Alternating Virtual Knots

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By
HOMAYUN KARIMI, M.SC.

A Thesis
Submitted to the school of Graduate Studies
in Partial Fulfilment of the Requirements
for the Degree
of Doctor of Philosophy

McMaster University

DOCTOR OF PHILOSOPHY (2018)	McMaster University
(Mathematics)	Hamilton, Ontario

TITLE:	Alternating Virtual Knots
AUTHOR:	Homayun Karimi, B.Sc. (Amirkabir University), M.Sc. (McMaster University)
SUPERVISOR:	Dr. Hans U. Boden
NUMBER OF PAGES:	1, 103

Abstract

In this thesis, we study alternating virtual knots. We show the Alexander polynomial of an almost classical alternating knot is alternating. We give a characterization theorem for alternating knots in terms of Goeritz matrices.

We prove any reduced alternating diagram has minimal genus, and use this to prove the first Tait Conjecture for virtual knots, namely any reduced diagram of an alternating virtual knot has minimal crossing number.

Acknowledgements

I would like to express my gratitude to my supervisor Prof. Hans U. Boden. He walked me through many different areas of Knot Theory and Low Dimensional Topology. The amount of support and encouragement which I have received from him is beyond words.

I would like to thank my parents for their support, Dr. Ian Hambleton and Dr. Andrew Nicas for the wonderful courses and being members of my Supervisory Committee, David Duncan for helping me in learning the basics of gauge theory, Lindsay White for providing TEXtemplate and computations of Alexander and Jones polynomials, and Amanda Tost for her encouragement during the final year.

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Chapter 1

Introduction

In knot theory, the objects are knots, (links, and other knotted objects such as braids, graphs, doodles, virtual knots, etc.), and the goal is to develop a precise and useful mathematical understanding of these objects. The key problems are, how can we best represent a knot (or link, or braid, etc.) mathematically? When are two representatives equivalent as knots (or links, or braids, etc.), and how to tell when they are different? This is the classification problem, and here invariants play the central role. The early pioneers such as P.G. Tait and C.N. Little studied the classification problem for knots without the aid of knot invariants, and in retrospect it is amazing that they were able to produce a tabulation of knots up to 10 crossings which effectively solved the classification problem for low-crossing knots without access to such technology. Of course these early results were empirical, being based on the many examples and the mathematical intuition they had gleaned. It was only through the later work of M. Dehn, H. Seifert, J. Alexander, and K. Reidemeister that their results were placed on a sound mathematical footing. For instance, Alexander's polynomial is remarkably strong for low-crossing knots, and it distinguishes a great many knots in these early tabulations.

Nevertheless, despite this early success and the many spectacular breakthroughs in knot theory that have occurred since then, the classification problem for knots remains an open and active area of research. In the late 1980s, the advent of the Jones polynomial enabled powerful and elegant solutions to the classification problem for alternating knots. Here, a knot is called *alternating* if it admits a diagram whose crossings alternate between over and under crossing as one travels around the knot. Up to seven crossings, all knots are alternating, and the first non-alternating knot in the table is 8_{19}. As a result, there are relatively few non-alternating knots of low-crossing number. Not surprisingly, in producing his first tabulations of knots, Tait was largely concerned with the classification problem for alternating knots.

This would appear to be an easier problem, since it concerns a subset of all knots. However, it is not immediately apparent whether a given knot is a member of this subset. For instance, it is obvious that the standard diagram for 8_{19} is not alternating, but it is at all not obvious that 8_{19} cannot be represented by any alternating diagram. Knot invariants help to address this question, but as mentioned, Tait did not have the benefit of invariants. Instead, he formulated three far-reaching conjectures which, when proved 100 years later, greatly facilitated the classification of alternating knots. The first two conjectures were settled through deep results of Kauffman, Murasugi, and Thistlethwaite, each giving an independent argument involving the newly discovered Jones polynomial [Kau87, Mur87a, Thi87]. Tait's third conjecture was solved a few years later in a joint paper by Menasco and Thistlethwaite [MT93]. Details on the Tait conjectures for classical knots will be given in Section 2.3, for now we note that collectively, the results of Kauffman, Menasco, Murasugi, and Thistlethwaite in [Men84, Kau87, Mur87a, Thi87, MT93] provide a strategy for classifying alternating knots and links. Indeed, this problem is much more tractable than the harder problem of classification of knots in general. For instance, the latter problem has been achieved for prime knots by computational methods up to 16 crossings in [HTW98], whereas the former has been solved up to 23 crossings, see the srticles [RFS04a, RFS04b] and the online program [RF06].

The Tait conjectures further suggest that alternating knots can be understood in terms of one suitably chosen representative diagram. For instance, for prime alternating knots, any minimal crossing diagram is necessarily alternating, and any two such diagrams representing the same knot are related by a sequence of flypes. Furthermore, a reduced alternating diagram represents a composite knot if and only if the diagram is visibly composite [Men84]. Consequently, for alternating knots, the crossing number is additive under connected sum, i.e., it satisfies $c(K \# J) = c(K) + c(J)$. This formula is conjectured to hold for all knots, but the general case is a difficult open problem that serves to illustrate how much more we know about alternating knots.

The resolution of the Tait conjectures, especially the third conjecture [MT91b, MT93], provide an algorithm for distinguishing alternating knots and links. Indeed, given a collection of alternating knot and/or link diagrams, one can use the results to determine which of the diagrams are prime, which are reduced, and ultimately which among the prime and reduced ones are equivalent.

What these results do not provide is a means for generating a complete collection of all alternating knots diagrams. We call that the problem of "generation". This is a combinatorial problem, and in fact it can be rephrased entirely as a problem about graphs using the following well-known correspondence.

Every classical knot diagram is checkerboard colorable, and a coloring determines a graph, called the *Tait graph*. The Tait graph has one vertex for each black region, and edges between vertices whenever the corresponding black regions meet at a crossing. Obviously, performing a crossing change has no effect on the Tait graph, thus the map from knot diagrams to Tait graphs is not faithful. However, it becomes faithful if one restricts attention to alternating knots, (see [Mur96] for more details). The same holds for alternating links, and in this way we see that the generation problem can be recast as a problem about graphs. This problem has been well-studied, and in fact an inductive scheme for generating all alternating knots (and links!) is contained in the papers [RFS04a, RFS04b, RF04].

Virtual knots were introduced by Kauffman in [Kau99], and they grew out of the study of quantum topology and finite type invariants of knots. They represent a natural generalization of classical knots to knots in thickened surfaces. In fact it is well known that any two classical knots that are equivalent as virtual knots are in fact equivalent as classical knots, thus classical knot theory embeds faithfully into virtual knot theory. This was first proved by Goussarov, Polyak, and Viro [GPV00], but it is also an immediate consequence of Kuperberg's theorem [Kup03]. Not surprisingly, many of the standard invariants of classical knots extend in a natural way to virtual knots, but in many cases, the extensions are unsatisfactory. For instance, for classical knots, the knot group G_K and its peripheral structure give a complete invariant [Wal68], whereas there are many nontrivial virtual knots whose knot group and peripheral structure are "trivial," meaning that $G_K \cong \mathbb{Z}$. In a similar way, the Jones polynomial $V_K(t)$ extends to virtual knots in a completely straightforward way, but again we find the resulting invariant is not nearly as powerful in the virtual setting as it was for classical knots. For instance, it is simple to construct nontrivial virtual knots with trivial Jones polynomial, whereas for classical knots it is an open problem whether the only knot with trivial Jones polynomial is the unknot. (Among alternating classical knots, the Jones polynomial is an unknot detector!)

We can ask about the generation problem for alternating virtual knots. Here we describe a way to generate all alternating virtual knot diagrams (as Gauss diagrams): Take $2n$ points and place them on a core circle, and then number them $1, 2, 3, \cdots, 2n$ as we go counterclockwise around the circle. Now we draw arrows, starting at the odd points and ending at the even points. This produces an "alternating pattern," and any choice of signs for the chords gives us an alternating virtual knot.

The pattern is determined by a special kind of partition of the set $\{1, 2, \cdots, 2n\}$, namely one with exactly n subsets, each of which contains 2 elements, one even and the other odd. Of course, not all the diagrams are unique, and note that we need to mod out by rotational symmetry.

In this thesis, we study properties of alternating virtual knots, motivated in part by problems 15 and 16 in [FIKM14]. View a virtual knot as a knot in a thickened surface, taken up to diffeomorphism and stablization of the surface. A knot diagram on a surface is called *alternating* if its crossings alternate between over and under as one travels around the knot, and a virtual knot is called alternating if it can be represented by an alternating knot diagram on a thickened surface.

Clearly with this definition, any knot that is alternating as a classical knot remains alternating when viewed as a virtual knot, but it is not immediately clear whether the converse is true.

Question 1.1. If K is non-alternating as a classical knot, does it remain non-alternating as a virtual knot?

Put another way, does there exist a classical knot which is not alternating but which admits an alternating virtual knot diagram?

Apart from the Tait Conjectures, there are numerous results about invariants of classical knots that are useful in recognizing whether or not a given knot admits an alternating diagram. That is because many classical invariants take a very special form on alternating knots. For example, Murasugi proved that the Alexander polynomial of an alternating knot has non-zero coefficients which are alternating in sign. Thus, any knot whose Alexander polynomial does not have this form cannot admit an alternating diagram.

The following theorem gives an analogue of Murasugi's result for alternating virtual knots which are almost classical (see Definition 3.10).

Theorem 1.2. *If L is an almost classical alternating link with a connected alternating diagram D, then its Alexander polynomial $\Delta_L(t)$ is alternating.*

The Khovanov homology is a very powerful invariant for classical knots, and in [Lee05], Lee provides a structure theorem for the Khovanov homology of alternating links. Given an alternating link L with signature σ, she proves that its Khovanov homology is supported in the two lines $j = 2i - \sigma \pm 1$, where i and j are homological and quantum degrees, respectively. In this thesis, we establish the following generalization of her result for alternating virtual links:

Proposition 1.3. *If D is a connected alternating virtual link diagram with genus g, and signatures $\sigma_\xi, \sigma_{\xi^*}$, then its Khovanov homology is supported in the $g+2$ lines:*

$$j = 2i - \sigma_{\xi^*} + 1, j = 2i - \sigma_{\xi^*} - 1, \dots, j = 2i - \sigma_\xi - 1.$$

One central problem in knot theory is to characterize alternating knots in a useful or effective way. The solution is to give necessary and sufficient conditions for a given knot to admit an alternating diagram. In [Gre17], Greene provides an answer to this question in terms of spanning surfaces, and thus he characterizes alternating classical knots as those admitting positive and negative definite spanning surfaces. We prove an analogue of Greene's theorem, summarized in the next theorem and giving necessary and sufficient conditions for a virtual knot to admit an alternating virtual knot diagram.

Theorem 1.4. *Suppose K is a connected checkerboard colorable virtual link with virtual genus $g_v(K)$. Then D is an alternating diagram for K if and only if D admits a checkerboard coloring ξ with dual coloring ξ^* such that*

1) $sg(D) = g_v(K)$,

2) $\sigma_\xi(D) - \sigma_{\xi^*}(D) = 2g_v(K)$,

3) $G_\xi(D)$ is negative definite (or empty) and $G_{\xi^*}(D)$ is positive definite (or empty), where G_ξ and G_{ξ^*} refer to the Goeritz matrices associated to the checkerboard colorings ξ and ξ^*, respectively (cf. Section 4.1).

Sometimes in virtual knot theory, one can define invariants which have no analogue in the classical case or are trivial on all classical knots. One example is the virtual genus of a knot, which is zero for all classical knots. Using homological parity projection, it follows that for alternating virtual knots, this invariant can be computed from any reduced alternating diagram.

Theorem 1.5. *Suppose D is a reduced alternating virtual knot diagram for a virtual knot K. Then D is a minimal genus diagram.*

As a consequence of this, in Corollary 5.29 we address Question 1.1 by showing that any classical knot which admits an alternating virtual knot diagram is necessarily alternating as a classical knot.

Finally we have the Tait Conjectures for virtual knots. Unfortunately, the Jones polynomial is not strong enough to give a virtual analogue of the first Tait Conjecture. Nevertheless using the previous theorem and a result of Adams et al. ([AFLT02, Theorem 1.1]), we obtain a positive solution to the first Tait Conjecture for virtual knots, as follows:

Theorem 1.6. *Let D be a reduced alternating knot diagram for a virtual knot K. Then D has minimal crossing number.*

The outline of this thesis is as follows: In Chapter 2 we briefly introduce classical knot theory and some of classical knot invariants including the Jones and Alexander polynomials, knot signature and Gordon and Litherland's method for computing the signature. Then we discuss alternating classical knots, state the Tait Conjectures, and outline a proof of the first and second Tait Conjectures.

In Chapter 3, we introduce virtual knot theory, various ways of describing them and some of their invariants. Then we discuss the checkerboard colorable knots, different ways of describing this notion and why all these descriptions are equivalent to each other. At the end we introduce parity projection, which we will use later in the proof of Theorem 1.5.

Although the notion of signature has been defined for checkerboard knots, its behaviour under operations on knot diagrams has not been fully explored. In Chapter 4, we describe the signatures for virtual knots and study the effect of taking mirror images and other operations on knot diagrams.

In Chapter 5, first we use the classical theorem of Bott and Mayberry to prove Theorem 1.2, and the weak form of the first Tait Conjecture for alternating virtual links. Then we state and prove Theorems 1.4, 1.5 and 1.6. We will also apply Theorem 1.4 to decide which almost classical knots up to 6 crossings are alternating.

We devote Chapter 6 in its entirety to Khovanov homology. In the first section, we discuss Khovanov homology and Rasmussen's invariant. Then we describe Tubbenhauer's approach to define Khovanov homology for virtual knots. We give one example of Khovanov homology of a virtual knot computed by this method. Then we prove Theorem 1.3.

We list some open problems related to alternating virtual knots in Chapter 7, and at the end in the appendices we list the computations for some of the invariants of alternating virtual knots up to six crossings.

Chapter 2

Classical Knot Theory

In this chapter, we present a brief introduction to the mathematical theory of knots and links. We recall definitions of classical invariants, including the knot group, Alexander module, Jones polynomial, and Seifert invariants. We also present the Tait conjectures for alternating knots and give a survey of the celebrated results of Kauffman, Murasugi, and Thistlethwaite.

2.1 Knots, Links and their Invariants

In this section, we review the basic notions for classical knots and links and introduce their invariants. We begin with a review of the Kauffman bracket and Jones polynomial. We then recall the knot group and Alexander module, as well as invariants derived from Seifert surfaces, including the determinant, signature, and nullity. We also review Gordon and Litherland's method of computing signatures of knots and links in terms of spanning surfaces and Goeritz matrices.

Classical knot theory studies the embeddings of S^1 in S^3.

Definition 2.1. A *link* L of m components is a subset of S^3 consisting of m disjoint, piecewise linear, simple closed curves. A link of one component is called a *knot*.

Definition 2.2. Two links L_1 and L_2 in S^3 are *equivalent* if there is an orientation preserving, piecewise linear homeomorphism $h : S^3 \to S^3$ such that $h(L_1) = L_2$.

We can project a link onto a plane. For each link component, we have a closed curve in the projection. In general, these curves have intersection points, but we can arrange the intersection points to be transverse double points. This is called a *regular projection*. To each intersection point, we record the extra information of which arc is above and which is below. This is called a *link diagram*.

For such diagrams, there are three moves, called *Reidemeister moves*. Two links in S^3 are equivalent if and only if their associated diagrams are related by a sequence of Reidemeister moves and planar isotopy.

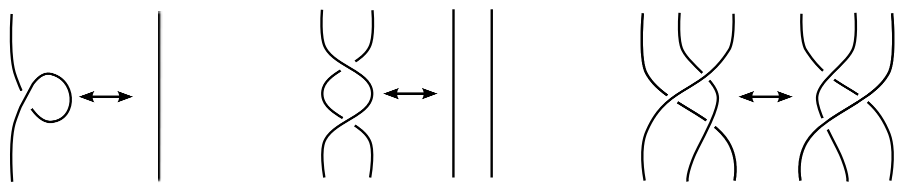

Figure 2.1: The classical Reidemeister moves.

Definition 2.3. A link $L \subset S^3$ with at least two components is called *split* if there is a 2-sphere in $S^3 \setminus L$ separating S^3 into two balls, each of which contains a component of L. A link diagram D in the plane is a *split diagram* if there is a simple closed curve in $\mathbb{R}^2 \setminus D$ separating $\mathbb{R}^2$ into two regions, each containing part of D, i.e. the diagram is disconnected. A link is split if and only if it admits a split diagram.

An *invariant* for links is a function which assigns an algebraic object (a number, a group, etc) to every link. This object should be "the same" for two equivalent links (equal numbers, isomorphic groups, etc). An example of an invariant for knots is the *crossing number*, defined as follows.

Definition 2.4. The minimal number of crossings taken over all diagrams of a given knot or link K is called the crossing number of K.

Using invariants, we can distinguish different links. In fact the only way to show two links are different is to find an invariant which assigns different objects to them. There are many different invariants for links.

Some of these invariants are very coarse. For example the Arf invariant, only assigns 0 or 1 to a knot. Some of them can be defined, only for knots. Some give us more information about knots and links. One way to test the strength of an invariant is to see if it detects the unknot. This means, if the invariant for a knot is the same as it is for the unknot, then that knot is equivalent to the unknot.

For example, a deep result of Kronheimer and Mrowka shows that Khovanov homology is an unknot detector [KM11]. It is an open problem whether there exists a nontrivial knot K whose Jones polynomial is trivial, i.e. whether the Jones polynomial is an unknot detector.

The Kauffman bracket and Jones polynomial

Let D be a diagram for a link L. For each crossing, there are two ways to resolve that crossing. One is called the 0-*smoothing*, and the other is the 1-*smoothing*, according to the following picture.

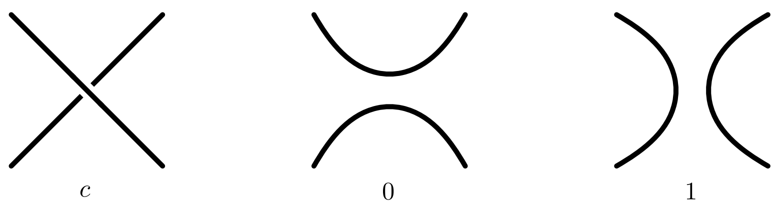

Figure 2.2: The 0- and 1-smoothing of a crossing.

Resolving all the crossings of D in both ways, we obtain 2^n states, and each state is a link diagram with no crossings. If s is a state, we denote by $|s|$, the number of cycles in this state. For a state s, let $i(s)$ be the number of 0-smoothings minus the number of 1-smoothings. Then *Kauffman bracket* is defined as follows:

$$\langle D \rangle = \sum_s \left(A^{i(s)}(-A^{-2} - A^2)^{|s|-1} \right),$$

with the sum taken over all states. One can show that the Kauffman bracket is invariant under Reidemeister two and three moves.

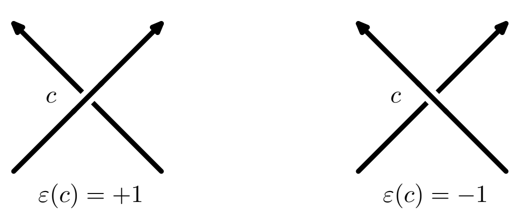

Figure 2.3: A positive and a negative crossing.

Definition 2.5. For a link diagram D, the *writhe* of D, denoted $w(D)$ is defined as $n_+(D) - n_-(D)$, where $n_+(D)$ and $n_-(D)$ are the number of positive and negative crossings in D, respectively.

The writhe $w(D)$ depends on the diagram and does not give a well-defined invariant of the underlying knot or link.

Definition 2.6. For a link L with diagram D, the Jones polynomial of L is given by

$$V_L(t) = (-A)^{-3w(D)} \langle D \rangle \big|_{t^{1/2}=A^{-2}} .$$

This normalization is chosen so that the right hand side of the above equation is invariant under all three Reidemeister moves. It follows that the polynomial $V_L(t)$ is independent of the diagram used.

Remark 2.7. If the link L has an odd number of components, $V_L(t)$ is a Laurent polynomial over the integers. If the number of components is even, $V_L(t)$ is $t^{1/2}$ times a Laurent polynomial (see [Jon85, Theorem 2]).

Definition 2.8. Given a Laurent polynomial P in one variable, let $M(P)$ denote its maximum degree and $m(P)$ its minimum degree. The *span* of P is defined to be the difference between the maximum and minimum degrees, i.e. $\operatorname{span} P(t) = M(P) - m(P)$.

Notice that if L has an even number of components, then by Remark 2.7, $t^{-1/2}V_L(t)$ is a Laurent polynomial and $\operatorname{span} V_L(t)$ means the span of $t^{-1/2}V_L(t)$.

The Knot Group and Alexander Module

Another important invariant of knots and links is the knot group, which is defined as the fundamental group of the *complement* of the knot or link. For a knot or a link K, this group is denoted G_K. Thus, $G_K = \pi_1(X_K)$ where X_K is the result of removing an open tubular neighborhood of K from S^3.

Here we describe the *Wirtinger presentation* of G_K. Let K be a knot or a link in S^3, and let D be a regular projection of it with n crossings. Enumerate the arcs of D by $x_1, \dots, x_m$ and the crossings by $c_1, \dots, c_n$.

For each crossing, labelled as in Figure 2.4, we have the relation $r_i = x_{\lambda(i)}x_{\nu(i)}^{-1}x_{\rho(i)}^{-1}x_{\nu(i)}$. The Wirtinger presentation for G_K is as follows.

$$G_K = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle.$$

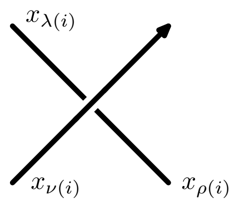

Figure 2.4: Arc labels at a crossing.

Here we are using the right-handed meridian convention. Notice that starting with a regular projection of a link K, we can find a link which is equivalent to K. Consider a small neighborhood of each crossing in the projection plane, and let v be a normal vector to the plane. Push the over-crossing arc into S^3 in the direction of v. We fix a base-point above the projection plane, i.e. in the v direction. Choose m loops, denoted $x_1, \ldots, x_m$, and suppose x_i goes under the i-th arc in a direction which forms a positive crossing. If we use the other direction, the convention is called left-handed meridian, and it leads to another presentation for G_K, which is isomorphic to the former. For this convention we have $r_i = x_{\lambda(i)} x_{\nu(i)} x_{\rho(i)}^{-1} x_{\nu(i)}^{-1}$.

In order to define the *Alexander module*, we briefly recall *Fox differentiation*. Let F_m be the free group on m generators, so elements of F_m are words in $x_1, \ldots, x_m$. For $j = 1, \ldots, m$, the Fox derivative $\partial/\partial x_j$ is an endomorphism of $\mathbb{Z}[F_m]$, the group ring, defined so that $\partial/\partial x_j(1) = 0$ and

$$\frac{\partial}{\partial x_j}(x_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Further, given words $w, z \in F_m$, the Fox derivative satisfies the Leibnitz rule:

$$\frac{\partial}{\partial x_j}(wz) = \frac{\partial}{\partial x_j}(w) + w \frac{\partial}{\partial x_j}(z).$$

These relations completely determine $\partial/\partial x_j$ on every word $w \in F_m$, and it is extended linearly to the group ring $\mathbb{Z}[F_m]$.

We use this to describe the construction of the Alexander module associated to a knot or link K. Let $G'_K = [G_K, G_K]$ and $G''_K = [G'_K, G'_K]$ be the first and second commutator subgroups, then the Alexander module is the quotient G'_K/G''_K. It is a finitely generated module over $\mathbb{Z}[t, t^{-1}]$, the ring of Laurent polynomials, and it is determined by the *Fox Jacobian matrix* A as follows. Here,

A is the $n \times m$ matrix with ij entry equal to $\frac{\partial r_i}{\partial x_j}\Big|_{x_1,\ldots,x_m=t}$. In particular, the Fox Jacobian is obtained by Fox differentiating the relations r_i with respect to the generators x_j and applying the abelianization map $x_j \mapsto t$ for $j = 1, \ldots, m$. We define the k-th *elementary ideal* $\mathcal{E}_k$ as the ideal of $\mathbb{Z}[t, t^{-1}]$ generated by all $(n-k) \times (n-k)$ minors of A.

The matrix A depends on the choice of a presentation for G_K, but the associated sequence of elementary ideals

$$\{0\} = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_n = \mathbb{Z}[t, t^{-1}]$$

does not.

For any knot or link K, the first elementary ideal $\mathcal{E}_1$ is a principal ideal, and the Alexander polynomial $\Delta_K(t)$ is defined as the generator of $\mathcal{E}_1$. The Alexander polynomial is well-defined up to multiplication by $\pm t^k$ for $k \in \mathbb{Z}$. It is obtained by taking the determinant of the *Alexander matrix*, which is the $(n-1) \times (n-1)$ matrix obtained by removing a row and column from A.

Seifert Invariants

Many useful invariants of knots and links can be defined in terms of *Seifert surfaces*. A Seifert surface for a link K is an orientable, compact, connected surface with boundary K. This allows us to define an invariant called the *Seifert genus* of K, denoted $g(K)$, to be the minimum genus over all the compact orientable surfaces in S^3 which cobound K.

One way to construct a Seifert surface is to apply *Seifert's algorithm* to a diagram for K. We take an oriented projection of the link, cut each crossing open in the manner that preserves orientation, attach disks to each resulting circle (the so-called *Seifert circuits*), and connect the disks with half-twisted bands at each crossing.

An orientable surface has trivial normal bundle (and trivial $[-1, 1]$-bundle) in S^3. Since the first Stiefel-Whitney class of the surface is zero, there is no obstruction to trivialize the normal bundle. Using the bicollaring $S \times [-1, 1]$ for S, and we can define the positive and negative push-offs for curves in S. We identify S with $S \times \{0\}$. For a curve in the surface, the positive push-off is a parallel copy in $S \times \{1\}$, and the negative push-off is the parallel copy in $S \times \{-1\}$, defined similarly.

The *Seifert pairing* is defined on $H_1(S) \times H_1(S)$, and takes (a, b) to $lk(a, b^+)$, where b^+ is the positive push-off of b, and lk denotes the linking number. Choose a basis for $H_1(S)$, and let V be the matrix of the pairing with respect to this basis. We call V a *Seifert matrix*.

Definition 2.9. Let L be a link with a connected Seifert surface S and Seifert matrix V according to a choice of basis for $H_1(S)$.
(i) The determinant $\det(V - tV^\tau)$ is equal to the Alexander polynomial defined earlier, thus setting $\Delta_L(t) = \det(V - tV^\tau)$ gives an alternative definition of the *Alexander polynomial*.
(ii) The absolute value of the Alexander polynomial at -1 is called the *determinant* of L:

$$\det(L) = |\Delta_L(-1)| = |\det(V + V^\tau)|.$$

Since the matrix $V + V^\tau$ is symmetric, it has real eigenvalues, and we define its signature, $\mathrm{sig}(V + V^\tau)$, to be the number of positive eigenvalues minus the number of negative eigenvalues. We also define its nullity to be the dimension of its kernel.

Definition 2.10. **(i)** Given an oriented link L, the *signature* of L is defined to be the signature of the matrix $V + V^\tau$ and is denoted $\sigma(L)$.
(ii) The *nullity* of L is defined to be $\mathrm{nullity}(V + V^\tau)$ and is denoted $N(L)$.

Note that every classical knot K has $\det(K)$ an odd integer, thus the nullity $N(K) = 0$. In that case, the signature $\sigma(K)$ is an invariant of knot concordance, which is introduced next.

Knot Concordance

Here we introduce the notion of smooth concordance for knots. Recall that an annulus is a 2-manifold A homormorphic to $S^1 \times [0, 1]$.

Definition 2.11. Two knots K_0 and K_1 are called *smoothly concordant* if there is a smoothly embedded annulus $A \to S^3 \times [0, 1]$ whose boundary is $-K_0 \times \{0\} \sqcup K_1 \times \{1\}$. A knot is *smoothly slice* if it is concordant to the unknot.

Notice that the Euler characteristic of the annulus $S^1 \times [0, 1]$ is zero (a genus zero surface with two boundary components). We can allow the surface to have a higher genus, and in that case, K_0 and K_1 are called *cobordant*. Any two knots are cobordant. To see this, take a Seifert surface for K_0 in $S^3 \times \{0\}$, and push the interior of it into the interior of $S^3 \times [0, 1]$, similarly for K_1 in $S^3 \times \{1\}$. The connected sum of the two surfaces provides a cobordism from K_0 to K_1.

When a knot K is slice, it bounds a smoothly embedded disk in D^4. We define the *slice genus* of K, denoted $g_4(K)$, to be the minimum genus of all the compact orientable surfaces in D^4 which bound K. It is clear that the slice genus $g_4(K)$ is less than or equal to the Seifert genus $g(K)$.

Similarly we say two oriented links L_0 and L_1 with m components are smoothly concordant if there is a smooth embedding of annuli $(\sqcup_{i=1}^m A_i \to S^3 \times [0,1]$ whose boundary is $-L_0 \times \{0\} \sqcup L_1 \times \{1\}$.

Any connected cobordism between two knots K_0 and K_1 can be described in terms of *elementary cobordisms*. In fact any connected cobordism S can be decomposed into union of a finite sequence of *births*, *deaths* and *saddles*.

A birth is a cobordism from the unknot to the empty set, a death is the cobordism from the empty set to the unknot. One can perform a saddle to arcs in a nontrivial knot or link. There are two types of saddles: (i) *joining* or *fusion type*, and (ii) *separating* or *fission type*. We can visualize a cobordism by a movie. We denote $S^3 \times \{t\}$ by S^3_t, then for a cobordism S, we can assume the height function $h : S \to [0,1]$ is Morse. For each t, $h^{-1}(t) = S^3_t \cap S$. At a regular value t, $h^{-1}(t)$ is a knot or link in S^3_t. Suppose S is a saddle between L_0 and L_1 with diagrams D_0 and D_1 respectively. To describe the saddle in terms of the movie, we choose two parallel arcs in D_0 and connect them by attaching a band as in Figure 2.5, this is called a *band move*.

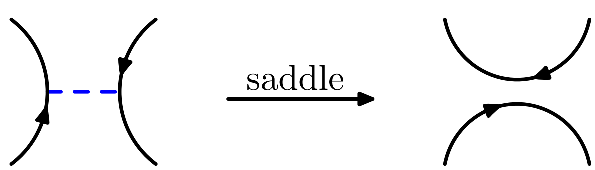

Figure 2.5: A band move.

