of 6 demand nodes and have a uniformly demand range of $20 - 50$.

Table A.1: Additional Results for (VRPI–1), $n = 5$

Q	K	Int. Prob.	Opt. Time	Heu. Time	Opt. Gap
200	3	U[0.01,0.05]	0.385	0.115	0.66%
200	3	U[0.01,0.10]	0.454	0.099	1.23%
200	3	U[0.01,0.20]	0.601	0.147	2.40%
200	3	U[0.01,0.40]	0.855	0.083	0.19%
100	4	U[0.01,0.05]	1.467	0.182	0.22%
100	4	U[0.01,0.10]	1.746	0.178	2.75%
100	4	U[0.01,0.20]	1.937	0.115	0.51%
100	4	U[0.01,0.40]	1.438	0.137	2.72%
60	6	U[0.01,0.05]	0.115	0.085	1.95%
60	6	U[0.01,0.10]	0.116	0.069	4.28%
60	6	U[0.01,0.20]	0.169	0.084	4.36%
60	6	U[0.01,0.40]	0.285	0.069	3.59%

Table A.2: Additional Results for (VRPI–1), $n = 6$

Q	K	Int. Prob.	Opt. Time	Heu. Time	Opt. Gap
200	3	U[0.01,0.05]	2.442	0.147	2.30%
200	3	U[0.01,0.10]	4.667	0.131	6.01%
200	3	U[0.01,0.20]	4.864	0.153	6.21%
200	3	U[0.01,0.40]	3.969	0.116	5.84%
100	4	U[0.01,0.05]	11.792	0.354	4.43%
100	4	U[0.01,0.10]	11.717	0.163	4.39%
100	4	U[0.01,0.20]	12.422	0.131	1.91%
100	4	U[0.01,0.40]	26.808	0.154	0.90%
60	6	U[0.01,0.05]	42.089	0.316	2.13%
60	6	U[0.01,0.10]	85.600	0.347	1.92%
60	6	U[0.01,0.20]	53.104	0.147	4.22%
60	6	U[0.01,0.40]	39.532	0.104	5.37%

Appendix B

Results for (VRPI–2)

The experiments in Table B.3 consist of 5 demand nodes. Rows 1-4 used instances where networks had a demand of 40-100 and the rest used demands 80-100. The experiments in Table B.4 consist of 6 demand nodes. Rows 1-4 used instances where networks had a demand of 40-100 and the rest used demands 80-100.

Table B.3: Additional Results for (VRPI–2), $n = 5$

Q	K	Int. Prob.	Opt. Time	Heu. Time	Opt. Gap
50	4	U[0.01,0.05]	2.848	0.717	0.11%
50	4	U[0.01,0.10]	3.426	0.785	0.02%
50	4	U[0.01,0.20]	4.301	0.669	0.01%
50	4	U[0.01,0.40]	2.143	0.668	0.11%
50	4	U[0.01,0.05]	3.147	0.686	0.19%
50	4	U[0.01,0.10]	3.381	0.716	0.00%
50	4	U[0.01,0.20]	4.779	0.702	0.04%
50	4	U[0.01,0.40]	1.383	0.647	0.06%
120	4	U[0.01,0.05]	2.395	0.653	0.82%
120	4	U[0.01,0.10]	5.895	0.820	0.08%
120	4	U[0.01,0.20]	2.555	0.769	0.09%
120	4	U[0.01,0.40]	4.721	0.769	0.85%

Table B.4: Additional Results for (VRPI–2), $n = 6$

Q	K	Int. Prob.	Opt. Time	Heu. Time	Opt. Gap
50	4	U[0.01,0.05]	28.547	0.815	0.89%
50	4	U[0.01,0.10]	23.478	0.848	0.03%
50	4	U[0.01,0.20]	18.882	0.765	0.06%
50	4	U[0.01,0.40]	21.882	0.762	0.14%
50	4	U[0.01,0.05]	14.860	0.756	0.02%
50	4	U[0.01,0.10]	16.276	0.732	0.02%
50	4	U[0.01,0.20]	24.941	0.816	0.10%
50	4	U[0.01,0.40]	29.679	0.785	0.06%
120	4	U[0.01,0.05]	37.654	0.818	0.02%
120	4	U[0.01,0.10]	13.345	0.779	0.03%
120	4	U[0.01,0.20]	17.068	0.838	0.16%
120	4	U[0.01,0.40]	11.617	0.732	0.00%

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