The Euler characteristic of a birth and a death is 1, and for the saddle, the Euler characteristic is -1. If S decomposes into b births, d deaths and s saddles, then $\chi(S) = b + d - s$. A cobordism S is a concordance if and only if $b + d = s$.

Spanning Surfaces and the Goeritz Matrix

In general, a *spanning surface* for L is an unoriented (and possibly non-orientable) compact, connected surface which cobounds L. Similar to Seifert's algorithm, we describe a way to construct spanning surfaces for a given link L.

Project the link L onto the surface of S^2, to obtain a projection D. Without specifying the over and under crossing information, D is just a 4-valent graph, and it divides S^2 into regions. A coloring of D is an assignment of two colors (black and white) to these regions, in a checkerboard manner, i.e. if two regions share an edge, they should have different colors. It is a well-known fact that any classical link diagram admits a checkerboard coloring, and in Section 3.3, we explain this from the point of view of virtual knot theory.

Now connect the black regions by half twisted bands at each crossing, the result is a spanning surface for L, which we call it the *black surface*. Similarly, we can define the *white surface*.

Definition 2.12. A diagram D for a knot K is called a *special diagram* if it admits a checkerboard coloring such that the black surface is oriented.

Every knot admits a special diagram. See Proposition 13.15 in [BZH14].

There is a convenient way to use spanning surfaces to compute the signature of a link introduced by Gordon and Litherland in [GL78].

For a crossing c, we have the following pictures:

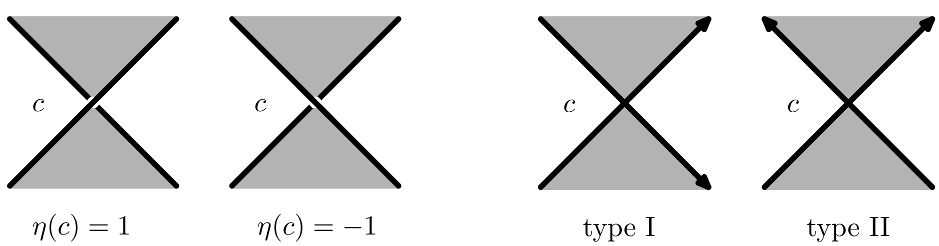

Figure 2.6: A type A and a type B crossing (left) and a type I and a type II crossing (right).

They are a type A crossing with $\eta(c) = +1$ and a type B crossing with $\eta(c) = -1$ crossing in a colored link diagram, and a type I and a type II crossing in an oriented, colored link diagram, respectively. We call $\eta(c)$ the *incidence number* of the crossing c. Thus, to each crossing, we have three binary quantities: the sign $\varepsilon(c)$, incidence number $\eta(c)$, and type. The next lemma implies that any two of these quantities determine the third.

Lemma 2.13. *For a crossing c, let $\varepsilon(c)$ be the sign of the crossing, and $\eta(c)$ be the incidence number. If $\varepsilon(c)\eta(c) = +1$, then the crossing is of type II, otherwise it is of type I.*

Proof. This Lemma follows from the Table 2.1. □

$\varepsilon(c)$	$\eta(c)$	type
$+$	$+1$	II
$+$	-1	I
$-$	$+1$	I
$-$	-1	II

Table 2.1: Table of sign, incidence number and type for a crossing c.

We enumerate the white regions of $S^2 \smallsetminus |D|$ by $X_0, X_1, \ldots, X_m$. Let $C(D)$ denote the set of all crossings of D. For each pair $i, j \in \{1, 2, \ldots, m\}$, let

$$C_{ij}(D) = \{c \in C(D) \mid c \text{ is adjacent to both } X_i \text{ and } X_j\}$$

and define

$$g_{ij} = \begin{cases} -\sum\limits_{c \in C_{ij}(D)} \eta(c), & \text{for } i \neq j, \\ -\sum\limits_{k=0; k \neq i}^{m} g_{ik}, & \text{for } i = j. \end{cases}$$

The pre-Goeritz matrix of D is defined to be the symmetric integral matrix $G'(D) = (g_{ij})_{0 \leq i,j \leq m}$, and the Goeritz matrix of D is denoted $G(D)$ and defined to be the principal minor $(g_{ij})_{1 \leq i,j \leq m}$ obtained by removing the first row and column from $G'(D)$.

In [GL78], Gordon and Litherland defined the correction term

$$\mu(D) = \sum_{c \text{ is type II}} \eta(c)$$

and established a formula for the signature of the link in terms of the signature of the Goeritz matrix and the correction term:

$$\sigma(L) = \operatorname{sig}(G(D)) - \mu(D).$$

2.2 Alternating Knots

In this section, we review classical results of Crowell, Murasugi, and Thistlethwaite on the Alexander and Jones polynomials of alternating knots and links.

We begin by recalling the definition of alternating knots and links.

Definition 2.14. A diagram D for a link L is *alternating* if, when traveling along each component, the over and under crossings alternate. A link L is *alternating* if it admits an alternating diagram.

The main question, due to Ralph Fox, is “What is an alternating knot?” This is a question about how to characterize alternating knots, and in [Gre17], Greene gives a beautiful answer in terms of spanning surfaces. Greene’s result provides a topological characterization of alternating knots, and in Section 5.2, we present a virtual analogue of Greene’s characterization for virtual alternating knots. There is also a topological characterization of alternating knot exteriors by Howie (see [How17]).

Another important question is how to determine whether a given knot is alternating? A number of knot invariants take a special form for alternating knots and links, and this can often be used to show that a given knot or link is not alternating. For instance, the Alexander polynomial and the Jones polynomial have special properties when computed for alternating knots and links, and each can be used to answer the second question. This is extremely useful as it allows us to use those invariants to determine whether a given knot or link is alternating.

Definition 2.15. Let D be a knot or link diagram. A crossing c in D is called *removable* if we can find a simple closed curve which intersects D only in the double point c. The diagram D is called *reduced*, if it has no removable crossings.

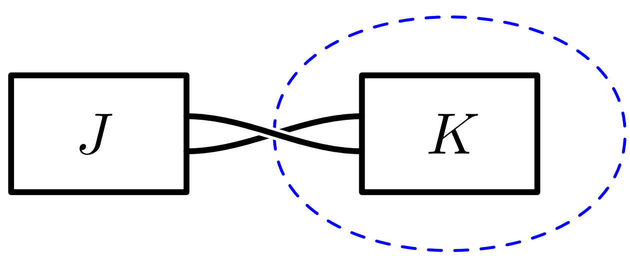

Figure 2.7: A removable crossing.

We can make a removable crossing disappear by rotating one side of the diagram 180 degrees.

In [Ban30], Bankwitz proved a nontriviality result for links admitting reduced alternating diagrams with at least one crossing. This was reproved by

Crowell in [Cro59] and more recently by Balister et al. in [BBRS01] using graph theoretic methods.

Proposition 2.16. *Suppose L is a link admitting a reduced alternating diagram with $n \geq 1$ crossings. Then $\det(L) \geq n$. In particular, L is a nontrivial link.*

Definition 2.17. Let $p(t) = \sum_{i=m}^{n} a_i t^i$ in $\mathbb{Z}[t, t^{-1}]$, where $m \leq n$ are integers. If for every i, a_i and a_{i+1} have opposite signs, then $p(t)$ is called an *alternating polynomial*. If in addition, for every $m \leq i \leq n$, $a_i \neq 0$, then $p(t)$ is called a *strongly alternating polynomial*.

Example 2.18. The polynomial $p_1(t) = t - 1 + t^{-1} + t^{-3}$ is alternating, but $p_2(t) = t - t^{-1} + t^{-3}$ is not.

In [Mur58], Murasugi proved the following theorem.

Theorem 2.19. *The Alexander polynomial $\Delta_K(t)$ of an alternating knot K is a strongly alternating polynomial with degree $2g$, where g denotes the Seifert genus of K.*

This theorem applies to show several families of knots are non-alternating. For example any torus knot $T_{p,q}$ with $p > q \geq 3$ is not alternating.

Example 2.20. For $K = T_{4,3}$, we have

$$\Delta_K(t) = \frac{(t^{12} - 1)(t - 1)}{(t^3 - 1)(t^4 - 1)} = 1 - t + t^3 - t^5 + t^6.$$

Therefore $T_{4,3}$ is not alternating.

In [Cro59] Crowell gave an independent proof of Theorem 2.19 and extended it to links. He also showed that the degree of the Alexander polynomial of a link is equal to twice its Seifert genus. In Section 5.1, we will outline his proof and generalize it to almost classical alternating knots.

We conclude this section by stating without proof three useful results due to Murasugi. The first theorem was proved in [Mur58], the second in [Mur65], and the third in [Mur89]. Please note that the third result was also proved independently by Thistlethwaite in [Thi87].

Theorem 2.21. *If D is an alternating diagram for a knot K, then the Seifert surface obtained by applying Seifert's algorithm has minimal genus. In other words, the genus of knot is realized by the genus of the given surface.*

Theorem 2.22. *If L is a special alternating link, i.e. if L admits a special alternating diagram, then $|\sigma(L)|$ is equal to the degree of $\Delta_L(t)$.*

Theorem 2.23. *The Jones polynomial of a non-split alternating link is alternating.*

2.3 The Tait Conjectures

Peter Guthrie Tait (1831-1901) is arguably the founding father of knot theory. He gave the first tabulation of knots up to 7 crossings in 1877 and was a close friend of James Clerk Maxwell (of the famous Maxwell's equations). Tait also corresponded frequently with William Rowan Hamilton, who invented the quaternions. In addition to his work in knot theory and topology, Tait made important contributions to combinatorics. He formulated a conjecture in graph theory (also known as the "Tait conjecture") which would have implied the four color theorem. (This conjecture was shown to be false by Tutte in [Tut46].) For more about Tait and his life, see [OR].

In this section, we recall three conjectures of Tait and the role played by the Jones polynomial in their resolution.

Conjecture 1: Any reduced diagram of a classical alternating link has minimal crossing number.

Conjecture 2: An amphicheiral alternating link has zero writhe.

Conjecture 3: Given any two reduced alternating diagrams D_1 and D_2 of an oriented, prime alternating link, D_1 can be transformed to D_2 by means of a sequence of flype moves (see Figure 2.8).

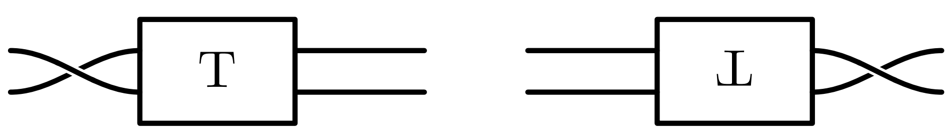

Figure 2.8: The flype move. In this picture T refers to a tangle diagram.

A Proof of the first two Tait Conjectures

For alternating links, the Jones polynomial has a special form. This property was used by Kauffman [Kau87], Murasugi [Mur87a] and Thistlethwaite [Thi87] to show the first Tait conjecture is true.

The second Tait conjecture was proved by Thistlethwaite [Thi88] and Murasugi [Mur87b].

The Tait flyping conjecture was proved by Thistlethwaite and Menasco [MT93]. The Tait flyping conjecture implies that any two reduced diagrams of the same alternating knot have the same writhe, and the second Tait conjecture, follows from this.

For simplified and shorter proofs of the first and second Tait Conjectures see [Tur87]. We outline the proof of the first and the second Tait Conjectures here. For more details, see [Lic97].

We denote by s_0, the all 0-smoothing state, and by s_1, the all 1-smoothing state.

Definition 2.24. The diagram D is called *plus-adequate*, if for any state s' with exactly one 1-smoothing, $|s_0| > |s'|$. The diagram D is called *minus-adequate* if, for any state s' with exactly one 0-smoothing, $|s_1| > |s'|$. A diagram is called *adequate* if it is both plus- and minus-adequate.

Plus-adequate means, at each crossing, two different cycles of s_0 meet. Similarly, minus-adequate means that at each crossing, two different cycles of s_1 meet.

Proposition 2.25. *A reduced alternating link diagram is adequate.*

Proof. Color the diagram, so that each crossing has $\eta(c) = +1$. Notice that by Lemma 5.18, all the crossings can have the same incidence number. The white regions corresponds to the cycles of s_0. If at a crossing, a white region meets itself, that crossing is removable. This means, the diagram is plus-adequate. The proof of minus-adequate is similar. □

Lemma 2.26. *Let D be a link diagram with n crossings.*
1) $M(\langle D\rangle) \leq n + 2|s_0| - 2$, and equality happens, if D is plus-adequate.
2) $m(\langle D\rangle) \geq -n - 2|s_1| + 2$, and equality happens, if D is minus-adequate.

Proof. Let $\langle D \,|\, s\rangle = A^{i(s)}(-A^{-2} - A^2)^{|s|-1}$, then $\langle D\rangle = \sum_s \langle D \,|\, s\rangle$. Then $M(\langle D \,|\, s_0\rangle) = n+2|s_0|-2$. If a state s', contains a 0-smoothing, and we change it to a 1-smoothing to obtain s'', then $i(s'') = i(s')-2$, and $|s''| = |s'|\pm 1$. Thus $M(\langle D \,|\, s''\rangle) = M(\langle D \,|\, s'\rangle)$, or $M(\langle D \,|\, s''\rangle) = M(\langle D \,|\, s'\rangle)-4$. This means, for any state s, $M(\langle D \,|\, s\rangle) \leq M(\langle D \,|\, s_0\rangle)$. If D is plus-adequate, and s' is a state with only one 1-smoothing, then $|s'| = |s_0| - 1$, and $M(\langle D \,|\, s'\rangle) = M(\langle D \,|\, s_0\rangle) - 4$. Therefore $M(\langle D\rangle) = M(\langle D \,|\, s_0\rangle) = n+|s_0|-2$. The proof for a minus-adequate diagram is similar. □

Corollary 2.27. *If D is an adequate diagram, then*

$$\begin{aligned} M(\langle D\rangle) - m(\langle D\rangle) &= (n + 2|s_0| - 2) - (-n - 2|s_1| + 2), \\ &= 2n + 2(|s_0| + |s_1|) - 4. \end{aligned}$$

Theorem 2.28. *If D is a connected, reduced, alternating, n-crossing link diagram for the link L, then* $\operatorname{span} V_L = n$.

Proof. We have

$$4 \operatorname{span} V_L = \operatorname{span} \langle D\rangle = M(\langle D\rangle) - m(\langle D\rangle).$$

The diagram D is alternating, so it has the boundary property (Definition 3.16), and $|s_0| + |s_1| = n + 2$ (Lemma 3.18). Combining this with Corollary 2.27, the result follows. □

Notice that, for any link L with n crossings, $\operatorname{span} V_L \leq n$.

Corollary 2.29. *The first Tait Conjecture is true.*

Proof. Let D be a connected, reduced, alternating diagram with n crossings for a link L. Then $\operatorname{span} V_L = n$. If D' is another diagram for L with n' crossings, then $n = \operatorname{span} V_L \leq n'$. □

Example 2.30. For the knot $K = 8_{19}$, $V_K(t) = t^3 + t^5 - t^8$. Then $\operatorname{span} V_K = 8 - 3 = 5 < 8$. If 8_{19} were alternating, then it would have a diagram with 5 crossings. Using other invariants, 8_{19} is different from any other knot with 5 crossings or less. Therefore 8_{19} is not alternating. Also according to [Lic97, Table 6.1], this knot has Alexander polynomial $\Delta_K(t) = 1 - t + t^3 - t^5 + t^6$ and $\det(K) = 1$. So Proposition 2.16 applies and shows K is not alternating. (One can also conclude this from Theorem 2.19.)

Definition 2.31. Let D be a link diagram, its *r-parallel* D^r, is the diagram in which each link component of D is replaced by r copies, all parallel in the plane, and each copy repeating the "over" and "under" behavior of the original link component.

Lemma 2.32. *If D is plus-adequate, then D^r is also plus-adequate. If D is minus-adequate, then D^r is also minus-adequate.*

Proof. The all 0-smoothing state $s_0 D^r$, is r parallel copies of $s_0 D$. Each cycle of $s_0 D^r$ runs parallel to a cycle of $s_0 D$, and there cannot be a crossing in which, a cycle of $s_0 D^r$ meets itself. □

Theorem 2.33. *Let D and E be diagrams with n_D and n_E crossings, respectively, for the same link L. Suppose D is plus-adequate, then $n_D - w(D) \leq n_E - w(E)$.*

Proof. Let $\{L_i\}$ be components of L, and D_i and E_i be sub-diagrams of D and E, corresponding to L_i. Choose non-negative integers μ_i and v_i, such that for any i, $w(D_i) + \mu_i = w(E_i) + v_i$. Change D to D_*, by changing each D_i to D_{*i}, by adding to D_i, μ_i positive kinks. Similarly, add v_i positive kinks to E_i. Note that D_* is still plus-adequate, and

$$w(D_{*i}) = w(D_i) + \mu_i = w(E_i) + v_i = w(E_{*i}).$$

It follows that $w(D_*) = w(E_*)$, because the writhe of a link diagram, is the sum of writhes of the components, and the sum of the signs of the crossings between different components. The second part, is a combination of linking numbers, so remains unchanged.

For any r, take D^r_* and E^r_*. Then $w(D^r_*) = r^2 w(D_*)$, because in r-parallel of a diagram, each crossing is replaced by r^2 crossings of the same sign. The diagrams D^r_* and E^r_*, are equivalent, so $V_{D^r_*}(t) = V_{E^r_*}(t)$, also their writhes are equal, therefore their bracket polynomials are equal, $\langle D^r_* \rangle = \langle E^r_* \rangle$. For any r, by Lemma 2.26, we have:

$$\begin{aligned} M(\langle E^r_* \rangle) &\leq (n_E + \sum_i v_i)r^2 + 2(|s_0 E| + \sum_i v_i)r - 2, \\ M(\langle D^r_* \rangle) &= (n_D + \sum_i \mu_i)r^2 + 2(|s_0 D| + \sum_i \mu_i)r - 2. \end{aligned}$$

If for every positive integer r, $ar^2 + br - 2 \leq cr^2 + dr - 2$ with $a, b, c, d \in \mathbb{Z}$ and b positive, then $a \leq c$. Suppose to the contrary that $a > c$, then $a \geq c + 1$. Choose $r > \max\{b, d\}$, then:

$$cr^2 + dr - 2 \leq cr^2 + r^2 - 2 = (c+1)r^2 - 2 \leq ar^2 - 2 < ar^2 + br - 2,$$

which is a contradiction. Therefore we have:

$$\begin{aligned} n_D + \sum_i \mu_i &\leq n_E + \sum_i v_i, \\ n_D - \sum_i w(D_i) &\leq n_E - \sum_i w(E_i). \end{aligned}$$

Once again, using the fact that the sum of the signs of crossings of distinct components, is determined by linking numbers of the components of L, the result follows. □

Corollary 2.34. *Let D and E be diagrams with n_D and n_E crossings, respectively, for the same link L.*

(i) *If D is plus-adequate, then the number of negative crossings of D is less than or equal to the number of negative crossings of E.*

(ii) *If D is minus-adequate, then the number of positive crossings of D is less than or equal to the number of positive crossings of E*

(iii) *An adequate diagram has the minimal number of crossings.*

(iv) *Two adequate diagrams of the same link (e.g. reduced alternating diagrams) have the same writhe.*

Proof. **(i)** Let, n_+ and n_- be the number of positive and negative crossings, respectively. We have

$$\begin{aligned} n_D - w(D) &\leq n_E - w(E), \\ n_+(D) + n_-(D) - (n_+(D) - n_-(D)) &\leq n_+(E) + n_-(E) - (n_+(E) - n_-(E)), \\ n_-(D) &\leq n_-(E). \end{aligned}$$

(ii) Use the negative kinks, in the proof of the theorem. It follows that

$$\begin{aligned} -n_D + \sum_i \mu_i \geq -n_E + \sum_i v_i &\implies n_D - \sum_i \mu_i \leq n_E - \sum_i v_i, \\ n_D + w(D) \leq n_E + w(E) &\implies n_+(D) \leq n_+(E). \end{aligned}$$

(iii) Follows from (i) and (ii).
(iv) From (iii), we have $n_D = n_E$. It follows from the theorem that

$$n_D - w(D) \leq n_E - w(E) \Rightarrow w(E) \leq w(D).$$

From (ii), we have

$$n_D + w(D) \leq n_E + w(E) \Rightarrow w(D) \leq w(E).$$

Therefore $w(D) = w(E)$. □

Therefore, this theorem proves the first two Tait Conjectures.

One application of the first Tait Conjecture is to show that the crossing number for alternating knots is additive under connected sum. In general it is not known whether the crossing number is additive under connected sum or not.

In [Mur89], Murasugi proves that if D is a reduced alternating link diagram for a non-split link L, then

$$M(V_L(t)) + m(V_L(t)) = w(D) - \sigma(L).$$

This means that we can read off the signature of a non-split alternating link from the Jones polynomial.

Chapter 3

Virtual Knot Theory

In this chapter, we provide an introduction to virtual knots and links. We begin with four equivalent definitions of virtual knots as equivalence classes of (i) virtual knot diagrams, (ii) Gauss diagrams, (iii) knots in thickened surfaces, and (iv) abstract link diagrams. We explain how to extend many of the invariants from classical to virtual knot theory, including the Jones polynomial and the knot group, and we recall a number of invariants of virtual knots, including the virtual knot group and virtual Alexander polynomial. In contrast to the situation for classical knots, not all virtual knots and links are checkerboard colorable. We prove a result relating checkerboard colorability to the virtual knot or link satisfying the boundary condition. We also introduce almost classical knots and links and review the general theory of parity and projection, due to Manturov.

Throughout this thesis, we will use decimal numbers to refer to virtual knots in Green's tabulation [Gre04].

3.1 Four Equivalent Definitions

Kauffman introduced the notion of *virtual knots and links* in [Kau99]. In this section, we review how to define virtual knots and links via virtual knot diagrams, Gauss diagrams, abstract link diagrams, and knots and links in thickened surfaces.

First description: We consider a collection of immersed closed curves in the plane, with a finite number of intersection points which all are double points. Record extra information at each intersection point by specifying which one of the two strands goes over the other (classical crossing), or we do not specify it by putting a small circle around the intersection point (virtual crossing). This is called a *virtual link diagram*. A virtual link is an equivalence class of virtual link diagrams modulo the *generalized Reidemeister moves* and the planar isotopy. The combination of classical and virtual Reidemeister moves are called the generalized Reidemeister moves. See Figures 2.1 and 3.1.

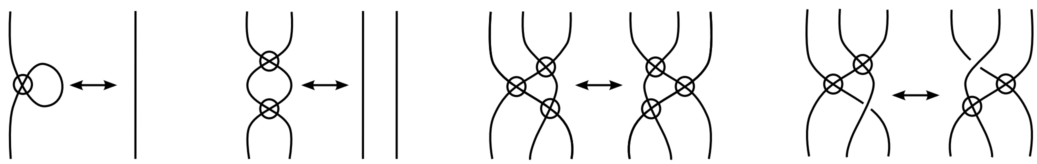

Figure 3.1: The virtual Reidemeister moves.

Second description: A *Gauss diagram* is a decorated trivalent graph consisting of one or more core circles, oriented counterclockwise, together with a finite collection of signed, directed chords connecting distinct pairs of points on the circles. Each core circle represents a knotted curve on a surface, and the directed chords, which are also called arrows, connect preimages of the classical crossings of the underlying immersed curve; they point from the over-crossing arc to the under-crossing arc, and their sign indicates the writhe of the classical crossing. A virtual knot or link is then an equivalence class of Gauss diagrams under the equivalence generated by the virtual Reidemeister moves.

In [Pol10], Polyak showed that all Reidemeister moves can be generated by the four moves $\Omega 1a$, $\Omega 1b$, $\Omega 2a$ and $\Omega 3a$ (see Figure 3.2). This observation facilitates defining new invariants of classical and virtual knots and links.

Starting with a virtual link diagram, we can associate a Gauss diagram to it. Number the classical crossings in an arbitrary order. Choose a point on each component, and a direction along which we start to travel that component. Now for each component draw a circle. If that component has n classical crossings, then choose $2n$ points on the associated circle. Choose a point on the circle as well, and start traveling counter-clockwise on the circle, while we are traveling on the link diagram component, along the chosen direction. Every time we pass a classical crossing on the diagram, we should pass a point on the circle. If we pass an under-crossing, put an arrow-tail on the associated point on the circle,

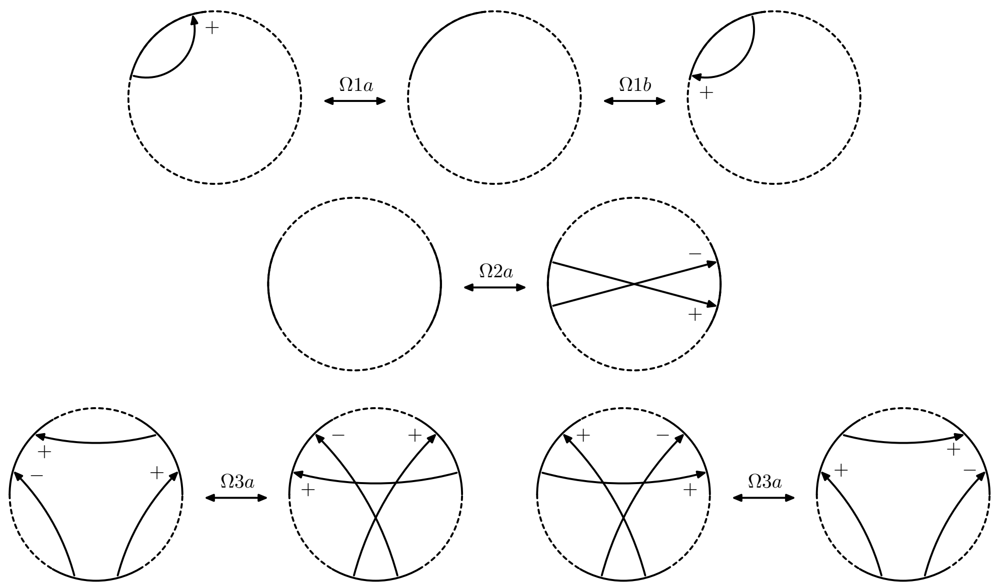

Figure 3.2: Reidemeister moves for Gauss diagrams.

otherwise we put an arrow-head. We also record the number of the classical crossing which we pass, beside the associated point on the circle. At the end, match the arrow-head and tails with the same number, and record the sign of the classical crossing beside the arrow-head or tail.

Example 3.1. Here we have the virtual diagram and the Gauss diagram for the knot 4.105.

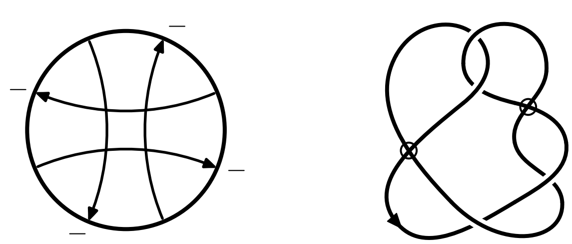

Figure 3.3: A Gauss diagram and virtual knot diagram for the almost classical knot $K = 4.105$.

Conversely, given a Gauss diagram D with n chords, enumerate the chords by $1, \ldots, n$. Draw n disjoint classical crossings in the plane and enumerate them by $c_1, \ldots, c_n$. For each chord i in D with the sign ε, assign suitable direction to the arcs of c_i in order to c_i has the sign ε. Start with a point on each core circle of D and go around counterclockwise. If you pass a chord's tail, in the corresponding classical crossing we have to go along the under-crossing arc. Passing a chord's head, we go along the over-crossing arc. Each core circle of D determines a component of the link diagram. Every time we cross an arc in a point other than the classical crossings, we need to put a circle around the intersection point to indicate it is a virtual crossing.

Third description:

Definition 3.2. Let Σ be a closed, oriented surface and $I = [0, 1]$. The product $\Sigma \times I$ is called a *thickened surface*. A link L in a thickened surface $\Sigma \times I$ is a finite collection of disjoint 1-dimensional submanifolds in the interior of $\Sigma \times I$ with each connected component is diffeomorphic to a circle.

Definition 3.3. *Stable equivalence* on links in thickened surfaces is generated by the following operations, which transform a given link L in a thickened surface $\Sigma \times I$ into a new link L' in a possibly different thickened surface $\Sigma' \times I$.
(i) Let $f : \Sigma \times I \to \Sigma' \times I$ be an orientation-preserving diffeomorphism sending the orientation class of Σ to that of Σ' (this implies that $f(\Sigma \times \{0\}) = \Sigma' \times \{0\}$ and $f(\Sigma \times \{1\}) = \Sigma' \times \{1\}$). The link $L' = f(L)$ in $\Sigma' \times I$ is said to be obtained from L in $\Sigma \times I$ by a *diffeomorphism*.
(ii) Let $h : S^0 \times D^2 \to \Sigma$ be the attaching region for a 1-handle that is disjoint from the image of L under projection $\Sigma \times I \to \Sigma$, then 0-surgery on Σ along h is the surface

$$\Sigma' = \Sigma \smallsetminus h(S^0 \times D^2) \cup_{S^0 \times S^1} D^1 \times S^1.$$

The link L' is the image of the link L in $\Sigma' \times I$, and we say that it is the link obtained from L by *stabilisation*.
(iii) *Destabilisation* is the inverse operation, and it involves cutting $\Sigma \times I$ along a *vertical annulus* A and attaching two copies of $D^2 \times I$ along the two annuli. In the resulting thickened surface, we keep only the components containing L. Note that in (iii), an annulus A in $\Sigma \times I$ is called vertical if there is an embedded circle $\gamma \subset \Sigma$ such that $A = \gamma \times I \subset \Sigma \times I$.

An equivalence class under the equivalence relation generated by (i), (ii), and (iii) above is called a virtual link.

Starting with a virtual link L in $\Sigma \times I$, consider the projection $p : \Sigma \times I \to \Sigma \times \{\frac{1}{2}\}$. By a small isotopy on L we can make sure $D = p(L)$ has only transverse double points. Choosing a point on each component of D and going around that component, we can read off the Gauss diagram. To obtain a virtual diagram we can use the Gauss diagram or we can project D onto a plane (again make sure the intersection points in the image are transverse). Each double point on D determines a classical crossing on the planar diagram. Any other double point is a virtual crossing.

Fourth description: Suppose S is a compact oriented surface with boundary. Let D be a link diagram on S with no virtual crossing. We denote by $|D|$, the graph obtained by replacing each classical crossing in D by a four-valent vertex. We say $P = (S, D)$ is an *abstract link diagram* (ALD), if $|D|$ is a deformation retract of S.

Let Σ be a closed, connected and oriented surface and $f : S \to \Sigma$ be an orientation preserving embedding. We call $(\Sigma, f(D))$ a *realization* of P.

Two abstract link diagrams (S_1, D_1) and (S_2, D_2) are said to be *abstract R-move equivalent*, if there are realizations $(\Sigma, f_1(D_1))$ and $(\Sigma, f_2(D_2))$ on the same surface Σ such that $f_1(D_1)$ differs from $f_2(D_2)$ by performing one Reidemeister move on F. Two abstract link diagrams (S_1, D_1) and (S_2, D_2) are abstract equivalent, if one can be changed into the other by a finite sequence of abstract R-moves.

A virtual link is an equivalence class of abstract link diagrams modulo abstract equivalence.

Associated with each virtual link diagram, we can construct an abstract link diagram (see [KK00]). We review that construction here.

Let D be a virtual link diagram with n classical crossings and $U_1, U_2, \ldots, U_n$ regular neighborhoods of the crossings of D. Put $W = \mathrm{cl}(\mathbb{R}^2 - \cup_{i=1}^n U_i)$. Thickening the arcs and loops of $D \cap W$, we obtain immersed bands and annuli in W whose cores are $D \cap W$. Their union together with $U_1, U_2, \ldots, U_n$ forms an immersed disk-band surface $N'(D)$ in the plane. Modifying $N'(D)$ as shown below at each virtual crossing of D, we obtain a compact oriented surface S_D embedded in $\mathbb{R}^3$, and a diagram $\tilde{D}$ on S_D corresponding to D. We call the pair $P = (S_D, \tilde{D})$ the *abstract link diagram associated with D*.

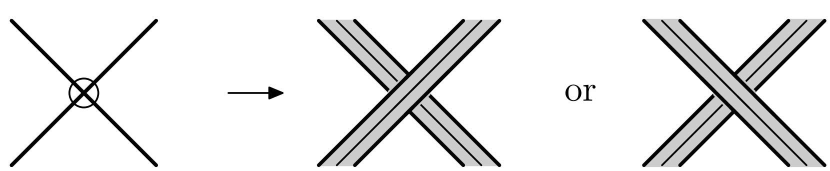

Figure 3.4: Modifying $N'(D)$ at a virtual crossing.

In [KK00], the authors show there is a bijection between the set of virtual link diagrams modulo the generalized Reidemeister moves and the set of abstract link diagrams modulo abstract equivalence.

The *supporting genus* of an ALD $P = (\Sigma, \tilde{D})$ is denoted by $sg(P)$, and is defined to be the minimal genus among the realization surfaces F of P. The supporting genus of a virtual link diagram D is defined to be the supporting genus of the ALD $P = (S_D, \tilde{D})$ associated with D and denoted by $sg(D)$. The *virtual genus* of a virtual link L is denoted by $g_v(L)$ and defined to be the minimum number among the supporting genus $sg(D)$, where D runs over all virtual link diagrams representing L.

Let L be a virtual link. A virtual link diagram D representing L such that $sg(D) = g_v(L)$ is called a *minimal diagram* of L.

All these four definitions are equivalent and we will use them interchangeably.

If we think of S^3 as $\mathbb{R}^3$ with a point at infinity, then a classical knot $K \subset S^3$ can be isotoped to be disjoint from the two points $\{0, \infty\}$. Thus, we can view it as a knot in the thickened surface $S^2 \times I$. The associated virtual knot is independent of the choice of isotopy, and we call such a knot classical. Notice that a classical knot diagram D (a virtual knot diagram with no virtual crossings) has supporting genus zero, thus any classical knot has virtual genus zero. Therefore a virtual knot is classical if and only if its virtual genus is zero. Kuperberg [Kup03, Theorem 1] proved a strong uniqueness result for minimal genus representatives. Namely, he showed that if $K \subset \Sigma \times I$ and $K' \subset \Sigma' \times I$ are two minimal genus representatives for the same virtual knot, then $K' = f(K)$ for some diffeomorphism $f : \Sigma \times I \to \Sigma' \times I$ as in (i) of Definition 3.3 above.

Definition 3.4. A virtual link is said to be *split*, if it admits a virtual link diagram D such that $|D|$ is not connected. Otherwise we say the virtual link is connected or non-split.

Definition 3.5. (i) Two given knots $K_0 \subset \Sigma_0 \times I$ and $K_1 \subset \Sigma_1 \times I$ in thickened surfaces are *virtually concordant* if there exists a connected oriented 3-manifold W with $\partial W \cong -\Sigma_0 \sqcup \Sigma_1$ and an annulus $A \subset W \times I$ cobounding $-K_0 \sqcup K_1$.
(ii) A knot $K \subset \Sigma \times I$ is called *virtually slice* if it is concordant to the unknot. Equivalently, the knot K is virtually slice if there exists a connected 3-manifold W with $\partial W \cong \Sigma$ and a 2-disk $\Delta \subset W \times I$ cobounding K. We call Δ a *slice disk* for K.

Remark 3.6. Note that a result of Boden and Nagel shows that a classical knot

K is virtually slice if and only if it is classically slice [BN17].

3.2 Invariants of Virtual Knots

In this section, we review a number of invariants of virtual knots and links. Some of the invariants, like the virtual crossing number and virtual genus, have no analogue in the theory of classical knots. Since the virtual genus was already discussed in the previous section, we recall the definition of the *crossing number*. Given a virtual knot or link diagram D, let $v(D)$ denote the number of virtual crossings of D. Then the virtual crossing number of a virtual knot or link K is defined to be

$$v(K) = \inf\{v(D) \mid D \text{ is a virtual knot diagram representing } K\}.$$

As with the virtual genus, we see that a virtual knot or link K is classical if and only if $v(K) = 0$.

Many of the standard invariants of classical knots and links extend in a straightforward way to virtual knots and links. For example, the Kauffman bracket $\langle D \rangle$ and Jones polynomial $V_K(t)$ can be defined for virtual knots and links in exactly the same way, but the resulting invariants are much less powerful than in the classical setting. Indeed, as we shall see, there are nontrivial virtual knots K with trivial Jones polynomial $V_K(t) = 1$, and the knot K can even be taken to be alternating. In fact, in chapter 6 we shall see the same phenomenon occurs for the Khovanov homology for virtual knots, which is to say that Khovanov homology is not an unknot detector for virtual knots.

The knot group is another powerful invariant of classical knots which generalizes in a natural way to virtual knots and links by means of Wirtinger presentations. As an invariant of classical knots, the knot group is an unknot detector, indeed the only classical knot K whose knot group is infinite cyclic is the trivial knot. In fact, Waldhausen's theorem implies that the knot group together with its peripheral structure is a *complete invariant* of classical knots, which is to say that two classical knots are equivalent if and only if they have isomorphic knot groups with equivalent peripheral structures. Unfortunately, these results fail in the virtual setting; one can construct nontrivial virtual knots with trivial knot group.

For a virtual knot K, one can mimic the construction of the Alexander module by regarding the quotient G'_K/G''_K as a module over $\mathbb{Z}[t, t^{-1}]$, This can be used to define elementary ideals and the Alexander polynomial for virtual knots and links. However, in contrast to the case of classical knots, the first elementary ideal may not be principal. One way to remedy the situation is to replace the elementary ideals $\mathcal{E}_k$ with the smallest principal ideal containing them. For instance, this would suggest a way to define an Alexander polynomial for a virtual knot K to be a generator of the principal ideal containing $\mathcal{E}_1$. However, since the knot group itself is only an invariant of the associated *welded knot*[1], the invariants one obtains in this way will not be very refined. Indeed, the Alexander polynomial for the vast majority of low-crossing virtual knots is trivial.

An alternative is to work with the virtual knot group VG_K, which was introduced in [BDG+15]. In particular, the virtual Alexander polynomial $H_K(s,t,q)$ is defined in terms of the elementary ideals of VG'_K/VG''_K, which is a finite dimensional module over $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}, q^{\pm 1}]$ called the *virtual Alexander module*. The virtual Alexander polynomial records information about the virtual crossing number; for example Theorem 3.4 of [BDG+15] implies that

$$q\text{-width } H_K(s,t,q) \leq 2v(K),$$

thus $H_K(s,t,q)$ provides a lower bound of the virtual crossing number of K.

Further, the virtual Alexander polynomial is intimately related to the generalized Alexander polynomial $G_K(s,t)$ of Sawollek and Silver-Williams, Propositions 3.8 and Corollary 4.8 of [BDG+15] show that each one of these polynomials determines the other:

$$G_K(s,t) = H_K(s,t,1) \qquad \text{and} \qquad H_K(s,t,q) = G_K(sq^{-1}, tq).$$

In order to study virtual knot concordance, it would be useful to have a virtual analogue of the knot signature. However, since virtual knots do not generally admit Seifert surfaces, the standard approach does not work to define signatures for virtual knots. One approach is to focus attention on almost classical knots, which consist exactly of virtual knots that do admit Seifert surfaces. The idea is to define signatures in the restricted setting of almost classical knot and to use parity projection to extend them to all virtual knots. This approach is carried out in [BCG17a].

[1] Two virtual knots or links are said to be *welded equivalent* if one can be obtained from the other by generalized Reidemeister moves plus the forbidden overpass, which is the move that exchanges two adjacent arrow-tails on a Gauss diagram.

An alternative approach which was developed by Im, Lee and Lee in [ILL10] is to generalize the method of Gordon and Litherland by defining Goeritz matrices and correction terms of checkerboard knots and links. We summarize their construction in Chapter 4, where the reader will find a full account of these invariants.

3.3 Checkerboard Knots

In this section, we introduce the notion of checkerboard coloring for virtual knots and links. We then recall the notion of the boundary property for virtual link diagrams [Dye16] and relate it to checkerboard colorability.

Definition 3.7. Given $P = (F, D)$, where F is a compact, connected, oriented surface and D is a link diagram on F, a *checkerboard coloring* ξ is an assignment, to each region of $F \smallsetminus |D|$, one of two colors, say black and white, such that any two adjacent regions sharing an edge of $|D|$ have different colors.

A virtual link diagram D is said to be *checkerboard colorable* if the associated ALD $P = (S_D, \tilde{D})$ admits a checkerboard coloring. A *checkerboard link* is a virtual link L which can be represented by a checkerboard colorable virtual link diagram.

Given a pair $P = (F, D)$ with checkerboard coloring ξ, define the dual checkerboard coloring ξ^* to be the one obtained from ξ by interchanging black and white regions. If a virtual link L is checkerboard colorable, then it admits two colorings which are dual to one another.

Note that, being checkerboard colorable depends only on the underlying *flat* virtual link diagram. A flat diagram of a link is the link projection, so we do not specify whether a classical crossing is an over- or under-crossing.

Example 3.8. The virtual knot 2.1 is not a checkerboard knot. The virtual knot 3.5 is a checkerboard knot (see Figure 3.5).

Definition 3.9. Suppose c is a chord in a Gauss diagram D, which we draw with c pointing up. We define the *index* of c as

$$\operatorname{ind}(c) = r_+(c) - r_-(c) + \ell_-(c) - \ell_+(c),$$

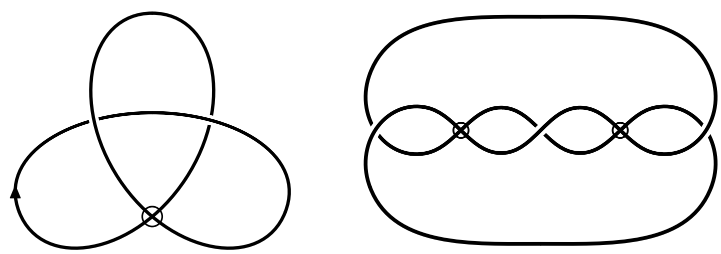

Figure 3.5: The virtual knots 2.1 (left) and 3.5 (right), notice that both have virtual genus 1.

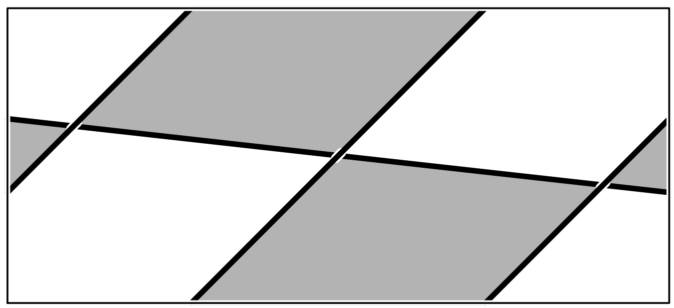

Figure 3.6: A checkerboard coloring of the knot 3.5 on a torus.

where $r_{\pm}(c)$ are the number of $\pm$-chords intersecting c and pointing to the right, and $\ell_{\pm}(c)$ are the number of $\pm$-chords pointing to the left.

Definition 3.10. Let D be a Gauss diagram. If $\operatorname{ind}(c) = 0$ for every chord c, then D is called an *almost classical diagram*. A virtual knot K which admits an almost classical diagram is called an *almost classical knot*.

Definition 3.11. Let D be a Gauss diagram. If $\operatorname{ind}(c) = 0 \pmod p$ for every chord c, then D is called a *mod p almost classical diagram*. A virtual knot K which admits a mod p almost classical diagram is called a *mod p almost classical knot*.

Observe that a virtual knot diagram D is checkerboard colorable if and only if every chord c of D has $\operatorname{ind}(c) = 0 \pmod 2$.

Remark 3.12. (i) A virtual knot is checkerboard if and only if it can be represented by a knot in $\Sigma \times I$ which bounds a spanning surface (possibly non-orientable).
(ii) A virtual knot is almost classical if and only if it can be represented by a knot in $\Sigma \times I$ which bounds an orientable spanning surface.

Suppose a Gauss diagram D for a knot K has n chords. Then these chords divide the core circle into $2n$ arcs. We start on one arc and label the right side of it by w (for white). As we go around the core circle counterclockwise, every time we pass a new arc we change the label from right side to the left and vice versa. Let c be a chord and assume when we pass the tail/head of c, we change the label from right side to the left. When we arrive to the head/tail of c, we should change the label from left to the right. This property should hold for all the chords in order for D to admit a checkerboard coloring. Notice this property holds if and only if the number of arcs at one side (hence at both sides) of c is odd for every chord c. Again this is true if and only if the number of chords intersecting c is even for every chord c. The latter means $\text{ind}(c) = 0 \pmod 2$ for every chord c.

In [Kam02], Kamada proves that a virtual link diagram is checkerboard colorable if and only if it can be made alternating by changing a finite number of classical crossings from over-crossing to under-crossing or vice versa.

Notice that Kamada's result implies that every classical link diagram admits a checkerboard coloring, and next we give an alternative, elementary proof of this well-known fact.

Suppose D is a classical diagram for a knot K. Pick a crossing c and smooth it in an oriented way, i.e. 0- or 1-smooth c if it is a positive or negative crossing respectively. The result is a two component link diagram. Every double point is either a self-intersection point of one of the components, or is a intersection point of the two components with each other. The self-intersection points correspond to chords on the Gauss diagram which are entirely on one side of the chord c. Now consider an intersection point between the two components. If we ignore one of the link components, then there is a simple closed curve (which is a subset of the second link component) which the first link component enters it at the intersection point, hence by the Jordan curve theorem, it has to leave the simple closed curve at some other intersection point. On the Gauss diagram this means if a chord intersects the chord c, there should be another intersecting chord. Therefore the mod 2 index of c is zero. This completes the proof.

We introduce the *coloring matrix* of a knot or link diagram. Given a knot or link diagram D, denote the classical crossings and arcs of D by $\{c_1, \cdots, c_k\}$ and $\{a_1, \cdots, a_k\}$ respectively. The $k \times k$ coloring matrix $M(D)$ of D can be defined as follows:

$$m_{ij}(D) = \begin{cases} 2, & \text{if } a_j \text{ is the over-crossing arc at } c_i, \\ 1, & \text{if } a_j \text{ is an under-crossing arc at } c_i, \\ 0, & \text{otherwise.} \end{cases}$$

In particular if a_j is the over-crossing and an under-crossing arc at c_i, then $m_{ij}(D) = 2 - 1 = 1$.

Definition 3.13. Let D be a virtual knot or link diagram with coloring matrix $M(D)$. The absolute value of a principal minor of $M(D)$ is called the *Alexander determinant* of D. It is also called the *knot determinant* of D.

It follows from [BGH$^+$17] that the Alexander determinant is well-defined for a checkerboard colorable knot or link L. We denote the Alexander determinant of L by $\det(L)$.

If we remove the sign of all the chords in a Gauss diagram, the remaining object is called a *Gauss pattern*, defined as follows.

Definition 3.14. A Gauss pattern is a trivalent graph with a core circle containing $2n$ distinct points and n oriented edges which connect the points in pairs (e.g. see Figure A.1 in Appendix A).

Proposition 3.15. *The Alexander determinant of a link depends only on the underlying Gauss pattern.*

Proof. Starting with a Gauss pattern, on each core circle enumerate the arcs which are between two consecutive arrow-heads. On the planar diagram for a knot this is the same, as we enumerate the arcs between two under-crossings. Now suppose D_1 and D_2 are two Gauss diagrams with the same Gauss pattern. Enumerate the arcs and chords of the Gauss pattern. It follows $M(D_1)$ and $M(D_2)$ are exactly the same matrices. In particular they have equal principal minors. □

Given a virtual link diagram D, for each classical crossing, we can resolve the crossing into a 0-smoothing or a 1-smoothing (see Figure 2.2).

If we resolve all the classical crossings, the resulting diagram is called a *state*. Then, a state is a virtual link diagram with only virtual crossings, i.e. it is an unknotted diagram of the unlink. For a link diagram with n classical crossings, we have 2^n states. In fact, once an ordering of the crossings $\{c_1, \dots, c_n\}$ has been fixed, the states are in one-to-one correspondence with binary strings of length n. For a given state s, the dual state is denoted $\bar{s}$ and it is the one obtained from s by changing all 0-smoothings to 1-smoothings, and vice versa. In other words, if s corresponds to the binary word with i-th entry $s_i \in \{0, 1\}$, then $\bar{s}$ corresponds to the binary word with i-th entry $\bar{s}_i = 1 - s_i$.

Definition 3.16. Let D be a virtual link diagram, and $(S_D, \tilde{D})$ be the abstract link diagram associated with D. Then D has the *boundary property* if there exists a state s_∂ such that $\partial S_D = s_\partial \cup \bar{s}_\partial$, where $\bar{s}_\partial$ is the dual state of s_∂.

The following lemma relates the boundary property to checkerboard colorability.

Lemma 3.17. *A virtual link diagram D has the boundary property if and only if it is checkerboard colorable.*

Proof. Suppose D has the boundary property and define a checkerboard coloring ξ as follows. Let the white regions be those with boundary a component of s_∂, and let the black regions be those with boundary a components of $\bar{s}_\partial$. This gives a checkerboard coloring ξ for D.

Conversely, suppose ξ is a checkerboard coloring of the abstract link diagram $(S_D, \tilde{D})$. Let s_∂ be the state obtained by performing 0-smoothing to all crossings c with $\eta(c) = +1$ and 1-smoothing to all crossings c with $\eta(c) = -1$, and let $\bar{s}_\partial$ be the dual state. Then it can be easily checked that $\partial S_D = s_\partial \cup \bar{s}_\partial$, therefore D has the boundary property. □

Since every alternating virtual link diagram is checkerboard colorable, it follows that every alternating virtual link diagram has the boundary property.

Let $|s_\partial|$ and $|\bar{s}_\partial|$ be the number of components of s_∂ and $\bar{s}_\partial$, respectively.

Lemma 3.18. *Suppose D is a virtual link diagram with n classical crossings. If S_D has genus g and D has the boundary property, then*

$$|s_\partial| + |\bar{s}_\partial| = n + 2 - 2g.$$

Proof. Attach disks to the boundary components of S_D to get a closed surface Σ. There is a cell decomposition on Σ, defined as follows: There is a one-to-one correspondence between the classical crossings of D and 0-cells, bands of S_D and 1-cells, and 2-disks that we attached to S_D and 2-cells. The Euler characteristic of Σ is $2 - 2g$. And the number of 0, 1 and 2-cells are n, $2n$ and $|s_\partial| + |\bar{s}_\partial|$, respectively. The lemma now follows. □

Remark 3.19. Different authors use different names for checkerboard colorable knots and links. For instance, in [KNS02], Kamada calls them *normal*. In [DKK17a], Dye refers to them as diagrams satisfying the boundary property, which we have seen is equivalent by Lemma 3.17. In [Rus17], Rushworth calls them *even diagrams*. Some authors refer to them as *mod-2 almost classical* diagrams.

3.4 Parity Projection

In [Man10], Manturov introduced the notion of parity, and this deep and important idea has led to some of the most striking and far-reaching results in virtual knot theory. In this section, we review the definition of parity and its associate projection. Note that parity is only defined for knots, not for links.

Let $\mathscr{D}$ be a *diagram category*, one whose objects are Gauss diagrams of knots and whose morphisms consist of compositions of Reidemeister moves.

Definition 3.20. A parity is a collection of functions $\{f_D \mid D \in \mathscr{D}\}$, where $f_D : \{\text{chords of } D\} \to \{0, 1\}$, satisfying in the following axioms:
Axiom 0: Under any Reidemeister move, the parity of any chord not participating in the move is unchanged.
Axiom 1: If $c \in D$ is a chord which occurs in a Reidemeister 1 move, then $f_D(c) = 0$.
Axiom 2: If $c_1, c_2 \in D$ are two chords which occur in a Reidemeister 2 move, then $f_D(c_1) = f_D(c_2)$.
Axiom 3: If $c_1, c_2, c_3 \in D$ are three chords which occur in a Reidemeister 3 move, and D' is the new Gauss diagram obtained, after applying the Reidemeister 3 move, then $f_D(c_i) = f_{D'}(c'_i)$, $\forall i$. Furthermore, either they are all even, all odd, or exactly two are odd.

A chord c is even, if $f_D(c) = 0$, and odd, if $f_D(c) = 1$.

Definition 3.21. The diagram obtained by removing all the odd chords from D is denoted $P_f(D)$ and is called the *projection* of D with respect to the parity f.

We denote by $P_f^k(D)$, the result of applying k times parity projection to D. Notice for any diagram D there is a positive number k such that $P_f^k(D) = P_f^{k+1}(D)$ and for this k we denote $P_f^k(D)$ by $P_f^\infty(D)$.

Proposition 3.22. *If two Gauss diagrams D_1 and D_2 are equivalent, then $P_f(D_1)$ and $P_f(D_2)$, are also equivalent.*

For a proof, see [Man10].

Definition 3.23. Let K be a knot diagram in Σ and c be a crossing of K, and γ be a simple closed curve on Σ. The oriented smoothing of K at c gives rise to two loops in Σ, which we denote by K_0 and K_1. We say the crossing c is even if either K_0 or K_1 intersects γ an even number of times, otherwise we say c is odd.

Manturov proves in [IMN14], that this defines a parity for knots in Σ. It is called *homological parity*. Let $P = P_\gamma$ denote the associated projection map, then $P(K)$ is the knot with all the odd crossings removed.

Let f be the mod 2 Gaussian parity, i.e. $f(c) = \text{ind}(c) \pmod 2$ for any chord $c \in D$. Then for any Gauss diagram D, the projection $P_f(D)$ is the Gauss diagram obtained by deleting all chords c of D with $\text{ind}(c) \neq 0 \pmod 2$. The following proposition is immediate.

Proposition 3.24. *For f and P_f as above, $P_f(D) = D$ if and only if D is checkerboard colorable.*

If f is the mod 2 Gaussian parity, then by the previous proposition, $P_f^\infty(D)$ is checkerboard colorable.

Proposition 3.25. *If K is a checkerboard knot and D is a diagram for K with minimal crossing number, then D is a checkerboard colorable diagram.*

Proof. Suppose D is not checkerboard colorable, then it contains a chord c with $ind(c) = 1 \pmod 2$. Therefore $P_f(D) \neq D$ and in fact $P_f(D)$ has fewer crossings. The knot K is checkerboard which means it has a checkerboard colorable diagram D'. The diagrams D and D' are equivalent and by Proposition 3.22, $P_f(D)$ and $P_f(D')$ are equivalent. On the other hand, by the previous proposition, $P_f(D') = D'$. Hence $P_f(D)$ is a diagram for K with fewer crossings than D, and this is impossible because D has minimal crossing number. □

The total Gaussian parity, denoted $f_{tot}(c)$, is defined by setting

$$f_{tot}(c) = \begin{cases} 1 & \text{if ind}(c) \neq 0 \\ 0 & \text{if ind}(c) = 0. \end{cases}.$$

It is not difficult to check that f_{tot} satisfies the parity axioms. Let P_{tot} denote the associated projection map on Gauss diagrams. Then an argument analogous to the preceding one shows that $P_{tot}(D) = D$ if and only if D is an almost classical diagram. We can also prove the following proposition similarly:

Proposition 3.26. *If K is an almost classical knot and D is a diagram for K with minimal crossing number, then D is an almost classical diagram.*

Proposition 3.27. *Suppose K is an almost classical knot and D is a diagram for K which has minimal crossing number. If D is a connected sum, then both of the factors are almost classical diagrams.*

Proof. By Proposition 3.26, D is an almost classical diagram, and it follows that $P_{tot}(D) = D$. Let $D = D_1 \# D_2$, and suppose c is a chord in D. Then c is a chord in one of the factors say D_1. It is clear from the definition of the index that $\text{ind}(c)$ as a chord in D is the same as $\text{ind}(c)$ as a chord in D_1. Hence $P_{tot}(D_1) = D_1$ and D_1 is almost classical. For D_2 the argument is similar. □

Remark 3.28. If D is an alternating Gauss diagram, P_{tot} might not be alternating. In Figure 3.7, the diagram on the left is an alternating diagram, but the projection on the right is not.

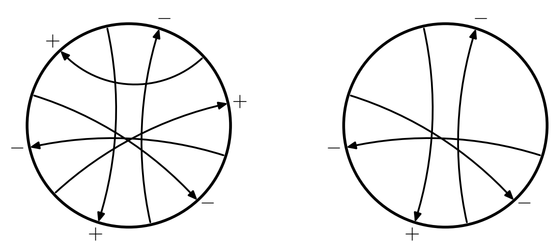

Figure 3.7: The projection of an alternating diagram.

Chapter 4

Signatures for Checkerboard Knots

In this chapter, we present virtual analogues of the signature, nullity, and determinants, which are invariants defined for checkerboard colorable virtual knots and links. We study the behavior of these invariants under connected sum of checkerboard long knots, showing that the signature is additive and the determinant is multiplicative. We relate the signature of virtual knots obtained by taking vertical and horizontal mirror images, and also under performing a crossing change.

4.1 Goeritz Matrices and Signatures

In this section, we introduce the signature and determinants of checkerboard knots and links, first defined by Im, Lee, and Lee in [ILL10]. Their definition is similar to the one developed by Gordon and Litherland in [GL78] and described in Section 2.1.

Let ξ be a checkerboard coloring for a pair $P = (F, D)$, where F is a closed, oriented and connected surface and D is a link diagram on F. We enumerate the white regions of $F_\xi \smallsetminus |D|$ by $X_0, X_1, \ldots, X_m$. Let $C(D)$ denote the set of all classical crossings of D on F. For each pair $i, j \in \{1, 2, \ldots, m\}$, let

$$C_{ij}(D) = \{c \in C(D) \mid c \text{ is adjacent to both } X_i \text{ and } X_j\}$$

and define

$$g_{ij} = \begin{cases} -\sum\limits_{c \in C_{ij}(D)} \eta(c), & \text{for } i \neq j, \\ -\sum\limits_{k=0; k\neq i}^{m} g_{ik}, & \text{for } i = j. \end{cases}$$

Just as for classical knots, the pre-Goeritz matrix of D is defined to be the symmetric integral matrix $G'_\xi(D) = (g_{ij})_{0\leq i,j\leq m}$, and the Goeritz matrix of D is the principal minor $G_\xi(D) = (g_{ij})_{1\leq i,j\leq m}$ obtained by removing the first row and column of $G'_\xi(D)$.

For any classical knot diagram, the Goeritz matrices are non-singular, but this does not hold in general for checkerboard colored virtual knot diagrams.

The correction term is defined by setting $\mu_\xi(D) = \sum\limits_{c \text{ is type II}} \eta(c)$.

Definition 4.1. For a link diagram D and a checkerboard coloring ξ, we define the signature as follows:

$$\sigma_\xi(D) = \text{sig}(G_\xi(D)) - \mu_\xi(D).$$

Remark 4.2. If L is a non-split checkerboard link represented by a diagram D of minimal genus and with coloring ξ, then the pair $\{\sigma_\xi(D), \sigma_{\xi^*}(D)\}$ of signatures is independent of the choice of virtual link diagram and gives a well-defined invariant of the virtual link L (see [ILL10]). Notice the fact that D should be a minimal genus diagram, is essential since using another diagram for L with different genus, one might obtain different signatures.

Example 4.3. We compute the signatures for the knot $K = 3.5$. The diagram D in Figure 3.5 has supporting genus equal to 1. On the other hand K is not classical, so D is a minimal genus diagram. Let ξ be the coloring in Figure 3.6, and let c_1, c_2 and c_3 be the crossings from left to right. There is only one white region for ξ, therefore $G_\xi(D)$ is the empty matrix and its signature is zero. The two crossings c_1 and c_3 are type II and they have $\eta = -1$, thus $\mu_\xi = -2$, and $\sigma_\xi = 2$. Now we consider ξ^* by changing the black and white color. There are two white regions for ξ^* and we have:

$$G'_{\xi^*} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix},$$

it follows $G_{\xi^*} = [2]$, and $\text{sig}(G_{\xi^*}) = 1$. On the other hand, only c_2 is a type II crossing for ξ^* and $\mu_{\xi^*} = -1$, as a result $\sigma_{\xi^*} = 2$.

In a similar way, one can also define analogues of the determinant for checkerboard links in terms of $\det_\xi(D) = |\det G_\xi(D)|$ and $\det_{\xi^*}(D) = |\det G_{\xi^*}(D)|$. In [ILL10], they prove that the pair $(|\det G_\xi(D)|, |\det G_{\xi^*}(D)|)$ is independent of the choice of virtual link diagram and gives a well-defined invariant of the checkerboard link L. We call this pair the *checkerboard determinants* of L. The nullities of L are defined to be $(N_\xi(L), N_{\xi^*}(L)) = (\text{nullity}(G_\xi(D)), \text{nullity}(G_{\xi^*}(D)))$.

Notice that in general, the checkerboard determinants $\det_\xi(K)$ and $\det_{\xi^*}(K)$ are not equal to one another, and not equal to $\det(K)$ for checkerboard knots and links.

Proposition 4.4. *For a non-split classical link L, the signatures, determinants and nullities are singletons. Also we have:*

$$|\Delta_L(-1)| = \det(L) = |\det_\xi(L)| = |\det_{\xi^*}(L)|.$$

Proof. For the first part see [ILL10]. For the second part, suppose V is a Seifert matrix for L, then $|\Delta_L(-1)| = |\det(V + V^\tau)|$. Combining [BZH14, Proposition 13.15] and [GL78, Theorem 1], the result follows. □

4.2 Signatures and Connected Sum

In this section, we study the behavior of the invariants of the previous section under connected sum. Since the operation of connected sum is not well-defined on round virtual knots, we will work with *long virtual knots*. Our main results are that, under connected sum, the signature is additive and the determinant is multiplicative for long checkerboard knots.

Recall that a long virtual knot diagram is a regular immersion of $\mathbb{R}$ in $\mathbb{R}^2$ which coincides with the x-axis outside of some compact set. Each double point is either a virtual crossing or a classical crossing, and classical crossings have over- and under-crossing arcs indicated as usual. A long virtual knot is defined to be an equivalence class of long virtual knot diagrams modulo the generalized Reidemeister moves. Long virtual knot diagrams are oriented from left to right.

If D_1 and D_2 are two long virtual knot diagrams, the connected sum $D_1\#D_2$ is defined to be the diagram obtained by concatenating them, with D_1 on the left and D_2 on the right. This gives a well-defined operation on long virtual knots, and in general, connected sum is not commutative.

Suppose D is a long virtual knot diagram which coincides with the x-axis outside of a closed ball B in $\mathbb{R}^2$. The closure of D, denoted $\widehat{D}$, is the round virtual knot diagram obtained as the union of $D \cap B$ and $\partial B \cap \{(x, y) \mid y \geq 0\}$.

An elementary argument shows that equivalent long virtual knot diagrams have equivalent closures. Thus, closure gives a well-defined map from long virtual knots to round virtual knots. A long virtual knot diagram is called *checkerboard colorable* if its closure is checkerboard colorable.

One can also view long virtual knots as round virtual knots with a choice of basepoint, which we take to be the point at infinity. For any checkerboard colorable long virtual knot diagram, there is a canonical choice of checkerboard coloring; it is the coloring with white region to the right of the basepoint.

We define the signature of the checkerboard long knot to be the signature of the corresponding round virtual knot with respect to the canonical checkerboard coloring of the long knot.

Proposition 4.5. *The signature of checkerboard long knots is additive under connected sum.*

Proof. Let D and D' be two checkerboard colorable long virtual knot diagrams, and let ξ and ξ' be the canonical checkerboard colorings of D and D'. Enumerate the white regions for D with $X_0, \ldots, X_m$, such that X_0 is the region containing the base-point. Similarly we have $Y_0, \ldots, Y_\ell$ for D', such that Y_0 contains the base-point of D'. The pre-Goeritz matrices for D and D' are the following matrices, respectively:

$$G'_\xi(D) = \begin{bmatrix} -x & u \\ u^\tau & A \end{bmatrix} \quad \text{and} \quad G'_{\xi'}(D') = \begin{bmatrix} -y & v \\ v^\tau & B \end{bmatrix},$$

where u and v are row vectors, and x and y are sum of the entries of u and v, respectively, and $A = G_\xi(D)$ and $B = G_{\xi'}(D')$ are the Goeritz matrices of D and D', respectively.

The white regions for $D\#D'$ are $Z = X_0 \cup Y_0, X_1, \ldots, X_m, Y_1, \ldots, Y_\ell$, where Z is the region to the right of the base-point of $D\#D'$. The pre-Goeritz matrix for $D\#D'$ is therefore given by

$$G'_{\xi\#\xi'}(D\#D') = \begin{bmatrix} -x-y & u & v \\ u^\tau & A & 0 \\ v^\tau & 0 & B \end{bmatrix}.$$

Since signatures are additive under block sum, i.e. since

$$\text{sig} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \text{sig}(A) + \text{sig}(B),$$

it follows that $\text{sig}\left(G_{\xi\#\xi'}(D\#D')\right) = \text{sig}\left(G_\xi(D)\right) + \text{sig}\left(G_{\xi'}(D')\right)$.

Every crossing of $D\#D'$ belongs to either D or D', and further it is clear that the incidence number and type of each crossing in $D\#D'$ with respect to $\xi\#\xi'$ is the same as it is in D (with respect to ξ) or in D' (with respect to ξ'). Consequently, it follows that the correction term is additive, i.e. that

$$\mu_{\xi\#\xi'}(D\#D') = \mu_\xi(D) + \mu_{\xi'}(D').$$

This completes the proof. □

Proposition 4.6. *The determinant of checkerboard long knots is multiplicative under connected sum.*

Proof. The result is immediate since

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \det(A)\det(B).$$

□

4.3 Virtual Unknotting Operations

In this section, we introduce the virtual unknotting operations, which consist of four moves which can be used to unknot any virtual knot diagram, and study their effect on the incidence number and type of the crossing.

We have four unknotting operations: Chord deletion (**cd**), crossing change (**cc**), sign change (**sc**) and orientaion reversal (**or**) (see [BCG17b]). On a Gauss diagram, we can show them as follows:

It is obvious that if we apply **cd** to a Gauss diagram with n chords, after at most $n-1$ times applying it, we get the unknot. Using all four operations, we can often unknot a given knot in fewer steps. For example, if on a diagram it happens for two chords that their heads and tails are next to each other, then depending on the signs of the chords and their directions, combining the other three operations would create two chords which can be removed using a Reidemeister 2 move.

Starting with a checkerboard diagram with a fixed crossing, if we apply any one of $\{\mathbf{cc}, \mathbf{sc}, \mathbf{or}\}$ to this crossing, the new diagram is again checkerboard. We will now examine the effect on the incidence number and type of the crossing.

Starting with a checkerboard colored diagram, if we apply **sc** to a chord c, then $\eta(c)$ remains the same, but its type changes. See Figure 4.2.

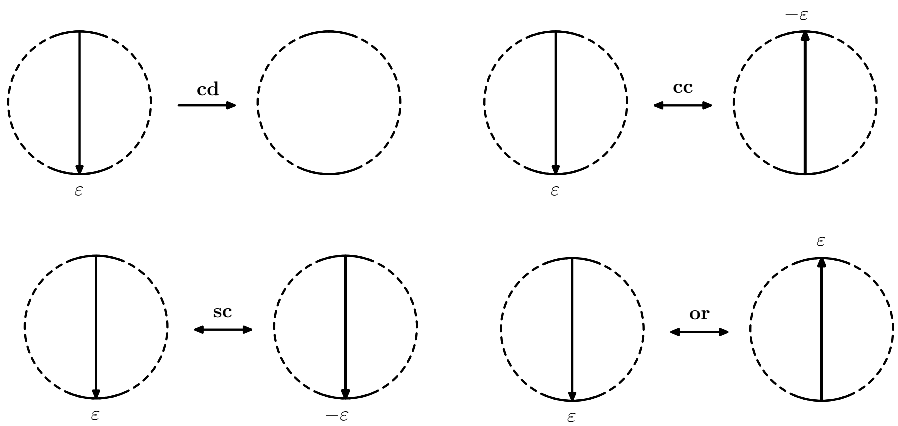

Figure 4.1: The unknotting operations.

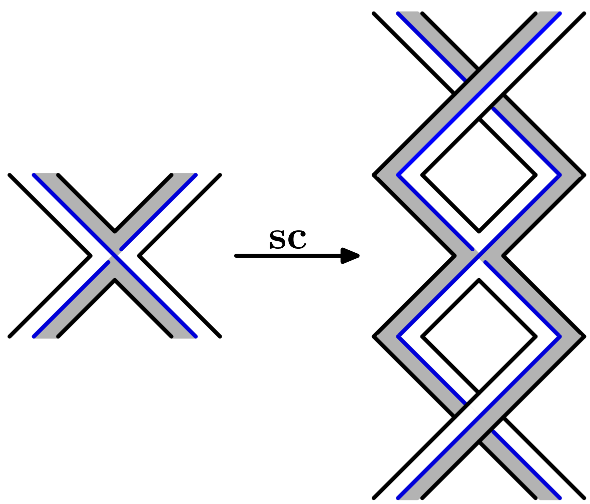

Figure 4.2: A coloring and the **sc** operation.

If we instead apply **or** to a chord c, then $\eta(c)$ changes sign and the type of the chord also changes, but the sign remains the same. See Figure 4.3.

Last but not least, if we apply **cc** to a chord c, then both the sign $\varepsilon(c)$ and the incidence $\eta(c)$ change signs, but the type remains the same. See Figure 4.4.

The table 4.1 summarizes the effect of applying **sc**, **or**, **cc** to the crossing c on its sign $\varepsilon(c)$, incidence number $\eta(c)$, and type.

Remark 4.7. Recall a flat knot is the knot diagram without the over- and under-

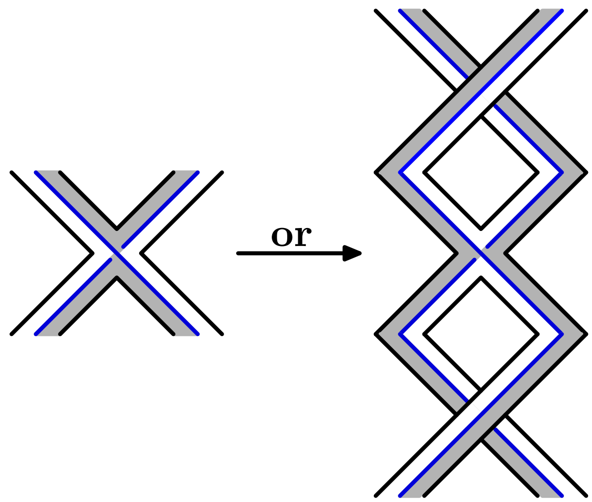

Figure 4.3: A coloring and the **or** operation.

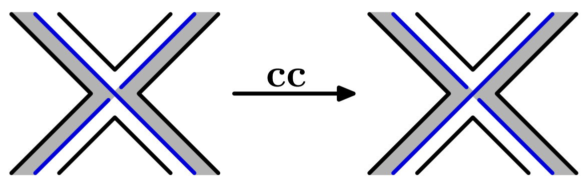

Figure 4.4: A coloring and the **cc** operation.

	sign	incidence	type
sc	$-\varepsilon(c)$	$\eta(c)$	$-\text{type}(c)$
or	$\varepsilon(c)$	$-\eta(c)$	$-\text{type}(c)$
cc	$-\varepsilon(c)$	$-\eta(c)$	$\text{type}(c)$

Table 4.1: The effect of applying **sc**, **or**, **cc** to the crossing c.

crossing information. More precisely, flat knots are equivalence classes of virtual knot diagrams up to Reideimeister moves and applications of **cc**. Flat knots can also be viewed as undecorated Gauss diagrams, but the interpretation of the arrow is different from that of a Gauss pattern. In a Gauss pattern, we interpret the arrow as pointing from over to under, whereas in flat knot theory one interprets the arrow as indicating which arc crosses from right to left.

Remark 4.8. If we apply **sc** to any chord in a Gauss diagram, then the Alexander determinant is unchanged. This follows from Proposition 3.15, since if we apply **sc** to any chord of a Gauss diagram, the underlying Gauss pattern is unchanged.

4.4 Signatures and Mirror Images

There are three involutions on virtual knots given by orientation reversal and horizontal and vertical mirror symmetry. In this section, we study the effect these involutions have on the signatures of the checkerboard knots.

We start with a virtual knot diagram D. In the Gauss diagram, if we apply **cc** to all chords, the result is called *vertical mirror image* of D, and we denote it by D^*. If we apply **sc** to all chords, the result is called *horizontal mirror image* of D, and we denote it by $D^\dagger$. We can also define $-D$, the *inverse* of D. Finally we can apply **or** to all chords of D, to obtain $D^{*\dagger}$.

Lemma 4.9. *If D is a minimal genus diagram, then so are $-D, D^*$ and $D^\dagger$.*

Proof. It is obvious that if D is a minimal genus diagram for K, then $-D$ is a minimal genus diagram for $-K$.

If $P = (S_D, \tilde{D})$ is the abstract link diagram associated with D, we place it inside $\{(x, y, z) \in \mathbb{R}^3 \mid y < 0\}$, in a way that the projection of $\tilde{D}$ on the xy-plane, is D. Now reflect P with respect to the plane $y = 0$. The result is the abstract link diagram associated with $D^\dagger$. This shows that if D is a minimal genus diagram, then $D^\dagger$ is also minimal genus.

Finally, if $P = (S_D, \tilde{D})$ is the abstract link diagram associated with D, we switch all the over-crossings and under-crossings in $\tilde{D}$, to obtain the abstract link diagram for D^*. It follows that if D is a minimal genus diagram, then D^* is also minimal genus. □

Suppose ξ is a checkerboard coloring of D and ξ^* is its dual coloring. Notice that a coloring is determined by the underlying flat knot. Therefore we can use the same notation for the colorings of the diagrams of the mirror images and the inverse knot.

The following picture, is a colored crossing in D (left), and $-D$ (right).

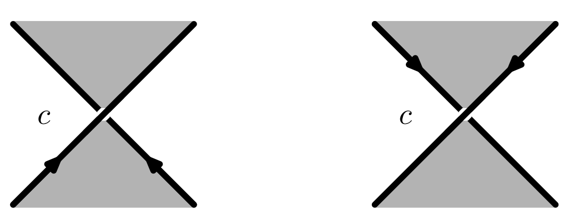

Figure 4.5: A colored crossing in D and $-D$.

Therefore, at each crossing, type and incidence number of that crossing is unchanged. Notice that black and white regions are also unchanged. Thus the two signatures for D and $-D$ are the same.

For D^*, based on section 4.3, at each crossing, the type is unchanged but the incidence number changes by a negative sign. As a result, both the Goeritz matrix and the correction term, are multiplied by -1. Thus $\sigma_\xi(D^*) = -\sigma_\xi(D)$ and $\sigma_{\xi^*}(D^*) = -\sigma_{\xi^*}(D)$.

For $D^\dagger$, we use the dual coloring ξ^*. Thus at each crossing, the type is unchanged but the incidence number changes by a negative sign. Therefore $\sigma_{\xi^*}(D^\dagger) = -\sigma_\xi(D)$, and $\sigma_\xi(D^\dagger) = -\sigma_{\xi^*}(D)$.

Similarly, for $D^{*\dagger}$, we find that $\sigma_{\xi^*}(D^{*\dagger}) = \sigma_\xi(D)$ and $\sigma_\xi(D^{*\dagger}) = \sigma_{\xi^*}(D)$.

We can summarize these observations, in the following proposition.

Proposition 4.10. *For a virtual knot K with checkerboard coloring ξ, the signatures of the mirror images and the inverse knot, are as follows:*

$$\begin{aligned}
(\sigma_\xi(-K), \sigma_{\xi^*}(-K)) &= (\sigma_\xi(K), \sigma_{\xi^*}(K)), \\
(\sigma_\xi(K^*), \sigma_{\xi^*}(K^*)) &= (-\sigma_\xi(K), -\sigma_{\xi^*}(K)), \\
(\sigma_\xi(K^\dagger), \sigma_{\xi^*}(K^\dagger)) &= (-\sigma_{\xi^*}(K), -\sigma_\xi(K)), \\
(\sigma_\xi(K^{*\dagger}), \sigma_{\xi^*}(K^{*\dagger})) &= (\sigma_{\xi^*}(K), \sigma_\xi(K)).
\end{aligned}$$

For virtual knots, since we have different mirror images, we can define different notions of a knot being *amphichiral*.

Definition 4.11. An unoriented virtual knot K is called *vertically amphichiral* if $K^* = K$, and is called *horizontally amphichiral* if $K^\dagger = K$. An oriented virtual knot K is called positive (negative) vertically amphichiral if $K^* = K$ ($K^* = -K$), and is called positive (negative) horizontally amphichiral if $K^\dagger = K$ ($K^\dagger = -K$).

We can use Proposition 4.10 to prove the following:

Proposition 4.12. *For a virtual knot K, if at least one of the signatures $\sigma_\xi(K)$ or $\sigma_{\xi^*}(K)$ is non-zero, then K is not positive vertically amphichiral. If $\sigma_\xi(K) + \sigma_{\xi^*}(K) \neq 0$, then K is not positive horizontally amphichiral.*

Now if we apply **cc** to one chord, we want to investigate how the signature changes.

Let D_+ be a minimal genus diagram with at least one positive crossing. We choose a positive crossing, and apply **cc** to that. It is not difficult to see that the new diagram is also minimal genus, and we denote it by D_-.

For classical links, both of the signatures are equal to the usual signature (see [GL78]). We can show the following relation holds between the signatures of D_+ and D_-:

$$\sigma(D_+) \leq \sigma(D_-) \leq \sigma(D_+) + 2.$$

There are several ways to show this. One way has been explained in [Liv04] for τ invariant, and the same argument applies with τ replaced by σ.

For checkerboard colorable virtual links, we have the following proposition (cf. [BZH14, Prop. 13.32]), which can be used to calculate the signature.

Proposition 4.13. *Let Q be a symmetric matrix of rank r over a field. There exists a chain of principal minors M_i ,$i = 0, 1, \ldots, r$, such that M_i is a principal minor of M_{i+1} and that no two consecutive determinants M_i and M_{i+1} vanish ($M_0 = 1$). For any such sequence of minors, $\sigma(Q) = \sum_{i=0}^{r-1} sign(M_i M_{i+1})$.*

Proposition 4.14. *Let D_+ and D_- be as above, and ξ be one of the colorings. Then we have*

$$\sigma_\xi(D_+) \leq \sigma_\xi(D_-) \leq \sigma_\xi(D_+) + 2.$$

Proof. We assume the Goeritz matrices are all non-singular. If they are not, we work with a non-singular sub-matrix, and the following proof works in that case as well. We denote by c, the crossing which we change from positive to negative, and consider four cases:

Case 1: Two different white regions meet at c, we call them X_0 and X_1, and c in both D_+ and D_- is a type II crossing.

In D_+, $\eta(c) = +1$, and in D_-, $\eta(c) = -1$. We have

$$\begin{aligned} \mu_\xi(D_+) &= \mu_\xi(D_-) + 2, \\ G_\xi(D_+) &= \begin{bmatrix} a+1 & * \\ * & A \end{bmatrix}, \\ G_\xi(D_-) &= \begin{bmatrix} a-1 & * \\ * & A \end{bmatrix}. \end{aligned}$$

If $\det(G_\xi(D_+))$ and $\det(G_\xi(D_-))$ have the same sign, then $\operatorname{sig}(G_\xi(D_+)) = \operatorname{sig}(G_\xi(D_-))$, hence $\sigma_\xi(D_-) = \sigma_\xi(D_+) + 2$. Now suppose the determinants have opposite signs, and assume the Goeritz matrices are $r \times r$. By Proposition 4.13, we have:

$$\begin{aligned}
\operatorname{sig}(G_\xi(D_+)) &= \sum_{i=0}^{r-2} \operatorname{sign}(M_i M_{i+1}) + \operatorname{sign}(\det(A)\det(G_\xi(D_+))).\\
\operatorname{sig}(G_\xi(D_-)) &= \sum_{i=0}^{r-2} \operatorname{sign}(M_i M_{i+1}) + \operatorname{sign}(\det(A)\det(G_\xi(D_-))),\\
&= \sum_{i=0}^{r-2} \operatorname{sign}(M_i M_{i+1}) + \operatorname{sign}(\det(A)\det(G_\xi(D_+)) - (\det(A))^2).
\end{aligned}$$

It follows, in this case $\operatorname{sig}(G_\xi(D_+)) = \operatorname{sig}(G_\xi(D_-)) + 2$, hence $\sigma_\xi(D_-) = \sigma_\xi(D_+)$.

Case 2: Same as case 1, except c is a type I crossing.

In D_+, $\eta(c) = -1$, and in D_-, $\eta(c) = +1$. We have

$$\begin{aligned}
\mu_\xi(D_+) &= \mu_\xi(D_-),\\
G_\xi(D_+) &= \begin{bmatrix} a-1 & * \\ * & A \end{bmatrix},\\
G_\xi(D_-) &= \begin{bmatrix} a+1 & * \\ * & A \end{bmatrix}.
\end{aligned}$$

By similar argument, if $\det(G_\xi(D_+))$ and $\det(G_\xi(D_-))$ have the same sign, then $\sigma_\xi(D_-) = \sigma_\xi(D_+)$, otherwise $\sigma_\xi(D_-) = \sigma_\xi(D_+) + 2$.

Case 3: Only one white region occurs at c, we call it X_0, and c is a type II crossing.

In this case, the corresponding Goeritz matrices are equal, but $\mu_\xi(D_+) = \mu_\xi(D_-) + 2$, therefore $\sigma_\xi(D_-) = \sigma_\xi(D_+) + 2$.

Case 4: Same as case 3, except c is a type I crossing.

In this case, the Goeritz matrices are equal, and $\mu_\xi(D_+) = \mu_\xi(D_-)$, thus $\sigma_\xi(D_-) = \sigma_\xi(D_+)$. □

Chapter 5

Alternating Virtual Knots

In this chapter, we introduce alternating virtual knots and links. Our main result is Theorem 5.19, which gives necessary and sufficient conditions for a virtual knot or link to be alternating in terms of its Goeritz matrices being positive and negative definite. We also prove a virtual analogue of the first Tait conjecture by adapting a result of [AFLT02]. about reduced alternating diagrams of knots on surfaces.

5.1 Alternating Virtual Knots

In this section, we recall the matrix-tree theorem of Bott and Mayberry [BM54], and use it to adapt Crowell's proof [Cro59] to show that the Alexander polynomial of any almost classical alternating link is alternating. We also outline the proof of [BBRS01, Theorem 2] and generalize it to alternating virtual links.

Definition 5.1. A virtual link diagram D is called *alternating* if the classical crossings alternate between over-crossing and under-crossing as you go around each component. A virtual link L is called *alternating* if it admits an alternating diagram.

The spectacular results concerning the Jones polynomial of classical alternating links are generally not true in the virtual case. For instance, the span of the Jones polynomial is not equal to the crossing number, and in fact there are alternating virtual knots with trivial Jones polynomial. For example, the knot $K = 6.90101$ is alternating and has Jones polynomial $V_K(t) = 1$. Further, the Jones polynomial is not necessarily alternating for alternating virtual knots.

For example, the knot $K = 5.2426$ in Figure 6.5 is alternating and has Jones polynomial $V_K(t) = 1/t^2 + 1/t^3 - 1/t^5$.

Let K be a virtual knot or link. We define the knot group G_K as in Section 2.1. We use Fox derivative to define the Jacobian matrix A. For virtual knots, the first elementary ideal $\mathcal{E}_1$ is not necessarily principal. We define the Alexander polynomial $\Delta_K(t)$ to be the generator of the smallest principal ideal containing $\mathcal{E}_1$. Since $\mathbb{Z}[t, t^{-1}]$ is a gcd domain, it is given by taking the gcd of all the $(n-1) \times (n-1)$ minors of A . If we remove the i-th row and j-th column of A we denote the corresponding minor by A_{ij}.

In [NNST12] and [BNW18], the authors showed for almost classical knots or links, $\mathcal{E}_1$ is principal, and the Alexander polynomial $\Delta_K(t)$ is given by taking the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing any row and any column from A.

Proposition 5.2. *For an almost classical knot or link L, the Alexander determinant* $\det(L)$ *is equal to* $|\Delta_L(-1)|$.

Proof. If D is a diagram for L, the coloring matrix $M(D)$ is exactly the matrix obtained from the Fox Jacobian matrix by replacing t with -1. □

Remark 5.3. For an almost classical knot K, the knot determinant $|\Delta_K(-1)|$ is an odd number (see [BGH+17]).

Proposition 5.4. *If L is a split checkerboard link then* $\det(L) = 0$.

Proof. Suppose $D = D_1 \cup D_2$ is a split checkerboard diagram for L. In each row of the coloring matrix the non-zero elements are either $2, -1, -1$ or $1, -1$. It follows the columns add up to zero. We consider a simple closed curve in the plane which separates D into two parts. It follows that the coloring matrix $M = M(D)$ admits a 2×2 block decomposition of the form

$$M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix},$$

where M_1 and M_2 are the coloring matrices for D_1 and D_2, respectively. Since $\det(M_1) = 0 = \det(M_2)$, it follows that the matrix obtained by removing a row and column from M also has determinant zero. □

We state the Bott-Mayberry theorem. For more details see [BZH14] and [BM54].

Let Γ be a finite oriented graph with vertices $\{c_i \mid 1 \le i \le n\}$ and oriented edges $\{u_{ij}^{\delta}\}$, such that c_i is the initial point and c_j the terminal point of u_{ij}^{δ}. Notice that δ enumerates the different edges from c_i to c_j. By a *rooted tree* (with root c_i) we mean a subgraph of $n-1$ edges such that every point c_k is terminal point of a path with initial point c_i. Let a_{ij} denote the number of edges with initial point c_i and terminal point c_j.

Theorem 5.5. *Let Γ be a finite oriented graph without loops ($a_{ii} = 0$). The principal minor H_{ii} of the graph matrix*

$$H(\Gamma) = \begin{bmatrix} (\sum_{k\neq 1} a_{k1}) & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & (\sum_{k\neq 2} a_{k2}) & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & (\sum_{k\neq n} a_{kn}) \end{bmatrix},$$

is equal to the number of rooted trees with root c_i.

Corollary 5.6. *Let Γ be a finite oriented loopless graph with a* valuation $f : \{u_{ij}^{\delta}\} \to \{-1, 1\}$ *on edges. Then the principal minor H_{ii} of the matrix $H = [b_{ij}]$, where*

$$b_{ij} = \begin{cases} \sum_{\delta} f(u_{ij}^{\delta}), & i \neq j, \\ -\sum_{k\neq i} b_{ki}, & i = j, \end{cases}$$

satisfies the following equation:

$$H_{ii} = \sum f(Tr(i)),$$

where the sum is to be taken over all c_i-rooted trees $Tr(i)$, and where

$$f(Tr(i)) = \prod_{u_{kj}^{\delta} \in Tr(i)} f(u_{kj}^{\delta}).$$

For a virtual link diagram, there are (at least) two ways one can associate a 4-valent graph. One way is to consider the diagram D itself. It has vertices for the classical and virtual crossings and edges running from one classical or virtual crossing to the next. This graph is planar. The other way to associate a graph is to consider vertices only for classical crossings. The key difference is that in general, this graph is not planar. For an alternating diagram D, we describe this graph and an orientation on it as follows:

Let D has classical crossings $c_1, \ldots, c_n$. The vertices of Γ are $c_1, \ldots, c_n$. At each vertex consider two out-going edges corresponding to the over-crossing arc, and two in-coming edges for the under-crossing arcs (see Figure 5.1). This is called the *source-sink orientation* or the *alternate orientation*. This orientation is possible because D is alternating, and an out-going edge at the vertex c_i, should be an in-coming edge for the adjacent vertex.

Remark 5.7. In general, any checkerboard colorable diagram D admits a source-sink orientation. In fact, a diagram is checkerboard colorable if and only if it admits a source-sink orientation (see [KNS02, Proposition 6]).

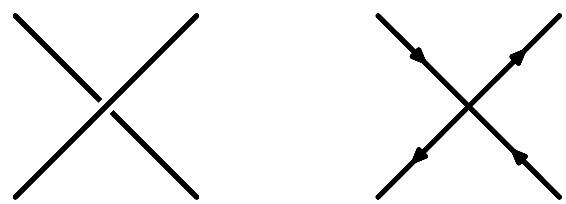

Figure 5.1: The source-sink orientation.

Theorem 5.8. *Let L be an almost classical alternating link with a connected alternating diagram D. The Alexander polynomial $\Delta_L(t)$ is alternating.*

Proof. For the unknot the result is obvious. Assume D has $n \geq 1$ classical crossings. Orient D and enumerate the crossings by $c_1, \dots, c_n$. Label the arcs by $g_1, \dots, g_n$. At the crossing c_i, suppose $g_{\nu(i)}$ is the over-crossing arc in the upward direction, and $g_{\lambda(i)}$ and $g_{\rho(i)}$ are the left and right under-crossing arcs, respectively. Define the relation $r_i = g_{\lambda(i)} g_{\nu(i)} g_{\rho(i)}^{-1} g_{\nu(i)}^{-1}$. Notice that we are using left-handed meridian convention.

Now consider the graph Γ associated with D, with the source-sink orientation on it. Label the edges by u_{ij}^{δ}. Define the valuation f as follows. At the crossing c_j, if u_{ij}^{δ} corresponds to $g_{\lambda(j)}$, then $f(u_{ij}^{\delta}) = 1$, and if it corresponds to $g_{\rho(j)}$, then $f(u_{ij}^{\delta}) = -t$.

Define the matrix H as in the Corollary 5.6. Notice that D is alternating and there is a one-to-one correspondence between the crossings of D and the set of over-crossing arcs. Therefore we can choose to label over-crossing arcs, such that $\nu(i) = i$. The matrix H is the transpose of the Jacobian matrix A. The Alexander polynomial $\Delta_L(t) = A_{ii} = H_{ii}$. By the Corollary 5.6,

$$H_{ii} = \sum \prod_{u_{kj}^{\delta} \in \mathrm{Tr}(i)} f(u_{kj}^{\delta}).$$

Since $f(u^\delta_{kj}) = 1, \text{or } -t$, the product $\prod_{u^\delta_{kj}\in \mathrm{Tr}(i)} f(u^\delta_{kj})$ is of the form $(-1)^l t^l$ and H_{ii} is an alternating polynomial. Therefore $\Delta_L(t)$ is alternating. □

Example 5.9. Up to 6 crossings, the following eight almost classical knots do not have alternating Alexander polynomial, therefore by Theorem 5.8 they are not alternating virtual knots.

K	$\Delta_K(t)$
5.2331	$t^2 - 1 + t^{-1}$
6.85091	$1 + t^{-1} - t^{-2}$
6.85774	$t - 1 + t^{-2}$
6.87548	$-t^2 + 2t + 1 - t^{-1}$
6.87875	$t + 1 - 2t^{-1} + t^{-2}$
6.89156	$2t - 1 - t^{-1} + t^{-2}$
6.89812	$t^2 - 2 + 2t^{-1}$
6.90099	$t - t^{-1} + t^{-3}$

Table 5.1: Almost classical knots with non-alternating Alexander polynomial.

The weak form of the first Tait Conjecture, namely that every knot having a reduced alternating diagram with at least one crossing is nontrivial, was first proved by Bankwitz [Ban30] in 1930; and since then, Menasco and Thistlethwaite [MT91a] and Andersson [And95] published simpler proofs. Here we outline the proof by Balister et al. [BBRS01] and generalize it to alternating virtual links. Notice that for alternating virtual knots, this result has been proved using a different method by Cheng in [Che15, Proposition 3.3].

Consider the graph Γ with vertices $\{c_1, \ldots, c_n\}$ as before.

Definition 5.10. The *outdegree* of the vertex c_i, denoted $d^+(c_i)$, is the number of edges of Γ with initial point c_i. The *indegree* of the vertex c_i, denoted $d^-(c_i)$, is the number of edges of Γ with terminal point c_i. Therefore

$$d^+(c_i) = \sum_{j=1}^{n} a_{ij} \quad , \quad d^-(c_i) = \sum_{j=1}^{n} a_{ji}.$$

Definition 5.11. A *walk* in a graph is an alternating sequence of vertices and edges, starting with a vertex c_i and ending with a vertex c_j. A walk is called a *trail* if all the edges in that walk are distinct. A *circuit* is a trail which starts and ends at a vertex c_i. An *Eulerian circuit* is a circuit which contains all the edges of Γ. A graph Γ is called *Eulerian* if it has an Eulerian circuit.

An Eulerian graph is necessarily connected and has $d^+(c_i) = d^-(c_i)$ for every vertex. Let $t_i(\Gamma)$ be the number of rooted trees with root c_i, then the BEST Theorem is as follows (see [vAEdB51] and [Bol98, Theorem 13]).

Theorem 5.12. *Let $s(\Gamma)$ be the number of Eulerian circuits of Γ, then*

$$s(\Gamma) = t_i(\Gamma)\prod_{j=1}^{n}(d^+(c_j) - 1)!$$

In particular, if Γ is a two-in two-out oriented graph, i.e., $d^+(c_i) = d^-(c_i) = 2$ for every i, then by Theorem 5.5 and 5.12,

$$s(\Gamma) = t_i(\Gamma) = H_{ii}, \quad \text{for every } i.$$

A vertex c of a graph Γ is an *articulation vertex* if Γ is the union of two nontrivial graphs with only the vertex c in common. In particular, a vertex incident with a loop is an articulation vertex. In [BBRS01] Balister et al. proved the following theorem:

Theorem 5.13. *Let Γ be a connected two-in two-out oriented graph with $n \geq 2$ vertices and with no articulation vertex. Then $s(\Gamma) \geq n$.*

Definition 5.14. Let Σ be a closed surface, and consider a link L in $\Sigma \times I$. Let D be the link projection in $\Sigma_0 = \Sigma \times \{0\}$. A crossing c of D is called *nugatory* if there is a simple closed curve on Σ_0 that intersects D exactly once at c.

Recall that associated with an alternating virtual link diagram D, there is an oriented two-in two-out graph Γ. If D has no nugatory crossings, then Γ has no articulation vertex.

Corollary 5.15. *Let K be an almost classical alternating knot, and D an alternating diagram for K, which has no nugatory crossings with $n \geq 2$ crossings. Then*

$$|\Delta_K(-1)| \geq n.$$

Proof. By the proof of Theorem 5.8, $|\Delta_K(-1)|$ counts $t_i(\Gamma)$ the number of rooted trees with root c_i in the oriented graph Γ, associated with the knot diagram D. By Theorem 5.12, $t_i(\Gamma) = s(\Gamma)$, and the result follows from Theorem 5.13. □

Theorem 5.16. *Let L be an alternating virtual link and D a connected alternating diagram with $n \geq 2$ classical crossings, which has no nugatory crossings. Then for the Alexander determinant of L we have:*

$$\det(L) \geq n.$$

Proof. Since D is alternating, we can repeat the proof of Theorem 5.8. By Theorem 5.6, the determinant of L counts the number of spanning trees which is equal to $s(\Gamma)$. The result follows from Theorem 5.13. □

Corollary 5.17. *Suppose a virtual link L admits a connected alternating diagram D with no nugatory crossings. If D has $n \geq 1$ classical crossings, then L is not split.*

Proof. By Theorem 5.16 $\det(L) \geq n$, in particular $\det(L) \neq 0$. The result follows from 5.4. □

5.2 Characterization of Alternating Virtual Links

In this section, we prove one of the main results in this thesis, which gives necessary and sufficient conditions for a virtual knot to admit an alternating virtual knot diagram.

Lemma 5.18. *Suppose D is a connected checkerboard colorable link diagram. Then D is alternating if and only if each crossing has the same incidence number, i.e. for any two crossings c_1 and c_2 of D, we have $\eta(c_1) = \eta(c_2)$.*

Proof. First assume D is alternating. Consider the Gauss diagram for D. On each core circle, start with the foot of one chord, it is between two arcs, travel counterclockwise and let the right side of the first arc and the left side of the second arc be black. Alter the coloring as you pass each arrow-head or tail. The diagram is alternating, thus at all the crossings we see one pattern. Namely, they are all type A crossings. Change the coloring, then all the crossings have type B.

Now suppose each crossing c has incidence number $\eta(c) = +1$. When we move counterclockwise on a core circle, at the tail of each arrow, first we see the black color on the right and then we see it on the left, and at the head of each arrow, first we see the black color on the left, then on the right. The diagram D is checkerboard colorable and passing each arrow-head or tail, we have to switch the color, thus the diagram should be alternating. If each crossing c has incidence number $\eta(c) = -1$, then switch the checkerboard coloring. □

Thus if D is an alternating diagram, then s_∂ is the state obtained from D by resolving all the crossings to 0-smoothing and $\bar{s}_\partial$ is all 1-smoothing state.

The next result is a generalization of Proposition 4.1. from [Gre17].

Theorem 5.19. *Suppose K is a connected checkerboard colorable virtual link with virtual genus $g_v(K)$. Then D is an alternating diagram for K if and only if D admits a checkerboard coloring ξ with dual coloring ξ^* such that*

1) $sg(D) = g_v(K)$,

2) $\sigma_\xi(D) - \sigma_{\xi^*}(D) = 2g_v(K)$,

3) $G_\xi(D)$ is negative definite (or empty) and $G_{\xi^*}(D)$ is positive definite (or empty).

Proof. Set $g = g_v(K)$ and suppose D has n real crossings.

First assume D is alternating. Then D is a minimal genus diagram (see Remark 5.21). Let ξ be the checkerboard coloring in which all the crossings are type B. Suppose $G_\xi(D)$ is not empty. Every crossing c of D has incidence number $\eta(c) = -1$ with respect to ξ. Therefore all the non-diagonal entries of $G_\xi(D)$ are non-negative. In this case negative crossings are type II. So if n_- is the number of negative crossings, then $\mu_\xi(D) = -n_-$. If $G_\xi(D) = [g_{ij}]_{1\leq i,j\leq m}$, and we have

$$[x_1 \cdots x_m]\,[g_{ij}]_{1\leq i,j\leq m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \sum_{1\leq i,j\leq m} g_{ij}x_ix_j.$$

$G_\xi(D)$ is symmetric, thus we have

$$\begin{aligned} \sum_{1\leq i,j\leq m} g_{ij}x_ix_j &= 2\sum_{1\leq i<j\leq m} g_{ij}x_ix_j + \sum_{i=1}^{m} g_{ii}x_i^2, \\ &= 2\sum_{1\leq i<j\leq m} g_{ij}x_ix_j - \sum_{i=1}^{m}\sum_{k=1,k\neq i}^{m} g_{ik}x_i^2 - \sum_{i=1}^{m} g_{i0}x_i^2, \\ &= -\sum_{1\leq i<j\leq m} g_{ij}(x_i - x_j)^2 - \sum_{i=1}^{m} g_{i0}x_i^2 \leq 0. \end{aligned}$$

Since $G_\xi(D)$ is not empty, there are more than one white regions. On the other hand D is connected , hence the associated ALD $P = (S_D, \tilde{D})$ is connected, as a result the two spanning surfaces are connected. Therefore, there exists at least one crossing which is between two distinct white regions. It follows that at least one of $g_{i0},\ i = 1, \cdots, m$ is non-zero. This shows the inequality above is strict and $G_\xi(D)$ is negative definite.

Every crossing c of D has incidence number $\eta(c) = +1$ with respect to the dual coloring ξ^*, thus by a similar argument, we see that $G_{\xi^*}(D)$ is positive definite.

Suppose we have β black disks and α white disks in ξ. All the crossings are type B, hence $|s_\partial| = \beta$ and $|\bar{s}_\partial| = \alpha$. The diagram D has the boundary property, therefore

$$\beta + \alpha = n + 2 - 2g. \tag{1}$$

Now $\text{sig}(G_\xi) = 1 - \alpha$, thus

$$\sigma_\xi = 1 - \alpha + n_-.$$

For ξ^*, we have β white disks and α black disks. The Goeritz matrix G_{ξ^*} is positive definite, thus $\text{sig}(G_{\xi^*}) = \beta - 1 = n + 2 - 2g - \alpha - 1$. In this coloring each positive crossing is type II, thus $\mu_{\xi^*}(D) = n_+$ and

$$\sigma_{\xi^*}(D) = n + 1 - 2g - \alpha - n_+ = n_- + 1 - 2g - \alpha.$$

Therefore $\sigma_\xi(D) - \sigma_{\xi^*}(D) = 2g$.

Conversely, assume $G_\xi(D)$ is negative definite and $G_{\xi^*}(D)$ is positive definite, and $\sigma_\xi(D) - \sigma_{\xi^*}(D) = 2g$. In ξ, let a be the number of type A crossings, and b be the number of type B crossings. Let a_+ and a_- be the number of positive and negative type A, and let b_+ and b_- be the number of positive and negative type B crossings, respectively. Then $a = a_+ + a_-$, $b = b_+ + b_-$. Also, we have $\mu_\xi(D) = a_+ - b_-$ and $\mu_{\xi^*}(D) = b_+ - a_-$. Thus, by (1), we have

$$\begin{aligned}
b - a &= (b_+ + b_-) - (a_+ + a_-), \\
&= \mu_{\xi^*}(D) - \mu_\xi(D), \\
&= \text{sig}(G_{\xi^*}) - \sigma_{\xi^*}(D) - \text{sig}(G_\xi) + \sigma_\xi(D), \\
&= \beta - 1 - \sigma_{\xi^*}(D) + \sigma_\xi(D) - 1 + \alpha, \\
&= \sigma_\xi(D) - \sigma_{\xi^*}(D) + n - 2g.
\end{aligned}$$

If $\sigma_\xi(D) - \sigma_{\xi^*}(D) = 2g$, then

$$a + b = n = b - a \Rightarrow a = 0,$$

and we have only type B crossings and by Lemma 5.18, the diagram is alternating.

If $G_\xi(D)$ or $G_{\xi^*}(D)$ or both are empty, then we can do the same calculations without any changes. □

Remark 5.20. We cannot eliminate the assumption $\sigma_\xi(D) - \sigma_{\xi^*}(D) = 2g$. For example the knot $K = 3.5$ (see Figure 3.5) has virtual genus equal to 1, and $\sigma_\xi(K) = \sigma_{\xi^*}(K) = 2$. The Goeritz matrix $G_\xi(K) = [2]$, which is positive definite, and $G_{\xi^*}(K)$ is empty, but K is not alternating.

Remark 5.21. In Section 5.3 we prove that any reduced alternating virtual knot diagram has minimal genus. This means for virtual knots, we can eliminate the first assumption from Theorem 5.19.

Corollary 5.22. *Suppose K is a connected alternating virtual link with r components, and D is an alternating diagram for K. Then the Goeritz matrices are non-singular, and the nullities of K both vanish.*

Example 5.23. For the classical knots, we can always rearrange the connected sum of two alternating diagrams to be alternating. For the virtual knot $K = 3.7$, there are a number of ways to form the connected sum $K\#K$. Some will be alternating diagrams and others will not. Do any of the non-alternating diagrams of $K\#K$ represent alternating virtual knots? The alternating connected sums that we know about are in the family 6.90101–6.90108. All of these are connected sums of two alternating knots, either 3.6 or 3.7. The connected sums which we should check, are among the knots 6.89187–6.89198. The first three virtual knots are connected sums with one side a trefoil, so each one can be rearranged to be alternating. The signatures for the others (6.89190–6.89197) are $(-2,-2),(-2,-2),(0,0),(-2,-2),(0,0),(-2,-2),(0,0),(0,0)$, respectively. It follows from these calculations and Theorem 5.19 that they do not represent alternating virtual knots. This situation is another instance where alternating virtual knots behave quite differently from classical alternating knots.

We use Theorem 5.19 to prove the analogue of the second Tait Conjecture for the virtual knots:

Theorem 5.24. *If K is a non-classical alternating virtual knot, then K is not positive amphichiral. If $|\sigma_\xi(K)| \neq |\sigma_{\xi^*}(K)|$, then K is not negative amphichiral.*

Proof. Since K is not classical, $g_v(K) \neq 0$ and by Theorem 5.19, $\sigma_\xi(K) - \sigma_{\xi^*}(K) = 2g_v(K)$, so the signatures cannot both be zero. It follows from Proposition 4.12 that K is not positive amphichiral. The second part follows directly from the same proposition. □

Application

Using the Jones polynomial it is possible to determine some virtual knots are not alternating. In [KNS02], Kamada et al. introduce a skein formula for the Jones polynomial of a virtual knot. Then they use this formula to find a condition in which the Jones polynomial of an alternating knot should satisfy.

Proposition 5.25. *Among 76 almost classical knots which have crossing number less than or equal to 6, there are 29 alternating knots.*

In this section, we will show how to deduce Proposition 5.25, and we use Theorem 5.19 and Corollary 5.22 to determine which almost classical knots (up to 6 crossings) admit alternating diagrams. There are a total of 76 such knots. The 29 knots in Table 5.2 admit alternating diagrams (1 with 3 crossings, 2 with 4 crossings, 5 with 5 crossings, and 21 with 6 crossings).

3.6	5.2439	6.90139	6.90185	6.90219
4.105	5.2445	6.90146	6.90194	6.90227
4.108	6.89187	6.90147	6.90195	6.90228
5.2426	6.89198	6.90150	6.90209	6.90232
5.2433	6.90109	6.90167	6.90214	6.90235
5.2437	6.90115	6.90172	6.90217	

Table 5.2: Alternating almost classical knots.

The signatures and the genus of the remaining knots are given in Table 5.3.

For the knots 6.72944, 6.73583, 6.75348, 6.77905, 6.77908, 6.77985, 6.85103 and 6.87875 at least one of the Goeritz matrices is singular. This means they have nullity greater than 0, and so by Corollary 5.22 they are not alternating. For the knot 6.78358, the Alexander determinant equals 5 and by Theorem 5.16 it is not alternating. Using these techniques we cannot decide whether the knots 6.87188, 6.87310 and 6.87859 are alternating or not (we will answer this question later in Lemma 5.35). For the remaining knots in the table, the absolute value of the difference between the two signatures is not equal to twice the genus of the corresponding knot. It follows that none of the knots in this table admit an alternating diagram.

Knot	Signatures	Genus
4.99	$(0,0)$	1
5.2012	$(0,2)$	2
5.2025	$(0,0)$	2
5.2080	$(2,2)$	1
5.2133	$(0,0)$	2
5.2160	$(0,0)$	1
5.2331	$(2,2)$	1
6.72507	$(0,2)$	2
6.72557	$(0,0)$	2
6.72692	$(0,2)$	2
6.72695	$(0,0)$	1
6.72938	$(0,2)$	2
6.72944	$(1,2)$	1
6.72975	$(0,0)$	2
6.73007	$(0,0)$	2
6.73053	$(2,2)$	1
6.73583	$(0,1)$	2
6.75341	$(0,0)$	1
6.75348	$(0,1)$	2
6.76479	$(0,2)$	2
6.77833	$(0,0)$	2
6.77844	$(0,0)$	2
6.77905	$(0,1)$	2
6.77908	$(-1,0)$	1

Knot	Signatures	Genus
6.77985	$(-1,0)$	2
6.78358	$(0,2)$	1
6.79342	$(0,0)$	2
6.85091	$(0,0)$	1
6.85103	$(-1,0)$	1
6.85613	$(0,2)$	2
6.85774	$(0,2)$	2
6.87188	$(0,2)$	1
6.87262	$(2,2)$	2
6.87269	$(0,0)$	1
6.87310	$(2,4)$	1
6.87319	$(0,0)$	2
6.87369	$(-2,0)$	2
6.87548	$(0,0)$	1
6.87846	$(2,2)$	1
6.87857	$(0,0)$	1
6.87859	$(-2,0)$	1
6.87875	$(-2,-1)$	1
6.89156	$(2,2)$	1
6.89623	$(0,0)$	1
6.89812	$(2,2)$	1
6.89815	$(0,0)$	1
6.90099	$(2,2)$	2

Table 5.3: Signatures and genus of some almost classical knots.

5.3 The Virtual Tait Conjecture

In this section, we introduce the Tait conjecture for virtual knots. The main result that is proved is, given an alternating virtual link L, if D is a reduced alternating virtual link diagram for L, then S_D has minimal genus. As a consequence, we deduce that D also has minimal crossing number.

The Jones polynomial is not an unknot detector for virtual knots. As an example the alternating virtual knot 3.7 has trivial Jones polynomial. Thus, it is not possible to deduce a virtual analogue of the Kauffman-Murasugi-Thistlethwaite theorem using the Jones polynomial.

Definition 5.26. Let Σ be a closed surface, and consider a link L in $\Sigma \times I$. Let D be the link projection on $\Sigma_0 = \Sigma \times \{0\}$. A crossing c of D is called a *removable crossing* if there is a trivial simple closed curve on Σ_0 that intersects D exactly once at c (a trivial curve is a curve that is homotopic to the constant curve). The projection D is *reduced* on Σ_0 if it has no removable crossings.

Notice if we smooth a removable crossing c in an oriented way, the part of D which lies inside the simple closed curve is a classical knot diagram. Consequently, we can twist along the curve to make the removable crossing disappear just as for classical links (see Figure 2.7).

Note that in a virtual link diagram, every removable crossing is nugatory, but not all nugatory crossings are removable. The point being, the simple closed curve in Definition 5.26 is required to be null-homotopic, whereas in Definition 5.14 it is not. For example, Figure 5.2 depicts a Gauss diagram of the virtual knot 5.37 which contains a crossing which satisfies this more general notion. In the next example, we show this crossing is not removable (in the sense of Definition 5.26).

Example 5.27. Consider the virtual knot $K = 5.37$ whose Gauss diagram appears in Figure 5.2. This virtual knot has a nugatory crossing in the sense of Dye's definition. The slice genus tables show that K has graded genus 2, which means the slice genus of K is less than or equal to 2. On the other hand the Rasmussen invariant of K is -4, which means the slice genus is greater than or equal to 2, so K has slice genus 2. If c was removable, it would be equivalent to a virtual knot with fewer crossings. There are only ten virtual knots with four or fewer crossings and with graded genus and slice genus equal to 2; they are 4.1, 4.3, 4.7, 4.9, 4.15, 4.29, 4.48, 4.61, 4.69, and 4.78.

However, K has Jones polynomial

$$V_K(t) = 1/t^2 + 1/t^3 - 2/t^4 + 1/t^5 + 2/t^{5/2} - 2/t^{7/2},$$

and this rules out all but 4.1, 4.3 and 4.7. Further, K has virtual Alexander polynomial

$$H_K(s,t,q) = (s^2t^3q - st^2q - s^3t^3 - 1 + s^3t^2q^{-1} - 2s^2tq^{-1} + sq^{-1})(1-st)(1-sq^{-1})(1-tq),$$

and direct comparison shows that K is not equivalent to 4.1, 4.3, and 4.7.

Let Σ be a compact surface, and $\pi : \Sigma \to \Sigma \times \{0\}$ be the projection map. Let $\pi(K)$ be a reduced alternating projection of an alternating knot in $\Sigma \times I$ and let $\pi(K')$ be an arbitrary projection (of the same knot). Then $c(\pi(K)) \leq c(\pi(K'))$ (See [AFLT02, Theorem 1.1]).

Note that Theorem 1.1 of [AFLT02] only considers the isotopy class of knots in a fixed thickened surface. The next theorem gives an extension of this result

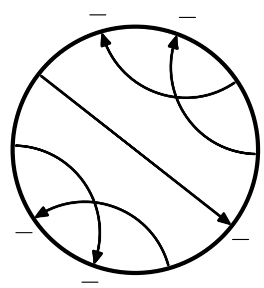

Figure 5.2: The Gauss diagram for 5.37.

for virtual knots, and it is proved by means of parity projection.

Theorem 5.28. *Suppose D is a reduced alternating virtual knot diagram for a virtual knot K. Then D is a minimal genus diagram.*

Proof. Let Σ be the Carter surface of D, and suppose to the contrary that genus of Σ is not minimal. We give the argument under the assumption that the genus of Σ is one more than the virtual genus of K, i.e. assuming that $g(\Sigma) = g_v(K) + 1$.

We will use $\mathfrak{K}$ to denote the knot in $\Sigma \times I$ associated to the virtual knot diagram D. Since $\mathfrak{K}$ is not a minimal genus representative of K, the set of knots in $\Sigma \times I$ which are isotopic to $\mathfrak{K}$ and which admit a destabilizing curve is non-empty. Thus, we can choose a knot $\mathfrak{K}'$ in $\Sigma \times I$ in this set, together with a destabilizing curve γ in Σ such that $\mathfrak{K}' \cap \gamma = \varnothing$. We can further choose $\mathfrak{K}'$ so that its projection $\pi(\mathfrak{K}')$ has minimal crossing number, where $\pi : \Sigma \times I \to \Sigma$. We will use D' to denote the virtual knot diagram determined by $\pi(\mathfrak{K}')$. Observe that $c(D') = c(\pi(\mathfrak{K}')) \geq c(\pi(\mathfrak{K})) = c(D)$ by [AFLT02, Theorem 1.1].

Let P_γ be homological parity projection, and we will apply this to the knots $\mathfrak{K}$ and $\mathfrak{K}'$ in the thickened surface $\Sigma \times I$.

Since the curve γ and diagram $\pi(\mathfrak{K}')$ do not intersect, it follows that all the crossings of $\mathfrak{K}'$ are even. Thus $P_\gamma(\mathfrak{K}') = \mathfrak{K}'$. On the other hand, since $\mathfrak{K}$ and $\mathfrak{K}'$ are isotopic knots in $\Sigma \times I$, it follows that $P_\gamma(\mathfrak{K})$ is equivalent to $P_\gamma(\mathfrak{K}') = \mathfrak{K}'$. Furthermore, it must be the case that $\mathfrak{K}$ contains an odd crossing, for otherwise we could destabilize $\mathfrak{K}$ along γ, contradicting the fact that Σ is the minimal genus surface for the virtual knot diagram D. If D'' denotes the virtual knot diagram determined by $P_\gamma(\mathfrak{K})$, then we have $c(D'') < c(D)$.

There are two cases, depending on the genus of the supporting surface of D''.

Case 1: The genus of the supporting surface of D'' equals $g(\Sigma) - 1$. Let $\mathfrak{K}''$

be a representative knot in $\Sigma' \times I$, where genus $g(\Sigma') = g(\Sigma) - 1$. Since the support genus of D' is also equal to $g(\Sigma)-1$, we can represent it by a knot $\widetilde{\mathfrak{K}}'$ in $\Sigma' \times I$. By Kuperberg's theorem, there is a diffeomorphism $\psi : \Sigma' \times I \to \Sigma' \times I$ carrying $\mathfrak{K}''$ to a knot isotopic to $\widetilde{\mathfrak{K}}'$. Notice that $\mathfrak{K}''$ and $\psi(\mathfrak{K}'')$ have the same underlying diagram, hence the same number of crossings. Thus, it follows that $c(\pi'(\mathfrak{K}'')) = c(\pi'(\psi(\mathfrak{K}''))) < c(D) \leq c(D')$, where $\pi' : \Sigma' \times I \to \Sigma'$, which contradicts our choice that D' had minimal crossing number.

Case 2: If the genus of the supporting surface of $P_\gamma(\mathfrak{K})$ equals $g(\Sigma)$, then it follows that $P_\gamma(\mathfrak{K})$ is represented by a knot in $\Sigma \times I$ which contains odd chords with respect to γ-homological parity. Thus, we can apply P_γ again, and the crossing number of the diagram of $P_\gamma^2(\mathfrak{K})$ will continue to decrease. If the genus of the supporting surface of $(P_\gamma)^2(\mathfrak{K})$ equals $g(\Sigma) - 1$, we argue as in the first case. Otherwise, after repeated application of P_γ, since each application removes chords, this process must terminate after some number of applications. In particular, it follows that, for some $n \geq 1$, $(P_\gamma)^n(\mathfrak{K})$ will not contain any odd chords with respect to γ-homological parity. Since $(P_\gamma)^n(\mathfrak{K})$ is equivalent to $\mathfrak{K}'$, it follows that the diagram of $(P_\gamma)^n(\mathfrak{K})$ must have supporting surface of genus equal to $g(\Sigma) - 1$, and we argue as before. □

Corollary 5.29. *If K is a classical knot which admits a virtual alternating diagram D, then K is a classical alternating knot.*

Proof. If D is a reduced virtual alternating diagram for K, then D is a minimal genus diagram. Since K is a classical knot, this implies that S_D has genus zero. Thus, D is a classical alternating diagram for K and the corollary follows. □

Theorem 5.30. *Let D be a reduced alternating knot diagram for a virtual knot K. Then D has minimal crossing number.*

Proof. Notice that if $g_v(K) = 0$, then D is a classical knot diagram and the result follows from the first Tait Conjecture for classical knots. Therefore we can assume $g_v(K) > 0$.

Throughout the argument, we will use $\mathfrak{K}$ to denote the associated knot in the thickened surface $\Sigma \times I$ associated to D. By the Theorem 5.28, D is a minimal genus diagram for K.

Let D' be a diagram for K with minimal crossing number, and use $\mathfrak{K}'$ to denote the associated knot in the thickened surface $\Sigma' \times I$. Using parity, one can show that, given any minimal crossing diagram for a virtual knot, its Carter surface has minimal genus (see Theorem 5 in [Man13]). From this, it follows that D' is also a minimal genus diagram for K.

By [Kup03, Theorem 1], there is a diffeomorphism $f : \Sigma \times I \to \Sigma' \times I$ such that $f(\mathfrak{K}) = \mathfrak{K}'$. We pick the same model surface for both Σ and Σ', which we denote by Σ, and we assume that $f : \Sigma \times I \to \Sigma \times I$ is the diffeomorphism with $f(\mathfrak{K}) = \mathfrak{K}'$. Notice that $f(\Sigma \times \{0\}) = \Sigma \times \{0\}$ (see Definition 3.3).

Let $\pi : \Sigma \times I \to \Sigma \times \{0\}$ be the projection map. We can assume $\pi(\mathfrak{K})$ and D have the same crossing number and $\pi(\mathfrak{K})$ is alternating and reduced. Similarly, we can assume $\pi(\mathfrak{K}')$ and D' have the same crossing number.

Let $f_0 : \Sigma \to \Sigma$ be the induced diffeomorphism, i.e. $f_0(x) = \pi(f(x, 0))$. For $(x, t) \in \Sigma \times I$, define $f'(x, t) = (f_0(x), t)$. This gives a diffeomorphism $f' : \Sigma \times I \to \Sigma \times I$ such that $f|_{\Sigma\times\{0\}} = f'|_{\Sigma\times\{0\}}$. Set $h = f \circ (f')^{-1}$. Notice that $h|_{\Sigma\times\{0\}}$ is the identity map.

Hatcher proved in [Hat76, Lemma 2], that if Σ is a compact surface other than S^2, then $\mathrm{PL}(\Sigma\times I,\ \mathrm{rel}\ \Sigma\times\{0\})$ is contractible. Here, $\mathrm{PL}(\Sigma\times I,\ \mathrm{rel}\ \Sigma\times\{0\})$ is the space of piecewise linear self-homeomorphisms of $\Sigma \times I$ that are the identity when restricted to $\Sigma\times\{0\}$. The corresponding statement in the smooth category, namely that $\mathrm{Diff}(\Sigma \times I,\ \mathrm{rel}\ \Sigma \times \{0\})$ is contractible, can be deduced from [Hat76, Lemma 2] and Hatcher's positive solution to the Smale Conjecture, that is, $\mathrm{Diff}(D^3, \partial D^3)$ is contractible [Hat83]. Since $h \in \mathrm{Diff}(\Sigma\times I,\ \mathrm{rel}\ \Sigma\times\{0\})$, there is a path of diffeomorphisms between h and the identity map. It follows that if $\mathfrak{K} \subset \Sigma \times I$ is a knot, then $\mathfrak{K}$ and $h(\mathfrak{K})$ are isotopic.

Now let $\mathfrak{K}'' = f'(\mathfrak{K})$ and $D'' = \pi(\mathfrak{K}'')$. It is clear from the definition of f' that

$$D'' = \pi(\mathfrak{K}'') = \pi(f'(\mathfrak{K})) = f_0(\pi(\mathfrak{K})) = f_0(D).$$

If we apply a diffeomorphism of $\Sigma \times \{0\}$ to an alternating knot diagram in $\Sigma\times\{0\}$, then we obtain another alternating knot diagram with the same crossing number. Thus D'' is an alternating diagram with the same crossing number as D. On the other hand

$$\mathfrak{K}' = f(\mathfrak{K}) = h(f'(\mathfrak{K})) = h(\mathfrak{K}''),$$

hence $\mathfrak{K}'$ and $\mathfrak{K}''$ are isotopic as knots in $\Sigma \times I$. Now by [AFLT02, Theorem 1.1.], $c(D'') \leq c(D')$, but D' has minimal crossing number, so $c(D'') = c(D')$. Therefore $c(D) = c(D')$, and D has minimal crossing number. □

Definition 5.31. Suppose D is a virtual link diagram and $(S_D, \tilde{D})$ the abstract link diagram associated with D. Consider the components of ∂S_D. A crossing is called *proper* if four different components of ∂S_D meet at that crossing. A diagram D is called *proper* if each crossing is proper.

Remark 5.32. Notice that if D is proper, then it has no nugatory crossings.

As an affirmative answer for Question 6.5 in [Kam04] in the case of virtual knots, we have:

Corollary 5.33. *If D is a proper alternating knot diagram for a virtual knot K, then D has minimal crossing number.*

Proof. The result follows from the Remark 5.32, Theorem 5.28, and [Kam04, Theorem 1.2]. □

Proposition 5.34. *If K is an almost classical knot that is alternating, then it admits an alternating almost classical diagram.*

Proof. Suppose D is a reduced alternating diagram for K, then by Theorem 5.30, it has minimal crossing number. The result follows from Proposition 3.26. □

In the next lemma, we will show that the three remaining almost classical knots from Proposition 5.25 are not alternating. In establishing the lemma, we will make use of Theorem 5.30 and Proposition 5.34.

Lemma 5.35. *The three almost classical knots* $6.87188, 6.87310$ *and* 6.87859 *are not alternating.*

Proof. Let $K_1 = 6.87188$, $K_2 = 6.87310$ and $K_3 = 6.87859$. We have $\det(K_2) = \det(K_3) = 7$. Suppose to the contrary that K_2 and K_3 admit reduced alternating diagrams. Then the diagrams would have minimal crossing number. By Table A.1, the only pattern with 6 or fewer crossings and determinant 7 is Θ_{5c}, but both K_2 and K_3 have minimal crossing number 6 and they cannot be equivalent to a 5 crossing alternating knot. Therefore they are not alternating.

Now $\det(K_1) = 9$ and if it were to admit a reduced alternating diagram, it would have one of the patterns Θ_{5a}, $\Theta_3 \# \Theta_3$ or Θ_{6g}. For a similar reason it cannot have the pattern Θ_{5a}. Suppose D is a reduced alternating diagram for K_1, then it has minimal crossing number and by Proposition 3.26, it must be an almost classical diagram. If D has the pattern $\Theta_3 \# \Theta_3$, then by Proposition 3.27, both factors must be almost classical which means up to mirror image the factors are the trefoil knot. It follows that D is a classical diagram, but the virtual genus of K_1 is 1, which is a contradiction.

It remains to show D does not have the pattern Θ_{6g}. The fact D is almost classical imposes restrictions on the signs of the chords, as we now explain.

Referring to the diagram for Θ_{6g} in A.1, list the chords $c_1, \ldots, c_6$ in counterclockwise order starting at the top (12 o'clock) and let $\varepsilon_1, \ldots, \varepsilon_6$ be their signs in D. Almost classicality implies that $\varepsilon_2 = \varepsilon_3$. For the other chords, there are two possibilities. Either $\varepsilon_1 = \varepsilon_4$ and $\varepsilon_5 = \varepsilon_6$, or $\varepsilon_1 = -\varepsilon_5$ and $\varepsilon_4 = -\varepsilon_6$. Each one of resulting almost classical knots is, up to mirror images, equivalent to one of the knots 6.90214, 6.90217, 6.90219 or 6.90227. However, 6.90227 is classical, and the Alexander polynomials of 6.90214 and 6.90217 are $3t^2 - 4t + 2$ and $t^3 - 4t^2 + 3t - 1$, respectively, which are different from $\Delta_{K_1}(t) = 2t^3 - 3t^2 + 3t - 1$. On the other hand, the Jones polynomial for K_1 is $t + 1/t - 1/t^2 + 2/t^3 - 1/t^4 - 1$, which is different from that of 6.90219, which equals 1. This completes the argument and shows that K_1 is not alternating. □

Chapter 6

Khovanov Homology

In this chapter, we introduce Khovanov homology, Lee homology and Rasmussen's invariant for classical and virtual knots. We calculate the Khovanov homology for the virtual knot 3.7. We generalize Lee's theorem about the Khovanov homology of alternating classical links to the virtual case.

6.1 Khovanov Homology for Classical Knots

In this section, we briefly introduce the Khovanov homology for classical knots and links. For more details see [Tur17] and [BN02].

Khovanov homology is a $(1+1)$-TQFT (*topological quantum field theory*), i.e. it is a functor from the category of compact 1-dimensional manifolds (a collection of circles) with morphisms compact and orientable, 2-dimensional cobordisms (surfaces) between them, into the category of graded vector spaces and graded linear maps.

Khovanov introduced the invariant for links in [Kho00]. It is a bigraded homology theory which can be defined and computed in a purely combinatorial way. Khovanov homology is a categorification of the Jones polynomial in that its graded Euler characteristic is equal to the unnormalized Jones polynomial. For a link L, we denote its Khovanov homology by $Kh^{*,*}(L)$, and we have

$$\widehat{\chi}(Kh^{*,*}(D)) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^j \dim Kh^{i,j}(D) = \widehat{V}_L(q).$$

Suppose D is a link diagram with n_+ positive crossings and n_- negative crossings. Let $n = n_+ + n_-$ and enumerate the crossings by $c_1, \ldots, c_n$. With

$\mathbb{Q}$ as the coefficient ring, we set $V = \mathbb{Q}\mathbf{1} \oplus \mathbb{Q}X$ to be the 2 dimensional vector space with basis $\{\mathbf{1}, X\}$. Setting the degree of $\mathbf{1}$ to be $+1$ and the degree of X to be -1 gives $V^{\otimes n}$ the structure of a graded vector space. This grading will be denoted j and called *vertical* or *quantum grading*.

If $W = \bigoplus_{m\in\mathbb{Z}} W_m$ is a graded vector space, then a *vertical grading shift* of W by ℓ is defined as $W\{\ell\} = \bigoplus_{m\in\mathbb{Z}} W'_m$, where $W'_m = W_{m-\ell}$.

We consider the cube of resolutions of D, which is an n-dimensional cube with 2^n vertices, one for each state. Here we denote states by $\alpha \in \{0,1\}^n$, which is a binary sequence of length n that indicates how each crossing has been resolved. Let r_α and k_α be the number of $\mathbf{1}$'s and cycles in α, respectively. Let $\mathcal{C}^{i,*}(D)$ be $\bigoplus V^{\otimes k_\alpha}\{r_\alpha + n_+ - 2n_-\}$, where we take the direct sum over all the states α with $r_\alpha = i + n_-$. Here i is called *horizontal* or *homological grading*.

If $\mathcal{C}(D) = \bigoplus_i \mathcal{C}^i(D)$, then a *horizontal grading shift* of $\mathcal{C}(D)$ by l is defined as $\mathcal{C}(D)[l] = \bigoplus_i \mathcal{C}'^i(D)$, where $\mathcal{C}'^i(D) = \mathcal{C}^{i-l}(D)$.

We define the Khovanov complex as $\mathcal{C}Kh(D) = \bigoplus_{i,j} \mathcal{C}^{i,j}(D)$. To define the differential d, we introduce the product and coproduct maps. Note that henceforth we will suppress the symbol $\otimes$ in writing elements of $V^{\otimes k}$.

$$\begin{aligned}
\Delta : V &\to V \otimes V, & m : V \otimes V &\to V.\\
\mathbf{1} &\mapsto \mathbf{1}X + X\mathbf{1} & \mathbf{1}\mathbf{1} &\mapsto \mathbf{1}\\
X &\mapsto XX & \mathbf{1}X &\mapsto X\\
& & X\mathbf{1} &\mapsto X\\
& & XX &\mapsto 0
\end{aligned}$$

We only define a map from a state α to a state α' if α' obtained from α by changing one 0 to 1. In that case, either two cycles of α merge into one cycle of α', or one cycle of α splits into two cycles of α'. In the first case, we use the product map m, and in the second, we use the coproduct map Δ. For all other cycles of α, we apply the identity. In order to write down all the maps, we fix once and for all an enumeration of the cycles in each state, and these are not changed throughout the calculations.

The last step is to assign negative signs to some of the maps. There are many ways to do that, but the homology groups for the different choices of signs are all isomorphic. Here we follow the sign convention of [BN02].

Suppose we change 0 to 1 in the m-th spot to obtain α' from α. In α, we count how many 1's we have before the m-th spot. If it is an odd number, we assign a negative sign to the associated map.

For a fixed j the map $d^2 : \mathcal{C}^{i,j} \to \mathcal{C}^{i+2,j}$ is zero and we obtain a bigraded homology theory denoted by $Kh^{*,*}(D)$.

In [Lee05], Lee constructs a new complex by modifying the maps Δ and m:

$$\begin{aligned} \Delta' : V &\to V \otimes V, & m' : V \otimes V &\to V. \\ \mathbf{1} &\mapsto \mathbf{1}X + X\mathbf{1} & \mathbf{11} &\mapsto \mathbf{1} \\ X &\mapsto \mathbf{11} + XX & \mathbf{1}X &\mapsto X \\ & & X\mathbf{1} &\mapsto X \\ & & XX &\mapsto \mathbf{1} \end{aligned}$$

This results in a new homology theory called Lee homology and denoted Lee(D). Notice the maps no longer preserve the homological degree, thus Lee homology is only graded rather being bigraded.

It turns out that Lee$(K) \cong \mathbb{Q} \oplus \mathbb{Q}$ for all knots, nevertheless as we will see the Lee homology contains a nontrivial and powerful invariant $s(K)$ called the *Rasmussen invariant*. This invariant was introduced by Rasmussen in [Ras10], and we briefly recall its definition.

The quantum degree defines a decreasing filtration on $\mathcal{C}Kh(K)$. This induces a filtration on Lee(K),

$$H_*(\mathcal{C}) = \mathcal{F}^n H_*(\mathcal{C}) \supset \mathcal{F}^{n+1} H_*(\mathcal{C}) \subset \ldots \supset \mathcal{F}^m H_*(\mathcal{C}).$$

For $x \in$ Lee(K), let $s(x)$ be the filtration degree of x, i.e. $s(x) = k$ if $x \in \mathcal{F}^k H_*(\mathcal{C})$ but x does not belong to $\mathcal{F}^{k+1} H_*(\mathcal{C})$. We define

$$\begin{aligned} s_{min}(K) &= \min\{s(x) \in \text{Lee}(K) \mid x \neq 0\}, \\ s_{max}(K) &= \max\{s(x) \in \text{Lee}(K) \mid x \neq 0\}. \end{aligned}$$

Rasmussen proves that $s_{max}(K) = s_{min}(K) + 2$ for all knots, and he defines $s(K) = s_{min}(K) + 1 = s_{max}(K) - 1$.

For a link L, the filtration on $\mathcal{C}Kh(L)$ induces a spectral sequence with E_0 term the Khovanov complex and $d_0 = d_{Kh}$. It follows that the E_1 term is $Kh^{*,*}(L)$. For every m, $d_m = 0$ unless m is a multiple of 4. As a result, for any $m \geq 0$, $E_{4m+1} \cong E_{4m+2} \cong E_{4m+3} \cong E_{4m+4}$. The E_∞ page is isomorphic to the Lee homology. For a knot K, it has two copies of $\mathbb{Q}$ which are placed on the y-axis. Their location indicates the filtration degree of the generators of the Lee homology. In particular the average of their y-coordinates is equal to $s(K)$.

In [Lee05], Lee proves that, for any alternating link L, its Khovanov homology $Kh^{*,*}(L)$ is supported in the two lines $j = 2i - \sigma(L) \pm 1$. As a result, in the spectral sequence $d_m = 0$ for every $m \geq 5$ and $E_\infty = E_5$. If K is an alternating knot, then the y-coordinates of the two surviving copies of $\mathbb{Q}$ are $-\sigma(K) \pm 1$. This implies that $s(K) = -\sigma(K)$.

If we have a cobordism S between two links L_0 and L_1, then S induces a map $\varphi'_S : \text{Lee}(L_0) \to \text{Lee}(L_1)$ with filtration degree equal to $\chi(S)$. We will describe the map φ'_S in a moment, but first notice that this implies that if K is a knot, then $|s(K)| \leq 2g_4(K)$. The same inequality holds for the knot signature $\sigma(K)$, and Lee's theorem tells us that, for alternating knots, the Rasmussen invariant $s(K)$ does not improve the bound on the 4-ball genus that one gets from the knot signature. However, for non-alternating knots, it is no longer true that $s(K) = -\sigma(K)$, and sometimes the Rasmussen invariant provides a better bound. It should further be noted that Rasmussen's invariant gives a lower bound on the smooth 4-ball genus, whereas the knot signature gives a bound on the topological 4-ball genus.

Example 6.1. For $K = 9_{42}$, $s(K) = 0$ and $\sigma(K) = 2$, thus the signature provides a better bound in this case. On the other hand, for the knot $K = 10_{132}$, $s(K) = -2$ and $\sigma(K) = 0$, so $s(K)$ gives a better bound for the 4-ball genus.

We now describe the map φ'_S, and since any cobordism decomposes into a sequence of elementary cobordisms, it suffices to define φ'_S for births, deaths, and saddles. In doing that, we will use the maps $\iota : \mathbb{Q} \to V$ $(1 \mapsto \mathbf{1})$ and $\varepsilon : V \to \mathbb{Q}$ $(\mathbf{1} \mapsto 0$ and $X \mapsto 1)$.

Note that an elementary cobordism is either a birth, a death, or a saddle. For a birth, we set $\varphi'_S = \iota$. For a death, we set $\varphi'_S = \varepsilon$.

As noted previously, a saddle can be either a fusion or joining saddle, or a fission or splitting saddle. For a fusion saddle, we use the product map m', and for a fission saddle we use the coproduct Δ'.

In general, the Rasmussen invariant is difficult to compute. However, the calculation simplifies for positive (or negative) knots, as we now explain.

Definition 6.2. A link is called *positive* if it admits a diagram with only positive crossings. Similarly, a links is *negative* if it admits a diagram with only negative crossings. .

If K is a positive knot with diagram D with n positive crossings, then the Rasmussen invariant is given by

$$s(K) = -k + n + 1,$$

where k is the number of cycles in the all 0-smoothing state of D [Ras10].

If K is a negative knot with diagram D with n negative crossings, then the (vertical) mirror image D^* has n positive crossings, and the all 0-smoothing state of D^* is the all 1-smoothing state of D. Since $s(K^*) = -k + n + 1$, and since the Rasmussen invariant satisfies $s(K^*) = -s(K)$ under taking mirror images, it follows that $s(K) = k - n - 1$.

6.2 Khovanov Homology for Virtual Knots

In this section, we briefly introduce the Khovanov homology for virtual knots and links.

The Khovanov homology for virtual knots and links, first was defined by Manturov in [Man04] and only with $\mathbb{Z}_2$ coefficients. Later, in [Man07], he defined the Khovanov homology with arbitrary coefficients. In [DKK17b] Dye, Kaestner and Kauffman reformulated Manturov's approach and used that to prove a number of results such as a large family of virtual knots with unit Jones polynomial is not classical. Tubbenhauer in [Tub14], used un-oriented TQFT's to define a Khovanov homology for virtual knots and links.

When one attempts to define a Khovanov theory for virtual knots, the major problem is the presence of the *single cycle smoothing* (see Figure 6.1). We need to assign a map to a single cycle smoothing, which we can do by assigning the zero map. In classical Khovanov theory, the signs of maps are chosen in a way to make each face of the cube of resolutions to be anti-commutative. Then this fact enables us to define a differential d satisfying $d^2 = 0$. For virtual knots, the existence of single cycle smoothings makes it more difficult to assign signs.

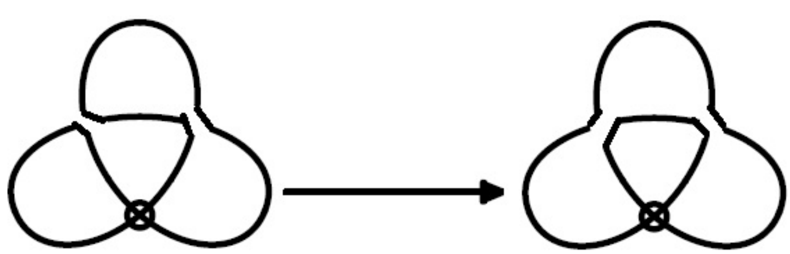

Figure 6.1: A single cycle smoothing, with induced map zero.

In what follows, we describe Tubbenhauer's method. We will cover the combinatorial definitions. To see the discussion about un-oriented TQFT's and for more detail, see [Tub14].

Let $\mathbb{Q}$ be the coefficient ring and $V = \mathbb{Q}\mathbf{1} \oplus \mathbb{Q}X$. Start with a virtual link diagram D with n classical crossings. Resolve all the crossings in both ways to obtain 2^n states. The Khovanov chain complex $\mathcal{C}(D)$ is defined as before, i.e. we assign $V^{\otimes k}$ to a state with k components. The degree of each element and the grading shifts are defined as before. Whenever two vertices of an edge in the cube of resolutions have the same number of states, then that indicates the presence of a single cycle smoothing. In that case, we assign the zero map to the edge. It remains to define the joining and splitting maps and the signs.

Choose orientations for the cycles of each state. Although we can do this in an arbitrary way, to have less complicated maps at the end, we use a spanning tree argument. Choose a spanning tree for the cube of the resolution and start with the rightmost vertices and choose orientations for the cycles of corresponding states. Now remove those vertices and again choose orientations for the rightmost vertices, in a compatible way. That means we compare the two vertices which are joined by an edge of the spanning tree, and orient the cycles of the left vertex as follows. For cycles which are not involved in the join, split or the single cycle smoothing, orient each cycle of the left vertex exactly like the corresponding cycle in the right vertex. For other cycles try to orient them in a way to have the most compatibility.

Choose an x-marker for each crossing and the corresponding 0- and 1-smoothings, as in Figure 6.2. We choose either x or x' and notice that up to rotating the diagram and the corresponding states, there are only these two ways to assign the x-markers.

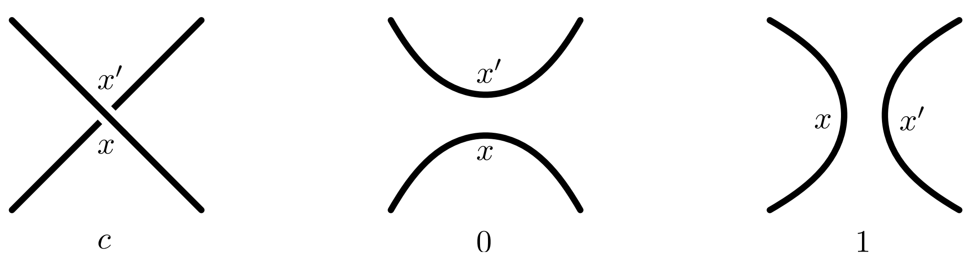

Figure 6.2: An x-marker for a crossing and the corresponding smoothings.

We define the sign of the non-zero maps as follows. By a spanning tree argument, number the cycles of each state. Suppose we have a joining map from a state s to another state s', and suppose s has $m+1$ cycles. Let the joining map, merges the cycles number a and b in s and the resulting cycle in

s' has number c, and let the cycle number a has the x-marker. In an (m+1)-tuple, put a first, then b and then place the remaining numbers in an ascending order. Let τ_1 be the permutation which takes $(1, 2, \cdots, m+1)$ to this $(m+1)$-tuple. Now in an m-tuple, put c first, and place the remaining numbers in an ascending order, and let τ_2 be the permutation which takes $(1, 2, \cdots, m)$ to this m-tuple. Define the sign of the joining map to be $\text{sign}(\tau_1)\text{sign}(\tau_2)$. The sign of the splitting map is defined, similarly.

To define the maps, we proceed as follows. Maps are defined between the two vertices of an edge of the cube of resolutions. For each edge the smoothing of only one of the crossings is different, and we define a map from the state with 0-smoothing to the state with 1-smoothing. If a cycle of the source state is disjoint from the smoothing change, assign the identity map to it, if its orientation agrees with the orientation of the corresponding cycle in the target state, otherwise assign negative of the identity map.

At a small neighborhood of the crossing, there are two parallel strings in each state. Notice that each cycle is oriented. Now if the map looks like $\downarrow\uparrow \to \rightleftarrows$, we decorate the four strings in the source and target state with a $+$ sign, and we call this decoration *standard*. We always rotate the states so the two strings in the source state are vertical, and the two strings in the target state are horizontal. Then we compare the orientation of each string with the orientation of the corresponding one in the standard decoration, if they agree, decorate that string with a $+$ sign, otherwise decorate it with a $-$ sign. We record the different cases in Table 6.1.

string	splitting map	string	joining map
$\downarrow\uparrow \to \rightleftarrows$	Δ^{+}_{++}	$\downarrow\uparrow \to \rightleftarrows$	m^{++}_{+}
$\downarrow\uparrow \to \rightrightarrows$	Δ^{+}_{-+}	$\uparrow\uparrow \to \rightleftarrows$	m^{-+}_{+}
$\downarrow\uparrow \to \leftleftarrows$	Δ^{+}_{+-}	$\downarrow\downarrow \to \rightleftarrows$	m^{+-}_{+}
$\downarrow\uparrow \to \leftrightarrows$	Δ^{+}_{--}	$\uparrow\downarrow \to \rightleftarrows$	m^{--}_{+}
$\uparrow\downarrow \to \rightleftarrows$	Δ^{-}_{++}	$\downarrow\uparrow \to \leftrightarrows$	m^{++}_{-}
$\uparrow\downarrow \to \rightrightarrows$	Δ^{-}_{-+}	$\uparrow\uparrow \to \leftrightarrows$	m^{-+}_{-}
$\uparrow\downarrow \to \leftleftarrows$	Δ^{-}_{+-}	$\downarrow\downarrow \to \leftrightarrows$	m^{+-}_{-}
$\uparrow\downarrow \to \leftrightarrows$	Δ^{-}_{--}	$\uparrow\downarrow \to \leftrightarrows$	m^{--}_{-}

Table 6.1: String decoration and corresponding maps.

Other cases happen only when we have a single cycle smoothing. We describe the map $\Delta^{a}_{bc}(v)$ as follows. Multiply v by a, apply Δ, then multiply the first component of the resulting tensor product by b and the second component by c. Notice that the first component of the tensor product, corresponds to the lower string or the string with the x-marker on it. Similarly, we define the map m_a^{bc}. First multiply the first component of the tensor product by b and the second component by c, then apply m, at the end multiply the result by a.

Remark 6.3. We know that every checkerboard colorable diagram admits a source-sink orientation ([KNS02, Proposition 6]). We can use this orientation to make all the decorations standard. In that case we only need the maps Δ_{++}^{+} and m_{+}^{++}.

Example 6.4. We compute the Khovanov homology for the knot $K = 3.7$. Figure 6.3 is a diagram for K and Figure 6.4 is the cube of resolutions.

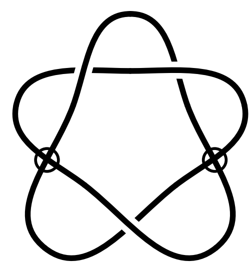

Figure 6.3: The knot $K = 3.7$.

The red dots are the x-markers. We enumerate the components of a state in a way that the one which has more x-markers in it be the first component. All the m maps are m_{+}^{++}, and Δ maps are Δ_{++}^{+}. A red arrow means the associated map has negative sign. All the maps are a single splitting or joining map except for the state which has 3 components in it. For this state the incoming map is $\Delta \otimes id$, and for the outgoing maps, the upper one is φ defined as $\varphi(a, b, c) = -m(a, c) \otimes b$, and the lower one is $-id \otimes m$.

The Khovanov complex is as follows:

$$V \otimes V\{-3\} \to V \oplus V \oplus \left(V^{\otimes 3}\right)\{-2\} \to \left(V^{\otimes 2}\right) \oplus \left(V^{\otimes 2}\right) \oplus \left(V^{\otimes 2}\right)\{-1\} \to V.$$

We record the basis elements of the chain complex in Table 6.2.

The image of each basis element is in Table 6.3.

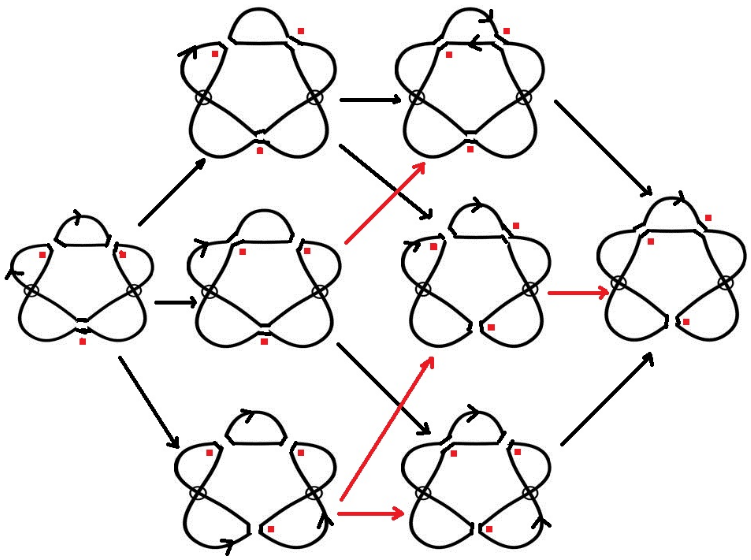

Figure 6.4: The cube of resolutions for $K = 3.7$.

$j \setminus i$	-2	-1	0	1
			$(\mathbf{11}, 0, 0)$	
1		$(0, 0, \mathbf{111})$	$(0, \mathbf{11}, 0)$	$\mathbf{1}$
			$(0, 0, \mathbf{11})$	
		$(\mathbf{1}, 0, 0)$	$(\mathbf{1}X, 0, 0)$	
		$(0, \mathbf{1}, 0)$	$(X\mathbf{1}, 0, 0)$	
-1	$\mathbf{11}$	$(0, 0, \mathbf{11}X)$	$(0, \mathbf{1}X, 0)$	X
		$(0, 0, \mathbf{1}X\mathbf{1})$	$(0, X\mathbf{1}, 0)$	
		$(0, 0, X\mathbf{11})$	$(0, 0, \mathbf{1}X)$	
			$(0, 0, X\mathbf{1})$	
		$(X, 0, 0)$		
	$\mathbf{1}X$	$(0, X, 0)$	$(XX, 0, 0)$	
-3		$(0, 0, \mathbf{1}XX)$	$(0, XX, 0)$	
	$X\mathbf{1}$	$(0, 0, X\mathbf{1}X)$	$(0, 0, XX)$	
		$(0, 0, XX\mathbf{1})$		
-5	XX	$(0, 0, XXX)$		

Table 6.2: The basis elements for the chain complex.

It is easy to check $d^2 = 0$. When we take the homology, two copies of $\mathbb{Q}$ survive, both in homological degree 0, one in quantum degree 1 and the other in quantum degree -1. Therefore the Khovanov homology of K is isomorphic to the Khovanov homology of the unknot.

In [DKK17b], Dye, Kaestner and Kauffman define Lee homology and the

$j \setminus i$	-2	-1	0	1
			$\mathbf{1}$	
1		$(0, -\mathbf{11}, -\mathbf{11})$	$-\mathbf{1}$	0
			$\mathbf{1}$	
		$(\mathbf{1}X + X\mathbf{1}, \mathbf{1}X + X\mathbf{1}, 0)$	X	
		$(-\mathbf{1}X - X\mathbf{1}, 0, \mathbf{1}X + X\mathbf{1})$	X	
-1	$(\mathbf{1}, \mathbf{1}, \mathbf{1}X\mathbf{1} + X\mathbf{11})$	$(0, -X\mathbf{1}, -\mathbf{1}X)$	$-X$	0
		$(0, -\mathbf{1}X, -\mathbf{1}X)$	$-X$	
		$(0, -X\mathbf{1}, -X\mathbf{1})$	X	
			X	
		$(XX, XX, 0)$		
	$(X, X, \mathbf{1}XX + X\mathbf{1}X)$	$(-XX, 0, XX)$	0	
-3		$(0, -XX, 0)$	0	
	$(X, X, XX\mathbf{1})$	$(0, 0, -XX)$	0	
		$(0, -XX, -XX)$		
-5	$(0, 0, XXX)$	$(0, 0, 0)$		

Table 6.3: The image of the basis elements.

Rasmussen invariant for virtual knots, and they show that Rasmussens invariant is an invariant of virtual knot concordance.

Example 6.5. Table B.1 lists the Rasmussen invariant for the alternating virtual knots up to six crossings. The three virtual knots 6.90115, 6.90150 and 6.90170 all have Rasmussen invariant equal to -2, and as a result we conclude that none of these virtual knots are slice.

In [BCG17a], Boden et al. define slice obstructions in terms of signatures of symmetrized Seifert matrices for almost classical knots, and as an application they show that neither 6.90115 nor 6.90150 are slice. However, the virtual knot 6.90170 is not almost classical, so the Rasmussen invariant not only provides an alternate method to show that 6.90115 and 6.90150 are not slice, it also shows that 6.90170 is not slice, which is a new result.

6.3 Khovanov Homology and Alternating Virtual Links

Following [Lee05], we seek a relation between Rasmussen's invariant and signatures of alternating virtual knots. If i is the homological degree and j is the quantum degree for Khovanov homology, then H-thinness for classical alternating knots means, $j = 2i - \sigma \pm 1$, where σ is the signature. This implies that $s = -\sigma$, where s is Rasmussen's invariant. On the other hand, not all virtual alternating knots are H-thin. For example the Khovanov polynomial for the knot $K = 5.2426$ depicted in Figure 6.5 is as follows:

$$\frac{1}{q^{11}t^3} + \frac{1}{q^9t^3} + \frac{1}{q^7t^2} + \frac{1}{q^5t^2} + \frac{1}{q^5} + \frac{1}{q^3},$$

which is supported in three lines $j = 2i - 1$, $j = 2i - 3$ and $j = 2i - 5$. Notice that from Table B.1 $(\sigma_{\xi^*}, \sigma_\xi) = (2, 4)$, and we can write the three lines as:

$$j = 2i - \sigma_{\xi^*} + 1, j = 2i - \sigma_{\xi^*} - 1, j = 2i - \sigma_\xi - 1.$$

In fact, instead of H-thinness we have:

Proposition 6.6. *If D is a connected alternating virtual link diagram with genus g, and signatures $\sigma_\xi, \sigma_{\xi^*}$, then its Khovanov homology is supported in $g + 2$ lines:*

$$j = 2i - \sigma_{\xi^*} + 1, j = 2i - \sigma_{\xi^*} - 1, \ldots, j = 2i - \sigma_\xi - 1.$$

Proof. Following [Lee05], we apply induction on the number of crossings. The base case is trivial. Let D be an alternating virtual link diagram with n crossings. 0 and 1 smooth the last crossing to obtain $D(*0)$ and $D(*1)$, respectively. We can easily see that they are alternating diagrams. Shift the Khovanov complex n_- horizontally, and $2n_- - n_+$ vertically. Denote the resulting complex by $\bar{C}(D)$ and its homology by $\bar{H}(D)$. We denote this shift by $\bar{C}(D) = C(D)[n_-]\{2n_- - n_+\}$. We have the following short exact sequence:

$$0 \to \bar{C}(D(*1))[+1]\{+1\} \to \bar{C}(D) \to \bar{C}(D(*0)) \to 0,$$

which gives a long exact sequence involving $\bar{H}(D)$, $\bar{H}(D(*0))$ and $\bar{H}(D(*1))[+1]\{+1\}$, which implies $\bar{H}(D)$ is supported inside $\bar{H}(D(*0))$ and $\bar{H}(D(*1))$.

It suffices to show that $\bar{H}(D)$ is supported in $g+2$ lines with y-intercepts of

$$-|s_\partial| + 2, -|s_\partial|, \cdots, -|s_\partial| - 2g$$

because after shifting back $\bar{H}(D)$, the result follows.

The all 0 state of D is the same as the all 0 state of $D(*0)$. Also the all 1 state of D is the same as the all 1 state of $D(*1)$. In the all 0 state of D, if we change the resolution of the last crossing from a 0-smoothing to a 1-smoothing, we obtain the all 0 state for $D(*1)$. Similarly, in the all 1 state of D, if we change the resolution of the last crossing from a 1-smoothing to a 0-smoothing, we obtain the all 1 state for $D(*0)$.

These three diagrams, all have the boundary property. $D(*0)$ and $D(*1)$, both have $n-1$ crossings. Thus we have:

$$\begin{aligned}
|s_\partial(D)| + |\bar{s}_\partial(D)| &= n+2-2g(D), \\
|s_\partial(D(*0))| + |\bar{s}_\partial(D(*0))| &= n+1-2g(D(*0)), \\
|s_\partial(D(*1))| + |\bar{s}_\partial(D(*1))| &= n+1-2g(D(*1)).
\end{aligned}$$

Using the above observations, we can rewrite the last two equations as:

$$\begin{aligned}
|s_\partial(D)| + |\bar{s}_\partial(D(*0))| &= n+1-2g(D(*0)), \\
|s_\partial(D(*1))| + |\bar{s}_\partial(D)| &= n+1-2g(D(*1)).
\end{aligned}$$

Since the genus is an integer, the first equation implies that $|\bar{s}_\partial(D(*0))|$ cannot be equal to $|\bar{s}_\partial(D)|$, so it is either one more, or one less. Similarly, $|s_\partial(D(*1))|$ is either one more, or one less than $|s_\partial(D)|$. Thus we have four different cases:

Case 1: $|\bar{s}_\partial(D(*0))| = |\bar{s}_\partial(D)| - 1$, $|s_\partial(D(*1))| = |s_\partial(D)| - 1 \;\Rightarrow\; g(D) = g(D(*0)) = g(D(*1))$.

We use the induction hypothesis. Since $|s_\partial(D(*0))| = |s_\partial(D)|$ and $g(D(*0)) = g(D)$, the y-intercepts of the lines for $D(*0)$, are:

$$-|s_\partial(D)|+2, -|s_\partial(D)|, \cdots, -|s_\partial(D)| - 2g(D).$$

The y-intercepts of the lines for $D(*1)[+1]\{+1\}$ are the y-intercepts of the lines for $D(*1)$ minus 1. Since $|s_\partial(D(*1))| = |s_\partial(D)| - 1$, the y-intercepts of the lines for $D(*1)[+1]\{+1\}$ and $D(*0)$ agree, and they are precisely the numbers that we are looking for. Thus the result follows in this case.

Case 2: $|\bar{s}_\partial(D(*0))| = |\bar{s}_\partial(D)| + 1$, $|s_\partial(D(*1))| = |s_\partial(D)| - 1 \;\Rightarrow\; g(D) = g(D(*0)) + 1 = g(D(*1))$.

In this case, there are $g(D)+1$ lines for $D(*0)$, and their y-intercepts are:

$$-|s_\partial(D)|+2, -|s_\partial(D)|, \cdots, -|s_\partial(D)| - 2g(D) + 2.$$

On the other hand for $D(*1)$, the y-intercepts are as before. Hence the union of the supports of $D(*0)$ and $D(*1)[+1]\{+1\}$ is again the desired $g(D)+2$ lines.

Case 3: $|\bar{s}_\partial(D(*0))| = |\bar{s}_\partial(D)| - 1$, $|s_\partial(D(*1))| = |s_\partial(D)| + 1 \;\Rightarrow\; g(D) =$

$g(D(*0)) = g(D(*1)) + 1$.

In this case, there are $g(D) + 1$ lines for $D(*1)[+1]\{+1\}$, and their y-intercepts are:

$$-|s_\partial(D)|, -|s_\partial(D)|, \cdots, -|s_\partial(D)| - 2g(D).$$

For $D(*0)$, we have the same $g(D) + 2$ line, as in case 1. As before, their union is the $g(D) + 2$ lines with the desired y-intercepts.

Case 4: $|\bar{s}_\partial(D(*0))| = |\bar{s}_\partial(D)| + 1$, $|s_\partial(D(*1))| = |s_\partial(D)| + 1 \;\Rightarrow\; g(D) = g(D(*0)) + 1 = g(D(*1)) + 1$.

Combining case 2 and 3, we see that the result follows. □

Corollary 6.7. *Classical alternating links are H-thin.*

The results of Kronheimer and Mrowka [KM11] imply that Khovanov homology is an unknot detector for classical knots. For virtual knots, this is not true even for alternating virtual knots. For example 3.7 has the trivial Khovanov homology ($\frac{1}{q} + q$). In fact, many alternating virtual knots have trivial Khovanov homology (which is supported in two lines). Therefore the previous result is very coarse.

Let D be a checkerboard virtual link diagram. Apply **or** to all crossings with $\eta = -1$. The result is a diagram in which $\eta = +1$ for each chord. Hence, by Lemma 5.18, the new diagram, which we call D_{alt}, is alternating.

However, an application of **or** does not change the Khovanov homology of the diagram (see [DKK17a]), thus it follows that D and D_{alt} have isomorphic Khovanov homology groups.

In particular, starting with any classical diagram, we can change it to an alternating virtual diagram with the same Khovanov homology. We can do the same, starting with any checkerboard colorable diagram.

Lemma 6.8. *Suppose D is a positive alternating virtual knot. Then $s(D) = -\sigma_{\xi^*}(D)$.*

Proof. We computed $\sigma_{\xi^*} = \beta - 1 - n_+$, where β is the number of all 0-smoothing state. For any positive knot K, we have $s(K) = 1 - \beta + n_+$ (see [DKK17a]). □

For a negative knot both Rasmussen's invariant and the signatures, are negative of the corresponding values for the vertical mirror image (positive knot). It follows that $s(K) = -\sigma_\xi(K)$. In general it is not true that Rasmussen's invariant is the negative of one of the signatures for alternating virtual knots. For example, the virtual knot 5.2427 is alternating (see Figure 6.6), has Rasmussen invariant $s(K) = -2$, and signatures $\sigma_\xi(K) = 4$ and $\sigma_{\xi^*}(K) = 0$.

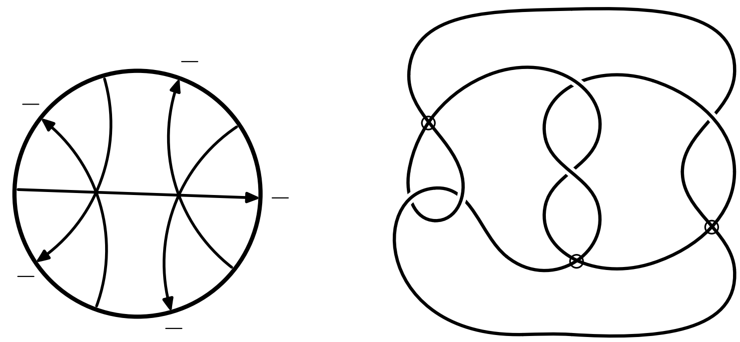

Figure 6.5: A Gauss diagram and virtual knot diagram for 5.2426.

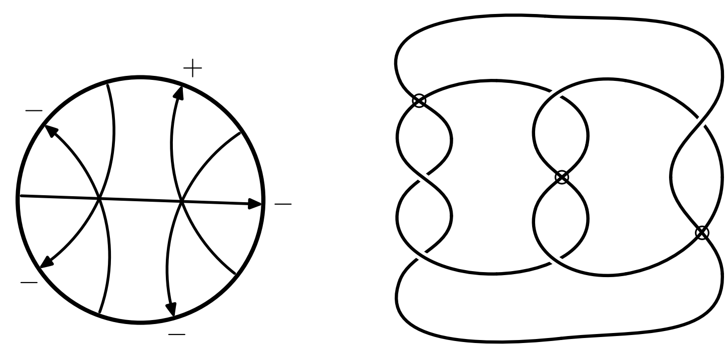

Figure 6.6: A Gauss diagram and virtual knot diagram for 5.2427.

Definition 6.9. The underlying Gauss pattern of an alternating virtual knot is called an *alternating pattern*.

Remark 6.10. If K is an alternating virtual knot, then by Proposition 3.15 the Alexander determinant of K depends only on the underlying alternating pattern.

Remark 6.11. Notice that some alternating patterns do not contain any classical knot diagrams. For example, the pattern Θ_{5a} in Figure A.1 has no classical diagrams in it.

Proposition 6.12. *If Θ is an alternating pattern, then it contains at least one almost classical knot.*

Proof. Suppose c is a chord in Θ and its arrow-head points upwards. The pattern is alternating which means the number of heads and tails on each side of c are equal. If we exclude the chords which don't intersect c, then at each side the number of remaining heads and tails are still equal. Therefore, the number of chords intersecting c with their arrow-heads to the left is equal to the number of chords intersecting c with their arrow-heads to the right. If we assign negative sign to each chord, the resulting Gauss diagram is almost classical. □

By assigning different signs to the chords of an alternating pattern with n chords, we obtain 2^n alternating knot diagrams (some of them might be equivalent to each other). We arrange them on the vertices of an n-dimensional cube. We can move from one vertex to another one by applying a sequence of sign change operations.

If D is an alternating pattern, label the chords by $1, \dots, n$. Denote each vertex by $D_{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}$, where ε_i is the sign of i-th chord. Consider $D_{++\cdots+}$, and find the states of resolution. Denote them by $S^{++\cdots+}_{i_1,i_2,\dots,i_n}$, where i_k is 0 or 1, according to whether we resolve the k-th chord to a 0-smoothing or a 1-smoothing. Then we have $S^{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n}_{i_1,i_2,\dots,i_n} = S^{++\cdots+}_{j_1,j_2,\dots,j_n}$, where if $\varepsilon_k = +$, then $j_k = i_k$, and if $\varepsilon_k = -$, then $j_k = 1 - i_k$.

In ordinary Khovanov homology, one assigns the map zero to each single cycle smoothing. Using different maps for single cycle smoothings, one can obtain a more refined version of Khovanov homology for virtual knots. For example, in [Rus17] Rushworth uses this approach to define a variant theory called doubled Khovanov homology. For virtual links whose cube of resolutions has no single cycle smoothings, then the doubled Khovanov homology is the direct sum of two copies of ordinary Khovanov homology. In that case, the doubled Khovanov homology is completely determined by the ordinary Khovanov homology and thus it contains no new information.

In [Rus17], Rushworth proves that for all checkerboard colorable diagrams, there is no single cycle smoothing. This fact follows also from [KNS02, Proposition 6]. Here we provide a different proof of that fact, from which it follows that for checkerboard links, the doubled Khovanov homology is the direct sum of two copies of ordinary Khovanov homology.

Lemma 6.13. *Let D be an alternating link diagram, and s_∂ be the all 0-smoothing state, and $\bar{s}_\partial$ the all 1-smoothing state. If we change one 0-smoothing to obtain the state s, the number of components of s_∂ and s, differs by one. Similar result holds for $\bar{s}_\partial$.*

Proof. Assume we change the smoothing in the last crossing. We consider $D(*1)$, which is an alternating diagram and has the boundary property. If D has c crossings, and g and g_1 are the genera for D and $D(*1)$ respectively, we have:

$$\begin{aligned} |s_\partial| + |\bar{s}_\partial| &= c + 2 - 2g, \\ |s| + |\bar{s}_\partial| &= c - 1 + 2 - 2g_1, \\ |s_\partial| - |s| &= 1 + 2(g_1 - g). \end{aligned}$$

Thus the difference is an odd number, and the result follows. The proof for the other case is similar. □

Proposition 6.14. *Let D be an alternating link diagram. Then there is no single cycle smoothing in the cube of resolutions for D.*

Proof. Assume we change a 0-smoothing of the state s to a 1-smoothing at the crossing c_i. If for all the other crossings, we have 0-smoothing in s, then this is the previous lemma. Otherwise, we apply **sc** to the crossings of D, which have been resolved to 1-smoothings in s. Call the new diagram D'. Since the state s is the all 0-smoothing state for D', the result follows from the previous lemma. □

Proposition 6.15. *Let D be a checkerboard colorable link diagram. Then there is no single cycle smoothing in the cube of resolutions for D.*

Proof. Assume we change one 0-smoothing of the state s to a 1-smoothing at the crossing c_i, and call the resulting state s'. First we consider D_{alt}. Let $C' = \{c_{i_1}, \ldots, c_{i_k}\}$ be the set of crossings of D which are changed to obtain D_{alt}. There are two cases. If c_i does not belong to C', then the edge with vertices s and s' corresponds exactly to an edge in the cube of resolutions for D_{alt}, and the result follows.

If $c_i \in C'$, then the same thing happens. The only difference is the direction of the map in D_{alt} is reversed, going from s' to s. The result still holds. □

Corollary 6.16. *If D is a checkerboard colorable link diagram, then the doubled Khovanov homology for D is the direct sum of two copies of the ordinary Khovanov homology for D.*

Chapter 7

Problem List and Further Studies

Suppose we have a checkerboard colored diagram. If we apply the unknotting operations on a chord, the question is, how exactly the signatures change. In particular if we start with an alternating diagram, we can study this question.

If we apply **sc** to a chord in an alternating diagram, then the resulting diagram is again alternating. Suppose D is an alternating diagram, with at least one positive crossing. Enumerate the crossings and suppose the last crossing is positive. We have:

$$\sigma_\xi = 1 - |\bar{s}_\partial(D)| + n_-(D) \ , \ \sigma_{\xi^*} = |s_\partial(D)| - 1 - n_+(D).$$

Apply **sc** to the last crossing to find D'. Then $s_\partial(D')$ is the same as $s_\partial(D(*1))$, and $\bar{s}_\partial(D')$ is $\bar{s}_\partial(D(*0))$. Similar to the Proposition 6.6, we have four cases. In each case the signatures of D' is the same as signatures of D, or one or both go up by 2.

Problem 7.1. For an alternating diagram, in terms of other invariants, like Alexander polynomials, determine when exactly the signatures change under the **sc** operation.

This problem is motivated by Conway's result about the signature of the classical knots in [Gil82], which states that if we apply **cc** to a positive crossing, then the knot signature goes up by 2, if $\Delta_K(-1)$ changes sign, otherwise the knot signature stays the same.

We can also analyze the behavior of the signatures under **sc** for a checkerboard colorable diagram.

Therefore we have the following problem:

Problem 7.2. Start with a checkerboard colorable diagram. For each unknotting operation (except **cd**), analyze the behavior of the signatures.

Problem 7.3. What is the relation between the Khovanov homologies of the knot diagrams with the same alternating pattern?

Related to the Problem 7.1, we have:

Problem 7.4. When we move between different vertices of the cube of an alternating pattern, determine how Rasmussen's invariant and the signatures change.

Problem 7.5. Do the signatures provide a lower bound on the slice genus of a knot? Is the signature for long virtual knots a concordance invariant?

Notice if the signatures were slice obstructions for checkerboard knots, then a direct consequence of Theorem 5.19 would be that non-classical alternating knots are never slice.

Problem 7.6. Suppose L is a virtual link and D is an alternating almost classical diagram for it. Apply Seifert's algorithm to find a Seifert surface for D. Does this surface have minimal genus?

Problem 7.7. Does Theorem 5.30 admit a converse if one assumes the knot K is prime, i.e. is every minimal crossing diagram of a prime alternating virtual knot is reduced and alternating?

There are (at least) two ways to state an analogue of the third Tait Conjecture for virtual knots. If the tangle diagram T (see Figure 2.8) is purely classical, then we can do a flype move and it will preserve the virtual alternating link type. It is not clear whether this move alone is powerful enough to pass from any minimal crossing diagram to any other. On the other hand, if the tangle diagram T is allowed to contain virtual crossings, then performing this kind of "virtual flype" will not always preserve the link type. For instance, the Kauffman flype is used in [Kam17] and leads to K-equivalence of virtual knots and links. The fact that "virtual flyping" can change the alternating virtual knot or link type was also noted by Zinn-Justin and Zuber in [ZJZ04].

Problem 7.8. Is there an analogue of the Tait flyping conjecture for alternating virtual knots?

This is Problem 15 in [FIKM14], see also [ZJ06].

Appendices

Appendix A

Alternating Patterns

Up to 6 crossings, there are 15 distinct alternating patterns. Figure A.1 shows their associated Gauss diagrams, and Table A.1 lists the Alexander determinants of the alternating virtual knots according to these patterns. Notice by Remark 6.10, the Alexander determinants depend only on the underlying pattern.

Alternating Pattern	Alexander Determinant
Θ_3	3
Θ_4	5
Θ_{5a}	9
Θ_{5b}	11
Θ_{5c}	7
Θ_{5d}	5
$\Theta_3 \# \Theta_3$	9
Θ_{6a}	15
Θ_{6b}	19
Θ_{6c}	17
Θ_{6d}	13
Θ_{6e}	19
Θ_{6f}	11
Θ_{6g}	9
Θ_{6h}	13

Table A.1: Alexander determinant for each alternating pattern.

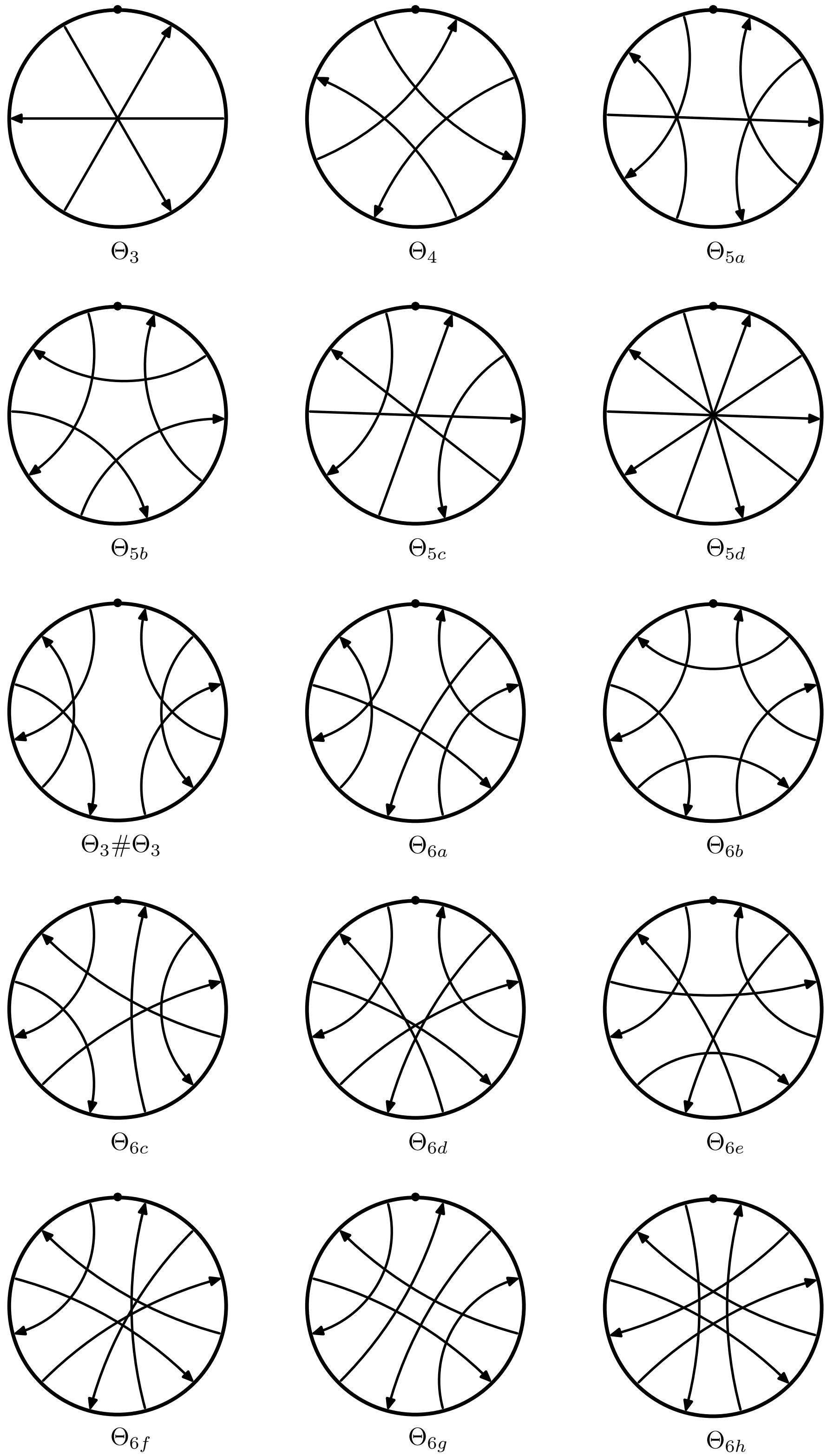

Figure A.1: Alternating patterns.

Appendix B

Alternating Virtual Knots and Their Invariants

The signatures and determinants are computed using a Mathematica program written by Micah Chrisman, and the Khovanov homology (unlisted) is computed using online Mathematica program written by Daniel Tubbenhauer. The Rasmussen's invariants are then computed by hand using Lee's spectral sequence.

Virtual knots are grouped according to their alternating patterns. For each pattern, we fix a base-point on the core circle of the Gauss diagram. Starting with the base-point, we travel counterclockwise. For knots with the same alternating pattern, instead of writing the entire Gauss code, we only list the signs of the chords. Boldface font is used to indicate that the knot is classical.

Virtual Knot	Alternating Pattern	Sign Sequence	Signatures $(\sigma^*_\xi, \sigma_\xi)$	Determinants $(\det^*_\xi, \det_\xi)$	Rasmussen Invariant
3.6	Θ_3	$---$	$(2,2)$	$(3,3)$	-2
3.7	Θ_3	$--+$	$(0,2)$	$(1,2)$	0
4.105	Θ_4	$----$	$(0,2)$	$(1,4)$	-2
4.106	Θ_4	$---+$	$(0,2)$	$(3,2)$	0
4.107	Θ_4	$--++$	$(-2,2)$	$(1,1)$	0
4.108	Θ_4	$-++-$	$(0,0)$	$(5,5)$	0
5.2426	Θ_{5a}	$-----$	$(2,4)$	$(4,5)$	-4
5.2427	Θ_{5a}	$----+$	$(0,4)$	$(2,1)$	-2
5.2428	Θ_{5a}	$---++$	$(0,2)$	$(7,2)$	0
5.2429	Θ_{5a}	$--+--$	$(2,4)$	$(8,1)$	-2
5.2430	Θ_{5a}	$--+-+$	$(0,2)$	$(5,4)$	0
5.2431	Θ_{5a}	$-+--+$	$(-2,2)$	$(1,3)$	0
5.2432	Θ_{5a}	$-+-+-$	$(-2,2)$	$(1,3)$	0
5.2433	Θ_{5b}	$-----$	$(0,4)$	$(1,5)$	-4
5.2434	Θ_{5b}	$----+$	$(0,4)$	$(3,1)$	-2

Virtual Knot	Alternating Pattern	Sign Sequence	Signatures $(\sigma^*_\xi, \sigma_\xi)$	Determinants $(\det^*_\xi, \det_\xi)$	Rasmussen Invariant
5.2435	Θ_{5b}	$---++$	$(0,2)$	$(8,3)$	0
5.2436	Θ_{5b}	$--+-+$	$(-2,2)$	$(4,1)$	0
5.2437	Θ_{5c}	$-----$	$(2,2)$	$(7,7)$	-2
5.2438	Θ_{5c}	$----+$	$(0,2)$	$(2,5)$	-2
5.2439	Θ_{5c}	$---++$	$(0,2)$	$(4,3)$	0
5.2440	Θ_{5c}	$--+--$	$(0,2)$	$(4,3)$	0
5.2441	Θ_{5c}	$--+-+$	$(-2,2)$	$(2,1)$	0
5.2442	Θ_{5c}	$--++-$	$(-2,2)$	$(2,1)$	0
5.2443	Θ_{5c}	$-++--$	$(-2,0)$	$(1,6)$	0
5.2444	Θ_{5c}	$-+++-$	$(-2,0)$	$(3,4)$	0
5.2445	Θ_{5d}	$-----$	$(4,4)$	$(5,5)$	-4
5.2446	Θ_{5d}	$----+$	$(2,4)$	$(4,1)$	-2
5.2447	Θ_{5d}	$---++$	$(0,2)$	$(3,2)$	0
5.2448	Θ_{5d}	$--+-+$	$(0,2)$	$(3,2)$	0
6.89187	$\Theta_3\#\Theta_3$	$------$	$(4,4)$	$(9,9)$	-4
6.89188	$\Theta_3\#\Theta_3$	$----+-$	$(2,4)$	$(6,3)$	-2
6.89189	$\Theta_3\#\Theta_3$	$----++$	$(0,2)$	$(3,6)$	-2
6.89198	$\Theta_3\#\Theta_3$	$+++---$	$(0,0)$	$(9,9)$	0
6.90101	$\Theta_3\#\Theta_3$	$--+--+$	$(0,4)$	$(4,1)$	0
6.90102	$\Theta_3\#\Theta_3$	$--+-+-$	$(0,4)$	$(4,1)$	0
6.90103	$\Theta_3\#\Theta_3$	$--+-++$	$(-2,2)$	$(2,2)$	0
6.90104	$\Theta_3\#\Theta_3$	$--++--$	$(0,4)$	$(4,1)$	0
6.90105	$\Theta_3\#\Theta_3$	$--++-+$	$(-2,2)$	$(2,2)$	0
6.90106	$\Theta_3\#\Theta_3$	$--+++-$	$(-2,2)$	$(2,2)$	0
6.90107	$\Theta_3\#\Theta_3$	$-+--+-$	$(0,4)$	$(4,1)$	0
6.90108	$\Theta_3\#\Theta_3$	$-+-+-+$	$(-2,2)$	$(2,2)$	0
6.90109	Θ_{6a}	$------$	$(2,4)$	$(7,8)$	-4
6.90110	Θ_{6a}	$-----+$	$(2,4)$	$(13,2)$	-2
6.90111	Θ_{6a}	$----+-$	$(0,4)$	$(2,4)$	-2
6.90112	Θ_{6a}	$----++$	$(0,4)$	$(5,1)$	-2
6.90113	Θ_{6a}	$---+--$	$(0,4)$	$(2,4)$	-2
6.90114	Θ_{6a}	$---+-+$	$(0,4)$	$(5,1)$	-2
6.90115	Θ_{6a}	$---++-$	$(0,2)$	$(4,11)$	-2
6.90116	Θ_{6a}	$---+++$	$(0,2)$	$(10,5)$	0
6.90117	Θ_{6a}	$--+---$	$(0,4)$	$(2,2)$	-2
6.90118	Θ_{6a}	$--+--+$	$(0,2)$	$(6,9)$	0
6.90119	Θ_{6a}	$--+-+-$	$(0,4)$	$(7,1)$	0
6.90120	Θ_{6a}	$--+-++$	$(-2,2)$	$(5,2)$	0
6.90121	Θ_{6a}	$--++--$	$(0,4)$	$(7,1)$	0
6.90122	Θ_{6a}	$--++-+$	$(-2,2)$	$(5,2)$	0
6.90123	Θ_{6a}	$--+++-$	$(0,2)$	$(12,3)$	0
6.90124	Θ_{6a}	$--++++$	$(-2,2)$	$(1,4)$	0
6.90125	Θ_{6a}	$-+----$	$(0,4)$	$(4,2)$	-2
6.90126	Θ_{6a}	$-+---+$	$(0,2)$	$(8,7)$	0
6.90127	Θ_{6a}	$-+--+-$	$(-2,4)$	$(1,1)$	0
6.90128	Θ_{6a}	$-+--++$	$(-2,2)$	$(4,3)$	0
6.90129	Θ_{6a}	$-+-+--$	$(-2,4)$	$(1,1)$	0
6.90130	Θ_{6a}	$-+-+-+$	$(-2,2)$	$(4,3)$	0
6.90131	Θ_{6a}	$-+-++-$	$(-2,2)$	$(3,2)$	0
6.90132	Θ_{6a}	$-+-+++$	$(-2,2)$	$(1,6)$	0
6.90133	Θ_{6a}	$-++---$	$(-2,2)$	$(6,1)$	0
6.90134	Θ_{6a}	$-++-+-$	$(-2,2)$	$(3,4)$	0

Virtual Knot	Alternating Pattern	Sign Sequence	Signatures $(\sigma^*_\xi, \sigma_\xi)$	Determinants $(\det^*_\xi, \det_\xi)$	Rasmussen Invariant
6.90135	Θ_{6a}	$-++-++$	$(-4,2)$	$(1,1)$	0
6.90136	Θ_{6a}	$-+++--$	$(-2,2)$	$(3,4)$	0
6.90137	Θ_{6a}	$-+++-+$	$(-4,2)$	$(1,1)$	0
6.90138	Θ_{6a}	$-+++++$	$(-4,0)$	$(2,4)$	2
6.90139	Θ_{6b}	$------$	$(2,4)$	$(12,9)$	-4
6.90140	Θ_{6b}	$-----+$	$(0,4)$	$(4,3)$	-2
6.90141	Θ_{6b}	$----++$	$(-2,4)$	$(1,1)$	0
6.90142	Θ_{6b}	$---+-+$	$(-2,2)$	$(11,1)$	0
6.90143	Θ_{6b}	$---+++$	$(-2,2)$	$(5,5)$	0
6.90144	Θ_{6b}	$--+--+$	$(0,4)$	$(12,1)$	0
6.90145	Θ_{6b}	$--+-++$	$(-2,2)$	$(4,4)$	0
6.90146	Θ_{6b}	$-++-+-$	$(-2,2)$	$(3,3)$	0
6.90147	Θ_{6c}	$------$	$(0,4)$	$(1,8)$	-4
6.90148	Θ_{6c}	$-----+$	$(0,4)$	$(3,4)$	-2
6.90149	Θ_{6c}	$----+-$	$(0,4)$	$(3,4)$	-2
6.90150	Θ_{6c}	$----++$	$(0,2)$	$(5,12)$	-2
6.90151	Θ_{6c}	$---+--$	$(0,4)$	$(3,2)$	-2
6.90152	Θ_{6c}	$---+-+$	$(0,4)$	$(8,1)$	0
6.90153	Θ_{6c}	$---++-$	$(0,4)$	$(8,1)$	0
6.90154	Θ_{6c}	$---+++$	$(0,2)$	$(13,4)$	0
6.90155	Θ_{6c}	$--+---$	$(0,2)$	$(5,2)$	-2
6.90156	Θ_{6c}	$--+--+$	$(0,4)$	$(1,1)$	0
6.90157	Θ_{6c}	$--+-+-$	$(-2,4)$	$(1,1)$	0
6.90158	Θ_{6c}	$--+-++$	$(-2,2)$	$(4,2)$	0
6.90159	Θ_{6c}	$--++--$	$(-2,2)$	$(7,1)$	0
6.90160	Θ_{6c}	$--++-+$	$(-2,2)$	$(4,4)$	0
6.90161	Θ_{6c}	$-+-+--$	$(-2,2)$	$(9,1)$	0
6.90162	Θ_{6c}	$-+-+-+$	$(-2,2)$	$(5,3)$	0
6.90163	Θ_{6c}	$-+-++-$	$(-2,2)$	$(5,3)$	0
6.90164	Θ_{6c}	$-+-+++$	$(-2,2)$	$(1,5)$	0
6.90165	Θ_{6c}	$-++---$	$(0,2)$	$(11,6)$	0
6.90166	Θ_{6c}	$-++--+$	$(-2,2)$	$(3,3)$	0
6.90167	Θ_{6d}	$------$	$(2,4)$	$(4,9)$	-4
6.90168	Θ_{6d}	$-----+$	$(2,4)$	$(10,3)$	-2
6.90169	Θ_{6d}	$----+-$	$(0,4)$	$(2,3)$	-2
6.90170	Θ_{6d}	$----++$	$(0,2)$	$(5,8)$	-2
6.90171	Θ_{6d}	$---++-$	$(0,2)$	$(5,4)$	0
6.90172	Θ_{6d}	$---+++$	$(0,0)$	$(13,13)$	0
6.90173	Θ_{6d}	$--+--+$	$(0,4)$	$(6,1)$	0
6.90174	Θ_{6d}	$--+-+-$	$(-2,4)$	$(1,1)$	0
6.90175	Θ_{6d}	$--+-++$	$(-2,2)$	$(3,3)$	0
6.90176	Θ_{6d}	$--++--$	$(-2,4)$	$(1,1)$	0
6.90177	Θ_{6d}	$--++-+$	$(-2,2)$	$(3,3)$	0
6.90178	Θ_{6d}	$--+++-$	$(-2,2)$	$(2,4)$	0
6.90179	Θ_{6d}	$-+---+$	$(0,4)$	$(6,1)$	0
6.90180	Θ_{6d}	$-+--+-$	$(-2,4)$	$(1,1)$	0
6.90181	Θ_{6d}	$-+-+-+$	$(-2,2)$	$(3,3)$	0
6.90182	Θ_{6d}	$-+-++-$	$(-2,2)$	$(2,4)$	0
6.90183	Θ_{6d}	$-++-++$	$(-2,2)$	$(2,2)$	0
6.90184	Θ_{6d}	$-++++-$	$(-4,0)$	$(1,4)$	2
6.90185	Θ_{6e}	$------$	$(0,4)$	$(1,9)$	-4
6.90186	Θ_{6e}	$-----+$	$(0,4)$	$(5,3)$	-2

Virtual Knot	Alternating Pattern	Sign Sequence	Signatures $(\sigma^*_\xi, \sigma_\xi)$	Determinants $(\det^*_\xi, \det_\xi)$	Rasmussen Invariant
6.90187	Θ_{6e}	$- - - - +-$	$(0,4)$	$(3,3)$	-2
6.90188	Θ_{6e}	$- - - - ++$	$(-2,4)$	$(1,1)$	0
6.90189	Θ_{6e}	$- - - + -+$	$(-2,2)$	$(8,1)$	0
6.90190	Θ_{6e}	$- - - + ++$	$(-2,2)$	$(4,5)$	0
6.90191	Θ_{6e}	$- - + - +-$	$(0,2)$	$(8,11)$	0
6.90192	Θ_{6e}	$- - + + --$	$(0,4)$	$(11,1)$	0
6.90193	Θ_{6e}	$- - + + -+$	$(-2,2)$	$(3,4)$	0
6.90194	Θ_{6e}	$- + + - +-$	$(0,2)$	$(16,3)$	0
6.90195	Θ_{6f}	$- - - - --$	$(2,4)$	$(3,8)$	-4
6.90196	Θ_{6f}	$- - - - -+$	$(0,4)$	$(2,2)$	-2
6.90197	Θ_{6f}	$- - - - +-$	$(0,4)$	$(2,2)$	-2
6.90198	Θ_{6f}	$- - - - ++$	$(-2,2)$	$(4,1)$	0
6.90199	Θ_{6f}	$- - - + -+$	$(-2,2)$	$(4,1)$	0
6.90200	Θ_{6f}	$- - - + ++$	$(-2,0)$	$(5,6)$	0
6.90201	Θ_{6f}	$- - + - --$	$(2,4)$	$(7,4)$	-2
6.90202	Θ_{6f}	$- - + - -+$	$(0,4)$	$(5,1)$	0
6.90203	Θ_{6f}	$- - + - +-$	$(0,4)$	$(5,1)$	0
6.90204	Θ_{6f}	$- - + - ++$	$(-2,2)$	$(2,3)$	0
6.90205	Θ_{6f}	$- - + + --$	$(0,4)$	$(5,1)$	0
6.90206	Θ_{6f}	$- - + + -+$	$(-2,2)$	$(2,3)$	0
6.90207	Θ_{6f}	$- - + + +-$	$(-2,2)$	$(2,3)$	0
6.90208	Θ_{6f}	$- - + + ++$	$(-4,0)$	$(1,3)$	2
6.90209	Θ_{6f}	$- + + - --$	$(2,2)$	$(11,11)$	-2
6.90210	Θ_{6f}	$- + + - -+$	$(0,2)$	$(8,3)$	0
6.90211	Θ_{6f}	$- + + - ++$	$(-2,0)$	$(5,6)$	0
6.90212	Θ_{6f}	$- + + + -+$	$(-2,0)$	$(5,6)$	0
6.90213	Θ_{6f}	$- + + + ++$	$(-4,-2)$	$(2,9)$	2
6.90214	Θ_{6g}	$- - - - --$	$(0,2)$	$(1,8)$	-2
6.90215	Θ_{6g}	$- - - - -+$	$(0,2)$	$(3,6)$	-2
6.90216	Θ_{6g}	$- - - - +-$	$(0,2)$	$(3,6)$	-2
6.90217	Θ_{6g}	$- - - - ++$	$(0,2)$	$(5,4)$	0
6.90218	Θ_{6g}	$- - - + -+$	$(0,2)$	$(5,4)$	0
6.90219	Θ_{6g}	$- - - + +-$	$(0,2)$	$(5,4)$	0
6.90220	Θ_{6g}	$- - - + ++$	$(0,2)$	$(7,2)$	0
6.90221	Θ_{6g}	$- - + - --$	$(0,2)$	$(5,4)$	0
6.90222	Θ_{6g}	$- - + - -+$	$(-2,2)$	$(3,1)$	0
6.90223	Θ_{6g}	$- - + - +-$	$(-2,2)$	$(3,1)$	0
6.90224	Θ_{6g}	$- - + - ++$	$(-2,2)$	$(2,2)$	0
6.90225	Θ_{6g}	$- - + + -+$	$(-2,2)$	$(2,2)$	0
6.90226	Θ_{6g}	$- - + + +-$	$(-2,2)$	$(2,2)$	0
6.90227	Θ_{6g}	$- + + - --$	$(0,0)$	$(9,9)$	0
6.90228	Θ_{6h}	$- - - - --$	$(0,2)$	$(1,12)$	-2
6.90229	Θ_{6h}	$- - - - -+$	$(0,2)$	$(5,8)$	-2
6.90230	Θ_{6h}	$- - - - ++$	$(-2,2)$	$(5,1)$	0
6.90231	Θ_{6h}	$- - - + -+$	$(-2,2)$	$(5,1)$	0
6.90232	Θ_{6h}	$- - - + +-$	$(0,2)$	$(9,4)$	0
6.90233	Θ_{6h}	$- - - + ++$	$(-2,2)$	$(2,2)$	0
6.90234	Θ_{6h}	$- - + - ++$	$(-2,2)$	$(3,3)$	0
6.90235	Θ_{6h}	$- - + + -+$	$(-2,2)$	$(3,3)$	0

Table B.1: Alternating knots and their invariants.

Appendix C

Table of Jones Polynomials

The computations of the Jones polynomial were performed in Matlab with a program written by Lindsay White.

Knot	Jones Polynomial	Knot	Jones Polynomial
3.6	$1/t + 1/t^3 - 1/t^4$	6.89188	$1/t + 1/t^3 - 1/t^4$
3.7	1	6.89189	$1/t + 1/t^3 - 1/t^4$
4.105	$1/t + 1/t^3 - 1/t^4$	**6.89198**	$-t^3 + t^2 - t + 3 - 1/t + 1/t^2 - 1/t^3$
4.106	1	6.90101	1
4.107	1	6.90102	1
4.108	$t^2 - t + 1 - 1/t + 1/t^2$	6.90103	1
5.2426	$1/t^2 + 1/t^3 - 1/t^5$	6.90104	1
5.2427	$1/t + 1/t^3 - 1/t^4$	6.90105	1
5.2428	$-t + 2 + 1/t^2 - 1/t^3$	6.90106	1
5.2429	$2/t - 1/t^2 + 1/t^3 - 2/t^4 + 1/t^5$	6.90107	1
5.2430	1	6.90108	1
5.2431	1	6.90109	$1/t^2 + 1/t^3 - 1/t^5$
5.2432	1	6.90110	$2/t - 2/t^2 + 2/t^3 - 2/t^4 + 2/t^5 - 1/t^6$
5.2433	$1/t^2 + 1/t^3 - 1/t^5$	6.90111	$1/t + 1/t^3 - 1/t^4$
5.2434	$1/t + 1/t^3 - 1/t^4$	6.90112	$1/t + 1/t^3 - 1/t^4$
5.2435	$-t + 2 + 1/t^2 - 1/t^3$	6.90113	$1/t + 1/t^3 - 1/t^4$
5.2436	1	6.90114	$1/t + 1/t^3 - 1/t^4$
5.2437	$1/t - 1/t^2 + 2/t^3 - 1/t^4 + 1/t^5 - 1/t^6$	6.90115	$t - 1 + 1/t - 1/t^2 + 2/t^3 - 1/t^4$
5.2438	$1/t + 1/t^3 - 1/t^4$	6.90116	$-t + 2 + 1/t^2 - 1/t^3$
5.2439	1	6.90117	$2/t - 1/t^2 + 1/t^3 - 2/t^4 + 1/t^5$
5.2440	1	6.90118	$t - 1/t^2 + 1/t^3$
5.2441	1	6.90119	1
5.2442	1	6.90120	1
5.2443	$t^2 - t + 1 - 1/t + 1/t^2$	6.90121	1
5.2444	1	6.90122	1
5.2445	1	6.90123	$t^2 - 2t + 2 - 1/t + 2/t^2 - 1/t^3$
5.2446	$1/t^2 + 1/t^4 - 1/t^5 + 1/t^6 - 1/t^7$	6.90124	$-t^3 + t^2 + 2 - 1/t$
5.2447	1	6.90125	$1/t + 1/t^3 - 1/t^4$
5.2448	1	6.90126	1
6.89187	$1/t^2 + 2/t^4 - 2/t^5 + 1/t^6 - 2/t^7 + 1/t^8$	6.90127	1

Knot	Jones Polynomial	Knot	Jones Polynomial
6.90128	1	6.90182	1
6.90129	1	6.90183	$t^2 - t + 1 - 1/t + 1/t^2$
6.90130	1	6.90184	$-t^4 + t^3 + t$
6.90131	$t^2 - t + 1 - 1/t + 1/t^2$	6.90185	$1/t^2 + 1/t^3 - 1/t^5$
6.90132	1	6.90186	$1/t + 1/t^3 - 1/t^4$
6.90133	1	6.90187	$2/t - 1/t^2 + 1/t^3 - 2/t^4 + 1/t^5$
6.90134	1	6.90188	1
6.90135	1	6.90189	1
6.90136	1	6.90190	1
6.90137	1	6.90191	$t - 1/t^2 + 1/t^3$
6.90138	$-t^4 + t^3 + t$	6.90192	$2 - 1/t - 1/t^3 + 1/t^4$
6.90139	$1/t^2 + 2/t^3 - 1/t^4 - 1/t^5 - 1/t^6 + 1/t^7$	6.90193	$t^2 - t + 1 - 1/t + 1/t^2$
6.90140	$2/t - 1/t^2 + 1/t^3 - 2/t^4 + 1/t^5$	6.90194	$2t^2 - 3t + 2 - 2/t + 3/t^2 - 1/t^3$
6.90141	1	6.90195	$1/t^2 + 1/t^4 - 1/t^5 + 1/t^6 - 1/t^7$
6.90142	$t - 1/t^2 + 1/t^3$	6.90196	$1/t + 1/t^3 - 1/t^4$
6.90143	1	6.90197	$1/t + 1/t^3 - 1/t^4$
6.90144	$2 - 1/t - 1/t^3 + 1/t^4$	6.90198	1
6.90145	$t^2 - t + 1 - 1/t + 1/t^2$	6.90199	1
6.90146	$2t^2 - 2t + 1 - 2/t + 2/t^2$	6.90200	1
6.90147	$1/t^2 + 1/t^3 - 1/t^5$	6.90201	$1/t + 1/t^3 - 1/t^4$
6.90148	$1/t + 1/t^3 - 1/t^4$	6.90202	1
6.90149	$1/t + 1/t^3 - 1/t^4$	6.90203	1
6.90150	$t - 1 + 1/t - 1/t^2 + 2/t^3 - 1/t^4$	6.90204	1
6.90151	$2/t - 1/t^2 + 1/t^3 - 2/t^4 + 1/t^5$	6.90205	1
6.90152	1	6.90206	1
6.90153	1	6.90207	1
6.90154	$t^2 - 2t + 2 - 1/t + 2/t^2 - 1/t^3$	6.90208	$-t^4 + t^3 + t$
6.90155	$1/t + 1/t^3 - 1/t^4$	**6.90209**	$t - 1 + 2/t - 2/t^2 + 2/t^3 - 2/t^4 + 1/t^5$
6.90156	1	6.90210	$t^2 - t + 1 - 1/t + 1/t^2$
6.90157	1	6.90211	1
6.90158	$t^2 - t + 1 - 1/t + 1/t^2$	6.90212	1
6.90159	1	6.90213	$-t^6 + t^5 - t^4 + 2t^3 - t^2 + t$
6.90160	1	6.90214	$1/t - 1/t^2 + 2/t^3 - 1/t^4 + 1/t^5 - 1/t^6$
6.90161	$t - 1/t^2 + 1/t^3$	6.90215	$1/t + 1/t^3 - 1/t^4$
6.90162	1	6.90216	$1/t + 1/t^3 - 1/t^4$
6.90163	1	6.90217	1
6.90164	$-t^3 + t^2 + 2 - 1/t$	6.90218	1
6.90165	$-1/t - 1/t^3 + 1/t^4$	6.90219	1
6.90166	$t^2 - t + 1 - 1/t + 1/t^2$	6.90220	$t^2 - t + 1 - 1/t + 1/t^2$
6.90167	$1/t^2 + 1/t^4 - 1/t^5 + 1/t^6 - 1/t^7$	6.90221	1
6.90168	$1/t - 1/t^2 + 2/t^3 - 1/t^4 + 1/t^5 - 1/t^6$	6.90222	1
6.90169	$1/t + 1/t^3 - 1/t^4$	6.90223	1
6.90170	$1/t + 1/t^3 - 1/t^4$	6.90224	1
6.90171	1	6.90225	1
6.90172	$-t^3 + 2t^2 - 2t + 3 - 2/t + 2/t^2 - 1/t^3$	6.90226	1
6.90173	1	**6.90227**	$t^2 - t + 2 - 2/t + 1/t^2 - 1/t^3 + 1/t^4$
6.90174	1	6.90228	$1/t - 2/t^2 + 3/t^3 - 1/t^4 + 2/t^5 - 2/t^6$
6.90175	1	6.90229	$1/t + 1/t^3 - 1/t^4$
6.90176	1	6.90230	1
6.90177	1	6.90231	1
6.90178	1	6.90232	$2 - 1/t - 1/t^3 + 1/t^4$
6.90179	1	6.90233	$t^2 - t + 1 - 1/t + 1/t^2$
6.90180	1	6.90234	1
6.90181	1	6.90235	1

Table C.1: Alternating virtual knots and their Jones polynomial.

Bibliography

[AFLT02] Colin Adams, Thomas Fleming, Michael Levin, and Ari M. Turner. Crossing number of alternating knots in $S \times I$. *Pacific J. Math.*, 203(1):1–22, 2002.

[And95] Peter Andersson. The color invariant for knots and links. *Amer. Math. Monthly*, 102(5):442–448, 1995.

[Ban30] Carl Bankwitz. über die Torsionszahlen der alternierenden Knoten. *Math. Ann.*, 103(1):145–161, 1930.

[BBRS01] Paul N. Balister, Béla Bollobás, Oliver M. Riordan, and Alexander D. Scott. Alternating knot diagrams, Euler circuits and the interlace polynomial. *European J. Combin.*, 22(1):1–4, 2001.

[BCG17a] Hans U. Boden, Micah Chrisman, and Robin Gaudreau. Signature and concordance of virtual knots, 2017. ArXiv/1708.08090, submitted.

[BCG17b] Hans U. Boden, Micah Chrisman, and Robin Gaudreau. Virtual knot cobordism and bounding the slice genus, 2017. ArXiv/1708.05982, published online 02 Feb 2018 in Experiment. Math.

[BDG+15] Hans U. Boden, Emily Dies, Anne Isabel Gaudreau, Adam Gerlings, Eric Harper, and Andrew J. Nicas. Alexander invariants for virtual knots. *J. Knot Theory Ramifications*, 24(3):1550009, 62, 2015.

[BGH+17] Hans U. Boden, Robin Gaudreau, Eric Harper, Andrew J. Nicas, and Lindsay White. Virtual knot groups and almost classical knots. *Fund. Math.*, 238(2):101–142, 2017.

[BM54] Raoul H. Bott and John P. Mayberry. Matrices and trees. In *Economic activity analysis*, pages 391–400. John Wiley and Sons, Inc., New York; Chapman and Hall, Ltd., London, 1954. Edited by Oskar Morgenstern.

[BN02] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Algebr. Geom. Topol.*, 2:337–370, 2002.

[BN17] Hans U. Boden and Matthias Nagel. Concordance group of virtual knots. *Proc. Amer. Math. Soc.*, 145(12):5451–5461, 2017.

[BNW18] Hans U. Boden, Andrew J. Nicas, and Lindsay White. Alexander invariants of periodic virtual knots. *Dissertationes Mathematicae*, pages 1–59, 2018.

[Bol98] Béla Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[BZH14] Gerhard Burde, Heiner Zieschang, and Michael Heusener. *Knots*, volume 5 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, extended edition, 2014.

[Che15] Zhiyun Cheng. Kauffman-Harary conjecture for alternating virtual knots. *J. Knot Theory Ramifications*, 24(8):1550046, 13, 2015.

[Cro59] Richard Crowell. Genus of alternating link types. *Ann. of Math. (2)*, 69:258–275, 1959.

[DKK17a] Heather A. Dye, Aaron Kaestner, and Louis H. Kauffman. Khovanov homology, Lee homology and a Rasmussen invariant for virtual knots. *J. Knot Theory Ramifications*, 26(3):1741001, 57, 2017.

[DKK17b] Heather A. Dye, Aaron Kaestner, and Louis H. Kauffman. Khovanov homology, Lee homology and a Rasmussen invariant for virtual knots. *J. Knot Theory Ramifications*, 26(3):1741001, 57, 2017.

[Dye16] Heather A. Dye. *An invitation to knot theory, virtual and classical*. CRC Press, Boca Raton, FL, 2016.

[FIKM14] Roger Fenn, Denis P. Ilyutko, Louis H. Kauffman, and Vassily O. Manturov. Unsolved problems in virtual knot theory and combinatorial knot theory. In *Knots in Poland III. Part III*, volume 103 of *Banach Center Publ.*, pages 9–61. Polish Acad. Sci. Inst. Math., Warsaw, 2014.

[Gil82] Cole A. Giller. A family of links and the Conway calculus. *Trans. Amer. Math. Soc.*, 270(1):75–109, 1982.

[GL78] Cameron McA. Gordon and Richard A. Litherland. On the signature of a link. *Invent. Math.*, 47(1):53–69, 1978.

[GPV00] Mikhael Goussarov, Michael Polyak, and Oleg Viro. Finite-type invariants of classical and virtual knots. *Topology*, 39(5):1045–1068, 2000.

[Gre04] Jeremy Green. A table of virtual knots, 2004. www.math.toronto.edu/drorbn/Students/GreenJ.

[Gre17] Joshua Evan Greene. Alternating links and definite surfaces. *Duke Math. J.*, 166(11):2133–2151, 2017. With an appendix by András Juhász and Marc Lackenby.

[Hat76] Allen E. Hatcher. Homeomorphisms of sufficiently large P^2-irreducible 3-manifolds. *Topology*, 15(4):343–347, 1976.

[Hat83] Allen E. Hatcher. A proof of the Smale conjecture, $\mathrm{Diff}(S^3) \simeq \mathrm{O}(4)$. *Ann. of Math. (2)*, 117(3):553–607, 1983.

[How17] Joshua A. Howie. A characterisation of alternating knot exteriors. *Geom. Topol.*, 21(4):2353–2371, 2017.

[HTW98] Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks. The first 1,701,936 knots. *Math. Intelligencer*, 20(4):33–48, 1998.

[ILL10] Young Ho Im, Kyeonghui Lee, and Sang Youl Lee. Signature, nullity and determinant of checkerboard colorable virtual links. *J. Knot Theory Ramifications*, 19(8):1093–1114, 2010.

[IMN14] Denis Petrovich Ilyutko, Vassily Olegovich Manturov, and Igor Mikhailovich Nikonov. Virtual knot invariants arising from parities. In *Knots in Poland. III. Part 1*, volume 100 of *Banach Center Publ.*, pages 99–130. Polish Acad. Sci. Inst. Math., Warsaw, 2014.

[Jon85] Vaughan F. R. Jones. A polynomial invariant for knots via von Neumann algebras. *Bull. Amer. Math. Soc. (N.S.)*, 12(1):103–111, 1985.

[Kam02] Naoko Kamada. On the Jones polynomials of checkerboard colorable virtual links. *Osaka J. Math.*, 39(2):325–333, 2002.

[Kam04] Naoko Kamada. Span of the Jones polynomial of an alternating virtual link. *Algebr. Geom. Topol.*, 4:1083–1101, 2004.

[Kam17] Naoko Kamada. Converting virtual link diagrams to normal ones. *Topology Appl.*, 230:161–171, 2017.

[Kau87] Louis H. Kauffman. State models and the Jones polynomial. *Topology*, 26(3):395–407, 1987.

[Kau99] Louis H. Kauffman. Virtual knot theory. *European J. Combin.*, 20(7):663–690, 1999.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[KK00] Naoko Kamada and Seiichi Kamada. Abstract link diagrams and virtual knots. *J. Knot Theory Ramifications*, 9(1):93–106, 2000.

[KM11] Peter B. Kronheimer and Tomasz S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.

[KNS02] Naoko Kamada, Shigekazu Nakabo, and Shin Satoh. A virtualized skein relation for Jones polynomials. *Illinois J. Math.*, 46(2):467–475, 2002.

[Kup03] Greg Kuperberg. What is a virtual link? *Algebr. Geom. Topol.*, 3:587–591 (electronic), 2003.

[Lee05] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197(2):554–586, 2005.

[Lic97] W. B. Raymond Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[Liv04] Charles Livingston. Computations of the Ozsváth-Szabó knot concordance invariant. *Geom. Topol.*, 8:735–742, 2004.

[Man04] Vassily O. Manturov. The Khovanov polynomial for virtual knots. *Dokl. Akad. Nauk*, 398(1):15–18, 2004.

[Man07] Vassily O. Manturov. Khovanov's homology for virtual knots with arbitrary coefficients. *Izv. Ross. Akad. Nauk Ser. Mat.*, 71(5):111–148, 2007.

[Man10] Vassily O. Manturov. Parity in knot theory. *Mat. Sb.*, 201(5):65–110, 2010.

[Man13] Vassily Olegovich Manturov. Parity and projection from virtual knots to classical knots. *J. Knot Theory Ramifications*, 22(9):1350044, 20, 2013.

[Men84] William Menasco. Closed incompressible surfaces in alternating knot and link complements. *Topology*, 23(1):37–44, 1984.

[MT91a] William Menasco and Morwen Thistlethwaite. A geometric proof that alternating knots are nontrivial. *Math. Proc. Cambridge Philos. Soc.*, 109(3):425–431, 1991.

[MT91b] William W. Menasco and Morwen B. Thistlethwaite. The Tait flyping conjecture. *Bull. Amer. Math. Soc. (N.S.)*, 25(2):403–412, 1991.

[MT93] William Menasco and Morwen Thistlethwaite. The classification of alternating links. *Ann. of Math. (2)*, 138(1):113–171, 1993.

[Mur58] Kunio Murasugi. On the genus of the alternating knot. I, II. *J. Math. Soc. Japan*, 10:94–105, 235–248, 1958.

[Mur65] Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[Mur87a] Kunio Murasugi. Jones polynomials and classical conjectures in knot theory. *Topology*, 26(2):187–194, 1987.

[Mur87b] Kunio Murasugi. Jones polynomials and classical conjectures in knot theory. II. *Math. Proc. Cambridge Philos. Soc.*, 102(2):317–318, 1987.

[Mur89] Kunio Murasugi. On invariants of graphs with applications to knot theory. *Trans. Amer. Math. Soc.*, 314(1):1–49, 1989.

[Mur96] Kunio Murasugi. *Knot theory and its applications.* Birkhäuser Boston, Inc., Boston, MA, 1996. Translated from the 1993 Japanese original by Bohdan Kurpita.

[NNST12] Takuji Nakamura, Yasutaka Nakanishi, Shin Satoh, and Yumi Tomiyama. Twin groups of virtual 2-bridge knots and almost classical knots. *J. Knot Theory Ramifications*, 21(10):1250095, 18, 2012.

[OR] John J. O'Connor and Edmund F. Robertson. Peter guthrie tait. www-history.mcs.st-andrews.ac.uk/Biographies/Tait.html.

[Pol10] Michael Polyak. Minimal generating sets of Reidemeister moves. *Quantum Topol.*, 1(4):399–411, 2010.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[RF04] Stuart Rankin and Ortho Flint. Enumerating the prime alternating links. *J. Knot Theory Ramifications*, 13(1):151–173, 2004.

[RF06] Stuart Rankin and Ortho Flint. *Knotilus*, a computer package for enumerating alternating knots and links, 2006. www.knotilus.math.uwo.ca.

[RFS04a] Stuart Rankin, Ortho Flint, and John Schermann. Enumerating the prime alternating knots. I. *J. Knot Theory Ramifications*, 13(1):57–100, 2004.

[RFS04b] Stuart Rankin, Ortho Flint, and John Schermann. Enumerating the prime alternating knots. II. *J. Knot Theory Ramifications*, 13(1):101–149, 2004.

[Rus17] William Rushworth. Doubled Khovanov homology, 2017. ArXiv/1704.07324.

[Thi87] Morwen B. Thistlethwaite. A spanning tree expansion of the Jones polynomial. *Topology*, 26(3):297–309, 1987.

[Thi88] Morwen B. Thistlethwaite. Kauffman's polynomial and alternating links. *Topology*, 27(3):311–318, 1988.

[Tub14] Daniel Tubbenhauer. Virtual Khovanov homology using cobordisms. *J. Knot Theory Ramifications*, 23(9):1450046, 91, 2014.

[Tur87] Vladimir G. Turaev. A simple proof of the Murasugi and Kauffman theorems on alternating links. *Enseign. Math. (2)*, 33(3-4):203–225, 1987.

[Tur17] Paul Turner. Five lectures on Khovanov homology. *J. Knot Theory Ramifications*, 26(3):1741009, 41, 2017.

[Tut46] William T. Tutte. On Hamiltonian circuits. *J. London Math. Soc.*, 21:98–101, 1946.

[vAEdB51] T. van Aardenne-Ehrenfest and Nicolaas G. de Bruijn. Circuits and trees in oriented linear graphs. *Simon Stevin*, 28:203–217, 1951.

[Wal68] Friedhelm Waldhausen. On irreducible 3-manifolds which are sufficiently large. *Ann. of Math. (2)*, 87:56–88, 1968.

[ZJ06] Paul Zinn-Justin. Alternating virtual link database, 2006. https://www.lpthe.jussieu.fr/ pzinn/virtlinks/.

[ZJZ04] Paul Zinn-Justin and Jean-Bernard Zuber. Matrix integrals and the generation and counting of virtual tangles and links. *J. Knot Theory Ramifications*, 13(3):325–355, 2004.

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